Giuseppe Primiero

LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE 10

Information and Knowledge

A Constructive Type-theoretical Approach

Springer

INFORMATION AND KNOWLEDGE

LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE

VOLUME 10

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal techniques like, for example, independence friendly logic, dialogical logics, multimodal logics, game theoretic semantics and linear logics, have the potential to cast new light no basic issues in the discussion of the unity of science.

This series provides a venue where philosophers and logicians can apply specific technical insights to fundamental philosophical problems. While the series is open to a wide variety of perspectives, including the study and analysis of argumentation and the critical discussion of the relationship between logic and the philosophy of science, the aim is to provide an integrated picture of the scientific enterprise in all its diversity.

Information and Knowledge

A Constructive Type-theoretical Approach

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Logic, which alone can give certainty,
is the instrument of proof;
intuition is the instrument of invention.
H. Poincaré, *La valeur de la Science*

. . . and he knows absolutely—knows it all the way,
deep as knowing goes, he feels the knowledge
start to hammer in his runner's heart—
that he is uncatchable.
D. De Lillo, *Underworld*

Information is not knowledge.
A. Einstein

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Introduction

This research is the result of a fruitful connection and provides a significant link between two topics of a logical and philosophical enquiry. It tries to provide a solution to the *problem of analyticity*: with this expression I understand, on the one hand, the essential nature of analytic truths and, on the other, the related explanation of the analytic nature of logical inference. The connection between these two sides of what will be referred to as the *Analyticity Principle*, can be briefly explained as follows: by *analytic truth* one understands in general a sentence whose content is logically true; by *logically true* one understands moreover truth independent from matters of fact or empirical data, a truth which is therefore established by logical criteria only. On this basis, it follows that a logical inference represents a purely analytic process, in opposition to its property of being able to produce knowledge, a situation which is exemplified by the conflicting notions of *validity* and *utility*. The question-begging topic of this research is therefore that of analyticity, the inspiring problem for which a solution is formulated in the present book. If analyticity represents the starting point of this research, the other part of its content is the result of a far more complex question; to represent the notion of *Information* in the context of logical calculi. The main idea of this research can therefore be formulated in the following terms: to find an intuitive and formally useful representation of the notion of information within a logical setting, in order to provide a clear formulation of the analyticity principle. The logical formulation is provided by the constructive version of Type Theory.

This research is thus part of a precise field of currently growing perspectives and theories, only recently explicitly recognized under the label of Philosophy of Information: by this term one refers to the critical investigation on the conceptual nature and on the basic principles of information, the determination of the relevant computational systems for such a notion, and the description of its use; it moreover expresses the philosophical formulation of problems related both to epistemology and technology. Therefore, the Philosophy of Information collects a wide range of philosophical investigations. Concerning the present research, the problem of analyticity represents the essential topic in the connection between logic and information. Information will be thus referred to as the conceptual term expressing the content of logical derivations: to provide a proper interpretation of such a content in a precise formal meaning is a first result of this research. The notion of information is in general defined according to either a result-based approach or rather an agent-based one: this obviously depends on the kind of philosophical conditions one is willing to satisfy or to consider relevant. The present interpretation is strongly influenced by the logical framework accepted, and thus it provides an all-invasive reformulation of the principles usually assumed to hold in the context of the various theories of information, in particular regarding the alethic value ascribed to such a notion. This depends on the description of the logical approach used, and of the results considered relevant to the proposed solution.

The logical and philosophical perspective accepted throughout this research is thus essential to the understanding of the notions involved, to the reformulation of the concept of information, and to the proposed solution to the problem of analyticity. The understanding of a logician's attitude towards truth and knowledge is at the very basis of an entire train of thoughts and of the choices about what reality and truth are, what it means to know, and obviously the consequences one is willing to accept from this. The constructive approach represents in this sense a way of accepting responsability for our own decisions, determining knowledge in terms of our own limits, and possibly establishing a dependence of our reality on the steps one chooses (or is able) to make, an approach which reflects also an ethics of knowledge. In this light, the constructive perspective provides an important and essential change: explaining information does not amount just to understanding what is expressed by a certain propositional content; rather, it is clarified also in terms of what is needed in order for a certain judgemental content to be formulated. Formally, this leads to ascribe a relevant role to the notion of assertion condition. Moreover, the resulting notion is developed in line with the logical elements and concepts furnished by the formalization; thus, it relays on a solid logical analysis.

The process of verifying an intuition may have more or less fruitful results, and it can even be wrong. To my mind this is exactly the role played by the formalism, to justify and prove whatever one *feels* could be the right model or the correct formulation of an idea. This process leads in the present case to Constructive Type Theory (CTT) as the framework which actually provides not only the formal but also the philosophical theory: the theory developed by Martin-Löf is in my opinion philosophically powerful and provides a high degree of conceptual awareness. The ability of developing a deep conceptual framework is essential to the work of logicians coming from philosophy, whereas mathematicians and computer scientists would value other properties in a theory. In this direction the role of the present research is twofold, showing a formal development for a certain theory and suggesting a theoretical extension of the epistemic analysis at the basis of the relevant philosophical logic. The formal development of this theory brings to recognize a deep and essential change in the epistemic background: the suggested extension of the constructive epistemology via the notion of information represents a step towards the perfect matching between the constructive philosophy and its formal logic, a second result of this work.

Chapter 1 introduces the foundation and formalization of CTT as the working framework: the theory is presented in its formal setting, but it mainly provides a new analysis of its philosophical themes. In particular, one will find some philosophical topics which are hardly considered in other introductions, whereas the technical and mathematical structure of CTT is well known and continuously developed. My aim here is to explore the theoretical possibilities of the theory, making reference to ancestors of the solutions proposed within CTT, directing attention to the epistemology and to the formal objects introduced for the problem at hand. In particular, the introduction of the *category/type* distinction and the explanation of the calculus of contexts is essential. CTT proposes a proper ontology, reflected in a hierarchical structure of types, forming in this way both the linguistic and the objectual levels of the structure we are speaking about. The hierarchical structure of types (and of their elements) can be thought of as a system (database) of informations available within the theory; in this sense, CTT reflects perfectly the notion of "ontology" as intended within computer science. The chapter ends by introducing the usually intended notion of information in the context of CTT: in this sense, CTT is a system which fully treats information, i.e. it is procedurally analytic, and it gives the ability of forgetting and recovering information in terms of an abstraction procedure. Nevertheless, this still refers to information only in a purely computational sense, whereas my aim here is to introduce an epistemic and formal description of that notion: this aim is obtained in the constructive setting by defining the essential difference from the concept of knowledge. The resulting notion is user-dependent and epistemically defined, it avoids the difficulties coming from the alethic nature imposed by the realistic approach and it presents an interesting and strong connection to meaning theory. The basis of such a formulation is contained in the strong commitment the constructivist owes to a notion of truth defined as existence of a proof: this in fact implies a stronger obligation in what he/she is disposed to accept, and eventually what he/she can later dismiss.

In Chapter 2, I shall present the problem (analyticity) and introduce the development of its possible solutions, up to the introduction of the notion of information. The analyticity principle is developed starting with the approaches of two great philosophers, Kant and Bolzano, in order to underline that the dichotomy between *act* and *content* (a central topic of the constructive approach in logic, in terms of the distinction between act of judgement and propositional content) is a natural theoretical consequence at the basis of the definitions of analyticity and analytic truth given by the two authors. Both conceptions aim at a description of scientific processes:

Kant connects analyticity to total uninformativeness of deductive processes, whereas Bolzano goes the other way round, using this notion to generalize the concept of validity in order to define derivability. The two authors characterize analyticity by the conceptual shift determined in the different definitions: thus, on the basis of the mentioned distinction between act and content, first the notion of meaning is introduced (mainly by referring to the work of Frege and Ayer) and finally the definition of analyticity is presented in terms of the notion of information (which historically is due to Hintikka, who takes over some Kantian insights). Chapter 2 finally leads to the mentioned epistemic description of the notion of information, based on a constructive reformulation of some basic principles: this means also to provide the conceptual lines along which a formal description of epistemic information within knowledge processes can proceed.

Chapter 3 introduces the formal structure which expresses the notion of information within CTT. Such a formalization does not present another framework to organize information within databases (one of the most basic applications provided by Informational Logics); it does not just draw a logical framework for some specific semantic approach to information. It rather furnishes a new topic in the philosophy of logic, especially for either analysis and representation of knowledge systems (i.e. for rational agents). In particular, the role that this formalization plays on the epistemic basis, and the related interpretation for rational agents will appear clearly. In such a frame the main concern will be to show how a knowledge frame, intended as the complete representation of an agent's knowledge content, can be extended, and how it can be updated by means of a formalized notion of information. The explanation and formal definition of this epistemic concept is therefore the core of the entire research: it is obtained by understanding the inner conceptual difference between the notions of knowledge and information, describing the latter in terms of an essential relation between the user, its epistemic state, and the conditions for stating knowledge. This description offers moreover the basis for a constructive model of dynamic reasoning. The ability of a rational agent to make use of informational contents rather than referring to explicitly proved contents allows for submission to revision: this can be seen as a procedure of type-checking, and it shows the essential connection to the decidability of forms of judgement. I maintain this result to be an important step towards the development of a systematic treatment of errors in the constructive setting, and it introduces the possibility of multi-agent systems, merging and decision-making processes, another area which needs to be faced within the community of logicians inspired by constructivism.[1] The structure of this chapter makes use of different conceptual references; in particular, it is based on a possible world semantics and it includes the typical formalization of Kripke models for

[1] See Primiero (2006).

intuitionistic logic. The development of the procedure of extension and updating is obtained by making explicit reference to the distinction between analytic and synthetic judgements in CTT, which obviously has a quite important consequence in determining what can be accounted as a synthetic procedure. This is the connecting point to the problem of analyticity for logical processes. The idea explained and supported throughout the research is that the relation of logical consequence formalized within CTT, i.e. derivability in the constructive sense, provides ways to formally clarify its synthetic nature.

This last topic is conceptually developed in Chapter 4 by showing how the principle of synthetic extension of logical reasoning regards essentially two aspects: (1) the structure of hypothetical reasoning and (2) the constructive notion of meaning. The distinction between knowledge and information is not just a question of formal expressions, rather it is reflected in the conceptual frame: knowledge is based on the analytic development of the derivations—it is therefore characterized by the property of correctness and it provides the meaning of the concepts involved; on the other hand, the substrate of these procedures will be shown to be synthetic, represented by the concept of information: it is characterized as a procedure of conceptual change, in terms of the meaningfulness of the notions involved. The resulting theory of meaning is a coherent extension of the normally intended intuitionistic one: it does not contradict the insight of the *meaning is use* slogan, and it provides moreover a complete understanding of those cases which appear problematic to this view. This is obtained by an epistemic description of some formal elements and their operations: the notions of presupposition (and therefore a reformulation of its theory), assumption, and type declaration are the core of the theory here presented. This last chapter completes therefore the essential aim of the research, i.e. to match a new reading of the formal structure of CTT with a conceptual interpretation of the notion of knowledge and meaning. The hope is that what is presented here proves useful to a complete understanding of the constructive philosophy of logic and to a general view on knowledge processes.

The possible developments of this epistemic description are various, and involve at least two important topics. The first concerns the nature of some particular kind of logical objects, which can be analysed directly in relation to their informational content: this applies in particular (at least in the constructive setting) to *abstract entities* as concepts, functions, and types. The sense in which the word “abstract” is here used is of a peculiar kind; it does not refer to abstraction as non-concrete, or non-definable, or unable to produce effects. The notion of information here considered can help in understanding the nature of such entities. This is to my mind an open field of research for extending the philosophical basis of constructivism. The second topic concerns more directly the philosophy of information in connection to its ethical problems and open questions: the present definition of information (and in particular its "weak" epistemic status) reveals the ethical consequences of the (here rejected) procedure of accepting informations as knowledge-contents to which truth-values can be ascribed. This intuition is particularly fruitful in describing phenomena of collective acquisition of "false information" via media and informational systems.

At the end of this introduction, I want to express gratitude to the people who have been my guidelines, in these years of formation, study, and personal growth: Giuseppe Roccaro, who introduced me to the beauty of logical reasoning, especially by the words of the Greeks and Latins—my knowledge and my scientific development owe a lot to him; Göran Sundholm, who during my year spent at Leiden Universiteit and since then until now made me see the other way than realism, gave me the comfortable feeling of studying something which exactly fits with my perspective on logic and other things—in the last years he has been the fruitful discussant of my ideas and the opponent of every word of mine, and has been for me a moral support and an incredible human help; I owe gratitude to Per Martin-Löf for having accepted to follow my studies during a semester spent at Stockholm University, posing crucial philosophical questions and illustrating the technicalities I needed to know in order to systematize this work—it has been a personal and human pleasure to know him and to learn from him. I am personally responsible for every conclusion I have drawn from papers and notes which he did not yet decide to publish; professor Leonardo Samoná, my PhD coordinator, has done everything possible to let me pursue further my studies in the best conditions, showing a great trust in me; Giovanni Sambin, in some short meetings, gave me his personal insight into constructivism as a way of doing and thinking.

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1
Constructive Type Theory: Foundation and Formalization

1.1 Philosophical Foundation

Constructive Type Theory has been developed by Per Martin-Löf in a series of papers and lectures since the 1970s: its first formulation, known as Intuitionistic Type Theory, was based on a strong impredicative axiom which allowed a type of all types being at the same time a type and an object of that type; it was abandoned after it was shown to lead to contradiction by Jean Yves Girard; the reformulation of the entire framework led to a strong predicative theory, which is now known as Constructive Type Theory (CTT). The theory has its theoretical core in the contribution by Brouwer and Heyting to Intuitionistic logic, and it is therefore built on a constructive epistemic framework, providing a new interpretation to many of the central notions of classical logic, such as those of *proposition*, *truth*, and *proof*. I will begin by presenting in this section some general aspects of the constructive type-theoretical approach, analysing in the next sections its formal structure. To start with, only a general theoretical description of such a logical approach will be given and later fully explained, especially in connection with the notions of *judgement* and *proof*. The main aim of the present chapter is thus to present the theoretical, logical, and formal basis of CTT: a philosophical analysis of the theory and the explanation of the elements allowing to reconsider the problem of analyticity in the light of the constructive framework will in turn justify the introduction of the notion of information within the epistemic description.

In the first instance, it should be stressed that the theoretical approach at the basis of CTT does not amount to a meta-mathematical interpretation: following Heyting's work, the theory starts instead by giving the constructive reading of the notion of *proposition*. It does not begin with a formal axiomatization and a mathematically formalized semantics: rather, one explains what a proposition is, what it means for a proposition to be true, and when one is allowed to assert the truth of a proposition, in order to verify what one can truthfully derive from it (i.e. which acts of inference preserve knowability of truth). Propositions are in turn explained in connection with the act of knowledge asserting them, namely, the act of *judgement*. In this first rather obvious sense, *meaning* is given within the type-theoretical framework in terms of computation, defining syntax to form canonical expressions, describing how assumptions-free judgements and hypothetical judgements (judgements made under assumptions) are formed: the meaning of each proposition will be given by the knowledge of a method to establish its truth. This systematization of the theory is based on the role given by Martin-Löf to logic and mathematics: logic is intended as the art of reasoning in a very old-fashioned sense, namely, the one intended by the Greeks and the Latins. Under this interpretation logic is complementary to mathematics, the latter being directed to prove theorems, whereas the activity of a logician is to build formal languages by means of forms of judgement and inference rules to obtain those theorems searched by mathematicians. Once logic is not only based on a purely formal interpretation but is also used as a proper theory of reasoning and knowledge, it regains its status as the foundation of scientific knowledge, connected both to philosophy and mathematics: logic is not just an empty formal structure in the Hilbertian style, but is rather thought of as an interpreted system, whose objects are filled with meanings.[1] This approach refers thus not just to a mathematical theory, but rather let us refer to it as a *logical framework*, in which different philosophical problems are investigated. At the same time, the framework is a useful and powerful technique for both mathematics (logic intended as proof-theory or meta-mathematics) and computer science (symbolism to design programming languages). It is an essential aim of this work to develop further the use of CTT as a theoretical and logical framework, in order to consider and to solve a specific epistemic problem.

1.2 Basic Epistemic Notions

Constructive Type Theory is to be presented first of all as a theory of expressions in the old sense, comparable to Aristotelian and Stoic logic. Aristotelian logic developed the forms of reasoning by means of judgements in the form "S is P", S being a schematic letter for the subject and P for the predicate, analysing all the possibilities composed by affirmation, negation, universal and particular judgements, and using syllogisms as forms of inference. This schema was completed by the Stoics, by introducing consequence as a form of judgement ("If A then B"), plus disjunction, conjunction, and negation. Aristotelian logic was pervasive and was in fact the only one until the 19th century; the work of Frege represents at the same time the first

[1] Martin-Löf (1993) presents this idea of the essential connection between logic and mathematics.

modern formalization for a logical calculus and the original ancestor for the notion of type[2]; on the other hand, Gentzen notoriously provided the first analysis made on the basis of sequents, using introduction and elimination rules. These essential notions of modern logic appear at some stage and with different roles in the formalization and methodology at the basis of CTT. I will start considering the epistemic notions used by the theory, developing them in connection with the proper logical structure and formalism. In later sections also the historical foundation will be presented.

The essential innovation given by the constructive approach is the new interpretation of the conceptual connections between the notions of:

- Proposition
- Truth
- Falsity
- Knowledge.[3]

These are key notions for the philosophical setting of the theory; their understanding relies on the concept of *judgement*, which allows the connection of the notion of proposition with those of truth and falsity, with affirmation and refutation being the form of construction of a judgement,[4] as follows:

- A is a proposition.
- A is true.
- A is false.

The notion of *judgement* is epistemically defined by saying *what it is that one must know in order to have the right to* make it: this means that from an epistemic perspective a judgement is a *piece of knowledge*. It is the aim of this research to explain what knowledge is, and which judgemental forms can be properly considered knowledge candidates in a constructive framework. This explanation is thus given according to the philosophical basis of Intuitionistic logic: at this stage the general notion of *evidence* can be used, as the one which (the Intuitionistic concept of) knowledge is based on. A sketch of the conceptual relation of these basic terms is the following:

evidence → (correct) judgement → knowledge

These are the basic epistemic notions, completed by their non-epistemic counterparts, namely, the notion of *proposition*, and the alethic notion ascribed to it, i.e. *truth* and *falsity*[5]:

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[2] Sundholm (1986) underlines in which sense the logic at the basis of Martin-Löf's Type Theory represents a return to the Fregean paradigm.
[3] Cf. Martin-Löf (1995).
[4] Martin-Löf (1995, p. 188).
[5] Cf. Martin-Löf (1995). The distinction between proposition and judgement will play an essential role throughout the formalization of the theory, and for the understanding of the philosophical problem introduced in Chapter 2. A first brief

Epistemic notions	Non-Epistemic notions
evidence	truth-maker
judgement	proposition
correctness	truth/falsity
knowledge	state of affairs

where *evidence* is to be intended as the basis on which a judgement is knowable or a proposition established as true (its proof). In turn, to give a proof of a proposition allows to assert the judgement which says that the proposition is true. This implies of course that in order to state that a certain proposition A is true, one has to construct its proof (say a), so that "A is true" is equal to "there exists a proof a of A". Of course the notion of *existence* which is used here to define the one of truth is something other than the notion of existence ruled by the existential quantifier[6]: it is related to the description of what was explained by Aristotle as existence of an essence, or by Frege as existence of an object which falls under a concept. According to this interpretation, the existential quantifier depends on the more primitive notion of existence, like when one affirms that

$$(\exists x \in A)B(x) \text{ is true} = \mathit{Proof}(\exists x \in A)B(x) \text{ exists},$$

a formula in which this distinction is obviously clarified by the presence both of the quantifier $\exists$ and of the verb "exists". It is only at this point, in virtue of the constructive explanation of existence as "instantiation" that classical logic is rejected.[7] Thus, the theory relies on the general *Verification Principle of Truth*, according to which truth is justified by the *existence* of a proof of the proposition, which makes the concept of truth for proposition no more primitive, but rather *defined*:

Principle 1.1 (Verification Principle of Truth) *The notion of truth is defined as existence of a proof (Truth = Proof + Existence).*

A summary of the crucial points of this (general) theoretical-foundational approach is the following[8]:

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explanation of this connection can be given here as follows, for the sake of clarity: asking what a proposition A is means nothing but asking what one needs to know in order to assert the judgement "A is a proposition". Here comes the Intuitionistic understanding of the notion of proposition, via the explanation of the meanings of the logical constants; given these explanations, a certain proposition A will be given by the set of its proofs. In this way, a proposition is defined by stipulating how its *canonical* proofs are formed.

[6] For this explanation cf. Sundholm (1993, 1994).

[7] It is relevant to underline the importance of the analysis developed by Martin-Löf (1991) relatively to the notion of logically possible existence and actual existence, a topic that will be reconsidered later. A formulation of existence as instantiation is given by Martin-Löf (1992).

[8] This list is extracted with some variations from Sundholm (1993).

1. Propositions are explained in terms of the proofs which are required for their truth.
2. Proofs are constructions.
3. Constructions are mathematical objects.
4. The theorem (justified judgement) "the proposition A is true", in its explicit form, sounds: "the construction a is a proof of A".
5. A theorem is explained by virtue of what is necessary to know in order to make that judgement.
6. Propositions have *provability conditions* (whereas judgements have *assertability conditions*).
7. Judgement and correctness for judgement are epistemic notions, proposition and truth/falsity for it are alethic notions.
8. Truth is given in terms of the existence of a proof.

After this presentation of the main framework of the theory, the analysis of its conceptual and formal basis follows: this will be done by starting from the philosophical problems endorsed by the theory, developing the logical formal structure, and paying particular attention to the Intuitionistic framework of the theory.

1.3 Types

1.3.1 *Constructive Notion of Type*

The notion of type in use within CTT has deep conceptual and formal roots in the history of logic.[9] The constructive notion of type can be possibly explained in connection to different general terms, all of them well known in the development of philosophical and mathematical logic, such as:

- Category
- Type (classical version)
- Sort
- Level

The notion of category obviously recalls first of all the use of this term in the Aristotelian logic (κατηγορία), and the form of predication conveyed by the judgemental form is essential to the understanding of the present framework, because it represents the essential root of the type-theoretical formulae. The corresponding Aristotelian notion represents the meaning-giving term in every well-formed predication: κατηγορία comes notoriously from the verb κατηγορείν, abbreviation for the long form κατά τινος ἀφορεύειν, "to say something about something". Within Aristotelian logic and metaphysics, there is an essential relation between what a being is, namely, its

[9] The background of the intuitionistic notion of type is presented by Martin-Löf (1987, 1993).

essence, and the predications being performed in relation to it: if essence corresponds to meaning, the latter is not just given by the category of substance (οὐσία, the first of the categories); rather categories determine all the meaningful predications which can be performed in relation to the subject involved. Thus, the (correct) forms of predication built up by the copula scheme "S is P"[10] are the ones which illustrate a thing's essence,[11] and categories are in this sense the way meaning is preserved. In relation to the mentioned connection between ontology and predication, Aristotle explains categories according to a twofold direction, as categories of "what is" and categories of "what is said".[12] The verb "to be" in its form "is" (copula) inside the Aristotelian form of judgement "S is P" is not a 2-place relation, but a way to attribute the category P to the subject S, and this suggests a rather obvious similarity to the notion of type as intended within CTT, in which the identity between propositional predication and set-theoretical properties fully and explicitly holds.[13] In particular, forms of predication for this theory correspond to instantiations of a certain type with one element, which means exactly that a certain individual belongs to a certain class: thus, the predication in the type-theoretical formalization will be in general represented by a subject predicated within a certain type. The connection between the Aristotelian notion of category and the constructive types is quite evident, both being essentially *meaning-giving structures*.[14]

The notion of category as intended by Aristotle is radically changed by Kant. The use Kant made of this term in the *Critique of Pure Reason* is related to a pure concept of understanding, which in turn corresponds to a form of judgement. The distinction with the Aristotelian notion of category is evident: the linguistic category is not extracted by being recognized in what there is (ontology), but rather from what is thought. On the other hand, the correspondence to categories as meaning-giving forms of expression is still entirely preserved under the Kantian view, and in turn it is even stricter with what later we will determine as proper categories of Type Theory.[15]

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[10] This one represents already a rough translation of the proper form conveyed by Greek language; in fact, ὑπάρχειν reads more exactly as "belonging" leading to a formulation of the judgemental form as "P belongs to S".

[11] Martin-Löf (1993, p. 38) refers to the connection between the Aristotelian τὰ σχήματα τῆς κατηγορίας and the syllogistic schemes of reasoning, a link which is expressed, for example in Metaphysics, book Δ.

[12] Aristotle (Cat, par. 2).

[13] This is the "propositions-as-sets" interpretation, to be introduced later.

[14] It is important here to underline that despite the mentioned similarity the use of the term "category" will be reserved later for a different kind of expression than what is intended by "type"; such a distinction will become natural by considering the question of method and particularly evident by means of the formalization. Cf. in particular Sections 1.3.4 and 1.5.3.

[15] Moreover, in connection to the Kantian philosophy of logic, CTT has a central point in explaining the difference between analytic and synthetic judgements,

The notion of category as a meaning-giving structure in the context of predication was explicitly restored by Husserl: expressions are considered by Husserl as meaningful signs, and meaning categories describe in turn as categories of the possible objects referred to by the expression, being also possible for a meaning category to be empty of real existing objects. Thus, the Husserlian system distinguishes clearly between *semantical* and *ontological categories*, by making the two levels already involved by the Aristotelian treatment more rigorous, where language is the way of referring to entities. Husserl considers both types of categories as essences, to be grasped by acts of thoughts; the study of essences is done in terms of essential insights on meanings and independently of the corresponding ontological kinds. Essences are distinguished between formal essences (*categories*), by means of which individuals are described, and material essences (*regions*), classifying entities according to their nature.[16] In the analysis of categories and types, the original link between the linguistic and the ontic regions will be restored, and this will directly determine the nature of the method and of the syntactic/semantic distinction for CTT.

A different use of the notion of type was notoriously due to Peirce,[17] who introduced the distinction between *token* and *type*. The latter term refers to the shape or form of something, whereas the former means the different occurrences of such a form. Referring to this terminology, the notion of type introduced by Russell[18] was somehow unlucky, referring to the word "type" in a different way: in fact, such an understanding of the term type has its own roots in the notion of *function*, essentially based on the Fregan understanding of this concept. CTT thus represents the evolution of the notion used by Frege, and our notion of type represents a strucure playing the role of categories and corresponding to formal rules holding for functions. In Section 1.6 the structure of the theory will introduce the notion of dependent object, and to this aim it is necessary to explain the technical connection of types with the structure of functions: a brief historical and technical introduction to the development of the notion of *function* will be given there. Meanwhile, it is here relevant for the clarification of types to give some insights on its intuitive notion: one generally refers to a function as a procedure that provides a value for each element given to it as input. The relation can be either a mathematical formula or a syntactic method, deterministic in that it has to produce always the same value on the basis of the same argument. Frege, on the Aristotelian assumption that the main category for each object is τό τι ἕν εἶναι, the *substance*, started by trying to use a unique universe, the one of objects (*Gegenstände*), and developed his system by making use of *functions*, to be able to go from objects to objects, and

something that will result later extremely important in our analysis. Cf. Martin-Löf (1994).

[16] Husserl (1913a,b).

[17] Peirce (1906).

[18] Russell (1908).

expanding the system by applying functions to functions: in the *Grundgesetze der Arithmetik*, Frege established that the formula $(\alpha_1, \dots, \alpha_n)$ is the "type" (using the modern terminology) of n-place functions, which gives rise to an object of a specific level (*Stufe*).[19] This means that if α_1 up to α_n are types, we can form a new type which collects all the previous ones (here we already introduce a formalization for such a predication):

$$\frac{\alpha_1 : type, \dots, \alpha_n : type}{(\alpha_1, \dots, \alpha_n) : type}$$

A schema of the correspondence with functions is the following:

$()$	object (*Gegenstand*)
$(())$	unary function
$(()())$	binary function
$(()\dots()_n)$	n-ary functions
	which take many functions into an object (*Wertverlauf*)

where clearly unary functions have objects as arguments, secondary functions have unary functions as arguments, and so on. Later, in his *Begriffschrift*, Frege introduced the judgeable contents (*beurteilbare Inhalte*), considering propositions inside the universe of objects: this gave rise to antinomies due to impredicativity. Russell (1903) presents a way out from the paradoxes generated by Frege's functional hierarchy, and in this sense it represents the natural ancestor of Type Theory. The Russellian type theory is related to the Fregean functional hierarchy by accounting the "simple" types, independently from the complexity of definition (so that it naturally reflects the order of "objects", "concepts", "second order concepts", etc.).[20] Russell presented the simple theory of types in two appendices (1903), then developed the ramified version (1908): in this new version the type of a function depends not only on the types of its arguments, but also on the types of entities referred to, and quantified over, by the function itself, i.e. through typing propositions. In the simple theory of types Russell modified the Fregean structure by defining:

- The type of the individual valued functionals
- The type of proposition-valued functions

Referring again to the previous case, when $\alpha_1, \dots, \alpha_n$ are types, the Russellian theory understands $[\alpha_1, \dots, \alpha_n]$ as a type too, i.e. n-ary propositional functions with types $\alpha_1, \dots, \alpha_n$ as arguments represent types themselves:

$$\frac{\alpha_1 : type, \dots, \alpha_n : type}{[\alpha_1, \dots, \alpha_n] : type}$$

---

[19] The level of a type is defined by Frege (1884) as the maximum of the levels of the argument types plus one.

[20] One should also remember the anticipation of the simple theory of type due to Schröder. Cf. Church (1976).

and the system is then enlarged adding clauses for relational types:

$[] = type$	e.g $\bot, \top$
$[[]] = type(type)$	unary function, e.g $\neg$
$[[],[]] = type((type),(type))$	binary function, e.g. $\wedge, \vee, \supset$
$[[[]]] = type(type(type))$	quantified function, e.g. $\forall, \exists$
$[[] \ldots []_n] = type(type) \ldots (type)_n$	type of n-ary functions/relations

Both structures, the Fregean and the Russellian, are powerful enough to express systems of a certain complexity, such as in the context of first-order logic, but not enough for more complex systems. A new notation for the theory of types was then introduced by Schönfinkel in 1924, based on the idea of representing functions of n arguments as a unary function having a value corresponding to a function with $n-1$ arguments, proceeding until one reaches the ground types (individuals and propositions). In this way, it is possible to give three clauses for forming types:

1. ι (for individuals) is a type.
2. o (for propositions) is a type.
3. If α and β are types, then $(\beta\alpha)$ is a type (with α for the argument type and β for the value type).

Accordingly the level is defined in the following way:

$$L(\iota) = L(o) = 0 \qquad L((\alpha\beta) = max(L(\alpha)+1, L(\beta)),$$

which represents the basic structure for Church's notation, and for the structures developed by Schütte, Curry, and Ajdukiewicz.[21] In general, the simple type structure makes it possible to type all the constants of first-order logic, while with dependent types of Intuitionistic type theory one is able to type even quantifiers whose domains vary.

According to Martin-Löf something is never an entity without being of a certain sort or kind, and each mathematical object is always typed: such types (as we will see later) are the source of the categories of predication, giving rise to the syntax and semantics of the theory. Whenever the notion of type is understood in this deep and broad philosophical aspect, being assimilated in a general and intuitive sense to a structure constituting a family of objects determined by any property, together with an equivalence relation, the resulting formal theory is of a specific kind: such a notion of type is conceptually prior to, and provides an interpretation for, other notions such as the one of proposition, or the mathematical ones of set, elements of a set, the set-valued functions over a given set, and predicates over a given set. Thus, a theory of types can be used to present a theory of sets, using variables ranging over sets and higher-order objects, but in fact by choosing to use the more general and basic interpretation of the notion

[21] Martin-Löf (1993) has treated the modern evolution of the notion of type.

of type, one understands the theory as a general logical framework able to formalize expressions, as it has been done at the beginning of this section: this kind of type theory is usually referred to as the *monomorphic version* of the theory, whereas starting by defining the types of sets (or propositions), the set-formation operations, and the proof rules for these sets, one considers a specific type and thus refers to the *polymorphic version*. In the monomorphic version the notion of set can therefore be intended in all of its generality, allowing to consider a logical procedure such as assumptions on sets not yet defined.[22] The monomorphic version of the theory allows for the introduction of different notions (sets, propositions, and similar) in terms of types; moreover, it leads to formalize derivations by means of metavariables ranging over formulae, and it requires the explicit formulation of all the information on which arguments are based: an application function on two sets will, for example, take two arguments in the polymorphic version (i.e. a function from A to B and an element in A), whereas the fully explicit formulation of the monomorphic version will take four arguments (respectively, the two sets A and B, the function from A to B, and finally the element in A).[23] Starting with his early work (1975), Martin-Löf has developed his type theory in a purely predicative way, so that second-order logic and simple type theory were not to be interpreted in it; the theory presented in his later publications (1982, 1984) is polymorphic and extensional, and the semantics given for the normalization procedure which lets an element be computated to its normal form provides a strong elimination rule, needed for propositional equality, in a way that judgemental equality is no longer decidable. In order to overcome this problem the monomorphic version is used, in which the equivalence relation needed by the definition of type and given in order to state the identity between objects within a certain type is decidable. Therefore, great attention has to be given to the notion of identity involved and to the formal rules for it. In Section 1.3.2 I proceed in defining the monomorphic notion of type, by considering the general expressions that will provide the basic relations between types and their objects.

1.3.2 *Definition*

The epistemic basis of CTT develops the notion of type in terms of its *definition*, by clarifying the relation between objects-of-types and types themselves. As it is well known, Aristotle underlined the strong connection between definition (ὁρισμός) and essence (τό τι ἔν εἶναι), the former being the expression which signifies the latter, its λόγος.[24] This amounts to a distinction in the clarification of the notion of definition itself:

[22] See, e.g. Nordström, Petersson and Smith (1990), pp. 137–138.
[23] In Section 1.9 we will insist more on the role of informational content for the distinction between the monomorphic and polymorphic versions.
[24] See, e.g. Arsitotle (Top, 101b39).

1. *Real definition* is intended as a genuine explanation of meaning.
2. *Nominal definition* is intended as an equational or identity definition.

To give a *real definition* means to express an analytic recollection of all the (definitional) properties of a term, whereas to give a *nominal definition* means to establish an equational definition between such a term and some other sign. Defined expressions receive meaning by a nominal definition, while primary expressions derive meaning from a real definition. This distinction is completely reflected within the type-theoretical framework: the definition of a type is given in terms of a meaning expression, being types of primary objects of the theory defined through the primary forms of judgement (the same is true for notions like *object* or *family of types*); on the other hand, definitions of other elements like *class*, *relation*, *connective*, *quantifier* are given in terms of defining equations.[25] For this reason, within the constructive type-theoretical framework a real definition is a concept explication, and can be understood as a *conceptual analysis*.[26]

The notion of type, obviously the first to be defined, is abstracted by the initial step of the theory, namely, by exposing a general theory of expressions. There are four forms of expressions introduced by the theory, asserting respectively that[27]:

1. A certain object is a type.
2. An expression is an element of that type.
3. Two expressions are the same inside the same type.
4. Two types are the same.

The semantics of type theory explains what these judgements mean. In this way, to introduce and define a type one must know:

1. What it means for an object to be of a certain type
2. What it means for two objects to be the same within a certain type

and they represent respectively what is called *application criterion* and *identity criterion*, according to the terminology introduced by Dummett (1973). The order in which these assertions are stated reflects the logical structure according to which the existence of a type comes *conceptually* before the assertion that something belongs to that type; nevertheless, clearly the *definition* of any type is given according to some object belonging to it. In this sense the form of expression

... is of the type ...

has to be preceded by (presupposes) the assertion that

... is a type,

[25] Martin-Löf (1993, pp. 60–61).
[26] Cf. Sommaruga (2000, p. 2). The formal treatment of the notion of identity, given in Section 1.4.1, will say more on this essential topic.
[27] Cf. Section 1.5 for further explanations and formalization.

where, for example, some α will take the place of the dots. This remark is necessary in order to introduce two problems:

1. Conceptual priority
2. Impredicativity

The first reflects the theoretical structure underlying the theory, which will be explained in the following paragraph; the second is the well-known problem caused by the Fregean hierarchical structure, avoided in Type Theory via the conceptual priority of types over objects belonging to types, and the essential introduction of the notion of category.

1.3.3 *Conceptual Priority*

The foundation and systematization of the theory is done by setting an order for the basic notions introduced, determining a *conceptual priority* among them.[28] Such a structure can be thought of as developing the Aristotelian πρότερον and ὕστερον κατὰ τὸν λόγον for the theory, the methodological and ontological distinction later translated by the scholastic tradition as *prior* and *posterior secundum rationem*. Involved in such a relation of order are of course the elements occurring in predications and the distinction between concepts defined or taken as primitive in the theory: this conceptual order determines a definitional order, established according to the nature of the objects to be defined; and finally, because a definition is an explanation of the essence (real definition), an order will hold also between essences. The following schema shows the sequence of priority between orders:

Conceptual order
↓
Definitional order
↓
Essential order

In the history of philosophy, in line with the mentioned Aristotelian distinction and its scholastic explanation, Augustine's *De Ordine* represents the medieval development of the Platonic inspired distinction between *ordo intellectum* and *ordo rerum*, whereas the Aristotelian tradition is followed by Thomas Aquinas.[29] These are the ancestors of the priority between orders holding in CTT, which takes into account the order of things and their definitions as distinguished from the order of concepts. The conceptual order within CTT thus establishes the priority between the basic logical concepts of

[28] Martin-Löf (1984, 1987, 1991, 1993).
[29] Martin-Löf (1993, pp. 61–65).

- *Proposition*
- *Truth*

and the mathematical ones of

- *Set*
- *Element of a set*
- *Function*

The first two notions are connected by the concept of existence, namely, via that of proof. Existence of truth in terms of evidence is moreover developed by introducing the classical distinction between the categories of actuality and potentiality; thus in turn truth is explained as *actual truth* and *potential truth*:

> *"Actual truth is knowledge dependent and tensed, whereas potential truth is knowledge independent and tenseless".*[30]

The actual truth of the proposition A, according to the Intuitionistic framework, presupposes a construction already obtained for A, while potential truth is the possibility to construct such a proof. Following the Aristotelian metaphysics, actuality precedes potentiality in the order of the real (i.e. in the order of entities).[31] The notion of actuality corresponds of course to the instantiation of an act performing and realizing truth: here one finds the first theoretical justification for defining the logical notions of *proposition* and *truth* upon a more fundamental one, precisely the notion of *judgement*, which immediately states the distinction between the act of judging and what is judged.

On the other hand, the mathematical concepts of *set* and *element of a set* are essential in that they represent an exact mathematical interpretation of the corresponding notions of *type* and *element belonging to a type*; the system is extended via the concept of *function*, which is the mathematical way to explain the relation between two elements belonging to equals (or different) types: on the basis of the Curry–Howard isomorphism (to be explained later in Section 1.5.7), the same is true respectively for the notions of *proposition* and *proof*. But in the first instance the order of conceptual priority holds between concepts and their definitions, i.e. the order can be established between two concepts[32]:

1. If the understanding of a concept presupposes the understanding of the other concept
2. If the definition of a concept refers back to the definition of the other concept

[30] Martin-Löf (1991, p. 143). About potentiality as possibility, in connection to the framework of CTT; cf. also Löhrer (2003).
[31] Martin-Löf (1990), mentions the Thomist formulation "Actus est prior potentia".
[32] Sommaruga (2000, p. 5).

In this sense, it is clear that one establishes an order between the concepts treated up to now, in the following way:

$$\textbf{judgement} \Longrightarrow \begin{array}{c}\textbf{proof} \searrow \\ + \\ \textbf{existence} \nearrow\end{array} \textbf{proposition} \Longrightarrow \textbf{truth}$$

The notion of *judgement* comes first because it should be understood as a "ground notion", explaining on its own the concept of *proposition* as its content; the concept of *proof* is considered as the (proof)object instantiating a demonstration act for the proposition contained in the judgement. The assertion performed in a judgement regards the truth of a proposition. Thus, the first schema has a second extension, that does not rely anymore on the specific content of a proposition with its proof object:

$$\begin{array}{cccc}\textbf{judging act} & \textbf{(demonstration)} \rightarrow & \textbf{evidence} & \rightarrow \textbf{correctness} \\ & & \downarrow & \\ & & \textbf{propositional} & \\ & & \textbf{content} & \end{array}$$

The *act of judging* establishes a *demonstration* (proof not intended as object) which furnishes *evidence* for a *propositional content*, and gives rise to *correctness* for proofs. The problem of definition and the structure of conceptual priority are thus essential to the theoretical frame of CTT, such that the theory represents an attempt to build each form of judgement starting only from the explanation of what a type is, and what it means for an object to be of a certain type. This represents the way in which types are defined and in which *categories* are introduced: the connection between forms of expression within the theory and the objects these expressions refer to is settled by the syntactic–semantic method.

1.3.4 *Method*

The starting point to explain the formal and theoretical structure of CTT is to give the definition of what a type is, namely, by answering the basic question "what is a type?" in terms of the other one "what does it mean to belong to such a type?". To answer these questions Martin-Löf develops a method which is called *syntactic–semantic*, consisting of two parts:

(a) *Syntactic*: the sense of a primary entity (in that it belongs to a certain type) is given by the process of composition of the formal expression which denotes such an entity.
(b) *Semantic*: the sense of that entity can be understood contextualizing the rules of composition applied to obtain the expression in the first part (a).

This method allows us to clarify the nature of mathematical objects by paying attention to the expressions denoting objects,[33] because these show exactly their meaning. Here one finds the connection between the notions of *definition*, *conceptual priority*, and *identity*, explained below: the relation between an expression and the object it signifies represents the *act of meaning* or *understanding*. For an object to come into being the expression by which that object is denoted is necessary: the formulation of such an expression, consisting in the predication of the object within a type, represents therefore the act of understanding the object. The connection between an object and its expression is thus a turning point for the method at the basis of the theory: a mathematical object is always expressed via the explanation of what is the type to which it belongs, and this brings us again to the conclusion that types come conceptually before objects, because the latter have an ontological status only if semantically typed, i.e. if their type has been previously declared. Also through the description of the syntactic–semantic method, the need clearly arises to justify the conceptual relation between the predication aptness of the type and its definition. The relation between the semantics and the syntax of the theory can be represented by a *General Principle of Meaning*, formulated in the following terms:

Principle 1.2 (General Principle of Meaning) *The relation between objects and expressions is given as follows: a certain object* a *is the meaning of the expression* "a"*; in the other direction,* "a" *is the expression denoting the object* a*.*

This principle reflects the natural direction from the ontological to the linguistic level.[34] Thus, the syntactic level goes from the object a to its expression *"a"*, and this means to consider the object in a purely formal way, the formalization consisting in divesting the object of sense, in the Hilbertian style. On the other hand, the process of endowing the expression *"a"* with sense means to give its content, referring to the object a. As the *General Principle* states, *"a"* is the expression of a, and a is the meaning of *"a"*, where an expression is obtained by the process of formalization. In such a process types are turned into type expressions, and objects into object expressions, so that the object set is turned into the category of set expressions (which is in turn its syntactic category). It is quite clear that the syntactic–semantic method is more than a simple distinction between syntax and semantics: the ontological basis on which the theory is

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[33] Martin-Löf (1987).

[34] Here Martin-Löf refers to the Husserlian approach, according to which *"in naturliche Einstellung wir sind gegenständlich gerichtet"*. Husserl considers the difference between the object a and its expression *"a"* by using respectively the expressions *Bedeutung* and *Ausdruck*. Moreover, in illustrating these notions during his lectures (1987), Martin-Löf refers to the Husserlian expressions *Syntaktische Kategorie* and *Bedeutungskategorie*, while in (1993) he uses the Husserlian terms *Sinnbeseelung* and *Sinnentleerung*.

built and the development of its linguistic level allow us to understand the entire method as nothing but a sort of duality recalling the philosophical distinction between *form* and *content*. In fact, the relation between "expression" and "content" and that between "object" and "type" can be thought of as a modern mathematical version of that ancient duality. Plato first introduced the distinction between εἶδος and ὄν, which was only an aspect of the all-invasive primary dichotomy between τὸ αἰσθητόν (the sensible) and τὸ νοητόν (the thought, or what belongs to it); for Aristotle the way from the ὕλη (matter) to the οὐσία (substance) is given inside the όν (being) through the essence, τό τι ἔν εἶναι, namely, referring to "things in that they are things" (τὸ ὄν ᾗ ὄν), an expression which in turn explains what ontology is about, and which we will take into account later. The mentioned dichotomy was then restored by the Scholastics in the terminology *materia* and *forma*, their connection giving rise to the *substantia*. Here the role of definition is particularly important, determining what really is the τὸ τί ἐστι (*quidditas*) — the being which really exists — the connection of form and content. Within the type-theoretical frame the relation between the construction and the object is given through the connection of form, represented by the type[35]: here we find the essential concept that mathematical objects are *objects of knowledge* which need to be expressed in order to be grasped. The syntactic–semantic method used by Martin-Löf is thus built up by the relation between the *expression* intended as object of a syntactic category and its *meaning*, i.e. the object for which the expression stands for, intended as a semantic category. The syntactic–semantic method is a way to state a new theory of essences, building a bridge between the semantics and a proper ontology. This method is enough for building up a theory of mathematical essences, given that in this interpretation a mathematical world of objects can exist only if expressed. The question which now naturally arises is the following: how many kinds of categories do we have? The answer is obtained by reducing the (classic) schema

$$\begin{array}{c}\textbf{Syntactic category}\\ \downarrow\\ \textbf{Semantic category}\\ \downarrow\\ \textbf{Objectual category}\end{array}$$

composed of three categories, to the following one

$$\textbf{Syntactic category} \leftrightarrow \textbf{Semantic category}$$

where the objectual category conflates into the semantic one because the object represents the meaning of the expression "a", i.e. a itself, and therefore

[35] Martin-Löf (1993, pp. 163–168). Another reference is made by Martin-Löf (1993), to the Heideggerian couple of terms *Zuhandenheit*, which explains the use of tools without paying attention to their formal structure, and *Vorhandenheit*, which instead refers to the use of tools on the basis of the knowledge of their form.

the semantics is actually the ontology the theory speaks about. This last point has a further explanation: ontology is intended not just as the science about the things of any world, such that these are objects of other sciences, e.g. physics. Ontology is all about "*things in that they are things*" (in terms of the Aristotelian definition): this means to take into account objects as they are defined, i.e. objects in terms of the concepts they express, or they are defined by. Thus, a proper object of ontology is a *defined object*, an object expressed with all its (essential) properties. Ontology in this sense, conceptually near to the Aristotelian way of understanding it, amounts to a study of objects with the concepts they contain, that in the type-theoretical setting means to express objects in terms of the types they belong to. Hence, we are again considering the only way objects can be taken into account, by referring to the expressions they are (correctly) predicated in: by means of language the syntax and the semantics of the theory are connected, and the study of the formal expressions of the theory introduces the *categories* of the theory (Section 1.5.3), already mentioned in connection to the question of meaning and the problem of impredicativity.

The introduced distinction between the syntactic and the semantic level of the method explains a basic distinction inside the notion of meaning,[36] namely:

– *Sameness of meaning*
– *Identity of meaning*

This distinction is of course of the greatest importance for the notion of synonymity and requires an explanation of the concept of identity, to be analysed in Section 1.4. What is relevant to underline at this point is that, on the basis of the conceptual priority, identity of meaning and even sameness of expressions (e.g. nominal definition) ultimately refer to identity of objects, as primary elements of the theory. The link between the syntactic and the semantic levels for the type-theoretical framework can be thought of as a two-way relation between objectual (or semantic) and syntactic categories:

objectual categories

a

$\downarrow \qquad \uparrow$

formalization **contentualization**

$\downarrow \qquad \uparrow$

"a"

syntactic categories

This schema is to be considered as a modified version of the one already proposed by Aristotle in the first chapter of *De Interpretatione*, where he explains the connection between the object, the related movement in the soul, and the expression for it, as follows:

[36] Martin-Löf (1987).

παθήματα
τῆς ψυχῆς
↙ ↖
σύμβολα → πράγματα

and which will be used as a basis by the Stoics[37] and the Scholastics.[38] This idea will be later endorsed by the well known "semiotic triangle" which states the relation between object, expression and meaning:

meaning
↙ ↖
expression → object

According to Martin-Löf, the new schema with only two elements (semantic/syntactic categories) includes the Saussurian relationship between *signifié* and *signifiant* inside the *signe*, and in relation with the *chose*:

signifié
↙ ↖
signifiant → chose

while in the case of the Fregean relation between *Bedeutung*, *Sinn*, and *Ausdruck*, the schema reverses its arrows, in the following way:

Sinn
↗ ↘
expresses *refers to*
↗ ↘
Ausdruck . . . **Bedeutung**

The triangle schema[39] shows the relation between the three essential stages, the *mental*, the *verbal*, and the *real*. The theoretical problem one needs to solve within CTT concerns either the necessity of establishing the third realm of concepts (e.g. as done by Frege) or the possibility of conflating together concepts (meaning, if linguistically intended) and objects, so as to make no categorical difference among them. The solution is given in a proper way by the notion of ontology explained above, which we present here as the *General Principle of Ontology*:

Principle 1.3 (General Principle of Ontology) *Categories of objects are actually categories of meaning, because essences of objects, i.e. things in that they are things, are expressed by concepts via their meaning.*

[37] They will change the words, using respectively τυγχάνω, σημαινόμενον, and σημαίνον.
[38] They will translate the schema with the following Latin terms: *res*, *passio/intentio/conceptus animae* or *intellectus*, and finally *nomen*. Martin-Löf gives references to Ockham, Boethius, Thomas (1993, p. 175–176).
[39] This schema was originally presented in Ogden and Richards (1923).

It is therefore essential at this point to introduce the topic of identity, both because it arose already in the conceptual framework of the theory and because it will be essential in introducing the formalization and the sort of type theory considered all along the rest of this chapter.

1.4 Identity

The definition of a type is hence given by explaining *what it means for an element to belong to a type* (*application criterion*) and *for two such elements to be identical within a type* (*identity criterion*). The notion of identity is thus clearly involved at the core of the theory, both for the definition of type and for the theory of expressions. Moreover, identity was also implied by the notion of synonymity by introducing sameness of meaning or identity of meaning for expressions, and this will be again a central topic in Chapter 2, where the problem of analyticity will be presented. It is therefore essential to present the theoretical treatment that CTT gives of the notion of identity.

The relation of identity between two expressions holds primarily when such a relation holds between their meanings (i.e. objects); therefore, identity of objects (i.e. identity related to the ontological level) comes before the identity related to the linguistic level (synonymity). The way these notions are introduced in the framework is in connection with the schema of relations between syntactic and semantic levels of the theory, presented by extending the schema of categories to a four-element schema, which introduces the identical elements. If the original schema presents a two-way relation between the object a and its expression "a", i.e. the relation between the propositional (or numerical) expression and the type expression via the object itself, the synonymity of two expressions referring to the same object, and the identity between objects themselves, can now be introduced. For example, one can take the two objects a and b, equal to each other inside the type α, and construct a schema including their expressions, "a" and "b" (Figure 1.1).[40]

In this schema the relation of *evaluation* corresponds essentially to computability; for example, it holds between the object $S(S(0))$ (Peano's classical axiomatic translation of the Arabic numeral) and 2 (or the relative expression). The process of evaluation is particularly important here, because it makes it possible to extend the previous schema, which included only two levels (syntactic-semantic), by introducing a third level represented

[40] The following schema is built up from different elements stated and explained by Martin-Löf (1993, pp. 187–192). The example for this schema presented by Martin-Löf refers to the mathematical object $2+2$ as the meaning (a), "$2+2$" as its expression ("a"), and the object 4 as the semantic value of $2+2$ (b) and "4" as the proper expression for it ("b"). This schema includes the semiotic triangle as its proper part: in that case the evaluation is made only referring to the object level (semantic level), not to the syntactic level.

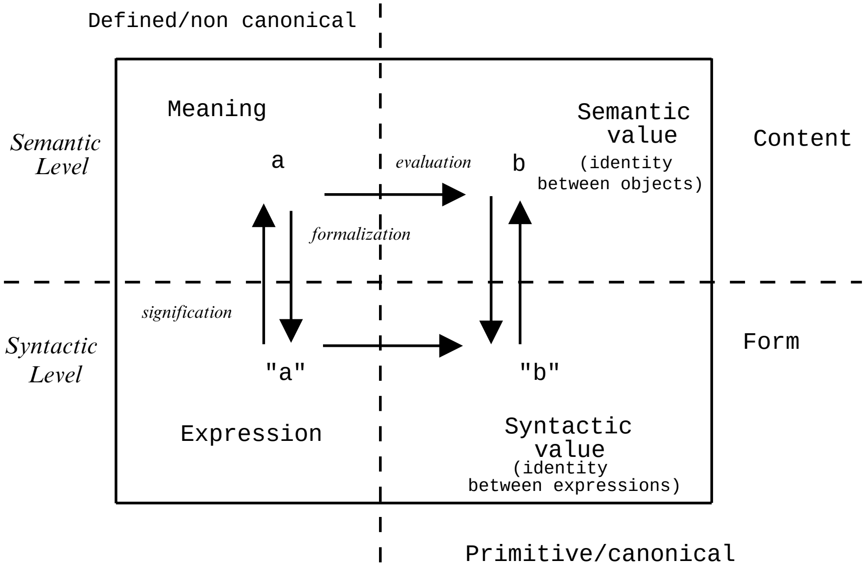

Figure 1.1. Schema including syntactical, semantical and identity relationships inside the type-theoretical framework.

by the identity relation: this is done in terms of *definition*. The relation of evaluation holds between the *definiens* and its *definiendum*, so that a is a defined term (always graspable through its type-expression "a"), which is evaluated as identical to b, the latter representing a primitive term. The definitional chain between a and b can be fulfilled by one step of computation (if a is a primitive term) or else by more steps. Moreover, a being the sense of "a", and representing b in this schema, the result of the evaluation process, namely the reference, one should be able to understand if sense and reference can be considered equal.[41] Thus, the relation represented by the horizontal arrow in the upper half of the schema is the relation of evaluation or computation, which states the identity of meaning, or synonymity. It is possible to state the equality between sense and reference only at the level of objects, while they are syntactically the same (i.e. concerning identity of expressions, in the lower half of the schema) only if both objects are primitive ones, i.e. the definitional chain has zero steps.

[41] This theme is developed by Martin-Löf (1993) in obvious connection with the *Sinn/Bedeutung* problem in Frege's writings (cf. Martin-Löf 2001): in this sense the Fregean idea that the *Sinn* corresponds to the object including its expression or mode of presentation (*Art des Gegebensein*) is particularly relevant; the object is identified via such expression, so that identity of senses results in identity of expressions (cf. also the Husserlian idea in *Ideen* that the logical meaning is given via a certain expression (Ausdruck)); for a detailed explanation of the Sense/Reference distinction in CTT, see Primiero (2004).

It is possible now to explain the identity criterion by distinguishing three versions of the concept of identity inside the theory[42]:

1. *Semantical identity criterion*, which corresponds to *definitional identity*, or identity between objects (=)
2. *Syntactical identity criterion*, which corresponds to *syntactically induced identity*, or identity between expressions (≡)
3. *Abstract* or *transcendental concept of identity*

Definitional identity (1) is introduced within the theory by the following rules:

- Reflexivity
- Symmetry
- Transitivity
- Substitution of identicals by identicals.

The rules for reflexivity, symmetry, and transitivity are the common rules holding in mathematics, and their formalization for objects belonging to types and for types themselves will be shown in Section 1.5.2; the fourth rule, the substitution of identicals by identicals, allows to state the equivalence between *definiens* and *definiendum* within a definition. These remarks complete the introduction of the notion of identity, in addition to what has already been explained in relation to identity for expressions. The role of identity for expressions is also relevant in connection to the informativeness of the evaluation procedure, and this is important for the task of a critical analysis of the synthetic nature of the logical system.[43]

1.4.1 *Definitional Identity vs. Syntactically Induced Identity*

The problem of definition is at the foundation of the type-theoretical framework, and certainly it presents a deep connection to the notion of identity. It has been explained how the notion of definition has to be understood, distinguishing between *real definition* and *nominal definition*. In the case of definitional identity, one is considering the level of nominal definitions: let us remember here that a definitional identity corresponds to a nominal definition, which is the way to obtain the meaning of defined expressions. A nominal definition represents then a stipulation for which no further justification is required, and it can be represented by the ancient couple "*definiendum* = *definiens*".[44] In the first instance, one distinguishes between definitional identity, which is a relation between linguistic expressions, and the relation of identity between the entities which are denoted by those expressions,

[42] Cf. also Sundholm (1999).
[43] Martin-Löf (1993, p. 236).
[44] Ibid. (1987).

i.e. their meanings.[45] The first kind of identity is given by the sign "$=_{def}$", so that when one refers to a relationship such as

$$a =_{def} b$$

it concerns the identity between two expressions. The second type of relation holds between the objects one is talking about, and it is formalized as

$$a = b.$$

The informal reasoning behind the interderivability of these two kinds of formulae is the following: given the identity axiom $a = a$, the formula $a =_{def} b$ implies that $(a = a) =_{def} (a = b)$, so that $a = a$ and $a = b$ have the same meaning and this immediately gives us the conclusion $a = b$.[46] On the other hand, the identity of the objects a and b is enough to state the identity of the respective expressions, so that it holds the formula $a =_{def} b$. Moreover, on the basis of the Intuitionistic approach, the validity of the judgement $a = b$ must of course correspond to the possession of its derivation. If such a closed derivation is supposed to hold, then it is clear that the identity $a = b$ implies the interconvertibility (formal counterpart of the informal "definitional equality") of the terms a and b. In this sense we can say that two derivations are interconvertible if, and only if, the proofs that they represent are identical, so that "identical" means in this context "provably identical". The relation of definitional identity is expressed by Martin-Löf according to three principles,[47] each of them having a formal counterpart, namely, a *conversion rule*:

1. Definitional equality between the *definiens* and its *definiendum*:

$$\text{redex conv convertum}(...)$$

2. Substitution of definitionally equal expressions for a variable in a given expression leads to definitionally equal expressions (preservation of definitional equality under substitution):

$$\frac{a \quad conv \quad c}{b[a] \quad conv \quad b[c]}$$

3. Definitional equality is reflexive, symmetric, transitive:

$$a \quad conv \quad a \qquad \frac{a \quad conv \quad b}{b \quad conv \quad a} \qquad \frac{a \quad conv \quad b \quad b \quad conv \quad c}{a \quad conv \quad c}$$

These are the general *formal* rules holding for the logical notion of identity and instantiated in the forms of judgements. Such rules allow the formulation of judgements either for a set-theoretical or for a propositional system,

[45] Ibid. (1975a, pp. 101–104).
[46] Ibid. (1975a, p. 102).
[47] Ibid. (1975a, p. 93).

via the expression of definitional identity between elements of a set and proof objects for propositions. In this way one understands the difference between definitional or semantical identity ($a = b$) and syntactically induced identity ($a =_{def} b$): the first one is much stronger than the second one, involving the ontological aspect of the theory; syntactical identity refers only to the identity of the formal expressions we use to express objects.[48]

1.4.2 *Identity as Theoretical Notion*

The identity issue in its philosophical and logical aspect has been widely considered since antiquity. It is clearly already present in Platonic dialogues, entering in the explanation of the relation of things to ideas; Aristotle brings this problem to its core, by considering the nature of *essence*, for which the notion of "being" is treated in connection to "predication", thus referring to the categories of "sameness", "otherness", and "contrariety".[49] Identity is explained in terms of predication when Aristotle says that two things are identical if all that is predicated (or predicable) of one of them, is predicable of the other.[50] The principle of identity ("*a being is what it is*") is thus obviously central to the Aristotelian philosophy, and it notoriously expresses the positive formulation of the basic principle of contradiction: "*a being cannot both be and not be at the same time and under the same respect*",[51] the logical principle *par excellence*, both a principle of knowledge and reality. Since these first formulations, the identity issue and the relation to definition was a central topic in Scholastics, particularly in Ockham, with the distinction between *definitio quid rei* and *definitio quid nominis*.[52] But one has to wait until Frege to find a fruitful connection for the development of the notion of identity with that of type: by using a unique universal type (the sort of all objects), Frege explained the question of identity both in terms of equality of content (*Inhaltgleichheit*) and as the binary relation of identity, respectively in the *Begriffschrift* and in the *Grundgesetze*; in *Über Sinn und Bedeutung* he explains the identity "=" as a relation between

[48] Note that within the Hilbertian formalistic perspective, the semantical identity criterion can be only reduced to this syntactic criterion, because all expressions are empty of their meaning. In Sections 1.5.4 and 1.5.5 the ground types for sets (*set*) and propositions (*prop*) will be introduced; the notion of identity presents the following meanings within those types:

- With $A = B : prop$ the concept of material equivalence between two propositions is intended.
- With $A = B : set$ the concept of equipotency between two sets is intended.

[49] Martin-Löf refers in particular to *Metaphysics*, book Δ, 1018a35-39; cf. Martin-Löf, (1993, pp. 41–42).
[50] Aristotle [Top] book 7, cap. 1, 152a 5–30.
[51] Cf. Aristotle [Metaph], book Γ, 1006a 3–5: "ἡμεῖς δὲ νῦν εἰλήφαμεν ὡς ἀδυνάτου ὄντος ἅμα εἶναι καὶ μὴ εἶναι."
[52] Ockham (1324), *Logica terminorum*, III, 26.

signs, in order to introduce identity of reference for distinct signs, so to justify the concept of *Sinn*. The notion of identity then occurs in a pivotal role in the famous Quinean slogan "no entity without identity",[53] which can be explained in terms of the following statements:

1. No entity without type
2. No type without semantical identity

The Quinean approach to identity is in order to avoid ambiguity: his interpretation of the sign "=" explains it as the extension from "is" (copula) to "is identical to", and the identity conditions are stated as the "division of reference". Thus, the problem of identity is stated in connection with synonymity for sentences (identity of meaning), analyticity, and informativeness of identity sentences.[54] According to Quine, identity assertions which can be considered true and useful are built up by different singular terms referring to the same thing,[55] and this explains the difference between forms of predication which are expressible as a function "Fa" and those for which the sign "=" is required. In particular, Quine holds that synonymity between sentences can be explained via the notion of analytic sentence.[56] In CTT the Quinean task to make these relations clear is actually obtained; the structure of reference and meaning is stated as follows[57]:

- An object expression stands for its meaning (which is the object itself).
- A type expression signifies its meaning, which is a type.
- Dependent or function objects or types have no value or reference.
- The reference of an expression whose meaning is a non dependent or functional object results from the evaluation of the meaning of the respective expression.

The identity of meaning of two expressions is given in terms of semantical identity: identity of meanings amounts of course to identity of the objects which represent the evaluation of those expressions. In general, in the case of CTT there is no type without semantical or intensional identity, and moreover also no such identity without type; then semantical identity is already typed.[58] The notions of identity and synonymity will first be formally considered in Section 1.5 and then taken into account again in Chapter 2: they represent the essential notions in order to clarify the idea of analyticity in logic and to identify the role of information for logical processes.

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[53] Quine (1958, 1960).
[54] Ibid. (1960, cap. 1, par. 14).
[55] Ibid. (1960, par. 24).
[56] For a definition of the notion of analyticity as intended by Quine, cf. Definition 2.10, in Section 2.2.4.
[57] Cf. Primiero (2004).
[58] Already by Curry's combinatorial logic one understands that semantical identity comes not *before* typing, but rather the two notions are to be considered simultaneously.

1.5 Formal Analysis of Types and Judgements

In Section 1.5.1 the ground types, namely, the type of sets and that of propositions, will be introduced as objects definable by monomorphic types; the forms of judgements in use within the theory and their formalization will be shown. Moreover, this will also allow for the introduction of another type, that of functions and its formalization. On the basis of the theoretical analysis developed above, the formal role of the notion of identity will be explained and formal rules for it introduced.

1.5.1 *Formalizing the Forms of Judgement*

The aim of this section is to introduce the formalization of the judgemental forms used in the type-theoretical framework. It will become clear why the notion of function and the notations explained above for this concept (those developed by Frege, Russell, and later by Church) are relevant to the notion of type considered here. I will first consider the formal expressions of the theory and try to explain how the formalization reflects the theoretical questions introduced above: conceptual priority, identity, and definition.

The essential questions for defining the formal objects of the theory are: What is a type? And what does it mean for an element to be of a certain type? According to the *definition* of type, and the explanation of mathematical objects already considered in the light of Intuitionistic logic, it is not possible to know what a type is and being at the same time in doubt about the properties of the objects belonging to that type: this obviously makes the theory more trustworthy from an epistemic point of view. There are then two complementary ways to follow, in order to get the definition of "type":

(a) To know a type is to know what an object of that type is.
(b) To know a type means knowing what it means for two objects of that type to be the same.

Accordingly, two main forms of judgements are obtained: the first will state that there is an element belonging to a certain type (a), the second that two objects are identical within the same type (b). These expressions will together furnish the *definition* for the type involved. According to the conceptual priority explained in Section 1.3.3, the type denoted by such a definition must be *meaningfully* stated before, i.e. its *meaningfulness* is a *presupposition* for those judgements to be done: in the conceptual order, the type comes before its definition.[59] Let us start by the formalization of type-expressions. The basic judgement

α **is a type**

[59] Cf. Martin-Löf (1993, pp. 31–32). The problem of priority between *type-declarations* and *type-definitions* is just introduced here: the classic solution provided by Martin-Löf in order to avoid impredicativity is presented in Section 1.5.3. A new critical treatment of the problem and a theoretical solution is presented throughout Chapter 4. For the formal presentation of presuppositions in CTT see also Primiero [forthcoming a].

is formalized as

$$\alpha : type. \tag{1.1}$$

This is an absolute judgement, corresponding to a presupposition for any other following judgement using the type α.[60] We can also state that two types are identical:

α and β are identical types

formally

$$\alpha = \beta : types. \tag{1.2}$$

In this case, we are presupposing respectively the declarations $< \alpha : type >$ and $< \beta : type >$. Any type declaration needs at this point to be defined, i.e. one needs to state what it means for an object a to be of the type α, i.e. to know the conditions under which one can assert

a is an object of the type α,

formalized as

$$a : \alpha. \tag{1.3}$$

This formula represents the application criterion, from which follows the identity criterion, the second condition in order to explain what the type α is: the latter consists of knowing when two objects of that type are the same. Obviously, when one knows what it means "being of the type α", one already knows what it means for two objects a and b that

a and b are identical objects of the type α,

formalized as

$$a = b : \alpha. \tag{1.4}$$

In order to make this last assertion, one needs of course to know before and respectively that $a : \alpha$, that $b : \alpha$, and finally (going backwards) that $\alpha : type$. Knowledge that two objects a and b are the same inside the type α also means to know if both are equal to a third object c, inside the same type α. In this way we obtain the three conditions for identity of types:

– Reflexivity
– Symmetry
– Transitivity

The rules for identical types state that:

- Given two identical types, an arbitrarily given object of one of the types will also be an object of the other type:

$$\frac{a : \alpha \quad \alpha = \beta : type}{a : \beta} \tag{1.5}$$

[60] In the following, when the judgement $\alpha : type$ works as a presupposition for another judgement (e.g. $a : \alpha$), the notation $< \alpha : type >$ will be used.

- Given two identical types, two identical objects of one of the types are identical objects of the other type:

$$\frac{a = b : \alpha \quad \alpha = \beta : type}{a = b : \beta.} \tag{1.6}$$

This is to be satisfied for all the objects of both the types in question. The properties holding for objects belonging to types will hold for types themselves. These rules allow clearly to state definitional equality between types.

1.5.2 *Formalizing Equality Rules*

The three identity conditions holding both for types and elements of types are formally presented in the remainder of this section.[61] Equality rules for elements of a monomorphic type are:

Reflexivity

$$\frac{a : \alpha}{a = a : \alpha} \tag{1.7}$$

Symmetry

$$\frac{a = b : \alpha}{b = a : \alpha} \tag{1.8}$$

Transitivity

$$\frac{a = b : \alpha \quad b = c : \alpha}{a = c : \alpha.} \tag{1.9}$$

Equality rules for types:

---

[61] In relation to the differences between the monomorphic and the polymorphic versions of type theory, it has been mentioned at the end of Section 1.3.1 that in the switch from the latter to the former the possibility of expressing a rule of extensional equality for sets with a strong elimination rule (a “too strong” one in fact) is lost: in the original semantics of Martin-Löf (1982, 1984) judgemental equility turns out to be more general than convertibility; in Nordström, Petersson and Smith (1990, pp. 60–61) rules of formation, introduction, and elimination for equivalence are provided, extensional with respect to substitution. The strong elimination rule used there does not express this extensionality based on structural induction; therefore, it is supported by a second Eq-elimination rule. By using both, one is able to derive an induction rule corresponding to the usual Id-elimination for the semantics of the polymorphic version of CTT. In Section 1.8.1, together with examples of rules for different sets definable in terms of types, the equality sets for the monomorphic version will also be considered.

Reflexivity

$$\frac{\alpha : type}{\alpha = \alpha : type} \tag{1.10}$$

Symmetry

$$\frac{\alpha = \beta : type}{\beta = \alpha : type} \tag{1.11}$$

Transitivity

$$\frac{\alpha = \beta : type \quad \beta = \gamma : type}{\alpha = \gamma : type} \tag{1.12}$$

1.5.3 *Categories*

Once the formalization for judgements is introduced, together with their equality rules, the structure of the theory is completed by the definition of the ground types. This leads to present the constructive notion of proposition and to explain its equivalence with the notion of set (the already mentioned Curry–Howard isomorphism). According to the Brouwer–Heyting–Kolmogorov (BHK) interpretation, instances of propositions, sets, and problems are actually instances of the same concept, i.e. rules valid in one case are valid also in the others. In CTT propositions, sets and problems represent the ground types of the theory: expressions involving propositions or sets as predicates are in fact particular instances of a unique form of predication, and the same is true for the predication of an element of a set and a proof of a proposition. This means to recognize two main expression forms within the theory:

$$\ldots : type, \tag{1.13}$$

$$\ldots : \alpha, \tag{1.14}$$

the second expression assuming implicitly that α has been introduced as a certain type (i.e. appearing on the left side of the colon in the first kind of expression). The first of these forms introduces types, in turn amounting either to a proposition, a set, or a problem; the second introduces an object of a certain type, respectively a proof, an element, or a solution. Both the expressions are generally formulated within contexts of assumptions, of the form

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n).$$

The formulation of a judgement under a context of assumptions leads to the expression of a hypothetical judgement; an empty context makes the judgement a categorical one.[62] We can formalize the previous forms of expression as follows:

$$type(\Gamma) \tag{1.15}$$

$$\alpha(\Gamma) \tag{1.16}$$

These expressions introduce what Martin-Löf calls the *categories* of the theory: namely, the first introduces the *category of types*, the second, the *category of objects of types*. The word *category* represents a general noun for the kind of predication structures used within the theory, in the light of the Aristotelian notion of category: *categories* are the meaning-giving structures of the theory, in terms of types and objects belonging to them. A form of judgement is nothing but a category of reasoning (logical sense) or of knowledge (philosophical sense), and to know means to make correct judgements in terms of such categories. Thus, for example, judgements falling under the first category are those like:

$$set : type$$
$$elem(\mathbb{N}) : type,$$

i.e. judgements stating that sets are types and that the elements of the set of natural numbers form a type. Such judgements say that something is of a certain category, they declare something to be a type. This is the proper sense in which an expression is called a *type-declaration*. Within the second kind of category fall those judgements declaring something to be of a *certain* type, e.g.

$$\mathbb{N} : set$$
$$0 : elem(\mathbb{N}),$$

i.e. judgements saying that natural numbers form a set, and that zero belongs to the type of the elements of natural numbers. These judgements represent a derived sense by which one refers to a type-declaration, namely the declaration of the type some element belongs to.

The idea of *category* is clearly given by abstraction from the type itself, in order to grasp those general forms of expression which are meaningful for the theory. It is exactly by introducing this notion of *category* as a form of expression that the problem of impredicativity for types is avoided. The relation between types and their definition has already been presented: a type is introduced by a type-declaration, such a judgement being in this way a presupposition for those judgements predicating objects within that type. The definition of the mentioned type is given exactly in terms of the latter judgements, representing the application and the identity criterion. In this formulation the notion of type itself could still be accounted as contradictory, in that its definition presupposes the concept, whereas only the

[62] The role of context will be widely clarified in Section 1.6, where hypothetical judgements are formally introduced.

introduction of the criteria of application and identity furnishes the meaning of such a concept. The introduction of the notion of *category* is required in order to avoid such a vicious circle: the meaning of expressions is distinguished from the meaning of the single types, in that the expressions refer to the meaning provided by the related category (as forms of expression), whereas types refer to the presence of a meaningful concept, introduced by the relevant presupposition. It is clear at this point that the concept of type escapes impredicativity by referring to the use of these meaning-giving structures, while on the other hand it is still necessary to clarify the nature of type-declaration and the definition of types in connection to the notion of meaning, namely, by explaining the nature of presuppositions.[63] To understand what a type is (and in turn what is one of its specifications, like *set* or *prop*), it is necessary only to grasp what an arbitrary object of that type is, i.e. one must understand which objects fall within that concept. As should be clear by now, to define the type *set* or *prop*, one needs to know respectively how canonical elements of a set can be formed, or how to show an effective construction for a proposition.[64] Once the categories are introduced, the notion of type is a primitive concept, introduced by the first general form of judgement (α is a type — formula 1.1). Such a judgement resumes thus all the different possible interpretations: it can in fact be read in different ways, after one states what the ground types are. In particular, we can give the following expressions as valid examples of the first form of category:

$$prop : type$$
$$set : type$$
$$prob : type$$

stating respectively the ground types of sets, propositions, and problems. They are all equivalent forms, coming from the definition of Intuitionistic logic, of constructive set theory, and from the reading of Kolmogorov (1932), according to which a problem is identified with the set of its solutions (the already mentioned BHK interpretation). On this basis, CTT was designed as a logic for mathematical reasoning, which through the computational content of constructive proofs can be used as a programming logic.[65] The forms of judgements

$$A : set$$
$$A : prop$$

are in fact different versions of the same form of expression, because a set is defined by explaining how its canonical elements are formed, while a proposition is defined by laying down the set of its proofs.[66]

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[63] This analysis is done in Section 3.2.1, and more extensively in Primiero (forthcoming a).
[64] Martin-Löf (1984, p. 22).
[65] For a development of this theory in terms of a programming language, cf. Nordström, Petersson and Smith (1990).
[66] In the following the formalization of the ground types and their rules will be presented. To this aim, remarks about the notation are needed: the type of

1.5.4 *Type* set

Sets are thus introduced as a ground type ($set : type$), and a certain set A ($A : set$) is known if one knows how to form canonical elements for this set ($a \in A$), and when two of its canonical elements are equal ($a = b \in A$), which represents the canonical definition of a type. Moreover, two sets are equal if a canonical element of one set is always a canonical element of the other set, and if two elements which are equal inside one of these sets are equal inside the other as well (equality for types). The notion of set has different possible interpretations:

– Class theory (where "class" is some subset of the universe of discourse)
– Cantor's set theory (where "set" is an intuitive description of the universe of discourse)[67]
– Formalized set theory (where "set" is an iterative or cumulative notion)[68]

In the type-theoretical framework proposed by Martin-Löf, the notion of set is defined according to a combination of logic and set theory, in which "set" is distinct both from class and iterative hierarchy, using instead the defining criteria. The rules stating that *set* is a type and that it is the type of a certain A, are the following:

$$set : type \quad set = set : type \tag{1.18}$$

$$\frac{A : set}{A : type} \quad \frac{A = B : set}{A = B : type} \tag{1.19}$$

A being a set, the elements of A define a type:

$$\frac{A : set}{El(A) : type} \quad \frac{A = B : set}{El(A) = El(B) : type} \tag{1.20}$$

That a is an element of the set A is formally expressed both by

$$a : El(A) \tag{1.21}$$

---

sets and that of propositions (*set* and *prop*) will always be represented by capital letters (second form of category); Greek letters will be used only for monomorphic types (first form of category); whereas the symbol $\in$ refers to set-theorethical expressions, in general the use of the colon $a : A$ is preferred, holding both for elements of sets and for proofs of propositions. Finally, the more common symbol $\forall$ instead of the proper Π is used also for sets, and this will be in common with the rule for the type *prop*, via the following definitional equality:

$$(\forall x \in A)B(x) =_{def} (\Pi x \in A)B(x) \tag{1.17}$$

Cf. Martin-Löf (1984, p. 32).

[67] Cantor (1878).

[68] Set theory has in fact also a type-theoretic interpretation and a related constructive version, introduced by Myhill (1975) and further explored, for example, by Aczel (1978, 1982, 1986) and Aczel and Rathjen (2001).

$$a \in A \tag{1.22}$$

To make an example of a set definable in terms of types, let us consider the set of natural numbers: one will need to make a judgement declaring such a collection of elements to be of a certain type, namely of the type *set*. The axioms used are exactly the type-theoretical counterpart of the first two Peano axioms, plus the type-declaration of $\mathbb{N}$ being a set:

$$\mathbb{N} : set; \quad 0 : \mathbb{N}; \quad \frac{a : \mathbb{N}}{s(a) : \mathbb{N}}$$

These are the formal rules for canonical elements of this set. By the identity criterion, we need to know when two elements of such a set are equal, starting from zero and using the successor rule:

$$0 = 0 : \mathbb{N}; \quad \frac{a = b : \mathbb{N}}{s(a) = s(b) : \mathbb{N}}$$

This represents a method which when executed yields a canonical element of the set as result, and correspondingly two elements are equal if the respective methods yield equal canonical elements. In Section 1.8 the computational rules for types will be formally and explicitly introduced, and some other examples will be provided for sets definable in terms of the monomorphic type theory. Once the type of sets is introduced, more attention can be given to the type of propositions.

1.5.5 *Type* prop

The ground type of propositions, *prop*, is explained by laying down the axiom

$$prop : type \tag{1.23}$$

and furnishing a justification for the following judgement:

$$A \text{ is a proposition, } A : prop$$

This judgement is explained by answering two questions: "what is a proposition?", which represents the application criterion, and "what does it mean for two propositions to be the same?", which corresponds to the identity criterion. In CTT the first question requires an epistemological analysis, which relies on the more general philosophical question: "what is it to know a proposition?" The classical solution and interpretation of the notion of proposition, given by Aristotle, is that "a proposition (ἀπόφανσις) is what can be true or false", and to know which is the case one has to know the state of affairs (ontology) to which the proposition refers, so that it is not the case that "the snow is white" is true because we affirm it, rather the other way round, i.e. the proposition is true if the snow actually happens to be white. In the history of modern logic, this has been notoriously translated by Boole as "a proposition is what has a truth-value, 1 or 0",[69] and this has been formally developed by the truth tables for connectives. Frege defined the concept of proposition in his *Grundgesetze der Arithmetik* on the basis of the "truth-conditions" for logical operators, developing later such conditions by considering the role of *Bedeutung*. The now common "truth-tables", introduced by Wittgenstein in his *Tractatus Logico-Philosophicus* and later also by Post and Lukasiewicz, can be summarized as follows:

Explanation of propositions in terms of truth-conditions

$$\frac{A\ \ true \quad B\ \ true}{A \wedge B\ \ true}$$

$$\frac{A\ \ true}{A \vee B\ \ true} \qquad \frac{B\ \ true}{A \vee B\ \ true}$$

$$\frac{\begin{array}{c}(A\ \ true)\\ B\ \ true\end{array}}{A \supset B}$$

$$\bot: false$$

$$\frac{\begin{array}{c}(x \in D)\\ P(x)\ \ true\end{array}}{(\forall x \in D)P(x)\ \ true} \qquad \frac{d \in D \quad P(d)\ \ true}{(\exists x \in D)P(x)\ \ true}$$

As known, by means of these tables it is possible to formalize the laws of classical logic with quantification over a finite domain; difficulties arise in the Boolean interpretation when one needs quantified propositions over infinite domains, such as in the following two laws:

$$\frac{A(x) : prop}{(\forall x)A(x) : prop} \qquad \frac{A(x) : prop}{(\exists x)A(x) : prop} \tag{1.24}$$

Moreover, in what we can refer to as the "Fregean–Wittgensteinean interpretation" of the notion of proposition another well-known problem arises: if only truth-conditions are needed in order to define a proposition, obviously all truths (such as all falsities) are identical propositions, because the principle of identity is based upon the truth-conditions, so that identity corresponds to material equivalence. On the basis of these remarks a general philosophical critique of classical logic, essentially regarding the role of the law of excluded middle, was developed by the Intuitionists, thus producing a new interpretation of the notion of proposition.

[69] This means that if we previously define a set like **Bool** by the domain $\{1, 0\}$, we can then define a proposition just as an element of that set.

1.5.6 *The Notion of Proposition for CTT*

The Intuitionistic approach, started by Brouwer and his pupil Heyting, introduced a new notion of proposition: according to this approach a proposition is now essentially intended as the result of a verification, or proof. This is the notion of proposition used within the framework of CTT. In this section I will present this topic, developing the Intuitionistic notion of *judgement* and *proof*, whose importance has already been illustrated at the very beginning of this chapter. In what follows, the formal and logical aspects will be explained more systematically.

According to the conceptual order described in Section 1.3.3, the judgement $\alpha : type$ (formula 1.1) is conceptually prior to the judgements providing its definition, namely $a : \alpha$ and $a = b : \alpha$ (formulae 1.3 and 1.4). Each entity inside the constructive framework is labelled by a type (according to the slogan "no entity without type"): in particular, each proposition is defined through the set of its proofs, which again are the main objects of the theory. In this sense, each form of judgement represents a type, and the semantics will be explained according to the theory of expressions already introduced. Once the conditions to recognize types are stated and propositions are considered on the basis of their proofs, one needs to state the criteria for defining proofs, so as to use proofs as a way to validate propositions, rejecting definition via truth-conditions. In order to understand what is really meant by the term *proposition* within the Intuitionistic type-theoretical framework, and what is intended by the process of *proving*, one needs to explain the relation between the notions of *proposition* and *judgement*.[70] Martin-Löf explains how the notion of formal proof has been developed by forgetting the judgement-proposition distinction, then operating formalistically upon the concept of immediate inference, obviously replacing the notion of proposition by that of formula.[71] The terms "proposition" and "judgement" clearly do not share the same meaning, and this distinction is pervasive both for the continental (German in particular, with the distinction between *Satz* and *Urteil*) and for the Anglo-Saxon philosophies. Aristotle in his *De Interpretatione* first introduced the notions of κατάφασις (affirmation) and απόφασις (negation) as the forms of expression investigated by logic. His distinction has survived in the history of logic for many centuries, at least until the debate between Kant and Bolzano. Frege assumed the single form of judgement introduced by Bolzano, "*A is true*", formalized as $\vdash A$: in this way he introduced the distinction between an affirmation and its content (*Urteilsinhalt*); Wittgenstein used the quite similar distinction between *Satz* and *Satzradikal*. Later Russell referred to the term *proposition*, whereas premises and conclusions

[70] Cf. Martin-Löf (1987).
[71] Cf. e.g. Martin-Löf (1996).

of a logical inference were named *sentences.*[72] After this conceptual development, a "proposition" is no longer the same as the greek πρότασις, now replaced by the term "judgement". The general distinction between "judgement" and "proposition" (as holding in the constructive framework) can be made clear in this way:

– *Proposition* is the content that logical operators work on, and what is held to be true.
– *Judgement* is performed by holding a proposition to be true.

When an inference is made, one is using propositions retained to be true, in order for truth to be preserved to the conclusion by means of the rules. This means that when one states a form of inference like the disjunction introduction, one needs to know not only that a proposition is true, but also that what is used as premises and conclusion are propositions (to which truth can be ascribed), namely, the premises "*A is a proposition*" and "*B is a proposition*", to conclude "*A or B is true*"[73]:

$$\frac{\begin{array}{c} A : prop \quad B : prop \\ A \quad true \end{array}}{A \vee B \quad true}$$

Clearly, expressions contained in an inference are judgements: an inference says that a proposition is true provided that some other propositions are true. In terms of the distinction between mediate and immediate proofs, which relies on the Aristotelian logic, one can make use of the concept of *immediately evident judgement*, which amounts to an *axiom*, evident by itself and not by virtue of any previously proved judgement. On the other hand, if a judgement is *not immediately evident*, i.e it follows from the mediation of a proof, it is a *theorem.* Obviously, the only proof for an axiom is the act of grasping its meaning, an intuitive one.[74] By means of the introduced difference between proposition and judgement, it is possible to explain the notion of proof which is developed in the type-theoretical

[72] Martin-Löf (1996, pp. 15–16).
[73] In this way Martin-Löf explains how to solve another difficulty, namely, that in the formalization one uses formulae, which are syntactic notions, whereas propositions represent semantic notions: in the process of forming an inference rule, even if semantic notions are involved, there must be no semantic conditions. This is solved by using the other two premises regarding the nature of the variables involved by the inference, namely that they are propositions. Cf. Martin-Löf (1996, p. 17).
[74] An analysis which clearly recalls the one made by Aristotle in *Posterior Analytics* and in *Metaphysics*, especially in the parts concerning the origin of scientific knowledge, stating the necessity for some *immediate knowledge*, in order to avoid the *regressus ad infinitum*: only postulating some principles which are grasped by intuition and which cannot be explained in terms of something else, one is able to construct a chain representing a justification for its conclusion. See *Posterior Analytics*, B, 19.

approach, not to be intended as a meta-mathematics (à la Hilbert), but as a theory of reasoning.[75] To explain the theory of proof one needs to introduce some basic terms and to explain their different meanings:

- **Demonstration**
- **Proof object**
- **Proof act**

The notion of judgement can be understood either as an act of knowing or as an object of knowledge. This difference is of the greatest importance, because judgements being the building elements of inferences, it is then clear that a proof developed through inferences should be considered as an act itself: to make an inference means to construct a proof, and the proof is at the same time a syntactic object and an act which makes a judgement evident, and for this reason known and understood. If *to know* means having knowledge, *to demonstrate* will mean to have or to construct a proof.[76] In this way the distinction between proposition and judgement amounts to an analysis of the respective kinds of proof in the following terms[77]:

proof of a proposition = verification

proof of a judgement = demonstration

The nature of proofs is the most relevant epistemological issue questioned by the Intuitionistic approach: it is essential to underline the possible definitions of "proof" as "act of proving" or "object" which counts as a proof,[78] introducing the difference between a *proof act* and a *proof object*. Proofs of the usual formulae in the Intuitionistic framework, as introduced by Heyting,[79] correspond to the proper ways of stating the meaning of propositions in terms of the conditions which are to be fulfilled by an object which serves as its proof. This can be explained by a translation of the preceding table for the explanation of propositions in terms of truth-conditions, as shown in the schema in the following page.

Interpreting expressions on an epistemic basis, each formula is explained in terms of its putative proof and one obtains a complete description of the notion of *knowledge* in terms of *proof*, where proof is intended as what establishes the truth of a proposition. In this sense, the notion of truth *simpliciter*, which was in use in the truth-conditional explanation of proposition, is brought back to the *Verification Principle of Truth* (**Principle 1.1**) introduced in Section 1.2, according to which the truth of a proposition is given by the existence of its proof, therefore changing truth-conditions

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[75] Cf. Martin-Löf (1996, p. 29).
[76] The identity between "to judge" and "to know" is essentially due to the Kantian analysis, which changed the Aristotelian definition of judgement as affirmation and denial. Cf. Martin-Löf (1996, p. 20).
[77] Ibid. (1993, p. 263).
[78] This distinction is fully explained and developed in Sundholm (1993).
[79] Heyting (1930, 1956).

Proposition	Explanation in terms of proof
$A \wedge B \quad true$	$\dfrac{a : A \quad b : B}{(a, b) : A \wedge B}$
	$\dfrac{a = c : A \quad b = d : B}{(a, b) = (c, d) : A \wedge B}$
$A \vee B \quad true$	$\dfrac{a : A}{l(a) : A \vee B} \quad \dfrac{b : B}{r(b) : A \vee B}$
	$\dfrac{a = c : A}{l(a) = (c) : A \vee B} \quad \dfrac{b = d : B}{r(b) = (d) : A \vee B}$
$A \rightarrow B \quad true$	$\dfrac{x : A \vdash b : B}{\lambda((x)b) : A \rightarrow B}$
	$\dfrac{x : A \vdash b = d : B}{\lambda((x)b) = \lambda((x)d) : A \rightarrow B}$
$(\forall x : A)B(x) \quad true$	$\dfrac{x : A \vdash b : B(x)}{\lambda((x)b) : (\forall x : A)B(x)}$
	$\dfrac{x : A \vdash b = d : B(x)}{\lambda((x)b) = \lambda((x)d) : (\forall x : A)B(x)}$
$(\exists x : A)B(x) \quad true$	$\dfrac{a : A \quad b : B(a)}{(a, b) : (\exists x : A)B(x)}$
	$\dfrac{a = c : A \quad b = d : B(a)}{(a, b) = (c, d) : (\exists x : A)B(x)}$
$\perp$	nothing

into proof-conditions. On the other hand, a proof is determined by the act of construction, in particular by Brouwer, and *proof constructions* are understood as mathematical objects.[80] This makes explicit the distinction between a *proof act* and a *proof object*: to demonstrate something, say α, one needs to carry out the construction c which constitutes a proof (object) for α, and which will explain its meaning; this in turn will allow us to state

[80] Heyting (1931, 1934).

that "α is true"[81]:

$$\frac{c : \alpha}{\alpha \quad true} \tag{1.25}$$

It should be clear now that the notion of *demonstration* is actually at a higher conceptual level than the one of proof (object/act); a third element makes the schema for the general notion of demonstration complete, together with the notions of proof act and proof object:

1. **Proof act**
2. **Proof object**
3. **Proof trace**

A trace consists in all the instructions that can be possibly written down to construct again the final proof object, and instantiated by the proofs given, for example, in journals or textbooks.[82] According to this threefold clarification of the notion of demonstration, one can say that a proof act is the way one goes from an enunciation to a proposition, the first referring to the content of a judgement before, and the second after it has been proved. In this way, the notion of *proof* is essential to introduce the new notion of judgement and to make clear the distinction with that of proposition. In other words, when an inference is performed, i.e. an instance of a proof act which gives rise to a proof object, judgements (premises) are used, and via an instance of the inference rules a judgement (conclusion) is made evident, i.e. one executes an act which makes the conclusion known, understood (while the proof itself constitutes the proof object). It should now be simple to distinguish between different judgements and to understand the basic role played by this notion, which of course is more fundamental than that of proposition: via a proof act one obtains a construction (proof object) which represents what one should know in order to be able to state a judgement, given that the meaning of a proposition is determined by what counts as its verification. Obviously, a judgement like

proof(A) exists

asserting the existence of a certain construction c for the proposition A is something different from asserting the demonstrability of a judgement like "A is true", because a proof (intended as a demonstration) and a proof object are not the same.[83] It is now quite clear why in the Intuitionistic

[81] As already mentioned, this passage represents an abstraction procedure in that it removes the content of the proof object: this topic will be reconsidered in terms of the notion of information in Section 1.9.

[82] The notion of proof trace, introduced by Martin-Löf, is clearly explained by vivid examples in Sundholm (1993).

[83] For the role played by these distinctions inside the constructive approach, cf. Sundholm (1998, 1998a). The role of proofs and the assumptions of truth will be a main point especially in the formal framework for the formalized notion of knowledge presented in Chapter 3.

interpretation the law of excluded middle ($A \vee \neg A$) is not valid: there is actually no general method to give a proof for this formula, while in Intuitionistic logic a proposition is true only when one finds a proof of it. This explanation allows the passage from the preceding form of judgement

A **is of the ground type** *prop*, $A : prop$,

to the new one

A **is true**, A *true*,

which presupposes the possession of a construction for such a proposition A. The constructive approach identifies the possibility of affirming "A true" with its verifiability or provability (its solution, if A is intended as a problem, recalling Kolmogorov's interpretation). This coincides obviously with having a proof for A.[84] This kind of analysis prefers to give up the two-valued semantics, rather than accepting that some definitional chain between a *definiens* and a *definiendum* can have some computational lacks (like in the case of quantification over infinite domains): in this sense, the new interpretation of the notion of proposition, and the related answer to the question "what is a proposition", consist in a rejection of the Boolean point of view, and it represents a development of the interpretation of propositions via truth-conditions.

Coming back to *prop* as a ground type, in order to state the axiom $prop : type$ one needs to verify the following rule:

$$\frac{A : prop}{A : type} \tag{1.26}$$

which says that if A is a proposition, then it is also a type; given that the assertion $A : type$ should be made on the basis of the knowledge of what is an object of such a type and what it means for two objects of that type to be the same, one can now fulfil such requests, considering proofs of propositions as objects of the propositional type, and consequently one already knows how such proofs are formed, and in turn it is possible to recognize identical proofs. This makes it possible even to state the criterion of identity between propositions, to show that the passage from *prop* to *type* preserves identity:

$$\frac{A = B : prop}{A = B : type} \tag{1.27}$$

which can now be read as a simple instance of the equality rule holding for types, where the identity is referred back to the proofs of the propositions (and this is something decidable, because one must be able to show such proofs).

[84] It is desirable to stress here that in the passage from $A : prop$ to $A : true$ one finds the distinction between the logic of analytic and synthetic judgements. The introduction of the problem of analyticity will obviously bring us to a reconsideration of this point.

1.5.7 *Propositions as Sets*

The ground types *set* and *prop* have been introduced and the related formal rules are going to be explained. These types are to be considered as equivalent, relying on the fact that CTT is based on the Curry–Howard correspondence between propositions and sets,[85] according to which a proposition is interpreted as a set whose elements are the proofs of the proposition. This correspondence can be interpreted as a way to reconsider set-theory in respect to logic[86]: the logical laws are reduced back to a set-theoretical interpretation, representing a complete isomorphism, since proposition and set are exactly the same notion. The equivalence can be stated as follows:

$$A : prop \textbf{ is equivalent to } A : set$$

$$a : A \textbf{ is equivalent to } a \in A$$

On the basis of the constructive explanation of propositions in terms of proof objects, it is possible to give a constructive explanation of all the logical connectives in terms of set theory as follows:

- $A \wedge B$ corresponds to the Cartesian product $A \times B$ of sets A and B, and to prove it one must obtain an ordered pair $< a, b >$ such that a is a proof of A and b is a proof of B.
- $A \vee B$ corresponds to the disjoint union $A + B$ of sets A and B, and to prove it one must obtain a proof $a : A$ or a proof $b : B$.
- $A \rightarrow B$ represents the set of functions from A to B, $A(B)$, and to prove it one must prove B, assuming A.
- $\neg A$ represents the empty set $\emptyset$; alternatively it can be seen as the set of functions from A to the impossible or to the absurdity: $A \rightarrow \perp$, which means that there are no proofs for A.

Moreover, one has to introduce the quantifiers, by using operations defined over family of sets:

- $(\exists x \in A)B(x)$, explained as the construction of an element a of the set A and a proof of $B[x/a]$, represents the disjoint union of a family of sets:
$$\sum_{x \in A} B_x = (\sum x \in A)B(x).$$
- $(\forall x \in A)B(x)$, explained as a function that to each element a in the set A gives a proof of $B[x/a]$, represents a Cartesian product of a family of sets:
$$\prod_{x \in A} B_x = (\prod x \in A)B(x)$$

One can at this point order the proposed explanation of propositions in terms of sets by means of the following schema:

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[85] Curry (1958), Howard (1980).

[86] In this sense the logicistic project of reducing all mathematical notions to logic fails exactly here.

Proposition	Set
$A \wedge B$	$A \times B$, Cartesian product
$A \vee B$	$A + B$, disjoint union
$A \rightarrow B$	$A(B)$, functions from A to B
$(\forall x \in A)B(x)$	$\Pi(A, B)$, cartesian product of a family of types $B(x)$, indexed on type $A, (x \in A)$
$(\exists x \in A)B(x)$	$\Sigma(A, B)$, disjoint union of a family of types $B(x)$, indexed on type $A, (x \in A)$
$\bot$	$\emptyset$, empty set

Finally the correspondence between the type *set* and the type *prop* can be formalized in terms of a stipulative definition:

$$\frac{set : type \quad prop : type}{prop = set : type} \tag{1.28}$$

Later, it will be shown that given the ground types *set* and *prop*, a formation rule will be the one for forming the *function type*, i.e. if α is a type and β is a type depending on a variable x ranging over the type α, one will be able to form the type of functions whose arguments are of the type α and whose value for a certain argument x is of the type β.[87]

1.6 Dependent Objects: Hypothetical Judgements

The basic structure of the theory analysed until now is represented by *categorical judgements*, i.e. judgements that do not depend on assumptions. The role of basic assertion conditions for these judgements in terms of presuppositions has been explained: in order to complete the development of the system, one needs to introduce judgements depending on assumptions, on whose basis inferences can be built; this amounts to extending categorical judgements to *hypothetical judgements*. Once hypothetical judgements are introduced, categorical ones can be intended as special cases of the former, where the number of hypotheses is equal to zero (judgements with empty contexts). On the basis of the equivalence between propositions and sets expressed by the Curry–Howard isomorphism, assumptions are explainable in two ways[88]:

1. An assumption is the declaration that a free variable ranges over some set, like in $x \in N$.
2. An assumption is the declaration that one has a construction for any proposition A $(x : A)$, which is then true.

[87] This explains why CTT is to be understood as a dependent type structure. Cf. Martin-Löf (1984, 1987, 1993).

[88] Nordström, Petersson and Smith (1990, p. 29).

The explanation is hence valid both for the type *set* and for the type *prop*, and in the next section it will be shown how the computational rules work on such types. Hypothetical judgements are also called *dependent types* because they represent objects depending on variables ranging over previously defined types. Thus, judgements formulated on the basis of assumptions constitute a way to include in the theory the old mathematical notion of function — an expression containing free variables, whose computation depends on those variables. In Section 1.3.1 the classical notion of function was briefly introduced and considered in its use up to Frege and Russell, especially regarding the influence it had on the development of the notion of type. Let us now consider very briefly how the mathematical notion was developed, in order to understand its connection to hypothetical judgements. The modern notion of *function* is given as a rule that assigns to every element x of a certain set X a unique element y of the set Y such that

$$y = f(x).$$

A function can also be represented by an ordered pair (x, y) with x belonging to the domain X and y to the co-domain Y, where there is a correspondence many-to-one between members of X and of Y; this notion of function has its roots in the work of Leibniz.[89] The notion had a new definition by Bernoulli in 1718 which was published after his death; it described a function in terms of a quantity composed in any manner by a variable and by any constant (first analytic expression):

> **Bernoulli's Definition of Function**: one calls function of a variable quantity one which is composed in some way by a variable and a constant quantity.

Euler, who introduced the $f(x)$ notation, gave a definition of function (1748), defining first a constant quantity as a value which is permanent, and a variable quantity as an indeterminate value: a function is thus an analytic expression composed by a variable quantity and from numbers or constant quantities:

> **Euler's Definition of Function I**: A function of a variable quantity is an analytic expression, which is formed in some way from the variable numeric quantity and from some number or from a constant numeric quantity.

The notion of function will be modified in a second definition by Euler himself (1755), where he says that functions are quantities depending on other quantities, such that as the second changes, so does the first:

> **Euler's Definition of Function II**: If some quantities so depend on other quantities that if the latter are changed the former

[89] Leibniz's definition of function was presented in his first work (1673) as well as his later publication (1684).

> undergo the change, then the former quantities are called functions of the latter. This denomination is of the broadest nature and comprises every method by means of which one quantity could be determined by others. If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way, or are determined by it, are called functions of it.

This was followed by Fourier's work (1822) where he introduced the Fourierseries, until the famous definition of Dirichlet (1837) representing the first definition in modern terms:

> **Dirichlet's Definition of Function**: If a variable y is related to a variable x, so that whenever a numerical value is assigned to x there is a rule according to which a unique value of y is said to be a function of the independent variable x.

In this definition the idea of dependency between quantities is clear, based on the determination of the second quantity by a first one in terms of some rule. The last part of this brief history is usually attributed to Frege, who gives in his *Funktion und Begriff* a notion quite independent from that of number: he uses the general term expression, which in the case of functions is constituted by a part which remains invariant and one which is replaceable, representing respectively the function and its argument; these two parts of the function object represent respectively the part which needs *to be saturated* and that which *saturates* an incomplete object.[90] This notion is quite near to the modern one, which allows for a function being taken as argument of another function. Dedekind was already going in the same direction, when he considered (1888) the notion of function in the following terms:

> **Dedekind's Definition of Function**: A function is a mapping system S in which each element s is associated with an object $\phi(s)$ generated by the mapping ϕ out of s.

The common point in the different definitions is the determination of function as a rule for computing. In its actual formal definition it is a function from a set of input values to a set of possible output values ($f : X \to Y$) in terms of a relation which is

- *Total*, i.e. for each input value there is at least one output value
- *Functional*, i.e. different input values can be assigned to one output value, but not the other way round

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[90] Frege (1891, p. 5):

Es komt mir darauf an, zu zeigen, dass das Argument nicht mit zur Funktion gehört, sondern mit der Funktion zusammen ein vollständiges Ganzes bildet; denn die Funktion für sich allein ist unvollständiges, ergänzungsbedürftig oder ungesättig zu nennen. Und dadurch unterscheiden sich die Funktionen von den Zahlen von Grund aus.

CTT provides an essential understanding of the notion of function for the theory of expressions. Its formal introduction shall be done in Section 1.7, by introducing a new ground type, $func$. In the first instance, the dependency between expressions is at the basis of hypothetical reasoning as composed by dependent objects: the due conceptual and formal analysis has to be provided.

1.6.1 *Judgements Depending on One Assumption*

The general formalization of a hypothetical judgement depending on one assumption is the following (dependent type):

$$\begin{array}{c}(x:\alpha)\\ \beta:type\end{array} \tag{1.29}$$

which means that for an arbitrary element x of the type α, β is a type provided by the substitution of x with a certain a in α.[91] Furthermore, if a_1 and a_2 are identical objects of the type α, β will be a type for both the substitution of x with any of a_1 and a_2. According to the equational definition by which $prop : type$ can be stated and by interpreting the types α and β as propositions (A and B), it follows that

$$\begin{array}{c}(x:A)\\ B:prop.\end{array}$$

The meaning of this kind of judgement is that for an *arbitrary* construction a of A, B is a proposition when a is substituted for x. By the identity criterion, it is possible to state when two types B and C are identical under the same assumption:

$$\begin{array}{c}(x:A)\\ B=C:prop\end{array}$$

so that $B[x/a : A]$ and $C[x/a : A]$ are identical types for an arbitrary object a of the type A. From the dependent type, one develops the dependent object, i.e. the judgement which states an object belonging to a type under some assumption:

$$\begin{array}{c}(x:\alpha)\\ b:\beta\end{array} \tag{1.30}$$

The identity criterion applies as follows:

$$\begin{array}{c}(x:\alpha)\\ b=c:\beta\end{array} \tag{1.31}$$

[91] The explanation for the type *set* given by Martin-Löf (1984, p. 16) is that β constitutes a family of sets over α.

1.6.2 *Judgements Depending on More Assumptions*

Judgements asserted under an arbitrary number of assumptions are now considered. Presupposing that α is a type, the set of assumptions of a hypothetical judgement represents the *context* under which such judgement is asserted.[92] A context is a collection of expression of the form:

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n), \tag{1.32}$$

each assumption depending on the preceeding assumptions in the context, i.e. each $x_n : \alpha_n$ depends on the assumptions from $x_1 : \alpha_1$ up to $x_{n-1} : \alpha_{n-1}$. The basic form of a hypothetical judgement is then:

$$A_1 \text{ true}, \ldots, A_n \text{ true} \Rightarrow A : prop$$

which says that A is a proposition under the assumption that $A_1, \ldots, A_n$ are all true propositions, and

$$A_1 \text{ true}, \ldots, A_n \text{ true} \Rightarrow A \text{ true}$$

which states the true proposition A under assumption of the truth of other propositions. Obviously, truth for assumptions is stated by presupposing that each of the expressions used is within the type *prop* and that each of them is true under the preceeding ones:

$$\begin{gathered} A_1 : prop \\ A_1 \quad true \Rightarrow A_2 : prop \\ \vdots \\ A_1 \quad true, \ldots, A_{n-1} \quad true \Rightarrow A_n : prop. \end{gathered}$$

The general meaning of such a form of judgement is a construction of the consequent, or thesis, if supplied by proofs of the hypotheses, or antecedents, i.e if proper substitutions are performed for the variables used within the context.[93] In the case of the second form of judgement, the preceding law is not sufficient and a new hypothesis is needed:

$$A_1 \text{ true}, \ldots, A_n \text{ true} \Rightarrow A : prop$$

The two forms of hypothetical judgement can be now generalized in respect to variables:

$$A_{(x_1,\ldots,x_m)} \text{ true}, \ldots, A_{(x_1,\ldots,x_m)} \text{ true} \Rightarrow_{x_1,\ldots,x_m} A_{(x_1,\ldots,x_m)} : prop$$

$$A_{(x_1,\ldots,x_m)} \text{ true}, \ldots, A_{(x_1,\ldots,x_m)} \text{ true} \Rightarrow_{x_1,\ldots,x_m} A_{(x_1,\ldots,x_m)} \text{ true}$$

This form of proof holds for any substitution of variables by any expression of the similarity of those variables.

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[92] Martin-Löf (1984, p. 19). This terminology was ideated by de Bruijn for the language program AUTOMATH, and then adopted by Martin-Löf; cf. de Bruijn (1970, 1980) and Martin-Löf (1993, p. 26).

[93] This terminology is obviously based on that of Gentzen's sequents calculus.

1.7 Introducing Functions

Hypothetical judgements are thus expressions of the form

$$\begin{array}{c}(x : \alpha)\\ \beta : type\end{array} \tag{1.33}$$

which can be intended as a relation between types. This kind of relation is an essential extension for the categorical structure of the system, in order to express implicational and quantified formulae. This means moreover to extend the ground types introduced up to now, namely *prop* and *set*, by means of the function type $func$, which represents all the functions from type α to type β taken as objects. If α is a type, the construction of a new type is possible by considering the latter as a family of sets over some $x : \alpha$, such that $\beta(x : \alpha)$ is also a type, $(x : \alpha)\beta : type$. A function can therefore be illustrated as the judgement regarding a certain object being a type ($\beta : type$) based on the previous assertion that any α is a type and that any x is an object or construction for that type $(x : \alpha)$.

It has already been explained briefly how the notion of function has been developed up to Frege's and Russell's contributions: it is now quite simple to consider Type Theory as a development of the Fregean functional structure, refined by Russell in the Simple Theory of Types. In order to introduce the function type for CTT, it is necessary to furnish its definition according to the criteria of application and identity. What does it mean to be a function type and for two objects of the type $func$ to be identical? The first step, already accomplished, in order to introduce a function type is to define a dependent type structure, on the basis of the ground types *set* and *prop*, and to establish the construction of elements for such types (namely, elements of sets and proofs for proposition). Thus, a function type, which is the new interpretation of the old-fashioned notion of function within the type-theoretical framework, corresponds to an object depending on certain variables, having a family of types as its parameter. The second step is the explanation of the formation rule for the function type, which again says that assuming that α is a type and that β is another type depending on a variable x ranging over α, then $(x : \alpha)\beta$ is also a type, discharging that assumption in the process:

$$\frac{\alpha : type \quad \begin{array}{c}(x : \alpha)\\ \beta : type\end{array}}{(x : \alpha)\beta : type} \tag{1.34}$$

The definition is completed by an identity rule as well, given that the function type preserves identity:

$$\frac{\alpha = \gamma : type \quad \begin{array}{c}(x : \alpha)\\ \beta = \delta : type\end{array}}{(x : \alpha)\beta = (x : \gamma)\delta : type} \tag{1.35}$$

The meaning of an instance of such a formation rule, is given in terms of application, obtaining an object like

$$f : (x : \alpha)\beta; \tag{1.36}$$

i.e. if one applies it to an arbitrary a which is an element of the type α, under substitution for x, such an object which belongs to the type β will be obtained; moreover, applications for a and b are equal in $\beta[a/x : \alpha]$ provided that a and b are equal inside α. This notion of application is nothing but to give to a modern notion of function its input, and in this way producing an instance.[94] The relation between the two notions of function (the old and the modern one), and the conceptual order between them, is established via the application and substitution rules: application is the mentioned way to give a function its input, whereas substitution does the same for functions in the old sense. The modern notion of function is obtained by abstraction from the respective old-fashioned function, and for this reason the second should come conceptually prior to the first.

To summarize, the old notion of function, namely

$$b : \beta(x : \alpha) \tag{1.37}$$

is explained by saying that the object b belongs to the type β provided that a certain a is substituted to x in the type α. On the other hand, the modern notion of function

$$f : (x : \alpha)\beta \tag{1.38}$$

says that there exists a function f which maps an object a into β whenever that a is provided in α. Both the explanations preserve identity, and this can be explicitly stated via the extensionality rule. The difference between the two notions is essentially that a function in the modern sense is supplied with an argument by application, while the old one receives its argument by substitution. Martin-Löf [95] underlines the necessity to have the old notion of function in order to state the new one and explains the meaning of the former in terms of the latter, according to the following schema[96]:

$$f : (x : \alpha)\beta \text{ means that } f(x) : \beta(x : \alpha)$$
$$f = g : (x : \alpha)\beta \text{ means that } f(x) = g(x) : \beta(x : \alpha)$$

[94] Martin-Löf (1987).
[95] Ibid. (1993, pp. 97–98).
[96] Ibid. (p. 119).

These two schemas, respectively for the old and for the modern notion, validate in a trivial way the following rules:

Argument Removal Rule:

$$\frac{\begin{matrix}(x:\alpha)\\ f(x):\beta\end{matrix}}{f:(x:\alpha)\beta} \tag{1.39}$$

which completes the explanation of the old notion of function; it is equivalent to the equality rule known as **Extensionality Rule**.[97] The old notion of function works then via substitution, which means that if one takes an arbitrary object a of the type α and substitutes a for x in $f(x)$ (with f not depending on x), one gets $f(a)$; if a and b are identical objects of the type α, and they are substituted for x in $f(x)$, identical objects of the type $\beta[a/x]$ will be obtained. This corresponds to the explanation of the modern notion of function, and in this sense the old notion is conceptually prior to the modern one.

Functional Abstraction:

$$\frac{\begin{matrix}(x:\alpha)\\ b:\beta\end{matrix}}{(x)b:(x:\alpha)\beta} \tag{1.40}$$

which shows that if b is an object of the type β, depending on the variable x of the type α, one is able to abstract b with respect to x, obtaining an object which is a function in the modern sense. In this way, the notion of function in the modern sense is defined independently from the first, by means of the rule for abstraction derived from the just mentioned schema.

In the next section computational rules for types will be introduced: in the section for the type $func$ it will be shown how application and abstraction rules work, i.e. how one gives rise to an object of such a type.

1.8 Computational Rules

The rules of the system are formulated in order to create and to operate within types. There are four essential rules[98]:

- *Formation rule*, to form types and to say when two types are the same
- *Introduction rule*, to introduce canonical elements of types and to say when two of these elements are equal (constructor)

[97] Cf. Section 1.8.2.
[98] Martin-Löf (1984, p. 24) Valentini (2000, p. 9).

- *Elimination rule*, to use those elements provided by the introduction rule, proving some property for them (selector)
- *Equality rule*, to state a way of computing with the elements of a type, via a definitional equation: their general formulation has already been given in the definition of the notion of type (cf. Section 1.3.1)

In Section 1.8.1 some applications for these rules are presented, in particular their general schema and the formulation holding for some of the operators within the type *set* and *prop*.[99]

1.8.1 *The System of Rules and Some Examples for set and prop*

Formation rule
This rule allows the formation of a new type from previously defined types. To such a rule is associated the related equational rule:

$$\frac{\alpha : type \quad \overset{(x:\alpha)}{\beta : type}}{(x:\alpha)\beta : type} \tag{1.41}$$

$$\frac{\alpha = \gamma : type \quad \overset{(x:\alpha)}{\beta = \delta : type}}{(x:\alpha)\beta = (x:\gamma)\delta : type.} \tag{1.42}$$

The equality criterion here states when two types built in this way are equal. For example, for *set* : *type* (i.e. sets definable in terms of types), the Π set (Cartesian product) of two different sets forms a new set:

$$\frac{A : set \quad \overset{(x:A)}{B : set}}{\Pi(A,B) : set} \qquad \frac{A = C : set \quad B = D : set}{\Pi(A,B) = (C,D) : set}$$

The corresponding formulation given *prop* : *type* is the following:

$$\frac{A : prop \quad \overset{(x:A)}{B : prop}}{(\forall x:A)B : prop} \qquad \frac{A = C : prop \quad \overset{(x:A)}{B = D : prop}}{(\forall x:A)B = (\forall x:C)D : prop}$$

Introduction Rule

Once it is stated that something is a *type*, if one wants to answer the question "what type is that?", one needs to know what it means for an arbitrary element to belong to that type, and this is given by the *introduction rule*, which tell us how a canonical element of a certain type is formed (as always the identity criterion states when two of such canonical elements are equal). The introduction rule represents the semantical explanation of the

[99] Other formulations of the logical rules are contained in Martin-Löf (1975, 1984, 1987, 1998), Valentini (2000).

formation rule, each introduction rule obviously presupposing the relative formation rule, which states the existence of the set for which canonical elements are defined.

For example, the case of a Π-introduction for the type *set* is equivalent to a λ-abstraction, saying that if x is a variable in the set A and $b(x)$ is a term of the set $B(x)$, then a canonical element in A can be abstracted which belongs to any of the elements in B:

$$\frac{A : set \quad B(x) : set \quad \begin{array}{c}(x : A)\\ b(x) : B(x)\end{array}}{\lambda((x)b(x)) : (\Pi x : A)B(x)} \tag{1.43}$$

$$\frac{\begin{array}{c}(x : A)\\ b(x) = d(x) : B(x)\end{array}}{\lambda((x)b(x)) = \lambda((x)d(x)) : (\Pi x : A)B(x)} \tag{1.44}$$

The equivalent case of the introduction rule for a new type *set* produced by the mentioned formation rule by the Cartesian product gives its elements in terms of ordered pairs, with the first element belonging to A and the second element belonging to B, as follows:

$$\frac{a \in A \quad b \in B}{< a, b >\in A \times B} \qquad \frac{a = c \in A \quad b = d \in B}{< a, b >=< c, d >\in A \times B}$$

The corresponding formulation for the type *prop* is the following ∀-formation:

$$\frac{\begin{array}{c}(x : A)\\ b(x) : B \quad true\end{array}}{(\forall x : A)B \quad true}$$

showing the construction $b(x)$ for $B(x)$: *prop*, which makes the proposition itself true.

Elimination Rule

The *elimination rule* corresponds to an application rule, i.e. it says how to define functions on the elements of the type obtained by the introduction rule. For example, the elimination rule for the Π-operator will produce a function on *all* the elements of the new type:

$$\frac{A : set \quad B(x) : set(x : A) \quad b : (\Pi x : A)B(x) \quad a : A}{c(A, B, b, a) : B(a)} \tag{1.45}$$

In this rule, provided that A and $B(x)(x : A)$ are declared as types such as in the previous rules, c represents an application function (sometimes called *ap*) which produces a canonical element for any ordered pair of elements a and b respectively in A and $B(x)(x : A)$. In this way a distinction is settled between a general canonical element and the canonical element $b(a) : B(a)$, given by the previous introduction rule $\lambda(x(b)) : (\Pi x : A)B(x)$. For *set* : *type* and starting from the formed Cartesian product, this results in a function on all the elements of the set $A \times B$, which is enough to explain how such a set works:

$$\frac{c \in A \times B \quad \begin{array}{c}(x \in A; x \in B)\\ d(x,y) \in C(<x,y>)\end{array}}{E(c,d) \in C(c)}$$

The formulation corresponding to the Π-operator for the type *prop* is a ∀-elimination:

$$\frac{a : A \quad (\forall x : A)B \quad true}{B[a/x] \quad true}$$

which shows a method that takes a construction a of A into a proof of $B(a)$.

Equality Rule

Finally, the *equality rule* explains how to compute the function c generated by the *elimination rule*, i.e. in order to determine its meaning:

$$\frac{a : A \quad b : B \quad \begin{array}{c}(x : A; y : B)\\ c(x,y) : C(<x,y>)\end{array}}{E(<a,b>, c) = d(a,b) : C(<a,b>)} \tag{1.46}$$

which gives the function $E(c,d)$ by evaluating the value of c on the arguments (A, B, a, b) of the elimination rule, and then obtains the identity with the method d always provided by the elimination rule. For the type *prop*, it corresponds to a ∀-identity:

$$\frac{\begin{array}{c}A : set \quad a : A\\ (A)B : prop \quad b : (x : A)B(x)\end{array}}{\forall E(A, B, \forall I(A, B, b), a) = b(a) : B(a)}$$

Rules can be shown for all other connectives, such as implication, existential quantifier, disjunction and conjuction (respectively function $A(B)$, Σ, $+$, for the type *set*). Laying down all the rules for connectives and quantifiers, one obtains the result that the semantics for the type *set* can be seen as the constructive explanation of propositions.[100]

1.8.2 *Rules for the type* Func

The first two rules for the type *func* are for application, in order to supply a function in the modern sense with its input, the second one working as an identity criterion:

[100] A complete translation of the rules of inference for connectives from Gentzen (1934) into the Intuitionistic type-theoretical setting is presented in Martin-Löf (1998, pp. 150–160).

Application and Identity

$$\frac{f : (x : \alpha)\beta \quad a : \beta}{f(a) : \beta[a/x]} \tag{1.47}$$

$$\frac{f : (x : \alpha)\beta \quad a = b : \alpha}{f(a) = f(b) : \beta[a/x]} \tag{1.48}$$

A rule can be formulated to show that two functions are equal when they are furnished with the same value in terms of application, and they are identical functions of type α to β; this means that they are equal for the same input, and that in general the type $func$ and the rules associated to it are identity preserving:

Identity Preserving

$$\frac{f = g : (x : \alpha)\beta \quad a : \alpha}{f(a) = g(a) : \beta[a/x]} \tag{1.49}$$

As previously mentioned, it is possible to show that the type $func$ is identity preserving also by stating the extensionality rule:

Extensionality

$$\frac{f : (x : \alpha)\beta \quad g : (x : \alpha)\beta \quad \begin{matrix}(x : \alpha)\\ f(x) = g(x) : \beta\end{matrix}}{f = g : (x : \alpha)\beta} \tag{1.50}$$

according to which if a function f in the type β applied to x is equal to a function g in the same type applied to the same x, under the assumption that x is of the type α, one concludes that f and g are identical functions of the type $(x : \alpha)\beta$. Martin-Löf[101] refers to it as the typed version of the ζ-rule in combinatorial logic. CTT represents in this way a sort of dependently typed lambda calculus or combinatorial logic, where the identity relations correspond to the convertibility relations. The role of typing for the identity relation is necessary, according to Martin-Löf, in order to avoid any meaningless expression.

In order to form an object of the type $func$, an abstraction rule and the associated identity rule are formulated: the abstraction rule works simply by abstracting a variable from an expression, and it obtains a function in the modern sense from one in the old sense:

[101] Martin-Löf (1993, p. 118).

Abstraction

$$\frac{\begin{array}{c}(x:\alpha)\\ b:\beta\end{array}}{(x)\beta:(x:\alpha)\beta} \tag{1.51}$$

which means that if b is an object of the type β, depending on a variable x ranging over the type α, then one may abstract b with respect to x, obtaining in this way an object of the function type. This rule is also said of *functional abstraction* and it is equal to Church λ-abstraction. The second rule is nothing but an explanation of the abstraction rule: to know that the preceding rule is correct, i.e. that $(x)\beta$ is a function of the type $(x:\alpha)\beta$, it must be shown that when this function is applied to any object of the type α, one gets an object of the type $\beta[a/x]$, a typed version of the β-conversion from combinatorial logic:

***β*-conversion**

$$\frac{a:\alpha \qquad \begin{array}{c}(x:\alpha)\\ b:\beta\end{array}}{((x)b)(a)=b[a/x]:\beta[a/x]} \tag{1.52}$$

The identity rule associated with the abstraction rule is expressed by the typed version of the ξ-rule from combinatorial logic, given in order to know that, provided that $(x:\alpha)$, and applied an abstraction $(x)b:\beta$ to identical functions f and g of the type β, their application will be identical:

***ξ*-rule**

$$\frac{\begin{array}{c}(x:\alpha)\\ f=g:\beta\end{array}}{(x)f=(x)g:(x:\alpha)\beta} \tag{1.53}$$

where the conclusion $(x)f=(x)g:(x:\alpha)\beta$ is obtained by the symmetry and transitivity of the identity relation. Both the abstraction rule (λ-rule) and the identity rule (ξ-rule) are derivable from the **Argument Removal Rule**. A last rule can be derived from the others:

***η*-rule**

$$\frac{\begin{array}{c}(x:\alpha)\\ b(x):\beta\end{array}}{((x)b(x))=b:(x:\alpha)\beta} \tag{1.54}$$

where x should not occur free in b: one concludes that the abstraction with respect to x of b applied to x is identical to b, as an object of the function type $(x : \alpha)\beta$, where the first step, namely, the derivation of $b(x)$ assuming that x is of the type α, is obtained by application, while in the conclusion the **Argument Removal Rule** and the **Extensionality Rule** are used.

1.9 Introducing Information

In order to introduce the notion of type as a ground notion on whose basis other logical notions such as proposition and set can be explained, the purely conceptual and theoretical analysis has been brought to the formal and technical aspects of this foundation in the framework of CTT. By relying on the more primitive notion of type, one uses a powerful conceptual frame, which has also some formal distinctions. The monomorphic version of the system introduced above, formulated on the priority of the notion of type over sets and propositions, presents an extremely significant difference in respect to the polymorphic version[102]: formally, the distinction between the two versions has been accounted in the first instance regarding the interpretability of different theories in the framework, involving questions about decidability and equality. Another sense in which the difference between the two frameworks can be translated concerns the formulation of the content of proofs (constructions) within the formalization: when working in a monomorphic version of the theory, it is always possible to build the proper derivation for each judgement one is referring to, because such judgement contains all the information required in order to reach this aim. This is made explicit by the notation when, for example, in the case of the Intuitionistic explanation of the logical constants, one writes down the Introduction-operator ($I-$, to the relative connective, e.g. $\wedge$) depending both on the propositions (A, B) and on the ordered pair of the relative constructions $(< a, b >)$. On the contrary, the relative notation for the polymorphic version reports only the propositions $(I \wedge (A, B))$, regardless of which construction refers to which proposition (provided of course that such constructions have been obtained). Inside the polymorphic version one can handle different types by the same syntactical proofs or derivations, while within the monomorphic version it is always made explicit which constructions work for each expression. This means that in general the polymorphic versions are syntactic simplifications obtained through deleting *type information* (information about the types one is speaking about). By such "information discharging"[103] the system does not lose any essential contentual information: this property is expressed by the constructive principle that the information at disposal within the theory should be always enough to let a mechanical

[102] The terminology for the polymorphic theory was introduced by the computer scientist Robin Milner. Polymorphic versions are presented by Martin-Löf (1975, 1982, 1984); Nordström, Petersson and Smith (1990), where there is also a short introduction to the monomorphic version (pp. 135–152); see also Salvesen (1989).
[103] Salvesen (1989).

type checker to produce effectively the well-formed expressions of our language.[104] Such an information discharging consists in ripping off from the theory any type information so that:

– It can be safely deleted because it is not necessary in order to reconstruct well-formed expressions.
– It can be practically and quickly restored if necessary.

The utility to build up a polymorphic theory consists in the possibility of handling types consistently, with no fear of losing reusability and flexibility.[105] This finds a great utility in programming languages, which are written without redundancy of information.[106] On the other hand, I have mentioned the problem one encounters in relation to the extensionality of equality rules, not to mention the loss of high generality provided by the interpretation of types as general structures of meaning in which it is possible to define both sets and propositions.

It appears extremely interesting that the distinction between monomorphic and polymorphic versions of the theory introduces in a first sense the notion of *information*, in terms of the general and wide understanding of this term, as the elements representing constructions within the derivations formalized for CTT. This notion can be accounted as computational information. Also the notions of discharging and retrieval of information in Type Theory are related to computational information. The theory can be described as a logical calculus which adopts notions and rules keeping total control of the amount of information contained in the different forms of judgement. It offers moreover a way to forget information, i.e. supposing $< A : set >$, the assertion of the judgement A *true* is done via the judgement $a : A$ by "forgetting" the information represented by the construction a. This way of getting rid of information can be accounted as a constructive one, because provided that there is necessarily a proof of the judgement that A is true, an element a such that $a : A$ can be always reconstructed. The operation of discharging part of the information contained in the data building up a derivation in Type Theory is considered as the essential way to build *abstract concepts*, i.e. it is a procedure of abstraction. Nevertheless, idealization does not correspond simply to abstraction from reality, in that it preserves truth[107]: in this way the intuition that the essence of constructivism is essentially linked to the notion of information is explicited. Constructivism does not consist of an a priori self-limitation to full information, rather to the awareness of the operations performed to build certain abstract concepts. This awareness should be interpreted in terms of a method to restore what was destroyed.

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[104] Martin-Löf (1993, p. 288).
[105] Salvesen (1989, p. 62).
[106] Nordström, Petersson and Smith (1990, p. 147).
[107] This is the essence of the "*Camerino Program*", developed by Sambin and Valentini (1998), Valentini (1998).

The research presented in the following chapters starts from the central role attributed to computational information in the formulation of the different versions of the thoery and the definition of type, but it provides the next step in the understanding of the constructive meaning of this notion. The main task consists in providing an epistemic description of the logical framework at the basis of CTT, and a definition of the role of information in it. In the polymorphic version of CTT, information corresponds to the object of abstraction from the flow of data of the theory; in the monomorphic version it is reduced to the content expressed by a logical derivation. An essential thesis of this research is that an epistemic distinction is needed among the notions of information and knowledge, and that it is possible and moreover natural to obtain it in a constructive framework. The basic aim is thus to present this distinction in terms of the formal structure of the monomorphic version of the theory, to describe the formal objects expressing informational contents, and to attribute to them the esential role of supporting procedures of knowledge. In this project it is central the philosophical idea that abstraction is to be understood in terms of information; in particular, two forms of abstraction will be considered, providing the logical elements needed for an epistemic definition of information: the first concerns abstraction in respect to specific contents of knowledge within judgements and the use of assumptions represented by "empty" constructions (in terms of variables); the second form of abstraction can be compared with the mentioned inference from $a : A$ to A *true*, and it suggests that a similar connection is to be found in the switch from $a : A$ to $A : type$, clarifying the use of presuppositions.[108] This epistemic analysis is the basis of the philosophical problem of analyticity for logical derivations: by considering the history and the different solutions to such a problem, by paying attention to some of the turning points in the history of logic, by considering the theoretical elements introduced in the constructive framework, it will appear completely natural to introduce our notion of information as a solution to the aforementioned problem of analyticity. In Chapter 2 the road to "information" will be open by the treatment of such a topic.

[108] See Primiero (2007a).

2
Analyticity and Information

In epistemology, the relation between science and method is of the greatest relevance, as already stated by Aristotle:

> *One should already be trained in how to accept statements, for it is absurd to be seeking science and at the same time the way of acquiring science.*[1]

Such a distinction is carefully developed in the *Posterior Analytics*, where logic is taken into account as a proper methodology of science. In connection to this methodological role, the root of the Aristotelian (classical) conception of logic is exposed to a crucial problem: logic as the methodological structure of scientific research and philosophical enquiry suffers from being considered alternatively as a method of justification and a tool of discovery. This is clearly an essential challenge for logicians interested in the philosophical foundation of the subject, and it represents the root of the so-called problem of analyticity. In the first part of this chapter, an introduction to the problem will be presented, by referring to some authors relevant to the understanding of different approaches. Such analysis does not pretend to be exhaustive: the aim is not to present a complete historical reconstruction of the concept and problem of analyticity,[2] rather to propose a perspective on such a problem. In particular, at the beginning of this chapter the different interpretations due to Kant and Bolzano will be considered as a critical starting point, providing a philosophical perspective on the problem. As a consequence, the core of the problem of analyticity will be reformulated in terms of the bearers of truth, i.e. by understanding to which entities analyticity itself must be ascribed. The philosophical problem will then be considered, in terms of the shift from an analysis in terms of conceptual content to one in terms of linguistic meaning. This represents a way to introduce the solution offered by Hintikka, the first

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[1] Aristotle (Metaph, $\alpha, 3, 995a$ 13–15):
διὸ δεῖ πεπαιδεῦσθαι πῶς ἕκαστα ἀποδεκτέον, ὡς ἄτοπον ἅμα ζετεῦν ἐπιστήμεν καὶ τρόπον ἐπιστήμες.

[2] For such a conceptual and historical study see the fascinating Proust (1989).

to explicitly present—on the basis of hints already present in Kant and Frege—a solution in terms of the concept of *information*. Hintikka's account misses nevertheless some conceptual constraints here considered essential, in particular the distinction between act and content of knowledge. The final aim of this chapter is to present an understanding of the notion of information in the light of an epistemic (constructive) knowledge system.

2.1 At the Origin of the Problem

The history of the problem of analyticity can essentially be developed on the basis of the philosophical distinction between *content* and *form*. This distinction is essential to define the truth-bearers in a theory of language and therefore it establishes in terms of which entities (either conceptual or linguistic) analyticity can actually be considered. The interpretation of the content/form distinction to the linguistic/epistemic range of arguments is moreover of the greatest importance for the debate already considered among classical and intuitionistic logic: it leads directly to the basic definition of the notions of *proposition* and *judgement*. Relying on these two concepts, one shall try to consider how the definition of analyticity has changed in different approaches. And for this project, the Kantian understanding of the procedures of analysis and synthesis is probably a good starting point to explain the modern perspective on the problem of analyticity: in fact, these procedures are essential to the whole critical theory of knowledge, and because one finds in the Kantian epistemology a complete treatment of the insights on this subject, already contained in Aristotle and essentially developed by authors in the Middle Ages and in the Modern Age.

2.1.1 *The Modern Origin of Analyticity: Kant*

The Kantian theory of knowledge, and in turn a possible understanding of his theory of meaning, can be given via a theory of representation: as post-Cartesian, his (missing) semantics (in other words, the way of conveying information) is substituted by a theory of concept and by a theory of judgement (by means of which concepts are understood and expressed). This amounts to an analysis of the concepts (*Zergliederung der Begriffe*) involved in our process of knowledge into their basic constituents.[3] This process of analysis aims at obtaining certain simple indefinable concepts;

[3] Kant (1800, sec. VIII, A, 95):

> *Wenn ich aber einen Begriff deutlich mache, so wächst durch diese blosse Zergliederung meine Erkenntnis ganz und gar nicht dem Inhalte nach. Dieser bleibt derselbe; nur die Form wird verändert, indem ich das, was in dem gegebenen Begriffe schon lag, nur besser unterscheiden oder mit klarerem Bewusstsein erkennen lerne. [...] so wird auch durch die blosse Aufhellung eines gegebenen Begriffes vermittelst der Analysis seiner Merkmale dieser Begriff selbst nicht im mindestens vermhert.*

once these concepts are reached, the process of analysis stops, i.e. when the reduction is accomplished, distinctness is achieved. The analysis of knowledge is in this way the proper method of philosophy by which, according to Kant, knowledge is clarified, not acquired. Starting by taking into account concepts and referring to the process of analysis in which they are involved, Kant turns to extend his idea about knowledge and analysis to the role of *judgements*: a *categorical judgement* states the relation between a subject and a predicate, by means of which two concepts are related. It follows the well-known distinction between *analytic* and *synthetic judgements* in the introduction of the *Critique of Pure Reason*[4]: the relation among subject and predicate in a (affirmative or negative) judgement is either of *containment* of the latter into the former, or it presents an *extension* of the former by means of the latter.[5] Within judgements of the form S *is* P, Kant is thus considering the relation between the two terms, namely how the predicate belongs to the subject: if the former is contained (*enthalten*) into the latter, the judgement will be analytic. In this case its role is that of showing something as already contained in the subject, what Kant refers to as an *Erläuterungsurteil*.[6] The relation of containment between subject and predicate defines analyticity for judgements, and correspondingly the process of analysis (or definition) explains the proper understanding of such a property, i.e. by establishing the relation of containment as its central feature. A first definition of this notion can at this point be formulated:

Definition 2.1 (Kant's Analyticity I) *A judgement is analytic when it is knowable from its own conceptual resources.*

This definition, which strictly determines judgements in terms of knowability of their contents, refers to analyticity as *knowledge/definition ex vi*

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[4] For a general introduction to the analytic/synthetic distinction cf. Rey (2003). We consider here Kant as the starting point for the modern treatment of the analytic/synthetic distinction. For the obvious presence of ancestors to the Kantian interpretation in the history of philosophy, see for example those referred to in Quine (1951, sec. 1). As mentioned above, the problem of analyticity is already present in Aristotle, all along the Aristotelian tradition in the Middle Ages, and at the origin of modern thought, e.g. in Leibniz. Nevertheless, the Kantian approach is in general acknowledged as one of the most relevant, especially if related to the role of Bolzano in this history.

[5] Kant (KrV, A, 6–7):

> *In allen* Urteile*, worinnen das Verhältnis eines Subjekts zum Prädikat gedacht wird, (wenn ich nur die bejahenden erwäge, denn auf die verneinenden ist die Anwendung leicht) ist dieses Verhältnis auf zweierlei Art möglich. Entweder das Prädikat B gehört zum Subjekt A als etwas, was in diesem Begriffe A (versteckterweise) enthalten ist; oder B liegt ganz ausser dem Begriff A, ob es zwar mit demselben in Verknüpfung steht. Im ersten Fall nenne ich das Urteil analytisch, im andern synthetisch.*

[6] Kant (KrV, A154/B193, 7–9):

> *Im analytischen Urteil bleibe ich bei dem gegebenen Begriffe, um etwas von ihm auszumachen. Soll es bejahend sein, so lege ich diesem Begriffe nur dasjenige bei, was in ihm schon gedacht war; soll es verneinend sein, so schliesse ich nur das Gegenteil desselben von ihm aus.*

terminorum, a formulation which results equivalent to the one of the *propositio per sé nota*.

The synthetic judgement, on the other hand, is defined by opposition to the analytic one: it refers to the case of the predicate being *completely outside* the concept referred to by the subject (*B liegt ganz ausser dem Begriff A*), representing in this way a judgement extending the content of knowledge provided by the subject alone (*Erweiterungsurteil*). This means that, in order to identify a synthetic judgement as true, one necessarily has to appeal to extra-conceptual resources, in opposition to analyticity as the property of being knowable out of its own conceptual terms. It has been clarified[7] that this last property does not correspond to the opposite of analyticity, so that this explanation of synthetic judgement let us infer two definitions of analytic judgement, via the assumption that concepts provide knowledge only through analysis. The two definitions of analytic judgements are the following:

1. True in virtue of definition and logic (analysis)
2. True in virtue of meaning

The distinction between analytic and synthetic judgements cannot therefore be uniquely grounded on conceptual resources. A second definition of analyticity presented by Kant in terms of the principle of contradiction, refers to the relation of the proposition to truth: the principle of contradiction is a necessary condition for truth, and a sufficient condition of analytic truth.[8] According to this explanation, a proposition is analytic when its negation is contradictory. This happens explicitly when the predicate of the judgement is part of the definition of the subject, and therefore it is directly connected to the definition in terms of the containment relation: in this way, moreover, all logical truths are analytic judgements. Under this interpretation, an analytic proposition is based on the (partial) *identity* of the concepts represented by the subject and the predicate[9]; on the other hand, the identity between the elements belonging to an analytic expression is a tautology only when such identity is explicit.

Kant's understanding of the notion of analyticity is thus based on two principles, completing each other. These two explanations of analytic sentences introduce Kant's critique of the role of logic, in particular referring to those statements held as true just in virtue of their logical form, namely analytic ones. Logically relevant expressions are of this kind, and synthetic judgements are grounded on a different principle, what Kant calls

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[7] See Coffa (1991, p. 16).

[8] Kant (KrV, A 151/B 190, 32–35):

> *wenn das Urteil analytisch ist, es mag nun verneinend oder bejahend sein, so muss dessen Wahrheit jederzeit nach dem Satze des Wiederspruchs hinreichend können erkannt werden.*

[9] Kant (1800, par. 36).

the highest principle of all synthetic judgements[10]: the formulation of a judgement able to provide new knowledge, requires the ability of extending the basic definitory relation between the subject and its inner predicates.[11] Possible extensions of knowledge are given therefore only by means of synthetic judgements, something which necessarily goes beyond the scope of logic. In this sense, the complete otherness ("*ganz anders*"), as opposed to the relation of containment, expresses what cannot be gained via conceptual analysis, i.e. something impossible to be found in the relation of disjointed concepts, and at the same time it requires the formulation of a non-purely logical relation. A clear reformulation of the Kantian notion of synthetic judgement is due to Berg (1999):

Definition 2.2 (Kant's Synthetic Judgement (Berg's Formulation)) *A judgement is synthetic when its subject and its predicate belong to different categories.*

In the Kantian epistemology, if analysis is the conceptual operation which produces analytic expressions, the only activity able to produce synthetic judgements is the *intuition*. Intuition (as non-purely logical, empirical operation) completes the structure of knowledge providing an answer to the crucial question: "*how are synthetic a priori (necessary) judgements possible?*".[12] According to the Kantian interpretation, purely conceptual knowledge is thus analytic. A concept which is not clearly or explicitly distinguished into its component concepts (say, e.g. "bachelor") has of course a relation of *identity* with the analysed connection of those components (e.g. "unmarried man"), i.e. the logical structure builds the contentual meaning conveyed by a concept. But the act of knowing these two concepts (the unanalysed "bachelor" and the analysed one, "unmarried man") are different, and only by setting a definitional identity such relation becomes tautological. This shows the relation between *conceptual knowledge* and *definitional knowledge*, where the latter is contentual, and the former is logical or formal:

conceptual knowledge ≡ logical knowledge
definitional knowledge ≡ contentual knowledge

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[10] Kant (KrV, A154/B193; A158/B197).

[11] Kant (KrV, A 154–5/B 193–4, 11–17):

> *In synthetischen Urteilen aber soll ich aus dem gegebenen Begriff hinausgehen, um etwas ganz anderes, als in ihm gedacht war, mit demselben im Verhältnis zu betrachten, welches daher niemals, weder ist, und wobei beim Urteile an ihm selbst weder die Wahrheit, noch der Irrtum angesehen werden kann.*

[12] Kant (KrV, B 143):

> *Die Erklärung der Möglichkeit synthetischer Urteile ist eine Aufgabe, mit der die allgemeine Logik gar nichts zu schaffen hat, die auch sogar ihren Namen nicht einmal kennen darf. Sie ist aber in einer trascendentalen Logik das wichtigste Geschäft unter allen und sogar das einzige, wenn von der Möglichkeit synthetischer Urteile a priori die Rede ist, imgleichen den Bedigungen und Umfange ihrer Gültigkeit.*

The two identities cannot be unified in the Kantian perspective, essentially because of the different types of *definition* involved. In the first of these two identities, logical knowledge is given in terms of a *nominal definition*, as the result of the act of analysis which furnishes the basic elements of the concept. Therefore this kind of knowledge does not correspond to what is intended by *definitional knowledge* as producing *contentual knowledge*: in this case the act of definition involved is that of a *real definition*, which constructs or produces the concept out of its basic elements (it is essentially the reversed process). Moreover, this also implies that the definition of analyticity for judgements is strictly related to a criterion of identity for concepts. According to Kant, therefore, conceptual knowledge is first of all to be considered in terms of the act of acquiring concepts, namely the *representation*: in particular, the case of analytic judgements requires to consider which is the proper relation between distinct/indistinct concepts and the related acts of knowledge. The relation between the different acts converging to different (but eventually identical) contents (like in the case of unanalysed concepts and analysed ones), shows the proper relevance of the act of defining in connection to the analytic-synthetic problem. Kant reaches a clarification of analyticity for judgements by distinguishing the different kinds of definition, i.e. by reducing the problem to the question "*what is the proper definition of a concept?*".[13] This distinction follows:

- In a *real definition*, the arising concept is an outcome of a construction by intuition (synthetic a priori concept); therefore a real definition is genetical; by means of an intuition, a construction is accomplished which takes into account all the elements of the concept involved, by taking a meaningful form.[14] This kind of definition has *completeness* as its main property, because conditions of pure intuition (as the faculty by which this kind of definition is obtained) are universality and necessity.[15]
- On the other hand, a *nominal definition* comes only after the concept has been "produced", and it presents only some distinctive features of the object. This definition is therefore a presentation of the features contained in the concept, and amounts to an analytic judgement, an *Erläuterungsurteil*. If, on the one hand, a nominal definition makes use of an analytic process, in dividing the whole in its parts, on the other hand, a real definition essentially proceeds by a synthetic method, adjoining the

[13] Kant treats the problem of definition itself in (KrV), *Transcendental Doctrine of Method*, ch. 1, sec. 1. Cf. in particular B755–760.
[14] Cf. Proust (1989, p. 41).
[15] Kant (KrV, B 747):

> *Es gibt zwar eine transzendentale Synthesis aus lauter Begriffen, die wiederum allein dem Philosophen gelingt, die aber niemals mehr als ein Ding uberhaupt betrifft, unter welchen Bedigungen dessen Wahrnehmung zur möglichen Erfahrung gehören könne. Aber in den mathematischen Aufgaben ist hievon und überhaupt von der Existenz gar nicht die Frage, sondern von den Eigenschaften der Gegenstände an sich selbst, lediglich so fern diese mit dem Begriffe derselben verbunden sind.*

essential parts of the concept.[16] In this difference, meaning is obviously and essentially involved: nominal definition proceeds regressively in the process of understanding the meaning,[17] and its method cannot therefore be exhaustive; on the contrary, a real definition takes into account exactly those elements necessary to give the (complete) meaning of the concept.

The distinction between the different kinds of definition reflects the different methodologies of philosophy and mathematics. While mathematical constructions are, according to Kant, the proper domain where definition represents a valid (complete) determination of the concept; philosophy, by means of definitions recollects the elements resulting from the process of analysis, those which are therefore retained to be sufficient in order to explain exhaustively the concept.[18] In this sense, the definition of analytic truth becomes the following:

Definition 2.3 (Kant's Analyticity II) *An analytic truth is such in virtue of definition and logic (analysis).*

It is by means of analytic judgements that one collects the elements necessary for the definition of a concept.[19] But it is actually relying on the act/object distinction that the connection between analyticity, definition, and concept can be at this point fully understood. A concept as the result of an act of representation, that is a *constructed concept*, is necessarily synthetically given; on the other hand, the description of the main characteristics of the concept, as in the case of the definition given by a dictionary, is instead to be intended as an analytic process, a matter of an analytic judgement, which does not express anything more than what already contained in the words expressing the concept. By means of this latter kind of judgement only identity is expressed (if explicit, the identity is obviously a tautology); the former kind of judgement instead, a synthetic judgement, accounts for the concept in its existence. To explain a concept by nominal

[16] For the distinction between nominal and real definition cf. Kant (1800, par. 106).

[17] Proust (1989, p. 43).

[18] Kant (KrV, B755–757):

> *Definieren soll, wie es der Ausdruck selbst gibt, eigentlich nur so viel bedeuten, als den ausfürlichen Begriff eines Dinges innerhalb seiner Grenzen ursprünglichen darstellen. Nach einer solchen Foderung kann ein empirischer Begriff gar nicht definiert, sondern nur explicirt werden. [...]. Anstatt des Ausdrucks: Definition, würde ich lieber den der Exposition brauchen [...]. Also bleiben keine andere Begriffe übrig, die zum Definieren taugen, als solche, die eine willkürliche Synthesis enthalten, welche a priori construirt werden kann; mithin hat nur die Mathematik Definitionen. [...] Philosophische Definitionen nur als Expositionen gegebener, mathematische aber als Constructionen ursprünglich gemachter Begriffe, jene nur analytisch durch Zergliederung (deren Vollständigkeit nicht apodiktisch gewiss ist), diese synthetisch zu Stande gebracht werden, und also den Begriff selbst machen, dagegen die ersteren i hn nur erklären.*

[19] Proust (1989, p. 46).

definition amounts therefore only to the empty job of performing substitutions of identical elements with the essential characterstics of the concept (which must of course be known before). This operation amounts to the understanding of what is analytic as "*true in virtue of meaning*", in such that meaning is exactly intended as expressed by these essential characteristics.[20]

According to Kant, analytic judgements are thus valid in virtue of the logical connection between subject and predicate, namely the containment of the second within the range of the first, or the belonging of the predicate to the essential characteristics of the subject. Logic, as containing only such kinds of definitional relations, is considered a rigorous tool to organize knowledge already acquired in a proper systematic way, being in fact not able to extend it.[21]

In this way, an essential distinction has been drawn between the act of construction (typically used by mathematical sciences) and the judgement as an object on which logic works by means of formal rules to analyse its content and derive truths analytically one from another. According to this view, Kant understood logic as a complete science, able only to proceed by analysis on its formal concepts. A different approach to the nature of logical procedure was to be introduced in terms of the distinction between the act and the object of knowledge: such an innovative view would have had effects on the interpretation of analyticity and on which conceptual or linguistic elements should be considered the bearers of truth. Bernard Bolzano's *Wissenschaftslehre* was for these reasons the way out of the Kantian view.

2.1.2 *Elements of the Bolzanian* Doctrine of Science

The Kantian critical approach explaining the role of logic for the theory of knowledge, furnished at the same time an essential account of analytic judgements. Along the Kantian interpretation, another explanation of knowledge processes was developed less than a century after, an account which refused the role given to intuition in the way Kant did, representing the beginning of the semantic conceptions of logic and truth, fully developed

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[20] In terms that will be introduced in Section 2.5.3, it is just a question of "know-how", whereas the constructive, synthetic part introducing existence and expressed by real definitions is essentially a "know-that". Proust (1989, p. 48), refers exactly to the nominal kind of definition as a "know-how", relating it to the Kantian term *Geschicklichkeit*, but without introducing any related term for real definitions.

[21] Kant (KrV, B171–172):

> *Die allgemeine Logik [*i.e. formal logic*] enthält gar keine Vorschriften für die Urteilskraft, und kann sie auch nicht enthalten. Denn da sie von allem Inhalte der Erkenntnis abstrahiert, so bleibt ihr nichts übrig, als das Geschäft, die blosse Form der Erkenntnis in Begriffen, Urteilen und Schlüssen analytisch auseinander zu setzen, und dadurch Formale Regeln alles Verstandengebrauchs zustande zu bringen.*

only in the 20th century. Within the scope of such a theory of knowledge, developed by Bolzano in his *Wissenschaftslehre*, it arises a new definition of analyticity, which represents a revolutionary step in the account of this concept. Here I propose an overview on his theory of knowledge, in order to develop the basic distinction with the Kantian understanding of analyticity.

The main elements to be taken into account in order to understand the notion of analyticity opposing the traditional Kantian approach, are at the same time the basis of the entire Bolzanian theory of knowledge, namely, the notion of representation and that of its content. This basic distinction is further elaborated by Bolzano. The realistic approach of the Bohemian philosopher is all contained is his notions of *Vorstellung an sich (representation in itself)* and of *Satz an sich (proposition in itself)*. A representation in itself is defined as that part of a proposition which is not itself a proposition, i.e. a constituent of a proposition which cannot be taken as a proposition by itself, and is to be distinguished from an idea in the normal sense of the word, namely, an idea possessed.[22] An idea in its ordinary sense corresponds to thinking or imaging or perceiving something, without stating the related judgement: in this sense a representation is always intended as something real, happening in a being, therefore reflecting its subjective aspect, i.e. it is a *thought*. These subjective thoughts are indeed actual instances of the objective representation, which exists but out of any time and any subject, therefore is unique (both in cases when no one is making that thought and when different beings are making it). The representation in itself is something totally different from the Kantian *Dinge an sich*[23]: even though independent from actual knowledge, it is not removed or separated from the human way of knowing; the representation is not the product or aim of knowledge, but actually the possibility of knowledge itself. Thus Bolzano distinguishes between a representation in us (Kant's determination of the soul) which exists differently in different minds, as different instances, from the representation in itself or objective representation which is instead unique (at least if the term itself is such).

On the other hand, a *Satz an sich/proposition in itself*, is an abstract non-linguistic proposition, distinguished from a proposition expressed in words, or statement, which is a way to state something and which falls under the principle of excluded middle. Whenever a proposition is not asserted, it will be called a *proposition in thought*.[24] A proposition in itself, as the content of a thought, will thus be nothing existing in reality. Instead it exists independently of any kind of mental or linguistic entity, in particular independently of enunciation, consciousness, and act of judging. The proposition in itself is therefore the way in which it is possible to maintain logic independent from thought and language and, on the other hand, to

[22] Bolzano (1837, par. 48).
[23] Cf. Proust (1989, p. 52).
[24] Bolzano (1837, par. 19).

let propositions have the same sense to different people. Compound propositions are built up by primitive operations on simple parts, and the way in which a complex proposition is built up by simple parts is expressed by a definitional chain. Strict identity between propositions amounts to sameness of their primitive forms and correspondingly statements are synonymous if they express the same proposition, while different statements could possibly refer to the same proposition in itself. Different forms of expression (especially in natural languages) have a unique reference in the language of concepts: the reducibility of the (multiple, infinite instances of the) former to the (uniqueness of the) latter is possible only by considering fixed the set of pure concepts. It remains to discover where and how identity can be justified, so to explain phenomena like tautologies.

Based on this description, it results obvious the step accomplished by Bolzano: the possible content of a representation is distinguished from the act by which it is asserted, namely a judgement will be the (actual) assertion of the truth of a proposition. A judgement always contains a proposition which is either in accord with the truth or it is not, producing in the first case a correct judgement, in the second an incorrect one. Moreover, the notion of judgement is always to be taken as the result of an act of judging, therefore not existing on its own, but related to the agent performing it.[25] Thus, Bolzano calls the proposition the *judgement's material*, performed by the act of judging, whose result (a judgement made) contains as many parts (or ideas) as contained in the proposition expressed: those ideas have to be properly connected to each other in order to obtain a correct judgement.[26] The importance of such a distinction is soon explained: on the basis of this theoretical frame, Bolzano is first of all reformulating the philosophical and linguistic vocabulary, in particular the Kantian one. The judgement/proposition distinction is essentially based on the clarification of the philosophical difference between act and content: every act of judging has a judgement as its result, with an asserted proposition as its content. The analytic/synthetic distinction will thus be applied to (expressed) propositions as the contents of judging acts (thus to *Sätze*), and according to these the relative judgements (*Urteile*) will also be called analytic or synthetic, simple or compound, true or false.[27] Moreover, a second important consequence follows: logic as the art of correct deductive reasoning always deals with judgements, and therefore with the related act of asserting truth for propositions. In logic, one deals with the connection between different judgements, which are used to build up scientific reasoning; whenever a judgement is done on the basis of other judgements, the former is said to be mediated or produced by the latter, this relation being

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[25] Ibid. (par. 34).
[26] Ibid. (par. 291).
[27] Ibid. (par. 294). The distinction act/content relatively to judgements in the Kantian philosophy is contained in the concept of *Urteilskraft*, central topic to all the three *Critiques*.

essentially the relation of consequence between premises and conclusion.[28] If the elements involved in inferential acts are judgements, the old perspective about how inferences are performed changes completely: judgements are always intended as the result of an act performed by an agent. This will also change the resulting value of performing inferences as a way to acquire knowledge.

2.1.3 *A New Concept of Analyticity (Against the Critical View)*

In order to explain the new conception of analyticity endorsed by Bolzano in his *Wissenschaftslehre*, it is essential to consider two main concepts of his doctrine, namely:

1. **Gegenständlichkeit**
2. **Gültigkeit**

In the light of the explanation of these notions, it will be possible to accomplish the next step towards an understanding of the relevance of analyticity in the context of scientific reasoning; this will finally bring us back to the role played by his innovative view on judgements and propositions in logic.

The connection between the elements of a proposition is reformulated by Bolzano in terms of the distinction between mediate and immediate connection. The canonical form of propositions presents a relational connection, expressed by substituting the role of the copula ("to be") by the primitive form of predication ("to have", roughly reflecting the Aristotelian form "to belong to"—ὑπάρχειν). In this way the common form "*x is y*" corresponds actually to the more general form "*x has y-ness*", or "*x has the property y*" ("*y-ness belongs to x*"). The x obviously corresponds to (the representation of) an object, and the y to (the representation of) a property. The connection between representations and real objects is thus the actual foundation of the Bolzanian theory of knowledge, a realist foundation at its core. The above-mentioned connection is referred to by Bolzano as *Gegenständlichkeit*: it refers to the proper correspondence between the representation of the subject and that of the object denoted by the proposition (which corresponds therefore to the modern relation of "reference" or "denotation"), where the object referred to by an idea (concept), must be clearly distinguished both from the idea in itself and from the mental idea.[29] The actual existence of a reference for the proposition represents

[28] Ibid. (par. 300).

[29] Ibid. (par. 49):

> *...(den Gegenstand, auf den sich eine Vortsellung bezieht, oder [...]) den gegenstand einer Vorstellung will ich gar sehr von ihr selbst, nicht nur von einer gedachten, sondern auch von den ihr zu Grunde liegenden Vorstellung an sich, unterschieden wissen, dergestelt, dass ich verlange, wenn eine gedachte Vorstellung einen oder keinen, oder mehre Gegenstände hat, auch den ihr zugehörigen objectiven Vorstellung einen oder keinen oder mehre Gegenstände, und zwar dieselben, beizulegen. Ich verstehe aber unter*

a sufficient criterion for propositional truth whereas, on the other hand, a proposition with a non-existent reference will simply be a false one. But just in order to be able to state the truth (or the falsity) of a proposition that (at this moment, or in this particular situation, for example) misses its reference, one must nonetheless be able to recognize the statement as meaningful, so to know what its subject and its predicate are. Meaningfulness appears therefore as the first condition for a proper predication to be accomplished and, in the realistic perspective held by Bolzano, a non-existent reference is something which cannot be recognized as meaningful. The second condition for predication is given by the correct connection between the object referred to by the representation of the subject and the property expressed by the representation of the predicate attributed to that subject; this relation amounts to a priority relation between the *ens* and its internal or defining properties. The nature of this predication is expressed by Bolzano in terms of the relation between a *concretum* and an *abstractum*.[30] The connection instantiated by a predication has a primitive relation to truth and falsity: according to his realistic view, Bolzano says that a proposition is true if the connection it expresses between subject and predicate is a suitable one.[31] The relation expressed by the predication is referred to real entities, and the suitability of the connection is expressed by the inclusion of the individual designated by the subject within those elements falling under the specification of the predicate.[32] Such a definition of truth is essentially based on the propositional reference to the object of the representations, and for any proposition the principle of excluded middle will hold eternally, i.e. each proposition will always be either true or false.[33] From this assumption, the Bolzanian explanation of the notion of analyticity follows.

In order to present his definition, it is necessary to introduce another property which, according to Bolzano, formally supports the concept

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dem Gegenstand einer Vorstellung jenes (bald existirende, bald nicht existirende) Etwas, von dem wir zu sagen pflegen, dass sie es vorstelle, oder dass sie die Vorstellung davon sein.

[30] Ibid. (par. 60). It is worth noting here how this second assertion condition amounts exactly to the conceptual order explained for CTT in Chapter 1; as in the case of the order of conceptual priority, the relation *abstractum-concretum* is also subject to a possible *regressus ad infinitum*, solved by Bolzano in terms of internal properties, restricting the possibility of changing function within the proposition, a solution which recalls the typed hierarchy created by Russell.

[31] Ibid. (par. 66):

Ist er nun wahr [...]: so können wir allerdings sagen, dass der durch X [...] vorgestellte Gegenstand zu gleicher Zeit auch, [...], durch die Vorstellung eines "Etwas, das die Beschaffenheit b hat" vorgestellt werde.

[32] The inclusion of classes is the interpretation given in Proust (1989) for the Bolzanian idea of true predication.

[33] Bolzano (1847, par. 147):

Jeder gegebene Satz nur eines von jenen Beiden allein und solches fortwährend: entweder wahr und dieses dann für immer, oder falsch und dieses abermals für immer.

of truth, namely the notion of *Gültigkeit* (*validity*). This concept is moreover essential in modifying the restrictions due to the property of *Gegenständlichkeit*. Bolzano explains the connection between truth and validity via the rejected hypothesis that one and the same proposition can ever be true once and false at another time, i.e. the principle of excluded middle will hold eternally for one and the same proposition:

Truth → Principle of Excluded Middle → Validity

A change of the truth-value could only be possible, according to Bolzano, considering ("*consciously and with the definite intention of becoming the more precisely acquainted with the nature of the given proposition*") a certain variation of the concepts involved in a proposition, therefore assuming that some of its elements are replaceable by different concepts. By doing this, what is obtained is not just the relation of a proposition to its truth, rather the relation between all the propositions derivable from that and their truths.[34] Properties of a certain (type of) proposition can be found by performing different operations of substitutions on ideas assumed as variable parts in that proposition. It is clear that under certain substitutions the truth-value will be maintained; performing some other substitutions the previous truth-value will change; and finally in some cases there will be no truth-value anymore, i.e. no denoted representation. Bolzano is thus after a criterion to select between all the possible substitutions performable for any given proposition, so to obtain only those propositions in some sense worthy, and in particular to know what relationship the set of true propositions emerging in this way has to the entire set.[35] This is the criterion of *Gültigkeit* or validity of a proposition: it amounts to the relation between the subset of true propositions derivable from a given one by substitution of some ideas, to the entire set of possible derivable propositions. When these possible substitutions provide only true propositions as result (the abovementioned subset corresponds entirely to the total set), the relation is that of *universal* or *total validity*; correspondingly, when the subset is empty and the possible performed substitutions give only false propositions as result, the relation is said to be of *universal* or *total invalidity* (countervalidity). In between these, it will be considered a variable probabilistic value. According to this explanation of what amounts clearly in the case of total validity to *necessary truths*, determined by variations on content, also an explanation

[34] Ibid. (par. 147):

> *Betrachten wir nämlich an einem gegebenen Satz nicht bloss, ob er selbst wahr oder falsch sein, sondern welch ein Verhalten zur Wahrheit auch alle diejenigen Sätze befolgen, die sich aus ihm entwickeln, wenn wir gewisse in ihm vorkommenden Vorstellungen als veränderlich annehmen, und uns erlauben, sie mit was immer für anderen Vorstellungen zu vertauschen: so leitet uns dies auf die Entdeckung mancher überaus merkwürdiger Beschaffenheiten der Sätze.*

[35] Ibid. (par. 147):

> *musst es uns zu wissen [...] besonders in welchem Verhältnis die Menge der wahren Sätze, die so erscheinen, zu der gesamten Menge stehet.*

of *analytic truth* will be derived: analyticity amounts to a complete generalization of validity. To formally define the notion of analytic proposition it is necessary at this point to determine how many elements in a proposition can be changed without such a proposition changing its truth-value, therefore determining it as analytic: a necessary and sufficient criterion for a proposition to be analytic is that even the substitution with whatsoever content of one idea in the proposition does not change its truth-value[36]; on the other hand, a synthetic proposition will be one in which there is not a single idea variable, without the truth-value of the proposition being changed. Analyticity is thus related to the substitubility of at least one of its elements: the main role in this definition is now played by the mentioned restriction on the kinds of possible substitutions and their result. The Bolzanian reformulation consists therefore in suggesting as constitutive of the notion of analyticity the simultaneous and uniform substitution of *Vorstellungen an sich* within a *Satz an sich*.[37] An informal definition of the Bolzanian idea of analyticity can therefore be formulated as follows:

Definition 2.4 (Bolzano's Analyticity) *A proposition is analytic when it is logically valid, i.e. when its truth-value is maintained whatever (suitable) substitutions are performed for (at least one of) its variable non-logical constituents ("come what may").*

This definition has, according to Bolzano, the advantage of including within the range of the analytic/synthetic distinction not only empty true propositions, but also false propositions and in general those including non-logical concepts. In this sense, the classical idea of logical analyticity is just a specific case of its first broader sense: it refers just to cases when the knowledge required in order to recognize the truth-value of the proposition is of the logical kind, i.e. the concepts representing the invariant parts of the proposition are all logical parts. This kind of expressions, e.g.

A **which is** B **is** B,

are considered logically analytic truths, propositions commonly recognized simply as analytic. Thus, analyticity is defined by means of the two principles of reference (*Gegenständlichkeit*) and validity (*Gültigkeit*) under

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[36] Ibid. (par. 148):

> *Wenn es aber auch nur eine einzige Vorstellung in einem Satz gibt, welche sich willkürlich abändern lässt, ohne die Wahr- oder Falschheit der selben zu stören; d.h. wenn alle Sätze, die durch den Austausch dieser Vorstellung mit beliebigen Andern zum vorscheine kommen, entweder insgesamt wahr oder ingesamt falsch sind, vorausgesetzt, dass sie nur Gegenständlichkeit haben [...] Ich erlaube mir also, Sätze dieser Art mit einem von Kant entlehnten Ausdrücke analytisch [...] zu nennen.*

[37] This operation of substitution can be formally represented as follows:

$$S(v_1, \ldots, v_n/w_1, \ldots, w_n)$$

where $v_1, \ldots, v_n$ are the substituted and $w_1, \ldots, w_n$ the substituting representations within the proposition S. Cf. Morscher (2003, p. 150).

restricted substitution (where the restriction is defined by the kinds of concepts involved within the proposition). The innovative step is represented by a completely new idea of analyticity, which will have a great influence on most of the following interpretations. This interpretation of logical truth is no longer based on the relation between concepts (containment), thus in turn on the notion of definition; rather, it is defined on the basis of truth independently of (non-logical) concepts involved. On the other hand, by referring to the notion of validity, the modern interpretation of analytic truth as (logically true) proposition empty of any possible meaning was just a step away: one needed only to reverse the given explanation of analytic knowledge, referring to the role of logical constants.

2.1.4 *Analyticity in Question: The Possibility of Knowledge*

Bolzano had as main focus in his *Wissenschaftslehre* the foundation of a knowledge system; the same is true of Kant's *Kritik der reinen Vernunft*, which aims to describe the possibility of knowledge. Both these masterpieces provide in turn an interpretation of the meaning of analyticity, in order to establish the principles determining the value of propositions in the context of scientific knowledge processes. Kant was the first in fully recovering the status of *formal logic*, re-establishing the central value of the Aristotelian notion of method, and providing the structure of pure reason, to be applied both in mathematics and in what he calls the *dogmatic* use (philosophy). For these disciplines, it is essential the way concepts are acquired, and it is exactly in formulating their distinction that Kant expresses the difference in terms of the dichotomy *form/content*, coming to the role of intuition. According to Kant, if mathematics considers concepts presented to the pure a priori intuition via construction (i.e. *in concreto*, but not empirically), philosophy on the other hand treats with concepts given a priori, containing the synthesis of possible non a priori intuitions.[38] This approach leads notoriously to problematic conclusions: in particular, the final separation between the philosophical and the mathematical methods actually means the possibility of every possible experience (i.e. the formulation of synthetic judgements) only in terms of the *transzendentale Sätze*, never given by construction of concepts, rather produced according to a priori concepts; these propositions contain the rule of any synthetic empirical unity of the experience, which actually makes experience meaningful because it connects things by concepts. To establish this possibility means

[38] Kant (KrV, A 719, 30–A 720, 3):

Alle unsere Erkenntnis bezieht sich doch zuletzt auf mögliche Anschauungen: denn durch diese allein wird ein Gegenstand gegeben. Nun enthält ein Begriff a priori (ein nicht empirischer Begriff) entweder schon eine reine Anschauung in sich, und alsdann kann er konstruiert werde; oder nichts als die Synthesis möglicher Anschauungen, die a priori nicht gegeben sind, und alsdann kann man wohl durch ihn synthetisch und a priori urteilen, aber nur diskursiv, nach Begriffen, und niemals intuitiv durch die Konstruktion des Begriffes.

in turn to be able to furnish a description of the pure method of concepts a priori, which needs to be purely analytic to be also universal, but in this way it is also totally uninformative. This produces the famous separation between logic and mathematics, a direction which can be intended as the opposite of the logicistic programme from the beginning of the 20th century. The problematic relation between mathematics and logic was already explicitly considered by Dedekind and, shortly after, Frege's logicistic thesis, in opposition to the Kantian view, stated that every arithmetical concept can be defined in purely logical terms, which makes them actually universal concepts, and that each theorem of arithmetic (because universal) can be proved using only the basic laws of logic. This approach surely determines the Fregean definition of analytic truth.

Kant's *Critique of Pure Reason* clearly represents for Bolzano the essential reference for the project of formulating the connection between logical procedure and knowledge. This connection is firstly grounded by the Bohemian priest in the link between analyticity and deducibility (*Ableitbarkeit*), using again the central notion of substitution, a solution which seems nowadays quite modern. The classical definition, based on the preservation of truth (the conclusion becomes true whenever the premises are true), is slightly modified by Bolzano, by saying that whenever a substitution of a component in the antecedents makes them true, the same substitution will make the conclusions (or derivable propositions) true.[39] This definition is strictly related to a contentual aspect of the theory of knowledge, and it suggests an interpretation in terms of models. Because judgements are the way to ascribe truth to propositions, clearly the previous definition of deducibility between propositions will amount to a definition of valid inference for judgements (as the proper way to acquire knowledge).[40] The relation of deducibility between judgements or inclusion between propositions is clearly linked to the notion of analyticity; analytic propositions are less informative precisely in the same way the conclusion of a deduction is such in relation to its premises, since more can be deduced from premises than from conclusions. In this sense, analytic propositions can be assumed as conclusions of possible deductive schemas, instead being excluded from the role of premises: in an analytically valid deduction, the reference of the conclusion is supposed to be already contained in that of

[39] Bolzano (1837, par. 155):

> *[...] und sage, dass die Sätze* $M, N, O, \ldots$ *ableitbar wären aus den Sätzen* $A, B, C, D, \ldots$ *hinsichtlich auf die veränderlichen Teile* $i, j, \ldots,$ *wenn jeder Inbegriff von Vorstellungen, der an der Stelle der* $i, j, \ldots,$ *die sämtlichen* $A, B, C, D, \ldots$ *wahr macht, auch die gesamten* $M, N, O, \ldots$ *wahr macht.*

[40] Ibid. (par. 300):

> *Ich sage aber, dass ein Urteil* M *durch ein oderetliche andere* $A, B, C, D, \ldots,$ *verursacht oder vermittelt werde, wenn die Ursache, dass wir das Urteil* M *fallen, darin liegt, dass wir so eben auch die Urteile* $A, B, C, D, \ldots,$ *gefallt. Häufig pflegt man die Handlung des Geistes, durch die er von den Urteilen* $A, B, C, D, \ldots,$ *zu dem Urteile* M *übergeht, einen Schluss oder ein Schliessen, auch wohl ein Folgern.*

the premises. Thus, the Bolzanian definition of deducibility copes perfectly well with a notion of analyticity which is no longer the Kantian one of containment, rather it is reformulated in terms of correctness of the form of reasoning.

This conceptual explanation of the two approaches to analyticity let to understand on which basis the Bolzanian account brings us further in the explanation of such a notion in terms of informativeness: let us start from reconsidering the *synthetic method* in Kant. Kant strongly supports the idea that extension of knowledge is exclusively provided by empirical (a priori) judgements characterized by the properties of *universality* and *necessity* (i.e. mathematical ones): universality can of course be intended either as *universal quantification* (which clearly for Bolzano would not have been a sufficient criterion to distinguish between a priori and empirical judgements) or in terms of *universal validity* (which would amount exactly to the notion formalized by Bolzano in terms of substitutions). The controversy between Kant and Bolzano about the concept of analyticity can be explained further insisting on the following connection:

analytic truths ↔ conceptual truths
↔ necessary (universally valid) truths

According to Bolzano the distinction between a priori and a posteriori judgements is based on their origin in terms of the act of judging, not (in the Kantian way) on their structure, and the same applies to the analytic/synthetic distinction for judgements. If an *analytic truth* is one such that it is determined by preservation of truth under the operation of substitution (in terms of every possible substitution) of non-logical parts (i.e. a logically valid proposition), the capability of recognizing definitionally equivalent terms (if p then p) requires the contentual knowledge of these terms. Thus, knowledge of the definition is the basis for both the accounts; but the Bolzanian notion of analyticity cannot logically follow from a modification of the Kantian one, rather, it actually rejects it. Let us consider the following cases:

1. On the one hand, there are analytic judgements which do not comply with the standard Kantian definition (e.g. *Every object is either A or not A*).
2. On the other hand, some synthetic judgements satisfy to the Kantian definition of analyticity (e.g. *Alexander, the son of the king of Macedonia, was king of Macedonia*).

According to Bolzano, the example presented at point 1 is an analytic judgement based on the principle of contradiction, but neither A nor $\neg A$ appear in the concept of *object* (i.e. the subject of the proposition); and the sentence contained at point 2 is synthetic, but the predicate is contained in the concept of the subject (namely, being king of Macedonia). According to

Kant, on the other hand, the second proposition clearly would not contradict his definition of analyticity, and the example presented at point 1 is a properly analytic judgement, because it is a formulation of the principle of non-contradiction, on which all analytic judgements are based. Thus, Kant would account two different versions of the notion of analyticity as corresponding to the Bolzanian one of logical analyticity: the first one accounts for the substitution of the *definiens* to the *definiendum* making explicit the inclusion of the predicate in the subject; the second simply follows from the principle of non-contradiction.[41] Clearly, analyticity was for Bolzano a property based on logical laws, essentially determined by necessity, and this represents the connection to the Kantian understanding of analytic truth. To complete the schema, a conceptual truth justifies a necessary truth (i.e. a necessary truth is one such that derives from a purely conceptual truth), one whose negation is in contradiction with such a conceptual truth. In the determination of the notion of *conceptual truth* based on necessity, is thus relevant the distinction between a priori/a posteriori judgements, reformulated in terms of the conceptual/empirical distinction for propositions. Also in this case the role of the act is essential: and when this has to be established for the concept of mathematical truth, Bolzano recalls the use of the *Anschauung an sich*, taking over the place of the Kantian *reine Anschauung*.

It should now be clear that the clarification of the distinction between the Kantian and the Bolzanian approaches to analyticity can be given essentially by considering the role of two activities:

- **Construction**
- **Intuition**

By analysing these two terms, it is possible to reformulate and explain the comparison introduced in this section, to finally come back to the conceptual development of the notion of analyticity. The Kantian *Vernunft* presents two different ways of proceeding, both are universal and a priori, each one established according to, respectively[42]:

1. *Forms* of intuition (space and time) — knowable a priori, and in which concepts are determined a priori
2. *Matter* or content, which is related to perception; it is given as determined only empirically, whereas its being a priori is only possible via the synthesis of possible perception.

In the first case, the reason works by means of given concepts, where perception brings elements to such concepts in the intuition (rational philosophical knowledge); in the second case, the reason works by constructing concepts (rational mathematical knowledge). Sensation is the material constituent of

[41] For this argumentation about the connection between the Kantian and the Bolzanian views, cf. Sebestik (2003).
[42] Kant (KrV, A 723–724).

cognition; concepts take the place of subject and predicate, and sensation represents the material of which intuitions are built of. In the process of *construction* concepts are referred a priori to intuition, which makes it possible to present them without empirical data (the way a geometer works).[43] Kant understands by the construction in the pure intuition (i.e. a priori) a sufficient criterion for the existence of mathematical objects: mathematicians have to prove that their combinations of concepts correspond to objects, and this is possible by presenting these objects a priori to the intuition.[44] Moreover, the difference between a priori and empiric intuition is at this point essential to a complete understanding of the resulting procedures and of the related known objects; according to Kant, the use of empiric imagination, by which we apprehend a concrete object, is the gate to consider concepts abstracted from the material aspects, and using thereafter only a method of construction: for example, the concept of number consists in the representation of the related method of construction, and such a representation is the schema of that concept.[45] In this way, the Kantian "intuitionism" (intended both as a theory of intuition and as a theory of meaning determined by construction) presents some clear connections to mathematical constructivism, especially in the central role attributed to these two procedures in the process of knowledge: the two activities need each other to be a complete procedure of knowledge. Nevertheless, this extremely important thesis is partially fruitless, or even incomplete and vicious by affecting the ontology, and separating intuition (in the interesting cases is *pure intuition*) from semantics.[46] Kant's theory of mathematical knowledge is essentially based therefore on a form of *intuitionistic semantics*,[47] but in fact the determination of meaning is given only by presenting concepts a priori in the *intuition*, which therefore represents something outside of the pure process of construction. This has an important consequence: one obtains a theory of concepts rather than a theory of properties, which in the

[43] Ibid. (A 713):

> *Einen Begriff aber Kostruiren, heisst: die ihm korrespondierende Anschauung a priori darstellen.*

[44] The role of intuition in the Kantian philosophy of knowledge has been addressed either in a "weak version" (ineliminability of intuition) or in a "strong version" (its central role); cf. Nef (2004, p. 124).

[45] Kant (KrV, A 140, 15–27):

> *wenn ich eine Zahl überhaupt nur denke [...] so ist dieses Denken mehr die Vorstellung einer Methode, einem gewissen Begriffe gemäss eine Menge [...] in einem Bilde vorzustellen [...]. Diese Vorstelllung nun von einem allgemeinen Verfahren der Einbildungskraft, einem Begriff sein Bild zu verschaffen, nenne ich das Schema zu diesem Begriffe.*

[46] This has been called the *Kantian error*, cf., e.g. Coffa (1991). On this point it is illuminating the following quotation from Nef (2004, p. 128):

> *[...] toute la faiblesse de la position kantienne se dévoile – pour expliquer la synthèse qui est au fondement des jugements synthétiques, il faut recourir à une intuition qui est elle-même [...] synthétique.*

[47] Cf. Nef (2004).

linguistic analysis leads to the Kantian definition of analyticity and to the thesis of the analyticity of logic, implying the exclusion of synthetic judgements from its method. The switch from the *Kritik* to the *Wissenschaftslehre* provides a more complete analyisis of representations, namely in the distinction between objective and subjective ones.[48] In the clarification of the act of judging, Bolzano recovers the role of pure intuition which Kant needed in support of constructions: for both authors, the process of knowledge is complete and synthetic, but Kant needs the distinction between formal and transcendental logic, which leaves the former a completely analytical science, whereas it is only by means of the categories of pure intellect that validity and utility for knowledge are saved.[49] The Kantian reduction presupposes, on the one hand, a doctrine of (pure) intuition; on the other hand, such an intuition is outside the formal development of logic. Bolzano, relying on the realistic definition of *Vorstellungen an sich* and *Sätze an sich*, develops a different approach to knowledge and to the logical relation of derivability, which also allows a new understanding of analytic expressions. The distinction between the two approaches is thus determined in the first instance by the description of intuition and construction. Bolzano understands the former as a type of representation (*Vorstellung*), the component of a proposition which is not itself a proposition. On the other hand, the object of intuition is determined in two species, internal and external[50]: such a distinction determines the type of logical function according to its object. Representations are therefore defined by the relation of the function to the object, whereas in Kant the same relation defines intuition.[51]

The second distinguishing characteristics among the two theories explicitly concerns the relation between this intuition and its semantic: Kant establishes a *semantic* relation to the object (intuition has an immediate relation to its object: some things *beziehen sich auf* other things). Bolzano provides a double interpretation for this relation: first, by introducing subordinated concepts in the link between a concept and its object; second, by

[48] Therefore, it is extremely interesting the third part of the *Wissenschaftslehre*, which concerns the proper epistemology.

[49] Kant (KrV, B81, 29- B82, 3):

> *Eine solche Wissenschaft, welche den Ursprung, den Umfang und die objektive Gültigkeit solcher Erkenntnisse bestimmte, würde transzendentale Logik heissen müssen, weil sie es bloss mit den Gesetzen des Verstandes und der Vernunft zu tun hat, aber lediglich, sofern sie auf Gegenstände a priori bezogen wird, und nicht, wie die allgemeine Logik, auf die empirischen sowohl Vernunfterkenntnisse ohne Unterschied.*

[50] Bolzano (1837, par. 286).

[51] Kant (KrV, B 304, 1–12):

> *Das Denken ist die Handlung, gegebene Anschauung auf einen Gegenstand zu beziehen. Ist die Art diesere Anschauung auf keinerlei Weise gegeben, so ist der Gegenstand bloss transzendental, und der Verstandesbegriff hat keinen anderen, als transzendentalen Gebrauch, nämlich die Einheit des Denkens eines Mannigfaltigen überhaupt. Durch eine reine Kategorie nun [. . .] wird kein Objekt bestimmt, sondern nur das Denken eines Objekt überhaupt, nach verschiedenen modis, ausgedrückt.*

suggesting an *epistemic* determination, i.e. the distinction between awareness and mediate knowledge.[52] Intuition in Kant is the singular complex image, whereas for Bolzano intuitions can be either simple or complex: the determination of the notion of intuition is thus for Bolzano given by a semantic relation (to the referred object) organized by an *epistemic* procedure, the formulation of representation and expression. A pure intuition is the objective representation given only by an actual representation of the referred object; an external object produces a representation which can be either an intuition or a concept; a pure concept as representation is produced only by objects which never actually exist.[53] A singular representation, i.e. an intuition in the Kantian sense, is a particular type of the first kind of representation, namely one which is referred at least and only to one object, when such an object actually exists. Thus for Bolzano an intuition is a singular and simple representation, and one having the effect of producing an immediate modification in us in the present instant.[54] Bolzano explicitly considers the Kantian theory of concepts construction (for the intuition) in his (1810), as well as reconsidering the role of pure intuition in establishing a philosophy of mathematics, namely for its capability of being source of synthetic judgements: he agrees that the possibility of synthetic judgements must be found in something other than the principle of contradiction, but it is unclear to him what would it mean the Kantian suggestion that this is provided by a *pure intuition* in the case of a priori judgements; the possibility of connecting a singular concept to the validity of its predicate for all those objects falling under the same concept is due exclusively to universality, which for Bolzano means to explicate the role of concept rather than intuition. This understanding of the role of concepts, determines mathematical truths as purely conceptual a priori, where intuition plays no role in their construction.[55]

The Bolzanian conception on intuition and representation is thus the result of the development of an entire ontological and epistemic system. On the basis of his understanding of analyticity in the context of logical deduction, Bolzano takes its anticritical view further, suggesting that there are a lot of synthetic propositions valid in logic, a result which was incomprehensible within the critical view. What is extremely important in the Bolzanian view on logic is that according to the mentioned definition

[52] Bolzano (1837, par. 77). George (2003, pp. 23–28) suggests this double distinction between the different approaches to the notion of intuition (*Anschauung*).
[53] George (2003, p. 33).
[54] Bolzano (1837, par. 72 and par. 300).
[55] Bolzano (1810, p. 183, par. 8):

> *[...] mathematics could best be defined as a science which deals with the general laws (forms) to which things must conform in their existence. By the word "things", I understand here not merely those which possess an objective existence independent of our awareness, but also those which simply exist among our ideas, either as individuals i.e. intuitions, or simply as general concepts, in other words, everything at all which can be an object of our perception.*

of analyticity for propositions, even the rules governing syllogisms can be intended as synthetic truths. In fact, even if the conclusion (e.g. "Caius is mortal") is analytic in respect to its premises ("If all men are mortal" and "if Caius is a man"), the strength of the rule itself is to be found in that it establishes the relation of deducibility, which actually extends the domain of the synthetic validity of the premises (i.e. producing an analytic truth). One finds here for the first time an analysis which takes into account the domain of validity of propositions in terms of instances, a formula which will be later considered to express a new formulation of the problem of analyticity, by means of a definition in terms of informativeness and individuals. The definition of analyticity for propositions given by Bolzano in terms of substitutions has quite an important consequence for the connection between analyticity and informativeness. The direct denotation of the elements referred to by the concepts involved in the proposition is no longer taken into account, rather one refers to validity in terms of possible substitutions under persistence of the truth-value: this obviously changes the way in which the informativeness of analytic truths is intended.

2.2 Analysis and Synthesis

The different roles Kant and Bolzano ascribe to analytic and synthetic propositions in the context of scientific knowledge are the heritage of the oldest conception about scientific method. Such a conception is to be found for the first time in a coherent and complete exposition in the Aristotelian *Organon*, where science is treated in terms of a methodological approach. The search for a definition of the notion of analyticity, accomplished in the Kantian and the Bolzanian treatises by recognizing the different and complementary activities of intuition and construction, was already at the core of the Aristotelian exposition of science. Aristotle named his theory of demonstration, and more generally his doctrine of science, *Analytics*: it is precisely an analysis, or reduction, or resolution of the process of proving the demonstrability guaranteed by the logical form, through the study of figures and modes of inference.[56] From the Aristotelian perspective, inferential methods are considered the way to provide methods of correct extension of knowledge. This is the epistemological assumption at the core of the *Prior Analytics*, where the ἐπιστήμε (science) is such because of the method on which it is built upon, namely the deductive method: this method is in turn grounded on the first unprovable principles, and according to this reading, proper knowledge is either noetic or deductive (a dichotomy which

[56] Aristotle (AnPri, I 32, 47a 2–5):

> *... if we examine the means by which syllogisms are produced, and possess the ability to invent them, and can also reduce the syllogisms when constructed to the figures previously described, our original understanding will be completed.*

reflects the roles of intuition and construction). Therefore, correctness for deductive processes is essential to the Aristotelian view on science in order to point out clearly the relationship between deductive method and first principles; on such a basis he establishes in turn the difference between the analytic and the synthetic method, by which science proceeds, as two ways of reasoning, from the principles and back to the principles.[57] What proceeds from the principles is synthetic, whereas the other direction is analytic, going backwards to the principles. The method of analysis, as a way back from what is derived to the principles used, clearly represents Aristotle's heritage from ancient mathematics, particularly from the geometrical application. Properly said, ανάλυσις is the method used by the Greek geometers, in finding the proof of a theorem or to reach a suitable solution to a problem. The method started by assuming what was sought, asking for the principles it comes from, and going back towards those already known. The way back from those principles to the theorem (to be proved) reverses the steps of the analytic process, and it consists obviously in what is called synthesis.[58] This way of proceeding is exactly what intended by Aristotle when he says that the last in the analysis is the first in the construction.[59] It is undoubtedly due to the geometrical origin of the analytic method the possibility of examples of logical inferences resulting in non-trivial conclusions. The Euclidean *Elements*, certainly the greatest example of the development of the geometrical science in the ancient Greece, anticipates in application the general study of the method of proof due to Aristotle (and according to some interpreters even to Plato).[60] According to the composition of this method, the role of analysis in solving mathematical problems is supported by a parallel use of the method of construction (κατασκευή in the Euclidean vocabulary), which is executed in the figure by which the theorem is to be proved. In general, while analysis is intended as the process backward from what is sought assumed as given (conclusion of a logical consequence) to its causes; synthesis is intended as moving along the line of a logical inference, from premises to conclusion, according to the explanation given by Aristotle.

Thus analytic and synthetic processes appear strictly related to the correct way of acquiring knowledge, and in turn to the question of logical deduction (involving the role of intuition) and of mathematical construction.

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[57] Aristotle (EthNic, I, 2 1095a 30–32):

μὴ λανθανέτω δ᾽ἡμᾶς ὅτι διαφέρουσιν οἱ ἀπὸ τῶν ἀρχῶν λόγοι καὶ οἱ ἐπὶ τὰς ἀρχάς.

[58] A study of the method of analysis in relation to its geometrical origin is presented in Hintikka and Remes (1974). The understanding of the analytic and synthetic methods in connection to the assumed and the sought, will be reconsidered in terms of the formalization of CTT in Chapter 4.

[59] Aristotle (EthNic, III, 5 1112 b 20–24):

τὸ ἔσχατον ἐν τῇ ἀναλύσει πρῶτον εἶναι ἐν τῇ γενέσει.

[60] A general presentation of the method of analysis in connection to its synthetic counterpart is given first in a text by Pappus, known as *Pappi Alexandrini Collectionis Quae Supersunt*. Cf. Hintikka and Remes (1974, ch. 2).

Aristotelian syllogistic was directed to propositional analysis, describing how scientific knowledge is developed and acquired: it explained how truth is preserved from premises to conclusions, according to forms of reasoning and figures of deduction. Aristotelian syllogistic copes essentially with the concept of *valid deduction*, revealing the empty structures according to which truth is preserved: the notion of *derivation* or *entailment*, main topic of the Aristotelian logical treatises, is related to the idea of preserving truth in its independence to the material content it refers to. This idea on the one hand brings us to a first concept of validity:

(I) Validity holding under any uniform substitution of terms

on the other hand, it involves a concept less independent from the nature of the elements contained in the sentences which are connected by the relation of derivability:

(II) Validity holding for uniform substitution of terms standing in an appropriate topical relationship, like that of genus and species[61]

It is here all *in nuce* the subtle distinction realized by the different interpretations due to Kant and Bolzano. On the one hand, independence from content relates to formal truth, where formality is connected to the conceptual (logical) elements involved in the proposition (thus also in connection to derivability); on the other hand, the idea of possible substitutions of terms plays the central role, explicitly in Bolzano under the restricting conditions illustrated, recalling the above-mentioned topical relationship. The problem of analyticity is thus strictly connected in the first case to the relation of consequence in terms of preservation of truth, in the second to a more specific relation of knowability of contents (definability). These approaches can now be described as follows:

(a) *Containment theory*: an inference is valid by virtue of the pure form of the sentences involved (*ex vi terminorum*)
(b) *Incompatibility theory*: a consequence holds by stating the incompatibility of the falsity of the conclusion with the truth of the premises; this definition is at the core of the notion of *logical truth*

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[61] These two notions of entailment will be developed by logicians in the Middle Ages in terms of the distinction between *perfect entailment* and *imperfect entailment*, as explained by Abelard in his *Dialectica*, Tractatus Tertius — Topica, Liber Primus *De Locis* (pp. 253–255):

> *inferentia alias perfecta est, alias imperfecta. Perfecta quidem est inferentia, cum ex ipsius antecedentis complexione consequentis veritas manifesta est et antecedentis constructio ita est disposita, ut in se consequentis quoque constructionem contineat, veluti in sillogismis aut in his ipoteticis que formas habent sillogismorum: ... Sunt autem alie inferentie, que imperfecte sunt, cum videlicet una tantum propositio antecedit, etsi de eisdem antecedentibus substracta una fiat ad ultimam inferentiam hoc modo .... Que quidem inferentie, quamvis imperfecte sint quantum ad antecedentis constructionem, tamen necessitatem ex rerum natura sepissime tenent veluti ista quam prius posuimus de 'animali' ad 'animatum' .... Perfectio itaque necessitatis etiam in his est inferentiis, non constructionis.*

For a treatment of these notions cf. Martin (1986).

Definition (a) amounts to the determination of a purely formal property, according to which it is possible to state the *validity of an inference*: this idea of inference as formal containment between conclusion and premises is at the core of the Kantian definition of analyticity, reflecting his understanding of analytic judgements. On the other hand, definition (b), by referring to the contradiction between the falsity of the consequence and the truth of the premises, is certainly a material interpretation, which defines the *holding of a consequence*. Material validity, in order not to fall into the trivial case of material implication, obviously requires a related operation of generalization, obtained by substitutions fulfilling the suitable places and respecting a certain relationship: this is based on the Bolzanian notion of derivability, by restriction on the operations of substitution, first by recognizing the right elements to be substituted, and then by considering the operation valid only under the condition that the elements substituted are of the same "type". Both the interpretations fulfil the general requirement of establishing a relation valid for all possible terms or concepts (*in omni termini*), i.e. to hold independently of non logical terms. What has been achieved in this further step, is another essential distinction in the clarification of the concept of analyticity: the act/content distinction is now interpreted as essential topic to a proper reformulation of the problem.

2.2.1 *Act and Content: A Foundational Distinction*

The development so far of the concept of analyticity has shown two interpretations, based on different understandings of the notion of entailment: the first explained according to formal containment, the second to a suitable material connection of the terms involved by the relation of logical derivation. In both cases, the relevant notion is the one of content to which the definitions refer to, namely in terms of the elements involved in analytic expressions. The bearers of truth and analyticity themselves have changed in the different explanations reported. In particular, it is the relation between *act* and *content* which needs to be reconsidered here.

It is quite evident in the Kantian line of argumentation that the relation between act and content is essential to the determination of conceptual, and in turn analytic knowledge. The core of the problem of definition is in fact to be recognized in the *act of representation*, intended as the way one understands and then defines concepts, corresponding to the mentioned difference between distinct and indistinct representations. On the other hand, the *act of construction* leads directly to a completely distinguished representation of concepts, producing new meaningful expressions. In other words, different acts of accessing concepts (*description/production*) allow to distinguish between analytic and synthetic methods of knowledge. Bolzano, proposing an epistemic interpretation of the notion of derivability, provides also a new idea of analyticity, based on substition procedures. The act/object distinction is the first step in this passage, and it leads to the semantic turn that will introduce meaning in the explanation of analyticity. This point has also another methodological relevance: Kant uses analytic/synthetic in connection to *Urteile* (judgements), namely because truth-values are ascribed to judgements, whereas *Aussage* (statements) and *Sätze* (propositions) do not include the relation to or the possession of such a truth-value. On the other hand, Bolzano takes into account the elements of propositions respectively with and without the act of judging: in this way a proposition exists both in itself and in its instances, namely when asserted, thus entering in a proper relation to truth. The act of asserting a proposition, by which the relation to truth is implied, gives rise to judgements which now contain propositions as their parts. Propositions become the proper bearers of truth, and one speaks of analytic or synthetic propositions as the content of analytic or synthetic judgements. This essential change of perspective will influence most of the following approaches on analyticity, and one needs to take it into account especially when such a distinction is not clearly made, or not made at all.

2.2.2 *Content and Meaning*

In terms of the conceptual distinction between act and content, Bolzano redefines the notion of analyticity for propositions and formulates his concept of derivability. In the rigorous determination of the notion of content he finds the essence of analyticity by means of substitution. The idea of restricted substitution will be considerably influenced by the introduction of the concept of meaning in this discussion. Such a step, essential here in the further determination of the different understandings of the notion of analyticity, has been accomplished first by Frege, in particular by developing the notion of content in connection with his theory of *Sinn* (meaning) and *Bedeutung* (reference).[62]

In Frege's logicism the problem of analyticity is foundational, and strictly connected to the question of definition. It is foundational because it corresponds to the essential presupposition of his entire project, namely to answer the question: "*are the laws of arithmetic synthetic a priori or analytic?*".[63] Frege's work relies on the idea of providing a justification to the foundations of arithmetic by showing that the laws of science, mathematics in particular, are derived from logical truths. This means of course that the notion of logical truth itself needs a definition upon which everything else can be founded. It is only referring to the notion of act of judgement that the Fregean perspective on analytic truths can be understood: Frege considers the distinction between synthetic and analytic as pertaining not to the content of judgements, rather to their justification, because when this

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[62] For a treatment of the Fregean relation of sense and reference within Constructive Type Theory, cf. Primiero (2004).

[63] Frege (1884, sec. II).

is missing, such is also the possibility of classifying judgements.[64] In terms of justifications therefore, an analytic truth is one such that provides by itself "*the ultimate ground upon which rests the justification for holding it to be true*".[65] The process of analysing the proof of a proposition (on whose basis such a proposition is maintained as true) will therefore show which elements define the proposition as analytic:

Definition 2.5 (Frege's Analyticity) *A proposition is analytic if it can be justified only in terms of general logical laws and definitions.*[66]

According to this explanation, the investigation must be further pursued on the notion of definition. The problem of definition is essentially at the core of the Fregean view on logic, and in its ultimate nature contains many of the implications of the entire Fregean system; moreover, it represents the key to understand how the question of content is developed in the context of analyticity.

The revolutionary treatment of content in the context of logical calculus is performed by Frege in his *Begriffschrift*: the conceptual language has its foundation precisely in the determination of the relation between the content of concepts, and therefore logic cannot possibly be considered only as a formal structure completely devoid of content, as later in the Hilbertian style. This step is accomplished by rejecting the classical propositional structure (the copula structure) and improving the Bolzanian step towards an exact understanding of the predicative relation. Notoriously, the Fregean move consists in reading the predicative structure as a functional schema, built by the couple *function + argument*, or else in terms of the *saturated/unsaturated* metaphor. An expression is therefore always a function (concept) filled with an argument (object), and having a (course of) truth-value(s) as its reference. By means of this structure it is possible to distinguish the relation between a concept and an object (subsumption of the latter under the former) from the relation between two concepts of different degrees (subordination of the lower within the higher).[67] Starting from this explanation of the predicative structure, Frege develops his entire theory of meaning and reference for singular terms (having objects as reference), indicative sentences (having truth-values as reference) and predicative expressions (with concepts as reference). According to this schema, an expression is meaningful if it satisfies the double requirement of expressing

[64] Ibid. (par. 3):

So hat man allgemein die Frage, wie wir zu dem Inhalte eines Urteils kommen, von der zu trennen, woher wir die Berechtigung für unsere Bahauptungen nehmen. Jene Unterscheidungen von apriori und aposteriori, synthetisch und analytisch betreffen nun nach meiner Auffassung nicht den Inhalt des Urteils, sondern die Berechtigung zur Urteilsfällung. Da, wo diese fehlt, fällt auch die Moglichkeit jener Eintheilung weg.

[65] Ibid. (par. 3).

[66] Ibid. (par. 3).

[67] Proust (1989, p. 117).

a thought and having a truth-value. The extensional reference of concepts is thus properly distinguished by its intensional value: on this basis Frege gives a clarification of what it really means for two concepts to be identical, and still to express different senses. Thus the notion of meaning arises in connection to the question of identity, and in turn of definition. The development of this concept within Frege's philosophy of logic is extremely difficult and it is solved properly only by the late *Über Sinn und Bedeutung*. A definition amounts:

- Either to a stipulation of identity between senses of a simple expression and a complex one: in this case it corresponds to a tautology, empty of content
- Or it is the result of constructing a concept by combining known elements.

This last distinction can be clearly connected to the analytic/synthetic topic. The main point in definitions is that they cannot correspond to simple equivalences, rather they have to develop a proper identity. A correct understanding of the notion of definition is by Frege developed in the time between the *Begriffschrift* and *Über Sinn und Bedeutung*: in the former work he maintains that identity is extrinsic, i.e. it is a relation between signs of objects and affects expressions rather than the thought; notoriously, the later work from 1892 opens with exactly the same problem, but suggesting a quite different view. In the theory of meaning and reference, signs have acquired a relation to the contents they denote, and identity is no longer a question of pure form. Real identity holds among those objects designated by the names staying on each side of the identity sign; the notion of meaning let instead to formulate identity between expressions, and this of course amounts to recognize synonymity. On the basis of this distinction, a new ad more interesting understanding of synthetic knowledge processes (and in turn of analyticity) may be provided[68]: thus, the extensional/intensional distinction is at the core of a correct understanding of the notion of identity as related to content.

With the introduction of the notion of meaning, Frege defines in a better way the idea itself of content identity: what does it mean for two meanings to be the same? By means of the distinction between *proposition in themselves* and *statements*, Bolzano was able to clarify the difference to be drawn between identity and synonymity: it is the identical content of propositions in themselves that synonymous expressions have in common. The notion of proposition in itself is replaced in Frege by the notion of *thought*, so that identity applies to thoughts as what *Sinne* refer to, namely their contents. Frege can thus reconsider the use of definition, so that analytic truths of the form "a = a" are analysed in relation to the grasp of

[68] The connection between analyticity of statements and the concept of synonymity is actually to be considered matter of a theory of meaning. For this connection see also the starting point of the analysis in Quine (1951).

truths of the form "a = b".[69] It is now essential to recognize that even analyticity of logical laws refers to the grasp of meanings they convey, those by which they are expressed. The identity expressed by a definition will be then both the intensional correspondence of senses, and the extensional one, amounting to substitutability *salva veritate*.[70] Once the notion of meaning has been introduced, it is clear that the next step is to consider logical truths as those persisting under any substitution: i.e. it can refer to logical terms (formulation *ex vi terminorum*), or to the substitution of non-logical terms having the same meaning, which of course let to obtain only truthful substitutions. The Fregean notion of analyticity (along the lines of the interpretation presented in Hintikka (1973)) now amounts to:

Definition 2.6 (Frege's Analyticity II) *A proposition is analytic if it is true* ex vi terminorum*, i.e. either on the basis of the meanings of its logical terms (logical truth) or on that of its non-logical terms (definition).*

In this definition the role of logical parts is completed by the content of expressions accounted in terms of meanings. These are contents determined by acts of definition and thus analyticity refers essentially to propositions.

Later, the Fregean notion of analyticity is reconsidered by Ayer (1936), who takes into account the relevant problems of the new stream in philosophy, inspired to the Fregean–Russellian turn, the origin of the later called "analytic philosophy". Among those problems, a relevant part is, obviously, devoted to the nature of philosophical analysis. Ayer tries to establish the connection between the notion of philosophical analysis and the method of definition, by considering if such analysis is entirely based on a collection of *explicit definitions*. By explicit definition one understands a substitution of a symbol by another one *having the same meaning*, i.e. again by *synonymity*, where of course this term reflects the equivalence of sentences in which symbols significantly occurring have been substituted (*salva veritate*). It seems clear that this topic reformulates the problem already considered in the Fregean analysis: equivalence is explained by saying that two sentences A and B, within the same language L, are equivalent if, and only if, given a certain other sentence C, for every set of arbitrary sentences $D_1, \ldots, D_n$, it is true that:

[69] Frege (1892, pp. 23–24):

> *Wenn sich das Zeichen "a" von dem Zeichen "b" nur als Gegenstand (hier durch die Gestalt) unterscheidet, nicht als Zeichen; das soll heißen: nicht in der Weise, wie es etwas bezeichnet: so würde der Erkenntniswert von a = a wesentlich gleich dem von a = b sein, falls a = b wahr ist. Eine Verschiedenheit kann nur dadurch zustande kommen, daßder Unterschied des Zeichens einem Unterschiede in der Art des Gegebenseins des Bezeichneten entspricht.*

[70] The Leibnizean principle *"Eadem sunt quorum unum potest substitui alteri salva veritate"* is clearly (either implicitly or explicitly) assumed all along the Fregean work. Cf. e.g. (1879, par. 8; 1884, par. 65; 1892, p. 35). For a treatment of the topic of identity and interchangeability both in Leibniz and Frege, cf. e.g. Angelelli (1967).

$D_1, \ldots, D_n \cup A$ entails C

$D_1, \ldots, D_n \cup B$ entails C[71]

Once again, the question of synonymity is linked to the notion of derivability, via a version of the incompatibility theory. To assert identity means therefore to state equality for two entailments, and this simply establishes the correspondence of an analytic statement (definition) with a logical truth, in terms of the substitution of the *definiens* by the *definiendum*. This provides a new connection between analytic statements and logical truths:

Definition 2.7 (Analyticity and Logical Truth I) *A true proposition is analytic if and only if it may be converted into a logically true one, by replacing its syntactically simple predicates by predicates which mean exactly the same thing.*

This identity between an analytic proposition and a logically true one explains why it was possible in the previous Fregean formulation to take into account both the logical and non-logical terms when formulating the notion of analyticity as truth *ex vi terminorum*. According to Ayer, the method of philosophy is not to be understood as working by means of explicit definitions, rather it works in terms of what he calls *definitions in use*, referring in this way directly to the Russellian theory of definite descriptions.[72] The philosophical explanation of a language is given by a clarification of the types of sentences occurring in it, and by an equivalence relation between them.[73] Among the types of sentences, stating the connection between analytic statements and logical truths, Ayer refers to the former as devoid of factual content, they say nothing.[74] On the other hand, logical truths are necessarily and universally true, but certainly no general statement which is filled with content can ever be neither necessary nor universal. This amounts to saying that if truths of logic are to be retained as both necessary and universal, they cannot show any content; therefore, it is impossible for them to be true, useful, or even surprising. This of course means to bring forth the Fregean explanation of truth in virtue of meaning, by intending as analytic only those expressions which are empty of content, so to be true under any substitution of equal terms.

[71] Ayer (1936, p. 60).
[72] Russell and Whitehead (1910, introduction ch. 3); Russell (1919, ch. 16). According to this theory, as known, every sentence containing a symbolic expression of the form "the so-and-so" can be translated into a sentence containing a subsentence asserting that one (and only one) object has a certain property, or else that no object has that property. It is interesting at this point to underline the conceptual relation between the explanation of definite description and the constructive approach to definition via computation and definitional chain. For more on this point cf. Section 1.4.
[73] Ayer (1936, p. 62).
[74] Ibid. (p. 79).

This interpretation of analyticity leads back to the purely formal aspect of logic, and can be found clearly in the Wittgensteinean *Tractatus*, with the obvious consequence that logical truths mean all exactly the same thing, by having only a unique truth-value. The process of interpreting analyticity as emptiness of meaning and therefore to complete uninformativeness was at this point completed.

The question on the definition of analytic expressions and the analyticity of logical relations has been thus explored to determine its essential nature in the context of a knowledge system, in particular in the process of increasing knowledge via logical derivability: it can be summarized as the connection between *truth* (assured by logical derivation), *content* (determining the meaningfulness of the act of knowledge), and *surprise* (in terms of the novelty provided by a certain conclusion).

2.2.3 *Analyticity Reconsidered: From Meaning to Information*

Ayer analyses the problem of analyticity somehow extending the Fregean explanation, essentially heading towards the description already expressed by Wittgenstein about the *Sätze der Logik* as purely analytic propositions. In Ayer's formulation analytic propositions provide no information about empirical situations, rather they are interesting in the way they illustrate the use of logical symbols[75]: the meaning of analytic propositions is to be found therefore in the convention governing our use of the logical terms contained in them. In particular, no new (contentual) information is furnished by such propositions and, accordingly, analytic derivations just handle propositions already contained in each other. Let us consider for example a figure of inference where both premises and conclusion are universals, like the following one:

All Bretons are Frenchmen
All Frenchmen are Europeans
All Bretons are Europeans[76]

Its correctness is established relatively to our use of the terms "if/then" and "all". According to the view here supported by Ayer, analyticity is to be interpreted as the *lack of informativeness* which distinguishes analytic from synthetic propositions. The core of the *Analyticity Principle* can be formulated precisely in the connection between logical form and informativeness:

Definition 2.8 (Analyticity and Information I) *A proposition p is analytic if it does not furnish any (contentual) information; an analytic derivation of p is such that the conclusion (p) is analytic in respect to the premises.*

[75] Ibid. (p. 79).
[76] Ibid. (p. 79).

The notion of *information* used here is to be explained in terms of contentual (empirical) meanings furnished by the non-logical parts. The connection between informativeness and synthetical propositions is, at this point, of the greatest importance, in order to analyse the supposed analytic nature of logical derivations.

Another way to formulate the role of analytically derived propositions is to say that, even though they will not furnish any new information regarding matters of fact, they enable the expression of all the information already contained in the premises in a *complete* way. The completeness of such a list is a way to gain *self-consistency* of a system of propositions: analytic methods will let derive everything contained in what we already know. In this sense a logical system, a logical derivation, or an inference, can be thought of as a way to discover and certify analytic propositions, whose relation is to be deducible one from another.[77] Nevertheless, this point of view is not in principle essential to describe analytic propositions, because the possibility itself of such a deduction is based on another property which can be considered primary for analytic proposition: each analytic proposition is self-evident in virtue of its form. This leads to the extension of the previous definition of analytic proposition based on the notion of logical truth, in terms of the logical form:

Definition 2.9 (Analyticity and Logical Truth II) *A proposition p is analytic if and only if every proposition having the same (logical) form as p is logically true.*

Analyticity is thus dependent only on the logical form of the expressions, and this is the reason that makes them not informative at all: this understanding of analytic propositions is related to their role in the context of a priori knowledge. The relation holding between logical truths, known a priori, and analytic ones is therefore here central. An a priori truth can be determined as independent from any real perception and any experience[78]; it has to be provided in terms of clearness and certainty, shown respectively by necessity and generality[79]:

A priori knowledge

- *Independent from perception and experience*
- *Clearness* ($\rightarrow$ *necessity*)
- *Certainty* ($\rightarrow$ *generality*)

Therefore, on the one hand, there is an inclusion among a priori knowledge and analytic one, because the former identifies all the knowledge which can be acquired by analytic propositions (tautologies). On the other hand, by

[77] Ayer (1936, p. 81).
[78] Kant (KrV, A 42/B 3).
[79] Ibid. (A 2; B 4).

the definition of analytic propositions in terms of their logical form, logical laws are also analytic. The combination of both these properties, makes an analytic truth such that it results from conceptual analysis, and only based on the definition of the concepts involved.[80]

The problematic step, at this point, is represented by holding that the notion of logical consequence preserves analyticity in the same sense that analytic truths are based only on logical terms. Hintikka has already underlined[81] that the identity or at least equivalence of these interpretations seems far from clear. It seems that logical derivations have actually lost their content. Analytic propositions are reduced to sentences whose truth is based only on their pure form and the foundational distinction between propositional content and judgements is forgotten, precisely because the notion of content has lost its relevance.

2.2.4 *Rejecting the Analytic/Synthetic Distinction: Quine*

Notoriously, some decades after this explicit connection with the idea of content as "information" due to Ayer, an attack against the validity of the notion of analyticity itself was raised by Quine (1951), suggesting to abolish the distinction analytic/synthetic, in particular for the concept of truth, on the basis of the unclear concepts referred to when speaking about analyticity. According to Quine we fail to have an empiricist clarification of the distinction analytic/synthetic, the only one which would not fall into a circular explanation: to believe in such a distinction is therefore just a "*metaphysical article of faith*". Quine explains exactly the two different senses of analytic proposition presented by the definitions above[82]:

– Either it is intended as a tautological assertion, true in virtue of the concepts involved (e.g. "No unmarried man is married"), and true under all reinterpretations of its concepts other than the logical ones (namely here "man" and "unmarried").
– Or, it is analyticity determined by the proposition being a logical truth, or it can be transformed into a logical truth by substitution of terms by means of synonymous terms ("No bachelor is married", which in fact can be turned into the first case by substitution of the terms "bachelor" by its synonymous "unmarried man", or else its definiendum), which recalls obviously the Fregean interpretation.[83]

Once again, the relation synonymity-definition is considered in the first instance by Quine as the core of the problem of analyticity.

[80] This (essentially Kantian) interpretation has been reconsidered by Hintikka (1973, pp. 126–127).
[81] Hintikka (1973).
[82] Cf. Burge (1992, pp. 4–10); Boghossian (1997, pp. 335–337); Pagin (2001, p. 7).
[83] Boghossian (1997) in fact calls it *Frege-analyticity*.

The Quinean distinction relies on the explanation of analyticity previously given by Carnap, which develops the central distinction between content and logical form. The idea at the basis of the Carnapian explanation of the notion of analyticity finds its origin in his *Der logische Aufbau der Welt*,[84] namely in the notion of structure: according to Carnap, it is possible, at least in principle, to translate statements about empiric content into structural statements, in which only the logical meaning plays a role. The description of these relations is clearly prior to the terms explained by the relations themselves: this means that no information is obtained by the content, other than what can already be obtained by its logical description. This analytic treatment of concepts is only possible under a clarification of the pertinent domain, i.e. under a semantic interpretation, by which it is possible to establish the net of logical relations defining the concepts. The development of this theory is contained in the second edition of *Meaning and Necessity*[85] in terms of the notion of *state-descriptions*, whose introduction in 1950 is the major innovation of the text dated 1946.[86] The relation of explication between statements is to be understood in terms of analyticity (i.e. an explicatum is similar to the term "*is analytic*"[87]) if it applies to all and only the sentences that are true in all possible state-descriptions, or independently of facts, where the holding in every state-description is defined by semantic rules of a language system. Shortly, Carnap defines logical truth as truth in all state-descriptions, or equivalently Leibnizean possible worlds (possible states of affairs). It is particularly relevant that the problem of analyticity is linked by Carnap to a theory of meaning, namely via the role analyticity plays in explication procedures. This builds also the relation to the content determining the meaning of a designator (meaningful expression) in terms of its intension, or the intensional entity assigned to it.

Quine identifies the root of the Carnapian theory of meaning with the analytic/synthetic distinction, and underlines that the solution proposed by Carnap of the notion of analyticity does not give any account of the notion of synonymity, used as *explanandum*. Quine considers first of all the relation *definiens-definiendum*, as clarification of the relation of synonymity, according to the following threefold explanation[88]:

(a) "*The definiens may be a faithful paraphrase of the definiendum into the narrower notation, preserving a direct synonymy as of antecedent*

[84] Carnap (1928).
[85] Ibid. (1956).
[86] Carnap (1950, p. 71), defines a *state-description* as follows:

> *A state-description for a system L [...] must state for every individual of L and for every property designated by a primitive predicate of L whether or not this individual has this property; and analogously for relations.*

[87] Cf. also Butrick (1970, pp. 22–29).
[88] Quine (1951, sec. II. *The Definition*).

usage": in this sense, definition goes back to instances of synonymity grounded on linguistic usage;

(b) "*The definiens may, in the spirit of explication, improve upon the antecedent usage of the definiendum*": definition is here expressed (following Carnap) by the term "explication", it is based on the improvement upon the definiendum obtained by supplementing its meaning; synonymity is intended in terms of contexts, and it is required that the context of the definiendum "*taken as a whole in its antecedent usage, be synonymous with the corresponding context of the definiens;*"[89]

(c) "*The definiendum may be a newly created notation, newly endowed with meaning here and now*": this is a case of synonymity by definition; it aims towards the equality of signs more than to a real explication of the concepts involved, therefore it appears often in the context of formalized languages.

This analysis suggests that the relation of synonymity is prior to that of definition: more precisely, the latter is based on the former, therefore it no longer represents a key to explain analyticity. This is obtained by maintaining synonymity primitive, and by introducing a concept of interchangeability relatively to *extensional* languages, i.e. languages in which two predicates being true of the same objects are interchangeable *salva veritate*.[90] This concept is still not enough to derive a proper concept of analyticity, being impossible to distinguish "synonymity by meaning alone" from "synonymity in matters of fact", under permutation of terms maintaining the truth of sentences. The notion of analyticity is still conceptually prior.

Analyticity treated within a formal language, in which a certain set of *semantic rules* has been established, makes the concept itself an irreducible character of this model. The analyticity of a proposition being equivalent to truth according to certain semantic rules[91] does not give any hint for the explanation of the notion, and therefore no conceptual separation has been yet formulated between analytic and synthetic proposition. The distinction itself turns out to be that article of unempirical faith referred to at the beginning. The problem can be reformulated in terms of finding the proper relation between what is defined as analytic and a theory of meaning: in the verificationist theory of meaning, an analytic statement is one being true no matter what the case is. This has as a consequence the impossibility of distinguishing between statements containing a factual component (emprirical) and those being recognizable as true only on the basis of the logical structure (i.e. the analytic ones). Thus, the notion of analyticity here rejected is that of a proposition "*true come what may*".[92] A few years later, Quine suggests to interpret the term "analytic" as a definition of the

[89] Ibid.
[90] Quine (1951, sec. III. *Interchangeability*).
[91] Ibid. (sec. IV. *Semantic Rules*).
[92] Ibid. (p. 43).

term “synonymous”,[93] but in this sense synonymous is referred to meaning, where matters of fact are considered as “collateral information”, not needed in order to establish the truth of the sentence:

Definition 2.10 (Quine’s Analyticity) *An analytic sentence is true only according to its meaning and independently from other external informations, so that it is actually analytic a logically true sentence, i.e. a sentence which involves only logical words. In this sense, two sentences are synonymous only if the formula built up from a bi-conditional containing the two sentences is analytic (where analytic means a conditional built up with the same sentence as antecedent and consequent, as in “if p then p”).*

It clearly appears by these passages, that whenever the notion of information is taken into account, it is mostly used to refer to matters of fact, distinguishing it from logical form or logical validity by virtue of form alone. The contrast between logical form and factual content is striking, and the role of the notion of information is switching from the former to the latter, but no clear account of it has been yet given. Even in more recent accounts of the problem, like in Boghossian (1997), a resolution of the problem of analyticity is given in terms of meaning and definition. Two concepts of analyticity are distinguished, by presupposing the underlying distinction between content and form. The first explication is that of *Metaphysical Analyticity*:

Definition 2.11 (Metaphysical Analyticity) *A statement S is metaphysically analytic iff S is true solely by virtue of its meaning.*

This definition refers obviously to those statements representing laws of logic, evident, and true by virtue of their meaning, in which content plays no role at all. On the other hand, statements involving our grasp of the meaning (therefore in terms of their content), refer to a different concept, namely *Epistemic Analyticity*:

Definition 2.12 (Epistemic Analyticity) *A statement S is epistemically analytic if, and only if, grasp of the meaning of S suffices for justified belief in the truth of S.*

This second category contains, according to Boghossian, an internal distinction:

- **Frege analyticity**: A statement analytic in this sense is transformable into a logical truth by substitution of synonyms for synonyms.[94]

[93] Quine (1960, cap. 1, par. 14).
[94] Cf. Boghossian (1997, p. 337).

- **Carnap analyticity**: A statement analytic in this sense is part of an implicit definition of certain of its component terms.[95]

The description provided by Boghossian is once again useful in recognizing a major distinction between the meaning of a sentence and the grasp of that meaning. Obviously, in this sense the metaphysical notion is still acceptable only if one does not rely entirely on the act/content distinction for any case of expression of a proposition. It seems evident at this point that a major role in the definition of analytic statements/propositions is due not only to the structure itself of the expression, but also to the act of understanding needed for grasping the meaning of that content: this is the result of stressing the foundational role of the act/object distinction. To pursue forth this topic means to furnish in turn a tentative constructive definition of analyticity in relation to a clarification of the notion of information, and possibly to understand the difference between analytic statements and analytic forms of inference.

2.2.5 *Towards a Constructive Notion of Analyticity*

The progress among the different definitions of analytic truth has been accomplished following the progressive shifting of the relevant roles of *form* and *content* (and the related *act* and *object*) for logical truth. Let us summarize such development in the following list:

- *Analytic truth =*
- *Truth in virtue of the relation of containment (Kantian analyticity) =*
- *True in virtue of the pure form =*
- *True come what may: true in every possible world, independently of what is the case, in every description of the state of affairs (from the Bolzanian analyticity to the Wittgensteinean notion of tautology) =*
- *Conceptual truth =*
- *Necessary truth =*
- *A priori truth =*
- *Proposition impossible to be false =*
- *Proposition whose negation is contradictory =*
- *Logical truth, also obtainable by means of substitution of synonyms by synonyms (Frege analyticity) =*
- *Resulting in a logical truth on the basis of a definitional chain (Carnap analyticity/Quine analyticity)*

The conceptual problems undergoing the understanding of the notion of analyticity are, in the first instance, due to an unclear use of the terms *proposition*, *statement*, and *judgement*, occurring sometimes with the same meaning. The different perspective provided by the constructive approach suggests a clearer understanding of those terms: it refers to a judgement

[95] Ibid. (p. 339).

as the act by which the truth of a proposition is asserted. According to this interpretation of the judgemental act, to be true corresponds to be knowable, and this of course allows for a reformulation of the entire notion of analyticity in the light of an epistemological account of truth. The epistemic notion of judgement is intertwined with that of knowledge in a double sense: the judgemental act consists in providing the elements for grasping the truth of a certain propositional content, but its explanation requires moreover the formulation of the elements needed *to be known* in order the judgement *to be made*, namely its *assertion conditions*.[96] The link between what one needs to know and the assertability of a judgement appears clearly. The role of these conditions in the formulation of justifications is extremely important: the analyticity of a proposition can be understood either in epistemic or metaphysical terms, provided the assumption that its truth is independent from our justified way of asserting it. Once, instead, the truth of a proposition is defined by the justified assertability of the judgement ascribing truth to it (i.e. its provability), it is clear that *"the grasp of the meaning"* of such a proposition explains entirely its truth, by relying on the act of assertion and its conditions. According to this description, within the constructive framework it is therefore essential to identify two elements involved in the epistemic schema, each concurring in explaining the notion of analyticity:

1. Explicitly proved judgemental knowledge
2. (Previosly acquired) implicitly proved knowledge

With this terminology one expresses an essential basic epistemic distinction for the constructive frame, namely one of the greatest importance. Every process of knowledge can be described as the acquisition of some true propositional content, by means of the formulation of the demonstration which makes it known: this process is explicit and relies on proofs. The part of this process which expresses the conditions of the explicit acquisition of knowledge, consists instead in the use of some other knowledge previously acquired, for which therefore basic proof-conditions are only implicitly satisfied. It is precisely in the light of this main distinction that a first conceptual clarification of a knowledge system will be presented later in this chapter, and then in the next one exposed in the formalized setting of Constructive Type Theory. According to this schema, an explanation will be provided of the last identity which completes the previous list, namely analyticity as a *lack of informativeness*. It is now precisely the complete *uninformativeness* which must be taken as a main point in our discussion about analyticity.

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[96] We present here the theoretical frame presupposed to an explicit formal treatment of the notion of assertion condition within CTT, to be presented in Chapter 3.

2.3 Informativeness of Derivations

At the end of the previous section, the problem at the root of all the different accounts of analyticity was introduced: the lack of a proper clarification of the concepts involved in this analysis, in particular the missing distinction between the concept of proposition and its truth on the one hand (intended as non-epistemic notions) and that of judgement and its correctness or derivability on the other (epistemic notions). I have just outlined this basic distinction within the constructive approach.[97] Once the distinction between the notions of judgement and proposition is made clear, a step further can be made in the clarification of the concept of analyticity, in particular as extended to inferences.

The laws of logic are obviously laws of truth, and on their basis it is possible to formulate the traditional concept of inference: provided certain true propositions, the antecedents, another proposition will be true, the conclusion. According to this explanation, if the laws of logic are purely analytic propositions, the relation of logical consequence holds between propositions involved in inferential processes preserving the property of being logically valid for these propositions: in this sense the validity of inferences is also based on preservation of analyticity. This point was at the core of the clear and provoking formulation of the problem of analyticity in terms of the so-called *Paradox of Inference*:

> *If in an inference the conclusion is not contained in the premises, it cannot be valid; and if the conclusion is not different from the premises, it is useless, but the conclusion cannot be contained in the premises and also possesses novelty; hence inferences cannot be both valid and useful.*[98]

This paradox explains the nature of the conclusion as a proposition that provides new knowledge in respect to its premises, and at the same time it is already contained in them, in other words a problem concerning the informativeness of logical inferences. Let us summarize such a relation:

- *Validity*: this property expresses the containment of the conclusion within the meaning-range of the premises.
- *Utility*: this property amounts to a certain novelty (to be thought of as “surprise”, “information”) that the deduced sentence should furnish compared to the knowledge contained in the premises.

Analytical containment of the conclusion into the premises (explanation of the concept of *validity*) and extension of knowledge (which represents the idea of *utility*) constitute the two elements of a contradictory relation, impossible (apparently) to be solved. The *Paradox of Inference* has introduced

[97] Cf. also Martin-Löf (1987).
[98] Cohen and Nagel (1934, p. 173). The tension between the validity and utility of inference was noted already by John Stuart Mill (1843).

these opposing elements that will now be constantly taken into account in our treatment of analyticity. The connection between the analyticity of propositions and that of derivations is obtained simply by recovering the traditional notion of analysis:

> *An argument step is analytic if (and only if) the conclusion is obtained by merely analysing what the premises give us.*[99]

If one takes into account the distinction between propositions and judgements introduced by the constructive account, obviously validity of inferences is to be understood differently than in our first analysis of the *Paradox*. By switching to the epistemic approach here suggested, one must consider the transition from the notion of *truth* (of propositions) to that of *knowability* (of judgements). Necessarily, one has now to spell out the idea of knowledge required by an analytic judgement and consequently by a step of inference between judgements. Analyticity for a judgement stating the truth of a proposition of the form "*S is P*", corresponds to knowing the judgement just by understanding the concepts contained in it, recognizing that P is already contained in the concept represented by S: i.e. by means of the conceptual resources provided by the judgement, one can grasp or understand it. In the case of the notion of inference as built up by judgements, its validity corresponds to being able to know the conclusion represented by the judgement J, once one knows the premises $J_1, ..., J_k$. Thus, it is according to this concept of knowledge that the related *principle of validity* is reformulated in its epistemic version:

Principle 2.13 (Principle of Validity) *In a valid inference, the knowability of the premises is transmitted to the knowability of the conclusion.*[100]

Considered in this way, the notion of validity for inferences (and in turn the principle of knowability) does not coincide with the containment of the conclusion in the range of the premises. This is extremely important, because it means that validity does not amount to (and does not conflate with) analyticity. Also the **Principle of Utility** for inferences must be reformulated according to this new explanation:

Principle 2.14 (Principle of Utility) *Knowledge of judgements representing the premises of an inference must contain information that leads to gain the new knowledge contained in the judgement representing the conclusion.*[101]

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[99] Hintikka (1973, p. 124).

[100] Martin-Löf refers to this, in a slight different way, as the **First Law of Knowability** (1995, p. 193).

[101] In particular, no absurdity can be known from known premises, therefore every judgement which is a premise of a valid inference furnishes some new knowledge to the agent. Martin-Löf (1995, p. 194) considers absolute consistency as the **Second Law of Knowability**: absurdity cannot be known to be true; the **Third Law** says that the unknowability of truth entails knowability of falsity.

By this second property, which defines valid inferences, it is clearly possible to reconsider the problem of analyticity to avoid the contradiction implicit in the *Paradox of Inference*. The argument sketched here must now be developed, to clarify the role of inferences in the (apparently paradoxical) relation between utility and validity: the key concept left to explain is that of *information*.

2.3.1 *Individuals and Degrees: Computing Information of Sentences*

The introduction of the notion of information in logic, especially in relation to the problem of analyticity, is mainly due to the work of the Finnish logician Jaakko Hintikka. I will partially follow his description to recognize how the introduction of the concept of information is linked to the problem of analyticity, to recognize which difficulties arise in his frame, and what else remains unclear.

Our analysis has identified two properties — *validity* and *utility* — defining *per contradictionem* analyticity for inferences. This explanation was given on the basis of the objects referred to when speaking about inference, truth and knowability: namely *judgements* rather than *propositions*. According to the constructive perspective, a possible understanding of analyticity is therefore grounded essentially on the *act of judging*, on its *object* and on the *product* of such an act. In its first understanding, as reported previously in our list, an analytic truth presents a certain number of concepts and their relations (*conceptual truth*): an analytic truth is such because it is established by the sole means of conceptual analysis, so that "analytic" and "conceptual" are essentially identical meanings.[102] Hintikka explicitly avoids considering which linguistic and conceptual elements are the bearers of truth, and consequently he dismisses the distinction between act and content.[103] It seems quite clear at this point that, on the contrary, the problem of analyticity appears clearer once the distinction between propositions and judgements is settled, and the latter are understood as the proper objects of knowledge acts. Actually, it is necessary to suggest a variation in the

[102] This first sense of the notion is called *Analyticity I* in Hintikka (1973). He develops this first explanation according to the following schema:

- Ia. analytic truth as definitional truth
- Ib. analytic truth as logically implied by the definitions of the terms it contains
- Ic. analytic truth as justified by logical laws and definitions (Frege's (1884) definition)
- Id. analytic truth as including all logical truths plus whatever truths one can obtain from them by replacing synonyms by synonyms (Quine's definition)

[103] Hintikka (1973, p. 125):

> *A premise is a sentence (proposition, judgement, etc. — the differences which there may be between these and other similar entities need not to detain us here) [...].*

first analysis of the concept of analyticity. Analytic truth needs to be reformulated by referring to the knowledge of a judgement assigning truth to a proposition on the basis of the concepts involved in it, and their relations. In this sense, it is not the truth of the proposition to be analytic/conceptual, rather our way of knowing it as true. Thus, the explanation of the first meaning of analyticity changes as follows[104]:

Definition 2.15 (Epistemic Analyticity I) *A judgement is analytically made if, and only if, the truth of the proposition asserted by that judgement is established (known) solely by means of the concepts involved, and of their relations.*

According to this first (revised) analysis, one has to consider the analytic value of establishing the truth of a proposition. The judgement A *true* will be known, i.e. it will be a justified assertion, whenever one knows that a proof of A exists (and how to formulate it). The knowledge of the judgement stating that such a demonstration exists is analytic knowledge and it gives rise to the synthetic judgement stating that the proposition A is true (A *true*). Furthermore, the relations between concepts are established by definition, so that it is simple to extend the notion of conceptual truth in terms of the definitions of those concepts (*analytic truth* = *conceptual truth* = *definitional truth*). Conceptual analysis amounts in effect to a definition of the concepts involved: but what a definition amounts to? On the one hand, a definition can be considered[105] as a chain of nominal definitions stating identity between expressions or synonymity (material supposition), by means of which it can also establish the identity between canonical and non-canonical objects: as already presented in Section 1.4, the relation between syntax and semantics, i.e. the relation between objects and their expressions (computation) is explained at this level, its extension being the relation of identity in terms of evaluation. On the other hand, the notion of definition can be intended as a *real definition*, an explanation of concepts which properly amounts to a conceptual analysis: it is therefore in this second sense that definition is involved by the problem of analyticity, by being analytic whatever follows logically by definitions of the concepts involved.[106] This view refers to the role that connectives and quantifiers play in the determination of the logical form of expressions and to the notion of synonymity.[107]

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[104] By numbering it by the Roman numeral we intend to distinguish it from the version presented by Boghossian, and referred to in Section 2.2.4.

[105] Cf. Section 1.3.2.

[106] Hintikka's sense *Ib.* of analyticity.

[107] In this sense, there are two approaches to the problem of analyticity: the idea by Hintikka that a solution must be given in terms of the explanation of the notion of informative content in terms of individuals; on the other hand, Dummett refers to rules relative to connectives and quantifiers, in order to establish the form and content of expressions. Both consider the "history" of a derivation, its logical steps

The constructive approach reformulates on this basis the notion of logical consequence. Under the classical approach, considering analytic whatever is logically implied by definitions of concepts means that the process of deriving a consequence must be itself analytic.[108] It follows immediately the assumption that the class of logical truths and of the propositions obtainable from them by substitution of synonyms by synonyms, i.e. of *definiens* by *definiendum*, is the class of analytic truths.[109] What should be reconsidered here is the supposed analyticity of logical derivations and, in turn, the analyticity of logical truths.[110] On the basis of the Kantian understanding of the concept of analytic and synthetic judgements, within the type theoretical setting the forms of categorical judgements $a : \alpha$ and $a = b : \alpha$ are analytic expressions, because such judgements are true solely on the basis of the terms involved, i.e. the object a is explicitly contained in the category of predication α. But a logical consequence from one of these judgements, for example, the judgement $a : \alpha$, is the knowledge of the *existence* of such an object a of the type α. This leads to the new judgement α *exists*, the knowledge of an object falling under the type involved being its semantic explanation. This can be formulated by the logical rule:

$$\frac{a : \alpha}{\alpha \quad exists} \tag{2.1}$$

which reflects the logical consequence from the judgement *A prop* to the judgement *A true* by means of the judgement *proof(A) exists*. The justification of the judgement stating the existence of the concept/type α is in terms of the construction of an object falling within such a concept, which makes that judgement evident: to give existence to the concept means to go out of the analysis of the concept itself, therefore this judgement is synthetic. This means as well that the process of abstraction considered in Section 1.9,

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in terms of the information acquired during its process. This represents a strong constructive principle, here reformulated for the formal language of CTT in order to develop the related notion of informativeness and to present the question of analyticity in new terms. The idea of having at disposal every step previously accomplished, to analyse the information acquired or lost, is justified only in a framework which take care to "register" every rule applied, and therefore the form of every sentence involved in terms of the construction justifying it, and this is obviously possible only when the definition of the logical operators is intended in a constructive way. The already mentioned *Forget-Restore Principle* by Valentini-Sambin, to be reconsidered again in this section, shows precisely this possibility of recovering information within CTT.

[108] Hintikka (1973, p. 127):

> *relations of logical consequence undoubtedly seem to be analytic in the unanalysed sense […], they seem to be based solely on the meanings of logical terms.*

[109] This corresponds obviously to a composition of the Fregean, Carnapian, and Quinean definitions of analyticity.

[110] For the following argument, cf. Martin-Löf (1994).

the so-called procedure of forgetting information by going from the judgement expressing a construction to be in a certain type to the judgement stating the truth of that type (or at a higher level the judgement stating that specific object to be a type), provides a switch from an analytic to a synthetic form of judgement. Logical truths in this sense are no longer to be considered analytic, and derivations are built from analytic judgements *synthesized* by means of a (new) construction:

$$\frac{a : \alpha \quad b : \beta}{c(a, b) : \Delta(\alpha, \beta).} \tag{2.2}$$

The traditional view on the validity of logical inferences introduces instead a quite different definition of analyticity, namely in terms of analysis by means of subsentences: an argument step whose conclusion is a subsentence of one of the premises is analytic.[111] It is here suggested the different view according to which one distinguishes analytic from synthetic judgements, intending the latter as existential judgements instantiated by means of constructions. Such a view can be introduced by the following definition of analyticity[112]:

Definition 2.16 (Epistemic Analyticity II) *A judgement is analytic if and only if the concept involved is expressed in terms of an individual instance (conceptual analysis). A judgement stating the truth of a concept involves its existence: this is a synthetic judgement, justified by the construction of such an instance. It follows that an argument step presenting in its conclusion the construction of the elements involved in the premises carries a synthetic value.*

In natural deduction, for example, it is simple to present a derivation which is synthetic in this sense, i.e. where the conclusion is clearly synthesized from the premises ($\&I(A, B)$): logical truths can be described by this kind of derivations.[113]

This relation between analytic and synthetic judgements is also clearly connected to a typically Kantian distinction. As already pointed out in Section 2.1, Kant holds both in the *Kritik der reinen Vernunft* and in the *Prolegomena*, that mathematics cannot be considered simply a process of

[111] Hintikka (1973, p. 134) refers to this as *Analyticity II*:

IIa. A proof is analytic if and only if all its steps are analytic in sense II.
IIb. A proof of F_2 from F_1 is analytic if and only if all the intermediate stages of it are sentences each of which is a subsentence of either F_1 or F_2.

[112] The connection between constructions and synthetic judgements will be reconsidered in Chapter 4.
[113] Martin-Löf (1992) shows this property for a law of propositional and predicate calculus: $A \supset (B \supset A\&B)$.

logical analytic deduction, because as a science it cannot be reduced to tautologies: it is instead the most clear example of the existence of a priori synthetic judgements. The synthetic nature of mathematical arguments is thus for Kant based on the use of constructions.[114] Constructions are to be understood as the exhibition a priori of an intuition of a general concept, an individual introduced in relation to a general concept of which it can be predicated.[115] A construction allows to consider a concept *in concreto*, namely to treat it in one of its instances, and in the type theoretical setting the explanation of the concept (α) is given precisely in terms of the element instantiating it ($a : \alpha$). In this sense, an inference between judgements of the form

$$\frac{A \quad true}{B \quad true} \tag{2.3}$$

is synthetic in that the concepts or propositions involved are always to be justified in terms of proper constructions ($a : A$ and $b : B$). This corresponds to consider the role of individuals in relation to the analyticity of inferences in terms of meaning explanation. Consider an instance of a type-theoretical rule, e.g. the conjunction introduction:

$$\frac{a : A \quad b : B}{\&I(A, B, a, b) : A\&B;}$$

the inferential step is analytic in virtue of the meaning explanation, i.e. *ex vi terminorum*[116]: the conclusion is in fact obtained via the constructions contained in the premises. But it is worth noting here that the object $A\&B$ is justified by a "new" construction, object or "individual", namely the one which is composed by the terms in the premises—(A, B, a, b): in this sense, there is a difference between the analyticity in terms of the meaning explanation and the one in terms of new individuals.

The notion of analyticity implied by conceptual analysis is obviously strictly related to the role of connectives and quantifiers: in the traditional interpretation, a conclusion can analyse only the individuals contained in

[114] Kant (1783, *Der transzendentalen Hauptfrage — Erster Teil*, par. 7):

die erste und oberste Bedingung ihrer Möglichkeit [der Mathematik]: nämlich, es muss ihr irgend eine reine Anschauung zum Grunde liegen, in welcher sie alle ihre Begriffe in concreto, und dennoch a priori darstellen, oder, wie man es nennt, sie konstruieren kann.

[115] Kant (KrV, A 713/B 741, 31–39):

Die philosophische Erkenntnis ist die Vernunfterkenntnis aus Begriffen, die mathematische aus der Konstruktion der Begriffe. Einen Begriff aber konstruiren, heisst: die ihm korrespondierende Anschauung a priori darstellen. Zur Konstruktion eines Begriffs wird also eine nicht empirische Anschauung erfordert, die folglich, als Anschauung, ein einzelnes Objekt ist, aber nichts destoweniger, als die Konstruktion eines Begriffs (einer allgemeinen Vorstellung), Allgemeingültigkeit für alle möglichen Anschauungen, die unter denselben Begriff gehören, in der Vorstellung ausdrücken muss.

[116] Hintikka's *Analyticity II*.

the premises, those falling under the scope of the quantifiers and related by connectives. The definition of the notion of analyticity is submitted to an extension by saying that an argument step is analytic when it does not introduce any new individual.[117] The number of individuals taken into account by a quantificational sentence is, according to Hintikka, given by the sum of the number of free singular terms and those individuals within the scope of the different quantifiers contained in the sentence. In his analysis, this computation of individuals will give back the *degree* of the sentence, while the maximum number of quantifiers is called the *depth* of the sentence.[118] At this point Hintikka explains the idea of the individuals involved by an inference, by saying that the content of a premise gives us a certain amount of information.[119] It is just applying this intuitive assumption to the core of the problem, that a new definition of analyticity is revealed:

Definition 2.17 (Hintikka's Analyticity) *A step of inference is analytic if it does not increase the information contained in the premises, i.e. if the information carried by the conclusion is no greater than the information carried by the premises. A sentence is therefore analytic if it does not convey any information.*[120]

According to this introduction of an intuitive notion of information, analytic truth is simply a variant of tautological truth: this sense of analyticity is clearly connected to the impossibility of extending knowledge by means of analytic sentences, again the sense in which Kant thought about analyticity (judgements of clarification vs. judgements of extension). If extension of knowledge is intended in relation to matters of fact, and in turn by "matters of fact" one understands the information conveyed by synthetic sentences; on the other hand, the information conveyed by analytic sentences must

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[117] Hintikka (1973, p. 136), *Analyticity III*, distinguished in the following different senses:

- IIIa. An analytic argument step cannot carry us from the existence of an individual to the existence of a different individual (Kant's sense).
- IIIb. An analytic argument step does not increase the number of individuals considered in their relation to each other.
- IIIc. In an analytic argument step the degree of the conclusion is no greater than the degree of at least one of the premises.
- IIId. An argument is analytic if, and only if, all of its steps are analytic in sense IIIc.
- IIIe. A proof of the sentence F_1 from sentence F_0 is analytic if all sentences occurring at intermediate stages have lower or equal degree than F_0 and F_1.

[118] This notion of *degree* and its computation is the one involved by the sense IIIc/IIIe of analyticity, mentioned in the previous footnote.

[119] Hintikka (1973, p. 146).

[120] Hintikka (1973, p. 146): *Analyticity IV*.

regard the meanings of our terms or other conceptual matters. The information carried by analytic sentences is conceptual or linguistic, as it was already suggested by Ayer,[121] and it is again to the notion of information-content that Hintikka is directed, rejecting as unsatisfactory this explanation. Which notion of information is implied by the development of a logical derivation? What do we mean when speaking about "information" in such a context? And which notion of information is implied by the concept of logical tautology (as used by Hintikka)? A satisfactory answer to each of these questions should explain the role of information within analytic judgements by maintaining, on the other hand, the role of constructions in connection to synthetic ones.

2.4 Different Notions of Information

The introduction of the notion of information in the context of logical studies is preceded by an intense and deep development of researches in various other fields. By the end of the 1950s the notion of Information enters into the field of scientific studies; this revolutionary step is due to C.E. Shannon with his works on the *Theory of Communication*, later also called *Mathematical Theory of Communication* or *Statistical Theory of Information*.[122] This approach introduces the *syntactic* notion of information, its aim was to provide a study on the communication channels, and by doing this, to introduce a mathematical representation of the flow of information over such channels; it therefore provided primarily a *measure* of the *information quantity* flowing on these channels. The Statistical Theory of Information deals explicitly only with the technical problem of communication, and does not take into account the problem of meaning.[123] Thus, obviously, the mathematical notion introduced to represent the concept of information was in this respect inadequate and incomplete. In particular, the problem of analyticity shows how the notion of meaning is deeply involved in the relation of containment and the notion of definability, being therefore a central topic once the idea of informativeness has been introduced in connection to the validity of derivations.

The semantic aspects of communication were soon explicitly endorsed by a new theory of information called *Semantic Theory of Information*. The

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[121] Hintikka (1973, p. 147):

> *Their apparent emptiness is merely emptiness of non-linguistic information.*

[122] Shannon (1948); Shannon and Weaver (1952). Shannon was probably highly and profoundly inspired by the work of Wiener on informational and cybernetic systems.

[123] Cf. Weaver (1952).

theory was first developed by Bar-Hillel and Carnap[124] and had already been treated informally by Popper[125]: this is generally known as *Theory of Weakly Semantic Information* (existing a later version called *Theory of Strongly Semantic Information*). The aim of such a theory was to deal with sentences in order to establish both "*how much*" and "*which*" information is expressed. Like in the syntactic/statistical theory, the amount of information produced by a (set of) sentence(s) is established in terms of probabilities, using the notion of entropy as explanation. The main distinction between the two kinds of theories is supposed to be the kind of probability under discussion.[126] The structure of the theory is based on a state description semantics, which determines the informative content of sentences according to the following basic principles:

1. $CONT(\sigma) =_{def}$ the set of all state descriptions inconsistent with σ
2. $CONT(\top) = MIN$, i.e. tautologies have minimal semantic content
3. $CONT(\bot) = MAX$, i.e. contradictions have maximal semantic content, also known as the Bar-Hillel–Carnap Paradox

On this basis, and especially by the necessity of rejecting the unacceptable consequences of the semantic content determined by the excluded state descriptions (particularly the situation described by the Bar-Hillel–Carnap Paradox), some revisions to this theory have been proposed. Hintikka's work falls under the studies of the Semantic Information Theory by extending the conceptual definition of information recognized in logical procedures: he develops two different notions of information, on the basis of the elements involved in the probabilistic computation. By this analysis, the limitations of such a theoretical approach will be made clear and, later on this basis, it will be shown that the concept of meaning used by the Semantic Information Theory is conceptually poor and misses the connection to the concept of truth.[127]

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[124] Cf. especially Bar-Hillel and Carnap (1952, 1953).
[125] Cf. for this origin Hintikka (1970).
[126] Hintikka (1970, p. 4). Both the theories are considered "hard sciences information theories". Among them one has to consider also the developments of the semantical approach due to Dretske (1981), Barwise and Perry (1983), and Devlin (1991); moreover, the Algorithmic Theory of Information, developed by Chaitin, Kolmogorov and Solomonov, belongs to such a group. A quite different approach is that from Brookes (1980). Finally a great impulse to the field has recently been given by Studies in Artificial Intelligence, particularly in connection to Information Retrieval Systems.
[127] An interesting history of the concept of information in its development and its connection to modern logic is presented in Dunn (2001), where relevant influences of the notion of information on other logical approaches are considered, such as boolean algebras, relevance logics, lattices.

2.4.1 *Conceptual vs. Contentual Information*

The explanation of analyticity in terms of meaning and (lack of) information represents at this point the core of the wide range of problems connected to this analysis. In the first instance, it has been explained how the whole analysis is subject to a vicious treatment due to the misunderstanding about propositions and judgements as bearers of truth. The distinction between conceptual and contentual information within logical sentences (formulae) introduced by Hintikka has already been mentioned. It is now this main distinction within the concept of information that will be taken into account: once this is done, it will be possible to extend the analysis to the constructive framework, in which the suggested difficulties will be clarified and a new interpretation of the problem will be presented.

The relation between information and analyticity has been presented via the connection of the sentences (propositions) and their logical form, in terms of conceptual relations. The logical form is intended in terms of quantifiers and logical operators. On the other hand, individuals falling under the scope of quantifiers and related by operators, establish its content: information conveyed is better understood as *information-content*.[128] This distinction leads back to the problem of analyticity for deductive reasoning: if the common view that logical truths are analytic is to be maintained, i.e. that they are empty of factual information or devoid of information-content, how is it possible to acquire knowledge by means of a logical derivation? Or, in other words, how can the derivation be useful at all? In the light of the role played by logical truths with respect to analyticity (and especially in terms of constructions) in the following sections I will present the two notions of information Hintikka defined (1973), and analyse the difficulties arising.

2.4.2 *Surface Information: Probability and Possible Worlds*

Hintikka suggests a first analysis of the concept of semantic information involved by propositional logic: atomic sentences and a certain number of connectives are defined, they provide *constituents* to build up all the possible worlds definable in that language. The concept of *constituent* goes back to Boole,[129] and the notion of probability on whose basis the concept of information is developed, refers to the work of Bar-Hillel and Carnap.[130] Semantic Information Theory assumes a purely logical interpretation of probability, its origin is to be found in the works of Boole and Jevons,[131] followed in the 20th century by the interpretations due to Wittgenstein

[128] Hintikka refers to this concept in his notion of *Analyticity IV*; Hintikka (1973, p. 147).
[129] Boole (1854, 1854a).
[130] Bar-Hillel and Carnap (1952, 1953), Bar-Hillel (1964), Carnap (1947, 1950).
[131] Boole (1854, 1854a), Jevons (1873).

and Keynes.[132] Probability, in this Boolean interpretation, depends on the state of our information about events, and it is described as a logical relation between propositions: the more the alternatives, the more probable the sentence which contains them, less the information acquired; the less the alternatives, the more the information acquired. The way one assigns "weights" (probabilities a priori) to different possibilities is particularly important. Carnap develops a double notion of probability, p^+ and p^*,[133] and using these he defines the already mentioned "state description semantics". Briefly, Carnap's p^+ notion of probability represents an extension to monadic first-order logic of the following propositional case: the notion of logical probability for a propositional logic starts from the concept of Boolean constituent, where given a sentence containing the constituents

$$(\pm)C_1 \wedge \ldots \wedge (\pm)C_k,$$

the number of possible different constituents will be 2^k. Every consistent sentence can be represented as a disjunction of all its constituents, one of them to be realized. This formal content describes the possible state of affairs of the "world" which the sentence belongs to. On the other hand, its content coincides with the *state-descriptions* incompatible with this world and excluded by the sentence. These constituents are ordered within the sentence h and the last is determined as width w of the sentence:

$$h = C_1 \vee C_2 \vee \ldots \vee C_{w(h)}.$$

The measure of probability of this constituent is then computed as the *ratio* between the width h and the total number of combinations within it:

$$p(h) = \frac{w(h)}{2^k} \tag{2.4}$$

from which the following measure of information is obtained:

$$inf(h) = -\log p(h) = -\log \frac{w(h)}{2^k} = k - \log w(h) \tag{2.5}$$

where the logarithm has base 2. The other way of computing the measure of information is by considering directly the numbers of alternatives excluded by the constituent, it is called the *content* of information:

$$cont(h) = \frac{2^k - w(h)}{2^k} = 1 - p(h). \tag{2.6}$$

The relation to the measure $inf(h)$ is the following:

$$inf(h) = \log(\frac{1}{1 - cont(h)}). \tag{2.7}$$

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[132] Wittgenstein (1922), Keynes (1921).
[133] Carnap (1950).

Within a monadic first-order calculus, similar cases of state-descriptions can be considered, conjunctions of any predicated individual, either asserted or negated:

$$\pm P_1(a_1) \wedge \ldots \wedge \pm P_k(a_1) \wedge \pm P_1(a_2) \wedge \ldots \wedge \pm P_k(a_2) \wedge \ldots$$

To any of these constituents is given an equal probability, which give rise to the measure p^+. In this case, a problem arises relatively to the assignment of a priori probabilities, given to different alternatives in an equal way and not to different kinds of alternatives. For this reason, Carnap introduces the *structure-descriptions* and the relative measure of probability p^*: equal probability weights are now given to disjunctions of structurally similar state descriptions, i.e. to state descriptions which can be submitted to permutations by changing names to individuals. Such disjunctions are the structure-descriptions.[134]

The semantic information carried by a sentence is then computed by summing the weights of all the state-descriptions excluded by such a sentence. Within the realistic approach supported by Hintikka, this amounts to an important problem: to consider the informativeness of a sentence in terms of the states excluded implies the tacit assumption that we know all the individuals belonging to the world we are treating with, and the size of our universe of discourse. The problem is mainly to be understood in terms of the size of the world, whether it is finite or infinite. The size of the universe of discourse is moreover a source of problems, if we consider that on the basis of a posteriori probabilities logical truths become less and less probable, being zero when the size of the world tends to infinity; informational content of logical falsehoods is decreasing as the world increases.

2.4.3 *Increasing Logical Information: Depth Information*

Hintikka develops therefore a different notion of information, consisting in a different way to ascribe an informational value to sentences. The first way assigns a value x such that $0 \leq x \leq 1$ to each *consistent* constituent of a state-description (therefore via the exclusion of incoherent constituents). Given these assignments, it is possible to define a notion of semantic information for sentences with the same terms, same predicative symbols, and the same depth of constituents. This kind of semantic information is called by Hintikka *depth information* (inf_{depth}).[135] Following the analysis of the Finnish logician, deductive reasoning increases non-trivially our information in a different way, by what he calls *surface information* (inf_{surf}). *Surface information* is computed not only on the basis of coherent constituents, rather upon constituents with a *fixed* number of singular terms, predicative symbols, and with a fixed depth, in which only trivially

[134] Cf. Hintikka (1970).
[135] Hintikka (1973, p. 186).

incoherent constituents have informative value = 0.[136] In both cases, the *degree* of the sentence is involved, i.e. the sum of the number of the free singular terms plus the number of layers of quantifiers (*depth* of the formula). Of the two notions introduced, the latter is computable, despite the absence of any decision procedure for the coherence of the quantificational theory, which implies the uncomputability of the first notion of information. Thus, inf_{surf} is, according to Hintikka, a perfectly objective, non-psychological notion, to represent the type of information added in developing a logical inference.[137] Surface information reveals uncertainty about the world, through recognition of incoherent constituents: in this sense it is information about reality, and therefore synthetic information. But because this information is computed upon the logical elements and the constituents are only contingent elements, inf_{surf} is also a form of conceptual information, i.e. it is a priori. Thus inf_{surf} represents the kind of information that can be actually increased by deductive inferences, because it takes into account the number of quantifiers and singular terms nested inside them, and actually involved in the process of reasoning. In other words, surface information is the only kind of information that allows for a proper treatment of the universe (therefore without the inconveniences arising with a non-fixed number of elements).

The treatment of the notion of informational content by Hintikka has shown some essential features connected to its explanation, concerning the notion of universe of discourse as the quantificational domain in which this information is definable and the nature of the elements involved in such a domain. The notion at hand requires a strong connection to the computability of the universe referred to, and moreover the restriction to propositional logic let to refer only to the amount of information "flowing" in the universe and based on the quantitative value given to the constituents of formulae, namely in terms of individuals. The question that naturally arises is whether this information can be considered as the content of judgements expressed by a rational agent. Moreover, the problem of meaning considered in this chapter seems to have been completely forgotten in this explanation: the content of statements has been replaced by a purely syntactical treatment of the individuals and the layers of quantifiers taken into account by logical derivations (i.e. meaning is determined only by logical form).

It seems therefore that the main questions raised by our analysis remain unattended by the framework built by Hintikka, and that a fully qualified notion of information has not yet been given for our framework. The last aim of this chapter is to give the conceptual description of a knowledge system to account for such a notion, to be formally described in the setting of Constructive Type Theory in Chapter 3.

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[136] It should be noted here that the principles governing the semantic theory of information and determining the Bar-Hillel–Carnap Paradox are completely abandoned.

[137] Hintikka (1973, pp. 187, 191).

2.5 Basic Elements of a Knowledge System

The problem of analyticity, developed throughout this chapter starting from the debate between Kant and Bolzano, was directed towards the introduction and analysis of the notion of information. The aim of this research consists essentially in providing a complete explanation of this concept within a knowledge system, based on its epistemic role and the clarification of its relation to the notion of truth. In this perspective, this final section will start by completing the reconstruction of the Semantic Information Theory, considering the basic principles of such an account: by suggesting the major revisions needed according to the general constructive approach (judgements vs. propositions/acts vs. states of affairs), one will consider in the following a description of a knowledge system for rational agents. The main goal is obviously to clarify the different epistemic aspects underlying the concept of knowledge, thus determining what is "information".

2.5.1 *Reconsidering the Semantic Approach*

The classic account of semantic information has been recently revised in terms of information as content.[138] This new interpretation substitutes the classical definition by introducing the description of *declarative objective semantic information* (DOS), which has the following properties:

1. An infon σ consists of a non-empty set (D) of data (d).
2. The data in (D) are well-formed.
3. The well-formed data in (D) are meaningful.

The new definition of semantic information refers to a meaningful entity independent from the mind, transferable through codification but independent both from the codification and the transmission.[139] To explain information as independent from the mind, as well as from its codification and transmission, means first to consider it as a medium entirely unaffected by the properties and influence of the user. Information is thus, in the first instance, simply *data*, in the interpretation given already by the Statistical Theory. This general definition includes a modality of syntactical information as well-formed entities or primary data. On the other hand, the semantical theory claims to take into account a semantical modality of such data, namely their *meaning*. The general definition considered is therefore that of *data* + *meaning*. Consequences of such a definition are that false information as well as tautologies convey information, even if they do not lead

[138] Floridi (2004).

[139] For this definition of Semantic Information and a full explanation of the principles involved cf. Floridi (2004, 2005).

to new knowledge; in this definition one ascribes truthfulness to semantic information, which means that contradictions have zero informative content, and that the semantic content of tautologies is minimum. A typology of information normally taken as exemplification of semantic content is called *factual information*: it is information of a declarative nature, it can be translated into a first-order logic, it is alethically valuable and expressible by judgements of the kind

a's being (of the type) F carries the information that b is G.

The definition of semantic information is then ruled by the following properties[140]:

TN. *Typological Neutrality*: the set of data (D) is composed by different types of information (primary data; metadata; operational data; derivative data).

TaxN. *Taxonomical Neutrality*: being a datum is an external property, information is a relational property.

GN. *Genetical Neutrality*: the well-formed formulae (δ) in the set of data (D), have a semantics independently of any interpreter/user.

AN. *Alethic Neutrality*: meaningful and well-formed data always qualify as information, no matter whether they represent or convey a truth or a falsehood.

These four principles state strong consequences for the notion of information one is dealing with in the semantic approach. The notion obtained by their formulation results in the complete rejection of the role of the user, information is therefore a result-based rather than a process-based approach, realistically based and committed to a classical logic framework. The formal interpretation of the notion of information provided for CTT in Chapter 3, obviously rejects such a model, and the description of a system of knowledge integrating such an epistemic notion can therefore be conceptually described by properties competing with those formulae for the DOS definition. In the following I will informally consider the relevant properties of the here intended epistemic notion of information[141]:

I. The constructive logical framework is built on a strong definition of knowledge and knowledge-contents, in terms of the judgement/ proposition distinction. In this model knowledge is given in terms of

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[140] Cf. Floridi (2005).

[141] A full analysis of the constructive notion of information on the basis of a precise reformulation of the mentioned principles is contained in Primiero (2007b). For an approach which aims as well to put the user back in the process of deduction, without being constructive, rather inspired to a framework of adaptive and dynamic logics, cf. Allo (2005), where the author argues that handling information-like objects is more fundamental than the notion of information itself.

proof-conditions (provability): the notion of information can alongside be described in terms of a different and more weak (or rather more basic) property, namely, that of meaningfulness. It is relevant for the constructive framework to associate different epistemic states of the agent to the notions of knowledge and information, according to the rational processes involved and the nature of the data processed. Information and informative content are expressed by meaningful data, where meaningfulness is property of all the primary data in the knowledge process: this means, in other words, that any propositional content to be asserted needs to satisfy in the first instance some assertability conditions, provided the content is meaningful to the agent. The satisfiability of these conditions does not imply yet the formulation of the relevant provability conditions, which are instead required for the justification of any knowledge content. It will be the aim of the formalization presented in Chapter 3 to describe a model in which these conditions are interpreted and clearly distinguished, so as to contain data performing each epistemic role, and accordingly describing the operations within the agent's knowledge process.

II. Data are the substrate of information; as such they are always received in relation to other data of analogous origin. Information expresses a relation between the incoming data and the receiver, the agent apt to receive and "read" such data. It is obviously essential for the theory to clarify the relation between the information and the agent. It is in relation to the operations performed by the agent that data are (or are not) defined as information, and in the constructive framework a datum is accepted by a certain agent as information under the previously established condition of standard meaningfulness. In the second instance, therefore, information is defined according to the epistemic value the agent attributes to the data occurring in the knowledge process: this means that the independence of the informational value of data from the user is rejected, and their value is entirely determined by the agent herself. Only the agent determines what is information to her, and what instead is knowledge, and this depends on the conditions she is able to satisfy for the content at hand.

III. The (supposed) independency of the meaning conveyed by the data from an interpreter or user involved in the process of communication is at this point a central topic. Semantic Information Theory defines data as meaningful independently from the presence of any user producing, receiving, or even understanding them (according to GN). This principle is formulated for a realistic conception of semantics, where the truth-value of propositions is determined by their relation to states of affairs, hence independently from the agent involved in the knowledge process. In the constructive setting such a principle can still be maintained, under a modified form: it says that the value of a set of informational data introduced in the knowledge process of an agent is independent only from *which* agent makes use of those data. This is stated in a similar way by the constructive semantics for propositions, being subject-independent despite their not being proof-independent. Such independence, both relatively to informational data and to the semantics of the proposition known, is to be understood as *indifference* relatively to which agent will perform the act of assuming information or stating the semantic value of a certain proposition: in fact, the epistemic value is determined by the satisfaction of proper conditions (meaningfulness or provability), which are to be given in a canonical form. This does not imply that the epistemic value exists independently of *any* agent. Once the derivable objects of our logical system are judgements with a propositional content, a first person perspective is tacitly endorsed, even in the case of epistemic states which are different from proper justified knowledge.[142] In such a perspective one necessarily relates the content of knowledge and information to an agent performing the acts of asserting a judgement or using the meaningful information: it is instead actually not relevant *who* the perfomer is.

IV. Assuming that only judgemental knowledge is able to convey truths or falsehoods, because only what is known can be properly judged, the epistemic counterpart played by informational contents can be determined by making explicit the definition of information for knowledge systems in terms of meaningful data. This is clarified by going back to the distinction between explicitly proved judgemental knowledge and (previously acquired) implicitly proved knowledge, introduced in Section 2.2.5. The notion of implicitly proved knowledge expresses precisely the conditions needed to be satisfied by any explicit knowledge acquired, i.e. the contents known in order such knowledge to be formulated: its *assertion conditions*. To state meaningfulness for a certain judgement J and the conditions under which it can be asserted, irrespective of whether J is actually known or not, are epistemic acts defined independently of the stated truth or falsity of such a judgement. An important consequence is here at hand: in order to maintain meaningfulness as a defining property of information, one needs to show that assertion conditions are also meaning-determiners; this will disclose the proper epistemic interpretation for the notion of information, possibly explaining how such a concept is to be distinguished from the notion of knowledge.

[142] The clarification of the importance of the first person perspective is due to B.G. Sundholm.

2.5.2 *Recollecting Perspectives on Information*

The restricted scope and aim of the Statistical Theory of Information has been initially extended in the direction of comprehending meaning. The principles ruling such an approach have been analysed in the light of our understanding of the related notion of knowledge. Before going further, in order to set the frame in which a new explanation of the concept of information can be given, it is at this point necessary to understand the proper role such a definition will play, and more generally the relation to the well-settled accounts already given in epistemology. In general, one can summarize the different directions in which the notion has been treated, as follows[143]:

1. A first approach is represented by insisting on the notion of probability based on prior knowledge. Such an account, on which the general notion of semantic information is explained, presents a definition of information content shaped along the following lines:

   Definition 2.18 (Conditional Probability Account) *To an agent with prior knowledge* $\boldsymbol{K}$*,* $\boldsymbol{r}$ *being* $\boldsymbol{F}$ *carries the information that* $\boldsymbol{s}$ *is* $\boldsymbol{G}$ *if and only if the conditional probability of* $\boldsymbol{s}$ *being* $\boldsymbol{G}$ *given that* $\boldsymbol{r}$ *is* $\boldsymbol{F}$ *is* 1 *(and less than* 1 *given* $\boldsymbol{K}$ *alone).*[144]

   Such understanding points out the difference determined by the flow of information between the internal contribution of a person's prior knowledge and the external contribution of objective probabilities.
2. Another definition of the notion of information is given by determining the truth-value of statements on the basis of a possible-worlds semantics; under such an account a definition of information content sounds as follows:

   Definition 2.19 (Possible Worlds Account) *To an agent with prior knowledge* $\boldsymbol{K}$*,* $\boldsymbol{r}$ *being* $\boldsymbol{F}$ *carries the information that* $\boldsymbol{s}$ *is* $\boldsymbol{G}$ *if in all possible worlds compatible with* $\boldsymbol{K}$ *and in which* $\boldsymbol{r}$ *is* $\boldsymbol{F}$*,* $\boldsymbol{s}$ *is* $\boldsymbol{G}$ *(and there is at least one possible world compatible with* $\boldsymbol{K}$ *in which* $\boldsymbol{s}$ *is not* $\boldsymbol{G}$*).*

   This idea has been developed in the light of the semantics of counterfactual conditionals, in order to establish the comparison and the compatibility between actual and prior knowledge.
3. The third kind of explanation refers explicitly to the connection between information and inference in terms of the relation between implicit and explicit knowledge. An account of it can be given in the following terms:

[143] Cf. Barwise and Seligman (1997).
[144] This definition reflects the one given in Dretske (1981). To formulate this and the following definitions, the above introduced concept of factual information has been used.

> **Definition 2.20 (Inferential Account)** *To an agent with prior knowledge **K**, **r** being **F** carries the information that **s** is **G** if the agent could legitimately infer that **s** is **G** from **r** being **F** together with **K** (but could not from **K** alone).*
>
> Such a definition has two main properties, distinguishing it from the previous ones: first, it takes into account a person's ability to infer; second, it refers to a relevant notion of background knowledge, which is not the same as a referring parameter for extracting possibilities.

This third account is in the present context the most relevant one: in fact, it uses a first person perspective and it is directly linked to the question of the informativeness of derivations, thus referring to the ground problems adressed here. The first is of course the problem of analyticity, which has been our starting point, and that is at the core of the understanding of inferential processes. The second problem directly connected with the informativeness of derivations is that of monotonicity of reasoning. In fact, it seems difficult to restrict the principle of monotonicity for a logical framework (persisting, e.g. under weakening rules), whereas revision of previous knowledge is highly probable, after addition of new informational content: one is here facing the wide and important problem of belief revision, on the philosophical level essentially connected to the explanation of error. It seems that it is in the relation between the notion of information and that of meaning that these problems have to be clarified, and at the basis of such a relation there is certainly the distinction between information and knowledge considered here. The main aim of this last section is to present an account of a knowledge system shaped on the structure of a logical framework and referring essentially to the third account of those presented in the list above. Later, it will become clear that once the theoretical explanation at the basis of this system is accepted, the resulting formalization will also correspond to an interpretation in terms of possible worlds.

2.5.3 *Knowledge: What, That, How*

It is useful to begin with a description of the different kinds of knowledge to consider. Hintikka's analysis provides an account of the notion of information in terms of the inner distinction between conceptual and contentual information; such an explanation is vitiated by the lack of a clear formulation of the information-bearers. In fact, formulae are involved, equally referring to propositions, statements or judgements: at the core of this description the essential distinction between the act and the content of expressions is missing, thus the notion of information content is also less clear. Moreover, even if related to a rather vague idea of content and supposed to be a conceptual kind of information, the notion described by

Hintikka is computed and defined in a purely syntactic way: this means that no role is played by meaning in the definition of information, rather only that informative contents are supposed to convey meanings. The constructive reformulation of the semantic theory suggests two basic principles to be assumed: first, the distinction between proper knowledge, expressed in judgemental form, and the assertion conditions for such knowledge; second, a serious account of the role of the agent, by using a first-person perspective.

The introduction of implicit knowledge in the way intended, as the counterpart of judgemental (explicit) knowledge, is less obvious than one might think at first: in all the relevant explanations of the notions of knowledge and information, it has always been assumed that knowledge is explicit, and information is usually conflated with the content of knowledge. The relation of implicit/explicit containment is obviously essential in the understanding of the relation between premises and conclusion of an inference. Often the former has been considered containing explicit knowledge, and the latter implicitly contained in the premises. This interpretation justifies in its essence the principle of analyticity for inferential processes. By referring to judgemental knowledge, it is instead possible to formulate implicit knowledge more clearly: such an expression refers to what contained in an agent's knowledge frame every time something is explicitly asserted; this knowledge refers therefore to the collection of assertions required (but not necessarily expressed) by the meaningful assertion of a judgement. In particular, it is clear that the meaningfulness of the concepts involved by the explicitly asserted judgements must be considered as implicit knowledge. For example, the true judgement "Venus is a planet" will require implicitly knowldge of the meaning of the term "planet". To introduce an expression that will be later formally defined, meaningfulness of concepts is a *presupposition* for judgements containing those concepts. By this explanation, the relation of containment between premises and conclusion is in fact inverted.

At this point it seems reasonable to compare the distinction here introduced between *implicit* and *explicit* knowledge with a classical epistemological description, namely the *knowledge-that/knowledge-how* distinction. *Knowledge-that* is normally ascribed to propositional knowledge, whereas *knowledge-how*, in its traditional account, is considered epistemic, non-propositional knowledge, thought to underline abilities or to ascribe them to a subject. This distinction is originally due to Ryle[145]: he considered the former as a relation between a thinker and a true proposition, the latter as an ability, intending respectively knowing that something is the case and knowing how to do something. This classic description has some variants, among them the one considering *knowledge-how* as a relation

[145] Ryle (1946); probably in the same light has to be seen the Russellian distinction between *knowledge by description* and *knowledge by acquaintance*.

between agents and actions[146]; recently this position has been rejected by Stanley and Williamson (2001), where ascriptions of *knowledge-how* are considered ascriptions of propositional knowledge, i.e. to say that someone knows "*how to F*", would mean always to ascribe to such a subject a particular kind of *knowledge-that*. This theory is explicitly based on the assumption that sentences of the *know-that* form relate subjects and propositions.

The extremely relevant distinction between *knowledge-how* and *knowledge-that* can be taken into account and someohow modified, by considering its role in the knowledge system introduced. Remembering in the first instance that propositional knowledge is here intended as judgemental content, our description of various forms of knowledge for a rational agent can be briefly presented as follows[147]:

- *Knowledge-that* amounts to knowledge of the truth of a proposition, i.e. knowledge that a proposition is true ("*A* is true"); the epistemic state derived by the possession of such knowledge is to be intended as producing proper or justified knowledge on the basis of the related proof objects.
- *Knowledge-how* will in turn corresponds to the ability of stating the truth of a certain proposition, in terms of knowing the related proper proof object for such a proposition; the *know-how* is, in this sense, the knowledge of the set of propositions making a certain other proposition true, or in other words it amounts to being able to lay down a demonstration for a proposition.
- By the previous point, to know how to lay down such conditions defining truth amounts to know what one has to do or to know for something to be judged; thus, *knowledge-what* is the implicit knowledge, stating the logical possibility for something to be known, in terms of its assertion conditions; it amounts to judgements stating conditions and in turn meaningfulness for other judgements to be made (to know what is possible to know/what is meaningful and can therefore be known).

According to this schema, which extends the usual dichotomy of *knowledge-that* and *knowledge-how*, the general concept of knowledge has to be clarified by three different epistemic states:

1. ***Knowledge-that* a proposition is true (knowledge of the truth)**
2. ***Knowledge-how* to demonstrate a judgement (knowledge of a demonstration)**
3. ***Knowledge-what* intended by a sentence (knowledge about assertion conditions)**

The distinction at this point is quite simple to clarify: to know the truth of a proposition (*know-that*), and to know the way in which a proposition can

[146] Carr (1979).
[147] Cf. also Martin-Löf (1996, p. 36).

be judged true (*know-how*), are both judgemental, explicit knowledge.[148] A rather different epistemic value is to be ascribed to the process of determining meaning and other truths expressing conditions for proper knowledge to be acquired. The latter can be characterized as *implicit knowledge*.

This first informal description of the properties of the epistemic states that we are introducing and distinguishing within a knowledge system for a rational agent has to be coherently presented and formally justified: this is obtained by a formal representation within the structure of Constructive Type Theory. It is the aim of Chapter 3 to show the relation between *informational states* and *knowledge states*: whereas the latter will serve as a term to express the explicit knowledge of a rational agent, in the former we will account both for meaningfulness as the essential property of information, and for conditions for dependent judgements, thus containing what has been here accounted for as implicit knowledge. In both cases they serve the same purpose, i.e. to complete the epistemic description of a knowledge system, by a conceptual understanding and a formal development of a rather intuitive notion of information.

[148] Which does not mean of course that everything in a demonstration must be explicitly stated. Rather, it amounts to the knowledge of the method by which every step of such a demonstration can be explicitly stated.

3
Formal Representation of the Notion of Information

3.1 CTT as the General Framework: Informal Description

The aim of this chapter is to introduce a formal description for a knowledge system, by which to represent and connect the different epistemic states suggested by the previous theoretical analysis. The core of this description is the basic distinction between the epistemic notions of *information* and *knowledge* and the formalization of related states. The focal point is obviously the formal introduction of the notion of information, according to the basic principles described in Section 2.6. On the basis of this description, a system of formal operations performed on different epistemic states is provided; they represent the actions of an agent on the informational contents towards the acquisition of new knowledge. The formal structure is that of Constructive Type Theory (CTT), introduced in Chapter 1.

In the spirit of the constructive philosophy, demonstrations of judgements (in the form of proof objects for the related propositional contents) are required in order for the propositional contents to be known and therefore asserted with a justification. Under this common description that reformulates the Verification Principle of Truth (Principle 1.1, Section 1.2), knowledge is defined simply in terms of justifications. On such an epistemic basis, it is possible to furnish the following definition.

Definition 3.1 (Knowledge State) *The notion of Knowledge is defined as an epistemic state towards a certain propositional content, expressed by a judgemental act and produced by the possession of a verification (proof) of the intended content. To possess a certain knowledge means for an agent to be in a certain knowledge state.*

This definition of knowledge as an epistemic state defines the strict notion produced by the constructive approach, determined by the rigid parameter of demonstration. *To know* in the proper sense means to be able to judge correctly, and a proved judgement is the expression of the epistemic state produced by the knowledge of the truth of a certain propositional content.

As suggested by the analysis in the previous chapter, it is nevertheless necessary to distinguish other epistemic states, presented as containing *implicit knowledge* supporting the judgemental one. To know something (to recognize it to be true, i.e. to be in a certain knowledge state) requires the knowledge of what one has to do (in order to make something, to judge something true); the latter is *logically possible knowledge*, expressing assertion conditions, i.e. judgement candidates stating conditions for judgements made. With this explanation in mind, a first definition of an epistemic state involving the notion of information can be introduced.

Definition 3.2 (Informational State) *The notion of Information is defined as the content of the epistemic state on which basis it is possible to acquire proper knowledge: it expresses the implicit contents needed in order to justify access to explicitly stated knowledge. To be in a certain knowledge state (Def. 3.1) means in general for an agent to formulate a set of informational contents, on which basis the former is obtained.*

The two epistemic states introduced and their formal and conceptual relations provide the basis to develop a knowledge framework.[1] The previous informal definitions need now to be formulated in terms of their content and role:

- *Informational State* represents the collection of assumptions and presuppositions on the basis of which a judgement is made by the agent; it is therefore the state or epistemic situation in which the judgement can be *understood* and *asserted*.
- *Knowledge State* represents the epistemic state composed by justified judgements, performed by the agent once assumptions and presuppositions for such judgements have been recognized (i.e. under the proper Informational State), and thus their conditions have been laid down.
- *Knowledge Frame* represents the collection of the Knowledge States acquired so far by the agent on the basis of the related Informational States; it consists, therefore, of all the knowledge and information actually possessed and used by the agent.

The notions of *Informational State* and *Knowledge State* both have a formal counterpart in CTT: the former will be explained in terms of expressions

[1] An important and interesting connection is represented by the explicit interpretation of informational states as *belief states*: this provides in turn a different reading of the concept of belief and its usual definition in terms of *justified knowledge*. An interpretation of the problem of Belief Revision for type-theoretical languages is given in Borghuis, Kamareddine, and Nederpelt (2002), whereas the first analysis of a belief revision system for the formalization of CTT, especially in the light of the connection between information and belief, is presented in Primiero (2006) and extended in Primiero (forthcoming b).

contained in *type theoretical contexts* for dependent judgements, and the latter in terms of knowledge acquired by judgements asserted on the basis of such contexts. The distinction between the mentioned epistemic states (information vs. knowledge) will thus be shown according to the formal description and roles of the following expressions:

- Presuppositions of the form $<\alpha : type>$, introducing concepts within the knowledge state
- Assumptions contained in contexts, i.e. collections of expressions of the form $(x_1 : \alpha_1, \ldots, x_n : \alpha_n)$ containing variables
- Judgements expressing knowledge in the form $a : \alpha$, i.e. expressions containing (canonical) proof objects

By means of these forms of expression, the formalization of CTT is able to represent all the information required to make a judgement of the form "*A is true*": this information refers in the first instance to presuppositions and assumptions needed by such a judgement, setting the conditions for the construction in A. Under this interpretation, information is defined as the content supporting the knowledge possessed and asserted by the agent, an idea which shall be conceptually and formally considered.

In the first instance a general description of the formal structure can be given in terms of the following properties:

- Every judgement asserted in CTT has its own presuppositions: in their basic formulation they introduce the concepts (types) contained in the judgement and for which predications can be performed; the introduction of a concept consists in performing an instance of the first form of category "$\ldots : type$"; such kind of expression will be called *type declaration*.
- Hypothetical judgements are formulated under collection of immediate assumptions (or *hypotheses*), stating conditions for the conclusion to be asserted; each assumption recalls one or more presuppositions, to be distinguished between direct and indirect; stating an assumption consists in performing an instance of the second form of category "$\ldots : \alpha$",[2] but using a variable in the place of a proper proof object.

The analysis of assumptions in CTT and their relevance to a study of knowledge processes leads to the intuitionistic notion of truth, particularly in connection to the interpretation due to Martin-Löf, in the new light of the treatment of knowledge and information considered here.

CTT as a framework for knowledge representation and informational systems provides some relevant advantages:

1. Concrete objects (demonstrations) always justify judgements, in the form of proof objects for propositional contents: in general, this means that the

[2] For the mentioned forms of categories, cf. Section 1.5.3.

agent has reasons to state his/her assertions, and the asserted judgements represent, in turn, the notion of *explicit knowledge*.
2. Whenever expressions of the system do not actually contain such objects, as in the case of assumptions of the kind $(x : \alpha)$, one refers to the already introduced notion of *implicit knowledge*, conceptually identified with *information*[3] contexts, by referring to knowledge which needs to be already possessed by the agent, contain information necessary to the actual knowledge state, and therefore one refers to them as "information-bearers", a role that also corresponds to that of "knowledge-candidates".
3. No constraint to a treatment of a finite quantity of information and a finite reasoning power is imposed: knowledge will be treated as ideally infinite by a proper (constructive) mathematical formalization.

A final remark left to be made concerns the notion of "false information", which also allows a better understanding of the connection here proposed between knowledge and information. Informational systems consider every transmission as a transmission of data (eventually disturbed by noise): this never affects the truth-value of the information in itself, which in fact answers to the principle of *Alethic Neutrality*. Information is not true nor false, but still one needs to explain the notion of "misinformation". In this respect, let us consider what happens to the element forming our Informational States, in terms of the following principles:

- The essential property of presuppositions is identified in terms of meaningfulness.
- Assumptions do not have a proper truth-value, because no proper proof conditions is satisfied for them; rather, a value of truthfulness is *assumed* to represent the basis for establishing the truth-value of other expressions (i.e. to get knowledge).

In both cases, our information-bearers do not ascribe truth or falsity directly to the concept of information, which is free from such a determination: presupposition of meaning and assumption of truth represent the context in which truth is proved for some other content. This implies something particularly important on the epistemic and philosophical point of view: information cannot be considered in its truth-value, such that it represents the basis on which the truth-value of something else is established (knowledge); information is used, assumed, conveyed, transmitted, modified, or preserved, and it always has a relation of dependence on the Knowledge Frame in which it is expressed. Every time the content of an Informational

[3] This description clearly differs from Borghuis, Kamareddine, and Nederpelt (2002, p. 475) where the knowledge contained in contexts is explicit, while implicit knowledge is expressed by the statements derivable by inference from such contexts.

State is taken by itself and submitted to verification, it turns into the content of a Knowledge State; on the other hand, this means that the truth-value of a Knowledge State changes whenever some information is assumed in a previous Informational State. From this point of view, the epistemic nature of information appears in all its importance, being absolutely necessary in the determination of proper (correct) knowledge, and clarifying in this way the misunderstanding contained in the expression "false information". This epistemic nature of information is strengthened by the role of the agent in determining relevant informational states: precisely because such states are defined by presupposition of meaningfulness and assumptions of truth, these properties can by definition vary from agent to agent, and even being in conflict among them. This exemplifies further the weak epistemic nature of information as conditional contents for the range of the knowledge state.

3.1.1 *Formalization of Knowledge and Information*

The aim of interpreting CTT as a framework for knowledge representation requires some "translations" of the formal and conceptual terms used within the theory.[4]

(a) *Types are general concepts*: This statement represents the main conceptual basis of the whole idea of using CTT as a language to formalize knowledge processes. Everything which can be expressed meaningfully in the theory, therefore considered as content carrying information, is defined in CTT as predicated within a certain type. This is also coherent with the notion of type itself as presented by Martin-Löf: a type is a meaning-object, introduced in the theory via a category of predication, and each object is typed. The equivalence between types and concepts means essentially that every judgement establishes the predication of a concept: in particular, a *type declaration* establishes the meaningfulness of a concept, by introducing a type as an element of the category of predication "... : *type*". This is formally defined by stating the rules for types, showing what it means for an object to belong to a type and for two objects to be the same within such a type. The distinction between the introduction and the definition of a type will be considered and analysed throughout the present and the following chapters. What is more important for the formal system is to show how every judgement in CTT is based on a series of presuppositions that always need a type declaration: i.e. every judgement containing α as type/predicate (like in

[4] This translation is partially based on that done by Borghuis, Kamareddine, and Nederpelt (2002), but some "meta-theoretical" properties are introduced which are essentially different from their approach.

$a : \alpha$), always presupposes a type declaration of the form $<\alpha : type>$, in order to make sense of the judgements following from it.

(b) *Proofs are justifications for judgements*: A proof within CTT can be considered as a "construction", containing all the information required to make a true judgement. Given that every judgement is asserted on the basis of the act of proving, in general it is always possible within CTT to have all the information which one needs in order to state a judgement of the form "*A is true*". This means that CTT is a logical framework in which information is always reconstructible and actually present to the agent.[5] On this basis, our aim here is to specify the use of the notion of information, by identifying the properties of the concept and recognizing which elements within proofs are information-bearers: in this sense the role of contexts and the understanding of the notion of construction is essential.

(c) *Contexts contain collections of assumptions for judgements*: As explained in Chapter 1, CTT extends the categorical part of the theory by introducing hypothetical judgements. In Chapter 2 it was suggested that assumptions can be taken into account as part of the implicit knowledge on the basis of which new knowledge is derived. Assumptions are assertion conditions for judgements, and the formal representation of the notion of information will be explained by referring to the content of contexts and the operations performed upon them.

3.1.2 *Contexts: Formal Explanation*

In this section I present the formal properties of contexts and explain their role, in order to clarify the formal structure on which we justify the introduction of the notions of knowledge and information. As known, the four forms of categorical judgement valid in CTT

$$\alpha : type \qquad \alpha = \beta : type \qquad a : \alpha \qquad a = b : \alpha$$

can be assumed as hypothetical, under contexts of the general form:

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n); \tag{3.1}$$

a categorical judgement can always be reduced to a hypothetical one under an empty context. Moreover, it is always the case that any context will have its own assumptions; the previous context Γ can be described as based on the context:

$$\Gamma' = (x_1 : \alpha_1, \ldots, x_{n-1} : \alpha_{n-1}). \tag{3.2}$$

[5] This was the general understanding of the notion of information for a constructive system introduced by Sambin and Valentini (1998) with the *forgetting-restore principle*, already mentioned at the end of Chapter 1; cf. also Valentini (1998).

The extension from Γ' to Γ by introduction of a new hypothesis represents one of the main operations in our formal system. Each assumption, being an expression on its own, has also one or more presuppositions, based primarly on (at least) a type declaration.[6]

To proceed in showing the formal structure of contexts, it must be remembered first that a canonical type α is *defined* by prescribing how a canonical object of that type is formed, as well as how two of its equal canonical objects are formed. The relation of equality is reflexive, symmetric, and transitive.[7] Given that α is a type ($\alpha : type$), $a : \alpha$ means that there is a canonical object (equal to) a of the type α (because a can also be in non-canonical form); and if α has a second object b, it is always possible to establish if the truth of the judgement $a = b : \alpha$ holds. These explanations, obviously valid for categorical judgements, are extended to hypothetical judgements by induction on the number of assumptions:

$$\begin{array}{c} (x_1 : \alpha_1, \ldots, x_n : \alpha_n) \\ \alpha : type \end{array} \tag{3.3}$$

means that

$$\alpha[a_1 \ldots a_n / x_1 \ldots x_n] : type \tag{3.4}$$

provided that

$$\begin{array}{c} a_1 : \alpha_1 \\ \vdots \\ a_n : \alpha_n[a_1 \ldots a_{n-1} / x_1 \ldots x_{n-1}]. \end{array} \tag{3.5}$$

This formalization explains the formal operations to be performed in order to validate a hypothetical judgement: such operations can be summarized as substitutions on contexts. That α is a type under the collection of assumptions $(x_1 : \alpha_1, \ldots, x_n : \alpha_n)$ simply means that one has to perform substitutions of the variable $x_1, \ldots, x_n$ with canonical objects $a_1, \ldots, a_n$, provided that these are proper objects of the involved types $\alpha_1, \ldots, \alpha_n$; and each substitution is performed assuming that the previous ones in the context have been performed.

The notion of context has its own formal representation: the calculus of contexts and their conceptual role are now briefly introduced.[8] Such an explanation takes into account two main notions:

- *Context*: formalized by Greek capital letters, a context declaration is of the form $\Gamma : context$; a general context is always represented by expressions contained within brackets.

[6] Cf. Section 3.2.1.

[7] See Sections 1.5.1 and 1.5.2.

[8] The calculus of environments and contexts is presented in a series of unpublished lectures, Martin-Löf (1991a). The formalization and the rules for contexts and environments is extracted and reformulated from there.

- *Environment*: represents the source from which one can choose proper elements (variables) for contexts; small Greek letters will be used to refer to them, corresponding to the relative context; each environment is therefore expressed within a certain context, like in $\gamma : \Gamma$.[9]

Both contexts and environments represent type objects in CTT:

$$context : type \tag{3.6}$$

$$environment : type \tag{3.7}$$

The formal rules concerning contexts and environments begin with the formation rules for contexts:

3.1.2.1 Context Formation Rules

$$(\ ) : context \qquad \frac{\begin{array}{c}\Gamma : context\\ \alpha\gamma : type(\Gamma)\end{array}}{(\Gamma, x : \alpha) : context} \tag{3.8}$$

The first rule is the declaration that contexts can be used as a type, i.e. *context* represents a meaningful predicate. The second rule means that given a context Γ, the formation of a certain type $\alpha : type$ under a certain environment γ belonging to Γ ($\gamma : \Gamma$) is allowed. Obviously, by the identity rule, given $\gamma = \delta : \Gamma$, the same would be true for $\alpha\gamma = \alpha\delta : type$. From the given premises, one concludes that $(\Gamma, x : \alpha)$ represents a new context, i.e. when Γ is extended via a new assumption $x : \alpha$, the variable x being new in Γ. This second rule shows first of all how to enlarge the context expressed in the first premise, a kind of operation which will be at the core of our formal analysis. The declaration $(\ ) : context$ in the first rule is restricted on the basis of the environment from which elements for the context can be extracted, but no restriction actually concerns the formation of environments themselves, like in $\gamma : (\ )$ and $\gamma = \delta : (\ )$. In order to know what $\Gamma : context$ means, one needs of course to consider other judgements of the form $\gamma : \Gamma$ and $\gamma = \delta : \Gamma$. Therefore, the previous conclusion of the formation rule $(\Gamma, x : \alpha) : context$ is known on the basis of some other implicit judgement, namely one of the form $\gamma : (\Gamma, x : \alpha)$, explaining that a certain environment γ exists which contains the proper elements for the context Γ, and that the couple "variable x, environment γ" belongs to the type $\alpha\gamma$, i.e. that $x : \alpha$, x is extracted from the environment γ ($x\gamma : \alpha\gamma$). That $\alpha\gamma : type$ holds is explained by the previous premise $\alpha : type(\Gamma)$. The general meaning of these rules is that the formation of a context is strictly associated with the

[9] Small Greek letters (γ) for environments are not to be confused in the following with α and β, which always represent types.

variables chosen, and in turn with the formation of a proper environment for that context. This is expressed by the following proposition.

Proposition 3.3 (Context Formation) *A collection of assumptions is coherent and can therefore be admissible as a context for knowledge if and only if its elements are chosen from an* environment *proper to the context.*

This principle states that the implicit knowledge expressed by a context contains the relevant information to a proper knowledge state: the restriction on the variables introduced in contexts amounts formally to the common restriction on the novelty of the variables chosen; out of the formalism it shall be considered their difference to the epistemic nature of proof objects. The formation rule goes one step further, namely in the application to environment.[10]

3.1.2.2 Environment Formation Rules

$$(\ ) : (\ ) : context \qquad \frac{\gamma : \Gamma \quad a : \alpha\gamma}{(\gamma, x = a) : (\Gamma, x : \alpha)} \tag{3.9}$$

The first formation rule, in which the first place is to be filled by a letter for an environment and the second by one for a context, is justified by its adoption as an axiom. The second formation rule is justified by the following two judgements:

$$\begin{gathered} (\gamma, x = a) : \Gamma \\ x(\gamma, x = a) : \alpha(\gamma, x = a) \end{gathered} \tag{3.10}$$

The definition of $\gamma : \Gamma$ justifies the first judgement; the second is instead derived by the following rules:

$$\mathbf{R1} \ \frac{\gamma : \Gamma \quad a : \alpha\gamma}{(\gamma, x = a) : \Gamma} \qquad \mathbf{R2} \ \frac{\gamma : \Gamma \quad a : \alpha\gamma}{(\gamma, x = a) = \gamma : \Gamma} \tag{3.11}$$

Each conclusion contained in these rules corresponds to one conclusion of the *rules of computation* for environments.

Computation Rule 1

$$x(\gamma, x = a) = a : \alpha\gamma \tag{3.12}$$

so that $\gamma : \Gamma$ means that $x\gamma : \alpha\gamma$ for every clause $x : \alpha$ asserted under context Γ. The conclusion of **R2** corresponds instead to the conclusion of the other rule of computation.

[10] The notion of *environment* is borrowed by computer science, to represent the logical translation of the notion of a computer memory. Martin-Löf (1991a) refers to the notion as invented by Peter Landin.

Computation Rule 2

$$x(\gamma, x = a) = x\gamma : \alpha\gamma = \alpha(\gamma, x = a) \tag{3.13}$$

The justification of these rules is particularly important: to say that both the computation rules correspond to the rules explained above does not avoid the circularity of the formalization, given that in those rules one makes use of environments. The circularity is instead avoided by showing how contexts are built up by successive *updatings* of the elements contained in it, explained by induction. This means that given the context

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n)$$

the justification of

$$(\gamma, x = a) = \gamma : (x_1 : \alpha_1, \ldots, x_n : \alpha_n)$$

goes by induction on n:

- $n = 0$, base case
- Step: it has to be justified for $(\gamma, x = a) : (x_1 : \alpha_1, \ldots, x_{n+1} : \alpha_{n+1})$

The calculus of contexts is finally built up by the following forms of judgements and updatings

Judgements	Updatings
$\Gamma : context$	
$\alpha : type/\Gamma$	$(\gamma, x = a) : (\Gamma, x : \alpha)$
$\gamma : \Gamma$	$(\gamma, x = a) : \Gamma$
$a : \alpha\gamma$	$(\gamma, x = a, x = b) : (\Gamma, x : \alpha)$

The notion of *updating* represents an essential formal operation within our description, which can be expressed as follows.

Proposition 3.4 (Updating) *Every collection of assumptions on which knowledge is based (its informational state), can properly be enlarged by definitional relations: their role is to extend or simply modify the assumed conditions by which the context is built up and on which knowledge is acquired. Such enlargements are formally represented by* updatings.

In general, the *updating* of an environment involves the inclusion of a definitional relation into the context, i.e. in the context some of the assumptions can be reformulated by stating a value for the variable. This is one of the operations that brings to the *inclusion* of contexts, formally represented by the following judgement:

$$\Gamma \leftarrow \Delta : context \tag{3.14}$$

This formula expresses the fact that the context Γ is included within the context Δ. This means that all the information contained in the context Γ is still available when switching to the new context Δ (or at least reconstructible in it), but not the other way round, so that something present in Δ is new in respect to Γ. The relation of inclusion between different contexts can be formally expressed as follows:

$$\begin{array}{c} \gamma : \Gamma, \quad a : \alpha\gamma \leftarrow \\ (\gamma, x = a) = \gamma : \Gamma \quad \leftarrow \\ (\gamma, x = a, y = b) = ((\gamma, x = a), y = b) = (\gamma, y = b) : (\Gamma, y : \alpha) \end{array} \tag{3.15}$$

where it is clear that starting from the given context Γ, and the element a in α under the environment γ, it is possible to extend the context with new assertions in the second and third line, relative to other variables (y satisfied by the object b). The *inclusion* of contexts corresponds obviously to an operation of *extension*: a context Γ is extended to a context Δ containing more than what is stated in the former. The relation of inclusion/extension of contexts is essential in the formal structure presented here, because it shows the possibility of updating informational states and extending knowledge states. Once contexts are considered as related to each other by such a relation of extension, one obtains a well-founded ordering between all the contexts. It is the aim of the following sections to explain the kind of structure such an ordering presents: this is provided by considering the operations extending a context into another one, the connection of these operations to the structure of syntax and semantics in CTT and to the analytic/synthetic distinction. A last important topic concerns the kind of mathematical structure involved in the representation of (eventually all) such relations.

3.2 Representation of Knowledge and Information

This section extends the analysis of the syntactic structure of CTT, in particular referring to the relation of extension from one context to another. Such an analysis provides the structure on which the epistemic interpretation of CTT in terms of knowledge and information can be performed. In a sentence, CTT will be considered as a framework for the representation of knowledge processes in which the operation of retrieving and extending information plays a central role. Such a representation of the notions of knowledge and information is achieved by referring to the role played by the different syntactical elements (expressions and operations) of the theory; these can be summarized as follows:

1. Presuppositions, which in the basic form are type declarations
2. Assumptions, contained in contexts for dependent types and objects (hypothetical judgements)
3. Operations of context extension
4. Categorical judgements

Each of these expressions, along with the related operations, plays a unique role in terms of the agent's knowledge processes. In the following, a description and an explanation of the theoretical issues involved in the mentioned expressions and operations is presented, in particular referring to the distinction between the notions of *presupposition* and *assumption*, and their epistemic relation to the notion of *premise* within the frame of CTT. The three notions are related to each other, but often their epistemic value is misunderstood. This distinction is essential to describe the different states of knowledge implied in CTT by type declarations, contexts (for dependent judgements), and categorical judgements. This topic leads to a complete understanding of the notion of information in use within the theory, and therefore to its formal description.

3.2.1 *Presuppositions*

The theory of judgement in CTT has been presented according to the order of conceptual priority. It has been explained how, according to the "order of concepts", the notion of type comes conceptually before that of an object belonging to it, whereas in the "order of the real", types are defined only in terms of proper predications, i.e. what it means for an object to belong to that type, and for two objects to be equal objects of that type. Every categorical judgement is therefore *conceptually* based on the judgement which states that the type involved is a proper category of predication: in other words, a predication of the form $a : \alpha$ is conceptually possible only in virtue of the previous judgement stating that $\alpha : type$, i.e. α is included in one of the *categories* of predication, which makes the predication meaningful. This judgement is a *presupposition* of the former: in general a *presupposition* is a judgement whose knowledge is necessary in order for some other judgement to be made, and in this specific sense it represents the judgement which states the condition of meaningfulness for another judgement.[11]

[11] Cf. Section 4.4.1 for an overview of the relations of our notion of presupposition with the branch of logic and philosophy of language known as "presupposition theory"; its treatment is restricted here to the formal analysis.

Formally, *every* basic judgement of the general form

$$a : \alpha \tag{3.16}$$

has the judgement

$$<\alpha : type> \tag{3.17}$$

as its presupposition, stating that α is a type, and therefore making it a concept *apt to be predicated* of objects. That the concept α is *predication apt* means that it is meaningful, and it is at disposal for predication which can be rightly or wrongly made, i.e. it can be either predication of objects truly belonging to the type α or not falling under it. *Meaningfulness* gives rise to predication apt, while one *has the right to predicate* an object of a type if and only if it actually belongs to the type: in this difference lies the extremely important distinction between the introduction of a type and its definition in terms of predications. This first explanation shows that the use of the expression *presupposition* refers essentially to *type declarations*, understanding by this expression the declaration that something is a type (i.e. that a concept belongs to the category of types[12]). In the sense introduced here, the role ascribed to type declarations is that of *meaning determining expressions* in a knowledge system, or briefly of *meaning declarations*: by setting a categorical judgement, one presupposes the declaration that the concepts contained in that expression are meaningful to the agent, i.e. the concept belongs to the agent's set of informations on which basis proper knowledge can be acquired.

Every judgement has of course its own presuppositions and this means that different forms of expression work as presuppositions for other judgements. A judgement may have other categorical judgements as presuppositions, these working as direct or immediate presuppositions for the judgement at hand, possibly recalling other presuppositions, the latter being indirect or mediate presuppositions:

(mediate) presuppositions
↑
immediate presuppositions
↑
judgement

The structure of presuppositions for a certain judgement can be explained as follows. Let Γ be a context of assumptions for a hypothetical judgement:

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n);$$

[12] In Chapter 4 the conceptual difference between a type declaration intended in this sense and the declaration that something is of a certain type will be further explained.

such a context has a number of judgements derivable on its basis; the structure of *presuppositions* for these judgements is given in the following schema.

Judgements	Presuppositions
$\emptyset$	none
$x : \alpha$	$\alpha : type$
(Γ) $\alpha : type$	$\Gamma : context$
(Γ) $\alpha = \beta : type$	(Γ), (Γ) $\alpha : type, \beta : type$
(Γ) $a : \alpha$	(Γ) $\alpha : type$
(Γ) $a = b : \alpha$	(Γ), (Γ) $a : \alpha, b : \alpha.$

According to this list, *presuppositions* take one of the following forms:

- *Type-declarations*
- *Categorical judgements*

Because presupposed categorical judgements also recall other presuppositions, and in the end the basic form will always be reduced to that of a *type declaration*, in general one can say that the *presupposition* of a certain judgement states the availability to predication (i.e. meaningfulness) of the category involved in such a judgement, or in general of the element which is on the right side of the colon in a following type-theoretical judgement. Type declarations represent in this sense meaning declarations within the theory, and for every categorical judgement there is at least one presupposition, namely the one stating the availability of the concept involved by the judgement.

3.2.2 *Assumptions*

The role of *assumptions* in CTT is strictly related to the formalization of hypothetical (dependent) judgements.[13] An *assumption* (or equivalently a *hypothesis*[14]) is basically an alethic notion, according to which a proposition is assumed to be true; on its basis the assertion of the truth of a certain consequent proposition is possible. In this sense a *hypothetical judgement* says that given the judgement "A *true*", a conditional judgement is of the form

$$A \quad true \Rightarrow B \quad true;$$

i.e. given the *assumed truth* of the antecedent proposition A, one infers the truth of the proposition B. Such a relation represents a dependent object: it contains an assertion condition for the consequent, namely the condition regarding the antecedent (truth of the proposition A). Under the propositions-as-sets correspondence, an assumption can be either the declaration of the set over which a free variable ranges or an ordinary logical assumption, i.e. the assumption that a variable works as a construction (proof object) for a certain proposition. In general, when expressed within assumptions, types act as ranges of free variables. Given this general explanation of dependent judgements, the analysis of their assertion conditions is of greatest interest. The definition of a non-dependent object $\alpha : type$ is, as known, given in terms of the application and the identity criteria.[15] The base case in order to explain a hypothetical judgement is that of a type declaration depending on one assumption (*dependent object*)[16]:

$$\begin{array}{c}(x : \alpha)\\ \beta : type\end{array} \tag{3.18}$$

In the case of such a dependent object $\beta : type(x : \alpha)$, it is not required that one actually *knows* an element a belonging to α: the definition criterion holds under substitution of the variable with such an object, but in the analysis of the conditions one needs only to assert such an element to be *knowable*. Knowability means *potentially known*, i.e. it expresses the possibility of finding a proper construction. In other words, this corresponds to the property of being assertable as true. In this sense, assumptions play an alethic role, and to formulate the condition of the consequent only the knowledge about something being a type and not proper knowledge of something in that type is supposed: this is because assuming A *true* requires the minimal presupposition $A : type$, whereas what is not required by the assumption is the proper formulation of the judgement $a : A$. Assumptions in use within

[13] Cf. Section 1.6.
[14] Therefore the related concepts are the same, and the operation on contexts presented by Ranta (1994) as "addition of hypothesis" is the same to the here formulated assumption-introduction.
[15] Cf. Section 1.5.1.
[16] Cf. Section 1.6.1.

CTT are then expressions stating the truth of a proposition, for which its truth-maker (proof object) is not formulated. This means that the assumption states the availability of a certain type to be used as a predicate, in the same sense in which type declarations work as presuppositions: only the actual substitution on the variable by means of a proper construction changes the epistemic status of this expression into a knowledge judgement. One is here only interested in the *information* received by such a predication, because it represents the assertion condition for the truth of the consequent proposition. Assumptions therefore recollect the cases of judgements assumed in order for some others to be made, on the basis of a variable declaration. The generalization of hypothetical judgements with one assumption leads to judgements with an arbitrary number of assumptions collected within *contexts*, where the meaning of the expressions contained there is explained by induction on the number n of assumptions.

A variant on the alethic notion of assumption is given by the operation of assuming something to be known: this has clearly an epistemic value and therefore it is often conflated with the notion of something needed to be known for something else to be known (previous notion of presupposition), or with the notion of something which is known and therefore brings knowledge of something else (common notion of premise). This variant can be labelled as a version of *epistemic assumption*.[17] To assume something to be known means in the first instance to make a stronger kind of assumption, i.e. it refers to assuming something to be *really true*, an assumption of a knowable judgement or else the assumption about possessing a proof object for a certain content: in this case what is presented in a context for a hypothetical judgement is an expression of the form $(a : A)$, from which a dependent judgement is built up. These expressions provide (in the same way alethic assumptions do) assertion conditions for stating dependent judgements. In natural deduction these expressions are equivalent to implications presenting closed derivations for the antecedent. Therefore, whereas in this last case of *epistemic assumptions* one is relying on the fact that it is really possible to provide a proof for the proposition used as antecedent, the case of properly alethic assumptions $(x : A)$ does not necessarily involve the same property.

A first operation on contexts can now be explained, in terms of a rule to introduce new assumptions; every type declaration $(\alpha_n : type)$ in this rule works as a *presupposition* for the judgement which follows, and it allows to formulate the judgement declaring a variable belonging to that type (in

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[17] Precisely this second kind of assumption resembles the commonly intended notion of presupposition as something required by the assertion of something else. Therefore, the distinction between these notions is of a remarkable importance. For the introduction of this distinction see Sundholm (2004).

this case $(x_n : \alpha_n)$), so that the last works as a proper *assumption*:

$$\frac{\begin{array}{c}\alpha_1 : type \\ \alpha_2 : type(x_1 : \alpha_1) \\ \vdots \\ \alpha_n : type(x_1 : \alpha_1, \ldots, x_{n-1} : \alpha_{n-1}) \\ \alpha : type(x_1 : \alpha_1, \ldots, x_{n-1} : \alpha_{n-1}, x_n : \alpha_n)\end{array}}{x : \alpha(x_1 : \alpha_1, \ldots, x_n : \alpha_n)} \tag{3.19}$$

Assertion conditions for the new dependent objects are to be understood once again in terms of the meaningfulness of the concept involved within the assumption: what is assumed consists in the truth of the proposition, knowledge about a concept rather than the analytic knowledge of something in that concept. This is also made clear by considering the presuppositions on the basis of which every assumption is stated, asserting the type as apt to be predicated.

The core of this epistemic analysis consists therefore in asserting that by taking into account the role of assumptions and of the related presuppositions, one is working in terms of *information* rather than in terms of *knowledge*. This argument shall be further justified in the rest of this research, supporting the simple observation that by means of these expressions no assertion is made regarding the elements involved in terms of predication, and only their truth is assumed in terms of the meaningfulness of their concepts. Type-theoretical contexts are thus considered in this framework as containing the presupposed information at the agent's disposal in order to acquire new knowledge. In the following section, a formal mathematical structure is introduced, in order to present coherently the idea of a system in which *extensions of knowledge* are essentially produced by *informational updatings*.

3.2.3 *Types and Meaning Declarations*

According to the basic distinction of judgement forms within CTT, the epistemic structure can be now illustrated, starting from type declarations, i.e. judgements of the form $\alpha : type$: their role consists essentially in the introduction of concepts within an agent's knowledge state. Appropriate justifications for this kind of statements are obviously represented by the rules defining the type: to know the meaning of a type means constructively to know how objects of such a type can be built up and how two objects of such a type can be found to be equal (application and identity criteria). Regarding the conceptual order instead, a type declaration is a presupposition in relation to those definitional expressions: it can be thought of as stating the general (meaning) conditions under which these objects are assertable. In this role, a type declaration expresses the condition of being apt to be predicated for a certain concept, and it does not amount yet to being rightly predicated (i.e. to have the right to predicate): in the rest of this work this condition will be referred to as *meaningfulness*. When presuppositions are involved, *assertion conditions* for judgements amount to *meaning conditions*, something different from the definition of meaning. For this reason, at the basis of a list of presuppositions for a judgement one always finds a type declaration, which can therefore be considered the foundational meaning level of predication. This is due to the conceptual priority of types over objects, which states the meaningfulness for the predication of objects. However, the satisfaction of the condition of meaningfulness is not to be understood as a proper knowledge state: only with a second step it is determined what it means to know that something belongs to a certain type.

Given this summary of the notion of type declaration, its epistemic role in the context of our system can be explained; presuppositions will be presented in the following as *informational meaningful data*, furnishing the meaningful concept for a judgement to be made:

$<\alpha : type>$
↓
Meaning is given to α
↓
$<\alpha : type>$ carries the information that
α is predication apt

According to this explanation of the notion of type declaration, the following principle holds:

Proposition 3.5 (Type Declarations) *A declaration of the kind* $<\alpha : type>$*, intended as a presupposition for other expressions, is an* informational expression*, in so far as such an expression is used or taken into account by the agent in his/her knowledge state without further justification (that means no analytic predication within this type being actually performed).*

Type declarations thus represent the introduction of a concept within the knowledge state of the agent: they express the acquisition of the information about a certain concept which can be used to make possibly meaningful predications, information gained which leads to knowledge if actually instantiated. At this stage, (true) knowledge is not yet involved, because it is still possible that the predication fails to be correct. Once the elements of a certain type are analytically predicated, one speaks of actual *knowledge* gained by the agent. It is already clear that a fruitful and complete presentation of the epistemic properties of the notion of information will be obtained in terms of a careful analysis of the relation between knowledge and the duality actual/potential: the mathematical structure used to represent the knowledge frame will make use of these important concepts.

This analysis leads to one of the more important problems concerning the theory: *knowing* a type is constructively explained by showing what the proper elements of such a type are, and when they are equal to others. A type declaration <α : *type*> represents the presupposition for any statement of the form $a : \alpha$, so that the former is conceptually prior to any statement involving an element of the type α.[18] It is not (rationally) possible to express a predication (therefore to express knowledge) without considering the predicate itself meaningful: in the declaration of something being a type, by means of the reference to the approapriate *category*, the type itself is something filled with meaning.[19] This is expressed in CTT by saying that what is found on the right-hand side of a type-theoretical judgement is *meaning referring*, while the element on the left-hand side of the colon is the instance of such a meaningful expression. An example will clarify the relation of meaning between elements of a predication. In the type-theoretical assertion

man:type

the concept "*man*" is considered as something meaningful in that it is an object of the category . . . : *type*, which allows predications to be accomplished; the concept "*man*" acquires the possibility of being used in a second form of judgement, one of the kind

Borja:man.

In this second judgement obviously the role of the concept "*man*" has changed, expressing now the meaningful concept predicated of the proper Spanish name "*Borja*", saying simply that "*Borja is a man*". In this second judgement, the concept "*man*" is used as the meaning-giving structure, by means of which the name "*Borja*" can be meaningfully used and understood. Finally, one can also state a definitional identity like the following:

my Spanish friend in Leiden = Borja:man

where the definite description "*my Spanish friend in Leiden*" acquires its meaning via the identity to the name "*Borja*", the latter being meaningful according to the previous predication. Let us now consider a mathematical example: the sentence referred to by the Gödel's First Incompleteness Theorem. This theorem states (without going into notorious details) the unprovability of a true sentence within a first-order-class mathematical theory. That means, in type-theoretical terms, that there is a sentence A, for which is known the assertability of the judgement "A true", but for which one cannot construct a certain a such that $a : A$, at least within the given theory (namely because A says that such an a does not exist). This is a case where the type-theoretical notation could give a judgement of the form $x : A$, without being ever able to show an actual a substituting such x. One needs to consider an extension of our theory in order to find such an element which can prove A. If this is the case, the concept "*man*" in the expression

[18] Martin-Löf recalls the *Universalia ante res* principle from the Scholastics.
[19] Cf. Section 1.5.3.

man : *type*, like the pseudo-Gödel's proposition A in the expression $x : A$, is meaningful in a sense that is different from "*man*" in *Borja* : *man* and eventually A in the sentence $\mathfrak{g} : A$ (where $\mathfrak{g}$ would be the construction obtained in a theory of a higher level). This analysis considers therefore every type declaration as an introduction of meaningful informational data within the knowledge state of an agent, which is used to acquire further knowledge in terms of judgements (i.e. eventually enlarging the whole knowledge frame). Type declarations allow for the introduction of information-bearers, or new knowledge candidates.

3.2.4 *Truth and the Role of Assumptions*

The role of assumptions is to express conditions under which other judgements are stated true or, in other words, to make explicit the antecedent conditions under which a certain consequent is asserted: assertion conditions for a dependent judgement are by definition dependent conditions. These conditions are thus expressed by assumptions of the form $(x : \alpha)$ collected in contexts. According to the analysis previously done, the value conveyed by this kind of assumptions is alethic and not epistemic, i.e. they are assumptions made concerning the truth of the predication, not an explicit declaration about knowledge of it: this is explained by saying that one is not constrained to know the judgement $a : \alpha$ when assuming $x : \alpha$, but rather that a possible consequent is stated under the assumed substitution of the variable present in the assumption by any object belonging to the type there expressed. This means, moreover, that if the consequent has to be true, such an object must be present, even if it is not known and not expressed in the condition. In other words, an assumption containing a variable expresses the information necessary to derive the new judgement, the assertion condition for the consequent: assumptions can thus be described as informational meaningful data, in that they do not present any proper knowledge gained by the agent, but rather the information which must be possessed in order for some truth to be evaluable. This is expressed by the following schema:

$(x : \alpha)$
↓
α (meaningful under presupposition $<\alpha : type>$)
is assumed as true
↓
$(x : \alpha)$ carries the information
that $\ldots [x/a] : type$ is true

according to which the expression of an assumption leads to the assumed truth of a predication containing that type (α) and thus conveying the information necessary in order to state the truth of another predication (of a dependent object). Here the term information underlines the value of the assumption, which does not express proper (actual) knowledge. On the basis of the explanation given for assumptions, the following proposition holds.

Proposition 3.6 (Assumptions) *Contexts containing assumptions are collections of* informational expressions*: their content conveys information about conditions for the knowledge gained by the agent. Such information is expressed as declarable truths by means of variables predicated in a type* $(\ldots(x : \alpha))$*; contexts can be extended by the introduction of new information and set conditions for the acquisition of knowledge.*

It will be an important aim of the following sections to illustrate the differences that occur in the extensions due respectively to information and knowledge. The alethic value involved in the concept of assumption leads straight to the analysis of the concept of truth holding in CTT. At this point the distinction introduced by Martin-Löf between potential and actual truth can be explained, by means of the technical role played by assumptions, in this way introducing the theoretical meaning of the notion of information. The notion of truth in the constructive philosophy is explained in terms of notions of existence of a proof, as expressed by the **Verificationist Principle of Truth** (Principle 1.1):

(a) *Truth simpliciter*: on this level the constructive notion of truth holds, according to which

$$A \text{ is true} = Proof(A) \text{ exists.}$$[20]

On the basis of this definition the constructive approach introduces different epistemological states of the definition of truth (but this is already true of Brouwer's Intuitionisms). In general, it holds the equivalence between truth and knowledge, i.e. truth in the constructive sense is interpreted as what is known because a proof of it is at our disposal, true being what has been demonstrated (Dummett's principle). This explanation amounts to an interpretation of the correcteness of the judgement candidate "A is true", in a twofold distinction:

(b) *Actual truth*: that a proposition A is *actually* true means that A has been proved, namely that a proof a of A has been constructed $(a : A)$, and this amounts to saying that A *is known to be true.*

Actuality therefore represents a state of our attitude towards truth, being actual truth provided by a proof object already constructed, and therefore known. This establishes the notion of truth from an epistemic perspective, in relation to our knowledge state: truth realized means that knowledge is acquired. Complementary to this notion of (actual) truth, a notion of

[20] Sundholm (2004, pp. 449–450).

potential truth is introduced. According to Martin-Löf,[21] potentiality in the intuitionistic sense is not potential existence in time (before, now, or later), but rather the potential existence of a proof. For example, when a proof which realizes the truth of a certain sentence is not possessed, one can still say that such a sentence is knowable, i.e. it is demonstrable (when a proof of it is constructed):

(c) *Potential truth*: that a proposition A is *potentially* true means that A is knowable, it can be proved, a proof of A can be constructed; this amounts to saying that the judgement A *is true* is demonstrable, or that the judgement A *has a proof* can be known, therefore that a proof of A can be found.

The distinction between actual and potential truth does not amount to a proper separation of cases within a general notion of truth: in the constructive framework only one explanation is given for the notion of truth, namely that of existence of a proof. This separation between actual and potential truth is just a way to take into account those cases where the agent is not actually in possession of the proof which realizes truth, still being able to take into account such a truth as knowable: they represent a descriptive method for the different epistemic states occurring in a knowledge system interpreted by a rational agent, whose approach to knowledge (and in turn to truth) is relatively more complex than the somehow rigid description furnished by the constructive definition of truth. And the notion of information here introduced aims at defining these cases. To express states of knowledge somehow "weaker" than that represented by proofs in the first instance resembles assumptions: when a hypothetical judgement is stated, assertion conditions for a certain judgement are expressed, conditions represented by the truth of another judgement which involves the same or another concept (type). In order to lay down this hypothetical judgement, one needs first to know the conditions under which the judgement can be stated. This does not amount to know properly the terms or elements involved in those conditions, i.e. to have actual constructions for the elements realizing the truth of the conditions: rather, one just assumes those conditions to be satisfied, thus considering the proposition involved as possibly known and therefore potentially true. In this sense it is quite clear that the notion of truth involved by an assumption is that of potential truth: when assuming $x : A$ in a certain context Γ ($\Gamma = (x : A)$), one is assuming the existence of a configuration (construction) able to actualize a certain proposition A or, in other words, assuming that one is able to show an actual proof object a for A. Such a configuration can be understood as the collection of conditions on the basis of which A is true. To make a judgement of the kind $a : A$ (even within a context) means instead to possess already a configuration which makes the proposition A true; therefore, the conclusion traced starting from

[21] Martin-Löf (1991).

it is the judgement "*A is proved/known to be true*". Clearly there is a deep epistemic difference between these two kinds of expressions.

It is therefore essential, in order to understand the role of assumptions and the epistemic meaning of contexts in general, to refer to the notion of potential truth. This notion will moreover be taken into account once again when the mathematical structure resulting from this epistemic description is analysed, especially in relation to the problem of defining the general notion of knowledge. The relation between actual and potential truth can be moreover specified: what is stated in a context is assumed as logically possible, so that assumptions express information regarding the assertion of the existence of those real configurations making a certain proposition true. Such configurations are type-theoretically understandable in a description which treats them as *states of possible worlds*,[22] and in the next section such a comparison will be rendered briefly in a formal structure.

This analysis spells out first of all the epistemic difference between the parts of a knowledge state represented by an expression of the form $b : \beta(x : \alpha)$. The notion of potential knowledge introduced above renders a special meaning to the role played by assumptions in our knowledge framework. When an instance of a proof object for a certain content is not yet presented, knowability (demonstrability) is involved, not yet knowledge (demonstration), but it is presupposed that one has all that is needed to find out and acquire such knowledge; therefore, this notion of information represents the agent's epistemic state before acquiring actual knowledge. It can be thought of as the possession of all the meaningful concepts of the theory and suppositions of truth, without necessarily being aware of (or taking into account) justifications for them. From this point of view, and given the conceptual explanation presented in the previous sections, it is possible to draw the following principle:

Proposition 3.7 (Information as meaningful Data) *The process of acquiring actual knowledge by a rational agent is based on using* informational meaningful data*, rendered formally and explained conceptually by type declarations and assumptions.*

This is an informal explanation of what an agent is able to recall in the set of knowledge contents as a starting point (assumed or presupposed knowledge) in order to gain new knowledge, or eventually as "*contributory information*" in some knowledge process which requires external data. In these cases there is no need for the agent to show canonical elements justifying such information: these contents are required to be meaningful, and therefore they have some other conditions to be fulfilled (the above-mentioned presuppositions). The next task is to give a more extended and formal analysis of these informational contents, and to consider the mathematical structure at the basis of the relation between information and knowledge.

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[22] Cf. e.g. Ranta (1991).

3.2.5 *Defining Information*

A more systematic introduction and explanation of the notion of information within a knowledge system for CTT is required at this point. According to what has been introduced throughout Sections 3.1 and 3.2, the formal elements considered are:

(a) Type declarations
(b) Assumptions within contexts
(c) Categorical (derived) judgements

Each of these expressions formally represents an element of the epistemic state in the agent's knowledge process: in such a process assumptions and presuppositions are explicitly considered as meaningful informational data (cases (a) and (b)); this is the information used by the agent in order to acquire actual knowledge (case (c)). This distinction is explicitly stated via the following propositions, starting with case (a):

Proposition 3.8 (Type Declarations II) *The collection of all judgements expressing type declarations of the form* $<\alpha : type>$ *working as (epistemic) presuppositions in a knowledge process are to be considered as meaningful informational data accepted by the agent; their role is to introduce concepts in order to make sense of the actual knowledge the agent is able to acquire.*

Case (b) considers the role of the following assumptions:

Proposition 3.9 (Assumptions II) *The collection of assumptions contained in contexts of the (basic) form* $(x_1 : \alpha_1, \dots, x_n : \alpha_n)$ *is to be considered as meaningful informational data accepted by the agent and stating (alethic) conditions for other judgements to be made.*

A general proposition stating the role of assumptions and presuppositions in the frame of a knowledge process can be now formulated:

Proposition 3.10 (Informational State) *All the expressions described by Propositions 3.8 and 3.9 represent information about meaningful concepts and concerning the hypothetical instantiation of such concepts the agent has at his/her disposal within a knowledge process. This collection contains the agent's implicit knowledge and expresses his/her* informational state *(**i-state**).*

The last case (c) refers instead to a different epistemic state: it is to be understood as a declaration of knowledge obtained via information processing. In this case one switches from meaningful presupposed information to actually possessed (proved) knowledge:

Proposition 3.11 (Categorical Judgements) *All justified judgements of the form $a : \alpha$ and $a = b : \alpha$ represent assertions using the content of informational states (Proposition 3.10) to derive knowledge of the concepts involved, supported by elements or proofs actually possessed and therefore known.*

From this first epistemic description of categorical judgements, it is possible to derive their role in the knowledge process:

Proposition 3.12 (Knowledge State) *The instances of categorical judgements referred to in Proposition 3.11 are to be thought of as the explicit knowledge acquired in the process and referred to as the agent's* knowledge state ***(k-state)****.*

According to these explanations of the agent's epistemic states – formulated in terms of **i-states** and **k-states** – it seems obvious to understand the former (Proposition 3.10) as related to the latter (Proposition 3.12): the epistemic notion of information is definable and graspable starting from, and as a part of, the notion of knowledge. Information is primarily explained as the set of conditions (meaningfulness and alethic conditions) on the basis of which knowledge is acquired. The informational state of an agent is the necessary basis upon which knowledge is acquired, and every **k-state** is (at least implicitly) based on an (at least one, possibly empty) **i-state**. Finally, the whole result of the knowledge process, as the sum of the information used and the knowledge acquired, can be expressed as follows:

Proposition 3.13 (Knowledge Frame) *A* knowledge frame ***(k-frame)*** *is the result of an entire knowledge process, executed by a rational agent on the basis of eventually many* ***i-states*** *from which several* ***k-states*** *are deduced. To be a proper* ***k-frame****, such a result needs to collect coherently the content of the different* ***k-states****.*

The following schema presents the relations between **i-states** and **k-states** in a **k-frame**:

$$\begin{array}{c} < \textit{i-state}_1 > \\ \downarrow \\ \textit{k-state}_1 \end{array}$$

$$\begin{array}{ccc} < \textit{i-state}_1 > & \ldots & < \textit{i-state}_n > \\ \downarrow & \downarrow & \downarrow \\ \textit{k-state}_1 & \ldots & \textit{k-state}_n \end{array}$$

$$\begin{array}{ccc} \textit{k-state}_1 & \ldots & \textit{k-state}_n \\ \searrow & \downarrow & \swarrow \\ & k\text{-}frame & \end{array}$$

On the basis of this interpretation, the following general declaration introduces the properties of information within the knowledge frame:

Proposition 3.14 (Structure of Information) *The structure of the agent's informational state represents an interconnected collection, in which information is accepted on the basis of the meaningfulness of the concepts involved, is accessible from previously acquired information and can be enlarged.*

The next step is the formal justification of such a proposition, especially in terms of the description of the operations performable on informational statements.

3.3 Contexts as Constructive Possible Worlds

Contexts in CTT express thus the conditions under which the asserted judgement represents new knowledge. This explanation of the role of formal contexts reflects the mentioned equivalence between contexts and possible worlds. It will be shown how the common mathematical structure handling a set of related possible worlds is equally useful to describe the connection between different **i-states** and their link to **k-states** in our epistemic model. There are essentially two ways of explicating the notion of possible world:

(a) By considering it as the maximal consistent set of sentences
(b) By invoking the notion of interpretation

Within propositional logic sets of propositions is the most natural candidate for representing possible worlds: a possible world can be, in the sense proposed by (a), the set of all propositional variables true in a given world, and therefore a set of atomic sentences. In order to avoid inconsistency and incompleteness, such a set must be determined as the maximal consistent set of sentences. This amounts to a syntactic definition. In the second sense, a possible world is just a complete interpretation concerning how the things can be in the widest possible interpretation (logical possibility). In a classical perspective, the first sense of possible world is just a special case of the second, because maximal consistency is gained by the widest possible description, just by cutting out the cases raising inconsistency within the world. Carnap's *state descriptions semantics* is the most common logical representation of this notion of a possible world.[23] State descriptions are built up by conjunctions of basic statements, containing each statement or its negation, but never both, and nothing else. A *possible world* then represents the semantic closure of a state description into the language (such that it has to maintain consistency). As a basic example, for a first-order

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[23] Carnap (1947, 9ff.; 1950, 70ff.).

predicative language one can think of a language containing only two predicates P and Q, and only one individual constant a; in a language built up by this vocabulary, a state description will contain the following list of expressions:

$$(Pa, Qa) \vee (Pa, \neg Qa) \vee (\neg Pa, Qa) \vee (\neg Pa, \neg Qa)$$

A constructive representation of possible worlds is sized in respect to what can be taken into account as "possible" in the proper sense of the word.[24] A quite natural simplification of such a state description is then to consider only those predicates which are actually instantiated within the description.[25] In this way one does not take into account those predicates not inhabited by any constant, but only express those assertions which are actually stated. The previous list is then reduced to one of the following:

$$\begin{gathered}(Pa, Qa), (Pa), (Qa)\\ (Pa, \neg Qa), (Pa)\\ (\neg Pa, Qa), (Qa)\\ (\neg Pa, \neg Qa)\\ \emptyset\end{gathered}$$

Let now our language be composed of the set of predicates $\alpha_1, \ldots, \alpha_k$, and let us consider the possibility that each of these predicates could be instantiated by a witness (construction) $a_1, \ldots, a_k$. If all these constructions are instantiated, the (actual) configuration of the resulting state description is the following:

$$a_1(\alpha_1), \ldots, a_k(\alpha_k)$$

On the other hand, if one takes into account seriously the notion of potentiality, the predicates are potentially instantiated only in terms of the relative constructions, which turn out to be variables. In this case it seems obvious that the state description would have the same structure of a context within CTT:

$$(x_1 : \alpha_1, \ldots, x_k : \alpha_k)$$

The idea at the basis of the formal development which follows is that the relations between contexts and the **k-states** obtained from them (i.e. the connection between information contained in contexts and the knowledge resulting from it) can be described by a possible-worlds semantics.[26] The

[24] Rescher's constructive approach to possible worlds starts with possible individuals and considers possible worlds with this population. The population of possible worlds then consists of possible individuals (of a fully individuated type). In such a description it is not the case that any and every collection of possible individuals constitutes a possible world. Cf., e.g. Rescher and Brandom (1979) or Rescher and Parks (1973), or for a more recent view Rescher (1999).

[25] Cf., e.g. Hintikka (1963, 1967).

[26] The same idea on the formal structure of contexts has been suggested first by Ranta (1991); his description is here applied to the concepts of knowledge and information.

essence of the analogy with a possible-worlds semantics is the following: contexts are possible worlds in which judgements are derived, so that each judgement stated true by the theory is true in a certain world, namely the one providing all the informational data needed in order to acquire the knowledge contained in that judgement (i.e. the information which expresses the conditions to verify the propositional content of the given judgement). This world is namely expressed by a context. Worlds can be enlarged via two formal operations:

1. *Informational updating*, which provides new information by means of operations performed on contexts
2. *Knowledge extension*, which provides new knowledge by logical derivation on the basis of some context.

These extensions of worlds are *potentially infinite*, but they can be represented only by finite states. A **k-frame** represents the agent's actual knowledge (world), based on a certain amount of information (i.e. holding in a certain context); in this description the agent acquires only and always a finite amount of information or knowledge at each stage, by considering finite approximations to the set of all the possible extensions his/her world can be submitted to. Nevertheless, even though the actual extensions are always finite, it is conceptually possible to take into account the collection of all the possible extensions of **k-states**, thus considering not the single extensions but rather the mathematical structure in its potential entirety: this would amount essentially to representing the totality of the agent's interconnected **k-frames**; this totality does not ever exist for a single agent, who possesses only finite parts of it. In other words, an agent can have access only to finitely representable approximations of such a totality. Therefore, the process of extending individual knowledge states is analysed within the totality of knowledge K intended as the entire (possible) world. The mathematical structure underlying this conceptual description of the acquisition of knowledge and information can be described in terms modelled after a Kripke semantics.

3.3.1 *Introducing Orderings: Kripke Models*

The idea of representing possible worlds associated to stages in a knowledge process leads to a further specification in terms of the mathematical properties of this structure: the extension of contexts intended as a possible worlds semantics can be simply and successfully formalized in terms of the structure of a Kripke semantics. Such a semantics is described by the so-called Kripke models, typically intended as orderings, i.e. non-empty partial orders for assignments of variables. Formally, a Kripke model is a certain structure of the following kind[27]:

$$M = (S, R, V)$$

in which S represents the collection of states, R is a relation between states, namely the partial relation of accessibility, and V is the valuation function, expressing the assignment of a variable to a certain state, intuitively making the variable valid at that state.

The knowledge system introduced above has the same kind of structure of a Kripke model, isomorphic to an agent knowledge system: the entirety of an agent's knowledge is contained in a **k-frame**, possibly composed by (eventually) many **k-states**, each expressing the validity of the formulae contained therein (knowledge possessed by the agent); a relation of accessibility or reachability is admitted between these different **k-states**, so that if a certain state k_2 is reachable from a preceding state k_1, then k_1 is embedded in k_2 and every formula holding in the former holds also in the latter ($k_1 \leftarrow k_2$). The relation of accessibility has to be thought of as the way to extend knowledge, maintaining coherence in the passage from one state to another; this represents the essential property (and limitation) of the standard model represented according to the constructive framework: the **k-frame** is properly extended only switching to states that are not in contradiction to previously acquired knowledge; in other words, monotonicity holds in a Kripke model for the collection of states S belonging to the model M. In such a model, when a formula A is true at a certain state S_n, it is said that S_n *forces* A ($S_n \models A$), with definition by induction on the construction of the formula A.[28] According to the property of monotonicity, therefore, if a certain formula is forced at one node, it is forced at all greater nodes. This actually shows that Kripke models are preserving structures. In our case this property would mean that what is acquired as knowledge content at any of the agent's states, should be maintained by every possible extension (i.e. under logical consequence), or that nothing incoherent with already acquired contents could be admitted. The problem is clearly represented by the admissibility of errors and procedures of revision: it can be solved by considering the operations performed on contexts referring to informational

[27] Kripke (1963, 1963a).

[28] A full definition for an intuitionistic semantics of a Kripke structure $M_i \models A$ is, for example, the following:

- If A is atomic, $M_i \vdash A$ iff $M_i \models A$; $M_i \not\models \bot$;
- If $A = a \wedge b$, $M_i \vdash A$ iff $M_i \vdash a$ and $M_i \vdash b$;
- If $A = a \vee b$, $M_i \vdash A$ iff $M_i \vdash a$ or $M_i \vdash b$;
- If $A = a \rightarrow b$, $M_i \vdash A$ iff for every $M_i \preceq M_{i+1}$, if $M_i \vdash a$, then $M_{i+1} \vdash b$;
- If $A = (\exists x)a(x)$, $M_i \vdash A$ iff there is a certain $b \in M_i$ such that $M_i \vdash a(b)$;
- If $A = (\forall x)a(x)$, $M_i \vdash A$ iff for all $M_i \preceq M_{i+1}$ and all $b \in M_{i+1}$, then $M_{i+1} \vdash a(b)$.

Cf., e.g. Buss (1993) for an intuitionistic treatment of Kripke structures.

statements, assumptions for judgements, and their direct presuppositions. This epistemic model offers therefore the possibility of taking into account the process of possible revision to which knowledge is often submitted. By introducing the notion of information as previously defined, one provides an extension of the constructive epistemic model by elements that can be properly submitted to revision in a type-checking operation performed by the agent. *Information* is defined by a weaker epistemic status; it allows for procedures of checking, without involving the much stronger notion of *knowledge*: contents already accepted as justified knowledge should be considered as *certain* knowledge, holding under monotonicity.[29]

3.4 The Knowledge Framework

3.4.1 *Updating Information, Extending Knowledge*

As already proposed by Ranta,[30] the extension of a context or interpretation of a context into a new one

$$\Gamma \leftarrow \Delta$$

can be described according to three cases:

(a) *Introduction of a concept* (type-declaration formulation)
(b) *Addition of a hypothesis* (assumption introduction)
(c) *Addition of a definition* (variable substitution)

These operations increase the knowledge content that the agent is able to derive[31]: the aim here is to extend this explanation by making explicit the increase of information which allows the acquisition of new actual knowledge. The three cases of accessibility relation between contexts formally correspond to mappings between one context and another containing new expressions.[32] The formal operations can be described as follows:

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[29] Among others, Jan Smith has recently treated in a series of lectures and notes the constructive properties of Kripke semantics. The following properties are partially studied according to his treatment. The property of monotonicity and the introduction of a revision procedure on this epistemic structure is proposed in Primiero (2006), where a double version of a *Restricted Monotonicity Principle* is introduced.
[30] Ranta (1994, pp. 145–147).
[31] Ibid., p. 145.
[32] A mapping is considered by Ranta (1988) as an association of all variables of the first context to variables of the second context. The mapping must then leave at least one new variable without any previously associated one in the new context, in order to be an actual enlargement of the former. Even though case (c) in the previous list is mostly a reduction of the number of expressions in the context, its explanation can be given in terms of an operation of information introduction producing knowledge content.

(a) The introduction of a concept is formally obtained by setting a type declaration introducing a new type into the context. [33]
Addition of a concept: the context

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n) \tag{3.20}$$

is extended to

$$\Delta = (x_1 : \alpha_1, \ldots, x_n : \alpha_n, < \alpha_{n+1} : \mathit{type} >). \tag{3.21}$$

In general, the introduction of a type is to be understood as a presupposition for using a new type in a following extension, e.g. the extension produced by the introduction of a hypothesis using that type. Nevertheless, it is important to identify the logical structure, which requires such an extension before any introduction of hypotheses containing new types is possible. This means conceptually that any introduction of hypotheses is always based on the proper presupposition which introduces the related concept.

(b) The second case shows the extension of a context by the addition of a *hypothesis* (or *assumption*), whose basic conditions are:

1. The presence of the proper presupposition introducing the type involved in the assumption (if the assumption uses a new type), i.e. what is expressed by the previous point (a)
2. The novelty of the variable used in the new assumption, as the normal condition imposed on the choice of variables

Addition of a hypothesis: the context

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n) \tag{3.22}$$

is extended to

$$\Delta = (x_1 : \alpha_1, \ldots, x_n : \alpha_n, x_{n+1} : \alpha_{n+1}). \tag{3.23}$$

When a context is enlarged by the introduction of a hypothesis $x_{n+1} : \alpha_{n+1}$, new meaningful information is introduced, on whose basis knowledge can be acquired, by derivation of some new judgement or even by substitution of the variable with some constant (cf. next case (c)).[34]

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[33] It is worth remembering here that the formal representation of a context should express the relation of dependency between assumptions; thus, a proper context should be written as follows:

$$\Gamma = x_1 : \alpha_1, \ldots, x_n : \alpha_n(x_1 : \alpha_1, \ldots, x_{n-1} : \alpha_{n-1})$$

saying that the set of assumptions $x_1, \ldots, x_n$ is based on the previous assumptions up to x_{n-1}. To simplify the formalization I will avoid explicitly expressing this condition.

[34] Cf. Ranta (1994, p. 145).

(c) The third case shows the enlargement of a context via the introduction of a *definition*.
Addition of a definition: the context

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n) \tag{3.24}$$

is extended to the new context

$$\Delta = (\Gamma, x_k = a : \alpha_k), \tag{3.25}$$

so that in the new context every occurrence of x_k (where the index k is such that $1 \leq k \leq n$) is replaced by $a : \alpha_k$, which can make the new context *shorter* than the first. Within the formalism this amounts to the disappearing of an expression of the kind x_k for an expression containing $a : \alpha_k$ having the variables $x_1, \ldots, x_n$ as its assumptions. This seems to show the growing of knowledge via the introduction of the constant a into the language: this operation furnishes the value of one of the variables, reducing the uncertainty within the context. This new element can be regarded as actual knowledge until part of the information (at least one hypothesis) on which it is based is disregarded and therefore the knowledge in question does not have enough compelling reasons to be accepted.

The formal operations introduced represent the way of *interpreting* one context into another, which results in the conceptual operation of *informational updating*.[35] The general operation of interpretation between contexts is expressed as follows:

- *Interpretation of a context into a new one*: the context

$$\Gamma = x_1 : \alpha_1, \ldots, x_n : \alpha_n(x_1, \ldots, x_{n-1}) \tag{3.26}$$

is extended to

$$\Delta = y_1 : \beta_1, \ldots, y_m : \beta_m(y_1, \ldots, y_{m-1}); \tag{3.27}$$

via the following sequence of definitions:

$$\begin{gathered} x_1 = f_1(y_1, \ldots, y_m) : \alpha_1 \\ \vdots \\ x_n = f_n(y_1, \ldots, y_m) : \alpha_n(f_1(y_1, \ldots, y_m) \ldots f_n(y_1, \ldots, y_m)) \end{gathered} \tag{3.28}$$

The interpretation can thus be obtained by one of the three operations considered setting the function between each element of the old context

[35] The operation of *informational updating* and the role of presuppositions will be further explained in Chapter 4, in connection to presupposition theory and meaning theory.

and each of the new one, or it can be considered as the translation of the elements of the old context into the elements of the new context, via a series of definitions. This amounts to *switching the type of information* used, normally considered as a way of learning, also for inductive inferential systems and analogical procedures.[36] In the most common case, the procedure of interpretation of contexts amounts to establishing the value of the variables, thus *completing* all the information (reducing uncertainty to zero), and providing all the knowledge implicitly contained in the **i-state**. The meaning of the formal structure interpreted as a knowledge process leads to the following observations concerning the interpretation of contexts:

- To interpret a context into a new one reveals that the new context has the same basis-structure of the old one: in this case the agent learns that he can operate on new information (provided by the new context) by considering valid the operations performed in the old framework.
- The new context can eventually show some properties holding also in the old context: this brings us to a "backward" process of acquiring knowledge (and can eventually be the starting point for a revision procedure).
- This procedure is also essential in revealing foundational changes in scientific paradigms: it is possible that the mapping from the elements of a certain context Γ to the elements of Δ shows some new concepts $\delta_1, \ldots, \delta_n$ which were not assumed as meaningful informational data in the previous context. This actually enlarges (at least at the level of understandable information) the set of concepts available to the agent.

The explanation of the mapping between contexts

$$f : \Gamma \leftarrow \Delta \tag{3.29}$$

introduces therefore the operation of *informational updating* as preliminary to that of *knowledge extension*. The necessary condition, in order for the extension to be a proper one, is that the new context shows some new *information* in respect to the previous one, either in terms of a new concept (type declaration of the form $<\alpha : type>$), or via new hypotheses (assumptions of the form $(x : \alpha)$), or via definitions completing the information contained in assumptions (via definitional equalities of the form $x = a : \alpha$). In this sense, the new context is an "*informational alternative*" of the previous one,[37] because such a context is able to furnish a larger **i-state**, potentially leading to a wider **k-frame**. The possibility that a knowledge extension is performed entirely within the same informational state, i.e. without any previous operation of informational updating being performed, means that the extension is potentially purely analytic.

Acquiring knowledge is therefore possible in our framework according to the cases shown, where each seems to explain one of the usual ways in

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[36] Cf. Jain and Stephan (2001).

[37] Ranta (1988, p. 147) speaks about "epistemic alternativeness", recalling Hintikka (1962).

which rational agents extend their set of informations, the basic enlargement of progressive finite states of total knowledge. The following section will show the mathematical structure underlying the operation of accessibility between contexts, represented as a relation between finite approximations of an infinite totality.

3.4.2 *The Structure of Knowledge*

The collection of contexts or **i-states** on the basis of which the agent develops his/her **k-states** has been formalized using a (constructive) possible-worlds semantics, defining the extension from context to context (mapping or interpretation) as the relation of accessibility between states holding within that semantics. These states are to be thought of as contained in a structure, whose nodes are represented by **i-states** and **k-states**, and which can be generalized by interpreting nodes as the different agents' **k-frames**: this allows for a change from a single agent to a multi-agent structure. This would represent the structure of all the agents' possible knowledge and information contents. It is the aim of this section to describe the mathematical structure of such a whole, according to a constructive treatment. All the elements expressed by a (constructive) state description are of course enumerable and, in general, state-description semantics considers all such descriptions as countable.[38] Each state description will be finite, thus having cardinality equal to a certain finite subset of ℕ. In a constructive logic, one obviously cannot speak of the set of all subsets of natural numbers, because there is no constructive way of taking into account a whole infinite set. But there is a mathematical formalization allowing the treatment of sets with infinite cardinality in terms of finite approximations to it. Such a treatment is taken into account here in order to make some general remarks on the complexity of this representation of a knowledge system.

The explanation of the relation between states and the description of the resulting total structure of knowledge can be considered holding on two levels: in the first instance, the theory describes the structure of all the possible **i-states** for a single agent, the accessibility relation between these states, and the acquisition of new knowledge states. In other words, one considers elements of the structure the predicates or types used in contexts by the agent, i.e. recognized as information, and the extension of this possible information is described by referring to the operations performable on contexts finally, one considers the formulation of **k-states** from initial **i-states**, by referring to the essential notion of logical (analytical) derivation, by the role of constructions for propositions, on which the constructive idea of truth and knowledge is based. This is what has been pursued up to now. In this second part, the structure will be presented in its generalization,

[38] Cf. Holm (2003).

representing the totality of all the possible information stored and still acquirable by the different rational agents, and the knowledge which can be produced on this basis. This generalization refers therefore to a merging of all the agents (the result of the different agents' epistemic states) and the nodes of this structure are some sort of "collective" knowledge states, their extensions due to an operation of informational updating.[39]

According to the constructive way of treating the infinite,[40] one can consider, in the first instance, the set of all contexts (**i-state**) as infinite, by taking into account each (collection) of them as a finite approximation to the infinite collection: this amounts to referring to the infinite as the non-terminating process generating a certain series.[41] Let us consider the set of all the **i-states** occurring in a certain system (individual or collective) as infinite, by referring to a function which expresses an accessibility relation from every state with cardinality n to its successor with cardinality $n + 1$. One obtains therefore in the first instance the entire structure of **i-states**, i.e. the structure obtained by operations of informational updating in which the function is expressed by any of the mentioned operations of interpretations of contexts. The collection of ordered **i-states**, each of them being

[39] The idea of this representation actually goes beyond the mathematical formalization, assuming a rather important philosophical perspective. Once the structure is presented as the formalization of a multi-agent system, it is clearly necessary to establish the nature of the agents belonging to it, i.e. it must be explicated on which criteria it is admissible for an agent to be part of such a system. This problem requires therefore a definition of the notion of system in which these agents are defined, the basic social structure able to define acceptability of knowledge, and to recognize the meaningfulness of information. If one is in search for multiple criteria of acceptability, the structure at hand must be one which is less invasive than a system as defined, for example, in natural sciences: one thinks here of different *collectives* with proper criteria of meaningfulness and shared procedures of provability; this leads to models admitting different cognitive contents and interpretative paradigms. Moreover, this idea of collective requires that the basic conditions for an agent to be part of it are made explicit: a starting point for expressing these criteria is monotonicity, as the principle allowing an extension from state to state to be accepted, reinterpreted as making an agent part of the collective (and thus of the shared informational and knowledge contents). In a collective there must be the individual capability (even necessity) to revise knowledge: also for this reason the criterion of monotonicity needs to become rather flexible, allowing changes of conceptual paradigm via the notion of information.

[40] Cf. Dummett (1977, p. 40):

> *In intuitionistic mathematics, all infinity is potential infinity: there is no completed infinite.*

[41] Ibid.:

> *to grasp the process which generates it, [...] to refer to such a structure is to refer to that process, and [...] to recognize the structure as being infinite is to recognize that the process will not terminate.*

accessible from the previous one, can be represented as follows:

$$I = \{i - state_0 \xrightarrow{f_0} i - state_1 \xrightarrow{f_1} \cdots \xrightarrow{f_k} i - state_{k+1} \ldots\} \tag{3.30}$$

In this model I is the totality of Information: it is useful (and coherent to the constructive description) to interpret this sequence as potentially infinite, which allows to consider the operation of informational updating as the different ways in which knowledge can be conveyed and transformed according to conditions and meaning. The infinity of the model is obviously given by the infinite applicability of the function f, defined by the above-mentioned operations of informational updating.

The second operation considered is that of knowledge extension, provided by the derivation of new judgements on the basis of contexts; the model can be considered as based on a single or on a multi-agent (collective) system: this produces a structure of updated knowledge states obtained by functions from one **k-state** to another. In this structure of knowledge every statement true in a certain context of finite cardinality (i.e. a judgement holding at a certain stage of the collection under a certain informational state of the previous model) will in general be true at any further stage of the set, satisfying in this way the property of monotonicity seen for Kripke models. This does not exclude the applicability of the revision procedure on the informational sequence mentioned above, on whose basis the knowledge sequence is obtained; in both the structures, any state represents an approximation to the total sequence. Therefore, in this second model, on the basis of extensions of **i-states** considered earlier and represented in this model by the function f, knowledge is also increased: this is shown in the formalization by the star function f^*, which refers to the operation of derivation. To extend **k-states** results in the process of producing the ideally infinite totality of knowledge K:

$$K = \{k - state_0 \xrightarrow{f_0^*} k - state_1 \xrightarrow{f_1^*} \cdots \xrightarrow{f_k^*} k - state_{k+1} \ldots\} \tag{3.31}$$

Every **i-state** pertains to a proper **k-frame** or theory, namely that containing those **k-states** for which it provides meaningful information (background knowledge); the extension of information introduced in the first model can be obtained in the scope of a single or of a limited number of **k-states**. This means that one can consider an extension of information moving only on a limited number of **k-states** which represents in this way its proper **k-frame**:

$$\underbrace{i - state \xrightarrow{f_n} i - state}_{k-frame_k} \tag{3.32}$$

The description of both structures implies the following properties:

1. A single agent's possible knowledge and possible information are always a finite quantity, a finite subset of an infinite sequence.

2. One defines the sequence which collects the whole possible knowledge (K) and the whole possible information (I), acquired by a collective (multi-agent system); these sets are produced by the infinite repetition of the functions corresponding to the operations of informational updating (from one **i-state** to another: f) and of knowledge extension (from **i-states** to a **k-state**: f^*).
3. The two sets have infinite cardinality: this means that the extension of information and the *possible* knowledge has in principle no upper bound.
4. Every finite collection of assertions expressed in contexts represents approximations to such an infinite description: this means that every *actual* stage of knowledge is finite, being an approximation to the possibly infinite one; such an actual state of the agent's knowledge expresses his/her proper knowledge at every stage.
5. A mathematical relation exists which explains the accessibility from a certain **k-frame** k_i to one of its successors k_{i+k} ($1 \leq i, k \leq n$, for $i, k, n \in \mathbb{N}$), based on a similar relation between the proper informational states for the two knowledge states; each successor presents therefore at least an informational updating or a knowledge extension.

The structure here introduced contains therefore a certain notion of infinity, of which every **i-state** and every **k-state** are a finite approximation. This mathematical structure is represented constructively by *choice sequences*, and it is expressed by the non-standard extension of CTT.[42] The coincidence between the structure analysed here and the non-standard extension of Type Theory is shown by the fact that the latter, like the former, is built by a relation of partial ordering and a functional relation between elements of the theory. Choice sequences are infinite sequences (of numerals or, more simply, binary units $0, 1$); the "law" which regulates such sequences is either a rule that gives effective outputs for every position of the sequence, or it consists in knowing a finite approximation of the sequence itself. The second case is the one referred to here. In this sense the following principle holds:

Proposition 3.15 (Principle of Open Data) *The truth of any statement made about a lawless sequence may depend only upon some initial segment of it: if some property ϕ holds for a lawless sequence ξ, then there exists an initial segment of ξ such that all lawless continuations of this sequence satisfy ϕ.*

In order to be constructively meaningful, discussions about lawless sequences must be interpreted by means of finite sequences. Therefore, the extension of every finite sequence to the infinite collection of all of them can be considered constructively meaningful. One deals always with finite sequences, for which a *spread law* is allowed which increases the number

[42] Martin-Löf (1990, 1990a, 1999).

of their elements, so that each of these sequences is a finite one, having an infinite sequence as its own limit. The law regulating the increasing of such a sequence towards its limit (spread law) coincides here with each of the steps possibly taken in order to finitely enlarge every sequence by means of meaningful information. The *whole* sequence which contains all contexts and all mappings between contexts (and in the second model the mapping between all **k-states**) pertains to the non-standard extension of CTT. One can therefore at this point summarize the ways in which the sequence can be extended. Let us start with a **k-state** k_1 in which the following context Γ holds:

$$\Gamma = (x_1 : \alpha_1, \ldots, x_k : \alpha_k) \tag{3.33}$$

A restriction is imposed concerning the moves which can be done at this stage in order to enlarge the context, therefore going towards a further state[43]:

(a) A mapping exists $\Gamma \overset{f_0}{\leftarrow} \Delta$ such that Δ contains every assumption expressed in the context Γ, plus the information given by the type declaration ($\beta : type$); informally this step shows that the introduction of a new concept (type) into the theory let the agent switching to a different **i-state**, which contains the condition for enlarging the starting **k-state**. If the presupposition ($\beta : type$) is expressed in Δ and missing in Γ, the informational extension is *synthetic*.
(b) A mapping exists $\Gamma \overset{f_1}{\leftarrow} \Delta$ such that Δ contains every assumption contained in Γ, plus the extension due to the mapping $\Gamma \overset{f_0}{\leftarrow} \Delta$ presented at the previous point (i.e. the introduction of the type declaration ($\beta : type$)), plus n new assumptions of the form ($x_1 : \beta, \ldots, x_n : \beta$); this is a *synthetic* informational extension.
(c) Eventually, the previous case can be completed by the derivation of n new judgements of the form ($b_1 : \beta, \ldots, b_n : \beta$); given ($\beta : type$) provided by the mapping $\Gamma \overset{f_0}{\leftarrow} \Delta$, this represents an *analytic* knowledge extension.
(d) The case where the context is enlarged by the introduction of a definition ($a = b : \alpha$) is also an *analytic* extension, a particular case of (c) (a new object is recognized as equal to another one).
(e) The case where the context is enlarged by the introduction of a statement of the form ($\alpha = \beta : type$) becomes a special case of (a) (a new concept is recognized as equal to another one).

In this way, one is able to explain the extension of information within the same concept (type) available at a certain **k-state**, which represents an

[43] The introduction of analytic and synthetic extensions is here essential to the reconsideration of the problem of analyticity presented in the final chapter. For the meaning of analytic and synthetic judgements in CTT cf. Martin-Löf (1994).

analytic extension. On the other hand, the extension of information given by new concepts is recognized by this model as a synthetic extension.

In the present model, the agent always has finite knowledge states; everything which is recognizable as knowledge content is based and justified on meaningful information possessed by the agent. Information expresses every meaningful concept contained in the agent's knowledge frame and every rational supposition the agent needs in order for his/her knowledge to be meaningful. No pre-imposed bound exists to the knowledge the agent can acquire, except for what is not definable in the theory. Finally, in its connection with analytic and synthetic judgements, the notion of information provides the conceptual link between extension of knowledge (utility) and logical derivability (validity). The model defining knowledge and information is thus completed and coherently presented. It is the aim of the final chapter to investigate the nature of information more deeply, especially in relation to its property of establishing meaningfulness for knowledge, in order to show how it can explain the notion of analyticity.

4
Constructive Philosophy of Information

4.1 An Extension for the Constructive Epistemology

The formal model presented in Chapter 3 has defined information as an epistemic concept within the representation of a knowledge system. Such a formal structure aims in the first instance to clarify the distinction between information and knowledge as epistemic states of a rational agent. The starting point for this analysis was the the concept of knowledge in the constructive approach, defined in terms of the basic notion of judgement: under this interpretation, judgements are pieces of knowledge, their collection representing an agent's actual knowledge. This has been expressed in the conceptual and formal structure by the term **knowledge frame**; such a frame is always a finite one, capable of extensions to any finite limit, and therefore potentially to the infinite. This frame is composed formally by **knowledge states** and extended in terms of the introduction of **informational states**.

By introducing the notion of information, one answers in the first instance the epistemological distinction considered in Chapter 2: knowledge can be in a rough sense divided as being explicitly and implicitly proved. In the same light, the account of knowledge expressed and represented by judgemental acts needs an internal distinction when considering that some judgements are obviously made by maintaining other judgements as background knowledge. Background knowledge is to be considered in this framework as a condition for other (explicit) knowledge to be acquired, and the expression *assertion condition* has been used accordingly with this meaning. To describe assertion conditions in terms of our formal notion of *information*, i.e. that one can consider the following relation of containment

informational background $\leftrightarrow$ **conditions for knowledge**,

one has to provide a common definition for *information* (both in the common understanding of the term, and in its formal development) and *assertion conditions* in the context of constructivism. With this aim, the development of the notion of information described in Chapter 2 focused on the development of the semantical interpretation, which historically was intended to represent the common notion, i.e. the information conveyed by a certain sentence, a certain state of affairs, or even a sign or an image. Nevertheless, it is not difficult to show that different concepts are conveyed by one and the same term. In particular, in order to understand which concept of information is at hand, it is extremely relevant to determine if its definition is based on an alethic or rather on an epistemic principle. It is common to say, for example, that different (eventually contrasting) informations come from different sources; sometimes one speaks of trustworthy information or unreliable information, true or false information. All of these expressions obviously link information to the notion of truth, whereas the aim of the more recent research was precisely to avoid such explicit connection.[1] In the epistemic formulation provided by the formalization of CTT, the distinction between information and knowledge is essentially based on the relation to conditions, and it has been explicitly maintained that truth and falsity are not ascribed to information: informational statements are therefore defined by explicit conditions of meaning for some contents (presuppositions) and by contents whose truth-value is assumed, in relation to expressing hypothetical knowledge (assumptions). Hence, according to the constructive perspective, the conceptual relation of judgements and truth to information is of a peculiar kind. Let us consider a classical example from the literature on information theory: the tossing of a (fair) coin. According to a probabilistic treatment, one says that the “information” conveyed by the tossing of such a coin can be extracted by a sample space of the kind:

$$X = \{tails, heads\}, \tag{4.1}$$

namely, by tossing the coin. This sample space consists of a collection of values, none yet ascribed to the involved variable. Once the coin has been tossed, one will find out which value extracted by the sample space will fulfil the variable, namely, by setting the identity between one of the possible values (say “tails”) and the variable:

$$X = tails\{tails, heads\} \tag{4.2}$$

It is common to refer to the fulfilment of the value variable as an operation providing the “information” that *tails* (or *heads*) is obtained as result; equivalently, one says that it provides “knowledge” about which is the result. The idea at the basis of the formalization presented in Chapter 3 was that the values contained in the sample space, corresponding to the agent’s state of knowing which values can be ascribed to the variable, are not epistemically equal to the value that the variable acquires out of those contained

[1] Cf. the mentioned Principle of Alethic Neutrality, Section 2.5.1.

in the sample space. In the first case, one does not possess any knowledge about the real value, rather only what the possible cases are: in this sense the epistemic state of the agent (his/her "information") is incomplete, i.e. the possible outcomes are known, not the actual one; obviously, evaluating the variable leads to completing this information.[2] The matching with the above-mentioned distinction between knowledge and its conditions is simply done: the sample space is nothing else than a collection of type-values for the variables, and the application to type theory is done in terms of contexts. Let us assume a given context containing two variables for which the type value is declared, i.e. let the type be ascribed to the values in the context

$$\Gamma = (x = Bool; y = Bool). \tag{4.3}$$

This expression obviously contains a certain amount of information, exactly as in the case of the sample space containing *tails* and *heads* as values. According to Chapter 3, extensions of contexts are considered as the formal operations by which information is increased towards acquisition of knowledge: one way to perform such an extension is, for example, by setting a definitional equality, i.e. to express a value for one (or more) of the variables involved, like in the following case, extending the context Γ to context Γ':

$$\begin{array}{c} \Gamma = (x = Bool; y = Bool) \\ f \uparrow x = x; \quad y = x \\ \Gamma' = (x = Bool) \end{array} \tag{4.4}$$

This extension is obtained by a function assigning the value of the variable y and identifying it with the variable x (in this case making the context shorter). In this way, our knowledge has obviously increased, as one normally would say, while our uncertainty has diminished. The next and last step will of course furnish a definite value out of the set *Bool* for the only remaining variable x, thus completing the information, determining our knowledge state, and emptying the context. In the epistemic operation described by the extension of contexts, there is clearly no difference in substance between *complete information* and *knowledge*. In this sense, the connection between information and uncertainty is also maintained: where every possible information has been given, knowledge is acquired, and no uncertainty is left, like in the case of the tossed coin. This treatment considers contexts as collections of expressions with yet undetermined quantities, thus corresponding to the sample space in the example. Complete information amounts to knowledge, hence explaining the epistemic difference

[2] This common way of speaking, about complete and incomplete information, is used, for example, in image recognizing, descriptions, and language analysis. According to such a common view, logic is a way to represent "complete information". It should be remarked, however, that this terminology does not solve the conceptual and terminological problem: one in fact could as well speak about "complete or incomplete knowledge".

between the two concepts. The definition here provided insists on the relational nature of information within the knowledge process:

- Knowledge is first of all obtained by setting conditions of meaningfulness, and to state meaningfulness for knowledge is an informational function.
- Truth of informational statements is accounted only as a basis for the truth of proper knowledge: information has not in itself a truth-value, rather it determines the context in which truth-values for knowledge are established.
- Information is never to be considered absolutely, rather it is defined as a part of knowledge processes: in this sense information has been identified as *implicitly proved knowledge*, representing the context in which some explicit knowledge is acquired.

Hence, information is properly speaking part of the knowledge process, but is not to be strictly identified with knowledge. The presence of a verification procedure defines knowledge contents, and it is in terms of constructions that the difference between information and knowledge can be explained further. Assumptions and presuppositions can be accounted as forming the informational background of the agent's knowledge state. Referring to assumptions and meaning declarations as forming such informational background presents the advantage of extending the constructive epistemology by giving an explanation for those elements whose nature is clearly distinct from that of proper knowledge, and for which a description was, until now, missing. The treatment of information here presented has for this reason in the first instance a relevant philosophical motivation. Moreover, the development of such an epistemic description is connected in a relevant way to the problem of analyticity and it is useful to formulate the related theory of meaning in new terms.

4.2 Information and Mathematics

If mathematics is considered as a purely analytic science, its truths being valid a priori, the knowledge it furnishes being tautological in principle, it would of course be natural to say that mathematics furnishes no information at all: this point of view seems to be rather objectionable in its essence, as the great amount of informative contents given by the different branches of mathematics is evident. This view is of course based on considering mathematics as implementing the analytic nature of logical deductions: in particular, it seems quite wrong to consider the content of mathematical theorems as self-evident once axioms for the relevant notions involved are displayed.[3] Kant saw mathematical reasoning as based on constructions which cannot be simply reduced to analytic methods: he recognized constructions as the property of mathematics despite the analyticity of pure

[3] Cf. Bernays (1965).

logic. In fact, the conclusion of a chain of inferences is not just part of the premises from which the chain starts, and the axioms are fruitful on the basis of a procedure of combinatorial application, rather than referring to pure analysis. In this sense, the notion of information is useful to refer to the introduction of new qualitative notions (concepts), supported and justified in terms of constructions.

On the basis of the formal structure of CTT analysed previously, one can consider these two elements as extending the purely analytic nature of mathematical reasoning:

1. The starting point for a demonstration always takes something for granted, namely, the concepts on which rules are applied: this level of objectivity is reflected by the term meaningfulness.
2. A second level of objectivity is given by the constructions instantiating existence for such concepts.

In this way two levels of awareness in accomplishing mathematical deductions are distinguished: the first level corresponds to the use of the meaning of the objects one is working with, knowledge of which is defined by the conceptual analysis that describes the essence of such objects. The analytic development of deductions is hence performed on the basis of these meaningfully given concepts (and eventually some extra data). Mathematical reasoning starts in this way by considering some concepts as given, whose meaning is explained by axioms and rules, developing what is self-evident in such concepts. In this sense, analyticity and a priori are restricted concepts: theorems implicitly contained in a set of premises represent something different from what is actually known starting from them (in particular, the latter answers to a quantitative standard and represents the analytic part of deductions). In the first instance, one starts by setting a qualitative notion (concepts taken in their meaningfulness) and the self-evident can be made explicit only in relation to some context of acquired information. Hence, a mathematical deduction develops (makes explicit) what is implicit in its assumptions, or in the axioms, according to the rules governing them. It is in this sense that mathematics treats information: a textbook for algebra or topology, for instance, contains a lot of "information", which does not represent knowledge for someone who does not possess the meaning of the notions involved (e.g. the first time one looks at such a textbook). One starts by giving definitions for those notions, referring to concepts already known, and describing the objects involved by their essential properties. The process of knowledge starts by setting information in the proper way, i.e. by filling terms with meaning. The concept of information here introduced is thus a qualitative notion, rather than a quantitative one. If the idea of Shannon's statistical approach to information was to reply to questions like "*how much information do we transmit/receive?*", the present analysis switches to a qualitative approach, furnishing an answer to a rather different question: "*which kind of information is available?*"

4.2.1 *From Analytic Method to the Analyticity of Logic*

The method of knowledge as already intended by Greek philosophy was structured on two complementary procedures: on the one hand, by the analytic method, intended to find out a proof or a solution to a given problem; on the other hand, by the synthetic part, with the role of justifying the related demonstration from what is known to what is sought. Each procedure is essentially justified by the other one: synthesis is built on the parts obtained by analysis, and the latter is shown to be correct in terms of the synthetic process. Synthesis thus amounts to construction and to proper deductive process, whereas analysis corresponds to resolution. The analytic method will remain at the core of scientific development in the whole history of science and therefore analyticity, as the essence of logical deduction, finds its roots in this method of acquiring scientific knowledge.[4] With Descartes, analysis and synthesis become the procedures of philosophical method, thus coming back to the essential insight of the Aristotelian method. The standard interpretation identifies analysis with reasoning backwards, which is usually called "*directional interpretation*" of analysis, that is to say it only proceeds by formally decomposing or loosing up propositions in their own constituents[5]:

Directional Interpretation of Analysis

	analysis		**synthesis**	
given	$\Rightarrow$	***components***	$\Rightarrow$	***sought***
	resolution		**composition**	

A different interpretation of analysis is called "*configurational analysis*": starting from what is given, the analytic method provides everything contained in the expressions involved; the method is then extended by the so-called *analytical hypothesis*, by means of which one restarts from the sought in order to perform a resolution. This extension of the analytic method allows for *auxiliary constructions* both on the given and on the sought (assumed as given): once the sought is obtained, the process of resolution justifies its assumption. In this way the analytic method becomes foundational for the deductive, synthetic method, it maintains its own original part, and it does not consist only in the resolution of what is actually given: the method is in fact extended by auxiliary constructions, which are not to be found in the simple resolution of the given.[6]

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[4] For a history of the methods of analysis and synthesis in connection to Type Theory, cf. Mäenpää (1993).

[5] Cf. Hintikka and Remes (1974). In natural deduction it amounts to using the subformula property.

[6] This interpretation is developed by Mäenpää (1993), and is explained in the constructive type-theoretical setting.

Configurational Interpretation of Analysis

	analysis	
given	$\Rightarrow$	***components***
$\Downarrow$	***analytical hypothesis***	$\Downarrow$
given sought	$\Rightarrow$	***justified sought***
	constructions	

To take into account the configurational interpretation of analysis means to reformulate the idea of analyticity itself, and the development of the notion of information goes in the same direction, showing hidden structures underlying the analytic development of logical reasoning: from the point of view suggested by this research, deductive reasoning uses auxiliary hypothetical judgements and extends its range of possible derivations by introducing new concepts, operations which essentially modify the purely analytic role of deductions.[7] Stressing the role of assumptions as conditions for (proper) constructions within hypothetical reasoning, and showing the relevance of meaning conditions for the dynamics of theories, amount to provide an extension for the construcitve epistemology, essentially based on the role of proofs in the definition of truth. In this respect, one could say that the analysis offered by the Paradox of Inference (cf. Chapter 2) is incomplete, because it does not take into account a central feature of deductions, namely, the role played by hypothetical reasoning in the construction of logical derivations and the kind of extension provided by the introduction of new meaningful concepts. These operations are synthetic procedures by which logical processes extend the informational content analysed in terms of proper constructions. There is thus a clear connection between the problem of analyticity (in the problematic formulation presented by the Paradox of Inference), the role of constructions and that of meaning, which can be further explored.

4.3 The Role of Constructions

Proof objects in CTT can be considered as formalizations corresponding to demonstration trees for classical logic. Each stage of the tree for a given proposition corresponds to a construction step in the proof object for type theory. Constructions are thus the interesting property of deductive processes and any step of construction is a step of demonstration (but it is not always the case that a step of demonstration can be considered as a step in a construction). Analysis and synthesis for constructions are quite

[7] In the following the expression "auxiliary constructions" is used to refer to the kind of constructions involved by assumptions.

naturally explained according to the relation with the introduction and elimination rules of natural deduction[8]: in general, synthesis of proof objects amounts to functional composition, and their analysis to functional decomposition.[9] Presenting in Chapter 1 the formal setting of the theory, the role of formation and computation rules in CTT was considered, besides introduction and elimination rules: a formation rule gives the composition of a proposition (type) from others by means of canonical constants called *constructors*; a computation rule shows the way to normalize proofs, reducing them to canonical form, making use of non-canonical constants called *selectors*. The deductive process leading to a conclusion by means of the constructors contained in the premises is synthetic; on the other hand, the backwards process, or reductive inference, analyses the sought construction into its constituents. The difference between introduction and elimination rules is that the former can either analyse a sought construction by reductive inference or synthesize a construction from those given by deductive inference, while the latter always analyse a given construction. These rules correspond therefore to a reformulation of the subformula principle: the proposition in the conclusion of an introduction rule is a functional composition of the propositions in the premises; conversely, the major premise of the special elimination rule for that proposition is a functional composition of the propositions contained in the conclusion.[10] The connection between deductive and reductive uses of inference is clearer when analysing deductions more closely. In particular, the notion of reductive analysis is explained when rules are modified in terms of the elements occurring in them: for example, the explication of the expression $A \supset B$ in terms of $B(x : A)$ is made by means of the introduction of the sought proof object b, rather than by the application of $b(x)$. This in general means that the role of constants and variables within proof objects is essential to describe analytic and synthetic processes. The introduced variables are dependent ones; their value is not yet determined but they cannot take any arbitrary value. The sequent calculi obtained by such reductions present rules holding in some context, and this is explicitly expressed in Type Theory. Precisely in terms of variables, and expressing premises and conclusions in contexts, auxiliary constructions are performed within analytic proofs.[11] This idea is of course in contrast with the cut elimination rule: such a rule means that only constituent propositions of the conclusion are needed for the development of the demonstration, and that eventually the use of a proposition which is not among the constituents can be executed only by means of a cut rule. This means that even in the analytic process it is not possible to avoid the use of auxiliary constructions. Auxiliary constructions essentially

[8] Cf. Mäenpää (1993, ch. 2).
[9] Ibid., p. 8.
[10] Ibid., pp. 45–46, where all the Introduction and Elimination rules for natural deduction calculi are analysed in order to show these properties.
[11] Ibid., pp. 73–77.

provide new constructions and concepts which are not trivially contained in what is sought. This, in turn, furnishes a new definition of what an analytic proposition is:

Definition 4.1 (Analyticity by Constructions) *A proposition is analytic if its proof does not make use of auxiliary constructions. A synthetic proposition is one which does not have such a proof.*[12]

In other words, this amounts to saying that an analytic proposition can be proved without using the cut rule.

Provided this explanation of analyticity related to constructions, the general explanation of a synthetic judgement can be given in terms of meaning: a synthetic judgement is such that its evidence does not rest entirely on the meaning provided by its component constructions.[13] In terms of the already presented treatment of analytic and synthetic judgements, CTT is a logic of analytic judgements, because it states judgements explicitly on the basis of their proof objects, and in this way it is complete and decidable. On the other hand, to show the connection between existential and synthetic judgements has a second essential consequence:

> *The logical laws in their usual formulation all say that an arbitrary proposition of a certain form is true, and the affirmative form of judgement, A is true, is a form of synthetic judgement.*[14]

This implies the quite startling conclusion that the laws of logic are synthetic a priori propositions. Hence, the interesting part of logic and mathematics is precisely given by the process of constructing objects to realize the concept involved: this is the role of analytic judgements, expressing

[12] Ibid., p. 82.

[13] This recalls again the Kantian explanation of the nature of synthetic judgements, whose basic components are given by the notions of *Erfahrung* and *Construktion*. The notion of *Erfahrung* is used in the *Kritik der reinen Vernunft* by Kant essentially without a definition; in the *Prolegomena* (Kant 1783, ch. 3, par. 5) it is defined as follows:

> *Erfahrung ist selbst nichts anders, als eine kontinuierliche Zusammenfügung (Synthesis) der Wahrnehmungen.*

In general, it is accounted as the structure of every possible mediation of the reason on the data offered by senses (*sinnliche Wahrnehmung*), the connection of these data being given according to universal and necessary laws of the intellect. In [KrV], A108 1–4, its role is presented in a similar way:

> *diese transzendentale Einheit der Apperzeption macht aber aus allen möglichen Erscheinungen, die immer in einer Erfahrung beisammen sein können, einen Zusammenhang aller dieser Vorstellungen nach Gesetzen.*

The notion of construction represents essentially the other way than analysis to get to concepts, which makes it the distinguishing method of mathematics from philosophy. Cf. Kant [KrV], B865, 9–11:

> *Alle Vernunfterkenntnis ist nun entweder die aus Begriffen, oder aus der Konstruktion der Begriffe; die erstere heisst philosophisch, die zweite mathematisch.*

[14] Martin-Löf (1994).

explicitly constructions for the types involved in the predication. It is a consequence of such a construction of the synthetic extension of knowledge given by the predication of truth related to the involved concept. This link between the analytic and the synthetic parts of knowledge processes is at the core of the epistemic model presented in Chapter 3, which identifies the main distinction between knowledge and information in the reference to justifications and conditions for asserting judgements. It is once again by the role of constructions that this epistemic distinction must be explained, and it is via constructions that analytic and synthetic extensions of knowledge are correctly described:

1. Constructions/justifications (proof objects) are required to realize extensions in terms of analytic judgements $(a : \alpha)$; these extensions provide proper knowledge to the agent's epistemic state.
2. Auxiliary constructions/conditions (expressed in terms of the variables contained in assumptions) are required to express hypothetical judgements in terms of expressions of the form $(x : \alpha)$; the extension of contexts by new assumptions requires the proper presuppositions to be formulated, i.e. the related meaning expression in the form $< \alpha : \mathit{type} >$ to be stated; these extensions are informational extensions (updatings) and they provide synthetic extensions to the derivation (definable only via analytic ones).

The use of contexts collecting hypotheses shows therefore the formulation of the conditions for some judgements achieved in the agent's knowledge state: these conditions can be interpreted as "auxiliary constructions". For example, the explanation of the hypothetical judgement

$$a : A(x_1 : A_1, \ldots, x_n : A_n) \tag{4.5}$$

is actually given by the judgement in which one has accomplished all the necessary substitutions of variables by proper proof objects, i.e. when the following inference is satisfied by the proper substitutions:

$$\frac{x_1 : A_1, \ldots, x_n : A_n}{a : A[a_1/x_1, \ldots, a_n/x_n]} \tag{4.6}$$

Whenever a hypothetical reasoning is performed, assumptions are taken into account in their alethic value, and their content is expressed as the information needed in order to know the given judgement. This, in turn, means that constructions involved by the conditions of a hypothetical reasoning are in this sense "auxiliary", supporting the proper construction contained in the derived expression $(a : A)$. The real meaning of such constructions is the information concerning the existence of the concept involved by the predication: the main consequence of such a view is that background information has a synthetic value.

4.4 Types and Categories of Information

In the first two sections of this chapter, the nature of assumptions has been considered: it has been suggested that their role is to provide a construction actually missing in the reasoning, in order to justify the acquisition of new knowledge, by analogy with the aforementioned "auxiliary constructions". In the case of assumptions in fact, one agrees in taking for granted a certain alethic value for some judgements. By this thesis, contexts are considered as sets of informational statements, the basic element defining our notion of information. Furthermore, throughout Chapter 3 and by the definitions therein, it is explicitly maintained that presuppositions for judgements, in particular those basic ones representing assertions introducing types within the frame (therefore named *type declarations*), are also part of the agent's informational state (cf. **Proposition 3.5** in Section 3.2.3 and **Proposition 3.8** in Section 3.2.5). Type declarations complete the information furnished by assumptions within contexts. This latter thesis, concerning the informational nature of presuppositions has not yet been fully justified, even though the basic reason to ascribe the same epistemic description to these expressions has already been considered: the (implicit) formulation of an expression of the kind $<\alpha : type>$ is always required by any judgement contained in a context (cf. Section 3.2.1). To formulate a context of assumptions means thus in the first instance to set the type of the possible values (sample space) for that context: in the mentioned example, the tossing of a coin is done under condition of a context of possible outcomes, whose type is given by declaring the set of values *heads* and *tails*. This kind of *type declaration* sets the *meaning* for the expressions predicating those values: types represent meaning objects, by which possible meaningful predications can be performed; the latter are possible only under the condition that suitable types have been declared.

4.4.1 *Overview on Presuppositions Theory and Dynamic Logics*

When referring to the notion of presupposition, it is obvious to consider the wide range of studies produced in the past decades, especially in Linguistics and Knowledge Theory, concerning the role of presuppositions in (natural) languages: these approaches provide formal developments of the basically intuitive notion that certain things can only be said if other things are taken for granted. Approaches on presuppositions are developed either on a static account of discourse analysis, or more recently on the basis of dynamic accounts, yielding to dynamic logics.[15] The latter, in particular, account for a modification of the context of evaluation by successive utterances (*inter-sentential dynamics*) or for dynamics of additional effects

[15] Cf. Muskens, van Benthem and Visser (1997).

produced by sub-sentential constituents (*intra-sentential dynamics*).[16] The goal of the theories of presupposition includes the determination of the special status of such propositions, and the formulation of an explanation concerning the reason and the conditions for this status to obtain. Classically, the definition of presupposition has been given either in a semantic flavour or in a pragmatic one. The semantic definition can be informally presented as follows:

Definition 4.2 (Semantic Notion of Presupposition) *One sentence semantically presupposes another if the truth of the second is a condition for the semantic value of the first to be true or false.*[17]

On the other hand, the classical pragmatic account originally due to Stalnaker[18] takes into account the role of the agent presupposing some sentence (speaker's presuppositions), in the following way:

Definition 4.4 (Pragmatic Notion of Presupposition) *A speaker presupposes that P at a given moment in a conversation just in case he is disposed to act, in his linguistic behaviour, as if he takes the truth of P for granted, and as if he assumes that his audience recognizes that he is doing so.*

The pragmatic account also refers more carefully to the influence the knowledge of some content produces on the situation in which that content is taken as known: thus, dynamic logics considers the influence of the context of knowledge on given situations—how actions change that situation and how the very same action is dependent on that situation for the change that it brings about.[19]

A first remark to be made in the comparison with the traditional analysis on presupposition concerns the epistemic nature of the notion here intended: in the framework introduced in Chapter 3, the treatment of presuppositions rejects the semantic approach (clearly based on a realistic account) and it modifies the pragmatic interpretation (which considers the role of the agent, but still referring to the truth of propositions). The epistemic approach can be summarized by the following proposition:

Definition 4.5 (Epistemic Notion of Presupposition) *A judgement J presupposes one or more judgements $J_1, \dots, J_n$ if J is a judgement-candidate only provided that $J_1, \dots, J_n$ are known, i.e. knowledge of the latter is a condition for the meaningfulness of the former.*

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[16] Cf. Beaver (1997).

[17] In Beaver (1997, p. 948), where its formal version is also presented:

Definition 4.3 (Strawsonian Presupposition) *ϕ presupposes ψ iff for all worlds w, if $[\phi]_w \in \{t, f\}$ then $[\phi]_w = t$. We write $\phi \gg \psi$.*

In this formula $[\phi]_w$ is the semantic valuation in a trivalent account of the formula ϕ with respect to the world w and t is the truth-value "true".

[18] Stalnaker (1974).

[19] Muskens, van Benthem, and Visser (1997, p. 590).

Changing perspective on what it means for an agent to presuppose something (in order to know something else) obviously restricts the range of expressions recognized as proper presuppositions, in particular distinguishing those taken in the alethic role. In the literature on presupposition theory, one refers to a rather wide range of elements, containing definite nouns, quantificational expressions, counterfactual conditions, intonational stress, up to categorial restriction, temporal modifiers, and so on: these expressions work in some context as presuppositions for other expressions to be formulated. The problem of defining the notion of presupposition amounts thus to knowing what complex sentences inherit from them, and why: i.e. which implications of simple sentences are also implications of sentences in which the simple constituent is embedded under negation, under a modal operator, or as antecedent in a conditional etc. This problem, called the "*presupposition projection problem*", is embedded in the larger problem of defining the meanings of complex sentences in terms of the meanings of constituents, called the "*projection problem for meanings*", which in turn is developed as the problem of compositionality.[20] Of course, essential to this analysis is the understanding of presuppositions as providing the context of meaning in which some assertion is true or can be asserted.[21] The semantic account introduces such understanding presenting presuppositions as worlds or models in which other propositions (those depending on them) can be evaluated, or else remain undefined (hence the need for a three-valued semantics, where a value "undefined" must be used).[22] This interpretation understands presuppositions essentially as a binary relation between sentences. Other accounts treat projection functions relatively to contexts, rendering presuppositions as a three-place relation between two sentences and a context of evaluation.[23]

The connection between the notion of presupposition and that of meaning is improved by dynamic logics: such logics use procedures of *context change potential*, i.e. the study of functions which allows describing the change of possibilities according to the change of contexts. This leads to the development of propositional logics which have a semantic notion of context change potential rather than of truth. In this way such logics develop in many cases procedures resembling those of reducing possibilities as a way to increase information, thus gaining complete information when only one possibility is left.[24] The aim of the context change potential and its connection to the

[20] Beaver (1997, p. 945–946).
[21] Clearly the formal notion of presupposition in CTT has in fact explicitly the same aim.
[22] Other possibilities are extensions like the method of *supervaluations* by van Fraassen (1969).
[23] Beaver (1997, p. 959).
[24] Cf. e.g. Stalnaker (1974); for a complete overview on Dynamic Logics cf. Muskens, van Benthem, and Visser (1997).

notion of information is really close to the constructive approach: by means of this analysis, the relation between semantics and admittance conditions is totally revised. One of the most significant and relevant approaches in this direction is due to Heim,[25] who, rather than viewing meaning as a static relation between language and truth (as in the classic semantical accounts), takes the meaning of an expression to be a method of updating the information state of communicating agents. Under this view, information states for agents are sets of possible worlds maintaining consistency under updating of the available information, every introduction of an atomic proposition provoking the elimination of all the world incompatible with it.[26] The role of presuppositions is at this point explained in terms of information: considering the relation of presupposition in terms of implication (*presuppositional implication*), updating a state with a formula which is presuppositionally implicated by such a state will add no information to it; and presuppositional implication between two formulae means that any updating with the first formula will produce a state for which updating with the second formula will add no information.[27] In other words: a presupposition added (explicitly) to a certain informational state does not add information if it is already (implicitly) contained in it; if a state is updated by a formula which is based on some (implicit) presupposition, adding the latter (explicitly) to the state will bring no new information. These intuitive properties are clearly maintained under our description of knowledge and informational states, which is similarly based on possible world semantics. Our formal operation of informational updating explains exactly in which sense an informational state can be extended or modified, properly in terms of introduction of hypotheses and related presuppositions or by setting a value for a variable, distinguishing it from the proper derivation of knowledge (extension). Such operations always bring the additional information provided by the related presuppositions, which explains why adding them explicitly in a second step will add no information at all, as it is simply shown by considering the following two updates:

$$\Gamma = (x_1 : \alpha_1, \dots, x_n : \alpha_n) \leftarrow \Gamma' = (x_1 : \alpha_1, \dots, x_n : \alpha_n, y : \beta) \tag{4.7}$$

$$\Gamma' \leftarrow \Delta = (x_1 : \alpha_1, \dots, x_n : \alpha_n, y : \beta < \beta : type >) \tag{4.8}$$

In the rest of this chapter the connection between presuppositions and conditions for meaningfulness will be explored: such a connection aims to restate the role of meaning as a part of the operation of informational updating for an agent's knowledge state, following what has been suggested in Chapter 3. In this way one is developing the intuition stated by Heim's

[25] Heim (1983).
[26] Heim describes extra clauses for updating in terms of conjuncted and negated propositions.
[27] Cf. Beaver (1997, pp. 970–971).

work, that meaning can be considered as a dynamic operator on the epistemic description of knowledge processes. Heim's structure allows for a clear distinction between *presupposition failure* and *updating with contradictory information*: on the basis of the constructive epistemic model, one is able to distinguish two corresponding cases of error, by the rather stronger epistemic distinction between errors produced by informational updating and by knowledge extension. This description recollects our distinction between *informational/knowledge error* and the two cases respectively considered in Heim's model, i.e. *presupposition failure/updating by contradictory information*[28]:

- **Informational error:** This is an error produced by one of the operations of informational updating, therefore involving either presuppositions or hypotheses, according to the following cases:

  Presupposition failure: This kind of error is at the basis of incoherent extensions of knowledge obtained via informational updating; it amounts formally to a missing or not well-formed type-declaration at the basis of an interpretation between contexts; informally, it corresponds to an informational updating of the agent's knowledge state on the basis of a missing concept.

  Updating by contradictory information: This kind of error is produced by an error in updating contexts via introduction of a new hypothesis (error about the stipulated variable), or eventually in the execution of the construction in a judgement derived from such an assumption.
- **Proper knowledge error:** This is an error produced by a misuse of the language, an error in predicating analytic extensions, i.e. in the formulation of judgements of the form $a : \alpha$, $a = b : \alpha$.

The comparison shows that the connection between the notion of information and the concept of meaning suggested by our formalization of informational states finds relevant connection and support in other theories. In order to develop this topic properly, the connection between presuppositions and analytic judgements has to be considered.

4.4.2 *Declaring and Explaining Meanings*

The relation between the meaning of a sentence and the provability of its truth has been turned in opposite directions by the switch from the realist truth-theoretical approach to the (anti-realist) proof-theoretical one. For example, in the intuitionistic approach, meaning is determined in terms of proofs, by rejecting *truth* as the key concept for the formulation of the

[28] For the introduction of our notion of error based on the epistemic model introduced in Chapter 3, cf. Primiero (2006).

theory of sense (i.e. where the content of a sentence is given by its truth-conditions).[29] The first consequence pointed out by Dummett, in holding that the meaning of a sentence is to be expressed in terms of the knowledge of such a meaning, is that a theoretical explanation of such a practical ability requires an account of the *implicit knowledge* of the speaker: a meaning theory will also require the agent to be able to state what it is to know some implicit part of his/her own mastery. The conditions for knowing the key concept contained in the theory of semantic value are expressed in a theory of sense. This implicit knowledge can of course be explained only in terms of the explicit use that the agent shows, and Dummett's argument maintains that using a bivalent concept of truth leaves the theory of sense incomplete, essentially due to undecidability problems. In this sense, the meaning of a component of the sentence rests on the meaning of the whole, whereas the classic distinction between theory of sense and theory of force lets us avoid problems related to the persistence of such a meaning in the different occurrences or kinds of sentences. Thus, on the one hand, the notion of meaning has been explained in terms of the rules regulating the deductive practice (involving the concept), which Prior has famously shown to be too broad an explanation, via the TONK-argument[30]; on the other hand, the already mentioned Paradox of Inference shows that a different problem comes from identifying meaning and proof, which means to render logic useless. The further explanation provided by the Natural Deduction Systems is essentially based on the *assumption* of the *meaningfulness* of the sentences or propositions occurring in the Introduction and Elimination rules: this means that knowledge of their proofs is already presupposed.[31]

For what concerns CTT, the relation between types and meaningful expressions is the essential step which brings us back to the idea of interpreted formal languages: this relation has been explained in terms of the notion of *category*, in Section 1.5.3, i.e. by introducing such a notion the concept of type can be grounded without circularity. The connection with meaning is explained by taking into account the role of types considered as meaning objects within predications; categories represent the meaningful forms of these possible expressions. Thus, in a first sense, *types are meanings*. But the notion of type involves an essential distinction, which can be explained by considering two different expressions:

- *"To declare a type to be meaningful"*
- *"Knowing what a type is"*

[29] This work has its philosophical basis in Dummett (1976), whereas the essential technique is represented by the normalisation procedures in natural deduction by Prawitz (1965). For a survey of the whole connection between proof theory and meaning, cf. Sundholm (1986).
[30] Prior (1960), Belnap (1962).
[31] Sundholm (1986, p. 489).

By explaining carefully this distinction, one can fully understand why type declarations are taken into account as informational statements. One can also consider this important point as expressing the basic extension of a meaning theory for CTT: proof-tables as determining meaning of sentences are completed by the information regarding the objects which need to be constructed in order to establish the truth of the propositions in question (their proof objects).[32]

To know what a certain type is means essentially the following: let us start by giving a type, and then one can ask what type is that; the answer to this question is naturally given by showing that one knows what it means for a certain object to belong to that type, and given two objects belonging to that type, one knows when two such objects are equal. When the two criteria, application and identity, are referred to canonical elements of the type, the rules needed in order to define the type are executed. Let us consider, for example, the type $\mathbb{N}$ of natural numbers: what type is that? To know such a type means first *to declare* that $\mathbb{N}$ is of a certain type, that it is a set, obviously by the presupposition that set is a type:

$$\begin{array}{c} < set : type > \\ \mathbb{N} : set \end{array} \tag{4.9}$$

To know such a set amounts then to knowing a first element (zero) belonging to the type and the successor rule in order to form other elements of this set:

$$0 : \mathbb{N} \quad \frac{a : \mathbb{N}}{s(a) : \mathbb{N}.} \tag{4.10}$$

The roles of the presupposition $< set : type >$ and of the declaration $\mathbb{N} : set$ (formula 4.9), on which are based the rules for defining what type $\mathbb{N}$ is, are the core of this discussion. The role of the mentioned presupposition is to introduce the notion of *set* within the collection of types, thus making it a predicable of the theory. The declaration about $\mathbb{N}$ being a set allows it to be recognized as a meaningful element, which can therefore be known. Its rules will say how such a set is known: the role of the declaration is thus to introduce the set in order to display how it can be known.

A different form of type introduction is instead represented by the declaration of a type which is not followed by its explanation (i.e. by the rules defining it). At least two different cases can be presented in which types are introduced as meaning objects without being defined by any rule; rather, their use is exclusively associated with assumptions and hypothetical reasoning:

1. Let α be a type

$$\alpha : type \tag{4.11}$$

[32] Sundholm (1986, p. 498).

and let β be a type under the assumption $(x : \alpha)$

$$\begin{array}{c}(x : \alpha)\\ \beta : type;\end{array} \tag{4.12}$$

then it holds that

$$\beta : type(x : \alpha); \tag{4.13}$$

the type α is in this case introduced or declared in the formula 4.11 in order to meaningfully set the assumption $(x : \alpha)$. No application or identity criterion is formulated for such a type, no knowledge is stated concerning any proper element belonging to it; but one acts as if those criteria were known (and in fact they have to be known if the reasoning performed has to be validated).

2. Assume X to be of a certain type

$$(X : set) \tag{4.14}$$

and use it as a presupposition to assume something in that set

$$(x : X); \tag{4.15}$$

under these assumptions, it is definitely possible to build a function type $X \to X$, and to substitute, for example, the type $\mathbb{N}$ in such a schema, thus obtaining a common function acting from natural numbers to natural numbers:

$$\begin{array}{c}(X : set) \quad ass.\\ (x : X) \quad ass.\\ X(x) : X \to X\\ (X)(x).x : (X \to X)(X : set)\end{array} \tag{4.16}$$

and introducing $\mathbb{N} : set$, one obtains

$$(x).x(X : set) : (\mathbb{N}) : (X \to X[X/\mathbb{N}] = (\mathbb{N} \to \mathbb{N})). \tag{4.17}$$

These examples show two different ways of taking into account the notion of type declaration. The first case considers explicitly a type declaration as the presupposition on whose basis an assumption can be formulated. This is the basic case referred to up to now, according to which it is presupposed that "*something is a type*", in order to be able to assume that "*something belongs to such a type*": in this way presuppositions are considered as belonging to the implicit knowledge of the agent, namely to the *informational state* on whose basis explicit knowledge is acquired. The second case presents a different kind of declaration (and also a different kind of abstraction): it is the case of "*something assumed to be of a certain type*" (e.g. a *set*). Also in this case no knowledge is declared, and the agent rather sets a type-assumption: one can say only that one assumes something to be of a certain type, and consequently there must be some elements belonging to it; thus, it is possible to assume such an element in our reasoning. In both cases, one is performing the same function, i.e. by means of a type (-variable) declaration the meaning of possible predications is settled. Such declarations *carry information* about the existence of elements performing the application criterion and the identity criterion appropriate for that type, i.e. the essential analytic judgements defining it. To set a clear terminology, one can distinguish between the following options:

(a) The explanation of a type, in terms of the rules governing it and by which the type can be known, gives the *meaning* of the elements involved in proper judgements (declarations of knowledge).
(b) The introduction of a type, in terms of a declaration of something being a type, expresses a basic condition of *meaningfulness* for further related predications.

It now appears quite clear that, in terms of the epistemic description presented, when type declarations are considered as presuppositions for assumptions or setting type-assumptions, they are to be explained as part of the informational states, according to the analysis expressed by the Proposition 3.8. Hence, three different cases can be recollected, in terms of which types are treated in relation to knowledge and informational states:

1. The judgement stating that something is a type $(\alpha : type)$: conditions for such a judgement *to be known* are represented by the application criterion $(a : \alpha)$ and the identity criterion $(a = b : \alpha(a : \alpha)(b : \alpha))$; to state properly such a judgement one needs therefore to be able to know these other judgements.
2. The statement declaring something to be a type, in the form "*Let α be a type*" $(<\alpha : type>)$ represents a presupposition for any assumption of the form $(x : \alpha)$; this second case shows the presupposition normally intended as essential to the formulation of assumptions: in order to assume something meaningfully one needs to state the conditions under which one performs such a predication, which is possible only if (at least) the essential condition of introducing the type is satisfied; obviously conditions for knowing it will amount to those exposed by the first case, namely, validity in terms of the application and identity criterion, but in this case it is not required that they are known.
3. An assumption that something is of a certain type, in the form "*Assume X to be a set*" $((X : set))$, on the basis of which it is possible to develop an entire hypothetical argument valid for some proper type (like $\mathbb{N}$) to be substituted for the variable; this third case amounts in turn to the introduction of an element (collection/predicate) by means of it belonging to the meaning category "$\ldots : type$", which is also the reason why the assumption can only be about a certain (specific) type, in order not to fall into contradiction. Of course, the expression by which one is assuming something to be of a certain type carries the same (hidden)

information considered for the second case, in terms of conditions for that to be a type. Even if one does not have proper or direct access to such knowledge, one is able to make use of it in terms of assumptions.

To compare case 1 with cases 2 and 3 means to show an undeniable epistemic difference between ways of introducing types and therefore to treat meanings within the knowledge frame of an agent. Here the distinction between knowledge and information is once again essential, so let us explain the priority relation between knowledge and meaning, and the proper connection between meaning and informational states.

4.4.3 *Meaning and Predication*

According to the explanation given in Chapter 1, the connection between syntax and semantics holding in CTT, relevant to the theoretical setting of the theory, can be summed up as follows:

1. Types are meaning-conferring objects.
2. The categories of the theory represent the forms in which meanings are expressed.
3. Formal expressions of the theory predicate types for objects, in this way endowing such objects with their meaning.

There is thus a connection between the level of expressions (both for objects and for types, namely, their signs) and the ontological level, by which one considers the proper object and the meaning in itself (what is referred to by the signs respectively of the object and of the type).[33] Thus, analytic judgements formulated within CTT express the proper meaning of an object, and in this sense they correspond clearly to *Erläuterungsurteile* in the Kantian sense, judgement clarifying the inner essence of the term involved, by making its meaning explicit. On the other hand, judgements stating existence for types represent *Erweiterungsurteile* in the Kantian terminology, synthetic judgements, because existence is not a property directly expressed by the type itself.

On the "types as meanings" interpretation, the clarification of types corresponds to that of meanings; in turn this correspondence is settled by saying that knowledge of meaning corresponds to knowledge about rightful predication involving that meaning (analytic judgements):

types as meanings:

→ definition of a type ("what type is it?")

→ knowledge (of the use) of the meaning (predication)

→ application and identity criterion

→ analytic judgements (showing inner elements of the type, representing proper predications involving that meaning)

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[33] For the syntactical–semantical method in CTT, cf. again Section 1.3.4.

If one instead takes into account the declaration of the type without its explanation, as it has been done in examples 2 and 3 in Section 4.4.2, the way of intending meaning as conveyed by such declarations reflects quite a deep difference: the declaration conveys information about the predicability aptness of that meaning. Obviously, when the declaration of the *meaningfulness* of a type is involved in a hypothetical judgement (in terms of presuppositions for the assumptions involved by the reasoning), the correctness of the reasoning based on those assumptions will eventually yield to existence of the types involved by the context of hypotheses[34]:

meaning(-fulness) of a type:

→ predication aptness of the type

→ presupposition for (analytic) judgements

→ (eventually) implying (synthetic) existence of the type (*under validity of the reasoning developed*)

Referring to meaning conditions (meaningfulness) as a part of the informational state occurring in an agent's knowledge process, one is not considering meaning as explanation and eventually analysis of concepts. Rather, the process involved is that of relating information to acquisition of knowledge, a process which is extremely important as a condition for the extension of knowledge, and characteristic of hypothetical reasoning: such a reasoning yields to the validity of the conclusion under the truth of what is expressed by the assumptions, and it moreover expresses the validity of those hypotheses if the conclusion turns out to be verified. To underline the epistemic distinction between proper knowledge and the information contained in the assumptions, let us consider once again the aforementioned difference in terminology:

- *Meaning*: Speaking about types as meanings, one is here considering the act of knowing a type in terms of its explication: under the set-as-proposition interpretation for types, this reflects the idea that knowing the meaning of a sentence corresponds, namely, to know how to state it truthfully, i.e. to know what it means for it to be true (or that knowing the meaning of a predicate amounts to know an object of which it can be truthfully predicated).
- *Meaningfulness*: By this term one refers to the act of taking into account a concept as a term of possible predication, such understanding being expressed by the presupposition involving the predicable type; in this sense one speaks about a concept as knowable and, referring to the type declaration in itself, one accounts even for possibly never known concepts, i.e. concepts which will never have a proper construction (empty concepts). This is evident in the case of hypothetical knowledge: meaningfulness of

[34] This particular topic will be reconsidered in Section 4.5, in order to finally reformulate in our own terms the problem of analyticity for logical arguments.

the types stated by the context is necessary to assume their constructions, even if such constructions could eventually not be found; this further explains why presuppositions are involved as background information by contexts considered as informational states.

Briefly, by *meaningfulness* one refers to the condition under which an assertion can be uttered correctly, whereas the explanation of *meaning* refers to the content of such an assertion, which in turn satisfies its provability conditions (and makes the assertion true). It is clear that the former condition (meaningfulness), i.e. the grasping of the concepts involved by the analytic development of logical reasoning, is taken into account only in terms of its contribution to the latter condition: the grasping of concepts is understood as supporting the provability condition of the sentences in which they occur, and it cannot be taken in itself, without its essential connection to the justification one gives in terms of predication.[35] This is also clear when one says that knowledge of the conclusion of an inference is properly given by knowledge of the premises *and* by understanding the meaning of the conclusion itself (i.e. the concepts involved). The notion of meaningfulness, considered as a part of the informational extension suggested throughout this work, answers exactly to the necessity of identifying a second level of conditions, reflected by the following (Fregean in essence) distinction[36]:

1. Knowing that a certain object is the referent of a name, given this name in a certain way (distinction between sense and reference)
2. Knowing the meaning of a sentence, on whose basis one can acquire knowledge of the truth of that sentence

In this distinction, one recognizes quite clearly how Frege understood and clarified for the first time the notion of information in terms of the distinction sense/reference; it is in fact in the way meaning is given to us that we can acquire information about it.[37] This also reflects the most natural and common way of understanding the notion of information: the sentence "the morning star is the evening star" furnishes information, whereas the tautological version "the morning star is the morning star" obviously does

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[35] This has already been clearly explained by Dummett (1977, ch. 7): what has been rarely underlined in the intuitionistic philosophy is the relevance that such a notion of meaningfulness plays in the determination of provability conditions, the discussion about its a priori character, and its connection to the analysis/synthesis problem.

[36] Such a distinction is considered by Dummett (1976, p. 85); it finds moreover a correspondence to what is spelt out in Section 2.6.3, in terms of the distinction between *knowledge-how* and *knowledge-what*.

[37] Dummett (1976, p. 85):

> *It relates therefore to the use of language to convey information.*

not (provided anyway that we understand what the morning star is). Our aim was to reveal the other side of the distinction just drawn, where information appears in the connection between meaning conditions and truth. In this respect, the informational content is considered in terms of coming to know the truth of a sentence, given that one knows the meaning of the concepts contained in such a sentence; hence, informational content depends on meaning(-fulness).[38] Hence, the epistemic description obtained by clarifying information as a conceptual extension of knowledge processes aims to build a theory of meaning which considers not only how truth for sentences is known, but also the basic conditions on which truth for sentences can be judged. In the constructive setting, the former is obtained by means of proof conditions, which were referred to in Chapter 2 as giving conditions for demonstrations (*knowledge-how*); on the other hand, truth conditions were extended by considering what else one must know in order to be able to judge something to be true, the sentence's assertion conditions (*knowledge-what*). The latter conditions were for Frege not to be ascribed to a theory of meaning, but rather to epistemology: the information conveyed by a sentence was represented by knowing that the conditions for the truth of the sentence are satisfied.[39] Such conditions state the agent's implicit awareness, making him able to predicate and recognize truths, whenever conditions for such truths are correctly provided. Once these latter conditions are laid down, the agent will be able to use the sentence in the appropriate ways: this will in turn give the *manifestation* of meaning.[40] This explanation of the meaning is performed by being able to lay down proper (eventually canonical) elements of which the concept involved is predicated: this amounts of course to actual knowledge and it corresponds to the analytic act of stating truthful predications for the concept. In Section 4.5, I shall reconsider the relation between constructions, meaningfulness, and the analytic development of logical reasoning. It is the final aim to provide a clear understanding of the notion of information in relation to logical knowledge, in order to see if such a concept is able to provide a new explanation of the problem of analyticity.

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[38] See once again Dummett (1976, p. 85):

> *A man can acquire no further information from learning the truth of a sentence of whose meaning he is unaware, and what information he does acquire will vary according to the particular meaning he attaches to it.*

[39] Famously Dummett already refers to such conditions as regarding a theory of sense. Dummett (1976, p. 89):

> *Even if the theory of reference merely states what has to be the case for a sentence to be true, should not the theory of sense state, not merely how we know the truth-condition of the sentence, but also how can we know, or on what basis we may judge, the sentence to be true?*

[40] Dummett (1977, pp. 373–375) refers at this point to the "meaning is use" slogan.

4.5 Information and Logical Knowledge

The introduction of the notion of information in the constructive epistemology makes use of the essential epistemic properties of hypothetical reasoning. The validity of such a kind of reasoning requires obviously that assumptions on which the conclusion is based are verified: this means that one must be able to furnish proper constructions for the types involved in the judgements contained in contexts if the reasoning has to be validated, i.e. the substitution of proper proof objects in place of the variables there contained must be accomplished, in order for the conclusion to be justified. If these substitutions can actually be performed, one also gets knowledge of the meanings there contained, having in fact produced analytic judgements. This means that the context

$$\Gamma = (x_1 : \alpha_1, \ldots, x_n : \alpha_n) \tag{4.18}$$

based on a set of proper presuppositions of the form

$$< \alpha_1 : type >, \ldots, < \alpha_n : type > \tag{4.19}$$

expresses the collection of hypotheses leading to a certain conclusion, for example, in the form

$$\frac{(x_1 : \alpha_1, \ldots, x_n : \alpha_n)}{a : \alpha.} \tag{4.20}$$

The meaning of such a form of reasoning is clearly that, in order to know the conclusion (i.e. to validate the argument), one needs to perform all the proper substitutions within the mentioned context:

$$(a_1/x_1 : \alpha_1, \ldots, a_n/x_n : \alpha_n) \tag{4.21}$$

If these substitutions can actually be performed, the conclusion can in fact be drawn from those hypotheses, and the inference from the premises to the conclusion is valid.[41] On the other hand, there is also another remark which can be made in relation to the performed substitutions: if proper constructions have been substituted for the variables, the types involved have been proved to be instantiated. In this way one is shifting from the *meaningfulness* of the concepts (given by the stated presuppositions) to the existence of the types, based on proper constructions:

$$\alpha_1 : exists, \ldots, \alpha_n : exists \tag{4.22}$$

This synthetic extension is obviously justified by knowing in the proper sense instances of the concepts. It must be clear that meaningfulness by itself does not lead directly to the synthetic judgements stating existence of types, and this is clear by considering the following simple example:

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[41] It should be noted that substitutions change hypotheses into premises, i.e. they become known judgements.

Let $\bot$ be a type (assumption)
and let $(x : \bot)$ (assumption);

under these assumptions the meaningfulness is maintained in order to perform a reasoning whatsoever, one which will not be ever validly instantiated (because of course the assumption of the reasoning is a contradiction). Even if it is possible to state such assumptions, because they are meaningful, one will never come to know that type in the proper sense, i.e. to provide a construction instantiating the absurd (according to the mentioned Second Law of Knowability: absurdity cannot be known to be true, cf. Section 2.3), and on this basis one is not able to construct any valid argument. The aforementioned property of CTT being a logic of analytic judgements can also be translated by saying that *it allows to identify all the information necessary in order to perform the reasoning.* A rather trivial example of this situation is given by some simple rules, like the conjunction introduction ($\&I$):

$$\frac{a : A \quad b : B}{\&I(A, B, a, b) : A\&B} \tag{4.23}$$

where the presence of constructions for A and for B allow the ordered pair $<a, b>$ to be built which justifies the judgement $A\&B$. The correctness of such a form of inference is based on its analytic nature. Let us consider now a rule of conjunction elimination ($\&E$). Such a form of inference can be obtained by extending the previous schema in one of the following two ways:

$$\frac{\dfrac{a : A \quad b : B}{\&I(A, B, a, b) : A\&B}}{\&E_1(A, B, \&I(A, B, a, b)) : A} \tag{4.24}$$

$$\frac{\dfrac{a : A \quad b : B}{\&I(A, B, a, b) : A\&B}}{\&E_2(A, B, \&I(A, B, a, b)) : B} \tag{4.25}$$

where obviously the first schema produces a projection for A and the second one for B, both out of the (already obtained) ordered pair $< a, b >$ contained in $\&I$. This progression seems thus to be completely analytic, and once again correct. This example is the best way to show what it means that the correctness of logical arguments is based on their analyticity, and that in this sense such arguments cannot furnish any "information": the correctness of logical derivations is due to the analytic construction of proof objects for the concepts involved. Such forms of reasoning are certainly the essential ones, but hardly can they be recognized as "interesting" and thus "informative". Relevant or interesting developments of logical knowledge are mainly based on the use of auxiliary constructions (of concepts and hypotheses), which in turn amounts to using informational bases, in order to support and provide conditions for knowledge. The final and essential aim of our epistemic analysis amounts to a clarification of informational states in the analytic development of logical knowledge, thus providing a real answer to the question: "what kind of information is produced by the development of a logical derivation?"

If one considers relevant and fruitful forms of reasoning (i.e. not just simply purely analytic derivations), mostly they make use of hypothetical reasoning: to reply to our question it is essential to consider the relation between the informational content and the analytic development of the derivation, namely, the actual process of extending knowledge. In the formal structure presented in Chapter 3, this amounts to considering the relation between *informational updatings* and *knowledge extensions*. Precisely in such a connection one can answer the previous question, by showing what information is given by logical derivations, and moreover in which sense the development of logical knowledge does not rest entirely on the analytic process. Moreover, it is in terms of the answer to such a question that the problem of analyticity is to be reformulated: analytic knowledge is supported by auxiliary steps of acquiring information which make logical processes interesting and fruitful; the role of these synthetic steps in the process of formulating logical reasoning is to support arguments by means of auxiliary constructions, here defined as *information*, and which extend the range of possible conclusions to be drawn. The sense and the correctness of analytic predications are based primarily on such *informational updatings*: they provide essential information about concepts in terms of their meaningfulness, introducing elements not necessarily already contained in the agent's knowledge state. The introduction of concepts is a synthetic extension under conditions of validity of the reasoning, namely, existence of those concepts is a result of proper instantiation of their constructions (thus, again, the analytic development furnishes proper justification). The main thesis supported by such a view is that updatings performed on background information reflect a synthetic value. At this point a new definition for informativeness of derivations can be presented, modelled after the previous one for analytic/synthetic judgements, and finally suggesting our solution to the problem of analyticity:

Definition 4.6 (Informative Derivations) *Interesting and informative logical derivations are developed starting from informational backgrounds. Knowledge extensions are produced on the basis of essentially synthetic updatings of this information: their content regards the meaningfulness of new concepts introduced within the agent's knowledge frame (type declarations) and the use of auxiliary constructions (assumptions). A whole logical process is thus in its essence not purely analytic.*

The suggested conclusion is that at the basis of inferential processes there is a core of informational states, and the operations of informational updatings are essentially synthetic: extensions of such informational states can be accounted in the first instance as synthetic creative acts, because they furnish all the nested predicables which can be extracted in an analytic procedure. Meaningfulness in terms of type declarations can be accounted as a part of this information. There is therefore a relevant sense in which inferential processes increase our knowledge, i.e. by supposing a synthetic informational basis from which analytic knowledge is developed. On the other hand, this means in turn that it is not possible simply to ascribe to information itself a synthetic nature: such ascription is possible only in relation to a proper knowledge state, in which such information is introduced and to which it is related. This is also in perfect agreement with the described nature of information: information is an epistemic notion providing conditions on which proper knowledge can be acquired; therefore, its definition is essentially related to the description of a knowledge system. Knowledge processes are thus essentially built on the use of external information, which allows to extend synthetically our knowledge frame.[42]

4.6 Final Epistemic Foundation for Information

Constructive epistemology has famously reduced the Socratic question "*what is truth?*" to its epistemic counterpart, namely, "*what does it mean to know the truth?*" The same revolution applies to the notion of meaning.[43] On the basis of this approach, there is the assumption that one cannot avoid taking into account the *act*, thus establishing the priority of the knowing subject on the object known. This was of course already one of the greatest debates in the history of philosophy since antiquity; it was clearly formulated in the modern age by Fichte in his *Einleitungen in die Wissenschaftslehere*, where he distinguished the sort of philosophy that begins with the "pure I" as "idealism" and that beginning with the "thing in itself" as "dogmatism", the two approaches taking as starting point respectively the act of knowing and the object known. But to appeal explicitly to such a position in logic was a brave step, taken in the 1930s by the Intuitionists, a theoretical revolution which produced a huge series of consequences for the entire conception of what logic is as a science. The philosophical notion of act has been later interpreted in terms of its precise mathematical counterpart, the notion of "construction" or "proof", which nonetheless still maintains a deep ethical suggestion. According

[42] Moreover, this understanding of the extensional nature of logical derivations agrees with the proper process of setting hypotheses: in scientific processes the verification of hypotheses is in fact to be considered as a main goal; one settles hypotheses and if the consequences of those hypotheses are verified, then the hypothesis itself is proved, thus providing new knowledge.

[43] Dummett (1976, p. 69):

> *Questions about meaning are best interpreted as questions of understanding: a dictum about what the meaning of a sentence consists in must be construed as a thesis about what is to* know *its meaning.*

to the constructive perspective, to know the truth means essentially to construct such a truth, having a proof or instance, or at least the certainty of the possibility of constructing such a proof. The question "*what is knowledge?*" becomes in this way a generalized way to ask "*how do we know?*", the essential step to truth being accomplished only by the aware subject. The responsibility of the act of knowing let us in this way also be aware of what is actually known; in other words, the validity of an act of knowing is proved by the subject's ability to lay down all the *information needed* to gain this knowledge, and also all the *information provided* by this act. In this passage the different senses undergoing our use of the term "information" are shown: knowledge consists not only in information gained, but also in information needed; thus, it is not only a reply to the question "*which information do we obtain by knowing this?*", but also a reply to the question "*which information do we need to get to know this?*". To provide answers to these questions in the light of the constructive perspective here suggested means to provide an explanation of the "flow of information" within knowledge processes. This expression accounts for those conditions which, in the practice of knowledge processes, do not always amount just to the certainty of proofs: in this respect the role that conjectures and assumptions play in supporting proofs is essential. The information flowing within logical reasoning is not just the content of the knowledge act supported by proofs: the practice of knowledge, out of its pure idealization, still appeals to weaker forms of data retrieval.

The development of a coherent and complete representation of knowledge processes is thus the first aim of the epistemic model obtained by extending the constructive epistemology via the introduction of the notion of information. In line with the literature on knowledge systems,[44] this research considers epistemic states as rational idealizations of psychological states. This model distinguishes in a proper way the two more common psychological states for rational (human) agents, defined as follows:

- *Belief*, the epistemic attitude produced by accepting *information* as a part of the proper knowledge state, possibly on its basis starting some knowledge process
- *Knowledge*, the epistemic attitude produced by the act of proving

To these states correspond two essential epistemic inputs:

- *Informational inputs* are intended to produce updatings on existing informational states; their nature is represented essentially by type declarations and auxiliary constructions, performing the operations of updating contexts.
- *Knowledge inputs* produce instead extensions of proper knowledge states.

[44] Cf., e.g. Gärdenfors (1988).

The condition for accepting informational updatings[45] is presented in terms of maintaining "*equilibrium under all forces of internal criticism*",[46] which means essentially that valid informational updatings produce coherent ("stable") extensions of knowledge, whereas incoherent extensions are produced by faulty updatings of the informational states, and they require procedures for checking and dismissing unreliable information. Proper knowledge maintains a fixed relation to the external world via the foundation of truth provided by the notion of proof; on the other hand, information has a weaker link to reality, provided only in terms of acceptability criteria by the agent, furnishing in this way auxiliary conditions for provability.

The notion of information here developed makes explicit the relevant portion of epistemic grounds (implicit ones) on which an expression depends. In this model, the basis of these epistemic grounds is represented by meaningfulness, as what is essentially presupposed by the judging agent, performing the act of knowing: to formally express a notion of information amounts in this way to set the formal and conceptual constraints to develop a form of moderate holism as intended by Quine (1981), where meaningfulness is considered as a primitive epistemic property.[47] The first goal reached in this way is to present a coherent epistemic model, which takes into account for the very first time in the constructive setting the complexity of the connection between information and knowledge, without collapsing the two notions into each other. A brief remark should be finally made on the relevance of such a distinction for the ethical analysis of information theory: our model in fact provides a theory of knowledge in which the dangerous implications of setting as identical the certainty provided by the notion of verification on the one hand and the value of what is assumed as true on the other hand are avoided. In the distinction between the information assumed or accepted, and the proved knowledge, one finds the space to revise and reject falsity and error. This obviously can be reconsidered in the light of our media systems, on which essentially our perception of facts and state of affairs is nowadays based: to be able to reconsider critically and eventually reject the information received (but not the knowledge that one can provably furnish) is a highly safe and appreciable behaviour.

[45] Explained as a *valid* informational updating in Primiero (2006).
[46] Gärdenfors (1988, pp. 9–10).
[47] Cf. Cozzo (2002).

Bibliography

Abelard (1118–1137) *Dialectica*, L.M. de Rijk (ed.), 2nd rev. edn, van Gorcum, Assen.

Aczel, P. (1978) The type theoretic interpretation of constructive set theory, in A. McIntyre, L. Pacholsky, and J. Paris (eds.), *Logic Colloquium '77*, North-Holland, Amsterdam, pp. 55–66.

Aczel, P. (1982) The type theoretic interpretation of constructive set theory: Choice principles, in S.S. Troelstra and D. van Dalen (eds.), *The E.L.J. Brouwer Centenary Symposium*, North-Holland, Amsterdam, pp. 1–40.

Aczel, P. (1986) The type theoretic interpretation of constructive set theory: Inductive definitions, in R. Barcan Marcus, G.J.W. Dorn and P. Weinegartner (eds.), *Logic, Methodolgy and Philosophy of Science VII*, North-Holland, Amsterdam, pp. 17–49.

Aczel, P. and Rathjen, M. (2001) Notes on Constructive Set Theory, Technical Report 40, Mittag-Leffler Institute, The Swedish Royal Academy of Sciences. Available at http://www.cs.man.ac.uk/∼petera/papers.html.

Allo, P. (2005) Being informative: Information as infromation handling, in Althoff, Dengel, Bergmann, Nick, Roth-Berghofer (eds.), *Proceedings of WM2005: Professional Knowledge Managment Experiences and Visions*, DFKI GmbH, pp. 579–586.

Allo, P. (2005a) Formalising semantic information: Lessons from logical pluralism, submission for E-CAP 2005 Proceedings. Available at http://homapages.vub.ac.be/∼pallo/2005/03/formalising-semantic-information.html.

Angelelli, I. (1967) On Identity and interchangeability in Leibniz and Frege, *Notre Dame Journal of Formal Logic*, VIII, (1–2).

Aristotle (Cat) *Categories*, trans. H.P Cooke and H. Tredennick, *Aristotle*, Vol. I, Loeb Classical Library, Harvard University Press, reprinted 2002, pp. 1–109.

Aristotle (AnPri) *Prior Analytics*, trans. H.P. Cooke, H. Tredennick, *Aristotle*, vol. I, Loeb Classical Library, Harvard University Press, reprinted in 2002, pp. 181–531.

Aristotle (AnPos) *Posterior Analytics*, trans. H.P. Cooke and H. Tredennick, *Aristotle*, Vol. II, Loeb Classical Library, Harvard University Press, reprinted 2002.

Aristotle (EthNic) *Nicomachean Ethics*, trans. H. Rackham, *Aristotle*, Vol. XIX, Loeb Classical Library, Harvard University Press.

Aristotle (Metaph) *Metaphysics*, translated with commentaries and glossary by H.G. Apostle, Indiana University Press, 1966.

Aristotle (Top) *Topics*, trans. H.P. Cooke and H. Tredennick, *Aristotle*, Vol. II, Loeb Classical Library, Harvard University Press, reprinted 2002.

Ayer, A.J. (1936) *Language, Truth and Logic*, Gollancz, London; 2nd edn., 1946, reprinted by Dover, New York.

Baral, C. and Zhang, Y. (2001) On the semantics of knowledge update, in Bernhard Nebel (ed.), *Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence, IJCAI 2001*, Seattle, Washington, USA. Morgan Kaufmann, pp. 97–102.

Bar-Hillel, J. (1964) *Language and Information*, Addison-Wesley and the Jerusalem Academic Press, Reading, MA.

Bar-Hillel, Y. and Carnap, R. (1952) An outline of a theory of semantic information, MIT Research Laboratory, Technical Report n. 247; reprinted in Bar-Hillel (1964).

Bar-Hillel, Y. and Carnap, R. (1953) Semantic information, *British Journal of the Philosophy of Science*, 4, 147–157.

Barwise, J. and Perry, J. (1983) *Situations and Attitudes*, MIT Press, Cambridge, MA.

Barwise, J. and Seligman, J. (1997) *Information Flow: The Logic of Distributed Systems*, Cambridge University Press, Cambridge.

Beaver, D.I. (1997) Presupposition, in J. Van Benthem and A. Ter Meulen (eds.), *Handbook of Logic and Language*, North-Holland, Elsevier, Amsterdam; The MIT Press, Cambridge, MA, pp. 939–1008.

Belnap, N.D. (1962) Tonk, Plonk and Plink, in *Analysis*, 22, Blackwell; reprinted in Strawson P.F. (ed.), *Philosophical Logic*, Oxford University Press (1967).

Berg, J. (1999) Kant über analytische und synthetische Urteile mit Berücksichtigung der Lehren Bolzanos, in Morscher, E. (ed.), *Bernard Bolzanos geistiges Erbe für das 21. Jahrhundert*, Akademia Verlag, Sankt Augustin.

Bernays, P. (1965) *Some empirical aspects of mathematics*, in Dockx, Bernays, pp. 123–128.

Bishop, E. (1967) *Foundations of Constructive Analysis*, McGraw-Hill, New York.

Bishop, E. (1970) Mathematical as a numerical language, in J. Myhll et al. (eds.), *Intuitionism and Proof Theory*, North-Holland, Amsterdam, pp. 53–71.

Bishop, E. and Bridges, D. (1985) *Constructive Analysis*, Springer, New York.

Boghossian, P.A. (1997), Analyticity, in B. Hale and C. Wright (eds.), *A Companion to the Philosophy of Language*, Blackwell, Oxford, pp. 331–368.

Bolzano, B. (1810) *Beiträge zu einer begründeteren Darstellung der Mathematik*; engl. trans. *Contributions to a better-grounded presentation of mathematics*, by W. Ewald, in *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Vol. I, Clarendon Press, Oxford.

Bolzano, B. (1830) *Einleitung zur Grössenlehere, Erste Begriffe der allgemeinen Grössenlehre – Von der mathematischen Lehrart*, Frommann, Stuttgart-Bad Cannstatt (1975).

Bolzano, B. (1837) *Wissenschaftslehere*, Sulzbach; herausgegeben von J. Berg, Friederich Fromann (Günther Holzboog), Stuttgart-Bad Cannstatt (1989); English translation, *Theory of Science* (J. Berg ed.), D. Reidel Publishing, Dordrecht, The Netherlands (1973).

Boole, G. (1854) *An Investigation of the Laws of Thought*, Mcmillan, London; Walton and Maberly, London.

Boole, G. (1854a) On a general method in the theory of probabilities, in *The Philosophical Magazine*, 4th series, 1854; 8th reprint in *Studies in Logic and Probabilities*, Open Court, La Salle (1952).

Borghuis, T., Kamareddine, F., and Nederpelt, R. (2002) Formalizing belief revision in type theory, *Logic Journal of the IGPL*, 10 (5), 461–500.

Bremer, M. and Cohnitz, D. (2004) *Information and Information Flow: An Introduction*, Ontos-Verlag.

Brookes, B.C. (1980) Foundations of information science: part I–IV, *Journal of Information Science*, 2(5).

Brillouin, L. (1956) *Science and Information Theory*, Academic Press, New York.

Brouwer, L.E.J. (1918) *Begründung der Mengenlehre unanhängig vom logischen Satz vom ausgeschlossen Dritten* - pt. 1: *Allgemeine Mengenlehre*, Koninklijke Nederlandse Akademie van Weteschappen, Verhandenlingen, 12(5).

Brouwer, L.E.J. (1925) Zur Begründung der intuitionistischen Mathematik I, *Mathematische Annalen* 93, 244–257.

Brouwer, L.E.J. (1926) Zur Begründung der intuitionistischen Mathematik II, *Mathematische Annalen* 95, 453–472.

Brouwer, L.E.J. (1975) *L.E.J. Brouwer. Collected Works*, Vol. I, *Philosophy and Foundations of Mathematics*, Heyting A. (ed.), North-Holland, Amsterdam.

Brouwer, L.E.J. (1981) *Brouwer's Cambridge Lectures on Intuitionism*, D. van Dalen (ed.), Cambridge University Press, Cambridge.

Bruijn de, N.G. (1970) The mathematical language AUTOMATH, its usage, and some of its extensions, in M. Laudet, et al. (eds.), *Symposium on Automatic Demonstration*, December 1980, Rocquencourt, Springer, Berlin, pp. 29–61.

Bruijn de, N.G. (1980) A survey of the project AUTOMATH, in J.P. Seldin and J.R. Hindley (eds.), *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Academic Press, New York, pp. 579–606.

Burge, T. (1992) Philosophy of language and mind 1950–1990, *The Philosophical Review*, 101, 3–51.

Buss, S.R. (1993) Intuitionistic validity in T-normal Kripke Structures, *Annals of Pure and Applied Logic*, 59, 159–173.

Butrick, R. (1970) *Carnap on Meaning and Analyticity*, Mouton, The Hague.

Cantor, G. (1878) Ein Beitrag zur Mannigfaltigkeitslehre, *Journ. für Reine u. Angew. Math.*, 84, 242–258.

Carnap, R. (1928) *Der logische Aufbau der Welt*, Weltkreis, Berlin 1928; English trans. *The Logical Structure of the World*, by R. George, University of California Press, Berkley and Los Angeles.

Carnap, R. (1947) *Meaning and Necessity: A Study in Semantics and Modal Logic*, University of Chicago Press, Chicago; 2nd edn., 1956.

Carnap, R. (1950) *Logical Foundations of Probability*, University of Chicago Press, Chicago; 2nd edn., 1963.

Carnap, R. (1970) *Philosophy of Logic*, Prentice-Hall Foundations of Philosophy Series, Englewood Cliffs, NJ.

Carr, D. (1979) The logic of knowing how and ability, *Mind*, LXXXVIII.

Church, A. (1976) Schröder's anticipation of the simple theory of types, *Erkenntnis*, 10 (October), 407–411.

Coffa, J.A. (1991) *The Semantic Tradition form Kant to Carnap: To the Vienna Station*, Cambridge, Cambridge University Press.

Cohen, L.J. (1992) *An Essay on Belief and Acceptance*, Clarendon Press, Oxford.

Cohen, M.R. and Nagel, E. (1934) *An Introduction to Logic and Scientific Method*, Routledge & Kegan Paul, London.

Cohen, L.J., Los, J., Pfeiffer, H., and Podewski, K.P. (eds.) (1982) *Logic, Methodology and Philosophy of Science IV*, North-Holland, Amsterdam; Pwn-Polish Scientific Publishers, Warszawa.

Coquand, T., Dybjer, P., Nordström, B., and Smith, J. (eds.) (2000) *Types for Proofs and Programs*, International Workshop TYPES '99, Lokeberg, Sweden, June 1999, Selected Papers, Lecture Notes in Computer Science 1956, Springer-Verlag.

Corsi, G. and Sambin, G. (eds.) (1991) *Atti del Congresso "Nuovi Problemi della Logica e della Filosofia della Scienza"* (in Italian), Viareggio, 8-13 gennaio 1990, Bologna, CLUEB.

Cozzo, C. (1994) *Meaning and Argument*, Almqvist & Wiksell, Stockholm.

Cozzo, C. (2002) Does epistemological holism lead to meaning-holism?, in *Topoi*, 21, Kluwer, Dordrecht, The Netherlands, pp. 25–45.

Danto, A.C. (1968) *Analytical Philosophy of Knowledge*, Cambridge University Press, Cambridge.

Dedekind, R. (1888) *Was sind und was sollen die Zahlen*, Braunschweig.

Devlin, K. (1991) *Logic and Information*, Cambridge University Press, Cambridge MA.

Pauli-Schimanovich, E., Kohler, W., and Stadler, F. (eds.) (1995) *The Foundational Debate: Complexity and Constructivity in Mathematcs and Physics*, Kluwer, Dordrecht, The Netherlands.

Dirichlet, G.L. (1837) Über die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinusreihen, in *Repertorium der Physik*, Vol. 1 pp. 152–174; reprint in L. Kronecker and L. Fuchs (eds.), *G. Lejeune Dirichlet's Werke*, 1889–1897, G. Reimer, Berlin, Vol. 1, pp. 133–160.

Dockx, S. and Bernays, P. (eds.) (1965) *Information and Prediction in Science*, Academic Press, New York and London.

Dretske, F. (1981) *Knowledge and the Flow of Information*, Bradfors Books, MIT Press.

Dummett, M. (1973) *Frege. Philosophy of Language*, 2nd edn., Duckworth, London.

Dummett, M. (1976) What is a Theory of Meaning (II), in Evans and McDowell (eds.), *Truth and Meaning*, Oxford 1976; republished in Dummett (1993), pp. 34–93.

Dummett, M. (1977) *Elements of Intuitionism*, Clarendon Press, Oxford 1977; 2nd edn., 2000.

Dummett, M. (1993) *The Seas of Language*, Clarendon Press, Oxford.

Dunn, J.M. (2001) The concept of information and the development of modern logic, in W. Stelzner and M. Stöckler (eds.), *Zwischen tradittioneller und moderner Logik: Nichtklassicher Ansätze*, Mentis Verlag, Paderborn, pp. 423–446.

Euler (1748) *Introductio in Analysis Infinitorum*, trans. by D. Blanton John (1990) *Introduction to Analysis of Infinite*, Books I and II, Springer, New York.

Euler (1755) *Institutiones Calculi Differentialis* in *Opera Omnia*, Series 1 *Opera Mathematica*, Vol. 10, G. Kowalewski (ed.), 1913, Birkhäuser; trans. D. Blanton John (2000) *Foundations of Differential Calculus*, Springer, New York.

Floridi, L. (ed.) (2002) *The Blackwell Guide to the Philosophy of Computing and Information*, Oxford, Blackwell.

Floridi, L. (2002a) What is the philosophy of information, *Metaphilosophy*, 1–2, 123–145.

Floridi, L. (2004) Outline of a theory of strongly semantic information, *Minds and Machines*, 14(2), 197–222.

Floridi, L. (2005) Is Information meaningful data?, *Philosophy and Phenomenological Research*, 70(2), 351–370.

Forcheri, P., Gentilini, P., and Molfino, M.T. (1999), *Informational Logic in Knowledge Representation and Automated Deduction*, AI Communications 12, pp. 185–205, IOS Press.

Fourier, J.-B. J. (1822) *Théorie analytique de la chaleur*, Paris, Didot 1822, reprint in G. Darboux (ed.) (1988) *Oeuvres de Fourier*, Vol. I, Gauthier-Villars, Paris.

Fourman, M.P. (1982) *Notions of Choice Sequence*, in Troelstra and van Dalen (eds.), pp. 91–105.

Franzel, T. (1987) *Provability and Truth*, Almqvist & Wiksell International, Stockholm.

Frege, G. (1879) *Begriffschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle.

Frege, G. (1884) *Die Grundlagen der Arithmetik*, Verlag Wilhelm Köbner, Breslau; English trans. *The Foundations of Arithmetic*, trans. by J.L. Austin, Basil Blackwell, Oxford (1950).

Frege, G. (1891) Funktion und Begriff, republished in Frege, *Funktion, Begriff, Bedeutung*, herausgegeben von M. Textor, Sammlung Philosophie 4, Vandenhoek & Ruprecht (2002), pp. 2–22.

Frege, G. (1892) Über Sinn und Bedeutung, *Zeitschrift für Philosophie und philosophische Kritik*, NF 100, pp. 25–50; republished in Frege, *Funktion, Begriff, Bedeutung*, herausgegeben von M.Textor, Sammlung Philosophie 4, Vandenhoek & Ruprecht (2002), pp. 23–46.

Gärdenfors, P. (1988) *Knowledge in Flux: modeling the dynamics of epistemic states*, MIT Press, Cambridge, MA.

Gentzen, G. (1934) Untersuchungen über das logische Schliessen, *Math. Zeitschrift*, 39, 176–210 and 405–431.

George, R. (2003) Intuitions, in *Philosophiques* 30/1, Printemps, pp. 19–46.

Geuvers, J.H. (1993) *Logics and Type Systems*, Ph.D. dissertation, Universiteit Nijmegen.

Harris, Z. (1988) *Language and Information*, Columbia University Press.

Heim, I. (1983) On the projection problem for presuppositions, in M. Barlow, D. Flickinger, and M. Westcoat (eds.), *Second Annual West Coast Conference on Formal Linguistics*, Stanford University, pp. 114–126; reprinted in S. Davis (ed.) *Pragmatics*, pp. 397–405, Oxford: Oxford University Press (1990).

Heyting, A. (1930) *Die Formalen regeln der Intuitionistischen Logik I*, Sitzungeberichte der Preussischen Akademie von Wissenschaften; English trans. in P. Mancosu (ed.), *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, Oxford.

Heyting, A. (1930a) *Sur la logique intuitionniste*, Academie Royale de Belgique, Bulletin de la classe de sciences 16.

Heyting, A. (1931) *Die intuitionistische Grundlegung der Mathematik*, Erkenntnis 2, pp. 106–115. English trans. in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings*, 2nd edn., Cambridge University Press (1983), pp. 52–61.

Heyting, A. (1934) *Mathematische Grundlagenforschung: Intuitionismus, Beweistheorie*, Julius Springer, Berlin, 1934.

Heyting, A. (1956) *Intuitionism: An Introduction*, North-Holland, Amsterdam.

Hintikka, J. (1962) *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, Cornely University Press, Ithaca, NY.

Hintikka, J. (1963) The modes of modality, *Acta Philos. Fennica*, 16, 65–82.

Hintikka, J. (1967) Individuals, possible worlds and epistemic logic, *Nous*, 1, 138–147.

Hintikka, J. (1970a) On semantic information, in Hintikka and Suppes (eds.) (1970), pp. 3–27.

Hintikka, J. (1970b) Surface information and depth information, in Hintikka and Suppes (eds.) (1970), pp. 263–297.

Hintikka, J. (1973) *Logic, Language Games and Information*, Oxford University Press, Oxford.

Hintikka, J., Niiniluoto, I., and Saarinen, E. (1987) *Essays on Mathematical and Philosophical Logic*, Reidel Publishing, Dordrecht, The Netherlands, 1987.

Hintikka, J. and Remes, U. (1974) *The Method of Analysis: Its Geometrical Origin and Its General Significance*, Reidel Publishing, Dordrecht, The Netherlands.

Hintikka, J. and Suppes, P. (eds.) (1970) *Information and Inference*, Reidel Publishing, Dordrecht, The Netherlands.

Hirsch, G. (1965) Sur la formation et la signification des Concepts, in Dockx, Bernays, pp. 111–120.

Holm, R. (2003) A constructive approach to state description semantics, *Journal of Applied Logic*, 1(1–2) 13–46.

Howard, W.A. (1980) The formulae-as-types notion of construction, in J.P. Seldin and J.R. Hindley (eds.) *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Academic Press, London, pp. 479–490.

Husserl, E. (1913a) *Ideas: General Introduction to Pure Phenomenology*, trans. W.R. Boyce Gibson, Collier Books 1962.

Husserl, E. (1913b) *Logical Investigations*, 2nd edn., 2 vols., trans. J.N. Findlay, Routledge, London, 2000.

Husserl, E. (1929) *Formale and transzendentale Logik: Versuch einer Kritik der logischen Vernunft*, sonderdruck aus Jahrbuch für Philosophie und phänomenologische Forschung, 10. Halle a.d.S. v–xiii; 1–298; Max Niemeyer Verlag, Halle (Saale), 1929; voll. I–II, Herausgegeben von Paul Janssen. Husserliana Studienausgabe, Vol. 1, Martinus Nijhoff Dordrecht, The Netherlands, 1977; Herausgegeben von Paul Janssen. Husserliana Studienausgabe. Vol. 2, Martinus Nijhoff, Dordrecht, The Netherlands, 1977; English trans. *Formal and Transcendental Logic: Attempt of a Critique of Logical Reason*, by Dorion Cairns, Martinus Nijhoff, The Hague, The Netherlands, 1969.

Israel, D., Perry, J. (1990) What is information, in P. Hanson (ed.), *Information, Language and Cognition*, Vancouver, University of British Columbia Press, pp. 1–19.

Jain, S. and Staphan, F. (2001) Learning by switching the type of information, in *Lectures notes in Computer Science*, vol. 2225; *Prooceedings of the 12th International Conference on Algorithmic Learning Theory*, Springer, London, pp. 205–218.

Jevons, W.S. (1873) *The Principles of Science*, Mcmillan, London; 2nd edn., 1878.

Kant, I. (KrV) *Kritik der reinen Vernunft*, new edn. (R. Schmidt herausgegeben), Felix Meiner Verlag, Hamburg 1956/1971.

Kant, I. (1783) *Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können*; engl. trans. *Prolegomena to any future metaphysics that will be able to come forward as science*, trans. and ed. Gary Hatfield, Cambridge University Press, Cambridge, 2004.

Kant, I. (1800) *Logik - ein Handbuch zu Vorlesungen*, herausgegeben von G.B. Jäsche, Leipzig, Verlag von Felix Meiner, dritte Auflage.

Katsuno, H. and Mendelzon, A.O. (1991) On the Difference between Updating a Knowledge Base and Revising it, in Allen, Fikes and Sandewall (eds.), *Principles of Knowledge Representation and Reasoning*, Morgan Kaufmann, San Mateo, California, pp. 387–394.

Keynes, J.M. (1921) *A Treatise on Probability*, Mcmillan, London.

Keller, P. (2002) *Information Flow: Logics for the (R)Age of Information*, Diploma thesis, Research Group on Theoretical Computer Science and Logic, ICM Institute of Computer Science and Applied Mathematics, University of Bern, Switzerland.

Khinchin, A.I. (1957) *Mathematical Foundations of Information Theory*, Dover Publications, New York.

Kolmogorov, A. (1932) Zur deutung der Intuitionistischen Logik, *Math. Zeitschrift*, 35, 58–65.

Krahmer, E. (1995) *Discourse and Presupposition*, Ph.D. dissertation, ITK/ TILDIL Dissertation Series, Univeristy of Tilburg.

Kripke, S. (1963) Semantical Considerations on Modal Logic, *Acta Philos. Fennica* 16, 83–94.

Kripke, S. (1963a) Semantical analysis of intuitionistic logic I, in J.N. Crossley and M.A.E. Dummett (eds.), *Formal Systems and Recursive Functions*, Proceedings of the Eighth Logic Colloquium at Oxford, July 1963, North-Holland, Amsterdam, pp. 92–129.

Kripke, S. (1972) Naming and Necessity, in D. Davidson and G. Harman (eds.), *Semantics of Natural Language*, Reidel, Dordrecht, pp. 253–255.

Laan, T. (1997) *The Evolution of Type Theory in Logic and Mathematics*, Ph.D. dissertation, Tecnische Universiteit Eindhoven, Printed and bound by Print Partners Ipskamp, Enschede.

Leibniz, G.W. (1673) *Methodus tangentium inversa seu de functionibus*, in *Mathematischen Schriften*, herausgegeben vom Leibniz Archiv der Niedersächischen Landesbibliothek, Hannover, dritter Band (1672–1676), pp. 193–201; available at `www.nlb-hannover.de/Leibniz/Leibnizarchiv/Veroeffentllchungen/VII3A.pdf`.

Leibniz, G.W. (1684) *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur et singular pro illis calculi genus*, Acta eruditorum, 466–473.

Löhrer, G. (2003) Possible proofs and method in metaphysics and logic, in H. Rott and V. Horak (eds.), pp. 221–252.

MacKay, D.M. (1969) *Information Mechanism and Meaning*, MIT Press, Cambridge, MAs and London.

Mäenpää, P. (1993) *The Art of Analysis: Logic and History of Problem Solving*, Helsinki, PhD thesis.

Mäenpää, P. (1997) From backward reduction to configurational analysis, in Otte, Panza, pp. 201–226.

Mäenpää, P. (1998) Analytic program derivation in type theory, in G. Sambin and J.M. Smith, pp. 113–126.

Martin, C.J. (1986) William's machine, *The Journal of Philosophy*, 83(10), 564–572.

Martin-Löf, P. (1970) On the notion of randomness, in A. Kino, J. Myhill, and R.E. Vesley (eds.), *Intuitionism and Proof Theory*, North-Holland, Amsterdam.

Martin-Löf, P. (1973) Hauptsatz for intuitionistic simple type theory, in P. Suppes et al. (eds.), *Logic, Methodology and Philosophy of Science IV*, North-Holland, Amsterdam, pp. 279–290.

Martin-Löf, P. (1975) An intuitionistic theory of types: predicative part, Logic Colloquium '73, North-Holland.

Martin-Löf, P. (1975a) About models for intuitionistic type theories and the notion of definitional equality, in S. Kanger (ed.), *Proceedings of the Third Scandinavian Logic Symposium*, North-Holland, Amsterdam, pp. 81–109.

Martin-Löf, P. (1982) Constructive mathematics and computer programming, in Cohen, Los, Pfeiffer, and Podewski (eds.), pp. 153–175.

Martin-Löf, P. (1984) *Intuitionistic Type Theory*, Bibliopolis, Napoli.

Martin-Löf, P. (1987) *Philosophical Implications of Type Theory*, notes of the lectures given at the Istituto di Filosofia, Facoltá di Lettere e Filosofia, Universitá degli Studi di Firenze, 15 March–15 May 1987.

Martin-Löf, P. (1987a) Truth of a proposition, evidence of a judgement, validity of a proof, *Synthese* 73, 407–420.

Martin-Löf, P. (1990) Choice sequences in type theory, unpublished notes by P. Mäenpaä and A. Ranta on a series of lectures given by Martin-Löf at Stockholm University.

Martin-Löf, P. (1990a) Mathematics of infinity, in P. Martin-Löf and G. Mints (eds.), pp. 146–197.

Martin-Löf, P. (1991) A Path from logic to metaphysics, in Corsi and Sambin (eds.), *Atti del Congresso "Nuovi Problemi della Logica e della Filosofia della Scienza"* (Italian), Viareggio, 8-13 gennaio 1990, Bologna, CLUEB, pp. 141–149.

Martin-Löf, P. (1991a) Contexts and environments, unpublished notes by P. Mäenpaä and A.Ranta of a series of lectures given by Martin-Löf at Stockholm University.

Martin-Löf, P. (1993) Philosophical aspects of intuitionistic type theory, unpublished notes by M. Wijers from lectures given at the Faculteit der Wijsbegeerte, Rijksuniversiteit Leiden.

Martin-Löf, P. (1994) Analytic and synthetic judgments in type theory, in P. Parrini (ed.), *Kant and Contemporary Epistemology*, Kluwer Academic, Dordrecht, pp. 87–99.

Martin-Löf, P. (1995) Verificationism then and now, in Pauli-Schimanovich et al. (eds.), pp. 187–196.

Martin-Löf, P. (1996) On the meanings of the logical constants and the justifications of the logical laws, *Nordic Journal of Philosophical Logic*, 1(1), 11–60, Scandinavian University Press.

Martin-Löf, P. (1998) An intuitionistic theory of types, in Sambin and Smith (eds.), pp. 127–172.

Martin-Löf, P. (1999) Nonstandard type theory, unpublished notes of a talk given at the Workshop on Proof and Computation, Matematisches Institut at Universität Mńchen 5–6 November 1999.

Martin-Löf, P. (2001) The sense/reference distinction in constructive semantics, unpublished transcription of a lecture given at a conference on Frege at Leiden University, 25 August 2001.

Martin-Löf, P. (2003) Are the objects of propositional attitude propositions in the sense of propositional and predicate logic?, transcript by G. Primiero and G. Sundholm of a Lecture given in Geneva on 19 December 2003 at the First League of European Research Universities, Philosophy Conference on *Formal Concepts and Properties*.

Martin-Löf, P. and Mints, G. (eds.) (1990) *COLOG-88*, Proceedings of the International Conference on Computer Logic, Tallinn, USSR, December 1988, Springer.

Mates, B. (1951) Analytic sentences, *Philosophical Review*, 60.

Morscher, E. (2003) La définition bolzanienne de l'analyticité logique, *Philosophiques* 30(1), 149–169.

Muskens, R., van Benthem, J., and Visser, A. (1997) Dynamics, in J. Van Benthem and A. Ter Meulen (eds.), *Handbook of Logic and Language*, North-Holland, Elsevier, Amsterdam, The MIT Press, Cambridge, Massachussets, pp. 587–648.

Myhill, J. (1975) Constructive set theory, *Journal of Symbolic Logic*, 40, 347–382, 1975.

Nef, F. (2004) *Qu'est-ce que la métaphysique?*, Gallimard, Paris 2004.

Nordström, B., Petersson, K., and Smith, J. (1990) *Programming in Martin-Löf's Type Theory: An Introduction*, Clarendon Press, Oxford.

Ockham, W. (1324) *Summa logicae*, ed. Ph. Boehner, G. Gál, and S.F. Brown, *Opera Philosophica*, I. St. Bonaventure, 1974; *Logica terminorum*, ital. trans. *Logica dei termini*, a cura di P. Müller, Rusconi, 1992.

Ogden, C.K. and Richards, I.A. (1923) *The Meaning of Meaning: A Study in the Influence of Language upon Thought and of the Science of Symbolism*, London.

Otte, M. and Panza, M. (eds.) (1997) *Analysis and Synthesis in Mathematics: History and Philosophy*, Kluwer, Dordrecht.

Pagin, P. (2001), A Quinean definition of synonymy, *Erkenntnis* 55, 7–32, Kluwer.

Pap, A. (1958) *Semantics and Necessary Truths*, Yale University Press, New Haven/London.

Peirce, C.S. (1906) Prolegomena to an Apology for Pragmaticism.

Prawitz, D. (1965) *Natural Deduction: A Proof-Theoretic Study*, Uppsala.

Prawitz, D. (1971) Ideas and results in proof-theory, in Fenstad J.E. (ed.), *Proceedings of the Second Scandinavian Logic Symposium*, North-Holland, Amsterdam.

Prawitz, D. (1987) Proofs and the meaning and completeness of the logical costants, in Hintikka, Niiniluoto, and Saarinen, pp. 25–40.

Primiero, G. (2004) *The Determination of Reference in a Constructive Setting*, in Giornale di Metafisica - Nuova Serie, Anno XXVI vol. 3, pp. 483–501.

Primiero, G. (2006) Belief revision in constructive type theory, in *The Logica 2005 Yearbook*, pp. 177–188, Filosofia, Czech Academy of Sciences, Prague.

Primiero, G. (2007a) On building abstract terms in type systems, in *The Logica 2006 Yearbook*, pp. 191–201, Filosofia, Czech Academy of Sciences, Prague.

Primiero, G. (2007b) *An Epistemic Constructive Definition of Information*, Logique & Analyse, issue December 2007, vol 50 (200).

Primiero, G. (forthcoming a) Presuppositions, Assumptions, Premises, manuscript.

Primiero, G. (forthcoming b) Type-theoretical Dynamics, forthcoming in *(Anti) Realism: The Realism-Antirealism Debate in the Age of Alternative Logics*, LEUS serie, Springer.

Prior, A.N. (1960) The runabout inference-ticket in *Analysis*, vol. 21, Blackwell; reprinted in P.F. Strawson (ed.), *Philosophical Logic*, Oxford University Press, 1967.

Proust, J. (1989) *Questions of Form: Logic and the Analytic Proposition from Kant to Carnap*, University of Minnesota Press, Minneapolis.

Quine, W.v.O. (1951) Two dogmas of empiricism, *The Philosophical Review*, reprinted in *From a Logical Point of View*, 3rd edn., Harvard University Press, Cambridge, Massachussets 1980.

Quine, W.v.O. (1960) *Word and Object*, Massachussets Institute of Technology, Cambridge.

Quine, W.v.O. (1981) Five milestones of empiricism, in *Theories and Things*, Harvard University Press, Cambridge, MA.

Ranta, A. (1988) Propositions as games as types, *Synthese*, 76, 377–395.

Ranta, A. (1991) Constructing possible worlds, *Theoria*, 57, 77–100.

Ranta, A. (1994) *Type-Theoretical Grammar*, Oxford University Press, Oxford.

Rey, G. (2003), *The Analytic/Synthetic Distinction*, The Stanford Encyclopedia of Philosophy (Fall 2003 edition), Edward N. Zalta (ed.); available at `http://plato.stanford.edu/archives/fall2003/entries/analytic-synthetic/`.

Rescher, N. (1999) How many possible Worlds are there?, in *Philosophy and Phenomenological Research*, 59, 403–420.

Rescher, N. and Brandom, R. (1979) *The Logic of Inconsistency: A Study in a Nonstandard Possible-World Semantics and Ontology*, Basil Blackwell, Oxford.

Rescher, N. and Parks, Z. (1973) Possible Individuals, Trans-Worlds Identity and Quantified Modal Logic, in *Nous*, 7, 330–350.

Rott, H. and Horak, V. (eds.) (2003) *Possibility and Reality*, Ontos Verlag, Frankfurt, London.

Russell, B. (1903) *Principles of Mathematics*, Cambridge University Press, Cambridge; reissue edition 1996, W.W. Norton.

Russell, B. (1908) Mathematical logic as based on the theory of types, *American Journal of Mathematics*, 30, 222–262; reprinted in J. van Hijenoort *From Frege to Gödel*, Harvard University Press, Cambridge, MA, 1967, pp. 152–182.

Russell, B. (1919) *Introduction to Mathematical Philosophy*, Allen & Unwin, London; Macmillian, New York.

Russell, B. and Whitehead, A.N. (1910, 1911, 1913) *Principia Mathematica*, 3 vols., Cambridge University Press, Cambridge; 2nd edn., 1925 (Vol. 1), 1927 (Vols. 2,3).

Ryle, G. (1946) Knowing how and knowing that, in *Proceedings of the Aristotelian Society*, Vol. XLVI, 1946; reprinted in *Collected Papers,* Vol. 2, Bernes and Nobles, New York, 1971, pp. 212–225.

Salvesen, A. (1989) *On Information Discharging and Retrieval in Martin-Löf's Type Theory*, Norwegian Computing Center, Report No. 803, Ph.D. Thesis, Institute of Informatics, University of Oslo.

Sambin, G. (1991) *Per una dinamica nei fondamenti* (Italian), in Corsi and Sambin (eds.), pp. 163–210.

Sambin, G. (2003) Some points in formal topology, *Theoretical Computer Science*, 305, 347–408.

Sambin, G. and Smith, J. (eds.) (1998) *Twenty-five Years of Constructive Type Theory*, Oxford Logic Guides, Clarendon Press, Oxford.

Sambin, G. and Valentini, S. (1998) Building up a toolbox for Martin-Löf Type Theory:subset theory, in G. Sambin, J. Smith (eds.), Oxford University Press, pp. 221–244.

Sebestik, J. (2003) La dispute de Bolzano avec Kant - Fragment d'un dialogue sur la connoissance mathématique, *Philosophiques* 30(1), 47–66.

Sequoiah-Grayson, S. (forthcoming) The scandal of deduction – Hintikka on the information yield of deductive inferences, *Journal of Philosophical Logic*, forthcoming.

Shannon, C.E. (1948) A mathematical theory of communication, *The Bell System Technical Journal*, 27; available at `http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf`.

Shannon, C.E. and Weaver, W. (1952) *The Mathematical Theory of Communications*, Urbana-Chicago-Lodon: University of Illinois Press.

Sommaruga-Rosolemos, G. (2000) *History and Philosophy of Constructive Type Theory*, Kluwer, Dordrecht, The Netherlands.

Stalnaker, R. (1974) Pragmatic presuppositions, in Munitz and Unger (eds.), *Semantics and Philosophy*, New York University Press, New York, pp. 197–213.

Stanley, J. and Williamson, T. (2001) Knowing how, *The Journal of Philosophy*, 98(8), 411–444; reprinted in P. Grim, P. Ludlow, and G. Mar (eds.), *The Philosopher's Annual* 26, CSLI Pubblications, Stanford, 2002, pp. 237–271.

Stuart Mill, J. (1843) *A System of Logic: Ratiocinative and Inductive, Being a Connected View of the Principles of Evidence and the Methods of Scientific Investigations*, 2 vol., Parker, London, 1846; revised Vol. I Harper, New York, 1846; revised again 1851–1872.

Sundholm, G.B. (1983) Constructions, Proofs and the Meaning of the Logical Constants, *Journal of Philosophical Logic*, 12, pp. 151–172.

Sundholm, G.B. (1983a) Systems of deduction, in: D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic I*, Reidel, Dordrecht, The Netherlands pp. 133–188.

Sundholm, G.B. (1986), Proof theory and meaning, in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic III*, Reidel, Dordrecht, The Netherlands, pp. 471–506.

Sundholm, G.B. (1993) *Questions of Proof*, Manuscripto(Campinas) 16, 47–70.

Sundholm, G.B. (1994) Existence, proof and truth-making: A perspective on the intuitionistic conception of truth, *Topoi* 13, 117–126.

Sundholm, G.B. (1997) Implicit epistemic aspects of constructive logic, *Journal of Logic, Language and Information* 6, 191–212.

Sundholm, G.B. (1998) Inference, consequence, implication: A constructivist's approach, *Philosophia Mathematica* 6, 178–194.

Sundholm, G.B. (1998a) Inference versus consequence, in *The Logica 1997 Yearbook*, Filosofia Publishers, Czech Academy of Science, Prague, pp. 26–35.

Sundholm, G.B. (1999) Identity: Propositional, criterial, Absolute, in *The Logica 1998 Yearbook*, Filosofia Publishers, Czech Academy of Science, Prague, pp. 20–26.

Sundholm, G.B. (2004) Antirealism and the roles of truth, in Niniluoto, Sintonen, and Wolenski (eds.), *Handbook of Epistemology* Kluwer, Dordrecht, The Netherlands, pp. 437–466.

Sundholm, B.G. (2006) A century of judgement and inference: 1836–1937, in L. Haparanta (ed.), *The History of Modern Logic*, Oxford University Press, Oxford.

Thompson, S. (1991) *Type Theory and Functional Programming*, Addison-Wesley.

Troelstra, A.S. (1982) On the origin and development of Brouwer's concept of choice sequence, in Troelstra and van Dalen (eds.), pp. 465–486.

Troelstra, A.S. (1996) Choice sequence: A retrospect, *Math. Logic Quarterly*, 9(1–2), 143–149.

Troelstra, A.S. and van Dalen, D. (eds.) (1982) *The L.E.J. Brouwer Centenary Symposium*, North-Holland, Amsterdam.

Troelstra, A.S. and van Dalen, D. (1988) *Constructivism in Mathematics: An Introduction*, 2 vols., North-Holland, Amsterdam.

Valentini, S. (1998) The forget-restore principle:a paradigmatic example, in G. Sambin and J. Smith (eds.), Oxford University Press, Oxford, pp. 275–283.

Valentini, S. (2000) *Meta-mathematical Aspects of Martin-Löf Type Theory*, Proefschrift in het gebied van de Natuurwetenschappen, Wiskunde en Informatica, Katholieke Universiteit Nijmegen.

van Atten, M.S.P.R. (1999) *Phenomenology of Choice Sequences*, Questiones Infinitae, Publications of the ZENO Institute of Philosophy, Vol. XXXI.

van Fraassen, B. (1969) Presuppositions, supervaluations, and free logic, in K. Lambert (ed.), *The Logical Way of Doing Things*, New Haven, T: Yale University Press, pp. 67–91.

Various authors (1992) *Bolzano's Wissenschaftslehere 1837-1987 - International Workshop*, Firenze, Olschki.

Veldman, W. (2004–05) *Intuïtionistische Wiskunde*, notes for Master Class Opleiding Wiskunde, Faculteit der Natuurwetenschappen, Wiskunde en Informatica, Studiejaar, Radboud Universiteit Nijmegen.

Villanueva, E. (ed.) (1990) *Information, Semantics & Epistemology*, Basil Blackwell, Cambridge.

von Baeyer, H.C. (2003) *Information: The New Language of Science*, Harvard University Press, Cambridge, MA.

Wang, H. (1974) *From Mathematics to Philosophy*, Routledge & Kegan Paul, London.

Weaver, W. (1952) Recent contributions to the mathematical theory of communications, in C.E. Shannon and W. Weaver, pp. 3–28.

Wittgenstein, L. (1922) *Tractatus logico-philosophicus*, Routledge, London.

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