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# Group theory and constellations

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Publibook, Paris, 2004



Éditions Publibook,
14, rue des Volontaires
75015 Paris – France
Tél : 33 (0)1 53 69 65 55

IDDN.FR.010.0101792.000.R.P.2003.035.40000

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# Preface

The name of constellation has been suggested to me by my friend and colleague Alexander Grossmann to denote a concept I introduced in 1974, ignoring at that time that Ettore Majorana was a precursor; in fact, in 1932, this great physicist published an article in the *Nuovo Cimento* [22] where he studied the evolution of the spin of a particle in a homogeneous magnetic field. He showed that, for a particle of spin $s$, the spin state was described by a set of $2s$ (not necessarily distinct) points on a sphere. Grossmann noted that such a set reminds the notion of a constellation of stars in the celestial sphere since it takes into account the possibility of stars to be in coincidence (double or multiple stars). Majorana demonstrated, in his paper, that the state evolutes in precessing regularly around a diameter parallel to the magnetic field. Constellations are now well-defined geometrical objects with nice properties.

My point of view was slightly different of Majorana's one; the problem I wanted to solve was to derive a classification of the states of a particle of spin $s$ according to the symmetry of the constellation they form, with respect to the spin rotation group. The states which have the maximal symmetry are those which are eigenstates of a component of the spin operator ($S_z$, for instance), with an eigenvalue $m$. They have an $SO(2, \mathbb{R})$ symmetry except when $m = 0$ (with $s$ an integer). In that case this symmetry is slightly higher; it becomes $O(2, \mathbb{R})$. All the other states have a finite symmetry described by some subgroup of the rotation group, the generic state having no symmetry at all. That means that its symmetry group is trivial: it reduces to the identity transformation.

Independently, in 1976, Ronald Shaw used the constellation concept in order to give a nice derivation of Petrov's classification of Einstein spacetimes and a description of the Wigner $3j$-symbols. He used, for that, a spinor description of states.

Later on, I discovered new and various applications of constellations: I made evident their link with the spin coherent states, with the electromagnetic field, with the generators of the Lorentz group, with the minimum uncertainty states for angular momentum, with the Clebsch-Gordan series of the rotation group. Before this abundance of matter and, thanks to Joshua Zak, I had the opportunity to teach constellations at the Technion in Haifa and I became tempted to devote a whole book on the subject. I must add that, in Haifa, I learnt, from Asher Peres, that the "Majorana representation" was also used by Roger Penrose in the context of the Bell inequalities.

I must say a few words about the precise way everything started for me. I was teaching quantum mechanics to undergraduate students, and in order to present the axioms of this theory, I decided to treat abstractly the case of a two-dimensional Hilbert space. Obviously, in discarding the phase of the vectors, I arrived at the sphere $S_2$ as the space of states. This sphere has a simple physical geometrical meaning when the Hilbert space is the one of the electron spin states. In other cases, usually referred to as quasi spin states, the sphere is only an abstract one. Then the following question came to my mind: does exist a simple geometrical way to describe the set of states associated with a Hilbert space of any dimension, for instance the set of states of any spin? I succeeded and I arrived at a classification of the (pure) states associated with an arbitrary spin $s$. Progressively, I became aware of the fact that constellations were not necessarily related with the only rotation group. They can be associated with the Lorentz group and other larger Lie groups, the projective orthogonal and symplectic complex groups in an arbitrary dimension. In all these cases, the sphere $S_2$ plays a particular role and that explains why mathematical physicists are referring to various spheres, namely the Riemann sphere, the celestial sphere, the Poincaré sphere, the sphere of spin electron states and the Bloch sphere. Behind that, Klein's ideas on geometry became visible, but from that point of view, there is only one sphere. This proves that mathematical physics must be distinguished from concrete physics. In an analogous way, from an abstract point of view there exists only one three-dimensional rotation group. However, mathematically, one must distinguish in the Euclidean group the rotation groups around different points and the class of all those subgroups. Klein's ideas are also needed in spinor theory. Spinors do not define a given geometry. In order to define a geometry, one must precise which group is acting on the set of spinors. It could be the $SU(2)$ group or the $SL(2, \mathbb{C})$ group or some other group. Even on the set of ray-spinors, we could make the rotation or the Lorentz group acting, defining in this way two distinct geometries.

As I learnt from René Thom, all manifolds of dimension two have a nice property in common, which explains the interest of constellations defined on the sphere $S_2$. Given a manifold $M$, one may consider the symmetrized product of $n$ copies of $M$. The necessary and sufficient condition for the product to be a manifold is that $M$ is of dimension two. This was an invitation for looking for generalizations of constellations to other manifolds than the sphere $S_2$, namely, the real plane, the projective real plane and the torus. Some informations are given about that in the present book.

# Chapter 1

# Group theory and geometry

Historically, the link between group theory and geometry was emphasized in the famous Erlangen's program proposed in the XIXth century by the German mathematician Felix Klein. The presentation of this program is considered generally as an important event in the history of mathematics. However, although the program is often mentioned in geometry textbooks, its content is not really taken into account, if we except books on projective geometry. A possible explanation of this fact is that, nowadays, geometry became a much richer domain: since it includes differential geometry, with a lot of modern notions such that fiber bundles, connections, etc. Moreover, geometry is to-day nothing else than the "trivial part" of the noncommutative geometry, a concept introduced by the French mathematician Connes. However, for our purpose we only need Klein's ideas.

## 1.1 The old definition of a group

In 1872, which is the year where the Erlangen program was introduced[1], a group was not defined as an abstract structure, but as a set of "transformations" (permutations) of some "space" (set) $M$ satisfying the following properties:

- the group contains the identity transformation $Id(M)$, which maps every element of $M$ on itself.
- if $g$ and $g'$ are transformations of the group, the transformation $g$ followed by the transformation $g'$ is also a transformation of the group denoted $g'.g$
- if $g$ is a transformation, its inverse $g^{-1}$, defined by $g^{-1}.g = g.g^{-1} = Id$, is also a transformation of the group.

As an example, the set $S(M)$ of all permutations of $M$ is a group. In fact, the first group explicitly studied as such in the mathematical literature was the group of permutations of the roots of an equation, the so-called Galois group. It permits the French mathematician Galois to say when a polynomial equation is solvable: it is when the Galois group is itself *solvable*, a term which will be explained later on. In the case where $M$ has 1, 2, 3, or 4 distinct elements, the group $S(M)$ is solvable. This term comes from the fact that equations of degrees one, two, three and four are said to be solvable.

[1]For a detailed version of the program, it is better to consult the French translation *Le programme d'Erlangen* [21]

In giving such an importance of group theory in geometry, Klein made real improvements. He permitted to classify the set of theorems and properties of Euclidean geometry according to their symmetry group, separating, for instance, the definition of a triangle from the definition of a right angled triangle. One of the main concepts he introduced was the one of subordinate geometry, which permitted him to discover the exact link between projective geometry and Euclidean geometry. It is well known that this last geometry is derived from projective geometry in supposing fixed the straightline (in the case of dimension two) at infinity. Klein insisted on the fact that such a derivation corresponds to a change of group. This new point of view makes clear that, in projective geometry, all straightlines are equivalent (there exists a projective transformation which maps a given straightline on another given straightline) and the choice of the one "at infinity" is not essential.

In the present introduction, we are not much concerned in history of mathematics, our next task is to present the Erlangen program in a modern language.

## 1.2 The modern definition of a group

A group $(G, .)$ is a set $G$ endowed with an inner binary composition law, denoted by a dot, and satisfying the following properties:

- the law is associative: $(g.g').g'' = g.(g'.g'')$
- there is a neutral element denoted by $e$, that is an element satisfying $e.g = g.e = g$ for any $g$
- each element g has an inverse belonging to the group and denoted $g^{-1}$. This inverse is such that $g.g^{-1} = g^{-1}.g = e$. This element is not necessarily distinct from $g$ itself. We note that $e$ is its own inverse. Any other element with this property is called an involution.

As it is easily checked, these axioms are direct consequences of the properties of transformations which were presented above. In particular, the set $S(M)$ has a group structure. More generally, any old fashioned group is a group.

From now on, we decide, for convenience, to denote by the letter $G$ the set as well as the group itself.

Although the modern approach makes a sharp distinction between a group and the way it acts on a space, we must recognize that, in mathematical literature, a group appears generally as a group which acts on some space $M$. It seems, at first sight, that if we give ourselves a (modern) group $G$ and a space on which it acts, the two definitions would coincide. This is not exactly true, essentially because, in the "action" of a group, there are perhaps many elements which act in the same way. If this is not the case, the group is said to act *effectively* We will examine this point later on.

If the group has a finite number of elements, this finite number is called the *order of the group* and the group is said to be a *finite group*.

A group is said to be Abelian[2] if any two elements $g$ and $h$ commute: $g.h = h.g$.

[2]From the name of the Norwegian mathematician Niels Henrik Abel.

## 1.3 Subgroup

Any subset $H$ of a group $G$ which has the structure of a group, *with respect to the same composition law*, is said to be a subgroup of $G$. As an example, if $N$ is a subset of $M$, $S(N)$ can be considered as a subgroup of $S(M)$, in supposing that each element of $M - N$ is invariant under $S(N)$. We write $H \leq G$ or $G \geq H$ to express that $H$ is a subgroup of $G$. If $H$ is distinct from $G$, we write $H < G$ or $G > H$.

A subgroup $H$ of $G$ is said to be *invariant* or *normal* if, for any $g \in G$, $g.H = H.g$. In that case, the equivalence relation in $G$

$$g \sim g', \quad \text{iff} \quad g.H = g'.H$$

has an interesting consequence: the set of equivalence classes form a group called the quotient group $G/H$. Note that $G$ is an invariant subgroup of $G$ itself. We note, in particular, that the quotient group $G/G$ has only one element.

We also note that any subgroup of an Abelian group is invariant.

Each group has two invariant subgroups called *improper subgroups*: first, the group $G$ itself, second, the trivial group $\{e\}$.

## 1.4 Homomorphism, isomorphism, automorphism

If there exists a mapping $\phi$ of a group $G$ into a group $G'$ which preserves the group law (*i.e.* such that the image of a product is the product of the images and the image of an inverse is the inverse of the image), $\phi$ is called a homomorphism. The kernel of a homomorphism (*i.e.* the set of elements which are mapped on the neutral element of $G'$) is a normal subgroup $Ker(\phi)$. The proof of that property is left to the reader. If $\phi$ is bijective, the homomorphism is called an isomorphism. The kernel of a isomorphism is the trivial subgroup $\{e\}$. In the general case, the image $\phi(G)$ is isomorphic to the quotient group $G/Ker(\phi)$.

By definition, an automorphism is an isomorphism of a group into itself.

**Remark** Usually, when one refers to a given group $G$, one means the class of all groups isomorphic to $G$. For instance, *the* rotation group denotes the class of groups isomorphic to $SO(3,\mathbb{R})$, the group of $3 \times 3$ orthogonal real matrices of determinant one, with the standard multiplication of matrices as an internal law. However, the physicist must be more prudent. He has, for instance, to make a distinction between 1) the rotation group around a given point in space, 2) the quotient group $E(3,\mathbb{R})/\mathrm{T}(3)$, where $E(3,\mathbb{R})$ denotes the real Euclidean group in three dimensions, and $T(3)$ denotes the three-dimensional translation group. As we already remarked, concrete physics is richer than mathematics.

**Example** The cyclic group $C_n$ of order $n$ is the group composed of plane rotations of angles $2\pi k/n$ , where $k = 0, 1, \ldots, n-1$. Let $\omega$ be a complex number such that $n$ is the lowest integer such $\omega^n = 1$. In that case, $\omega$ is called a $n^{th}$ primitive root of one. The set $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ has a group structure with respect to the multiplication law. It is isomorphic to the cyclic group $C_n$. Let $m$ be a divisor of $n$. The subset $\{1, \omega^m, \omega^{2m}, \ldots\}$ is a subgroup of $C_n$. This subgroup is isomorphic to the cyclic group $C_{n/m}$. Note that the cyclic groups are all Abelian.

## 1.5 Simple group, solvable group

We write $H \ll G$ to signify that $H$ is an invariant subgroup of $G$ which is not contained in another invariant subgroup of $G$, except $G$ itself. Suppose that we may write a *composition series*, that is a relation of the form:

$$\{e\} \ll H_1 \ll H_2 \ll \ldots \ll H_n \ll G. \tag{1.1}$$

The quotient groups $Q_i = H_{i+1}/H_i$ are known as the prime quotient groups. We note that a given group may have many distinct composition series. However, the set of $Q_i$'s does not depend on the composition series (Jordan and Hoelder theorem)[3].

A finite group $G$ is said to be *simple* if it possesses the (unique) composition series $\{e\} \ll G$. In other words, the only invariant subgroups of $G$ are $G$ itself and $\{e\}$. In particular, the cyclic group $C_p$, for $p$ prime, is simple.

A group $G$ is said to be *solvable* if all the $Q_i$ 's are cyclic.

## 1.6 Action of a group on a set

We say that a group $G$ acts on a set $M$ if there exists a homomorphism $\phi$ from $G$ in $S(M)$. We already said that the kernel $Ker(\phi)$ (the set of elements which are mapped on $Id(M)$) form a normal subgroup. Any element $k$ of $Ker(\phi)$ acts as $e$ and any element of type $kg$ or $gk$ acts as $g$.

If $e$ is the only element which is mapped on $Id(M)$, one says that $G$ acts on $M$ *effectively*. When it is not the case, one can verify that it is the quotient group $G/Ker(\phi)$ which acts effectively. As we shall see, the notion of effective action plays a fundamental role in relating the old and the modern definition of groups.

**Example** Consider the group $SL(2,\mathbb{C})$, that is the group of complex $2 \times 2$ matrices with determinant one[4]. One can make this group acting on the complete Cauchy plane, that is the Cauchy plane with the point at infinity.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \longrightarrow \frac{az+b}{cz+d} \tag{1.2}$$

We intend to show that this group does not act effectively. Indeed, there is an element of $SL(2,\mathbb{C})$, other than the unit matrix, which acts as the unit matrix. This element is such that $z = \frac{az+b}{cz+d}$, whatever is $z$. It is such that $c = 0$, $d = a$, and $b = 0$. One obtains, apart the unit matrix 1, the matrix -1. One verifies that these two elements form an invariant subgroup, denoted $Z_2$. Since the elements g and -g act the same way, the group which acts effectively is the quotient group $SL(2,\mathbb{C})/Z_2$.

Let us mention another definition. A group $G$ is said to act *freely* on a set $M$ if $M$ has no fixed point ($G$ is "free" to move any point of $M$). As an example, the reader will verify that the rotation group $SO(3,\mathbb{R})$ acts freely on the sphere. It means that given an arbitrary point on this sphere, there exists a rotation which moves it. This is not the case for the subgroup of plane rotations $SO(2,\mathbb{R})$ acting on the sphere. Two antipodal points are invariant.

[3]The analogy with the decomposition of natural integers into prime factors is evident.

[4]The letter $L$ stands for *linear*, the letter $S$ for *special* (determinant equal to one).

## 1.7 Action of a group on itself

The mathematicians consider the three following ways for a group to act on itself:

- left action: $g$ transforms $g'$ into $g.g'$
- right action: $g$ transforms $g'$ into $g'.g^{-1}$
- conjugation action: $g$ transforms $g'$ into $\phi(g)g' = g.g'.g^{-1}$.

The fact that the left and right actions define subgroups of the permutation group $S(G)$ is known, for finite groups, as the Cayley theorem. It is a simple matter to show that they are effective actions.[5]

Let us denote by $C(G)$ the set of elements of $G$ which commute with every element of $G$. This set is a subgroup called the *center* of $G$. The conjugation action is effective if and only if $C$ is trivial ($C = \{e\}$). If it is not the case, it is the quotient group $G/C$ which act effectively.[6]

## 1.8 Orbits and strata

A group acts *transitively* on $M$ if for any couple of points $x, y \in M$, there exists a transformation mapping $x$ on $y$. As examples, the rotation group acts transitively on the sphere and the Euclidean group acts transitively on the three-dimensional real space.

When a group acts transitively on a space $M$, this space is said to be a *homogeneous space* of the group.

We note that the action of a group on itself by left or right translations is transitive. It follows that a group is a homogeneous space of itself in two ways (the left and the right ones). The word *homogeneous* seems unappropriate in this case since there is a distinguished element, namely the neutral element. Section 1.16 will permit to clarify this apparent contradiction.

In a more general case, we have to define *orbits*. Two points $x$ and $y$ belong to the same *orbit* if there is a transformation which maps $x$ on $y$. Each orbit is a homogeneous space. The set of orbits form a partition of the space $M$ (the union of orbits is $M$ itself and the intersection of two arbitrary orbits is empty). When the action is transitive, $M$ is the unique orbit. For two points, to belong to a given orbit is an equivalence relation; an orbit is an equivalence class. One checks that homogeneity of a space $M$ means equivalence of all points of $M$.

**Example** The action of plane rotations around the point $O$ of $\mathbb{R}^2$ divides the real plane in orbits which are labelled by a positive number $R$, the radius of a given circular orbit ($R = 0$ corresponds to a trivial orbit $\{O\}$).

The action of $G$ on itself by conjugation is neither transitive since the element $e$ is always mapped on itself (fixed point). In this peculiar action, the orbits have a special name; they are called *conjugacy classes*. $\{e\}$ is the trivial conjugacy class. Two subsets or two elements which belong to the same orbit are said to be conjugate.

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[5] In those actions, there is no fixed point. The group acts *freely*.

[6] If $C = G$, the group $G$ is Abelian.

The set of elements which map a given point $x$ on itself is a subgroup $G_x$ called the *stability* subgroup of $x$. Other names are *stabilizer*, *isotropy subgroup* and *little group*.

Two points $x$ and $y$ which lie on a same orbit have conjugate stability subgroups. It means that there exists an element $g$ such that $g.G_x.g^{-1} = G_y$. The proof is easy. Since $x$ and $y$ lie on the same orbit, there exists an element $g$ of $G$ such that $\phi(g)x = y$. From $\phi(G_x)x = x$, one gets $\phi(g)\phi(G_x)\phi(g^{-1})y = y$.

By definition, two points which have conjugate stability subgroups lie on a same *stratum*.[7] The points of a given stratum are said to be of the same type. It is clear that a stratum is a union of orbits of the same type. In the last example of plane rotations, we have two strata: first, the point $O$, second the plane minus the point $O$. Every physicist knows another example, the one concerning the action of the connected Lorentz group on the four-momentum space. The strata are six in number:

- the zero four-momentum,
- the set of all future time-like four-momenta,
- the set of all past time-like four-momenta,
- the set of future light-like four-momenta,
- the set of all past light-like four-momenta,
- the set of all space-like four-momenta.

It is important to note that the action of the connected Poincaré group on this space leads to the same strata. This is due to the fact that translations do not act (the Poincaré group does not act effectively on the four-momentum space; it is its quotient by the translation subgroup which acts effectively; this quotient is isomorphic to the connected Lorentz group). Note also that the Poincaré group acts transitively on the Minkowski space-time. This is a good reason to conclude that, geometrically, the four-momentum space cannot be confused with the Minkowski space-time.

## 1.9 Geometries

The main idea of Erlangen's program can be stated in the following way. A geometry is a system $(G, M, \phi)$, where $\phi$ is an isomorphism from $G$ to some subgroup of $S(M)$. The group $G$ is supposed to act *effectively* and *transitively* on the set $M$. The set $M$ is called the geometrical space. Abusively, when $G$ does not act effectively (that is when $\phi$ has a non trivial kernel) one can say that the system $(G, M, \phi)$ still defines a geometry. However, usually this is an accepted definition only when there exists an orbit of $G$ which is dense in $M$. To be more precise, we will refer to an elementary geometry each time the group $G$ acts effectively and transitively on the set $M$.

Generally, for the sake of simplicity, the elements of $M$ are called *points*.

We now have two problems to examine. The first problem consists in determining when two systems define the same geometry, the second one in studying the possible relationships between distinct geometries.

[7]The concept of stratum is due to the French mathematician René Thom. It was popularized with the present meaning by the French physicist Louis Michel.

For the moment, we only give the answer to the first question. It is quite simple. Two geometries $(G, M, \phi)$ and $(G', M', \phi')$ are identical if and only if 1) $G$ and $G'$ are isomorphic, 2) the groups $S(M)$ and $S'(M')$ are isomorphic, 3) $\phi(G)$ and $\phi'(G')$ are conjugate subgroups of $S(M)$.

Let us examine briefly some examples of geometries in physics. When we speak of the geometry of the Minkowski space, we have in mind either the Lorentz group as the group associated with it or the Poincaré group. These geometries are distinct. In the same way, the two-dimensional spinor space does not define a unique geometry. We have to know which group (the group $SU(2)$ or the group $SL(2,\mathbb{C})$) is concerned. Conversely, given a group, say the Lorentz group, many distinct geometries may be constructed, depending on the space on which the group is acting. It could be the Minkowski space, but also the four-momentum space, the two-dimensional or the four-dimensional spinor space, the celestial sphere, the set of all electromagnetic fields, etc. In all these examples the role of the group and the role of the space are clear. But what about $\phi$? Let us examine a case where $\phi$ must be necessarily fixed in order to know the geometry we are concerned. The $SL(2,\mathbb{C})$ group acts in the two following ways on a spinor space:

$$\Lambda \in SL(2,\mathbb{C}), \quad \psi \in \mathbb{C}^2, \quad \psi \to \Lambda\psi \ \text{ or } \ \psi \to \Lambda^{*-1}\psi. \tag{1.3}$$

Usually these geometries are said to apply on two kinds of spinors.

## 1.10 Figures

We adopt the standard following simple definition: *A figure is a subset of M.* However, there are objects which are sometimes called figures and which do not satisfy that definition; we can give, as examples, the case of an oriented circle and the case of a cube (as a set of eight vertices, twelve sides, and six faces) in Euclidean geometry. We have also in mind other objects, namely the constellations. We will group these objects and the ordinary figures under the name of *generalized figures*. The only requirement for an object to be a generalized figure is that we can make the group $G$ acting on it, the action being deduced from the action on $M$.

Two generalized figures $F$ and $F'$ are said to be congruent if there exists an element $g \in G$ which maps $F$ on $F'$. It is not difficult to check that the congruence relation is an equivalence relation. Let us give some simple examples of generalized figures.

- The simplest generalized figure after the point is a couple of two (*not necessarily distinct*) points (in a given order). The set of all couples is the Cartesian product $M^2 = M \times M$. If the group $G$ acts transitively on $M^2$, we say that the action of $G$ is 2-transitive on $M$. Similarly, we could define the set of triples (resp. quadruples, etc.) we would denote by $M^3 = M \times M \times M$ (resp. $M^4 = M \times M \times M \times M$, etc.) and define correspondingly the 3-transitive (resp. 4-transitive, etc.) action. As an example, one has the oriented triangles (including the degenerate ones) in plane Euclidean geometry.

- Another simple set of figures is the set $M{*}^2$ of pairs of (distinct) points. A pair is different from a couple in that its elements are distinct and not ordered. The two couples $(x, y)$ and $(y, x)$, where $x \neq y$, correspond to the unique pair $\{x, y\}$. There is no difficulty to introduce the sets $M{*}^n$ for $n > 2$. A figure of $M{*}^n$ is a subset of $n$ (distinct) points. We may mention, as an example, the set of nondegenerate[8] triangles in plane Euclidean geometry.

- The next set of generalized figures we want to introduce is not classical, but quite convenient. It is the set $C_2(M) = \underline{M \times M}$ of pairs of *not necessarily distinct* points. This set is called the set of *constellations of order two* on space $M$ or the *ciel*[9] of order two. It is a simple matter to generalize this definition and introduce the cieux of order three, four, etc., that is $C_3(M) = \underline{M \times M \times M}$, $C_4(M) = \underline{M \times M \times M \times M}$, etc. Obviously, the ciel of order one $C_1(M)$ can be identified with $M$ itself and the ciel of order zero $C_0(M)$ with the empty set. The choice of these words is induced by the idea of constellations of the celestial sphere (constellations may involve multiple stars). The celestial sphere, which will be defined later, is an example of a manifold satisfying an interesting property mentioned in the preface: it is a two-dimensional manifold and the necessary and sufficient condition for the cieux built on $M$ to be manifolds is that $M$ is of dimension two. That is why we will be interested in constellations on two-dimensional manifolds and, especially, on the sphere $S_2$.

According to what we have said about the Erlangen program, if we want to study the geometry of $S_2$, we are obliged to decide which group is acting transitively on it. A possible obvious choice is the rotation group but one could prefer a larger group, for instance one of the Lorentz groups. There also exist intermediate choices. We will see the relationship between all these geometries and the spinor geometries.

## 1.11 Simply related geometries

We introduce the three following definitions.

**Definition 1.1 (Derived geometry)** *Let $(G, M)$ be a geometry*[10]*, $F$ a generalized figure and $GF$ the congruence class of $F$. The geometry $(G, GF)$ will be called a* derived geometry *of the geometry $(G, M)$.*

As an example, if $G$ is the Euclidean group, $M$ the real 3-dimensional affine space, and $F$ a circle, the geometry $(G, GF)$ is the geometry where the space is the set of all circles of $M$.

**Definition 1.2 (Subordinate geometry)** *Let $(G, M)$ be a geometry, $H$ a subgroup of $G$. Let us consider the action of $H$ on $M$. If $H$ does not act transitively on $M$, the space $M$ splits into orbits of $H$ and every system $(H, M')$, where $M'$ is an orbit, is a new geometry. Such a geometry will be called a* subordinate geometry.

The most interesting case is when $M$ contains a dense orbit of $H$. Then we speak of a *strict* subordinate geometry .

**Example** $G$ is the three-dimensional rotation group and $M$ the ordinary sphere $S_2$. The action is effective since any nontrivial rotation moves at least one point. It is transitive

[8]Triangles with angles equal to 180° and 0° are not discarded.
[9]*Ciel* means *sky* in French (pronunciation: *sjɛl*). The plural of *ciel* is *cieux*.
[10]In order to simplify the text, we decide to omit the letter $\phi$ in the system defining a geometry.

since one can map any point on any other point with the aid of a rotation. The system $(G, S_2)$ defines a geometry. Consider the stability subgroup $H$ of a point, say the North pole (the Earth language is used for convenience). $H$ is the one-dimensional rotation group around the pole axis. The sphere $S_2$ is a union of orbits (the parallels). These parallels are circles except the North and the South poles which are just points. The system $(H, S_2)$ decomposes into subordinate geometries of two types: $(H, C)$ and $(H, P)$, where $H$ is the one-dimensional rotation group, $C$ a circle and $P$ a point.[11] Obviously, $(H, P)$ has no interest. The group $H$ acts effectively and freely on $C$.

**Example of a strict subordinate geometry** Consider the case studied in Section 1.6, namely the group $SL(2, \mathbb{C})$ acting on the complete Cauchy plane. The subgroup which leaves invariant the point at infinity is made of the transformations of the type:

$$z \longrightarrow az + b.$$

This defines a four-dimensional subgroup of $SL(2, \mathbb{C})$ which has the Cauchy plane as a homogeneous space. The Cauchy plane is dense in the complete Cauchy plane.

**Definition 1.3 (Subgeometry)** *A subordinate geometry is called a* subgeometry *when the corresponding subgroup $H$ acts transitively on $M$.*

**Example** $G$ is the proper real similitude group in two dimensions $S_+(2, \mathbb{R})$, that is the group of matrices of the form

$$\begin{pmatrix} c\cos\phi & -c\sin\phi & a \\ c\sin\phi & c\cos\phi & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } a, b, c \text{ real and } c > 0. \tag{1.4}$$

The space $M$ is the real plane $\mathbb{R}^2$. The action is described by the matrix multiplication:

$$\begin{pmatrix} c\cos\phi & -c\sin\phi & a \\ c\sin\phi & c\cos\phi & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} c(x\cos\phi - y\sin\phi) + a \\ c(x\sin\phi + y\cos\phi) + b \\ 1 \end{pmatrix} \tag{1.5}$$

It is a simple matter to check the effectiveness and the transitivity of the action. The proper Euclidean subgroup $H = E_+(2, \mathbb{R})$ is made of those matrices for which $c = 1$. Its action is also effective and transitive. We have two distinct geometries $(G, M)$ and $(H, M)$ with the same space $M$. The geometry $(H, M)$ is a subgeometry of $(G, M)$).

Where does the difference between those two geometries lie? Clearly not in the action on $M$ itself. Let us examine the action on figures in the general case. It is clear that if two figures are congruent under $H$, they are congruent under $G$. Generally, a class of $G$-congruent figures split into subclasses in the geometry $(H, M)$. In our example, let us consider the set of circles. The similitude group acts transitively on this set. That is why we can say that in this geometry there is *only one category of circles.* In Euclidean geometry, this set splits into subsets, each subset contains circles of *a given radius.*

[11] The two poles are *fixed points* in the action of $H$ on $M$.

## 1.12 A strict subordinate geometry

Let us give an alternative definition of a strict subordinate geometry. Consider a figure $F$ in the geometry $(G, M)$. Let $H$ be the stability subgroup of $F$. It could happen that $H$ acts transitively on the set $M' = M - F$, where $M'$ is dense in $M$. In that case, one says that $F$ is the absolute which permits to go from the geometry $(G, M)$ to the geometry $(H, M')$. This last geometry is said to be a *strict subordinate* of the geometry $(G, M)$.

Before giving a historical example of such a situation, let us give the definition of a geometrical *invariant*. It is a property of a figure, which is invariant under the group action. This implies that all congruent figures have this property in common. Care! it does not follow that an invariant characterizes the congruence class.

We intend to illustrate all the notions we have introduced with the classical example of the two-dimensional real projective geometry (the reader will generalize without difficulty the following discussion to the $n$-dimensional case). The space of this geometry is the projective real plane $P_2(\mathbb{R})$ and the group is $PGL(3, \mathbb{R})$. The elements of $P_2(\mathbb{R})$ are equivalence classes of elements of $\mathbb{R}^3 - \{(0,0,0)\}$ defined by the relation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} ax \\ ay \\ az \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.6}$$

where $a$ is any non zero real number. A convenient known notation for the equivalence class of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is $x : y : z$.

For almost all elements, one has $z \neq 0$ and the equivalence relation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} z^{-1}x \\ z^{-1}y \\ 1 \end{pmatrix}$$

defines a map of this set of elements on the plane $\mathbb{R}^2$. This proves that $P_2(\mathbb{R})$ is "a little bit more than" $\mathbb{R}^2$.

The elements of the group $PGL(3, \mathbb{R})$ are the non singular $3 \times 3$ real matrices with the equivalence relation

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \sim \begin{pmatrix} ag_{11} & ag_{12} & ag_{13} \\ ag_{21} & ag_{22} & ag_{23} \\ ag_{31} & ag_{32} & ag_{33} \end{pmatrix} \tag{1.7}$$

where $a$ is any non zero real number. The action of the group is the one defined by matrix multiplication (left action). The space $P_2(\mathbb{R})$ is a two-dimensional manifold and the group $PGL(3, \mathbb{R})$ is an eight-dimensional continuous group.

Let us consider some sets of figures. We start with the set of straightlines. A straightline is the set of points defined by an equation of the type $mx + ny + pz = 0$, with $m, n, p$ not all zero. One checks that this definition is compatible with the equivalence relation (1.2). There is an interesting fact. Since a straightline is defined by a triple $(m, n, p)$, i.e. a row matrix, and since $(m\ n\ p)$ and $(am\ an\ ap)$ define the same straightline, we see that the set of straightlines also defines a projective real plane, called the dual. The group

$PGL(3,\mathbb{R})$ acts on the dual on the right. This duality is a basic property of projective geometry.

The space $P_2(\mathbb{R})$ is a homogeneous space of $PGL(3,\mathbb{R})$ and its dual is also a homogeneous space. We know that the stabilizers of two elements of a homogeneous space are conjugate subgroups. Let us look for the stabilizer of the straightline (0 0 1). We have to solve the matrix equation

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

with $a \neq 0$. One obtains $g_{31} = g_{32} = 0$ and $g_{33} \neq 0$. The stabilizer is the set of matrices

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{pmatrix}$$

Due to Eq. (1.3), this set of matrices is isomorphic to the group of non singular matrices

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with } g_{11}g_{22} - g_{12}g_{21} \neq 0 \tag{1.8}$$

Duality permits us to give an alternative description of this fact. The equation of the straightline (0 0 1) is $z = 0$. This straightline is the subset of points of type $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. We check that the subgroup of matrices (1.8) acts transitively on this subset.

Let us call the straightline $z = 0$ the ***absolute*** and consider the space obtained in taking away the absolute from $P_2(\mathbb{R})$. This new space contains all points of the type $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $z \neq 0$. We already saw that the equivalence relation (1.2) permits us to write any element of this space in a unique way in the form. We arrive at a new geometry where the group is the one defined by Eq. (1.8) (the one which preserves the absolute). The action is defined by the matrix multiplication

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} g_{11}x + g_{12}y + g_{13}z \\ g_{21}x + g_{22}y + g_{23}z \\ 1 \end{pmatrix} \tag{1.9}$$

One recognizes the two-dimensional real affine geometry, the group of which is the affine group in two dimensions.

Let us give a résumé of our results. We have stated algebraic definitions of two geometries. They are

- **The two-dimensional real projective geometry**.
  Group: $PGL(3,\mathbb{R})$ Space: the projective real plane $P_2(\mathbb{R})$

- **The two-dimensional affine real geometry**.
  Group: $Aff(2,\mathbb{R})$ Space: the real plane $\mathbb{R}^2$.

The group $Aff(2, \mathbb{R})$ is a six-dimensional subgroup of $PGL(3, \mathbb{R})$ and the real plane $\mathbb{R}^2$ is obtained from the projective real plane $P_2(\mathbb{R})$ in throwing away a given straightline. $Aff(2, \mathbb{R})$ is the stabilizer of this straightline. We note that if we had chosen another straightline than the absolute, we would arrive at a conjugate subgroup of $Aff(2, \mathbb{R})$.

We can give an interpretation of these results in ordinary geometry where the absolute is called the straightline at infinity of the real plane. This interpretation is related with the theory of perspective. In fact, projective geometry came from the theory of perspective.[12]

Let us consider the image of a landscape given by a dark room. To simplify the description, we suppose that this dark room is a quite large opaque cubic house. The vertical wall $W$ in front of the landscape has a small hole at its midpoint $O$. An observer standing in the dark room would see the reversed image of the landscape on the opposite wall $W'$. To simplify the geometrical description, we will suppose these two walls infinitely large. For the observer, the state of a point $A'$ of $W'$ is the image of some point $A$ lying on the line $\Delta$ which goes through the points $A'$ and $O$. Since he is ignoring what is going outside, it is better, for him, to say that $A'$ is the image of the whole line $\Delta$.[13] The problem is to know if the mapping of $W'$ on the set of straightlines containing $O$ is a one-to-one mapping (a bijection). The answer is *no*. Every straightline which contains $O$ cuts the plane $W'$ in a single point except the straightlines which belong to $W$. Because mathematicians do not like exceptions, they prefer to invent the *points at infinity* in order to have a general statement and say that "*every* straightline which contains $O$ cuts the plane $W'$ in a single point (the image); in particular, if the straightline belongs to $W$, the image is a point at infinity".

We note that the straightlines which belong to a given plane have their images on a straightline of $W'$. It follows that the set of *points at infinity* form a *straightline at infinity.* With these extra points at infinity added, the plane $W'$ is said to be *complete*. It is the *projective plane* $P_2(\mathbb{R})$.The bijection which exists between the complete plane and the set $S$ of straightlines through $O$ permits to identify these two sets. Clearly, the group $GL(3, \mathbb{R})$ acts on $S$. Then it acts on $W'$. The systems $(GL(3, \mathbb{R}), \mathrm{S})$ and $(GL(3, \mathbb{R}), \mathrm{W}')$ are equivalent and seem to define a given geometry. However, the group $GL(3, \mathbb{R})$ does not act *effectively*. This means that it has superfluous elements, namely elements which act as the identity operator. We already say that these superfluous elements of a group $G$ (in a given action) form a normal subgroup $H$ and that it is the quotient group $G/H$ which acts effectively. Here the superfluous elements are the diagonal matrices of $GL(3, \mathbb{R})$. They form a subgroup isomorphic to $\mathbb{R}^*$ (the multiplicative group of nonzero real numbers). The group which acts effectively is the quotient group $GL(3, \mathbb{R})/\mathbb{R}^*$. This quotient group can be used as a definition of the so-called projective group $PGL(3, \mathbb{R})$.

The group $PGL(3, \mathbb{R})$ has an element which maps a given straightline of $S$ on any other straightline, even a straightline of $W$. It is equivalent to say that this group may map any point of $W'$ on any other point, even at infinity. We know that the affine group $Aff(2, \mathbb{R})$ cannot do it. It maps the straightline at infinity onto itself. This straightline is the corresponding absolute figure.

[12]The exact link between projective geometry and the theory of perspective is presented in details in H. Bacry, *La symétrie dans tous ses états*, chapter 4, Paris, Vuibert, 2000.

[13]We are concerned with mathematics. Obviously, the observer knows that $A$ cannot lie anywhere on $\Delta$; it lies on a half-line having $O$ as an origin.

| Groups | $PGL(3,\mathbb{R})$ | $Aff(2,\mathbb{R})$ | $S_+(2,\mathbb{R})$ | $E_+(2,\mathbb{R})$ |
|---|---|---|---|---|
| Spaces | $P_2(\mathbb{R})$ | $Aff(2,\mathbb{R})$ | $S(2,\mathbb{R})$ | $E(2,\mathbb{R})$ |
| Name of the geometry (two-dimensional real) | projective group | affine group | proper similitude group | proper Euclidean group |
| Invariants: | | | | |
| Distance | no | no | no | yes |
| Angle | no | no | yes | yes |
| Ratio of two segments | no | no | yes | yes |
| Collinearity of three points | yes | yes | yes | yes |
| Ratio of two collinear segments | no | yes | yes | yes |
| Cross ratio of four collinear points | yes | yes | yes | yes |
| Excentricity of a conic | no | no | yes | yes |
| Parallelism of straightlines | no | yes | yes | yes |
| Non degenerate conics | one class | three classes: ellipses hyperbolas, parabolas | infinite number of classes | infinite number of classes |

Table 1.1: Some related geometries

There are other interesting figures in the real plane, namely the conics. A conic in $W'$ is the image of a quadratic cone in $S$. The group $PGL(3,\mathbb{R})$ may map any quadratic cone onto any other one. It follows that it may map any conic in $W'$ onto any other one. The group $Aff(2,\mathbb{R})$ cannot do it. It maps ellipses on ellipses, hyperbolas on hyperbolas and parabolas on parabolas. Why? Because an ellipse does not cut the straightline at infinity, a hyperbola cuts it in two points and a parabola is tangent to it. One deduces that there is only one kind of conic in the real projective plane geometry, but there are three kinds of conics in the real affine plane geometry. We verify here a general fact: when the group becomes smaller, a congruence class may split into subclasses.

To conclude this section, we put in a table a set of informations about four of the groups we have considered, namely the groups of the chain $PGL(3,\mathbb{R}) > \mathrm{Aff}(2,\mathbb{R}) > \mathrm{S}_+(2,\mathbb{R}) > \mathrm{E}_+(2,\mathbb{R})$. The first group acts on the projective real plane $P_2(\mathbb{R})$, the three other ones on the real plane $\mathbb{R}^2$. We will note that the "no" is more frequent on the left handside. More generally, the number of invariants increases when the group is smaller and smaller.

## 1.13 Classification of all elementary geometries associated with a group

In order to classify the elementary geometries associated with a given group $G$, we have to examine more carefully the notion of homogeneous space. We recall the reader that a homogeneous space is a set on which the group acts transitively.

Two elements $g$ and $g'$ are said to be $H$-equivalent if $g^{-1}.g'$ belongs to a given subgroup $H$. This relation is reflexive since $g^{-1}.g = e$ belongs to $H$; moreover it is also symmetric since $g^{-1}.g' \in H$ implies $(g^{-1}.g')^{-1} = g'^{-1}.g \in H$; and transitive since the conditions $g^{-1}.g' \in H$ and $g'^{-1}.g'' \in H$ imply $g^{-1}.g'.g'^{-1}.g'' = g^{-1}.g'' \in H$. The classes of the partition defined by this equivalence relation are called *left cosets* because, as we shall see, they are related with the left action of $G$ on itself. (Another equivalence relation will lead to the definition of *right cosets*. It is the one we obtain in replacing the product $g^{-1}.g'$ by $g.g'^{-1}$ in the definition).

Let us show that the left cosets are the subsets $g.H$. First, if $k$ and $k'$ are elements of $g.H$, one has $k = g.h$ and $k' = g.h'$ where $h$ and $h'$ belong to $H$. Therefore $k^{-1}.k' = h^{-1}.h'$ belongs to $H$ and $k$ and $k'$ are equivalent. Conversely, suppose $k^{-1}.k' \in H$, $k$ belonging to some subset $g.H$ (say $k = g.h$). This implies that we have $h^{-1}.g^{-1}.k' \in H$ or $k' \in g.H$. We have proved in this way that the subsets $g.H$ are the equivalence classes.

We now examine the set $G/H$ of all left cosets. The group $G$ is acting on this set by left translations in a transitive way. The stabilizer of the coset $H$ is $H$ itself. The stabilizer of the coset $g.H$ is the conjugate subgroup $g.H.g^{-1}$. One sees, in this way, that each class of conjugate subgroups defines a geometry associated with $G$. Conversely, if $(G, M)$ is a geometry and $H$ the stabilizer of the point $x$, the stabilizer of the point $g.x$ is $g.H.g^{-1}$. We arrive at the conclusion that a geometry is uniquely described by a group $G$ and a class of conjugate subgroups of $G$. In order to illustrate these notions, we will consider the case of two finite groups, namely the groups $O_+$ and $S_3$ .

## 1.14 The group $O_+$

Let us examine the group $O_+$ of the 24 rotations which leave invariant a cube, considered either as the set of its eight vertices or as the set of its six faces. It is clear that this group acts transitively on the cube. Each of the 24 rotations belongs to one of the following five conjugacy classes.

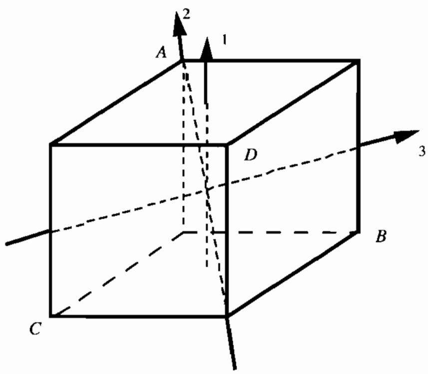


Figure 1.1: The 24 rotations of the cubic group $O_+$

- Class 1. The neutral element (null rotation)

- Class 2. Six rotations of 90° (around an arrow like 1 on Figure 1.14)
- Class 3. Three rotations of 180° (arrow 1)
- Class 4. Eight rotations of 120° (arrow 2)
- Class 5. Six rotations of 180° (arrow 3)

The stabilizer of the vertex $A$ is the cyclic group $C_3$ generated by a rotation of 120° around the axis 2. It follows that the set of vertices is the homogeneous space $O_+/C_3$. The stabilizer of a face is the cyclic group $C_4$ generated by a rotation of 90° around the axis 1. It follows that the set of faces is the homogeneous space $O_+/C_4$. We define in this way two geometries. The first one has a space really made of points. If, for the second one, we hesitate to call points the six faces of the cube, we could replace the cube by its "dual" polyhedrum, namely the octahedron, the vertices of which are the midpoints of the faces. We will not do it.

Let us find the congruence classes of pairs of points (segments) in both geometries.

- Geometry $(O_+, O_+/C_3)$: the space has 8 points and there are $\frac{8\times7}{2} = 28$ segments. If the edges of the cube have length one, the set of 28 segments is made of three congruence classes: *i*) 12 segments of length one (the edges), *ii*) 12 segments of length $\sqrt{2}$, and *iii*) 4 segments of length $\sqrt{3}$. We check that the length is an invariant, a property which characterizes the rotation group.
- Geometry $(O_+, O_+/C_4)$: the space has 6 elements (the faces) and there are $\frac{6\times5}{2} = 15$ pairs of faces. This set decomposes in *iv*) a congruence class of 12 pairs of adjacent faces (or, equivalently, 12 edges) and *v*) a congruence class of 3 pairs of parallel faces.

Each congruence class among the five ones we have derived can be used to define a geometry. All of them are new but two of them are identical (*i* and *iv*). The reader is invited to examine these geometries and to build other ones by himself. He could also study the geometry of the regular tetrahedron defined by the four vertices $A$, $B$, $C$, $D$ of Figure 1.14.

## 1.15 The group $S_3$

Let us now examine our second example, namely $S_3$, the permutation group[14] of three objects. This group has the following geometrical interpretation. Consider a regular triangle with vertices denoted 1, 2, 3. This triangle is invariant under the symmetries with respect to each height of the triangle and the three rotations (angles 0, 120°, and 240°) around the center of mass of the three vertices. The group $S_3$ is of order six ($3! = 6$). Its elements are $e$ (the neutral element), the transpositions (12), (23), (31) (where (12) permute the vertices 1 and 2 and leaves the vertex 3 fixed), and the two cyclic permutations (123) (it maps vertex 1 onto vertex 2, vertex 2 onto vertex 3 and vertex 3 onto vertex 1) and its inverse (132). The group $S_3$ has six subgroups of order 6, 3, 2 and 1 (the order

[14]We use the decomposition of a permutation in independent cycles. As an example (245)(16) means that the object 2 is mapped on the object 4, which is mapped on the object 5, and the object 5 is mapped on the object 2, the objects 1 and 6 are exchanged. The cycles of order one are omitted (the object 3 is mapped on itself).

of a subgroup is always a divisor of the order of the group). The set of all subgroups of a group has a lattice structure in relation with the inclusion relation. In our case, this structure is described in Figure 1.15. It is easy to check that the three subgroups of order

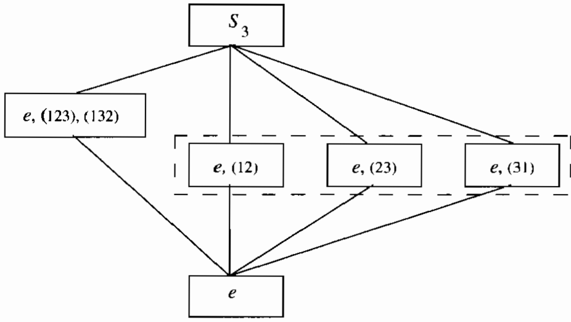


Figure 1.2: The lattice of subgroups of the group $S_3$. The three subgroups of order two are conjugate. (If we put these three conjugate subgroups in a unique case, we check that the lattice is still a lattice. However, this is not a general property.)

three are conjugate. It follows that there are four geometries associated with the group $S_3$. The geometry associated with the coset space $S_3/S_3$ has only one point. It is a very poor geometry! the only figure is this point itself. The geometry associated with the coset space $S_3/A_3$, where $A_3$ is the alternate subgroup $\{e,\ (123),\ (132)\}$, is also poor since there are only two points. This means however that figures exist: apart the two points, say $x$ and $y$, which are trivial figures, we have the two couples $(x, y)$ and $(y, x)$ and the pair $\{x, y\}$ (the whole space). The geometry associated with $S_3/\{e, (12)\}$ is the "natural" geometry attached with the group $S_3$. Its space has three points and the group $S_3$ acts in permuting these three points. Let us denote these points by the letters $x$, $y$, $z$. The main figures are the three points, the three pairs, the six couples, the unique subset of order three (the whole space). But we have more complex figures as pairs of couples, the figures composed of a point and a pair, and so on.

The richest geometry is the one associated with the coset space $S_3/\{e\}$. It is the geometry described by the system $(S_3, S_3)$ with $S_3$ acting on itself by left translations. The simplest figures are, apart the six points, the thirty couples, the fifteen pairs (subsets of order two), etc. It is a simple exercise to prove that the fifteen pairs form four distinct congruence classes, three with three elements each and one with six elements. A congruence class with three elements is such that the union of the three elements is the whole space. The twenty subsets of order three (triangles) form three congruence classes with six elements and one of two elements. The two last triangles are complementary in that their union is the whole space. Each other congruence class is made of three pairs of complementary triangles. All these results can be obtained in setting $x_1 = e$, $x_2 = (12)$, $x_3 = (23)$, $x_4 = (31)$, $x_5 = (123)$ and $x_6 = (132)$. The action of the group is described by Table 1.15. One readily verifies on this table that the two triangles $\{x_1, x_5, x_6\}$ and $\{x_2, x_3, x_4\}$ form a single congruence class.

We must underline that, although the construction of the six points space was made with the aid of the elements of the group which has a privileged element (the unit element),

| | $e$ | $(12)$ | $(23)$ | $(31)$ | $(123)$ | $(132)$ |
|---|---|---|---|---|---|---|
| $x_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| $x_2$ | $x_2$ | $x_1$ | $x_6$ | $x_5$ | $x_4$ | $x_3$ |
| $x_3$ | $x_3$ | $x_5$ | $x_1$ | $x_6$ | $x_2$ | $x_4$ |
| $x_4$ | $x_4$ | $x_6$ | $x_5$ | $x_1$ | $x_3$ | $x_2$ |
| $x_5$ | $x_5$ | $x_3$ | $x_4$ | $x_2$ | $x_6$ | $x_1$ |
| $x_6$ | $x_6$ | $x_4$ | $x_2$ | $x_3$ | $x_1$ | $x_5$ |

Table 1.2: Action of $S_3$ on itself

the space is *homogeneous*. This seems to be a paradoxal situation. Homogeneity means that we cannot distinguish between the six points as we cannot distinguish between the elements of any congruence class. Since we are in the case where the group is acting on itself, it is interesting to define the geometry in the opposite way, that is in giving an axiomatic definition of the space first and deduce the group action. A space defined with the aid of these new axioms will be called a *generalized affine space* (*g.a.s.*)[15], a notion we are going to introduce in the next section. We will show in Chapter 3 why this notion is more physical than the one of a group.

## 1.16 Generalized affine space

A *generalized affine space* is a set endowed with an inner ternary law

$$(a, b, c) \to abc$$

satisfying the two following conditions:

$$aab = baa = b, \quad \text{(axiom 1)}$$
$$ab(cde) = (abc)de = a(dcb)e. \quad \text{(axiom 2)}$$

The group which acts on this space is defined with the aid of couples (left translations[16]) written as $\overleftarrow{ab}$. The point $b$ is the origin and the point $a$ the end of the couple. Two translations $\overleftarrow{ab}$ and $\overleftarrow{cd}$ are equivalent if $\forall x,\ abx = cdx$. A translation is equivalent to a unique translation with a given origin $o$. Indeed $abx = ab(oox) = (abo)ox$ implies that $\overleftarrow{ab}$ is equivalent to $\overleftarrow{(abo)o}$. The translations, up to an equivalence, is the group we are looking for. The neutral element is $\overleftarrow{oo}$. The inverse of $\overleftarrow{ao}$ is $\overleftarrow{oa}$, which is equivalent to $\overleftarrow{(oao)o}$. The product of translations $\overleftarrow{bo}\overleftarrow{ao}$ equals $\overleftarrow{(boa)o}$, which is a very simple result. One verifies easily the associativity of this product.

The expression *generalized affine space* comes from the following property. In the case of an ordinary affine space, the product $abc$ is the point $d$ such that $abcd$ is a parallelogram. The ordinary affine space is commutative. Left and right translations are equivalent: $\overleftarrow{xab} = \overleftarrow{bax}$.

[15]Such an idea arose in 1978 during a discussion with the French physicist Alexander Grossmann. This work is unpublished.

[16]The right translations can be defined with the aid of couples of the type applied on the right handside.

## 1.17 Geometry and automorphisms

Geometries are usually defined in introducing first the space $M$ and providing it with some structure. In doing so, we introduce in an implicit way the group $Aut(M)$ of all automorphisms, that is the group of transformations which preserve that structure. A geometry $(G, M)$ will be then defined with the aid of a mapping $f: G \to Aut(M)$. Let us give examples:

| **Structure of the space $M$** | **Name of an automorphism** |
|---|---|
| Topological space | homeomorphism |
| Differentiable manifold | diffeomorphism |
| Metric space | isometry |
| Vector space | automorphism (bijective endomorphism) |
| Complex Hilbert space | unitary operator |
| Real Hilbert space | real orthogonal operator |
| Symplectic manifold | symplectomorphism |

Table 1.3: Names of automorphisms for each type of space

## Exercises

**1.** Prove that, if $g$, $g'$, $h$ are elements of a group $G$, the equality $g.h = g'.h$ implies $g = g'$.

**2.** Prove that a group has only one neutral element.

**3.** Prove that $(g^{-1})^{-1} = g$ and that $(g.h)^{-1} = h^{-1}.g^{-1}$.

**4.** Prove the following theorem: a subset $H$ of a group $G$ is a subgroup of $G$ if and only if, for any two elements $h$ and $h'$ of $H$, $h.h'^{-1}$ belongs to $H$.

**5.** If $\phi$ is a homomorphism of a group $G$ in a group $G'$, the element $\phi(e)$ is he neutral element of $G'$.

**6.** Let $\phi$ be a homomorphism of a group $G$ in a group $G'$. Verify that $Ker(\phi)$ is an invariant subgroup.

**7.** Check that the quotient group $G/\{e\}$ is isomorphic to $G$ itself.

**8.** Prove that every cyclic group is solvable. Relate this property with the factorization of the polynomial $x^n - 1$.

**9.** Prove that the group $S_n$ is solvable if $n =$ 1, 2 or 3.

**10.** What is the Galois group of the equation $x^3 - 4x^2 + 5x - 2 = 0$?

**11.** Show that any group $G$ acts freely on itself by left or right translations.

**12.** Find a fixed point in the conjugation action of $G$ on itself. Show that the set of fixed points in this action is the center of $G$. Show that the center of an Abelian group is the group itself.

**13.** Prove that, for two arbitrary elements $x, y$ of a group $G$, $xy$ is conjugate of $yx$.

**14.** Find the orbits of the group $SL(2, \mathbb{C})$ when it acts on the space $\mathbb{C}^2$ (Hint: find how it acts on the null spinor and on the spinor).

**15.** Find the orbits of the group $S_+(2, \mathbb{R})$ on the plane $\mathbb{R}^2$.

**16.** Check that the tetrahedron $ABCD$ in Fig. 1.14 is regular. Call $T_+$ the subgroup of $O_+$ composed of those rotations which leaves this tetrahedron invariant. How many subgroups are conjugate to $T_+$? Which are they?

**17.** Join the centers of adjacent faces in Fig. 1.14. Count the number of faces, edges and vertices of the polyhedron obtained in this way (octahedron). How these numbers are they related to the ones of the cube? Do the same for the regular tetrahedron.

**18.** Construct the lattice of subgroups of the group $S_4$.

**19.** Study the lattice of subgroups for the symmetry group $H$ of a regular hexagonal. Show that the lattice of subgroups of $S_3$ is a sublattice of it.

**20.** Find the five $g.a.s.$ with one, two, three and four elements.

**21.** One wants to draw a circle in perspective. Show that it could be represented by an ellipse, a hyperbola or a parabola, depending on the position of the circle.

**22.** Examine the one-dimensional camera and define the space $P_1(\mathbb{R})$, the real one-dimensional projective space and the group $PGL(2, \mathbb{R})$. Show that it is homeomorphic to the circle $S_1$.

**23.** Show that the set of straighthlines of the real two-dimensional affine space is homeomorphic to an open Möbius strip. Hint: consider the space parametrized by the coordinates $a, b, c$ of the straightline $ax + by + c = 0$, where $(a, b) \neq (0, 0)$ and $(a, b, c)$ is equivalent to $(\lambda a, \lambda b, \lambda c)$ for $\lambda \neq 0$ .
Show that adding the straightline at infinity compactifies the Möbius strip in identifying

all points of its border.

**24.** Find the main differences between projective geometry and perspective theory (Hint: think of a machine permitting the resolution of perspective).

# Chapter 2

# On some Lie groups

Here, we intend to give general informations concerning some Lie groups, more precisely the groups $O(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, and some subgroups of the similitude group $S(n,\mathbb{R})$, namely the Euclidean group $E(n,\mathbb{R})$, the orthogonal group $O(n,\mathbb{R})$, and the Thales group $Th(n,\mathbb{R})$.

## 2.1 The orthogonal complex group $O(n,\mathbb{C})$

Let $\psi$ be a vector of $\mathbb{C}^n$, with components $\psi^a$. We define on $\mathbb{C}^n$ a scalar product as follows:

$$(\psi,\phi)=\sum_{a=1}^{n}\psi^a\phi^a. \tag{2.1}$$

A transformation $O$ is said to be orthogonal if it preserves the scalar product, i.e. if, for arbitrary $\psi$ and $\phi$, one has

$$(O\psi,O\phi)=(\psi,\phi), \tag{2.2}$$

that is

$$O^TO=Id, \tag{2.3}$$

where the symbol $T$ denotes the transposed operator and $Id$ is the identity operator.

Instead of the canonical basis of $\mathbb{C}^n$, which satisfies $(e_a,e_b)=\delta_{ab}$, we may prefer an arbitrary one $f_a$. One sets

$$(f_a,f_b)=g_{ab}, \tag{2.4}$$

where $g_{ab}=g_{ba}$. Then, instead of (2.3), we get

$$O^TgO=g. \tag{2.5}$$

Conversely, given an arbitrary symmetric tensor $g$, Eq. (2.5) defines a group isomorphic to $O(n,\mathbb{C})$, provided the equation

$$\mathrm{Det}(g-\lambda Id)=0 \tag{2.6}$$

has only non-zero roots ($g$ non-degenerate).

If we impose the condition

$$\mathrm{Det}(O)=1, \tag{2.7}$$

we arrive at a subgroup of $O(n,\mathbb{C})$, namely the group $SO(n,\mathbb{C})$, known as the special complex orthogonal group.

We note the following interesting case where the tensor $g$ is of an antidiagonal kind, with alternate values $\pm 1$. It is clear that, for $g$ to be symmetric, this corresponds to an odd value of $n$.

## 2.2 Real forms of the group $O(n,\mathbb{C})$

Suppose that we impose the tensor $g$ and the operators $O$ to be real. We arrive at a subgroup of $O(n,\mathbb{C})$. These subgroups are not all isomorphic. The class the corresponding group belongs to depends on the roots of Eq. (2.6). Suppose that this equation has $p$ positive and $q$ negative roots; if $pq \neq 0$, the group we arrive at is denoted by $O(p,q)$. We note that $O(p,q)$ is isomorphic to $O(q,p)$. Whenever $p$ or $q$ is zero, the group is denoted by $O(n,\mathbb{R})$ and is called the real orthogonal group. Whenever $p$ or $q$ equals one, and $n \geq 2$, the group is called the generalized Lorentz group (the Lorentz group corresponds to $n = 4$). For $pq \neq 0$, and $n = 5$, the two corresponding non isomorphic groups are known as the de Sitter groups.

Obviously, one may again impose the restriction $\mathrm{Det}(O) = 1$. Then we define the groups $SO(p,q)$ and $SO(n,\mathbb{R})$. This last group is known as the real rotation group in $n$ dimensions.

The topology of these groups is the following one. The group $O(n,\mathbb{R})$ is two-sheeted. The connected component (*i.e.* the set of elements connected with the identity transformation) is the group $SO(n,\mathbb{R})$. The other sheet may be obtained by multiplying all elements of $SO(n,\mathbb{R})$ by a diagonal matrix with $n-1-2k$ times the value 1 and $2k+1$ times the value -1 ($k$ has an arbitrary value). This matrix is referred as a parity transformation. When $n$ is odd, this matrix is usually chosen as the operator - Id.

As it will be shown for the Lorentz group, the group $O(n-1,1)$ is four-sheeted, and its subgroup $SO(n-1,1)$ is two-sheeted. For more information, the reader is referred to the chapter devoted to the Lorentz group.

## 2.3 The symplectic groups

Let $\sigma$ be a non-degenerate antisymetric tensor of dimension $n$ and of order two. The set of complex matrices $S$ satisfying the condition

$$S^T \sigma S = \sigma \tag{2.8}$$

form a group called the complex symplectic group $Sp(n,\mathbb{C})$. The antisymmetric nature of $\sigma$ implies that $n$ is even. More precisely, the groups associated with different $\sigma$'s are all isomorphic. Two interesting facts must be underlined:

- all the matrices of this group have determinant equal to one,
- supposing that the matrices $S$ are real defines a unique subgroup called the real symplectic group and denoted $Sp(n,\mathbb{R})$.

We may impose $\sigma$ to be antidiagonal with alternate values $\pm 1$. We have shown that, for $n$ odd, the group is the orthogonal group $O(n,\mathbb{C})$. We see that, for $n$ even, it is the symplectic group $Sp(n,\mathbb{C})$.

## 2.4 The homothesis, translation, and Thales groups

Let $\lambda$ a non-zero complex number. The matrices $\lambda$ Id acting on $\mathbb{C}^n$ form a group called the complex homothesis group $H(n,\mathbb{C})$. It is isomorphic to the multiplicative group of non-zero complex numbers, a group of real dimension two. If we impose $\lambda$ to be real, we define the one-dimensional real homothesis group $H(n,\mathbb{R})$, a subgroup of $H(n,\mathbb{C})$.

The translation group $T(n,\mathbb{C})$ associated with $\mathbb{C}^n$ can be defined in the following way. Consider all elements of $\mathbb{C}^{n+1}$ of the form

$$\psi = \begin{pmatrix} \psi \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ . \\ . \\ \psi_n \\ 1 \end{pmatrix} \tag{2.9}$$

that is elements with the $(n+1)$th coordinate equal to one. The group $T(n,\mathbb{C})$ is the set of matrices of the form

$$T_\phi = \begin{pmatrix} Id & \phi \\ 0 & 1 \end{pmatrix} \tag{2.10}$$

It acts on $\psi$ as follows:

$$T_\phi \psi = \begin{pmatrix} \psi + \phi \\ 1 \end{pmatrix}. \tag{2.11}$$

Clearly, this group is the translation group of $\mathbb{C}^n$.

We note that the homothesis group $H(n,\mathbb{C})$ can be defined alternatively in the following way

$$H_\lambda = \begin{pmatrix} \lambda Id & 0 \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

The group generated by the translation and the homothesis groups will be called the Thales group and will be denoted $Th(n,\mathbb{C})$. (The notation $Th$ may recall that $T$ is for translations and $h$ for homothesis). It is a group of $n+1$ complex dimension. When the translations and the homothesis are real, we arrive at the real Thales subgroup $Th(n,\mathbb{R})$, a group of real dimension $n+1$.

## 2.5 The Euclidean and the similitude groups

What we have written for the homothesis group as a set of $(n+1)\times(n+1)$ matrices can be done for the orthogonal groups $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$. Then, the following groups can be defined:

- The complex Euclidean group $E(n,\mathbb{C})$, generated by $O(n,\mathbb{C})$ and $T(n,\mathbb{C})$
- The special complex Euclidean group $SE(n,\mathbb{C})$, generated by $SO(n,\mathbb{C})$ and $T(n,\mathbb{C})$.
- The real Euclidean group $E(n,\mathbb{R})$, generated by $O(n,\mathbb{R})$ and $T(n,\mathbb{R})$.
- The proper Euclidean group $SE(n,\mathbb{R})$, generated by $SO(n,\mathbb{R})$ and $T(n,\mathbb{R})$.

- The complex similitude group $S(n, \mathbb{C})$, generated by $E(n, \mathbb{C})$ and $H(n, \mathbb{C})$, equivalently generated by $Th(n, \mathbb{C})$ and $O(n, \mathbb{C})$, equivalently generated by $Th(n, \mathbb{C})$ and $SO(n, \mathbb{C})$.
- The real similitude group $S(n, \mathbb{R})$, generated by $E(n, \mathbb{R})$ and $H(n, \mathbb{R})$, equivalently generated by $Th(n, \mathbb{R})$ and $O(n, \mathbb{R})$. If $n$ is odd, this group is equivalently generated by $Th(n, \mathbb{R})$ and $SO(n, \mathbb{R})$.

## Exercises

**1.** Find the real forms of the group defined by an antidiagonal tensor $g$ with alternate values $\pm 1$.

**2.** Prove the last proposition of section 2.5.

# Chapter 3

# The rotation group $SO(3, \mathbb{R})$

We do not intend to present to the reader a mathematical definition of this group. It is more interesting to define it from concrete physics, that is from pure kinematical arguments concerning the movement of a rigid body. As we will see, the set of possible movements of such a body is intimately related to the connected Euclidean group in three dimensions $SE(3, \mathbb{R})$, hereafter denoted $E_+(3, \mathbb{R})$. If we fix a point of our rigid body, we are left with a rotation *subgroup* of $E_+(3, \mathbb{R})$. If, instead of that point, we fix another point, we arrive at a conjugate subgroup. The conjugacy class of these subgroups can be seen as describing the possible *orientations* of the body. This conjugacy class is nothing else than the quotient group $E_+(3, \mathbb{R})/\mathrm{T}(3, \mathbb{R})$, where $T(3, \mathbb{R})$ is the translation subgroup. In order to analyze these links, we need to have at our disposal convenient parametrizations of these groups. Before giving a description of the usual parametrizations of the rotation group, we want to give a qualitative description of the two following unusual sets: the set of possible positions of a rigid body, and the set of possible ways, for a rigid body, to go from one position to another. We will present the groups which are related with these sets.

## 3.1 The Euclidean group and its covering

Let us start with the simplest case where the position of a very small rigid body is only described by the point where it lies, ignoring its dimensions and its orientation There is an infinite number of paths from an initial position $A$ to a final one $B$. Such a path is conveniently described by a function $M(t)$ where $t$ runs from 0 to 1 and $M(0) = A$, $M(1) = B$.Two paths $M(t)$ and $N(t)$ can be considered as equivalent if $M(0) = N(0) = A$ and $M(1) = N(1) = B$. The corresponding equivalence class is described by the couple $(A, B)$, often written $\overrightarrow{AB}$. Note that the two paths $M(t)$ and $N(t)$ have the property that they can be deformed one into the other in a *continuous way*. One can define the following equivalence relation between the couples: $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are said to be equivalent if $ABB'A'$ is a parallelogram. The new equivalence classes are called *translations*. They are elements of a three-dimensional vector space and the group of translations is an Abelian group $T(3)$.

Note that we started with the notion of the set of positions of a small body, the affine space, a set in which there is no privileged point, and we arrived at a set of vectors which has a group structure with a neutral element, the null vector. In modern mathematics, one prefers to define first the vector space, then the affine space. One sees that physics privileges the opposite way. In fact, there is a way to reconcile mathematics and physics. For this purpose, we proceed as follows.

We start with the affine space. Let $A$, $B$, $C$ be three arbitrary (not necessarily distinct) points. By definition, the product $ABC$ is the point $D$ such that $ABCD$ is a parallelogram. We readily verify by a simple geometrical construction the axioms of what we call a generalized affine space (*g.a.s.*):

$$AAB = BAA = B \qquad \text{(condensation axiom)} \tag{3.1}$$

$$AB(CDE) = A(DCB)E = (ABC)DE \qquad \text{(skew-associativity axiom).} \tag{3.2}$$

This g.a.s. is said to be commutative; by that, we mean: $ABC = CBA$. The left and right actions of the translation group are identical: $A\overrightarrow{BC} = \overleftarrow{CB}A$ implies $= \overrightarrow{BC} = \overleftarrow{CB}$.

We have thus related the homogeneous space of all positions of a small body to the translation group in three dimensions $T(3)$. Formally, we have derived a group from a *g.a.s.* We intend to do a similar construction for the set of positions of an extended rigid body and derive the connected Euclidean group $E_+(3)$. It is important to underline the two main differences between the two cases. First, we are loosing the commutativity property. The non commutativity is shown on Fig. 3.1, where the initial position is represented by a frame $Oxyz$. Two rotations of 90°, namely $R_z$ and $R_x$, are performed, in the first case in the order $R_z$, $R_x$ and in the second case in the opposite order. We check that the resulting rotations are different.

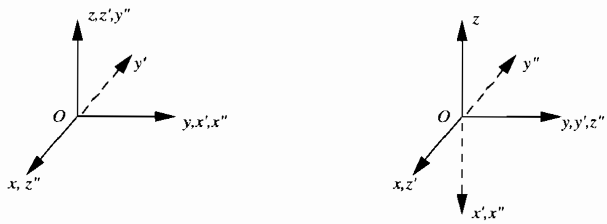


Figure 3.1: The frame $Oxyz$ becomes $Ox'y'z'$ under the first rotation ($R_z$ on the left, $R_x$ on the right), then $Ox''y''z''$ under the second one ($R_x$ on the left, $R_z$ on the right).

The other difference is a topological one (homotopy). Let us consider the set of continuous trajectories from an initial position to a final position of the rigid body. We can choose two distinct equivalence relations which will lead us to two distinct groups instead of one in the previous case.

- Two continuous paths are equivalent if they have the same initial position $A$ and the same final position $B$. An equivalence class is written $\overrightarrow{AB}$ and is called a *Euclidean motion.* In that case, we obtain the connected Euclidean group $E_+(3)$. The quotient group $E_+(3)/T(3)$ is isomorphic to the rotation group $SO(3,\mathbb{R})$, the group of real orthogonal $3 \times 3$ matrices of determinant one.

- Two continuous paths are equivalent if they have the same initial and final positions and they can be continuously transformed one into each other. This condition splits the class $\overrightarrow{AB}$ in two subclasses; the group we obtain is the *double covering* of $\overline{E_+(3)}$. The quotient group $\overline{E_+(3)}/T(3)$ is isomorphic to the group $SU(2)$, itself the double covering of $SO(3,\mathbb{R})$.

To prove this property rigorously we need parametrizations of the Euclidean group and the rotation group. However, there is an experimental way of checking that. Perform a continuous *orientation change* of angle $2\pi$ of a rigid body; it means that you perform a rotation in ignoring a possible translation motion; at the end, the body is brought to its initial position. To be more concrete, put, for instance, a book on your right hand, your arm standing horizontally, the thumb in the back direction. Perform a $2\pi$ rotation around a vertical axis, ignoring a possible translation motion of the hand. It is clear that your arm is not back to its initial position. This means that this final position is not in the neighbourhood of the initial position. Now, if you iterate the same rotation you are back to the initial situation. A rotation of $4\pi$ is equivalent to the null rotation! We are going to obtain that result with the aid of a more mathematical argument. For that, we need a parametrization of the rotation group.

## 3.2 First parametrization of the rotation group

For a physicist, a rotation is defined by an oriented axis or a unit vector $\mathbf{u}$ and an angle $\phi$ satisfying, say, $0\leq\phi\leq\pi$. If we denote such a rotation by $R_{\mathbf{u}}(\phi)$, we readily see that we have to identify $R_{\mathbf{u}}(\pi)$ with $R_{-\mathbf{u}}(\pi)$. The rotation $R_{\mathbf{u}}(\phi)$ acts on a vector $\mathbf{r}$ as follows

$$R_{\mathbf{u}}(\phi)\mathbf{r} = \mathbf{r} + \sin\phi\mathbf{u}\times\mathbf{r} + (1-\cos\phi)\mathbf{u}\times(\mathbf{u}\times\mathbf{r}). \tag{3.3}$$

It is easy to check this formula in choosing successively $\mathbf{r}$ collinear then orthogonal to $\mathbf{u}$. If we decide that $\mathbf{u}$ defines the $z$ direction and if we set $\mathbf{r} = (x,y,z)$, we see that the rotation $R_{\mathbf{u}}(\phi)$ is described by the matrix

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

This proves, in particular, that the trace of $R_{\mathbf{u}}(\phi)$ is a function of $\phi$. It is $1+2\cos\phi$.

Our parametrization by $\mathbf{u}$ and $\phi$ shows us that the set of rotations is a ball $B$ of radius $\pi$, in which antipodal points are identified. If, instead of $\phi$, we choose the variable $\tan\frac{\phi}{2}$, we get, instead of a ball, the whole space, *including the points at infinity*. We see, in this way, that the rotation group is isomorphic, as a manifold, to the projective space $P_3(\mathbb{R})$. The surface of the ball has been mapped, by the change of variable, on the plane at infinity.[1]

The parametrization of the rotation group by $\phi$ and $\mathbf{u}$ proves that the group is connected (there is no proper subset which is both open and closed). It is clear that the space we just described is also the set of positions of a rigid body which has a fixed point. A motion like the one we have spoken about (the book on a hand) is represented by a

[1] We know that it is a plane. In fact, the curvature of the sphere is going to zero.

closed path in $P_3(\mathbb{R})$ or $B$ . Now we can check that there are two distinct classes of closed paths. The first (resp. second) one is made of paths which "cross" an even (resp. odd) number of times the plane at infinity. A small closed path in a neighbourhood of $O$ is equivalent to the trivial path described by the point $O$ alone. The $2\pi$-rotation of the hand is described by a diameter of $B$, with end points identified (topology of a circle). It is, for instance, the *closed* path $AOA'$. Without cutting the circle, it is impossible to map it continuously on the trivial closed path $O$. If we perform a second rotation of angle $2\pi$, the "total" path $AOA'BOB'$ can be transformed continuously into the path $AOA'A'OA$, which is equivalent to the trivial path. Because it has two distinct classes of closed paths, the rotation group is said to be 2-connected. It is clear that we can compose closed paths.

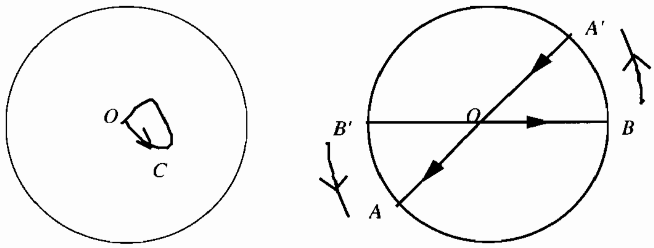


Figure 3.2: The path $C$ is a trivial path. The path $OAA'OBB'O$ is also trivial. To show it, move $B$ towards $A'$ ($B'$ towards $A$)

The path $AxA$ followed by the path $AyA$ gives the path $AxAyA$. If we call 0 the class of trivial closed paths and by 1 the other class, we arrive at a group structure, with the composition law:

$$0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0. \tag{3.5}$$

This group is isomorphic to the additive group of relative integers modulo 2 (replace 0 by *even* and 1 by *odd*). This group is, by definition, the first homotopy group of the rotation group.

## 3.3 Matrices

The isomorphism between the rotation group and the group $SO(3, \mathbb{R})$ ($S$ for *special*, $O$ for *orthogonal*) is well known. We are going to make this group acting on the space $S$ of $2 \times 2$ Hermitian[2] traceless matrices. This set is isomorphic to $\mathbb{R}^3$ since such a matrix can be written, with the aid of Pauli matrices, in the form

$$\sigma.\mathbf{r} = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{3.6}$$

Let us consider the group $SU(2)$. It is the group of unitary matrices of determinant one, *i.e.* matrices of the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \text{with } \mathrm{Det}(U) = |a|^2 + |b|^2 = 1. \tag{3.7}$$

[2]From the name of Hermite.

If we set $a = \xi + i\eta$ and $b = \zeta + i\tau$, we see that the group $SU(2)$ is homeomorphic to the real sphere $S^3$ of equation $\xi^2 + \eta^2 + \zeta^2 + \tau^2 = 1$.

We intend to make the group $SU(2)$ acting on $S$ as follows. Denote by $U$ an element of $SU(2)$ and $H$ an element of $S$. We have

$$U \, : \, H \longrightarrow UHU^*. \tag{3.8}$$

We check that it is an action on $S$. Indeed, $UHU^*$ is Hermitian

$$(UHU^*)^* = UHU^*,$$

and traceless

$$\mathrm{Tr}(UHU^*) = \mathrm{Tr}(U^*UH) = \mathrm{Tr}(H) = 0.$$

It preserves the quadratic form $x^2 + y^2 + z^2 = -\mathrm{Det}(H)$. It follows that this transformation is orthogonal. Because the unit matrix $I$ cannot be transformed in its opposite $-I$, we are sure that this transformation is a rotation. Moreover, all rotations are implemented since the unit vector $(0, 0, 1)$ can be mapped on an arbitrary unit vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$; indeed,

$$\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{-i\phi/2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

We have proved the homomorphism

$$SU(2) \longrightarrow SO(3, \mathbb{R}). \tag{3.9}$$

The kernel is composed of the matrices $Id$ and $-Id$. Denoting the group $\{Id, -Id\}$ by $Z_2$, we have proved the following isomorphism

$$SU(2)/Z_2 \sim SO(3, \mathbb{R}), \tag{3.10}$$

which implies that the matrices $U$ and $-U$ act in the same way on $S$. This property has a topological description. The matrices $U$ and $-U$ lie on the same diameter of the sphere $S_3$. Note that these two points define a straightline going through the center of the sphere. We learnt in Chapter 1 that the set of these straightlines define the projective real space $P_3(\mathbb{R})$. We verify in this way that the group $SO(3, \mathbb{R})$ has the topology of $P_3(\mathbb{R})$.

**Remarks**

1) The group $SU(2)$ is simply connected. It means that all closed paths in it are trivial or, in other words, that its first homotopy group is trivial.[3]

2) When two connected groups $G$ and $G'$ are related by a homomorphism such that $G \to G'$, and $G/H \sim G'$, one says that $G$ is a covering of $G'$. If $G$ is simply connected, it is called the *universal covering*. In that case, $H$ is the first homotopy group of $G'$. Eq. 3.10 provides us with an example of this property.

[3]The sphere $S^n$ has a trivial first homotopy group for any $n \geq 2$. The only spheres which have a group structure are $S^1$ (the group $U(1)$) and $S^3$ ( the group $SU(2)$).

## 3.4 The density matrix and the Hilbert space of states of dimension two

Let us show that the space of states associated with a Hilbert space of dimension two is a sphere $S_2$. Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ be a normalized spinor ($|\psi_1|^2 + |\psi_2|^2 = 1$). Since the state is defined up to a phase, one can set $\psi = e^{i\lambda} \begin{pmatrix} |\psi_1| e^{-i\phi/2} \\ |\psi_2| e^{i\phi/2} \end{pmatrix}$, (where $0 \leq \phi < 2\pi$), which is equivalent to $\begin{pmatrix} |\psi_1| e^{-i\phi/2} \\ |\psi_2| e^{i\phi/2} \end{pmatrix}$. Now, one can set $|\psi_1| = \cos\theta/2$ and $|\psi_2| = \sin\theta/2$, with $0 \leq \theta \leq \pi$. One verifies that the angles $\theta$ and $\phi$ parametrize the sphere $S_2$. It is not difficult to prove that the Cartesian coordinates of this sphere are

$$X = \psi^+ \sigma_1 \psi, \qquad Y = \psi^+ \sigma_2 \psi, \qquad Z = \psi^+ \sigma_3 \psi, \tag{3.11}$$

where the $\sigma_i$'s are the Pauli matrices.

In fact, each point of the ball associated with this sphere has a physical interpretation. It represents a density matrix state. We know that such a state is described by a positive matrix of trace equal to one. One may write it as

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{3.12}$$

The positivity condition implies that

$$\mathrm{Tr}(\rho^2) \leq 1, \tag{3.13}$$

the equal sign corresponding to the case of a pure state. This condition reads

$$x^2 + y^2 + z^2 \leq 1, \tag{3.14}$$

which proves our affirmation.

## 3.5 The plane rotation group

It is the group of matrices , where $\phi$ runs from 0 to $2\pi$. We have $R(0) = R(2\pi)$. It is isomorphic to the group $U(1)$ of complex numbers of modulus one. The corresponding manifold is the unit circle.

All these matrices are simultaneously diagonalized by the complex transformation defined by $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ . They become $R'(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$

The universal covering group of $U(1)$ is the additive group of real numbers. The homomorphism is described by

$$f : \mathbb{R} \to \mathrm{U}(1), \qquad \mathrm{f(x)} = \mathrm{x} - 2\pi \left[\frac{\mathrm{x}}{2\pi}\right] \tag{3.15}$$

where [ ] means "integral part". The kernel of this homomorphism is the group of numbers of the form $2\pi n$, where $n$ is an integer. This group is isomorphic to the group of integers, the group $\mathbb{Z}$.

According to what we have said in Section 3.3, we must expect that the first homotopy group of $U(1)$ is isomorphic to $\mathbb{Z}$. The proof of that fact is left to the reader. We underline the peculiar property that the group $U(1)$ is its own covering. Indeed, the homomorphism

$$\exp(i\phi) \to \exp(im\phi),$$

where $m$ is an integer has the cyclic group $Z_m$ as a kernel. It is an isomorphism if and only if $m = \pm 1$.

## 3.6 Generators (infinitesimal rotations)

Eq. (3.3) can be written in a matrix way in using the fact that multiplication by $\mathbf{u}$, namely the operator $(\mathbf{u}\times)$ is described by the matrix

$$\begin{aligned} A &= \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_y \\ -u_y & u_x & 0 \end{pmatrix} \\ &= u_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + u_y \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + u_z \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.16}$$

It is a simple exercise to derive the matrix associated with $R_{\mathbf{u}}(\phi)$. It is

$$R_{\mathbf{u}}(\phi) \simeq I + \sin\phi A + (1 - \cos\phi)A^2. \tag{3.17}$$

Let us examine the case where $\phi$ is small. If we neglect the terms of order higher than one for $\phi$, we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \phi \left( u_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + u_y \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + u_z \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

It follows that one can write, as physicists do,

$$R_{\mathbf{u}}(\phi) = Id - i\phi(\mathbf{u}.\mathbf{L}) \tag{3.18}$$

where the "scalar product" $\mathbf{u}.\mathbf{L}$ represents the expression $u_xL_x + u_yL_y + u_zL_z$, the operators $L_x$, $L_y$, $L_z$ being represented by the Hermitian matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

The operators $L_x$, $L_y$, $L_z$ are called the generators of the rotation group associated with the orthogonal axes $Ox$, $Oy$, $Oz$. We verify the following commutation relations

$$[L_x, L_y] = iL_z, \quad [L_y, L_z] = iL_x, \quad [L_z, L_x] = iL_i. \tag{3.20}$$

We note that the set of operators -$i\phi(\mathbf{u}.\mathbf{L})$ is represented by the (real) vector space of real antisymmetric matrices. We have

$$[-i\phi(\mathbf{u}.\mathbf{L}), -i\phi'(\mathbf{u}'.\mathbf{L})] = -i\phi\phi'((\mathbf{u} \times \mathbf{u}').\mathbf{L}).$$

The commutator of two real antisymmetric matrices is a real antisymmetric matrix. The commutator defines a Lie algebra structure, characterized by the properties

$$[X,Y] = -[Y,X] \qquad \text{(antisymmetry)} \tag{3.21}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \text{(Jacobi identity).}^4 \tag{3.22}$$

The matrix $A$ of Eq. (3.16) has a nice property: for $n$ a positive integer, one has

$$A^{2n+1} = (-)^n A, \quad A^{2n+2} = (-)^n A^2. \tag{3.23}$$

It follows that Eq. (3.17) reads

$$\begin{aligned} R_{\mathbf{u}}(\phi) &\sim I + \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots\right) A + \left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \dots\right) A^2 \\ &= (\phi A)^0 + \frac{(\phi A)^1}{1!} + \frac{(\phi A)^2}{2!} + \frac{(\phi A)^3}{3!} + \dots = e^{\phi A}. \end{aligned} \tag{3.24}$$

One deduces, from (3.16),

$$R_{\mathbf{u}}(\phi) = \exp(-i\phi(\mathbf{u}.\mathbf{L})). \tag{3.25}$$

We have to check the convergence of the series. It is a simple matter to show that the convergence of the standard series of $\cos\phi$ and $\sin\phi$ for any value of $\phi$ guarantees this convergence. Eq. (3.25) proves that there is a mapping from the Lie algebra of the rotation group and the rotation group itself. This mapping is known as the *exponential mapping.* Its image is the group itself.

Let us add a word about notation. The rotation group being isomorphic to the group $SO(3,\mathbb{R})$, the corresponding Lie algebra is usually denoted by $so(3,\mathbb{R})$.

## 3.7 The canonical generators of $SO(3)$

We can always choose an orthonormal basis in such a way that a given rotation of angle $\phi$ is described by the matrix of Eq. (3.4). For $\phi \neq 0$, it is a simple matter to check that the only eigenvectors are the ones collinear to $\mathbf{u}$ , with eigenvalue 1 (the axis of a rotation is fixed). However, if we complexify the space, we get three orthogonal eigendirections associated with the eigenvalues $\exp(-i\phi)$, 1 and $\exp(i\phi)$. These directions are , $\begin{pmatrix} x \\ ix \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ and $\begin{pmatrix} x \\ -ix \\ 0 \end{pmatrix}$, respectively. According to Eq. (3.25), the matrix associated with $R_{\mathbf{u}}(\phi)$ and $\mathbf{u}.\mathbf{L}$ for $\mathbf{u}$ in the $z$ direction are

$$R_{\mathbf{u}}(\phi) \sim \begin{pmatrix} \exp(-i\phi) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i\phi) \end{pmatrix} \quad \text{and} \quad \mathbf{u}.\mathbf{L} = L_z \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The unitary matrix $\begin{pmatrix} a & -ia & 0 \\ 0 & 0 & -1 \\ -a & -ia & 0 \end{pmatrix}$, where $A = 1/\sqrt{2}$ realizes this transformation which maps the expressions (3.19) into

$$L_x \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The matrices $L_+ = L_x + iL_y$, $L_- = L_x - iL_y$ and $L_z$ are called (improperly) the canonical generators of the rotation group $SO(3)$.

## 3.8 The generators of $SU(2)$

Eqs (3.16) and (3.18) permit to verify that a rotation of angle $\phi$ ($0 \leq \phi \leq \pi$) around the unit vector $\mathbf{u}$ is described by one of the two unitary matrices

$$U_{\mathbf{u}}^{\pm} = \pm \begin{pmatrix} \cos\frac{\phi}{2} - iu_z \sin\frac{\phi}{2} & -i(u_x - iu_y)\sin\frac{\phi}{2} \\ -i(u_x + iu_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + iu_z \sin\frac{\phi}{2} \end{pmatrix} \tag{3.26}$$

If we enlarge the range of the values taken by $\phi$ (namely $0 \leq \phi \leq 2\pi$), we can suppress the sign $\pm$ but we have to give a meaning to the matrices $U_{\mathbf{u}}(\phi)$ for $\phi = 2\pi$ whatever is $\mathbf{u}$.

Let us introduce the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.27}$$

one can write these matrices in a form similar to (3.18), namely

$$U_{\mathbf{u}}(\phi) = \cos\frac{\phi}{2} 1 - i(\sigma.\mathbf{u})\sin\frac{\phi}{2}. \tag{3.28}$$

**Remarks**

1) When $\phi$ is small, we get $U_{\mathbf{u}}(\phi) = 1 - i(\sigma.\mathbf{u})$. The Pauli matrices form a basis for the Lie algebra of $SU(2)$. If we divide them by two, they obey the commutation relations (3.20). It follows that the Lie algebras $su(2)$ and $so(3, \mathbb{R})$ of $SU(2)$ and $SO(3, \mathbb{R})$ are isomorphic.

2) The exponential mapping $\exp(-i(\sigma.\mathbf{u})\frac{\phi}{2})$ maps the Lie algebra on the group $SU(2)$.

3) We have $\mathrm{Tr}(U_{\mathbf{u}}(\phi)) = 2\cos\frac{\phi}{2} = \frac{\sin\phi}{\sin\frac{\phi}{2}}$. Note that we have obtained earlier, for a rotation, the relation $\mathrm{Tr}(R_{\mathbf{u}}(\phi)) = 1 + 2\cos\phi = 4\cos^2\frac{\phi}{2} - 1 = \frac{\sin\frac{3\phi}{2}}{\sin\frac{\phi}{2}}$. These formulas are special cases of the general formula $\frac{\sin\frac{D\phi}{2}}{\sin\frac{\phi}{2}}$ where $D$ is the dimension of the irreducible representation (representation of spin $\frac{D-1}{2}$).

## 3.9 The Cayley mapping

Let $C$ be a $3\times 3$ antisymmetric real matrix ($C = -\tilde{C}$). Then the matrix $R = \frac{1-C}{1+C}$, where 1 denotes the identity matrix, is orthogonal. Indeed

$$\tilde{R}R = \frac{1-\tilde{C}}{1+\tilde{C}}\frac{1-C}{1+C} = \frac{1+C}{1-C}\frac{1-C}{1+C} = 1. \tag{3.29}$$

Since $1+C$ and $1-C$ are transposed matrices, they have the same determinant and $\mathrm{Det}(R) = 1$. Therefore, $R$ is a rotation. This mapping is known as the Cayley mapping.

Conversely, given a rotation $R$, the matrix $C = \frac{1-R}{1+R}$, whenever it makes sense, is antisymmetric. $C$ is not defined iff $R$ has -1 as an eigenvalue, that is iff $R$ is a rotation of angle $\pi$. It follows that the Cayley mapping maps the Lie algebra of the rotation group on the set of rotations of angle less than $\pi$. This set is represented by the open ball of Fig. 3.2, a set which is obviously homeomorphic to the three-dimensional real space.

A similar calculation can be made for the group $SU(2)$. Suppose $K$ is a traceless antiHermitian matrix, that is $K^* = -K$ and $\mathrm{Tr}(K) = 0$. Let us prove that $U = \frac{1-K}{1+K}$ belongs to $SU(2)$. It is easy to check that $U$ is unitary:

$$U^*U = \frac{1-K^*}{1+K^*}\frac{1-K}{1+K} = \frac{1+K}{1-K}\frac{1-K}{1+K} = 1. \tag{3.30}$$

In order to prove that $\mathrm{Det}(U) = 1$, we use the Cayley-Hamilton formula

$$X^2 - \mathrm{Tr}(X)X + \mathrm{Det}(X)1 = 0, \tag{3.31}$$

for the two matrices $1\pm K$. Since $\mathrm{Tr}(1\pm K) = \mathrm{Tr}(1)\pm\mathrm{Tr}(K) = 2$,

$$(1\pm K)^2 - 2(1\pm K) + \mathrm{Det}(1\pm K)1 = 0.$$

Therefore $\mathrm{Det}(1\pm K)1 = 1 - K^2$ and $\mathrm{Det}(1+K) = \mathrm{Det}(1-K)$ from which we get

$$\mathrm{Det}(U) = \frac{\mathrm{Det}(1-K)}{\mathrm{Det}(1+K} = 1.$$

Conversely, let us give ourselves a matrix $U$ of $SU(2)$. We have

$$\left(\frac{1-U}{1+U}\right)^* = \frac{1-U^*}{1+U^*} = \frac{1-U^{-1}}{1+U^{-1}} = \frac{U-1}{U+1} = -\frac{1-U}{1+U}.$$

This is an antiHermitian matrix. Moreover, we have

$$\mathrm{Tr}\left(\frac{1-U}{1+U}\right) = \mathrm{Tr}\left(\frac{1-U^*}{1+U^*}\right)^* = -\mathrm{Tr}\left(\frac{1-U}{1+U}\right) = 0.$$

We have to look for the meaning of the expressions $\frac{1-K}{1+K}$ and $\frac{1-U}{1+U}$. In order to determine it, we consider the case where $K$ and $U$ are diagonal. The only diagonal traceless anti-Hermitian matrices are $\pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The matrix $\frac{1-K}{1+K}$ is always defined. According to Eq. (3.7), the only diagonal unitary matrix of determinant one with eigenvalue $-1$ is the matrix $-1$. It follows that the mapping maps the Lie algebra $su(2)$ on $SU(2) - \{-1\}$.

## 3.10 $SO(3,\mathbb{R})$ as the quotient $U(2)/U(1)$

One can make the group $U(2)$ acting on the space $S$ of Hermitian $2\times 2$ matrices as follows.

$$V \in U(2) \ : H \to VHV^*. \tag{3.32}$$

The matrix $V$ satisfies the relation $V^*V = 1$. It follows that $\mathrm{Det}(V)$ is of modulus one, say $\exp(i\phi)$. This permits us to set $V = \pm\exp(i\phi/2)U$, where $U$ is of determinant one. We may write:

$$V = \pm\exp(i\phi/2)\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \text{with } |a|^2 + |b|^2 = 1. \tag{3.33}$$

It acts as the identity if it leaves unchanged the three Pauli matrices. It is a simple matter to prove that, in that case, $V$ is of the form

$$V = \pm\exp(i\phi/2)Id.$$

These matrices form a group isomorphic to $U(1)$. It follows that $SO(3,\mathbb{R})$ is isomorphic to the quotient $U(2)/U(1)$.

## 3.11 The geometry on the sphere $S_2$

According to the definition of a geometry, the expression "geometry of the sphere" is unprecised except if we say which group is acting on the sphere and how it acts. Here we want to make the rotation group $SO(3,\mathbb{R})$ acting in the ordinary way. We may call this geometry $[SO(3,\mathbb{R}), \mathrm{S}_2]$. We already know that the rotation group acts transitively and effectively on the sphere. What we want to examine is the action of the group on some figures. First, circles, then oriented circles, and finally figures made of two points (couples, turns and chords).

**Circles** Obviously two circles are congruent if they have the same radius $r$. The radius takes all the values between zero and $R$, the radius of the sphere. For $r = 0$, we get points of $S_2$; for $r = R$, we get geodesics (great circles). Let us first consider the generic case corresponding to a circle of radius $r$ with $0<r<R$. Choosing a circle of radius $r$ is equivalent to indicating its center. Consider the orbit of circles of radius $r$. Since the set of centers is the sphere of radius $\sqrt{R^2 - r^2}$, we see that the geometry is still the spherical geometry. The set of such centers can be interpreted as the set of rotations of angle $\theta$, with $0 < \theta < \pi$ (small circles). For $r = 0$, we get $S_2$ itself as an orbit, and again the spherical geometry. For $r = R$, we get the great circles which have the same center $O$ (the center of the sphere). Let us find the stabilizer of a great circle, say the equator. Clearly, this stabilizer contains the rotation group $SO(2,\mathbb{R})$ around the poles, but it contains also all rotations of angle $\pi$ around a diameter of the equator. This proves that there exists a subgroup of $SO(3,\mathbb{R})$ which contains $SO(2,\mathbb{R})$. In the matrix form, these transformations are

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

| | $d=0$ | $d\in]0,\pi R[$ | $d=\pi R$ |
|---|---|---|---|
| $S_2\times S_2$ Orbits of couples | $S_2$ | $P_3(\mathbb{R})$ | $S_2$ |
| $P_3(\mathbb{R})$ Orbits of turns (or circles) | $S_2$ | $S_2$ | $P_2(\mathbb{R})$ |
| $\underline{S_2\times S_2}$ Orbits of chords | $S_2$ | $SO(3,\mathbb{R})/C_2$ | $P_2(\mathbb{R})$ |
| Orbits of oriented circles | $S_2$ | $S_2$ | $S_2$ |

Table 3.1: Orbits of circles, oriented circles, couples, turns, and chords.

and

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A look on the effect of the $\pi$ rotation on the equator shows that this group is isomorphic to $O(2,\mathbb{R})$. We will see later on that the orbit $SO(3,\mathbb{R})/\mathrm{O}(2,\mathbb{R})$ is isomorphic to the real projective plane $P_2(\mathbb{R})$. We obtain, in this way, the geometry $[SO(3,\mathbb{R}),\mathrm{P}_2(\mathbb{R})]$.

**Oriented circles** Let us first consider a small oriented circle (even of zero radius). We can associate with it both its center and a unit vector to indicate the orientation. The orbit is a sphere. Let us now find the stabilizer of an oriented great circle, say the oriented equator. Clearly, it is a subgroup of the stabilizer of the equator itself, that is, according to what we have just proved, a subgroup of $O(2,\mathbb{R})$. It is not difficult to verify that this stabilizer is $SO(2,\mathbb{R})$. We will now show that the orbit $SO(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is isomorphic to the sphere $S_2$; it follows that every geometry is of the type $[SO(3,\mathbb{R}),\mathrm{S}_2]$.

**Couples** Let us consider the action of the rotation group on the direct product $\Sigma = S_2\times S_2$, that is the set of ordered pairs of points (couples) on $S_2$. It is a four-dimensional manifold. In general, there is a unique geodesic (great circle) which links the two points of a couple (generic case). The exceptions are when the two points are the same (null couples) or the ends of a diameter (diameter couples). Let us denote by $\Sigma^*$ the subset of $\Sigma$ made of all generic couples. It is easy to see that the stabilizer of a generic couple is trivial. Each orbit of $\Sigma^*$ is of the type $SO(3,\mathbb{R})$, that is $P_3(\mathbb{R})$. Let us call *length of a couple* the shortest geodesic distance between the head and the tail of it. This length is an invariant. It is a number $d$ lying in the interval $]0,\pi R[$. The set $\Sigma^*$ is the union of orbits parametrized by $d$.

A null couple ($d=0$) corresponds to a single point of $S_2$; it has, as a stabilizer, the corresponding $SO(2,\mathbb{R})$ subgroup. The associated orbit is of the type $SO(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$. We prove, in this way that $SO(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is isomorphic to $S_2$.

In the same way, one can prove that a diameter couple ($d=\pi R$) has the same type of orbit. We conclude that $\Sigma = S_2\times S_2$ has the decomposition given in Table 3.11:

**Turns**[5] Let us consider a generic couple, say an equatorial couple $(M,N)$; there is a rotation of angle $\theta$ around the poles which maps the head $M$ on the tail $N$; such a rotation

[5]For more details, consult the book by L. C. Biedenharn and J. D. Louck.

can be represented by an axial vector **OK** of length $\theta R/\pi$, this vector pointing towards the pole $P$ defined by the fact that the triplet **OM**, **ON**, **OP** is direct. We note that the point $K$ describes not only the couple $(M, N)$ but also any other equatorial couple which can be deduced from it by a rotation around the poles (sliding along the geodesics). We say that two couples are equivalent in that case and that they define a single *turn*. It is clear that to any rotation of angle $\theta$, with $0 < \theta < \pi$ corresponds a unique generic turn. We see that the set of generic turns is identical to the set of rotations of angle different from zero or $\pi$. This set is equivalent to the set of small circles (see above). By continuity, we identify the set of turns with the rotation group itself. We already saw that this set can be described by the interior of the sphere and the sphere itself where two diametrically opposite points have to be identified. It follows that two distinct diameter couples are not equivalent (except if they are opposite), in contradiction with the fact that they can be transformed one into the other through a rotation along a geodesic[6].

We just show that the action of $SO(3, \mathbb{R})$ on turns is an action of the rotation group on itself. Since this action transforms a rotation into a rotation of the same angle, we got a way of visualizing the conjugation action.

Let us examine the stabilizers of turns. The generic turn **OK** has the group $SO(2, \mathbb{R})$ as a stabilizer. Its orbit is a sphere; the invariant of such an orbit is the length $d$. A null turn lies on a point of $S_2$. Its orbit is also a sphere (the sphere $S_2$ itself); it corresponds to the value $d = 0$. A diameter turn is identical to its opposite. Its stabilizer is the group $O(2, \mathbb{R})$ and the corresponding orbit ($d = \pi R$) is isomorphic to $P_2(\mathbb{R})$. We get, therefore, the decomposition of the space of turns $P_3(\mathbb{R})$ indicated in Table 3.11.

**Chords** The set of chords is the set of constellations of order two on $S_2$,that is $\underline{S_2 \times S_2}$. Null chords coincide with points of $S_2$; therefore they lie on an orbit isomorphic to $S_2$. Diameters coincide with diameter turns; therefore they lie on an orbit isomorphic to $P_2(\mathbb{R})$. We only have to study the generic case. It is clear (make a drawing) that such a chord is invariant under a group $C_2$ with two elements generated by a $\pi$-rotation around the diameter orthogonal to the chord. All results about couples, turns and chords are indicated in Table 3.11.

[6] This is a departure from what is made in the Biedenharn and Louck book. For us, there are many "scalar" turns, each scalar turn being associated with a diameter.

## Exercises

**1.** Consider a regular $n$- polygonal of vertices $A_1$, $A_2$,..., $A_n$ and a matter point which can only have the $A_i$'s as positions. Construct the corresponding *g.a.s.* and deduce from it the cyclic group $C_n$.
**2.** With the aid of Eq. (3.28), prove the relation

$$\begin{aligned} U_{\mathbf{u}}(\phi)\sigma.\mathbf{r}U_{\mathbf{u}}(\phi)^{-1} &= \sigma.[R_{\mathbf{u}}(\phi)\mathbf{r}] \\ &= \sigma.\mathbf{r} + \sin\phi\mathbf{u}\times\mathbf{r} + (1-\cos\phi)\mathbf{u}\times(\mathbf{u}\times \mathrm{r})]. \end{aligned}$$

## Problem

We define a set of transformations $T(a_0, \mathbf{a})$ on a three-dimensional real space $S$, where $a_0$ is real and $\mathbf{a}$ a vector of $S$, such that $a_0^2 + \mathbf{a}^2 \neq 0$. The action is as follows:

$$\mathbf{r} \to \mathbf{r}' = (a_0^2 - \mathbf{a}^2)\mathbf{r} + 2a_0\mathbf{a}\times\mathbf{r} + 2(\mathbf{a}.\mathbf{r})\mathbf{a}$$

**a.** What is the action of $T(a_0, \mathbf{a})$ on $\mathbf{a}$? on a vector orthogonal to $\mathbf{a}$? Show that $T(a_0, \mathbf{a})$ is a rotation followed by a positive dilatation. Find the parameters and the axis of the transformation.
**b.** Verify that the $T(a_0, \mathbf{a})$'s form a group $G$. Does it act effectively? If not, find the group $G'$ which acts effectively.
**c.** We now define the group $\Gamma$ as the group of $2\times 2$ complex matrices $\Lambda$ acting on a spinor space and leaving the scalar product invariant up to a nonzero factor $\lambda$.

$$< \Lambda\psi|\Lambda\phi >= \lambda < \psi|\phi >, \text{ for arbitrary } |\psi > \text{ and } |\phi > .$$

Compute $\Lambda^+\Lambda$ and show that $\lambda$ cannot take an arbitrary value.
**d.** Prove that $\Lambda$ is of the form

$$\Lambda = \exp(i\gamma) \text{ , with } 0\leq\gamma < 2\pi.$$

**e.** One makes $\Gamma$ acting on $S$ as follows. One associates with each vector $\mathbf{r}$ the matrix $\sigma.\mathbf{r}$. The action of the group $\Gamma$ is given by

$$\sigma.\mathbf{r} \to \Lambda(\sigma.\mathbf{r})\Lambda^+$$

Verify that it is an action. Does $\Gamma$ act effectively? If not, find the group Γ' which does.
**f.** Prove that there exists a homomorphism

$$\Gamma' \to G'$$

and find its kernel.

# Chapter 4

# The subgroups of $SO(2,\mathbb{R})$ and $SO(3,\mathbb{R})$, polygonals and polyhedrons

## 4.1 Finite subgroups of $SO(2,\mathbb{R})$

Since the group $SO(2,\mathbb{R})$ is Abelian, each of its elements forms a single conjugacy class. This means that every subgroup has no other conjugate subgroup except itself.

For a shorthand, we will denote by $\theta$ the rotation of angle $\theta$, with

$$0 \leq \theta < 2\pi.$$

The group law (called the sum) will be denoted by the sign #. It is defined by the relation

$$\theta_1 \# \theta_2 = \theta_1 + \theta_2 - \text{Int}\left[\frac{\theta_1 + \theta_2}{2\pi}\right] 2\pi, \tag{4.1}$$

where the symbol Int [...] means "integral part". More generally,

$$\theta_1 \# \theta_2 \# \theta_3 \# \ldots \# \theta_n = \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n - \text{Int}\left[\frac{\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n}{2\pi}\right] 2\pi.$$

We will denote by $n\theta$ the sum of $n$ elements equal to $\theta$:

$$n\theta = \theta \# \theta \# \theta \# \ldots \# \theta. \tag{4.2}$$

Let $G$ be a finite subgroup of $SO(2,\mathbb{R})$ and denote by $\theta$ its smallest element. The order of $\theta$ will be the smallest integer $n$ such that $n\theta = 0$. It is easy to prove that : $\theta = \frac{2\pi}{n}$.

Let us now show that the subgroup $G$ only contains as elements the rotations 0, $\theta$, $2\theta$, $3\theta$, ..., $(n-1)\theta$. In fact, suppose that it contains some element $\theta'$ such that

$$k\theta < \theta' < (k+1)\theta.$$

Since $G$ is a group, $\theta' - k\theta$ must also belong to $G$, that is

$$0 < \theta' - k\theta < \theta,$$

which is impossible since $\theta$ denotes the smallest rotation of $G$.

We just have shown that any finite subgroup of $SO(2,\mathbb{R})$ is defined by some positive integer $n$. The corresponding subgroup is called the cyclic group $C_n$ and its order is equal to $n$ :

$$C_n = \left\{0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, \dots, (n-1)\frac{2\pi}{n}\right\}. \tag{4.3}$$

The group $SO(2,\mathbb{R})$ can also be considered as the multiplicative group of all complex numbers of modulus one. The group $C_n$ is isomorphic to the subgroup of $SO(2,\mathbb{R})$ which contains the $n$ complex numbers $z$ such that $z^n = 1$. These numbers can be written $\exp(2\pi ik/n)$, where $k = 1, 2, ..., n$. Among them, there are, by definition, $\phi(n)$ primitive roots of unity. Each primitive root is a generator of the group and corresponds to a unique oriented polygonal in the unit circle of the Cauchy plane. The function $\phi(n)$ is known as the Euler totient function. It is the number of numbers less than $n$ and prime to $n$. One has

$$\begin{aligned}
\phi(1) &= \phi(2) = 1 \\
\phi(3) &= \phi(4) = \phi(6) = 2 \\
\phi(5) &= 4 \\
\phi(7) &= 6, \text{ etc...}
\end{aligned}$$

For $p$ prime, $\phi(p) = p-1$. As a consequence, there exists four oriented regular pentagonals in the unit circle $|z| = 1$. The proofs of the following propositions are left to the reader.

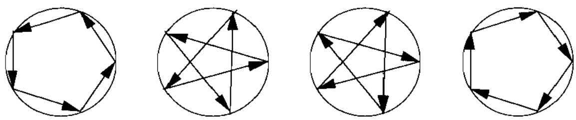

Figure 4.1: The four oriented regular pentagonals.

**Proposition 4.1** *$C_n$ contains $C_m$ as a subgroup if and only if $m$ divides $n$.*

**Proposition 4.2** *$C_n$ has no proper subgroup if and only if $n$ is prime.*

**Proposition 4.3** *The mapping $z \longrightarrow z^m$ is an automorphism of $C_n$ if and only if $m$ is relatively prime to $n$.*

**Proposition 4.4** *$\phi(n)$ is even, except for $n = 1$ and 2.*

## 4.2 Subgroups of $SO(3,\mathbb{R})$

Our intention is not to give the way one can construct the subgroups of $SO(3,\mathbb{R})$. There are quite good books where the reader could find information on that subject. We prefer to present here a rational description of the list of these subgroups. If we except the trivial subgroup $\{e\}$, each subgroup is the symmetry group of some figure. These figures can be classified in seven categories.

1. The regular pyramidals (the cyclic groups $C_n$, $n = 2, 3, 4, 5, \dots$).

2. The circular cone (the group $SO(2, \mathbb{R})$).

3. The regular prisms (the dihedral groups $D_n$, $n = 2, 3, 4, 5, \dots$).

4. The circular cylinder (the group $O_2(\mathbb{R})$).

5. The regular tetrahedron (the group $T$).

6. The cube (the regular hexahedron) or the regular octahedron (the group $O_+$).

7. The regular dodecahedron or the regular icosahedron (the group $Y_+$).

1 and 2. The cyclic group $C_n$ is composed of the $n$ rotations which leave invariant the regular $n$-pyramidal, that is a right pyramidal with a regular $n$-polygonal as a base. The case $n = 2$ corresponds to a degenerate pyramidal (an isoceles triangle). The $n$-pyramidal has $n + 1$ faces ($n$ isoceles triangles and the polygonal base), $2n$ sides ($n$ of a given length and $n$ of another given length), and $n + 1$ vertices. In the limit where $n$ goes to infinity, the group becomes the $SO(2, \mathbb{R})$ group, namely the rotation group in one dimension. This group could also be denoted $C_\infty$. All these groups are Abelian. We note that if $m$ divides $n$, $C_m$ is a subgroup of $C_n$.

3 and 4. The regular $n$-prism has $n + 2$ faces ($n$ rectangles and two regular $n$-polygonals as bases), $3n$ sides, and $2n$ vertices. The dihedral group $D_n$ has $2n$ elements and has $C_n$ as a subgroup. The other elements are rotations of angle $\pi$ which permute the two bases. In the case $n = 2$, the prism is degenerate in a rectangle. The only group which is Abelian is $D_2$. When $n$ goes to infinity, one gets the group $D_\infty$, which is isomorphic to the group $O(2, \mathbb{R})$. The group $D_n$ has two generators: a generator of $C_n$ and a $\pi$-rotation permuting the two bases of the prism.

5 to 7. The other subgroups are associated with one of the five Platon regular convex polyhedrons, namely the tetrahedron (four triangular faces, six sides, four vertices), the hexahedron or cube (six square faces, twelve sides, eight vertices), the octahedron (eight triangular faces, twelve sides, six vertices), the dodecahedron (twelve pentagonal faces, thirty sides, twenty vertices), and the icosahedron (twenty triangular faces, thirty sides, twelve vertices).

We note that each pyramidal, prism or Platon polyhedron obeys the following Euler rule: $F + V - S = 2$, where $F, V, S$ are the number of faces, of vertices, and of sides, respectively. This rule is a general one; it is valid for any convex[1] polyhedron in our three-dimensional space.

The Platon solids have the following property: if we join the centers of faces, we get again a Platon solid with the same symmetry. The number of sides is unchanged; the number of faces becomes the number of vertices and the number of vertices the number of faces. Under such a transformation, the tetrahedron becomes a tetrahedron; the cube gives rise to an octahedron and the octahedron to a cube; the dodecahedron gives rise to an icosahedron and an icosahedron to a dodecahedron. That is why the five Platon solids correspond to three groups only.

The group $T$ has 12 elements. Among the twenty-four permutations of the four vertices, only twelve are rotations (the identical rotation, eight rotations of angle 120°, and three rotations of 180°). They are the twelve even permutations of the four vertices. Note that the other twelve permutations - the odd ones - are not orthogonal transformations. The

[1] A polyhedron is said to be convex if given any face, it is located entirely on one side.

group $T$ contains four subgroups isomorphic to $C(3)$ associated with each vertex, and three subgroups isomorphic to $C(2)$ associated with each pair of opposite (orthogonal) sides.

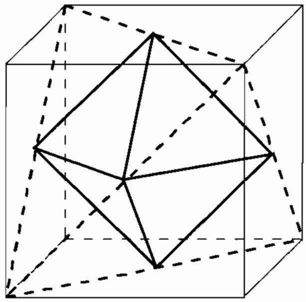

Figure 4.2: Cube, tetrahedron and octahedron.

The group $O_+$ (the symmetry group of the cube and the octahedron) has 24 elements. It is a subgroup of $O$, the complete symmetry group of the cube, that is the group with forty-eight elements we examine in a previous chapter. The other 24 elements are improper rotations (orthogonal transformations of determinant -1). There are four subgroups of $O_+$ isomorphic to $T$ (as shown on Fig. 4.2, each diagonal of the cube corresponds to two tetrahedrons with the same symmetry). This permits to see that $O_+$ contains $\underline{8}$ rotations around a diagonal of angle 120°. To each set of parallel faces corresponds a subgroup isomorphic to $C(4)$, which means that we have $3 \times 2 = \underline{6}$ rotations of angle 90° and $\underline{3}$ rotations of angle 180°. To each pair of opposite sides corresponds a subgroup isomorphic to $C(2)$, that is $\underline{6}$ rotations of angle 180°. If we add the identical rotation, we check that the number of elements of $O_+$ equals $8 + 6 + 3 + 6 + 1 = 24$.

The group $Y_+$ (the symmetry group of the icosahedron and the dodecahedron) has 60 elements. It is a subgroup of the group $Y$, a group of order 120, which includes improper rotations. Let us consider a icosahedron. To each pair of parallel faces corresponds a subgroup isomorphic to $C(3)$, that is $10 \times 2 = \underline{20}$ rotations of angle 120°. To each pair of opposite vertices corresponds a subgroup isomorphic to $C(5)$, that is $6 \times 2 = \underline{12}$ rotations of 72° and $6 \times 2 = \underline{12}$ rotations of 144°. To each pair of opposite sides corresponds a subgroup isomorphic to $C(2)$, that is $\underline{15}$ rotations of angle 180°. If we add the identical rotation, we arrive at a total of $20 + 12 + 12 + 15 + 1 = 60$ elements.

**Some relations** $s$: length of a side, $r$: radius of the inscribed sphere, $R$: radius of the circonscribed sphere.

$$2r\tan(\pi/n) = s\tan(\beta/2) \quad ,$$
$$2R = s\tan(\beta/2)\tan(\pi/m) \quad .$$

| | Tetrahedron | Cube | Octahedron | Dodecahedron | Icosahedron |
|---|---|---|---|---|---|
| Number of vertices ($V$) | 4 | 8 | 6 | 20 | 12 |
| Number of faces ($F$) | 4 | 6 | 8 | 12 | 20 |
| Number of sides ($S$) | 6 | 12 | 12 | 30 | 30 |
| Sides converg. to a vertex ($m$) | 3 | 3 | 4 | 3 | 5 |
| $\tan(\pi/m)$ | $\sqrt{3}$ | $\sqrt{3}$ | 1 | $\sqrt{3}$ | $\sqrt{5-2\sqrt{5}}$ |
| Sides of a face ($n$) | 3 | 4 | 3 | 5 | 3 |
| $\tan(\pi/n)$ | $\sqrt{3}$ | 1 | $\sqrt{3}$ | $\sqrt{5-2\sqrt{5}}$ | $\sqrt{3}$ |
| Angle of two adj. sides ($\alpha$) | 120° | 90° | 120° | 108° | 120° |
| $\cos\alpha$ | 0.5 | 0 | 0.5 | $\frac{1-\sqrt{5}}{4}$ | 0.5 |
| $\tan(\alpha/2)$ | $\frac{1}{\sqrt{3}}$ | 1 | $\frac{1}{\sqrt{3}}$ | $\sqrt{1+\frac{2}{\sqrt{5}}}$ | $\frac{1}{\sqrt{3}}$ |
| $\beta$ (angle adj. faces) | $70°32'$ | 90° | $109°28'$ | $116°34'$ | $138°11'$ |
| $\cos\beta$ | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | $-\frac{1}{\sqrt{5}}$ | $-\frac{\sqrt{5}}{3}$ |
| $\tan(\beta/2)$ | $\frac{1}{\sqrt{2}}$ | 1 | $\sqrt{2}$ | $\frac{1+\sqrt{5}}{2}$ | $\frac{3+\sqrt{5}}{2}$ |

Table 4.1: Some general properties

## Exercises

**1.** The group $C_n$ is isomorphic to the additive group of integers modulo $n$.
**2.** Find a geometrical proof of the fact that $\phi(n)$ is even for $n > 3$.
**3.** Prove the identity:

$$n = \sum_{d|n} \phi(d),$$

where the sum is taken on all divisors of $n$.
**4.** Show that, if $n$ and $m$ are relatively prime, the totient Euler function verifies $\phi(n)\phi(m) = \phi(nm)$. Any function satisfying this property is called an *arithmetical function*. Deduce from that property that, for $n = p^a q^b r^c$, where $p, q, r$ are prime numbers, $\phi(n) = n(1 - 1/p)(1 - 1/q)(1 - 1/r)$
**5.** Show that the homogeneous space $SO(2,\mathbb{R})/C_n$ is a group isomorphic to $SO(2,\mathbb{R})$. In the case where $n = 2$, this homogeneous space is isomorphic to $P_1(\mathbb{R})$.
**6.** Make $SO(2,\mathbb{R})$ acting on the set of chords of a circle. Classify the orbits and find the strata (they are two in number).
**7.** Answer the same question by considering, instead of he chords, the set of all inscribed triangles.
**8.** Draw the part of the lattice of subgroups of $SO(3,\mathbb{R})$ which implies the group $C_n$ for $n = 1$ to 13. This can be drawn without intersecting lines.
**9.** Show that a rotation of a cube is an even permutation of the vertices.
**10.** Prove that the group $O_+$ is isomorphic to the permutation group $S_4$ (Hint: make the group acting on the set of diagonals of the cube.
**11.** The even permutations of $S_n$ is, by definition the alternate group $A_n$ of $n$ elements. Show that $T$ is isomorphic to $A_3$.
**12.** Prove that $C_n$ is the invariance group of a regular oriented $n$-gonal and that $D_n$ is the invariance group of a regular $n$-gonal.
**13.** Proof of a Legendre theorem. Consider an arbitrary convex polyhedron; denote by $F_n$ the number of faces with $n$ sides. We have: $F = F_3 + F_4 + F_5 + \ldots$
a. Prove that $2S = 3F_3 + 4F_4 + 5F_5 + \ldots$ ; deduce that $F_3 + F_5 + F_7 + \ldots$ is even.
b. Use the Euler identity in order to prove the inequalities:

$$V \geq 2 + \frac{F}{2} \quad \text{and} \quad S \geq \frac{3}{2}F.$$

c. Prove that $2S \geq 3V$.
d. Prove the inequality

$$3F_3 + 2F_4 + F_5 \geq 12 + (F_7 + 2F_8 + 3F_9 + \ldots).$$

Deduce that every polyhedron has necessarily faces which are either triangular, quadrilateral or pentagonal.
e. Count the number of hexagonals and pentagonals of a football. Compute the number of faces, vertices and sides. Check the Euler identity and the last inequality.
**14.** Construct the snub cube described in the figure 4.2. Its invariance group is not $O$, but only$O_+$. It has $F = 36$ faces: $F_4 = 6$, and $F_3 = 32$. It follows, from the preceding exercise that $S = 60$. The Euler formula gives $V = S - F + 2 = 24$. Show that the 24

vertices lie on a single orbit of the cubic group $O_+$, that the square faces form an orbit of the type $O_+/C_4$, and that the 32 regular triangles form two orbits: the trivial orbit (triangles named B) and, for the eight triangles named A, the orbit $O_+/C_3$. Verify the inequalities of Exercise 12.
Check that the opposite face of a square is a square. Compare to the cube case.

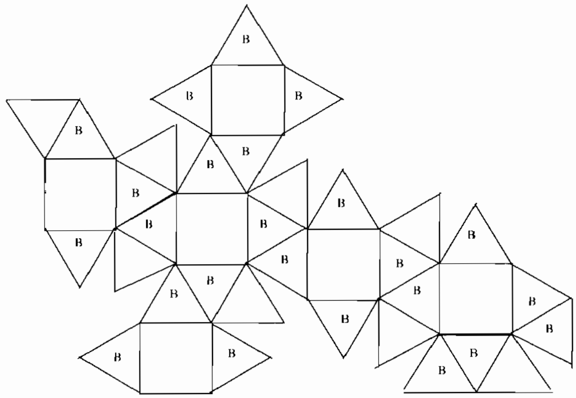


Figure 4.3: The snub cube.

# Chapter 5

# The Möbius group

## 5.1 The projective complex line $P_1(\mathbb{C})$

The complex line $\mathbb{C}$ is of complex dimension one, but because it is of real dimension two, the complex line is commonly referred to as the complex plane![1] It is better to call it the Cauchy plane.

If we add the point at infinity to the complex line, we get the extended Cauchy plane which is, in fact, homeomorphic to a sphere, the Riemann sphere. A rigorous way of constructing the extended Cauchy plane is the following. Let $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be a nonzero element of $\mathbb{C}^2$, that is a nonzero spinor[2]. We introduce an equivalence relation between the set of nonzero spinors:

$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is equivalent to $\psi' = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}$ if there exists a complex number $\lambda$ such that $\psi' = \lambda\psi$.

The projective complex line $P_1(\mathbb{C})$ is, by definition, the set of equivalence classes of spinors. This set is isomorphic to $\overline{\mathbb{C}}$ and to the Riemann sphere. Let us show it. Consider the class of the spinor $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. If $z_2 \neq 0$, a representative of its class can be obtained in multiplying this spinor by $z_2^{-1}$; it is $\begin{pmatrix} z \\ 1 \end{pmatrix}$ with $z = z_2^{-1} z_1$. These classes are parametrized by a complex number $z$. Note that this spinor is also equivalent to $\begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}$ When $z_1$ tends to zero, it becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the point at infinity $\infty$ which compactifies the plane and transforms it into $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We are interested in two ways of constructing the Riemann sphere. We start in describing the algebraic method which makes use of the Pauli matrices. For that purpose,

[1]If we do want to associate in a rigorous way the words *complex* and *plane* to denote it, we can use the expression*the real plane of complex numbers.*

[2]The word "spinor" is not well defined since it is associated with various geometries: there are rotation spinors, Lorentz spinors, Dirac spinors.

one decides to express the spinor $\psi$ in another way. We define:

$$|z_1|^2 + |z_2|^2 = \rho^2,$$
$$z_2/z_1 = \tan(\theta/2)\exp(i\phi),$$

with $0 \leq \theta \leq \pi$ and $\phi$ runs from zero to $2\pi$.

One is led to write

$$z_1 = \rho\cos(\theta/2)\exp[(-i\phi + i\alpha)/2], \tag{5.1}$$
$$z_2 = \rho\sin(\theta/2)\exp[(i\phi + i\alpha)/2]. \tag{5.2}$$

Two spinors with the same values of $\theta$ and $\phi$ are equivalent. Then one associates with the spinor $\psi$ the three-vector:

$$\mathbf{R} = \psi^+ \sigma \psi.$$

The components of this vector are

$$X = \rho\sin\theta\cos\phi, \quad Y = \rho\sin\theta\sin\phi, \quad Z = \rho\cos\theta. \tag{5.3}$$

Since $\rho$ is arbitrary, one may set $\rho = 1$. We obtain the spherical coordinates of the Riemann sphere.

Let us now turn to an important geometrical construction, the so-called *stereographic projection*. We consider a sphere of radius one and center $O$. We define on this sphere a North pole $N$ and we identify the equatorial plane with the Cauchy plane. The real and the imaginary axes are called the $X$-axis and the $Y$-axis, respectively; the vector $\mathbf{ON}$ defines the $Z$-axis. These three axes define spherical coordinates on the sphere. Let $z$ be a point of the Cauchy complex plane. The straightline joining $z$ to $N$ cuts the sphere at the point of coordinates $\theta, \phi$ such that

$$z = \cot(\theta/2)\exp(i\phi). \tag{5.4}$$

We note that the South pole (resp. North pole) corresponds to $z = 0$ (resp. $z = \infty$).

The link between the algebraic and the geometrical approaches is given by

$$z = z_1/z_2. \tag{5.5}$$

## 5.2 The homographic transformations

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an invertible $2\times 2$ complex matrix, that is an element of the group $GL(2,\mathbb{C})$, the general linear group ($ad - bc \neq 0$). One can make this group acting on the Riemann sphere $\overline{\mathbb{C}}$ or $P_1(\mathbb{C})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; : \; z \rightarrow \frac{az+b}{cz+d}. \tag{5.6}$$

Let us examine if the action is effective. The equation $z = \frac{az+b}{cz+d}$, whatever is $z$, implies

$$c = 0, \; a = d, \; b = 0.$$

These conditions define the group $\mathbb{C}^*$ of invertible diagonal scalar complex matrices[3], a group which could also be denoted[4] $GL_{diag}(1,\mathbb{C})$ (it is the center of the group $GL(2,\mathbb{C})/\mathbb{C}^*$). It follows that the group which acts effectively is $GL(2,\mathbb{C})/\mathbb{C}^*$.

**Remarks**

1) It is convenient sometimes to define homographic transformations in replacing $GL(2,\mathbb{C})$ by $SL(2,\mathbb{C})$ (see Section 1.6). Because $Z_2 < \mathbb{C}^*$, one arrives at the isomorphisms:

$$GL(2,\mathbb{C})/\mathbb{C}^* \sim \mathrm{SL}(2,\mathbb{C})/\mathrm{Z}_2 \quad \text{or,}$$

$$GL(2,\mathbb{C})/\mathrm{GL}_{\mathrm{diag}}(1,\mathbb{C}) \sim \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}_{\mathrm{diag}}(1,\mathbb{C})$$

This group is known as the Möbius group. Since it will be shown to be isomorphic to the connected Lorentz group, we will also denote it by $L$.

2) A Möbius transformation of $\overline{\mathbb{C}}$ is a bijection.

3) The group $GL(2,\mathbb{C})$ is of dimension four, the group $SL(2,\mathbb{C})$ of dimension three.

4) The Möbius group acts transitively on . Therefore the sphere is a homogeneous space of the Möbius group. We are going to prove that

$$\overline{\mathbb{C}} = L/S(2,\mathbb{R})$$

where $S(2,\mathbb{R})$ is the real similitude group $S(2,\mathbb{R})$, generated by translations, rotations and dilatations.

5) The subgroup of $GL(2,\mathbb{C})$ which leaves invariant the point at the infinity is the group of transformations $z' = az + b$ $(a \neq 0)$. Hint: write (5.6) in the form $\frac{1}{z} = \frac{c+d/z}{a+b/z}$ . If we set $z = x + iy$, $z' = x' + iy'$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, one obtains

$$\begin{aligned} x' &= a_1 x - a_2 y + b_1 \\ y' &= a_2 x + a_1 y + b_2 \end{aligned} \tag{5.7}$$

The matrices of $SL(2,\mathbb{C})$ corresponding to these transformations are of the form $\pm\begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix}$.

The subgroup of $GL(2,\mathbb{C})$ which leaves invariant an arbitrary point is conjugate, therefore isomorphic, to the group $S(2,\mathbb{R})$.

6) The most general bijection of on which is conform and direct is a Möbius transformation[5].

---

[3] $\mathbb{C}^*$ is put for $\mathbb{C} - \{0\}$.

[4] The index "diag" avoids a confusion with other isomorphic subgroups such as the ones which transform $z$ into $az$ or into $1/cz$.

[5] For a proof, see L. R. Ford, *Automorphic Functions*, page 3.

## 5.3 Fixed points of a Möbius transformation

Generally, this problem is presented as follows. The point $z$ is a fixed point if

$$cz^2 - (a - d)z - b = 0.$$

- If $c \neq 0$ and $(a-d)^2 + 4bc \neq 0$, we have two fixed points.
- If $c \neq 0$ and $(a-d)^2 + 4bc = 0$, we have one fixed point.
- If $c = 0$ and $a - d \neq 0$, there are two fixed points and, among them $z = \infty$.
- If $c = 0$ and $a - d = 0$, there is one fixed point: $z = \infty$.
- If $c = A - d = b = 0$, all points are fixed (identity transformation). If we except this last case, the number of fixed points is either one or two.

These results can be simplified if we remember that $\overline{\mathbb{C}}$ is the Riemann sphere. Instead of speaking of fixed points, it is more natural to claim that any Möbius transformation leaves a chord (or a constellation of order two) invariant. After all, from the group theoretical point of view, the Riemann sphere is a homogeneous space and the point at infinity has no privileged role[6]: whatever is the point we discard from a sphere, we are left with a set which has the topology of the plane. Then, we have the following simplified statement.

**Theorem 5.1** *Any Möbius transformation leaves a constellation of order two invariant.*

## 5.4 The real similitude group $S(2,\mathbb{R})$

The relationship between $S(2,\mathbb{R})$ and $L$ is of the same type as that between the Euclidean and the projective group. In both cases, the subgroup is derived in fixing a figure as an absolute. It is the point at infinity in the Möbius case, the straightline at infinity in the projective case. This is not surprising since we have shown that the Möbius group was also a projective group.

We can decide to find the subgroup of $S(2,\mathbb{R})$ which leaves a point of $\mathbb{C}$ fixed. We are free to choose the point and we take $z = 0$. The condition

$$0 = a.0 + b$$

gives $b = 0$. Eqs (5.7) tell us that the group is the one of $2 \times 2$ real matrices of the form

$$\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} = \sqrt{a_1^2 + a_2^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This is the direct product of the dilatation group $D(1,\mathbb{R})$ by the rotation group $SO(2,\mathbb{R})$.We deduce from that result that the set of couples of a sphere $S_2$ is the homogeneous space $L/[D(1,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})]$.

There is an interesting subgroup of $S(2,\mathbb{R})$, namely the Euclidean group $E(2,\mathbb{R})$ defined by the transformations

$$z' = az + b, \quad \text{with } |a| = 1.$$

[6]Obviously, in the framework of functions of a complex variable, the point of view is different.

It has the property of acting on $\mathbb{C} \times \mathbb{C}$ in preserving the distance $|z_1 - z_2|$. The similitude group acts on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ in preserving the 3-points ratio:

$$V(z_1, z_2, z_3) = \frac{z_1 - z_3}{z_2 - z_3}, \tag{5.8}$$

that is the ratio of lengths $\left|\frac{z_1-z_3}{z_2-z_3}\right|$ and the angle $\arg V = \arg(z_1 - z_3) - \arg(z_2 - z_3)$. Note that if two of the three points are identical, the 3-point ratio takes one of the three values 1, $\infty$, 0.

**Application** We readily see that a necessary and sufficient condition for three points to lie on a straightline is that $V$ is real. One deduces that the equation of a straightline joining $z_1$ and $z_2$ is given by

$$\frac{z - z_2}{z_1 - z_2} = \frac{z^* - z_2^*}{z_1^* - z_2^*}$$

or

$$i(z_1^* - z_2^*)z - i(z_1 - z_2)z^* + (iz_1 z_2^* - iz_2 z_1^*) = 0.$$

It means that a straightline is given by an equation of the form

$$B^* z + Bz^* + C = 0, \quad \text{with } C \text{ real.} \tag{5.9}$$

We intend to show that a permutation of the three numbers $z_1$, $z_2$, $z_3$ affects generally the value of the three point ratio. It is not difficult to get the following results:

$$\left\{\begin{array}{llll} e & V(z_1, z_2, z_3) & = & \lambda \\ (12) & V(z_2, z_1, z_3) & = & \frac{1}{\lambda} \\ (23) & V(z_1, z_3, z_2) & = & 1 - \lambda \\ (13) & V(z_3, z_2, z_1) & = & \frac{\lambda}{\lambda-1} \\ (123) & V(z_2, z_3, z_1) & = & 1 - \frac{1}{\lambda} \\ (132) & V(z_3, z_1, z_2) & = & \frac{1}{1-\lambda} \end{array}\right. \tag{5.10}$$

It follows that a nondegenerate triangle does not define uniquely the 3-point ratio; instead, one can associate with any nondegenerate triangle a set of six points (a hexagonal) contained in $\mathbb{C}$ where the points 0 and 1 are discarded. The knowledge of one of these six points implies the knowledge of the other ones. We note that this set of six points can be reduced to a set of three or two points in three cases:

- One of the three $z_i$ is the midpoint of the two other ones. The set reduces to the three numbers: -1, $\frac{1}{2}$, 2.
- The $z_i$'s form a regular triangle. The set reduces to two numbers: $\frac{1 \pm \sqrt{3}}{2}$
- If two of the $z_i$ are equal, the six numbers reduce to three, namely 0, 1, $\infty$.

**Application** the geometrical construction of the hexagonal. We know that the 3-point ratio $V(z_1, z_2, z_3)$ is invariant by a similitude. Since there exists a similitude which maps $(z_1, z_2, z_3)$ on $(z, 1, 0)$, we construct this oriented triangle. This determines uniquely the number $z$. But $V(z_1, z_2, z_3) = V(z, 1, 0) = z$. We are left with the construction of the points $1/z$, $1 - z$, $z/(z - 1)$, $1 - 1/z$ and $1/(1 - z)$.

## 5.5 The cross ratio (or biratio)

We come back to the homographic transformations. We have the following lemma.

**Lemma 5.1** *Given two nondegenerate oriented triangles* $T = (z_1, z_2, z_3)$ *and* $T' = (z_1', z_2', z_3')$*; there is a unique homographic transformation which maps* $T$ *on* $T'$*.*

*Proof* Let us first prove that there is at most one such transformation. Suppose that they are two in number, say $g$ and $g'$. One would have $g'^{-1}g(z_i) = z_i$. According to Section 3.3, the only transformation which has more than two fixed points is the identity transformation. It follows that $g = g'$.
Now, the transformation defined by the following equality

$$\frac{(z' - z_1')(z_2' - z_3')}{z' - z_2')(z_1' - z_3')} = \frac{(z - z_1)(z_2 - z_3)}{z - z_2)(z_1 - z_3)} \tag{5.11}$$

maps trivially each $z_i$ on $z_i'$, even if one of them equals $\infty$. ■

By definition, the cross ratio of four complex numbers is given by the formula

$$W(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4} \tag{5.12}$$

**Theorem 5.2** *If each of the quartets* $(z_1, z_2, z_3, z_4)$ *and* $(z_1', z_2', z_3', z_4')$ *are made of four distinct numbers and if*

$$W(z_1, z_2, z_3, z_4) = W(z_1', z_2', z_3', z_4'),$$

*there exists a unique Möbius transformation which maps each* $z_i$ *on the corresponding* $z_i'$*.*

*Proof* The transformation (5.11) maps $(z_1, z_2, z_3)$ on $(z_1', z_2', z_3')$ and conserves the cross ratio. ■

**Consequence** Any $W(z_1, z_2, z_3, z_4)$ is equal to some $W(z_1', z_2', z_3', \infty) = V(z_1', z_2', z_3')$. This implies that under a permutation of the four numbers $z_i$ , the cross ratio takes the six values mentioned above (Eq. (5.10)). The only problem is to classify the permutations of the permutation group $S_4$. The calculation gives the following results:

$$\begin{array}{llllll}
\lambda & : & e & (12)(34) & (13)(24) & (14)(23) \\
1/\lambda & : & (12) & (34) & (1423) & (1324) \\
1-\lambda & : & (23) & (2431) & (2134) & (14) \\
\lambda/(\lambda-1) & : & (13) & (1432) & (24) & (1234) \\
1-1/\lambda & : & (123) & (243) & (142) & (134) \\
1/(1-\lambda) & : & (132) & (143) & (342) & (124)
\end{array} \tag{5.13}$$

## 5.6 Harmonic conjugation

The value $\lambda = -1$ plays a peculiar role. When it takes this value, the numbers $z_1$, $z_2$, $z_3$, $z_4$ are said to be *harmonically conjugate.* We see that this value of the cross section is obviously invariant under a subgroup of $S_4$ of order eight; this subgroup contains the permutations of the first two rows of Eq. (5.13). This subgroup has two generators, namely (12) and (1423) since

$$\begin{array}{lcl} e & = & (12)(12) \\ (12)(34) & = & (1423)(1423) \\ (13)(24) & = & (1423)(12) \\ (14)(23) & = & (12)(1423) \\ (12) & = & (12) \\ (34) & = & (12)(1423)(1423) \\ (1423) & = & (1423) \\ (1324) & = & (1423)(1423)(1423) \end{array}$$

The harmonic conjugation condition can be written in an alternative way:

$$\mathrm{Perm}\begin{pmatrix} z_1 - z_3 & z_1 - z_4 \\ z_2 - z_3 & z_2 - z_4 \end{pmatrix} = 0,$$

where Perm denotes the permanent of the matrix[7]. Note that the permanent of the most general $2 \times 2$ matrix is invariant under the only following six permutations of the entries (among $4! = 24$):

- $I$: identical transformation,
- $P_r$: permutation of the rows,
- $P_c$: permutation of the columns,
- $P_{d_1}$: symmetry with respect to the main diagonal (rows replaced by columns and *vice versa*),
- $P_{d_2}$: symmetry with respect to the other diagonal,
- $P_rP_c = P_{d_1}P_{d_2}$: symmetry with respect to both diagonals.

Moreover, when this permanent is null, it is invariant under eight changes of signs of entries (among $2^4 = 16$):

- $C$: no change of signs
- $C_{ri}$: change of signs of the $i^{th}$ row entries,
- $C_{ci}$: change of signs of the $i^{th}$ column entries,
- $C_{tot} = C_{r_1}C_{r_2} = C_{c_1}C_{c_2}$: change of signs of all entries,
- $C_{d_1} = C_{r_1}C_{c_2}$: change of signs of the main diagonal entries,
- $C_{d_2} = C_{r_2}C_{c_1}$: change of signs of the second diagonal entries.

[7]The permanent of a matrix is defined in the same way the determinant is defined, except that instead of alternate signs, all signs are plus signs.

In combining these transformations (i.e. in taking the direct product), we get a group of $6 \times 8 = 48$ transformations (among a total of $24 \times 16 = 384$). Let us now express the elements of the group of order eight in terms of the permutations and changes of signs.

$$\begin{array}{lcl} e & = & IC \\ (12)(34) & = & P_r P_c \\ (13)(24) & = & P_{d_1} C_{tot} \\ (14)(23) & = & P_{d_2} C_{tot} \\ (12) & = & P_r \\ (34) & = & P_c \\ (1423) & = & P_{d_1} P_c C_{tot} \\ (1324) & = & P_{d_1} P_r C_{tot} \end{array}$$

## 5.7 Geometrical interpretation of the harmonic conjugation

The number $\arg W$ in Definition (5.12) is easily interpreted. One has

$$ArgW = \arg(z_1 - z_3) - \arg(z_2 - z_3) - \arg(z_1 - z_4) + \arg(z_2 - z_4).$$

It is the angle $\theta_3 - \theta_4$ of Fig. 5.1. When $W$ takes the value $-1$, this difference is equal to $\pi$, which implies that the quadrilateral $(z_1, z_3, z_2, z_4)$ is inscribable in a circle (see Fig. 5.1). Such a quadrilateral is said to be harmonic. Moreover, we have:

$$|z_1 - z_3||z_2 - z_4| = |z_2 - z_3||z_1 - z_4|$$

This is a well known property of harmonic quadrilaterals: the products of the lengths of

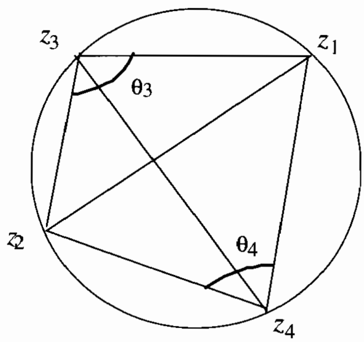


Figure 5.1: $ArgW = \theta_3 - \theta_4$

opposite sides are equal.

## 5.8 Harmonic conjugation and constellation language

The constellation language is more appropriate whenever the harmonic conjugation property is concerned. Indeed, if the cross product equals minus one, we may write

$$\text{Perm}\begin{pmatrix} z_1 - z_3 & z_1 - z_4 \\ z_2 - z_3 & z_2 - z_4 \end{pmatrix} = 0,$$

it is simpler to say that the constellations of order two $\{z_1, z_2\}$ and $\{z_2, z_4\}$ are harmonically conjugate or orthogonal. We write, in that case:

$$\text{Perm}\ (\{z_1, z_2\}, \{z_2, z_4\}) = 0.$$

This property will be generalized later on to constellations of arbitrary order.

## 5.9 Circles of $P_1(\mathbb{C})$

The equation of the circle of center $z_0$ and radius $\mathbb{R}$ in $\mathbb{C}$ is of the form

$$|z - z_0|^2 = R^2,$$

that is

$$\alpha|z|^2 + \beta z^* + \beta^* z + \gamma = 0 \tag{5.14}$$

with $\alpha \neq 0$, $\alpha$ and $\gamma$ real, $\alpha z_0 + \beta = 0$, $\alpha^2 R^2 = |\beta|^2 - \alpha\gamma$.

In order to extend the notion of circle to $P_1(\mathbb{C})$, we have to define a circle going through the point at infinity. We note that Eq. (5.14) can be written

$$\alpha + \beta/z + \beta/z + \gamma/|z|^2 = 0. \tag{5.15}$$

For $z = \infty$, this equation imposes $\alpha = 0$. According to Eq. (5.9), Eq. (5.14) describes an arbitrary straightline in $\mathbb{C}$. It becomes, by definition a circle by adjoining to this straightline the point at infinity. Such a generalized circle will be called a “circle”, with inverted commas.

**Theorem 5.3** *The Möbius group acts transitively on the set of “circles” of $P_1(C)$.*

*Proof* The proof can be obtained in replacing $z$ in Eq. (5.14) by $(az + b)/(cz + d)$. ■

One can also use the following lemma:

**Lemma 5.2** *Any element of the Möbius group can be written as the product*

$$g = shs'$$

*where $s$ and $s'$ are similitudes and $h$ is the transformation mapping $z$ on $1/z$ (inversion).*

*Proof* The lemma is proved by the identity

$$\begin{pmatrix} (bc - ad)/c & a/c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.16}$$

Since similitudes transform circles into circles and since the inversion does transform a circle into a circle or a straightline, the theorem is proved. ■

The space of "circles" is of dimension three. Let us find the stabilizer of the straight "circle" $\mathrm{Im}\,(z) = 0$, that is $z = z^*$. It is obtained in setting

$$(az + b)(cz + d)^* = (cz + d)(az + b)^*$$

It follows that $ac^*$, $ad^* + bc^*$, and $bd^*$ are real numbers. Let us decompose all entries in real and pure imaginary parts as follows

$$a = a' + ia'' \ , \quad b = b' + ib'' \ , \quad c = c' + ic'' \ , \quad d = d' + id''.$$

One obtains the conditions

$$a'' = \lambda a' \ , \quad c'' = \lambda c' \ , \quad b'' = \mu b' \ , \quad d'' = \mu d' \ , \quad (\lambda - \mu)(a'd' - b'c') = 0.$$

Because the determinant $ad - bc$ is not zero, we are sure that $a'd' - b'c'$ does not vanish. Therefore $\lambda = \mu$. We conclude that the subgroup of $GL(2, \mathbb{C})$ which stabilizes this "circle" is composed of all matrices of the form

$$(1 + i\lambda) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

We can suppose that $1 + i\lambda$ is of modulus one. The stabilizer is the direct product $U(1) \times GL(2, \mathbb{R})$ and the set of "circles" $\Gamma$ is

$$\Gamma \sim GL(2, \mathbb{C})/\mathrm{U}(1) \times \mathrm{GL}(2, \mathbb{R}). \tag{5.17}$$

We verify that this set is of dimension three (3 = the order of the group $GL(2, \mathbb{C})$, that is 8, - the order of the group $U(1) \times GL(2, \mathbb{R})$, that is 5).

If, instead of $GL(2, \mathbb{C})$, we start with $SL(2, \mathbb{C})$, we arrive at a similar result:

$$\Gamma \sim SL(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{R}). \tag{5.18}$$

## 5.10 Circles of the Riemann sphere

We are going to show that the stereographic projection maps circles on "circles". This has an important consequence: the Möbius group may be defined as the group acting on the Riemann sphere in transforming circles into circles.

Let $\mathbf{u}_0$ be a unit vector of coordinates $(\theta_0, \phi_0)$. A circle having $\mathbf{u}_0$ as an axis is of the form $\mathbf{u}.\mathbf{u}_0 = C$ (constant), that is

$$\sin\theta \sin\theta_0 (\cos\phi \cos\phi_0 + \sin\phi \sin\phi_0) + \cos\theta \cos\theta_0 = C. \tag{5.19}$$

Let $|z - z_0|^2 = R^2$ be a circle in the Cauchy plane. Using Eq. (5.4), this equation becomes

$$[\cot(\theta/2) \exp(i\phi) - z_0]\, [\cot(\theta/2) \exp(-i\phi) - z_0^*] = R^2. \tag{5.20}$$

One can identify Eqs (5.19) and (5.20) in setting

$$z_0 = \frac{\sin\theta_0}{C - \cos\theta_0} \exp(i\phi_0),$$

and

$$R^2 = \frac{1 - C^2}{(C - \cos\theta_0)^2},$$

which proves our assertion.

## Exercises

**1.** Verify that $\Gamma \sim SL(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R})$.

**2.** Find the subgroup of $SU(2,\mathbb{C})$ which stabilizes the unit circle $|z|^2 = 1$. Show that it is the group of $2 \times 2$ unimomdular matrices $U$ verifying $U^* g U = g$, where $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This group is denoted $SU(1,1)$. Deduce that

$$\Gamma \sim SL(2,\mathbb{C})/\mathrm{SU}(1,1)$$

and the isomorphism of $SL(2,\mathbb{R})$ and $SU(1,1)$.

**3.** Prove that for an isoceles triangle, the 3-point ratio is of modulus one.

# Chapter 6

# The Lorentz group and the celestial sphere

## 6.1 The celestial sphere and the map of the sky

We propose to make a distinction between the *celestial sphere*, a pure geometrical object where points are undistinguishable and the *map* of the sky, which corresponds to the celestial sphere where stars are represented. Such a distinction is necessary: as we will show, the celestial sphere is constructed from the light cone alone but the map of the sky involves the light emitted by the stars; its geometrical construction involves their worldlines in relation with the past light cone, the summit of which being the lens of a camera at a given time (an event).

In fact we introduce the three following maps:

- The instantaneous map: it is the picture of the sky taken at a given time by a terrestrial observer.

- The instantaneous Foucault map: it is the picture of the sky taken at a given time by a terrestrial observer confortably seated on a Foucault pendulum.

- The standard map of fixed stars. It will be defined later on.

## 6.2 The instantaneous map

*At a given time*, a terrestrial observer can be considered as associated with a Lorentz frame. In such a frame, an event in space-time has four coordinates, namely $t$, $x$, $y$, $z$. Suppose that our terrestrial observer takes a picture of the sky and that the optical center of his camera is at the origin $O$ of the Lorentz frame. The film is supposed to be itself spherical, of center $O$. For convenience, the figure is made in a simplified three-dimensional space-time. We consider the worldlines $D$ and $D'$ of two stars. They cut the past light cone at events $M$ and $M'$ . If $OM$ and $OM'$ lie on the same light-like straightline, the images of the stars are superposed (double star). Note that we always take pictures in the past (retarded waves).

## 6.3 The instantaneous Foucault map and the standard map

The map is now supposed to be taken from an observer seated on a Foucault pendulum. In such a map, the daily rotation of the earth has no effect. Let us take the word *instantaneous* with an approximative sense; this means that we observe the stars during, say a few minutes. The sun, the moon, and the planets are not represented. Their movement is too complicated compared to the one of the stars. In a good approximation, our observer has the feeling that stars are fixed on the celestial sphere. In fact, this is not strictly true for two reasons. The first one is that the earth is moving around the sun; the second is that stars are not rigorously fixed one with respect to each other; however their movement is imperceptible for almost all of them, especially those which are very far from the earth. They are called the *fixed stars*. We restrict ourselves to these stars and examine the problem of the earth movement. Let us denote by $F$ one of the Lorentz frames associated with the fixed stars. In this frame, all stars worldlines are straight and parallel. In a good approximation, the worldline of the sun is parallel to them. The map constructed in $F$ is the standard map.

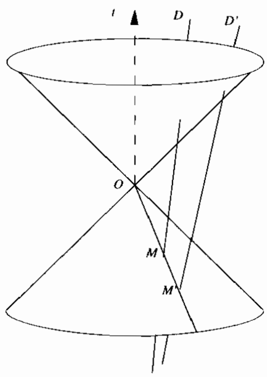


Figure 6.1: The stars corresponding to wordlines $D$ abd $D'$ seem to be superposed ($O$, $M$, $M'$ on the same light-like straightline).

It is clear that in such a frame the earth is not fixed, its worldline is not a straightline but a helix around the straight worldline of the sun[1], with the year as a period. This fact explains why the instantaneous Foucault map of the celestial sphere differs from the standard map. The helicoidal wordline is responsible of two effects. First, the earth has not the same position at two different times of the year, say $t_1$ and $t_2$; however, because the path covered by the earth is small compared to the distance of the stars, only the positions of the closest stars are perturbed; it is the parallax phenomenon, a phenomenon which does not affect the fixed stars. Second, at each time, the worldline of the earth has

[1]In order to get a better approximation, one can choose the straightworldline of the center of mass of the solar system.

a different direction in space-time. At time $t_1$, the earth can be considered as a Galilean frame different from what it is at time $t_2$. The corresponding boost $B(t_1, t_2)$ describes the phenomenon known as the aberration of fixed stars: each fixed distant star seems to have a small elliptic apparent trajectory with the year as a period (the boost $B(t_1, t_2)$ is the identity transformation for $|t_1 - t_2|$ an integral number of years. The ellipse would be a circle for a polar star, a straight segment for equatorial ones. The apparent diameter of the major axis is the same for all stars and only depends of the ratio $v/c$ (speed of the earth divided by the speed of light). This small phenomenon was used by Bradley in 1727 to evaluate the speed of light[2].

The things can be said in other words, we mean group theoretical words: the boosts $B(t_1, t_2)$ form a continuous set of Lorentz transformations one can make acting on the instantaneous Foucault map. If our observer left his seat on the Foucault pendulum and sits on an ordinary chair, these boosts must be combined with rotations. We are making a (non periodic) continuous set of Lorentz transformations acting on the instantaneous map. Group theoretical transformations are involved just by *wait and see!*

## 6.4 The celestial sphere

Up to now, we were concerned with maps, that is with camera *pictures*. These are *physical* objects. Strictly speaking, if we perform a Lorentz transformation on the map, the map will change drastically, for the same reason that two photographies of a given landscape taken at the same time from two different places are not related by a Euclidean transformation. Indeed, in a landscape, a bird leg can be hidden behind a tree on one picture and visible on the other one. However we already saw that, if we discard optical phenomena, we may define a *geometrical* object, namely the projective space, on which the Euclidean group acts. The same may be done here.

The natural mathematical object we have to introduce involves the Lorentz group action on the light cone. Stars worldlines are ignored. This object is the *celestial sphere* defined as the set of light-like straight worldlines. As in projective geometry, we do not care of the sense of propagation of light; the future and the past half-cones play the same role. It is clear that one can make the Lorentz group acting on the celestial sphere.

Our aim is to show that the Lorentz group is isomorphic to the Möbius group and that its action on the celestial sphere is equivalent to the action of the Möbius group on the Riemann sphere. Since we know that the Lorentz group acts also on the instantaneous map of fixed stars, on such a map, the isomorphism of the Möbius and the Lorentz groups has the following consequence: if four distant fixed stars lie on a given circle at time $t$, they will lie also on a circle at any time $t'$.

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[2]The aberration of fixed stars is a purely relativistic effect; it would vanish if $c$ was infinite; Bradley uses the propagation of light in the æther to explain it.

## 6.5 The Lorentz group and the circles of the celestial sphere

The events $M$ and $M'$ have coordinates $(t, x, y, z)$ and $(t', x', y', z')$ with the properties

$$\frac{t'}{t} = \frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} \tag{6.1}$$

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2 = 0. \tag{6.2}$$

The last equation can be written

$$1 - X^2 - Y^2 - Z^2 = 0. \tag{6.3}$$

where $X$ (resp. $Y$, $Z$) denotes either the ratio $x/t$ (resp. $y/t$, $z/t$) or the ratio $x'/t'$ (resp. $y'/t'$, $z'/t'$).

Our camera description of the light cone is the relativistic counterpart of what we have done in defining projective geometry. The difference between the two approaches is that we have taken now into account the fact that light propagates at a finite speed. When $c$ tends to infinity, the past and the future half light cones become not distinguishable and light rays are just straightlines in the $x, y, z$ space. Eq. (6.3) defines the celestial sphere. If we are able to prove that the Lorentz group conserves the circles of it, this would prove that the Möbius group is isomorphic to the Lorentz group. Since the Lorentz group conserves the sphere, it is enough to verify that it conserves planes in order to prove that it maps circles on circles.

Note that the equivalence relation Eq. (6.1) can be used for any direction in space-time. It is easy to check that the sets of time-like, light-like, space-like directions are represented by

$$\begin{array}{ll} \text{time-like directions:} & X^2 + Y^2 + Z^2 < 1, \\ \text{light-like directions:} & X^2 + Y^2 + Z^2 = 1, \\ \text{space-like directions:} & X^2 + Y^2 + Z^2 > 1. \end{array} \tag{6.4}$$

Let $aX + bY + cZ + d = 0$ the equation of a plane. In homogeneous coordinates, this equation reads $ax + by + cz + dt = 0$. We know that Lorentz transformations are linear; therefore they transform the equation $ax + by + cz + dt = 0$ into an equation of the form $a'x + b'y + c'z + d't = 0$, that is a plane into a plane. Since the Lorentz group acts on $\mathbb{R}^3$ in preserving planes and the celestial sphere, a Lorentz transformation is a Möbius transformation. The problem is to know if we get in this way all Möbius transformations.

It is well known that the complete Lorentz group is the union of four continuous sheets:

- $L_+^\uparrow$ : the connected Lorentz group,
- $L_-^\uparrow$ : the connected Lorentz group $\times$ parity,
- $L_+^\downarrow$ : the connected Lorentz group $\times$ time reversal,
- $L_-^\downarrow$ : the connected Lorentz group $\times$ parity $\times$ time reversal.

The subgroups of the complete Lorentz group which contain the connected subgroup as a subgroup are:

- $L_{compl} = L_+^\uparrow \cup L_-^\uparrow \cup L_+^\downarrow \cup L_-^\downarrow$, the *complete* Lorentz group,
- $L^\uparrow = L_+^\uparrow \cup L_-^\uparrow$, the *orthochronous* Lorentz group,
- $L_{unimod} = L_+^\uparrow \cup L_+^\downarrow$, the *enantiochronous* Lorentz group,
- $L_+ = L_+^\uparrow \cup L_-^\downarrow$, the *unimodular* Lorentz group,
- $L_+^\uparrow$, the *connected* Lorentz group.

By continuity arguments, it is clear that the Möbius group contains the connected Lorentz group and, therefore, is isomorphic to one of those five groups.

## 6.6 Spinors and light rays

Let $\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be a nonzero spinor and $\psi^+ = (z_1^* \ z_2^*)$ be its adjoint. We can associate with $\psi$ the $2 \times 2$ Hermitian matrix of zero determinant and positive trace:

$$H = \psi\psi^+ = \begin{pmatrix} |z_1|^2 & z_1 z_2^* \\ z_1^* z_2 & |z_2|^2 \end{pmatrix} = \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix}, \tag{6.5}$$

$$\text{Det}H = T^2 - X^2 - Y^2 - Z^2 = 0, \tag{6.6}$$

$$\text{Tr}H = 2T > 0. \tag{6.7}$$

Let $\Lambda$ be an $SL(2,\mathbb{C})$ matrix. The matrix $\Lambda H \Lambda^+$ is also Hermitian and of null determinant. This proves that the group $SL(2,\mathbb{C})$ acts on the set of $2 \times 2$ Hermitian matrices of null determinant, or the set of future light-like four-vectors.

Instead of a nonzero spinor we can use a ray-spinor, that is a spinor defined up to a complex factor ($\psi$ equivalent to $\lambda\psi$, with $\lambda \neq 0$). We would make the group $SL(2,\mathbb{C})$ acting on the set of future light-like four-vectors up to a positive factor, that is on the celestial sphere. Let us examine if it acts effectively on it. We have to look for those $SL(2,\mathbb{C})$ matrices which transform every ray-spinor into itself. We have to solve the equation

$$\Lambda\psi = \lambda\psi, \quad \text{for all } \psi.$$

If $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this equation reads

$$az_1 + bz_2 = \lambda z_1, \quad cz_1 + dz_2 = \lambda z_2,$$

that is

$$az_1 z_2 + bz_2^2 = cz_1^2 + dz_1 z_2, \quad \text{whatever are } z_1 \text{ and } z_2,$$

or

$$b = c = 0, \ a = d.$$

One conclude that the group which acts effectively is $SL(2,\mathbb{C})/\mathrm{Z}_2$. This proves that this group is isomorphic to the Möbius group. In order to know which of the five Lorentz groups we are concerned with, we have to see if parity $\Pi$, time reversal $\Theta$ and $\Pi\Theta$ are implemented.

Suppose $\Pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. One must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} = \begin{pmatrix} T-Z & -X+iY \\ -X+iY & T+Z \end{pmatrix} \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}, \tag{6.8}$$

whatever are $T$, $X$, $Y$, $Z$. It is easy to check that this implies $a = b = c = d = 0$, which is incompatible with $\Pi^2 = Id$.

Analogous calculations for $\Theta$ and $\Pi\Theta$ gives

$$\Theta \text{ does not exist}, \quad \Pi\Theta = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.9}$$

$\Pi\Theta$ leaves all ray-spinors invariant. It follows that the group which acts effectively on the celestial sphere is the connected Lorentz group $L_+^\uparrow$ . Then we have proved that the Möbius group is isomorphic to $L_+^\uparrow$ , a fact which could be related to the connectedness of the Möbius group.

## 6.7 The isomorphism of $SL(2,\mathbb{C})$ and $Sp(2,\mathbb{C})$

Let us introduce the alternate form of the determinant on $\mathbb{C}^2$, that is the antisymmetric tensor $\epsilon_{ab}$, with components

$$\epsilon_{12} = -\epsilon_{21} = 1, \ \epsilon_{11} = \epsilon_{22} = 0.$$

This tensor induces a symplectic product on $\mathbb{C}^2$ as follows:

$$(\psi, \phi) = -(\phi, \psi) = (\epsilon\psi)^T\phi = \epsilon_{ab}\psi^b\phi^a.$$

We define the covariant components of a spinor by the relation:

$$\psi_a = \epsilon_{ab}\psi^b. \tag{6.10}$$

It follows that

$$\psi_1 = \psi^2, \ \psi_2 = -\psi^1. \tag{6.11}$$

We may write:

$$(\psi, \phi) = \psi_a\phi^a = \psi_1\phi^1 + \psi_2\phi^2 = \psi^2\phi^1 - \psi^1\phi^2 = -\psi_1\phi_2 + \psi_2\phi_1.$$

Let $\Lambda$ be an $Sp(2,\mathbb{C})$ transformation, that is a transformation satisfying

$$(\Lambda\psi, \Lambda\phi) = (\psi, \phi). \tag{6.12}$$

We have:

$$(\Lambda\psi, \Lambda\phi) = (\epsilon\Lambda\psi)^T\Lambda\phi = \psi^T\Lambda^T\epsilon^T\Lambda\phi = \psi^T\epsilon^T\phi.$$

This relation must be verified whatever are the spinors $\psi$ and $\phi$. One gets the condition

$$\Lambda^T\epsilon^T\Lambda = \epsilon^T. \tag{6.13}$$

It is easy to verify that this relation is equivalent to $\mathrm{Det}(\Lambda) = 1$. This proves the isomorphism of $Sp(2,\mathbb{C})$ and $SU(2,\mathbb{C})$.

## 6.8 The Lorentz group and the set of oriented circles of the celestial sphere

Let us consider the oriented unit circle defined by the points $(-1, i, 1)$ in the extended Cauchy plane, that is by the spinors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, given in this order, and let us find the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, of $SL(2, \mathbb{C})$ which maps it into the opposite circle $(1, i, -1)$. One obtains

$$-a + b = -c + d, \quad ia + b = i(ic + d) \quad a + b = -c - d \tag{6.14}$$

that is $a = 0$, $b = -1$, $c = 1$, $d = 0$. This transforms the four-vector $\begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix}$, into

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} T-Z & -X+iY \\ -X-iY & T+Z \end{pmatrix}$$

one recognizes the action of the parity operator $\Pi$. We deduce that the Lorentz group which acts effectively on the set of oriented circles is the group $L^{\uparrow}$.

## 6.9 The polar decomposition of an $SL(n, \mathbb{C})$ matrix

Let $\Lambda$ be a matrix of $SL(n, \mathbb{C})$; it is easy to check that the matrix $\Lambda\Lambda^*$ is Hermitian with positive eigenvalues (positive matrix). $\Lambda\Lambda^*$ can be diagonalized with the aid of a matrix $X$; its positive square root $H$ is uniquely defined as follows:

$$\Lambda\Lambda^* = H^2, \quad XH^2X^{-1} = H_0^2, \quad H = X^{-1}H_0X.$$

Let us set

$$U = H^{-1}\Lambda, \tag{6.15}$$

The matrix $U$ is unitary since $UU^* = H^{-1}\Lambda\Lambda H^{-1} = Id$.

The relation $\Lambda = HU$ is called the polar decomposition because it generalizes the polar decomposition of a complex number (take $n = 1$). Now, consider the matrix $\Lambda^*\Lambda$. For analogous reasons, it is the square of a Hermitian matrix $K$. We have

$$\Lambda\Lambda^*\Lambda = H^2\Lambda = \Lambda K^2,$$

that is

$$K^2 = \Lambda^{-1}H^2\Lambda = (K\Lambda)^{-1}H_0^2X\Lambda$$

and $K$ is uniquely determined by

$$K = \Lambda^{-1}X^{-1}H_0X\Lambda = \Lambda^{-1}H\Lambda = U^*H^{-1}H\Lambda.$$

This provides us with a second polar decomposition

$$\Lambda = UK. \tag{6.16}$$

which has the same unitary part as the first one. We note that $H$ and $K$ have the same spectrum.

## 6.10 The group $SO(3,\mathbb{C})$

In the present section, we intend to show the isomorphism of the Lorentz group and the complex group $SO(3,\mathbb{C})$. This isomorphism is obtained as follows. Let $\mathbf{F}$ be a $2\times 2$ complex matrix of zero trace. We can write

$$\mathbf{F} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = x\sigma_1 + y\sigma_2 + z\sigma_3 \tag{6.17}$$

where $x$, $y$, $z$ are complex numbers.

A matrix $g$ of $SL(2,\mathbb{C})$ transforms $\mathbf{F}$ into $g\mathbf{F}g^{-1}$, another $2\times 2$ complex traceless matrix with the same determinant $-x^2-y^2-z^2$. This defines an action of $SL(2,\mathbb{C})$ on $\mathbb{C}^3$. It is easy to prove that the group which acts effectively is the group $SL(2,\mathbb{C})/Z_2$. In fact:

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{implies} = d,\ b = c \\
\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{implies } b = 0, \\
\text{Det}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} &= 1 \text{ implies } a = \pm 1.
\end{aligned}$$

Note that the transformation

$$\begin{aligned}
\mathbf{F}' &= \mathbf{F} - i\gamma\mathbf{v}\times\mathbf{F} - [\gamma^2/(1+\gamma)]\mathbf{v}\times(\mathbf{v}\times\mathbf{F}) \\
&= \gamma\mathbf{F} - i\gamma\mathbf{v}\times\mathbf{F} - [\gamma^2/(1+\gamma)](\mathbf{v}.\mathbf{F})\mathbf{v},
\end{aligned} \tag{6.18}$$

where $\mathbf{v}$ is a real vector such that $\mathbf{v}^2 \leq 1$ and $\gamma = (1-\mathbf{v}^2)^{1/2}$, preserves $\mathbf{F}^2$. This proves that this transformation is a complex rotation (compare it to Formula (2.1)). Let us define the real and imaginary parts of the vector $\mathbf{F}$ in setting $\mathbf{F} = \mathbf{B} - i\mathbf{E}$. The transformation reads

$$\begin{aligned}
\mathbf{B}' &= \mathbf{B} - \gamma\mathbf{v}\times\mathbf{E} - [\gamma^2/(1+\gamma)]\mathbf{v}\times(\mathbf{v}\times\mathbf{B}) \\
&= \gamma\mathbf{B} - \gamma\mathbf{v}\times\mathbf{E} - [\gamma^2/(1+\gamma)](\mathbf{v}.\mathbf{B})\mathbf{v}
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
\mathbf{E}' &= \mathbf{E} + \gamma\mathbf{v}\times\mathbf{B} - [\gamma^2/(1+\gamma)]\mathbf{v}\times(\mathbf{v}\times\mathbf{E}) \\
&= \gamma\mathbf{E} + \gamma\mathbf{v}\times\mathbf{B} - [\gamma^2/(1+\gamma)](\mathbf{v}.\mathbf{E})\mathbf{v}
\end{aligned} \tag{6.20}$$

One recognizes the way the magnetic and the electric vectors transform under a boost of speed $\mathbf{v}$. The link with the usual notation $F^{\mu\nu}$ is the following

$$B^i = 1/2\epsilon^{ijk}F_{jk}, \quad E^i = F^{0i} \tag{6.21}$$

The $SO(3,\mathbb{C})$ invariant $\mathbf{F}^2$ decomposes into the two Lorentz invariants:

$$\Delta(\mathbf{F}) = \mathbf{B}^2 - \mathbf{E}^2 = 1/2F^{\mu\nu}F_{\mu\nu} \tag{6.22}$$

$$\Phi(\mathbf{F}) = 2\mathbf{B}.\mathbf{E} = 1/4\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} \tag{6.23}$$

The action of $SO(3,\mathbb{C})$ on $\mathbb{C}^3$ decomposes into actions on complex spheres $\mathbf{F}^2$ = constant. If $\mathbf{F}^2$ is not zero, the group which acts on the complex sphere is $SO(3,\mathbb{C})/\mathrm{SO}(2,\mathbb{C})$; since $SO(2,\mathbb{C})$ is of real dimension two, the corresponding complex sphere is of real dimension four. We are left with the case where $\mathbf{F}^2 = 0$. Take, for instance, $\mathbf{F} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, which corresponds to $\mathbf{B} = (1,0,0)$ and $\mathbf{E} = (0,-1,0)$. The stabilizer is still a subgroup of complex dimension one; it is the subgroup of $SO(3,\mathbb{C})$ matrices of the form

$$\begin{pmatrix} 1+2f^2 & 2if^2 & -2if \\ 2if^2 & 1-2f^2 & 2f \\ 2if & -2f & 1 \end{pmatrix}. \tag{6.24}$$

This corresponds to triangular matrices of $SL(2,\mathbb{C})$:

$$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$$

Indeed:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{implies } a = d,\ c = 0.$$

## 6.11 The Lie algebra of $SL(2,\mathbb{C})$

We have shown that a transformation of $SU(2)$ may always be written as the exponential of a linear real combination of Pauli matrices. We could think that, since the group $SL(2,\mathbb{C})$ is the complex form of $SU(2)$, an arbitrary element of $SL(2,\mathbb{C})$ could be written as the exponential of a linear complex combination of Pauli matrices. Such a property would mean that given an arbitrary complex unimodular matrix: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there would exist a $\mathbf{F} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$ such that $g = \exp(\mathbf{F})$. Let us compute $\exp(\mathbf{F})$.

$$\begin{aligned} \exp(\mathbf{F}) &= \sum_{n=0}^{\infty} \frac{F^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{R^{2n}}{(2n)!} + \frac{R^{2n+1}}{(2n=1)!}\frac{\mathbf{F}}{R}\right] \\ &= \cosh R + \frac{\sinh R}{R}\mathbf{F} \\ &= \begin{pmatrix} \cosh R + z\sinh(R)/R & \sinh(R)(x-iy)/R \\ \sinh(R)(x+iy)/R & \cosh R - \sinh(R)z/R \end{pmatrix}, \end{aligned} \tag{6.25}$$

where $R^2 = x^2 + y^2 + z^2$. We readily check that $\mathrm{Det}(\exp(\mathbf{F})) = 1$. Since $R$ is a complex number, we can state $R = R_1 + iR_2$, and

$$\begin{aligned} \cosh R &= \frac{\exp(R_1 + iR_2) + \exp(-(R_1 + iR_2))}{2} \\ &= \cosh(R_1)\cos(R_2) + i\sinh(R_1)\sin(R_2). \end{aligned}$$

Similarly,

$$\sinh R = \sinh(R_1)\cos(R_2) + i\cosh(R_1)\sin(R_2).$$

It is not difficult to check that any element of $SL(2,\mathbb{C})$ can be put in the form of some $\exp(\mathbf{F})$.

## 6.12 The Poynting vector

The Poynting vector $\mathbf{P}$ reads, in terms of $\mathbf{F}$ and $\mathbf{F}^*$, as the vector product

$$\mathbf{P} = \frac{i}{2}\mathbf{F} \times \mathbf{F}^*. \tag{6.26}$$

It is related to the electromagnetic energy density

$$\rho = \frac{1}{2}\mathbf{F}.\mathbf{F}^* = \mathbf{F}^*.\mathbf{F}. \tag{6.27}$$

We have the following lemma:

**Lemma 6.1** *There exists a Lorentz frame in which the Poynting vector is zero if and only if* $\mathbf{F}^2 \neq 0$.

*Proof* Suppose that $\mathbf{P} \neq \mathbf{0}$. It is easy to prove that, under a boost of speed $\mathbf{v}$ in the $\mathbf{P}$ direction, $\mathbf{P}$ and $\rho$ are transformed as follows:

$$\mathbf{P}' = (2\gamma^2 - 1)\mathbf{P} - 2\gamma^2\rho\mathbf{v},$$

$$\rho' = (2\gamma^2 - 1)\rho - 2\gamma^2\mathbf{P}.\mathbf{v}.$$

This proves that $\mathbf{P}$ conserves the same direction. Let $P$ be the length of $\mathbf{P}$, $v$ the algebraic value of $\mathbf{v}$. Expressing $\gamma$ in terms of the speed, we get the following condition for $\mathbf{P}'$ to vanish.

$$Pv^2 - 2\rho v + P = 0.$$

This is an equation with $v$ as an unknown. It has a physical solution if it has a root $v$ such that $|v| < 1$. The two roots have a positive sum ($\rho$ and $P$ are positive) and a product equal to one. It follows that the two roots are positive. If they are distinct, one of them is less than one and it is possible to make $\mathbf{P}'$ vanishing. If they are equal, they are equal to one, it is impossible to make $\mathbf{P}'$ vanishing. Let us examine this peculiar case. If $\theta$ denotes the angle between the magnetic and electric field, the last equation reads

$$B^2 - 2EB\sin\theta + E^2 = 0.$$

This equation has a real solution if and only if $\theta = \pi/2$ and $E = B$. This situation corresponds to $\mathbf{F}^2 = 0$. The lemma is proved. We note that, if $\mathbf{F}^2 = 0$, a boost in the Poynting direction leaves $\mathbf{P}$ and $\rho$ unchanged. Moreover $P = \rho$. ■

**Remark** Define the vector $\mathbf{U}$ as the vector $\mathbf{U} = \mathbf{P}/\rho$. If $\mathbf{U}$ is not a unit vector, there exists a Lorentz frame such that its transform $\mathbf{U}'$ vanishes. One has:

$$\mathbf{U}' = \frac{(1+\mathbf{v}^2)\mathbf{U} - 2\mathbf{v}}{1 + \mathbf{v}^2 - 2\mathbf{U}.\mathbf{v}} = \mathbf{0}. \tag{6.28}$$

We deduce that

$$1 - \mathbf{U}^2 = \left(\frac{1-\mathbf{v}^2}{1+\mathbf{v}^2}\right) > 0 \tag{6.29}$$

Then we have the following corollary.

**Corollary 6.1** *The vector* $\mathbf{U} = \mathbf{P}/\rho$ *obeys the inequality* $\mathbf{U}^2 \leq 1$, *the equality occuring when and only when* $\mathbf{F}^2 = 0$.

## Exercises

**1.** Prove that

$$\psi^a \frac{\partial}{\partial \phi^a} = \phi^a \frac{\partial}{\partial \psi^a}.$$

**2.** Define the inverse of the tensor $\epsilon$. Find its (contravariant) components.

**3.** Show that the formula $\Lambda^T \epsilon^T \Lambda = \epsilon^T$ proves that the representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ is equivalent to its contragredient.

**4.** Find the orbits of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$. Give the stabilizers.

**5.** Find the orbits on the set of nonzero spinors defined up to a phase. Prove that the stabilizer is unique and isomorphic to the Thales group $Th(2,\mathbb{R})$, i.e. the group generated by the translations and homotheties in $\mathbb{R}^2$.

**6.** A Dirac spinor is an element of $\mathbb{C}^4$. The group $SL(2,\mathbb{C})$ acts reducibly on this space, according to the sum $\Lambda \oplus (\Lambda +)^{-1}$. Show that there are two invariants associated with a given spinor $\psi$. They can be written $\psi^+ \gamma_0 \psi$ and $\psi^+ \gamma_0 \gamma_5 \psi$, where $\gamma_0$ and $\gamma_5$ are two square matrices satisfying $\gamma_0^2 = -\gamma_5^2 = 1$. Show that the stabilizer of a spinor for which the two invariants are zero (are both different of zero)is isomorphic to $\mathbb{R}^2$ (is trivial).

**7.** Show that the stabilizer of the Lorentz group acting on the projective Dirac space is isomorphic to the Thales group $Th(2,\mathbb{R})$, i.e. the group generated by the translations and homotheties in $\mathbb{R}^2$.

## Problem

We consider the group $SL(n,\mathbb{C})$ of all $n \times n$ complex matrices of determinant one and its action on $M_n$, the set of all $n \times n$ Hermitian matrices, defined as follows:

$$\Lambda \in SL(n,\mathbb{C}),\ \mathrm{H} \in \mathrm{M_n},\ \mathrm{H} \longrightarrow \Lambda \mathrm{H} \Lambda^+$$

**a.** Check that the group acts. Does it act effectively? transitively? freely?

**b.** Use the property that every Hermitian matrix is diagonizable by a unitary matrix and the polar decomposition to find the orbits and the strata of $M_n$.

**c.** We choose $n = 2$. What are the elements of $M_2$ which describe a density matrix? Find the interpretation of the orbits on such matrices.

# Chapter 7

# Axiomatics of spherical constellations

The spherical constellations are those defined on the sphere $S^2$. The geometry depends on the group which is chosen to act on them. Such a group could be a one which acts on $S^2$ itself. Among those possible groups, there are the Lorentz group, the group $O(3)$, the rotation group $SO(3)$.

**Definition 7.1 (Constellation)** *$f$ is called a* constellation *if it is a mapping of the two-dimensional real sphere $S_2$ in $I\!N$ such that $f^{-1}(I\!N^*)$ is a finite set. Here, $I\!N^*$ stands for $I\!N - \{0\}$. The trivial constellation $e$ is the zero mapping.*

**Definition 7.2** *Let $u$ be an element of $S_2$; $f(u)$ is called the* multiplicity *of $u$.*

**Definition 7.3** *The* order *of a constellation $f$ is the number*

$$\omega(f) = \sum_{m \in S_2} f(u) \tag{7.1}$$

**Definition 7.4** *The* apparent order *of a constellation is defined as*

$$\omega'(f) = Card[f^{-1}(I\!N^*)]. \tag{7.2}$$

*Obviously*

$$\omega'(f) \leq \omega(f). \tag{7.3}$$

**Definition 7.5** *The* degree of degeneracy *of a constellation is defined as*

$$\Delta(f) = \omega(f) - \omega'(f). \tag{7.4}$$

*If $\Delta(f) = 0$, the constellation is said to be* non degenerate.

**Definition 7.6 (Ciel)** *The set of all constellations of order $n$ is called the* ciel *of order $n$ and is denoted $\mathcal{C}_n$. The ciel of order zero is called the* empty ciel. *Note that $\mathcal{C}_1$ can be identified with the sphere itself.*
*The union of all $\mathcal{C}_n$'s, from $n = 0$ to $\infty$ is called the* united ciel *and denoted by $\mathcal{C}$.*

**Definition 7.7 (Star)** *A* star *is a constellation of apparent order one. A star is said to be simple, double triple,..., or of multiplicity $n$ if its apparent order equals $1, 2, 3, \ldots, n$.*

**Proposition 7.1** *There exists a partial ordering on the united ciel defined by*

$$f \leq g \quad \text{iff} \quad \forall u \in S_2, \; f(u) \leq g(u). \tag{7.5}$$

*This relation defines on the united ciel a lattice structure.*

**Definition 7.8** *Given two constellations $f$ and $g$, one defines the union by the relation*

$$(f \cup g)(u) = f(u) + g(u) \tag{7.6}$$

**Proposition 7.2** *The union defines an Abelian semi-group structure on the united ciel*

$$\forall f, g \in \mathcal{C}, \exists f \cup g \in \mathcal{C}. \tag{7.7}$$

*The neutral element is the trivial constellation $e$.*

**Definition 7.9** *One can associate with a given constellation $f$ a Young diagram in classifying the multiplicities in the decreasing order. The number of squares is given by the order $\omega(f)$, the number of rows by the apparent order $\omega'(f)$. A one-column diagram corresponds to a nondegenerate constellation, a one-row diagram to a star. A Young diagram defines a* class *of constellations. A constellation of order 4 belongs to one of the five following classes.*

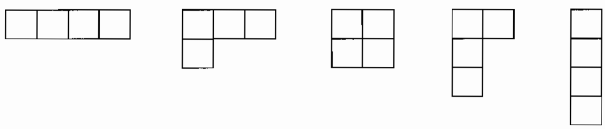


Figure 7.1: Classes of constellations of order 4

**Proposition 7.3** *Let $G$ be a group acting on the real sphere. We can make it acting on the united ciel as follows*

$$\forall f \in \mathcal{C}, \; \forall u \in S_2, \; \forall g \in G \qquad [\rho(g) f](u) = f(g^{-1} u) \tag{7.8}$$

*It is a simple matter to verify that the group leaves the classes invariant. Equivalently, the classes are subsets of orbits.*

**Theorem 7.1** *The ciel of order $n$ can be identified, as a manifold*

1) *with the symmetrized product $S_2^n / S_n$ , where $S_n$ is the symmetric group with $n$ elements,*

2) *with the set of ray-polynomials of degree $\leq n$, homogeneous with one complex variable,*

3) *with the set of ray-polynomials of degree $n$ , homogeneous in two complex variables,*

4) *with the projective space $\boldsymbol{C}^{n+1} / \boldsymbol{C}^*$, where $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$,*

5) *with the symetric space $SU(n+1)/U(n)$,*

6) *with the set of all oriented ellipses of* $I\!R^{n+1}$*, centered at the origin, having a given value of* $a^2 + b^2$ *(sum of the squares of the axes).*

*Proof of Theorem 7.1 - 1)* The space $S_2^n/S(n)$ is composed of "sets" $\{u_1, u_2, \ldots, u_n\}$, where the $u_i$'s are not necessarily distinct. The identification readily follows. ■

*Proof of Theorem 7.1 - 2)* Consider a polynomial of degree $\leq n$. It can be written in the following way:

$$p_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n. \tag{7.9}$$

Suppose its degree is $n$. Then it can be factorized as follows:

$$p_n(z) = a_0(z - z_1)^{n_1}(z - z_2)^{n_2} \ldots (z - z_k)^{n_k}, \tag{7.10}$$

with

$$\sum_{i=1}^{k} n_j = n. \tag{7.11}$$

Suppose, for instance, that $n = 8$, $k = 4$, and $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $n_4 = 2$. We can associate with it the "set" $\{z_1, z_1, z_1, z_2, z_2, z_3, z_4, z_4\}$, which characterizes a constellation by a stereographic projection on the complex line (Riemann sphere). Its order is 8 and its apparent order is 4.
Whenever the degree of the polynomial is less than $n$, say $n'$, our factorization is still valid in replacing $n - n'$ roots by $\infty$. Conversely, given a constellation of order $n$, one can associate a polynomial of degree less than or equal to $n$, *up to a factor*, since $p_n(z)$ and $\lambda p_n(z)$ have the same roots.

**Remark** Although $\mathcal{C}_n$ is the set of ray-polynomials in one variable of degree less than or equal to $n$, it is not true that $\mathcal{C}$ is the set of all ray-polynomials. ■

*Proof of Theorem 7.1 - 3)* Alternatively, let us consider a homogeneous polynomial of degree one with two complex variables, namely $az_1 + bz_2$. Suppose it is defined up to a factor. One can associate with it a complex number in the extended complex line. This complex number is $b/a$ (if $a$ is zero, it is infinite).
Any homogeneous polynomial of degree $n$ reads:

$$p_n(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \ldots a_n z_2^n. \tag{7.12}$$

Because it is defined up to a factor, it can be factorized in a unique way in homogeneous polynomials of degree one up to a factor:

$$p_n(z^1, z^2) = a_0(\alpha_1 z_1 + \alpha_2 z_2)(\beta_1 z_1 + \beta_2 z_2)(\gamma_1 z_1 + \gamma_2 z_2) \ldots (\lambda_1 z_1 + \lambda_2 z_2). \tag{7.13}$$

It follows that the set of homogeneous polynomials defined up to a factor can be identified with $S_2/S_n$. ■

*Proof of Theorem 7.1 - 4)* Any element of $\mathbb{C}^{n+1}$ has the sequence form $(a_0, a_1, a_2, \ldots, a_n)$, with which we associate the polynomial $p_n(z) = a_0 z^n + a_1 (z)^{n-1} + \ldots + a_n$. The correspondance is one-to-one. It is also one-to-one for the classes of $\mathbb{C}^{n+1}/\mathbb{C}^*$ (ray space or projective space)

$$\lambda \in \mathbb{C}^*, \ (\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, ..., \mathrm{a}_n) \sim (\lambda \mathrm{a}_0, \lambda \mathrm{a}_1, \lambda \mathrm{a}_2, \ldots, \lambda \mathrm{a}_n) \tag{7.14}$$

and the classes of polynomials

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_n \sim \lambda (a_0 z^n + a_1 z^{n-1} + \ldots + a_n). \tag{7.15}$$

■

*Proof of Theorem 7.1 - 5)* Let us make the group $SU(n+1)$ acting on $\mathbb{C}^{n+1}$. Let us show that this group acts transitively on $\mathbb{C}^{n+1}/\mathbb{C}^*$. To prove it, it is sufficient to show that, given two unit vectors $|a_1>$ and $|b_1>$, there exists a unitary transformation mapping $|a_1>$ on $|b_1>$. Let us construct an orthonormal basis $|a_i>$ $(i = 1, 2, \ldots, n)$ with $|a_1>$ as the first vector, and an orthonormal basis $|b_i>$ with $|b_1>$ as the first vector. It is easy to check that the operator

$$U = \sum_{ij} \delta_{ij} |b_j >< a_i|$$

is unitary and maps $|a_1>$ on $|b_1>$.

Let us look for the stabilizer of an element of $\mathbb{C}^{n+1}/\mathbb{C}^*$, say the element $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$, which we decide to write shortly $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is a simple matter to see that the stabilizer is of the form $U = \begin{pmatrix} a & B \\ 0 & D \end{pmatrix}$, where $B$ is a row matrix with $n$ elements and $D$ a square $n \times n$ matrix. The matrix $U$ belonging to $SU(n)$ obeys

$$|a|^2 = 1, \ B = 0, \ D^+ D = Id, ^* = \mathrm{Det} D.$$

This proves that the stabilizer is isomorphic to $U(n)$. It follows that

$$\mathbb{C}^{n+1}/\mathbb{C}^* \sim \mathcal{C}_n \sim \mathrm{SU(n+1)/U(n)}. \tag{7.16}$$

■

We deduce the following corollaries:

**Corollary 7.1** *$SU(n+1)$ acts transitively on $\mathcal{C}_n$.*

**Corollary 7.2** *$\mathcal{C}_n$ is a symplectic manifold. This property is a consequence of the Kostant-Souriau theorem according to which each coadjoint orbit of a Lie group has a canonical symplectic structure. Since the group $SUi(n+1)$ is semi-simple, we may identify adjoint and coadjoint orbits. Adjoint orbits are the ones associated with the action of the group on its Lie algebra. The Lie algebra of $SU(n+1)$ is the set of Hermitian traceless $(n+1) \times (n+1)$-matrices. Let $H$ be the diagonal Hermitian matrix with $n$ times the eigenvalue $a$, namely,*

$$H = \begin{pmatrix} -na & 0 \\ 0 & aId \end{pmatrix}$$

*Proof of Corollary* Consider the adjoint action of a matrix $U$

$$H \to UHU^+ = \begin{pmatrix} \alpha & B \\ C & D \end{pmatrix} \begin{pmatrix} -na & 0 \\ 0 & aId \end{pmatrix} \begin{pmatrix} \alpha^* & C^+ \\ B^+ & D^+ \end{pmatrix}$$

The unitary character of $U$ implies that

$$|\alpha|^2 + BB^+ = 1,$$
$$C = -\frac{DB^+}{\alpha^*}.$$

The condition $UHU^+ = H$ implies

$$B = 0, \ |\alpha| = 1, \quad \text{and } DD^+ = Id.$$

It follows that the stabilizer is isomorphic to $U(n)$. ■

*Proof of Theorem 7.1 - 6)* Let us consider the Hamiltonian of the $n+1$ dimensional harmonic isotropic oscillator

$$H = \frac{p_1^2 + p_2^2 + \ldots + p_{n+1}^2 + x_1^2 + x_2^2 + \ldots + x_{n+1}^2}{2} = \frac{\mathbf{p}^2 + \mathbf{x}^2}{2}. \tag{7.17}$$

The differential equation of the movement is

$$\frac{d^2\mathbf{x}}{dt^2} = -\mathbf{x}, \tag{7.18}$$

the solution of which is

$$\mathbf{x} = \mathbf{A}\cos t + \mathbf{B}\sin t = \mathrm{Re}\,[(\mathbf{A} + i\mathbf{B})\exp(-it)], \tag{7.19}$$

where $\mathbf{A}$ and $\mathbf{B}$ are vectors of $\mathbb{R}^n$. The corresponding trajectory is an ellipse of equation $x^2 + y^2 = 1$ in the basis $\mathbf{A}$, $\mathbf{B}$. Each solution is uniquely defined by a complex vector of $\mathbb{C}^{n+1}$. It follows that there is a one-to-one mapping between the set of solutions and $\mathbb{C}^{n+1}$.

We have

$$\mathbf{p} = \frac{d\mathbf{x}}{dt} = -\mathbf{A}\sin t + \mathbf{B}\cos t.$$

It follows that the energy is given by

$$E = \frac{\mathbf{p}^2 + \mathbf{x}^2}{2} = \mathbf{A}^2 + \mathbf{B}^2 = ||\mathbf{A} + i\mathbf{B}||.$$

We readily see that changing the energy corresponds to changing the norm of the vector $\mathbf{A} + i\mathbf{B}$. Changing the phase of this vector is equivalent to multiplying the vector $\mathbf{A} + i\mathbf{B}$ by $\exp(i\tau)$, that is to time translating. It follows that multiplying the vector $\mathbf{A} + i\mathbf{B}$ by a factor is equivalent to changing both energy and time beginning of the motion. Now fixing the energy of the movement and leaving undetermined the time beginning defines uniquely the elliptic trajectory. If we choose the vectors $\mathbf{A}$ and $\mathbf{B}$ along the axes of the ellipse, we arrive at the conclusion that the space $\mathcal{C}_n = \mathbb{C}^{n+1}/\mathbb{C}^*$ can be interpreted as the set of ellipses in $\mathbb{R}^{n+1}$, centered at the origin and obeying the relation $\mathbf{A}^2 + \mathbf{B}^2 = \mathit{constant}$. ■

# Chapter 8

# Ray-polynomials and constellations

## 8.1 Ray-polynomials of degree two

Let us consider an element $a : b : c$ of $P_2(\mathbb{C})$ and an arbitrary element $x : y$ of $P_1(\mathbb{C})$. One may associate with them what we call a ray-polynomial of degree two, namely:

$$P(a : b : c, x : y) = ax^2 + bxy + cy^2. \tag{8.1}$$

We call roots of this ray-polynomial the elements $x : y$ of $P_1(\mathbb{C})$ which makes $P(a : b : c, x : y)$ vanishing. If we fix the values of $a$, $b$, $c$, we see that this is a homogeneous polynomial of degree two.

We note that the discriminant $\Delta = b^2 - 4ac$ of the ray-polynomial is a projective quantity (i.e. defined up to a factor). If $\Delta = 0$, we have $x = y$ as a root, that is $x : y = 1 : 1$. If $\Delta \neq 0$, we have $x : y = x : \frac{1 \pm \sqrt{\Delta}}{2} x$. We do not say that a ray-polynomial has two roots, we prefer to state that a ray-polynomial has a constellation of order two as a root, the *apparent* order being one whenever $\Delta = 0$. If we remind that $P_1(\mathbb{C})$ can be identified with a sphere of radius one, we see that a ray-polynomial of degree two is associated with a chord of that sphere, the length $d$ of this chord satisfying: $0 \leq d \leq 2$.

Constellations of order two can also be considered as a special case of a complex projective quadric. Such a quadric in $P_{n-1}(\mathbb{C})$ is defined by the equation:

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = 0, \tag{8.2}$$

where, clearly, the square matrix is an element of a projective space of dimension $n^2 - 1$. In such a language, constellations of order two appear as complex projective quadrics of dimension zero.

More generally, constellations of any order $d$ can be called alternatively either complex projective surfaces of dimension zero and of degree $d$ or ray-polynomials of degree $d$ with two complex variables (or on $P_1(\mathbb{C})$).

## 8.2 Geometries of constellations of order two

We already saw that one can make the rotation group and the Lorentz group acting on the sphere. According to the Erlangen program, the geometry depends on which group is

| Length | Stability subgroup | Stratum | Dimension of the stratum |
|---|---|---|---|
| $d=0$ | $SO(2,\mathbb{R})$ | $S_2{\sim}P_1(\mathbb{C})$ | 2 |
| $0<d<2$ | $C_2$ | | 4 |
| $d=2$ | $O(2,\mathbb{R})$ | $P_2(\mathbb{R})$ | 2 |

acting. Let us start with the rotation group $SO(3,\mathbb{R})$. It is clear that the length of the chord is an invariant. When the chord is of zero length (say the North pole), the stability subgroup is isomorphic to $SO(2,\mathbb{R})$ (the rotation group around the poles). When the chord is a diameter (say the North-South diameter), the stability subgroup is isomorphic to $O(2,\mathbb{R})$. This subgroup contains the rotation subgroup $SO(2,\mathbb{R})$ and any $\pi$-rotation around an arbitrary equatorial diameter. In the generic case (that is, for instance, a chord orthogonal to the North-South axis of length $d\neq 0,\ 2$), the stability subgroup is isomorphic to the cyclic subgroup $C_2$ of order two around the North-South axis. We conclude that the four-dimensional space contains three strata: We see that the diameters can be considered as *real* constellations of order two. Each stratum is a union of orbits, as described in Section 3.11.

Let us now examine the case where the group is the $SO(3,\mathbb{C})$ group. We intend to show that the action of this group on $S_2\times S_2$ can be deduced from the one on $\mathbb{C}^3$ as follows. Let $\mathbf{F}$ be an element of $\mathbb{C}^3$ defined by

$$\mathbf{F}=\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \tag{8.3}$$

where $x$, $y$, $z$ are complex numbers. Let us denote by $\hat{\mathbf{F}}$ the class of vectors $\mathbf{F}$ defined up to a complex factor. $\hat{\mathbf{F}}$ is an element of $P_2(\mathbb{C}){\sim}\mathbb{C}^3/\mathbb{C}$. Clearly, the group $SO(3,\mathbb{C})$ acts on $P_2(\mathbb{C})$, as it was shown in Section 6.10.

Now, on can associate with any element $\hat{\mathbf{F}}$ a constellation of order two $\{z_1,z_2\}$ as follows:

$$\hat{\mathbf{F}}=\begin{pmatrix} z_1+z_2 & -2 \\ 2z_1z_2 & -(z_1+z_2) \end{pmatrix} \tag{8.4}$$

provided $x-iy$ is not zero. We note that

$$\hat{\mathbf{F}}\begin{pmatrix} 1 \\ z_1 \end{pmatrix}=(z_2-z_1)\begin{pmatrix} 1 \\ z_1 \end{pmatrix} \tag{8.5}$$

$$\hat{\mathbf{F}}\begin{pmatrix} 1 \\ z_2 \end{pmatrix}=(z_2-z_1)\begin{pmatrix} 1 \\ z_2 \end{pmatrix} \tag{8.6}$$

It follows that the constellation is invariant under the action of $\hat{\mathbf{F}}$.

Let us examine the commutator of two elements:

$$[\hat{\mathbf{F}},\hat{\mathbf{F}}']=2(z_1+z_2-z_1'-z_2')\begin{pmatrix} Z_1+Z_2 & -2 \\ 2Z_1Z_2 & -(Z_1+Z_2) \end{pmatrix} \tag{8.7}$$

where $Z_1$ and $Z_2$ are the roots of the following polynomial[1]:

$$(z_1+z_2-z_1'-z_2')Z^2-2(z_1z_2-z_1'z_2')Z+(z_1'+z_2')z_1z_2-(z_1+z_2)z_1'z_2', \tag{8.8}$$

[1]or, in our language, $\{Z_1,Z_2\}$ is the constellation root of this polynomial.

that is

$$Z_{1,2} = \frac{z_1 z_2 - z_1' z_2' \pm \sqrt{(z_1 - z_1')(z_2 - z_1')(z_1 - z_2')(z_2 - z_2')}}{z_1 + z_2 - z_1' - z_2'} \tag{8.9}$$

The constellation $\{Z_1, Z_2\}$ has a very simple geometrical interpretation, due to the properties:

$$\frac{(Z_1 - z_1)(Z_2 - z_2)}{Z_1 - z_2)(Z_2 - z_1)} = \frac{(Z_1 - z_1')(Z_2 - z_2')}{Z_1 - z_2')(Z_2 - z_1')} = -1. \tag{8.10}$$

This means that the constellation $\{Z_1, Z_2\}$ is harmonically conjugate to the constellations $\{z_1, z_2\}$ and $\{z_1', z_2'\}$. It is a simple matter to prove that those two conditions uniquely defines the constellation $\{Z_1, Z_2\}$. We arrive at a simple connection between commutators and harmonically conjugate constellations.

Let us now examine the following peculiar cases:

- The constellations $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ are non degenerate (*i.e.* of apparent order two). Since the cross ratios of Eq. (8.10) are equal, we know that there exists a homographic transformation which maps the points $Z_1$, $Z_2$, $z_1$, $z_2$ on the points $Z_1$, $Z_2$, $z_1'$, $z_2'$, respectively. This is equivalent to say that there exists a transformation mapping the constellation $\{z_1, z_2\}$ on the constellation $\{z_1', z_2'\}$ generated by the constellation $\{Z_1, Z_2\}$ (that is, with $\{Z_1, Z_2\}$ as a fixed constellation).

- Let us now suppose that $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ are real constellations, that is they correspond to diameters of the Riemann sphere ($z_1 z_2^* + 1 = z_1' z_2'^* + 1 = 0$). The commutator of these two constellations is the diameter perpendicular to the given diameters. This corresponds to the vector product in the three-dimensional real space.

- One of the two constellations, say $\{z_1', z_2'\}$ is degenerate; this means that we may write $z_1' = z_2' = z'$. One obtains:

$$\{Z_1, Z_2\} = \left\{ \frac{2z_1 z_2 - (z_1 + z_2)z'}{z_1 + z_2 - 2z'}, z' \right\} \tag{8.11}$$

- Both constellations are degenerate. One can set: $z_1 = z_2 = z$ and $z_1' = z_2' = z'$. Then

$$\{Z_1, Z_2\} = \{z, z'\} \tag{8.12}$$

- Suppose that $z_2 = z_2'$. The commutator of $\{z_1, z_2\}$ and $\{z_1', z_2\}$ is just $\{z_2, z_2\}$.

- Whenever $z_1 + z_2 = z_1' + z_2'$, Eq. (8.8) is of degree one. Since we are working with projective spaces, we must say that one of the two roots is infinite. The commutator of $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ is $\{\infty, (z_1 + z_2)/2\}$.

We can make the following statement:

**Proposition 8.1** *With each ray of the complex Lie algebra of $SL(2, \mathbb{C})$ can be associated a constellation of order two denoted $\{z_1, z_2\}$. The constellation $\{Z_1, Z_2\}$ associated with the commutator $[\{z_1, z_2\}, \{z_1', z_2'\}]$ is the one given by the cross product conditions:*

$$\text{Perm}\ (\{z_1, z_2\}, \{Z_1, Z_2\}) = \text{Perm}\ (\{Z_1, Z_2\}, \{z_1', z_2'\}) = 0, \tag{8.13}$$

*where* Perm *represents the* permanent *of the matrix:*

$$\begin{pmatrix} z_1 - Z_1 & z_1 - Z_2 \\ z_2 - Z_1 & z_2 - Z_2 \end{pmatrix} \tag{8.14}$$

Let us precise that a permanent is computed as a determinant, except that all signs are plus signs. It is symmetric with respect to the $z_i$'s and with respect to the $Z_i$'s. As we are going to show, our proposition generalizes the concept of harmonically conjugate constellations to an arbitrary order. It shows also that this concept is intimately associated with the notion of constellations.

## 8.3 Geometry of constellations of arbitrary order

Let us consider two constellations of order $n$, namely

$$\{z_1, z_2, \ldots, z_n\} \text{ and } \{Z_1, Z_2, \ldots, Z_n\}. \tag{8.15}$$

They will be said to be harmonically conjugate (h.c.) if their permanent product, defined as

$$(z, Z) = (-)^n (Z, z) = \frac{1}{n!}\text{Perm} \begin{pmatrix} z_1 - Z_1 & z_1 - Z_2 & \ldots & z_1 - Z_n \\ z_2 - Z_1 & z_2 - Z_2 & \ldots & z_2 - Z_n \\ \ldots & \ldots & \ldots & \ldots \\ z_n - Z_1 & z_n - Z_2 & \ldots & z_n - Z_n \end{pmatrix} \tag{8.16}$$

with

$$\text{Perm}\,(z_i - Z_j) = \sum_{permut} (z_1 - Z_{\sigma_1})(z_2 - Z_{\sigma_2}) \ldots (z_n - Z_{\sigma_n}), \tag{8.17}$$

with the summation extending to all permutations $\sigma$ of the $z_i'$, is zero.

It is clear that the permanent product is symmetric with respect to the $z_i$'s and with respect to the $Z_i$'s. It follows that the permanent product is of the form

$$(z, Z) = \sum_{k=0}^{n} \lambda_k S_k(z_1, z_2, \ldots, z_n) S_{n-k}(Z_1, Z_2, \ldots, Z_n), \tag{8.18}$$

where $S_k(z_1, z_2, \ldots, z_n)$ denotes the symmetric function of order $k$, namely

$$S_k(z_1, z_2, \ldots, z_n) = \sum_{\sigma} z_{\sigma_1} z_{\sigma_2} \ldots z_{\sigma_k}, \tag{8.19}$$

and the $\lambda_k$ are constants to be determined. It is easy to obtain them in supposing all $z_i$ equal to $z$ and all$Z_i$ equal to $Z$. We get

$$(z, Z) = (z - Z)^n = \sum_{k=0}^{n} (-)^k \begin{pmatrix} n \\ k \end{pmatrix} z^k Z^{n-k}.$$

Since

$$S_k(z_1, z_2, \ldots, z_n) = \begin{pmatrix} n \\ k \end{pmatrix} z^k \text{ and } S_{n-k}(Z_1, Z_2, \ldots, Z_n) = \begin{pmatrix} n \\ k \end{pmatrix} Z^{n-k},$$

one obtains

$$\lambda_k = \frac{(-)^k}{\begin{pmatrix} n \\ k \end{pmatrix}}. \tag{8.20}$$

**Remarks**

1) Two constellations of order $n$ which have more than $n/2$ stars in common are trivially h.c. (This readily follows from the fact that more than a quarter of the matrix would be composed of zeroes).

2) Two constellations of order one are h.c. if and only if they are equal.

3) Any constellation of odd order is self h.c. This property follows from the fact that any antisymmetric matrix of odd dimension has a null permanent.

4) Missing star property: Given two constellations $z = \{z_1, z_2, \ldots, z_n\}$, of order $n$ and the other $Z = \{Z_1, Z_2, \ldots, Z_{n-1}\}$ of order $n-1$, there exists a unique star $Z_n$ such that
$$(\{z_1, z_2, \ldots, z_n\}, \{Z_1, Z_2, \ldots, Z_{n-1}, Z_n\}) = 0.$$
This follows from the fact that this equation is of degree one in $Z_n$.

## Exercise

**1.** A constellation $\{a, b, c, d\}$ of order 4 can be considered as the union of two constellations of order 2 in six ways. Show that if $x$ denotes the value of the cross-ratio of one pair, the other cross-ratios have the values $1-x$, $1/x$, $1-1/x$, $1/(1-x)$, and $1-1/(1-x)$. Show that these values are distinct except when $x$ takes one of the values $-1$, 2, or $-1/2$ (harmonic constellations) or one of the two values $-1 \pm i\sqrt{3}/2$ (antiharmonic constellations). Show that the function

$$f(x) = \frac{(x^2 - x + 1)^3}{27x^2(x-1)^2} - \frac{1}{9}$$

characterizes the constellation of order 4. Check that

$$f(x) = \{ [(a-b)(b-c)(c-d)(d-a)]^3 + [(a-c)(c-b)(b-d)(d-a)]^3 + [(a-b)(b-d)(d-c)(c-a)]^3 \} / [(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)]^2.$$

Express $f(x)$ in terms of the variable $u = x + 1/x$.

# Chapter 9

# Projective classical groups

We may consider that we have defined in Chapter 8 the permanent product of two constellations on $\mathbb{C}$. Whenever this product is zero, we say that the constellations are harmonically conjugate. Then, we have shown that the notion of harmonic conjugation may be extended to constellations on the extended plane, that is on $P_1(\mathbb{C})$. We will see that the sky of constellations of order $n$ is isomorphic to $P_n(\mathbb{C})$. The harmonic conjugation defines a symmetric or antisymmetric scalar product on the projective space, according to the parity of $n$. This permits to define the projective groups $PO(n+1,\mathbb{C})$ if $n$ is even and $PSp(n+1,\mathbb{C})$ if $n$ is odd. These two groups have the Lorentz group and the rotation group $SO(3,\mathbb{R})$ as subgroups. Some applications are obtained about the Clebsch-Gordan products of constellations.

## 9.1 Harmonic conjugation of constellations on $\mathbb{C}^* \sim \mathrm{P}_1(\mathbb{C})$

Let $A$ be an element of $P_n(\mathbb{C})$ represented by a nonzero complex column of $\mathbb{C}^{n+1}$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{pmatrix} \sim \begin{pmatrix} \lambda a_0 \\ \lambda a_1 \\ \dots \\ \lambda a_n \end{pmatrix} \tag{9.1}$$

where $\lambda$ is a nonzero complex number.

We associate with $A$ the constellation $Z = [z_1, z_2, \dots, z_n]$, where the $z_i$ are the roots of the polynomial

$$a_0 z^n - \sqrt{\begin{pmatrix} n \\ 1 \end{pmatrix}} a_1 z^{n-1} + \sqrt{\begin{pmatrix} n \\ 2 \end{pmatrix}} a_2 z^{n-2} - \dots + (-)^n a_n. \tag{9.2}$$

That constellation is defined on $P_1(\mathbb{C})$ since whenever $a_0 = a_1 = \dots = a_k = 0$ with $a_{k+1} \neq 0$, $k+1$ of those roots are infinite.

The relationship between $P_n(\mathbb{C})$ and constellations on $P_1(\mathbb{C})$ defines a bijection between $P_n(\mathbb{C})$ and the sky $\mathcal{C}_n$ of constellations of order $n$ on $P_1(\mathbb{C})$. Then we can write

$$P_n(\mathbb{C}) = \mathrm{C_n}(\mathrm{P_1}(\mathbb{C})). \tag{9.3}$$

We must interpret the harmonic conjugation in $P_n(\mathbb{C})$. For this purpose, we introduce the following $(n+1)\times(n+1)$-antidiagonal matrix $g$:

$$g = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -(-)^n & 0 & \dots & 0 & 0 \\ (-)^n & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{9.4}$$

and the bilinear form $(A', A) = A'^t g A$ associated with it. It is now a simple matter to prove that

$$(A', A) = \sum_{k=0}^{n} a_0 a'_0 (-)^k S_k(z_1, z_2, \dots, z_n) S_{n-k}(z'_1, z'_2, \dots, z'_n) / \begin{pmatrix} n \\ k \end{pmatrix}, \tag{9.5}$$

which is simply

$$(A', A) = a_0 a'_0 (Z', Z). \tag{9.6}$$

We see that orthogonality defined by the matrix $g$ is equivalent to harmonic conjugation. It follows that the groups which preserve the harmonic conjugation in $P_n(\mathbb{C})$ is identical with the linear group on $\mathbb{C}^{n+1}$ which preserves the bilinear from $g$. This group is isomorphic to $PO(n+1,\mathbb{C})$ if $n$ is even and to $PSP(n+1,\mathbb{C})$ if $n$ is odd. They are the projective complex orthogonal group and the projective complex symplectic group, respectively.

## 9.2 The Möbius (Lorentz) group

The connected Möbius group is the group of homographic transformations

$$z \longrightarrow \frac{az+b}{cz+d}, \tag{9.7}$$

that is the projective linear group $PGL(2,\mathbb{C}) \sim \mathrm{PSp}(2,\mathbb{C}) \sim \mathrm{PO}(3,\mathbb{C})$ acting on $P_1(\mathbb{C})$.

The full Möbius group is two-sheeted. It contains homographic and antihomographic transformations:

$$z \longrightarrow \frac{az^*+b}{cz^*+d}, \tag{9.8}$$

where $z^*$ is the complex conjugate of $z$. The full Möbius group is the group of all holomorphic and antiholomorphic mappings which map the set of circles and straightlines into itself. It is easy to verify that the Lorentz group including parity or time-reversal is isomorphic to the full Möbius group.

Any Möbius transformation may be considered as a sequence of transformations of the following kinds:

- translations $z \to z + \alpha$
- dilations $z \to \lambda z$
- inversion $z \to 1/z$

and, eventually,

- a complex conjugation $z \to z^*$

It is a very simple matter to verify that such transformations on constellations preserve the harmonic conjugation. It follows that the homographic transformations of the Möbius group form a subgroup of $PO(n+1, \mathbb{C})$ or $PSP(n+1, \mathbb{C})$. The representation is irreducible as it follows from the classical work by Bargmann on the rotation group and polynomials. This representation is often denoted by $D_{j_0}$, with $n = 2j$.

Clearly the antihomographic transfromations act antilinearly on $P_n(\mathbb{C})$. It follows that we can consider $D_{j_0}$ as a corepresentation[1]

## 9.3 The projective unitary group $PU(n+1)$ and the rotation group $SO(3)$

Let us consider a Hilbert space $H$ of dimension $n+1$. The pure states associated with it are not vectors but rays. They form the projective space $P(H)$ isomorphic to $P_n(\mathbb{C})$. The canonical transformations of $H$ are the unitary operators; they form a group $U(H)$ isomorphic to $U(n+1)$. It is the group of linear transformations which preserves the Hermitian sacalar product $< \psi_1|\psi_2 >$ *up to a phase*, that is which preserves the expression $| < \psi_1|\psi_2 > |$ or, equivalently, the quantity

$$\frac{| < \psi_1|\psi_2 > |}{\sqrt{< \psi_1|\psi_1 >< \psi_2|\psi_2 >}} \tag{9.9}$$

The group action of $U(H)$ on $H$ (or of $U(n+1)$ on $\mathbb{C}^{n+1}$) induces an action on the set of states $P_n(\mathbb{C})$. The group which acts on $P_n(\mathbb{C})$ is usually denoted $PU(n+1)$. It can be defined either as the group preserving the vanishing of the permanent product or as the one preserving the orthogonality of states.

This last definition invites us to express the orthogonality of states in terms of constellation language. For this purpose, we introduce the notion of antipodal constellation.

**Definition 9.1** *The antipodal constellation of* $[z_1, z_2, \ldots, z_n]$ *is the constellation* $[-z_1^{*-1}, -z_2^{*-1}, \ldots, -z_n^{*-1}]$

The expression *antipodal* is justified by the stereographic projection relating $P_1(\mathbb{C})$ and the Riemann sphere $S^2$. Two constellations are antipodal if they are symmetric with respect to the center of the sphere.

**Definition 9.2** *A constellation will be said to be real if it is equal to its antipodal.*

This denomination follows from the following property. If we identify antipodal points on $S^2$, we get the projective space $P_2(\mathbb{R})$. Then, any real constellation of order $2n$ will appear as a constellation of order on $P_2(\mathbb{R})$. In other words $P_{2n}(\mathbb{R})$ is the sky of order $n$ of $P_2(\mathbb{R})$.

We now have the following proposition.

[1]A corepresentation is a representation where some elements of the group are represented antilinearly.

**Theorem 9.1** *Two constellations $Z$ and $Z'$ of order n are associated with orthogonal states if and only if $Z'$ is harmonically conjugate of $\tilde{Z}$ , where $\tilde{Z}$ denotes the antipodal constellation of $Z$.*

*Proof* Let $|\psi>$ be a representative of $Z$ in $\mathbb{C}^{n+1}$ and $|\psi'>$ a representative of $Z'$.

$$|\psi>= \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{pmatrix}, \quad |\psi'>= \begin{pmatrix} a'_0 \\ a'_1 \\ \dots \\ a'_n \end{pmatrix}.$$

According to Eq. (9.5), we have

$$\sum_{k=0}^{n} a_0 a'_0 (-)^{n-k} S_{n-k}(z_1, z_2, \dots, z_n) S_k(z'_1, z'_2, \dots, z'_n) / \begin{pmatrix} n \\ k \end{pmatrix} = 0 \tag{9.10}$$

Since

$$S_k(z'_1, z'_2, \dots, z'_n) = (-)^{n-k} S_{n-k}(-z_1'^*, -z_2'^*, \dots, -z_n'^*) z_1'^* z_2'^* \dots z_n'^*.$$

The theorem follows ■

**Remarks**

1) In the case where $n = 1$, harmonic conjugation is equivalent to equality. It follows that orthogonality coincides with antipodality, as it is well known for spin 1/2 states and for polarisaion states of the photon on the Poincaré sphere.

2) The orthogonality relation may be written in the permanent form as follows

$$< Z|Z' >= 0 \sim \text{Perm}\,[1 + z_i'^* z_j] = 0$$

3) The group which preserves both harmonic conjugation and orthogonality is the intersection of $PU(n+1)$ with $PO(n+1, \mathbb{C})$ or $PSp(n+1, \mathbb{C})$. It is isomorphic to $PO(n+1, \mathbb{R})$ or $PSp(n+1, \mathbb{R})$.

   Let us examine the $SO(3, \mathbb{R})$ subgroup of the Lorentz group. It is the group of homographic transformations of the form

$$z \longrightarrow \frac{az + b}{-b^* z + a^*}$$

   By performing that transformation on $Z$ and $Z'$, the permanent of $1 + z_i'^* z_j$ is simply multiplied by a factor. It follows that $SO(3, \mathbb{R})$ preserves the orthogonality property and that $D_{j_0}$ is a unitary representation of $SO(3, \mathbb{R})$.

4) When $n$ is even, that is when states are those of integral spins, the representation of $SO(3, \mathbb{R})$ is real since $SO(3, \mathbb{R}) \subset \text{PO}(n+1, \mathbb{R})$. We note that real constellations only appear in those representations.

5) We have the following corollary. Two real constellations of order two are harmonic conjugate if and only if their corresponding diameters are perpendicular.

# Chapter 10

# The spherical rotation constellations

## 10.1 Irreps of $SO(2,\mathbb{R}) \sim \mathrm{U}(1)$

Strictly speaking, the group $SO(2,\mathbb{R})$ is the group of all real unimodular orthogonal $2 \times 2$-matrices, that is the matrices of the form:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{10.1}$$

where $0 \leq \phi < 2\pi$.

These matrices are irreducible on the real field of numbers, but they can be reduced on the complex number field. In fact, a change of basis puts them in the form

$$\begin{pmatrix} \exp(i\phi) & 0 \\ 0 & \exp(-i\phi) \end{pmatrix} \tag{10.2}$$

This proves that the group $SO(2,\mathbb{R})$ is isomorphic to the group $U(1)$.

$U(1)$ is an Abelian group. We know that such a group has only irreducible representations (irreps) of dimension one. It is, moreover, a compact group. It follows that all its irreps are unitary. These two conditions show that the element $\exp(i\phi)$ is represented by a number of modulus one, say $\exp[im(\phi)]$, where $m$ is a function to be defined. The neutral element correspond to $\phi = 0$. We must have

$$\exp[im(0)]\exp[im(\phi)] = \exp[im(\phi)], \quad \text{for any value of } \phi.$$

This implies that

$$\exp[im(0)] = 1, \quad \text{that is } m(0) = 0. \tag{10.3}$$

The function $m(\phi)$ must satisfy the condition

$$m(\phi) + m(\phi') = m(\phi + \phi'),$$

which means that for $\phi'$ infinitesimal,

$$m(\phi) + m(d\phi) = m(\phi) + \frac{dm(\phi)}{d\phi} d\phi.$$

Let us choose $\phi = 0$. We get

$$m(0) + \frac{dm(0)}{d\phi} d\phi = \frac{dm(\phi)}{d\phi} d\phi$$

Since $m(0) = 0$, one gets $\frac{dm(\phi)}{d\phi} = constant$. Let us denote this constant by $m$. We obtain in integrating:

$$m(\phi) = m\phi + constant. \tag{10.4}$$

The constant is zero (take $\phi = 0$). It follows that the element $\exp(i\phi)$ is represented by $\exp(im\phi)$.

Since the element $\phi = 2\pi$ must have the same representative than the one for which $\phi = 0$, the exponential $\exp(2im\pi)$ equals 1, which means that $m$ is an integer.

An irrep of $U(1)$ is then characterized by the choice of an integer $m$. We will call such a representation $D_m$. We already saw that the group $U(1)$ has two faithful irreps associated with the values $m = 1$ and $-1$. For them, the kernel of the representation contains one element, the neutral one. If $m = 0$, we get the trivial representation. Its kernel is the group itself. For another integral value $m$, the kernel of the representation contains $m$ elements, namely the elements $\phi = 0, \frac{2\pi}{m}, \frac{4\pi}{m}, \ldots, \frac{2\pi(m-1)}{m}$.

If we except the trivial representation, all the irreps are complex representations. In order to obtain the irreducible *real* representations, we have to take the direct sum of the irreps $D_m$ and $D_{-m}$. We get the set of matrices

$$D_m + D_{-m}: \quad R(\phi) \rightarrow \begin{pmatrix} \cos m\phi & -\sin m\phi \\ \sin m\phi & \cos m\phi \end{pmatrix} \tag{10.5}$$

## 10.2 Irreps of $SU(2)$

Here we follow the paper by Bargmann on this subject. The group $SU(2)$ acts on the space $\mathbb{C}^2$. Let $\zeta$ and $\zeta'$ two vectors of $\mathbb{C}^2$, with components $\xi, \eta$ and $\xi', \eta'$, respectively. The inner product is

$$\zeta^*.\zeta' = \xi^*\xi' + \eta^*\eta'. \tag{10.6}$$

Let $U$ be a matrix of $SU(2)$. It satisfies

$$U^+U = Id, \text{ that is } U^{-1} = U^+, \text{ and Det}U = 1.$$

The inverse matrix of the unimodular matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Its adjoint is $\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$. It follows that a matrix of $SU(2)$ is of the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \text{with } |a|^2 = |b|^2 = 1. \tag{10.7}$$

We may set

$$a = \cos(\phi/2) - ib_3 \sin(\phi/2), \; b = -i(b_1 + ib_2)\sin(\phi/2), \tag{10.8}$$

where $b_1$, $b_2$, $b_3$ are real quantities satisfying $b_1^2 + b_2^2 + b_3^2 = 1$, and $\phi$ runs from 0 to $4\pi$.

For $b_1 = b_2 = 0$, $b_3 = 1$, the matrix $U$ may be written

$$U = \begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix} \tag{10.9}$$

corresponding to the subgroup $U(1)$ with $\phi$ running from 0 to $4\pi$.

Let us introduce the Hilbert space $H$ of entire analytic functions $f(\zeta)$, with the following inner product:

$$(f,g) = \int \phi^{-2} f(\zeta)^* g(\zeta) \exp(-\zeta^*.\zeta') d\mu(\zeta), \tag{10.10}$$

where

$$d\mu(\zeta) = d(\mathrm{Re}\,\xi) d(\mathrm{Im}\,\xi) d(\mathrm{Re}\,\eta) d(\mathrm{Im}\,\eta). \tag{10.11}$$

Let us find the expression of the inner product of $f$ and $g$ in the expansion coefficients of their power series. For that, we compute the inner product $(\xi^a\eta^b, \xi^{a'}\eta^{b'})$ and we introduce polar coordinates $\xi = r\exp(i\alpha)$, $\eta = s\exp(i\beta)$. We have

$$d\mu(\zeta) = rsdrdsd\alpha d\beta,$$

and

$$\begin{aligned}(\xi^a\eta^b, \xi^{a'}\eta^{b'}) &= \int \pi^{-2} r^{a+a'+1} s^{b+b'+1} e^{i(a'-a)\alpha + i(b'-b)\beta} e^{-(r^2+s^2)} drdsd\alpha d\beta \\ &= 0, \text{ if } a \neq a' \text{ or } b \neq b' \\ &= a!b!, \text{ if } a = a' \text{ and } b = b' \end{aligned} \tag{10.12}$$

It follows that an orthonormal complete set is given by the set of monomials . $\frac{\xi^a\eta^b}{\sqrt{a!b!}}$.

If we write the expansion

$$f(\zeta) = \sum_{a,b} f_{a,b} \frac{\xi^a\eta^b}{\sqrt{a!b!}}, \tag{10.13}$$

one obtains:

$$(f,f) = \sum_{a,b} |f_{a,b}|^2. \tag{10.14}$$

Note that the function $f$ belongs to $H$ if and only if this sum is finite.

The sets $H_j$ of all homogeneous polynomials of degree $2j$ ($j = 0, 1/2, 1, 3/2, \ldots$) form orthogonal subspaces. A polynomial of $H_j$ satisfies

$$(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta})f = 2jf. \tag{10.15}$$

**Coherent states** They are labelled by elements of $\mathbb{C}^2$. Let $z$ be an element of $\mathbb{C}^2$. We associate with it the following function of $H$:

$$e_z(\zeta) = \exp(z^*\zeta). \tag{10.16}$$

It is a simple matter to verify that coherent states belong to $H$. We have

$$(e_z, e_{z'}) = \exp(z'^*.z). \tag{10.17}$$

**Creation and annihilation operators** We consider the polynomial basis in $H$ and the four following operators acting on them:

- creation operators: $\xi$, $\eta$ : multiplication by $\xi$, by $\eta$ ;

- annihilation operators: $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta}$ (partial derivations).

They satisfy the commutation rules:

$$\left[\frac{\partial}{\partial\xi},\xi\right]=\left[\frac{\partial}{\partial\eta},\eta\right]=1, \tag{10.18}$$

all other commutators being zero.

Let us show that $\xi$ and $\frac{\partial}{\partial\xi}$ are adjoint operators.

$$(\xi^a\eta^b,\xi.\xi^{a'}\eta^{b'})=(\xi\eta^b,\xi^{a'+1}\eta^{b'})=a!\delta_{a,a'+1}$$

$$(\frac{\partial}{\partial\xi}\xi^a\eta^b,\xi^{a'}\eta^{b'})=a(\xi^{a-1}\eta^b,\xi^{a'}\eta^{b'}=a.(a-1)!\delta_{a-1,a'}$$

We check that these two inner products are equal.

**Unitary operators $T_U$** One can associate with each unitary operator $U$ on $\mathbb{C}^2$, a unitary operator $T_U$ on $H$, as follows

$$(T_Uf)(\zeta)=f(U^+\zeta) \tag{10.19}$$

Because the measure in (10.8) is invariant under $U$, the operators $T_U$ provide a unitary representation of the group $SU(2)$.

It is easy to check that the subspace $H_j$ of homogeneous polynomials of degree $2j$ is invariant under the transformations $T_U$ and that the restrictions to the $H_j$ 's subspaces, being of different dimensions are inequivalent. Now, an orthonormal basis of $H_j$ is provided by the functions

$$v_m^j=\frac{\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!(j-m)!}},\ m=j;j-1,j-2,\ldots,-j. \tag{10.20}$$

This representation of $SU(2)$ will be denoted by $D_j$. We will admit the irreducibility of those representations and their completeness.

Note that if $U$ is given by (10.7), one obtains

$$T_Uv_m^j=\frac{(a\xi-b^*\eta)^{j+m}(b\xi+a^*\eta)^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \tag{10.21}$$

We saw that if $b=0$, $U$ has the form (10.9) corresponding to a subgroup $U(1)$. One gets

$$T_Uv_m^j=\exp(-im\phi)v_m^j. \tag{10.22}$$

## 10.3 The irreps of $SO(3)$

In Chapter 3 we have proved the isomorphism (Eq. (3.9))

$$SU(2)/Z_2 \sim SO(3,\mathbb{R}).$$

It follows that not all irreps of $SU(2)$ are irreps of $SO(3,\mathbb{R})$. For an irrep of $SU(2)$ to be an irrep of $SO(3,\mathbb{R})$, it is necessary that the matrices $Id$ and $-Id$ are represented by the identity operator. We have, from Eq. (10.21),

$$T_{Id}v_m^j=\frac{\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!(j-m)!}},$$

and

$$T_{-Id}v^j_m = \frac{(-\xi)^{j+m}(-\eta)^{j-m}}{\sqrt{(j+m)!(j-m)!}} = (-)^{2j}T_{Id}v^j_m.$$

We see that we obtain an irrep of $SO(3)$ provided $2j$ is an even integer, that is $j$ is an integer.

## 10.4 Spin states and Vilenkin representation

We remind the reader that a state in quantum mechanics is a ray of the Hilbert space of states. It follows that a state of spin $j$ is represented by a homogeneous polynomial of degree $2j$ in two complex variables *up to a factor*, that is by a ray-polynomial. A ray-polynomial is uniquely defined by its $2j$ complex roots, those roots being possibly infinite. Using a stereographic projection, we see that a state of spin $j$ is represented by a spherical constellation of order $2j$.

We readily note that this description creates a close relationship between the rotation group and spin states, even when spin is half an integer. Said in another way, the rotation group acts effectively on spin *states*, although it does not act effectively on *spinors*.

In the Bargmann representation, the Lie algebra has as a basis the following values

$$J_1 = \xi\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\xi},\ J_2 = -i\xi\frac{\partial}{\partial\eta} + i\eta\frac{\partial}{\partial\xi},\ J_3 = \frac{1}{2}(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}). \tag{10.23}$$

Vilenkin considers a homogeneous polynomial of degree $2j$, say $f(\xi,\eta)$ as a polynomial of degree less than or equal to $2j$, in setting it $\xi^{2j}f(1,\eta/\xi)$. Denoting the variable $\eta/\xi$ by $z$ and $f(1,z)$ by $\Phi(z)$, the basis (10.23) becomes:

$$J_1 = \frac{1}{2}(1-z^2)\frac{d}{dz} + jz,\ J_2 = -\frac{i}{2}(1+z^2)\frac{d}{dz} + ijz,\ J_3 = j - z\frac{d}{dz}. \tag{10.24}$$

The scalar product becomes

$$(\Phi(z),\Phi'(z)) = \frac{(2j+1)!}{\pi}\int\frac{\Phi(z)^*\phi'(z)}{(1+|x|^2+|y|^2)^{2j+2}}dxdy, \tag{10.25}$$

where $z = x+iy$.

The orthonormal basis is given by the monomials

$$u_m = \frac{z^{j-m}}{\sqrt{(j+m)!(j-m)!}},\ m = j, j-1, j-2, \ldots, -j. \tag{10.26}$$

The corresponding states[1] are usually denoted $|jm>$. They are known as the canonical states. The associated constellation is composed of $2j$ stars, $2m$ stars on the North pole $(z=0)$, $2j-2m$ stars on the South pole( $z=\infty$).

[1]States are normalized vectors up to a phase factor.

## 10.5 The projective complex Lie algebra and constellations

Let us introduce the following basis in the complex Lie algebra:

$$J_3 = j - z\frac{d}{dz},\ J_+ = J_1 + iJ_2 = \frac{d}{dz},\ J_- = J_1 - iJ_2 = -z^2\frac{d}{dz} + 2jz. \tag{10.27}$$

Note the action of these operators on the canonical basis

$$J_3|jm >= m|jm >, \tag{10.28}$$

$$J_+|jm >= \sqrt{(j - m(j + m + 1)}|jm + 1 >, \tag{10.29}$$

$$J_+|jj >= 0, \tag{10.30}$$

$$J_-|jm >= \sqrt{(j + M)(j - m + 1)}|jm - 1 >, \tag{10.31}$$

$$J_-|j - j >= 0 \tag{10.32}$$

The state $|jm>$ is unchanged under the operator $J_3$, as it is unchanged under any operator of the form $\exp(-i\phi J_3)$, that is under rotations around the third axis. This property has a trivial geometrical interpretation. Obviously, the stabilizer of this state is isomorphic to $SO(2, \mathbb{R})$ except in one case: the state $|jj/2>$, when j is an integer; such a state is also invariant under any rotation of angle $\pi$ around an equatorial axis; then, its stabilizer is isomorphic to the group $O(2, \mathbb{R})$.

The operator $J_+$ transfers a star from the South to the North pole, the operator $J_-$ does the converse. Obviously, whenever there is no star to be transferred, the action is impossible and the result is zero, namely the constellation of order zero.

These results can be generalized. First, we note that to be an eigenstate of $J_3$ is a projective property in the sense that an eigenstate of $J_3$ is also an eigenstate of any operator of the type $\lambda J_3$, where $\lambda$ is an arbitrary complex number. Two linear complex combinations of the form $aJ_1 + bJ_2 + cJ_3$ and $\lambda(aJ_1 + bJ_2 + cJ_3)$ have common eigenstates. We may associate with such a three-dimensional complex ray $\mathbf{F}$ a constellation of order two. We do it as follows. We take as matrices $2J_1$, $2J_2$, $2J_3$ their representatives in spin 1/2 representation, that is the Pauli matrices. Then, we write

$$\sigma(\mathbf{F}) = \begin{pmatrix} c & a - ib \\ a + ib & -c \end{pmatrix}. \tag{10.33}$$

We associate with it the constellation $\{z_1; z_2\}$ as follows:

$$a - ib = -2,\ a + ib = 2z_1z_2,\ c = z_1 + z_2.$$

Then,

$$\sigma(\mathbf{F}) = (z_1 + z_2)\sigma_3 + z_1z_2\sigma_- - \sigma_+ = \begin{pmatrix} z_1 + z_2 & -2 \\ 2z_1z_2 & -(z_1 + z_2) \end{pmatrix} \tag{10.34}$$

We note that $\mathrm{Det}[\sigma(\mathbf{F})] = -(z_1 - z_2)^2$ is zero if and only if the constellation is degenerate. We also note that its eigenvalues are $\pm(z_1 - z_2)$. They are distinct if and only if the constellation is not degenerate.

The ray **F** will be said to be real if one of its representative is real. It is easy to show that, in such a case, $z_1 \neq z_2$. A real ray satisfies the condition: $\sigma(\mathbf{F})^+ = \lambda\sigma(\mathbf{F})$ (Hermitian up to a complex factor $\lambda$). Let us prove that, in that case, the constellation lies on a diameter. Take the Hermitian conjugate of (10.34).

$$\begin{aligned}\sigma(\mathbf{F})^+ &= (z_1^* + z_2^*)\sigma_3 + z_1^* z_2^* \sigma_+ - \sigma_- \\ &= z_1^* z_2^* [(\frac{1}{z_1^*} + \frac{1}{z_2^*})\sigma_3 + \sigma_+ - \frac{1}{z_1^* z_2^*}\sigma_-.\end{aligned}$$

By comparaison with Eq. (10.34), we get the conditions for $\sigma(\mathbf{F})$ to be Hermitian up to a factor:

$$z_1 + z_2 = -\left(\frac{1}{z_1^*} + \frac{1}{z_2^*}\right),$$

$$z_1 z_2 = -\frac{1}{z_1^* z_2^*}.$$

It is a simple matter to prove that they are equivalent to the unique condition:

$$z_1 z_2^* + 1 = 0, \tag{10.35}$$

which means that the stars $\{z_1\}$ and $\{z_2\}$ are orthogonal, i.e. diametrally opposite on the sphere. We arrive at the following theorem.

**Theorem 10.1** *The constellation of order two associated with a complex ray* **F** *lies*

1) *on a point, if $\sigma(\mathbf{F})$ is degenerate, that is if Det$[\sigma(\mathbf{F})] = 0$,*

2) *on a chord, if $\sigma(\mathbf{F})$ is non degenerate,*

3) *on a diameter, if $\sigma(\mathbf{F})$ is Hermitian up to a factor. In this last case, the constellation is said to be real.*

Let us state another theorem.

**Theorem 10.2** *The following assertions hold*

1) *The constellations associated with the eigenstates of $\sigma(\mathbf{F})$ are $\{z_1\}$ and $\{z_2\}$, if $\sigma(\mathbf{F})$ is non degenerate and associated with the constellation $(z_1, z_2)$. These eigenstates are orthogonal (and their associated constellations diametrally opposite) if and only if $(z_1, z_2)$ is real $(z_1 z_2^* + 1 = 0)$.*

2) *The constellation associated with the eigenstate of $\sigma(\mathbf{F})$ is $(z)$ if $\sigma(\mathbf{F})$ is degenerate and associated with the constellation $(z, z)$.*

*These two results can be stated in the following way:*

> *The eigenconstellations of $(z_1, z_2)$ are those of order one which can be defined with the aid of the stars $(z_1)$ or $(z_2)$*

*Proof* The proof of this theorem is quite simple.

$$\sigma(\mathbf{F})\begin{pmatrix}1\\z_1\end{pmatrix}=\begin{pmatrix}z_1+z_2 & -2\\ 2z_1z_2 & -(z_1+z_2)\end{pmatrix}\begin{pmatrix}1\\z_1\end{pmatrix}=(z_2-z_1)\begin{pmatrix}1\\z_1\end{pmatrix}$$
$$\sigma(\mathbf{F})\begin{pmatrix}1\\z_2\end{pmatrix}=\begin{pmatrix}z_1+z_2 & -2\\ 2z_1z_2 & -(z_1+z_2)\end{pmatrix}\begin{pmatrix}1\\z_2\end{pmatrix}=(z_1-z_2)\begin{pmatrix}1\\z_2\end{pmatrix}$$

If $z_1 \neq z_2$, we get two distinct eigenstates. If $z_1 = z_2$, one cannot distinguish between the two equations, the unique eigenvalue is zero. ■

Let us examine the three following cases. For that purpose, we write

$$\sigma(\mathbf{F})=\frac{1}{2}\begin{pmatrix}1+z_1/z2 & -2/z_2\\ 2z_1 & -(1+z_1/z_2)\end{pmatrix}. \tag{10.36}$$

1) $z_1 = 1,\ z_2 = -1,\quad \sigma(\mathbf{F}) = \sigma_1,$

2) $z_1 = i,\ z_2 = -i,\quad \sigma(\mathbf{F}) = \sigma_2,$

3) $z_1 = 0,\ z_2 = \infty,\quad \sigma(\mathbf{F}) = \sigma_3,$

In Fig. 10.5 we see the geometrical interpretation of those formulas. To be an eigenstate of the matrix $\sigma_i$ means to be invariant under rotations around the $i^{th}$ axis.

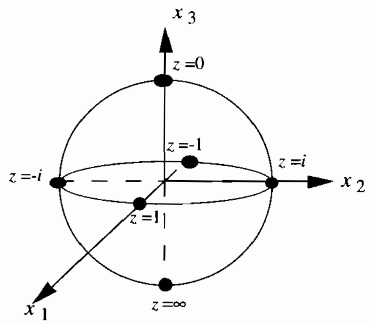



Figure 10.1: Geometrical interpretation of the eigenstates of the Pauli matrices. The stereographic projection is taken from the South pole on the equatorial plane.

## 10.6 Eigenconstellations of $J(F)$ in representation of spin $j$

The above results can be generalized in replacing the spin 1/2 representation $\sigma(\mathbf{F})$ by the representation $J(\mathbf{F})$ of spin $j$. The result is the following.

**Theorem 10.3** *Given a constellation $(z_1, z_2)$ associated with the matrix $\sigma(\mathbf{F})$, the eigen-constellations of its representative in the spin $j$ representation are the constellations of order $2j$ of the form $(z_1, z_1, \ldots, z_1, z_2, z_2, \ldots, z_2)$. If $z_1 \neq z_2$, they are $2j+1$ in number* *and apparent order 1 or 2. If $z_1 = z_2$, there is a unique constellation of apparent order 1 which will be called a Bloch constellation.*

This simple result is proved as follows. Let

$$J(\mathbf{F}) = (z_1 + z_2)J_3 + z_1 z_2 J_- - J_+ \tag{10.37}$$

be the representative of $\sigma(\mathbf{F})$ in the representation of spin $j$. It is a simple calculation to verify that the state

$$|\psi_\lambda >= \sum_{m=-j}^{j} \sum_{k=0}^{j-m} \sqrt{\frac{(j+m)!(j-m)!}{(2j)!}} \frac{(j-\lambda)!(j+\lambda)!}{k!(j-\lambda-k)!(j-m-k)!(\lambda+m+k^*!)} z_1^{j-m-k} z_2^k |jm>,$$

(where $\lambda$ is a half-integer obeying $0 \le \lambda \le j$ ) is an eigenstate

$$J(\mathbf{F})|\psi_\lambda >= \lambda(z_2 - z_1)|\psi_\lambda > .$$

We know that a state of spin $j$ is described by a constellation of order $2j$. The relationship is the following one. Let

$$|\psi >= S_0|j > +S_1|j-1> +S_2|j-2> + \ldots + \S_{2j}|-j> \tag{10.38}$$

be such a state. The associated constellation is $(z_1, z_2, \ldots, z_{2j})$, where the $z_i$ are defined by the relations

$$\begin{cases} S_0 = 1 \\ S_1 = \frac{z_1+z_2+\ldots+z_{2j}}{\sqrt{2j}} \\ S_2 = \frac{z_1z_2+z_1z_3+\ldots+z_{2j-1}z_{2j}}{\sqrt{j(2j-1)}} \\ \ldots \\ S_p = \begin{pmatrix} 2j \\ p \end{pmatrix}^{-1/2} \sum z_{j_1} z_{j_2} \ldots z_{j_p} \\ \ldots \\ S_{2j} = z_1 z_2 \ldots z_{2j} \end{cases} \tag{10.39}$$

where the summation is made on all combinations (elementary functions). We verify that, for $j = 1/2$, we are back to the relations already written.

It is not difficult to show, after a small combination calculation, that the state $|\psi_\lambda >$ is represented by the constellation $(z_1, z_1, \ldots, z_1, z_2, \ldots, z_2)$, where the multiplicity of $z_1$ is $2\lambda$ and the one of $z_2$ is $2j - 2\lambda$. In the peculiar case where $z_1 = z_2$, all the $|\psi_\lambda >$'s collapse in a single state with the corresponding eigenvalue zero. Such a state has a nice property which we are going to describe.

## 10.7 Rushin-Ben-Arieh property of Radcliffe-Bloch states

A state of spin $j$ associated with a star constellation can be parametrized by spherical coordinates $\theta$, $\phi$. It will be denoted $|j\theta\phi >$. Such states were introduced by Radcliffe under the name of *spin coherent states*. Such a name is in agreement with the Perelomov general definition of coherent states associated with a Lie group. According to Perelomov, a set of coherent states is associated with a homogeneous space of the group. Moreover, it is also in agreement with the restricted definition of coherent states we have proposed, namely the homogeneous space must be canonically symplectic (coadjoint orbit) in order to have a classical interpretation. It is the case for the Radcliffe spin coherent states, the sphere being a canonical symplectic orbit of the group $SO(3,\mathbb{R})$. The classical model of spin associated with this structure was discovered by Kramers, then studied by the Author. A modern description can be found in a book by Souriau.

The sphere of spin coherent states may be seen as the sphere known by physicists as the Bloch sphere. Two physicists, Rushin and Ben-Arieh, have shown an interesting physical property of such states. They proved the following property for an arbitrary state

$$\Delta J_1 \Delta J_2 \geq \frac{1}{2}| < J_3 > | \tag{10.40}$$

where $\Delta J_i = \sqrt{< (J_j - < J_j >)^2 >}$ and the signs $< \ldots >$ denotes the mean value of an operator in an arbitrary state $|\psi >$. In particular,

$$< J_3 > = < \psi | J_3 | \psi > \tag{10.41}$$

The Radcliffe-Bloch state minimalizes the inequality (10.40).

## 10.8 Spherical constellations and classification of pure spin states

Pure spin states of the electron (spin 1/2) are simple to classify. They all look the same. An electron in a pure state has always its spin up. We only have to specify in which direction its spin is pointing. If it is in an eigenstate of $\sigma_3$, it is said to point in North direction or in South direction, according to its eigenvalue. We can always state that it is pointing in a direction which may be called afterwards the North direction. More generally, we know that the set of spin states form a sphere and it is always possible to say that the spin is up in some direction. Obviously, we prefer to choose three axes $Ox$, $Oy$, $Oz$ on that sphere which are related to physical objects such as a magnetic field and to admit that the direction of the spin is along some axis defined by a unit vector $\mathbf{n}$ with components $\sin\theta\cos\phi$ along $Ox$, $\sin\theta\sin\phi$ along $Oy$ and $\cos\theta$ along $Oz$. Then the spin is in eigenstate of $\sigma.\mathbf{n}$, with a positive eigenvalue. We note that its state is invariant under the subgroup $SO(2,\mathbb{R})$ of rotations around $\mathbf{n}$ and the set of states is the manifold $SO(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, that is the sphere $S_2$. We understand now why we say that all the spin states look the same; they all have the subgroup $SO(2,\mathbb{R})$ as a stability subgroup.

We intend to classify in an analogous way pure spin states for any value of the spin. Let us start, as an example with spin one states. Spin one states are described by constellations of order two on the sphere. In the general case, a constellation of order two is composed of two stars pointing in directions $\mathbf{n}$ and $\mathbf{n}'$ and the stability subgroup of such a constellation is a subgroup generated by the rotation of angle $\pi$ around the direction $\mathbf{n}+\mathbf{n}'$. It is the cyclic group $C_2$ with two elements. There are also two particular cases, the one where the vectors $\mathbf{n}$ and $\mathbf{n}'$ are equl and the case where $\mathbf{n}+\mathbf{n}' = \mathbf{0}$. In the first case, the stability subgroup is $SO(2,\mathbb{R})$; in the second, it is $O(2,\mathbb{R})$ because we have to take into account all the rotations around diameters orthogonal to $\mathbf{n}$. Therefore, there exist three kinds of spin states of spin one, namely:

- The generic case. The set of such states is $SO(3,\mathbb{R})/C_2$. The apparent order of the associated constellations is two.

- The degenerate case. The corresponding set is $SO(3,\mathbb{R})/SO(2,\mathbb{R})$. The apparent order of the associated constellations is one.

- The special case. The corresponding set is $SO(3,\mathbb{R})/O(2,\mathbb{R})$. The apparent order of the associated constellations is two. Note that they are real constellations.

Let us see how the so-called canonical states are relatedto this classification. The degenerate case corresponds to the addition of two spin one-half "spin up". This is the case of a spin noted $|11>$ or $|1-1>$. The special case corresponds to the addition of a spin one-half "spin up" and a spin one-half "spin down", that is a spin of the kind $|10>$. We note that the canonical spin states are the most symmetric states, but that these states are exceptionnal ones: their symmetry is quite large.

Before investigating higher values of the spin, we note other interesting properties of canonical states in relationship with constellations description. They are shown on Figure 10.8, where the vector $\mathbf{n}$ is chosen to point in the North direction. We note the three following properties:

1) All these states are eigenstates of the operator $J_3$. This implies that they are also eigenstates of the operator $\exp(-i\phi J_3)$, for an arbitrary value of $\phi$. This property may be stated in another way: *these states are invariant under the group $SO(2,\mathbb{R})$* generated by $J_3$. This is an obvious statement if we look for the constellatrion description of these states.

2) It is well known that we go from the state $|1-1>$ to the state $|10>$ and from the state $|10>$ to the state $|11>$ by making the operator $J_+$ acting. But the way this operator is acting is obvious in the constellation language. It arises one star in the North direction. Once all stars are "up", the operator $J_+$ becomes powerless. We readily note that the operator $J_-$ operates in the opposite way, pushing the stars "down".

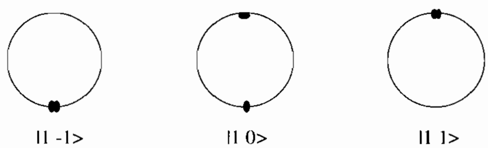



Figure 10.2: Canonical states

3) The number of canonical states is the number of ways to put two stars at the end of a diameter, that is three.

Those three properties may be generalized to the case of spin $j$ states. We state them without proof.

**Theorem 10.4** *Canonical states are described by constellations of order $2j$ which are invariant under rotations around a given diameter. They are $2j+1$ in number. The* *operator* $J_+$ *transforms the constellation* $\{2m, 2j - 2m\}$ *into the constellation* $\{2(m + 1), 2(j - m - 1)\}$. *If* $j$ *is half an integer, these states belong to* $j + 1/2$ *distinct orbits isomorphic to* $S_2$ *(*$SO(2, \mathbb{R})$ *as a stability subgroup). If* $j$ *is an integer, these states belong to* $j$ *distinct orbits isomorphic to* $S_2$ *and one to an orbit isomorphic to* $SO(3, \mathbb{R})/O(2, \mathbb{R})$.

The following question arises: what is the space of orbits of spin one states? In order to answer that question, we must note that the angle between the vectors $\mathbf{n}$ and $\mathbf{n}'$ is an invariant under rotations. It follows that the space of orbits is isomorphic to the interval $[0, \pi]$, as shown on Fig. 10.8.

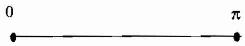



Figure 10.3: Space of orbits of spin one

We now turn to the classification of orbits of spin states for an arbitrary value of spin $j$. Our way of solving this problem consists in answering the question: Given a closed subgroup $H$ of $SO(3, \mathbb{R})$, how many points can be put on the sphere in such a way that the associated constellation has $H$ as a stabilizer?

Let us first consider a stabilizer of the type $C_n$, the cyclic subgroup of order $n$. Let $\Delta$ be the axis of rotations of angles $2\pi m/n$ which form the group $C_n$. It is clear that the stars associated with a constellation invariant under $C_n$ are necessarily either on the axis $\Delta$ itself or at the vertices of a certain number of regular $n-$polygons perpendicular to $\Delta$. For $C_n$ to be the stabilizer of a state, it must be the maximal subgroup which leaves the state invariant. This implies that the constellation must contain at least one $n-$polygon, since without any polygon the stabilizer would contain $SO(2, \mathbb{R})$ as a subgroup. Because the number of stars on $\Delta$ is unlimited, a necessary and sufficient condition for $C_n$ to be the stabilizer of some state of spin $j$ is $2j \geq n$.

Let us now examine the case of dihedral groups $D_n$ as stabilizers. Let $\Delta$ be the $n-$axis of symmetry and $\Delta$ the corresponding diameter. The $2j$ points must be situated on $\Delta$ in even number and on the vertices of $n-$polygonals, with $n \geq 2$. We must distinguish between the two following cases:

1) There are nonequatorial polygons. Their number is necessarily even, due to the symmetry properties of $D_n$. This corresponds to a number of stars which is a multiple of $2n$. Since the number of stars in the equatorial plane is a multiple of $n$ and the number of stars on $\Delta$ is even, we get the condition:

$$2j = 2na + 2nb + 2c, \tag{10.42}$$

where $a \geq 1$, $b \geq 0$, $c \geq 0$.

2) All polygons are equatorial. We get, in that case,

$$2j = nb + 2c, \tag{10.43}$$

with $b \geq 1$, $c \geq 0$. In fact, this result is not valid when $n = 2$ because ir corresponds to a situation where the symmetry is larger if $b = 1$ and $c = 0$ (no star on $\Delta$ or if $b = c = 1$

| **Stability subgroup** | **Representations** |
|---|---|
| $O(2,\mathbb{R})$ | $j$ integer |
| $SO(2,\mathbb{R})$ | all |
| $C_n$ | $2j \geq n$ |
| $D_2$ | $j$ integral (except $j = 1$). |
| $D_4$ | $j$ integral (except $j = 1$ and 3). |
| $D_n$ for $n > 2$ (except $n = 4$) | $2j = n + nb + 2c$ ($b$ and $c \geq 0$) |
| $T$ | $j$ integral (except 1 and 3) |
| $O$ | $j = 4a + 3b$ |
| $Y$ | $j = 10a + 6b$ |

Table 10.1: Classification of pure spin states.

(symmetry of the square) and when $n = 4$, $b = c = 1$ (symmetry of the octahedron). We arrive at the results presented in Table 10.8.

We are left with the polyhedron subgroups. The situation is quite simple for the octahedron group $O$ and the icosahedron group $Y$. For $O$, the $2j$ stars must be at the vertices of an octahedron and/or a cube. Therefore $O$ is a stabilizer for all values of $2j$ of the kind $2j = 8a + 6b > 0$, for $a$ and $b$ nonnegative integers. In the case of $Y$, the $2j$ stars must lie on the vertices of a icosahedron and/or a dodecahedron. It follows that $Y$ is a stabilizer for $2j = 20a + 12b > 0$. In order to study the case of the tetrahedron group $T$, we must remind that $T$ is a subgroup of $O$. In order to have $T$ as a stability subgroup, $2j$ must be of the form $8a + 6b + 4c = 2(4a + 3b + 2c)$, with $a \geq 0$, $b \geq 0$, $c \geq 1$. This includes all integral values of $j$, except 1 and 3.

## 10.9 The set of spin 3/2 states

The reader will verify by himself the following results:

1) First orbit: completely degenrate constellations. They are of the type $|3/2\ 3/2>$. The little group is $SO(2)$.

2) Second orbit: constellations of apparent order two. They are of the type $|3/2\ 1/2>$. Since the little group is also $SO(2)$, these two orbits form a single stratum.

3) Third kind of orbits: constellations of the type: with two equal angles (different of

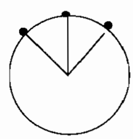


Figure 10.4: Third kind of orbits

$2\pi/3$). These orbits form a stratum with litlle group $C^2$.

4) Fourth kind of orbits. Constallations represented by a regular triangle. The corresponding stratum has $C^3$ as a little group.

5) The fifth orbit of large regular triangles, that is regular triangles having its center of mass at the origin of the sphere. The corresponding stratum has $D(3)$ as a little group.

6)The sixth kind of orbits corresponding to non regular triangles. The little group of this stratum is trivial. It only contains the identity transformation.

## 10.10 Remarks on spin coherent states

This kind of states have been introduced by Radcliffe in 1971. Their Clebsch-Gordan coefficients have been studied by Bellissard and Holtz. This set of states may be introduced using the general scheme defined by Perelomov from group theory. He defined such a set as the set of states belonging to some orbit in a given representation of a group. If we adopt such a definition, any orbit of constellations we have founded will constitute a set of coherent states. Unfortunately, from this point of view, any state is a coherent state. In a paper written in collaboration with Alexander Grossmann and Joshua Zak, we show that the Radcliffe states obey a special condition: the corresponding orbit is symplectic, which permit to state that coherent states may be considered as classical ones, a condition obeyed by the standard harmonic oscillator coherent states. The Radcliffe states are, indeed, the ones for which the little group is $SO(2)$, that is the degenerate constellations of the type $[z, z, z, \ldots, z]$, the orbit being the two-dimensional sphere $S^2 \sim SO(3)/SO(2)$.

It has been shown by Ruschin and Ben-Aryeh that the Radcliffe states are minimum uncertainty states for angular momentum operators, i.e.

$$\Delta J_1 \Delta J_2 = \frac{1}{2} J_3. \tag{10.44}$$

For all other states, we have, instead of (10.44),

$$\Delta J_1 \Delta J_2 > \frac{1}{2} J_3. \tag{10.45}$$

## 10.11 Clebsch-Gordan product of states

The product of representations of spin $j$ and spin $j'$ is given by

$$D_j \times D_{j'} = D_{|j-j'|} + D_{|j-j'+1|} + \ldots + D_{j+j'}. \tag{10.46}$$

In particular,

$$D_1 \times D_1 = D_0 + D_1 + D_2. \tag{10.47}$$

Suppose we start with two real constellations, one in the $x-$direction, the second in the $y-$direction. The three product states appearing in the right hand of (10.47) will be:

- the trivial constellation of order zero
- the real constellation in the $z-$direction
- the union of the two constellations, that is the constellation of order four with stars on the diameters $x$ and $y$.

Note that the real constellation in the $z-$direction is the two fixed points of the Lorentz transformation which maps the first constellation ( the one in the $x-$direction) on the second (the one in the $y-$direction).

More generally, the state $D_{j+j'}$ of Eq. (10.46) will be always the union of the constellations associated with $D_j$ and $D_{j'}$ .

The general case is obtained as follows. Suppose we want to write the product of the following constellations

$$Z = [z_1, z_2, \ldots, z_{2j}], \tag{10.48}$$

$$Z' = [z'_1, z'_2, \ldots, z'_{2j'}], \tag{10.49}$$

We define the set of new constellations of order $n$:

$$Z^{\#} = [z_1, z_2, \ldots, z_{2j}, z, z, \ldots, z], \tag{10.50}$$

$$Z'^{\#} = [z'_1, z'_2, \ldots, z'_{2j}, z, z, \ldots, z]. \tag{10.51}$$

$Z^{\#}$ differs from $Z$ by the adding of $n-2j$ times the star $z$ and $Z'^{\#}$ differs from $Z'$ by the adding of $n-2j'$ times the same star $z$.we impose $Z^{\#}$ and $Z'^{\#}$ to be harmonically conjugate, a condition which has the form of an equation of degree $n-2j$ in $z$. The corresponding constellation is the one composed of the roots of that equation. It is easy to show that for $n$ larger than $2j+2j'$, the two constellations $Z^{\#}$ and $Z'^{\#}$ are trivially harmonically conjugate, whatever is the value of $z$.

## 10.12 Coherent Senitzky states

This kind of states may be considered as generalized Radcliffe states defined as follows.

$$|\alpha_1, \alpha_2, \ldots, \alpha_n >= \frac{(\alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n)^h}{\sqrt{h!}(|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2)^{h/2}}$$

$$= \sum_{h_1, h_2, \ldots, h_n} \sqrt{\frac{h!}{h_1! h_2! \ldots h_n!}} \frac{\alpha_1^{h_1} + \alpha_2^{h_2} + \ldots + \alpha_n^{h_n}}{(|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2)^{h/2}} |h_1, \ldots, h_n >,$$

where $|h_1, \ldots, h_n>$ denotes a Bargmann state.

Because $|\alpha_1, \alpha_2, \ldots, \alpha_n >= |\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n>$ for any value of $\lambda$ (except zero), we see that the coherent Senitzky states form the orbit $SU(n)/U(n-1) \sim P_{n-1}(\mathbb{C})$. They are constellations of oder $n-1$.

## Exercises

**1.** Use Vilenkin representation in order to find the eigenstates of $aJ_1 + bJ_2 + cJ_3$, where $a, b, c$ are arbitrary complex numbers. If $a^2 + b^2 + c^2 \neq 0$, the eigenstate is of the form $(z + z_1)^A(z + z_2)^{2j-A}$, where a is an integer satisfying $0 \leq A \leq 2j$. It corresponds to a constellation $\{z_1, z_1, \ldots, z_1, z_2, z_2, \ldots, z_2\}$ of order $2j$ and apparent order two, made of two stars of respective orders $A$ and $2j - A$.

If $a^2 + b^2 + c^2 = 0$, there is only one eigenstate, of the form $[(a + ib)z + c]^{2j}$ , with eigenvalue zero, corresponding to a degenerate constellation of order $2j$. If $a + ib$ is zero, the constellation is a star at infinity.

**2.** Show that the states satisfying condition (10.44) are - if we except the two states $|JJ>$ and $|J - J>$ - of the type

$$\exp(bJ_3)\exp\{-i(\pi/2)J_2\}|JM>,$$

where $b$ is some real number.

**3.** Construct the constellations of the product

$$D_1 \times D_n$$

where the first constellation is composed of the North-South stars and the second constellation is described by a regular equatorial $2n$-polygonal (with $n \geq 2$).

Show that

$D_{n-1}$ is the trivial constellation,

$D_n$ is the regular equatorial $2n$--polygonal constellation obtained form the original one by a rotation of angle $\pi/2n$,

$D_{n+1}$ is the union of the two original constellations.

**4.** Write the scalar product of two Senitzky states.

# Chapter 11

# The finite irreps of the Lorentz group

## 11.1 The representations of $SO(4,\mathbb{R})$

There is a certain relationship between the irreducible representations of the compact group $SO(4,\mathbb{R})$, which are all finite-dimensional, and the finite-dimensional irreducible representations of $SL(2,\mathbb{C})$. The reason is that their Lie algebras are real forms of the same complex Lie algebra, namely the one of $SO(4,\mathbb{C})$. One of the main difference between the two groups is that $SO(4,\mathbb{R})$ is compact and $SL(2,\mathbb{C})$ is not. This explains why this last group has also infinite-dimensional irreps. We must underline that we are not concerned with this kind of representations in the present book.

Let us show that $SO(4,\mathbb{R})$ is isomorphic to the group $\frac{SU(2)\times SU(2)}{Z_2}$. In order to prove it, let us look at the action of $SU(2)\times SU(2)$ on the matrices of the form

$$X\left(\begin{array}{cc} t+iz & ix+y \\ ix-y & t-iz \end{array}\right). \tag{11.1}$$

We note that $\mathrm{Det}(X)=x^2+y^2+z^2+t^2$ is the square length of the vector $(x,y,z,t)$ . The matrices $X$ satisfy the property

$$X^{+}=(\mathrm{Det}X)^{-1}X^{-1}. \tag{11.2}$$

Let $U$, $V$ be two matrices of $SU(2)$ and let us examine the matrix $UXV^{+}$. It is easy to verify that the determinant and the property (11.2) are conserved. Since the transformation leaves the length unchanged, it may be interpreted as a rotation of the group $SO(4,\mathbb{R})$. It follows that there exists a homomorphism:

$$SU(2)\times SU(2)\rightarrow SO(4,\mathbb{R}) \tag{11.3}$$

The kernel of this homomorphism is obtained by solving the equation $UXV^{+}=X$, for an arbitrary $X$. In setting

$$U=\left(\begin{array}{cc} a & b \\ -b^{*} & a^{*} \end{array}\right),\ V=\left(\begin{array}{cc} a' & b' \\ -b''^{*} & a''^{*} \end{array}\right), \tag{11.4}$$

and writing the condition in the form

$$UX=XV, \tag{11.5}$$

one gets $a = a'$ and $b = b' = 0$. The kernel is made of the couple

$$(U, V) = \{(Id, Id), (-Id, -Id)\}. \tag{11.6}$$

This proves the isomorphism

$$SO(4, \mathbb{R}) \sim \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{Z}_2} \tag{11.7}$$

The group $SU(2) \times SU(2)$ is the covering group of $SO(4, \mathbb{R})$. The covering group of a rotation group is called the spin group. If we denote by $D_{j0}$ the irrep of dimension $2j+1$ of the first $SU(2)$ group and by $D_{0j}$ the same irrep for the second $SU(2)$ group, we see that the general irrep of the spin group may be denoted $D_{jj'} = D_{j0} + D_{0j'}$. It is of dimension $(2j+1)(2j'+1)$.

It is a simple matter to see that the irreps of $SO(4, \mathbb{R})$ are those $D_{jj'}$ , where $j$ and $j'$ are both integers or half integers. In particular, the vector representation of dimension four is $D_{1/2\ 1/2}$.

Let us consider the subgroup $SO(3, \mathbb{R})$ of $SO(4, \mathbb{R})$. It leaves invariant the matrix $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$. It is easy to prove that the couples $(U, V)$ which leave this matrix invariant are of the type $(U, U)$. It follows that the representation $D_{jj'}$ is the direct sum:

$$D_{|j-j'|} + D_{|j-j'|+1} + D_{|j-j'|+2} + \ldots + D_{+j'j}. \tag{11.8}$$

of irreps of $SO(3, \mathbb{R})$.

## 11.2 Finite dimensional irreps of $SL(2, \mathbb{C})$

Let us consider the action of $SU(2, \mathbb{C})$ on the space of Hermitian matrices defined as follows.

Let $H = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$ be the Hermitian matrix associated with the space-time vector $(t, x, y, z)$. We have $\mathrm{Det} H = t^2 - x^2 - y^2 - z^2$. Given $\Lambda$ , a matrix of $SL(2, \mathbb{C})$, we describe its action on $H$ as $\Lambda H \Lambda^+$. Such an action preserves the determinant and the Hermitian character of $H$. This property proves the homomorphism

$$SL(1, \mathbb{C}) \rightarrow \mathrm{L}_+^\uparrow \tag{11.9}$$

The kernel is obtained in looking for $SL(2, \mathbb{C})$ transformations which preserve an arbitrary matrix $H$. In order to find them, we set

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } ad - bc = 1, \quad \text{and } \Lambda H = H(\Lambda^+)^{-1}. \tag{11.10}$$

We get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}$$

We obtain the conditions:

$$b = c = 0, \ a = d \ \ real. \tag{11.11}$$

It follows that the kernel is the couple $\{Id, -Id)\}$. We obtain the isomorphism

$$L_+^{\uparrow} \sim SL(2,\mathbb{C})/Z_2. \tag{11.12}$$

This relation is analogous to (11.7).

The finite dimensional irreps of $SL(2,\mathbb{C})$ are also labelled by two indices $j$ and $j'$. If $J_1$, $J_2$, $J_3$, $K_1$, $K_2$, $K_3$ are the representatives of the generators of the group, we have

$$\begin{cases} (\mathbf{J}^2 - \mathbf{K}^2) + 2i\mathbf{J}.\mathbf{K} = j(j+1) \\ (\mathbf{J}^2 - \mathbf{K}^2) - 2i\mathbf{J}.\mathbf{K} = j'(j'+1) \end{cases} \tag{11.13}$$

We note that $j = j'$ if and only if $\mathbf{J}.\mathbf{K} = 0$.

Only irreps for which $j + j'$ is an integer are irreps of $L_+^{\uparrow}$.

## 11.3 Finite dimensional irreps of $L$

The vectors $\mathbf{J}$ and $\mathbf{K}$ transform differently under the parity operation. This can be shown if we use Lorentz indices for these vectors

$$J_i = \frac{1}{2}\epsilon_{\iota\varphi\kappa} M_{jk}, \quad K_i = M_{0i}, \qquad i, j, k = 1, 2, 3. \tag{11.14}$$

Let $\Pi$ be the parity operator. We have

$$\Pi M_{jk} \Pi^{-1} = M_{jk}, \quad \Pi M_{0i} \Pi^{-1} = -M_{0i},$$

that is

$$\Pi \mathbf{J} \Pi^{-1} = \mathbf{J}, \quad \Pi \mathbf{K} \Pi^{-1} = -\mathbf{K}. \tag{11.15}$$

That is why $\mathbf{J}$ is called an axial vector (parity +1) and $\mathbf{K}$ a polar vector (parity -1). In the same way, the magnetic vector $\mathbf{B}$ is an axial vector and the electric vector $\mathbf{E}$ is a polar vector.

It follows that the parity operator permute the two invariants $(\mathbf{J}^2 - \mathbf{K}^2) + 2i\mathbf{J}.\mathbf{K}$ and $(\mathbf{J}^2 - \mathbf{K}^2) - 2i\mathbf{J}.\mathbf{K}$. Two cases occur:

1) $j \neq j'$. The representation $D_{jj'} = D_{jj'} + D_{j'j}$ is irreducible.

2) $j = j'$. There are two irreducible representations $D_{jj}^+$ and $D_{jj}^-$.

Let us examine some examples. The electromagnetic field corresponds to the six-dimensional representation $D_{10}$. Space-time vectors (like the energy-momentum four-vector) correspond to the four-dimensional representation $D_{1/21/2}^-$.

## Exercise

**1.** Compute $j$ and $j'$ for spinor representations:

$$\mathbf{J} = \frac{1}{2}\sigma, \quad \mathbf{K} = \pm\frac{i}{2}\sigma,$$

where $\sigma$ are the Pauli matrices.

# Chapter 12

# Petrov's classification of curvature tensors

## 12.1 The curvature tensor

The following physical application lies in the geometry of Lorentz constellations. It concerns the classification of curvature tensors for space-time in general relativity. Such a tensor $\Omega$ is a real one, with components $R_{\alpha\beta\gamma\delta}$ having the following symmetry properties:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\beta\alpha\delta\gamma}, \tag{12.1}$$

and satisfying

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0. \tag{12.2}$$

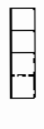
where space-time indices $\alpha$, $\beta$, $\gamma$, $\Delta$ run from zero to three. Two tensors which are multiple one from each other are said to be of the same type.

The number of different components of a curvature tensor equals 20. Indeed, the corresponding space can be seen as the symmetric product of the 6-dimensional representation $D_{10}$ of the Lorentz group by itself. One has

$$\begin{array}{ccccccccccccccc} D_{10} & \times & D_{10} & = & D_{00}^{+} & + & D_{00}^{-} & + & D_{10} & + & D_{20} & + & D_{11}^{+} & + & D_{11}^{-} \\ & & & & (s) & & (s) & & (a) & & (s) & & (s) & & (a) \\ 6 & \times & 6 & = & \underline{1} & + & 1 & + & 6 & + & \underline{10} & + & \underline{9} & + & 9, \end{array}$$

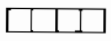
where the symmetric (resp. antisymetric) character is mentioned together with the dimension of the representations. Eq. (12.2) concerns the vanishing of a pseudoscalar quantity, that is the representation $D_{00}^{-}$. We are left with the twenty components associated with the representations: $D_{00}^{+}$, $D_{20}$, and $D_{11}^{+}$.

In the vacuum, the Ricci tensor $R_{\alpha\gamma} = g^{\beta\delta}R_{\alpha\beta\gamma\delta}$ is zero. Discarding this trivial case is equivalent to discard the $D_{00}^{+} + D_{11}^{+}$ part of the tensor, that is to only retain the $D_{20}$ part. The Petrov classification of curvature tensors concerns the classification of tensors of type $D_{20}$. This five-dimensional representation corresponds to constellations of order 4. We know that they are of five types:

$I_a$:  $[a, b, c, d]$,

$I_b$:  $[a, a, b, b]$,

$II_a$:  $[a, a, b, c]$,

$II_b$:  $[a, a, a, a]$,

$III$:  $[a, a, a, b]$.

We intend to give in the next sections an alternative derivation of the Petrov classification.

## 12.2 The Lorentz group as a subgroup of $SO(3,3)$

Under the mathematical point of view, this section could be entitled: "the group $SO(3,\mathbb{C})$ as a subgroup of $SO(3,3)$". The contents of this chapter may be generalized easily to another dimension than 3, and seen as $SO(n,\mathbb{C})$ as a subgroup of $SO(n,n)$. However, we have in mind applications to physics, a fact which justifies our choice.

We have shown that the Lorentz group is isomorphic to $SO(3,\mathbb{C})$, in using the action of this group on the space $\mathbb{C}^3$. An element of $\mathbb{C}^3$ is physically denoted $\mathbf{F}$, in order to remind that $\mathbf{F}$ is related to the electromagnetic field by the relation $\mathbf{F} = \mathbf{B} - i\mathbf{E}$, where $\mathbf{B}$ and $\mathbf{E}$ represent the magnetic and the electric vectors, respectively. The $SO(3,\mathbb{C})$ invariant associated with $\mathbf{F}$ is given by $\mathbf{F}^2 = \mathbf{B}^2 - \mathbf{E}^2 - 2i\mathbf{B}.\mathbf{E}$. We intend to show that the group $SO(3,3)$ is characterized by the two separate invariants $\mathbf{B}^2 - \mathbf{E}^2$ and $2\mathbf{B}.\mathbf{E}$, from which it follows that $SO(3,\mathbb{C})$ is a subgroup of $SO(3,3)$.

Let us consider the space $\mathbb{R}^6$, the elements of which are of the form $\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}$. The two invariants may be written as

$$\left\{ \begin{array}{l} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = \mathbf{B}^2 - \mathbf{E}^2 \\ \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = 2\mathbf{B}.\mathbf{E} \end{array} \right. \tag{12.3}$$

The two symmetric tensors have the same signature, namely $(+\ +\ +\ -\ -\ -)$. It follows that the Lorentz group can be seen in two ways as a subgroup of $SO(3,3)$. In other words, it is the subgroup of $SO(3,3)$ which preserves a second scalar product with the same signature. We note that the matrix is the parity operator, since under this transformation the vector $\mathbf{B}$ is unchanged (axial vector) and the vector $\mathbf{E}$ changes sign (polar vector).

The $D_{20}$ part of the tensor $\Omega$ is symmetric with respect to each scalar product. It means that it must satisfy the properties:

$$\Omega = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} P^T & R^T \\ Q^T & S^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}^T \begin{pmatrix} P^T & R^T \\ Q^T & S^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}$$

It is easy to deduce that $\Omega$ is of the form

$$\Omega = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}, \tag{12.4}$$

with $P$ and $Q$ symmetric, that is with six real components each. That is a total of twelve components. In the basis $\mathbf{B} - i\mathbf{E}$, $\mathbf{B} + i\mathbf{E}$, the matrix $\Omega$ is simpler. We have

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \mathbf{B} - i\mathbf{E} \\ \mathbf{B} + i\mathbf{E} \end{pmatrix},$$

$$\Omega = \frac{1}{4} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} P + iQ & 0 \\ 0 & P - iQ \end{pmatrix}.$$

It is clear that $D^+_{00}$ and $D^-_{00}$ correspond to the following scalar tensors

$$D^+_{00} \qquad (\mathbf{B}^2 - \mathbf{E}^2) \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}, \qquad P = 1, \; Q = 0,$$

$$D^-_{00} \qquad (2\mathbf{B}.\mathbf{E}) \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}, \qquad P = 0, \; Q = 1.$$

The ten other components (those of $D_{20}$) correspond to $\mathrm{Tr}(P) = \mathrm{Tr}(Q) = 0$.

## 12.3 Petrov's classification

The group $SO(3, \mathbb{C})$ does not act transitively on the set of $3 \times 3$-dimensional symmetric complex matrices $P + iQ$. Not all these matrices are diagonizable. Those which are diagonizable are said to be of Petrov's type $I$. One has to distinguish two subclasses:

$I_a$: Three distinct eigenvalues.

$I_b$: A double eigenvalue.

The case of three equal eigenvalues corresponds to the zero matrix and must be discarded as an unsignificant one from the physical point of view.

Petrov's types $II_a$ and $II_b$ correspond to matrices which can be put in the respective Jordan forms

$$(P + iQ)_{diag.} = \begin{pmatrix} p & 1 & 0 \\ 0 & p & 0 \\ 0 & 0 & -2p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0 \end{pmatrix}$$

Petrov's type $III$ corresponds to matrices of the type

$$(P + iQ)_{diag.} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

## Exercise

**1.** Find the $SO(3)$ subclasses of the Petrov classes. Show that the stabilizers are the following ones:

$I_a$: $[a, b, c, d]$, $D(4)$, $T$, $C(2) \times C(2)$,

$I_b$: $[a, a, b, b]$, $O(2) \times O(1, 1)$,

$II_a$: $[a, a, b, c]$, $C(2)$,

$II_b$: $[a, a, a, a]$, $S(2)$,

$III$: $[a, a, a, b]$, $SO(2) \times SO(1, 1)$.

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# Index

Henri BACRY

# Group theory and constellations

Recherches

The Great Bear is apparently composed of seven stars. However one of them is double. We may say that this constellation is made of eight not necessarily distinct stars. Mathematically, we define a constellation of order n as a set of n not necessarily distinct points on a manifold M. From a physical point of view, a pure state of spin s is shown to be a constellation of order 2s on the sphere S, as proved by Majorana. If the 2s points lie on a diameter, the corresponding state is invariant under a rotation around the diameter There are 2s + 1 ways of putting the 2s points on a diameter. They correspond to the eigenstates of the angular momentum along the diameter.

Mathematically, constellations are objects which have something to do with projective representations of classical groups. Moreover the notion of harmonically conjugate set of points on the sphere is shown to be more simple if we use constellations instead of set of points and harmonically conjugate constellations is defined whatever is the order of them.

Some applications are discussed as Clebsch-Gordon coefficients, Petrov classification of Einstein espaces, the Møbius and Lorentz groups, spinors, and so on.

Henri Bacry is an amerit professor of theoretical physics at the Université de la Méditerranée. He is the Author of many papers in group theory applied to physics (space groups, space-time groups, particle physics groups) and some papers in mathematics.

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ISBN 274830305-9