MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Linear Groups

The Accent on Infinite Dimensionality

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix}$$

Martyn R. Dixon
Leonid A. Kurdachenko
Igor Ya. Subbotin

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Linear Groups

The Accent on Infinite Dimensionality

Monographs and Research Notes in Mathematics

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Linear Groups

The Accent on Infinite Dimensionality

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Preface

The theory of linear groups is one of the oldest and most developed theories of modern mathematics. The footprint of such groups can be found in almost all mathematical (and many non-mathematical) disciplines. Their connections with other disciplines, including the natural sciences, are diverse and non-trivial. Linear groups appeared in the form of geometric transformations in F. Klein's Erlangen Program and were also one of the sources for the emergence of the general theory of groups. Their universal applicability and penetration into areas quite different from each other prompted J. G. Thompson to say: "I believe in a heliocentric universe, the sun of which is linear groups." (see [69]). The utility of the matrix apparatus made it possible for a very deep development of the theory of finite dimensional linear groups, also called matrix groups. There is a vast array of articles and books devoted to the study of finite dimensional linear groups, far too numerous to mention. Many interesting results were obtained, which contributed to progress both in the general theory of groups and in many other fields. However, our goal here is not to make a schematic overview of the theory of finite dimensional linear groups; there are a sufficient number of excellent monographs devoted to them (see, for example, [20, 35, 36, 37, 42, 56, 80, 86, 177, 197, 202] to name but a few), as well as many review articles devoted to various aspects of the theory. In this book, we are more concerned with demonstrating some approaches to the study of infinite dimensional linear groups and showing how effective these approaches have been.

At the outset we need to clarify the concept. Usually by a "finite dimensional linear group" we mean a group G which is isomorphic to some subgroup of the group of linear transformations of a finite dimensional vector space. Therefore, it is natural to call a group G an *infinite dimensional linear group* if it is isomorphic to some subgroup of the group of linear transformations of an infinite dimensional vector space. It should be understood that the same group can act both on a finite dimensional vector space and also on an infinite dimensional one.

The theory of infinite dimensional linear groups is fundamentally different from the theory of finite dimensional linear groups. As for finite dimensional linear groups, one can consider matrices of linear transformations with respect to various bases. These matrices will be infinite, but have a finiteness property, since only a finite number of basis elements are involved in the decomposition of each vector space element; then each column of such an infinite matrix has only a finite set of nonzero coefficients. However, the theory of infinite matrices is far from being as well established as the theory of matrices of finite size, although the beginnings of this theory can be traced back to the 19th century. Infinite matrices and determinants were introduced into analysis by Poincaré in 1884 in the discussion of the well-known Hill's equation. The rapid development of the theory of linear spaces of infinite dimension began at the beginning of the 20th century. The foundations were laid mainly by the studies of Ivar Fredholm and Vito Volterra. They considered the theory of linear equations with an infinite number of equations and unknowns using the limit representation of linear equations with a finite number of equations and unknowns when the number of equations and unknowns becomes infinite. This led to the development of the theory of integral equations. On the other hand, the works of David Hilbert, John von Neumann, Erhard Schmidt and Friges Riesz on the theory of integral equations gave impetus to the development of the theory of linear spaces of infinite dimension. This led to the creation of the theory of Banach and Hilbert spaces. Some overview of this topic can be found in the book of R. G. Cooke [34] and the article of M. Bernkopf [11]. However, an exhaustive search of all these areas shows that the main object of study was infinite matrices over the fields of real and complex numbers, and this study was carried out using the very well established apparatus of mathematical analysis. A study of the algebraic properties of groups of infinite matrices over arbitrary fields began relatively recently (see the book of W. Holubowski [82]). In the bibliography we give some works related to this topic.

We have a similar situation with infinite dimensional linear groups. There has been significant progress, not for groups of arbitrary linear transformations, but for those that have some additional properties that make it possible to use the tools from analysis. For fields that are not endowed with topological structures the situation is different and the difference from finite-dimensional linear groups manifests itself from the beginning. If A is a vector space over a field F on which a group G of linear transformations acts and if A has finite dimension, then A has a finite series of G-invariant subspaces

$$0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$$

whose factors are G-chief. The factors $G/C_G(A_j/A_{j-1})$ are irreducible linear groups when $1 \leq j \leq n$ and the stabilizer $\bigcap_{1\leq j\leq n} C_G(A_j/A_{j-1})$ of this series is a nilpotent subgroup. However if the vector space A has infinite dimension, then A is not guaranteed to have a finite G-chief series. In general, A has a linearly ordered series of G-invariant subspaces, whose factors are G-chief. The factors $G/C_G(A_j/A_{j-1})$ are again irreducible linear groups, but the stabilizer of this series already has properties very far from nilpotency. As we will see in Chapter 1, this stabilizer may contain nonabelian free subgroups. In addition, the structure of infinite dimensional irreducible linear groups is noticeably more complex than the structure of finite dimensional irreducible groups. There is a further significant difference. Not every group can play the role of a group of linear transformations of a finite dimensional vector space.

On the other hand, every group is isomorphic to a subgroup of a group of linear transformations of some vector space having infinite dimension. This already shows that the study of infinite dimensional linear groups must be carried out under certain natural limitations.

As for finite dimensional linear groups, one of the important ideas is that of an (infinite dimensional) irreducible linear group. In Chapter 2, we take a fairly detailed look at the structure of infinite dimensional irreducible linear groups. In Chapter 2, we examine in sufficient detail the structure of abelian infinite dimensional irreducible linear groups, as well as the structure of infinite dimensional irreducible linear groups under conditions that are significantly weaker than commutativity.

When studying infinite dimensional linear groups it is natural to use the vast experience gained in the study of finite dimensional linear groups. In particular, if the condition of finite dimensionality is involved, then the following new interesting classes of structure of infinite dimensional linear groups arise.

Let F be a field, let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If a group G is irreducible, then every proper G-invariant subspace of A is trivial. In particular such a subspace has finite dimension. In this way, if every proper G-invariant subspace has finite dimension, we obtain a generalization of the notion of an irreducible linear group.

Dually, if a group G is irreducible, then every non-zero G-invariant subspace of A coincides with A. In particular, such a G-invariant subspace has finite codimension. Thus, if every non-zero G-invariant subspace has finite codimension, we obtain a generalization of the idea of an irreducible linear group.

These two types of infinite dimensional linear group are studied in Chapter 3.

Let B be a finite dimensional vector space and K be a group of its linear transformations. An infinite dimensional linear group G can easily be constructed as follows. Let C be an infinite dimensional vector space and let $A = B \oplus C$. The group K acts on the subspace trivially, which is to say that $f(c) = c$ for each $f \in K$ and each $c \in C$. It is then possible quite naturally to extend the action of the group K onto the whole space A and in this way form the group G which acts trivially on C, but like K on B. The group G is an infinite dimensional linear group such that $A/C_A(G)$ and $[G, A]$ have finite dimension. Thus the infinite dimensional linear group G is very close to being finite dimensional in the sense that $A/C_A(G)$ and $[G, A]$ are both finite dimensional. However, it should immediately be noted that, in general, the fact that $A/C_A(G)$ is finite dimensional does not imply that $[G, A]$ is finite dimensional and neither is the reverse true. The question of when this is true is discussed in Chapter 4. Quite broad generalizations of this particular topic are also discussed.

We say that a linear group G has *finite central dimension* if the quotient space $A/C_A(G)$ has finite dimension. We noted above that linear groups having finite central dimension are close to being finite dimensional linear groups in some sense. It seems clear that the presence of a "large" set of subgroups having finite central dimension should have a significant impact on the structure of an infinite dimensional linear group. This is confirmed by studies conducted in the framework of the theory of finitary linear groups. A linear group is called *finitary* if each of its finitely generated subgroups has finite central dimension. The theory of finitary linear groups is already one of the more well established theories of infinite dimensional linear groups yielding many interesting and deep results. In our book, we barely touch this topic, but it is deserving of a separate independent book. Nevertheless, in our bibliography we provide links to some important works devoted to finitary groups.

In Chapter 5 we consider those linear groups, in which the family of subgroups having infinite central dimension satisfies the classical finiteness conditions of minimality and maximality. Here we also consider linear groups that are in some sense dual to the finitary groups, namely those linear groups in which every subgroup that is not finitely generated has finite central dimension.

In Chapter 6 we consider a situation that is in many respects the opposite of that occurring in Chapter 2. If G is a group of linear transformations of the vector space A and G acts irreducibly on A, then the only G-invatiant subspaces of A will be the space A itself and the zero subspace. On the other hand, if we consider the opposite situation, that is we assume that every subspace of A is G-invariant, then it is easy to see that the group G is abelian and isomorphic to some subgroup of the multiplicative group of the field over which the vector space A is considered. This shows that the family of G-invariant subspaces, more precisely its size and location, has an essential influence on the structure of the group G. Chapter 6 is devoted to the study of the various ways that G is impacted by this influence.

This book is intended for students who have studied standard university courses in abstract and linear algebra and learned the basics of group theory and matrix theory. We hope that it will be useful to both graduate students and experts in group theory.

We would like to extend our sincere appreciation to The University of Alabama (Tuscaloosa, USA) and National University (California, USA) for their support of this work.

The authors would also like to thank their families for all their love and much needed support while this work was in progress. An endeavor such as this is made lighter by the joy that they bring. Finally, it is a pleasure to thank the staff of our publishers for their co-operation and dedication.

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Chapter 1

Essential Toolbox

Let F be a field and let A be a vector space over F. A function $f : A \longrightarrow A$ is called a *linear transformation* (or *F-endomorphism*) of A if

$$f(x + y) = f(x) + f(y) \text{ and } f(\alpha x) = \alpha f(x),$$

for all elements $x, y \in A$ and elements $\alpha \in F$. The set $\mathbf{End}(F, A)$ of all linear transformations of A is an associative F-algebra with identity in which the operations are addition, composition of linear transformations and scalar multiplication by elements of the field F. The group $GL(F, A)$ of all invertible linear transformations (also called non-singular transformations or F-automorphisms) of A is called the *General Linear Group* of A. The subgroups of $GL(F, A)$ are called linear groups. If $f, g \in \mathrm{End}(F, A)$ then we shall write fg for composition (maps will generally be written on the left). Thus $fg(a) = f(g(a))$ for all $a \in A$.

In this chapter we collect together concepts and results that traditionally arise when linear groups are first studied. Most of these concepts are studied in the theory of finite dimensional linear groups, or as extensions or analogues of ideas that have arisen in that theory. At the same time, we show here that their role need not be so significant in the study of infinite dimensional linear groups; the situations arising in the theory of infinite dimensional linear groups are much more diverse and have a different level of complexity. Approaches that are effective in the study of finite dimensional groups turn out to be far from being so effective in the study of infinite dimensional linear groups.

Although this book is mostly self-contained the reader may find other monographs useful. Rings and commutative algebra are the topics of consideration in [2, 17, 18, 19, 21, 22, 29, 61, 91, 93, 101, 133, 146, 176, 186, 192]. General and specific aspects of group theory are considered in [6, 31, 59, 64, 65, 66, 87, 88, 89, 92, 96, 185, 188, 190, 215, 219]. General Algebra texts include [16, 131, 153]. We often will be concerned with groups satisfying finiteness conditions and here mention the articles of V. S. Charin and D. I. Zaitsev [28], S. N. Chernikov [30], L. S. Kazarin and L. A. Kurdachenko [99], A. I. Maltsev [143], D. I. Zaitsev [223] and D. I. Zaitsev, M. I. Kargapolov and V. S. Charin [226].

Linear Groups and Generalized Matrices

Let $\mathcal{B} = \{b_\lambda | \lambda \in \Lambda\}$ be a basis for the vector space A. If f is a linear transformation of A, then $f(b_\lambda) = \sum_{\mu\in\Lambda} \tau_{\mu\lambda} b_\mu$, for some $\tau_{\mu\lambda} \in F$, where for each λ only finitely many of the coefficients $\tau_{\mu\lambda}$ are non-zero. Then $M_{\mathcal{B}}(f) = [\tau_{\mu\lambda}]_{\lambda,\mu\in\Lambda}$ is an infinite dimensional matrix with coefficients in F, where each column has only finitely many non-zero coefficients. The matrix $M_{\mathcal{B}}(f)$ is called the *matrix of the linear transformation* f relative to the basis $\mathcal{B}$. We let $\mathrm{Mat}_\Lambda(F)$ denote the set of infinite dimensional matrices, every column of which has at most finitely many non-zero entries, whose coefficients are indexed by the set Λ. On the set $\mathrm{Mat}_\Lambda(F)$ we introduce operations of addition, multiplication and scalar multiplication by elements of F as follows. Let $R = [\rho_{\lambda\mu}]_{\lambda,\mu\in\Lambda}, S = [\sigma_{\lambda\mu}]_{\lambda,\mu\in\Lambda} \in \mathrm{Mat}_\Lambda(F)$ and let $\alpha \in F$. For all $\lambda, \mu \in \Lambda$, set

$$R + S = [\kappa_{\lambda\mu}]_{\lambda,\mu\in\Lambda}, \text{ where } \kappa_{\lambda\mu} = \rho_{\lambda\mu} + \sigma_{\lambda\mu}$$
$$RS = [\kappa_{\lambda\mu}]_{\lambda,\mu\in\Lambda}, \text{ where } \kappa_{\lambda\mu} = \sum_{\nu\in\Lambda} \rho_{\lambda\nu}\sigma_{\nu\mu}$$
$$\alpha R = [\eta_{\lambda\mu}]_{\lambda,\mu\in\Lambda}, \text{ where } \eta_{\lambda\mu} = \alpha\rho_{\lambda\mu}.$$

Since each column of the matrix S has only finitely many non-zero entries the sum in the second of these equations is a finite sum, so the operation of multiplication is well-defined. Using essentially the same arguments used for finite dimensional matrices, it is possible to show that with respect to these operations, the set $\mathrm{Mat}_\Lambda(F)$ is an algebra over the field F. As in the finite dimensional case, once we have chosen a basis for A we obtain the usual identifications between $\mathbf{End}(F, A)$ and $\mathrm{Mat}_\Lambda(F)$ as follows.

Theorem 1.1. *Let A be a vector space over the field F and let Λ be a set such that the cardinality of Λ is $\boldsymbol{dim}_F(A)$. Then the F-algebras $End(F, A)$ and $Mat_\Lambda(F)$ are isomorphic.*

Proof. Let $\mathcal{B} = \{b_\lambda | \lambda \in \Lambda\}$ be a basis for the vector space A. We define a function
$$\Phi : \mathbf{End}(F, A) \longrightarrow \mathrm{Mat}_\Lambda(F) \text{ by } \Phi(f) = M_{\mathcal{B}}(f)$$
for all $f \in \mathbf{End}(F, A)$. To show that Φ is injective let $f, g \in \mathbf{End}(F, A)$ and $f \neq g$. We write $M_{\mathcal{B}}(f) = [\tau_{\lambda\mu}]_{\lambda,\mu\in\Lambda}, M_{\mathcal{B}}(g) = [\rho_{\lambda\mu}]_{\lambda,\mu\in\Lambda}$. If $f(b_\lambda) = g(b_\lambda)$, for all $\lambda \in \Lambda$, then $f(a) = g(a)$ for all $a \in A$, so $f = g$. Hence there exists $\kappa \in \Lambda$ such that $f(b_\kappa) \neq g(b_\kappa)$. However $f(b_\kappa) = \sum_{\mu\in\Lambda} \tau_{\mu\kappa} b_\mu$ and $g(b_\kappa) = \sum_{\mu\in\Lambda} \rho_{\mu\kappa} b_\mu$. Since $f(b_\kappa) \neq g(b_\kappa)$ it follows that there exists $\nu \in \Lambda$ such that $\tau_{\nu\kappa} \neq \rho_{\nu\kappa}$ and hence $M_{\mathcal{B}}(f) \neq M_{\mathcal{B}}(g)$.

Furthermore, Φ is surjective which we show as follows. Let $M = [\sigma_{\lambda\mu}]_{\lambda,\mu\in\Lambda} \in \mathrm{Mat}_\Lambda(F)$ and for each $\lambda \in \Lambda$ let $c_\lambda = \sum_{\mu\in\Lambda} \sigma_{\mu\lambda} b_\mu$. If $a \in A$, then $a = \sum_{\lambda\in\Lambda} \alpha_\lambda b_\lambda$, for certain $\alpha_\lambda \in F$. Let h be the transformation defined by $h(a) = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$ and let $d = \sum_{\lambda \in \Lambda} \delta_\lambda b_\lambda$, for certain δ_λ. If $\gamma \in F$, then

$$h(a+d) = \sum_{\lambda \in \Lambda} (\alpha_\lambda + \delta_\lambda) c_\lambda = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda + \sum_{\lambda \in \Lambda} \delta_\lambda c_\lambda = h(a) + h(d) \text{ and}$$

$$h(\gamma a) = \sum_{\lambda \in \Lambda} \gamma \alpha_\lambda c_\lambda = \gamma \left(\sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda \right) = \gamma h(a),$$

so the function h is linear. Clearly we have $h(b_\lambda) = c_\lambda = \sum_{\mu \in \Lambda} \sigma_{\mu\lambda} b_\mu$, so M is the matrix of the linear transformation h relative to the basis B. Hence $\Phi(h) = M$.

Finally, the function Φ is a homomorphism, which we show as follows. Let

$$f, g \in \mathbf{End}(F, A), \gamma \in F, M_{\mathcal{B}}(f) = [\tau_{\lambda\mu}]_{\lambda,\mu \in \Lambda}, M_{\mathcal{B}}(g) = [\rho_{\lambda\mu}]_{\lambda,\mu \in \Lambda}.$$

Then

$$\begin{aligned}
(f+g)(b_\lambda) = f(b_\lambda) + g(b_\lambda) &= \sum_{\mu \in \Lambda} \tau_{\mu\lambda} b_\mu + \sum_{\mu \in \Lambda} \rho_{\mu\lambda} b_\mu = \sum_{\mu \in \Lambda} (\tau_{\mu\lambda} + \rho_{\mu\lambda}) b_\mu; \\
fg(b_\lambda) = f(g(b_\lambda)) &= f \left(\sum_{\mu \in \Lambda} \rho_{\mu\lambda} b_\mu \right) = \sum_{\mu \in \Lambda} \rho_{\mu\lambda} f(b_\mu) \\
&= \sum_{\mu \in \Lambda} \rho_{\mu\lambda} \left(\sum_{\kappa \in \Lambda} \tau_{\kappa\mu} b_\kappa \right) = \sum_{\mu \in \Lambda} \sum_{\kappa \in \Lambda} \rho_{\mu\lambda} \tau_{\kappa\mu} b_\kappa \\
&= \sum_{\kappa \in \Lambda} \left(\sum_{\mu \in \Lambda} \tau_{\kappa\mu} \rho_{\mu\lambda} \right) b_\kappa \text{ and} \\
(\gamma f)(b_\lambda) = \gamma(f(b_\lambda)) &= \gamma \left(\sum_{\mu \in \Lambda} \tau_{\mu\lambda} b_\mu \right) = \sum_{\mu \in \Lambda} \gamma \tau_{\mu\lambda} b_\mu.
\end{aligned}$$

These equations in turn show that

$$\Phi(f+g) = \Phi(f) + \Phi(g), \Phi(fg) = \Phi(f)\Phi(g) \text{ and } \Phi(\gamma f) = \gamma \Phi(f),$$

so Φ is an F-algebra homomorphism, hence an isomorphism, giving the result. □

This theorem shows that we can study the F-algebra $\mathrm{Mat}_\Lambda(F)$ of all infinite dimensional matrices of dimension $|\Lambda|$ over the field F instead of the F-algebra $\mathbf{End}(F, A)$, as the need arises. Likewise, instead of studying the group $GL(F, A)$ we can study the group of all invertible elements of the F-algebra $\mathrm{Mat}_\Lambda(F)$. In the case when the F-dimension of A is finite, Theorem 1.1 is very significant, since the theory of finite dimensional matrices is well established.

The use of this matrix apparatus enabled finite dimensional linear algebra to grow into a rich area of mathematics. The same can be said about the theory of finite dimensional linear groups, which is essentially a theory of finite dimensional matrices. The theory of matrix groups is of course a fundamental area of group theory. However the literature concerned with the algebraic theory of infinite dimensional matrices is sparse. There are some sporadic results for the case when $\Lambda = \mathbb{N}$, the set of natural numbers. In particular, here we do not have useful tools, such as the determinant of a matrix, which shows that we cannot rely solely on the matrix apparatus to study infinite dimensional linear groups. As we shall see later, effective approaches that have provided significant progress in the theory of finite dimensional linear groups cannot be relied on.

Some interesting papers, giving further insight into the theory of infinite dimensional matrices, include the work of A. Bier [12, 13], A. Bier and W. Holubowski [14], W. Holubowski and R. Slowik [84], W. Holubowski, I. Kashuba and S. Zurek [83], X. Hou [85], A. Marcoci, L. Marcoci, L. E. Personn, and N. Popa [144], M. P. Sedneva [189], P. N. Shivakumar and K. C. Sivakumar [194], P. N. Shivakumar, K. C. Sivakumar and Y. Zhang [195] and R. Slowik [164, 165, 166, 167, 168, 169, 170]

g-Invariant Subspaces, Factors and Modules

Let A be a vector space over a field F. If M is a subset of A, then we let FM denote the subspace spanned by M.

If G is a subgroup of $GL(F, A)$, then for each element $a \in A$ we define the *G-orbit a^G of a* by

$$a^G = \{f(a) | f \in G\}.$$

We note that if e is the identity transformation of A, then $e(a) = a \in a^G$. Furthermore, if $b \in a^G$, then $b = f(a)$ for some element $f \in G$. Let $c \in b^G$ be an arbitrary element, so there exists an element $g \in G$ such that $c = g(b)$. Then $c = g(b) = g(f(a)) = (gf)(a) \in a^G$, so that $b^G \subseteq a^G$. On the other hand $a = f^{-1}(b)$ and by repeating the argument above we have $a^G \subseteq b^G$, so $a^G = b^G$. From this it follows easily that $a^G = b^G$ if and only if $b \in a^G$.

This implies that if the orbits a^G and b^G are different, then $a^G \cap b^G = \emptyset$. Thus the set of G-orbits forms a partition of the vector space A.

If a is an element of the vector space A, then we define the *centralizer of a* to be

$$C_G(a) = \{g \in G | g(a) = a\}.$$

It is easy to prove that $C_G(a)$ is a subgroup of G. Furthermore, if $b \in a^G$, then $b = f(a)$, for some $f \in G$ and if $z \in C_G(a)$, then

$$fzf^{-1}(b) = fz(f^{-1}(b)) = fz(a) = f(z(a)) = f(a) = b.$$

Consequently $fC_G(a)f^{-1} \leq C_G(b)$. On the other hand, let $w \in C_G(b)$. Then $w(b) = b$, so $w(f(a)) = f(a)$ and it follows that

$$f^{-1}(w(f(a))) = f^{-1}(w(b)) = f^{-1}(b) = a.$$

Therefore $f^{-1}wf \in C_G(a)$, so $f^{-1}C_G(b)f \leq C_G(a)$. It is now easy to see that $f^{-1}C_G(b)f = C_G(a)$, whenever $b = f(a)$.

If M is an arbitrary subset of A, then define the *centralizer of* M to be $C_G(M) = \bigcap_{a \in M} C_G(a)$. If M is the union of certain G-orbits, then $C_G(M)$ is a normal subgroup of G. To see this, let $g \in G, x \in C_G(M), a \in M$. Then $g(a) \in M$ and we have

$$(g^{-1}xg)(a) = g^{-1}(x(g(a))) = g^{-1}(g(a)) = a,$$

so that $g^{-1}xg \in C_G(M)$ also.

For each fixed element $a \in A$, there is a natural mapping $\Delta : G \longrightarrow A$ defined by $\Delta(g) = g(a)$, for each $g \in G$. Clearly $\mathbf{Im}\,(\Delta) = a^G$. We may then define a relation R on G by defining $g\,R\,h$ if and only if $\Delta(g) = \Delta(h)$ and it is easy to show that R is an equivalence relation. If $\Delta(g) = \Delta(h)$, then $g(a) = h(a)$, so $g^{-1}h(a) = a$ and hence $g^{-1}h \in C_G(a)$. Consequently, $gC_G(a) = hC_G(a)$. Conversely, if $gC_G(a) = hC_G(a)$, then $h = gz$, for some element $z \in C_G(a)$ and we have $h(a) = g(z(a)) = g(a)$. Hence the set of R-equivalence classes is precisely the set of right cosets $\{gC_G(a)|g \in G\}$. For fixed $a \in A$, the mapping $gC_G(a) \longmapsto g(a)$, for $g \in G$, is therefore a bijection. Furthermore, if a^G is finite, then $C_G(a)$ has finite index in G and $|a^G| = |G : C_G(a)|$, the well-known orbit-stabilizer theorem.

Continuing with the notation above, a subspace V of A is called G-*invariant* if $g(v) \in V$, for each element $v \in V$ and each element $g \in G$. Thus V is G-invariant if and only if V is a union of certain G-orbits and, by our remark above, $C_G(V)$ is a normal subgroup of G in this case.

On the other hand, if H is a subgroup of G, then we define the *centralizer of* H *in* A by

$$C_A(H) = \{a \in A|h(a) = a \text{ for all } h \in H\}.$$

It is easy to see that $C_A(H)$ is a subspace of A and if H is a normal subgroup of G, then $C_A(H)$ is a G-invariant subspace. To see this let $g \in G, h \in H, a \in C_A(H)$. Then $y = g^{-1}hg \in H$ and we have

$$h(g(a)) = gg^{-1}hg(a) = gy(a) = g(y(a)) = g(a).$$

It is also easy to see that if $g \in G$, then $C_A(H^g) = g^{-1}(C_A(H))$. For, if $b \in g^{-1}(C_A(H))$, then $b = g^{-1}(c)$, for some $c \in C_A(H)$. If $x \in H^g$, then $x = g^{-1}hg$, for some $h \in H$. Then

$$x(b) = (g^{-1}hg)(b) = (g^{-1}hg)(g^{-1}(c)) = g^{-1}(h(c)) = g^{-1}(c) = b.$$

Then $b \in C_A(H^g)$.

Conversely, if $b \in C_A(H^g)$, then

$$g(b) = g(x(b)) = g(g^{-1}hg(b)) = gg^{-1}(h(g(b)) = h(g(b))$$

so that $g(b) \in C_A(H)$. Hence $b \in g^{-1}(C_A(H))$.

Let $\zeta_G(A) = C_A(G)$. Then our work above shows that $\zeta_G(A)$ is a G-invariant subspace called the *G-center* of A and it is easy to see that indeed

$$\zeta_G(A) = \{a \in A | C_G(a) = G\} = \{a \in A | g(a) = a \text{ for all } g \in G\}.$$

We next consider two generalizations of this idea. Let

$$\mathbf{FO}_G(A) = \{a \in A | a^G \text{ is finite}\} = \{a \in A || G : C_G(a)| \text{ is finite}\}$$
$$\mathbf{FDO}_G(A) = \{a \in A | Fa^G \text{ has finite dimension in } A\}.$$

It is easy to see that both $\mathbf{FO}_G(A)$ and $\mathbf{FDO}_G(A)$ are G-invariant subspaces of A.

If V is a subspace of A, then we define the *invariator of V in G* by

$$\mathbf{Inv}_G(V) = \{g \in G | g(v) \in V \text{ for every element } v \in V\}.$$

Again it is easy to see that $\mathbf{Inv}_G(V)$ is a subgroup of G and that V is G-invariant if and only if $\mathbf{Inv}_G(V) = G$. Clearly also the intersection of an arbitrary family of G-invariant subspaces is a G-invariant subspace. Thus the invariator of the subspace V is an analogue of the normalizer of a subgroup.

If M is a subset of A, let GM be the intersection of all the G-invariant subspaces containing M. Then GM is the smallest G-invariant subspace containing M. We call the subspace GM the *G-invariant closure of M*.

If U, V are subspaces of A such that $U \leq V$, then the *centralizer of V/U* is

$$C_G(V/U) = \{g \in G | g(v) \in v + U \text{ for every } v \in V\}.$$

From this definition it is clear that $C_G(V/U) \subseteq \mathbf{Inv}_G(V)$. Furthermore, $C_G(V/U)$ is a subgroup of G. For, if $g, h \in C_G(V/U), v \in V$, then $h(v) = v + u$, for some element $u \in U$ and we have

$$gh(v) = g(h(v)) = g(v + u) = g(v) + g(u) \in (v + U) + (u + U) = v + U.$$

Hence $gh \in C_G(V/U)$. Similarly $g^{-1} \in C_G(V/U)$.

Additionally, if U, V are G-invariant subspaces of A, then $C_G(V/U)$ is a normal subgroup of G. To see this, choose elements $g \in G, h \in C_G(V/U), v \in V$, so that $g(v) \in V$ and $h(g(v)) = g(v) + u$, for some $u \in U$. We have

$$\begin{aligned}(g^{-1}hg)(v) &= g^{-1}(h(g(v)) = g^{-1}(g(v) + u) = g^{-1}(g(v)) + g^{-1}(u) \\ &= v + g^{-1}(u) \in v + U,\end{aligned}$$

which verifies our claim that $C_G(V/U)$ is a normal subgroup when V, U are G-invariant.

If $G = C_G(V/U)$, then V/U is called a *G-central factor* and if $G \neq C_G(V/U)$, then V/U is called a *G-eccentric factor*.

In the study of linear groups it is often useful to use the language of modules, as in the finite dimensional case. Generally, all our modules will be left modules. As usual the group ring of a group G over a ring R is denoted by RG. Interested readers can consult any one of the excellent monographs [1, 23, 54, 55, 60, 63, 97, 145, 172, 171, 173, 174, 191] and [193] to obtain further information concerning group rings and modules.

We may think of the vector space A as a (left) FG-module as follows. For an arbitrary element $y = \alpha_1 g_1 + \cdots + \alpha_n g_n \in FG$, where $\alpha_i \in F, g_i \in G$, $1 \leq i \leq n$, then for every element $a \in A$ we let

$$ya = (\alpha_1 g_1 + \ldots \alpha_n g_n)a = \alpha_1 g_1(a) + \ldots \alpha_n g_n(a).$$

Then the family of G-invariant subspaces is precisely the family of FG-submodules of A and for each $M \subseteq A$, the G-invariant closure of M is precisely the FG-submodule generated by M.

If V is a subspace of A, then *the annihilator of V in FG* is

$$\mathbf{Ann}_{FG}(V) = \{x \in FG | xv = 0, \text{ for all } x \in V\}.$$

It is easy to see that $\mathbf{Ann}_{FG}(V)$ is a subalgebra of FG; indeed it is a left ideal of FG. If V is an FG-submodule of A, then $\mathbf{Ann}_{FG}(V)$ is an ideal of FG. To see this, let $x \in FG, y \in \mathbf{Ann}_{FG}(V), v \in V$. Then $xv \in V$, so

$$(xy)v = x(yv) = x0 = 0 \text{ and } (yx)v = y(xv) = 0.$$

Dually, if Y is a subalgebra of FG, then we define the *annihilator of Y in A* by

$$\mathbf{Ann}_A(Y) = \{a \in A | ya = 0 \text{ for all } y \in Y\}.$$

It is easy to see that this is a subspace of A. Furthermore, if Y is an ideal of FG, then $\mathbf{Ann}_A(Y)$ is an FG-submodule of A because we have $yg \in Y$ for all $g \in G, y \in Y$ and

$$y(g(a)) = (yg)(a) = 0,$$

for all $a \in \mathbf{Ann}_A(Y)$.

If $g \in G$ and $g(a) = a$, for some $a \in A$, then $(g-1)(a) = 0$. We let $\omega(FG)$ denote the ideal of FG generated by the elements of the form $g-1$, for $g \in G$. This ideal is called the *augmentation ideal* of FG. We note that if $a \in \zeta_G(A)$, then $(g-1)(a) = 0$ for every element $g \in G$, so $\zeta_G(A) = \mathbf{Ann}_A(\omega(FG))$.

A notion which is dual to the G-center is that of the subspace generated by the elements $(g-1)a$, for $a \in A, g \in G$ and we denote this subspace by $[G, A]$, the *G-commutator subspace of A*.

We shall often write $(g-1)a$ as $[g, a]$. If $g_1 \ldots, g_n$ are elements of G and $a \in A$ then we write the complex commutator $[g_n, [g_{n-1}, [\ldots [g_1, a]] \ldots]$ as $[g_n, g_{n-1}, \ldots, g_1, a]$.

We next show that $[G, A]$ is a G-invariant subspace of A. Let $g, x \in G, a \in A$ and let $u = xgx^{-1}$. Then $xg = ux$ and we have

$$x(g-1)a = (xg-x)a = (ux-x)a = (u-1)(xa) \in [G, A].$$

We note that the factor $A/[G, A]$ is G-central. Furthermore, if V is an FG-submodule of A such that A/V is G-central, then $[G, A] \leq V$.

Hypercentral Action

Starting with the G-center we can construct the *upper G-central series*

$$0 = \zeta_{G,0}(A) \leq \zeta_{G,1}(A) \leq \dots \zeta_{G,\alpha}(A) \leq \zeta_{G,\alpha+1}(A) \leq \dots \zeta_{G,\gamma}(A)$$

of A, where

$$\zeta_{G,1}(A) = \zeta_G(A), \zeta_{G,\alpha+1}(A)/\zeta_{G,\alpha}(A) = \zeta_G(A/\zeta_{G,\alpha}(A)),$$

for all ordinals α,

$$\zeta_{G,\lambda}(A) = \bigcup_{\beta<\lambda} \zeta_{G,\beta}(A), \text{ for all limit ordinals } \lambda \text{ and}$$

$\zeta_G(A/\zeta_{G,\gamma}(A)) = 0$.

It follows easily from the definition that $[G, \zeta_{G,\alpha+1}(A)] \leq \zeta_{G,\alpha}(A)$ for each ordinal $\alpha < \gamma$.

The last term $\zeta_{G,\gamma}(A)$ of the series is called the *upper G-hypercenter of A* and is denoted by $\zeta_{G,\infty}(A)$.

The ordinal γ is called the *G-central length of A* and is denoted by $\mathbf{zl}_G(A)$.

An ascending series of G-invariant subspaces

$$0 = A_0 \leq A_1 \leq \dots A_\alpha \leq A_{\alpha+1} \leq \dots A_\gamma$$

is said to be *G-central* if $[G, A_{\alpha+1}] \leq A_\alpha$ for each ordinal $\alpha < \gamma$. It is easy to see that if A has a G-central series of length γ, then $\mathbf{zl}_G(A) \leq \gamma$.

We say that a vector space (or FG-module) A is *G-hypercentral* if $A = \zeta_{G,\infty}(A)$ and we then say that G *acts hypercentrally on* A. We also say that G acts *stably* on A (we will introduce the term stabilizer later).

We note that A is G-hypercentral if and only if A has an ascending series of G-invariant subspaces, the last term of which is A, and for which G acts trivially on each of the factors.

If A is G-hypercentral and has a finite upper central series, then we say that A *is G-nilpotent*. In this case we obtain the following result.

Theorem 1.2. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that A is G-nilpotent. Then* *(i) G is a nilpotent group of class at most $\boldsymbol{zl}_G(A) - 1$;*

(ii) if $\boldsymbol{char}(F) = p$, a prime, then G is a p-group and has finite exponent at most $p^{\mathbf{zl}_G(A)-1}$;

(iii) if $\boldsymbol{char}(F) = 0$, then G is torsion-free.

Proof. (i) Let

$$0 = C_0 \leq C_1 \leq \cdots \leq C_n = A$$

be the upper central series of A. We use induction on $n = \mathbf{zl}_G(A)$. Suppose first that $n = 2$. If $a \in A, g, h \in G$ we have

$$ga = a + c, ha = a + d,$$

for certain elements $c, d \in C_1$. Then

$$(gh)(a) = g(h(a)) = g(a + d) = g(a) + g(d) = a + c + d,$$
$$(hg)(a) = h(g(a)) = h(a + c) = h(a) + h(c) = a + d + c.$$

Since this is true for every element $a \in A$, we have $gh = hg$, so G is abelian.

Now assume that $n > 2$ and that the result is true for all G-nilpotent vector spaces whose G-central length is less than n. Let $X = C_G(C_{n-1}) \cap C_G(A/C_1)$. Then for arbitrary elements $x \in X, a \in A$ we have $xa = a + c_0$, for some element $c_0 \in C_1$. If g is an element of G, then $ga = a + c_1$, for some element $c_1 \in C_{n-1}$. Then we obtain

$$(gx)(a) = g(x(a)) = g(a + c_0) = g(a) + g(c_0) = a + c_1 + c_0$$
$$(xg)(a) = x(g(a)) = x(a + c_1) = x(a) + x(c_1) = a + c_0 + c_1$$

Since this is true for every element $a \in A$, we have $gx = xg$, which shows that $X \leq \zeta(G)$. By the induction hypothesis, $G/C_G(C_{n-1})$ and $G/C_G(A/C_1)$ are nilpotent of class at most $n - 2$. Using Remak's theorem we have the embedding

$$G/X \longrightarrow G/C_G(C_{n-1}) \times G/C_G(A/C_1)$$

which implies that G/X is nilpotent of class at most $n - 2$. Since $X \leq \zeta(G)$ we deduce that G is nilpotent of class at most $n - 1$. This proves the first statement.

(ii) and (iii). We also use induction on n to prove the second statement. Suppose first that $n = 2$. Then, for each $a \in A, g \in G$, we have $ga = a + c$, where $c \in C_1$. Clearly,

$$(g^2)(a) = g(g(a)) = g(a + c) = g(a) + g(c) = a + 2c$$

and it is easy to see that, by induction, $(g^n)(a) = a + nc$. Thus if $\mathbf{char}(F) = p$, then $g^p a = g^p(a) = a + pc = a$ for all $a \in A$, so we have $g^p = 1$. If $\mathbf{char}(F) = 0$, then $g^m \neq 1$, for all $m \in \mathbb{Z}$, which shows that G is torsion-free.

Assume that $n > 2$ and that our assertion has been proved for all G-nilpotent vector spaces the G-central length of which is less than n.

Let $\mathbf{char}(F) = p$. Then the induction hypothesis implies that $G/C_G(A/C_1)$ is a p-group of exponent $r \leq p^{n-2}$. It follows that $g^r \in C_G(A/C_1)$ for all $g \in G$. Hence for each $a \in A$ we have $g^r a = a + c$, for some element $c \in C_1$. Repeating the arguments above we deduce that $((g^r)^p)a = a + pc = a$, which is true for all $a \in A$, so we have $g^{rp} = 1$ and hence G is a p-group of exponent at most p^{n-1}.

Suppose now that $\mathbf{char}(F) = 0$ and assume that there exists a nontrivial element $x \in G$ and a natural number s such that x has order s. By the induction hypothesis, we see that $G/C_G(A/C_1)$ is torsion-free. Hence $x \in C_G(A/C_1)$. Then, for all $a \in A$, we have $xa = a + c$, where $c \in C_1$. Again repeating the procedure above, we deduce that $x^n a = a + nc$ for all $n \in \mathbb{Z}$. In particular $a = 1a = x^s a = a + sc$ so that $sc = 0$. Since the additive group of A is torsion-free, $c = 0$. Hence $xa = a$ and, as this is true for all $a \in A$, we have $x = 1$. This proves the result. □

We let $UT_n(F)$ denote the group of unitriangular $n \times n$ matrices. The following corollary of the above result is well-known.

Corollary 1.3. *Let F be a field and let G be a subgroup of $UT_n(F)$. Then G is nilpotent. Furthermore, if $\mathbf{char}(F) = p$, a prime, then G is a p-group of exponent at most p^{n-1}; if $\mathbf{char}(F) = 0$, then G is torsion-free.*

Next, we obtain some of the standard closure properties concerning G-nilpotency and G-hypercentrality.

Lemma 1.4. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$.*

(i) If A is G-hypercentral (respectively G-nilpotent) and H is a subgroup of G, then A is also H-hypercentral (respectively H-nilpotent);

(ii) if A is G-hypercentral (respectively G-nilpotent) and B is a G-invariant subspace of A, then B is G-hypercentral (respectively G-nilpotent);

(iii) if A is G-hypercentral (respectively G-nilpotent) and B is a G-invariant subspace of A, then A/B is G-hypercentral (respectively G-nilpotent);

(iv) if B, C are G-invariant G-nilpotent subspaces of A, then $B + C$ is G-nilpotent and $\mathbf{zl}_G(B + C) = \max\{\mathbf{zl}_G(B), \mathbf{zl}_G(C)\}$;

(v) if B, C are G-invariant G-hypercentral subspaces of A, then $B + C$ is G-hypercentral;

(vi) let H, K be subgroups of G such that $K \leq N_G(H)$. If A is H-hypercentral and K-hypercentral, then A is HK-hypercentral;

(vii) let H, K be subgroups of G such that $K \leq N_G(H)$. If A is H-nilpotent and K-nilpotent, then A is HK-nilpotent;

(viii) *let H be a subgroup of G and let $g \in G$. If A is H-hypercentral (respectively H-nilpotent), then A is H^g-hypercentral (respectively H^g-nilpotent).*

Proof. The proofs of Statements (i)-(v) are quite clear. To prove (vi) let

$$0 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\gamma = A$$

be the upper H-central series of A. Let $h \in H, x \in K$. Since $K \leq N_G(H)$, $x^{-1}hx = h_1 \in H$. If $c \in C_1 = \zeta_H(A)$ is arbitrary, then

$$h(xc) = (hx)c = (xh_1)c = x(h_1c) = xc,$$

so that $xc \in C_1$. Hence C_1 is K-invariant. Let

$$0 = D_0 \leq D_1 \leq \dots D_\alpha \leq D_{\alpha+1} \leq \dots D_\tau = A$$

be the upper K-central series of A and let $E_{1,\alpha} = C_1 \cap D_\alpha$, for $\alpha \leq \tau$. Then each subspace $E_{1,\alpha}$ is HK-invariant for $\alpha \leq \tau$ and the ascending series

$$0 = E_{1,0} \leq E_{1,1} \leq \dots E_{1,\alpha} \leq E_{1,\alpha+1} \leq \dots E_{1,\tau} = C_1$$

is an HK-central series for C_1.

Again let $h \in H, x \in K$ so that $x^{-1}hx = h_1 \in H$. If $c \in C_2$ is arbitrary, then

$$h(xc) = (hx)c = (xh_1)c = x(c + c_0) = xc + xc_0,$$

for some element $c_0 \in C_1$. By our work above $xc_0 \in C_1$ so that $h(xc) \in xc + C_1$ and this shows that $xc \in C_2$. Hence C_2 is K-invariant.

Let $E_{2,\alpha} = C_1 + (C_2 \cap D_\alpha)$, for $\alpha \leq \tau$. Then the subspaces $E_{2,\alpha}$ are HK-invariant for $\alpha \leq \tau$ and the ascending series

$$\begin{aligned} 0 &= E_{1,0} \leq E_{1,1} \leq \dots E_{1,\alpha} \leq E_{1,\alpha+1} \leq \dots E_{1,\tau} = C_1 \\ &= E_{2,0} \leq E_{2,1} \leq \dots E_{2,\alpha} \leq E_{2,\alpha+1} \leq \dots E_{2,\tau} = C_2 \end{aligned}$$

is an HK-central series of C_2. Using similar arguments and transfinite induction we prove that A has an ascending HK-central series of HK-invariant subspaces the last term of which is A. It follows that A is HK-hypercentral.

The proof of (vii) is similar to that given in (vi).

To prove (viii) we again consider the upper central series

$$0 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\tau = A$$

of A. It is easy to see that $g^{-1}C_\alpha$ is an H^g-invariant subspace and we have already shown above that $g^{-1}C_1 = C_A(H^g)$. Using transfinite induction it can be shown that

$$0 = g^{-1}C_0 \leq g^{-1}C_1 \leq \dots g^{-1}C_\alpha \leq g^{-1}C_{\alpha+1} \leq \dots g^{-1}C_\tau = A$$

is an upper H^g-central series. This completes the proof. □

Let $\{A_\lambda | \lambda \in \Lambda\}$ be a family of vector spaces over a field F and let $B = \underset{\lambda \in \Lambda}{\mathrm{Cr}}\, A_\lambda$ be their Cartesian product. For each $\lambda \in \Lambda$, let G_λ be a subgroup of $GL(F, A_\lambda)$ and let G be a subgroup of $\underset{\lambda \in \Lambda}{\mathrm{Cr}}\, G_\lambda$. We can define a natural action of G on B as follows: if $(a_\lambda)_{\lambda \in \Lambda} \in B, (g_\lambda)_{\lambda \in \Lambda} \in G$ then set

$$(g_\lambda)_{\lambda \in \Lambda}((a_\lambda)_{\lambda \in \Lambda}) = (g_\lambda a_\lambda)_{\lambda \in \Lambda} = (g_\lambda (a_\lambda)_{\lambda \in \Lambda}).$$

In this way we can think of G as a subgroup of $GL(F, B)$. Of course we can think of $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$, the corresponding direct sum of the A_λ, as a subspace of B and the natural action of G on B restricts to an action of G on A, so that indeed G is also a subgroup of $GL(F, A)$. The proof of the following result is rather easy and is omitted.

Lemma 1.5. *Let $\{A_\lambda | \lambda \in \Lambda\}$ be a family of vector spaces over a field F and for each $\lambda \in \Lambda$, let G_λ be a subgroup of $GL(F, A_\lambda)$. Let $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ and let G be a subgroup of $\underset{\lambda \in \Lambda}{Cr}\, G_\lambda$. If each A_λ is G_λ-hypercentral, then A is G-hypercentral.*

Our next result is well-known in the group theoretical case, being analogous to the well-known result in nilpotent and hypercentral groups (see [184, 5.2.1], for example).

Lemma 1.6. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If A is G-hypercentral and B is a non-zero G-invariant subspace of A, then $B \cap \zeta_G(A) \neq 0$.*

Proof. Let

$$0 = Z_0 \leq Z_1 \leq \ldots Z_\alpha \leq Z_{\alpha+1} \leq \ldots Z_\gamma = A$$

be the upper G-central series of A. Since $B \neq 0$, there is a least ordinal α such that $B \cap Z_\alpha \neq 0$. If α is a limit ordinal, then we have, by the definiton of α,

$$B \cap Z_\alpha = B \cap \left(\bigcup_{\beta < \alpha} Z_\beta \right) = \bigcup_{\beta < \alpha} (B \cap Z_\beta) = 0,$$

which is a contradiction. Hence $\alpha - 1$ exists and $B \cap Z_{\alpha-1} = 0$, by definition of α. However, we then obtain $[G, B \cap Z_\alpha] \leq B \cap Z_{\alpha-1} = 0$ which implies that $B \cap Z_\alpha \leq \zeta_G(A)$. In particular $B \cap \zeta_G(A) \neq 0$, as required. □

Next we obtain the following characterization of G-hypercentral vector spaces, again reminiscent of a result of S. N. Chernikov (see [183, Theorem 2.19], for example).

Lemma 1.7. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Then G acts G-hypercentrally on A if and only if for each $a \in A$ and every countable subset $\{x_n | n \in \mathbb{N}\}$ of elements of G there exists a natural number k such that $[x_k, x_{k-1}, \ldots, x_1, a] = 0$.*

Proof. Let
$$0 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma = A$$
be the upper G-central series of A and let α be the least ordinal such that $a \in Z_\alpha$. To prove necessity, we use transfinite induction, the assertion being clear if α is a natural number, so we suppose that α is infinite. Clearly α is not a limit ordinal, so that $\alpha - 1$ exists. Then $a_1 = [g_1, a] \in Z_{\alpha-1}$ and the induction hypothesis implies that there is a natural number k such that

$$0 = [g_k, \dots, g_2, a_1] = [g_k, \dots, g_2, [g_1, a]] = [g_k, \dots, g_1, a],$$

as required.

To prove sufficiency we first show that $\zeta_G(A) \neq 0$. We suppose the contrary and let $\zeta_G(A) = 0$. Let $0 \neq a \in A$ be arbitrary. Since $a \notin \zeta_G(A)$, there is an element $g_1 \in G$ such that $[g_1, a] \neq 0$. Since $\zeta_G(A) = 0$ we have $[g_1, a] \notin \zeta_G(A)$. Hence there exists $g_2 \in G$ such that $[g_2, [g_1, a]] = [g_2, g_1, a] \neq 0$. Similar arguments and mathematical induction show that we can construct an infinite subset $\{g_n | n \in \mathbb{N}\}$ of elements of G such that $[g_n, g_{n-1}, \dots, g_1, a] \neq 0$ for all $n \in \mathbb{N}$. This provides the contradiction sought and it follows that $\zeta_G(A) \neq 0$.

Next we note that the conditions of the lemma are inherited by each quotient space of A by a G-invariant subspace. Since $\zeta_G(A)$ is such a G-invariant subspace we may apply transfinite induction to deduce that A has an upper central series, the last term of which is A. Hence A is G-hypercentral, as required. $\square$

Corollary 1.8. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Then G acts G-hypercentrally on A if and only if, for each countable subgroup H of G, every cyclic FH-submodule of A is H-hypercentral.*

Proof. If A is G-hypercentral, then Lemma 1.4 shows that A is H-hypercentral for every subgroup H of G. Using Lemma 1.4 again we see that every FH-submodule (and hence every cyclic FH-submodule) of A is H-hypercentral.

Conversely, assume that for each countable subgroup H of G every cyclic FH-submodule of A is H-hypercentral, but suppose that, to the contrary, A is not G-hypercentral. Let Z denote the upper hypercenter of A, so $Z \neq A$ and hence $A/Z \neq 0$. Then $\zeta_G(A/Z) = 0$. Since we carry out all subsequent reasoning in the quotient space A/Z we may replace A/Z by A and hence assume that $Z = 0$. It will be clear that this does not limit the argument's generality.

Using the arguments from the proof of Lemma 1.7 we can find an element $d \in A$ and a countable subset $\{g_n | n \in \mathbb{N}\}$ of G such that $[g_n, \dots, , g_1, d] \neq 0$ for each natural number n. Let $H = \langle g_n | n \in \mathbb{N}\rangle$, a countable subgroup of G. Then the FH-submodule D generated by d is H-hypercentral. However, Lemma 1.7 then implies that there is a natural number k such that $[g_k, \dots, g_1, d] = 0$, giving us a contradiction, which proves the result. $\square$

Our next result is analogous to the well-known result that if G is a finitely generated group and H is a subgroup of finite index, then H is also finitely generated.

Lemma 1.9. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A is a finitely generated FG-module. If B is an FG-submodule of A such that $\mathbf{dim}_F(A/B)$ is finite, then B is finitely generated as an FG-module.*

Proof. Let $M = \{g_1, \dots, g_k\}$ be a finite generating set for G and without loss of generality assume that $g_i^{-1} \in M$, for each i. Let $\{a_1, \dots, a_t\}$ be a finite generating set for A, so that $A = FGa_1 + \cdots + FGa_t$. Also let $c_1, \dots, c_n$ be elements of A such that $\{c_1 + B, \dots, c_n + B\}$ is a basis for A/B. Let $Fc_1 + \cdots + Fc_n = C$ and note that $A = C \oplus B$. Let $\mathrm{pr}_B : A \longrightarrow B$ and $\mathrm{pr}_C : A \longrightarrow C$ be the canonical projections of A on B and C respectively. Let E denote the FG-submodule generated by

$$\{\mathrm{pr}_B(a_j), \mathrm{pr}_B(g_r c_m) | 1 \leq j \leq t, 1 \leq m \leq n, 1 \leq r \leq k\}.$$

Clearly E lies in B and is a finitely generated FG-submodule. If $v \in E + C$ is arbitrary, then $v = u + c$ for some elements $u \in E, c \in C$. For certain elements $\alpha_1, \dots, \alpha_n \in F$ we may write $c = \alpha_1 c_1 + \cdots + \alpha_n c_n$ and we have

$$\begin{aligned} g_r v &= g_r u + g_r c = g_r u + g_r(\alpha_1 c_1 + \dots \alpha_n c_n) \\ &= g_r u + \alpha_1 g_r c_1 + \cdots + \alpha_n g_r c_n \\ &= g_r u + \alpha_1 \mathrm{pr}_B(g_r c_1) + \dots \alpha_n + \mathrm{pr}_B(g_r c_n) \\ &+ \alpha_1 \mathrm{pr}_C(g_r c_1) + \cdots + \alpha_n \mathrm{pr}_C(g_r c_n) \in E + C \end{aligned}$$

It follows that $E + C$ is an FG-submodule of A and the equality $a_j = \mathrm{pr}_B(a_j) + \mathrm{pr}_C(a_j)$ implies that a_j is an element of $E + C$, for $1 \leq j \leq t$. Therefore,

$$E + C = A = B + C$$

and since E is an FG-submodule of B the Dedekind modular law implies that $B = E$, so that B is a finitely generated FG-submodule of A, as required. □

Our next result gives some conditions under which a G-hypercentral module is actually G-nilpotent-by-finite dimensional.

Proposition 1.10. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A is a finitely generated FG-module. If $\zeta_{G,\infty}(A)$ has finite codimension in A, then $\mathbf{zl}_G(A)$ is finite.*

Proof. Let

$$0 = Z_0 \leq Z_1 \leq \dots Z_\alpha \leq Z_{\alpha+1} \leq \dots Z_\gamma = \zeta_{G,\infty}(A) \leq A$$

be the upper G-central series of A so that γ is the G-central length. Since $\mathbf{dim}_G(A/\zeta_{G,\infty}(A))$ is finite, Lemma 1.9 implies that $\zeta_{G,\infty}(A)$ is finitely generated as an FG-module. This implies further that γ is not a limit ordinal.

If γ is infinite then we may write $\gamma = \tau + n$, for some limit ordinal τ and some natural number n. Let $M = \{v_1, \ldots, v_l\}$ be a finite subset of Z_γ which generates Z_γ as an FG-module. Since $Z_\gamma/Z_{\gamma-1}$ is the G-center of $A/Z_{\gamma-1}$ we have

$$Z_\gamma/Z_{\gamma-1} = F(v_1 + Z_{\gamma-1}) + \cdots + F(v_l + Z_{\gamma-1})$$

and hence, in particular, $Z_\gamma/Z_{\gamma-1}$ has finite dimension at most l. Then Lemma 1.9 implies that $Z_{\gamma-1}$ is finitely generated as an FG-submodule. These arguments can then be applied to $Z_{\gamma-1}$ to show that $Z_{\gamma-2}$ is also finitely generated and inductively we see that Z_τ is a finitely generated FG-module.

We note also that A/Z_τ is finite dimensional as an F-space. Let $W = \{w_1, \ldots, w_m\}$ be a finite subset of Z_τ which generates Z_τ as an FG-submodule. Since $Z_\tau = \bigcup_{\beta<\tau} Z_\beta$ there exists $\beta(j) < \tau$ such that $w_j \in Z_{\beta(j)}$ for $1 \leq j \leq m$. Let σ be the greatest ordinal in the set $\{\beta(1), \ldots, \beta(m)\}$, so that $w_j \in Z_\sigma$ for $1 \leq j \leq m$. Since Z_σ is an FG-submodule of Z_τ it follows that $Z_\sigma = Z_\tau$ and since $\sigma < \tau$, we obtain a contradiction, which shows that γ is finite. □

The following corollary is analogous to the well-known fact that a finitely generated hypercentral group is nilpotent.

Corollary 1.11. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A is a finitely generated FG-module. If A is G-hypercentral, then A is G-nilpotent and has finite F-dimension.*

Proof. By Proposition 1.10, with $\zeta_{G,\infty}(A) = A$ we deduce that $\mathbf{zl}_G(A)$ is finite, so A is G-nilpotent and the proof of Proposition 1.10 shows that $\mathbf{dim}_F(A)$ is finite. □

When A is a vector space over a field F and G is a subgroup of $GL(F, A)$, we say that A is *G-locally nilpotent* if for every finitely generated subgroup H of G and every finite subset M of A the FH-submodule generated by M, $(FH)M$, is H-nilpotent.

We say that A is *G-locally (finite dimensional)* if for every finitely generated subgroup H of G and every finite subset M of A, the FH-submodule $(FH)M$ has finite F-dimension. Our next result is an immediate consequence of Corollary 1.11.

Corollary 1.12. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If A is G-hypercentral, then A is G-locally nilpotent and G-locally (finite dimensional).*

For the next result we need some more notation and terminology. If $\mathcal{M}$ is a family of sets we shall write $\bigcup \mathcal{M}$ to mean $\bigcup\{M : M \in \mathcal{M}\}$ and likewise we let $\bigcap \mathcal{M}$ denote $\bigcap\{M : M \in \mathcal{M}\}$.

If $\mathfrak{X}$ is a class of groups, then the *$\mathfrak{X}$-residual* of a group G is defined to be

$$G^{\mathfrak{X}} = \bigcap\{N \triangleleft G | G/N \in \mathfrak{X}\}.$$

If the set $\mathrm{Res}_{\mathfrak{X}}(G) = \{N \triangleleft G | G/N \in \mathfrak{X}\}$ has a least element L, then $L = G^{\mathfrak{X}}$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$, but in general $G/G^{\mathfrak{X}}$ need not be an $\mathfrak{X}$-group.

When $\mathfrak{X} = \mathfrak{A}$, the class of abelian groups, then the $\mathfrak{A}$-residual of a group G is precisely the derived subgroup G' and $G/G^{\mathfrak{A}} \in \mathfrak{A}$, in this case.

More generally, if $\mathfrak{X} = \mathfrak{N}_c$, the class of nilpotent groups of nilpotency class at most c, then the $\mathfrak{N}_c$-residual is precisely $\gamma_{c+1}(G)$ and again $G/G^{\mathfrak{N}_c} \in \mathfrak{N}_c$.

For a class of groups $\mathfrak{X}$, a group G is called a *residually* $\mathfrak{X}$*-group* if, for each non-trivial element $g \in G$, there is a normal subgroup H_g such that $g \notin H_g$ and $G/H_g \in \mathfrak{X}$. It is easy to prove that G is a residually $\mathfrak{X}$-group if and only if $\cap \mathrm{Res}_{\mathfrak{X}}(G) = 1$.

We shall denote the class of residually $\mathfrak{X}$-groups by $\mathbf{R}\mathfrak{X}$. If $\mathfrak{X} = \mathfrak{F}$, the class of all finite groups, then we obtain the familiar class $\mathbf{R}\mathfrak{F}$ of *residually finite groups*. If p is a prime, we let $\mathfrak{F}_p$ denote the class of all finite p-groups and in this way we obtain the class $\mathbf{R}\mathfrak{F}_p$ of residually $\mathfrak{F}_p$-groups. It is well known that every free group is residually $\mathfrak{F}_p$, for each prime p, a theorem originally due to K. Iwasawa [90].

It is easy to see that $G^{\mathfrak{X}}$ is always a characteristic subgroup of G and that $G/G^{\mathfrak{X}}$ is always a residually $\mathfrak{X}$-group. For example when $\mathfrak{X} = \mathfrak{F}$, the class of all finite groups, then $G^{\mathfrak{X}} = G^{\mathfrak{F}}$ is the finite residual of G and $G/G^{\mathfrak{F}}$ is residually finite.

Theorem 1.13. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F, A)$. If A is G-hypercentral, then G is residually nilpotent. Moreover,*

(i) if $\mathbf{char}(F) = p$, a prime, then $G \in \mathbf{R}\mathfrak{F}_p$;

(ii) if $\mathbf{char}(F) = 0$, then G is residually (torsion-free nilpotent).

Proof. Let $\mathcal{L}$ be the family of all finitely generated FG-submodules of A. If $B \in \mathcal{L}$, then Corollary 1.11 shows that B is G-nilpotent.

(i) Suppose first that $\mathbf{char}(F) = p$, a prime. Then Theorem 1.2 implies that $G/C_G(B)$ is a nilpotent p-group and since it is finitely generated it is finite. Since A is generated by the elements of $\mathcal{L}$ it follows that $\bigcap_{B\in\mathcal{L}} C_G(B) = 1$ and hence $G \in \mathbf{R}\mathfrak{F}_p$.

(ii) If F has characteristic 0, then Theorem 1.2 shows that $G/C_G(B)$ is a torsion-free nilpotent group and Remak's theorem implies that G embeds in $\underset{B\in\mathcal{L}}{\mathrm{Cr}}\, G/C_G(B)$. The result then follows since a Cartesian product of nilpotent groups is residually nilpotent. □

We now show that for groups G having the structure indicated in the preceding theorem, there are vector spaces on which they act G-hypercentrally.

Theorem 1.14. *Let G be a group with a descending series of normal subgroups*

$$G = H_0 \geq H_1 \geq \cdots \geq H_n \geq H_{n+1} \geq \cdots \bigcap_{n\in\mathbb{N}} H_n = 1$$

whose factors are finite p-groups for some fixed prime p. Then there is a vector space A over the prime field $\mathbb{F}_p$ *such that A is G-hypercentral and* $\boldsymbol{zl}_G(A) = \omega$*, the first infinite ordinal. Furthermore,* $A = \bigoplus_{n\in\mathbb{N}} A_n$*, where* A_n *is a finite G-invariant subspace of A, for all* $n \in \mathbb{N}$*.*

Proof. For each natural number n, let $K_n = G/H_n$, which is a finite p-group by hypothesis and let $K = \underset{n\in\mathbb{N}}{\mathrm{Cr}}\, K_n$. Define a mapping $f : G \longrightarrow K$ by $f(g) = (gH_n)_{n\in\mathbb{N}}$ for each element $g \in G$. Since $\bigcap_{n\in\mathbb{N}} H_n = 1$, Remak's Theorem implies that G is isomorphic to the subgroup $L = \mathbf{Im}\,(f)$ of K. Let $\langle a\rangle$ be a cyclic group of order p and let $W_n = \langle a\rangle \wr K_n$, the standard wreath product, for each $n \in \mathbb{N}$. Let A_n be the base group of this wreath product. Then $W_n = A_n \rtimes K_n$, where A_n is an elementary abelian p-group and $C_{K_n}(A_n) = 1$, for each $n \in \mathbb{N}$. We use additive notation for each A_n. Since W_n is a finite p-group it is nilpotent, so for each n, A_n has a finite K_n-central series

$$0 = Z_{n,0} \leq Z_{n,1} \leq \dots Z_{n,s(n)} = A_n,$$

for some natural number $s(n)$. Let $A = \bigoplus_{n\in\mathbb{N}} A_n$. We can define an action of K on A as follows. For each $(c_n)_{n\in\mathbb{N}} \in A$ and $(x_n)_{n\in\mathbb{N}} \in K$ then set

$$(x_n)(c_n) = (x_n^{-1} c_n x_n)_{n\in\mathbb{N}}.$$

It is easy to see that $C_K(A) = 1$. We can think of A as a vector space over $\mathbb{F}_p$, so the fact that $C_K(A) = 1$ implies that K is isomorphic to a subgroup of $GL(\mathbb{F}_p, A)$. Setting $C_k = \oplus_{n\in\mathbb{N}} Z_{n,k}$ for each natural number k we see that the series

$$0 = C_0 \leq C_1 \leq \dots \leq C_k \leq C_{k+1} \leq \dots \bigcup_{n\in\mathbb{N}} C_n = A$$

is K-central. Lemma 1.4 shows that A is also G-hypercentral. This gives us the appropriate example. □

A class of groups $\mathfrak{X}$ is called a *formation* when the following conditions are satisfied:

(F1) if $G \in \mathfrak{X}$ and $H \lhd G$, then $G/H \in \mathfrak{X}$;

(F2) if $K, L \lhd G$ and $G/K, G/L \in \mathfrak{X}$, then $G/(K \cap L) \in \mathfrak{X}$.

It is easy to see that the class of all finite groups and the class of nilpotent groups are examples of formations.

Lemma 1.15. *Let* $\mathfrak{X}$ *be a formation of groups and let G be a countable group. If* $G \in \mathbf{R}\mathfrak{X}$*, then G has a descending series of normal subgroups*

$$G = H_0 \geq H_1 \geq \dots \geq H_n \geq H_{n+1} \geq \dots \bigcap_{n\in\mathbb{N}} H_n = 1$$

whose factors belong to $\mathfrak{X}$*.*

Proof. Let $\{g_n | n \in \mathbb{N}\}$, be the set of nontrivial elements of G. Since $G \in \mathbf{R}\mathfrak{X}$ it follows that for each $n \geq 1$, there is a normal subgroup K_n of G such that $g_n \notin K_n$ and $G/K_n \in \mathfrak{X}$. Clearly $\bigcap_{n\in\mathbb{N}} K_n = 1$ and if we set $H_n = K_1 \cap \cdots \cap K_n$, then $G/H_n \in \mathfrak{X}$, since $\mathfrak{X}$ is a formation. Clearly $H_{n+1} \leq H_n$ for all n and $\bigcap_{n\in\mathbb{N}} H_n = \bigcap_{n\in\mathbb{N}} K_n = 1$, as required. □

Corollary 1.16. *Let G be a countable group. If $G \in \mathbf{R}\mathfrak{F}_p$ for some prime p, then there is a vector space A over $\mathbb{F}_p$ such that A is G-hypercentral and $\boldsymbol{zl}_G(A) = \omega$, the first infinite ordinal. Furthermore, $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite G-invariant subspace of A, for each $n \in \mathbb{N}$.*

Proof. This follows from Theorem 1.14 and Lemma 1.15 because the class of finite p-groups is a formation. □

By the result of K. Iwasawa [90] mentioned above a finitely generated free group G has the property that $G \in \mathbf{R}\mathfrak{F}_p$ for each prime p. This yields the following corollary immediately.

Corollary 1.17. *Let G be a finitely generated free group. Then for each prime p there exists a vector space A over $\mathbb{F}_p$ such that A is G-hypercentral and $\boldsymbol{zl}_G(A) = \omega$, the first infinite ordinal. Furthermore, $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite G-invariant subspace of A, for each $n \in \mathbb{N}$.*

To complete this list of consequences of Theorem 1.14 we also state the following obvious consequence, since finitely generated groups are countable, which gives examples of periodic groups G acting on G-hypercentral spaces.

Corollary 1.18. *Let G be a finitely generated residually finite p-group for some prime p. Then there exists a vector space A over $\mathbb{F}_p$ such that A is G-hypercentral and $\boldsymbol{zl}_G(A) = \omega$, the first infinite ordinal. Furthermore, $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite G-invariant subspace of A, for each $n \in \mathbb{N}$.*

There are many examples of infinite finitely generated residually finite p-groups. Important easily described examples of such groups include those constructed by R. I. Grigorchuk [67]. As we can see, such groups can be realized as linear groups acting hypercentrally on a vector space of countable dimension over $\mathbb{F}_p$.

We next wish to consider analogues of triangular and unitriangular subgroups of $\mathrm{Mat}_{\mathbb{N}}(F)$. A matrix $M = [\mu_{kj}]_{k,j\in\mathbb{N}}$ is called *upper triangular* if $\mu_{k,j} = 0$, whenever $k > j$. Thus M has the form

$$\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \cdots & \mu_{1,n} & \mu_{1,n+1} \cdots \\ 0 & \mu_{22} & \mu_{23} \cdots & \mu_{2,n} & \mu_{2,n+1} \cdots \\ 0 & 0 & \mu_{33} & \cdots & \mu_{3,n+1} \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

Using exactly the same arguments used in the finite dimensional case, it can be shown that the subset $T_{\mathbb{N}}(F)$ of all upper triangular matrices is a subalgebra of

$\mathrm{Mat}_{\mathbb{N}}(F)$. It is not hard to see that an upper triangular matrix $M = [\mu_{kj}]_{k,j\in\mathbb{N}}$ is invertible if and only if $\mu_{kk} \neq 0$ for all natural numbers n. It follows that the subset $T_{\mathbb{N}}(F)$ consisting of all invertible upper triangular matrices is a subgroup of the group of units of the ring $\mathrm{Mat}_{\mathbb{N}}(F)$.

An upper triangular matrix $M = [\mu_{kj}]_{k,j\in\mathbb{N}}$ is called *unitriangular* if also $\mu_{k,k} = 1$ for all $k \in \mathbb{N}$. It follows at once that all unitriangular matrices are invertible and the subgroup $UT_{\mathbb{N}}(F)$ of all unitriangular matrices is a subgroup of $T_{\mathbb{N}}(F)$.

As we saw in Corollary 1.3 the group of all finite dimensional unitriangular matrices is nilpotent. Using the results just obtained we now show that the structure of the group $UT_{\mathbb{N}}(F)$ is considerably more complicated.

Theorem 1.19. *Let G be a countable group. If $G \in \mathbf{R}\mathfrak{F}_p$, for some prime p, then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(F)$.*

Proof. By Corollary 1.16 there exists a vector space A over $\mathbb{F}_p$ such that G acts G-hypercentrally on $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite G-invariant subspace. Let

$$0 = Z_{n,0} \leq Z_{n,1} \leq \dots Z_{n,s(n)} = A_n$$

be the upper central series of A_n, for each $n \in \mathbb{N}$ and certain $s(n) \in \mathbb{N}$. We choose a basis $\mathcal{B}_n$ of A_n as follows. Choose an arbitrary basis for $Z_{n,1}$ and extend this to a basis for $Z_{n,2}$. This new basis extends to a basis for $Z_{n,3}$ and we repeat this procedure until we obtain the basis $\mathcal{B}_n$ of A_n that we require. It follows from this construction that the restriction to A_n of an element of G is a unitriangular matrix relative to this basis. We let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, which is a basis for A since $A = \bigoplus_{n\in\mathbb{N}} A_n$. The matrix of every element of G is a block-triangular matrix, having the form

$$\begin{pmatrix} X_1 & 0 & 0 & \dots \\ 0 & X_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots 0 & \dots X_n & \dots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

where each X_i is an $m(i) \times m(i)$ upper triangular matrix for certain integers $m(i) \in \mathbb{N}$.

Theorem 1.1 now implies that G is isomorphic to some subgroup of $UT_{\mathbb{N}}(\mathbb{F}_p)$, which completes the proof.

$\square$

The next result follows directly from Theorem 1.19 and the fact that a finitely generated free group G has the property that $G \in \mathbf{R}\mathfrak{F}_p$ for every prime p by the result of K. Iwasawa [90].

Corollary 1.20. *Let G be a finitely generated free group. Then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(\mathbb{F}_p)$ for every prime p.*

We mention also the following two results the first of which appeared in the paper of A. S. Olijnyk and V. I. Sushchansky[158] and is of course a special case of Corollary 1.20. The second is due to R. I. Grigorchuk, Y. Leonov, V. Nekrashevych and V. I. Sushchansky[68].

Corollary 1.21. *Let G be a finitely generated free group. Then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(\mathbb{F}_2)$.*

Corollary 1.22. *Let G be a finitely generated residually finite p-group for some prime p. Then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(\mathbb{F}_p)$.*

In particular, we see that $UT_{\mathbb{N}}(\mathbb{F}_p)$ contains periodic subgroups which are not locally finite such as the Grigorchuk group. It was proved in [68] that $UT_{\mathbb{N}}(\mathbb{F}_p)$ contains entire series of subgroups whose properties are very far from nilpotency. This is all in contrast to the case of periodic subgroups of $GL_n(F)$ which are locally finite, a well-known result of I. Schur (see [202, 9.1], for example).

The situation with infinite dimensional unitriangular groups over fields of characteristic 0 is by no means simpler, as we shall see.

Theorem 1.23. *Let G be a group having a descending series of normal subgroups*

$$G = H_0 \geq H_1 \geq \cdots \geq H_n \geq H_{n+1} \geq \cdots \bigcap_{n\in\mathbb{N}} H_n = 1$$

whose factors are finitely generated torsion-free nilpotent groups. Then there exists a vector space A over the field $\mathbb{Q}$ of rational numbers such that A is G-hypercentral and $\boldsymbol{zl}_G(A) = \omega$. Furthermore, $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite dimensional G-invariant subspace of A, for each $n \in \mathbb{N}$.

Proof. For each natural number n, let $K_n = G/H_n$, which is a finitely generated torsion-free nilpotent group by hypothesis. Let $K = \underset{n\in\mathbb{N}}{\mathrm{Cr}}\, K_n$. Define a mapping $f : G \longrightarrow K$ by $f(g) = (gH_n)_{n\in\mathbb{N}}$ for each element $g \in G$. Since $\bigcap_{n\in\mathbb{N}} H_n = 1$, Remak's Theorem implies that G is isomorphic to the subgroup $L = \mathbf{Im}\,(f)$ of K.

The group K_n is isomorphic to some subgroup of $UT_m(\mathbb{Z})$ for some natural number m, depending upon n (see [95, Theorem 17.2.5], for example). Hence there is a free abelian subgroup A_n such that $\mathbf{Aut}\,(A_n)$ contains K_n. The natural semidirect product $W_n = A_n \rtimes K_n$ is a nilpotent group. Thus A_n is K_n-nilpotent and $C_{K_n}(A_n) = 1$, for each $n \in \mathbb{N}$. We use additive notation for each A_n. Then, for each $n \in \mathbb{N}$, A_n has a finite K_n-central series

$$0 = Z_{n,0} \leq Z_{n,1} \leq \ldots Z_{n,s(n)} = A_n,$$

for some natural number $s(n)$.

Let $A = \bigoplus_{n\in\mathbb{N}} A_n$. We can define an action of K on A as follows. For each $(c_n)_{n\in\mathbb{N}} \in A$ and $(x_n)_{n\in\mathbb{N}} \in K$ set

$$(x_n)(c_n) = (x_n^{-1} c_n x_n)_{n\in\mathbb{N}}.$$

It is easy to see that $C_K(A) = 1$. We can extend the action of K on A in a natural way to $B = \bigoplus_{n\in\mathbb{N}} B_n$, where B_n is the $\mathbb{Q}$-divisible envelope of A_n. Then the series

$$0 = C_{n,0} \le C_{n,1} \le \cdots \le C_{n,s(n)} = B_n$$

is a K_n-central series of B_n, where $C_{n,j}$ is the $\mathbb{Q}$-divisible envelope of $Z_{n,j}$, for each $n \in \mathbb{N}$.

We can think of B as a vector space over $\mathbb{Q}$ and, as $C_K(A) = 1$, we can think of K as a subgroup of $GL(\mathbb{Q}, B)$. Let $E_k = \bigoplus_{n\in\mathbb{N}} C_{n,k}$. Then the series

$$0 = E_0 \le E_1 \le \cdots \le E_k \le E_{k+1} \le \cdots \bigcup_{n\in\mathbb{N}} E_n = B,$$

is K-central and Lemma 1.4 shows that B is also G-hypercentral.

□

Corollary 1.24. *Let G be a finitely generated free group. Then there is a vector space A over the field $\mathbb{Q}$ such that A is G-hypercentral and $\boldsymbol{zl}_G(A) = \omega$. Furthermore, $A = \bigoplus_{n\in\mathbb{N}} A_n$, where A_n is a finite dimensional G-invariant subspace of A, for each $n \in \mathbb{N}$.*

Proof. By a theorem of Magnus (see [184, 6.1.10], for example) the lower central series of G has length ω and terminates in the trivial group. Furthermore the factors of this series are finitely generated torsion-free nilpotent groups (see [130, §36] for example) and then Theorem 1.23 gives the result. □

By following the proof of Theorem 1.19 we may also deduce the following fact.

Corollary 1.25. *Let G be a group having a descending series of normal subgroups*

$$G = H_0 \ge H_1 \ge \cdots \ge H_n \ge H_{n+1} \ge \cdots \bigcap_{n\in\mathbb{N}} H_n = 1$$

whose factors are finitely generated torsion-free nilpotent groups. Then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(\mathbb{Q})$.

Corollary 1.26. *Let G be a finitely generated free group. Then G is isomorphic to a subgroup of $UT_{\mathbb{N}}(\mathbb{Q})$.*

We note that by repeating the arguments in the results above we can obtain complete analogues of Corollaries 1.25 and 1.26 for the group $UT_{\mathbb{N}}(\mathbb{Z})$. In particular we see that $UT_{\mathbb{N}}(\mathbb{Z})$ contains all finitely generated free groups, a result obtained by W. Holubowski [81]. All this serves to emphasize the fact that things will generally be very different for infinite dimensional linear groups as opposed to the finite dimensional case.

Unipotent and Algebraic Elements

When A is a vector space over a field F and G is a subgroup of $GL(F,A)$ we define the *lower G-central series of A*,

$$A = \gamma_{G,1}(A) \geq \gamma_{G,2}(A) \geq \dots \gamma_{G,\alpha}(A) \geq \gamma_{G,\alpha+1}(A) \geq \dots \gamma_{G,\kappa}(A),$$

by

$$\begin{aligned}\gamma_{G,2}(A) &= [G,A] = [G,\gamma_{G,1}(A)],\\ \gamma_{G,\alpha+1}(A) &= [G,\gamma_{G,\alpha}(A)], \text{ for all ordinals } \alpha\\ \gamma_{G,\lambda}(A) &= \bigcap_{\beta<\lambda} \gamma_{G,\beta}(A) \text{ for all limit ordinals } \lambda \text{ and}\\ \gamma_{G,\kappa}(A) &= [G,\gamma_{G,\kappa}(A)].\end{aligned}$$

It clearly follows from the definition that $\gamma_{G,\alpha}(A)/\gamma_{G,\alpha+1}(A) \leq \zeta_G(A/\gamma_{G,\alpha+1}(A))$ for each ordinal $\alpha < \kappa$.

The last term $\gamma_{G,\kappa}(A)$ of this series is called the *lower G-hypocenter of A* and is denoted by $\gamma_{G,\infty}(A)$.

We say that the FG-module A is *G-hypocentral* if $\gamma_{G,\infty}(A) = 0$.

Now suppose that A has a finite G-central series

$$0 = C_0 \leq C_1 \leq \dots \leq C_{n-1} \leq C_n = A.$$

We have $C_1 \leq \zeta_G(A) = \zeta_{G,1}(A)$. Suppose inductively that we have proved $C_j \leq \zeta_{G,j}(A)$ for all $j \leq k$. Since $C_{k+1}/C_k \leq \zeta_G(A/C_k)$ we have

$$(C_{k+1} + \zeta_{G,k}(A))/\zeta_{G,k}(A) \leq \zeta_G(A/\zeta_{G,k}(A)) = \zeta_{G,k+1}(A)/\zeta_{G,k}(A)$$

so $C_{k+1} \leq \zeta_{G,k+1}(A)$ and hence $C_n \leq \zeta_{G,n}(A)$ which shows that

$$\mathbf{zl}_G(A) \leq n.$$

Furthermore, $\gamma_{G,2}(A) = [G,A] = [G,C_n] \leq C_{n-1}$. Suppose inductively that $\gamma_{G,j}(A) \leq C_{n-j+1}$ for all $j \leq k$. Then

$$\gamma_{G,k+1}(A) = [G,\gamma_{G,k}(A)] \leq [G,C_{n-k+1}] \leq C_{n-k}$$

and in particular we have $\gamma_{G,n+1}(A) = 0$. Thus in this case the length of the lower central series is at most n. It follows that if A is G-nilpotent, then the length of the upper central series and the length of the lower central series coincide. However when the lengths of the upper central series or the lower central series are infinite, the situation is very different. We illustrate this with the following example.

Example 1.27. *Let A be a vector space with countable basis $\{b_n | n \in \mathbb{N}\}$. Define a linear transformation g of A by the infinite matrix*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \dots \end{pmatrix}$$

Thus $gb_1 = b_1$ and $gb_{n+1} = b_n + b_{n+1}$, for each $n \in \mathbb{N}$. Let $G = \langle g \rangle$. It is clear that g is an F-automorphism and that A has upper G-central series

$$0 = \zeta_{G,0}(A) \leq \zeta_{G,1}(A) \leq \dots \zeta_{G,n}(A) \leq \zeta_{G,n+1}(A) \leq \dots \zeta_{G,\omega}(A) = A,$$

where

$$\zeta_{G,n}(A) = Fb_1 \oplus \dots \oplus Fb_n \text{ for all } n \in \mathbb{N}.$$

We see also that $A = [G, A]$, so $\gamma_{G,1}(A) = \gamma_{G,2}(A)$.

On the other hand we define the linear transformation h of A by the infinite matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \dots \end{pmatrix},$$

so $hb_n = b_n + b_{n+1}$ for all $n \in \mathbb{N}$. Setting $H = \langle h \rangle$ we note that h is an F-automorphism of A, which has the lower H-central series

$$A = \gamma_{H,1}(A) \geq \gamma_{H,2}(A) \geq \dots \gamma_{H,n}(A) \geq \gamma_{H,n+1}(A) \geq \dots \gamma_{H,\omega}(A) = 0$$

where $\gamma_{H,n}(A) = \oplus_{j \geq n} Fb_j$, for $n \in \mathbb{N}$. However in this case $\zeta_H(A) = 0$.

Continuing with our general discussion, when A has a finite G-central series

$$0 = C_0 \leq C_1 \leq \dots \leq C_{n-1} \leq C_n = A,$$

then it is easy to see that $(g-1)^n A = 0$, for all $g \in G$. This prompts the following definition.

An element $g \in GL(F, A)$ is called *unipotent* if there is a natural number n such that $(g-1)^n = 0$, the zero linear transformation. Thus the element g is unipotent if and only if the vector space A is $\langle g \rangle$-nilpotent.

If A is a G-nilpotent vector space, then every element of G is unipotent. Of course the converse is not true. Later we give an example showing this, but first we note some important properties enjoyed by unipotent elements.

We introduce some notation that will be useful later. Let g be an element of finite order in the group G, say

$$|g| = p_1^{k_1} \dots p_n^{k_n}, \text{ where } p_1, \dots, p_n$$

are distinct primes. Let $\Pi(g) = \{p_1, \ldots, p_n\}$. Let T be the set of elements of finite order in G and let

$$\Pi(G) = \bigcup_{g \in T} \Pi(g).$$

If π is a set of primes, then we say that G is a *π-group* if G is periodic and $\Pi(G) \subseteq \pi$. Let $\mathbb{P}$ denote the set of all primes and let $\pi' = \mathbb{P} \setminus \pi$. We say that G is a *π'-group* if G is periodic and $\Pi(G) \cap \pi = \emptyset$. As usual when $\pi = \{p\}$, a single prime, then we denote π' by p'. We note that if G is an abelian group, then the set of elements of finite order in G is a subgroup, denoted by $\mathbf{Tor}(G)$.

Lemma 1.28. *Let A be a vector space over the field F.*

(i) If $u \in GL(F, A)$ is unipotent, then u^g is also unipotent for all $g \in GL(F, A)$.

(ii) If $u, v \in GL(F, A)$ are unipotent and $u^v \in \langle u \rangle$, then uv is unipotent.

*(iii) If **char**$(F) = p$, a prime, and $u \in GL(F, A)$ is a p-element, then u is unipotent.*

*(iv) If **char**$(F) = p$, a prime, and $u \in GL(F, A)$ is a unipotent p'-element, then $u = 1$.*

*(v) If **char**$(F) = p$, a prime, and $u \in GL(F, A)$ is a unipotent element of finite order, then u is a p-element.*

Proof. (i) If $u \in GL(F, A)$ is unipotent, then A is $\langle u \rangle$-nilpotent. By Lemma 1.4(viii) it follows that A is also $\langle u^g \rangle$-nilpotent for all $g \in G$, so u^g is also unipotent.

(ii) Since u, v are unipotent it follows that A is both $\langle u \rangle$- and $\langle v \rangle$-nilpotent. By Lemma 1.4(vii) A is therefore $\langle u, v \rangle$-nilpotent and Lemma 1.4(i) implies that A is $\langle uv \rangle$-nilpotent. It follows that uv is unipotent.

(iii) Let $|u| = p^n = k$ and let C^k_j denote the binomial coefficient $k!/j!(k-j)!$. Then

$$(u-1)^k = u^k - C^k_1 u^{k-1} + C^k_2 u^{k-2} + \cdots + (-1)^{k-1} C^k_{k-1} u + (-1)^k.$$

Clearly p divides each of the coefficients C^k_j for $2 \le j \le k-1$. If $p = 2$, then for all $a \in A$ we deduce

$$(u-1)^k a = u^k a + 1a = a + a = 2a = 0$$

and if $p \neq 2$, then

$$(u-1)^k a = u^k a - 1a = a - a = 0,$$

which proves the result.

(iv) Let

$$0 = C_0 \leq C_1 \leq \cdots \leq C_{n-1} \leq C_n = A$$

be the upper $\langle u \rangle$-central series of A and suppose that $C_1 = C_A(u) \neq A$. Then C_2 contains an element a such that $ua \neq a$ and for this element we have $ua = a + c$ for some $c \in C_1$. Since $ua \neq a$ we have $c \neq 0$. Let q be the order of u so that $a = u^q a = a + qc$ from which we deduce that $qc = 0$. However $pc = 0$ and since $\gcd(p, q) = 1$ we have $c = 0$. This contradiction proves the result.

(v) There is a p-element x and a p'-element y such that $u = xy$ and $x, y \in \langle u \rangle$. Using this and Lemma 1.4(i) we deduce that A is $\langle y \rangle$-nilpotent. By (iv) we see that $y = 1$, so $u = x$, a p-element. □

Example 1.29. *Let p be a prime such that $p > 10^{75}$. In [161] A. Yu. Olshanskii constructed a p-group G whose proper subgroups all have prime order p. Let F be a field of prime characteristic p and consider the group algebra FG of G over F. As usual the additive group A of FG is a vector space over F and since $C_G(A) = 1$ we can think of G as a subgroup of $GL(F, A)$. Every element of G has order p, so Lemma 1.28(iii) shows that every element of G is unipotent. Of course the group G is very far from being a nilpotent or even a generalized nilpotent group.*

We note that a product of two unipotent linear transformations is not unipotent in general. That this is so follows using the next simple example.

Example 1.30. *Let A be a 2-dimensional vector space over the prime field $\mathbb{F}_3$ with basis $\{a_1, a_2\}$. Let*

$$x = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$$

define linear transformations of A, each of order 3. It follows that x, y are unipotent by Lemma 1.28(iii), but their product relative to the basis $\{a_1, a_2\}$ is

$$xy = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Clearly $(xy)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is an element of order 2, so xy has order 4. Lemma 1.28(iv) shows that xy cannot be unipotent.

A natural question arises:

- In which linear groups does the set of unipotent elements form a subgroup?

We shall give a partial answer to this question.

We recall that a subgroup H of a group G is called *subnormal* if there is a finite series

$$H = H_0 \leq H_1 \leq H_{n-1} \leq H_n = G$$

in which $H_{i-1} \lhd H_i$ for $1 \leq i \leq n$.

For each subgroup H of G we can construct the *normal closure series*

$$G = H_0 \geq H_1 \geq \dots H_\alpha \geq H_{\alpha+1} \geq \dots H_\gamma$$

as follows: let

$$\begin{aligned} H_1 &= H^G, \\ H_{\alpha+1} &= H^{H_\alpha} \text{ for all ordinals } \alpha, \\ H_\lambda &= \bigcap_{\beta<\lambda} H_\beta \text{ for all limit ordinals } \lambda \text{ and} \\ H_\gamma &= H^{H_\gamma}. \end{aligned}$$

If a subgroup H is subnormal in G, then the normal closure series of H is finite and H is the last term of the series.

Proposition 1.31. *Let A be a vector space over a field F. If $u, v \in GL(F, A)$ are unipotent elements and $\langle u, v\rangle$ is a nilpotent group, then A is $\langle u, v\rangle$-nilpotent.*

Proof. Let $V = \langle u, v\rangle$. Since u, v are unipotent it is evident that A is both $\langle u\rangle$- and $\langle v\rangle$-nilpotent. The group V is nilpotent so $\langle u\rangle$ is subnormal in V and we let

$$V = U_0 \geq U_1 \geq \cdots \geq U_{n-1} \geq U_n = \langle u\rangle$$

be the normal closure series for $\langle u\rangle$. The proof is by induction on the length of this series. If $\langle u\rangle \lhd V$, then A is $\langle u\rangle\langle v\rangle$-nilpotent using Lemma 1.4(vii), as required. Suppose now that $n > 1$. It is clear that $U_{n-1} = \langle U_n^x | x \in U_{n-2}\rangle$. Consider the ascending series

$$U_n^{x_1} \leq U_n^{x_1}U_n^{x_2} \leq \cdots \leq U_n^{x_1}U_n^{x_2}\dots U_n^{x_k} \leq \dots,$$

where the elements $x_j \in U_{n-2}$ are distinct for $j \in \mathbb{N}$. Since V is a finitely generated nilpotent group, it satisfies the maximal condition on subgroups and hence there is a natural number m such that $U_{n-1} = U_n^{x_1}\dots U_n^{x_m}$. Using Lemma 1.4(viii) we see that A is $U_n^{x_j}$-nilpotent for $1 \leq j \leq m$. Repeated use of Lemma 1.4(vii) shows that A is U_{n-1}-nilpotent. However $V = \langle U_{n-1}, v\rangle$ and the normal closure series for U_{n-1} has length $n-1$, so we may apply the induction hypothesis to it. It follows that A is V-nilpotent. □

The next corollary follows immediately.

Corollary 1.32. *Let A be a vector space over a field F. If $u, v \in GL(F, A)$ are unipotent elements and $\langle u, v\rangle$ is a nilpotent group, then uv is also unipotent.*

It is now an easy matter to show that the set of unipotent elements of a locally nilpotent group forms a normal subgroup.

Theorem 1.33. *Let A be a vector space over a field F and let G be a locally nilpotent subgroup of $GL(F, A)$. Then the set of all unipotent elements of G is a normal subgroup of G.*

Proof. If u is a unipotent element of G, then A is $\langle u \rangle$-nilpotent. Since $\langle u \rangle = \langle u^{-1} \rangle$, then u^{-1} is also unipotent. If also v is a unipotent element of G, then $\langle u, v \rangle$ is nilpotent, so uv is unipotent, by Corollary 1.32. Finally, if u is unipotent and $g \in G$, then u^g is unipotent by Lemma 1.28(i). This completes the proof. □

Unipotent elements are an important particular case of the following type of linear transformation. A linear transformation $h \in \mathbf{End}_F(A)$ is called *algebraic over the field F* if h is a root of some polynomial $f_h(X) \in F[X]$. Naturally our main focus will be concerned with elements of $GL(F, A)$ which are algebraic over F.

If $g \in GL(F, A)$ is algebraic over F, then $f_g(g) = 0$, the zero linear transformation. Let $Y = \langle y \rangle$ be an infinite cyclic group. The function $\vartheta : f(y) \mapsto f(g)$ is a homomorphism from the group algebra FY to the subalgebra S_g of $\mathbf{End}_F(A)$ generated by g. The fact that $f_g(g) = 0$ implies that $f_g(y) \in \mathbf{Ann}_{FY}(S_g)$, so that $\mathbf{Ann}_{FY}(S_g) \neq 0$. Furthermore,

$$S_g \cong FY/\mathbf{Ann}_{FY}(S_g)$$

so that $\mathbf{dim}_F(S_g)$ is finite. Conversely, if $\mathbf{dim}_F(S_g)$ is finite, then again using the isomorphism $S_g \cong FY/\mathbf{Ann}_{FY}(S_g)$ we deduce that $\mathbf{Ann}_{FY}(S_g) \neq 0$ and hence there exists a polynomial $f(X) \in F[X]$ such that $f(g)$ is the zero linear transformation.

The following result gives some of the basic properties of algebraic elements. We recall that a field F is locally finite if for every finite subset M of F, there exists a finite subfield K containing M.

Lemma 1.34. *Let A be a vector space over a field F.*

(i) *An element $g \in GL(F, A)$ is algebraic over F if and only if the subalgebra S_g of $\boldsymbol{End}_F(A)$ generated by g is a finite dimensional F-space.*

(ii) *If $g \in GL(F, A)$ is algebraic over F, then each element of $\langle g \rangle$ is algebraic over F.*

(iii) *If $g \in GL(F, A)$ is algebraic over F, then g^v is also algebraic over F for each element $v \in GL(F, A)$.*

(iv) *If $u, v \in GL(F, A)$ are algebraic elements such that $uv = vu$, then the subalgebra of $\boldsymbol{End}_F(A)$ generated by u and v has finite F-dimension. Each element of $\langle u, v \rangle$ is algebraic over F and in particular uv is algebraic.*

(v) *If $u_1, \ldots, u_n \in GL(F, A)$ are algebraic over F and $\langle u_1, \ldots, u_n \rangle$ is abelian, then the subalgebra of $\boldsymbol{End}_F(A)$ which is generated by $u_1, \ldots, u_n$ has finite F-dimension. In particular every element of $\langle u_1, \ldots, u_n \rangle$ is algebraic over F.*

(vi) *Let H be a subgroup of $GL(F, A)$ and let $g \in GL(F, A)$ be an element such that $H^g = H$. If g is algebraic over F and S_H, the subalgebra of $\boldsymbol{End}_F(A)$ generated by H, has finite F-dimension, then the subalgebra generated by $H\langle g \rangle$ has finite F-dimension.*

(vii) *If $g \in GL(F, A)$ has finite order, then g is algebraic over F.*

(viii) *If F is a locally finite field, then each element g of $GL(F, A)$ which is algebraic over F has finite order.*

(ix) *If $g \in GL(F, A)$ is such that g^n is algebraic over F for some natural number n, then g is also algebraic over F.*

Proof. The proof of (i) occurs just before the statement of the lemma and Statement (ii) is a direct consequence. Statement (iii) follows from the fact that if $h \in \mathbf{End}_F(A)$, the mapping $h \longmapsto hv$ is an F-automorphism of $\mathbf{End}_F(A)$.

(iv) We note that the subalgebra S_u, generated by u, is generated as an F-subspace by the elements u^n, for $n \in \mathbb{Z}$. Since u is algebraic over F there is a natural number k such that the set $\{u^0 = 1, u, u^2, \ldots, u^k\}$ forms an F-basis for S_u. Similarly, there is a natural number m such that the set $\{v^0 = 1, v, v^2, \ldots, v^m\}$ forms an F-basis for S_v, the subalgebra generated by v. Let $S_{\langle u,v \rangle}$ be the subalgebra of $\mathbf{End}_F(A)$ generated by S_u and S_v. Clearly, $\langle u, v \rangle$ is abelian so every element of $S_{\langle u,v \rangle}$ has the form $\sum_{1 \le j \le r} x_j y_j$ for certain elements $x_j \in S_u, y_j \in S_v$ and $r \in \mathbb{N}$ and it is then once again easy to see that every element of $S_{\langle u,v \rangle}$ is an F-linear combination of elements of the form $u^s v^t$, where $0 \le s \le k, 0 \le t \le m$. It follows that $S_{\langle u,v \rangle}$ has finite F-dimension and (i) implies that every element of $\langle u, v \rangle$, in particular uv, is algebraic over F.

Statement (v) follows from (iv) by induction.

(vi) Let $\mathcal{B} = \{b_1, \ldots, b_m\}$ be an F-basis for S_H. Since $H^g = H$ we also have $(S_H)^g = S_H$. Since g is algebraic over F, there is a natural number d such that $\{1, g, g^2, \ldots, g^d\}$ forms an F-basis for the subalgebra S_g generated by g. Let $S_g S_H$ denote the vector space generated by all products xy, where $x \in S_g, y \in S_H$. It is easy to see that $S_g S_H$ is generated as an F-vector space by the elements $g^j b_s$, for $0 \le j \le d, 1 \le s \le m$. Hence $S_g S_H$ has finite F-dimension. Let $x \in S_g, y \in S_H$ and consider yx. We have

$$x = \sum_{0 \le j \le d} \lambda_j g^j, y = \sum_{1 \le s \le m} \mu_s b_s,$$

where $\lambda_j, \mu_s \in F$ for $0 \leq j \leq d, 1 \leq s \leq m$. Then

$$yx = \sum_{1 \leq s \leq m, 0 \leq j \leq d} \mu_s \lambda_j b_s g^j.$$

Since $g^{-j} S_H g^j = S_H$ for all j we see that

$$g^{-j} b_s g^j = \sum_{1 \leq i \leq m} \nu_i b_i, \text{ for certain } \nu_i \in F, 1 \leq i \leq m.$$

It follows that $b_s g^j = \sum_{1 \leq i \leq m} \nu_i g^j b_i \in S_g S_H$, for all s, j such that $0 \leq j \leq d, 1 \leq s \leq m$. Consequently, $yx \in S_g S_H$. Now if u, v are arbitrary elements of $S_g S_H$, then

$$u = \sum_{1 \leq i \leq k} x_i y_i, v = \sum_{1 \leq i \leq r} w_i z_i,$$

where $x_j, w_i \in S_g, y_j, z_i \in S_H$, for $1 \leq j \leq k, 1 \leq i \leq r$. Hence we have

$$uv = \left(\sum_{1 \leq j \leq k} x_j y_j \right) \left(\sum_{1 \leq i \leq r} w_i z_i \right) = \sum_{1 \leq j \leq k, 1 \leq i \leq r} x_j y_j w_i z_i$$

and this is clearly an element of $S_g S_H$ since $y_j w_i \in S_g S_H$ by our work above. Hence $S_g S_H$ is a subalgebra of $\mathbf{End}_F(A)$ and furthermore $S_g S_H = S_{H\langle g \rangle}$, the subalgebra of $\mathbf{End}_F(A)$ generated by $H\langle g \rangle$. In particular, it follows that $S_{H\langle g \rangle}$ is of finite F-dimension.

(vii) If g has order k, then clearly g is a root of the polynomial $X^k - 1 \in F[X]$. Hence g is algebraic over F.

(viii) Since g is algebraic over F it is the root of a polynomial $f_g(X) \in F[X]$. Let

$$f_g(X) = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \cdots + \alpha_n X^n, \text{ where } \alpha_0, \ldots, \alpha_n \in F.$$

Let K be the subfield of F generated by the elements $\alpha_0, \ldots, \alpha_n$. Then K is a finite field since F is locally finite. Let $R = K\langle y \rangle$ be the group algebra of an infinite cyclic group $\langle y \rangle$ over K. Then A is an R-module if we define $yd = gd$, for each $d \in A$. The R-submodule Rd generated by d is isomorphic to $R/\mathbf{Ann}_R(d)$ and the choice of g implies that $f_g(g)d = 0$, so $f_g(y) \in \mathbf{Ann}_R(d)$. It is easy to see that $\mathbf{dim}_K(R/\mathbf{Ann}_R(d)) \leq n$ and hence $\mathbf{dim}_K(Rd) \leq n$. On the other hand Rd is the $\langle g \rangle$-invariant subspace, generated by d in the K-space A. If $|K| = q$, it follows that $|Rd| \leq q^n$ and hence the $\langle g \rangle$-orbit, $d^{\langle g \rangle}$, has at most $r = q^n$ elements. However, $|d^{\langle g \rangle}| = |\langle g \rangle : C_{\langle g \rangle}(d)|$, so that $g^r \in C_{\langle g \rangle}(d)$. Since this is true for an arbitrary element $d \in A$, we have $g^r \in C_{\langle g \rangle}(A) = 1$, so g has finite order as required.

(ix) Since $h = g^n$ is algebraic over F, the subalgebra S_h of $\mathbf{End}_F(A)$ generated by the linear transformation h has finite dimension over F by (i). If S_g is the subalgebra of $\mathbf{End}_F(A)$ generated by g, then it is easy to see that

$$S_g = S_h + gS_h + \cdots + g^{n-1} S_h.$$

It follows that $\mathbf{dim}_F(S_g)$ is finite and (i) implies that g is algebraic over F. $\square$

We note that, as with unipotent elements, a product of two algebraic linear transformations need not in general be algebraic. This can be seen from the following simple example.

Example 1.35. *Let $G = X \rtimes D$ be the infinite dihedral group, where $X = \langle x\rangle$ is an infinite cyclic group, $D = \langle d\rangle$ is a group of order 2 and $x^d = x^{-1}$. Let F be an arbitrary field and let FG be the group algebra of G over F. The additive group A of FG is a vector space over F. Since $C_G(A) = 1$ we may think of G as a subgroup of $GL(F, A)$. Put $B = FX$, so that $A = B \oplus dB$. Since $\boldsymbol{Ann}_{FX}(A) = 0$, x cannot be algebraic. However $(dx)^2 = 1$ and Lemma 1.34(vii) implies the elements d, dx are algebraic. But $x = d(dx)$, which illustrates that the product is not algebraic.*

Nevertheless, as in the case of unipotent elements we have the following result, whose proof is similar to that of Proposition 1.31.

Proposition 1.36. *Let A be a vector space over a field F. If $u, v \in GL(F, A)$ are algebraic and $\langle u, v\rangle$ is nilpotent, then the subalgebra $S_{\langle u,v\rangle}$ of $\boldsymbol{End}_F(A)$ generated by the subgroup $\langle u, v\rangle$ has finite F-dimension. In particular, every element of $\langle u, v\rangle$ is algebraic over F.*

Proof. Let $V = \langle u, v\rangle$ and note that $\langle u\rangle$ is subnormal in V since V is nilpotent. Let

$$V = U_0 \geq U_1 \geq \cdots \geq U_{n-1} \geq U_n = \langle u\rangle$$

be the normal closure series for $\langle u\rangle$. The proof is by induction on the length of this series. If $\langle u\rangle \triangleleft V$, then the result follows from Lemma 1.34(i) and (vi).

Suppose now that $n > 1$. It is clear that $U_{n-1} = \langle U_n^x | x \in U_{n-2}\rangle$. Consider the ascending series

$$U_n^{x_1} \leq U_n^{x_1} U_n^{x_2} \leq \cdots \leq U_n^{x_1} U_n^{x_2} \dots U_n^{x_k} \leq \dots,$$

where the elements $x_j \in U_{n-2}$ are arbitrary but distinct, for $j \in \mathbb{N}$. Since V is a finitely generated nilpotent group, it satisfies the maximal condition and hence there is a natural number m such that $U_{n-1} = U_n^{x_1} \dots U_n^{x_m}$. Using Lemma 1.34(v) and (iii) we see that the subalgebra of $\mathbf{End}_F(A)$ generated by U_{n-1} has finite F-dimension. However, $V = \langle U_{n-1}, v\rangle$, so using the same argument, after finitely many steps we deduce that the subalgebra of $\mathbf{End}_F(A)$ generated by V has finite F-dimension. Lemma 1.34(i) shows that every element of V is algebraic over F. □

Corollary 1.37. *Let A be a vector space over a field F. If the elements $u, v \in GL(F, A)$ are algebraic and if $\langle u, v\rangle$ is a nilpotent subgroup, then uv is also an algebraic element over F.*

Corollary 1.38. *Let A be a vector space over a field F and let $u_1, \dots, u_n \in GL(F, A)$ be algebraic elements over F. If $\langle u_1, \dots, u_n\rangle$ is a nilpotent subgroup, then the subalgebra of $\boldsymbol{End}_F(A)$ generated by $\langle u_1, \dots, u_n\rangle$ has finite F-dimension.*

Proof. The proof follows using a simple induction argument, Proposition 1.36 and Lemma 1.34(vi). □

The next result, analogous to Theorem 1.33, appears in the paper of M. P. Sedneva [189].

Theorem 1.39. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If G is locally nilpotent, then the subset K of all elements of G which are algebraic over F is a normal subgroup of G. Furthermore, A is K-locally (finite dimensional).*

Proof. If u is an F-algebraic element of G, then Lemma 1.34(ii) implies that u^{-1} is also F-algebraic. If v is also an F-algebraic element of G, then the subgroup $\langle u, v\rangle$ is nilpotent and uv is an algebraic element by Corollary 1.37. Also if $g \in G$, then u^g is F-algebraic by Lemma 1.34(iii). This proves that K is a normal subgroup of G.

Let H be a finitely generated subgroup of K. We may assume that H is infinite. By Corollary 1.38, the subalgebra S_H of $\mathbf{End}_F(A)$ generated by H has finite F-dimension. Let FH be the group algebra of H over F and define the function $\Psi : FH \longrightarrow S_H$ as follows. Each element z of FH is an F-linear combination of the elements of H and we define $\Psi(z)$ to be the same linear combination, but as an element of $\mathbf{End}_F(A)$. It is easy to see that Ψ is an epimorphism so that $S_H \cong FH/\ker(\Psi)$. Since H is infinite and $\mathbf{dim}_F(S_H)$ is finite it follows that $\ker(\Psi) \neq 0$. If $w \in \ker(\Psi)$, then $\Psi(w)$ is the zero endomorphism.

Now let $d \in A$ be an arbitrary element and let D be the H-invariant closure of the F-subspace Fd. Then D is a cyclic FH-submodule of the FH-module A and $D \cong FH/\mathbf{Ann}_{FH}(d)$. It is easy to see that $\ker(\Psi) \leq \mathbf{Ann}_{FH}(d)$, so $FH/\mathbf{Ann}_{FH}(d)$ has finite F-dimension. It follows that $\mathbf{dim}_F(D)$ is also finite.

If M is a finite subset of A, then $HM = \sum_{d\in M} FHd$. By our work above, each FH-subspace FHd has finite F-dimension and the finiteness of M implies that HM also has finite dimension. This proves the result. □

An important special case of algebraic F-automorphisms concerns the *finitary linear transformations.* Let $g \in G$ be arbitrary and consider the mapping $\tau_g : A \longrightarrow A$ defined by $\tau_g(a) = (g-1)a$, for $a \in A$. This mapping is clearly linear and $\mathbf{Im}\,(\tau_g) = (g-1)A$ is the F-subspace of A generated by the elements $(g-1)a$. Furthermore, $\mathbf{ker}(\tau_g) = C_A(g)$, so that

$$(g-1)A = \mathbf{Im}\,(\tau_g) \cong A/\mathbf{ker}(\tau_g) = A/C_A(g).$$

This isomorphism is easily seen to be an FZ-isomorphism, where $Z = C_A(g)$.

A linear transformation $g \in GL(F, A)$ is called *finitary* if $A/C_A(g)$ has finite F-dimension. Equivalently, this means $\mathbf{dim}_F(g-1)A$ is finite. We denote the subset of $GL(F, A)$ consisting of finitary linear transformations by $FGL(F, A)$. Since $C_A(g) = C_A(g^{-1})$ and $C_A(g) \cap C_A(f) \leq C_A(gf)$ for all elements $f, g \in GL(F, A)$ it follows that $FGL(F, A)$ is a subgroup of $GL(F, A)$.

Furthermore, if g is finitary and $x \in GL(F, A)$, then g^x is also finitary, so that $FGL(F, A) \lhd GL(F, A)$. We note also that every finitary F-automorphism g is F-algebraic. In fact, since $\mathbf{dim}_F(A/C_A(g))$ is finite there is a polynomial $f_g(X) \in F[X]$ such that $f_g(g)$ induces the zero endomorphism on $A/C_A(g)$. Thus, for all elements $a \in A$, $f_g(g)a \in C_A(g)$ and hence $(g-1)f_g(g)a = 0$. Hence $(g-1)f_g(g)$ is the zero linear transformation. Thus g is a zero of $(X-1)f_g(X) \in F[X]$.

Although the class of finitary linear groups forms an important class of linear groups, this class and its properties will not play a major role in this book. Many articles have been written concerning the class of finitary linear groups including the work of V. V. Belyaev [9], S. Black [15], B. Bruno, M. Dalle Molle and F. Napolitani [24], B. Bruno and R. E. Phillips [25], J. I. Hall and B. Hartley [72], F. Leinen [136, 137, 135], F. Leinen and O. Puglisi [138, 139, 140], M. S. Lucido [141], U. Meierfrankenfeld [147], U. Meierfrankenfeld, R. E. Phillips and O. Puglisi [148], O. Puglisi [179, 182, 181, 180], and B. A. F. Wehrfritz [203, 205, 206, 204, 207, 208, 209, 210, 211, 213, 212, 214, 216, 217, 218]. Some other papers in this interesting branch of group theory will be mentioned occasionally throughout this book.

Series of G-Invariant Subspaces

Suppose, as usual, that A is a vector space over the field F and that $G \leq GL(F, A)$. Let C, D be G-invariant subspaces of A such that $D \leq C$. The factor C/D is called a *G-chief factor* if, whenever B is a G-invariant subspace such that $D \leq B \leq C$, it follows that either $B = D$ or $B = C$.

We recall that a non-zero module A over a ring R is called *simple*, if every non-zero submodule of A coincides with A.

When we think of A as an FG-module, this means that a G-chief factor C/D is a simple FG-module.

A subgroup G of $GL(F, A)$ is called *irreducible* if A contains no proper non-zero G-invariant subspaces. In this case A is G-chief and a simple FG-module.

Suppose that A has finite dimension over F. If A contains a proper non-zero G-invariant subspace then we can choose a non-zero G-invariant subspace B_1 of least dimension. It is clear by this choice that B_1 is a simple FG-module, so B_1 is G-chief. If $B_1 \neq A$, then in the quotient space A/B_1, we may choose a non-zero G-invariant subspace B_2/B_1 of least dimension, so that B_2/B_1 is G-chief. Continuing in this fashion, since A has finite dimension, we can construct a finite series

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n = A \tag{1.1}$$

of G-invariant subspaces whose factors are G-chief. It follows that the groups $G/C_G(B_j/B_{j-1})$ are irreducible, for $1 \leq j \leq n$. Let

$$Z = C_G(B_1) \cap C_G(B_2/B_1) \cap \cdots \cap C_G(B_n/B_{n-1}).$$

Then the series (1.1) is Z-central and Theorem 1.2 shows that Z is nilpotent. By Remak's theorem, G/Z is isomorphic to a subgroup of the direct product

$$G/C_G(B_1) \times G/C_G(B_2/B_1) \times \cdots \times G/C_G(B_n/B_{n-1})$$

of finitely many irreducible groups.

Thus immediately, two important trends in the the theory of finite dimensional linear groups can be identified, namely the theory of irreducible linear groups and the theory of stability groups.

If the vector space A has infinite dimension over F, then such an approach is no longer quite so practical since the situation here is much more complicated, as we shall see. As a matter of fact, finite series of G-invariant subspaces will not always occur in this case. Indeed, the following much more extensive systems of subgroups are more likely to occur.

Let $\mathcal{S}$ be a family of G-invariant subspaces of A, linearly ordered by inclusion. Then $\mathcal{S}$ is called a *complete system* if it satisfies the condition:

- For every subfamily $\mathcal{L}$ of $\mathcal{S}$ the intersection and union of all members of $\mathcal{L}$ belong to $\mathcal{S}$.

Proposition 1.40. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If $\mathcal{S}$ is a linearly ordered family of G-invariant subspaces of A, ordered by inclusion, then $\mathcal{S}$ can be extended to a complete system.*

Proof. We may assume that $\mathcal{S}$ is not complete, so we may assume that $\mathcal{S}$ contains a subset $\mathcal{M}$ such that either $\bigcup \mathcal{M} \notin \mathcal{S}$ or $\bigcap \mathcal{M} \notin \mathcal{S}$. Let $\mathcal{S}_0 = \mathcal{S}$ and suppose that $U = \bigcup \mathcal{M} \notin \mathcal{S}$. Set $\mathcal{S}_1 = \mathcal{S}_0 \bigcup \{U\}$. The subfamily $\mathcal{S}_1$ is linearly ordered. To see this let $B, C \in \mathcal{S}_1$. If $B, C \in \mathcal{S}_0$, then either $B \leq C$ or $C \leq B$, because $\mathcal{S}_0$ is linearly ordered. On the other hand, suppose that $B \in \mathcal{S}_0$ and $C = U$. If $M \leq B$ for all $M \in \mathcal{M}$, then $C = U = \bigcup \mathcal{M} \leq B$ as required. If there exists $M_1 \in \mathcal{M}$ and $M_1 \nleq B$, then $B \leq M_1$, since $\mathcal{S}_0$ is linearly ordered. Thus $B \leq \bigcup \mathcal{M} = U = C$ and this shows that $\mathcal{S}_1$ is linearly ordered.

Suppose now that $\bigcup \mathcal{M} \in \mathcal{S}$, but $V = \bigcap \mathcal{M} \notin \mathcal{S}$. In this case we let $\mathcal{S}_1 = \mathcal{S}_0 \bigcup \{V\}$ and again show that $\mathcal{S}_1$ is linearly ordered. Suppose that $B, C \in \mathcal{S}_1$. If $B, C \in \mathcal{S}_0$, then either $B \leq C$ or $C \leq B$, since $\mathcal{S}_0$ is linearly ordered. Suppose that $B \in \mathcal{S}_0$ and $C = V$. If $B \leq M$ for every $M \in \mathcal{M}$, then $B \leq \bigcap \mathcal{M} = V = C$. If there exists $M_1 \in \mathcal{M}$ such that $B \nleq M_1$, then $M_1 \leq B$, as $\mathcal{S}_0$ is linearly ordered. However $\bigcap \mathcal{M} \leq M_1$, so $C = V \leq B$.

Suppose now that we have constructed such families $\mathcal{S}_\beta$ for all ordinals $\beta < \alpha$. If $\alpha - 1$ exists and $\mathcal{S}_{\alpha-1}$ is not complete, then $\mathcal{S}_{\alpha-1}$ contains a subset $\mathcal{M}$ such that either $\bigcup \mathcal{M} \notin \mathcal{S}_{\alpha-1}$ or $\bigcap \mathcal{M} \notin \mathcal{S}_{\alpha-1}$. In this case we construct a family $\mathcal{S}_\alpha$ as above. Thus we may suppose that α is a limit ordinal. In this case set $\mathcal{S}_\alpha = \bigcup_{\beta<\alpha} \mathcal{S}_\beta$. Let B, C be arbitrary subspaces in the family $\mathcal{S}_\alpha$. Then there are ordinals $\beta, \nu < \alpha$ such that $B \in \mathcal{S}_\beta$ and $C \in \mathcal{S}_\nu$. We have either $\beta \leq \nu$ or $\nu \leq \beta$. In the first case, $B, C \in \mathcal{S}_\nu$ and in the second case $B, C \in \mathcal{S}_\beta$. Since $\mathcal{S}_\nu$ and $\mathcal{S}_\beta$ are linearly ordered, it follows that $\mathcal{S}_\alpha$ is also linearly ordered. This process will terminate at some ordinal γ. As a result, $\mathcal{S}_\gamma$ is a complete family. □

Let $\mathcal{S}$ be a family of G invariant subspaces of A. Then a pair $(B, C) \in \mathcal{S}$, where $B \leq C$, is a *jump* of $\mathcal{S}$ if it satisfies the condition:

- For every element $D \in \mathcal{S}$ such that $B \leq D \leq C$, either $B = D$ or $D = C$.

If the pair (B, C) is a jump of $\mathcal{S}$, then C/B is called a *factor of the system* $\mathcal{S}$.

Suppose now that $\mathcal{S}$ is a complete system of G-invariant subspaces of A. For each $0 \neq a \in A$ let

$$V_a = \bigcup\{V \in \mathcal{S} | a \notin V\} \text{ and}$$
$$\Lambda_a = \bigcap\{V \in \mathcal{S} | a \in V\}$$

Since $\mathcal{S}$ is a complete system it follows that $V_a, \Lambda_a \in \mathcal{S}$ and since $\mathcal{S}$ is a linearly ordered system, either $V_a \leq \Lambda_a$ or $\Lambda_a \leq V_a$. However, $a \in \Lambda_a$ and $a \notin V_a$ so we must have $V_a \leq \Lambda_a$ and it is then easy to see that the pair (V_a, Λ_a) is a jump of the system $\mathcal{S}$. Conversely, let (B, C) be a jump of the system $\mathcal{S}$. Choose $y \in C \setminus B$. From the definitions of V_y and Λ_y we see that $B \leq V_y$ and $\Lambda_y \leq C$. Since $y \in C$ it follows that $V_y \neq C$ and since (B, C) is a jump, we deduce that $B = V_y$. Also, since $\Lambda_y \neq B$ and (B, C) is a jump, we deduce that $\Lambda_y = C$.

When A is a vector space over the field F and G is a subgroup of $GL(F, A)$, a family $\mathcal{S}$ of G-invariant subspaces of A is called a *Kurosh-Chernikov system*, if it satisfies the conditions

(KC1) $0, A \in \mathcal{S}$;

(KC2) $\mathcal{S}$ is linearly ordered and complete.

Families of this kind were originally introduced in the classical article of A. G. Kurosh and S. N. Chernikov [132] concerning groups with operators. We observe that vector spaces are examples of groups with operators.

If $\mathcal{S}, \mathcal{R}$ are families of G-invariant subgroups, linearly ordered by inclusion, then we say that $\mathcal{R}$ is a *refinement* of $\mathcal{S}$ if every term of $\mathcal{S}$ is also a term of $\mathcal{R}$ (thus, $\mathcal{S} \subseteq \mathcal{R}$). If $\mathcal{S}$ is a proper subset of $\mathcal{R}$, then $\mathcal{R}$ is called a *proper refinement* of $\mathcal{S}$.

Proposition 1.41. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that $\mathcal{S}$ is a complete system of G-invariant* *subspaces of A. If $\mathcal{R}$ is a proper refinement of $\mathcal{S}$, which is also a complete system, then $\mathcal{R}$ contains a term B and $\mathcal{S}$ has a jump (D, E) such that $D \lneqq B \lneqq E$.*

Proof. Since $\mathcal{R}$ is a proper refinement of $\mathcal{S}$, then we can choose a G-invariant subspace B such that $B \in \mathcal{R} \setminus \mathcal{S}$. Let

$$V_B = \bigcup\{X \in \mathcal{S} | B \nleqq X\} \text{ and}$$
$$\Lambda_B = \bigcap\{X \in \mathcal{S} | B \leq X\}$$

Since $\mathcal{S}$ is a complete system $V_B, \Lambda_B \in \mathcal{S}$. Since $\mathcal{R}$ is linearly ordered, we easily see that $V_B \leq B \leq \Lambda_B$. Suppose that $C \in \mathcal{S}$ and that $V_B \leq C \leq \Lambda_B$. Since $\mathcal{R}$ is linearly ordered, either $C \leq B$ or $B \leq C$. In the first case, since $B \notin \mathcal{S}$, we deduce that $B \nleqq C$, so $C \leq V_B$ and hence $C = V_B$. In the second case we see similarly that $C = \Lambda_B$. It follows that (V_B, Λ_B) is a jump of the system $\mathcal{S}$. Then since $B \notin \mathcal{S}$ we deduce that

$$B \neq V_B = D \text{ and } B \neq \Lambda_B = E.$$

This completes the proof. □

The Kurosh-Chernikov system $\mathcal{S}$ is called *G-chief* if it has no proper refinement.

Corollary 1.42. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that $\mathcal{S}$ is a Kurosh-Chernikov system of G-invariant subspaces of A. Then $\mathcal{S}$ is G-chief if and only if every factor of $\mathcal{S}$ is G-chief.*

Proof. Suppose first that $\mathcal{S}$ is G-chief and that it has a jump (B, C) such that C/B is not a G-chief factor. Then there is a G-invariant subspace D such that $B < D < C$. Let $\mathcal{S}_1 = \mathcal{S} \bigcup \{D\}$. By Proposition 1.40, $\mathcal{S}_1$ can be extended to a complete system $\mathcal{R}$ which is clearly a Kurosh-Chernikov system. Since $D \in \mathcal{R}$ we deduce that $\mathcal{R} \neq \mathcal{S}$, which is a contradiction.

Conversely, suppose that every factor of $\mathcal{S}$ is G-chief, but that $\mathcal{S}$ is not G-chief. Then there is a proper refinement $\mathcal{S}_1$ of $\mathcal{S}$. By Proposition 1.40, $\mathcal{S}_1$ can be extended to a complete system $\mathcal{R}$. Clearly $\mathcal{R}$ is a Kurosh-Chernikov system and since $\mathcal{S}_1 \neq \mathcal{S}$ we have $\mathcal{R} \neq \mathcal{S}$. Using Proposition 1.41 we see that $\mathcal{R}$ contains a term B and $\mathcal{S}$ has a jump (D, E) such that $D \lneqq B \lneqq E$. On the other hand the factor E/D is G-chief, so either $D = B$ or $B = E$ and hence $B \in \mathcal{S}$, again a contradiction. This completes the proof.

□

Theorem 1.43. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Every family of G-invariant subspaces of A, linearly ordered by inclusion, can be extended to a G-chief Kurosh-Chernikov system.*

Proof. Let $\mathcal{S}$ be a family of G-invariant subspaces of A, linearly ordered by inclusion. Then $\mathcal{S}\bigcup\{0, A\}$ is also linearly ordered and by Proposition 1.40 $\mathcal{S}\bigcup\{0, A\}$ can be extended to a complete system $\mathcal{S}_1$. Clearly $\mathcal{S}_1$ is a Kurosh-Chernikov system and if $\mathcal{S}_1$ is G-chief, then the result follows. Suppose, therefore, that $\mathcal{S}_1$ is not a G-chief system. Then Corollary 1.42 shows that $\mathcal{S}_1$ has a jump (B, C) such that C/B is not a G-chief factor. It follows that there is a G-invariant subspace D of A such that $B \lneqq D \lneqq C$. By Proposition 1.40 again the linearly ordered system $\mathcal{S}_1\bigcup\{D\}$ can be extended to a complete system $\mathcal{S}_2$, which is clearly a Kurosh-Chernikov system.

Suppose that we have already constructed an ascending chain

$$\mathcal{S} = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \dots \mathcal{S}_\beta \subseteq \mathcal{S}_{\beta+1} \subseteq \dots$$

of Kurosh-Chernikov systems for all ordinals $\beta < \alpha$ such that $\mathcal{S}_\beta$ is not G-chief for all $\beta < \alpha$. If $\mathcal{S}_{\alpha-1}$ exists, then as above, Corollary 1.42 shows that $\mathcal{S}_{\alpha-1}$ has a jump (U, V) such that V/U is not a G-chief factor. Hence there is a G-invariant subspace W of A such that $U \lneqq W \lneqq V$ and a further application of Proposition 1.40 shows that $\mathcal{S}_{\alpha-1}\bigcup\{W\}$ can be extended to a complete system $\mathcal{S}_\alpha$, which is clearly a Kurosh-Chernikov system. On the other hand, if α is a limit ordinal, we let $\mathcal{R} = \bigcup_{\beta<\alpha} \mathcal{S}_\beta$, which is easily seen to be linearly ordered. Again by Proposition 1.40 $\mathcal{R}$ can be extended to a complete system $\mathcal{S}_\alpha$, which once again is a Kurosh-Chernikov system.

This process of construction of the system $\mathcal{S}_\alpha$ must terminate at some ordinal γ and then $\mathcal{S}_\gamma$ is the required G-chief system. □

We next consider some important types of Kurosh-Chernikov systems $\mathcal{S}$. Suppose first that $\mathcal{S}$ is a totally ordered ascending family of subspaces. Then $\mathcal{S} \setminus \{0\}$ has a minimal element A_1. In turn $\mathcal{S} \setminus \{0, A_1\}$ has a minimal element A_2 and so on. If $\mathcal{S}$ is finite, then this process will terminate after finitely many steps and we obtain a finite series

$$0 = A_0 \leq A_1 \leq \dots \leq A_k \leq A_{k+1} = A$$

of G-invariant subspaces. On the other hand if $\mathcal{S}$ is infinite, then we obtain an ascending series of the type

$$0 = A_0 \leq A_1 \leq \dots \leq A_n \leq A_{n+1} \leq \dots$$

when two possibilities arise. Either

$$\mathcal{S} \setminus \{0, A_n | n \in \mathbb{N}\} = A \text{ or } \mathcal{S} \setminus \{0, A_n | n \in \mathbb{N}\} \neq A.$$

Since $\mathcal{S}$ is complete, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$ and we note that $\bigcup_{n\in\mathbb{N}} A_n \neq A_j$, for each $j \in \mathbb{N}$. In the first case we have that $\bigcup_{n\in\mathbb{N}} A_n = A$. In the second case, the family $\mathcal{S} \setminus \{0, A_n | n \in \mathbb{N}\}$ is non-empty and we choose an arbitrary term B of this system. Then $A_j \leq B$ for each $j \in \mathbb{N}$, so $\bigcup_{n\in\mathbb{N}} A_n \leq B$. Since

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S} \setminus \{0, A_n | n \in \mathbb{N}\},$$

it follows that $\bigcup_{n\in\mathbb{N}} A_n$ is a minimal element of the family $\mathcal{S} \setminus \{0, A_n | n \in \mathbb{N}\}$. We set $A_\omega = \bigcup_{n\in\mathbb{N}} A_n$ and using the arguments above we construct an ascending series

$$\begin{aligned} 0 = A_0 \le A_1 \le \cdots \le A_n \le A_{n+1} \le \ldots A_\omega \le A_{\omega+1} \le \ldots \\ \le A_\alpha \le A_{\alpha+1} \le \ldots A_\gamma = A, \end{aligned}$$

where for limit ordinals ν we have $\bigcup_{\beta<\nu} A_\beta = A_\nu$.

On the other hand, if a Kurosh-Chernikov system $\mathcal{S}$ is a descending totally ordered system, then we obtain in similar fashion a descending series

$$\begin{aligned} A = A_0 \ge A_1 \ge \cdots \ge A_n \ge A_{n+1} \ge \ldots A_\omega \ge A_{\omega+1} \\ \ge \ldots A_\alpha \ge A_{\alpha+1} \ge \ldots A_\gamma = 0 \end{aligned}$$

where for limit ordinals ν we now have $A_\nu = \bigcap_{\beta<\nu} A_\beta$.

The Kurosh-Chernikov system $\mathcal{S}$ is called *G-central*, if every factor is G-central. We note that we have already discussed one special case of this, namely ascending G-central series.

Now let $\mathcal{S}$ be a Kurosh-Chernikov system and let $\mathcal{T}$ be the set of all jumps of $\mathcal{S}$. For each jump (B, C) of $\mathcal{S}$ we let

$$\mathrm{St}_G(\mathcal{S}) = \bigcap_{(B,C)\in\mathcal{T}} C_G(C/B).$$

This subgroup $\mathrm{St}_G(\mathcal{S})$ is called the *stability group of the system* $\mathcal{S}$ or the *stabilizer of* $\mathcal{S}$. With this definition, $\mathcal{S}$ is a $\mathrm{St}_G(\mathcal{S})$-central system and conversely, if a Kurosh-Chernikov system is G-central, then $G = \mathrm{St}_G(\mathcal{S})$.

As in the finite dimensional case, Remak's theorem implies the embedding of the factor group $G/\mathrm{St}_G(\mathcal{S})$ into the Cartesian product $\underset{(B,C)\in\mathcal{T}}{\mathrm{Cr}}\, G/C_G(C/B)$. In particular, if $\mathcal{S}$ is G-chief, then each of the factor groups $G/C_G(C/B)$ is irreducible. However there are many differences with the finite dimensional case here.

The most obvious difference is that in general the set of components in the Cartesian product will be infinite.

In this case, we are no longer dealing with subgroups of a direct product, but with subgroups of a Cartesian product. This alone makes the task of studying such groups much more complicated.

The situation with the subgroup $\mathrm{St}_G(\mathcal{S})$ is also much more complicated here. As we noted above, in the case when A has finite dimension, this subgroup is always nilpotent. But as we have seen above, the stabilizer of an ascending G-central series can contain free groups, as well as other types of very exotic groups. Thus in the case when A has infinite dimension, $\mathrm{St}_G(\mathcal{S})$ may be very far from being nilpotent.

In this chapter we have given some of the basic tools and definitions that will prove useful for the study of infinite dimensional linear groups. We have also tried to illustrate some of the differences with the theory of finite dimensional linear groups. Next we discuss irreducible groups.

Chapter 2

Irreducible Linear Groups

In this chapter we shall consider the important topic of *irreducible linear groups*. We first obtain the following important corollary of Theorem 1.43.

Theorem 2.1. *Let A be vector space over a field F and let G be a subgroup of $GL(F, A)$. Then A has a G-chief Kurosh-Chernikov system $\mathcal{S}$.*

Proof. The family $0 \leq A$ of G-invariant subspaces of A is linearly ordered by inclusion, so Theorem 1.43 implies that this family can be extended to a G-chief Kurosh-Chernikov system $\mathcal{S}$. □

According to Corollary 1.42 every factor V/U of the G-chief Kurosh-Chernikov system $\mathcal{S}$ is G-chief. Then $G/C_G(V/U)$ is an irreducible linear subgroup of $GL(F, V/U)$. In this way, as we mentioned in Chapter 1, we obtain two important types of linear groups, namely irreducible linear groups and the stability group of the system $\mathcal{S}$. As we saw in Chapter 1, in contrast to the case of finite dimensional linear groups, stability groups are very far from being nilpotent and their structure can be very complicated. The theory of finite dimensional linear groups has evolved over many decades, resulting in a large array of articles and a very advanced, sophisticated theory. For infinite dimensional irreducible linear groups the situation is also significantly different from the finite dimensional case where the use of powerful techniques from matrix theory is not always possible. Consequently the list of infinite dimensional irreducible linear groups and results relating to such groups is not so broad.

The Construction of Simple FG-modules

Let F be a field, let A be a vector space over F and let G be a subgroup of $GL(F, A)$. Then we can think of A as an FG-module and if G is an irreducible linear group, then A is a simple FG-module. There is a vast literature concerned with the description of finite dimensional (in particular, finite) irreducible groups. Of course here we are interested in the case when $\mathbf{dim}_F(A)$ is infinite and, in this case, G will itself be infinite.

We start with the following easily proved well-known classical result, which immediately indicates approaches to the description of irreducible linear groups.

Proposition 2.2 (Schur's Lemma). *Let R be a ring and let A be a simple R-module. Then $\mathbf{End}_R(A)$ is a division ring.*

Proof. Let $f \in \mathbf{End}_R(A)$. Then $\mathbf{ker}(f)$ is an R-submodule of A and if f is a non-zero mapping, then $\mathbf{ker}(f) \neq A$. Since A is a simple R-module, it follows that $\mathbf{ker}(f) = 0$, so f is a monomorphism. Similarly $\mathbf{Im}(f)$ is an R-submodule of A. Since f is non-zero, it follows that $\mathbf{Im}(f) \neq 0$ and the simplicity of A implies that $\mathbf{Im}(f) = A$. Consequently the mapping f is also an epimorphism, so f is an automorphism of A as required. $\square$

We here recall that if G is an abelian group, then $\mathbf{Tor}(G)$ is the subgroup of elements of G that have finite order.

Corollary 2.3. *Let R be an integral domain, let G be a group and let A be a simple RG-module. Let $I = \mathbf{Ann}_{RG}(A)$. Then*

(i) The center C/I of the ring RG/I is an integral domain;

(ii) The group $\zeta(G/C_G(A))$ is isomorphic to the multiplicative group of a field. In particular, the torsion subgroup $\mathbf{Tor}(\zeta(G/C_G(A))$ is a locally cyclic group which is a p'-group if $p = \mathbf{char}(R) > 0$.

Proof. (i) Let $c \in C$ and define the function

$$\tau_c : A \longrightarrow A \text{ by } \tau_c(a) = ca, \text{ for all } a \in A.$$

The definition of C makes it easy to see that τ_c is an RG-endomorphism of A. Now let

$$\Phi : C \longrightarrow \mathbf{End}_{RG}(A) \text{ be defined by } \Phi(c) = \tau_c, \text{ for all } c \in C.$$

Let c, d be elements of C and let a be an element of A. We have

$$\begin{aligned} \tau_{c+d}(a) &= (c+d)a = ca + da = \tau_c(a) + \tau_d(a) \text{ and} \\ \tau_{cd}(a) &= (cd)a = c(da) = c\tau_d(a) = \tau_c(\tau_d(a)) = (\tau_c \circ \tau_d)(a). \end{aligned}$$

It follows that $\Phi(c+d) = \Phi(c) + \Phi(d)$ and $\Phi(cd) = \Phi(c) \circ \Phi(d)$, so that Φ is a ring homomorphism. If $\tau_c(a) = ca = 0$ for all $a \in A$, then $c \in \mathbf{Ann}_{RG}(A) = I$. If a is an arbitrary element of A, then for all $c \in C$ and every RG-endomorphism f we have

$$(f \circ \tau_c)(a) = f(\tau_c(a)) = f(ca) = cf(a) = \tau_c(f(a)) = (\tau_c \circ f)(a),$$

which shows that $f \circ \tau_c = \tau_c \circ f$. Hence the center Z of $\mathbf{End}_{RG}(A)$ contains $\mathbf{Im}(\Phi)$. By Proposition 2.2, $\mathbf{End}_{RG}(A)$ is a division ring and hence Z is a field.

Consequently it follows that $\mathbf{Im}(\Phi) \cong C/I$ is an integral domain. Clearly the characteristics of Z and R are the same. This completes the proof of (i).

(ii) Now let
$$Y/C_G(A) = \zeta(G/C_G(A))$$
and let τ_y be the RG-endomorphism defined above. The function Φ induces a mapping
$$\Psi : Y \longrightarrow U(Z),$$
where $\Psi(y) = \tau_y$ for $y \in Y$ and $U(Z)$ is the unit group of Z. We know that $\Psi(gh) = \Psi(g) \circ \Psi(h)$ for all $g, h \in Y$, so that Ψ is a group homomorphism. Furthermore, if $y \in \mathbf{ker}(\Psi)$, then $\tau_y(a) = ya = a$ for all $a \in A$, so that $y \in C_G(A)$. Hence $\mathbf{ker}(\Psi) = C_G(A)$ and it follows that $Y/C_G(A)$ is isomorphic to a subgroup of $U(Z)$. To finish the proof of (ii) we note that the torsion subgroup of the mutiplicative group of a field of characteristic p is a locally cyclic p'-group (see the book of G. Karpilovsky [98, Proposition 4.1], for example). □

We can obtain yet more information in the case when the group G is abelian. We shall use some of the notation established in the proof of Corollary 2.3 in the proof of the next result. In the abelian case of course $C = RG$ and $G/C_G(A)$ is isomorpic to the multiplicative group of a field.

Corollary 2.4. *Let F be a field, let G be an abelian group and let A be a simple FG-module. Then*

(i) $\mathbf{End}_{FG}(A)$ *is a field extension of* F*;*

(ii) If $f \in \mathbf{End}_{FG}(A)$*, then* $f = \tau_c$*, for some element* $c \in FG$*;*

(iii) $A \cong \mathbf{End}_{FG}(A)$*, as* FG*-modules.*

Proof. We first prove (ii). As in the proof of Corollary 2.3, we let $\tau_c : A \longrightarrow A$ be the FG-endomorphism of A defined by $\tau_c(a) = ca$, if $c \in FG$ and $a \in A$. We let $0 \neq a \in A$ be arbitrary and let $f \in \mathbf{End}_{FG}(A)$. Since A is a simple FG-module we have $A = FGa$ and hence there exists $c \in FG$ such that $f(a) = ca$. If $b \in A$, then there also exists $y \in FG$ such that $b = ya$ and we have, since G is abelian,
$$f(b) = f(ya) = yf(a) = y(ca) = (cy)a = c(ya) = cb = \tau_c(b).$$
Hence (ii) follows.

Next we prove (i) and note that if α is an element of F, then the function τ_α is an FG-endomorphism of A and the mapping $\xi : \alpha \longmapsto \tau_\alpha$ is a ring monomorphism from F into $\mathbf{End}_{FG}(A)$. Hence $F \cong \mathbf{Im}(\xi)$ is a subfield of $\mathbf{End}_{FG}(A)$.

Let $u, v \in \mathbf{End}_{FG}(A)$. The work above shows that $u = \tau_c, v = \tau_d$ for certain elements $c, d \in FG$. Then
$$u \circ v = \tau_c \circ \tau_d = \tau_{cd} = \tau_{dc} = v \circ u$$

and hence $\mathbf{End}_{FG}(A)$ is commutative. Since $\mathbf{End}_{FG}(A)$ is a division ring, by Proposition 2.2, it follows that it is a field. Hence (i) follows.

Finally let $d \in A$ be fixed. Since A is a simple FG-module it follows that $A = (FG)d$. Then for an arbitrary element $a \in A$, there is an element $x \in FG$ such that $a = xd$. We now define

$$\Omega : A \longrightarrow \mathbf{End}_{FG}(A) \text{ by } \Omega(a) = \tau_x.$$

We need to check that this map is well-defined. To see this let $y \in FG$ be another element such that $a = yd$. Then $(y - x)d = 0$, so $z = y - x \in \mathbf{Ann}_{FG}(d)$. Since FG is a commutative ring we have $z \in \mathbf{Ann}_{FG}(A)$ so that $\tau_z = 0$. We now have

$$\tau_y = \tau_{x+z} = \tau_x + \tau_z = \tau_x,$$

as required.

Let b be a further element of A, so that $b = wd$ for some element $w \in FG$. We have

$$\begin{aligned}\Omega(a+b) &= \Omega(xd + wd) = \Omega((x+w)d) = \tau_{x+w} = \tau_x + \tau_w \\ &= \Omega(a) + \Omega(b).\end{aligned}$$

If $v \in FG$ is arbitrary, then for each $f \in \mathbf{End}_{FG}(A)$ we have

$$vf(a) = f(va) \text{ for all } a \in A. \tag{2.1}$$

On the other hand,

$$(\tau_v \circ f)(a) = \tau_v(f(a)) = vf(a). \tag{2.2}$$

Then by Equations (2.1) and (2.2) we have

$$\Omega(va) = \Omega(v(xd)) = \Omega((vx)d) = \tau_{vx} = \tau_v \circ \tau_x = v\tau_x = v\Omega(a).$$

Thus Ω is an FG-homomorphism. By (ii) Ω is an FG-epimorphism. However A is simple, so $\mathbf{ker}(\Omega) = 0$ and hence Ω is an FG-isomorphism. $\square$

Of course, our ultimate goal in this chapter is to study the structure of irreducible linear groups but it is useful to look at the process for constructing some simple FG-modules. Corollary 2.4 shows one method of how to make such constructions.

We next note some elementary properties of modules. The following result is quite standard and well-known.

Lemma 2.5. *Let R be a ring and let A be an R-module. If $\mathcal{M}$ is a non-empty family of minimal R-submodules of A and B is the submodule generated by all members of $\mathcal{M}$, then B is the direct sum of some of the elements of $\mathcal{M}$.*

Proof. Let $M_1 \in \mathcal{M}$ and let S be another element of $\mathcal{M}$. Then $M_1 \cap S = M_1$ or $M_1 \cap S = 0$, since M_1 is a minimal submodule. In the former case we have $M_1 \leq S$, so $M_1 = S$ since S is a minimal submodule. Thus if $B \neq M_1$, then there exists a submodule $M_2 \in \mathcal{M}$ such that $M_1 \cap M_2 = 0$. We then let

$$L_2 = M_1 + M_2 = M_1 \oplus M_2.$$

If $L_2 \neq B$, then there exists $M_3 \in \mathcal{M}$ such that $M_3 \nleq L_2$. Since M_3 is a minimal submodule, then $L_2 \cap M_3 = 0$. It follows that

$$L_3 = L_2 + M_3 = L_2 \oplus M_3 = M_1 \oplus M_2 \oplus M_3.$$

Let $\alpha > 3$ be an ordinal and suppose that for each $\beta < \alpha$ we have constructed a family $\{M_\lambda | \lambda \leq \beta\}$ of elements of $\mathcal{M}$ such that the submodule L_β generated by $\{M_\lambda | \lambda \leq \beta\}$ is a direct sum of the M_λ, for $\lambda \leq \beta$ and $\{M_\lambda | \lambda \leq \beta\} \subseteq \{M_\lambda | \mu \leq \delta\}$ whenever $\beta \leq \delta < \alpha$. Suppose first that α is not a limit ordinal. Then $\nu = \alpha - 1$ exists and if $L_\nu = B$ then the result follows since $L_\alpha = L_\nu$ in this case. Suppose that $L_\nu \neq B$. Then $\mathcal{M}$ contains a minimal submodule M_α such that L_ν does not contain M_α. Since M_α is a minimal submodule we have $L_\nu \cap M_\alpha = 0$ and it follows that

$$L_\alpha = L_{\alpha-1} + M_\alpha = L_{\alpha-1} \oplus M_\alpha.$$

Clearly, L_α is the direct sum of the M_β for $\beta \leq \alpha$.

Suppose next that α is a limit ordinal and let L_α be the submodule generated by the M_β for $\beta < \alpha$. In this case $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$. Suppose that L_α is not a direct sum of the M_β for $\beta < \alpha$. Then there exists $\lambda < \alpha$ such that

$$M_\lambda \cap \langle M_\beta | \beta \neq \lambda, \beta < \alpha \rangle \neq 0.$$

Let $x \neq 0$ lie in this intersection and note that $x = y_1 + \cdots + y_m$, where $y_j \in M_{\beta_j}$ for certain $\beta_j \neq \lambda, \beta_j < \alpha$. There is an ordinal $\mu < \alpha$ such that $\lambda, \beta_1, \ldots, \beta_m \leq \mu$ and it follows that $x, y_1, \ldots, y_m \in L_\mu$, contradicting the fact that $L_\mu = \bigoplus_{\beta \leq \mu} M_\beta$. This contradiction shows that L_α is the direct sum of the submodules M_β for $\beta < \alpha$.

It is now clear by transfinite induction that there exists an ordinal γ such that $L_\gamma = B$. □

Lemma 2.6. *Let R be a ring, let G be a group and let A be an RG-module. Suppose that H is a normal subgroup of G and let B be an RH-submodule of A.*

(i) If $g \in G$, then gB is an RH-submodule. Moreover

- *(ia) If B is a Noetherian RH-module, then so is gB;*
- *(ib) If B is an Artinian RH-module, then so is gB;*
- *(ic) If B is a simple RH-module, then so is gB;*

(ii) If A is a simple RG-module and B is a simple RH-module, then $A = \bigoplus_{g \in S} gB$ for some subset S of G. In particular, A is a semisimple RH-module.

Proof. (i) Since H is normal in G we have $gRH = RHg$ for each element $g \in G$. Therefore

$$RH(gB) = (RHg)B = (gRH)B = g(RHB) = gB,$$

which shows that gB is an RH-module.

(ia) Let $\{C_n | n \in \mathbb{N}\}$ be an ascending chain of RH submodules of gB. Then $\{g^{-1}C_n | n \in \mathbb{N}\}$ is an ascending chain of RH-submodules of B and since B is Noetherian there is a natural number t such that $g^{-1}C_n = g^{-1}C_t$ for all $n \geq t$. It follows that $C_n = C_t$ for all $n \geq t$ which gives the result.

(ib) has a similar proof to (ia).

(ic) Let C be a non-zero RH-submodule of gB and let $0 \neq c \in C$. If $g^{-1}c = 0$, then

$$c = 1c = (gg^{-1})c = g(g^{-1}c) = 0,$$

a contradiction which shows that $g^{-1}C \neq 0$. By (i), $g^{-1}C$ is an RH-submodule of B and hence $g^{-1}C = B$, since B is a simple RH-module. Then $C = g(g^{-1}C) = gB$ is a simple RH-module.

(ii) Let $\mathcal{M} = \{gB | g \in G\}$ and let D be the RH-submodule generated by all elements of $\mathcal{M}$. If $d \in D$, then

$$d = g_1b_1 + \cdots + g_kb_k$$

for certain elements $g_j \in G, b_j \in B$ with $1 \leq j \leq k$ and if $x \in G$, then we have

$$xd = x(g_1b_1 + \ldots g_kb_gk) = xg_1b_1 + \ldots xg_kb_k \in D.$$

Since $B \neq 0$, it follows that D is a non-zero RG-submodule of A, so $D = A$, by the simplicity of A. We now apply Lemma 2.5 to obtain the result. □

In the following result we introduce some further notation. Accordingly, for the group G we let $\mathbf{Alg}\,(G)$ be the set of all elements of G which are algebraic over F.

Lemma 2.7. *Let A be a vector space over a field F and let G be an abelian subgroup of $GL(F, A)$. Then $\boldsymbol{Alg}\,(G)$ is a subgroup of G and $G/\boldsymbol{Alg}\,(G)$ is torsion-free.*

Proof. By Theorem 1.39, $\mathbf{Alg}\,(G)$ is a subgroup of G. Furthermore, Lemma 1.34(ix) implies that $G/\mathbf{Alg}\,(G)$ is torsion-free. □

If F is a field, A is a vector space over F and G is an irreducible abelian subgroup of $GL(F, A)$, then we can think of A as a simple FG-module such that $C_G(A) = 1$. As above we let $\mathbf{Alg}\,(G)$ denote the set of all elements of G which are algebraic over F.

By Corollary 2.3 the group G is isomorphic to a subgroup of the multiplicative group of some field extension of F. Then the subgroup $\mathbf{Alg}\,(G)$ is isomorphic to a subgroup of the multiplicative group of an algebraic field extension F_a of F. We identify $\mathbf{Alg}\,(G)$ with its image in F_a and let $F_1 = F[\mathbf{Alg}\,(G)]$ be the subring of F_a generated over F by $\mathbf{Alg}\,(G)$. Since every element of $\mathbf{Alg}\,(G)$ is algebraic over F it follows that F_1 is a subfield of F_a. Using the usual technique from field theory we can show that the field extension of F by $\mathbf{Alg}\,(G)$ is uniquely defined up to isomorphism.

In the algebra $E = \mathbf{End}_F(A)$ we consider the subalgebra $\mathbf{End}_{FG}(A)$ of all FG-endomorphisms of A. Corollary 2.4 shows that A is isomorphic to the FG-module $\mathbf{End}_{FG}(A)$. On the other hand, Corollary 2.4 also shows that $\mathbf{End}_{FG}(A)$ is a field extension of F. In what follows we identify A with its image in E. Let $K = \mathbf{Alg}\,(G)$. In E we also consider the subring $A_1 = F[K]$, generated over F by the linear group K. As we noted above A_1 is a subfield of the field $\mathbf{End}_{FG}(A)$. We can also think of A_1 as a module over the group ring FK. Let B be an FK-submodule of A_1 and let $0 \neq b \in B$, so

$$b = \alpha_1 g_1 + \cdots + \alpha_n g_n$$

where $\alpha_j \in F, g_j \in K$ for $1 \leq j \leq n$. Since A_1 is a subfield, $b^{-1} \in A_1$ and we may write

$$b^{-1} = \beta_1 h_1 + \cdots + \beta_m h_m$$

where $\beta_j \in F, h_j \in K$ for $1 \leq j \leq m$. Since B is an FK-submodule we have

$$1 = b^{-1}b = (\beta_1 h_1 + \dots \beta_m h_m)b \in B.$$

It follows that A_1 is a simple FK-submodule and Lemma 2.6 implies that there is a subset S of G such that $A = \bigoplus_{g \in S} gA_1$.

We may also think of A as an F_1G-module. Let K_1 be the set of all elements of G which are algebraic over F_1. If $g \in K_1$, then g is algebraic over F_1. On the other hand F_1 is an algebraic extension of F. By the properties of algebraic extensions g is algebraic over F and it follows that $K_1 = K$. If $t \in G \setminus K$, then Lemma 1.34(vii) shows that t has infinite order. Let A_2 denote the subring $F_1\langle t\rangle$ of E, generated over F_1 by the infinite cyclic group $\langle t\rangle$. As usual we consider A as a module over the group ring $R = F_1\langle t\rangle$. The fact that t is not algebraic over F_1 implies that $\mathbf{Ann}_R(A_1) = 0$ and it follows that $A_2 \cong F_1\langle t\rangle$. Since $\mathbf{End}_{FG}(A)$ is a field, it contains F_2 the field of fractions of A_2. It therefore follows that F_2 is isomorphic to the field of fractions of a group ring of an infinite cyclic group over F_1.

Now we can consider also A as a module over the group ring F_2G. Let K_2 be the set of elements of G which are algebraic over F_2. We can then repeat the previous constructions again. Such constructions with the use of transfinite induction lead us finally to the module A. As we can see, the process of constructiong simple modules over an abelian group G is very similar to the process of constructing field extensions; as with the latter there are two main types-algebraic and transcendental.

On the Structure of Irreducible Abelian Groups

Let us return to our main task, that of clarifying the structure of irreducible linear groups. The first natural step is the description of irreducible abelian groups. In turn, it makes sense to divide this stage into parts which are associated with the presence in the group of both algebraic and transcendental elements.

Lemma 2.8. *Let A be a vector space over the field F and let G be an infinite irreducible abelian subgroup of $GL(F, A)$.*

(i) If $\boldsymbol{char}(F) = 0$, then $|\boldsymbol{Alg}(G)| \leq |F|$;

(ii) If F is a locally finite field, then $\boldsymbol{Alg}(G) = \boldsymbol{Tor}(G)$ is finite or countable. In particular, this is the case when F is finite;

(iii) If $\boldsymbol{char}(F) = p$, a prime, and F is not locally finite, then $|\boldsymbol{Alg}(G)| \leq |F|$.

Proof. Let $G_1 = 1$ and inductively define, for $n \geq 2$,

$$G_n = \{g \in G \setminus (G_1 \cup \cdots \cup G_{n-1}) | \text{ there is a polynomial of degree } n, \\ f_g(X) \in F[X], \text{ such that } g \text{ is a root of } f_g(X)\}.$$

Clearly, if F is infinite, then $|G_n| \leq |F|$, for every $n \geq 2$ and if F is finite, then G_n is also finite for all $n \geq 1$. We have $\mathbf{Alg}\,(G) = \bigcup_{n\in\mathbb{N}} G_n$. Hence if F is infinite, then $|\mathbf{Alg}\,(G)| \leq |F|$. If F is infinite and locally finite, then F is countable, in which case $\mathbf{Alg}\,(G)$ is finite or countable. If F is finite, then $\mathbf{Alg}\,(G)$ is finite or countable.

If F is locally finite, then Lemma 1.34(viii) shows that $\mathbf{Alg}\,(G)$ is periodic and Lemma 1.34(vii) implies that $\mathbf{Alg}\,(G) = \mathbf{Tor}(G)$. The result follows. □

We need some concepts from the theory of abelian groups, the first being an analogue of a linearly independent subset of a vector space.

Let G be an abelian group. A subset X of G, consisting of elements of infinite order, is said to be $\mathbb{Z}$-*independent* or simply *independent* if, given distinct elements $x_1, \ldots, x_n$ in X and integers $k_1, \ldots, k_n$, the relation $x_1^{k_1} \ldots x_n^{k_n} = 1$ implies that $x_j = 1$ for $1 \leq j \leq n$.

We note that if $X = \{x_\lambda | \lambda \in \Lambda\}$ is a $\mathbb{Z}$-independent subset of G, then $\langle X \rangle = \underset{\lambda\in\Lambda}{\mathrm{Dr}} \langle x_\lambda \rangle$ is a direct product of infinite cyclic groups. Conversely, if $H = \underset{\lambda\in\Lambda}{\mathrm{Dr}} \langle c_\lambda \rangle$, where c_λ is an element of infinite order for every $\lambda \in \Lambda$, then the subset $\{c_\lambda | \lambda \in \Lambda\}$ is $\mathbb{Z}$-independent.

Zorn's Lemma implies that a $\mathbb{Z}$-independent subset of an abelian group is always contained in some maximal $\mathbb{Z}$-independent subset. By analogy with vector space theory, we have the following fundamental result. The reader can find a proof in the book of L. Fuchs [58, Chapter III], or any one of a number of other excellent books.

Proposition 2.9. *Let G be an abelian group.*

(i) If G has an infinite $\mathbb{Z}$-independent subset, then all maximal $\mathbb{Z}$-independent subsets of G have the same cardinality;

(ii) If G has a finite maximal $\mathbb{Z}$-independent subset M, then each maximal $\mathbb{Z}$-independent subset S of G is finite and $|S| = |M|$;

(iii) If X is a maximal $\mathbb{Z}$-independent subset of G, then the factor group $G/\langle X\rangle$ is periodic. Conversely, if Y is a $\mathbb{Z}$-independent subset of G such that $G/\langle Y\rangle$ is periodic, then Y is a maximal $\mathbb{Z}$-independent subset of G.

Let G be an abelian group. The cardinality of a maximal $\mathbb{Z}$-independent subset of G is called the *$\mathbb{Z}$-rank* or *torsion-free rank* of G, denoted by $\mathbf{r}_{\mathbb{Z}}(G)$. If G has a finite maximal $\mathbb{Z}$-independent subset, then we shall say that G has finite $\mathbb{Z}$-rank.

We note that $\mathbf{r}_{\mathbb{Z}}(G) = \mathbf{r}_{\mathbb{Z}}(G/\mathbf{Tor}(G))$ for every abelian group G. In the book [58] the notation $\mathbf{r}_0(G)$ is used instead of $\mathbf{r}_{\mathbb{Z}}(G)$. However we use the notation $\mathbf{r}_0(G)$ for a more general invariant of a group G, which then need not be abelian.

We observe from the definition that the concept of $\mathbb{Z}$-rank is not really useful for periodic groups. For periodic abelian groups we introduce a different notion of rank which is also based on the concept of dimension. We note that if p is a prime, then in every abelian group G the set of elements of p-power order forms a subgroup denoted by $\mathbf{Tor}_p(G)$.

Let p be a prime and let n be a non-negative integer. If P is an abelian p-group, then the *n-layer* of P is the subgroup

$$\mathbf{\Omega}_n(P) = \{a \in P | a^{p^n} = 1\}.$$

.

It is clear that $\mathbf{\Omega}_{n+1}(P)/\mathbf{\Omega}_n(P)$ is an elementary abelian p-group, for each $n \geq 0$, which can therefore be thought of as a vector space over the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. In particular, $\mathbf{\Omega}_1(P)$ is such a vector space.

Using this notation the *p-rank* of P is defined to be the dimension of $\mathbf{\Omega}_1(P)$ over $\mathbb{F}_p$. More generally, if G is an arbitrary abelian group then we define the *p-rank* of G to be the the p-rank of $\mathbf{Tor}_p(G)$. The p-rank of G is denoted by $\mathbf{r}_p(G)$.

We note that if $G = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, C_\lambda$ where C_λ is a cyclic p-group or a Prüfer p-group, then $\mathbf{r}_p(G) = |\Lambda|$.

Also we note that for an abelian p-group G, $\mathbf{r}_p(G)$ is finite if and only if G is a Chernikov group.

Our next result gives precise conditions for an infinite abelian subgroup G of $GL(n, F)$ to be irreducible, at least when the elements of G are algebraic over the field F.

Theorem 2.10. *Let F be a field and let G be an infinite abelian group.*

*(i) Let **char** $(F) = 0$ and suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F. If every element of G is algebraic over F, then **Tor**(G) is locally cyclic and $|G| \leq |F|$.*

*Conversely, suppose that **Tor**(G) is locally cyclic and $|G| \leq |F|$. Then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$ and the elements of G are algebraic over F.*

*(ii) Let F be locally finite and let **char** $(F) = p$ for some prime p. Suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F. If every element of G is algebraic over F, then G is a locally cyclic p'-group.*

Conversely, suppose that G is a locally cyclic p'-group. Then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$ and the elements of G are algebraic over F.

*(iii) Let F be of prime characteristic p, but not locally finite. Suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F. If every element of G is algebraic over F, then **Tor**(G) is a locally cyclic p'-sugbroup and $|G| \leq |F|$.*

*Conversely, suppose that **Tor**(G) is a locally cyclic p'-group and $|G| \leq |F|$. Then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$ and the elements **Tor**(G) are algebraic over F.*

Proof. Suppose that G is irreducible and that each element of G is algebraic over F. If **char** $(F) = 0$, then Lemma 2.8 implies that $|G| \leq |F|$. Furthermore, if F is locally finite, then G is periodic and if F is of prime characteristic but not locally finite, then $|G| \leq |F|$.

Using Corollary 2.3(ii) we see that **Tor**(G) is a locally cyclic group and if **char** $(F) = p$, a prime, then **Tor**(G) is a p'-group. If F is locally finite, then Lemma 1.34(viii) shows that G is periodic, so that $G = \mathbf{Tor}(G)$ in this case. This proves necessity in (i), (ii), (iii).

Conversely, suppose first that F is not locally finite, that $|G| \leq |F|$ and that **Tor**(G) is a periodic locally cyclic group. Moreover suppose that if **char** $(F) = p$, a prime, then **Tor**(G) is a p'-group. Let K be the algebraic closure of F. Then $|K| = |F|$. Also $U(K)$ is a divisible abelian group and **Tor**$(U(K))$ is a divisible locally cyclic group such that $\Pi(\mathbf{Tor}(U(K))) = \mathbb{P}$, whenever **char** $(F) = 0$ and $\Pi(\mathbf{Tor}(U(K)) = p'$ whenever **char** $(F) = p$ (see [98, Chapter 4, Proposition 4.1], for example). Let D be the divisible hull of G. The existence of D follows from [58, Section 24]. Again by [58, Section 24] we have $\mathbf{r}_p(D) = \mathbf{r}_p(G)$ for each prime p and $\mathbf{r}_{\mathbb{Z}}(D) = \mathbf{r}_{\mathbb{Z}}(G)$. It follows that $D = T \times V$ where T is a periodic locally cyclic group such that $\Pi(T) = \Pi(\mathbf{Tor}(G))$ and V is torsion-free such that $\mathbf{r}_{\mathbb{Z}}(V) = \mathbf{r}_{\mathbb{Z}}(G)$. It follows that $D \cong U(K)$ and hence G is isomorphic to a subgroup of $U(K)$. As usual, we identify G with its image in $U(K)$ and hence assume that G is a subgroup of $U(K)$.

Let $A = F[G]$ be the subring of K, generated over F by G. Since every element of G is algebraic over F, A is a subfield of K. We may think of A as a module over the group ring FG. By the above arguments we deduce that A is a simple FG-module. Finally, suppose that $C_G(A) \neq 1$ and let $1 \neq g \in C_G(A)$. Then $ga = a$ for all elements $a \in A$. We choose $a \neq 0$, so $(g-1)a = 0$. Since $g - 1 \neq 0$ we obtain a contradiction since A contains no zero divisors. This contradiction shows that $C_G(A) = 1$.

For each $g \in G$ let $\tau_g : A \longrightarrow A$ be the mapping defined by $\tau_g(a) = ga$, for each $a \in A$. We have already seen that τ_g is a linear transformation of the F-vector space A and the mapping $\Phi : G \longrightarrow GL(F, A)$ defined by $\Phi(g) = \tau_g$ for $g \in G$ is a group homomorphism. As above $\ker(\Phi) = C_G(A)$ and since $C_G(A) = 1$ we deduce that G is isomorphic to a subgroup of $GL(F, A)$. It is clear that this subgroup is irreducible and by construction, every element of G is algebraic over F.

Suppose now that the field F is locally finite. Again let K denote the algebraic closure of F. Then K is uniquely determined by the characteristic p, so that it contains every locally finite field of characteristic p. Further $U(K)$ is a divisible locally cyclic group with $\Pi(U(K)) = \mathbb{P} \setminus \{p\}$ (see [98, Chapter 4, Proposition 4.1], for example). Then G is isomorphic with a subgroup of $U(K)$. Again we may identify G with its image in $U(K)$ so that we may assume G is a subgroup of $U(K)$.

We may now prove (ii) by using the same arguments as those used previously, so we omit the details. $\square$

Moving to the case when an irreducible group contains non-algebraic elements, we begin with the following important assertion.

Proposition 2.11. *Let F be a field, let A be a vector space over F and let G be an infinite abelian subgroup of $GL(F, A)$. Suppose that G is irreducible. If $G \neq \mathbf{Alg}(G)$, then $G/\mathbf{Alg}(G)$ has infinite $\mathbb{Z}$-rank. In particular, if the $\mathbb{Z}$-rank of G is finite, then every element of G is algebraic over F. Moreover, if F is locally finite, then G is periodic.*

Proof. Suppose, for a contradiction, that $G/\mathbf{Alg}(G)$ has finite $\mathbb{Z}$-rank. Let $H = \mathbf{Alg}(G)$ and let K be the algebraic closure of F. As we saw above the subalgebra $L = F[H]$ is a subfield of $\mathbf{End}_F(A)$. We can think of A as an LG-module and since A is a simple FG-module it is also simple as an LG-module.

Let $g \in G \setminus \mathbf{Alg}(G)$ and let $D = L\langle x\rangle$ be the group algebra of the infinite cyclic group $\langle x\rangle$ over the field L. Then D is a principal ideal domain and the set of all the maximal ideals of D is infinite. As usual we can think of A as a DG-module defining the action of x on A by $xa = g(a)$, for each element $a \in A$. Using [115, Corollary 1.16] we see that $\mathbf{Ann}_D(A)$ is a maximal ideal of D and since D is a principal ideal domain we have $\mathbf{Ann}_D(A) = f(x)D$, for some irreducible polynomial $f(x) \in L[x]$. Then g is a root of $f(x)$, so that g is algebraic over L. Since L is algebraic over F, it follows that g is algebraic over F. Then $g \in \mathbf{Alg}\,(G)$ giving the contradiction sought. It follows that $G/\mathbf{Alg}\,(G)$ has infinite $\mathbb{Z}$-rank.

In particular, if G has finite $\mathbb{Z}$-rank, then $G/\mathbf{Alg}\,(G)$ also has finite $\mathbb{Z}$-rank and it follows that then every element of G is algebraic over F. If, in addition, F is locally finite, then Lemma 1.34(viii) implies that G is periodic. $\square$

We shall need the following auxiliary result whose proof can be found in [58, Theorem 14.14], for example.

Lemma 2.12. *Let G be an abelian group and let H be a subgroup of G such that G/H is free abelian. Then $G = H \times C$, for some subgroup C.*

We now have the following generalization of Theorem 2.10.

Theorem 2.13. *Let F be a field and let G be an infinite abelian group.*

(i) Let $\boldsymbol{char}\,(F) = 0$ and suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F. If $G \neq \boldsymbol{Alg}(G)$, then $G/\boldsymbol{Alg}(G)$ has infinite $\mathbb{Z}$-rank, $\boldsymbol{Tor}(G)$ is locally cyclic and $|\boldsymbol{Alg}(G)| \leq |F|$.

Conversely, if $\boldsymbol{Tor}(G)$ is locally cyclic and H is a subgroup of G such that $\boldsymbol{Tor}(G) \leq H$, G/H is a torsion-free group of infinite $\mathbb{Z}$-rank and $|H| \leq |F|$, then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$ and $\boldsymbol{Alg}(G) = H$.

(ii) Let F be locally finite and let $\boldsymbol{char}\,(F) = p$ for some prime p. Suppose that G is a non-periodic irreducible subgroup of $GL(F, A)$ for some vector space A over F. Then $\mathbf{r}_{\mathbb{Z}}(G)$ is infinite and $\boldsymbol{Tor}(G)$ is a locally cyclic p'-group.

Conversely if $\mathbf{r}_{\mathbb{Z}}(G)$ is infinite and $\boldsymbol{Tor}(G)$ is a locally cyclic p'-group, then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$.

(iii) Let F be infinite of prime characteristic p and not locally finite. Suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F such that $G \neq \boldsymbol{Alg}(G)$. Then $G/\boldsymbol{Alg}(G)$ has infinite $\mathbb{Z}$-rank, $\boldsymbol{Tor}(G)$ is a locally cyclic p'-subgroup and $|\boldsymbol{Alg}(G)| \leq |F|$.

Conversely, if $\boldsymbol{Tor}(G)$ is a locally cyclic p'-group and H is a subgroup of G such that $\boldsymbol{Tor}(G) \leq H$, G/H is a torsion-free group of infinite $\mathbb{Z}$-rank and $|H| \leq |F|$, then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$, and $\boldsymbol{Alg}(G) = H$.

Proof. Suppose that G is irreducible. Using Corollary 2.3(ii) we see that $\mathbf{Tor}(G)$ is locally cyclic and if F has characteristic the prime p, then $\mathbf{Tor}(G)$ is a p'-group. Lemma 2.7 shows that, whatever the characteristic of F, $G/\mathbf{Alg}\,(G)$ is torsion-free and Proposition 2.11 shows that $G/\mathbf{Alg}\,(G)$ has infinite $\mathbb{Z}$-rank.

If $\mathbf{char}\,(F) = 0$, then Lemma 2.8 implies that $|\mathbf{Alg}\,(G)| \leq |F|$.

If F is locally finite, then $\mathbf{Alg}\,(G)$ is periodic. If we suppose that G has finite $\mathbb{Z}$-rank, then Proposition 2.11 shows that G is periodic. Using Lemma 1.34 we deduce that every element of G is algebraic over F, giving us a contradiction. Hence G has infinite $\mathbb{Z}$-rank.

If F is of prime characteristic and not locally finite, then $|\mathbf{Alg}\,(G)| \leq |F|$, by Lemma 2.8. Thus the conditions given in (i), (ii) and (iii) are necessary.

Conversely, suppose first that F is not locally finite, and let H be a subgroup of G such that $|H| \leq |F|$, $\mathbf{Tor}(G) \leq H$ and $\mathbf{Tor}(G)$ is a periodic locally cyclic group. Suppose that G/H is torsion-free. Moreover suppose that if $\mathbf{char}\,(F) = p$, a prime, then $\mathbf{Tor}(G)$ is a p'-group. Let K be the algebraic closure of F. Repeating the argument from the proof of Theorem 2.10 we see that H is isomorphic to some subgroup of $U(K)$. Without loss of generality we may assume that H is a subgroup of $U(K)$.

Let $B = F[H]$ be the subring of K generated over F by H. Since every element of H is algebraic over F we see see that B is a subfield of K.

In the torsion-free group G/H we choose a maximal $\mathbb{Z}$-independent subset $\{u_\lambda H | \lambda \in \Lambda\}$. Then

$$S/H = \langle u_\lambda H | \lambda \in \Lambda \rangle = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\,\langle u_\lambda H\rangle$$

is a free abelian group and G/S is periodic. By Lemma 2.12

$$S = H \times V, \text{ where } V = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\,\langle v_\lambda\rangle \cong S/H$$

is free abelian. Let $R = B[X_\lambda | \lambda \in \Lambda]$ be the polynomial ring in the variables X_λ, $\lambda \in \Lambda$. Clearly R is a unique factorization domain. Let C be the field of fractions of R. Then $U(C) = U(B) \times S_1$, where $S_1 = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\,\langle w_\lambda\rangle$. In particular, $S_1 \cong V$. By construction $U(B)$ contains H. Without loss of generality we may assume that $U(C)$ contains S and that $U(C) = U(B) \times V$. Since $B = F[H]$ and $C = B[S_1] = B[V]$ we deduce that $C = F[S]$.

Finally, let P be the algebraic closure of C and let K be the algebraic closure of F. Then $|P| = |C|$. Also $U(P)$ is a divisible abelian group and $|U(P)| = |S|$. Furthermore, $\mathbf{Tor}(U(K)$ is a divisible locally cyclic subgroup such that $\Pi(\mathbf{Tor}(U(K))) = \mathbb{P}$, whenever $\mathbf{char}\,(F) = 0$ and $\Pi(\mathbf{Tor}(U(K)) = p'$ whenever $\mathbf{char}\,(F) = p$ (see [98, Chapter 4, Proposition 4.1], for example). Let D be the divisible hull of G, whose existence follows from [58, Section 24]. Again by [58, Section 24] we have $\mathbf{r}_p(D) = \mathbf{r}_p(G)$ for each prime p and $\mathbf{r}_{\mathbb{Z}}(D) = \mathbf{r}_{\mathbb{Z}}(G)$. By the choice of S the divisible hull of S coincides with the divisible hull of D. It follows that D is isomorphic to $U(P)$. Hence without loss of generality we may assume that G is a subgroup of $U(P)$.

Let $A = C[G]$ be the subring of K, generated over C by G. Since every element of G is algebraic over F, A is a subfield of K. We may think of A as a module over the group ring FG. By the above arguments we deduce that A is a simple FG-module and that $C_G(A) = 1$. As in the proof of Theorem 2.10, we can treat the elements of G as linear transformations of the vector space A and in this way G becomes an irreducible linear group of the vector space A.

Suppose now that the field F is locally finite. Suppose that $\mathbf{Tor}(G)$ is a locally cyclic p'-group where $\mathbf{char}\,(F) = p$ and $\mathbf{r}_{\mathbb{Z}}(G)$ is infinite. Again let K denote the algebraic closure of F. Repeating the arguments from the proof of Theorem 2.10 we deduce that $\mathbf{Tor}(G)$ is isomorphic to some subgroup of $U(K)$ and without loss of generality we may assume that $\mathbf{Tor}(G)$ is a subgroup of $U(K)$.

Let $B = F[\mathbf{Tor}(G)]$ be the subring of K generated over F by $\mathbf{Tor}(G)$. As above we see that B is a subfield of K. We can repeat the arguments above and construct a vector space A over the field F in such a way that G becomes an irreducible subgroup of $GL(F, A)$. This completes the proof. □

On the Structure of Non-Abelian Irreducible Groups

The description of arbitrary non-abelian irreducible (infinite dimensional) linear groups is considerably more difficult than for abelian groups and for some classes of groups is generally too complicated. The description of abelian irreducible groups together with Corollary 2.3 quite clearly shows us the scope of the possibilities. It is natural, therefore, to continue the study of irreducible infinite dimensional linear groups in those classes of groups that are in one sense or another close to abelian. Some such classes are the classes of nilpotent and hypercentral groups. The description given here will not be quite so detailed as for abelian groups.

The following result gives a reduction to the case of abelian groups.

Proposition 2.14. *Let F be a field, let G be an infinite hypercentral group and let $Z = \zeta(G)$. If there exists a simple FZ-module C such that $C_Z(C) = 1$, then there exists a simple FG-module A such that $C_G(A) = 1$.*

Proof. Let

$$B = FG \otimes_{FZ} C,$$

a left FG-module. Of course B is a vector space over F and there is a natural way to identify C with an FZ-submodule of B. If S is a transversal to Z in G, then it is well-known that

$$B = \oplus_{x \in S} xC.$$

Theorem 2.1 shows that B has a G-chief Kurosh-Chernikov system $\mathcal{S}$. Let A be some factor of this system, so that A is a simple FG-module. Lemma 2.5 shows that

$$A = \oplus_{x \in Y} xC \text{ for some subset } Y \text{ of } S.$$

Suppose that $C_G(A) \neq 1$. Since G is hypercentral and $C_G(A)$ is a normal subgroup of G, it follows that $C_G(A) \cap Z = C_Z(A) \neq 1$ (see [52, Lemma 1.2.2], for example). If $x \in Y$, then

$$C_Z(A) \leq C_Z(xC) = C_Z(C) = 1$$

and we obtain a contradiction, which shows that $C_G(A) = 1$. The result follows. □

Taking Theorems 2.10 and 2.13 together with Proposition 2.14 we obtain the following result.

Theorem 2.15. *Let F be a field and let G be an infinite hypercentral group.*

(i) Suppose that F is not locally finite. If G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F, then $\boldsymbol{Tor}(\zeta(G))$ is locally cyclic if F is of characteristic 0 and a locally cyclic p'-group if F has characteristic $p > 0$.

Conversely, suppose that $\boldsymbol{Tor}(\zeta(G))$ is locally cyclic (and also a p'-group if $\boldsymbol{char}\,(F) = p$ is prime). Suppose further that there is a subgroup H of $\zeta(G)$ such that $\boldsymbol{Tor}(\zeta(G)) \leq H$ and $\zeta(G)/H$ is torsion-free of infinite rank with $|H| \leq |F|$. Then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$.

(ii) Let F be locally finite and let $\boldsymbol{char}\,(F) = p$ for some prime p. Suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A over F. Then $\boldsymbol{Tor}(\zeta(G))$ is a locally cyclic p'-group. Moreover, if $\mathbf{r}_{\mathbb{Z}}(G)$ is finite, then $\zeta(G)$ is periodic.

Conversely, if $\boldsymbol{Tor}(\zeta(G))$ is a locally cyclic p'-group, then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$.

Proof. Suppose that G is an irreducible subgroup of $GL(F, A)$ for some vector space A. Using Corollary 2.3 we deduce that $\mathbf{Tor}(\zeta(G))$ is locally cyclic. Furthermore, if $\mathbf{char}\,(F) = p$ is prime, then $\mathbf{Tor}(\zeta(G))$ is a p'-subgroup. If F is a locally finite field and if G has finite $\mathbb{Z}$-rank, then repeating the arguments from the proof of Proposition 2.11 and using [115, Corollary 1.21] we deduce that $\zeta(G)$ is periodic. This proves the first part of each of (i) and (ii).

Conversely, let G be a hypercentral group and suppose that $\mathbf{Tor}(\zeta(G))$ is locally cyclic (and a p'-group, if $\mathbf{char}\,(F) = p$ is prime). Let $Z = \zeta(G)$. Then Theorems 2.10 and 2.13 show that there is a simple FZ-module B such that $C_Z(B) = 1$. Proposition 2.14 allows us to construct a simple FG-module A such that $C_G(A) = 1$. Consequently, G is isomorphic to a subgroup of $GL(F, A)$ and G is irreducible, because A is a simple FG-module. □

We note that the paper of L. A. Kurdachenko and I. Ya. Subbotin [122] discusses a more general situation than that considered here, namely necessary and sufficient conditions are obtained for the existence of simple DG-modules when D is a Dedekind domain and G is a hypercentral group.

One other generalization of the class of abelian groups which has received considerable attention is the class of FC-groups, first introduced by R. Baer [3]. This class of groups is a common natural extension of both the class of abelian groups and the class of finite groups. We now give some details concerning FC-groups.

Let G be a group, let $x \in G$ and let

$$x^G = \{x^g = g^{-1}xg | g \in G\}.$$

Then $C_G(x^G)$ is a normal subgroup of G. Let

$$\mathbf{FC}(G) = \{x \in G | x^G \text{ is finite}\} = \{x \in G | G/C_G(x^G) \text{ is finite}\}.$$

It is easy to see that $\mathbf{FC}(G)$ is a characteristic subgroup of G, which is called the *FC-center* of G. A group G is called an *FC-group* if $G = \mathbf{FC}(G)$.

The class of FC-groups has often featured in the theory of infinite groups, one reason being that this well-studied class has a rich and extensive theory. Most of the results obtained concern periodic FC-groups and many authors have made significant contributions to this part of the theory as reflected in the books of Yu. M. Gorchakov [64] and M. J. Tomkinson [199]. The survey article of M. J. Tomkinson should also be mentioned here [200]. The theory of non-periodic FC-groups also has interesting parallels with the theory of abelian groups, many of the main results being obtained in a series of papers of L. A. Kurdachenko. These results were covered in the survey article [163] of J. Otal and N. N. Semko.

The progress achieved in the theory of FC-groups naturally leads to the study of extensions of this class. One of the first such extensions was the class of groups with Chernikov conjugacy classes which we now discuss.

We say that a group G has *Chernikov conjugacy classes* or that G is a *CC-group* if $G/C_G(x^G)$ is a Chernikov group for each $x \in G$. The class of CC-groups was introduced by Ya. D. Polovitsky in the paper [178]. The theory of CC-groups is not so well established as the theory of FC-groups; some of the main results obtained can be found in the survey article [46] of M. R. Dixon and L. A. Kurdachenko. Our immediate goal is to consider irreducible linear CC-groups. However, we first give the following general result, proved in the works of B. Hartley [77] and D. I. Zaitsev [222].

Proposition 2.16. *Let R be a ring, let G be a group, let H be a normal subgroup of G and let A be a simple RG-module. If $|G : H|$ is finite, then A contains a simple RH-submodule B and there is a finite subset S of G such that $|S| \leq |G : H|$, where $A = \bigoplus_{g \in S} gB$.*

Proof. By Lemma 2.6, it suffices to show that A contains a simple RH-submodule. Let S be a transversal to H in G. If $T \subseteq S$ and B is an RH-submodule of A, then we write

$$T \sim B$$

to indicate that $TB = \bigoplus_{x \in T} xB = A$, but $T_1 B \neq A$ for every proper subset T_1 of T. Since A is a simple RG-module, we note that $A = GB = SB$. Hence, for an RH-submodule B, there exists $T \subseteq S$ such that $T \sim B$. The idea of the proof is to show the existence of an RH-submodule U of A and a subset $S_0 \subseteq S$ such that $S_0 \sim V$ for each RH-submodule V of U. It will then follow that the submodule U is simple.

The construction of the pair (U, S_0) will be carried out inductively as follows. Put $B_1 = A$ and choose $S_1 \subseteq S$ such that $S_1 \sim B_1$. Suppose that there exists a proper RH-submodule $B_2 < B_1$ such that $S_1 \nsim B_2$. For B_2 we may choose a subset $S_2 \subseteq S$ such that $S_2 \sim B_2$. Proceeding in this way, we can construct a descending chain of non-zero RH-submodules $B_1 \geq B_2 \geq \cdots \geq B_i$ and a collection of subsets $S_1, S_2, \ldots, S_i$ of S which satisfy

$$B_j \neq B_{j+1}, S_j \sim B_j, S_j \nsim B_{j+1} \text{ for } j < i.$$

We claim that the subsets S_j are all distinct. For, suppose that there are indices k, m with $k < m$ and $S_k = S_m$. Then B_m is a proper submodule of B_k whereas $S_k \sim B_k$ and $S_k = S_m \sim B_m$. Then

$$S_k B_{k+1} \geq S_k B_m = S_m B_m = A.$$

However, if T is a proper subset of S_k, then $TB_{k+1} \leq TB_k \neq A$.

Summing up, $S_k B_{k+1} = A$ but $TB_{k+1} \neq A$, which means that $S_k \sim B_{k+1}$, a contradiction which proves our claim. Hence the subsets S_j are all distinct. Since S is finite, there are only finitely many such subsets S_j so this process terminates at some point. This means that the RH-submodule B_n and the subset S_n satisfy the following condition:

- if C is a proper non-zero RH-submodule of B_n, then $S_n \sim C$.

Evidently the condition above expresses the fact that $S_n C = A$ but $TC \neq A$ for every proper subset T of S_n, so the pair (B_n, S_n) satisfies the property we seek and our next goal is to finish the proof of the simplicity of B_n.

Suppose that $S_n = \{x_1 \ldots, x_t\}$ and define

$$K_j = x_j B_n \cap \left(\sum_{m \neq j} x_m B_n \right) \text{ for } 1 \leq j \leq t.$$

By Lemma 2.6 each K_j is an RH-submodule of A, so $L_j = x_m^{-1} K_j$ is also an RH-submodule. Since $L_j \leq B_n$ we have $x_m L_j \leq x_m B_n$, whenever $m \neq j$.

Thus $x_mL_j = K_j \leq \sum_{m\neq j} x_mB_n$ and it follows that

$$S_nL_j = \sum_{1\leq m\leq t} x_mL_j \leq \sum_{m\neq j} x_mB_n.$$

Since $S_n \sim B_n$ we have $S_nL_j \neq A$ and $S_n \nsim L_j$. From the choice of B_n we deduce that $L_j = 0$, so $K_j = 0$. This implies that

$$A = S_nB_n = \bigoplus_{1\leq m\leq t} x_mB_n.$$

Under these conditions, assume that B_n contains a proper non-zero RH-submodule C. Then

$$S_nC = \sum_{1\leq m\leq t} x_mC \neq \sum_{1\leq m\leq t} x_mB_n = A,$$

which contradicts the choice of B_n. This contradiction shows that B_n is a simple RH-module and the result follows. □

As a corollary we obtain an extension, in the infinite case, of the well-known Clifford's theorem (see [32]).

Corollary 2.17. *Let R be a ring, let G be a group and let H be a normal subgroup of G of finite index. If A is a semisimple RG-module, then A is a semisimple RH-module.*

We next obtain a result which first appeared in the paper of S. Franciosi, F. de Giovanni and L. A. Kurdachenko [57].

Proposition 2.18. *Let A be a vector space over the field F and let G be an irreducible subgroup of $GL(F, A)$. If P is a G-invariant elementary abelian p-subgroup of $\boldsymbol{FC}(G)$ for some prime p, then $\boldsymbol{char}(F) \neq p$ and P contains a subgroup J such that $|P/J| = p$, where $\boldsymbol{core}_G\, J = 1$.*

Proof. Let M be a minimal G-invariant subgroup of P. Since $P \leq \mathbf{FC}(G)$ it follows that M is finite and hence $H = C_G(M)$ has finite index in G. By Proposition 2.16 there exists a finite subset X of G and a simple FH-submodule B such that $A = \bigoplus_{x\in X} xB$. Since M is a normal subgroup of G and $C_G(A) = 1$ it follows that $C_A(M)$ is a proper FG-submodule of A, so that $C_A(M) = 0$. If $M \leq C_H(B)$, then $xMx^{-1} \leq C_H(xB)$ for each $x \in X$. However M is a normal subgroup of G, so we obtain $M \leq C_H(xB)$ for each $x \in X$ and as $A = \bigoplus_{x\in X} xB$ we have $M \leq C_H(A) = 1$. This contradiction shows that $M \nleq C_H(B)$ and Corollary 2.3 implies that the characteristic of F is not p.

Let K be a finite G-invariant subgroup of P, let $0 \neq a \in A$ and $A_1 = FKa$. Since K is finite, $\mathbf{dim}_F(A_1)$ is also finite. Therefore A_1 contains a simple FK-submodule B. Let $C = FPB = \sum_{x\in P} xB$.

Let $\mathcal{L}$ be a local system of P consisting of the finite nontrivial G-invariant subgroups of P that contain K and let $L \in \mathcal{L}$. Since P is abelian and $C = \sum_{x \in P} xB$ we have $C_L(B) = C_L(C)$. Let $E = FLB$, so $E = \bigoplus_{1 \leq m \leq t} E_m$ where E_m is a simple FL-submodule for $1 \leq m \leq t$ (see [116, Corollary 5.15]). By Corollary 2.3 $L/C_L(E_1)$ is a cyclic group and therefore $|L/C_L(E_1)| = p$. Since $C_L(B) = C_L(C)$ we have $C_L(E_1) = C_L(C)$ so that $|L/C_L(C)| = p$. As this is true for each $L \in \mathcal{L}$ we have $|P/C_P(C)| = p$. Let $J = C_P(C)$ and $U = \mathbf{core}_G\, J$. Since A is a simple FG-module it follows that $A = FGB$. Therefore, for every $d \in A$ we have $d = \sum_{1 \leq j \leq r} x_j b_j$ for certain $b_j \in B, x_j \in FG$ for $1 \leq j \leq r$. Let u be an element of U and note that $x_i U = U x_i$, as U is normal in G, so $ux_j = x_j u_j$ for some element $u_j \in U$ for $1 \leq j \leq r$. Hence

$$\begin{aligned} ud = u\left(\sum_{1 \leq j \leq r} x_j b_j\right) &= \sum_{1 \leq j \leq r} (ux_j) b_j = \sum_{1 \leq j \leq r} (x_j u_j) b_j \\ &= \sum_{1 \leq j \leq r} x_j (u_j b_j) = \sum_{1 \leq j \leq r} x_j b_j = d. \end{aligned}$$

It follows that $\mathbf{core}_G\, J = U \leq C_G(A) = 1$, as required. □

We next need some information concerning properties of CC-groups. The following result was proved in the work of Ya. D. Polovitsky [178].

Proposition 2.19. *Let G be a CC-group, let $x \in G$ and let $X = \langle x \rangle^G$. If x has finite order, then X is a Chernikov group. If x has infinite order, then X contains a normal Chernikov subgroup T such that $X/T = \langle xT \rangle$.*

The proof of this result can be found in the book [116, Theorem 3.13].

Our next result can be found in the work of L. A. Kurdachenko and J. Otal [113].

Proposition 2.20. *Let G be a CC-group. Then G contains a normal subgroup U satisfying the following conditions.*

(i) $U = \underset{\lambda \in \Lambda}{\mathrm{Dr}}\, S_\lambda$, where S_λ is either a finite minimal normal subgroup of G or S_λ is an infinite cyclic normal subgroup for each $\lambda \in \Lambda$;

(ii) $H \cap U$ is nontrivial for every nontrivial normal subgroup H of G.

Proof. Let S be the subgroup of G generated by all the minimal G-invariant subgroups of $\mathbf{FC}(G)$. By Proposition 2.19 the minimal normal subgroups of a CC-group are finite, so it follows that S is generated by all the finite minimal normal subgroups of G. Repeating almost verbatim the proof of Lemma 2.5 we deduce that, for some index set Δ, $S = \underset{\mu \in \Delta}{\mathrm{Dr}}\, S_\mu$ where S_μ is a finite minimal normal subgroup of G for $\mu \in \Delta$. If S satisfies condition (ii), then we set $U = S$. Otherwise, there exists a nontrivial normal subgroup H such that $H \cap S = 1$. Let $1 \neq h \in H$ and $Q = \langle h \rangle^G$. Since G is a CC-group Proposition 2.19 implies that either Q is Chernikov or Q contains a G-invariant Chernikov subgroup R such that Q/R is infinite. If R is nontrivial, then R contains a finite G-invariant minimal subgroup and the choice of S implies that $R \cap S \neq 1$, a contradiction. Thus $R = 1$, so Q is an infinite cyclic subgroup. Let $S_1 = S \times Q$. If S_1 satisfies (ii), then define $U = S_1$. If S_1 does not satisfy (ii), then we proceed in the same way using transfinite induction. This completes the proof. □

As usual we let the *socle* of a group G be the join of all the minimal normal subgroups of the group G and denote this unique subgroup by $\mathbf{Soc}(G)$. It is well-known that in this case $\mathbf{Soc}(G)$ is then the direct product of certain minimal normal subgroups of G (see [183, Lemma 5.23], for example and also Lemma 2.5). Furthermore, the *abelian socle* of the group G is the join of all the abelian minimal normal subgroups of G; this is denoted by $\mathbf{Soc}_{\text{ab}}(G)$ and is the direct product of certain abelian minimal normal subgroups.

Any subgroup U of a group G satisfying the conditions

(i) $U = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, S_\lambda$, where S_λ is either a finite minimal normal subgroup of G or S_λ is an infinite cyclic normal subgroup for each $\lambda \in \Lambda$ and

(ii) $H \cap U$ is nontrivial for every nontrivial normal subgroup H of G

is called a *quasi-socle* of G. We shall use the notation $\mathbf{QSoc}(G)$ for such a quasi-socle, but note that even in a CC-group there may be numerous such subgroups. In contrast with the usual socle, which is uniquely determined, the existence of a quasi-socle in an arbitrary group is doubtful, although Proposition 2.20 shows there is at least one such quasi-socle in every CC-group. For a CC-group G we always have $\mathbf{Soc}(G) \leq \mathbf{QSoc}(G)$.

Let $\mathbf{Soc}(G) = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, S_\lambda$ where S_λ is a minimal normal subgroup of G for each $\lambda \in \Lambda$. Let $\Delta = \{\lambda \in \Lambda | S_\lambda \text{ is abelian}\}$ and let $\mathbf{Soc}_{\text{ab}}(G) = \underset{\lambda\in\Delta}{\mathrm{Dr}}\, S_\lambda$.

Before proving the next result we need to discuss the non-abelian version of the $\mathbb{Z}$-rank.

Let G be a group which has an ascending series whose factors are either infinite cyclic or periodic. If the number of infinite cyclic factors is finite then the group G is said to have *finite* 0*-rank*. The 0-rank of the group G is the number of infinite cyclic factor groups in the series and is denoted by $\mathbf{r}_0(G)$. Thus if the number of infinite cyclic factors is exactly r then $\mathbf{r}_0(G) = r$. If no such integer r exists then we shall say that G has infinite 0-rank. If G has no such ascending series the 0-rank is undefined.

We note at once that this definition does not depend on the choice of ascending series (see [52, Lemma 2.2.3], for example).

However the definition is equivalent to the following apparently weaker version: the group G has finite 0-rank $\mathbf{r}_0(G) = r$ if G has a finite subnormal series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r (see [52, Corollary 2.4.2], for example).

The following theorem gives some structural characteristics of infinite dimensional irreducible CC-groups.

Theorem 2.21. *Let A be a vector space over the field F and let G be an irreducible subgroup of $GL(F, A)$. If G is a CC-group, then*

(i) $\boldsymbol{Soc}_{ab}(G)$ *is a p'-subgroup, where p is the characteristic of F;*

(ii) $\boldsymbol{Soc}_{ab}(G)$ *contains a subgroup J such that $\boldsymbol{Soc}_{ab}(G)/J$ is locally cyclic and $\boldsymbol{core}_G\, J = 1$;*

(iii) if G has finite 0-rank, then every element of its FC-center is algebraic over F;

(iv) if G has finite 0-rank and F is locally finite, then every quasi-socle of G coincides with the socle of G.

Proof. Since G is a CC-group, Proposition 2.19 shows that

$$\mathbf{Soc}(G) \leq \mathbf{FC}(G).$$

Suppose that F has prime characteristic p. Then Proposition 2.18 shows that $\mathbf{Soc}_{\mathrm{ab}}(G)$ is a p'-group. This proves (i). On the other hand, let q be a prime and let S_q be the Sylow q-subgroup of $\mathbf{Soc}_{\mathrm{ab}}(G)$. Again by Proposition 2.18 we see that S_q contains a subgroup J_q such that $|S_q/J_q| = q$ and $\mathbf{core}_G\, J_q = 1$. Let $\pi = \Pi(\mathbf{Soc}_{\mathrm{ab}}(G))$ and let $J = \underset{q\in\pi}{\mathrm{Dr}}\, J_q$. Then $\mathbf{Soc}_{\mathrm{ab}}(G)/J$ is locally cyclic and is a p'-subgroup of G if $\mathbf{char}\,(F) = p$. It is clear that $\mathbf{core}_G\, J = 1$, which proves (ii).

Next to prove (iii) suppose that G has finite 0-rank. Let d be an arbitrary element of $\mathbf{FC}(G)$, let $D = \langle d\rangle^G$ and $H = C_G(D)$. Then the normal subgroup H has finite index in G. If d has finite order, then by Lemma 1.34(vii) d is algebraic over F. Thus we may suppose that d has infinite order. We have $H \cap D \leq \zeta(D)$ and $D/(H \cap D)$ is finite. It follows that there is a natural number k such that $d^k = z \in H \cap D$ and this clearly shows that $z \in \zeta(H)$. By Proposition 2.16 there exists a finite subset Y of G and a simple FH-submodule B such that

$$A = \oplus_{y\in Y} yB.$$

Lemma 2.6 shows that yB is an FH-submodule for every $y \in Y$. Let $\langle x\rangle$ be an infinite cyclic group and let $K = F\langle x\rangle$ be the group ring of $\langle x\rangle$ over the field F. We make yB into a K-module by defining $xc = zc$ for each $c \in yB$. Since G is a CC-group, the normal closure of every finite subset of G is Chernikov-by-(finitely generated abelian) by Proposition 2.19 and hence every finitely generated subgroup of G is center-by-finite. In particular the CC-group G is locally (polycyclic-by-finite). Using [115, Corollary 1.16] we deduce that $\mathbf{Ann}_K(yB)$ is a maximal ideal of K. Since $F\langle x\rangle$ is a Principal Ideal Domain there is an irreducible polynomial $g_y(x)$ over F such that $\mathbf{Ann}_K(yB) = g_y(x)K$ and we let $g(x) = \Pi_{y\in Y} g_y(x)$. Since $g_y(z)a = 0$ for each element $a \in yB$ it follows that $g(z)a = 0$ for all elements $a \in A$. Thus the linear transformation z is a root of the polynomial $g(x)$, whose coefficients belong to F, so that z is algebraic over F. Using Lemma 1.34(ix) we see that d is also algebraic over F. This completes the proof of (iii).

Finally to prove (iv), suppose that F is a locally finite field and G has finite 0-rank. Assume, for a contradiction, that some quasi-socle of G does not coincide with the socle of G. In particular, this means that G contains a normal infinite cyclic subgroup C. Set $H = C_G(C)$, so that $|G/H| \leq 2$. Proposition 2.16 shows that A contains a simple FH-submodule B such that $A = B \oplus uB$ for some $u \in G \setminus H$. If $C \cap C_H(B) \neq 1$, then

$$\begin{aligned} C \cap C_H(uB) &= C \cap uC_H(B)u^{-1} = uCu^{-1} \cap uC_H(B)u^{-1} \\ &= u(C \cap C_H(B))u^{-1} \neq 1. \end{aligned}$$

Then $C \cap C_G(A) \neq 1$, a contradiction which shows that $C \cap C_H(B) = 1$ and in this case

$$C \cong CC_H(B)/C_H(B) \leq \zeta(H/C_H(B)),$$

which now gives a contradiction with [115, Corollary 1.21]. This final contradiction gives the result. □

The assertions (i), (ii) and (iv) first appeared in the paper [113] of L. A. Kurdachenko and J. Otal under more general hypotheses concerning modules over the group ring DG, where D is a Dedekind domain.

Our next result also appeared in [113].

Lemma 2.22. *Let F be a field and let $S = \underset{\lambda\in\Lambda}{Dr}\, S_\lambda$, where S_λ is a finite non-abelian simple group for each $\lambda \in \Lambda$. Then there is a simple FS-module A such that $C_S(A) = 1$.*

Proof. Let γ be the order type of Λ. We show by transfinite induction that S has an ascending series of normal subgroups

$$1 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\gamma = S$$

such that $C_{\alpha+1} = C_\alpha \times S_{\alpha+1}$ where $S_{\alpha+1} = S_{\lambda(\alpha)}$ for some $\lambda(\alpha) \in \Lambda$, whenever $0 \leq \alpha < \gamma$ and C_α admits a simple FC_α-module A_α such that $C_{C_\alpha}(A_\alpha) = 1$.

If $\gamma = 1$, then $S = C_1$ is a finite non-abelian simple group and [53, Lemma B.10.2] shows that there is a faithful, simple FS-module A_1 in this case, so the induction starts.

Let $\alpha > 1$ and suppose that the result has been proved for all ordinals $\beta < \alpha$. Suppose first that α is not a limit ordinal and let $\beta = \alpha - 1$. Then by induction there is a simple FC_β-module V such that $C_{C_\beta}(V) = 1$. Also there is a simple $FS_{\beta+1}$-module B such that $C_{S_{\beta+1}}(B) = 1$. Let $U = V \otimes_F B$. By [53, Corollary B.1.12] U is an $F(C_\beta \times S_{\beta+1})$-module. The well-known properties of tensor products imply that U is a semisimple FC_β-module and that $U = \bigoplus_{x\in X} xV$ for some subset X of $S_{\beta+1}$. Let A be an $FC_{\beta+1}$-composition factor of U so that A is a simple $FC_{\beta+1}$-module. Let $L = C_{C_{\beta+1}}(A)$. Since U is a semisimple FC_β-module,

$$A \cong \oplus_{x \in X_1} xV \text{ for some subset } X_1 \subseteq X.$$

This implies that $L \cap C_\beta = 1$ since $C_{C_\beta}(V) = 1$. Similarly, U is a semisimple $FS_{\beta+1}$-module and $U = \bigoplus_{y \in Y} yB$ for some subset $Y \subseteq C_{\beta+1}$. In turn we have

$$A \cong \oplus_{y \in Y_1} yB \text{ for some subset } Y_1 \subseteq Y.$$

Since $C_{S_{\beta+1}}(B) = 1$ we have $L \cap S_{\beta+1} = 1$. In particular we have $[L, S_{\beta+1}] = [L, C_\beta] = 1$ which means that $L \leq \zeta(C_{\beta+1})$. However $C_{\beta+1}$ is a direct product of finite nonabelian simple groups, so has a trivial center. Hence

$$L = C_{S_{\beta+1}}(A) = 1.$$

Setting $C_\alpha = C_\beta \times S_\alpha$ and $A = A_\alpha$ we see that the result holds in this case.

Now let α be a limit ordinal. For every $\beta < \alpha$ there is a simple FC_β-module A_β such that $C_{C_\beta}(A_\beta) = 1$. It follows from the previous part of the proof that we can choose the module A_β such that $A_\beta \leq A_{\beta+1}$ for all $\beta < \alpha$. Let A_α be the injective limit of the A_β for $\beta < \alpha$. Then $A_\alpha = \bigcup_{\beta<\alpha} D_\beta$ where $D_\beta \cong A_\beta$ for all $\beta < \alpha$ and we may regard A_α as an FC_α-module. Let E be an FC_α-submodule of A_α and assume that $E \neq 0$. Then there is a least ordinal δ such that $E \cap D_\delta \neq 0$. It is clear that δ is not a limit ordinal. Since $E \cap D_\delta$ is a non-zero FC_δ-submodule of the simple FC_δ-module D_δ we have $D_\delta = E \cap D_\delta$ so $D_\delta \leq E$. By Lemma 2.6 we have

$$D_{\delta+1} = \oplus_{z \in Z} zD_\delta \text{ for some (finite) subset } Z \subseteq C_{\delta+1}.$$

Since E is an $FC_{\delta+1}$-submodule and $D_\delta \leq E$ it follows that $zD_\delta \leq E$ for each $z \in Z$. Therefore $D_{\delta+1} \leq E$. An easy induction implies that $D_\beta \leq E$ for all $\beta < \alpha$ and hence $A_\alpha = \bigcup_{\beta<\alpha} D_\beta \leq E$. From this it follows that A_α is a simple FC_α-module.

Finally, let $L = C_{C_\alpha}(A_\alpha)$. For all $\beta < \alpha$ we have

$$L \cap C_\beta = C_{C_\beta}(A_\alpha) \leq C_{C_\beta}(A_\beta) = 1$$

and since $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ it follows that $L = 1$ and so $C_{C_\alpha}(A_\alpha) = 1$ which completes the induction.

For $\alpha = \gamma$ we obtain a simple FS-module $A = A_\alpha$ such that $C_S(A) = 1$. □

Our final result in this chapter gives conditions under which a CC-group can be an irreducible subgroup of $GL(F, A)$ over some field F.

Theorem 2.23. *Let F be a field and let G be an infinite CC-group. Let Q be a subgroup of $\boldsymbol{Soc}_{ab}(G)$ such that $\boldsymbol{Soc}_{ab}(G)/Q$ is locally cyclic and $\mathbf{core}_G\, Q = 1$.*

(i) *Let F be locally finite of characteristic the prime p and suppose that $\boldsymbol{Soc}_{ab}(G)/Q$ is a p'-subgroup. If G has finite 0-rank and every quasi-socle of G coincides with the socle, then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$;*

(ii) *Suppose F is not locally finite and that if $\boldsymbol{char}(F) = p$ is a prime, then $\boldsymbol{Soc}_{ab}(G)/Q$ is a p'-subgroup. Then there exists a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$;*

(iii) *Let F be locally finite of characteristic the prime p and suppose that $\boldsymbol{Soc}_{ab}(G)/Q$ is a p'-subgroup. If G has infinite 0-rank, then there is a vector space A over F such that G is isomorphic to an irreducible subgroup of $GL(F, A)$.*

Proof. (i) Let $S = \mathbf{Soc}(G)$ and let $R = \mathbf{Soc}_{\mathrm{ab}}(G)$. Then

$$S = R \times T$$

where T is a direct product of finite non-abelian simple groups. By Theorem 2.10 there exists a simple FR-module B such that $C_R(B) = Q$ and by Lemma 2.22 there exists a simple FT-module C such that $C_T(C) = 1$. Let $U = B \otimes_F C$ and consider U as an $F(R \times T)$-module using [53, Corollary B.1.12]. Using well-known properties of the tensor product we see that U is semisimple as an FR-module and there exists $X \subseteq S$ such that $U = \bigoplus_{x \in X} xC$. Choose an FS-composition factor E of U. Then E is a simple FS-module and we let $L = C_S(E)$. It follows that there is a subset X_1 of X such that $E \cong \bigoplus_{x \in X_1} xC$. This implies that

$$L \cap T = C_T(E) \leq C_T(C) = 1$$

and hence $L \leq C_S(T)$. Since T is a direct product of finite non-abelian simple groups we also have $C_S(T) = R$, so $L \leq R$. Similarly, there are subsets $Y_1 \subseteq Y \subseteq S$ such that

$$U = \oplus_{y \in Y} yB \text{ and } E \cong \oplus_{y \in Y_1} yB.$$

We deduce that $L = C_R(B) = Q$.

Form $V = FG \otimes_{FS} E$ and take A to be an FG-composition factor of V. Of course A is a simple FG-module. Once again there are subsets $Z_1 \subseteq Z \subseteq G$ such that

$$V = \oplus_{z \in Z} zE \text{and } A \cong \oplus_{z \in Z_1} zE.$$

Assume, for a contradiction, that $C_G(A) \neq 1$. Since every quasi-socle coincides with S Proposition 2.20 implies that $C_S(A) = C_G(A) \cap S \neq 1$. However

$$C_S(A) \leq C_S(gE) = gC_S(E)g^{-1} = gQg^{-1}, \text{ for each } g \in G.$$

Hence

$$C_S(A) \leq \bigcap_{g \in G} gQg^{-1} = \mathbf{core}_G\, Q = 1,$$

which is a contradiction. It follows that $C_G(A) = 1$.

(ii) Suppose now that the field F is not locally finite and that $\mathbf{Soc}_{\mathrm{ab}}(G)/Q$ is a p'-group whenever F has prime characteristic p. Let $S = \mathbf{QSoc}(G)$ be a quasi-socle of G. Write

$$S = \mathbf{Soc}_{\mathrm{ab}}(G) \times T \times D,$$

where T is a direct product of finite non-abelian simple groups and D is a direct product of G-invariant infinite cyclic groups. Let

$$R = \mathbf{Soc}_{\mathrm{ab}}(G) \times D.$$

Since $\mathbf{Soc}_{\mathrm{ab}}(G)/Q$ is a locally cyclic subgroup of the abelian group R/Q Theorem 2.10 implies the existence of a simple FR-module B such that $C_R(B) = Q$ and by Lemma 2.22 there is a simple FT-module C such that $C_T(C) = 1$. Let $U = B \otimes_F C$ and consider U as an $F(R \times T)$-module using [53, Corollary B.1.12]. Let E be an FS-composition factor of U and note that E is a simple FS-module. As above we can prove that $C_S(E) = Q$.

Form $V = FG \otimes_{FS} E$ and take A to be an FG-composition factor of V so that A is a simple FG-module. We may again write

$$V = \oplus_{z \in Z} zE \text{ and } A \cong \oplus z \in Z_1 zE$$

for certain subsets $Z_1 \subseteq Z \subseteq G$. If H is a nontrivial normal subgroup of G, then Proposition 2.20 implies that

$$H \cap \mathbf{QSoc}(G) = H \cap \mathbf{Soc}(G) \neq 1.$$

Assume, for a contradiction, that $C_G(A) \neq 1$. Then $C_S(A) \neq 1$.We obtain

$$C_S(A) \leq C_S(gE) = gC_S(E)g^{-1} = gQg^{-1}, \text{ for each } g \in G.$$

Hence $C_S(A) \leq \bigcap_{g \in G} gQg^{-1} = 1$, which is the contradiction sought. Hence $C_G(A) = 1$.

The proof of (iii) is similar. □

We note that in the paper [113] of L. A. Kurdachenko and J. Otal the question of the existence of simple DG-modules was studied in the more general situation when D is a Dedekind domain and G is a CC-group.

We have confined ourselves here to an exposition of certain of those results which give a fairly clear picture as to our current understanding of this problem. We note also that the question of the existence of simple DG-modules over a hyperfinite group G (with D a Dedekind domain) was considered in the paper [123] of L. A. Kurdachenko and I. Ya. Subbotin. The question of the existence of simple FG-modules over a soluble group of finite special rank was discussed in the paper [201] of A. V. Tushev. Some sufficient conditions for the existence of simple FG-modules were considered in the paper [198] of F. Szechtman.

Chapter 3

Linear Groups Which Are Close to Irreducible

In Chapter 2 we considered infinite dimensional irreducible linear groups. The concrete description of such groups turned out to only be possible for groups sufficiently close to being abelian and as we saw earlier a definitive study of infinite dimensional linear groups is really presently only possible in the presence of certain restrictions. The question arises as to what type of restrictions will be useful here. This situation is similar to the one that occurred during the time when the theory of infinite groups was in its infancy. At that time the theory of finite groups was already quite well established and many of its fundamental results had been obtained. It was therefore natural to use the experience already accumulated in the theory of finite groups when trying to study infinite groups. Thus, for example, in the theory of infinite groups a large amount of work has been done concerning groups with finiteness conditions where many of the results of finite group theory have been used to generalize that theory. In this way group theory has been enriched with many new fundamental results and techniques.

The theory of finite dimensional linear groups is one of the most developed algebraic theories, so it is natural to rely on this theory when building the theory of infinite dimensional linear groups and in this way we come to an approach based on finite dimensional conditions. The effectiveness of this approach has already been demonstrated in the theory of finitary linear groups (see the survey paper of R. E. Phillips [175], for example). In this book we give other approaches and results that are obtained as a result of the application of finite dimensional conditions. In this chapter we consider several types of linear groups which are, in one sense or another, close to irreducible.

Quasi-Irreducible Linear Groups

We shall, as usual, let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If the group G is irreducible, then every proper G-invariant subspace of A is trivial and, in particular, has finite dimension. Thus a linear group G whose action on the (infinite dimensional) vector space A has the property

- every proper G-invariant subspace of A has finite dimension

can be considered as a generalization of an irreducible linear group. In this case suppose that A_1 is a proper G-invariant subspace of A. Then A_1 has finite F-dimension and we may then consider A/A_1. If this is not G-chief, then there is a finite dimenional G-invariant subspace A_2 such that $A_1 \lneqq A_2$ and we may then consider A/A_2. Either this is G-chief or there is a finite dimensional G-invariant subspace A_3 strictly containing A_2 and so on. We may therefore construct an ascending chain

$$A_1 \leq A_2 \leq \cdots \leq A_n \leq \dots$$

of G-invariant subspaces, each finite dimensional. There are therefore two situations that arise here for an infinite dimensional linear group G:

(i) Every proper G-invariant subspace of A has finite dimension and A is the ascending union of certain proper G-invariant subspaces, or

(ii) A contains a proper G-invariant finite dimensional subspace B such that the factor A/B is G-chief.

A linear group G that satisfies condition (i) is said to be *quasi-irreducible.*

The study of quasi-irreducible linear groups was begun by D. I. Zaitsev in the paper [223] for the case when F is a finite field. The case when F is an arbitrary field was considered in the papers of L. A. Kurdachenko and I. Ya. Subbotin [124, 125]. In turn these results were generalized by L. A. Kurdachenko [104] to the case of modules over group rings RG when R is a Dedekind domain. We shall now discuss the structure of quasi-irreducible linear groups as determined in the articles [104, 124, 125]. We also mention here the papers [102, 121] of L. A. Kurdachenko and I. Ya. Subbotin.

We begin with the following result which is immediate from the definition of a quasi-irreducible group.

Lemma 3.1. *Let A be an (infinite dimensional) vector space over the field F and let G be a subgroup of $GL(F, A)$. If G is quasi-irreducible, then A does not decompose into a direct sum of two proper G-invariant subspaces.*

This situation is analogous to that of quasifinite groups, those (infinite) groups in which all proper subgroups are finite.

When F is a field, A is a vector space over F and G is a subgroup of $GL(F, A)$, then we let $\mathfrak{M}$ be the family of all minimal G-invariant subspaces of A. We let $\mathbf{Soc}_G(A)$ denote the subspace generated by all subspaces in the family $\mathfrak{M}$. Using Lemma 2.5 we see that $\mathbf{Soc}_G(A) = \bigoplus_{\lambda\in\Lambda} A_\lambda$ for some index set Λ and A_λ is a minimal G-invariant subspace for each $\lambda \in \Lambda$. The subspace $\mathbf{Soc}_G(A)$ is called the *G-socle* of the space A.

Corollary 3.2. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If G is quasi-irreducible, then $\boldsymbol{Soc}_G(A)$ is a proper G-invariant subspace of A.*

Proof. Suppose that A contains a proper non-zero G-invariant subspace B. Then B is finite dimensional over F. It follows that B contains a non-zero minimal G-invariant subspace S and hence $\mathbf{Soc}_G(A)$ is itself non-zero. Since $\mathbf{Soc}_G(A)$ is a direct sum of certain minimal G-invariant subspaces, Lemma 3.1 shows that $\mathbf{Soc}_G(A)$ must be a proper subspace. □

Lemma 3.3. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If G is quasi-irreducible, then $\boldsymbol{Soc}_H(A) = A$ for every nontrivial finite normal subgroup H of G.*

Proof. We note immediately that $B = \mathbf{Soc}_H(A)$ is non-zero. To see this, let a be a nontrivial element of A. Since H is finite, the H-invariant subspace X generated by the element a has finite dimension, so X contains a minimal H-invariant subspace and hence $\mathbf{Soc}_H(A)$ is non-zero as claimed. Let $\mathfrak{M}$ be the family of all minimal H-invariant subspaces of A. If $C \in \mathfrak{M}$ and g is an arbitrary element of G, then Lemma 2.6 implies that gC is a minimal H-invariant subspace of A. Hence $gC \in \mathfrak{M}$. Since $\mathbf{Soc}_H(A)$ is generated by all subspaces in the family $\mathfrak{M}$, the subspace B is G-invariant.

Suppose, for a contradiction, that $B \neq A$. In this case B is a proper G-invariant subspace of A so it is finite dimensional and there is an F-subspace C of A such that $A = B \bigoplus C$. Let

$$C_0 = \bigcap_{h \in H} hC.$$

For each element $h \in H$ we have $A/hC = hA/hC \cong A/C$ from which it follows that

$$\mathbf{dim}_F(A/hC) = \mathbf{dim}_F(A/C) = \mathbf{dim}_F(B)$$

is finite. Since H is finite the subspace C_0 therefore has finite codimension in A and in particular C_0 is non-zero. The construction of C_0 implies that it is H-invariant and since $C_0 \leq C$ we have that $C_0 \cap B = 0$. Let c_1 be an arbitrary element of C_0 and let C_1 be the H-invariant subspace generated by c_1. Since H is finite we once again have that $\mathbf{dim}_F(C_1)$ is finite and hence C_1 contains a non-zero minimal H-invariant subspace C_2. However $C_2 \leq \mathbf{Soc}_H(A) = B$. On the other hand $C_2 \leq C_0$ and $C_0 \cap B = 0$, so that $C_2 = 0$, which is the contradiction sought. The result now follows. □

Corollary 3.4. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If G is quasi-irreducible, then $C_A(H) = 0$ and $[H, A] = A$ for every nontrivial finite normal subgroup H of G.*

Proof. Immediately Lemma 3.3 shows that $\mathbf{Soc}_H(A) = A$, so that $A = \bigoplus_{n \in \mathbb{N}} C_n$, where C_n is a minimal H-invariant subspace of A, for each $n \in \mathbb{N}$. Since $[H, C_n]$ is also H-invariant we have the following two possibilities: either $[H, C_n] = C_n$ or $[H, C_n] = 0$. Thus we obtain the direct decomposition

$$A = C_A(H) \oplus [H, A].$$

An application of Lemma 3.1 leads immediately to the equations $C_A(H) = 0$ and $[H, A] = A$, as required. □

Corollary 3.5. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If G is quasi-irreducible and B is a proper non-zero G-invariant subspace of A, then $C_H(B) = 1$ for every nontrivial finite normal subgroup H of G.*

Proof. Since the subspace B is G-invariant it follows that $C_G(B)$ is a normal subgroup of G. Then the subgroup $C_H(B) = C_G(B) \cap H$ is also a normal subgroup of G. Since $B \le C_A(C_H(B))$, Corollary 3.4 shows that $C_H(B)$ must be trivial, as required. □

Corollary 3.6. *Let A be a vector space over the field F, let G be a subgroup of $GL(F, A)$ and let H be a nontrivial normal subgroup of G which has an ascending series of G-invariant subgroups*

$$1 = H_0 \le H_1 \le \dots H_\alpha \le H_{\alpha+1} \le \dots H_\gamma = H$$

whose factors are finite. If G is quasi-irreducible and B is a proper non-zero G-invariant subspace of A, then $C_H(B) = 1$.

Proof. We proceed by induction on α. If $\alpha = 1$, then the result follows using Corollary 3.5. Let $\alpha > 1$ and suppose we have already proved that $1 = H_\beta \cap C_H(B)$ for all $\beta < \alpha$. Let $C_\alpha = H_\alpha \cap C_H(B)$. If α is a limit ordinal, then

$$H_\alpha = \cup_{\beta<\alpha} H_\beta$$

and it follows that

$$C_\alpha = \cup_{\beta<\alpha}(C_\alpha \cap H_\beta) = \cup_{\beta<\alpha} C_\beta = 1.$$

If α is not a limit ordinal, then let $L = H_{\alpha-1}$. Suppose that $C_\alpha \ne 1$. Then $L \cap C_\alpha = 1$ and we have

$$C_\alpha \cong C_\alpha/(L \cap C_\alpha) \cong (LC_\alpha)/L \le H_\alpha/L.$$

It follows that C_α is finite which gives us a contradiction with Corollary 3.5 since $B \le C_A(C_\alpha)$. Consequently $C_\alpha = 1$ and setting $\alpha = \gamma$ gives us $C_H(B) = 1$ which is the desired result. □

We recall from Chapter 2 that if G is a group and $x \in G$, then

$$x^G = \{x^g = g^{-1}xg | g \in G\} \text{ and } \mathbf{FC}(G) = \{x \in G | x^G \text{ is finite}\},$$

this latter characteristic subgroup of G being the *FC-center* of G.

Beginning with the FC-center we may construct the *upper FC-central series* of the group G, which is a series of characteristic subgroups of G,

$$1 = C_0 \le C_1 \le \dots C_\alpha \le C_{\alpha+1} \le \dots C_\gamma,$$

defined by

$$\begin{aligned} C_1 &= \mathbf{FC}(G), \\ C_{\alpha+1}/C_\alpha &= \mathbf{FC}(G/C_\alpha) \text{ for ordinals } \alpha, \\ C_\lambda &= \bigcup_{\beta<\lambda} C_\beta \text{ for limit ordinals } \lambda. \end{aligned}$$

The last term C_γ of this series is called the *upper FC-hypercenter* of the group G and if $G = C_\gamma$, then G is called *FC-hypercentral*. The reader is referred to [183] for further information concerning FC-hypercentral groups.

We note that if $x \in \mathbf{FC}(G)$ and the element x has finite order, then the normal closure $\langle x \rangle^G$ is finite (see [52, Proposition 1.5.2], for example). Thus if H is a G-invariant periodic subgroup of $\mathbf{FC}(G)$, then H has an ascending series of G-invariant subgroups whose factors are finite. In turn this fact implies that if H is such a periodic G-invariant subgroup of the upper FC-hypercenter of G, then H has an ascending series of G-invariant subgroups whose factors are finite.

Let G be a group and π be some set of primes. If $\mathcal{S}$ is a family of periodic normal π-subgroups of G, then clearly the subgroup generated by all subgroups in the family $\mathcal{S}$ is also a π-group. It follows that every group G has a greatest normal π-subgroup which we denote by $\mathbf{O}_\pi(G)$.

We let $T_n(F)$ denote the group of $n \times n$ upper triangular matrices over the field F and recall that $UT_n(F)$ is the group of all unitriangular $n \times n$ matrices.

Corollary 3.7. *Let A be a vector space over the field F, let G be a subgroup of $GL(F, A)$ and let H be a nontrivial periodic G-invariant subgroup of the upper FC-hypercenter of G. If G is quasi-irreducible, then H is an abelian-by-finite subgroup having finite special rank. Furthermore, if $\boldsymbol{char}(F) = p$ is a prime, then $\boldsymbol{O}_p(H)$ is trivial.*

Proof. Let B be a proper non-zero G-invariant subspace of A. By Corollary 3.6 $C_H(B) = 1$, so we may think of H as a subgroup of $GL_n(F)$ where n is the F-dimension of B. By Kargapolov's theorem (see [202, Corollary 9.31], for example) H is soluble-by-finite and by Maltsev's theorem [202, Theorem 3.6] H contains a normal subgroup S of finite index with the property that $g^{-1}Sg \leq T_n(F_1)$ for some finite field extension F_1 of F and some element g of G. Let

$$U = g^{-1}Sg \cap UT_n(F_1) \text{ and } V = gUg^{-1}.$$

Then V is normal in S. If $\mathbf{char}\,(F) = p$ is prime, then $UT_n(F_1)$ is a bounded nilpotent p-group. If we suppose that the subgroup U is nontrivial, then H contains a finite nontrivial G-invariant p-subgroup P. But in this case it is easy to see that $C_A(P) \neq 0$, which is a contradiction to Corollary 3.4. If $\mathbf{char}\,(F) = 0$, then $UT_n(F_1)$ is a torsion-free nilpotent group and in this case the subgroup U is also trivial, since H is periodic. Hence in all cases $U = 1$. It follows that $g^{-1}Sg$ is isomorphic to a subgroup of $T_n(F_1)/UT_n(F_1)$. However,

$$T_n(F_1)/UT_n(F_1) \cong \underbrace{U(F_1) \times \cdots \times U(F_1)}_{n}.$$

The fact that the periodic subgroups of the multiplicative group of a field are locally cyclic (see [98, Proposition 4.4.1], for example) then implies that S is an abelian group of finite special rank. □

Our next result is then very easy to deduce.

Corollary 3.8. *Let A be a vector space over the field F, let G be a hypercentral subgroup of $GL(F, A)$ and let H be a nontrivial periodic G-invariant subgroup of the upper hypercenter of G. If G is quasi-irreducible, then H is an abelian-by-finite subgroup of finite special rank. Furthermore, if $\boldsymbol{char}(F) = p$ is prime, then H is a p'-subgroup.*

Proof. The upper FC-hypercenter contains the upper hypercenter so we may simply apply Corollary 3.7. □

As a particular case of Corollary 3.8 we take H to be the torsion subgroup of G.

Corollary 3.9. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. If G is quasi-irreducible, then $\boldsymbol{Tor}(G)$ is an abelian-by-finite subgroup of finite special rank. Moreover, if $\boldsymbol{char}(F) = p$ is prime, then $\boldsymbol{Tor}(G)$ is a p'-group.*

Let $S = \mathbf{Soc}_G(A)$. By Corollary 3.2 S is a proper non-zero G-invariant subspace of A in the case when G is quasi-irreducible and, in particular, S has finite F-dimension, n, say. If $C_G(S) = 1$, then we may think of G as being a subgroup of $GL_n(F)$ and indeed G is completely reducible, since S is a finite direct sum of minimal G-invariant subspaces.

The following series of results was obtained in the work of L. A. Kurdachenko [104] in a more general setting. We have here adapted these results for the case of linear groups. In the case when $C_G(\mathbf{Soc}_G(A))$ is trivial the following result can be obtained for locally radical groups. We recall that a group G is *radical* if it has a (possibly infinite) ascending series of characteristic subgroups, starting at the identity, each of whose factors is locally nilpotent. Then the group is *locally radical* if every finitely generated subgroup is radical.

Proposition 3.10. *Let A be a vector space over the field F and let G be a locally radical subgroup of $GL(F, A)$. If G is quasi-irreducible and $C_G(\boldsymbol{Soc}_G(A))$ is trivial, then G is an abelian-by-finite group.*

Proof. As we noted above, we may think of G as being a completely reducible subgroup of $GL_n(F)$, where $n = \mathbf{dim}_F(\mathbf{Soc}_G(A))$. In this case G is soluble (see [52, Corollary 1.4.9], for example) and the theorem of Maltsev (see [202, Lemma 3.5], for example) implies that G is abelian-by-finite. □

It is appropriate to now give the following definitions. Let R be an integral domain and let A be an R-module. Let

$$\mathbf{Tor}_R(A) = \{a \in A | \mathbf{Ann}_R(a) \neq 0\}.$$

It is easy to see that $\mathbf{Tor}_R(A)$ is an R-submodule of A and we call $\mathbf{Tor}_R(A)$ the *R-periodic part* of the module A. We say that A is *periodic as an R-module* or, simply, that A is *R-periodic*, if $\mathbf{Tor}_R(A) = A$. In this case it follows that $\mathbf{Ann}_R(a) \neq 0$, for each element $a \in A$. We say that A is *R-torsion-free* if $\mathbf{Tor}_R(A) = 0$.

We define the *R-assassinator of A* to be the set

$$\mathbf{Ass}_R(A) = \{P | P \text{ is a prime ideal of } R \text{ such that } \mathbf{Ann}_A(P) \neq 0\}.$$

If U is an ideal of R, then we set

$$A_U = \{a \in A | U^n a = 0 \text{ for some natural number } n\}.$$

It is easy to see that A_U is an R-submodule of A.

The R-submodule A_U is called the *U-component of A*. If $A = A_U$, then A is called a U-module. Furthermore, let

$$\Omega_{U,k}(A) = \{a \in A | U^k a = 0 \text{ for this fixed } k\}.$$

It is easy to see that $\Omega_{U,k}(A)$ is an R-submodule of A, that

$$\Omega_{U,1}(A) \leq \Omega_{U,2}(A) \leq \cdots \leq \Omega_{U,k}(A) \leq \cdots \leq A_U \text{ and that}$$
$$A_U = \bigcup_{k \in \mathbb{N}} \Omega_{U,k}(A).$$

If D is a Dedekind domain, then we can obtain a "primary decomposition" for D-periodic modules. We quote the following well-known result, the proof of which can be found in the monograph [119, Corollary 3.8] of L. A. Kurdachenko, N. N. Semko and I. Ya. Subbotin for example.

Proposition 3.11. *Let D be a Dedekind domain and let A be a D-module. Suppose that A is D-periodic and let $\pi = \boldsymbol{Ass}_D(A)$. Then*

(i) $A = \bigoplus_{P \in \pi} A_P$;

(ii) if B is a D-submodule of A, then

$$(A/B)_P = (A_P + B)/B \text{ and } A/B = \oplus_{P \in \pi}(A_P + B)/B.$$

If G is a group and $1 \neq x \in \zeta(G)$ we shall let $\langle t_x \rangle$ denote an infinite cyclic group and for the field F we shall let $D(x)$ denote the group algebra $F\langle t_x \rangle$ of $\langle t_x \rangle$ over F. We note that $D(x)$ is a principal ideal domain. Now let A be a vector space over F and let G be a subgroup of $GL(F, A)$. We can define an action of t_x on A by setting

$$t_x a = xa \text{ for all } a \in A.$$

We can extend this action in a natural way to an action of the ring $D(x)$ on A and there is then a natural action of the group ring $D(x)G$ on A. Thus we may consider A as a left $D(x)G$-module.

Lemma 3.12. *Let A be a vector space over the field F and let G be a quasi-irreducible subgroup of $GL(F, A)$. If g is an arbitrary element of $\zeta(G)$, then either $\boldsymbol{Ann}_{D(g)}(A)$ is a maximal ideal of the ring $D(g)$ or $yA = A$ for every element $y \in D(g)$.*

Proof. Let x be an arbitrary element of the ring $D(g)$. Since $g \in \zeta(G)$, the subspace xA is G-invariant. Suppose, for a contradiction, that the subspace xA is proper and non-zero. Then xA has finite dimension. The mapping $\phi : A \longrightarrow A$ defined by $\phi(a) = xa$ for each $a \in A$ is a $D(g)G$-endomorphism of A such that $\mathbf{Im}\,(\phi) = xA$ and $\mathbf{ker}(\phi) = \mathbf{Ann}_A(x)$. We remark that $\mathbf{ker}(\phi)$ is also a G-invariant subspace and our assumption implies that it is a proper subspace. Hence $\mathbf{ker}(\phi)$ also has finite dimension. Using the isomorphism

$$A/\mathbf{ker}(\phi) \cong \mathbf{Im}\,(\phi) = xA$$

we deduce that A has finite dimension. This is the required contradiction and it follows that either $xA = A$ or $xA = 0$.

If $xA = 0$ for some $0 \neq x \in D(g)$ then the $D(g)$-module A is periodic. It follows that $A = \bigoplus_{P\in\pi} A_P$, where A_P is the non-zero P-component of A and $\pi = \mathbf{Ass}_{D(g)}(A)$, by Proposition 3.11. The fact that $g \in \zeta(G)$ implies that each P-component A_P is a G-invariant subspace of A. Then Lemma 3.1 shows that $A = A_P$ for some prime ideal P of the ring $D(g)$. Since $D(g)$ is a principal ideal domain, the ideal P is generated by some polynomial $f_P(t_g)$. Furthermore, every prime ideal of a principal ideal domain is maximal, from which it follows that the polynomial $f_P(t_g)$ is irreducible. By what we proved above the subspace $f_P(t_g)A$ must be zero and it follows that $\mathbf{Ann}_{D(g)}(A) = P$ is a maximal ideal of $D(g)$. The result follows. □

We shall also need the following technical result.

Lemma 3.13. *Let R be a ring and let A be an R-module. Suppose that $A = A_1 \oplus \cdots \oplus A_n$ where A_j is a simple R-module for each $1 \leq j \leq n$. If $\boldsymbol{Ann}_R(A_i) = \boldsymbol{Ann}_R(A_j)$ for all i, j, then for each non-zero element $a \in A$ the R-submodule Ra is isomorphic to A_j for each $1 \leq j \leq n$.*

Proof. Let

$$a = a_1 + \cdots + a_n, \tag{3.1}$$

where $a_j \in A_j$ for $1 \leq j \leq n$. Without loss of generality we may suppose that $a_1 \neq 0$. Set $D = A_2 \oplus \cdots \oplus A_n$ and let $B = Ra$. Then $(B + D)/D$ is a non-zero submodule of A/D. The isomorphism $A/D \cong_R A_1$ shows that A/D is a simple R-module and it follows that $A/D = (B+D)/D$. Hence $A = B + D$. Using the isomorphism $(B+D)/D \cong_R B/(B \cap D)$ we see that $A_1 \cong_R B/(B \cap D)$. Let $c \in B \cap D$ so that $c = ya$ for some element $y \in R$. Thus

$$c = ya_1 + \cdots + ya_n.$$

On the other hand $ya_1 = 0$ since $c \in D$, so that $y \in \mathbf{Ann}_R(a_1)$. Since A_1 is a simple R-module and $a_1 \neq 0$ it follows that $\mathbf{Ann}_R(a_1) = \mathbf{Ann}_R(A_1)$. Thus, by hypothesis, $y \in \mathbf{Ann}_R(A_j)$ for each j with $1 \leq j \leq n$. Hence $c = 0$, so $B \cap D = 0$ and it follows that $B \cong_R A_1$. By choosing a such that $a_i \neq 0$ for each i in (3.1) we may deduce that $A_i \cong A_j$ for all i, j. This completes the proof. □

The next result follows immediately.

Corollary 3.14. *Let R be a ring, let A be an R-module and let Λ be an index set. Suppose that $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ where A_λ is a simple R-module for each $\lambda \in \Lambda$. If $\boldsymbol{Ann}_R(A_\lambda) = \boldsymbol{Ann}_R(A_\mu)$ for all $\lambda, \mu \in \Lambda$, then for each non-zero element $a \in A$ the R-submodule Ra is isomorphic to A_λ for each $\lambda \in \Lambda$.*

We next consider the case when G is abelian.

Lemma 3.15. *Let A be a vector space over the field F and let G be an abelian quasi-irreducible subgroup of $GL(F, A)$. Then the group G contains an element y such that $\boldsymbol{Ann}_{D(y)}(A) = 0$.*

Proof. Suppose that $\mathbf{Ann}_{D(x)}(A) \neq 0$ for all $x \in G$. Let

$$S_1 = \mathbf{Soc}_G(A) \text{ and } S_2/S_1 = \mathbf{Soc}_G(A/S_1).$$

If $0 \neq a \in S_2$, then $(FG)a \cap S_1 \neq 0$ and hence there is an element $u \in FG$ such that $0 \neq ua \in S_1$. Let

$$u = \alpha_1 x_1 + \cdots + \alpha_m x_m$$

for certain elements $\alpha_1, \ldots, \alpha_m \in F, x_1, \ldots, x_m \in G$. By our assumption $\mathbf{Ann}_{D(x_1)}(A) \neq 0$ and Lemma 3.12 implies that in this case $\mathbf{Ann}_{D(x_1)}(A) = P_1$ is a maximal ideal of $D(x_1)$. Let

$$F_1 = D(x_1)/P_1,$$

a field and note that A is an F_1G-module. Since A has non-zero annihilator in $D(x_2), A$ has non-zero annihilator P_2 in the group ring $F_1\langle t_2 \rangle$ where $t_2 = t_{x_2}$. Using Lemma 3.12 again we see that P_2 is a maximal ideal of the ring $F_1\langle t_2 \rangle$. In other words it follows that $\mathbf{Ann}_{F\langle x_1, x_2 \rangle}(A)$ is a maximal ideal of the ring $F\langle x_1, x_2 \rangle$. Using the same arguments, after finitely many steps we deduce that $\mathbf{Ann}_{F\langle x_1 \ldots, x_m \rangle}(A)$ is a maximal ideal of the ring $F\langle x_1, \ldots, x_m \rangle$. Hence the cyclic $F\langle x_1, \ldots, x_m \rangle$-submodule generated by the element a is simple. Since S_1 is G-invariant it is an $F\langle x_1 \ldots, x_m \rangle$-submodule so that $F\langle x_1, \ldots, x_m \rangle a \cap S_1 = 0$. On the other hand $0 \neq ua \in F\langle x_1, \ldots, x_m \rangle a \cap S_1$ and we obtain a contradiction. This proves the lemma. □

Corollary 3.16. *Let A be a vector space over the field F and let G be an abelian quasi-irreducible subgroup of $GL(F, A)$. Then G is not periodic.*

Proof. Suppose, to the contrary, that G is periodic and let x be an arbitrary element of G. Suppose that x has order n. Then $t_{x^n} - 1 \in \mathbf{Ann}_{D(x)}(A)$ and, in particular, $\mathbf{Ann}_{D(x)}(A) \neq 0$. Since this is true for all elements of the group G, we obtain a contradiction with Lemma 3.15. □

Proposition 3.17. *Let A be a vector space over the field F and let G be a quasi-irreducible subgroup of $GL(F, A)$. Suppose that H is a normal subgroup of finite index in G and let $\{g_1, \ldots, g_t\}$ be a transversal to H in G. Then A contains an H-invariant subspace B satisfying the following properties:*

(i) H is a quasi-irreducible subgroup of $GL(F, B)$;

(ii) $A = g_1B + \cdots + g_tB$;

(iii) H is isomorphic to a subgroup of

$$H/(g_1^{-1}C_H(B)g_1) \times \cdots \times H/(g_t^{-1}C_H(B)g_t).$$

Proof. Let $\mathfrak{M}$ denote the family of all infinite dimensional H-invariant subspaces of A. If U is a proper G-invariant subspace of A, then U has finite dimension over F. Furthermore, it then follows that A satisfies the minimal condition on G-invariant subspaces. Since the subgroup H has finite index in G, the space A also satisfies the minimal condition on H-invariant subspaces by a result of J. S. Wilson (see [116, Theorem 5.2], for example). Therefore the family $\mathfrak{M}$ has a minimal element B. By the choice of B, every proper H-invariant subspace of B has finite dimension over F. Since G is a quasi-irreducible subgroup of $GL(F, A)$, the space A has an ascending series of proper G-invariant subspaces

$$C_1 \leq C_2 \leq \cdots \leq C_n \leq \ldots$$

such that

$$A = \bigcup_{n \in \mathbb{N}} C_n.$$

Furthermore, each subspace C_n has finite dimension over F. It follows that

$$B = \bigcup_{n \in \mathbb{N}} (B \cap C_n).$$

The space B has infinite dimension and hence $B \neq B \cap C_n$ for all natural numbers n. Furthermore the subspaces $B \cap C_n$ are H-invariant and again the choice of B shows that every proper H-invariant subspace of B

is finite dimensional. It follows that H is a quasi-irreducible subgroup of $GL(F, B)$.

The mapping $A \longmapsto g_i A$ for $a \in A$ is an F-isomorphism and clearly H-invariant subspaces are mapped to H-invariant subspaces. Thus H is a quasi-irreducible subgroup of $GL(F, g_i B)$ for $1 \leq i \leq t$. Since $g_1 B + \cdots + g_t B$ is an infinite dimensional G-invariant subspace of A we deduce that

$$A = g_1 B + \cdots + g_t B.$$

From the equality $C_H(g_i B) = g_i^{-1} C_H(B) g_i$ we also see that

$$\bigcap_{1 \leq i \leq t} g_i^{-1} C_H(B) g_i = C_H(A) = 1,$$

so Remak's theorem gives an embedding of H into

$$H/(g_1^{-1} C_H(B) g_1) \times \cdots \times H/(g_t^{-1} C_H(B) g_t).$$

This completes the proof. □

Lemma 3.18. *Let A be a vector space over the field F and let G be an abelian quasi-irreducible subgroup of $GL(F, A)$. If g is an arbitrary element of G, then the $D(g)$-module A is periodic.*

Proof. By the definition of quasi-irreducible group the vector space A has a family $\{A_n | n \in \mathbb{N}\}$ of G-invariant subspaces such that $A = \bigcup_{n \in \mathbb{N}} A_n$ where each subspace A_n has finite dimension over F. If a is an arbitrary element of A, then there exists a natural number k such that $a \in A_k$. Since the subspace A_k is G-invariant, $(F\langle g \rangle)a = D(g)a \leq A_k$. By definition $D(g)$ is the group ring of an infinite cyclic group to that it has infinite dimension over F. Then the isomorphism

$$D(g)a \cong_{D(g)} D(g)/\mathbf{Ann}_{D(g)}(a)$$

together with the fact that $\mathbf{dim}_F(A_k)$ is finite imply that $\mathbf{Ann}_{D(g)}(a)$ is non-zero. Consequently A is periodic as a $D(g)$-module, as required. □

We now require some further notation and terminology.

Let R be an integral domain and let A be an R-module. We say that A is *R-divisible* if $xA = A$ for every non-zero element $x \in R$.

Let D be a Dedekind domain and let A be a simple D-module. Then $A \cong D/P$ for some maximal ideal P. We note that D/P^k and P/P^{k+1} are isomorphic as D-modules, for all $k \in \mathbb{N}$ (see [119, Corollary 1.28], for example). In particular, the D-module D/P^k is embedded in the D-module D/P^{k+1} for each $k \in \mathbb{N}$. Hence we may consider the injective limit of the family of D-modules $\{D/P^k | k \in \mathbb{N}\}$ and we denote this injective limit by C_{P^∞}.

The D-module C_{P^∞} is called a *Prüfer P-module* (which is reminiscent to the reader of the Prüfer p-group, C_{p^∞}, for the prime p).

It follows from its construction that C_{P^∞} is a P-module and moreover

$$\mathbf{\Omega}_{P,k}(C_{P^\infty}) \cong_D D/P^k \text{ for all } k \in \mathbb{N}.$$

Furthermore,

$$\mathbf{\Omega}_{P,k+1}(C_{P^\infty})/\mathbf{\Omega}_{P,k}(C_{P^\infty}) \cong (D/P^{k+1})/(P/P^{k+1}) \cong D/P.$$

In particular, $\mathbf{\Omega}_{P,1}(C_{P^\infty}) \cong D/P$ is simple. Also, if C is a proper D-submodule of C_{P^∞}, then there exists a natural number k such that $C = \mathbf{\Omega}_{P,k}(C_{P^\infty})$. Similarly, if $b \notin \mathbf{\Omega}_{P,k-1}(C_{P^\infty})$, then $C = bD$.

We note that if A is an FG-module, then the intersection $\mathbf{mon}_{FG}(A)$ of all non-zero FG-submodules of A is called the *FG-monolith* of A. If the module A has a non-zero FG-monolith, then we say that A is an *FG-monolithic module.*

We also observe that a Prüfer P-module is monolithic and its monolith coincides with $\mathbf{\Omega}_{P,1}(C_{P^\infty})$.

Lemma 3.19. *Let A be a vector space over the field F and let G be an abelian quasi-irreducible subgroup of $GL(F, A)$. Then the group G contains an element g such that*

$$A = C_1 \oplus \cdots \oplus C_n$$

where C_j is a Prüfer P-module for some maximal ideal P of the ring $D(g)$, for $1 \leq j \leq n$.

Proof. By Lemma 3.15 the group G contains an element g such that $\mathbf{Ann}_{D(g)}(A) = 0$. Lemma 3.12 implies that in this case the $D(g)$-module A is $D(g)$-divisible. Using Lemma 3.18 we see that A is $D(g)$-periodic so that $A = \bigoplus_{P\in\pi} A_P$, where A_P is the non-zero P-component of A and $\pi = \mathbf{Ass}_{D(g)}(A)$ by Proposition 3.11. Since G is abelian each P-component A_P is a G-invariant subspace of A. Then Lemma 3.1 implies that $A = A_P$ for some prime ideal P of the ring $D(g)$ and since in a principal ideal domain every prime ideal is a maximal ideal, it follows that P is a maximal ideal of $D(g)$. Thus P is generated by some irreducible polynomial $f_P(t_g)$. Hence $\mathbf{dim}_F D(g)/P$, which is the degree of $f_P(t_g)$, is finite. Since $\mathbf{Ann}_{D(g)}(A) = 0$ it follows that $\mathbf{\Omega}_{P,1}(A)$ is a proper $D(g)$-submodule of A. Therefore the subspace $\mathbf{\Omega}_{P,1}(A)$ has finite F-dimension. On the other hand there is an index set Λ such that

$$\mathbf{\Omega}_{P,1}(A) = \oplus_{\lambda\in\Lambda} U_\lambda \text{ where } U_\lambda \cong_{D(g)} D(g)/P$$

for each λ in the set Λ.

Hence the index set Λ must be finite so we may assume that $\Lambda = \{1, \ldots, n\}$ for some natural number n. Finally (using [119, Theorem 5.26], for example) we deduce that $A = \bigoplus_{1\leq j\leq n} C_j$ where C_j is a Prüfer P-module for $1 \leq j \leq n$. $\square$

We can now obtain the first main theorem of this section which gives the structure of locally radical quasi-irreducible subgroups G in the case when the centralizer of the G-socle is trivial. Much of the hard work has already been done.

Theorem 3.20. *Let A be a vector space over the field F and let G be a locally radical quasi-irreducible subgroup of $GL(F, A)$. If $C_G(\boldsymbol{Soc}_G(A)) = 1$, then the following assertions hold:*

(i) G contains a normal abelian subgroup U of finite index;

(ii) the torsion subgroup $\boldsymbol{Tor}(U)$ of the subgroup U has finite special rank;

(iii) if $\boldsymbol{char}(F) = p > 0$, then $O_p(G) = 1$;

(iv) the vector space A contains a U-invariant subspace B such that U is a quasi-irreducible subgroup of $GL(F, B)$;

(v) the subgroup U contains an element x of infinite order such that B is P-periodic for some maximal ideal P of the ring $F\langle x\rangle$. Moreover,

$$B = C_1 \oplus \cdots \oplus C_n$$

where C_j is a Prüfer P-module, for $1 \leq j \leq n$;

(vi) $A = B \bigoplus g_1 B \bigoplus \cdots \bigoplus g_t B$, where $\{1, g_1, \ldots, g_t\}$ is a transversal to U in G.

Proof. Statement (i) follows from Proposition 3.10 and statements (ii), (iii) follow from Corollary 3.7. Using Proposition 3.17 we see that the vector space A contains a U-invariant subspace B such that U is a quasi-irreducible subgroup of $GL(F, B)$. By Lemma 3.15 $U \backslash C_U(B)$ contains an element x such that $\mathbf{Ann}_{F\langle x\rangle}(B) = 0$. Clearly the element x has infinite order. By Lemma 3.19

$$B = C_1 \oplus \cdots \oplus C_n$$

where C_j is a Prüfer P-module for some maximal ideal P of the ring $D(x)$ for $1 \leq j \leq n$. The statement (vi) follows from Proposition 3.17. □

We now consider the case when $C_G(\mathbf{Soc}_G(A))$ is nontrivial.

Lemma 3.21. *Let A be a vector space over the field F and let G be a hypercentral quasi-irreducible subgroup of $GL(F, A)$. If $C_G(\boldsymbol{Soc}_G(A))$ is nontrivial, then $C_G(\boldsymbol{Soc}_G(A)) \cap \zeta(G)$ contains an element x of infinite order such that A is $F\langle x\rangle$-periodic and $\boldsymbol{Ass}_{F\langle x\rangle}(A) = \{(x-1)F\langle x\rangle\}$.*

Proof. Let $S_1 = \mathbf{Soc}_G(A)$ and $Z = C_G(S_1) \cap \zeta(G)$. Since $C_G(S_1)$ is a normal nontrivial subgroup of the hypercentral group G it follows that $Z \neq 1$. Suppose that Z contains an element $z \neq 1$ of finite order. Since $S_1 \leq C_A(z)$, $C_A(z) \neq 0$

and we obtain a contradiction with Corollary 3.4. This contradiction shows that Z is a torsion-free subgroup. Let $1 \neq x \in Z$. The mapping $\phi : A \longrightarrow A$ defined by $\phi(a) = (x-1)a$, for each $a \in A$, is an FG-endomorphism of A such that $\mathbf{ker}(\phi) = C_A(x)$ and $\mathbf{Im}\,(\phi) = (x-1)A = [x, A]$. Let $S_2/S_1 = \mathbf{Soc}_G(A/S_1)$. By Corollary 3.2 S_2/S_1 is a proper G-invariant subspace of A/S_1 so

$$S_2/S_1 = E_1/S_1 \oplus \cdots \oplus E_k/S_1$$

where E_j/S_1 is a minimal G-invariant subspace of A/S_1 for $1 \leq j \leq k$. Suppose that S_1 does not contain $\phi(E_j) = [x, E_j]$ for some j. Then

$$\phi(E_j) \cong E_j/(E_j \cap \mathbf{ker}(\phi)).$$

Since E_j/S_1 is a minimal G-invariant subspace of A/S_1 it follows that either $E_j \cap \mathbf{ker}(\phi) = S_1$ or $E_j \cap \mathbf{ker}(\phi) = E_j$. But in the latter case we then have $[x, E_j] = 0$, contradicting our assumption. It follows that $E_j \cap \mathbf{ker}(\phi) = S_1$ and hence that $\phi(E_j) \cong E_j/S_1$. In particular, $\phi(E_j)$ is a minimal G-invariant subspace of A. However in this case we have $\phi(E_j) \leq \mathbf{Soc}_G(A) = S_1$. This contradiction shows that $[x, E_j] \leq S_1$ for all j such that $1 \leq j \leq k$ and it follows that S_1 contains $[x, S_2]$. Furthermore let

$$S_3/S_2 = \mathbf{Soc}_G(A/S_2), \ldots, S_{n+1}/S_n = \mathbf{Soc}_G(A/S_n), \ldots.$$

Using similar arguments to those used above we can see that $S_{n+1} \neq S_n$ and $[x, S_{n+1}] \leq S_n$ for each $n \in \mathbb{N}$. Consequently $\bigcup_{n\in\mathbb{N}} S_n$ is an infinite dimensional G-invariant subspace of A which means that $A = \bigcup_{n\in\mathbb{N}} S_n$. This further implies that A is periodic as an $F\langle x\rangle$-module. Moreover, the set $\mathbf{Ass}_{F\langle x\rangle}(A)$ consists of only one maximal ideal, generated by the element $(x-1)$. □

We now need some further notation and terminology.

If D is a Dedekind domain and A is a simple D-module, then $A \cong D/P$ for some maximal ideal P and we let $\phi_k : D/P^{k+1} \longrightarrow D/P^k$ denote the canonical epimorphism. We can consider the projective limit of the family of D-modules $\{D/P^k | k \in \mathbb{N}\}$ and denote this projective limit by $D(P^\infty)$.

The following useful result was proved by L. A. Kurdachenko and H. Smith [120]. Again, it is rather reminiscent of what happens in the case of the group C_{p^∞}.

Proposition 3.22. *Let D be a Dedekind domain, let P be a maximal ideal of D and let A be a Prüfer P-module over D. Then the ring of endomorphisms of A is isomorphic to $D(P^\infty)$. Furthermore, $D(P^\infty)$ is a principal ideal domain and the set of all non-zero ideals of $D(P^\infty)$ is precisely the set $\{P^n D(P^\infty) | n \in \mathbb{N}\}$.*

Proof. Choose an element $y \in P \setminus P^2$ and consider the mapping $\phi : A \longrightarrow A$ defined by $\phi(a) = ya$. Let $A_n = \mathbf{\Omega}_{P,n}(A)$ so that $yA_{n+1} = A_n$ for each $n \in \mathbb{N}$. The submodule A_1 is simple, so there is a non-zero element a_1 such that $A_1 = Da_1$ and $\mathbf{Ann}_D(a_1) = P$. The equality $yA_2 = A_1$ shows that there is an element a_2 such that $a_1 = ya_2$. Thus $a_2 \notin A_1$ from which it follows that $Da_2 = A_2$ and $\mathbf{Ann}_D(a_2) = P^2$. Continuing in this way we can choose a set of elements $\{a_n | n \in \mathbb{N}\}$ with the following properties:

$$A = \sum_{n \in \mathbb{N}} Da_n,$$
$$A_n = Da_n \leq Da_{n+1} = A_{n+1},$$
$$ya_1 = 0, ya_{n+1} = a_n, \mathbf{Ann}_D(a_n) = P^n$$

so that $Da_n = \mathbf{\Omega}_{P,n}(A) \cong D/P^n$ for all $n \in \mathbb{N}$.

Let ϕ be an arbitrary endomorphism of the D-module A. Then $P^n\phi(a_n) = 0$ and it follows that $\phi(a_n) \in Da_n$. Thus $\phi(a_n) = \alpha_n a_n$ for some element $\alpha_n \in D$. If β_n is a further element of D with the property that $\alpha_n a_n = \beta_n a_n$ then

$$0 = (\alpha_n - \beta_n)a_n \text{ so } \alpha_n - \beta_n \in \mathbf{Ann}_D(a_n) = P^n$$

and in particular $\alpha_n + P^n = \beta_n + P^n$ for each $n \in \mathbb{N}$.

Now define a mapping

$$\Theta : \mathbf{End}_D(A) \longrightarrow \underset{n \in \mathbb{N}}{\mathrm{Cr}}\, D/P^n$$

by $\Theta(\phi) = (\alpha_n + P^n)_{n \in \mathbb{N}}$. We have

$$\alpha_n a_n = \phi(a_n) = \phi(ya_{n+1}) = y\phi(a_{n+1}) = y(\alpha_{n+1}a_{n+1}) = \alpha_{n+1}a_n.$$

It follows that $(\alpha_{n+1} - \alpha_n)a_n = 0$ so that $\alpha_{n+1} - \alpha_n \in \mathbf{Ann}_D(a_n) = P^n$ which gives $\alpha_n + P^n = \alpha_{n+1} + P^n$. It follows that $\Theta(\phi) \in D(P^\infty)$.

Let ψ be a further endomorphism of A and let $\Theta(\psi) = (\beta_n + P^n)_{n \in \mathbb{N}}$. It is easy to see that

$$(\phi + \psi)(a_n) = (\alpha_n + \beta_n)a_n \text{ and } \phi\psi(a_n) = (\alpha_n\beta_n)a_n.$$

It follows that

$$\Theta(\phi + \psi) = \Theta(\phi) + \Theta(\psi) \text{ and } \Theta(\phi\psi) = \Theta(\phi)\Theta(\psi)$$

which shows that the mapping Θ is a ring homomorphism.

On the other hand, let $(\gamma_n + P^n)_{n \in \mathbb{N}} \in D(P^\infty)$. We can define a mapping $\chi : A \longrightarrow A$ by $\chi(a_n) = \gamma_n a_n$ for each $n \in \mathbb{N}$. It is easy to check that χ is an endomorphism of A and $\Theta(\chi) = (\gamma_n + P^n)_{n \in \mathbb{N}}$. It follows that $\mathbf{Im}\,(\Theta) = D(P^\infty)$ and it is easily seen that $\mathbf{ker}(\Theta) = 0$. We have therefore shown that

$$\mathbf{End}_D(A) \cong \mathbf{Im}\,(\Theta) = D(P^\infty).$$

Let $(\lambda_n + P^n)_{n \in \mathbb{N}}$ be an invertible element of the ring $D(P^\infty)$. Then there is an element $(\mu_n + P^n)_{n \in \mathbb{N}} \in D(P^\infty)$ such that

$$(\lambda_n + P^n)_{n \in \mathbb{N}}(\mu_n + P^n)_{n \in \mathbb{N}} = (1 + P^n)_{n \in \mathbb{N}}.$$

Hence, $\lambda_n\mu_n + P^n = 1 + P^n$ for each $n \in \mathbb{N}$ and in particular this implies that $\lambda_1 \notin P$.

Conversely, let $(\nu_n + P^n)_{n\in\mathbb{N}}$ be an element of $D(P^\infty)$ such that $\nu_1 \notin P$. Since P is a maximal ideal of D we have $\nu_1 D + P = D$. We also have $\nu_n + P^m = \nu_m + P^m$ for each $n \geq m$ and in particular $\nu_n \notin P$ for all $n \in \mathbb{N}$. Again we have $\nu_n D + P = D$ and it follows that $\nu_n D + P^n = D$ (see [119, Lemma 1.18], for example). Hence there are elements $\tau_n \in D, \xi_n \in P^n$ such that $\nu_n\tau_n + \xi_n = 1$ and we have

$$1 + P^n = \nu_n\tau_n + \xi_n + P^n = (\nu_n + P^n)(\tau_n + P^n).$$

Since the inverse element is unique we have $\tau_n + P^m = \tau_m + P^m$ for each $n \geq m$, so $(\tau_n + P^n)_{n\in\mathbb{N}} \in D(P^\infty)$. Hence

$$U(D(P^\infty)) = \{(\nu_n + P^n)_{n\in\mathbb{N}} \in D(P^\infty) | \nu_1 \notin P\}.$$

Let $(\beta_n + P^n)_{n\in\mathbb{N}}$ be a non-zero element of $D(P^\infty)$ and let the natural number t be chosen such that $\beta_1, \ldots, \beta_t \in P$ but $\beta_{t+1} \notin P$. Note that $P^t D(P^\infty) = y^t D(P^\infty)$. To see this let $(\sigma_n + P^n)_{n\in\mathbb{N}} \in D(P^\infty)$ and let $x \in P^t$. Then $P^t = y^t D + P^{t+m}$ for all $m \in \mathbb{N}$ (see [119, Proposition 1.24], for example), so we obtain

$$x = y^t z_{tm} + w_{tm} \text{ where } z_{tm} \in D, w_{tm} \in P^{t+m}.$$

Then

$$\begin{aligned}
x((\sigma_n + P^n)_{n\in\mathbb{N}}) &= (x(\sigma_n + P^n))_{n\in\mathbb{N}} = (x\sigma_n + P^n)_{n\in\mathbb{N}} \\
&= ((y^t z_{tm} + w_{tm})\sigma_n + P^n)_{n\in\mathbb{N}} \\
&= (0, \ldots 0, (y^t z_{tm} + w_{tm})\sigma_{t+1} + P^{t+1}, \ldots, \\
&\quad (y^t z_{tm} + w_{tm})\sigma_{t+m} + P^{t+m}, \ldots) \\
&= (0, \ldots, 0, y^t z_{tm}\sigma_{t+1} + P^{t+1}, \ldots, \\
&\quad y^t z_{tm}\sigma_{t+m} + P^{t+m}, \ldots) \in y^t D(P^\infty)
\end{aligned}$$

because $w_{tm} \in P^{t+m}$ for $m \in \mathbb{N}$.

From the equality $P^t D(P^\infty) = y^t D(P^\infty)$ we see that

$$(\beta_n + P^n)_{n\in\mathbb{N}} = y^t((\gamma_n + P^n)_{n\in\mathbb{N}})$$

where $(\gamma_n + P^n)_{n\in\mathbb{N}}$ is an invertible element of $D(P^\infty)$. We remark that the D-module $D(P^\infty)$ is torsion-free (see [157, Proposition 14], for example) so the representation of

$$(\beta_n + P^n)_{n\in\mathbb{N}} = y^t((\gamma_n + P^n)_{n\in\mathbb{N}}),$$

where $(\gamma_n + P^n)_{n\in\mathbb{N}}$ is an invertible element of $D(P^\infty)$, is unique.

If $(\delta_n + P^n)_{n\in\mathbb{N}}$ is a further non-zero element of $D(P^\infty)$, then again for some natural number m we have

$$(\delta_n + P^n)_{n\in\mathbb{N}} = y^m((\eta_n + P^n)_{n\in\mathbb{N}}),$$

where $(\eta_n + P^n)_{n\in\mathbb{N}}$ is an invertible element of $D(P^\infty)$. Thus

$$(\beta_n + P^n)_{n\in\mathbb{N}}(\delta_n + P^n)_{n\in\mathbb{N}} = y^{t+m}((\gamma_n + P^n)_{n\in\mathbb{N}})(\eta_n + P^n)_{n\in\mathbb{N}} \neq 0.$$

Hence $D(P^\infty)$ is an integral domain.

Let U be a non-zero ideal in the ring $D(P^\infty)$ and let

$$0 \neq (\beta_n + P^n)_{n\in\mathbb{N}} \in U, \text{ where } (\beta_n + P^n)_{n\in\mathbb{N}} = y^t((\gamma_n + P^n)_{n\in\mathbb{N}}),$$

and $(\gamma_n + P^n)_{n\in\mathbb{N}}$ is an invertible element of $D(P^\infty)$. It follows that $y^t D(P^\infty) \leq U$. However $y^t D(P^\infty) = P^t D(P^\infty)$ is the projective limit of the family $\{D/P^{t+m} | m \in \mathbb{N}\}$ and $D(P^\infty)/P^t D(P^\infty) \cong D/P^t$. The set of all ideals of D/P^t is $\{D/P^t, P/P^t, P^{t-1}/P^t, 0\}$ (this follows from [119, Corollary 1.28] for example). Hence $U/P^t D(P^\infty)$ is isomorphic to P^{t-k}/P^t for some natural number k. In other words, $U = P^{t-k}D(P^\infty) = y^{t-k}D(P^\infty)$. This means that $D(P^\infty)$ is a principal ideal domain and the set of all non-zero ideals of $D(P^\infty)$ is precisely $\{P^n D(P^\infty) | n \in \mathbb{N}\}$. □

For hypercentral quasi-irreducible subgroups of $GL(F, A)$ in which $C_G(\mathbf{Soc}_G(A)) \neq 1$ we obtain a result with some features similar to those occurring in Theorem 3.20.

Theorem 3.23. *Let A be a vector space over the field F and let G be a hypercentral quasi-irreducible subgroup of $GL(F, A)$. If $C_G(\boldsymbol{Soc}_G(A))$ is nontrivial, the following assertions hold:*

(i) G is abelian-by-finite;

(ii) the torsion subgroup $\boldsymbol{Tor}(G)$ has finite special rank;

(iii) if $\boldsymbol{char}(F) = p$ is a prime, then $\boldsymbol{Tor}(G)$ is a p'-group;

(iv) $\boldsymbol{Tor}(G) \cap \zeta(G)$ is locally cyclic;

(v) $C_G(\boldsymbol{Soc}_G(A)) \cap \zeta(G)$ contains an element x of infinite order such that A is $F\langle x\rangle$-periodic and $\boldsymbol{Ass}_{F\langle x\rangle}(A) = \{P\}$ where P is the ideal of $F\langle x\rangle$ generated by $x - 1$;

(vi) $A = C_1 \bigoplus \cdots \bigoplus C_n$ where C_j is a Prüfer P-module for $1 \leq j \leq n$.

Proof. Lemma 3.21 shows that $C_G(\mathbf{Soc}_G(A)) \cap \zeta(G)$ contains an element x of infinite order such that A is $F\langle x\rangle$-periodic and $\mathbf{Ass}_{F\langle x\rangle}(A) = \{P\}$ where P is the ideal of $F\langle x\rangle$ generated by the element $x - 1$. Using Lemma 3.12 we deduce that A is $F\langle x\rangle$-divisible. The equality $\mathbf{\Omega}_{P,1}(A) = C_A(x)$ shows that $\mathbf{\Omega}_{P,1}(A)$ is a proper submodule of A. Since $x \in \zeta(G)$ the subspace $\mathbf{\Omega}_{P,1}(A)$ is G-invariant and therefore has finite F-dimension. On the other hand there is an index set Λ such that

$$\mathbf{\Omega}_{P,1}(A) = \oplus_{\lambda\in\Lambda} U_\lambda \text{ where } U_\lambda \cong_{F\langle x\rangle} F\langle x\rangle/P \text{ for } \lambda \in \Lambda.$$

Hence the index set Λ must be finite so we may assume that $\Lambda = \{1, \dots, n\}$ for some natural number n. Hence (using [119, Theorem 5.26], for example) we deduce that $A = \bigoplus_{1 \le j \le n} C_j$ where C_j is a Prüfer P-module for $1 \le j \le n$.

Let $R = D(P^\infty)$, let K be the field of fractions of R and let R_0 be the R-module K/R. Set $A^* = \mathrm{Hom}_R(A, R_0)$. Then A^* is a free R-module and the R-rank of A^* is precisely n by a result of B. Hartley [78, Lemma 1.2]. By [78, Lemma 2.1] $B = A^* \otimes_R K$ is a simple KG-module. By Zassenhaus's theorem (see [202, Theorem 3.7], for example) G is soluble and Maltsev's theorem (see [202, Lemma 3.5], for example) implies that G is abelian-by-finite. Furthermore, Corollary 2.3 shows that G satisfies (iv).

Statement (i) now follows from Proposition 3.10 and statements (ii), (iii) follow from Corollary 3.7. This completes the proof. □

We note that indeed Theorem 3.23 can be extended to the case when G is an FC-hypercentral group.

Let us now return to the situation that we began this section with. Earlier we were interested in groups that were close to being irreducible in some sense. The situation we have just been considering was concerned with quasi-irreducible groups, those subgroups G of $GL(F, A)$ with the property:

- Every proper G-invariant subspace of A has finite dimension and A is an ascending union of certain proper G-invariant subspaces.

We now consider the second of the situations that arose earlier, namely we suppose that A contains a proper finite dimensional G-invariant subspace B such that the factor A/B is G-chief.

Lemma 3.24. *Let A be a vector space over the field F and let G be an abelian subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the factor A/B is G-chief. If the subspace B has finite F-dimension and A/B has infinite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Suppose first that A contains a non-zero G-invariant subspace C such that $C \cap B = 0$. In this case $(C + B)/B$ is a non-zero G-invariant subspace of A/B. Since A/B is G-chief it follows that $A/B = (C + B)/B$ and hence $A = B + C = B \oplus C$.

We therefore suppose that every non-zero G-invariant subspace of A contains B. In this case B is the FG-monolith of A. We again consider for each element $x \in G$ the group algebra $D(x) = F\langle t_x \rangle$ of an infinite cyclic group $\langle t_x \rangle$ over the field F. We define an action of the element t_x on A by $t_x a = xa$ for all $a \in A$. This action can be extended naturally to an action of the ring $D(x)$ on A.

Choose a non-zero element b of B. Since the subspace B is G-invariant, $(F\langle t_x \rangle)b \le B$. We note that the group ring $F\langle t_x \rangle$ has infinite F-dimension. Then the isomorphism $(F\langle t_x \rangle)b \cong (F\langle t_x \rangle)/\mathbf{Ann}_{F\langle t_x \rangle}(b)$ together with the fact that $\mathbf{dim}_F(B)$ is finite imply that b has a non-zero annihilator in the ring

$F\langle t_x\rangle$. This means that the $D(x)$-periodic part of A is non-zero also. Indeed setting $\mathbf{Tor}_{D(x)}(A) = T$ we have $B \leq T$. Also $\mathbf{Ann}_{D(x)}(b) = \mathbf{Ann}_{D(x)}(B) = P$ is a maximal ideal of the ring $D(x)$. Since G is an abelian group, the subspace T is G-invariant and since the factor A/B is G-chief, it follows that either $T = B$ or $T = A$.

Suppose first that $T = B$. Then there is a $D(x)$-submodule U of A such that $A = T \oplus U$. This submodule U is $D(x)$-torsion-free so that PU is non-zero. We have $PA = PU \leq U$ from which it follows that $PA \cap B = PA \cap T = 0$. Clearly PA is G-invariant and also $PA = PU$ so $PA \neq 0$. We therefore obtain a contradiction which shows that $T = A$ so the vector space A is periodic as a $D(x)$-module.

In this case $A = \bigoplus_{Q\in\pi} A_Q$ where A_Q is the non-zero Q-component of A and $\pi = \mathbf{Ass}_{D(x)}(A)$ by Proposition 3.11. Since G is abelian each Q-component A_Q is a G-invariant subspace of A. Since A is FG-monolithic and $B \leq A_P$ we have $A = A_P$.

Suppose that $\mathbf{\Omega}_{P,1}(A) \neq A$. Then it must be the case that $\mathbf{\Omega}_{P,1}(A) = B$ since $\mathbf{\Omega}_{P,1}(A)$ is G-invariant. Since $\mathbf{\Omega}_{P,2}(A)$ is G-invariant it then follows that $\mathbf{\Omega}_{P,2}(A) = A$. We know that $D(x)$ is a principal ideal domain so it contains an element y such that $P = yD(x)$ and the equality $\mathbf{\Omega}_{P,2}(A) = A$ shows that $y^2 a = 0$ for each element $a \in A$. Therefore $ya \in \mathbf{\Omega}_{P,1}(A)$ for each $a \in A$. Let $\rho : A \longrightarrow A$ be the mapping defined by $\rho(a) = ya$ for $a \in A$. Since G is abelian, ρ is an FG-endomorphism of A and as we saw above $\mathbf{\Omega}_{P,1}(A)$ contains $yA = \mathbf{Im}\,(\rho)$. Clearly $\mathbf{ker}(\rho) = \mathbf{\Omega}_{P,1}(A)$ so that

$$yA = \mathbf{Im}\,(\rho) \cong A/\mathbf{ker}(\rho) = A/\mathbf{\Omega}_{P,1}(A).$$

The inclusion $yA \leq \mathbf{\Omega}_{P,1}(A) = B$ implies that $A/\mathbf{\Omega}_{P,1}(A)$ has finite F-dimension. On the other hand, $A/\mathbf{\Omega}_{P,1}(A) = A/B$ has infinite dimension by hypothesis, a contradiction which proves that in fact $\mathbf{\Omega}_{P,1}(A) = A$. It therefore follows that $\mathbf{Ann}_{D(x)}(A) = P$ is a maximal ideal of $D(x)$ and this is true for every element $x \in G$.

Let $d \in A \setminus B$ and let $D = (FG)d$ be the G-invariant subspace generated by the element d. Since A is FG-monolithic, $B \leq D$ and it follows that there is an element $u \in FG$ such that $0 \neq ud \in B$. Let

$$u = \alpha_1 x_1 + \ldots \alpha_m x_m$$

for certain elements $\alpha_1, \ldots, \alpha_m \in F, x_1, \ldots, x_m \in G$. From what we proved above $\mathbf{Ann}_{D(x_1)}(A)$ is a maximal ideal of the ring $D(x_1)$. Let F_1 be the field $D(x_1)/\mathbf{Ann}_{D(x_1)}(A)$, so we can consider A as an F_1G-module. This module is again F_1G-monolithic and B is its F_1G-monolith. By the arguments used above A is annihilated by some maximal ideal of the group ring $F_1\langle x_2\rangle$ and it follows that $\mathbf{Ann}_{F\langle x_1,x_2\rangle}(A)$ is a maximal ideal of the ring $F\langle x_1, x_2\rangle$. Using these arguments we see that after finitely many steps $\mathbf{Ann}_{F\langle x_1,\ldots,x_m\rangle}(A)$ is a maximal ideal of the ring $F\langle x_1, \ldots, x_m\rangle$. By Corollary 3.14 the cyclic $F\langle x_1, \ldots, x_m\rangle$-submodule generated by the element d

is simple. As B is G-invariant it is an $F\langle x_1, \ldots, x_m\rangle$-submodule and hence $(F\langle x_1, \ldots, x_m\rangle)d \cap B = 0$ since $d \notin B$. On the other hand

$$0 \neq ud \in (F\langle x_1, \ldots, x_m\rangle)d \leq B$$

and we obtain a contradiction which proves the lemma. □

We now extend this result to the case when G is abelian-by-finite.

Corollary 3.25. *Let A be a vector space over the field F and let G be an abelian-by-finite subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the factor A/B is G-chief. If the subspace B has finite F-dimension and A/B has infinite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Let H be a normal abelian subgroup of G of finite index. Then both B and A/B contain minimal H-invariant subspaces, denoted by C and D/B respectively such that $B = \bigoplus_{1 \leq j \leq n} x_j C$ and $A/B = \bigoplus_{1 \leq m \leq k} y_m D/B$ (see [116, Theorem 5.5], for example). We note that C has finite F-dimension and that D/B has infinite F-dimension. It follows that the vector space A has a finite series

$$\begin{aligned} 0 = B_0 \leq C = B_1 \leq B_2 \leq \cdots \leq B_n = B \leq D = B_{n+1} \\ \leq \cdots \leq B_{n+k} = A \end{aligned}$$

of H-invariant subspaces whose factors are H-chief, the factors B_j/B_{j-1} have finite dimension for $1 \leq j \leq n$ and the factors B_m/B_{m-1} have infinite dimension for $n+1 \leq m \leq n+k$. Let $E = B_{n-1}$ and consider the quotient space A/E. Its H-invariant subspace B_{n+1}/E satisfies the conditions of Lemma 3.24 and it follows from this result that B_{n+1}/E contains an H-invariant subspace E_1/E such that $B_{n+1}/E = B_n/E \bigoplus E_1/E$. In particular, the factor B_{n+1}/E_1 is H-chief and has finite dimension. Next we consider the factor B_{n+2}/E_1. Again using Lemma 3.24 we see that B_{n+2}/E_1 contains an H-invariant subspace E_2/E_1 such that $B_{n+2}/E_1 = B_{n+1}/E_1 \bigoplus E_2/E_1$. In particular the factor B_{n+2}/E_2 is H-chief and has finite F-dimension. Repeating these arguments we see that after finitely many steps we obtain a proper H-invariant subspace E_k such that A/E_k has finite dimension. Let $\{g_1, \ldots, g_t\}$ be a transversal to H in G. The F-isomorphism $A/E_k \cong A/g_jE_k$ holds for each j with $1 \leq j \leq t$ and hence A/g_jE_k is finite dimensional. Hence $L = \bigcap_{1 \leq j \leq t} g_jE_k$ has finite codimension in A and it is clearly G-invariant. Since E_k is a proper subspace and since $L \leq E_k$, it follows that L is also a proper subspace. Since B is a minimal G-invariant subspace of A it follows that $B \cap L = 0$ or $B \leq L$. If the latter, then L/B is a proper G-invariant subspace of A/B, a contradiction since A/B is G-chief. Hence $B \cap L = 0$. However $(B + L)/B$ must be A/B, so $A = B + L = B \oplus L$. This proves the result. □

We next obtain the case when G is hypercentral from these results.

Proposition 3.26. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the factor A/B is G-chief. If the subspace B has finite F-dimension and A/B has infinite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Let $H = C_G(B)$ and suppose that the normal subgroup H is nontrivial. Since G is hypercentral it follows $H \cap \zeta(G) \neq 1$ and we choose an element $1 \neq z \in H \cap \zeta(G)$. Then $B \leq C_A(z)$. Since $z \in \zeta(G)$, the subspace $C_A(z)$ is G-invariant and it follows that $B = C_A(z)$ since A/B is G-chief. Consider the mapping $\phi : A \longrightarrow A$ defined by $\phi(a) = (z-1)a = [z, a]$ for all $a \in A$. The fact that $z \in \zeta(G)$ implies that ϕ is an FG-endomorphism of A. It follows that $\mathbf{Im}\,(\phi) = [z, A]$ and $\mathbf{ker}(\phi) = C_A(z)$ are G-invariant. Since $C_A(z) = B$ we have, by the first isomorphism theorem,

$$[z, A] = \mathbf{Im}\,(\phi) \cong A/\mathbf{ker}(\phi) = A/B.$$

It follows that $[z, A]$ is a G-invariant subspace of A of infinite dimension. Furthermore, as ϕ is an FG-endomorphism of A, $C = [z, A]$ is a minimal G-invariant subspace. Hence $B \cap C = 0$ so $A = B \bigoplus C$.

Assume now that $C_G(B)$ is trivial. In this case we may consider G as a subgroup of $GL_n(F)$ where $n = \mathbf{dim}_F(B)$. Then G is soluble by Zassenhaus's theorem (see [202, Theorem 3.7] , for example) and Maltsev's theorem (see [202, Lemma 3.5], for example) then implies that G is abelian-by-finite. The result now follows upon applying Corollary 3.25. □

Theorem 3.27. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a finite dimensional G-invariant subspace B such that the factor A/B is G-chief. Then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Since B has finite F-dimension it has a finite series

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$$

whose factors are G-chief. We use induction on n, the case $n = 1$ being covered by Proposition 3.26.

Suppose that $n > 1$ and that the result is true for the quotient space A/B_1. Then A/B_1 contains a G-invariant subspace D/B_1 such that $A/B_1 = B/B_1 \oplus D/B_1$. Since the factor D/B_1 is G-chief and B_1 is a minimal G-invariant subspace, Proposition 3.26 implies that D contains a G-invariant subspace C such that $D = B_1 \oplus C$. Then we have

$$A = B + D = B + B_1 + C = B + C$$

and $B \cap C = 0$ since $D \cap B = B_1$. Hence $A = B \oplus C$. □

Corollary 3.28. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that every proper G-invariant subspace has finite F-dimension. Then either G is an irreducible or a quasi-irreducible subgroup of $GL(F, A)$.*

Proof. Suppose that G is neither irreducible nor quasi-irreducible. As we have seen earlier A then contains a non-zero G-invariant subspace B which is finite dimensional over F and such that A/B is G-chief. However Theorem 3.27 shows that then the vector space A contains a G-invariant subspace C such that $A = B \oplus C$. Thus C is a proper G-invariant subspace of A which has infinite dimension, contradicting our hypotheses. This contradiction proves the result. □

To conclude this section we note that Theorem 3.27 is no longer true when we step outside the realm of hypercentral groups even in the case when G is soluble. The corresponding counterexample was given by D. I. Zaitsev in the paper [222] and we now give his construction.

Let P be an elementary abelian p-group for some prime p and suppose that P is finite or countably infinite. We may think of P as a vector space over the field $\mathbb{F}_p$ with p elements. If P is finite, we can choose an irreducible cyclic p'-subgroup H in the linear group $GL(\mathbb{F}_p, P)$; if P is countably infinite, then we can choose an irreducible quasicyclic p'-subgroup H in the linear group $GL(\mathbb{F}_p, P)$ (see Chapter 2). We let $G = P \rtimes H$ be the natural semidirect product of these two groups.

Suppose that there is a non-zero bilinear form

$$\Phi : P \times P \longrightarrow \mathbb{F}_p$$

satisfying the condition

$$\Phi(x^h, y^h) = \Phi(x, y) \text{ for all elements } x, y \in P \text{ and } h \in H. \tag{3.2}$$

This allows us to prove the following technical result where we use the notation established above.

Proposition 3.29. *There is a vector space A over the prime field $\mathbb{F}_p$ satisfying the following conditions:*

(i) G is a subgroup of $GL(\mathbb{F}_p, A)$;

(ii) A contains a finite G-invariant subspace K such that the factor A/K is G-chief;

(iii) there is no G-invariant subspace B such that $A = K \bigoplus B$.

Proof. Let P' be an elementary abelian p-group isomorphic to P. We denote the image of an element a of P under this isomorphism by a'. Further, we let P_0 denote the additive group of the field $\mathbb{F}_p$.

Let $A = P_0 \bigoplus P'$ and define an action of the group G on A as follows. If $a \in A$ and $g \in G$, then write

$$a = x_0 + x' \text{ where } x_0 \in P_0, x' \in P',$$
$$g = hy \text{ where } y \in P, h \in H.$$

Then put

$$ga = x_0 + \Phi(x, y) + (hxh^{-1})',$$

where Φ is a bilinear form satisfying (3.2). The action of the element g on A is linear. Indeed, let $c = z_0 + z'$ be a further element of A where $z_0 \in P_0, z' \in P'$. Then

$$a + c = (x_0 + z_0) + x' + z' = x_0 + z_0 + (x + z)'$$

and we obtain

$$\begin{aligned} g(a + c) &= (x_0 + z_0) + \Phi(x + z, y) + (h(x + z)h^{-1})' \\ &= x_0 + \Phi(x, y) + (hxh^{-1})' + z_0 + \Phi(z, y) + (hyh^{-1})' \\ &= ga + gc \end{aligned}$$

Furthermore, let g_1, g_2 be arbitrary elements of the group G, so that $g_1 = h_1y_1, g_2 = h_2y_2$, for some elements $y_1, y_2 \in P, h_1, h_2 \in H$. We have

$$g_1g_2 = h_1y_1h_2y_2 = (h_1h_2)(h_2^{-1}y_1h_2y_2)$$

so that

$$(g_1g_2)a = x_0 + \Phi(x, (h_2^{-1}y_1h_2y_2)) + (h_1h_2x(h_1h_2)^{-1})'.$$

Since the bilinear form Φ is linear in the second argument and taking into account the property (3.2) we have

$$\Phi(x, h_2^{-1}y_1h_2y_2) = \Phi(x, h_2^{-1}y_1h_2) + \Phi(x, y_2) = \Phi(h_2xh_2^{-1}, y_1) + \Phi(x, y_2).$$

Thus

$$(g_1g_2)a = x_0 + \Phi(h_2xh_2^{-1}, y_1) + \Phi(x, y_2) + (h_1h_2x(h_1h_2)^{-1})'. \tag{3.3}$$

On the other hand

$$\begin{aligned} g_1(g_2a) &= g_1(x_0 + \Phi(x, y_2) + (h_2xh_2^{-1})') \\ &= x_0 + \Phi(x, y_2) + \Phi(h_2xh_2^{-1}, y_1) + (h_1h_2xh_2^{-1}h_1^{-1})' \\ &= x_0 + \Phi(x, y_2) + \Phi(h_2xh_2^{-1}, y_1) + (h_1h_2x(h_1h_2)^{-1})', \end{aligned} \tag{3.4}$$

which shows that $(g_1g_2)a = g_1(g_2a)$ upon comparing (3.3) and (3.4). Hence G is a subgroup of $GL(\mathbb{F}_p, P)$.

Clearly $K = P_0$ is a finite G-invariant subspace of A. Indeed, $ga = a$ for each element $a \in K$ and $g \in G$. Now consider the quotient space A/K. For each element $a = x_0 + x' \in A$, where $x_0 \in P_0, x' \in P'$ and $g = hy \in G$, where $y \in P, h \in H$, we have

$$g(a + K) = ga + K = (x_0 + \Phi(x, y) + (hxh^{-1})') + K = (hxh^{-1})' + K.$$

Hence it follows that $P = C_G(A/K)$ and the action of G on A/K is equivalent to the action of H on P. This implies that the factor A/K is G-chief.

Suppose that A contains a G-invariant subspace B such that $A = K \oplus B$. Since K is G-central we have $[G, A] \leq B$. Since Φ is a non-zero form P contains elements x_1, y_1 such that $\Phi(x_1, y_1) \neq 0$. Then

$$[y_1, x_1'] = (y_1 - 1)x_1' = y_1x_1' - x_1' = \Phi(x_1, y_1) + x_1' - x_1' = \Phi(x_1, y_1).$$

Since K has order p, it follows that $K \leq [G, A]$, so $K \leq B$, a contradiction which proves the result. □

We turn now to a concrete construction of the group G and the vector space A. The general construction of irreducible abelian subgroups of $GL(F, A)$ was given in Chapter 2. However specific details will be important for constructing the corresponding bilinear form, so we shall give all the steps here.

Let F be the algebraic closure of the prime field $\mathbb{F}_2$. In the polynomial ring $\mathbb{F}_2[X]$ consider the polynomial $X^2 + X + 1$ which is irreducible over $\mathbb{F}_2$ and let h_1 be a root of this polynomial in the field F. Then $h_1^3 = 1$. Let P_1 be an extension of the field $\mathbb{F}_2$ obtained by adjoining the element h_1. Since $h_1 \in F$ we have $P_1 = \mathbb{F}_2[h_1]$.

In the polynomial ring $P_1[X]$ consider the polynomial $X^3 + h_1$. This polynomial is irreducible over the field P_1 (see [134, Chapter VIII, Theorem 16], for example). Let P_2 be an extension of the field P_1 obtained by adjoining a root h_2 of the polynomial $X^3 + h_1$ to P_1. We note that the order of the element h_2 in the multiplicative group of the field F is 3^2.

In the polynomial ring $P_2[X]$ we consider the polynomial $X^3 + h_2$ and let h_3 denote a root of this polynomial in the field F. We note that $h_3 \notin P_2$. This follows from the fact that $X^9 + h_1$ is irreducible over the field P_1 (see [134, Chapter VIII, Theorem 16], for example). Let $P_3 = P_2[h_2]$.

Continuing in this way and using induction, we construct an ascending sequence

$$P_1 \leq P_2 \leq \cdots \leq P_n \leq$$

of finite fields such that P_{n+1} is an extension of the field P_n obtained by adjoining a root h_{n+1} of the polynomial $X^3 + h_n$.

Let $P = \bigcup_{n \in \mathbb{N}} P_n$. Then P is an infinite locally finite field and the subset $\{h_n | n \in \mathbb{N}\}$ of P generates a quasicyclic 3-subgroup H in the multiplicative group of the field P. The additive group of the field P is an infinite elementary abelian 2-group. Thus we may consider P as a vector space over the field $\mathbb{F}_2$ and H as a subgroup of $GL(\mathbb{F}_2, P)$. As we saw in Chapter 2, H is an irreducible subgroup of $GL(\mathbb{F}_2, P)$. Now we can construct the natural semidirect product G of the additive group of the field P and the multiplicative subgroup H of

P. We note that P is a minimal normal abelian subgroup of G and that the group G is metabelian.

Now we must define the bilinear form $\Phi : P \times P \longrightarrow \mathbb{F}_2$. We can think of the field P_n as a vector space over $\mathbb{F}_2$ and the subset $S_n = \{h_n^k | 0 \leq k < 2 \cdot 3^{n-1}\}$ is a basis of this vector space. We have a natural ascending sequence

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \ldots$$

and the union $S = \bigcup_{n \in \mathbb{N}} S_n$ of this sequence is a basis of the vector space P. As usual, it will suffice to determine the form values on the elements in this basis S and then extend the action with the help of the bilinearity property to all other pairs.

Put

$$\Phi(u, v) = \begin{cases} 1 \text{ if } v = u \text{ or } v = uh_1 \\ 0 \text{ in all other cases.} \end{cases}$$

We show that for the form Φ defined in this way the condition

$$\Phi(xh, yh) = \Phi(x, y) \text{ for all elements } x, y \in P \text{ and } h \in H \tag{3.5}$$

holds.

Note that the action of the subgroup H on P which in their semidirect product is recorded as conjugation, is, in the field P, the multiplication of the elements from H by the elements of P, so the condition shown in (3.5) looks a little different from that given in (3.2). To prove the condition given in (3.5) it suffices to establish that

$$\Phi(uh, vh) = \Phi(u, v) \text{ for all elements } u, v \in S \text{ and } h \in H. \tag{3.6}$$

In turn, the equality(3.6) follows once we have established the following:

$$\Phi(h_n^{k+1}, h_n^{t+1}) = \Phi(h_n^k, h_n^t) \text{ for } n \geq t, 0 \leq k, t < 2 \cdot 3^{n-1}.$$

This is not difficult to check using the definition given above for the form Φ. We illustrate how such a check is carried out in the case when $k + 1 < 2 \cdot 3^{n-1}, t + 1 = 2 \cdot 3^{n-1}$ and leave the remaining cases for the reader. We know that h_n is a root of the polynomial $X^{2m} + X^m + 1$, where $m = 3^{n-1}$. Therefore $h_n^{t+1} = h_n^m + 1$. It follows that

$$\Phi(h_n^{k+1}, h_n^{t+1}) = \Phi(h_n^{k+1}, h_n^m + 1) = \Phi(h_n^{k+1}, h_n^m) + \Phi(h_n^{k+1}, 1).$$

Since $h_n^{k+1} \neq 1$, the definition of the form Φ shows that $\Phi(h_n^{k+1}, 1) = 0$. Furthermore, the definition of Φ shows also that the equality

$$\Phi(h_n^{k+1}, h_n^m) = 1$$

is possible only in the case when $k+1 = m = 3^{n-1}$, that is if $k = 3^{n-1} - 1$. On the other hand, since in our case $k < t$, we have $h_n^k \neq h_n^t$ and the definition of the form Φ then yields that $\Phi(h_n^k, h_n^t) = 1$ holds only when $h_n^t = h_n^k h_1$. Since $h_1 = h_n^m$ and $t = 2 \cdot 3^{n-1} - 1$, this implies that $k = 3^{n-1} - 1$. This proves that $\Phi(h_n^{k+1}, h_n^{t+1}) = \Phi(h_n^k, h_n^t)$ for $k+1 < 2 \cdot 3^{n-1}, t+1 = 2 \cdot 3^{n-1}$.

To complete the construction, it now remains to use Proposition 3.29.

Almost Irreducible Linear Groups

We turn now to a very natural dual situation to the ones just discussed. As usual we let F be a field, let A be a vector space over F and let G be a subgroup of $GL(F, A)$. When the group G is irreducible every non-zero G-invariant subspace of A coincides with A; in particular every G-invariant subspace has finite codimension. This therefore provides a very natural dual object to study. Consequently we now discuss infinite dimensional linear groups G acting on a vector space A so that

- every non-zero G-invariant subspace of A has finite codimension.

A group with this property can be considered as yet a further generalization of an irreducible linear group. In this situation of course if we take a finite intersection of G-invariant subspaces $A_1, A_2, \dots, A_n$ then each of these has finite codimension in A and hence $\bigcap_{1 \le i \le n} A_i$ also has finite codimension in A. When we discuss infinite such intersections there are two cases that will arise:

(i) every non-zero G-invariant subspace of A has finite codimension and the intersection of all the non-zero G-invariant subspaces is zero;

(ii) A contains a minimal non-zero G-invariant subspace B of finite codimension.

In the second case B will be the intersection of the G-invariant subspaces of A and in this case A is *FG-monolithic*. In the first case A will not be an FG-monolithic space.

If a linear group G satisfies condition (i) above then the linear group G will be called *almost irreducible*.

Almost irreducible infinite dimensional linear groups were studied in the paper [126] of L. A. Kurdachenko and I. Ya. Subbotin (see also the survey paper [102]). In the paper [121] of L. A. Kurdachenko and I. Ya. Subbotin these results were generalized to the case of modules over a group ring RG where R is a Dedekind domain. Here we obtain the structure of almost irreducible linear groups found in [126]. The reader should compare the results here with those obtained in the last section.

Proposition 3.30. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Let H be a normal subgroup of finite index in G and let X be a transversal to H in G. Then A contains* *an H-invariant subspace B of infinite codimension such that the following conditions hold:*

(i) H is an almost irreducible subgroup of $GL(F, A/xB)$ for all elements x of X;

(ii) $\bigcap_{x\in X} xB = 0$ and $\bigcap_{x\in X} x^{-1}C_H(A/B)x = 1$;

(iii) H is isomorphic to a subgroup of $\underset{x\in X}{Dr}(H/x^{-1}C_H(A/B)x)$.

Proof. Let

$$\mathfrak{M} = \{U | U \text{ is an } H - \text{invariant subspace of } A \text{ such that } A/U \text{ has infinite dimension over } F\}.$$

Since A has infinite dimension over F, $0 \in \mathfrak{M}$ so that the family $\mathfrak{M}$ is non-empty. If W is a non-zero G-invariant subspace of A, then the quotient space A/W has finite dimension over F. It follows that every ascending series of G-invariant subspaces beginning with W cannot be infinite and hence A satisfies the maximal condition on G-invariant subspaces. The fact that the normal subgroup H has finite index in G implies that A satisfies the maximal condition for H-invariant subspaces (see [116, Theorem 5.3], for example). When we order the set $\mathfrak{M}$ by inclusion we see that $\mathfrak{M}$, has a maximal element B and the choice of B implies that every H-invariant subspace of A containing B but distinct from B has finite codimension in A. In other words, every non-zero H-invariant subspace of A/B has finite codimension in A/B. For each fixed $x \in X$ the mapping $a \longmapsto xa$ for $a \in A$ is an F-isomorphism, so every non-zero H-invariant subspace of A/xB has finite codimension in A/xB. This is true for each element $x \in X$.

Clearly the intersection $\bigcap_{x\in X} xB$ is a G-invariant subspace of B and since B has infinite codimension this intersection also has infinite codimension. It follows that $\bigcap_{x\in X} xB$ must be the zero subspace.

Suppose that the factor space A/B is FH-monolithic and let M/B denote the FH-monolith of A/B. Then M/B has finite codimension in A/B. The F-isomorphism

$$A/M \cong xA/xM = A/xM$$

shows that the H-invariant subspace xM has finite codimension in A for each element $x \in X$. Since the subset X is finite it follows that the intersection $M_0 = \bigcap_{x\in X} xM$ has finite codimension in A. In particular $M_0 \neq 0$. Of course M_0 is a G-invariant subspace. The equality $\bigcap_{x\in X} xB = 0$ implies that

$$\bigcap_{x\in X} (xB \cap M_0) = 0.$$

There is an FH-isomorphism

$$M_0/(M_0 \cap xB) \cong_{FH} (M_0 + xB)/xB.$$

Since H is a normal subgroup of G, it is easy to prove that xM/xB is the FH-monolith of A/xB. In particular xM/xB is the smallest H-invariant subspace of A/xB. Since we clearly have $(M_0+xB)/xB \leq xM/xB$ it follows that either

$$(M_0 + xB)/xB = xM/xB \text{ or } (M_0 + xB)/xB = 0.$$

In the former case the factor $(M_0 + xB)/xB$ is H-chief. The equality $\bigcap_{x\in X}(xB \cap M_0) = 0$ together with Remak's theorem gives the embedding

$$M_0 \longrightarrow \oplus_{x\in X} M_0/(M_0 \cap xB),$$

which shows that the FH-module M_0 is isomorphic to some FH-submodule of a semisimple FH-module. Then M_0 is itself a semisimple FH-module (see [116, Corollary 5.6], for example) so that

$$M_0 = V_1 \oplus \cdots \oplus V_k$$

for certain minimal H-invariant subspaces V_j of M_0 (for $1 \leq j \leq k$). If we suppose that V_j has finite dimension over F, then $\mathbf{dim}_F(xV_j)$ is also finite for each $x \in X$. Since the subset X is finite the non-zero G-invariant subspace $\sum_{x\in X} xV_j$ also has finite dimension over F which is contrary to our hypotheses. This contradiction shows that V_j is infinite dimensional over F. If D is a non-zero G-invariant subspace of A, then D has finite codimension. It follows that the intersection $D\cap V_j$ has finite codimension in V_j and hence $D\cap V_j \neq 0$. It is clear that the subspace $D \cap V_j$ is H-invariant. Since V_j is H-chief, we deduce that $D \cap V_j = V_j$ and hence $V_j \leq D$. This is true for each V_j so we obtain the inclusion $M_0 \leq D$. This is true for every non-zero G-invariant subspace. Since the interersection of the non-zero G-invariant subspaces is zero we deduce that $M_0 = 0$ and we obtain a contradiction. This contradiction shows that A/B is not FH-monolithic and it follows that H is an almost irreducible subgroup of $GL(F, A/B)$. For each $x \in X$ the mapping $a \longmapsto xa$ for $a \in A$ is an F-isomorphism and hence H is an almost irreducible subgroup of $GL(F, A/xB)$ for all $x \in X$ also.

Finally $C_H(A/xB) = x^{-1}C_H(A/B)x$ for every $x \in X$, so

$$\cap_{x\in X}(x^{-1}C_H(A/B)x) = \cap_{x\in X} C_H(A/xB) = C_H(A) = 1$$

and Remak's theorem leads again to an embedding

$$H \longrightarrow \underset{x\in X}{\mathrm{Dr}}\, H/x^{-1}C_H(A/B)x.$$

The result follows. $\square$

Lemma 3.31. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If B, C are non-zero G-invariant subspaces of A, then $B \cap C \neq 0$.*

Proof. Suppose the contrary and let $B \cap C = 0$. Since B is non-zero and G-invariant it follows that B has finite codimension in A. The F-isomorphism

$$C \cong C/(C \cap B) \cong (C + B)/B$$

then shows that C has finite dimension over F giving a contradiction which proves the result. □

Lemma 3.32. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Then every non-zero G-endomorphism of A is a G-monomorphism.*

Proof. Let $\phi : A \longrightarrow A$ be a non-zero G-endomorphism. Then the subspaces $\mathbf{Im}\,(\phi)$ and $\mathbf{ker}(\phi)$ are G-invariant and also of course $\mathbf{Im}\,(\phi) \cong A/\mathbf{ker}(\phi)$. If we assume that $\mathbf{ker}(\phi) \neq 0$, then $\mathbf{ker}(\phi)$ has finite codimension in A. It follows that $\mathbf{Im}\,(\phi)$ has finite dimension over F. This means that $\mathbf{Im}\,(\phi)$ must be trivial. Hence $\mathbf{ker}(\phi) = A$ contradicting the fact that ϕ is non-zero. Hence $\mathbf{ker}(\phi) = 0$ and the result follows. □

Corollary 3.33. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If $1 \neq z \in \zeta(G)$, then $C_A(z) = 0$.*

Proof. Since z is an element of the center of G, the mapping $\phi : A \longrightarrow A$ defined by $\phi(a) = (z-1)a$ for each $a \in A$ is a G-endomorphism of A. This is a non-zero G-endomorphism and Lemma 3.32 implies that $\mathbf{ker}(\phi) = C_A(z) = 0$, as required. □

Corollary 3.34. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If U is a nontrivial normal abelian subgroup of G such that its centralizer $C_G(U)$ has finite index in G, then $C_A(U) = 0$.*

Proof. We suppose for a contradiction that $C = C_A(U) \neq 0$. Let $H = C_G(U)$ and let X be a transversal to H in G. By Proposition 3.30 A contains an H-invariant subspace B of infinite codimension such that H is an almost irreducible subgroup of $GL(F, A/xB)$ for each $x \in X$. Since the subgroup U is normal in G, the subspace C is G-invariant. Then the assumption $C \neq 0$ implies that A/C has finite dimension over F. The fact that B has infinite codimension implies that the H-invariant subspace $(C + B)/B$ is non-zero. Since U is abelian H contains U. We have $(C + B)/B \leq C_{A/B}(u)$ for each $u \in U$ and in particular $C_{A/B}(u)$ is non-zero. Since $u \in \zeta(H)$, Corollary 3.33 implies that $u \in C_H(A/B)$. Since this is true for each $u \in U$ we see that $U \leq C_H(A/B)$ and we then have

$$U = x^{-1}Ux \leq x^{-1}C_H(A/B)x$$

for each element $x \in X$. Hence

$$U \leq \bigcap_{x \in X} x^{-1}C_H(A/B)x.$$

By Proposition 3.30 the last intersection is trivial which gives the required contradiction. It follows that $C_A(U) = 0$. $\square$

Corollary 3.35. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If G contains a nontrivial finite normal soluble subgroup S, then $C_A(S) = 0$.*

Proof. Suppose, for a contradiction, that $C_A(S) \neq 0$. Since S is a soluble normal subgroup it contains a G-invariant abelian subgroup U and, since U is finite, $C_G(U)$ has finite index in G. Then Corollary 3.34 shows that $C_A(U) = 0$. On the other hand, $C_A(S) \leq C_A(U)$, giving the desired contradiction. It follows that $C_A(S) = 0$. $\square$

Proposition 3.36. *Let A be a vector space over the field F of prime characteristic p and let G be a finite p-subgroup of $GL(F, A)$. Then A is G-nilpotent.*

Proof. Let g be an arbitrary element of the group G. Since F has characteristic p and g is a p-element, Lemma 1.28 implies that the element g is unipotent. However a finite p-group is nilpotent so Proposition 1.31 implies that A is G-nilpotent. $\square$

Corollary 3.37. *Let A be a vector space over the field F of prime characteristic p and let G be an almost irreducible subgroup of $GL(F, A)$. Then G contains no nontrivial finite normal p-subgroups.*

Proof. Suppose, for a contradiction, that P is a nontrivial finite normal p-subgroup of the group G. Then Proposition 3.36 shows that A is P-nilpotent. In particular the subspace $C_A(P)$ is non-zero. On the other hand, Corollary 3.35 shows that this subspace must be zero, which is the contradiction sought. The result follows. $\square$

We recall that if π is a set of primes, then $\mathbf{O}_\pi(G)$ is the largest normal π-subgroup of the group G.

Corollary 3.38. *Let A be a vector space over the field F of prime characteristic p and let G be an almost irreducible subgroup of $GL(F, A)$. Let V be the upper FC-hypercenter of G. Then $\boldsymbol{O}_p(G) \cap V$ is trivial.*

Proof. Suppose the contrary and let $\mathbf{O}_p(G) \cap V$ be nontrivial. Then the intersection $\mathbf{O}_p(G) \cap V \cap \mathbf{FC}(G)$ is also nontrivial (by [116, Corollary 3.14], for example). Let $1 \neq x$ be an arbitrary element of $\mathbf{O}_p(G) \cap \mathbf{FC}(G)$. Then the subgroup $P = \langle x \rangle^G$ is finite and nontrivial (see [52, Proposition 1.5.2], for example). Of course P is a p-subgroup and this contradicts Corollary 3.37. This contradiction shows that $\mathbf{O}_p(G) \cap V$ is trivial. $\square$

The next result follows immediately since the upper hypercenter is contained in the upper FC-hypercenter.

Corollary 3.39. *Let A be a vector space over the field F of prime characteristic p and let G be an almost irreducible subgroup of $GL(F, A)$. Then the upper hypercenter of G contains no p-elements.*

Corollary 3.40. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Suppose that G is a locally soluble FC-hypercentral group and that B is a proper non-zero G-invariant subspace of A such that $C_G(A/B) \neq 1$.*

(i) If $\textbf{\textit{char}}(F) = p$*, a prime, then* $\mathbf{O}_{p'}(C_G(A/B)) = 1$.

(ii) If $\textbf{\textit{char}}(F) = 0$*, then* $\textbf{\textit{Tor}}(C_G(A/B)) = 1$.

Proof. It is convenient to handle both proofs at the same time. If F has characteristic p, then let Q denote the subgroup $\mathbf{O}_{p'}(C_G(A/B))$ and if $\mathbf{char}\,(F) = 0$, then let Q denote the subgroup $\mathbf{Tor}(C_G(A/B))$. Suppose the contrary and that the subgroup Q is nontrivial. Then $Q \cap \mathbf{FC}(G)$ is also nontrivial (see [116, Corollary 3.16], for example). Let $1 \neq x$ be an arbitrary element of the subgroup $Q \cap \mathbf{FC}(G)$. Then the subgroup $\langle x \rangle^G$ is a finite nontrivial group (see [52, Proposition 1.5.2], for example). By the choice of Q, $\langle x \rangle^G$ is a p'-subgroup when $\mathbf{char}\,(F) = p$. Since G is locally soluble $\langle x \rangle^G$ contains a nontrivial G-invariant abelian subgroup K. Since K is a finite group (which is a p'-group when F has characteristic p) Maschke's theorem implies that there is a K-invariant subspace C such that $A = B \oplus C$ (see [116, Corollary 5.14], for example). If y is an arbitrary element of K, then $(y - 1)A \leq B$ and in particular we have

$$(y - 1)C \leq B \cap C = 0.$$

It follows that the subspace $C_A(K)$ contains C and, in particular, $C_A(K)$ is non-zero. This contradicts Corollary 3.33 and the result now follows. $\square$

Corollary 3.41. *Let A be a vector space over the field F of prime characteristic p and let G be an almost irreducible subgroup of $GL(F, A)$. Suppose that G is a hypercentral group. If B is a non-zero G-invariant subspace of A such that $C_G(A/B)$ is nontrivial, then the subgroup $C_G(A/B)$ is torsion-free.*

Proof. It follows from Corollary 3.39 that $C_G(A/B)$ contains no p-elements. Corollary 3.40 shows that $C_G(A/B)$ contains no p'-elements. Hence $C_G(A/B)$ is torsion-free. $\square$

Corollary 3.42. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Suppose that G is a locally soluble FC-hypercentral group. If B is a non-zero G-invariant subspace of A such that $C_G(A/B)$ is nontrivial, then $C_G(A/B) \cap \textbf{\textit{FC}}(G)$ is torsion-free.*

Proof. We note that $\mathbf{Tor}(\mathbf{FC}(G))$ contains all the elements of the FC-center of G that have finite order (see [52, Corollary 1.5.3], for example). Furthermore $\mathbf{Tor}(\mathbf{FC}(G))$ is normal in G.

Suppose, for a contradiction, that $C_G(A/B) \cap \mathbf{Tor}(\mathbf{FC}(G))$ is nontrivial. The inclusion $C_G(A/B) \cap \mathbf{Tor}(\mathbf{FC}(G)) \leq \mathbf{Tor}(C_G(A/B))$ implies that the subgroup $\mathbf{Tor}(C_G(A/B))$ is nontrivial. If the field F has characteristic 0, then Corollary 3.40 implies that $\mathbf{Tor}(C_G(A/B))$ is trivial and we obtain the desired contradiction in this case.

Suppose therefore that the field F has prime characteristic p. If $1 \neq x$ is an arbitrary element of $\mathbf{Tor}(\mathbf{FC}(G))$, then the subgroup $X = \langle x \rangle^G$ is finite (see [52, Proposition 1.5.2], for example). Let Y be a minimal G-invariant subgroup of X. Since G is locally soluble it follows that X is soluble and hence Y is an elementary abelian q-subgroup for some prime q. Corollary 3.38 implies that $q \neq p$. However Corollary 3.40 shows that the subgroup $C_G(A/B) \cap \mathbf{Tor}(\mathbf{FC}(G))$ contains no normal p'-subgroups and we obtain the contradiction sought. It follows that $C_G(A/B) \cap \mathbf{Tor}(\mathbf{FC}(G)) = 1$. As we noted earlier, the factor group $\mathbf{FC}(G)/\mathbf{Tor}(\mathbf{FC}(G))$ is torsion-free and hence $C_G(A/B) \cap \mathbf{FC}(G)$ is torsion-free. □

We point out the following subsidiary result concerning irreducible subgroups over finite dimensional vector spaces.

Lemma 3.43. *Let A be a finite dimensional vector space over the field F and let G be an irreducible subgroup of $GL(F, A)$. If G is an FC-hypercentral group, then G is abelian-by-finite.*

Proof. Since A has finite dimension over F it follows from [152, Theorem 2] that G is nilpotent-by-finite. In turn it follows that G is abelian-by-finite (by [52, Theorem 1.4.11], for example). □

Corollary 3.44. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If G is a locally soluble FC-hypercentral group, then $\boldsymbol{Tor}(G)$ is an abelian-by-finite subgroup of finite special rank. Furthermore, if the field F has prime characteristic p, then $\boldsymbol{O}_p(\boldsymbol{Tor}(G)) = 1$.*

Proof. Since G is not irreducible, the vector space A contains a proper nonzero G-invariant subspace B. Then the quotient space A/B has finite dimension over F. Hence there is a maximal G-invariant subspace M which contains B. Let $H = C_G(A/M) \cap \mathbf{Tor}(G)$. If we assume that the subgroup H is nontrivial, then $H \cap \mathbf{FC}(G)$ is also nontrivial (see [116, Corollary 3.16], for example) and we obtain a contradiction with Corollary 3.42. Hence $H = 1$ and in this case we can think of $\mathbf{Tor}(G)$ as a subgroup of the finite dimensional linear group $GL_n(F)$ where $n = \mathbf{dim}_F(A/M)$. Moreover, since M is a maximal G-invariant subspace of A, it follows that $\mathbf{Tor}(G)$ is an irreducible subgroup of $GL_n(F)$. Lemma 3.43 implies that in this case $\mathbf{Tor}(G)$ is abelian-by-finite.

Corollary 3.38 shows that $\mathbf{O}_p(\mathbf{Tor}(G))$ is trivial. Finally, let V be a normal abelian subgroup of finite index in $\mathbf{Tor}(G)$. Then we have

$$A/M = C_1/M \oplus \cdots \oplus C_k/M$$

where C_j/M is a minimal V-invariant subspace of A/M for $1 \leq j \leq k$. Using Corollary 2.3 we see that the factor group $V/C_V(C_j/M)$ is a locally cyclic p'-group for $1 \leq j \leq k$. The equality

$$C_V(C_1/M) \cap \cdots \cap C_V(C_k/M) = C_V(A/M) = 1$$

together with Remak's theorem gives the embedding

$$V \longrightarrow V/C_V(C_1/M) \times \cdots \times V/C_V(C_k/M).$$

From this it follows that V has finite special rank. Since V has finite index in $\mathbf{Tor}(G)$ it follows that $\mathbf{Tor}(G)$ also has finite special rank. The result follows. □

Since hypercentral groups are FC-hypercentral and locally nilpotent (so locally soluble) the next result is immediate.

Corollary 3.45. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If the group G is hypercentral, then $\mathbf{Tor}(G)$ is an abelian-by-finite subgroup of finite special rank and the subgroup $\zeta(\mathbf{Tor}(G))$ is locally cyclic. Moreover, if F has prime characteristic p, then $\mathbf{Tor}(G)$ is a p'-group.*

Lemma 3.46. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. If $1 \neq x$ is an element of the center $\zeta(G)$ such that $x \in C_G(A/B)$ for some proper non-zero G-invariant subspace B, then A, considered as an $F\langle x\rangle$-module, is torsion-free.*

Proof. Suppose the contrary and let the $F\langle x\rangle$-periodic part of A be non-zero. Since $x \in \zeta(G)$, the subspace $\mathbf{Tor}_{F\langle x\rangle}(A)$ is G-invariant. The hypotheses imply that $\mathbf{Tor}_{F\langle x\rangle}(A)$ has finite codimension in A. The fact that $F\langle x\rangle$ is a principal ideal domain implies that the $F\langle x\rangle$-module $A/\mathbf{Tor}_{F\langle x\rangle}(A)$ must be torsion-free. On the other hand it is clear that every $F\langle x\rangle$-torsion-free module has infinite dimension over F. Consequently $A = \mathbf{Tor}_{F\langle x\rangle}(A)$.

Then $A = \oplus_{P\in\pi} A_P$ where A_P is the non-zero P-component of A and $\pi = \mathbf{Ass}_{F\langle x\rangle}(A)$ by Proposition 3.11. Since $x \in \zeta(G)$ it follows that for each ideal U of the ring $F\langle x\rangle$ the U-component A_U of A is G-invariant and also each of the subspaces $\mathbf{\Omega}_{U,n}(A)$ is G-invariant. Lemma 3.31 implies that $\pi = \{P\}$ for some maximal ideal P of the ring $F\langle x\rangle$. Since $F\langle x\rangle$ is a principal ideal domain there is an element $y \in P$ such that $P = yF\langle x\rangle$. Let $L = \mathbf{\Omega}_{P,1}(A)$. Since the G-invariant subspace L is non-zero, the quotient space A/L has finite dimension over F.

Suppose that $A \neq L$. Then also $K = \mathbf{\Omega}_{P,2}(A) \neq L$. Let $\rho : K \longrightarrow L$ be the mapping defined by $\rho(a) = ya$ for all $a \in K$. Since $x \in \zeta(G)$ the mapping ρ is an FG-homomorphism.Therefore $\mathbf{Im}\,(\rho) = yK$ and $\mathbf{ker}(\rho)$ are G-invariant subspaces of A. Clearly $L = \mathbf{ker}(\rho)$ and by the first isomorphism theorem we have

$$yK = \mathbf{Im}\,(\rho) \cong_{FG} K/\mathbf{ker}(\rho) = K/L.$$

This isomorphism shows that the G-invariant subspace yK has finite dimension over F. On the other hand, $yK \neq 0$, so the vector space A contains a non-zero finite dimensional G-invariant subspace of finite codimension, yielding a contradiction. This contradiction shows that $A = \mathbf{\Omega}_{P,1}(A)$ and hence $PA = 0$.

Suppose now that the ideal P does not contain the element $x - 1$. Since P is a maximal ideal of $F\langle x\rangle$ we deduce that

$$(x - 1)F\langle x\rangle + P = F\langle x\rangle.$$

Hence in the ring $F\langle x\rangle$ there exist elements u, v such that

$$(x - 1)u + yv = 1.$$

The definition of x implies that $(x - 1)A \leq B$ and hence $(x - 1)A \neq A$. Let a be an element such that $a \in A \setminus (x - 1)A$. We have

$$\begin{aligned} a &= 1 \cdot a = ((x - 1)u + yv)a = (x - 1)ua + yva \\ &= (x - 1)ua \in (x - 1)A, \end{aligned}$$

since $PA = 0$ and we obtain a contradiction, which proves that $x - 1 \in P$. Since the ideal $(x - 1)F\langle x\rangle$ is maximal we have $(x - 1)F\langle x\rangle = P$. We showed above that $PA = 0$ and hence $(x - 1)A = 0$. This means that $x \in C_G(A) = 1$, again a contradiction which proves the result. $\square$

Next we need the following ideas analogous to those in abelian group theory (see [58] for example).

Let D be a Dedekind domain, let A be a D-module and let $x \in D$. The submodule B of A is said to be *x-pure* if $xB = B \cap xA$. Of course it is always the case that $xB \leq B \cap xA$.

If L is an ideal of D and B is an x-pure submodule of A for every $x \in L$, then B is called an *L-pure submodule*. We say that B is a *pure submodule* if it is D-pure.

Now let P be a maximal ideal of D. The submodule B is called a *P-basic submodule* if it satisfies the following conditions:

(i) $B = B_1 \oplus B_2$, where B_1 is a direct sum of cyclic P-submodules and B_2 is a projective D-submodule;

(ii) $P(A/B) = A/B$;

(iii) B is a P-pure submodule of A.

We note that each module over a Dedekind domain contains a P-basic submodule for every maximal ideal P of D (see [119, Theorem 9.15], for example). We shall need another technical result.

Lemma 3.47. *Let D be a principal ideal domain and P a maximal ideal of D. Suppose that A is a D-module such that the quotient module A/PA has finite dimension r over the field D/P. If A is D-torsion-free, then for each natural number n*

$$A/P^nA = B_{1,n} \oplus \cdots \oplus B_{r,n}$$

where each direct summand $B_{j,n}$ is isomorphic to D/P^n for $1 \leq j \leq r$. Furthermore, if $\mathbf{\Omega}_{P,1}(A/P^nA) = L$, then the submodules $B_{j,n}$ can be chosen such that $B_{j,n-1} = (B_{j,n} + L)/L$ for $1 \leq j \leq r$.

Proof. Let n be a fixed natural number. Let C be a P-basic submodule of A. The equality $PC = C \cap PA$ implies that

$$C/PC = C/(C \cap PA) \cong (C + PA)/PA.$$

It follows that the factor C/PC considered as a vector space over the field D/P has finite dimension $m \leq r$. Since every projective module over a principal ideal domain is free, then the fact that A is D-torsion-free implies that C is a free submodule having D-rank m. It follows that

$$C/P^nC = E_{1,n} \oplus \cdots \oplus E_{m,n}$$

where $E_{j,n} \cong D/P^n$ for $1 \leq j \leq m$. Since C/P^nC is a P-module, then being P-pure, it is pure (see [119, Lemma 8.2], for example). It follows that

$$A/P^nC = C/P^nC \oplus R/P^nC$$

for some D-submodule R (see [119, Theorem 8.11], for example). We have

$$R/P^nC \cong (A/P^nC)/(C/P^nC) \cong A/C.$$

It follows that $P(R/P^nC) = R/P^nC$ which in turn implies that

$$P^n(R/P^nC) = R/P^nC.$$

Hence we obtain $P^n(A/P^nC) = R/P^nC$. On the other hand

$$P^n(A/P^nC) = (P^nA + P^nC)/P^nC = P^nA/P^nC$$

and we obtain

$$\begin{aligned} A/P^nA &\cong (A/P^nC)/(P^nA/P^nC) = (A/P^nC)/P^n(A/P^nC)) \\ &= (C/P^nC \oplus R/P^nC)/(R/P^nC) \cong C/P^nC. \end{aligned}$$

The result now follows. $\square$

Proposition 3.48. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Suppose that $1 \neq x$ is an element of $\zeta(G)$ such that, as an $F\langle x\rangle$-module, A is torsion-free. Then F can be embedded in a field K and the vector space A can be embedded in a vector space R over the field K such that*

(i) R has finite dimension over K;

(ii) the action of G on A can be extended to an action of G on R such that G is an irreducible subgroup of $GL(K, R)$.

Proof. Since G is almost irreducible A contains a proper non-zero G-invariant subspace Q. Then Q has finite codimension over F and it follows that $\mathbf{Ann}_{F\langle x\rangle}(A/Q)$ is non-zero. Then the group ring $F\langle x\rangle$ contains a maximal ideal P such that $PA \neq A$. Since A is $F\langle x\rangle$-torsion-free, PA is non-zero and also G-invariant since x is in the center of G. Then the quotient space A/PA has finite dimension over F and hence over the field $F\langle x\rangle/P$. Let $\mathbf{dim}_{F\langle x\rangle/P}(A/PA) = r$. Then

$$A/PA = B_{1,1} \oplus \cdots \oplus B_{r,1} \text{ where } B_{j,1} \cong F\langle x\rangle/P \text{ for } 1 \leq j \leq r.$$

Using Lemma 3.47 we obtain that

$$A/P^nA = B_{1,n} \oplus \cdots \oplus B_{r,n}$$

where each direct summand B_{jn} is isomorphic to $F\langle x\rangle/P^n$ for $1 \leq j \leq r$. Let

$$\phi_{n,m} : A/P^nA \longrightarrow A/P^mA \text{ for } m \leq n$$

be the canonical epimorphism. Then

$$\{A/P^nA, \phi_{n,m} | m \leq n; m, n \in \mathbb{N}\}$$

is a projective family of the quotient spaces A/P^nA and we can then construct the projective limit V of this family. Then V is an $F\langle x\rangle$-module and the mapping

$$a \longmapsto (a + P^nA)_{n\in\mathbb{N}} \text{ for all } a \in A \tag{3.7}$$

is a homomorphism of A into V. Lemma 3.47 implies that $P^nA \neq P^{n+1}A$ for each natural number n and it follows that the intersection $\bigcap_{n\in\mathbb{N}} P^nA$ has infinite codimension in A. Since G is an almost irreducible subgroup it follows that $\bigcap_{n\in\mathbb{N}} P^nA = 0$. By Remak's theorem it follows that the homomorphism defined in (3.7) is an embedding of A into V. We identify A with its image under this homomorphism.

By Lemma 3.47 $\phi_{n,m}(B_{j,n}) = B_{j,m}$ for $1 \leq j \leq r, m \leq n$. Elementary properties of projective limits imply that

$$V = W_1 \oplus \cdots \oplus W_r,$$

where W_j is the projective limit of the family $\{B_{j,n}, \phi_{n,m} | m \leq n; n \in \mathbb{N}\}$.

Clearly, each direct summand W_j is isomorphic to the projective limit of the projective family $\{F\langle x\rangle/P^n, \phi_{n,m}|m \leq n, m, n \in \mathbb{N}\}$. Denote this latter projective limit by D.

We next define an action of the group on the vector space V as follows. For each $(\alpha_n)_{n\in\mathbb{N}} \in V$ we have $\alpha_n = a_n + P^n A$ for a suitable element $a_n \in A$ and $\phi_{n,m}(\alpha_n) = \alpha_m$ for $m \leq n$. If g is an arbitrary element of the group G, then we set

$$g\alpha_n = ga_n + P^n A \text{ for all } n \in \mathbb{N}$$

and

$$g(\alpha_n)_{n\in\mathbb{N}} = (g\alpha_n)_{n\in\mathbb{N}}.$$

We have

$$\begin{aligned}\phi_{n,m}(g\alpha_n) &= \phi_{n,m}(ga_n + P^n A) = \phi_{n,m}(g(a_n + P^n A))\\ &= g\phi_{n,m}(a_n + P^n A)\\ &= (g(a_m + P^m A)) = (ga_m + P^m A) = g(\alpha_m),\end{aligned}$$

so $g(\alpha_n)_{n\in\mathbb{N}} \in V$. Since $A \leq V$ and $C_G(A) = 1$, we have $C_G(V) = 1$ so we can think of G as a subgroup of $GL(F, V)$.

The action of $F\langle x\rangle$ on V can be naturally extended to an action of D on V. We note that $P^nV = V_n$ is the projective limit of the projective family

$$\{A/P^{n+t}A|t \in \mathbb{N}\} = V_n$$

(see [157, Theorem 18], for example) and the quotient V/P^nV is isomorphic to A/P^nA for each natural number n (see [157, 9.5, Corollary of Theorem 4], for example).

In Proposition 3.22 it was proved that the group of all invertible elements of the ring D coincides with the set of all elements $(x_n)_{n\in\mathbb{N}}$ for which $x_1 \neq 0$ and that D is a principal ideal domain in which the set of all non-zero ideals is precisely the set $\{P^nD|n \in \mathbb{N}\}$. Since $D/P^nD \cong F\langle x\rangle/P^nF\langle x\rangle$ for all natural numbers n, each proper quotient ring of D is finitely generated and periodic.

Let U be a non-zero D-submodule of V and let $U_1/U = \mathbf{Tor}_D(V/U)$. Since the D-rank of V is r it follows that $\mathbf{r}_D(U_1) = \mathbf{r}_D(U) = q < r$. The facts that D is a principal ideal domain and V is a finitely generated D-module together imply that V is a Noetherian D-module. In particular, the quotient module U_1/U is finitely generated. Using the fact that every proper quotient ring of D has finite dimension over F we deduce that U_1/U has finite dimension over the field F.

In the D/P-vector space A/PA choose an arbitrary basis

$$\{e_1 + PA, \ldots, e_r + PA\}.$$

Then the elements $e_1, \ldots, e_r$ generate the D-module V (see [157, 9.5, Theorem 18], for example). It follows that $A + P^nV = V$ for each natural number n.

Since $V/P^nV \cong A/P^nA$ for all natural numbers n it follows that $A \cap P^nV = P^nA$ for all $n \in \mathbb{N}$. Therefore,

$$V/U_1 = (A + U_1)/U_1 + (P^nV + U_1)/U_1.$$

From the obvious inequality $P^n(V/U_1) = (P^nV + U_1)/U_1$ we obtain

$$\begin{aligned}
&(V/U_1)/P^n(V/U_1)\\
&= ((A + U_1)/U_1 + (P^nV + U_1)/U_1)/(P^nV + U_1)/U_1)\\
&\cong ((A + U_1)/U_1)/((A + U_1)/U_1 \cap (P^nV + U_1)/U_1)\\
&= ((A + U_1)/U_1)/((P^nA + U_1)/U_1)\\
&= (A + U_1)/U_1)/(P^n(A + U_1)/U_1).
\end{aligned}$$

If we assume that $U_1 \cap A = 0$, then $(A + U_1)/U_1 \cong A$ and hence

$$\begin{aligned}
((A + U_1)/U_1)/P^n((A + U_1)/U_1) &\cong A/P^nA \\
&\cong \underbrace{F\langle x\rangle/P^n \oplus \cdots \oplus F\langle x\rangle/P^n}_{r}
\end{aligned} \tag{3.8}$$

On the other hand, $\mathbf{r}_D(V/U_1) = r - q < r$. Since V is a free D-module we have $V = U_1 \oplus M$ for some free D-submodule M so we have

$$(V/U_1)/(P^n(V/U_1)) \cong M/P^nM$$

and

$$M/P^nM \cong \underbrace{F\langle x\rangle/P^n \oplus \cdots \oplus F\langle x\rangle/P^n}_{r-q}.$$

This contradiction to (3.8) shows that $U_1 \cap A \neq 0$.

If U is a non-zero DG-submodule of V, then clearly U_1 is a DG-submodule of V. Then $U_1 \cap A$ is a non-zero DG-submodule of A so that the quotient space $A/(U_1 \cap A)$ has finite dimension over F. However V/U_1 is D-torsion-free and in particular it is $F\langle x\rangle$-torsion-free. It follows that $U_1 \cap A = A$ and hence $U_1 = V$. Since U_1/U has finite dimension over F, it follows that V/U has finite dimension over F. In particular V/U is D-periodic.

Let $R = V \otimes_D K$ where K is the field of fractions of the integral domain D. We immediately notice that the dimension of R over K coincides with the D-rank of V. In particular, R has finite dimension over K.

If S is a non-zero G-invariant subspace of the K-vector space R, then the intersection $S \cap V$ is a non-zero DG-submodule of V. As we saw above, in this case the quotient $V/(S \cap V)$ is D-periodic. It follows that

$$S = (S \cap V) \otimes_D K = V \otimes_D K = R.$$

Clearly, $C_G(V) = C_G(R) = 1$. Therefore, G is an irreducible subgroup of $GL(K, R)$. This completes the proof. □

Corollary 3.49. *Let A be a vector space over the field F and let G be an almost irreducible subgroup of $GL(F, A)$. Suppose that $1 \neq x$ is an element of $\zeta(G)$ such that as an $F\langle x\rangle$-module A is torsion-free. If G is an FC-hypercentral group, then G is abelian-by-finite.*

Proof. By Proposition 3.48 the field F can be extended to a field K and the vector space A can be extended to a vector space R over the field K in such a way that the action of the group G can be extended to an action of G on R so that G becomes a subgroup of $GL(K, R)$. Furthermore G is an irreducible subgroup of $GL(K, R)$. Using Lemma 3.43 we deduce that the group G is abelian-by-finite. □

Now we can describe the structure of almost irreducible hypercentral groups.

Theorem 3.50. *Let A be a vector space over the field F and let G be an almost irreducible hypercentral subgroup of $GL(F, A)$. Then G satisfies the following conditions:*

(i) G is abelian-by-finite;

(ii) the torsion subgroup T of the group G has finite special rank;

(iii) $T \cap \zeta(G)$ is a locally cyclic group;

(iv) if the field F has prime characteristic p, then T is a p'-subgroup.

Proof. The assertions (ii)-(iv) follow directly from Corollary 3.45. Since A is not FG-monolithic, it contains a proper G-invariant subspace B such that the quotient space A/B has finite dimension over F. It follows that A/B contains a maximal G-invariant subspace M/B.

If $C_G(A/M) = 1$, then the group G is isomorphic to some subgroup of $GL_k(F)$, where $k = \mathbf{dim}_F(A/M)$. Moreover, the choice of M shows that we may consider G as an irreducible subgroup of $GL_k(F)$. Then Lemma 3.43 implies that G is abelian-by-finite.

Suppose now that $C_G(A/M)$ is nontrivial. Since G is hypercentral the intersection $C_G(A/M) \cap \zeta(G)$ is also nontrivial. Let $1 \neq x \in C_G(A/M) \cap \zeta(G)$. Then Lemma 3.46 implies that the vector space A considered as an $F\langle x\rangle$-module is $F\langle x\rangle$-torsion-free. Then Corollary 3.49 shows that G is an abelian-by-finite group. □

This theorem allows the following generalization.

Theorem 3.51. *Let A be a vector space over the field F and let G be an almost irreducible locally soluble FC-hypercentral subgroup of $GL(F, A)$. Then G satisfies the following conditions:*

(i) G is abelian-by-finite;

(ii) the torsion subgroup T of the group G has finite special rank;

(iii) $T \cap \zeta(G)$ is a locally cyclic group;

(iv) if the field F has prime characteristic p, then T is a p'-subgroup.

Proof. The assertions (ii)-(iv) follow directly from Corollary 3.44. Since A is not FG-monolithic, it contains a proper G-invariant subspace B such that the quotient space A/B has finite dimension over F. It follows that A/B contains a maximal G-invariant subspace M/B.

If $C_G(A/M) = 1$, then the group G is isomorphic to some subgroup of $GL_k(F)$, where $k = \mathbf{dim}_F(A/M)$. Moreover, the choice of M shows that we can consider G as an irreducible subgroup of $GL_k(F)$. Then Lemma 3.43 implies that G is abelian-by-finite.

Suppose now that $C_G(A/M)$ is nontrivial. Then Corollary 3.42 implies that the intersection $C_G(A/M) \cap \mathbf{FC}(G)$ is torsion-free. However a torsion-free FC-group is abelian (see [52, Corollary 1.5.10], for example), so $C_G(A/M) \cap \mathbf{FC}(G)$ is abelian. Let $1 \neq v \in C_G(A/M) \cap \mathbf{FC}(G)$ and let $V = \langle v \rangle^G$. Since $v \in \mathbf{FC}(G)$, the subgroup $U = C_G(V)$ has finite index in G. The fact that V is abelian implies that $V \leq U$. Let X be a transversal to U in G. Clearly G is an almost irreducible subgroup of $GL(F, M)$. Then Proposition 3.30 implies that M contains a U-invariant subspace B such that $U/C_U(M/xB)$ is an almost irreducible subgroup of $GL(F, M/xB)$ for each element $x \in X$, $\bigcap_{x \in X} xB = 0$ and $\bigcap_{x \in X} x^{-1} C_U(A/B)x = 1$. Since the subspace M has finite codimension, $U/C_U(A/xB)$ is an almost irreducible subgroup of $GL(F, A/xB)$ for each element $x \in X$. The choice of v implies that $v \in C_U((A/xB)/(M/xB))$ and $v \in \zeta(U)$. Using Lemma 3.46 we deduce that the subspace A/xB considered as an $F\langle x \rangle$-module is $F\langle x \rangle$-torsion-free. Then Corollary 3.49 implies that $U/C_U(A/xB)$ is an abelian-by-finite group. Furthermore, U is isomorphic to some subgroup of $\underset{x \in X}{\mathrm{Dr}}\, U/(x^{-1}C_U(A/B)x)$. Each direct summand here is abelian-by-finite. Since X is finite the direct product is itself abelian-by-finite and hence U is abelan-by-finite. Since the index $|G : U|$ is finite, G is also abelian-by-finite. □

We return now to the beginning of this section where we began the discussion of infinite dimensional linear groups satisfying the property that every non-zero G-invariant subspace of A has finite codimension and we recall that two cases arose. The case when G is non-FG-monolithic (that is when G is almost irreducible) has just been discussed. We turn now to consider the second of the situations that can arise, namely we shall suppose that A contains a minimal G-invariant subspace B such that the quotient space A/B has finite dimension over F.

Lemma 3.52. *Let A be a vector space over the field F and let G be an abelian subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the quotient A/B is G-chief. If the subspace B has* *infinite dimension over F and A/B has finite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Suppose that A contains a non-zero G-invariant subspace C such that $C \cap B = 0$. In this case $(C+B)/B$ is a non-zero G-invariant subspace of A/B. Since A/B is G-chief we have $A/B = (C+B)/B$ so $A = B + C = B \bigoplus C$.

We therefore suppose that every non-zero G-invariant subspace of A contains B. In this case B is the FG-monolith of A. We again consider for each element $x \in G$ the group algebra $D(x) = F\langle t_x \rangle$ of an infinite cyclic group $\langle t_x \rangle$ over the field F. We define an action of the element t_x on A by $t_x a = xa$ for all $a \in A$. This action can be extended naturally to an action of the ring $D(x)$ on A.

In A choose a non-zero element $a \notin B$. We note that the group ring $F\langle t_x \rangle$ has infinite dimension over F. By the first isomorphism theorem we have

$$D(x)(a+B) \cong D(x)/\mathbf{Ann}_{D(x)}(a+B)$$

and since $\mathbf{dim}_F(A/B)$ is finite it follows that the coset $a+B$ has non-zero annihilator in the ring $D(x)$. This means that the quotient space A/B is $D(x)$-periodic. Since A/B has finite dimension, it is a periodic finitely generated module over the principal ideal domain $D(x)$. Hence there is an element $y \in D(x)$ such that y annihilates the quotient A/B.

Suppose first that A is $D(x)$-torsion-free. Since G is abelian the map

$$\rho : A \longrightarrow A \text{ defined by } \rho(a) = ya$$

for all $a \in A$ is an FG-endomorphism of A. It follows that $\mathbf{Im}(\rho) = yA$ is a G-invariant subspace of A. Since A is $D(x)$-torsion-free we have $\mathbf{ker}(\rho) = 0$ so that

$$yA = \mathbf{Im}(\rho) \cong A/\mathbf{ker}(\rho) \cong A.$$

On the other hand, the choice of the element y gives the inclusion $yA \leq B$. Since B is a minimal G-invariant subspace it follows that either $yA = B$ or $yA = 0$. The latter possibility contradicts our assumption that A is $D(x)$-torsion-free. Hence $yA = B$ so that yA is a minimal G-invariant subspace of A which contradicts the fact that $yA \cong A$.

This contradiction shows that A is not $D(x)$-torsion-free. Then its $D(x)$-periodic part $T = \mathbf{Tor}_{D(x)}(A)$ is non-zero. Since G is abelian the subspace T is a G-invariant subspace and since B is the FG-monolith of A we have $B \leq T$. Since the quotient space A/B is G-chief it follows that either $T = B$ or $T = A$.

Suppose first that $T = B$. Then $T = \oplus_{Q \in \pi} A_Q$ where A_Q is the non-zero Q-component of A and $\pi = \mathbf{Ass}_{D(x)}(A)$ by Proposition 3.11. Since G is an abelian group each Q-component A_Q is a G-invariant subspace of A. Since the vector space A is FG-monolithic we deduce that $T = A_P$ for some maximal ideal P of the ring $D(x)$. Again the fact that G is abelian implies the subspace $\mathbf{\Omega}_{P,1}(T)$ is a non-zero G-invariant subspace of A_P. On the other hand $A_P = B$

is a minimal G-invariant subspace from which it follows that $B = \mathbf{\Omega}_{P,1}(A_P)$ Furthermore, $P = \mathbf{Ann}_{D(x)}(T)$. It follows from [119, Proposition 8.9] that there is a $D(x)$-submodule U of A such that $A = T \oplus U$. The submodule U is $D(x)$-torsion-free so that PU is a non-zero subspace. Since $PA = PU \leq U$ we have

$$PA \cap B = PA \cap T = 0. \tag{3.9}$$

Since G is abelian, PA is a G-invariant subspace of A and $PA \neq 0$ since $PA = PU$. Hence $B \leq PA$ and (3.9) gives us a contradiction which shows that $T = A$, so A is a periodic $D(x)$-module.

Using the arguments above we see that $A = A_P$ for some maximal ideal P of the ring $D(x)$. As above $\mathbf{\Omega}_{P,1}(A)$ is a non-zero G-invariant subspace of A and if $\mathbf{\Omega}_{P,1}(A) \neq A$, then $\mathbf{\Omega}_{P,1}(A) = B$. Since the second layer $\mathbf{\Omega}_{P,2}(A)$ is also G-invariant it then follows that $\mathbf{\Omega}_{P,2}(A) = A$. We know that $D(x)$ is a principal ideal domain so it contains an element z such that $P = zD(x)$. The equality $\mathbf{\Omega}_{P,2}(A) = A$ shows that $z^2 a = 0$ for each element a of the space A. It follows that $za \in \mathbf{\Omega}_{P,1}(A)$ for each $a \in A$. Consider the mapping

$$\tau : A \longrightarrow A \text{ defined by } \tau(a) = za \text{ for } a \in A.$$

Since G is abelian, τ is an FG-endomorphism of A and, as we have seen above, $\mathbf{\Omega}_{P,1}(A)$ contains $zA = \mathbf{Im}(\tau)$. Clearly $\mathbf{ker}(\tau) = \mathbf{\Omega}_{P,1}(A)$ so that

$$zA = \mathbf{Im}(\tau) \cong A/\mathbf{ker}(\tau) = A/\mathbf{\Omega}_{P,1}(A) = A/B.$$

This isomorphism shows that the G-invariant subspace zA has finite dimension so $zA \neq B$. On the other hand, the inequality $A \neq \mathbf{\Omega}_{P,1}(A)$ implies that zA is non-zero contradicting the fact that B is a minimal G-invariant subspace of A. It therefore follows that $\mathbf{\Omega}_{P,1}(A) = A$ and hence $\mathbf{Ann}_{D(x)}(A) = P$ is a maximal ideal of $D(x)$, which is true for all $x \in G$.

The last part of the proof now follows almost verbatim the last part of the proof of Lemma 3.24 so we omit it. □

With a bit more effort we can now obtain a similar result when the group G is abelian-by-finite.

Corollary 3.53. *Let A be a vector space over the field F and let G be an abelian-by-finite subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the quotient A/B is G-chief. If the subspace B has infinite dimension over F and A/B has finite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Let V be a normal abelian subgroup of G of finite index. Then both B and A/B contain minimal V-invariant subspaces, denoted by C and D/B respectively such that $B = \oplus_{1 \leq j \leq n} x_j C$ and $A/B = \oplus_{1 \leq j \leq k} y_j D/B$ (see [116, Theorem 5.5], for example) for certain elements $x_i, y_j \in A$ with $1 \leq i \leq n$,

$1 \le j \le k$. We note that C has infinite dimension over F and that D/B has finite dimension over F. It follows that A has a finite series

$$0 = B_0 \le C = B_1 \le B_2 \le \cdots \le B_n = B \le D = B_{n+1} \le \ldots$$
$$\le B_{n+k} = A$$

of V-invariant subspaces whose factors are V-chief, the factors B_j/B_{j-1} have infinite dimension for $1 \le j \le n$ and the factors B_m/B_{m-1} have finite dimension for $n+1 \le m \le n+k$.

Let $E = B_{n-1}$ and consider the factor D/E. Using Lemma 3.52 we see that D/E contains a V-invariant subspace E_1/E such that $D/E = B/E \oplus E_1/E$. In particular E_1/E is V-chief of finite dimension over F. Furthermore, the factors D/B and E_1/E are FV-isomorphic. Next we consider the factor E_1/B_{n-2}. Again using Lemma 3.52 we see that E_1/B_{n-2} contains a V-invariant subspace E_2/B_{n-2} such that $E_1/B_{n-2} = E_2/B_{n-2} \oplus E/B_{n-2}$. Then E_2/B_{n-2} is a V-invariant subspace that is V-chief and of finite dimension over F. Repeating these arguments we see that after finitely many steps we obtain a non-zero V-invariant subspace E_k which has finite dimension over F. Let $\{g_1, \ldots, g_t\}$ be a transversal to V in G. The F-isomorphism $E_k \cong g_j E_k$ holds for each j with $1 \le j \le t$ and hence $g_j E_k$ is finite dimensional. Clearly the subspace $L = \sum_{1 \le j \le t} g_j E_k$ is a finite dimensional G-invariant subspace. Since the subspace E_k is non-zero and since $E_k \le L$, we have $L \neq 0$. However B is a minimal G-invariant subspace and has infinite F-dimension, so $L \cap B = 0$. Since A/B is G-chief, it follows that $A/B = (L + B)/B$, so $A = L + B = L \oplus B$ as required. $\square$

Next we prove the result for hypercentral groups.

Proposition 3.54. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B such that the quotient A/B is G-chief. If the subspace B has infinite dimension over F and A/B has finite dimension, then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Let $H = C_G(A/B)$ and suppose that the subgroup H is nontrivial. Since G is hypercentral it follows that $H \cap \zeta(G) \neq 1$. Choose an element $1 \neq z \in H \cap \zeta(G)$. This choice of z means that $(z-1)A = [z, A] \le B$. Now consider the mapping

$$\tau : A \longrightarrow A \text{ defined by } \tau(a) = (z-1)a \text{ for all } a \in A.$$

The fact that $z \in \zeta(G)$ implies that τ is an FG-endomorphism of A. It follows that the subspaces $\mathbf{Im}\,(\tau) = [z, A]$ and $\mathbf{ker}(\tau) = C_A(z)$ are G-invariant subspaces of A and by the first isomorphism theorem we have

$$[z, A] = \mathbf{Im}\,(\tau) \cong A/\mathbf{ker}(\tau) = A/C_A(z).$$

If we suppose that $C_A(z) = 0$, then we obtain the isomorphism $[z, A] \cong_G A$. On the other hand, the inclusion $[z, A] \le B$ and the fact that B is a minimal

G-invariant subspace of A yield a contradiction. This contradiction shows that the subspace $C_A(z)$ is non-zero. Again using the fact that B is a minimal G-invariant subspace of A, we see that either $B \cap C_A(z) = 0$ or the subspace $C_A(z)$ contains B. In the former case the fact that A/B is G-chief implies that $A = B \oplus C_A(z)$ and the result then follows.

Suppose then that $B \leq C_A(z)$. In this case, since the factor A/B is G-chief and since $C_A(z) \neq A$, it follows that $B = C_A(z)$. In this case we have $[z, A] \cong_G A/B$ which implies that the G-invariant subspace $[z, A]$ has finite dimension. On the other hand, as we have seen above, B contains $[z, A]$. Since B is a minimal G-invariant subspace of A of infinite dimension, the subspace $[z, A]$ must be zero. But then $z \in C_G(A) = 1$ and we obtain a contradiction which shows that this case is impossible.

Assume now that the subgroup $H = C_G(A/B)$ is trivial. In this case we may consider G as a subgroup of $GL_n(F)$ where $n = \mathbf{dim}_F(A/B)$. Then G is soluble by the theorem of Zassenhaus (see [202, Theorem 3.7], for example) and hence G is abelian-by-finite by the theorem of Maltsev (see [202, Lemma 3.5], for example). Corollary 3.53 now gives the result. □

Theorem 3.55. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace B of infinite dimension such that the quotient A/B is finite dimensional. Then A contains a G-invariant subspace C such that $A = B \oplus C$.*

Proof. Since A/B has finite dimension over F it follows that A has a finite series

$$0 = B_0 \leq B_1 = B \leq B_2 \leq \cdots \leq B_{n+1} = A$$

whose factors are G-chief and such that B_j/B_{j-1} has finite F-dimension for $2 \leq j \leq n$. We use induction on n. If $n = 1$, the result follows from Proposition 3.54.

Suppose that $n > 1$ and that the result is true for the G-invariant subspace B_n. Thus B_n contains a G-invariant subspace D such that $B_n = B \oplus D$. Since $D \cong B_n/B \leq A/B$ it follows that the subspace D has finite dimension over F. In the quotient space A/D the minimal G-invariant subspace $B_n/D \cong B$ has infinite dimension and the factor A/B_n is G-chief of finite dimension. By Proposition 3.54 we deduce that A/D contains a G-invariant subspace C/D of finite dimension over F such that $A/D = B_n/D \oplus C/D$. We have

$$A = B_n + C = (B + D) + C = B + C.$$

Since C has finite dimension and B is a minimal G-invariant subspace, $C \cap B = 0$ and we obtain $A = B \oplus C$. □

Corollary 3.56. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that every non-zero G-invariant subspace of A has finite codimension over F. Then G is either an irreducible or an almost irreducible subgroup of $GL(F, A)$.*

Proof. Suppose that G is neither irreducible nor almost irreducible. As we saw above, in this case A contains a non-zero proper minimal G-invariant subspace B having infinite dimension over F such that the quotient space A/B has finite dimension over F. However Theorem 3.55 shows that then the vector space A contains a G-invariant subspace C such that $A = B \oplus C$. Thus C is a proper G-invariant subspace of A having finite dimension, which is a contradiction. The result follows. □

In concluding this section we note that Theorem 3.55 is no longer true when we leave the realm of hypercentral groups, even in the soluble case. The corresponding counterexample was constructed by D. I. Zaitsev [222]. We describe this example now.

Let P be an elementary abelian p-group which may be finite or countably infinite, written additively. We may think of P as a vector space over the prime field $\mathbb{F}_p$. If P is finite we may choose an irreducible cyclic p'-subgroup H in the linear group $GL(\mathbb{F}_p, P)$. If P is a countably infinite group we may choose an irreducible quasicyclic p'-subgroup H in the linear group $GL(\mathbb{F}_p, P)$ (see Chapter 2). Let $G = P \rtimes H$ be the natural semidirect product of these two groups.

Let P' be an elementary abelian p-group isomorphic to P. We denote the image of an element a of P under this isomorphism by a'.

Let $A = P' \oplus \langle z \rangle$ where $\langle z \rangle$ is a cyclic group of order p. Define an action of the group G on A in the following way:

$$\begin{gathered}\text{If } a \in A, \text{ then } a = x' + kz \text{ where } x' \in P', 0 \leq k < p;\\ \text{if } g \in G, \text{ then } g = yh \text{ where } y \in P, h \in H \text{ and let}\\ ga = (h(x + ky)h^{-1})' + kz\end{gathered}$$

The action of the element g on A is linear. Indeed, let $c = u' + tz$ be another element of A where $u' \in P', 0 \leq t < p$. Then

$$a + c = (x' + u') + (k + t)z$$

where $k + t$ is reduced modulo p and we obtain

$$\begin{aligned}g(a + c) &= (h((x + u) + (k + t)y)h^{-1})' + (k + t)z\\ &= (h(x + ky)h^{-1})' + kz + (h(u + ty)h^{-1})' + tz\\ &= ga + gc\end{aligned}$$

Furthermore, let g_1, g_2 be arbitrary elements of the group G, so $g_1 = h_1y_1, g_2 = h_2y_2$, where $y_1, y_2 \in P, h_1, h_2 \in H$. We have

$$g_1g_2 = h_1y_1h_2y_2 = (h_1h_2)(h_2^{-1}y_1h_2 + y_2),$$

so that

$$\begin{aligned}g_2a &= (h_2(x + ky_2)h_2^{-1})' + kz,\\ g_1(g_2a) &= (h_1(h_2(x + ky_2)h_2^{-1} + ky_1)h_1^{-1})' + kz \qquad (3.10)\\ &= (h_1h_2(x + ky_2)h_2^{-1}h_1^{-1} + h_1(ky_1)h_1^{-1})' + kz\end{aligned}$$

and

$$\begin{aligned}(g_1g_2)a &= (h_1h_2(x + k(h_2^{-1}y_1h_2 + y_2))(h_1h_2)^{-1})' + kz\\ &= (h_1h_2(x + ky_2)h_2^{-1}h_1^{-1} + h_1h_2h_2^{-1}(ky_1)h_2h_2^{-1}h_1^{-1})'\\ &\quad + kz\\ &= (h_1h_2(x + ky_2)h_2^{-1}h_1^{-1} + h_1(ky_1)h_1^{-1})' + kz.\end{aligned} \tag{3.11}$$

Comparing the equations (3.10) and (3.11) we see that

$$(g_1g_2)a = g_1(g_2a)$$

which shows that G is a subgroup of $GL(\mathbb{F}_p, P)$.

Now be more specific and let P be an infinite countable elementary abelian p-group and let H be a quasicyclic q-group where q is a prime and $q \neq p$. As above, P as a vector space over the prime field $\mathbb{F}_p$and we can construct the natural semidirect product G of the groups P and H. Clearly the subspace P' of A is G-invariant. Note also that P' has codimension 1. Suppose that A contains a G-invariant subspace C such that $A = P' \oplus C$. Then $C = \mathbb{F}_pc$ for some element $c \in A$. We have $c = v' + sz$ for some element $v \in P$ and integer s such that $0 \leq s < p$. Clearly $s \neq 0$.

Let $0 \neq w \in P$. Then

$$wc = (v + sw)' + sz = v' + sw' + sz.$$

Since the subspace C is G-invariant, $wc \in C$, so $v' + sw' + sz \in C$. Then there is a natural number r with $0 \leq r < p$ such that $v' + sw' + sz = r(v' + sz)$. It follows that $r = 1$ and $sw' = 0$. This means that $w' = 0$ or $w = 0$ and we obtain a contradiction.

G-Contrainvariant Subspaces and G-Core-Free Subspaces

In this section we discuss a further generalization of the notion of irreducibility. In order to motivate the first topic in this section we recall that for an abstract group G, the subgroup H of G is called *contranormal* in G if $G = H^G$, the normal closure of H in G. Thus contranormality is essentially the exact opposite of normality. Contranormal subgroups have made various appearances in the literature. We here cite just one paper, [129], of L. A. Kurdachenko, I. Ya. Subbotin and T. Velichko, which discusses contranormal subgroups.

Suppose now that the group G is a subgroup of $GL(F, A)$ where, as usual, A is a vector space over the field F. If B is a subspace of A, then we may form some very natural G-invariant subspaces of the vector space A associated with B. First we have the G-invariant subspace generated by B. We saw in Chapter 1 that this was the intersection of all the G-invariant subspaces containing B, which we denoted by GB and which is the smallest G-invariant subspace of G containing B. Secondly there is also the subspace

$$\mathbf{core}_G\, B = \cap_{g\in G} gB$$

which is the largest G-invariant subspace of A contained in B. Of course B is G-invariant if and only if $GB = B = \mathbf{core}_G\, B$. In the case when the group G is irreducible and B is non-zero, then $GB = A$ and $\mathbf{core}_G\, B = 0$. We are therefore led quite naturally to the following definitions.

A subspace B of the vector space A is called G-*contrainvariant* if $GB = A$.

Furthermore, we say that the subspace B is G-*core-free* if $\mathbf{core}_G\, B = 0$.

These types of subspace occur naturally, as we noted above, in the case when the group G is irreducible. In such a case we have for a non-zero subspace B the inclusion $B \leq GB$. Hence $GB = A$ so that B is a G-contrainvariant subspace. This will be true for each non-zero subspace of G.

In the case when the group G is irreducible we have for a proper subspace B the inclusion $\mathbf{core}_G\, B \leq B$. Hence $\mathbf{core}_G\, B = 0$ so that B will be a G-core-free subspace. This is true for each proper subspace of G.

On the other hand, suppose that the vector space A contains no proper G-contrainvariant subspaces. If M is a maximal subspace of A, then in this case $M \leq GM \neq A$ which implies that $M = GM$. Thus in this case every maximal subspace of A is G-invariant. Now let a be an arbitrary non-zero element of the vector space A. Then there is an index set Λ and some set of elements $\{a_\lambda | \lambda \in \Lambda\}$ which forms a basis for A and such that the element a is included in this basis, say $a = a_\mu$ for some $\mu \in \Lambda$. For each index $\kappa \in \Lambda$ with $\kappa \neq \mu$, let

$$S_\kappa = \{\lambda \in \Lambda | \lambda \neq \mu, \lambda \neq \kappa\}.$$

Let D_κ denote the subspace of A generated by the subset $\{a_\mu\} \cup S_\kappa$. Then it is clear that D_κ is a subspace of A of codimension 1 and that D_κ is a maximal subspace of A. Thus when A contains no proper G-contrainvariant subspaces D_κ is a G-invariant subspace of A. Furthermore, we have $Fa = Fa_\mu = \bigcap_{\kappa\neq\mu} D_\kappa$ so the subspace Fa is a G-invariant subspace of A. Thus every one-dimensional subspace of A is G-invariant and hence every subspace of A is G-invariant.

Thus if the vector space A contains no proper G-invariant subspaces, then every proper subspace of A is G-contrainvariant. On the other hand if the vector space A contains no proper G-contrainvariant subspaces, then every subspace of A is a G-invariant subspace.

Assume now that the vector space A contains no proper G-core-free subspaces. Let a be an arbitrary non-zero element of the vector space A. Then the containment $\mathbf{core}_G\, Fa \leq Fa$ and the fact that the subspace Fa cannot be G-core-free imply that $\mathbf{core}_G\, Fa = Fa$. Thus, in this case, every one-dimensional subspace of A is G-invariant and, as above, this fact implies that every subspace of A is G-invariant.

So, on the one hand, if the vector space A contains no proper G-invariant subspaces, then every proper subspace is G-core-free and on the other hand if the vector space A contains no proper G-core-free subspaces then every subspace of A is G-invariant.

Thus if the group G is an irreducible subgroup of $GL(F, A)$, then there are only two types of subspaces. The subspaces 0 and A are G-invariant and all other subspaces are G-contrainvariant (respectively G-core-free). Because of this fact it is of interest to study those subgroups G of $GL(F, A)$ with the property that all subspaces of A are either G-invariant or G-contrainvariant. Dually, so to speak, we can also consider those subgroups G of $GL(F, A)$ all of whose subspaces are either G-invariant or G-core-free. As we have seen, these groups are generalizations of irreducible groups.

We note that if A is a vector space of dimension 2, then every subspace is either G-invariant or G-contrainvariant and likewise every subspace is either G-invariant or G-core-free. Indeed, in this situation, if G is irreducible, then, as we saw above, each proper subspace is G-contrainvariant and G-core-free. If G is not irreducible, then every proper non-zero subspace B has dimension 1. If B is not a G-invariant subspace, then the G-invariant subspace GB generated by B must coincide with the entire space A. Hence B is G-contrainvariant. Furthermore, the G-core of the subspace B does not coincide with B and hence must be zero. Therefore the subspace B is G-core-free.

We begin this section with the following useful result (which appeared as [49, Lemma 1.2] in the paper of M. R. Dixon, L. A. Kurdachenko and J. Otal).

Lemma 3.57. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspace Fa is G-invariant for all elements $a \in A$. Then for every element $g \in G$ there exists an element $\tau_g \in F$ such that $ga = \tau_g a$ for each element $a \in A$.*

Proof. Let $\{a_\lambda | \lambda \in \Lambda\}$ be a basis for A over F and let g be a fixed element of G. Since Fa_λ is G-invariant for every $\lambda \in \Lambda$ it follows that there exists $\tau_\lambda \in F$ such that $ga_\lambda = \tau_\lambda a_\lambda$. Suppose that there are indices $\lambda, \mu \in \Lambda$ such that $\tau_\lambda \neq \tau_\mu$ and let $b = a_\lambda + a_\mu$. Then

$$gb = g(a_\lambda + a_\mu) = ga_\lambda + ga_\mu = \tau_\lambda a_\lambda + \tau_\mu a_\mu.$$

On the other hand, the subspace Fb is also G-invariant, so we have

$$gb = \gamma b = \gamma(a_\lambda + a_\mu) = \gamma a_\lambda + \gamma a_\mu$$

for some element $\gamma \in \Lambda$. Since the elements a_λ, a_μ are linearly independent we have $\tau_\lambda = \gamma = \tau_\mu$, which is a contradiction. This contradiction shows that $\tau_\lambda = \tau_\mu$ for all $\lambda, \mu \in \Lambda$ and we denote this common value by τ_g. This completes the proof. $\square$

The next corollary then shows that the structure of a group G acting on a vector space A all of whose subspaces are G-invariant is rather restricted. This result also appeared in [49].

Corollary 3.58. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspace Fa is G-invariant for all elements $a \in A$. Then G is isomorphic to a subgroup of the multiplicative group of the field F.*

Proof. By Lemma 3.57 for each element g of G there exists an element $\tau_g \in F$ such that $ga = \tau_g a$ for each element $a \in A$. Let $U(F)$ denote the group of units of F and consider the mapping $\vartheta : G \longrightarrow U(F)$ defined by $\vartheta(g) = \tau_g$. It is easy to see that ϑ is a homomorphism from G into $U(F)$ and clearly $\mathbf{ker}(\vartheta) = C_G(A) = 1$. Hence ϑ is a monomorphism, as required. $\square$

We now turn to the study of infinite dimensional linear groups G acting on a vector space A in such a way that the subspaces of A are all either G-invariant or G-contrainvariant. In such cases, subspaces all of whose subspaces are G-invariant arise quite naturally.

Lemma 3.59. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-contrainvariant. If B is a proper G-invariant subspace of A, then every subspace of B is G-invariant.*

Proof. Let C be a subspace of B. Then $GC \leq B \neq A$ so C is not G-contrainvariant. Therefore C is G-invariant and the result follows. $\square$

Corollary 3.60. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-contrainvariant. Then one of the following holds:*

(i) *G is irreducible;*

(ii) *every subspace of A is G-invariant;*

(iii) *A contains a maximal G-invariant subspace M such that every subspace of M is G-invariant.*

Proof. We may assume that G is not irreducible otherwise (i) holds so we may let B be a proper non-zero G-invariant subspace of the vector space A. By Lemma 3.59 every subspace of B is G-invariant. If B is not a maximal G-invariant subspace of A, then A contains a proper G-invariant subspace C such that $B \leq C$ and $B \neq C$. Again Lemma 3.59 shows that every subspace of C is G-invariant. By repeating this process we can see that either A is the union of an ascending series of proper G-invariant subspaces or A contains a maximal G-invariant subspace M. In the former case let a be an arbitrary element of A. Then there is a proper G-invariant subspace D of A containing a and Lemma 3.59 again implies that every subspace of D is G-invariant. In particular, the subspace Fa is G-invariant. Thus every one-dimensional subspace of A is G-invariant and from this it follows that every subspace of A is G-invariant so that (ii) holds. In the latter case, Lemma 3.59 implies that every subspace of M is G-invariant so (iii) holds. The result follows. $\square$

We next give another generalization of the G-center, $\zeta_G(A) = C_A(G)$, of the vector space A. Let

$$\mathbf{Norm}_A(G) = \{a \in A | g(a) \in Fa \text{ for each element } g \in G\}.$$

It is clearly the case that $C_A(G)$ is a subset of $\mathbf{Norm}_A(G)$ and that $\mathbf{Norm}_A(G)$ is a G-invariant *subset* of A. Indeed for every element a of $\mathbf{Norm}_A(G)$ it is clear that Fa is a G-invariant subspace of A.

We note however that the subset $\mathbf{Norm}_A(G)$ is not in general a subspace of the vector space A. This can be seen from the following easy example.

Let $F = \mathbb{F}_5$, the prime field consisting of 5 elements and let A be a vector space over F with basis $\{a_1, a_2\}$. Let $G = \langle g \rangle$ be a cyclic group of order 4 and define an action of G on A by

$$ga_1 = 2a_1, ga_2 = -a_2.$$

It is clear from this definition that $a_1, a_2 \in \mathbf{Norm}_A(G)$. However

$$g(a_1 + a_2) = ga_1 + ga_2 = 2a_1 - a_2 \notin F(a_1 + a_2)$$

so that $a_1 + a_2 \notin \mathbf{Norm}_A(G)$.

The following theorem, though, shows that in the case when all the subspaces are either G-invariant or G-contrainvariant, then such examples can only occur in the dimension 2 case.

Theorem 3.61. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-contrainvariant. Then*

(i) if $\boldsymbol{Norm}_A(G) = 0$, then G is irreducible;

(ii) if $\boldsymbol{Norm}_A(G) = A$, then every subspace of A is G-invariant;

(iii) if $\boldsymbol{Norm}_A(G)$ is non-zero, but is not a subspace, then A has dimension 2.

Proof. (i) Suppose first that $\mathbf{Norm}_A(G) = 0$ and suppose that D is a proper non-zero G-invariant subspace of A. Let d be an arbitrary element of D. Since $\mathbf{Norm}_A(G) = 0$, the subspace Fd is not G-invariant. The hypotheses imply that Fd is G-contrainvariant. On the other hand $Fd \leq D \neq A$ giving us a contradiction. This contradiction shows that G is irreducible.

(ii) Suppose now that $\mathbf{Norm}_A(G) = A$. Then every one dimensional subspace of A is G-invariant, so that in this case every subspace of A is G-invariant.

(iii) Finally, suppose that $\mathbf{Norm}_A(G)$ is nonempty but is not a subspace. In this case $\mathbf{Norm}_A(G)$ contains two elements a, b such that there exists an element $g \in G$ with the properties that $ga \in Fa, gb \in Fb$ but $g(a + b) \notin F(a + b)$. Then there exist elements $\alpha, \beta \in F$ such that $g(a + b) = ga + gb = \alpha a + \beta b \notin F(a + b)$. By the definition of $\mathbf{Norm}_A(G)$ the subspaces Fa and

Fb are G-invariant and it follows that their sum $Fa + Fb$ is also G-invariant. However, the subspace $F(a + b)$ is not G-invariant. By hypothesis, $F(a + b)$ is G-contrainvariant so that $G(F(a + b)) = A$. However since $Fa + Fb$ is G-invariant we have $A = G(F(a + b)) \leq Fa + Fb$. Thus $A = Fa + Fb$ so that $\mathbf{dim}_F(A) = 2$. This completes the proof. $\square$

As we saw in Corollary 3.60 if A is a vector space which is not irreducible whose non-G-invariant subspaces are G-contrainvariant and if there is at least one G-contrainvariant subspace, then the subspace M defined in that corollary satisfies $M \subseteq \mathbf{Norm}_A(G)$. Then when A has F-dimension greater than 2 we must also have $M = \mathbf{Norm}_A(G) = M$. Our next few results will use the notation of Corollary 3.60 and in particular $\mathbf{Norm}_A(G)$ will be a maximal G-invariant subspace such that every subspace is also G-invariant. The results become particularly strong when the group G is hypercentral.

Lemma 3.62. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the subspaces of A which are not G-invariant are G-contrainvariant. If $\boldsymbol{dim}_F(A) \neq 2$ and if $M = \boldsymbol{Norm}_A(G) \neq A$, then M is a maximal G-invariant subspace of A. Furthermore either*

(i) $C_M(G) = 0$ or

(ii) the following properties hold

- *(a) $C_M(G) = M$ and $C_{A/M}(G) = A/M$ so that $\boldsymbol{dim}_F(A/M) = 1$;*
- *(b) G is abelian;*
- *(c) if $\boldsymbol{char}(F) = 0$, then G is torsion-free and if $\boldsymbol{char}(F) = p$ for some prime p, then G is an elementary abelian p-group;*
- *(d) if e is an element of A such that $e \notin M$, then $[G, Fe] = M$.*

Proof. It follows from Theorem 3.61 and the hypotheses that M is a maximal G-invariant subspace of A and by Corollary 3.60 every subspace of M is G-invariant. Suppose, for a contradiction, that $C = C_M(G)$ is a proper non-zero subspace of M. In M there is a complement subspace D to the subspace C, so that $M = C \oplus D$. Our hypotheses imply that C, D are both G-invariant subspaces of A. Since $C \neq M$, by assumption, it follows that D is also non-zero. Since $C_G(D) \neq G$ there is an element $g \notin C_G(D)$. Then D has an element $d \neq 0$ such that $gd \neq d$. Since $d \in M$, the subspace Fd is G-invariant and it follows that there is an element $\alpha \in F$ such that $\alpha \neq 1$ and $gd = \alpha d$. In C we choose a non-zero element c. Then $gc = c$. Now consider the subspace $F(c+d)$. The containment $F(c+d) \leq M$ implies that $F(c+d)$ is a G-invariant subspace. Then $g(c + d) = \gamma(c + d)$ for some element $\gamma \in F$. On the other hand

$$g(c + d) = gc + gd = c + \alpha d$$

so we obtain $c + \alpha d = \gamma c + \gamma d$. It follows that $\alpha = \gamma = 1$ and we obtain a contradiction. This contradiction shows that either the G-invariant subspace C is zero or $C = M$.

Consider the case when $M \leq C_A(G)$. Since $C_A(G)$ is G-invariant and A/M has no proper G-invariant subspaces we deduce that $M = C_A(G)$ or $A = C_A(G)$. However in the latter case we have $C_G(A) = G$ and we obtain a contradiction. Hence $M = C_A(G)$. Since G is a hypercentral group it has nontrivial center $\zeta(G)$. Let z be a nontrivial element of $\zeta(G)$ and let $\xi_z : A \longrightarrow A$ be the mapping defined by the rule $\xi_z(a) = [z, a] = (z - 1)a$ for all $a \in A$. Since $z \in \zeta(G)$ the mapping ξ_z is an FG-endomorphism of A. It follows that the subspaces $\mathbf{Im}(\xi_z)$ and $\mathbf{ker}(\xi_z)$ are G-invariant. Since $G = C_G(M)$ it follows that the subspace $\mathbf{ker}(\xi_z)$ contains the subspace M. If we suppose that $M \lneqq \mathbf{ker}(\xi_z)$, then the fact that A/M has no proper nontrivial G-invariant subspaces implies that $\mathbf{ker}(\xi_z) = A$. This means that $z \in C_G(A) = 1$, contradicting the choice of the element z. This contradiction therefore implies that $M = \mathbf{ker}(\xi_z)$ and the first isomorphism theorem gives

$$\mathbf{Im}(\xi_z) \cong_{FG} A/\mathbf{ker}(\xi_z) = A/M. \tag{3.12}$$

It follows that $K_z = \mathbf{Im}(\xi_z)$ is a simple FG-submodule of A. Therefore $K_z \cap M = 0$ or $K_z \leq M$.

We consider the first of these two cases. Since A/M is a simple FG-module it follows that $A = M \oplus K_z$. Since K_z is a proper G-invariant subspace of A, Lemma 3.59 implies that every subspace of K_z is G-invariant. Since K_z is a simple FG-module it follows that $\mathbf{dim}_F(K_z) = 1$. Hence $K_z = Fb_z$ for some element $b_z \in K_z$. Since $M = \mathbf{ker}(\xi_z)$ it follows that K_z is not contained in $\mathbf{ker}(\xi_z)$ and hence $zb_z = \nu b_z$ for some element $1 \neq \nu \in F$. Let $d \in M$ be an arbitrary non-zero element. Since the subspaces Fd and Fb_z are G-invariant the sum $Fd + Fb_z$ is also a G-invariant subspace. Suppose, for a contradiction, that $Fd + Fb_z$ is a proper subspace of A. Then Lemma 3.59 shows that every subspace of $Fd + Fb_z$ is also G-invariant. Hence

$$z(d + b_z) = \lambda(d + b_z) = \lambda d + \lambda b_z$$

for some element $\lambda \in F$. On the other hand we have

$$z(d + b_z) = zd + zb_z = d + \nu b_z.$$

It follows that $\lambda = \nu = 1$ giving us a contradiction. This contradiction shows that $Fd + Fb_z = A$ so that A is of F-dimension 2 contrary to our hypotheses.

For the second case we have $K_z \leq M$. Since $K_z \cong A/M$, by (3.12), it follows that K_z is a simple FG-module. On the other hand, the equality $C_G(M) = G$ implies that K_z has dimension 1. Again let b_z be an element of K_z such that $K_z = Fb_z$. Then $gb_z = b_z$ for all elements $g \in G$ and (3.12) implies that the factor A/M is G-central, so $G = C_G(A/M)$.

Choose an element $e \in A$ such that $e \notin M$. If we suppose that $(z-1)e = 0$, then since $A = M + Fe$ we deduce that $z \in C_G(A) = 1$, contradicting the choice of z. It therefore follows that $(z-1)e$ is non-zero. Since $(z-1)e \in M$ we see that the subspace Fe is not G-invariant. Hence Fe is a G-contrainvariant subspace.

The equality $G = C_G(A/M)$ implies that $(g-1)e \in M$ for every element $g \in G$. We next define the mapping $\delta : G \longrightarrow M$ by $\delta(g) = (g-1)e$ for each $g \in G$. If g, y are arbitrary elements of the group G, then we have

$$(gy-1) = (g-1) + (y-1) + (g-1)(y-1)$$

so that

$$\delta(gy) = (gy-1)e = (g-1)e + (y-1)e + (g-1)(y-1)e.$$

Since $(y-1)e \in M$ it follows that $(g-1)(y-1)e = 0$ and we obtain

$$\delta(gy) = (g-1)e + (y-1)e = \delta(g) + \delta(y).$$

Hence the mapping δ is a homomorphism from the group G into the additive group of the vector space A. It is clear that $\mathbf{ker}(\delta) = C_G(e)$. The equalities $A = M + Fe$ and $G = C_G(M)$ together imply that $C_G(e) = C_G(A) = 1$. Thus we have the isomorphism $G \cong \mathbf{Im}(\delta)$ so that the group G is abelian. Moreover, if $\mathbf{char}\,(F) = 0$, then G is torsion-free and if $\mathbf{char}\,(F)$ is the prime p, then G is an elementary abelian p-group. It also follows that

$$\mathbf{Im}(\delta) = [G, e] = \{(g-1)e | g \in G\}$$

is a G-invariant subspace of A, because $[G, e] \leq M$. Then the subspace $[G, Fe] = F[G, e]$ is a G-invariant subspace of M. Suppose that $U = [G, Fe] \lneqq M$ so that the quotient space M/U is non-zero. If g is an arbitrary element of G and a is an arbitrary element of A, then $a = x + \mu e$ for some element $x \in M$ and some element $\mu \in F$. Then we obtain

$$\begin{aligned} g(a+U) &= g(x + \mu e + U) = (gx + \mu ge) + U = (x + \mu e + \mu(g-1)e) + U \\ &= x + \mu e + U = a + U. \end{aligned}$$

From this it follows that $C_G(M/U) = G$. We also have

$$FG(e+U) = F(e+U) = (Fe+U)/U \neq A/U.$$

However we already saw that the subspace Fe is G-contrainvariant, so that $G(Fe) = A$ and hence $FG(e+U) = A/U$ which gives us a contradiction. This contradiction shows that $U = [G, Fe] = M$ and this completes the proof. $\square$

We next give an example to show that the situation described in Lemma 3.62 can actually arise.

In our example we let F be an arbitrary field and let A be a vector space over F of countable dimension. Let $\mathcal{B} = \{a_n | n \in \mathbb{N}\}$ be a basis for A. Let S

be the subset of matrices $[\alpha_{jm}]_{j,m\in\mathbb{N}}$ whose coefficients $\alpha_{jm}\in F$ satisfy the following conditions:

$$\begin{aligned}\alpha_{j,1}&=0 \text{ for all } j>n \text{ for some } n\\ \alpha_{jj}&=1 \text{ for all } j\\ \alpha_{jm}&=0 \text{ whenever } m>1 \text{ and } j\neq m.\end{aligned}$$

Thus the matrices in the set S have the form

$$\begin{pmatrix} 1 & 0 & 0\dots & 0 & 0 & 0\dots\\ \alpha_{21} & 1 & 0\dots & 0 & 0 & 0\dots\\ \alpha_{31} & 0 & 1 & \dots & 0 & 0\dots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

Let $A=[\alpha_{jm}]_{j,m\in\mathbb{N}}, B=[\beta_{jm}]_{j,m\in\mathbb{N}}\in S$ and let $AB=[\gamma_{jm}]_{j,m\in\mathbb{N}}$. Then

$$\begin{aligned}\gamma_{jj}&=1 \text{ for all } j,\\ \gamma_{j,1}&=\sum_{k\in\mathbb{N}}\alpha_{jk}\beta_{k,1}=\alpha_{j,1}\beta_{11}+\alpha_{jj}\beta_{j,1}=\alpha_{j,1}+\beta_{j,1} \text{ for } j>1,\\ \gamma_{jm}&=\sum_{k\in\mathbb{N}}\alpha_{jk}\beta_{km}=0 \text{ for } j\neq m \text{ and } m>1.\end{aligned}$$

Hence $AB\in S$. From the identities above we easily see that if $A^{-1}=[\sigma_{jm}]_{j,m\in\mathbb{N}}$, then $\sigma_{jj}=1$ for all $j\in\mathbb{N}$, $\sigma_{j,1}=-\alpha_{j,1}$ whenever $j>1$ and $\sigma_{jm}=0$ whenever $m>1$ and $j\neq m$. It follows that $A^{-1}\in S$. Let G be the subset of linear transformations of A whose matrices relative to the basis $\mathcal{B}$ belong to S. Our work above shows that G is a subgroup of $GL(F,A)$.

Let $M=\bigoplus_{n>1}Fa_n$. It follows that $C_G(M)=G$. In particular, M is a G-invariant subspace of A having codimension 1 in A. Hence M is a maximal G-invariant subspace of the vector space A and every subspace of M is G-invariant. Let d be an element of A such that $d\notin M$. Then

$$d=\sigma_1a_1+\sigma_2a_2+\dots\sigma_ka_k \text{ where } \sigma_1\neq 0.$$

Let g_j be the element of G whose matrix relative to the basis $\mathcal{B}$ has the form

$$\begin{pmatrix} 1 & 0 & 0\dots & 0 & 0 & 0\dots\\ 0 & 1 & 0\dots & 0 & 0 & 0\dots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0\dots\\ \alpha_{j,1} & 0 & \dots & 0 & 1 & 0\dots\\ 0 & 0 & \dots & 0 & 0 & 1\dots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

where $\alpha_{j,1} = 1$, for $j > 1$. Then $(g_j - 1)d = \sigma_1 a_j$. It follows that $a_j \in G(Fd)$ for $j > 1$ and hence $a_1 \in G(Fd)$ also. We therefore obtain $G(Fd) = A$. Hence the subspace Fd is G-contrainvariant. Now if C is a subspace of A such that M does not contain C, then C contains an element $d \notin M$. From what we proved above we have $G(Fd) = A$. Then $A = G(Fd) \leq C$ so C is G-contrainvariant. Thus every subspace of A is either G-invariant or is G-contrainvariant which yields the desired example.

We recall from Chapter 1 the definition of the invariator of a subspace V in G.

$$\mathbf{Inv}_G(V) = \{g \in G | g(a) \in V \text{ for every element } a \in V\}.$$

If B is an F-subspace of the vector space A, then the *norm of B in G* is defined to be the set

$$\mathbf{Norm}_G(B) = \bigcap_{b \in B} \mathbf{Inv}_G(Fb).$$

We observe that $\mathbf{Norm}_G(B)$ coincides with the intersection of the invariators of all the F-subspaces of B. Hence if C is an F-subspace of B, then $\mathbf{Norm}_G(B) \leq \mathbf{Inv}_G(C)$. In particular we have

$$\mathbf{Norm}_G(A) = \{x \in G | xFa = Fa \text{ for each } a \in A\}.$$

We remark that the invariator of a subspace is analogous to the normalizer of a subgroup. In abstract group theory the norm of a group in G is the intersection of the normalizers of all the subgroups of G.

Corollary 3.63. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Then the center of G contains $\mathbf{Norm}_G(A)$.*

Proof. Let g be an arbitrary element of $\mathbf{Norm}_G(A)$. By Lemma 3.57 there is an element $\tau_g \in F$ such that $ga = \tau_g a$ for each element a of A. If x is an arbitrary element of G, then

$$(xg)a = x(\tau_g a) = \tau_g(xa) \text{ and } (gx)a = g(xa) = \tau_g(xa)$$

so that $(gx)a = (xg)a$ for all $a \in A$. Thus $(g^{-1}x^{-1}gx(a)) = a$ and since this is valid for all elements $a \in A$ we have $g^{-1}x^{-1}gx \in C_G(A) = 1$. It follows that $gx = xg$ and hence $g \in \zeta(G)$. This proves the result. $\square$

If $N = \mathbf{Norm}_G(A)$, then every one-dimensional subspace of A is N-invariant and it then follows that every subspace of A is N-invariant. As we noted above in Corollary 3.58, there is a homomorphism $\vartheta : N \longrightarrow U(F)$ such that $\mathbf{ker}(\vartheta) = C_N(A) = 1$. In particular N is isomorphic to a subgroup of the multiplicative group of the field F. Let $\vartheta(x) = \alpha_x$ for each element x of N. Then we have $xa = \alpha_x a$ for each element $a \in M$ by Lemma 3.57.

Lemma 3.64. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the* *subspaces of A which are not G-invariant are G-contrainvariant. Suppose that $\boldsymbol{dim}_F(A) \neq 2$. If $M = \boldsymbol{Norm}_A(G) \neq A, C_M(G) = 0$ and $C_G(M) \neq 1$, then the following assertions hold:*

(i) $\boldsymbol{dim}_F(A/M) = 1$ *and* $C_G(M) = C_G(A/M)$;

(ii) $G/C_G(M)$ *is isomorphic to a subgroup of the multiplicative group of* F *and* $G/\boldsymbol{Norm}_G(A)$ *is isomorphic to a subgroup of the additive group of the subspace* M;

(iii) $C_G(M) \cap \boldsymbol{Norm}_G(A) = 1$ *and* G *embeds in* $G/C_G(M) \times G/\boldsymbol{Norm}_G(A)$. *In particular,* G *is abelian. Furthermore, if* $\boldsymbol{char}(F) = 0$, *then* $\boldsymbol{Tor}(G)$ *is a locally cyclic subgroup. If* $\boldsymbol{char}(F) = p$ *is prime, then* $\boldsymbol{Tor}(G) = P \times Q$ *where* P *is an elementary abelian* p*-group and* Q *is a locally cyclic* p'*-group;*

(iv) *for each* $g \in G$ *there exists* $\alpha_g \in F$ *such that* $ga = \alpha_g a$ *for each* $a \in M$;

(v) *if* $e \in A$ *and* $e \notin M$, *then* $(g - \alpha_g)e \in M$ *for all* $g \in G$;

(vi) *if* $e \in A$ *and* $e \notin M$, *then* $G(Fe) = A$.

Proof. The condition $C_M(G) = 0$ implies that $C_G(M) \neq G$. Since $C_G(M) \neq 1$ and the group G is hypercentral, the intersection $C_G(M) \cap \zeta(G)$ is also not trivial. Let z be a nontrivial element in $C_G(M) \cap \zeta(G)$ and consider the mapping $\xi_z : A \longrightarrow A$ defined by $\xi_z(a) = [z, a] = (z-1)a$ for each $a \in A$. Since $z \in \zeta(G)$ the mapping ξ_z is an FG-endomorphism of A. It follows that the subspaces $\mathbf{Im}(\xi_z)$ and $\mathbf{ker}(\xi_z)$ are G-invariant. The choice of z implies that the submodule M is contained in $\mathbf{ker}(\xi_z)$. Suppose that $M \lneqq \mathbf{ker}(\xi_z)$. In this case, since A/M contains no proper nontrivial G-invariant subspaces by Lemma 3.62 it follows that $\mathbf{ker}(\xi_z) = A$. However this means that $z \in C_G(A) = 1$, contrary to the choice of the element z. This contradiction implies that $\mathbf{ker}(\xi_z) = M$ and we obtain, by the first isomorphism theorem,

$$\mathbf{Im}(\xi_z) \cong A/\mathbf{ker}(\xi_z) = A/M.$$

Thus $K_z = \mathbf{Im}(\xi_z)$ is a simple FG-submodule of A and there are then the two possibilities that either $K_z \cap M = 0$ or $K_z \leq M$. The arguments of Lemma 3.62 show that in the first case we have $\mathbf{dim}_F(A) = 2$, contrary to our hypotheses and in the second case we have

$$\mathbf{dim}_F(A/M) = \mathbf{dim}_F(K_z) = 1.$$

Let e be an element of A such that $e \notin M$. Suppose that $(z-1)e = 0$. In this case since $A = M + Fe$ it follows that $z \in C_G(A) = 1$. This contradiction shows that $b_z = (z-1)e \in K_z$ is non-zero. Since $b_z \in M$ we see that the subspace Fe is not G-invariant and hence Fe is a G-contrainvariant subspace.

Let $x \in C_G(A/M)$. Then $xe = e + c$ for some element $c \in M$ and since $z \in \zeta(G)$ we have

$$\begin{aligned} xb_z &= x((z-1)e) = ((z-1)x)e = (z-1)(xe) = (z-1)(e+c) \\ &= (z-1)e + (z-1)c = b_z. \end{aligned}$$

It follows that $x \in C_G(b_z) = C_G(Fb_z) = C_G(K_z)$. Conversely, let $x \in C_G(K_z)$. Since the quotient module A/M has dimension 1 we have $xe = \mu e + c_1$ for some $\mu \in F$ and $c_1 \in M$. In this case we obtain

$$\begin{aligned} b_z &= xb_z = x((z-1)e) = (z-1)(xe) = (z-1)(\mu e + c_1) \\ &= (z-1)(\mu e) + (z-1)c_1 = \mu b_z. \end{aligned}$$

It follows that $\mu = 1$, so that $x \in C_G(A/M)$ and we obtain the equality $C_G(K_z) = C_G(A/M)$. Since $K_z \leq M$ we have $C_G(M) \leq C_G(K_z)$. Suppose that $C_G(M) \lneqq C_G(K_z)$ so that $C_G(K_z)$ contains an element y such that $y \notin C_G(M)$. Then M contains an element a_0 such that $ya_0 \neq a_0$. Since the subspace Fa_0 is G-invariant we have $ya_0 = \nu a_0$ for some element $\nu \in F$ such that $\nu \neq 1$. Then

$$y(a_0 + b_z) = ya_0 + yb_z = \nu a_0 + b_z. \tag{3.13}$$

On the other hand the subspace $F(a_0 + b_z)$ is G-invariant and hence there is an element $\sigma \in F$ such that

$$y(a_0 + b_z) = \sigma(a_0 + b_z) = \sigma a_0 + \sigma b_z. \tag{3.14}$$

It follows from (3.13) and (3.14) that $\nu = \sigma = 1$ and we obtain a contradiction, which proves that actually we have $C_G(M) = C_G(K_z)$. Furthermore we deduce that $C_G(M) = C_G(A/M)$.

Since every subspace of M is G-invariant there is a homomorphism $\vartheta : G \longrightarrow U(F)$ such that $\mathbf{ker}(\vartheta) = C_G(M)$ by Corollary 3.58. In particular, the factor group $G/C_G(M)$ is isomorphic to a subgroup of the multiplicative group of the field F. Using the argument from the proof of Lemma 3.62 we see that the subgroup $C_G(M)$ is abelian. Furthermore we note that if $\mathbf{char}\,(F) = 0$, then $C_G(M)$ is torsion-free whereas if $\mathbf{char}\,(F)$ is the prime p, then it is an elementary abelian p-group.

Let $\vartheta(g) = \alpha_g$ for each element $g \in G$. Since every subspace of M is G-invariant it follows from Lemma 3.57 that for every element of the group G and for every element a of the space M we have $ga = \alpha_g a$. The isomorphism $A/M \cong Fb_z$ together with $gb_z = \alpha_g b_z$ shows that $(g - \alpha_g)e \in M$.

Let Λ be an index set and let $\{a_\lambda | \lambda \in \Lambda\}$ be a basis for the subspace M. Set $a_1 = e$ and attach this element to the basis of the subspace M just defined to obtain a basis of the entire space A. Let $\Delta = \Lambda \cup \{1\}$.

Let Γ be the set of finitary matrices $[\chi_{\mu\nu}]_{\mu,\nu\in\Delta}$ whose coefficients have the following properties:

$$\begin{aligned}\chi_{\mu,1} &= 0 \text{ for all but finitely many indices } \mu\\ \chi_{\nu\nu} &= \alpha \text{ for some non-zero element of } F\\ \chi_{\mu\nu} &= 0 \text{ whenever } \mu \neq 1 \text{ and } \mu \neq \nu.\end{aligned}$$

Let $X_1 = [\gamma_{\mu\nu}]_{\mu,\nu\in\Delta}, X_2 = [\sigma_{\mu\nu}]_{\mu,\nu\in\Delta}$ be two arbitrary matrices from the set Γ and let $X_1X_2 = [\kappa_{\mu\nu}]_{\mu,\nu\in\Delta}$. Then

$$\begin{aligned}\kappa_{\mu\mu} &= \alpha_{\mu\mu}\sigma_{\mu\mu}\\ \kappa_{\mu,1} &= \sum_{\tau\in\Delta}\gamma_{\mu\tau}\sigma_{\tau,1} = \gamma_{\mu,1}\sigma_{11} + \gamma_{\mu\mu}\sigma_{\mu,1} \text{ whenever } \mu \neq 1,\\ \kappa_{\mu\nu} &= \sum_{\tau\in\Delta}\gamma_{\mu\tau}\sigma_{\tau\nu} = \gamma_{\mu\mu}\sigma_{\mu\nu} + \gamma_{\mu\nu}\sigma_{\nu\nu} \text{ whenever } \nu \neq 1, \mu \neq \nu, \text{ so that}\\ \kappa_{\mu\nu} &= 0 \text{ whenever } \mu \neq \nu, \text{ and } \mu \in \Lambda.\end{aligned}$$

It follows that $X_1X_2 \in \Gamma$.

Using these equalities we also see that if $X_1^{-1} = [\eta_{\mu\nu}]_{\mu,\nu\in\Delta}$, then

$$\begin{aligned}\eta_{\mu\mu} &= \gamma_{\mu\mu}^{-1} \text{ for all } \mu \in \Delta,\\ \eta_{\mu,1} &= -\gamma_{11}^{-2}\gamma_{\mu,1} \text{ for all } \mu \in \Lambda,\\ \eta_{\mu\nu} &= 0 \text{ whenever } \mu \neq \nu \text{ for } \nu \in \Lambda.\end{aligned}$$

It follows that $X_1^{-1} \in \Gamma$ and hence Γ is a group with the usual multiplication of finitary matrices. Indeed it is easy to see that Γ is itself a product of two subgroups Σ and Θ. Here Σ is the subgroup consisting of all scalar matrices of the form αE, where $\alpha \in F$ and Θ is the subset of all matrices $[\chi_{\mu\nu}]_{\mu,\nu\in\Delta}$ such that

$$\begin{aligned}\chi_{\mu,1} &= 0 \text{ for all but finitely many indices } \mu\\ \chi_{\mu\mu} &= 1 \text{ for all } \mu \in \Delta\\ \chi_{\mu\nu} &= 0 \text{ whenever } \nu \in \Lambda \text{ and } \mu \neq \nu.\end{aligned}$$

We notice immediately that Σ is a subgroup that is isomorphic to a subgroup of the multiplicative group of the field F and the subgroup Θ is isomorphic to the additive group of the vector subspace M. Since every matrix of Σ is a scalar matrix, Σ is contained in the center of the group Γ and hence Γ is an abelian group.

For each element $g \in G$ we have

$$ga_1 = \alpha_g a_1 + \sum_{\lambda\in\Lambda}\alpha_{\lambda,1}a_\lambda,$$

where all but finitely many of the coefficients $\alpha_{\lambda,1}$ are zero, so this really is a finite sum and

$$ga_\lambda = \alpha_g a_\lambda \text{ for all } \lambda \in \Lambda.$$

For each element $g \in G$ we define an infinite finitary matrix $D_g = [\gamma_{\mu\nu}]_{\mu,\nu\in\Delta}$ whose coefficients satisfy the following conditions:

$$\begin{aligned}\gamma_{\lambda,\lambda} &= \alpha_g \text{ for all } \lambda \in \Delta\\ \gamma_{\lambda,1} &= \alpha_{\lambda,1} \text{ for all } \lambda \in \Lambda\\ \gamma_{\lambda\mu} &= 0 \text{ whenever } \lambda \neq \mu, \mu \in \Lambda.\end{aligned}$$

If y is another element of G such that $ya_1 = \alpha_y a_1 + \sum_{\lambda\in\Lambda} \beta_{\lambda,1} a_\lambda$, then

$$\begin{aligned}(gy)a_1 = g(ya_1) &= g(\alpha_y a_1 + \sum_{\lambda\in\Lambda} \beta_{\lambda,1} a_\lambda) = \alpha_y(ga_1) + \sum_{\lambda\in\Lambda} \beta_{\lambda,1} g a_\lambda\\ &= \alpha_y\alpha_g a_1 + \sum_{\lambda\in\Lambda} \alpha_y \alpha_{\lambda,1} a_\lambda + \sum_{\lambda\in\Lambda} \beta_{\lambda,1}\alpha_g a_\lambda\\ &= \alpha_y\alpha_g a_1 + \sum_{\lambda\in\Lambda} (\alpha_y\alpha_{\lambda,1} + \alpha_g\beta_{\lambda,1}) a_\lambda.\end{aligned}$$

Let $D_y = [\sigma_{\mu\nu}]_{\mu,\nu\in\Delta}$ and $D_g D_y = [\kappa_{\mu\nu}]_{\mu,\nu\in\Delta}$. Then

$$\begin{aligned}\kappa_{\nu\nu} &= \alpha_g\alpha_y \text{ for all } \nu \in \Delta\\ \kappa_{\mu,1} &= \sum_{\tau\in\Delta} \gamma_{\mu\tau}\sigma_{\tau,1} = \gamma_{\mu\mu}\sigma_{\mu,1} + \gamma_{\mu,1}\sigma_{11}\\ &= \alpha_g\beta_{\mu,1} + \alpha_y\alpha_{\mu,1} \text{ whenever } \nu \neq 1,\\ \kappa_{\mu\nu} &= 0 \text{ whenever } \mu \neq \nu, \nu \in \Lambda.\end{aligned}$$

It follows that $D_g D_y = D_{gy}$.

We now define a mapping $\Phi : G \longrightarrow \Gamma$ by

$$\Phi(g) = D_g \text{ for each } g \in G.$$

From what we proved above we have $\Phi(gy) = D_{gy} = D_g D_y = \Phi(g)\Phi(y)$ so that the mapping Φ is a group homomorphism. It is clear that this mapping is also a monomorphism and since the group Γ is abelian it follows that the group G is likewise abelian.

Let H be the inverse image of the subgroup Σ under Φ. Then it is clear that

$$H = \{h \in G : he = \alpha_h e\}.$$

Since $ha = \alpha_h a$ for each element $a \in M$ at the equality $he = \alpha_h e$ implies that $H = \mathbf{Norm}_G(A)$. Therefore G/H is isomorphic to the subgroup $\Gamma/\Sigma \cong \Theta$ so that G/H is isomorphic to a subgroup of the additive group of the vector space M. In a similar manner the inverse image of the subgroup Θ under Φ is $C_G(A/M) = C_G(M)$. It is clear that $H \cap C_G(A/M) = 1$ and this implies that $\mathbf{Norm}_G(A) \cap C_G(M) = 1$. Using Remak's theorem we obtain an embedding of G into $G/\mathbf{Norm}_G(A) \times G/C_G(M)$.

Suppose that the field F has characteristic 0. Since $G/\mathbf{Norm}_G(A)$ is isomorphic to a subgroup of the additive group of the vector space M it follows that $G/\mathbf{Norm}_G(A)$ is torsion-free. Also the factor group $G/C_G(M)$ is isomorphic to a subgroup of the multiplicative group of the field F so the torsion subgroup of this group is locally cyclic (see [115, Corollary 2.2], for example). Hence in this case $\mathbf{Tor}(G)$ is locally cyclic.

Suppose now that the field F has characteristic p for some prime p. Since $G/\mathbf{Norm}_G(A)$ is isomorphic to a subgroup of the additive group of the vector space M it follows that $G/\mathbf{Norm}_G(A)$ is an elementary abelian p-group. In this case the torsion subgroup of the multiplicative group of the field F is a locally cyclic p'-group (see [115, Corollary 2.2], for example). Then we see that $\mathbf{Tor}(G) = P \times Q$ for some elementary abelian p-group P and a locally cyclic p'-subgroup Q. This completes the proof. $\square$

The proof of Lemma 3.64 indicates how to construct a group G satisfying the hypotheses of that lemma.

Next we discuss the case when $C_G(M) = 1$.

Lemma 3.65. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the subspaces of A which are not G-invariant are G-contrainvariant. Suppose that $\boldsymbol{dim}_F(A) \neq 2$. If $M = \boldsymbol{Norm}_A(G) \neq A, C_M(G) = 0$ and $C_G(M) = 1$, then the following assertions hold:*

(i) $\boldsymbol{dim}_F(A/M) = 1$ and $C_G(M) = C_G(A/M) = 1$;

(ii) G is isomorphic to a subgroup of the multiplicative group of F so G is abelian;

(iii) if $\boldsymbol{char}(F) = 0$, then $\boldsymbol{Tor}(G)$ is a locally cyclic subgroup. If $\boldsymbol{char}(F) = p$ is prime, then $\boldsymbol{Tor}(G)$ is a locally cyclic p'-group;

(iv) for each $g \in G$ there exists $\alpha_g \in F$ such that $ga = \alpha_g a$ for each $a \in M$;

(v) if $e \in A$ and $e \notin M$, then $(g - \alpha_g)e \in M$ for all $g \in G$;

(vi) if $e \in A$ and $e \notin M$, then $G(Fe) = A$;

(vii) if F is locally finite, then every subspace of A is G-invariant.

Proof. As we have seen before, since every subspace of M is G-invariant it follows that there is a homomorphsim $\vartheta : G \longrightarrow U(F)$ such that $\mathbf{ker}(\vartheta) = C_G(M)$ using Corollary 3.58. The fact that $C_G(M) = 1$ implies that G is abelian and that (ii) holds. In particular, if the field F has characteristic 0, then $\mathbf{Tor}(G)$ is a locally cyclic group (see [115, Corollary 2.2], for example). If $\mathbf{char}\,(F) = p$, a prime, then $\mathbf{Tor}(G)$ is a locally cyclic p'-group (again see [115, Corollary 2.2], for example). Hence (iii) follows.

Let $\vartheta(g) = \alpha_g$ for each element $g \in G$. Since every subspace of M is G-invariant we have that $ga = \alpha_g a$ for every element $g \in G$ and each element $a \in M$ using Lemma 3.57. This proves (iv).

Let $\rho_g : A \longrightarrow A$ be the mapping defined by

$$\rho_g(a) = (g - \alpha_g)a \text{ for all } a \in A.$$

Since G is abelian, ρ_g is an FG-endomorphism of A and it follows that the subspaces $\mathbf{Im}(\rho_g)$ and $\mathbf{ker}(\rho_g)$ are G-invariant. The definition of ρ_g implies that M is contained in $\mathbf{ker}(\rho_g)$. Suppose that $M \lneqq \mathbf{ker}(\rho_g)$. Since A/M is a simple FG-module it then follows that $A = \mathbf{ker}(\rho_g)$ which means that $ga = \alpha_g a$ for all elements $a \in A$. Therefore every one-dimensional subspace of A is $\langle g \rangle$-invariant and hence every subspace is $\langle g \rangle$-invariant. Our hypotheses imply that A contains a subspace which is not G-invariant so it follows that there is an element $v \in G$ such that $\mathbf{ker}(\rho_v) \neq A$ and hence $\mathbf{ker}(\rho_v) = M$. By the first isomorphism theorem we have

$$\mathbf{Im}(\rho_v) \cong A/\mathbf{ker}(\rho_v) = A/M$$

which shows that $B_v = \mathbf{Im}(\rho_v)$ is a simple FG-submodule. There are then two possibilities, namely that $B_v \cap M = 0$ or that $B_v \leq M$.

Suppose first that $B_v \cap M = 0$ and that $\mathbf{dim}_F(M) \geq 2$. Let $d \in M$. Then the subspace $D = Fd$ is G-invariant. The direct sum $B_v \oplus D$ is a proper G-invariant subspace of A and Lemma 3.59 implies that every subspace of $B_v \oplus D$ is likewise G-invariant. Lemma 3.59 and the fact that B_v is a simple FG-submodule imply that $\mathbf{dim}_F(B_v) = 1$ and hence $B_v = Fb_v$ for some element $b_v \in B_v$. As we have already remarked, for each element $g \in G$, there is an element $\alpha_g \in F$ such that $ga = \alpha_g a$ for all elements $a \in M$. Since $\mathbf{dim}_F(B_v) = 1$ we have $gb_v = \beta b_v$ for some element $\beta \in F$. As every subspace of $B_v \oplus D$ is G-invariant we have

$$g(b_v + d) = \gamma(b_v + d) = \gamma b_v + \gamma d$$

for some element $\gamma \in F$. On the other hand

$$g(b_v + d) = gb_v + gd = \beta b_v + \alpha_g d.$$

If follows that $\gamma = \alpha_g = \beta$ so that $gb_v = \alpha_g b_v$. This is true for each element $g \in G$ and in particular $vb_v = \alpha_v b_v$. This means that $b_v \in \mathbf{ker}(\rho_v)$ and that $B_v \leq \mathbf{ker}(\rho_v) = M$ giving us a contradiction. This contradiction shows that $\mathbf{dim}_F(M) = 1$ in this case.

Since A/M is a simple FG-module we have $B_v + M = A$, so $A = B_v \oplus M$. Lemma 3.59 implies that every subspace of B_v must be G-invariant. Since B_v is a simple FG-module it has dimension 1. Hence in this case $\mathbf{dim}_F(A) = 2$, contrary to our hypotheses. Hence $B_v \leq M$.

Now consider the case when $\mathbf{Im}(\rho_v) = B_v \leq M$. The facts that every subspace of M is G-invariant and that B_v is a simple FG-module imply that

B_v has dimension 1. The isomorphism $B_v \cong A/M$ shows that $\mathbf{dim}_F(A/M) = 1$ also and (i) follows. Choose an element $e \in A$ such that $e \notin M$. Then $ve = \alpha_v e + b_v$ for some element b_v of M such that $Fb_v = \mathbf{Im}(\rho_v)$. The choice of v implies that $b_v \neq 0$ and since $(v - \alpha_v)e = b_v \in M$ we deduce that the subspace Fe is not G-invariant. Hence Fe is a G-contrainvariant subspace, so (vi) follows.

Let g be an arbitrary element of the group G. Since $\mathbf{dim}_F(A/M) = 1$ we have $ge = \mu e + c$ for some $\mu \in F$ and element $c \in M$. Then, since G is abelian,

$$\begin{aligned} gb_v &= g((v - \alpha_v)e) = (g(v - \alpha_v))e = ((v - \alpha_v)g)e \\ &= (v - \alpha_v)(ge) = (v - \alpha_v)(\mu e + c) = \mu b_v. \end{aligned}$$

On the other hand since $b_v \in M$ it follows that $gb_v = \alpha_g b_v$ and hence $\mu = \alpha_g$. Thus $ge = \alpha_g e + c$ so that (v) follows.

Finally assume that the field F is locally finite and let the characteristic of F be the prime p. Since G is isomorphic to a subgroup of the multiplicative group of the field F it is a locally cyclic p'-subgroup (see [115, Corollary 2.2], for example). Let $H = \{h \in G | he = \alpha_h e\}$ and let $\{a_\lambda | \lambda \in \Lambda\}$ be a basis for M. As in the proof of Lemma 3.64 we see that $H = \mathbf{Norm}_G(A)$. For every element $g \in G$ we have, for certain scalars $\alpha_\lambda \in F$,

$$\begin{aligned} g^2 e &= g(\alpha_g e + \sum_{\lambda\in\Lambda} \alpha_\lambda a_\lambda) = \alpha_g g(e) + g(\sum_{\lambda\in\Lambda} \alpha_\lambda a_\lambda) \\ &= \alpha_g(\alpha_g e + \sum_{\lambda\in\Lambda} \alpha_\lambda a_\lambda) + \sum_{\lambda\in\Lambda} \alpha_g \alpha_\lambda a_\lambda = \alpha_g^2 e + 2\sum_{\lambda\in\Lambda} \alpha_g \alpha_\lambda a_\lambda, \\ g^3 e &= g(\alpha_g^2 e + 2\sum_{\lambda\in\Lambda} \alpha_g \alpha_\lambda a_\lambda) = \alpha_g^2(ge) + 2\sum_{\lambda\in\Lambda} \alpha_g^2 \alpha_\lambda a_\lambda \\ &= \alpha_g^2(\alpha_g e + \sum_{\lambda\in\Lambda} \alpha_\lambda a_\lambda) + 2\sum_{\lambda\in\Lambda} \alpha_g^2 \alpha_\lambda a_\lambda \\ &= \alpha_g^3 e + 3\sum_{\lambda\in\Lambda} \alpha_g^2 \alpha_\lambda a_\lambda \text{ and so on.} \end{aligned}$$

Finally we see that

$$g^p e = \alpha_g^p e + p\sum_{\lambda\in\Lambda} \alpha_g^{p-1} \alpha_\lambda a_\lambda = \alpha_g^p e.$$

It follows that $g^p \in H$. On the other hand g is a p'-element so that $\langle g \rangle = \langle g^p \rangle \in H$ and hence $G = \mathbf{Norm}_G(A)$. Thus (vii) follows. □

We summarize the results of the preceding lemmas in the following theorem.

Theorem 3.66. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-contrainvariant. Suppose that $\mathbf{dim}_F(A) > 2$ and* *that* $\boldsymbol{Norm}_A(G) = M \neq A$*. Suppose that* $C_M(G) = 0$*. Then the following statements hold:*

(i) every subspace of M *is* G*-invariant;*

(ii) $\boldsymbol{dim}_F(A/M) = 1$ *and* $C_G(M) = C_G(A/M)$*;*

(iii) for each element $g \in G$ *there exists an element* $\alpha_g \in F$ *such that* $ga = \alpha_g a$ *for each element* $a \in M$*;*

(iv) if $e \in A$ *and* $e \notin M$*, then* $(g - \alpha_g)e \in M$*;*

(v) if $e \in A$ *and* $e \notin M$*, then* $G(Fe) = A$*.*

Furthermore one of the following holds:

(a) if $C_G(M) = C_G(A/M) = G$*, then* G *is abelian. Also if* $\boldsymbol{char}(F) = 0$*, then* G *is torsion-free and if* $\boldsymbol{char}(F) = p$*, a prime, then* G *is an elementary abelian* p*-group;*

(b) if $G \neq C_G(M) = C_G(A/M) \neq 1$*, then* $G/C_G(M)$ *is isomorphic to a subgroup of the multiplicative group of the field* F *and* $G/\boldsymbol{Norm}_G(A)$ *is isomorphic to a subgroup of the additive group of the subspace* M*,* $C_G(M) \cap \boldsymbol{Norm}_G(A) = 1$ *and* G *embeds in* $G/C_G(M) \times G/\boldsymbol{Norm}_G(A)$*. In particular* G *is abelian; Furthermore, if* $\boldsymbol{char}(F) = 0$*, then* $\boldsymbol{Tor}(G)$ *is a locally cyclic group. If* $\boldsymbol{char}(F) = p$*, a prime, then* $\boldsymbol{Tor}(G) = P \times Q$ *where* P *is an elementary abelian* p*-group and* Q *is a locally cyclic* p'*-group;*

(c) if $C_G(M) = C_G(A/M) = 1$*, then* G *is isomorphic to a subgroup of the multiplicative group of* F*. In particular* G *is abelian. Furthermore, if* $\boldsymbol{char}(F) = 0$*, then* $\boldsymbol{Tor}(G)$ *is a locally cyclic subgroup. If* $\boldsymbol{char}(F) = p$ *is prime, then* $\boldsymbol{Tor}(G)$ *is a locally cyclic* p'*-group. Finally if* F *is locally finite, then every subspace of* A *is* G*-invariant.*

We turn now to the consideration of the second of the situations noted in the introduction of this section, namely we shall study those vectors spaces A in which each subspace is either G-invariant or G-core-free. To begin we prove an easy result which is dual to Lemma 3.59.

Lemma 3.67. *Let* A *be a vector space over the field* F *and let* G *be a subgroup of* $GL(F, A)$*. Suppose that the subspaces of* A *which are not* G*-invariant are* G*-core-free. If* B *is a non-zero* G*-invariant subspace of* A*, then every subspace of* A *containing* B *is* G*-invariant.*

Proof. Suppose that C is a subspace of A that contains B. Then $B \leq \mathbf{core}_G\, C$ and in particular $\mathbf{core}_G\, C$ is non-zero. Hence C must be G-invariant as required. $\square$

Corollary 3.68. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. If B is a non-zero G-invariant subspace of A, then every subspace of A/B is G-invariant.*

Corollary 3.69. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. If B, C are non-zero G-invariant subspaces of A such that $B \cap C = 0$, then every subspace of B is G-invariant.*

Proof. Let b be an arbitrary element of the subspace B. Then $G(Fb) \leq B$. On the other hand, Corollary 3.68 shows that every subspace of A/C is G-invariant. Hence if g is an arbitrary element of the group G, then $g(b + C) \in F(b + C)$. Therefore there are elements $\beta \in F, c \in C$ such that $gb = \beta b + c$. Since B is G-invariant, $gb = \beta b + c \in B$. From $\beta b \in B$ we deduce that $c \in B$. Hence $c \in B \cap C = 0$, so $c = 0$. Thus we see that $gb \in Fb$ for each element $g \in G$. It follows that the subspace Fb is G-invariant. Thus every subspace of B is G-invariant as required. □

Corollary 3.70. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. Suppose also that $\boldsymbol{dim}_F(A) > 2$. If B, C are both non-zero G-invariant subspaces of A such that $B \cap C = 0$, then every subspace of A is G-invariant.*

Proof. Let a be an arbitrary non-zero element of the vector space A. The equality $B \cap C = 0$ implies that either $a \notin B$ or $a \notin C$. Suppose that $a \notin B$. If $a \in C$, then Corollary 3.69 implies that Fa is G-invariant. We may therefore assume that $a \notin C$ also. By Corollary 3.68 the subspace $F(a + B)$ is G-invariant and if g is an arbitrary element of G, then we obtain $ga = \alpha a + b_0$ for some elements $\alpha \in F, b_0 \in B$. The choice of a implies that $Fa \cap B = 0$. Suppose that $\mathbf{dim}_F(B) > 1$. Then B contains a non-zero subspace B_1 such that $B = Fb_0 \oplus B_1$. By Corollary 3.69 every subspace of B is G-invariant and again by Corollary 3.68 we see that the subspace $F(a + B_1)$ is G-invariant. Hence $ga = \alpha a + b_0 \in Fa + B_1$. It follows that $b_0 \in (Fa + B_1) \cap Fb_0 = 0$. Hence $ga \in Fa$ for each element $g \in G$, so that Fa is G-invariant in this case.

Suppose now that $\mathbf{dim}_F(B) = 1$ and suppose that $0 \neq b_0 \in B$. Then $B = Fb_0$. Since $a \notin C$, the subspace $F(a + C)$ of the quotient space A/C is non-zero. By Corollary 3.68 this subspace is G-invariant so that $ga = \alpha a + b_0 \in Fa + C$. It follows that $\alpha a + b_0 = \sigma a + c$ for some elements $c \in C, \sigma \in F$. Then $(\alpha - \sigma)a = c - b_0$. The equality $B \cap C = 0$ implies that $c - b_0 \neq 0$ so that $\alpha - \sigma \neq 0$ from which it follows that $a \in B + C$. Thus $a = \beta b_0 + c_0$ for some elements $c_0 \in C, \beta \in F$. Suppose that $\mathbf{dim}_F(C) > 1$. Then C contains a non-zero subspace C_1 such that $C = Fc_0 \oplus C_1$. Using Corollary 3.69 we see that the subspaces Fc_0 and C_1 are both G-invariant. Since $Fb_0 = B$ is also G-invariant it follows that $B + Fc_0$ is G-invariant. Then $C_1 \cap (B + Fc_0) = 0$. Again using

Corollary 3.69 we see that every subspace of $B + Fc_0$ is G-invariant. It follows that Fa is a G-invariant subspace of A.

Finally, suppose that $\mathbf{dim}_F(B) = \mathbf{dim}_F(C) = 1$. Since $\mathbf{dim}_F(A) > 2$, there exists an element $a \notin B+C$. By Corollary 3.68 the subspace $(Fa+B)/B$ is a G-invariant subspace of A/B and likewise $(Fa + C)/C$ is a G-invariant subspace of A/C. If g is an arbitrary element of the group G we again obtain that $ga = \alpha a + b_0$ for certain elements $\alpha \in F, b_0 \in B$. If we assume that $b_0 \neq 0$, then by repeating the arguments above we see that $a \in B + C$, contradicting the choice of a. This contradiction again shows that the subspace Fa is G-invariant. Since this is true for each element a of the space A we see that every subspace of A is G-invariant. □

A very easy induction allows us to obtain the following result.

Corollary 3.71. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. Suppose also that $\boldsymbol{dim}_F(A) > 2$. If $B_1, \dots, B_n$ are non-zero G-invariant subspaces of A such that $B_1 \cap \cdots \cap B_n = 0$, then every subspace of A is G-invariant.*

The following generalization of the previous corollary to infinite index sets is a little bit more complicated.

Corollary 3.72. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. Suppose also that $\boldsymbol{dim}_F(A) > 2$. If Λ is an index set and $\{B_\lambda | \lambda \in \Lambda\}$ is a family of non-zero G-invariant subspaces of A such that $\bigcap_{\lambda \in \Lambda} B_\lambda = 0$, then every subspace of A is G-invariant.*

Proof. If the index set Λ is finite, then the result follows from Corollary 3.71. Therefore we may suppose that $\bigcap_{\lambda \in \Delta} B_\lambda \neq 0$ for all finite subsets Δ of Λ. Suppose that the vector space A has an element a such that the subspace Fa is not G-invariant. In this case the group G contains an element g such that $ga \notin Fa$. The equality $\bigcap_{\lambda \in \Lambda} B_\lambda = 0$ implies that there is an index $\mu \in \Lambda$ such that the subspace B_μ does not contain a. Since $B_\mu \neq 0$, Corollary 3.68 implies that the subspace $(Fa + B_\mu)/B_\mu$ is a G-invariant subspace of A/B_μ. It follows that $ga = \alpha a + b_\mu$ for certain elements $\alpha \in F, b_\mu \in B_\mu$. Furthermore, our assumption concerning a implies that the subspace Fa does not contain the element b_μ. There exists an index $\nu \in \Lambda$ such that the subspace B_ν does not contain b_μ. Since $Fa \cap B_\mu = 0$ and $Fb_\mu \cap B_\nu = 0$ it follows that $(Fa + B_\mu) \cap B_\mu \cap B_\nu = 0$. Our remark above implies that the G-invariant subspace $D = B_\mu \cap B_\nu$ is non-zero and hence Corollary 3.68 implies that every subspace of the quotient space A/D is G-invariant. On the other hand,

$$g(a + D) = ga + D = (\alpha a + b_\mu) + D = (\alpha a + D) + (b_\mu + D) \notin (Fa + D)/D$$

and we obtain a contradiction. This contradiction proves that every one-dimensional subspace of A is G-invariant and hence every subspace of A is G-invariant. This completes the proof. □

Suppose that A is vector space such that $\mathbf{dim}_F(A) > 2$, $G \leq GL(F, A)$ and that the non-G-invariant subspaces of A are G-core-free. If we suppose that A contains a subspace which is not G-invariant, then Corollary 3.72 shows that the space A must be FG-monolithic. Conversely, if we suppose that $\mathbf{mon}_G(A) \neq 0$, then A must contain a subspace which is not G-invariant.

Lemma 3.73. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the subspaces of A which are not G-invariant are G-core-free. If $M = \boldsymbol{mon}_G(A) \neq 0$ and $C_G(A/M) = G$, then the following assertions hold:*

(i) $C_G(M) = G$ *so that* $\boldsymbol{dim}_F(M) = 1$*;*

(ii) G *is abelian; if also* $\boldsymbol{char}(F) = 0$*, then* G *is torsion-free and if* $\boldsymbol{char}(F) = p$*, a prime, then* G *is an elementary abelian* p*-group;*

(iii) if E *is a subspace of* A *which is not* G*-invariant, then* $E \cap M = 0$*.*

Proof. Since the group G is hypercentral it has nontrivial center $\zeta(G)$. Let $z \in \zeta(G)$ be a nontrivial element and let $\xi_z : A \longrightarrow A$ be the mapping defined by $\xi_z(a) = [z, a] = (z-1)a$ for $a \in A$. Since $z \in \zeta(G)$ the mapping ξ_z is an FG-endomorphism of A and it follows that the subspaces $\mathbf{Im}(\xi_z)$ and $\mathbf{ker}(\xi_z)$ are G-invariant. Because $G = C_G(A/M)$ we deduce that the subspace M contains $\mathbf{Im}(\xi_z)$. Since M is the FG-monolith of A it is a simple FG-submodule and if we suppose that $\mathbf{Im}(\xi_z) \lneqq M$, then it follows that $\mathbf{Im}(\xi_z) = 0$. However this means that $z \in C_G(A) = 1$, contradicting the choice of the element z. This contradiction shows that $M = \mathbf{Im}(\xi_z)$ and the first isomorphism theorem implies that

$$M = \mathbf{Im}(\xi_z) \cong_{FG} A/\mathbf{ker}(\xi_z).$$

Since A is not a simple FG-module it follows that $\mathbf{ker}(\xi_z)$ is non-zero and hence it contains M. If u is an arbitrary element of M and g is an arbitrary element of G, then $u = (z - 1)a$ for some element $a \in A$ and $ga = a + c$ for some element $c \in M$, by hypothesis. Hence we have, since $z \in \zeta(G)$,

$$\begin{aligned} gu &= g((z-1)a) = (g(z-1))a = ((z-1)g)a = (z-1)(ga) \\ &= (z-1)(a+c) = (z-1)a + (z-1)c = (z-1)a = u, \end{aligned}$$

so that $G = C_G(M)$. Since M is simple we have that $\mathbf{dim}_F(M) = 1$. The arguments from the proof of Lemma 3.62 show that the group G is then abelian. Furthermore, if $\mathbf{char}\,(F) = 0$, then G is torsion-free and if $\mathbf{char}\,(F) = p$, a prime, then G is an elementary abelian p-group.

Finally let E be a subspace of A which is not G-invariant. The equality $G = C_G(A/M)$ implies that every subspace of A/M is G-invariant. Therefore every subspace of A which contains M is a G-invariant subspace. Since E is a subspace of A which is not G-invariant, the intersection $E \cap M$ must be 0. The result follows. $\square$

We next give an example which shows that the situation described in Lemma 3.73 can happen.

To do this let F be an arbitrary field and let A be a vector space over F having countable dimension, with basis $\mathcal{B} = \{a_n | n \in \mathbb{N}\}$. Let S be the subset of matrices $[\alpha_{jm}]_{j,m\in\mathbb{N}}$ whose coefficients $\alpha_{jm} \in F$ satisfy the following conditions:

$$\begin{aligned} \alpha_{jj} &= 1 \text{ for all } j, \\ \alpha_{1,m} &\neq 0 \text{ for only finitely many } m > 1, \\ \alpha_{jm} &= 0 \text{ whenever } j > 1 \text{ and } j \neq m. \end{aligned}$$

Thus the matrices in the set S have the form

$$\begin{pmatrix} 1 & \alpha_{12} & \alpha_{13}\dots & \alpha_{1,n} & 0 & 0\dots \\ 0 & 1 & 0\dots & 0 & 0 & 0\dots \\ 0 & 0 & 1 & \dots & 0 & 0\dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

Let $A = [\alpha_{jm}]_{j,m\in\mathbb{N}}, B = [\beta_{jm}]_{j,m\in\mathbb{N}} \in S$ and let $AB = [\gamma_{jm}]_{j,m\in\mathbb{N}}$. Then

$$\begin{aligned} \gamma_{jj} &= \sum_{k\in\mathbb{N}} \alpha_{jk}\beta_{kj} = 1 \text{ for all } j, \\ \gamma_{1m} &= \sum_{k\in\mathbb{N}} \alpha_{1,k}\beta_{km} = \alpha_{11}\beta_{1m} + \alpha_{1,m}\beta_{mm} = \alpha_{1,m} + \beta_{1,m}, \\ \gamma_{jm} &= \sum_{k\in\mathbb{N}} \alpha_{j,k}\beta_{k,m} = 0 \text{ whenever } j > 1, \text{ and } j \neq m \end{aligned}$$

Hence $AB \in S$. From the identities above we easily see that if $A^{-1} = [\sigma_{jm}]_{j,m\in\mathbb{N}}$, then $\sigma_{jj} = 1$ for all $j \in \mathbb{N}$, $\sigma_{1,m} = -\alpha_{1,m}$ whenever $m > 1$ and $\sigma_{jm} = 0$ whenever $j > 1$ and $j \neq m$. It follows that $A^{-1} \in S$.

Let G be the subset of linear transformations of A whose matrices relative to the basis $\mathcal{B}$ belong to S. Our work above shows that G is a subgroup of $GL(F, A)$.

Let $M = Fa_1$. It follows that $C_G(M) = G = C_G(A/M)$. In particular, M is a G-invariant subspace of A having dimension 1. Also every subspace of A which contains M is G-invariant. Let B be a non-zero subspace of A and suppose that $B \cap M = 0$. If b is a non-zero element of B, then

$$b = \beta_1 a_1 + \beta_2 a_2 + \dots \beta_k a_k \text{ where some } \beta_j \neq 0 \text{ for } 2 \leq j \leq k.$$

Let $m \geq 2$ be the least natural number such that $\beta_m \neq 0$. Let g_m be the element of G whose matrix relative to the basis $\mathcal{B}$ has the following form

$$\begin{pmatrix} 1 & 0 & 0\dots & 0 & \alpha_{1,m} & 0\dots \\ 0 & 1 & 0\dots & 0 & 0 & 0\dots \\ 0 & 0 & 1 & \dots & 0 & 0\dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots 0 & 1 & 0 & 0\dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

where $\alpha_{1,m} = 1$. Then $(g_j - 1)b = \beta_m a_1$. It follows that $a_1 \in G(Fb)$ so every G-invariant subspace of A contains M. Hence M is the FG-monolith of A. Indeed a subspace of A is G-invariant if and only if it contains M. Let D be a subspace of A which does not contain M. Since $\mathbf{dim}_F(M) = 1$ this means that $D \cap M = 0$. If we suppose that $\mathbf{core}_G\, D \neq 0$, then since $\mathbf{core}_G\, D$ is a G-invariant subspace we obtain the contradiction $M \leq \mathbf{core}_G\, D \leq D$. This contradiction shows that the G-core of D is zero.

We next consider the situation when $C_G(A/M) \neq G$.

Lemma 3.74. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the subspaces of A which are not G-invariant are G-core-free. If $M = \boldsymbol{mon}_G(A) \neq 0$ and $1 \neq C_G(A/M) \neq G$, then the following assertions hold:*

(i) $\boldsymbol{dim}_F(A/M) = 1$ *and* $C_G(M) = C_G(A/M)$;

(ii) $G/C_G(M)$ *is isomorphic to a subgroup of the multiplicative group of F and* $G/\boldsymbol{Norm}_G(A)$ *is isomorphic to a subgroup of the additive group of M;*

(iii) $C_G(M) \cap \boldsymbol{Norm}_G(A) = 0$ *and G embeds in* $G/C_G(M) \times G/\boldsymbol{Norm}_G(A)$. *In particular G is abelian. Also, if* $\boldsymbol{char}(F) = 0$, *then* $\boldsymbol{Tor}(G)$ *is a locally cyclic group. If* $\boldsymbol{char}(F) = p$, *a prime, then* $\boldsymbol{Tor}(G) = P \times Q$ *where P is an elementary abelian p-group and Q is a locally cyclic p'-group;*

(iv) if $0 \neq e \in M$, then for each $g \in G$ there exists $\alpha_g \in F$ such that $ge = \alpha_g e$;

(v) $(g - \alpha_g)a \in M$ for each $g \in G$ and each $a \in A$;

(vi) if E is a subspace of A which is not G-invariant, then $E \cap M = 0$.

Proof. By Corollary 3.68 every subspace of A/M is G-invariant. As in the proof of Lemma 3.62 we can prove that either $C_{A/M}(G) = A/M$ or $C_{A/M}(G) = 0$ so the hypotheses imply that $C_{A/M}(G) = 0$.

Since the group G is hypercentral, the intersection $C_G(A/M) \cap \zeta(G)$ is nontrivial and we choose a nontrivial element z of $C_G(A/M) \cap \zeta(G)$. Let

$\xi_z : A \longrightarrow A$ be the mapping defined, as usual, by $\xi_z(a) = [z,a] = (z-1)a$ for each $a \in A$. As usual, since $z \in \zeta(G)$ the mapping ξ_z is an FG-endomorphism of A and it follows that the subspaces $\mathbf{Im}(\xi_z)$ and $\mathbf{ker}(\xi_z)$ are G-invariant. Since $z \in C_G(A/M)$ the G-invariant subspace M contains $\mathbf{Im}(\xi_z)$. Since M is the FG-monolith of A it is a simple FG-module and hence if we suppose that $\mathbf{Im}(\xi) \lneqq M$, then $\mathbf{Im}(\xi_z) = 0$, giving the contradiction that $z \in C_G(A) = 1$. This contradiction shows that $M = \mathbf{Im}(\xi_z)$. Since G is not irreducible, $\mathbf{ker}(\xi_z)$ is non-zero and hence it contains M. It follows that every subspace of $A/\mathbf{ker}(\xi_z)$ is G-invariant. The first isomorphism theorem gives

$$M = \mathbf{Im}(\xi_z) \cong_{FG} A/\mathbf{ker}(\xi_z)$$

and hence $A/\mathbf{ker}(\xi_z)$ is a simple FG-module which is possible only if $\mathbf{dim}_F(A/\mathbf{ker}(\xi_z)) = 1$. Then $\mathbf{dim}_F(M) = 1$ also. Hence the factor group $G/C_G(M)$ is isomorphic to a subgroup of the multiplicative group of the field F.

Since every subspace of the quotient space A/M is G-invariant, Corollary 3.58 implies that there exists a homomorphism $\vartheta : G \longrightarrow U(F)$ such that $\mathbf{ker}(\vartheta) = C_G(A/M)$. Hence the factor group $G/C_G(A/M)$ is isomorphic to a subgroup of the multiplicative group of the field F. Let $\vartheta(g) = \alpha_g$ for each element $g \in G$. Then for each $g \in G$ we have $g(a+M) = \alpha_g(a+M)$ for each element $a \in A$ by Lemma 3.57. Let $d \in A$ be an element such that $d \notin \mathbf{ker}(\xi_z)$. Then $e = (z-1)d \neq 0$ and hence $Fe = M$. We have $gd = \alpha_g d + c$ for some element $c \in M$. Then since $z \in \zeta(G)$ we have

$$\begin{aligned} ge &= g((z-1)d) = (g(z-1))d = ((z-1)g)d = (z-1)(gd) \\ &= (z-1)(\alpha_g d + c) = (z-1)(\alpha_g d) + (z-1)c = \alpha_g((z-1)d) = \alpha_g e. \end{aligned}$$

In particular, if $g \in C_G(M)$, then $\alpha_g = 1$ from which it follows that $g(a+M) = a + M$ and $g \in C_G(A/M)$, so that $C_G(M) \leq C_G(A/M)$. As in the proof of Lemma 3.73 we may also prove that $C_G(A/M) \leq C_G(M)$ from which it follows that $C_G(M) = C_G(A/M)$.

Using the proof of Lemma 3.62 we see that the subgroup $C_G(A/M)$ is abelian; moreover, if $\mathbf{char}\,(F) = 0$, then $C_G(A/M)$ is torsion-free and if $\mathbf{char}\,(F)$ is the prime p, then $C_G(A/M)$ is an elementary abelian p-group.

Let Λ be an index set and choose a basis $\{a_\lambda + M | \lambda \in \Lambda\}$ in the quotient space A/M. Set $a_1 = e$ and let $\Delta = \Lambda \cup \{1\}$ to obtain a basis $\{a_\lambda | \lambda \in \Delta\}$ of the entire space A. Let Γ be the set of matrices $[\chi_{\mu\nu}]_{\mu,\nu\in\Delta}$ whose coefficients have the following properties:

$$\begin{aligned} \chi_{1,\nu} &= 0 \text{ for only finitely many indices } \nu \\ \chi_{\nu\nu} &= \alpha \text{ for some non-zero element } \alpha \in F \\ \alpha_{\mu\nu} &= 0 \text{ whenever } \mu \neq 1 \text{ and } \mu \neq \nu. \end{aligned}$$

Let $X_1 = [\gamma_{\mu\nu}]_{\mu,\nu\in\Delta}, X_2 = [\sigma_{\mu\nu}]_{\mu,\nu\in\Delta}$ be two arbitrary matrices from the set Γ and let $X_1X_2 = [\kappa_{\mu\nu}]_{\mu,\nu\in\Delta}$. Then

$$\begin{aligned}
\kappa_{11} &= \gamma_{11}\sigma_{11},\\
\kappa_{\nu\nu} &= \sum_{\mu\in\Delta}\gamma_{\nu\mu}\sigma_{\mu\nu} = \gamma_{\nu\nu}\sigma_{\nu\nu} \text{ whenever } \nu \neq 1,\\
\kappa_{1,\nu} &= \sum_{\tau\in\Delta}\gamma_{1,\tau}\sigma_{\tau\nu} = \gamma_{11}\sigma_{1,\nu} + \gamma_{1,\nu}\sigma_{\nu\nu},\\
\kappa_{\mu\nu} &= \sum_{\tau\in\Delta}\gamma_{\mu\tau}\sigma_{\tau\nu} = 0 \text{ whenever } \mu \neq 1, \mu \neq \nu
\end{aligned}$$

It follows that $X_1X_2 \in \Gamma$.

Using these equalities we can also see that if $X_1^{-1} = [\eta_{\mu\nu}]_{\mu,\nu\in\Delta}$, then

$$\begin{aligned}
\eta_{\nu\nu} &= \gamma_{\nu\nu}^{-1} = \alpha^{-1} \text{ for all } \nu \in \Delta,\\
\eta_{1,\nu} &= -\alpha^{-2}\gamma_{1,\nu} \text{ for all } \nu \in \Lambda,\\
\eta_{\mu\nu} &= 0 \text{ whenever } \mu \neq \nu \text{ and } \mu \neq 1.
\end{aligned}$$

It follows that $X_1^{-1} \in \Gamma$ and hence Γ is a group with the usual multiplication of finitary matrices. Indeed it is easy to see that Γ is itself a product of two subgroups Σ and Θ. Here Σ is the subgroup consisting of all scalar matrices of the form αE, where $\alpha \in F$ and Θ is the subset of all matrices $[\chi_{\mu\nu}]_{\mu,\nu\in\Delta}$ such that

$$\begin{aligned}
\chi_{\nu\nu} &= 1 \text{ for all } \nu \in \Delta \text{ and}\\
\chi_{1,\nu} &\neq 0 \text{ for all but finitely many } \nu \in \Delta\\
\chi_{\mu\nu} &= 0 \text{ whenever } \mu \neq 1 \text{ and } \mu \neq \nu
\end{aligned}$$

We notice immediately that Σ is a subgroup that is isomorphic to a subgroup of the multiplicative group of the field F and the subgroup Θ is isomorphic to the additive group of the vector space A/M. Since every matrix of Σ is a scalar matrix, Σ is contained in the center of the group Γ and hence Γ is an abelian group.

For each element $g \in G$ we have

$$ga_1 = \alpha_g a_1 \text{ and } ga_\lambda = \alpha_g a_\lambda + \alpha_{1,\lambda}a_1 \text{ for } \lambda \in \Lambda.$$

For each element $g \in G$ we define an infinite finitary matrix $D_g = [\gamma_{\mu,\nu}]_{\mu,\nu\in\Delta}$ whose coefficients satisfy the following conditions:

$$\begin{aligned}
\gamma_{\lambda\lambda} &= \alpha_g \text{ for all } \lambda \in \Delta\\
\gamma_{1,\lambda} &= \alpha_{1,\lambda} \text{ for all } \lambda \in \Lambda\\
\gamma_{\lambda\mu} &= 0 \text{ whenever } \lambda \neq \mu, \lambda, \mu \in \Lambda.
\end{aligned}$$

If y is another element of G such that

$$ya_1 = \alpha_y a_1 \text{ and } ya_\lambda = \alpha_\lambda a_\lambda + \beta_{1,\lambda} a_1$$

for $\lambda \in \Lambda$, then

$$\begin{aligned}
(gy)a_1 &= g(ya_1) = g(\alpha_y a_1) = \alpha_y(ga_1) = \alpha_y(\alpha_g a_1) = (\alpha_y \alpha_g)a_1 \\
&= (\alpha_g \alpha_y)a_1, \\
(gy)a_\lambda &= g(ya_\lambda) = g(\alpha_y a_\lambda + \beta_{1,\lambda} a_1) = g(\alpha_y a_\lambda) + g(\beta_{1,\lambda} a_1) \\
&= \alpha_y(ga_\lambda) + \beta_{1,\lambda}(ga_1) = \alpha_y(\alpha_g a_\lambda + \alpha_{1,\lambda} a_1) + \beta_{1,\lambda}(\alpha_g a_1) \\
&= (\alpha_y \alpha_g)a_\lambda + \alpha_y \alpha_{1,\lambda} a_1 + \beta_{1,\lambda} \alpha_g a_1 \\
&= (\alpha_g \alpha_y)a_\lambda + (\beta_{1,\lambda} \alpha_g + \alpha_y \alpha_{1,\lambda})a_1 \text{ for } \lambda \in \Lambda.
\end{aligned}$$

It follows that $D_g D_y = D_{gy}$.

We now define a mapping $\Phi : G \longrightarrow \Gamma$ by

$$\Phi(g) = D_g \text{ for each } g \in G.$$

From what we proved above we have $\Phi(gy) = D_{gy} = D_g D_y = \Phi(g)\Phi(y)$ so that the mapping Φ is a group homomorphism. It is clear that this mapping is also a monomorphism and since the group Γ is abelian it follows that the group G is likewise abelian.

Let H be the inverse image of the subgroup Σ under Φ. Then it is clear that

$$H = \{h \in G : ha_\lambda = \alpha_h a_\lambda \text{ for each } \lambda \in \Lambda\}.$$

Hence $H = \mathbf{Norm}_G(A)$. Therefore G/H is isomorphic to the subgroup $\Gamma/\Sigma \cong \Theta$ so that G/H is isomorphic to a subgroup of the additive group of the vector space A/M. In a similar manner the inverse image of the subgroup Θ under Φ is $C_G(A/M) = C_G(M)$. It is clear that $H \cap C_G(A/M) = 1$ and this implies that $\mathbf{Norm}_G(A) \cap C_G(M) = 1$. Using Remak's theorem we obtain an embedding of G into $G/\mathbf{Norm}_G(A) \times G/C_G(M)$.

Suppose that the field F has characteristic 0. Since $G/\mathbf{Norm}_G(A)$ is isomorphic to a subgroup of the additive group of the vector space A/M it follows that $G/\mathbf{Norm}_G(A)$ is torsion-free. Also the factor group $G/C_G(A/M)$ is isomorphic to a subgroup of the multiplicative group of the field F so it follows that the torsion subgroup of this group is locally cyclic (see [115, Corollary 2.2], for example). It follows that in this case $\mathbf{Tor}(G)$ is locally cyclic.

Suppose now that the field F has characteristic p for some prime p. Since $G/\mathbf{Norm}_G(A)$ is isomorphic to a subgroup of the additive group of the vector space A/M it follows that $G/\mathbf{Norm}_G(A)$ is an elementary abelian p-group. In this case the torsion subgroup of the multiplicative group of the field F is a locally cyclic p'-group (see [115, Corollary 2.2], for example). Then we see that $\mathbf{Tor}(G) = P \times Q$ for some elementary abelian p-group P and a locally cyclic p'-subgroup Q.

The last statement of the lemma is proved in precisely the same way as was done in Lemma 3.73. □

One way of constructing a group G satisfying these hypotheses is evident from the proof.

We next consider the case when $C_G(A/M) = 1$.

Lemma 3.75. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that G is not irreducible and that the subspaces of A which are not G-invariant are G-core-free. If $M = \boldsymbol{mon}_G(A) \neq 0$ and $C_G(A/M) = 1$, then either $\boldsymbol{dim}_F(A) = 2$ or the following assertions hold:*

(i) $\boldsymbol{dim}_F(A/M) = 1$ *and* $C_G(M) = 1$;

(ii) *G is isomorphic to a subgroup of the multiplicative group of F; in particular G is abelian;*

(iii) *if* $\boldsymbol{char}(F) = 0$, *then* $\boldsymbol{Tor}(G)$ *is a locally cyclic group; if* $\boldsymbol{char}(F) = p$, *a prime, then* $\boldsymbol{Tor}(G)$ *is a locally cyclic p'-subgroup;*

(iv) *if $0 \neq e \in M$, then for each $g \in G$ there exists $\alpha_g \in F$ such that $ge = \alpha_g e$;*

(v) $(g - \alpha_g)a \in M$ *for each $g \in G$ and each $a \in A$;*

(vi) *if E is a subspace of A which is not G-invariant, then $E \cap M = 0$;*

(vii) *if F is locally finite, then every subspace of A is G-invariant.*

Proof. As we have seen before, since every subspace of A/M is G-invariant, Corollary 3.58 implies that there is a homomorphsim $\vartheta : G \longrightarrow U(F)$ such that $\mathbf{ker}(\vartheta) = C_G(A/M)$. The fact that $C_G(A/M) = 1$ implies that G is abelian. In particular, if the field F has characteristic 0, then $\mathbf{Tor}(G)$ is a locally cyclic group (see [115, Corollary 2.2], for example). If $\mathbf{char}\,(F) = p$, a prime, then $\mathbf{Tor}(G)$ is a locally cyclic p'-group (again see [115, Corollary 2.2], for example).

Let $\vartheta(G) = \alpha_g$ for each element $g \in G$. Since every subspace of A/M is G-invariant Lemma 3.57 shows that we have $g(a+M) = \alpha_g(a+M) = \alpha_g a + M$ for every element $g \in G$ and each element $a \in A$. Let $\rho_g : A \longrightarrow A$ be the mapping defined by

$$\rho_g(a) = (g - \alpha_g)a \text{ for all } a \in A.$$

Since G is abelian, ρ_g is an FG-endomorphism of A and it follows that the subspaces $\mathbf{Im}(\rho_g)$ and $\mathbf{ker}(\rho_g)$ are G-invariant. The definition of ρ_g implies that M contains $\mathbf{Im}(\rho_g)$. Since M is the FG-monolith of A it is a simple FG-module. If we assume that $\mathbf{Im}(\rho_g) \lneqq M$, then $\mathbf{Im}(\rho_g) = 0$. This means that $ga = \alpha_g a$ for each element $a \in A$ Therefore every one-dimensional subspace of

A is $\langle g\rangle$-invariant and hence every subspace is $\langle g\rangle$-invariant. Our hypotheses imply that A contains a subspace which is not G-invariant so it follows that there is an element $v \in G$ such that $\mathbf{Im}(\rho_v) \neq 0$ and hence $\mathbf{Im}(\rho_v) = M$. By the first isomorphism theorem we have

$$M = \mathbf{Im}(\rho_v) \cong A/\mathbf{ker}(\rho_v). \tag{3.15}$$

Since A is not a simple FG-module, $\mathbf{ker}(\rho_v) \neq 0$. Hence M is contained in $\mathbf{ker}(\rho_v)$ and it follows that every subspace of $A/\mathbf{ker}(\rho_v)$ is G-invariant. Then the isomorphism (3.15) shows that $A/\mathbf{ker}(\rho_v)$ is a simple FG-module which is only possible if $\mathbf{dim}_F(A/\mathbf{ker}(\rho_v)) = 1$. Then $\mathbf{dim}_F(M) = 1$.

Choose an element d such that $d \notin \mathbf{ker}(\rho_v)$. Then $e = (v - \alpha_g)d \neq 0$ and hence $Fe = M$. We have $gd = \alpha_g d + c$ for some element $c \in M$. Then since G is abelian we have

$$\begin{aligned}
ge &= g((v-\alpha_g)d) = (g(v-\alpha_g))d = ((v-\alpha_g)g)d = (v-\alpha_g)(gd) \\
&= (v-\alpha_g)(\alpha_g d + c) = (v-\alpha_g)(\alpha_g d) + (v-\alpha_g)c \\
&= \alpha_g((v-\alpha_g)d) = \alpha_g e.
\end{aligned}$$

In particular, $g \in C_G(M)$ if and only if $\alpha_g = 1$, which is equivalent to $g(a+M) = a+M$ which is itself equivalent to $g \in C_G(A/M)$ so that $C_G(M) = C_G(A/M) = 1$.

Assertion (vi) is proved in the same way as was done in Lemma 3.73. The last statement of the lemma is proved in the same way as was done in Lemma 3.65. $\square$

All the results above can be summarized in the following theorems.

Theorem 3.76. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. Then*

(i) if $\boldsymbol{mon}_G(A) = 0$, then every subspace of A is G-invariant;

(ii) if $\boldsymbol{mon}_G(A) = A$, then G is irreducible.

Proof. The first assertion follows from Corollary 3.72, while the second one is clear. $\square$

A fairly detailed description of the group G when $0 \neq \mathbf{mon}_G(A) = M \neq A$ can also be given as follows.

Theorem 3.77. *Let A be a vector space over the field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that the subspaces of A which are not G-invariant are G-core-free. If $0 \neq \boldsymbol{mon}_G(A) = M \neq A$, then*

(i) every subspace of A/M is G-invariant;

(ii) $\boldsymbol{dim}_F(M) = 1$ and $C_G(M) = C_G(A/M)$;

(iii) *if* $0 \neq e \in M$, *then for each* $g \in G$ *there exists* $\alpha_g \in F$ *such that* $ge = \alpha_g e$;

(iv) $(g - \alpha_g)a \in M$ *for each* $g \in G$ *and each* $a \in A$;

(v) *if* E *is a subspace of* A *which is not* G*-invariant, then* $E \cap M = 0$.

(vi) *if* $C_G(M) = C_G(A/M) = G$, *then* G *is abelian. Moreover, if* $\boldsymbol{char}(F) = 0$, *then* G *is torsion-free; if* $\boldsymbol{char}(F) = p$, *a prime, then* G *is elementary abelian;*

(vii) *if* $G \neq C_G(M) = C_G(A/M) \neq 1$, *then* $G/C_G(M)$ *is isomorphic to a subgroup of the multiplicative group of the field* F *and* $G/\boldsymbol{Norm}_G(A)$ *is isomorphic to a subgroup of the additive group of the subspace* M, $C_G(M) \cap \boldsymbol{Norm}_G(A) = 1$ *and* G *embeds in* $G/C_G(M) \times G/\boldsymbol{Norm}_G(A)$. *In particular* G *is abelian; furthermore, if* $\boldsymbol{char}(F) = 0$, *then* $\boldsymbol{Tor}(G)$ *is a locally cyclic group. If* $\boldsymbol{char}(F) = p$, *a prime, then* $\boldsymbol{Tor}(G) = P \times Q$ *where* P *is an elementary abelian* p*-group and* Q *is a locally cyclic* p'*-group;*

(viii) *if* $C_G(M) = C_G(A/M) = 1$, *then* G *is isomorphic to a subgroup of the multiplicative group of* F. *In particular* G *is abelian. Furthermore, if* $\boldsymbol{char}(F) = 0$, *then* $\boldsymbol{Tor}(G)$ *is a locally cyclic subgroup. If* $\boldsymbol{char}(F) = p$ *is prime, then* $\boldsymbol{Tor}(G)$ *is a locally cyclic* p'*-group. Finally if* F *is locally finite, then every subspace of* A *is* G*-invariant.*

Chapter 4

The Central Dimension of a Linear Group and Its Generalizations

Central and Augmentation Dimension

In this chapter we consider linear groups that are in some sense close to finite dimensional ones. As usual we let G be a subgroup of $GL(F, A)$ and we let $Z = C_A(G) = \zeta_G(A)$. As we saw earlier Z is a G-invariant subspace of A and G acts trivially on Z, so that in effect G acts on the quotient space A/Z. This leads to the following definition.

We say that the *central dimension* of G is the dimension of the quotient space $A/\zeta_G(A)$. We denote the central dimension of the linear group G by $\mathbf{centdim}_F(G)$.

In particular, if $\mathbf{centdim}_F(G) = \mathbf{dim}_F(A/\zeta_G(A))$ is finite, then we shall say that G has *finite central dimension.* Otherwise we say that G has *infinite central dimension.* In some sense then, saying that a group G has finite central dimension is analogous to saying that the index of the center of an infinite group is finite.

We immediately note the following simple properties of groups with finite central dimension. We let $\mathbf{codim}_F(B)$ denote the codimension of a subspace B of A.

Lemma 4.1. *Let F be a field, let A be a vector space over F and let G, K be subgroups of $GL(F, A)$.*

(i) If G, K have finite central dimension, then $G \cap K$ and $\langle G, K\rangle$ have finite central dimension. Furthermore,

$$\boldsymbol{centdim}_F(G \cap K) \leq \boldsymbol{centdim}_F(G) + \boldsymbol{centdim}_F(K) \text{ and}$$
$$\boldsymbol{centdim}_F(\langle G, K\rangle) \leq \boldsymbol{centdim}_F(G) + \boldsymbol{centdim}_F(K).$$

(ii) If K has finite central dimension, then K^g has finite central dimension for each $g \in G$.

(iii) *If G has finite central dimension and $\boldsymbol{char}(F) = p$ is prime, then G contains an elementary abelian normal p-subgroup L such that G/L is isomorphic to some subgroup of $GL_n(F)$, where $n = \boldsymbol{centdim}_F(G)$.*

(iv) *If G has finite central dimension and $\boldsymbol{char}(F) = 0$, then G contains a torsion-free abelian normal subgroup L such that G/L is isomorphic to some subgroup of $GL_n(F)$, where $n = \boldsymbol{centdim}_F(G)$.*

Proof. (i) We have $C_A(G) \cap C_A(K) \le C_A(G \cap K)$ and $C_A(G) \cap C_A(K) = C_A(\langle G, K\rangle)$. However

$$\mathbf{codim}_F(C_A(G) \cap C_A(K)) \le \mathbf{codim}_F(C_A(G)) + \mathbf{codim}_F(C_A(K))$$

so (i) follows.

(ii) We have $C_A(K^g) = g^{-1}(C_A(K))$, so

$$\mathbf{codim}_F(C_A(K^g)) = \mathbf{codim}_F(C_A(K))$$

and (ii) follows.

To prove (iii) and (iv), let $L = C_G(A/C_A(G))$ so that G/L is isomorphic to a subgroup of $GL(F, A/C_A(G))$ which is precisely $GL_n(F)$, where $n = \mathbf{centdim}_F(G) = \mathbf{dim}_F A/C_A(G)$. If $x \in L$, then $x(a) = a + c$, for some element $c \in C_A(G)$ and hence $x(a) - a = (x-1)(a) \in C_A(G)$, for each $a \in A$. If $y \in L$, then

$$(xy)(a) = x(y(a)) = x(a + (y-1)(a)) = a + (x-1)a + (y-1)(a)$$

and in precisely the same way we also have $(yx)(a) = a+(y-1)a+(x-1)(a)$. Since this is true for each $a \in A$ we have $xy = yx$, so L is abelian. If $L \ne 1$, then for every element $1 \ne x \in L$, there is an element $d \in A$ such that $x(d) \ne d$ and hence $d_1 = (x-1)(d) \ne 0$. We have $x^2(d) = d + 2(x-1)(d) = d + 2d_1$ and, by induction on n, $x^n(d) = d + nd_1$. Hence if $\mathbf{char}(F) = p$, a prime, then

$$x^p(d) = d + pd_1 = d$$

for all $d \in A$ and therefore $x^p = 1$. If $\mathbf{char}(F) = 0$, then $nd_1 \ne 0$, for all natural numbers n, so that $x^n \ne 1$ for all such n. In the former case L is an elementary abelian p-group; in the latter case L is torsion-free. □

Thus we immediately see that groups of finite central dimension contain large abelian normal subgroups of the type indicated in the lemma. The following quantative result can easily be deduced.

Corollary 4.2. *Let A be a vector space over the field F and let $G = \langle g_1, \dots, g_n\rangle$ be a finitely generated subgroup of $GL(F, A)$. If $\boldsymbol{codim}_F(C_A(g_i)) = c_i$ for $1 \le i \le n$, then $\boldsymbol{centdim}_F(G) \le c_1 + \cdots + c_n$.*

Let $g \in GL(F, A)$. Then $g - 1$ is a linear transformation of A and $\mathbf{ker}(g - 1) = C_A(g), \mathbf{Im}(g-1) = (g-1)A = [g, A]$. Thus we have

$$[g, A] = \mathbf{Im}(g-1) \cong A/\mathbf{ker}(g-1) = A/C_A(g).$$

The special case where $\mathbf{dim}_F(A/C_A(g))$ is finite is particularly important. As we saw in Chapter 1 an element $g \in GL(F, A)$ is called *finitary* if $\mathbf{dim}_F(A/C_A(g))$ is finite. In particular, this means that the cyclic group $\langle g \rangle$ has finite central dimension in this case.

A subgroup G of $GL(F, A)$ is called *finitary* if each element of G is finitary. By using Lemma 4.1 we see that a group G is finitary if and only if every finitely generated subgroup of G has finite central dimension. Finitary linear groups have been studied in some detail in a number of papers (see for example the respective papers of R. E. Phillips and J. I. Hall[175, 70] and also the paper of J. I. Hall[71]).

The isomorphism $[g, A] \cong A/C_A(g)$ implies that $\mathbf{dim}_F(A/C_A(g)) = \mathbf{dim}_F([g, A])$. Thus the element g is finitary if and only if $\mathbf{dim}_F([g, A])$ is finite. In general, if G is an arbitrary subgroup of $GL(F, A)$, then G acts trivially on $A/[G, A]$ so that G really acts on $[G, A]$. We recall that $[G, A]$ is the subspace generated by all elements of the form $(g - 1)a$, for all $a \in A$ and $g \in G$. Considering A as a module for the group ring FG, then the subring $\omega(FG)$ is generated by all elements of the form $g - 1$, for $g \in G$ is, as we saw in Chapter 1, a two sided ideal of FG called the augmentation ideal of FG. Using this approach $[G, A]$ is then the FG-submodule $\omega(FG)A$.

We define the *augmentation dimension* of a group G to be the dimension of the subspace $[G, A]$ and we denote this by $\mathbf{augdim}_F(G)$.

In particular, if $\mathbf{augdim}_F(G) = \mathbf{dim}_F([G, A])$ is finite, then we shall say that the group G has *finite augmentation dimension.*

As the following result shows, the properties of the augmentation dimension are similar to those of the central dimension.

Lemma 4.3. *Let A be a vector space over the field F and let G, K be subgroups of $GL(F, A)$.*

(i) If G, K have finite augmentation dimension, then $G \cap K$ and $\langle G, K \rangle$ have finite augmentation dimension. Furthermore,

$$\boldsymbol{augdim}_F(G \cap K) \leq \boldsymbol{dim}_F([G, A] \cap [K, A]) \text{ and}$$
$$\boldsymbol{augdim}_F(\langle G, K \rangle) \leq \boldsymbol{augdim}_F(G) + \boldsymbol{augdim}_F(K).$$

(ii) If K has finite augmentation dimension, then the subgroup K^g has finite augmentation dimension for each element $g \in G$.

(iii) If G has finite augmentation dimension and $\boldsymbol{char}(F) = p$ is prime, then G contains an elementary abelian normal p-subgroup L such that G/L is isomorphic to some subgroup of $GL_n(F)$, where $n = \boldsymbol{augdim}_F(G)$.

(iv) If G has finite augmentation dimension and $\boldsymbol{char}(F) = 0$, then G contains a torsion-free abelian normal subgroup L such that G/L is isomorphic to some subgroup of $GL_n(F)$, where $n = \boldsymbol{augdim}_F(G)$.

Proof. (i) If x is an element of $G \cap K$, then

$$(x-1)a \in [G,A] \text{ and } (x-1)a \in [K,A]$$

for all $a \in A$. It follows that $[G \cap K, A] \leq [G,A] \cap [K,A]$ and hence

$$\mathbf{augdim}_F(G \cap K) = \mathbf{dim}_F([G \cap K, A]) \leq \mathbf{dim}_F([G,A] \cap [G,K]).$$

Since G acts trivially on $A/[G,A]$ the subspace $[G,A] + [K,A]$ is G-invariant and similarly $[G,A] + [K,A]$ is K-invariant. Clearly every element of $\langle G, K\rangle$ acts trivially on $A/([G,A] + [K,A])$. Therefore

$$[\langle G, K\rangle, A] \leq [G,A] + [K,A]$$

and hence

$$\mathbf{augdim}_F(\langle G, K\rangle) \leq \mathbf{augdim}_F(G) + \mathbf{augdim}_F(K)$$

so (i) follows.

(ii) Let $x \in K$. For each element $a \in A$ we have

$$\begin{aligned}(g^{-1}xg - 1)a &= (g^{-1}xg - g^{-1}g)a = g^{-1}xg(a) - (g^{-1}g)(a) \\ &= g^{-1}(xg(a) - g(a)) = g^{-1}((x-1)g(a)) \\ &\in g^{-1}([K,A]).\end{aligned}$$

It follows that $[K^g, A] \leq g^{-1}([K,A])$ and, in particular, $\mathbf{dim}_F([K^g, A])$ is finite.

To prove (iii) and (iv), let $L = C_G([G,A])$, so G/L is isomorphic to a subgroup of $GL(F, [G,A])$. This latter group is nothing more than $GL_n(F)$, where $n = \mathbf{augdim}_F(G)$. Now we may use the same arguments as are used in the proof of Lemma 4.1(iii), (iv) to obtain the result. □

Once again it is evident that a subgroup of a group with finite augmentation dimension also has finite augmentation dimension.

If x, y are arbitrary elements of $GL(F, A)$, we have

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1)$$

and it follows that

$$[xy, A] = [x,A] + [y,A].$$

Using mathematical induction we deduce that $[x^n, A] \leq [x, A]$ for each natural number n. Furthermore, $x^{-1} - 1 = -(x-1)x^{-1}$, which implies that $[x^{-1}, A] = [x, A]$. Hence we obtain the inclusion $[x^n, A] \leq [x, A]$ for all integers n. In turn it follows that $[\langle x\rangle, A] = [x, A]$.

Again, since $[g, A] \cong A/C_A(g)$ we see that

$$\mathbf{centdim}_F(\langle g\rangle) = \mathbf{augdim}_F(\langle g\rangle)$$

for every element $g \in GL(F, A)$.

With this notation established we observe the following corollary to our work above.

Corollary 4.4. *Let A be a vector space over the field F and let $G = \langle g_1, \dots, g_n\rangle$ be a finitely generated subgroup of $GL(F, A)$.*

(i) If $\boldsymbol{dim}_F([g_i, A]) = c_i$ for $1 \le i \le n$, then $\boldsymbol{augdim}_F(G) \le c_1 + \cdots + c_n$.

(ii) If $\boldsymbol{codim}_F(C_A(g_i)) = c_i$ for $1 \le i \le n$, then $\boldsymbol{augdim}_F(G) \le c_1 + \cdots + c_n$.

(iii) If $\boldsymbol{dim}_F([g_i, A]) = c_i$ for $1 \le i \le n$, then $\boldsymbol{centdim}_F(G) \le c_1 + \cdots + c_n$.

Proof. (i) As we noted above

$$[G, A] = [g_1, A] + \cdots + [g_n, A]$$

from which it follows that

$$\mathbf{augdim}_F(G) = \mathbf{dim}_F([G, A]) \le \sum_{1 \le i \le n} \mathbf{dim}_F([g_i, A]) = c_1 + \cdots + c_n.$$

This proves (i).

(ii) We also have

$$\begin{aligned}\mathbf{augdim}_F(\langle g_i\rangle) = \mathbf{centdim}_F(\langle g_i\rangle) &= \mathbf{codim}_F(C_A(\langle g_i\rangle)) \\ &= \mathbf{codim}_F(C_A(g_i)),\end{aligned}$$

so (ii) follows from (i)

(iii) follows from these equations and Corollary 4.2. □

We also make the following observation.

Corollary 4.5. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let $g_1, \dots, g_n$ be elements of G.*

(i) If $\boldsymbol{centdim}_F(G) = c$ is finite, then

$$\boldsymbol{augdim}_F(\langle g_1, \dots, g_n\rangle) \le nc;$$

(ii) If $\boldsymbol{codim}_F(G) = d$ is finite, then $\boldsymbol{centdim}_F(G) \le nd$.

In particular, we see that if G is a finitely generated subgroup of $GL(F, A)$, then Corollary 4.4 implies that $\mathbf{centdim}_F(G)$ is finite if and only if $\mathbf{augdim}_F(G)$ is finite. However, the following two examples show that for arbitrary subgroups of $GL(F, A)$ this is no longer the case.

Example 4.6. *Let F be a field and let A be a vector space over F with basis $\{a, c_n | n \in \mathbb{N}\}$. Define F-isomorphisms g_n of A by*

$$g_n(a) = a + c_n, g_n(c_k) = c_k \text{ for all } k \in \mathbb{N},$$

extending these actions linearly in the usual way.

We have for all n, k

$$\begin{aligned}(g_n g_k)(c_m) &= g_k g_n(c_m) \text{ for all } m,\\ g_n g_k(a) &= g_n(a + c_k) = a + c_n + c_k = g_k g_n(a)\end{aligned}$$

and it follows in particular that $g_n^t(a) = a + tc$ for each integer t. From these equations we see that the group $G = \langle g_n | n \in \mathbb{N}\rangle$ is abelian. Indeed G is free abelian if ***char***$(F) = 0$ *and G is an infinite elementary abelian p-group if* ***char***$(F) = p$*, a prime. It is clear that $\zeta_G(A) = \bigoplus_{k\in\mathbb{N}} Fc_k$ which implies that* ***centdim***$_F(G) = 1$*. On the other hand it is also clear that $[G, A] = \bigoplus_{k\in\mathbb{N}} Fc_k$, so that in this case* ***augdim***$_F(G)$ *is infinite.*

Example 4.7. *Let F be a field and let A be a vector space over F with basis $\{a, c_n | n \in \mathbb{N}\}$. Define F-isomorphisms g_n of A by*

$$g_n(a) = a, g_n(c_n) = a + c_n, g_n(c_k) = c_k \text{ for all } k \neq n$$

and extend these actions linearly in the usual way.

Then

$$\begin{aligned}g_n g_k(a) &= g_k g_n(a) = a,\\ g_n g_k(c_m) &= g_k g_n(c_m) = c_m, \text{ whenever } m \notin \{k, n\},\\ g_k g_n(c_n) &= g_k(a + c_n) = a + c_n \text{ and}\\ g_n g_k(c_n) &= g_n(c_n) = a + c_n \text{ whenever } k \neq n,\\ g_n^t(c_n) &= ta + c_n, \text{ for each integer } t.\end{aligned}$$

It follows that $G = \langle g_n | n \in \mathbb{N}\rangle$ is abelian. Indeed G is free abelian if ***char***$(F) = 0$ *and G is an infinite elementary abelian p-group if* ***char***$(F) = p$*, a prime. It is clear that $\zeta_G(A) = Fa$ and this implies that* ***centdim***$_G(G)$ *is infinite. On the other hand it is also clear that $[G, A] = Fa$ so that in this case* ***augdim***$_F(G) = 1$.

Given the intimate relationship between, on the one hand, the center $\zeta(G)$ of an abstract group G and, on the other, the derived subgroup G' (see, for example, [183, Chapter 4]) we might hope to expect that some similar such relationship exists between the central and augmentation dimensions. We formulate this as the following two questions which arise naturally in connection with the work above.

- Let G be a subgroup of $GL(F, A)$. Suppose that **centdim**$_F(G)$ is finite. For which groups G does it then follow that **augdim**$_F(G)$ is finite?

- Let G be a subgroup of $GL(F, A)$. Suppose that $\mathbf{augdim}_F(G)$ is finite. For which groups G does it then follow that $\mathbf{centdim}_F(G)$ is finite?

The answers to these questions have been obtained in the paper [50] of M. R. Dixon, L. A. Kurdachenko and J. Otal. We now give an account of the main results in this work.

In the above examples, the group G was either an infinite elementary abelian p-group or a free abelian group with infinite $\mathbb{Z}$-rank. In the latter case the factor group G/G^p is an infinite elementary abelian p-group for each prime p. One might hope that in the absence of such infinite sections positive answers to the two questions might then be obtained and this is indeed the case.

Let p be a prime. We say that a group G has *finite section p-rank* $\mathbf{sr}_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and *there is* an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

There is a very large literature pertaining to groups of finite section p-rank and detailed information concerning groups with finite section p-rank can be found in the book [52].

We shall also work with another numerical invariant of a group, which is closely related to the section p-rank. The group G has *finite special rank* $\boldsymbol{r}(G) = r$ if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property. If there is no such integer r then G is said to be of *infinite special rank*.

The general concept of special rank (and also the term "special rank") was introduced by A. I. Maltsev [142]. Again, the number of research articles concerned with special rank is very large and essential information associated with groups of finite special rank can also be found in [52]. We shall require some of the properties of groups with finite special rank and also of groups with finite section p-rank during this chapter.

Proposition 4.8. *Let A be a vector space over the field F and let G, K be subgroups of $GL(F, A)$ such that K is a subgroup of G of finite special rank r. Then*

(i) If $\boldsymbol{centdim}_F(G) = c$ is finite, then $\boldsymbol{augdim}_F(K) \leq rc$;

(ii) If $\boldsymbol{augdim}_F(G) = d$ is finite, then $\boldsymbol{centdim}_F(K) \leq rd$.

Proof. (i) If L is an arbitrary finitely generated subgroup of K, then L contains elements $x_1, \dots, x_m$ such that $L = \langle x_1, \dots, x_m \rangle$, where $m \leq r$. Corollary 4.5 shows that

$$\mathbf{dim}_F([L, A]) \leq mc \leq rc.$$

Next choose a finitely generated subgroup V of K such that $\mathbf{dim}_F([V, A])$ is maximal. If L is again an arbitrary finitely generated subgroup of K, then there is a finitely generated subgroup U of K such that $\langle V, L \rangle \leq U$. It follows that $[V, A], [L, A] \leq [U, A]$ so that $\mathbf{dim}_F([V, A]) \leq \mathbf{dim}_F([U, A])$. On the other hand, the choice of V ensures that $\mathbf{dim}_F([V,A]) = \mathbf{dim}_F([U,A])$ and hence $[V,A] = [U,A]$. It follows that $[L,A] \leq [V,A]$ and since this is true for each finitely generated subgroup of K we deduce that $[V,A] = [K,A]$. In particular,

$$\mathbf{augdim}_F(K) = \mathbf{dim}_F([K,A]) \leq rc,$$

as required.

(ii) The proof of this statement is completely analogous to the one given in (i), so is omitted. □

The following result is a variant of the Tits Alternative (and can be found, for example, in the book [202, Corollary 10.17] of B. A. F. Wehrfritz).

Lemma 4.9. *Let F be a field of prime characteristic p and let A be a vector space over F. Let G be a subgroup of $GL(F,A)$. Suppose that U,V are G-invariant subspaces of A such that $U \leq V$, $\mathbf{dim}_F(V/U)$ is finite and $C_G(V/U)$ is an elementary abelian p-group. If G contains no non-cyclic free subgroups, then G contains normal subgroups P and K such that $P \leq K$ satisfying the following conditions.*

(i) P is a bounded p–subgroup;

(ii) K/P is abelian and

(iii) G/K is locally finite.

Proof. Let $C = C_G(V/U)$. This is a normal subgroup of G. Suppose also that $\mathbf{dim}_F(V/U) = d$. Then G/C is isomorphic to some subgroup of $GL_d(F)$. We claim that G/C has no non-cyclic free subgroups. For, if S/C is a non-cyclic free subgroup of G/C, then it is well-known that $S = C \rtimes S_1$ (see [130, §52], for example). In this case S_1 is a non-cyclic free subgroup contradicting the hypothesis on G. Using the Tits Alternative for finite dimensional linear groups (see [202, Corollary 10.17]) we see that G/C contains a soluble normal subgroup R/C such that G/R is locally finite. The fact that C is elementary abelian implies that R is also soluble. Since R/C is a soluble subgroup of $GL_d(F)$, it follows that there are normal subgroups P_0, K_0 with $P_0 \leq K_0$ of R such that P_0/C is isomorphic to a subgroup of $UT_d(F)$, K_0/P_0 is abelian and R/K_0 is finite of order at most $\mu(d)$, where μ is the Maltsev function (see [202, Theorem 3.6], for example). Since $\mathbf{char}(F) = p > 0$, $UT_d(F)$ is a bounded nilpotent p-subgroup, so that P_0/C is also a bounded nilpotent p-subgroup. Since C is an elementary abelian p–subgroup, P_0 is a bounded p-subgroup.

For each element $g \in G$,

$$R/K_0^g = R^g/K_0^g \cong R/K_0$$

is finite of order at most $\mu(d)$. Let $K = \mathbf{core}_G\, K_0 = \bigcap_{g \in G} K_0^g$. From Remak's theorem we obtain the embedding $R/K \hookrightarrow \mathrm{Cr}_{g\in G} R/K_0^g$. The latter group is locally finite since it is a Cartesian product of finite groups of bounded order (see [52, Lemma 2.5.3], for example) and hence G/K is locally finite also because an extension of a locally finite group by a locally finite group is locally finite (see [52, Corollary 1.2.14], for example). This proves (iii).

Let $P = \mathbf{core}_G P_0 = \bigcap_{g\in G} P_0^g$. Then P is a subgroup of K since $P_0 \leq K_0$ and again using Remak's theorem we obtain an embedding of K/P in the Cartesian product $\underset{g\in G}{\mathrm{Cr}}\, KP_0^g/P_0^g$. This latter group is a subgroup of $\underset{g\in G}{\mathrm{Cr}}\, K_0^g/P_0^g$ and since K_0/P_0 is abelian, K_0^g/P_0^g is abelian, for each $g \in G$. Hence $\underset{g\in G}{\mathrm{Cr}}\, K_0^g/P_0^g$ is also abelian and it follows that K/P is likewise abelian which proves (ii). Finally, since $P \leq P_0$ and P_0 is a bounded p-group, it follows that P is also a bounded p-subgroup which proves (i). □

We next obtain an analogous result for the characteristic 0 case.

Lemma 4.10. *Let F be a field of characteristic* 0 *and let A be a vector space over F. Let G be a subgroup of $GL(F, A)$. Suppose that U, V are G-invariant subspaces of A such that $U \leq V$, $\boldsymbol{dim}_F(V/U) = d$ is finite and $C_G(V/U)$ is a torsion-free abelian subgroup. If G contains no non-cyclic free subgroups, then G has normal subgroups C, L, R such that $C \leq L \leq R$ satisfying the following conditions:*

(i) C is a torsion-free abelian subgroup;

(ii) L/C is torson-free nilpotent;

(iii) L/C is of class at most $d - 1$;

(iv) R/L is abelian;

(v) G/R is finite.

Proof. Let $C = C_G(V/U)$. Then C is a normal subgroup of G such that G/C is isomorphic to some subgroup of $GL_d(F)$. As in the proof of Lemma 4.9 we may prove that G/C contains no non-cyclic free subgroups. Then G/C contains a soluble normal subgroup S/C such that G/S is finite (by [202, Corollary 10.17], for example). Since C is torsion-free abelian it follows that S is soluble. Since S/C is a soluble subgroup of $GL_d(F)$ we see, using [202, Theorem 3.6], that S contains normal subgroups H_1, H_2 with the property that $C \leq H_1 \leq H_2$ such that H_1/C is a torsion-free nilpotent group of nilpotency class at most $d - 1$, H_2/H_1 is abelian and S/H_2 is finite of order at most $\mu(d)$. Since $|G : H_2|$ is finite we let R be the core of H_2 in G, so G/R is also finite. Also we let L be the core of H_1 in G and note that $C \leq L \leq R$, since C is normal in G. Since L is a subgroup of K_1 it follows that L/C is nilpotent of class at most $d - 1$ and, as in the proof of Lemma 4.9, R/L is abelian. The result follows. □

According to Lemmas 4.1 and 4.3, the hypotheses in Lemmas 4.9 and 4.10 will hold in particular when G has finite central dimension or finite augmentation dimension.

As in Chapter 1 $\Pi(G)$ denotes the set of primes occurring as divisors of the orders of the elements of G having finite order.

Proposition 4.11. *Let A be a vector space over a field F and let G be a locally finite subgroup of $GL(F, A)$. Suppose that either*

(i) $\boldsymbol{codim}_F(\zeta_G(A))$ is finite, or

(ii) $\boldsymbol{dim}_F([G, A])$ is finite.

If $\boldsymbol{char}(F) \notin \Pi(G)$ then $A = \zeta_G(A) \bigoplus [G, A]$.

Proof. Suppose first that $\mathbf{codim}_F(\zeta_G(A)) = c$ is finite. Let $\mathcal{L}$ be the local family consisting of all finite subgroups of G. If $K \in \mathcal{L}$, then it follows that $A = \zeta_K(A) \oplus [K, A]$, by [116, Corollary 5.16], for example. Clearly $\zeta_G(A)$ is a subspace of $\zeta_K(A)$, and this implies that $\mathbf{dim}_F([K, A]) \leq c$. Let V be a finite subgroup of G such that $\mathbf{dim}_F([V, A])$ is maximal. If $S \in \mathcal{L}$, then there exists a finite subgroup $W \in \mathcal{L}$ such that $\langle S, V \rangle$ is a subgroup of W, so $[V, A] \leq [W, A]$. Hence $\mathbf{dim}_F([V, A]) \leq \mathbf{dim}_F([W, A])$ and the choice of V implies that $\mathbf{dim}_F([V, A]) = \mathbf{dim}_F([W, A])$. Since V is a subgroup of W we have $[V, A] = [W, A]$. Thus $[S, A]$ is a subspace of $[V, A]$ and since S is an arbitrary finite subgroup of G, we have $[V, A] = [G, A]$. Then

$$A = \zeta_V(A) \oplus [V, A] = \zeta_V(A) \oplus [G, A].$$

If W is a finite subgroup of G containing V, then $\zeta_W(A) \leq \zeta_V(A)$ and $[W, A] = [V, A]$, so we obtain

$$\begin{aligned}\zeta_V(A) &= \zeta_V(A) \cap (\zeta_W(A) \oplus [W, A] = \zeta_W(A) \oplus (\zeta_V(A) \cap [W, A]) \\ &= \zeta_W(A) \oplus (\zeta_V(A) \cap [V, A]) = \zeta_W(A).\end{aligned}$$

Since W is an arbitrary finite subgroup it follows that $\zeta_G(A) = \zeta_V(A)$ so we obtain $A = \zeta_G(A) \oplus [G, A]$.

For the case when $\mathbf{dim}_F([G, A]) = c$ is finite the arguments given above again work with some minor modifications and once again we obtain the direct decomposition $A = \zeta_G(A) \oplus [G, A]$. The details in this case are left to the reader. □

We can immediately deduce the following corollary using the definitions. This shows that for certain types of locally finite groups the finiteness of $\mathbf{centdim}_F(A)$ is equivalent to the finiteness of $\mathbf{augdim}_F(A)$.

Corollary 4.12. *Let A be a vector space over the field F and let G be a locally finite subgroup of $GL(F, A)$. Suppose that $\boldsymbol{char}(F) \notin \Pi(G)$ if $\boldsymbol{char}(F)$ is a prime.*

(i) If $\boldsymbol{centdim}_F(A)$ is finite, then $\boldsymbol{augdim}_F(A)$ is finite;

(ii) If $\boldsymbol{augdim}_F(A)$ is finite, then $\boldsymbol{centdim}_F(A)$ is finite.

In either case $\boldsymbol{centdim}_F(A) = \boldsymbol{augdim}_F(A)$.

We next need some preliminary results concerning finite dimensional linear groups.

Proposition 4.13. *Let A be a vector space of dimension n over the field F and let G be a periodic abelian subgroup of $GL(F, A)$. Suppose that if $\boldsymbol{char}(F)$ is prime, then $\boldsymbol{char}(F) \notin \Pi(G)$. Then G has special rank at most n.*

Proof. There is a direct decomposition $A = A_1 \oplus \cdots \oplus A_t$, where A_j is a simple FG-submodule of A for $1 \leq j \leq t$ (see [202, Corollary 1.6], for example). Since A_j is a simple FG-module Corollary 2.3 implies that $G/C_G(A_j)$ is locally cyclic, for $1 \leq j \leq t$. Of course, we have $\bigcap_{1\leq j\leq t} C_G(A_j) = 1$ so Remak's theorem implies that G is isomorphic to a subgroup of $G/C_G(A_1) \times \cdots \times C/C_G(A_t)$. This latter group has special rank $t \leq n$. Consequently, G has special rank at most n also. □

Corollary 4.14. *Let A be a vector space of dimension n over the field F and let G be a periodic q-subgroup of $GL(F, A)$ for some prime q. Suppose that $\boldsymbol{char}(F) \neq q$. Then G has special rank at most $\frac{1}{2}(5n^2 + n)$.*

Proof. Since periodic subgroups of finite dimensional linear groups are locally finite (see [202, Theorem 9.1], for example) it follows that the group G is locally finite. If U is an arbitrary abelian subgroup of G, then Proposition 4.13 shows that U has finite special rank at most n. It follows from [149] that G has special rank at most $\frac{1}{2}n(5n + 1)$. □

Corollary 4.15. *Let A be a vector space of dimension n over the field F of prime characterstic p and let G be a periodic subgroup of $GL(F, A)$. Suppose that G has finite section p-rank r. Then G is an abelian-by-finite group of finite special rank. Furthermore the special rank of G is at most*

$$\frac{1}{2}n(5n + 1) + r + 1.$$

Proof. Once again we note that by [202, Theorem 9.1] the group G is locally finite. Let V be an arbitrary finite subgroup of G and let S_q be a Sylow q-subgroup of V, for some prime $q \neq p$. Then Corollary 4.14 shows that S_q has finite special rank at most $\frac{1}{2}n(5n+1)$. If S_p is a Sylow p-subgroup of V, then S_p has special rank at most r (see [52, Corollary 6.1.9], for example). It follows from [52, Corollary 6.3.16] that V has special rank at most $\frac{1}{2}n(5n+1)+r+1$ and since V is an arbitrary finite subgroup of G, we see that the group G also has finite special rank at most $\frac{1}{2}n(5n + 1) + r + 1$. Since a linear group of finite special rank is abelian-by-finite by [202, Theorem 10.9], G is also abelian-by-finite. □

We return now to infinite dimensional linear groups.

Corollary 4.16. *Let A be a vector space over the field F of prime characteristic p and let G be a periodic subgroup of $GL(F, A)$. Suppose that G has finite section p-rank r. Suppose that one of the following two conditions holds.*

(i) $\boldsymbol{centdim}_F(G) = \boldsymbol{codim}_F(\zeta_G(A)) = n$ *is finite, or*

(ii) $\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G, A]) = n$ *is finite.*

Then G has finite special rank at most

$$\frac{1}{2}n(5n + 1) + r + 1.$$

Proof. We use Lemma 4.1 in case (i) and Lemma 4.3 in case (ii) to deduce that G contains a normal elementary abelian p-subgroup L such that G/L is isomorphic to some subgroup of $GL_n(F)$. Let V be an arbitrary finite subgroup of G. If S_q is a Sylow q-subgroup of V for some prime $q \neq p$, then $L \cap S_q = 1$, so that S_q is isomorphic to a subgroup of $GL_n(F)$. Corollary 4.14 then implies that S_q has special rank at most $\frac{1}{2}n(5n + 1)$. If S_p is a Sylow p-subgroup of V, then S_p has special rank at most r (see [52, Corollary 6.1.9], for example). It follows that V has special rank at most $\frac{1}{2}n(5n+1)+r+1$ by [52, Corollary 6.3.16] and since V is an arbitrary subgroup of G, it follows that G itself has finite special rank at most $\frac{1}{2}n(5n + 1) + r + 1$. The result follows. □

We now prove some of the main results of this chapter. First we need the following technical result.

Lemma 4.17. *Let G be a group with finite section p-rank for some prime p. Then no section of G contains a non-cyclic free group.*

Proof. Suppose that G contains a non-cyclic free subgroup. Then G contains a non-cyclic free subgroup S of free rank 2 and $S_1 = [S, S] = S'$ is a free group of countably infinite free rank (see [130, $ 36], for example). Then $U = S/S_1$ is a free abelian group of countably infinite $\mathbb{Z}$-rank and U/U^p is an infinite elementary abelian p-group which yields a contradiction to the hypotheses. This contradiction shows that G contains no non-cyclic free groups. Since the property of having finite section p-rank is closed under taking subgroups and homomorphic images the result follows. □

We show next that when G has finite section p-rank r and finite central dimension c, then G has finite augmentation dimension, bounded by an easily defined function of r and c.

Theorem 4.18. *Let A be a vector space over the field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that* $\boldsymbol{centdim}_F(G) = \boldsymbol{codim}_F(\zeta_G(A)) = c$ *is finite. If G has finite section p-rank r, then* $\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G, A])$ *is finite. Furthermore, there exists a function κ_1 such that*

$$\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G, A]) \leq \kappa_1(c, r).$$

Proof. Lemma 4.17 shows that G contains no non-cyclic free subgroups.

It is clear that $C_G(A/\zeta_G(A))$ is an elementary abelian p-group. Since $A/\zeta_G(A)$ is finite dimensional, Lemma 4.9 implies that G has normal subgroups $P \leq K$ such that P is a bounded p–subgroup, K/P is abelian and G/K is locally finite. Clearly we may write

$$\mathbf{Tor}(K/P) = T/P = P_1/P \times Q/P,$$

where P_1/P is the p-component of T/P and Q/P is the corresponding p'-component. If H is an arbitrary periodic subgroup of G, then H is an extension of the soluble normal subgroup $H \cap K$ by the locally finite group $H/(H \cap K) \cong HK/K$. Since a periodic soluble group is locally finite, it follows that H is likewise locally finite (see [52, Corollary 1.2.14], for example). Since P_1 is a locally finite p-group of finite section p-rank r, P_1 has finite special rank at most r (see [52, Corollary 6.1.9], for example). Then Proposition 4.8 shows that $\mathbf{dim}_F([P_1, A]) \leq rc$. Let $D_1 = [P_1, A]$, so that P_1 is a subgroup of $C = C_G(A/D_1)$. We note that $G/C \leq GL(F, A/D_1)$ and certainly TC/C is a p'-group so, applying Proposition 4.11 to TC/C we deduce that

$$A/D_1 = \zeta_{TC/C}(A/D_1) \oplus [TC/C, A/D_1]. \tag{4.1}$$

Clearly,

$$(\zeta_G(A) + D_1)/D_1 \leq \zeta_{TC/C}(A/D_1) \tag{4.2}$$

so we have

$$\mathbf{dim}_F([T, A/D_1]) \leq c$$

by (4.1) and (4.2). Let $D_2/D_1 = [T, A/D_1]$. Then

$$\begin{aligned}\mathbf{dim}_F(D_2) &\leq \mathbf{dim}_F(D_1) + \mathbf{dim}_F(D_2/D_1)\\ &\leq \mathbf{dim}_F(D_2) \leq rc + c = (r+1)c.\end{aligned}$$

Furthermore $T \leq C_G(A/D_2)$ so $[T, A] \leq D_2$.

The factor group K/T is torsion-free abelian of finite section p-rank, so K/T has finite $\mathbb{Z}$-rank by [52, Proposition 3.2.3]. Also $\mathbf{r}_{\mathbb{Z}}(K/T) = \mathbf{sr}_p(K/T) \leq r$ by [52, Lemma 3.2.4]. It follows that K/T has finite special rank and $\mathbf{r}(K/T) = \mathbf{r}_{\mathbb{Z}}(K/T) \leq r$ (see [52, Lemma 6.1.14], for example). Applying Proposition 4.8 to $KC_G(A/D_2)/C_G(A/D_2)$ we obtain that $\mathbf{dim}_F([K, A/D_2]) \leq rc$. Put $D_3/D_2 = [K, A/D_2]$ so that

$$\begin{aligned}\mathbf{dim}_F(D_3) &\leq \mathbf{dim}(D_2) + \mathbf{dim}_F(D_3/D_2)\\ &\leq c + rc + rc = c(1 + 2r).\end{aligned}$$

Furthermore $K \leq C_G(A/D_3)$ so $[K, A] \leq D_3$.

Corollary 4.16 applied to the locally finite group $G/C_G(A/D_3)$ shows that G/K has finite special rank at most $\frac{1}{2}c(5c+1)+r+1$. We apply Proposition 4.8 again to give us the estimate

$$\mathbf{dim}_F([G, A/D_3]) \leq \frac{1}{2}(5c^2 + c + 2r + 2)c = \frac{1}{2}(5c^3 + c^2 + 2c + 2rc).$$

Let $D/D_3 = [G, A/D_3]$. Then G acts trivially on A/D, so $[G,A]$ is a subspace of D and

$$\begin{aligned}\mathbf{dim}_F([G,A]) \le \mathbf{dim}_F(D) &\le \mathbf{dim}_F(D/D_3) + \mathbf{dim}_F(D_3)\\ &\le \frac{1}{2}(5c^3 + c^2 + 2c + 2rc) + c(1+2r)\\ &= \frac{1}{2}(5c^3 + c^2 + 4c + 6rc).\end{aligned}$$

Hence here we define

$$\kappa_1(c,r) = \frac{5c^3 + c^2 + 6rc + 4c}{2}.$$

□

We use a similar technique in the characteristic 0 case and give complete details for this case. In this case the function of d and r that we obtain is very easily defined.

Theorem 4.19. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F,A)$. Suppose that*

$$\boldsymbol{centdim}_F(G) = \boldsymbol{codim}_F(\zeta_G(A)) = d$$

is finite. If G has finite section p-rank r for some prime p, then $\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G,A])$ is finite. Furthermore, there exists a function κ_2 such that

$$\boldsymbol{dim}_F([G,A]) \le \kappa_2(d,r).$$

Proof. Once again, Lemma 4.17 shows that G contains no non-cyclic free subgroups.

Since $A/\zeta_G(A)$ is finite dimensional, Lemma 4.10 implies that G has normal subgroups $C \le L \le R$ such that C is torsion-free abelian, L/C is torsion-free nilpotent, R/L is abelian and G/R is finite.

Since C is torsion-free abelian of finite section p-rank it follows from [52, Lemma 3.2.4] that C has finite $\mathbb{Z}$-rank and that $\mathbf{r}_{\mathbb{Z}}(C) = \mathbf{sr}_p(C) \le r$. Hence C has finite special rank and also $\mathbf{r}(C) = \mathbf{r}_{\mathbb{Z}}(C) \le r$, using [52, Lemma 6.1.14]. Thus Proposition 4.8 shows that $D_1 = [C,A]$ has finite dimension and indeed $\mathbf{dim}_F(D_1) \le rd$. Note also that C is a subgroup of $C_G(A/D_1)$.

Since L/C is torsion-free nilpotent of finite section p-rank it follows that L/C has finite 0-rank and $\mathbf{r}_0(C) = \mathbf{sr}_p(C) \le r$, by [52, Lemma 3.2.24]. Lemma 6.2.2 of [52] then shows that L/C has finite special rank and that $\mathbf{r}(L/C) = \mathbf{r}_0(L/C) \le r$. Proposition 4.8 allows us to deduce that $\mathbf{dim}_F([L, A/D_1]) \le rd$. Setting $D_2/D_1 = [L, A/D_1]$ we have that

$$\mathbf{dim}_F(D_2) \le rd + rd = 2rd$$

and we note that $L \le C_G(A/D_2)$.

Let $T/L = \mathbf{Tor}(R/L)$. We apply Proposition 4.11 to T/L to obtain

$$A/D_2 = \zeta_T(A/D_2) \oplus [T, A/D_2]$$

and since $(\zeta_G(A) + D_2)/D_2 \leq \zeta_T(A/D_2)$ we have

$$\mathbf{dim}_F([T, A/D_1]) \leq d.$$

Let $D_3/D_2 = [T, A/D_2]$ so that $\mathbf{dim}_F(D_3) \leq d + 2rd$ and $T \leq C_G(A/D_3)$.

The factor R/T is torsion-free abelian and as before R/T has finite special rank at most r. Again applying Proposition 4.8 we see that $\mathbf{dim}_F([R, A/D_3]) \leq rd$. Let $D_4/D_3 = [R, A/D_3]$ so that $\mathbf{dim}_F(D_4) \leq d + 2rd + rd = d + 3rd$ and $R \leq C_G(A/D_4)$.

Finally G/R is finite and Proposition 4.11 shows that

$$A/D_4 = \zeta_G(A/D_4) \oplus [G, A/D_4].$$

Since $(\zeta_G(A) + D_4)/D_4 \leq \zeta_G(A/D_4)$ we have $\mathbf{dim}_F([G, A/D_4]) \leq d$. Let $D/D_4 = [G, A/D_4]$, so that

$$\mathbf{dim}_F(D) \leq d + d + 3rd = 2d + 3rd.$$

We have $[G, A] \leq D$ and setting

$$\kappa_2(r, d) = 2d + 3rd$$

it follows that $\mathbf{dim}_F([G, A]) \leq \kappa_2(r, d)$ as required. □

These theorems can be regarded as analogues for linear groups of the well-known group theoretical theorem known as Schur's Theorem (although Schur did not prove this result–see the paper [128] of L. A. Kurdachenko. and I. Ya. Subbotin on this subject). This theorem first appeared in the work of B. H. Neumann [154] and can be stated as follows.

- If the center of a group G has finite index, then the derived subgroup of G is finite.

The converse of this result is false since there are infinite groups with finite center and finite derived subgroup. However P. Hall [73] proved

- If the derived subgroup of a group G is finite, then the second center of G has finite index.

The finiteness condition of section p-rank is quite strong. In the presence of this finiteness condition it is possible to obtain a stronger result for linear groups than just an analogue of Hall's theorem quoted above. The proofs of the following two theorems are analogous to the proofs of Theorems 4.18 and 4.19, so we shall not go into them in detail, but highlight the specifics of the use of the augmentation dimension.

Theorem 4.20. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that $\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G, A]) = c$ is finite. If G has finite section p-rank r, then $\boldsymbol{centdim}_F(G) = \boldsymbol{codim}_F(\zeta_G(A))$ is finite. Furthermore,*

$$\boldsymbol{codim}_F(\zeta_G(A)) \leq \kappa_1(c, r).$$

Proof. Lemma 4.17 shows that G contains no non-cyclic free subgroups.

Since $[G, A]$ is finite dimensional, Lemma 4.9 implies that G has normal subgroups $P \leq K$ such that P is a bounded p–subgroup, K/P is abelian and G/K is locally finite. We may write

$$\mathbf{Tor}(K/P) = T/P = P_1/P \times Q/P,$$

where P_1/P is a p-group and Q/P is a p'-group. As in the proof of Corollary 4.15 we note that every periodic subgroup of G is locally finite.

As in the proof of Corollary 4.15 the group P_1 has finite special rank at most r and Proposition 4.8 implies that $\mathbf{codim}_F(\zeta_{P_1}(A)) \leq rc$. Let $D_1 = \zeta_{P_1}(A)$ and note that since P_1 is normal in G the subspace D_1 is G-invariant and $P_1 \leq C_G(D_1)$. We apply Proposition 4.11 to the group T/P_1 to see that $D_1 = \zeta_T(D_1) \oplus [T, D_1]$. Since $[T, D_1] \leq [G, A]$ we have

$$\mathbf{dim}_F(D_1/\zeta_T(D_1)) \leq c.$$

Let $D_2 = \zeta_T(D_1)$. Note that D_2 is a G-invariant subspace such that $T \leq C_G(D_2)$ and $\mathbf{codim}_F(D_2) \leq c + rc$.

As in the Proof of Theorem 4.18 we see that K/T has finite special rank and $\mathbf{r}(K/T) = \mathbf{r}_{\mathbb{Z}}(K/T) \leq r$. Using Proposition 4.8 we deduce that $\mathbf{dim}_F(D_2/\zeta_K(D_2)) \leq rc$ and setting $D_3 = \zeta_K(D_2)$ we have

$$\mathbf{codim}_F(D_3) \leq c + rc + rc = c(1 + 2r).$$

Furthermore, $K \leq C_G(D_3)$.

Again, as in the proof of Theorem 4.18, we see that G/K has finite special rank at most $\frac{1}{2}c(5c+1)+r+1$. A further application of Proposition 4.8 gives us the estimate

$$\mathbf{dim}_F(D_3/\zeta_G(D_3)) \leq \frac{1}{2}(5c^3 + c^2 + 2c + 2rc).$$

Put $Z = \zeta_G(D_3)$. Then $Z \leq \zeta_G(A)$ and furthermore,

$$\begin{aligned}\mathbf{codim}_F(\zeta_G(A)) &\leq \frac{1}{2}(5c^3 + c^2 + 2c + 2rc) + c(1 + 2r)\\ &= \frac{5c^3 + c^2 + 4c + 6rc}{2} = \kappa_1(r, c).\end{aligned}$$

□

We can also obtain the following result the proof of which differs little from that of Theorem 4.19 so this time we omit the proof.

Theorem 4.21. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that $\boldsymbol{augdim}_F(G) = \boldsymbol{dim}_F([G, A]) = d$ is finite. If G has finite section p-rank r for some prime p, then $\boldsymbol{centdim}_F(G) = \boldsymbol{codim}_F(\zeta_G(A))$ is finite . Furthermore,*

$$\boldsymbol{codim}_F(\zeta_G(A)) \leq \kappa_2(c, r).$$

Hypercentral Dimension

The theorem concerning groups whose centers have finite index, mentioned above, has significant generalizations. Indeed the interested reader should refer to [52, Chapters 1 and 7] where abstract group theoretic analogues of some of the results we present next can be found. The first of these generalizations was obtained in the work of R. Baer [4]. Interestingly, Baer did not give a direct proof of this result which now bears his name: in the article [4] Baer notes that this result can be obtained from Zusatz zum Endlichkeitssatz of this paper. The result itself appears for the first time with a separate proof in the famous lectures of P. Hall (see [74, 75]). A natural next step in the context of linear groups is to obtain an analogue of this result of Baer's, using the same restrictions that were used to prove the analogue of the theorem of B. H. Neumann.

Theorem 4.22. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that there is a natural number n such that $\boldsymbol{codim}_F(\zeta_{G,n}(A)) = c$ is finite. If G has finite section p-rank r, then $\boldsymbol{dim}_F(\gamma_{G,n+1}(A))$ is finite and there is a function κ_3 such that $\boldsymbol{dim}_F(\gamma_{G,n+1}(A)) \leq \kappa_3(c, r, n)$.*

Proof. Let

$$0 = Z_0 \leq Z_1 \leq \cdots \leq Z_n$$

be a segment of the upper G-central series of A. The proof is by induction on n, the case $n = 1$ being Theorem 4.18, in which case $\kappa(c, r, 1) = \kappa_1(c, r)$.

Assume now that $n > 1$ and suppose the natural inductive hypothesis holds. In other words assume that $\gamma_{G,n}(A/Z_1) = D/Z_1$ has finite dimension at most $\kappa_3(c, r, n)$. Then $Z_1 \leq \zeta_G(D)$ so the codimension of $\zeta_G(D)$ in D is at most $\kappa_3(c, r, n)$. Applying Theorem 4.18 to the subspace D we see that $[G, D]$ has finite dimension at most $\kappa_1(\kappa_3(c, r, n), r)$. Since $\gamma_{G,n}(A) \leq D$ we have $\gamma_{G,n+1}(A) = [G, \gamma_{G,n}(A)] \leq [G, D]$. It follows that $\gamma_{G,n+1}(A)$ has finite dimension at most

$$\kappa_3(c, r, n + 1) = \kappa_1(\kappa_3(c, r, n), r)$$

as required. □

We note that the function κ_3 is defined recursively by

$$\kappa_3(c,r,1) = \kappa_1(c,r) \text{ and}$$
$$\kappa_3(c,r,n+1) = \kappa_1(\kappa_3(c,r,n),r) \text{ for } n \geq 1.$$

For the characteristic 0 case we also have the following result, obtained in a similar manner, so we omit the proof:

Theorem 4.23. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F,A)$. Suppose that there is a natural number n such that $\boldsymbol{codim}_F(\zeta_{G,n}(A)) = d$ is finite. If G has finite section p-rank r for some prime p, then $\boldsymbol{dim}_F(\gamma_{G,n+1}(A))$ is finite and there is a function κ_4 such that $\boldsymbol{dim}_F(\gamma_{G,n+1}(A)) \leq \kappa_4(d,r,n)$*

The function κ_4 is defined in an analogous manner to the way κ_3 was defined and one can see from the proof that κ_4 is defined recursively by

$$\kappa_4(d,r,1) = \kappa_2(d,r) \text{ and}$$
$$\kappa_4(d,r,n+1) = \kappa_2(\kappa_4(d,r,n),r).$$

We note immediately the following pair of dual results.

Theorem 4.24. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F,A)$. Suppose that there is a natural number n such that $\boldsymbol{dim}_F(\gamma_{G,n+1}(A)) = c$ is finite. If G has finite section p-rank r, then $\boldsymbol{codim}_F(\zeta_{G,n}(A))$ is finite and $\boldsymbol{codim}_F(\zeta_{G,n}(A)) \leq \kappa_3(c,r,n)$.*

Proof. Let
$$A = L_1 \geq L_2 \geq \cdots \geq L_n \geq L_{n+1}$$
be a segment of the lower G-central series, so $\gamma_{G,j}(A) = L_j$ for each j. We prove the result by induction on n, the case $n = 1$ being Theorem 4.20 with $\kappa_5(c,r,1) = \kappa_1(c,r)$.

Assume that $n > 1$ and suppose inductively that the result holds in the subspace L_2. Then $\gamma_{G,n}(L_2) = \gamma_{G,n+1}(A)$, so that $\gamma_{G,n}(L_2)$ has finite dimension c. Applying the induction hypothesis to L_2 we see that $Z = \zeta_{G,n-1}(L_2)$ satisfies $\mathbf{dim}_F(L_2/\zeta_{G,n-1}(L_2)) \leq \kappa_5(c,r,n)$. We note that $\zeta_{G,n-1}(L_2) \leq \zeta_{G,n-1}(A)$. Since $L_2 = [G,A]$ we deduce that $[G,A/Z]$ has finite dimension at most $\kappa_5(c,r,n)$. We now apply Theorem 4.20 to deduce that $C/Z = \zeta_G(A/Z)$ has finite codimension at most $\kappa_1(\kappa_5(c,r,n),r)$ in A/Z. Of course $C \leq \zeta_{G,n}(A)$ and setting

$$\kappa_5(c,r,n+1) = \kappa_1(\kappa_5(c,r,n),r)$$

we see that the result follows since it is clear that $\kappa_5(c,r,n) = \kappa_3(c,r,n)$. □

We can obtain the following result in the same way and again omit the proof.

Theorem 4.25. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that there is a natural number n such that $\boldsymbol{dim}_F(\gamma_{G,n+1}(A)) = d$ is finite. If G has finite section p-rank r, then $\boldsymbol{codim}_F(\zeta_{G,n}(A))$ is finite. Furthermore,*

$$\boldsymbol{codim}_F(\zeta_{G,n}(A)) \leq \kappa_4(d, r, n).$$

When A is a vector space over a field F and G is a subgroup of $GL(F, A)$, let $\mathcal{N}$ (and respectively $\mathcal{L}$) denote the family of G-invariant subspaces X of A such that A/X is G-nilpotent (respectively G-locally nilpotent). The intersection of all members of the family $\mathcal{N}$ (respectively $\mathcal{L}$) is called the *G-nilpotent residual* (respectively *G-locally nilpotent residual*) of A.

We denote the G-nilpotent residual of A by $A^{\mathfrak{N}}$ and the G-locally nilpotent residual of A by $A^{\mathbf{L}\mathfrak{N}}$.

We note that if there is a natural number k such that $\mathbf{zl}_G(A/B) \leq k$, for each $B \in \mathcal{N}$, then $A^{\mathfrak{N}} = \gamma_{G,k+1}(A)$, but that this is not true in general. Indeed, in general $A/A^{\mathfrak{N}}$ need not be G-nilpotent as can be seen from the following example.

Example 4.26. *Let F be a field and let $G = \langle g \rangle$ be an infinite cyclic group. Let A be the additive group of the group ring FG, which can then be thought of as a vector space over F so that G can be thought of as a subgroup of $GL(F, A)$. Let J be the augmentation ideal of FG, so $J = (g - 1)FG$. Then $A/J^nA = A/J^n$ is G-nilpotent for every natural number n. It is not difficult to see however that $\bigcap_{n\in\mathbb{N}} J^n = 0$ and that $\zeta_G(A) = 1$.*

It follows from Theorems 4.22 and 4.23 that under the finiteness condition of finite section p-rank that if $\mathbf{codim}_F(\zeta_{G,n}(A))$ is finite, then so is $\mathbf{dim}_F(\gamma_{G,n+1}(A))$. In turn, this implies that the G-nilpotent residual of A also has finite dimension. As can be seen from the proofs of the above theorems, the function that bounds the dimension of $\gamma_{G,n+1}(A)$ is quite cumbersome. On the other hand, the bounds on the dimension of the G-nilpotent residual of A are much more straightforward, as we shall see.

To study this further we need the following specific direct decomposition of vector spaces, which will play an important role in the coming proofs.

Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. A G-invariant subspace E of A is called *G-hypereccentric* if it has an ascending series

$$0 = E_0 \leq E_1 \leq \dots E_\alpha \leq E_{\alpha+1} \leq \dots E_\gamma = E$$

of G-invariant subspaces of A such that each factor $E_{\alpha+1}/E_\alpha$ is G-eccentric and G-chief for every $\alpha < \gamma$.

We say that A has the *$Z(G)$-decomposition* if

$$A = \zeta_{G,\infty}(A) \oplus \eta_{G,\infty}(A),$$

where $\eta_{G,\infty}(A)$ is the maximal G-hypereccentric G-invariant subspace of A. This concept was introduced by D. I. Zaitsev [225] for abelian normal subgroups of groups.

We note that if such a $Z(G)$-decomposition exists, then the subspace $\eta_{G,\infty}(A)$ contains every G-invariant G-hypereccentric subspace of A and, in particular, it is unique.

To see this, let B be a G-hypereccentric, G-invariant subspace of A and let $E = \eta_{G,\infty}(A)$. If $(B+E)/E$ is non-zero, it contains a non-zero minimal G-invariant subspace U/E. Since $(B+E)/E \cong B/(B \cap E)$ it follows that U/E is G-isomorphic to some G-chief factor of B. Hence $G \neq C_G(U/E)$. On the other hand, $(B+E)/E \leq A/E \cong \zeta_{G,\infty}(A)$, so that $G = C_G(U/E)$. This contradiction shows that $B \leq E$. Hence $\eta_{G,\infty}(A)$ contains every G-hypereccentric, G-invariant subspace of A as claimed and clearly it is then unique.

We now give some results giving conditions for the existence of a $Z(G)$-decomposition in a vector space, which will be needed in the sequel. These results were obtained in the paper [114] of L. A. Kurdachenko and J. Otal for the more general case of modules over group rings.

Lemma 4.27. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. If $A/\zeta_G(A)$ is G-chief and G-eccentric, then A contains a minimal G-invariant subspace D such that $A = \zeta_G(A) \oplus D$.*

Proof. Since G is hypercentral, $\zeta(G) \neq 1$ and we let $1 \neq z \in \zeta(G)$. The function $\xi_z : A \longrightarrow A$ defined by $\xi_z(a) = (z-1)a$ for all $a \in A$ is linear with $\mathbf{ker}(\xi_z) = C_A(z)$ and $\mathbf{Im}(\xi_z) = [z, A]$. If $g \in G$, then we have, since $z \in \zeta(G)$,

$$g\xi_z(a) = g(z-1)a = (gz-g)a = (zg-g)a = (z-1)(ga) = \xi_z(ga).$$

If $c \in C_A(z)$, then

$$z(gc) = (zg)c = (gz)c = g(zc) = gc,$$

so that $gc \in C_A(z)$. It follows that $\mathbf{ker}(\xi_z)$ and $\mathbf{Im}(\xi_z)$ are G-invariant subspaces of A. Since $[g, \zeta_G(A)] = 0$ for each $g \in G$, we have $\zeta_G(A) \leq \mathbf{ker}(\xi_z)$. This inclusion and the fact that $A/\zeta_G(A)$ is G-chief implies that $\zeta_G(A) = \mathbf{ker}(\xi_z)$. The isomorphism

$$A/\zeta_G(A) = A/\mathbf{ker}(\xi_z) \cong_G \mathbf{Im}(\xi_z)$$

shows that $\mathbf{Im}(\xi_z)$ is G-chief and G-eccentric. We set $D = \mathbf{Im}(\xi_z)$. Since D is G-eccentric we have $D \cap \zeta_G(A) = 0$. Also $(D + \zeta_G(A))/\zeta_G(A)$ is a non-zero G-invariant subspace of $A/\zeta_G(A)$. This implies that $(D + \zeta_G(A))/\zeta_G(A) = A/C$ and hence $A = D \oplus \zeta_G(A)$, as required. $\square$

From Lemma 4.27 there immediately follows our next technical result.

Corollary 4.28. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace C satisfying the following conditions.*

(i) C is G-nilpotent;

(ii) A/C is G-chief and G-eccentric.

Then A contains a minimal G-invariant subspace D such that

$$A = C \oplus D.$$

The duals of these results also hold.

Lemma 4.29. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a minimal G-invariant subspace D satisfying the following conditions.*

(i) $[G, A] \leq D$;

(ii) D is G-chief and G-eccentric.

Then A contains a G-invariant subspace C such that $A = C \oplus D$.

Proof. Since G is hypercentral, $\zeta(G) \neq 1$ and we let $1 \neq z \in \zeta(G)$, so that the function $\xi_z : A \longrightarrow A$ defined by $\xi_z(a) = (z-1)a$ for all $a \in A$ is linear with $\mathbf{ker}(\xi_z) = C_A(z), \mathbf{Im}(\xi_z) = [z, A]$. As in the proof of Lemma 4.27 $\mathbf{ker}(\xi_z)$ and $\mathbf{Im}(\xi_z)$ are G-invariant subspaces of A. We have $[z, A] \leq D$ and the minimality of D implies that either $[z, A] = D$ or $[z, A] = 0$. In the latter case $z \in C_G(A) = 1$, contradicting the choice of z. Thus $\mathbf{Im}(\xi_z) = D$. Suppose, for a contradiction, that $[z, D] = 0$. Then $D \leq \mathbf{ker}(\xi_z)$. Since $G \neq C_G(D)$ there exist elements $d \in D, g \in G$ such that $gd \neq d$. Also $\mathbf{Im}(\xi_z) = D$ so there exists $b \in A$ such that $d = \xi_z(b)$. But $[z, A] = D \leq \mathbf{ker}(\xi_z)$ so $[g, b] \in \mathbf{ker}(\xi_z)$ and hence, because $g \in \zeta(G)$,

$$\begin{aligned} gd &= g\xi_z(b) = g(z-1)b = (z-1)gb = \xi_z(gb) = \xi_z(b + (g-1)b) \\ &= \xi_z(b) + \xi_z((g-1)b) = \xi_z(b) = d. \end{aligned}$$

This contradiction shows that $[z, D] \neq 0$. Since $z \in \zeta(G)$, we can show, as above, that $[z, D]$ is a G-invariant subspace of D, so $[z, D] = D$. Thus $[z, A] = [z, D]$. If $a \in A \setminus D$, then there exists $d_1 \in D$ such that $(z-1)a = (z-1)d_1$ and it follows that $(z-1)(a - d_1) = 0$. By the choice of a we have $a \neq d_1$ and since $a = d_1 + (a - d_1)$ we have $A = D + C_A(z)$. We have noted that $C_A(z)$ is a G-invariant subspace of A. Thus $D \cap C_A(z)$ is a G-invariant subspace of D so that either $D \cap C_A(z) = 0$ or $D \cap C_A(z) = D$. As we saw above $D \cap C_A(z) = D$ is impossible, so we have $A = D \oplus C$, where $C = C_A(z)$. $\square$

The following two results can be immediately deduced.

Corollary 4.30. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace D satisfying the following conditions.*

(i) A/D is G-nilpotent;

(ii) D is G-chief and G-eccentric.

Then A contains a G-invariant subspace C such that $A = C \oplus D$.

Corollary 4.31. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace D satisfying the following conditions.*

(i) $[G, A] \leq D$;

(ii) D has a finite series of G-invariant subspaces whose factors are G-chief and G-eccentric.

Then A contains a G-invariant subspace C such that $A = C \oplus D$.

Now suppose that for our vector space A over the field F that A has a finite series of G-invariant subspaces whose factors are G-chief, where as usual $G \leq GL(F, A)$. An application of the Jordan-Hölder theorem shows that any two such series have the same length and the same set of factors, up to isomorphism and hence the length of such a series is an invariant of A, which we denote by $\mathbf{chl}_G(A)$. It is clear that if A has finite F-dimension, then $\mathbf{chl}_G(A) \leq \mathbf{dim}_F(A)$.

Corollary 4.32. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. If A has a finite G-chief series of G-invariant subspaces, then A has the $Z(G)$-decomposition.*

Proof. We prove the result by induction on $\mathbf{chl}_G(A)$. If we have $\mathbf{chl}_G(A) = 1$, then A is a minimal G-invariant subspace and the result is then clear.

Suppose that $\mathbf{chl}_G(A) > 1$ and that the result is true for all G-invariant subspaces B such that $\mathbf{chl}_G(B)$ is smaller than $\mathbf{chl}_G(A)$. Since A has a G-chief series it contains a G-invariant subspace B such that A/B is a G-chief factor and it is clearly the case that

$$\mathbf{chl}_G(A) = \mathbf{chl}_G(B) + \mathbf{chl}_G(A/B).$$

Naturally $\mathbf{chl}_G(B) < \mathbf{chl}_G(A)$. By the induction hypothesis B has the $Z(G)$-decomposition, $B = E \oplus C$, where C is the upper G-hypercenter of B and E is a G-hypereccentric, G-invariant subspace.

Suppose first that A/B is G-eccentric and consider A/E. Since $B/E = (E \oplus C)/E \cong_G C$, Corollary 4.28 shows that A contains a minimal G-invariant subspace D such that $E \leq D$, D/E is G-eccentric and $A/E = B/E \oplus D/E$. This implies that

$$A = B + D = (E + C) + D = C + D.$$

Also $C \cap D \leq E$ so that

$$C \cap D = (C \cap D) \cap E = C \cap E = 0.$$

Since D/E is G-eccentric, D is a G-hypereccentric, G-invariant subspace, so A has the $Z(G)$-decomposition.

Suppose now that A/B is G-central and consider A/C. In this case $B/C = (E \oplus C)/C \cong_G E$ and Corollary 4.31 shows that A contains a minimal G-invariant subspace Z such that $C \leq Z$, Z/C is G-central and $A/C = B/C \oplus Z/C$. Then we have $A = B + Z$ and since $B = E + C$ we obtain $A = E + Z$. Since $E \cap Z \leq B \cap Z \leq C$ we also have

$$E \cap Z = (E \cap Z) \cap C = E \cap C = 0.$$

Since Z/C is G-central, Z is the G-hypercenter of A and in this case also A has the $Z(G)$-decomposition. $\square$

We now extend this result further.

Proposition 4.33. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace C satisfying the conditions:*

(i) C is G-nilpotent;

(ii) A/C has a finite G-chief series.

Then A has the $Z(G)$-decomposition. Furthermore,

$$\boldsymbol{chl}_G(\eta_{G,\infty}(A)) \leq \boldsymbol{chl}_G(A/C).$$

In particular, if $\boldsymbol{dim}_F(A/C)$ is finite, then $\boldsymbol{dim}_F(\eta_{G,\infty}(A))$ is finite and $\boldsymbol{dim}_F(\eta_{G,\infty}(A)) \leq \boldsymbol{dim}_F(A/C)$.

Proof. Let B be the upper G-hypercenter of A. It is clear that $C \leq B$. Since $\mathbf{chl}_G(A/C)$ is finite, $\mathbf{zl}_G(B)$ is also finite, as is $\mathbf{chl}_G(A/B)$. If $B = 0$, then the result follows from Corollary 4.32, so we assume that $B \neq 0$ and use induction on $\mathbf{chl}_G(A/B)$. If $A = B$, then A is G-nilpotent and the result then holds.

Hence we may suppose that $A \neq B$. Since $\mathbf{chl}_G(A/B)$ is finite A/B contains a minimal G-invariant subspace D/B, which is G-eccentric. By Corollary 4.28 D contains a minimal G-invariant subspace E such that E is G-eccentric and $D = B + E$. Consider A/E. If K/E is the upper G-hypercenter of A/E, then $D/E = (B + E)/E \leq K/E$ and it follows that $\mathbf{chl}_G((A/E)/(K/E)) < \mathbf{chl}_G(A/B)$. By the induction hypothesis A/E has the $Z(G)$-decomposition $A/E = K/E \oplus L/E$ where $L/E = \eta_{G,\infty}(A/E)$. By Corollary 4.30 K contains a G-invariant subspace Y such that $K = Y \oplus E$. Since it is clear that $Y \cap L = 0$ we have $A = K + L = Y + E + L = Y \oplus L$. This completes the proof. $\square$

The following dual result holds and its proof, modulo some minor details, repeats the proof of Proposition 4.33 and we omit this here.

Proposition 4.34. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace C satisfying the conditions:*

(i) A/C is G-nilpotent ;

(ii) C has a finite G-chief series.

Then A has the $Z(G)$-decomposition. Furthermore, $\boldsymbol{chl}_G(\eta_{G,\infty}(A)) \leq \boldsymbol{chl}_G(C)$. In particular, if $\boldsymbol{dim}_F(C)$ is finite, then $\boldsymbol{dim}_F(\eta_{G,\infty}(A))$ is finite and $\boldsymbol{dim}_F(\eta_{G,\infty}(A)) \leq \boldsymbol{dim}_F(C)$.

In the case when the linear group G is hypercentral we obtain the following result directly from Proposition 4.33.

Theorem 4.35. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that there is a natural number n such that $\boldsymbol{codim}_F(\zeta_{G,n}(A)) = c$ is finite. Then the G-nilpotent residual has finite dimension at most c.*

There is also the following natural converse.

Theorem 4.36. *Let A be a vector space over a field F and let G be a hypercentral subgroup of $GL(F, A)$. Suppose that there is a natural number n such that $\boldsymbol{dim}_F(\gamma_{G,n}(A)) = c$ is finite. Then*

$$\boldsymbol{codim}_F(\zeta_{G,m}(A)) \leq c,$$

for some natural number m.

We return now to the question of finding the dimension of the G-nilpotent residual, using the results obtained above. The results we now obtain first appeared in the paper of M. R. Dixon, L. A. Kurdachenko and J. Otal [51]. The following result is reminiscent of the well-known group theoretical fact that the chief factors of a locally nilpotent group are central. The reader will observe other analogues as we progress.

Lemma 4.37. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that A is G-locally nilpotent. If B, C are G-invariant subspaces of A such that $C \leq B$ and B/C is G-chief, then B/C is G-central.*

Proof. Without loss of generality $C = 0$, so that B is a minimal G-invariant subspace of A. Since the subspace $[G, B]$ is also G-invariant it follows that $[G, B] = 0$ or $[G, B] = B$. Suppose, for a contradiction, that $[G, B] = B$. Then there exist $b \in B, y \in G$ such that $(y-1)b = d \neq 0$. Since $d \in B$, the minimality of B implies that $B = (FG)d$ and hence there is a finite subset S of G such that $b \in (FS)d$. Let $H = \langle y, S\rangle$ and $K = (FH)d$. Then $d = [y, b] \in [H, K]$. The subspace $[H, K]$ is H-invariant, so $K = (FH)d \leq [H, K]$. Hence $K = [H, K]$. Here H is a finitely generated group and K is a finitely generated H-module so the hypotheses imply that K is H-nilpotent. But then $K \neq [H, K]$, which gives the contradiction sought. Therefore $[G, B] = 0$ and hence $B \leq \zeta_G(A)$. □

Corollary 4.38. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that A is G-locally nilpotent. If B is a finite dimensional G-invariant subspace of A, then there is a natural number k such that $k \leq$ $\boldsymbol{dim}_F(B)$ and $B \leq \zeta_{G,k}(A)$.*

Proof. We use induction on $\mathbf{dim}_F(B)$. Since B has finite dimension, there is a finite series of G-invariant subspaces whose factors are G-chief. Let

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$$

be such a series. Lemma 4.37 shows that $B_1 \leq \zeta_G(A)$. Clearly $\mathbf{dim}_F((B + \zeta_G(A))/\zeta_G(A)) < \mathbf{dim}_F(B)$ and the induction hypothesis implies the existence of a natural number t such that

$$(B + \zeta_G(A))/\zeta_G(A) \leq \zeta_{G,t}(A/\zeta_G(A))$$

and $t \leq \mathbf{dim}_F(((B + \zeta_G(A))/\zeta_G(A))$. Thus $B \leq \zeta_{G,t+1}(A)$ and $t + 1 \leq$ $\mathbf{dim}_F(B)$, as required. □

Lemma 4.39. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that there is a natural number k such that $\boldsymbol{codim}_F(\zeta_{G,k}(A)) = d$ is finite. If G has finite special rank r, then the G-nilpotent residual $L = A^{\mathfrak{N}}$ of A has finite dimension at most $d(r + 1)$. Furthermore, A/L is G-nilpotent.*

Proof. Let

$$0 = Z_0 \leq Z_1 \leq \cdots \leq Z_k = Z = \zeta_{G,k}(A)$$

be the relevant part of the upper G-central series of A. By Theorem 1.2 $G/C_G(Z)$ is nilpotent. Let $C = C_G(Z)$ and let $D_1 = [C, A]$. Then $Z \leq$ $\zeta_C(A)$ and so $\mathbf{centdim}_F(C) = \mathbf{dim}_F(A/\zeta_C(A)) \leq d$. By Proposition 4.8, $\mathbf{augdim}_F(C) = \mathbf{dim}_F(D_1) \leq dr$. Note that $C \leq C_G(A/D_1)$. It follows that $G/C_G(A/D_1)$ is also a nilpotent group. The quotient space $(A/D_1)/((Z + D_1)/D_1)$ has finite F-dimension and so has a finite G-composition series. By Proposition 4.33 A/D_1 has the $Z(G)$-decomposition, that is

$$A/D_1 = Y/D_1 \oplus E/D_1,$$

where Y/D_1 is the upper G-hypercenter of A/D_1 and E/D_1 is a G-hypereccentric, G- invariant subspace. Since $(Z + D_1)/D_1 \leq Y/D_1$, E/D_1 has finite dimension at most d. It follows that E has finite dimension and

$$\mathbf{dim}_F(E) \leq dr + d = d(r + 1).$$

Since A/E is G-nilpotent we have $L \leq E$ and then $\mathbf{dim}_F(L) \leq d(r + 1)$.

Finally we show that A/L is G-nilpotent. Let $k = \mathbf{dim}(E/L)$ and let $t = \mathbf{zl}_G(A/E)$. Let B be a G-invariant subspace of A such that A/B is G-nilpotent. Of course $L \leq B$ and since $(E + B)/B \cong E/(E \cap B)$ we have

$$\mathbf{dim}_F((E + B)/B) \leq k.$$

Using Corollary 4.38 we deduce that $(E+B)/B \leq \zeta_{G,k}(A/B)$. Hence $\mathbf{zl}_G(A/B) \leq k+t$ and this implies the inclusion $\gamma_{G,k+t+1}(A) \leq B$. Since this is true for each such G-invariant subspace B it follows that $\gamma_{G,k+t+1}(A) \leq L$ also and hence A/L is G-nilpotent, as claimed. □

The significant point about this result is that the dimension of the G-nilpotent residual is bounded by a function that depends only on the codimension of $\zeta_{G,k}(A)$ and the special rank of G, not on the integer k. This will be rather important in what follows.

Corollary 4.40. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F,A)$. Suppose also that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If A is finitely generated as an FG-module and G has finite special rank r, then the G-nilpotent residual L of A has finite dimension at most $d(r+1)$. Furthermore, A/L is G-nilpotent.*

Proof. By Proposition 1.10 $\mathbf{zl}_G(A)$ is finite and Lemma 4.39 immediately gives the result. □

Corollary 4.41. *Let A be a vector space over a field F and let G be a finitely generated subgroup of $GL(F,A)$. Suppose also that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If G has finite special rank r, then the G-locally nilpotent residual L of A has finite dimension at most $d(r+1)$. Furthermore, A/L is G-hypercentral.*

Proof. Put $Z = \zeta_{G,\infty}(A)$. Since A/Z has finite dimension, there exists a finite subset M such that $A = FM + Z$. Let $\mathcal{D}$ be the family of all finitely generated FG–submodules of A containing M. If $B \in \mathcal{D}$, then $Z \cap B \leq \zeta_{G,\infty}(B)$ and so $\mathbf{dim}_F(B/\zeta_{G,\infty}(B)) \leq d$. It follows from Corollary 4.40 that the G-nilpotent residual $L(B)$ of B has finite dimension at most $d(r+1)$ and $B/L(B)$ is FG–nilpotent.

Let $C \in \mathcal{D}$ be such that $B \leq C$. Since $C/L(C)$ is G-nilpotent, $B/(B \cap L(C))$ is also G-nilpotent so $L(B) \leq B \cap L(C)$. In particular we have $L(B) \leq L(C)$.

Let $K \in \mathcal{D}$ be an FG-submodule chosen such that $\mathbf{dim}_F L(K)$ is maximal. If $C \in \mathcal{D}$ and $K \leq C$ we have $L(K) \leq L(C)$, and the choice of K implies that $L(K) = L(C)$.

Let S be an arbitrary finite subset of A and let T be the finitely generated submodule of A generated by M, S and K. Clearly $T \in \mathcal{D}$, K is an FG-submodule of T and our work above shows that $L(T) = L(K)$. Thus $T/L(K)$ is G–nilpotent. Then $A/L(K)$ is G-locally nilpotent and hence $L \leq L(K)$. In particular we have $\mathbf{dim}_F(L) \leq d(r+1)$.

Since $(Z+L)/L \leq \zeta_{G,\infty}(A/L)$ it follows that $\zeta_{G,\infty}(A/L)$ has finite codimension in A/L. By Corollary 4.38 we deduce that

$$(A/L)/\zeta_{G,\infty}(A/L)$$

is G-nilpotent and hence A/L is G-hypercentral, as required. □

The following theorem is concerned with an important special case when the linear group G has finite special rank. As we shall see, in this case the estimation of the dimension of the G-locally nilpotent residual is particularly straightforward.

Theorem 4.42. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If G has finite special rank r, then the G-locally nilpotent residual L of A has finite dimension at most $d(r + 1)$. Furthermore, A/L is G-hypercentral.*

Proof. If H is a finitely generated subgroup of G, then Corollary 4.41 shows that the H-locally nilpotent residual $L(H)$ of A has finite dimension at most $d(r + 1)$. If K is also a finitely generated subgroup of G such that $H \leq K$, then clearly $L(K)$ is H-invariant and Corollary 4.41 again shows that $A/L(K)$ is K-hypercentral. Therefore Lemma 1.4(i) shows that $A/L(K)$ is also H-hypercentral and it follows that $L(H) \leq L(K)$.

Let T be a finitely generated subgroup of G such that $\mathbf{dim}_F(L(T))$ is maximal. If V is a finitely generated subgroup of G containing T, then by our work above $L(T) \leq L(V)$ and the choice of T shows that $L(T) = L(V)$. It follows that $L(T)$ is V-invariant. Let $g \in G$ be arbitrary and set $U = \langle g, T\rangle$. As above $L(T)$ is U-invariant and in particular $g(L(T)) = L(T)$ for all $g \in G$. Hence $L(T)$ is G-invariant.

Now let M be an arbitrary finite subset of A and let W be an arbitrary finitely generated subgroup of G. Let $B = (FW)M$ be the FW-submodule generated by M. Set $Y = \langle W, T\rangle$ and $B_1 = (FY)M$. Since $L(Y) = L(T)$ it follows that $A/L(T)$ is Y-hypercentral and Corollary 1.12 implies that $A/L(T)$ is Y-locally nilpotent. In particular, $(B_1 + L(T))/L(T)$ is Y-nilpotent. Lemma 1.4(i) then shows that $(B_1 + L(T))/L(T)$ is W-nilpotent and hence $A/L(T)$ is G-locally nilpotent. Thus $L = L(T)$ and $\mathbf{dim}_F(L) \leq d(r + 1)$. As in the proof of Corollary 4.41 we can show that A/L is G-hypercentral. □

The Positive Characteristic Case

In the previous section we established results in the case when the group G has finite special rank. Now we wish to extend these ideas, using a similar method of proof in the case when the linear group G has finite section p-rank, but we have thought it best to consider the case of characteristic 0 and the case of non-zero characteristic separately. Quite often the proofs are very similar in each case, so we shall often omit the proofs when this happens.

Lemma 4.43. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that there is a natural number k such that $\boldsymbol{codim}_F(\zeta_{G,k}(A)) = d$ is finite. If G has finite section p-rank r,* *then the G-nilpotent residual L of A has finite dimension at most*

$$\frac{1}{2}(5d^3 + d^2 + 6d + 6rd).$$

Furthermore, A/L is G-nilpotent.

Proof. Let

$$0 = Z_0 \leq Z_1 \leq \cdots \leq Z_k = Z$$

be a segment of the upper G-central series of A. By Theorem 1.2 $G/C_G(Z)$ is nilpotent. Let $C = C_G(Z)$ and let $D_1 = [C, A]$. Then $Z \leq \zeta_C(A)$ and so the hypotheses imply that $\mathbf{dim}_F(A/\zeta_C(A)) \leq d$. By Theorem 4.18, $\mathbf{dim}_F(D_1) \leq \kappa_1(d, r)$. Note that $C \leq C_G(A/D_1)$ so $G/C_G(A/D_1)$ is also a nilpotent group. The quotient space $(A/D_1)/((Z + D_1)/D_1)$ has finite F-dimension so has a finite G-composition series. Proposition 4.33 implies that A/D_1 has the $Z(G)$-decomposition, so

$$A/D_1 = Y/D_1 \oplus E/D_1,$$

where Y/D_1 is the upper G-hypercenter of A/D_1 and E/D_1 is a G-hypereccentric, G-invariant subspace. Since $(Z + D_1)/D_1 \leq Y/D_1$, E/D_1 has finite dimension at most d and it follows that E has finite dimension with $\mathbf{dim}_F(E) \leq \kappa_1(d, r) + d$. Since A/E is G-nilpotent we have $L \leq E$ and then

$$\begin{aligned} \mathbf{dim}_F(L) \leq \mathbf{dim}_F(E) \leq \kappa_1(d, r) + d &= \frac{1}{2}(5d^3 + d^2 + 4d + 6rd) + d \\ &= \frac{1}{2}(5d^3 + d^2 + 6d + 6rd). \end{aligned}$$

The proof that A/L is G-nilpotent follows in exactly the same way as in the proof of Lemma 4.39 and we omit the proof. □

Corollary 4.44. *Let A be a vector space over a field F of prime characteristic p and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If A is finitely generated as an FG-module and G has finite section p-rank r, then the G-nilpotent residual L of A has finite dimension at most*

$$\frac{1}{2}(5d^3 + d^2 + 6d + 6rd).$$

Furthermore, A/L is G-nilpotent.

Proof. By Proposition 1.10 it follows that $\mathbf{zl}_G(A)$ is finite and then the result follows upon applying Lemma 4.43 □

Further, an almost verbatim repetition of the proof of Corollary 4.41 yields the next result.

Corollary 4.45. *Let A be a vector space over a field F of prime characteristic p and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If G has finite section p-rank r, then the G-locally nilpotent residual L of A has finite dimension at most $\frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$. Furthermore, A/L is G-hypercentral.*

The proof of Theorem 4.42 can be repeated to give:

Theorem 4.46. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If G has finite section p-rank r, then the G-locally nilpotent residual L of A has finite dimension at most*

$$\frac{1}{2}(5d^3 + d^2 + 6d + 6rd).$$

Furthermore, A/L is G-hypercentral.

We now consider the dual problem:

- Suppose that A is a vector space which contains a G-invariant subspace B of finite dimension, such that A/B is G-nilpotent (respectively G-hypercentral). Is it the case that the upper G-hypercenter of A has finite codimension?

We now show that with the additional information concerning finite section p-rank the answer is positive.

The following lemma now plays a central role.

Lemma 4.47. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and that A/B is G-nilpotent. If G has finite section p-rank r, then the upper G-hypercenter Z of A has finite codimension at most*

$$\frac{1}{2}(5d^3 + d^2 + 6d + 6rd).$$

Furthermore, $\boldsymbol{zl}_G(A)$ is finite.

Proof. Let

$$B = Z_0 \leq Z_1 \leq \cdots \leq Z_k = A$$

be a series in A such that $Z_1/B = \zeta_G(A/B)$ and $Z_{j+1}/Z_j = \zeta_G(A/Z_j)$, for $1 \leq j \leq k-1$. By Theorem 1.2 $G/C_G(A/B)$ is nilpotent. We let $C = C_G(A/B)$ so that $[C, A] \leq B$. Then $\mathbf{augdim}_F(C) = \mathbf{dim}_F([C, A]) \leq d$. Using Theorem 4.20 we deduce that $D = \zeta_C(A)$ has finite codimension in A at most

$$\kappa_1(d, r) = \frac{1}{2}(5d^3 + d^2 + 4d + 6rd).$$

Clearly D is G-invariant and $G/C_G(D)$ is nilpotent. Furthermore, $D/(D \cap B)$ is G-nilpotent and $D/(D \cap B)$ is finite dimensional so Proposition 4.34 implies that D has a $Z(G)$-decomposition and we write $D = Y \oplus E$, where Y is the upper G-hypercenter of D and E is a G-invariant G-hypereccentric subspace. Then E is the G-nilpotent residual of D. Since $D/(D \cap B)$ is G-nilpotent we have $E \leq D \cap B$ and hence $\mathbf{dim}_F(E) \leq d$. Then $\mathbf{dim}_F(D/Y) = \mathbf{dim}_F(E) \leq d$. Since D is G-invariant we have $Y \leq \zeta_{G,\infty}(A)$ so that $\zeta_{G,\infty}(A)$ has codimension at most

$$\frac{1}{2}(5d^3 + d^2 + 4d + 6rd) + d = \frac{1}{2}(5d^3 + d^2 + 6d + 6rd),$$

as required.

The proof also shows that $\mathbf{zl}_G(A)$ is finite. □

We can obtain further results using a now familiar pattern of proof and we leave many of the details for the reader.

Corollary 4.48. *Let A be a vector space over a field F of prime characteristic p and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and that A/B is G-hypercentral. If A is finitely generated as an FG-module and G has finite section p-rank r, then the upper G-hypercenter Z of A has finite codimension at most $\frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$. Furthermore, $\boldsymbol{zl}_G(A)$ is finite.*

Proof. We use Corollary 1.11 to show that A/B is G-nilpotent and the result then follows by Lemma 4.47. □

Corollary 4.49. *Let A be a vector space over a field F of prime characteristic p and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and that A/B is G-hypercentral. If G has finite section p-rank r, then the upper G-hypercenter Z of A has finite codimension at most $\frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$.*

Proof. Since B has finite dimension, there is a finite subset M such that $B = FM$. If D is a finitely generated FG-submodule of A containing M, then $B \leq D$ and D/B is G-hypercentral. Using Corollary 4.48 we see that the upper G-hypercenter $\zeta_{G,\infty}(D)$ of D has finite codimension in D at most $\frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$.

Suppose that U is another finitely generated FG-submodule of A that contains D. As above, the upper G-hypercenter $\zeta_{G,\infty}(U)$ of U satisfies

$$\mathbf{codim}_F(\zeta_{G,\infty}(U)) \leq \frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$$

and clearly $\zeta_{G,\infty}(D) \leq \zeta_{G,\infty}(U)$ for all such U.

We may therefore choose a finitely generated FG-submodule K containing B such that $\mathbf{dim}_F(K/\zeta_{G,\infty}(K))$ is maximal. If once more U is a finitely generated FG-submodule of A containing K, then we have

$$\begin{aligned} K/\zeta_{G,\infty}(K) &= K/(K \cap \zeta_{G,\infty}(U)) \cong (K + \zeta_{G,\infty}(U))/\zeta_{G,\infty}(U) \\ &\leq U/\zeta_{G,\infty}(U). \end{aligned}$$

Since $\mathbf{dim}_F(U/\zeta_{G,\infty}(U)) \leq \mathbf{dim}_F(K/\zeta_{G,\infty}(K)$ by the choice of K we deduce that

$$U/\zeta_{G,\infty}(U) = (K + \zeta_{G,\infty}(U))/\zeta_{G,\infty}(U)$$

and hence $U = K + \zeta_{G,\infty}(U)$.

Let $\mathcal{L}$ be the family of all finitely generated FG-submodules of A containing K. Clearly $A = \bigcup \mathcal{L}$ and it is easy to see that

$$\bigcup_{U \in \mathcal{L}} \zeta_{G,\infty}(U) = \zeta_{G,\infty}(A) = Z.$$

Then $A/Z = \bigcup_{U \in \mathcal{L}}(U + Z)/Z$. However

$$U + Z = K + \zeta_{G,\infty}(U) + Z = K + Z$$

for each $U \in \mathcal{L}$ so we obtain $A/Z = (K+Z)/Z$ and $A = K+Z$. Consequently, since $K \cap Z = \zeta_{G,\infty}(K)$, we have

$$\begin{aligned}\mathbf{dim}_F(A/Z) &= \mathbf{dim}_F((K+Z)/Z) = \mathbf{dim}_F(K/(K \cap Z)) \\ &= \mathbf{dim}_F(K/\zeta_{G,\infty}(K)) \leq \frac{1}{2}(5d^3 + d^2 + 6d + 6rd),\end{aligned}$$

as required. $\square$

We conclude this section with the following result which provides the solution to the dual problem posed earlier.

Theorem 4.50. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\mathbf{dim}_F(B) = d$ is finite and that A/B is G-hypercentral. If G has finite section p-rank r, then the upper G-hypercenter Z of A has finite codimension at most*

$$\frac{1}{2}(5d^3 + d^2 + 6d + 6rd).$$

Proof. Let H be an arbitrary finitely generated subgroup of G. Clearly B is H-invariant. Since A/B is G-hypercentral, Lemma 1.4(i) shows that A/B is H-hypercentral and Corollary 4.49 shows that the upper H-hypercenter $\zeta_{H,\infty}(A)$ has finite codimension in A at most $\frac{1}{2}(5d^3 + d^2 + 6d + 6rd)$.

Let K be a finitely generated subgroup of G such that

$$\mathbf{codim}_F(\zeta_{K,\infty}(A)) = k$$

is maximal. If V is a finitely generated subgroup of G containing K then of course $\mathbf{dim}_F(A/\zeta_{V,\infty}(A)) \leq k$. On the other hand $\zeta_{V,\infty}(A) \leq \zeta_{K,\infty}(A)$ so that $\mathbf{dim}_F(A/\zeta_{V,\infty}(A)) \geq \mathbf{dim}_F(A/\zeta_{K,\infty}(A))$ and the way we chose K then implies that $\mathbf{dim}_F(A/\zeta_{V,\infty}(A)) = \mathbf{dim}_F(A/\zeta_{K,\infty}(A))$. But $\zeta_{V,\infty}(A) \leq \zeta_{K,\infty}(A)$ so we deduce that $\zeta_{V,\infty}(A) = \zeta_{K,\infty}(A)$ and this is true for all such V. It follows that $\zeta_{G,\infty}(A) = \zeta_{K,\infty}(A)$ and the result now follows. $\square$

The Characteristic Zero Case

In this section we discuss the Characteristic 0 cases of the results obtained in the last section and for the most part we omit the proofs. The proof of the next result is also essentially the same as that given for the proof of Lemma 4.43, but we use Theorem 4.19 rather than Theorem 4.18

Lemma 4.51. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that there is a natural number k such that $\boldsymbol{codim}_F(\zeta_{G,k}(A)) = d$ is finite. If G has finite section p-rank r, for some prime p, then the G-nilpotent residual L of A has finite dimension at most $3d + 3rd$. Furthermore, A/L is G-nilpotent.*

Once again, it is noteworthy that the dimension of the G-nilpotent residual is dependent only on the codimension of $\zeta_{G,k}(A)$ and the section p-rank of G, not on k. The proof here depends upon the fact that $\kappa_2(d, r) = 2d + 3rd$.

The following result, analogous to Corollary 4.44, may be deduced by applying Proposition 1.10, but in this case we use Lemma 4.51 to obtain the final result.

Corollary 4.52. *Let A be a vector space over a field F of characteristic 0 and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If A is finitely generated as an FG-module and G has finite section p-rank r, for some prime p, then the G-nilpotent residual L of A has finite dimension at most $3d + 3rd$. Furthermore, A/L is G-nilpotent.*

As with Corollary 4.45 a proof similar to that of Corollary 4.41 gives us the next result.

Corollary 4.53. *Let A be a vector space over a field F of characteristic 0 and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that $\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$ is finite. If G has finite section p-rank r, for some prime p, then the G-locally nilpotent residual L of A has finite dimension at most $3d + 3rd$. Furthermore, A/L is G-hypercentral.*

Finally, in the study of this particular problem, we can also prove, using the usual techniques:

Theorem 4.54. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that*

$$\boldsymbol{codim}_F(\zeta_{G,\infty}(A)) = d$$

is finite. If G has finite section p-rank r, for some prime p, then the G-locally nilpotent residual L of A has finite dimension at most $3d+3rd$. Furthermore, A/L is G-hypercentral.

For the dual problem we were concerned with in the previous section, we also have, using a similar method of proof to that given in Lemma 4.47 the following results in the characteristic 0 case.

Lemma 4.55. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and A/B is G-nilpotent. If G has finite section p-rank r, for some prime p, then the upper G-hypercenter Z of A has finite codimension at most $3d + 3rd$. Furthermore, $\boldsymbol{zl}_G(A)$ is finite.*

We note again here that the significant point is that the codimension of the upper hypercenter is bounded by a function that depends only upon the F-dimension of $\gamma_{G,k}(A)$ and the section p-rank of G, not on the number k.

As with Corollary 4.48 we have:

Corollary 4.56. *Let A be a vector space over a field F of characteristic 0 and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and A/B is G-hypercentral. If A is finitely generated as an FG-module and G has finite section p-rank r, for some prime p, then the upper G-hypercenter Z of A has finite codimension at most $3d + 3rd$. Furthermore, $\boldsymbol{zl}_G(A)$ is finite.*

For the characteristic 0 case there is a similar proof to that given in Corollary 4.49, but the estimate for the codimension is of course different.

Corollary 4.57. *Let A be a vector space over a field F of characteristic 0 and let G be a finitely generated subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and A/B is G-hypercentral. If G has finite section p-rank r, for some prime p, then the upper G-hypercenter Z of A has finite codimension at most $3d + 3rd$.*

Finally we give the theorem which answers the dual question posed earlier, but for the characteristic 0 case. The proof is similar to that given in Theorem 4.50.

Theorem 4.58. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$. Suppose that A contains a G-invariant subspace B such that $\boldsymbol{dim}_F(B) = d$ is finite and A/B is G-hypercentral. If G has finite section p-rank r, for some prime p, then the upper G-hypercenter Z of A has finite codimension at most $3d + 3rd$.*

Chapter 5

Linear Groups Saturated with Subgroups of Finite Central Dimension

In Chapter 4 we studied those linear groups having finite central dimension in some detail. It is intuitively clear that the presence of a "large" collection of subgroups that have finite central dimension should have a significant impact on the structure of a linear group. Since we are dealing with infinite objects, the question naturally arises as to what should be understood by the term "large collection". A similar question arose in the theory of abstract groups during the transition from the study of finite groups to the study of infinite groups, when infinite groups were in their infancy. With this as a model one can simply use the extensive experience established in the theory of infinite groups in order to use various approaches in the study of infinite dimensional linear groups, as we shall do in this chapter.

Linear Groups whose Proper Subgroups have Finite Central Dimension

One of the "largest" families of a group G is the family of all proper subgroups of G. The study of groups whose proper subgroups have some fixed property $\mathcal{P}$ began with the classical work of R. Dedekind [40]. In this work he considered those finite groups whose proper subgroups are normal. Shortly afterwards, in their famous paper [150], G. Miller and H. Moreno described those finite groups all of whose proper subgroups are abelian. In this setting, we need to mention the remarkable article [187] due to O. Yu. Schmidt, which completely describes those finite groups all of whose proper subgroups are nilpotent. In order to develop the theory of infinite groups further, the following problem of O. Yu. Schmidt was very significant:

- Describe those infinite groups whose proper subgroups are finite.

This problem was formulated during the twentieth century in the 1930's. This was the time when the theory of infinite groups began to develop intensively at Moscow University. Schmidt's problem turned out to be the starting point of a whole new branch of the theory of infinite groups, whose effects are still felt today and the essence of which was the study of groups with different minimal conditions.

A whole series of articles was devoted to solving the Schmidt problem and many important results were established. Some early "positive" results were obtained, two highlights being the works of M. I. Kargapolov [94] and P. Hall and C. R. Kulatilaka [76]. They showed that an infinite locally finite group whose proper subgroups are finite is a Prüfer p-group for some prime p. These results were not easy to come by. In general the situation was rather complicated and until 1980 no other examples were known of infinite groups all of whose proper subgroups are finite. Then, in the article [160], A. Yu. Olshanskii finally constructed an infinite periodic group whose proper subgroups all have prime order. Then, in the paper [161], A. Yu. Olshanskii constructed an infinite p-group, where p is a prime greater than 10^{75}, all of whose proper subgroups have order p.

Based on such considerations the following natural problem can therefore be posed for infinite dimensional linear groups:

- Describe the infinite dimensional linear groups whose proper subgroups have finite central dimension.

This problem was considered in the paper [43] of M. R. Dixon, M. J. Evans and L. A. Kurdachenko where a somewhat more general situation was considered.

As usual we let A be an infinite dimensional vector space over a field F. Let G be a subgroup of $GL(F, A)$ and let $\mathcal{L}_{icd}(G)$ denote the family of subgroups of G having infinite central dimension. We say that G satisfies the *minimal condition on subgroups of infinite central dimension*, which we shorten to *min-icd*, if the family of subgroups, ordered by inclusion, satisfies the minimal condition.

In particular, if every proper subgroup of the infinite dimensional linear group G has finite central dimension, then $\mathcal{L}_{icd}(G)$ just consists of the group G.

The paper [43] was devoted to the study of infinite dimensional linear groups satisfying the condition min-icd and in this section we shall discuss the results of this work. We also mention here the paper of M. R. Dixon, M. J. Evans and L. A. Kurdachenko [41] and M. R. Dixon and L. A. Kurdachenko [45].

We begin with the following easy result which will be used repeatedly in what follows.

Lemma 5.1. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. If G has finite central dimension, then each subgroup H of G*

also has finite central dimension and furthermore

$$\boldsymbol{centdim}(H) \leq \boldsymbol{centdim}(G).$$

Proof. It is clear that $C_A(G) \leq C_A(H)$, so that

$$\mathbf{centdim}(H) = \mathbf{dim}_F(A/C_A(H)) \leq \mathbf{dim}_F(A/C_A(G)) = \mathbf{centdim}(G).$$

□

The next lemma is also very easy to prove.

Lemma 5.2. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which satisfies min-icd. Then the following hold.*

(i) If H is a subgroup of G, then H also satisfies min-icd;

(ii) If K is a normal subgroup of G and if K has infinite central dimension, then the quotient group G/K satisfies the minimal condition on all subgroups.

Proof. Condition (i) is clear.

To prove condition (ii) suppose the contrary and suppose that G has a strictly descending chain of subgroups

$$L_1 > L_2 > L_3 > \cdots > L_n > L_{n+1} > \ldots$$

all containing K. Then there is a natural number m such that the subgroup L_m has finite central dimension. Since $K \leq L_m$ Lemma 5.1 implies that K also has finite central dimension, which gives a contradiction that proves the result. □

Next we obtain a couple of results which often are of use when direct products are involved.

Lemma 5.3. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which satisfies min-icd. Let Λ be an index set and let X, H be subgroups of G satisfying the following conditions:*

(i) Λ is infinite;

(ii) $X = \underset{\lambda\in\Lambda}{Dr} X_\lambda$, where X_λ is a nontrivial H-invariant subgroup of X, for each $\lambda \in \Lambda$;

(iii) $H \cap X \leq \underset{\lambda\in\Gamma}{Dr} X_\lambda$ for some subset Γ of Λ.

If the subset $\Omega = \Lambda \setminus \Gamma$ is infinite then H has finite central dimension.

Proof. Since the subset Ω is infinite it has an infinite strictly descending chain

$$\Omega_1 \supsetneq \Omega_2 \supsetneq \dots \Omega_n \supsetneq \Omega_{n+1} \supsetneq \dots$$

consisting of subsets of Ω. Since $H \cap \underset{\lambda\in\Omega}{\mathrm{Dr}}\, X_\lambda = 1$, it follows that the chain of subgroups

$$\langle H, X_\lambda | \lambda \in \Omega_1\rangle \geq \langle H, X_\lambda | \lambda \in \Omega_2\rangle \geq \dots \geq \langle H, X_\lambda | \lambda \in \Omega_n\rangle \geq \dots$$

is also strictly descending. It follows that there is a natural number d such that $\langle H, X_\lambda | \lambda \in \Omega_d\rangle$ has finite central dimension. Lemma 5.1 then shows that H also has finite central dimension which gives the required result. $\square$

Lemma 5.4. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which satisfies min-icd. Let Λ be an index set and let H, K be subgroups of G satisfying the following conditions:*

(i) K is a normal subgroup of H;

(ii) There are subgroups H_λ of G such that $K \lneqq H_\lambda$ for all $\lambda \in \Lambda$ and $H/K = \underset{\lambda\in\Lambda}{Dr}\, H_\lambda/K$

(iii) Λ is infinite.

Then H has finite central dimension.

Proof. Choose infinite disjoint subsets Γ and Δ of Λ with the property that $\Lambda = \Gamma \cup \Delta$. Let

$$U/K = \underset{\lambda\in\Gamma}{\mathrm{Dr}}\, H_\lambda/K \text{ and } V/K = \underset{\lambda\in\Delta}{\mathrm{Dr}}\, H_\lambda/K.$$

Since the subset Δ is infinite there is a strictly descending chain

$$\Delta_1 \supsetneq \Delta_2 \supsetneq \dots \Delta_n \supsetneq \Delta_{n+1} \supsetneq \dots$$

consisting of subsets of Δ. This gives rise to the chain of subgroups

$$\langle U, H_\lambda | \lambda \in \Delta_1\rangle \geq \langle U, H_\lambda | \lambda \in \Delta_2\rangle \geq \dots \geq \langle U, H_\lambda | \lambda \in \Delta_n\rangle > \dots$$

which is also strictly descending. Then the condition min-icd shows that there is a natural number m such that $\langle U, H_\lambda | \lambda \in \Delta_m\rangle$ has finite central dimension. Lemma 5.1 shows that the subgroup U likewise has finite central dimension. Using similar arguments with V replacing U we see also that V has finite central dimension. However $H = UV$ and Lemma 4.1 shows that the subgroup H then has finite central dimension, which is what we wanted to prove. $\square$

We next show that the elements of infinite order in a group satisfying min-icd are quite well-behaved, in the sense that the cyclic subgroup such an element generates always has finite central dimension.

Lemma 5.5. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which satisfies min-icd. Then every infinite cyclic subgroup of G has finite central dimension.*

Proof. Let g be an element of G of infinite order and let p, q be distinct primes. Let

$$u_1 = g^p, u_{n+1} = u_n^p, v_1 = g^q, v_{n+1} = v_n^q \text{ for } n \in \mathbb{N}.$$

Then the infinite descending chains

$$\langle u_1 \rangle \geq \langle u_2 \rangle \geq \cdots \geq \langle u_n \rangle \geq \dots \text{ and } \langle v_1 \rangle \geq \langle v_2 \rangle \geq \cdots \geq \langle v_n \rangle \geq \dots$$

are strictly descending so there are natural numbers m, n such that the subgroups $\langle u_m \rangle$ and $\langle v_n \rangle$ have finite central dimension. Since the greatest common divisor of p^m and q^n is 1 it follows that $\langle g \rangle = \langle u_m \rangle \langle v_n \rangle$. This equality and Lemma 4.1 together imply that $\langle g \rangle$ has finite central dimension. □

We pause our discussion of infinite dimensional linear groups briefly to obtain a well-known technical lemma, which was alluded to in Chapter 3.

Lemma 5.6. *Let G be an abelian group. If G is not a product of two proper nontrivial subgroups, then either G is a cyclic p-group or G is Prüfer p-group for some prime p.*

Proof. Suppose that there is a prime p such that $G \neq G^p$. Since G/G^p is an elementary abelian p-group it follows that G contains a proper subgroup H such that the index of H in G is p. Then G/H is a cyclic group, say $G/H = \langle xH \rangle$. If we assume that $\langle x \rangle$ is a proper subgroup, then $G = \langle x \rangle H$ and we obtain a contradiction. Hence in this case we have $G = \langle x \rangle$. Since an infinite cyclic subgroup is clearly the product of two proper nontrivial subgroups x must have finite order and clearly also in this case G must be a cyclic p-group.

Suppose then that $G = G^p$ for all primes p. Then $G = G^n$ for all natural numbers n and it follows that G is a divisible abelian group. In this case we have $G = \underset{\lambda \in \Lambda}{\mathrm{Dr}}\, D_\lambda$ where, for each $\lambda \in \Lambda$, D_λ is a Prüfer p-group for some prime p or D_λ is isomorphic to the additive group of rational numbers, $\mathbb{Q}$ (see [58, Theorem 23.1] for example). Our hypothesis implies that then G must be either a Prüfer p-group for some prime p or $G \cong \mathbb{Q}$. However in the latter case, G is a product of two proper nontrivial subgroups. Hence, in this case, G must be a Prüfer p-group. □

We shall apply this result immediately in the next lemma.

Lemma 5.7. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which satisfies min-icd. Suppose that G has infinite central dimension. Then G/G' is a Chernikov group.*

Proof. If $G = G'$, then the result is immediate so we may assume that $G \neq G'$. Suppose, for a contradiction, that G/G' is not Chernikov and let

$$\mathcal{S} = \{H \leq G | H/H' \text{ is not Chernikov and } \mathbf{centdim}(H) \text{ is infinite}\}.$$

Then $\mathcal{S}$ is nonempty since $G \in \mathcal{S}$. Furthermore, since G satisfies min-icd it follows that the family $\mathcal{S}$ satisfies the minimal condition so it has a minimal element D, say. In particular D/D' is not Chernikov.

Suppose that U, V are proper subgroups of D such that

$$U \neq D', V \neq D', D = UV \text{ and } U \cap V = D'.$$

Lemma 4.1 implies that at least one of these subgroups, U say, has infinite central dimension. Since U is a proper subgroup of D, the minimal choice of D implies that U/U' is a Chernikov group. Hence the isomorphism $U/D' \cong (U/U')/(D'/U')$ shows that the quotient group U/D' is also Chernikov. Since U has infinite central dimension it follows from Lemma 5.2 that the abelian group D/U is also Chernikov. Hence the quotient group D/D' is Chernikov, contrary to the choice of D. This argument shows that D/D' cannot be decomposed into a product of proper nontrivial subgroups. It follows from Lemma 5.6 that D/D' is a Prüfer p-group or is a cyclic p-group for some prime p, which yields a final contradiction to the definition of D as the minimal element of $\mathcal{S}$. The result follows. $\square$

Next we define a radical which will prove to be very useful in the following discussion.

Let G be a subgroup of $GL(F, A)$ and let $\mathbf{Fin}(G)$ denote the subset of G consisting of all elements g of the group G for which the cyclic subgroup $\langle g \rangle$ has finite central dimension. We first show that $\mathbf{Fin}(G)$ is a normal subgroup of G.

Lemma 5.8. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. The set $\boldsymbol{Fin}(G)$ is a normal subgroup of G.*

Proof. Clearly $\mathbf{Fin}(G)$ is nonempty. If x, y are elements of $\mathbf{Fin}(G)$, then the subgroups $\langle x \rangle, \langle y \rangle$ have finite central dimension. Lemma 4.1 implies that $\langle x, y \rangle$ also has finite central dimension and Lemma 5.1 shows that then $\langle xy \rangle$ has finite central dimension. In particular this implies that $xy \in \mathbf{Fin}(G)$ and since it is clear that $x^{-1} \in \mathbf{Fin}(G)$ it follows that $\mathbf{Fin}(G)$ is a subgroup of G.

Next, if $x \in \mathbf{Fin}(G)$ and g is an arbitrary element of G, then again using Lemma 4.1 we deduce that the subgroup $\langle x \rangle^g = \langle x^g \rangle$ has finite central dimension so that $x^g \in \mathbf{Fin}(G)$. Hence $\mathbf{Fin}(G)$ is a normal subgroup of G. $\square$

We shall call the normal subgroup $\mathbf{Fin}(G)$ the *finitary radical* of the linear group G.

Corollary 5.9. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Then $\boldsymbol{Fin}(G)$ is the largest finitary subgroup of G.*

Proof. From the definition of $\mathbf{Fin}(G)$ it follows that $\mathbf{Fin}(G)$ is a finitary linear group. It is also clear that $\mathbf{Fin}(G)$ contains all finitary subgroups of G. □

We shall also need the following property of finitary linear groups.

Lemma 5.10. *Let A be a vector space over the field F and let G be a periodic subgroup of $GL(F, A)$. If G is finitary, then G is locally finite.*

Proof. Let M be a finite subset of G and let $K = \langle M \rangle$. Lemma 4.1 implies that K has finite central dimension. It follows that the subspace $C_A(K)$ has finite codimension n in A and hence $K/C_K(A/C_A(K))$ is isomorphic to some subgroup of $GL_n(F)$. Since $K/C_K(A/C_A(K))$ is finitely generated and periodic it must be finite (see [202, 9.1], for example). It follows that the subgroup $C_K(A/C_A(K))$ is also finitely generated (see [52, Proposition 1.2.13], for example). If F is of characteristic the prime p, then Lemma 4.1 shows that $C_K(A/C_A(K))$ is an elementary abelian p-group which, being finitely generated, is therefore finite. Hence in this case, K is finite.

If, on the other hand, F has characteristic 0, then Lemma 4.1 shows that the subgroup $C_K(A/C_A(K))$ must be torsion-free. However our hypothesis that G is periodic then shows that $C_K(A/C_A(K))$ is trivial. Hence in this case also K is finite. The result follows. □

Lemma 5.11. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which has infinite central dimension. If G satisfies the condition min-icd, then either G is periodic or G is a finitary linear group.*

Proof. We suppose to the contrary that G is neither periodic nor finitary. Let

$$\mathcal{S} = \{H \leq G | H \text{ is not finitary and not periodic}\}.$$

Since G is an element of $\mathcal{S}$ the set $\mathcal{S}$ is nonempty.

If a subgroup H is not finitary then there is an element $x \in H$ such that the cyclic subgroup $\langle x \rangle$ has infinite central dimension. Lemma 5.1 shows that the subgroup H also has infinite central dimension. It follows that $\mathcal{S} \subseteq \mathcal{L}_{icd}(G)$. Since the family $\mathcal{L}_{icd}(G)$ satisfies the minimal condition the family $\mathcal{S}$, ordered by inclusion, therefore has the minimal condition. Let D denote a minimal element of the set $\mathcal{S}$ and let $L = \mathbf{Fin}(D)$. Since D is not periodic it has an element h of infinite order and Lemma 5.5 implies that the subgroup $\langle h \rangle$ has finite central dimension, so $h \in L$. It follows that the subgroup L is nontrivial. Suppose that S is a proper subgroup of D such that $L \leq S$. Then S is not periodic and the inequality $S \neq D$ implies that S must be finitary so that $S \leq L$. It follows that L is a maximal subgroup of D. We know from Lemma 5.8 that L is a normal subgroup of D, so the factor group D/L is cyclic of prime order q.

Let g be an arbitrary element of $D \setminus L$ so that $D = L\langle g \rangle$. If we assume that the subgroup $\langle g \rangle$ has finite central dimension, then $\langle g \rangle$ is finitary so

Corollary 5.9 shows that $g \in L$. Hence $\langle g \rangle$ has infinite central dimension. In particular, Lemma 5.5 shows that the element g has finite order. Since L is not periodic it contains an element a of infinite order. The fact that $\mathbf{centdim}_F(\langle g \rangle)$ is infinite together with Lemma 5.1 shows that the subgroup $\langle g, a \rangle$ has infinite central dimension. Then the minimal choice of D implies that $\langle g, a \rangle = D$. Since the index $|D : L|$ is finite and D is finitely generated it follows that L is also finitely generated (see [52, Proposition 1.2.13] for example). Using Lemma 4.1 again we deduce that L also has finite central dimension. Hence the subspace $C_A(L)$ has finite codimension in A. Since L is normal in D, the subspace $C_A(L)$ is D-invariant. Set $K = C_D(A/C_A(L))$. Then Lemma 4.1 shows that K is a normal subgroup of D. Furthermore, if F has characteristic p, then $K \cap L$ is an elementary abelian p-group for some prime p, whereas if F has characteristic 0, then $K \cap L$ is a torsion-free abelian subgroup. In addition, D/K is isomorphic to some subgroup of $GL_n(F)$ where $n = \mathbf{dim}_F(A/C_A(L))$. Since D/K is a finitely generated finite dimensional linear group D/K is residually finite (by [202, Theorem 4.2], for example). Let

$$V = \bigcap\{U \leq D||D : U| \text{ is finite}\},$$

the finite residual of D. It follows that K contains the subgroup V.

Let U be a proper normal subgroup of D of finite index, so that U is not periodic since $\langle g \rangle$ has infinite central dimension. Lemma 5.1 shows that the subgroup $U\langle g \rangle$ also has infinite central dimension. Since D is a minimal element of $\mathcal{S}$ and U is non-periodic we have $D = U\langle g \rangle$ and hence the factor group D/U is abelian. Consequently $D' \leq U$. This latter inclusion is valid for all normal subgroups of D having finite index so $D' \leq V$ and hence D/V is abelian. Lemma 5.7 shows that in this case D/V is a Chernikov group. On the other hand the group D/V is residually finite so it follows that D/V is finite. Hence D/K is also finite. The finiteness of D/K and D/L implies, using Remak's theorem, that $D/(L \cap K)$ is likewise finite. In turn since D is finitely generated it follows that $L \cap K$ is finitely generated (see [52, Proposition 1.2.13] again, for example). If F has prime characteristic p, then the finitely generated elementary abelian group $L \cap K$ is finite and in this case the group D is finite which is a contradiction. On the other hand, if F has characteristic 0, then $M = L \cap K$ is a finitely generated torsion-free abelian group. Let p be an arbitrary prime and define subgroups

$$M_0 = M, M_{j+1} = M_j^p \text{ for all } j \in \mathbb{N}.$$

Then, for each $j \in \mathbb{N}$, M_j is a normal subgroup of D and $M_j\langle g \rangle \neq M_{j+1}\langle g \rangle$. Hence the series

$$M_0\langle g \rangle \geq M_1\langle g \rangle \geq \dots \geq M_j\langle g \rangle \geq M_{j+1}\langle g \rangle \geq \dots$$

is strictly descending so there is a natural number t such that the subgroup $M_t\langle g \rangle$ has finite central dimension. Lemma 5.1 shows that in this case the subgroup $\langle g \rangle$ has finite central dimension and we obtain a contradiction. This final contradiction proves the result. $\square$

We need a further technical group theoretic result.

Lemma 5.12. *Let G be a group, let g be an element of G and let B be an infinite elementary abelian p-subgroup for some prime p. Suppose that*

(i) B is $\langle g \rangle$-invariant and that

(ii) $g^n \in C_G(B)$ for some natural number n.

Then B contains a subgroup of the form $C = \underset{j \in \mathbb{N}}{Dr}\, C_j$, where C_j is a finite $\langle g \rangle$-invariant subgroup of C for each $j \in \mathbb{N}$.

Proof. Let $1 \neq c_1 \in B$ and let $C_1 = \langle c_1 \rangle^{\langle g \rangle}$. Since $g^n \in C_G(B)$, the subgroup C_1 is finite and clearly C_1 is $\langle g \rangle$-invariant. Since B is an elementary abelian p-group, there exists a subgroup B_1 of B such that $B = C_1 \times B_1$. Of course B_1 need not be $\langle g \rangle$-invariant. However, since $g^n \in C_G(B)$, the family $\{B_1^x | x \in \langle g \rangle\}$ is finite. For every $x \in \langle g \rangle$ we have $B/B_1^x = B^x/B_1^x \cong B/B_1$ and, in particular, B_1^x has finite index in B. Let

$$\{B_1^x | x \in \langle g \rangle\} = \{D_1, \ldots, D_k\} \text{ and } E_1 = D_1 \cap \cdots \cap D_k.$$

Then the subgroup E_1 is $\langle g \rangle$-invariant, $C_1 \cap E_1 = 1$ and the index $|B : E_1|$ is finite. In particular, the subgroup E_1 is infinite and we now let $1 \neq c_2 \in E_1$. Set $C_2 = \langle c_2 \rangle^{\langle g \rangle}$ so that, as with C_1, we have C_2 finite and $\langle g \rangle$-invariant. The choice of C_2 shows that $C_1 \cap C_2 = 1$. Since B is elementary abelian, there is a subgroup B_2 of B such that $B = (C_1 \times C_2) \times B_2$. Using similar arguments to those used above we next obtain a subgroup

$$E_2 = \bigcap_{x \in \langle g \rangle} B_2^x$$

which is infinite and $\langle g \rangle$-invariant. Furthermore, $(C_1 \times C_2) \cap E_2 = 1$. These arguments can now be repeated and in this way we construct a family $\{C_m | m \in \mathbb{N}\}$ of finite $\langle g \rangle$-invariant subgroups of B such that

$$\langle C_m | m \in \mathbb{N} \rangle = \underset{m \in \mathbb{N}}{\mathrm{Dr}}\, C_m,$$

as required. $\square$

Lemma 5.13. *Let A be a vector space over the field F and let G be a periodic subgroup of $GL(F, A)$ which has infinite central dimension. If G satisfies the condition min-icd, then either G has the minimal condition on all subgroups or G is a finitary linear group.*

Proof. Suppose, for a contradiction, that G is neither finitary nor satisfies the minimal condition on all subgroups. Let

$$\mathcal{S} = \{H \leq G | H \text{ is neither finitary nor has the minimal condition on subgroups}\}.$$

Since $G \in \mathcal{S}$ it follows that $\mathcal{S}$ is non-empty.

Let H be a subgroup in the family $\mathcal{S}$. Then H is not finitary so it follows that H has infinite central dimension by Lemma 5.1. Hence $\mathcal{S}$ is a subset of $\mathcal{L}_{icd}(G)$. Since the family $\mathcal{L}_{icd}(G)$ satisfies the minimal condition, the family $\mathcal{S}$, ordered by inclusion, also satisfies the minimal condition. Let D be a minimal element of $\mathcal{S}$. We first show that D is locally finite. Let $L = \mathbf{Fin}(D)$. Since D does not have the minimal condition on subgroups it has an infinite strictly descending chain

$$D_1 > D_2 > D_3 \dots D_j > D_{j+1} > \dots$$

of subgroups. Since G satisfies the condition min-icd, there is a natural number n such that D_n has finite central dimension. In particular we have $D_n \leq L$ and hence L does not satisfy the minimal condition on subgroups since D_n doesn't have this property. If S is a proper subgroup of D such that $L \leq S$, then again it follows that S also does not satisfy the minimal condition on subgroups. The minimal choice of D implies that S must be finitary, so Corollary 5.9 implies that $S \leq L$. This shows that L is a maximal subgroup of D and, since L is normal in G by Lemma 5.8, it follows that the factor group D/L has prime order q for some prime q. However the periodic finitary linear group L is necessarily locally finite by Lemma 5.10 so that D is also locally finite. Choose an arbitrary element $g \in D \setminus L$. Then $D = L\langle g \rangle$. If we suppose that the subgroup $\langle g \rangle$ has finite central dimension, then Corollary 5.9 shows that $g \in L$, which is a contradiction. Hence the subgroup $\langle g \rangle$ has infinite central dimension.

We may write $g = g_1 g_2$ for some q-element g_1 and q'-element g_2 where $g_1 g_2 = g_2 g_1$. Since D/L is a q-group it follows that $g_2 \in L$ and also that $D = \langle g_1 \rangle L$. Therefore without loss of generality we may suppose that g is a q-element. If $C_L(g)$ does not satisfy the minimal condition on subgroups, then it contains an abelian subgroup B which also doesn't satisfy the minimal condition on subgroups (see [100, Theorem 5.8], for example). We may then write

$$B = \underset{p \in \Pi(B)}{\mathrm{Dr}} B_p$$

where B_p is the p-component of B for each prime $p \in \Pi(B)$. Then either there exists a prime p such that the lower layer $\mathbf{\Omega}_1(B_p)$ is infinite or the set $\Pi(B)$ is infinite. In the latter case each subgroup B_p is $\langle g \rangle$-invariant and an application of Lemma 5.3 implies that the subgroup $\langle g \rangle$ has finite central dimension which gives a contradiction. If the subgroup $\mathbf{\Omega}_1(B_p)$ is infinite for some prime p, then Lemma 5.12 implies that it contains an infinite subgroup of the form $\underset{n \in \mathbb{N}}{\mathrm{Dr}}\, C_n$

for certain finite $\langle g\rangle$-invariant subgroups C_n of B_p for $n \in \mathbb{N}$. In this case Lemma 5.3 can again be applied and we once again deduce that $\langle g\rangle$ has finite central dimension, so the same contradiction ensues. This contradiction proves that $C_L(g)$ satisfies the minimal condition on subgroups and since it is locally finite it is a Chernikov group (again see [100, Theorem 5.8], for example). It follows from a theorem of B. Hartley [79] that L contains a normal locally soluble subgroup T of finite index. Since D/L is finite we may assume without loss of generality that T is normal in D. The finiteness of $|L : T|$ implies that T does not satisfy the minimal condition so that, by a theorem of D. I. Zaitsev [221], T contains a $\langle g\rangle$-invariant abelian subgroup C which also does not satisfy the minimal condition on subgroups. The arguments above may then be applied to C and we once again arrive at a contradiction. This final contradiction proves the result. □

We may now obtain the first main result of this chapter which follows immediately given our build-up.

Theorem 5.14. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ which has infinite central dimension. If G satisfies the condition min-icd, then either G has the minimal condition on all subgroups or G is a finitary linear group.*

Proof. Indeed, if G is not periodic, then Lemma 5.11 implies that G is a finitary linear group. If G is periodic, then we may apply Lemma 5.13. □

It is well known that locally soluble groups that are finite dimensional linear groups are soluble (see, for example, the book of B. A. F. Wehrfritz [202, Corollary 3.8]) For linear groups having finite central dimension this is also the case as is shown by the next result.

Lemma 5.15. *Let A be a vector space over the field F and let G be a locally soluble subgroup of $GL(F, A)$. If G has finite central dimension, then G is soluble.*

Proof. Let $C = C_A(G)$ so that C has finite codimension. Let n be the F-dimension of A/C and let $H = C_G(A/C)$. Then we may consider G/H as a locally soluble subgroup of $GL_n(F)$. By a theorem of Zassenhaus (see, for example, [202, Corollary 3.8]), G/H is soluble. Lemma 4.1 shows that H is abelian and hence G is soluble. □

Next we extend the scope of Lemma 5.15. The next natural step in this study of groups with many subgroups of finite central dimension concerns locally soluble linear groups with min-icd. To begin this study we first recall the following important concept, which has already been touched upon in Chapter 1.

Let G be a group and let $\mathcal{S}$ be a family of subgroups of G. Then $\mathcal{S}$ is called a *Kurosh-Chernikov system* if it satisfies the following conditions:

(KC1a) $G \in \mathcal{S}$;

(KC2b) for each pair A, B of subgroups of $\mathcal{S}$ either $A \le B$ or $B \le A$. Thus $\mathcal{S}$ is linearly ordered by inclusion;

(KC3c) for every subfamily $\mathcal{L}$ of $\mathcal{S}$ the intersection of all elements of $\mathcal{L}$ and the union of all elements of $\mathcal{L}$ belong to $\mathcal{S}$. In particular, for each nontrivial element g of the group G, let

$$V_g = \bigcup\{H \in \mathfrak{S} | g \notin H\},$$
$$\Lambda_g = \bigcap\{H \in \mathfrak{S} | g \in H\}.$$

Then $V_g, \Lambda_g \in \mathcal{S}$.

(KC4d) for each nontrivial element g of the group G the subgroup V_g is normal in Λ_g.

The factor groups Λ_g/V_g are called the *sections* of the system $\mathcal{S}$.

If every subgroup of $\mathcal{S}$ is normal in G, then $\mathcal{S}$ is called a *normal Kurosh-Chernikov system*. In this case the factor groups V_g/Λ_g are called the *factors* of the system $\mathcal{S}$.

Such families were introduced by A. G. Kurosh and S. N. Chernikov in their fundamental classical article [132]. Further information concerning series in groups can be found in [132] and [183]. In particular in §6 of the paper[132] it was shown that a locally soluble group has a normal Kurosh-Chernikov system whose factors are abelian. We use this statement to prove the following result.

Lemma 5.16. *Let A be a vector space over the field F and let G be a locally soluble subgroup of $GL(F, A)$ that has infinite central dimension. If G satisfies the condition min-icd, then either G is soluble or G has an ascending series of normal subgroups*

$$1 = S_0 \le S_1 \le \cdots \le S_\omega = \bigcup_{n \in \mathbb{N}} S_n \le G$$

such that S_n has finite central dimension, and the factor S_{n+1}/S_n is abelian for $n \ge 0$. Moreover, in this second case, G/S_ω is a soluble Černikov group.

Proof. As we noted above G has a normal Kurosh-Chernikov system $\mathcal{S}$ whose factors are abelian. Suppose that every nontrivial element of $\mathcal{S}$ has infinite central dimension. Then the set $\mathcal{S}$, linearly ordered by inclusion, satisfies the minimal condition. Thus the family of all nontrivial members of $\mathcal{S}$ has a least element H. It follows from the definition of a Kurosh-Chernikov system that this least element H is a normal abelian subgroup. Since H has infinite central dimension, Lemma 5.2 shows that the factor group G/H satisfies the minimal condition for all subgroups and since G/H is locally soluble it is a Chernikov group (see [130, §59], for example). In particular G/H is soluble and since H is abelian the group G is soluble.

Now suppose that $\mathcal{S}$ has a nontrivial member having finite central dimension. Let M denote the union of all members of $\mathcal{S}$ which have finite central dimension. Since $\mathcal{S}$ is linearly ordered, M is a normal subgroup of G. Let

$$\mathcal{L} = \{X \in \mathcal{S} | M \lneqq X\}.$$

Then every element of $\mathcal{L}$ has infinite central dimension. Moreover, if $X, Y \in \mathcal{S}$ where Y has finite central dimension and X has infinite central dimension, then Lemma 5.1 shows that Y does not contain X. Since $\mathcal{S}$ is linearly ordered it follows that $Y \leq X$. This is true for each element Y of $\mathcal{S}$ having finite central dimension, so that $M \leq X$. Since G satisfies min-icd the set $\mathcal{L}$ (which is ordered by inclusion) satisfies the minimal condition. Since $\mathcal{L}$ is completely ordered it follows that $\mathcal{L}$ has a least element R. It follows that R is the intersection of all members of $\mathcal{S}$ having infinite central dimension. It is not difficult to see that $M = V_g$ and $R = \Lambda_g$ for each element $g \in R \setminus M$. It follows that R/M is abelian. As above G/R is a Chernikov group so that G/R is soluble and hence G/M is soluble.

If M has finite central dimension, then Lemma 5.15 implies that M is soluble and hence G is also soluble.

Suppose, therefore, that M has infinite central dimension. Then Lemma 5.2 implies that the factor group G/M satisfies the minimal condition on all subgroups and hence is a Chernikov group. If $Y \in \mathcal{S}$, $Y \leq M$ and $Y \neq M$, then Y has finite central dimension and therefore is soluble by Lemma 5.15. Then

(a) either there is a natural number m such that the derived length of Y is at most m for every $Y \in \mathcal{S}$ satisfying $Y \lneqq M$, or

(b) for each natural number d there is an element Y of $\mathcal{S}$ such that $Y \lneqq M$ and Y has derived length at least d.

In case (a) M is soluble of derived length at most d so that G is also soluble. In case (b) for each natural number k we let

$$S_k = \bigcup \{Y \in \mathcal{S} | Y \lneqq M, Y \text{ has derived length at most } k\}.$$

Then $S_k \in \mathcal{S}$ has derived length at most k and furthermore $\bigcup_{k \in \mathbb{N}} S_k = S_\omega = M$. This completes the proof. □

We recall from Chapter 1 that the intersection of all the subgroups of finite index in a group G is called the finite residual denoted by $G^{\mathfrak{F}}$. A group G is called $\mathfrak{F}$-*perfect* if G contains no proper subgroups of finite index. Thus G is $\mathfrak{F}$-perfect if and only if $G = G^{\mathfrak{F}}$.

Lemma 5.17. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ that has infinite central dimension. If G is a finitary linear group satisfying the condition min-icd, then $G/G^{\mathfrak{F}}$ is finite.*

Proof. Let us suppose for a contradiction that $G/G^{\mathfrak{F}}$ is infinite. Then G has an infinite descending series of normal subgroups

$$G \geq G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} \geq \ldots,$$

such that G/G_i finite is for each $i \geq 1$. Since G satisfies the condition min-icd, there exists a natural number k such that the subgroup G_k has finite central dimension. The finiteness of G/G_k implies that $G = HG_k$ for some finitely generated subgroup H. Since G is finitary Lemma 4.1 shows that H has finite central dimension and using Lemma 4.1 again we see that the group G is also of finite central dimension, which is a contradiction to the hypotheses. This contradiction proves that the factor group $G/G^{\mathfrak{F}}$ is finite. □

We omit the proof of the following very easily deduced corollary.

Corollary 5.18. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ that has infinite central dimension. If G is a finitary linear group satisfying the condition min-icd, then the finite residual $G^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect.*

We next show that the subgroup S_ω that appears in Lemma 5.16 is soluble.

Lemma 5.19. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ that has infinite central dimension. Suppose that G has an ascending series of normal subgroups*

$$1 = S_0 \leq S_1 \leq \cdots \leq S_n \leq \cdots \leq \bigcup_{n\geq 1} S_n = G,$$

in which each subgroup S_n has finite central dimension and each of the factors S_{n+1}/S_n is abelian for all natural numbers n. If G satisfies the condition min-icd, then G is soluble.

Proof. Since the vector space $A/C_A(S_k)$ is finite dimensional, A has a finite series of G-invariant subspaces

$$A = A_0 \geq A_1 \geq \cdots \geq A_{n(k)} = C_A(S_k),$$

such that each factor A_j/A_{j+1} is a G-chief factor for $0 \leq j \leq n(k) - 1$. The quotient space $A/C_A(S_{k+1})$ also has finite dimension so that the quotient $C_A(S_k)/C_A(S_{k+1})$ is also finite dimensional. The series above can be extended using the finite series of G-invariant subspaces

$$A = A_0 \geq A_1 \geq \cdots \geq A_{n(k)} \geq \cdots \geq A_{n(k+1)} = C_A(S_{k+1}),$$

each factor of which is a G-chief factor. Considering that $G = \bigcup_{k\in\mathbb{N}} S_k$ we may use an easy induction to see that A has a descending series of G-invariant subspaces

$$A = A_0 \geq A_1 \geq A_2 \geq \cdots \geq A_\omega = \bigcap_{n\in\mathbb{N}} A_n = C_A(G)$$

such that the factors A_{n-1}/A_n are G-chief factors for all $n \in \mathbb{N}$.

Each of the factors A/A_n is finite dimensional. This implies that $G/C_G(A/A_n)$ is isomorphic to some locally soluble subgroup of $GL_{m(n)}(F)$ where $m(n)$ is the F-dimension of the space A/A_n. By a theorem of H. Zassenhaus $G/C_G(A/A_n)$ is soluble (see [202, Corollary 3.8], for example). The series

$$G = C_G(A/A_0) \geq C_G(A/A_1) \geq \cdots \geq C_G(A/A_n) \geq \dots$$

may be strictly descending or there may be a natural number t such that $C_G(A/A_{t+n}) = C_G(A/A_t) = L$ for all natural numbers n. In the former case there exists a natural number k such that $C_G(A/A_k)$ has finite central dimension. Lemma 5.15 implies that in this case the group $C_G(A/A_k)$ is soluble and since $G/C_G(A/A_k)$ is also soluble, it follows that G is soluble.

In the latter case we see that for all elements $a \in A$ and $g \in L$ we have $(g-1)a \in A_{t+n}$ for each $n \in \mathbb{N}$. It follows that

$$(g-1)a \in \bigcap_{n\in\mathbb{N}} A_{t+n} = C_A(G).$$

In particular we have $L \leq C_A(A/C_A(G))$. By Theorem 1.2 L is abelian and since $G/L = G/C_G(A/A_t)$ is soluble we have that G itself is soluble. □

The following theorem can now be read off from Lemmas 5.16 and 5.19.

Theorem 5.20. *Let A be a vector space over the field F and let G be a locally soluble subgroup of $GL(F, A)$ that has infinite central dimension. If G satisfies the condition min-icd, then G is soluble.*

We now obtain some of the further structure of linear groups with min-icd. First we have

Lemma 5.21. *Let A be a vector space over the field F and let G be an infinite abelian Chernikov q-subgroup of $GL(F, A)$ for some prime q. If G is finitary, then $q \neq$ **char** (F).*

Proof. Suppose the contrary and let $q =$ **char** (F). We recall that every q-subgroup of $GL_n(F)$ is bounded and nilpotent for each natural number n (see, for example, [202, 9.1]). This shows in particular that $\mathbf{dim}_F(A)$ is infinite. Without loss of generality we may suppose that G is a divisible Chernikov group. We let $G_k = \mathbf{\Omega}_k(G)$. We note that G_k is finite for each natural number k and that $G = \bigcup_{k\in\mathbb{N}} G_k$. Since G is finitary $C_k = C_A(G_k)$ has finite codimension in A. Furthermore $C_m \leq C_k$ whenever $m \geq k$ and we therefore have a descending chain of G-invariant subspaces

$$A = C_0 \geq C_1 \geq C_2 \geq \dots C_k \geq C_{k+1} \geq \cdots \bigcap_{k\in\mathbb{N}} C_k = C_A(G).$$

We may think of $G/C_G(A/C_k)$ as a q-subgroup of $GL_{n(k)}$ where $n(k)$ is the codimension of C_k in A for each $k \in \mathbb{N}$. By our earlier remark each q-subgroup of $GL_{n(k)}(F)$ is a bounded nilpotent subgroup. On the other hand G is divisible, so it follows that G has no nontrivial bounded factor group. Therefore, it follows that $G \leq C_G(A/C_k)$ which means that $[G, A] \leq C_k$. Since this is true for every natural number k we have

$$[G, A] \leq \bigcap_{k \in \mathbb{N}} C_k = C_A(G).$$

Consequently, $G \leq C_G(A/C_A(G))$. By Theorem 1.2 G is an elementary abelian q-group and we obtain a contradiction, which proves the result. □

Corollary 5.22. *Let A be a vector space over the field F and let G be a finitary subgroup of $GL(F, A)$. Suppose that G contains a normal subgroup K having finite central dimension such that G/K is a divisible Chernikov group. If F has characteristic the prime p, then G/K is a p'-group.*

Proof. Suppose the contrary and let G/K have the nontrivial p-component P/K. Then P/K is also divisible. Since the quotient space $A/C_A(K)$ has finite dimension, we can think of the quotient group $P/C_P(A/C_A(K))$ as a subgroup of $GL_n(F)$, where n is the F-dimension of $A/C_A(K)$. However every p-subgroup of $GL_n(F)$ is a bounded nilpotent group, by [202, 9.1] for example, and it is easy to deduce the inclusion $P \leq C_G(A/C_A(K))$.

Now consider the G-invariant subspace $C_A(K)$ and note that we may consider P/K as a finitary subgroup of $GL(F, C_A(K))$. Lemma 5.21 shows that $P \leq C_G(C_A(K))$. Now the inclusions $P \leq C_G(A/C_A(K))$ and $P \leq C_P(C_A(K))$ together with Theorem 2.1 imply that P is an elementary abelian p-group which is a contradiction. The result is proved. □

Our next sequence of results deals with (locally soluble)-by-finite groups and ultimately with locally generalized radical groups.

Lemma 5.23. *Let A be a vector space over the field F and let G be a (locally soluble)-by-finite subgroup of $GL(F, A)$ having infinite central dimension. Suppose that G satisfies the condition min-icd and that G is not a Chernikov group.*

(i) If F has characteristic p, then G contains a normal bounded nilpotent p-subgroup H such that the quotient group G/H is Chernikov and the divisible part of G/H is a p'-group;

(ii) if F has characteristic 0, then G contains a normal nilpotent torsion-free subgroup H such that the quotient group G/H is Chernikov.

In any case H has finite central dimension.

Proof. A (locally soluble)-by-finite group satisfying the minimal condition on subgroups is periodic and is therefore locally finite. Furthermore, locally finite groups with the minimal condition on subgroups are Chernikov (see [100, Theorem 5.8], for example). Since G is not Chernikov, by hypothesis, it doesn't satisfy the minimal condition on subgroups. Hence we may apply Theorem 5.14 to deduce that G is a finitary linear group. Let L be a normal locally soluble subgroup which has finite index in G. Lemma 4.1 shows that L has infinite central dimension and Theorem 5.20 implies that L is soluble. Let

$$L = D_0 \geq D_1 \geq \cdots \geq D_n \geq D_{n+1} = 1$$

be the derived series of L. Clearly, there is a natural number m such that D_m has infinite central dimension, but D_{m+1} has finite central dimension. Lemma 5.7 implies that the factor group D_m/D_{m+1} is Chernikov and Lemma 5.2 implies that the factor group L/D_m satisfies the minimal condition for all subgroups. However L/D_m is soluble so locally finite and [100, Theorem 5.8] shows that it is Chernikov also. Thus L/D_{m+1} is Chernikov, by [52, Corollary 1.3.6] for example.

Let $K = D_{m+1}$. The fact that K is normal in G implies that the subspace $C_A(K)$ is G-invariant. Since the quotient space $A/C_A(K)$ has finite dimension A has a finite series of G-invariant subspaces

$$A = A_0 \geq A_1 \geq \cdots \geq A_t = C_A(K)$$

such that the factors A_j/A_{j+1} are G-chief factors for $0 \leq j \leq t-1$. Consequently we may think of $L/C_L(A_j/A_{j+1})$ as a soluble irreducible subgroup of $GL_{n(j)}(F)$ where $n(j) = \mathbf{dim}_F(A_j/A_{j+1})$. Using a theorem of A. I. Maltsev we deduce that $L/C_L(A_j/A_{j+1})$ is abelian-by-finite for each j (see [202, Lemma 3.5] for example). Since each of the subspaces A_j is G-invariant $C_L(A_j/A_{j+1})$ is a normal subgroup of G for $0 \leq j \leq t-1$. Let

$$U = C_L(A_0/A_1) \cap C_L(A_1/A_2) \cap \cdots \cap C_L(A_{t-1}/A_t).$$

Clearly the subgroup U is normal in G and Remak's theorem gives the embedding

$$L/U \longrightarrow L/C_L(A_0/A_1) \times L/C_L(A_1/A_2) \times \cdots \times L/C_L(A_{t-1}/A_t)$$

which shows that L/U is also abelian-by-finite. Let L_1 be a normal subgroup of L having finite index in L such that $U \leq L_1$ and L_1/U is abelian. Since L is finitary and has infinite central dimension an application of Lemma 4.1 shows that L_1 also has infinite central dimension. Then Lemma 5.7 implies that the factor group L_1/L_1' is Chernikov, so that L_1/U is likewise Chernikov. In turn it also follows that L/U is a Chernikov group.

Let $H = K \cap U$ so that H is a normal subgroup of G. Again using Remak's theorem we have

$$G/H \longrightarrow G/K \times G/U$$

and we deduce that the factor group G/H is Chernikov. Furthermore, the series

$$A = A_0 \geq A_1 \geq \cdots \geq A_t = C_A(K)$$

is H-central.

Suppose that the characteristic of F is the prime p. Then we may apply Theorem 2.1 to deduce that H is a bounded nilpotent p-group. Let R/H denote the divisible part of G/H. Then Corollary 5.22 shows that H is a p'-group.

On the other hand, if F has characteristic 0, then Theorem 2.1 this time implies that H is a torsion-free nilpotent group. This completes the proof. □

The results we have proved above provide the basis for the following description of certain (locally soluble)-by-finite infinite dimensional linear groups. Already we know from Theorem 5.14 that (locally soluble)-by-finite groups satisfying the condition min-icd are either Chernikov or are finitary (indeed, the hypotheses concerning locally soluble groups can be weakened considerably). In the following result we obtain the finitary structure further. We let $A \rtimes B$ denote the semidirect product of the groups A and B. We first consider the case when F has prime characteristic.

Theorem 5.24. *Let A be a vector space over the field F of prime characteristic p and let G be a (locally soluble)-by-finite subgroup of $GL(F, A)$ having infinite central dimension. Suppose that G satisfies the condition min-icd and that G is not a Chernikov group. Then G has a series of normal subgroups $P \leq D \leq G$ satisfying the following conditions:*

(i) P is a bounded nilpotent p-group having finite central dimension;

(ii) $D = P \rtimes Q$ for some non-trivial divisible Chernikov p'-subgroup Q;

(iii) Q has infinite central dimension

(iv) P satisfies min-Q (the minimal condition on Q-invariant subgroups),

(v) G/D is finite.

In particular G is a nilpotent-by-abelian-by-finite group and satisfies the minimal condition on normal subgroups.

Proof. Lemma 5.23 shows that G contains a normal bounded nilpotent p-subgroup P of finite central dimension such that the factor group G/P is Chernikov. Let D/P be the divisible part of G/P. Lemma 5.23 also implies that D/P is a p'-group. By [100, 1.D.4 Lemma], D contains a p'-subgroup Q such that $D = P \rtimes Q$. If we suppose that Q is trivial, then the subgroup P has finite index in G. In this case $G = PH$ for some finitely generated subgroup H of G. Theorem 5.14 shows that G is a finitary linear group. It follows that H has finite central dimension and Lemma 4.1 implies that G also has finite central dimension, contradicting the choice of G. Hence we must have $Q \neq 1$. If we assume that Q has finite central dimension, then Lemma 4.1 implies that D has finite central dimension and the arguments used above can now be used once again to deduce the contradiction that G has finite central dimension. Thus Q has infinite central dimension. Suppose that P has the strictly descending chain

$$P_1 > P_2 > \dots P_n > P_{n+1} > \dots \tag{5.1}$$

of Q-invariant subgroups. Lemma 5.1 shows that each of the subgroups P_nQ has infinite central dimension. Since $P \cap Q = 1$ we obtain the strictly descending chain

$$P_1Q > P_2Q \dots P_nQ > P_{n+1}Q > \dots$$

of subgroups each having infinite central dimension. This contradicts the assumption min-icd and hence it follows that a descending chain like (5.1) cannot exist. Thus P satisfies the condition min-Q.

Finally suppose that G has a descending chain

$$K_1 \geq K_2 \geq \dots \geq K_n \geq K_{n+1} \geq \dots$$

of normal subgroups. Then there is a descending chain

$$K_1P \geq K_2P \geq \dots \geq K_nP \geq K_{n+1}P \geq \dots.$$

Since G/P is a Chernikov group it satisfies the minimal condition for all subgroups and hence there is a natural number t such that $K_tP = K_{t+n}P$ for all natural numbers n. Next we consider the descending chain

$$P \cap K_1 \geq P \cap K_2 \geq \dots \geq P \cap K_n \geq P \cap K_{n+1} \geq \dots$$

and note that $P \cap K_n$ is Q-invariant since both P and K_n are normal in G. Since P satisfies min-Q it follows that there is a natural number m such that $P \cap K_m = P \cap K_{m+n}$ for all natural numbers n. Let l be the maximum of the two integers m and t. Then we have

$$K_l = K_l \cap K_lP = K_l \cap K_{l+n}P = K_{l+n}(P \cap K_l) = K_{l+n}(P \cap K_{l+n}) = K_{l+n}$$

for all natural numbers n. It follows that G satisfies the minimal condition on normal subgroups. □

As an application of Theorem 5.24, we deduce the following result concerning (locally soluble)-by-finite groups with infinite central dimension in which all proper subgroups have finite central dimension.

Corollary 5.25. *Let A be a vector space over the field F of prime characteristic p and let G be a (locally soluble)-by-finite subgroup of $GL(F, A)$ having infinite central dimension. Suppose that every proper subgroup of G has finite central dimension. Then G is a Prüfer q-group for some prime q.*

Proof. The fact that every proper subgroup of G has finite central dimension implies that G satisfies min-icd. If G is not a Chernikov group, then G satisfies all the conditions of Theorem 5.24 and according to this theorem the subgroup

Q is nontrivial and has infinite central dimension. We deduce that $G = Q$, which is a contradiction, since Q is Chernikov. Hence G must be a Chernikov group. If the divisible part D of G is a proper subgroup of G, then there is a finite subgroup H such that $G = HD$. Both H and D have finite central dimension and Lemma 4.1 implies that G too has finite central dimension. This contradiction implies that $G = D$. Furthermore if $D = D_1 \times D_2$ and both D_1, D_2 are nontrivial, then Lemma 4.1 can again be applied to obtain a contradiction. It follows that G must be a Prüfer group. □

We next consider the case when the field F has characteristic 0 where the situation is much simpler, as the following theorem shows.

Theorem 5.26. *Let A be a vector space over the field F of characteristic* 0 *and let G be a (locally soluble)-by-finite subgroup of $GL(F, A)$ having infinite central dimension. Suppose that G satisfies the condition min-icd. Then G is a Chernikov group.*

Proof. We suppose that the result is false and that G is a counterexample. Then Theorem 5.14 implies that G is a finitary linear group. Let

$$\mathcal{S} = \{H \leq G | H \text{ is non-Chernikov and has infinite central dimension}\}.$$

Then $\mathcal{S}$ is a nonempty set since $G \in \mathcal{S}$. It is clear that $\mathcal{S}$ is a subset of $\mathcal{L}_{icd}(G)$ and since the family $\mathcal{L}_{icd}(G)$ satisfies the minimal condition, the family $\mathcal{S}$ also satisfies the minimal condition. Hence $\mathcal{S}$ has a minimal element and we denote this minimal element by D. Since the subgroup D is also (locally soluble)-by-finite it contains a locally soluble normal subgroup L which has finite index in D. Since D is a finitary linear group it is easy to see using Lemma 4.1 that L must be of infinite central dimension. Since D is not Chernikov and $|D : L|$ is finite it follows that L is not Chernikov so that the minimal choice of D implies that $L = D$. Hence D is locally soluble and Theorem 5.20 implies that D is soluble.

We now use Lemma 5.23 to deduce that D contains a normal nilpotent torsion-free subgroup K such that D/K is a Chernikov group. Since the subgroup D is not Chernikov it follows that K is nontrivial. Let D_1/K be the divisible part of D/K. Since D/D_1 is finite we may argue as above and deduce that $D = D_1$. Hence D/K is a divisible Chernikov group. Suppose now that D/K is not a Prüfer group and write

$$D/K = Q_1/K \times Q_2/K$$

where Q_1/K and Q_2/K are both nontrivial. Clearly the subgroups Q_1, Q_2 are not Chernikov and the minimality of D implies that both the groups Q_1, Q_2 are of finite central dimension. Then $D = Q_1Q_2$ also has finite central dimension by Lemma 4.1, which is contrary to the choice of D. This contradiction shows that D/K is a Prüfer q-group for some prime q.

Let S be a proper subgroup of D with the property that $SK = D$. Since K is not Chernikov, the minimal choice of D implies that K has finite central dimension and Lemma 4.1 then implies that S has infinite central dimension. Again the minimal choice of D implies that S must be a Chernikov group. In particular, S is periodic so that $S \cap K = 1$. Then

$$S \cong S/(S \cap K) \cong SK/K = D/K$$

which shows that S is an (abelian) Prüfer group. On the other hand if $SK \neq D$, then $S/(S \cap K) \cong SK/K$ is finite and therefore S is nilpotent-by-finite. It follows from [26, Theorem 2.5] that D is periodic and this gives us a contradiction. This contradiction completes the proof. Hence G is a Chernikov group. □

Just as in the positive characteristic case, when every proper subgroup of the (locally soluble)-by-finite group G has finite central dimension the group G is Prüfer.

Corollary 5.27. *Let A be a vector space over the field F of characteristic 0 and let G be a (locally soluble)-by-finite subgroup of $GL(F, A)$ having infinite central dimension. Suppose that every proper subgroup of G has finite central dimension. Then G is a Prüfer q-group for some prime q.*

Proof. By Theorem 5.26 G is a Chernikov group. Using virtually the same arguments as in the proof of Corollary 5.25 we see that G is a Prüfer group. □

Theorems 5.24 and 5.26 show that locally soluble groups satisfying the condition min-icd are locally finite. Therefore the next natural step in the study of groups with min-icd is the study of such locally finite groups. In the next result we use the terminology Q is a *Sylow q-subgroup* of the group G to merely mean that Q is a maximal q-subgroup of G.

Lemma 5.28. *Let A be a vector space over the field F and let G be a locally finite subgroup of $GL(F, A)$ having infinite central dimension. Let Q be a Sylow q-subgroup of G for some prime q. Suppose that G satisfies the condition min-icd.*

(i) If F has characteristic 0, then Q is a Chernikov group.

(ii) If F has prime characteristic p and $q \neq p$, then Q is Chernikov.

Proof. Suppose for a contradiction that the result is false so Q is not Chernikov. Since Q is a locally finite group it does not have the minimal condition on subgroups (see [100, Theorem 5.8] for example) so Q has an infinite strictly descending chain of subgroups

$$Q_1 > Q_2 > \ldots Q_n > Q_{n+1} > \ldots.$$

Since G satisfies min-icd there is a natural number k such that the subgroup $D = Q_k$ has finite central dimension and the choice of D shows that D does not satisfy the minimal condition on all subgroups. Let $C = C_A(D)$ so that the quotient space A/C has finite dimension and let m denote this F-dimension. As usual we may think of $D/C_D(A/C)$ as a subgroup of $GL_m(F)$. Since $D/C_D(A/C)$ is a q-group it is a Chernikov group (by [202, 9.1], for example). If F has characteristic 0, then $C_D(A/C)$ is abelian and torsion-free by Lemma 4.1. Since D is locally finite, this implies that $C_D(A/C)$ is trivial so that D is a Chernikov group. However this contradicts the fact that D does not have the minimal condition on subgroups. Hence the result follows in this case.

On the other hand, if F has prime characteristic p, then Lemma 4.1 implies that $C_D(A/C)$ is an elementary abelian p-group. If $q \neq p$, then again $C_D(A/C)$ is trivial and again D is a Chernikov group. This final contradiction proves the result. □

Theorem 5.14 shows that the case of finitary linear groups is a basic case which needs to be understood. We therefore need some results concerning finitary linear groups. Preliminary to this we make note of the following well known results. Let n be a natural number and let G be a subgroup of $GL_n(F)$ containing a soluble subgroup H. Then there is a function $\zeta : \mathbb{N} \longrightarrow \mathbb{N}$ such that the derived length of H is at most $\zeta(n)$ (in other words the soluble subgroups of G are of bounded derived length, bounded in terms of the dimension of the corresponding space–see the book of B. A. F. Wehrfritz [202, Theorem 3.7], for example). From this it follows, first, that every locally soluble subgroup of G is soluble and secondly that a subgroup of G generated by some family of normal locally soluble subgroups is also soluble. In turn this means that G has a largest normal (locally) soluble subgroup. As we see next, this second property is also enjoyed by finitary linear groups. We prove next the following result of R. E. Phillips [175, Theorem 8.2.2]

Proposition 5.29. *Let A be a vector space over the field F and let G be a finitary subgroup of $GL(F, A)$. Then*

(i) If H, K are normal locally soluble subgroups of G, then HK is a locally soluble subgroup of G.

(ii) The subgroup of G generated by an arbitrary family of normal locally soluble subgroups of G is locally soluble.

(iii) The group G has a largest normal locally soluble subgroup L.

(iv) If H, K are subgroups of G such that H is a normal locally soluble subgroup of K and if K/H is locally soluble, then K is a locally soluble subgroup.

(v) The quotient group G/L contains no nontrivial normal locally soluble subgroups.

Proof. (i) Let $\{g_1, \dots, g_n\}$ be an arbitrary finite subset of HK and write $g_i = x_i y_i$ for certain elements $x_i \in H, y_i \in K$. Since H is normal in G we have $x_i^{y_j} \in H$ for $1 \leq i, j \leq n$. Hence H contains the subgroup $\langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle}$, so that this subgroup is locally soluble. On the other hand we clearly have the inclusion

$$\langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle} \leq \langle x_1, \dots, x_n, y_1, \dots, y_n \rangle.$$

Since a finitely generated subgroup of a finitary linear group has finite central dimension it follows that $\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ has finite central dimension. Lemma 5.1 shows that the subgroup $\langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle}$ also has finite central dimension. By Lemma 5.15 it follows that $\langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle}$ is soluble and of course is $\langle y_1, \dots y_n \rangle$-invariant. Since $\langle y_1, \dots y_n \rangle \leq K$ it follows that $\langle y_1, \dots y_n \rangle$ is soluble and hence the product

$$\langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle} \langle y_1, \dots, y_n \rangle$$

is also soluble. Finally since we have

$$\langle g_1, \dots, g_n \rangle \leq \langle x_1, \dots, x_n \rangle^{\langle y_1 \dots, y_n \rangle} \langle y_1, \dots, y_n \rangle$$

the subgroup $\langle g_1, \dots g_n \rangle$ is soluble. Hence the subgroup HK is locally soluble.

(ii) Let $\mathcal{S}$ be a family of normal locally soluble subgroups of G and let D denote the subgroup generated by all the subgroups in the family $\mathcal{S}$. Let $\{g_1, \dots, g_n\}$ be an arbitrary finite subset of D. Then there is a finite subfamily $\{H_1, \dots H_m\}$ of $\mathcal{S}$ such that $\{g_1, \dots, g_n\} \subseteq H_1 H_s \dots H_m$. Using (i) and an easy induction argument we deduce that the subgroup $H_1 \dots H_m$ is locally soluble. Therefore its finitely generated subgroup $\langle g_1 \dots g_n \rangle$ is soluble and hence D is locally soluble.

(iii) This follows directly from (ii).

(iv) Again let $\{g_1, \dots, g_n\}$ be an arbitrary finite subset of K and let $U = \langle g_1, \dots, g_n \rangle$. Since U is a finitely generated subgroup of the finitary group G it follows that U has finite central dimension. Since K/H is locally soluble UH/H is soluble. Furthermore, by Lemma 5.1 $U \cap H$ has finite central dimension and since $U \cap H$ is locally soluble it is soluble by Lemma 5.15. However $UH/H \cong U/(U \cap H)$ so $U/(U \cap H)$ is soluble and an extension of a soluble group by a soluble group is soluble. Hence U is soluble which proves (iv).

(v) This follows directly from (iv). $\square$

If G is a group, then the largest normal locally soluble subgroup of G is called the *locally soluble radical* of G, if such a subgroup exists. The previous result shows that a finitary linear group contains such a locally soluble radical. On the other hand a well-known example due to P. Hall shows that not all groups have a locally soluble radical (see [183, Theorem 8.19.1] for example.

The next result occurs as [175, Theorem 8.5] in the paper of R. E. Phillips. Its proof is omitted since it requires a number of specific concepts and results that are not entirely relevant to this topic.

Proposition 5.30. *Let A be a vector space over the field F and let G be a finitary subgroup of $GL(F, A)$. Let L be the locally soluble radical of G. Then G/L is a finitary subgroup of $GL(F, B)$ for some vector space B.*

For the proof of the next result we shall require, among other things, the fact that the outer automorphism group of a simple group of Lie type is soluble. We denote the outer automorphism group of a group G by $\mathbf{Out}(G)$.

Theorem 5.31. *Let A be a vector space over the field F and let G be a locally finite subgroup of $GL(F, A)$ having infinite central dimension. If G satisfies the condition min-icd, then G is a soluble-by-finite group.*

Proof. Theorem 5.26 and Lemma 5.28 show that it is enough to prove that G is a (locally soluble)-by-finite group. Theorem 5.14 implies that either G is a finitary linear group or that G satisfies the minimal condition on all subgroups. In the latter case, since G is locally finite, it is a Chernikov group (see [100, Theorem 5.8] for example). Of course in this case G is soluble-by-finite. Consequently we may suppose that G is a finitary linear group.

Assume first that F has characteristic 0. Then Lemma 5.28 implies that the Sylow p-subgroups of G are Chernikov for all primes p. It follows by an important theorem of V. V. Belyaev [7] that G is then (locally soluble)-by-finite and the result follows in this case.

Thus we may assume that F has prime characteristic $p > 0$ and we suppose for a contradiction that G is not (locally soluble)-by-finite. Let

$$\mathcal{S} = \{H \leq G | \mathbf{centdim}_F H \text{ is infinite and } H \text{ is not (locally soluble)-by-finite}\}.$$

Then $\mathcal{S}$ is non-empty since $G \in \mathcal{S}$ and $\mathcal{S}$ is a subset of $\mathcal{L}_{icd}$ which satisfies the minimal condition. Hence the family $\mathcal{S}$ also satisfies the minimal condition and so has a minimal element. Let D be a minimal element of $\mathcal{S}$.

Suppose first, for a contradiction, that D contains a proper subgroup H of infinite central dimension. By the minimal choice of D the subgroup H must be (locally soluble)-by-finite. Using Theorem 5.24 we deduce that H contains a Chernikov p'-subgroup of infinite central dimension. An application of Lemma 4.1 enables us to deduce that H contains a Prüfer q-subgroup Q having infinite central dimension, where $q \neq p$. Suppose, for a contradiction, that Q is a normal subgroup of D. Then Lemma 5.2 implies that the quotient group D/Q satisfies the minimal condition on subgroups. Since D/Q is locally finite it is a Chernikov group (see [100, Theorem 5.8], for example). Hence D is a Chernikov group and we obtain a contradiction to the choice of D. It follows that Q is not a normal subgroup of D and in particular this means that $C_D(Q) \neq D$. Let

$$Q = \langle y_n | y_1^q = 1, y_{n+1}^q = y_n, n \in \mathbb{N} \rangle$$

and consider the descending chain of centralizers

$$C_D(y_1) \geq C_D(y_2) \geq \cdots \geq C_D(y_n) \geq C_D(y_{n+1}) \geq \ldots.$$

Since $Q \leq C_D(y_n)$ for each n, Lemma 5.1 shows that each of the subgroups $C_D(y_n)$ has infinite central dimension for all natural numbers n. However G satisfies the condition min-icd so there is a natural number t such that

$$C_D(y_{t+n}) = C_D(y_t) \text{ for all } n \in \mathbb{N}.$$

It follows that $C_D(y_t) = C_D(Q)$ and in particular we have that Q is normal in $C_D(y_t)$. Repeating the arguments used above we deduce that $C_D(y_t)$ is a Chernikov group. A result of B. Hartley [79] shows that D is soluble-by-finite and we obtain a contradiction which proves that every proper subgroup of D has finite central dimension.

Suppose that D contains a proper subgroup W of finite index. Then D contains a proper normal subgroup of finite index and since G is locally finite there is a finite subgroup Y such that $D = VY$. Since D is finitary, Y has finite central dimension. However since V is a proper subgroup of D it also has finite central dimension. Lemma 4.1 also then shows that D has finite central dimension, giving a contradiction which shows that D is an $\mathfrak{F}$-perfect group.

Let S be the locally soluble radical of D, whose existence follows from Proposition 5.29. Then, according to Proposition 5.30, D/S is also a finitary linear group and Proposition 5.29 shows that D/S has trivial locally soluble radical. By Theorem A of a paper of V. V. Belyaev [8] D/S has a minimal normal subgroup M/S of one of the following types:

(i) M/S is a direct product of (isomorphic) simple groups that are either finite or of Lie type;

(ii) M/S is isomorphic to an alternating group on some infinite set or

(iii) M/S is a simple group of classical type.

We consider each of these possibilities for M/S in turn, starting with case (i). Let

$$M/S = \operatorname*{Dr}_{\lambda \in \Lambda} M_\lambda/S,$$

where M_λ/S is a simple group that is finite or of Lie type and $M_\lambda/S \cong M_\mu/S$ for each $\lambda, \mu \in \Lambda$. We note that since M_λ/S is a simple group the set of primes dividing the orders of its elements contains more than one prime. Let $q \in \Pi(M_\lambda/S)$ and suppose that $q \neq p$. If the index set Λ is infinite, then M/S contains an infinite elementary abelian q-subgroup. It follows that M contains a countable subgroup which is not Chernikov by [100, 1.D.4 Lemma]. On the other hand Lemma 5.28 shows that each Sylow q-subgroup of D is Chernikov and this gives a contradiction which establishes that the set Λ is finite.

Now using the results of [10], we deduce that D/S acts as a finitary permutation group on the set of simple factors of M/S. If the kernel of this action is denoted by T/S, then the factor group D/T is finite because the index set Λ is finite. Since D is $\mathfrak{F}$-perfect T cannot be a proper subgroup of D so that $D = T$ and hence D/S acts trivially on the set of factors of M/S. Consequently M/S is simple. If M/S is finite then $C_{D/S}(M/S)$ has finite index in D and we obtain $D/S = C_{D/S}(M/S)$ so that M/S is abelian, a contradiction.

Hence in this case M/S is of Lie type and therefore $\mathbf{Out}(M/S)$ is soluble. In other words the factor group

$$(D/S)/(M/S \times C_{D/S}(M/S)) \text{ is soluble.}$$

Suppose that $D/S \neq M/S$. In this case we note that $D/S \neq M/S \times C_{D/S}(M/S)$, otherwise D would be generated by two proper subgroups, both having finite central dimension, and Lemma 4.1 then implies that D has finite central dimension, a contradiction. Consequently $D \neq D'$ and by Lemma 5.7 D/D' is a Chernikov group. Since D is $\mathfrak{F}$-perfect D/D' is divisible. Then Corollary 5.22 shows that D/D' is a p'-group. In particular D contains a normal subgroup D_1 such that D/D_1 is a divisible Chernikov q-group for some prime $q \neq p$. Then D contains a countable q-subgroup Q such that $D = QD_1$ (see [100, 1.D.4 Lemma] for example). Since D is not a q-group, Q is a proper subgroup of D. Hence both Q and D_1 have finite central dimension and Lemma 4.1 implies that D also has finite central dimension, which is impossible. This contradiction shows that $D/S = M/S$ and that D/S is itself simple. Since D/S is $\mathfrak{F}$-perfect it is infinite so it must be of Lie type.

Let $C = C_A(S)$. Since S is a proper subgroup of D it has finite central dimension. It follows that C has finite codimension in the vector space A and hence C has infinite dimension. We may consider $D/C_D(C)$ as a finitary subgroup of $GL(F, C)$. By definition $S \leq C_D(C)$ and we suppose for a contradiction that $C_D(C) \neq S$. Since D/S is simple this implies that $D = C_D(C)$. Hence $C = C_A(D)$ which means that D has finite central dimension. This contradiction proves that $C_D(C) = S$. We have $C_C(g) = C_C(gS)$ for each element $g \in D$. But D is finitary, so $C_C(g)$ has finite codimension in A. We now apply [10, Theorem 4.5] to deduce that $C = B \oplus E$, where B has finite dimension over F and $E \leq C_C(D/S)$. Since S acts trivially on both B and E, they are both D-invariant and $E \leq C_C(D)$. It follows that D has finite central dimension and we conclude that case (i) cannot occur.

In Cases (ii) and (iii) M/S is isomorphic to a non-linear simple locally finite group. It follows by the classification of finite simple groups and [100, 4.8 Theorem] that M/S contains, for each prime $q \neq p$, an infinite elementary abelian q-subgroup. This contradicts [100, 1.D.4 Lemma] and Lemma 5.28 as above. The proof is complete. □

To conclude this section, we note that the results obtained above from the work of M. R. Dixon, M. J. Evans and L. A. Kurdachenko [43] can be strengthened. To do this we use two further results from the paper of R. E. Phillips [175]. Indeed Theorems 5.6.1 and 5.6.2 of that paper allow us to prove the following result which might be compared to the usual Tits Alternative (see the book of B. A. F. Wehrfritz [202, Theorem 10.16], for example).

Proposition 5.32. *Let A be a vector space over the field F and let G be a finitary subgroup of $GL(F, A)$. If G contains no non-abelian free subgroups then G contains a normal locally soluble subgroup L such that*

(i) L' is locally nilpotent;

(ii) G/L is locally finite.

Furthermore G/L is a finitary subgroup of $GL(F, B)$ for some vector space B.

Using this result and Theorems 5.20 and 5.31 we can obtain the following result, because a locally generalized radical group contains no non-abelian free subgroups.

Theorem 5.33. *Let A be a vector space over the field F and let G be a locally generalized radical subgroup of $GL(F, A)$ having infinite central dimension. If G satisfies the condition min-icd, then G is a soluble-by-finite group.*

We continue by giving examples to show that the groups arising in Theorems 5.24 and 5.26 actually exist as subgroups of $GL(F, A)$ having infinite central dimension. Indeed every Prüfer group can be realized as a subgroup of infinite central dimension in a suitable $GL(F, A)$ in such a way that every proper subgroup of the Prüfer group has finite central dimension.

Let P be a prime field, so that $P = \mathbb{Z}_p$ has prime order p or $P = \mathbb{Q}$, the field of rational numbers. Let K denote the algebraic closure of P. Then the multiplicative group of K contains a Prüfer p-subgroup Γ where p is the (prime) characteristic of P (see the book of G. Karpilovsky [98, Chapter 4 and Theorem 61] for example). Let F be the subfield of K obtained by adjoining the elements of Γ to P. Thus F coincides with the subring $P[\Gamma]$ generated by Γ over P and every element of F is algebraic over P. We set

$$\Gamma = \langle \gamma_n | \gamma^q = 1, \gamma_{n+1}^q = \gamma_n, n \in \mathbb{N} \rangle.$$

Let A be a vector space over F of countably infinite dimension and let

$$\{a_n | n \in \mathbb{N}\}$$

be an ordered basis for A. Define a linear transformation g_n of A relative to the basis $\{a_n | n \in \mathbb{N}\}$ using the following infinite dimensional matrix $M(g_n) = [\alpha_{jk}]_{j,k\in\mathbb{N}}$ where

$$\begin{aligned}
&\alpha_{1,1} = \gamma_n, \alpha_{2,2} = \gamma_{n-1}, \dots, \alpha_{n,n} = \gamma_1,\\
&\alpha_{j,j} = 1 \text{ for all } j > n\\
&\alpha_{i,j} = 0 \text{ for all } i \neq j.
\end{aligned}$$

Thus

$$M(g_n) = \begin{pmatrix} \gamma_n & 0 & 0 & 0\dots & 0 & 0 & 0\dots \\ 0 & \gamma_{n-1} & 0 & \dots & 0 & 0 & 0\dots \\ 0 & 0 & \gamma_{n-2} & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0\dots & \gamma_1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0\dots & 1 & 0 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \end{pmatrix}.$$

Clearly g_n is an F-isomorphism of A, so $g_n \in GL(F, A)$ and furthermore we have

$$g_1^q = 1, g_{n+1}^q = g_n, n \in \mathbb{N}.$$

Let

$$Q = \langle g_n | n \in \mathbb{N} \rangle.$$

Then Q is a Prüfer q-subgroup and it is easy to see that

$$C_A(g_n) = \oplus_{j>n} F a_j \text{ and } C_A(Q) = 0,$$

so that Q has infinite central dimension and every proper subgroup of Q has finite central dimension.

Suppose now that the characteristic of the field F is the prime p. Let E_{jk} denote the matrix in which every coefficient is 0 except for the coefficient in the jth row and kth column which is 1. It is easy to see that, as in the finite dimensional case, the following multiplication rule holds for such matrices:

$$E_{jk}E_{mt} = \begin{cases} 0 \text{ if } k \neq m \\ E_{jt} \text{ if} k = m. \end{cases}$$

Let $j \neq k$ and define a transvection $t_{jk}(\alpha) = E + \alpha E_{jk}$ for all $j, k \in \mathbb{N}$ and $\alpha \in F$, where E is the identity matrix. We have

$$t_{jk}(\alpha)t_{jk}(\beta) = t_{jk}(\alpha + \beta) \text{ for all } \alpha, \beta \in F.$$

It follows that the set $T_{jk} = \{t_{jk}(\alpha) | \alpha \in F\}$ is a subgroup under matrix multiplication and this subgroup is isomorphic to the additive group of the field F. In particular, in this case, T_{jk} is an infinite elementary abelian p-subgroup. Define a linear transformation $b_{jk}(\alpha)$ of A relative to the ordered basis $\{a_n | n \in \mathbb{N}\}$ by using the matrix $t_{jk}(\alpha)$ for $\alpha \in F$. Clearly $b_{jk}(\alpha)$ is an F-isomorphism of A, so that $b_{jk}(\alpha) \in GL(F, A)$ and the set

$$B_{jk} = \{b_{jk}(\alpha) | \alpha \in F\}$$

is a subgroup of $GL(F, A)$. It is easy to see that $B_{jk} \cong T_{jk}$ and in particular B_{jk} is an infinite elementary abelian p-subgroup. From the definition we have $C_A(B_{jk}) = \oplus_{m \neq j} F a_m$, so that B_{jk} has finite central dimension.

Furthermore, we have

$$M(g_2)^{-1}t_{12}(1)M(g_2) = t_{12}(\gamma_2^{-1}\gamma_1) \text{ and in general}$$
$$M(g_{n+1})^{-1}t_{12}(1)M(g_{n+1}) = t_{12}(\gamma_{n+1}^{-1}\gamma_n) \text{ for all } n \in \mathbb{N}.$$

Since $F = P[\Gamma]$ it is easy to see that F coincides with the subring obtained by adjoining the elements $\gamma_{n+1}^{-1}\gamma_n$ for $n \in \mathbb{N}$ and it follows that

$$T_{12} = M(g_n)^{-1}T_{12}M(g_n).$$

Hence T_{12} is generated by all elements of the form $M(g_n)^{-1}t_{12}(1)M(g_n)$ for $n \in \mathbb{N}$. Since every element of T_{12} has the form $t_{12}(\alpha)$ for some $\alpha \in F$ it follows, using similar reasoning, that the subgroup T_{12} is generated by all elements of the form $M(g_n)^{-1}t_{12}(\alpha)M(g_n)$ for $n \in \mathbb{N}$. It is therefore evident that B_{12} is Q-invariant and indeed B_{12} is a minimal Q-invariant subgroup. Then the natural semidirect product $B_{12}Q$ is a linear subgroup of $GL(F, A)$ satisfying the conditions of Theorem 5.24. By doing similar computations we obtain the following identities. We have

$$M(g_{n+1})^{-1}t_{13}(1)M(g_{n+1}) = t_{13}(\gamma_{n+1}^{-1}\gamma_{n-1})$$
$$M(g_{n+1})^{-1}t_{23}(1)M(g_{n+1}) = t_{23}(\gamma_{n+1}^{-1}\gamma_n)$$

Furthermore,

$$\begin{aligned}
t_{12}(\alpha)^{-1}t_{13}(\beta)t_{12}(\alpha) &= (E - \alpha E_{12})(E + \beta E_{13})(E + \alpha E_{12}) \\
&= (E - \alpha E_{12})(E + \beta E_{13} + \alpha E_{12}) \\
&= E + \beta E_{13} + \alpha E_{12} - \alpha E_{12} \\
&= E + \beta E_{13} = t_{13}(\beta); \\
t_{23}(\alpha)^{-1}t_{13}(\beta)t_{23}(\alpha) &= (E - \alpha E_{23})(E + \beta E_{13})(E + \alpha E_{23}) \\
&= (E - \alpha E_{23})(E + \beta E_{13} + \alpha E_{23}) \\
&= E + \beta E_{13} + \alpha E_{23} - \alpha E_{23} \\
&= E + \beta E_{13} = t_{13}(\beta); \\
[t_{12}(\beta), t_{23}(\alpha)] &= (E - \beta E_{12})(E - \alpha E_{23})(E + \beta E_{12})(E + \alpha E_{23}) \\
&= (E - \beta E_{12})(E + \beta E_{12} + \alpha\beta E_{13}) \\
&= E + \beta E_{12} + \alpha\beta E_{13} - \beta E_{12} \\
&= E + \alpha\beta E_{13} = t_{13}(\alpha\beta)
\end{aligned}$$

Let $U = \langle B_{13}, B_{12}, B_{23}\rangle = (B_{13}B_{12})B_{23}$. From the relations above we deduce that $B_{13} \leq \zeta(U)$ and $U' \leq B_{13}$ so that U is a nonabelian subgroup that is nilpotent of class 2. Also U is Q-invariant and U has a series

$$1 \leq B_{13} \leq B_{12}B_{13} \leq U$$

of Q-invariant subgroups whose factors are Q-chief. Thus the natural semidirect product UQ is a linear subgroup of $GL(F, A)$ satisfying the conditions of Theorem 5.24.

As we saw in the constructions above an important role is played by the fact that the multiplicative group of the field F contains a divisible Chernikov subgroup. This condition is indeed essential as the following last few results of this section show.

Proposition 5.34. *Let A be a vector space over the field F and let G be an irreducible soluble subgroup of $GL(F, A)$. If A has finite dimension, then G contains a normal subgroup V such that*

(i) V has finite index in G;

(ii) $V \cong U_1 \times \cdots \times U_n$ where, for $i = 1, \ldots, n$, U_i is isomorphic to the multiplicative group of some finite field extension K of F.

Proof. Since G is irreducible and soluble a theorem of Maltsev (see [202, Lemma 3.5], for example) shows that G contains a normal abelian subgroup H of finite index. By Clifford's theorem (see [202, Theorem 1.7] A contains a minimal H-invariant subspace L such that

$$A = \bigoplus_{x \in S} xL$$

for some finite subset S of the group G. Since

$$\cap_{x \in S} C_G(xL) = C_G(A) = 1,$$

Remak's theorem implies that there is an embedding of H into $\operatorname{Dr}_{x \in S} G/C_G(xL)$.

Let $X = xL$ and let E be the algebraic closure of the field F. Let $Y = X \otimes_F E$. If M is a nontrivial H-invariant subspace of Y, then $M \cap X$ is a nontrivial H-invariant subspace of X. It follows from the minimality of X that $M \cap X = X$ and

$$Y = X \otimes_F E = (M \cap X) \otimes_F E \leq M.$$

Thus Y is a minimal H-invariant subspace. Since H is abelian and E is algebraically closed it follows that $\mathbf{dim}_E(Y) = 1$ (see [202, Corollary 1.3], for example) and hence $Y = Eu$ for some element $u \in Y$. Let $v_1, \ldots, v_t$ be a basis for the F-space X. Then there are elements $\tau_1, \ldots, \tau_t \in E$ such that $v_j = \tau_j u$ for $1 \leq j \leq t$. Let K denote the extension of the field F obtained by adjoining the elements $\tau_1, \ldots, \tau_k$. Each element τ_j is algebraic over F, so K is a finite extension of F. By our work above $U = X \otimes_F K$ has dimension 1 and it follows that the factor group $H/C_H(X)$ is isomorphic to some subgroup of the multiplicative group of the field K. Since $\bigcap_{x \in S} C_H(xL) = 1$, Remak's theorem gives the embedding $H \leq \operatorname{Dr}_{x \in S} H/C_H(xL)$. □

Let F be a field. If the field F satisfies the following condition,

(RE) For each finite extension E of F the multiplicative group of E contains no periodic divisible subgroups,

then we say that F has the condition (RE).

Theorem 5.35. *Let A be a vector space over the field F and let G be a locally generalized radical subgroup of $GL(F, A)$ having infinite central dimension. Suppose that the field F satisfies the condition (RE). If G satisfies the condition min-icd, then G is a Chernikov group.*

Proof. Theorem 5.33 shows that G is a soluble-by-finite group. Suppose, for a contradiction, that G is not a Chernikov group. Then Theorem 5.26 implies that the field F has prime characteristic p. Theorem 5.24 implies that G is a finitary linear group. Furthermore, G contains a Prüfer q-subgroup D having infinite central dimension where $p \neq q$. Let

$$D = \langle d_n | d_1^q = 1, d_{n+1}^q = d_n, n \in \mathbb{N} \rangle$$

and let $D_n = \langle d_n \rangle$ for $n \in \mathbb{N}$. Since G is finitary, the subspace $C_n = C_A(D_n)$ has finite codimension for each $n \in \mathbb{N}$. The fact that D is abelian implies that the subspace C_n is D-invariant. The finiteness of $\mathbf{dim}_F(A/C_n)$ implies that A has a finite series

$$A = A_0 \geq A_1 \geq \cdots \geq A_k = C_n$$

of D-invariant subspaces such that the factors A_{j-1}/A_j are D-chief factors for $1 \leq j \leq k$. Proposition 5.34 shows that the factor group $D/C_D(A_{j-1}/A_j)$ is isomorphic to some periodic subgroup of the multiplicative group of some finite field extension of F. If we suppose that $D \neq C_D(A_{j-1}/A_j)$, then the factor group $D/C_D(A_{j-1}/A_j)$ is divisible. On the other hand the multiplicative group of the field F contains no nontrivial divisible subgroups by the condition (RE), a contradiction which shows that $D = C_D(A_{j-1}/A_j)$. This is true for each j with $1 \leq j \leq k$. Then Theorem 1.2 implies that $D/C_D(A/C_n)$ is a bounded p-group, a contradiction which shows that also $D = C_D(A/C_n)$ and hence we have $[D, A] \leq C_n = C_A(D_n)$. This is true for each natural number n and hence $[D, A] \leq \bigcap_{n\in\mathbb{N}} C_A(D_n)$. However $D = \bigcup_{n\in\mathbb{N}} D_n$ which implies that $\bigcap_{n\in\mathbb{N}} C_A(D_n) = C_A(D)$. Hence $[D, A] \leq C_A(D)$. Using Theorem 1.2 again we deduce that $D/C_D(A)$ is an elementary abelian p-group, a final contradiction which proves the result. □

Here the question naturally arises whether the condition that the multiplicative group of the field F should contain no divisible subgroups is sufficient. The first natural example that gives a negative answer to this question is the field $\mathbb{R}$ of all real numbers. The torsion subgroup of the multiplicative group of $\mathbb{R}$ is a cyclic group of order 2 while the torsion subgroup of the multiplicative group of the field $\mathbb{C}$ of all complex numbers is divisible. Of course $\mathbb{C}$ is a finite extension of $\mathbb{R}$. However, this example does not quite correspond to our situation since, as we have seen above, the linear groups satisfying the condition min-icd which are not Chernikov arise only in the case when the field F has prime characteristic, so this is the case of interest and not the characteristic 0 case. We now present a series of results from a paper of L. A. Kurdachenko and I. Ya. Subbotin [127] illustrating this.

Proposition 5.36. *Let F be a field of prime characteristic p. Suppose that the torsion subgroup of the mulipicative group of F is finite. If E is a finite field extension of F, then the torsion subgroup of the multiplicative group of E is also finite.*

Proof. Let P be the prime subfield of F. We may assume that E is a normal extension of F and that the extension E is a separable extension of F; if not, then we can replace E by $F(T)$ where T is the torsion subgroup of the multiplicative group of E which is a separable Galois extension. Let G be the Galois group of this extension.

Consider $E^* = P(T)$. Then $F^* = F \cap E^* = P(F \cap T)$ and this, by assumption, is a finite field. We also see that F^* is the fixed field of G acting on E^*, so E^* is a finite Galois extension of F^*. Hence E^* is finite, so the torsion subgroup of the multiplicative group of E is finite. □

Corollary 5.37. *Let A be a vector space over the field F of prime characteristic p and let G be a locally generalized radical subgroup of $GL(F, A)$ having infinite central dimension. Suppose that the torsion subgroup of the multiplicative group of F is finite. If G satisfies the condition min-icd, then G is a Chernikov group.*

Corollary 5.38. *Let A be a vector space over the finite field F and let G be a locally generalized radical subgroup of $GL(F, A)$ having infinite central dimension. If G satisfies the condition min-icd, then G is a Chernikov group.*

Next we require a further small technical result.

Proposition 5.39. *Let F be a field. Suppose that the p-component of the multiplicative group of the field F is finite and nontrivial for each prime p. Suppose that the order of the 2-component is at least 4. If E is a finite field extension of F, then the p-component of the multiplicative group of the field E is finite for all primes p.*

Proof. Let S_p and R_p denote the respective p-components of the multiplicative groups of F and E. Then by [33, Lemma 4.2] R_p/S_p is finite. It follows that R_p is finite for all primes p. □

These results give examples of certain fields for which the answer to the question posed above is positive. In the general case the answer is negative, as the following example shows, which was provided to the authors by R. Guralnick, to whom we are deeply indebted.

Proposition 5.40. *There exist fields E and F satisfying the following conditions:*

(i) E and F are locally finite fields;

(ii) the p-component of the multiplicative group of the field F is finite for each prime p;

(iii) E is a finite field extension of F;

(iv) there exists a prime q such that the q-component of the multiplicative group of the field E is a Prüfer q-subgroup, but the q-component of the multiplicative group of the field F is trivial;

(v) the p-component of the multiplicative group of the field E is finite for all primes $p \neq q$.

Proof. Let p and q be distinct primes such that q and $p-1$ are relatively prime. For each natural number e, let $m_e = q^e$ and let F be the union of the finite fields of order p^{m_e}. The choice of q ensures that the q-component of the multiplicative group of F is trivial.

Let α be a root of the polynomial $X^q - 1$ and let E be the field obtained by adjoining α to F, so $E = F[\alpha]$. Let E_0 be the subfield of E obtained by adjoining the element α to the prime subfield of F. Suppose that $|E_0| = p^d$. Then $p^d \equiv 1 \pmod q$ and E contains a subfield of degree $s_e = dq^e = dm_e$ over the prime field for each $e \in \mathbb{N}$. However q^e divides $p^{s_e} - 1$ for all $e \in \mathbb{N}$ and so the q-component E_q of the multiplicative group of the field E is infinite. This means that E_q is a Prüfer q-subgroup.

We note that the r-component F_r of the multiplicative group of F is finite for each prime r. If r divides $p^{m_e} - 1$, with e minimal, then we choose no more powers of r for any larger e and so the multiplicative group of F is residually finite. However the multiplicative group of E is not residually finite, since it contains a Prüfer q-subgroup.

Next we prove that the r-component E_r of the multiplicative group of E is finite for each prime $r \neq q$. Choose the smallest s_e such that the field of order p^{s_e} contains rth roots of unity (or 4th roots of unity if $r = 2$). If no such field exists then our claim follows. We then use a series of extensions of degree q and obtain no further r-torsion (see the paper of J.-L. Colliot-Thélène, R. M. Guralnick and R. Wiegand[33, Lemma 4.2]). □

Linear Groups with the Maximal Condition on Subgroups of Infinite Central Dimension

In this section we will continue the study of linear groups in which the family of all subgroups having infinite central dimension is "very small" in some sense. More concretely, in this section we shall study those linear groups in which the family $\mathcal{L}_{icd}(G)$ satisfies the maximal condition. The maximal condition on ordered sets is dual to the minimal condition. Earlier in the chapter we saw how to handle the condition min-icd using the analogy with the problem of S. N. Chernikov concerning groups with the minimal condition for all subgroups. As we saw earlier a significant portion of the groups in the class of linear groups satisfying min-icd also satisfied the minimal condition for all subgroups. For groups satisfying the maximal condition on all subgroups R. Baer also formulated the problem of whether such groups would be polycyclic-by-finite. The maximal condition is somewhat more intractible than the minimal condition, one reason being that groups with the maximal condition need no longer be periodic. Thus it is fair to say that the problem of S. N. Chernikov received more attention. A negative solution to the Baer problem was also obtained in the work of A. Yu. Olshanskii in [159]. Nevertheless, the maximal condition has played a very important role in the development of the theory of infinite groups. The study of groups with the maximal condition on various types of subgroup enriched the theory of infinite groups with ideas, techniques, constructions and diverse, interesting and deep results.

Let G be a subgroup of $GL(F, A)$. We say that G satisfies the *maximal condition on subgroups of infinite central dimension*, which we shorten to *max-icd*, if the family $\mathcal{L}_{icd}(G)$, ordered by inclusion, satisfies the maximal condition.

Linear groups satisfying the condition max-icd were studied in the paper [127] of L. A. Kurdachenko and I. Ya. Subbotin and it is to this that we now turn attention. As we shall see, the situation here is much more diverse (as is often typically the case for the maximal conditions).

As usual we start with some simple properties. First we give a group theoretic result illustrating that the solution to Baer's problem is positive for generalized radical groups.

Lemma 5.41. *Let G be a locally generalized radical group. If G satisfies the maximal condition on all subgroups, then G is polycyclic-by-finite.*

Proof. Clearly, every group which satisfies the maximal condition on subgroups is finitely generated, so G is a finitely generated generalized radical group. Hence G contains a series

$$1 = L_0 \leq L_1 \leq \dots L_\alpha \leq L_{\alpha+1} \leq \dots L_\gamma = G$$

of normal subgroups whose factors are either locally nilpotent or locally finite. The fact that G satisfies the maximal condition on all subgroups implies that the ordinal γ is finite and every factor of this series is finitely generated. In particular, every locally finite factor of this series is finite. Let $T = \mathbf{Tor}(G)$, the maximal normal torsion subgroup of G. Then T is locally finite and hence is finite. Without loss of generality we may suppose that $T = 1$. Since G satisfies the maximal condition for all subgroups, G contains a normal polynilpotent subgroup D such that the Hirsch-Plotkin radical of G/D is trivial. Suppose that G/D is infinite and let K/D be the largest normal locally finite subgroup of G/D. By our remark above K/D is finite and it follows by our assumption that $K/D \neq G/D$. Then the Hirsch-Plotkin radical R/K of G/K is a nontrivial nilpotent group. By the choice of K, R/K must be torsion-free. Since K/D is finite we deduce that $C_{G/D}(K/D)$ has finite index in G/D. Hence

$$C/D = C_{G/D}(K/D) \cap R/D$$

has finite index in R/D and it follows that C/D is nontrivial. Certainly $K/D \cap C/D \leq \zeta(C/D)$ and we have

$$\begin{aligned}(C/D)/(K/D \cap C/D) &\cong (C/D)(K/D)/(K/D) \leq (R/D)/(K/D) \\ &\cong R/K\end{aligned}$$

which shows that the factor group $(C/D)/\zeta(C/D)$ is nilpotent. Hence C/D is nilpotent and since C/D is nontrivial we obtain a contradiction to the definition of the subgroup D. This contradiction shows that G/D is finite. By definition D has a finite series of normal subgroups whose factors are nilpotent. Since every such factor is finitely generated it is polycyclic. It follows that D is polycyclic, so that G is polycyclic-by-finite. □

Lemma 5.42. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd.*

(i) If H is a subgroup of G, then H satisfies max-icd.

(ii) If G has an infinite strictly ascending chain of subgroups

$$H_1 \lneqq H_2 \lneqq \cdots \lneqq H_n \lneqq H_{n+1} \ldots$$

then each subgroup H_n in the chain has finite central dimension.

(iii) If the subgroup H has infinite central dimension, then the set of subgroups

$$\{S | H \leq S \leq G\}$$

satisfies the maximal condition; in particular, if H is a normal subgroup of G, then the factor group G/H satisfies the maximal condition on all subgroups.

Proof. (i) is clear.

(ii) Since G satisfies max-icd, there is a natural number t such that each subgroup H_m for $m \geq t$ has finite central dimension. Lemma 5.1 shows that H_j also has finite central dimension for $j \leq t$.

(iii) Since H has infinite central dimension, Lemma 5.1 implies that every subgroup of G containing H has infinite central dimension. It follows that the family $\{S | H \leq S \leq G\}$ satisfies the maximal condition. □

The following corollaries are generally quite easy consequences of this result, but they highlight many natural situations.

Corollary 5.43. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that K is a locally generalized radical subgroup of G. If H is a normal subgroup of K having infinite central dimension, then the section K/H is a polycyclic-by-finite group.*

Corollary 5.44. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. If H, K are subgroups of G such that H is normal in K and the section K/H does not satisfy the maximal condition for all subgroups, then H has finite central dimension.*

Corollary 5.45. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. If H, K are subgroups of G such that H is normal in K and the section K/H is an infinite locally finite group, then H has finite central dimension.*

Proof. If a locally finite group X satisfies the maximal condition on all subgroups, then X must be finitely generated and hence finite. In particular, since K/H is infinite it does not satisfy the maximal condition on subgroups so the result follows by Corollary 5.44 □

Corollary 5.46. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that H, K are subgroups of G such that H is normal in K and the section K/H is a locally nilpotent group. If K/H is not finitely generated, then H has finite central dimension.*

Proof. If X is a group which satisfies the maximal condition on all subgroups, then X must be finitely generated. Thus in our situation, if K/H is not finitely generated, then it does not satisfy the maximal condition for all subgroups, so Corollary 5.44 can again be applied. □

Corollary 5.47. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that H, K are subgroups of G such that H is normal in K and $K/H = \underset{\lambda \in \Lambda}{Dr}\, K_\lambda/H$ where $K_\lambda \neq H$ for each $\lambda \in \Lambda$ and Λ is an infinite index set. Then H has finite central dimension.*

Corollary 5.48. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that H is a subgroup of G having infinite central dimension. If K is an H-invariant subgroup of G such that $H \cap K$ satisfies the maximal condition for all subgroups, then K satisfies the maximal condition for H-invariant subgroups.*

Proof. If the subgroup K satisfies the maximal condition on subgroups, then the result is clear. Therefore suppose that K has a strictly ascending series

$$L_1 \lneqq L_2 \lneqq \dots L_n \lneqq L_{n+1} \lneqq \dots$$

of H-invariant subgroups. Next form the ascending chain of subgroups

$$HL_1 \leq HL_2 \leq \dots \leq HL_n \leq HL_{n+1} \leq \dots.$$

Suppose that there is a natural number m such that $HL_m = HL_n$ for all natural numbers $n \geq m$. Since $H \cap K$ satisfies the maximal condition on subgroups, there is a natural number t such that $H \cap L_t = H \cap L_n$ for all $n \geq t$. Without loss of generality we may assume that t is the maximum of m and t. Let $n \geq t$. Then the inclusion $L_t \leq L_n$ implies, using the Dedekind law, that

$$L_n = L_t(H \cap L_n) = L_t(H \cap L_t) = L_t$$

which contradicts our assumption concerning the chain $\{L_n | n \in \mathbb{N}\}$. This contradiction shows that the chain $\{HL_n | n \in \mathbb{N}\}$ is strictly ascending. But in this case Lemma 5.41 shows that each of the subgroups HL_n has finite central dimension and Lemma 5.1 implies that the subgroup H also has finite central dimension. This gives us a contradiction, which proves the result. □

We proceed to obtain results analogous to those obtained for linear groups satisfying min-icd.

Lemma 5.49. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. If G is not finitely generated, then G is a finitary linear group.*

Proof. Let L be an arbitrary finitely generated subgroup of G. Since G is not finitely generated, there exists an element $g_1 \notin L$. Let $L_1 = \langle L, g_1 \rangle$ which is also finitely generated. Thus, since G is not finitely generated, there exists an element $g_2 \notin L_1$. We continue with similar arguments and construct a strictly ascending chain

$$L_1 \lneqq L_2 \lneqq \cdots \lneqq L_n \lneqq L_{n+1} \lneqq \ldots$$

of finitely generated subgroups. By Lemma 5.41 each subgroup in this chain has finite central dimension. In particular $\mathbf{centdim}_F(L)$ is finite. Thus every finitely generated subgroup of G has finite central dimension and hence if $g \in G$, then $A/C_A(g)$ is finite dimensional. This means that G is a finitary linear group. □

Lemma 5.50. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that L and H are subgroups of G satisfying the following conditions:*

(i) $L = \underset{\lambda \in \Lambda}{Dr}\, L_\lambda$ where L_λ is a nontrivial H-invariant subgroup of L for each $\lambda \in \Lambda$;

(ii) $H \cap L \leq \underset{\lambda \in \Omega}{Dr}\, L_\lambda$.

If the set $\Gamma = \Lambda \backslash \Omega$ is infinite, then the subgroup H has finite central dimension.

Proof. Since the subset Γ is infinite there is an infinite strictly ascending chain

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \Gamma_{n+1} \subseteq \ldots$$

of infinite subsets. Since it is clear that $H \cap \underset{\lambda \in \Gamma}{\mathrm{Dr}}\, L_\lambda = 1$ we obtain the following strictly ascending chain of subgroups

$$\langle H, L_\lambda | \lambda \in \Gamma_1 \rangle \lneqq \langle H, L_\lambda | \lambda \in \Gamma_2 \rangle \lneqq \cdots \lneqq \langle H, L_\lambda | \lambda \in \Gamma_n \rangle \lneqq \dots.$$

Since G satisfies the condition max-icd it follows that there is a natural number d such that the subgroup $\langle H, L_\lambda | \lambda \in \Gamma_d \rangle$ has finite central dimension. By Lemma 5.1 we deduce that H also has finite central dimension, as required. □

Lemma 5.51. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that K, H are subgroups of G such that K is a normal subgroup of H and $H/K = B/K \times C/K$. If neither of the sections B/K and C/K satisfy the maximal condition on subgroups, then the subgroup H has finite central dimension.*

Proof. The isomorphism $H/B \cong (H/K)/(B/K) \cong C/K$ shows that H/B also does not satisfy the maximal condition on subgroups. Corollary 5.44 implies that the subgroup B has finite central dimension. By the same argument the subgroup C also has finite central dimension. However the equality $H = BC$ and Lemma 4.1 together show that H has finite central dimension, as required. □

Corollary 5.52. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that K, H are subgroups of G such that H is a normal subgroup of K and $K/H = \underset{\lambda \in \Lambda}{Dr}\, K_\lambda/H$ where $K_\lambda \neq H$ for all $\lambda \in \Lambda$. If the index set Λ is infinite, then K has finite central dimension.*

Proof. The fact that Λ is infinite ensures that there are infinite subsets Γ, Δ of Λ with the properties that

$$\Gamma \cup \Delta = \Lambda \text{ and } \Gamma \cap \Delta = \emptyset.$$

It follows that Γ and Δ each have an infinite strictly ascending chain

$$\Gamma_1 \subsetneqq \Gamma_2 \subsetneqq \cdots \subsetneqq \Gamma_n \subsetneqq \Gamma_{n+1} \subsetneqq \dots$$
$$\Delta_1 \subsetneqq \Delta_2 \subsetneqq \cdots \subsetneqq \Delta_n \subsetneqq \Delta_{n+1} \subsetneqq \dots$$

of infinite subsets. Let

$$U/H = \underset{\lambda \in \Gamma}{\mathrm{Dr}}\, L_\lambda/H \text{ and } V/H = \underset{\lambda \in \Delta}{\mathrm{Dr}}\, L_\lambda/H.$$

Then we obtain two strictly ascending chains of subgroups

$$\langle L_\lambda | \lambda \in \Delta_1 \rangle \lneqq \langle L_\lambda | \lambda \in \Delta_2 \rangle \lneqq \cdots \lneqq \langle L_\lambda | \lambda \in \Delta_n \rangle \lneqq \dots$$

and

$$\langle L_\lambda | \lambda \in \Gamma_1 \rangle \lneqq \langle L_\lambda | \lambda \in \Gamma_2 \rangle \lneqq \cdots \lneqq \langle L_\lambda | \lambda \in \Gamma_n \rangle \lneqq \dots.$$

It follows that the sections U/H and V/H do not satisfy the maximal condition on subgroups. The equality $K/H = U/H \times V/H$ together with Lemma 5.51 shows that K has finite central dimension. □

Lemma 5.53. *Let A be a vector space over the field F and let G be a locally generalized radical subgroup of $GL(F, A)$ satisfying max-icd. Then the factor group $G/\mathbf{Fin}(G)$ is polycyclic-by-finite.*

Proof. Without loss of generality we may suppose that G has infinite central dimension, otherwise $G = \mathbf{Fin}(G)$. Furthermore, if G is not finitely generated, then Lemma 5.49 implies that G is a finitary linear group and in this case also $G/\mathbf{Fin}(G)$ is trivial. Hence we may also assume that G is finitely generated and this implies that G is a generalized radical group. Let

$$1 = L_0 \leq L_1 \leq \dots L_\alpha \leq L_{\alpha+1} \leq \dots L_\gamma = G$$

be an ascending series of normal subgroups whose factor groups are locally nilpotent or locally finite. Since G is finitely generated it follows that the ordinal γ is not a limit ordinal. Hence γ is either finite or $\gamma = \beta + k$ for some limit ordinal γ and some natural number k. In the latter case Lemma 5.42 implies that L_β is the union of an ascending chain of subgroups each having finite central dimension and it follows that $L_\beta \leq \mathbf{Fin}(G)$. If the subgroup L_β has infinite central dimension, then the factor group G/L_β is polycyclic-by-finite by Corollary 5.43.

Suppose then that L_β has finite central dimension and suppose also that $L_{\beta+k-1}$ has finite central dimension. Then $L_{\beta+k-1} \leq \mathbf{Fin}(G)$. Also the factor group $G/L_{\beta+k-1}$ is finitely generated. If it is locally finite, then it is finite and hence $G/\mathbf{Fin}(G)$ is finite. If $G/L_{\beta+k-1}$ is locally nilpotent, then it is a finitely generated nilpotent group and hence polycyclic so that $G/\mathbf{Fin}(G)$ is polycyclic. Assume now that there is a natural number $m < k - 1$ such that $L_{\beta+m}$ has finite central dimension while $L_{\beta+m+1}$ has infinite central dimension. Corollary 5.43 implies that the factor group $G/L_{\beta+m+1}$ is polycyclic-by-finite. If $L_{\beta+m+1}/L_{\beta+m}$ is finitely generated, then as above it is either finite or polycyclic and in either case it follows that $G/L_{\beta+m}$ is also polycyclic-by-finite. If $L_{\beta+m+1}/L_{\beta+m}$ is not finitely generated, then one of Corollary 5.45 or Corollary 5.46 applies to show that $L_{\beta+m+1} \leq \mathbf{Fin}(G)$ so that $G/\mathbf{Fin}(G)$ is polycyclic-by-finite.

Therefore we may suppose that γ is finite, say $\gamma = k$ for some natural number k. Suppose that the subgroup L_{k-1} has finite central dimension. Then $L_{k-1} \leq \mathbf{Fin}(G)$ and as above we deduce that either G/L_{k-1} is finite or polycyclic. Assume next that there is a natural number $m < k - 1$ such that the subgroup L_m has finite central dimension and the subgroup L_{m+1} has infinite central dimension. Corollary 5.43 implies that the factor group G/L_{m+1} is polycyclic-by-finite. If L_{m+1}/L_m is finitely generated, then as above it is either finite or polycyclic and in either case it follows that G/L_m is also polycyclic-by-finite. If L_{m+1}/L_m is not finitely generated, then one of Corollary 5.45

or Corollary 5.46 applies to show that $L_{m+1} \leq \mathbf{Fin}(G)$ so that $G/\mathbf{Fin}(G)$ is polycyclic-by-finite. $\square$

The next step in this discussion is to obtain a detailed picture of the structure of soluble linear groups satisfying the condition max-icd. We need the following group theoretic result first.

Lemma 5.54. *Let G be a free abelian group of infinite 0-rank. Then G contains a subgroup H such that $G/H = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, C_n/H$ where C_n/H is a Prüfer p-group for some prime p, for each $n \in \mathbb{N}$.*

Proof. Suppose first that $G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, X_n$ where, for each n, $X_n = \langle g_n\rangle$ is an infinite cyclic group. For the prime p let $H = \langle g_1^p, g_2^p g_1^{-1}, \ldots, g_{n+1}^p g_n^{-1}\rangle$. It is clear that G/H is a Prüfer p-group.

Again let $G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, X_n$ where, for each n, $X_n = \langle g_n\rangle$ is an infinite cyclic group. Since $\mathbb{N}$ is infinite we may choose a family $\{\Gamma_n | n \in \mathbb{N}\}$ of infinite subsets Γ_n such that $\mathbb{N}$ is the disjoint union of the sets Γ_n. Then $G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_n$ where $G_n = \underset{j\in\Gamma_n}{\mathrm{Dr}}\, X_j$. Then, as we saw in the first paragraph, G_n contains a subgroup H_n such that G_n/H_n is a Prüfer p-group, for each $n \in \mathbb{N}$. We then let $H = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, H_n$.

Finally let Λ be an infinite set and let $G = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, X_\lambda$ where $X_\lambda = \langle g_\lambda\rangle$ is an infinite cyclic group for each $\lambda \in \Lambda$. Since Λ is infinite it contains a countably infinite subset Γ and we let $\Delta = \Lambda \setminus \Gamma$. Also we set $K = \underset{\lambda\in\Gamma}{\mathrm{Dr}}\, X_\lambda, L = \underset{\lambda\in\Delta}{\mathrm{Dr}}\, X_\lambda$ so that $G = K \times L$. By our work above K contains a subgroup D such that K/D is a direct product of countably many copies of Prüfer p-groups. To complete the proof we set $H = D \times L$. $\square$

We also need the following result of L. A. Kurdachenko [103]. For this result we need some knowledge concerning the *basic* subgroups of an abelian group and refer the reader to the book of L. Fuchs [58] for terminology.

Lemma 5.55. *Let G be an abelian p-group for some prime p. If the factor group G/G^p is finite, then $G = B \times D$ for some finite subgroup B and some divisible subgroup D.*

Proof. Let B be a basic subgroup of G. Thus B is a subgroup of G satisfying the conditions

(i) B is a direct product of cyclic p-subgroups;

(ii) B is a pure subgroup of G;

(iii) G/B is divisible.

The existence of such subgroups follows, for example, from the results of [58, Section 33]. Since B is a pure subgroup of G we have that $B^p = B \cap G^p$ and it then follows that

$$B/B^p = B/(B \cap G^p) \cong BG^p/G^p \leq G/G^p$$

and the finiteness of G/G^p implies that B/B^p is also finite. Since B is a direct product of cyclic p-groups it follows that this direct product has only finitely many factors which implies that B is finite. The fact that B is a pure subgroup implies that $G = B \times D$ for some subgroup D (see [58, Theorem 27.5], for example) and the isomorphism $D \cong G/B$ allows us to deduce that D is divisible. □

Lemma 5.56. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. If G has infinite central dimension and $G \neq G'$, then either the factor group G/G' is finitely generated or G contains a subgroup S such that $G' \leq S$, the factor S/G' is finitely generated and G/S is a Prüfer p-group for some prime p.*

Proof. Let $D = G'$ and suppose that the abelianization G/D is not finitely generated. Let $T/D = \mathbf{Tor}(G/D)$ and in the factor group G/D choose a maximal $\mathbb{Z}$-independent set of elements $\{u_\lambda T | \lambda \in \Lambda\}$. Then the factor $U/T = \langle u_\lambda T | \lambda \in \Lambda\rangle$ is free abelian and indeed $U/T = \operatorname*{Dr}_{\lambda \in \Lambda} \langle u_\lambda T\rangle$. Furthermore the factor group $(G/T)/(U/T) \cong G/U$ is periodic. Assume that the set Λ is infinite. Then Lemma 5.54 implies that U/T contains a subgroup Y/T such that U/Y is a direct product of countably many Prüfer p-groups. Then $G/Y = U/Y \times W/Y$ for some subgroup W/Y (see [58, Theorem 21.2], for example). Thus G/W is a direct product of countably many Prüfer p-groups. Corollary 5.52 implies that in this case the group G must have finite central dimension, a contradiction which shows that the set Λ must be finite. Consequently, G/D has finite 0-rank.

If the set $\Pi(G/U)$ is infinite, then using Corollary 5.52 we deduce once again that G has finite central dimension. This contradiction shows that the set $\Pi(G/U)$ is finite. Let S_p/U be the p-component of G/U for each prime p and let

$$\Theta = \{p \in \Pi(G/U) | S_p/U \text{ is infinite}\}.$$

Lemma 5.51 implies that $\Theta = \{p\}$ for some prime p. Let $P/U = \operatorname*{Dr}_{q \neq p} S_q/U$. Then $G/P \cong S_p/U$ is infinite and P/U is finite. Suppose that the factor group $(G/P)/(G/P)^p$ is infinite. Then it is an infinite elementary abelian p-group and Corollary 5.52 implies that G must have finite central dimension. This contradiction shows that $(G/P)/(G/P)^p$ is finite. Lemma 5.55 implies that $S_p/U = C_p/U \times K_p/U$, where K_p/U is finite and C_p/U is a divisible p-group. The isomorphism $G/P \cong S_p/U$ together with Lemma 5.51 implies that C_p/U is a Prüfer p-group. Hence the subgroup $S/U = P/U \times K_p/U$ is finite, so that S/U is finitely generated and G/S is a Prüfer group as required. □

Corollary 5.57. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying max-icd. Suppose that G has infinite central dimension and $G \neq G'$. If the factor group G/G' is not finitely generated, then G has an ascending series*

$$G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \leq \cdots \bigcup_{n \in \mathbb{N}} G_n = G$$

consisting of normal subgroups having finite central dimension. Furthermore, G is finitary.

Proof. Since G is not finitely generated it is clear from Lemma 5.49 that G is finitary. In any case by Lemma 5.56 G contains a normal subgroup S such that $G' \leq S$, the factor S/G' is finitely generated and the factor group G/S is a Prüfer group. Define the subgroups G_n by $G_n/S = \boldsymbol{\Omega}_n(G/S)$ for each $n \in \mathbb{N}$. We then obtain a strictly ascending series of normal subgroups

$$S = G_0 \lneqq G_1 \lneqq G_2 \lneqq \cdots \lneqq G_n \lneqq G_{n+1} \lneqq \ldots$$

such that $\bigcup_{n \in \mathbb{N}} G_n = G$. By Lemma 5.42 each of the subgroups G_n has finite central dimension. The result follows. □

Lemma 5.58. *Let A be a vector space over the field F and let G be a finitary subgroup of $GL(F, A)$ satisfying max-icd. If G has infinite central dimension, then every subgroup of G having finite index has infinite central dimension.*

Proof. Suppose the contrary and let H be a subgroup of G of finite index having finite central dimension. Let $K = \mathbf{core}_G\, H$. Then K has finite index in G also. By Lemma 5.1 K has finite central dimension. Clearly there is a finitely generated subgroup L of G such that $G = LK$. Since G is finitary, L has finite central dimension. However then Lemma 4.1 implies that G has finite central dimension and we obtain a contradiction which proves the result. □

We need some information concerning finite dimensional irreducible linear groups next.

Lemma 5.59. *Let A be a finite dimensional vector space over the field F and let G be an irreducible Prüfer q-subgroup of $GL(F, A)$ for some prime q. Suppose that $q \neq \boldsymbol{char}(F)$ whenever $\boldsymbol{char}(F)$ is prime. If the multiplicative group of the field F contains a Prüfer q-subgroup, then $\boldsymbol{dim}_F(A) = 1$.*

Proof. Let $G = \langle g_n | g_1^q = 1, g_{n+1}^q = g_n, n \in \mathbb{N}\rangle$ and let R be a Prüfer q-subgroup of the multiplicative group of F. Let $f : G \longrightarrow R$ be an isomorphism between these two Prüfer groups. Let $\chi_n(X)$ denote the characteristic polynomial of the linear transformation induced on A by the element g_n, for $n \in \mathbb{N}$. By the Cayley Hamilton theorem g_n is a root of $\chi_n(X)$ and it follows that $\lambda_n = f(g_n)$ is also a root of $\chi_n(X)$. Thus the vector space A has a non-zero element a such that $g_n a = \lambda_n a$. In turn, it follows that the F-subspace

$A(\lambda_n) = \{b \in A | g_n b = \lambda_n b\}$ is non-zero. Since G is abelian the subspace $A(\lambda_n)$ is G-invariant and it follows that $A = A(\lambda_n)$, since A is simple. Thus for each element $a \in A$ we have $F\langle g_n \rangle a = Fa$. This is valid for each natural number n so the equation $G = \cup_{n\in\mathbb{N}}\langle g_n \rangle$ implies that $(FG)a = Fa$, for each element $a \in A$. Since A is simple as an FG-module, $A = (FG)a = Fa$ for every $0 \neq a \in A$. Consequently $\mathbf{dim}_F(A) = 1$. □

As we mentioned earlier when we study infinite dimensional linear groups two cases arise quite naturally, one of these being when the corresponding group is not finitely generated. We now consider this situation further.

Theorem 5.60. *Let A be a vector space over the field F and let G be a soluble subgroup of $GL(F, A)$ satisfying max-icd. Suppose that G has infinite central dimension and that the factor group G/G' is not finitely generated. Then G contains normal subgroups K, V such that $K \leq V$ satisfying the following conditions:*

(i) G/V is a Prüfer q-group for some prime q, the factor group G/K is abelian-by-finite and V/K is finitely generated. Furthermore K is nilpotent and the vector space A is K-nilpotent;

(ii) if $\boldsymbol{char}(F) = p$ a prime, then $q \neq p$ and K is a bounded p-subgroup;

(iii) if $\boldsymbol{char}(F) = 0$, then K is torsion-free;

(iv) there exists a field extension E of the field F such that the multiplicative group of the field E contains a Prüfer q-subgroup;

(v) the subspace $C = C_A(V)$ has finite codimension in A;

(vi) C contains a G-invariant subspace B having a direct decomposition $B = \bigoplus_{n\in\mathbb{N}} B_n$, where B_n is a G-invariant subspace of A having finite dimension for each natural number n. Furthermore, $[G, C] \leq B$ and $\boldsymbol{dim}_E(B_n) = 1$ for each $n \in \mathbb{N}$.

Proof. Since G is not finitely generated it follows that G is finitary by Lemma 5.49. By Lemma 5.56, G contains a normal subgroup S such that $G' \leq S$, the factor S/G' is finitely generated and the factor group G/S is a Prüfer q-group for some prime q. Lemma 5.42 shows that the subgroup S has finite central dimension. Let $C = C_A(S)$ so that the subspace C has finite codimension in A. Since S is a normal subgroup of G, the subspace C is G-invariant. Let $V = C_G(C)$. Then V is a normal subgroup of G and it is clear that $S \leq V$. If $V = G$, then G must have finite central dimension, contrary to our hypotheses. Hence V is a proper subgroup of G and it follows that the factor group V/S is finite. Furthermore, G/V is a Prüfer q-group so that V also has finite central dimension. The finiteness of $\mathbf{dim}_F(A/C)$ implies that the series $C \leq A$ can be refined to a series

$$C = C_1 \leq C_2 \leq \cdots \leq C_n = A$$

such that the factors C_{j+1}/C_j are G-chief for $1 \leq j \leq n-1$. Using Maltsev's theorem (see [202, Lemma 3.5], for example) we deduce that the factor groups $G/C_G(C_{j+1}/C_j)$ are abelian-by-finite for $1 \leq j \leq n-1$. Let

$$H = C_G(C_2/C_1) \cap C_G(C_3/C_2) \cap \cdots \cap C_G(C_n/C_{n-1}).$$

Using Remak's theorem we deduce that the factor group G/H is isomorphic to a subgroup of the direct product $\underset{1 \leq j \leq n-1}{\mathrm{Dr}} G/C_G(C_{j+1}/C_j)$ and it is then clear that G/H is likewise abelian-by-finite. Let

$$K = H \cap V.$$

Again using Remak's theorem we obtain an embedding of the factor group G/K in the group $G/H \times G/V$ and this implies that G/K is also abelian-by-finite. Let U/K be a normal abelian subgroup of G/K such that U has finite index in G. Lemma 5.58 implies that the subgroup U has infinite central dimension. Lemma 5.56 further implies that U contains a subgroup W such that $U' \leq W$, W/U' is finitely generated and U/W is a Prüfer q-group. Since U is normal in G its derived subgroup U' is also normal in G. The finiteness of G/U implies that W/U' has finitely many conjugates in G and it follows that the normal closure W^G/U' is finitely generated. Replacing W by W^G we may assume that W is normal in G and Lemma 5.42 shows that the subgroup W has finite central dimension. Since U/K is abelian we have $U' \leq K$. This inclusion and the fact that $WK/K \cong W/(W \cap K)$ yield that WK/K is finitely generated. Moreover the factor group G/WK is Prüfer-by-finite. Suppose, for a contradicition, that $VW/WK = V(WK)/WK$ is infinite. Then the subgroup VW must have finite index in G and Lemma 5.58 implies that in this case VW must have infinite central dimension. On the other hand Lemma 4.1 implies that the subgroup VW has finite central dimension giving the contradiction sought. It follows that VW/WK is finite. Since WK/K is finitely generated it follows that VW/K is finitely generated and, in particular, V/K is finitely generated.

By construction, every element of K acts trivially on each factor of the series

$$C = C_1 \leq C_2 \leq \cdots \leq C_n = A.$$

Using Theorem 1.2 we deduce that K is nilpotent. Furthermore we also deduce that if $\mathbf{char}\,(F) = 0$, then K is torsion-free, whereas if $\mathbf{char}\,(F) = p$, for the prime p, then K is a bounded p-subgroup. Also if $\mathbf{char}\,(F) = p$, then Corollary 5.22 implies that $q \neq p$. This completes the proof of (i), (ii), (iii) and (v).

We now prove (iv). Suppose that the multiplicative group of no finite field extension of F contains a Prüfer q-subgroup. Let

$$G/V = \langle d_m V | d_1^q V = V, d_{m+1}^q V = d_m V, m \in \mathbb{N} \rangle \cong C_{q^\infty}.$$

Set $D_m = \langle V, d_m \rangle$ for each natural number m. Since G is finitary, the subspace $A_m = C_A(D_m) \leq C$ has finite codimension in A for each natural number m

and it is clear that A_m is G-invariant. From the finiteness of $\mathbf{dim}_F(A/A_m)$ we deduce that A has a finite series

$$A = E_0 \geq E_1 \geq \cdots \geq E_k = A_m$$

of G-invariant subspaces such that the factors E_{j-1}/E_j are G-chief for $1 \leq j \leq k$. Proposition 5.34 shows that the factor group $G/C_G(E_{j-1}/E_j)$ is isomorphic to some periodic subgroup of the multiplicative group of a field F_0, for some finite extension F_0 of the field F. The equality $V = C_G(C)$ implies that $V \leq C_G(E_{j-1}/E_j)$ for $1 \leq j \leq k$. Suppose, for a contradiction, that $G \neq C_G(E_{j-1}/E_j)$. Then the factor group $G/C_G(E_{j-1}/E_j)$ must be a Prüfer q-group. However by our hypothesis the multiplicative group of the field F_0 contains no nontrivial Prüfer groups. This contradiction shows that $G = C_G(E_{j-1}/E_j)$ for each j such that $1 \leq j \leq k$. Then Theorem 1.2 implies that the factor group $G/C_G(A/A_m)$ is a bounded p-group whenever $\mathbf{char}\,(F) = p$ and torsion-free whenever $\mathbf{char}\,(F) = 0$. This again yields a contradiction unless $G = C_G(A/A_m)$. It follows that $[G, A] \leq A_m = C_A(D_m)$. This is true for each natural number m and hence we have

$$[G, A] \leq \bigcap_{m \in \mathbb{N}} C_A(D_m).$$

The equality $G = \cup_{m \in \mathbb{N}} D_m$ implies that $\cap_{m \in \mathbb{N}} C_A(D_m) = C_A(G)$ so that $[G, A] \leq C_A(G)$. Again using Theorem 1.2 we deduce that $G/C_G(A)$ must be an elementary abelian p-group or a torsion-free group, which is the final contradiction. This contradiction shows that there exists a finite field extension E of the field F such that the multiplicative group of E contains a Prüfer subgroup.

Finally, we prove (vi). The subspace C has the direct decomposition $C = A_m \oplus [D_m, C]$ for every natural number m (see [116, Corollary 5.16], for example). Since the subgroup D_m is normal in G, both factors of this decomposition are G-invariant for all $m \in \mathbb{N}$. As we saw above, the subspace A_m has finite codimension in C so that the subspace $[D_m, C]$ has finite dimension for all $n \in \mathbb{N}$. We have

$$\begin{aligned} C &= [D_1, C] \oplus A_1 = [D_1, C] \oplus [D_2, A_1] \oplus A_2 \\ &= [D_1, C] \oplus [D_2, A_1] \oplus \cdots \oplus [D_m, A_{m-1}] \oplus A_m = \ldots \end{aligned}$$

It follows that

$$[D_m, C] = [D_1, C] \oplus [D_2, A_1] \oplus \cdots \oplus [D_m, A_{m-1}]$$

for all $m \in \mathbb{N}$. Let

$$B = [D_m, C] = [D_1, C] \oplus [D_2, A_1] \oplus \cdots \oplus [D_m, A_{m-1}] \oplus \ldots.$$

If $x \in G$, then there is a natural number m such that $x \in D_m$. It follows that $[x, C] \leq [D_m, C] \leq B$ so that $[G, C] \leq B$. On the other hand B is a direct sum of countably many G-invariant subspaces, each of which has finite dimension. Each of these direct summands can be decomposed in turn into a direct sum of minimal G-invariant subspaces (see [116, Corollary 5.15], for example). It follows that $B = \oplus_{m\in\mathbb{N}} B_m$ for certain minimal G-invariant subspaces B_m of C, for each $m \in \mathbb{N}$. Using Lemma 5.59 we deduce that $\mathbf{dim}_F(B_m) = 1$ for each $m \in \mathbb{N}$. This completes the proof. $\square$

The previous theorem is a slightly refined version of [127, Theorem A] due to L. A. Kurdachenko and I. Ya. Subbotin.

Our next result reduces the case when G is a soluble group which is not finitely generated and satisfies the condition max-icd to Theorem 5.60. For its proof we need the following definition.

A soluble group G is called an A_3*-group* if G has a finite series

$$1 = D_) \leq D_1 \leq \cdots \leq D_{n-1} \leq D_n = G$$

of normal subgroups whose factor groups are either finite groups or abelian Chernikov groups or torsion-free abelian groups having finite 0-rank. This definition is due to A. I. Maltsev [143].

Theorem 5.61. *Let A be a vector space over the field F and let G be a soluble subgroup of $GL(F, A)$ satisfying max-icd. Suppose that G has infinite central dimension and that G is not finitely generated. Then G contains normal subgroups $K \leq W \leq G$ satisfying the following conditions:*

(i) G/W is finitely generated and abelian-by-finite;

(ii) W/K is abelian;

(iii) W contains a subgroup L such that L/K is finitely generated and W/L is a Prüfer q-group for some prime q;

(iv) K is nilpotent and the vector space A is K-nilpotent.

Proof. By Lemma 5.49 G is a finitary linear group. Let

$$1 = S_0 \leq S_1 \leq \cdots \leq S_{n-1} \leq S_n = G$$

be the derived series of G. Since G is not finitely generated, there is a natural number t such that G/S_t is polycyclic, while S_t/S_{t-1} is not finitely generated. Let $H = S_t$. The facts that G/S_t is finitely generated and G is finitary together with Lemma 4.1 show that the subgroup S_t has infinite central dimension. Since the factor group S_t/S_{t-1} is not finitely generated, Lemma 5.56 implies that S_t contains a subgroup L with $S_{t-1} \leq L$, L/S_{t-1} is finitely generated and S_t/L is is a Prüfer q-group for some prime q. It follows that the factor group G/S_{t-1} is a soluble A_3-group. Then G/S_{t-1} contains a nilpotent normal subgroup D/S_{t-1} such that G/D is a finitely generated abelian-by-finite group by [143, Theorem 4]. Since G/S_{t-1} is not finitely generated, D/S_{t-1} also is not finitely generated. Then $(D/S_{t-1})/(D/S_{t-1})' \cong D/D'S_{t-1}$ is also not finitely generated (see [183, Corollary of Theorem 2.26], for example). It follows that the factor group D/D' is not finitely generated.

By Theorem 5.60 D contains normal subgroups U, V such that $V \leq U, D/U$ is finite, U/V is abelian and A is V-nilpotent. Let $t = |D/U|$ and set $W = D^t$. Since the factor D/W is a bounded soluble group, D has a finite series of G-invariant subgroups

$$W = W_0 \leq W_1 \leq \cdots \leq W_{n-1} \leq W_n = D$$

whose factors are elementary abelian. Suppose, for a contradiction, that D/W is infinite. Then there is a natural number m such that D/W_m is finite, but W_m/W_{m-1} is infinite. This choice implies that the factor group G/W_m is finitely generated and it follows that the subgroup W_m has infinite central dimension. On the other hand the fact that W_m/W_{m-1} is an infinite elementary abelian group together with Corollary 5.52 implies that the subgroup W_m has finite central dimension, yielding the desired contradiction. It follows that D/W is finite. The choice of W yields that $W \leq U$. Then $W' \leq U' \leq V$. Set $K = W'$. Then K is a G-invariant subgroup and since $K \leq V$ it follows that the vector space A is K-nilpotent. Since the subgroup W has infinite central dimension, Lemma 5.56 shows that W satisfies condition (iii). Finally the fact that G/W is abelian-by-finite follows, for example, from Lemma 2.4.4 of the book [52]. □

The theorem we have just obtained is a slightly refined version of [127, Theorem B] due to L. A. Kurdachenko and I. Ya. Subbotin.

The next natural problem to study is that of finitely generated groups satisfying the maximal condition on subgroups having infinite central dimension. Here we also have two cases.

Theorem 5.62. *Let A be a vector space over the field F and let G be a finitely generated soluble subgroup of $GL(F, A)$ satisfying max-icd. Suppose that G has infinite central dimension but that its finitary radical has finite central dimension. Then the following conditions hold:*

(i) G contains a normal subgroup U such that G/U is polycyclic;

(ii) there is a natural number m such that $(x-1)^m = 0$ for each element x of U, so that every element of U is unipotent and the subgroup U is nilpotent;

(iii) if $\boldsymbol{char}(F) = 0$, then U is torsion-free; if $\boldsymbol{char}(F) = p$, a prime, then U is bounded;

(iv) U satisfies the maximal condition for $\langle g\rangle$-invariant subgroups for each element $g \notin \boldsymbol{Fin}(G)$.

Proof. Let $C = C_A(\mathbf{Fin}(G))$ so that C has finite codimension in A. Then A has a finite series of G-invariant subspaces

$$0 = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_m = A$$

such that the factors $C_2/C_1, \dots, C_m/C_{m-1}$ have finite dimension and are G-chief. Applying Maltsev's theorem (see [202, Lemma 3.5], for example) we deduce that the factor groups

$$G/C_G(C_2/C_1), \dots, G/C_G(C_m/C_{m-1})$$

are abelian-by-finite and hence polycyclic, because G is finitely generated and soluble. Let

$$U = C_G(C_1) \cap C_G(C_2/C_1) \cap C_G(C_3/C_2) \cap \cdots \cap C_G(C_m/C_{m-1}).$$

Since $\mathbf{Fin}(G) \leq C_G(C_1)$, Lemma 5.53 shows that the factor group $G/C_G(C_1)$ is polycyclic. By Remak's theorem we obtain the embedding

$$G/U \leq G/C_G(C_1) \times G/C_G(C_2/C_1) \times \cdots \times G/C_G(C_m/C_{m-1})$$

which shows that the factor group G/U is polycyclic. Each element of U acts trivially on the each of the factors C_{j+1}/C_j for $0 \leq j \leq m-1$, so we can apply Theorem 1.2 to obtain (ii) and (iii). Finally let $g \notin \mathbf{Fin}(G)$. Then the cyclic subgroup $\langle g \rangle$ has infinite central dimension. Using Corollary 5.48 we deduce that U satisfies the maximal condition on $\langle g \rangle$-invariant subgroups. This completes the proof. □

Theorem 5.63. *Let A be a vector space over the field F and let G be a finitely generated soluble subgroup of $GL(F, A)$ satisfying max-icd. Suppose that G and its finitary radical both have infinite central dimension. Then G contains a normal subgroup L satisfying the following conditions:*

(i) G/L is abelian-by-finite;

(ii) L has infinite central dimension and the factor L/L' is not finitely generated;

(iii) $L \leq \boldsymbol{Fin}(G)$;

(iv) the subgroup L satisfies the maximal condition for $\langle g \rangle$-invariant subgroups for each element $g \notin \boldsymbol{Fin}(G)$.

Proof. Let

$$1 = D_0 \leq D_1 \leq \cdots \leq D_n = G$$

be the derived series of G. If G is polycyclic, then its finitary radical $\mathbf{Fin}(G)$ is also polycyclic and Lemma 4.1 shows that $\mathbf{Fin}(G)$ has finite central dimension, contradicting the hypothesis. Hence there is a natural number m such that the factor group G/D_m is polycyclic but the factor D_m/D_{m-1} is not finitely generated. Repeating the arguments of the proof of Theorem 5.61 we can construct a normal subgroup L such that G/L is abelian-by-finite and L/L' is not finitely generated. In particular L is not finitely generated, so that Lemma 5.49 and Corollary 5.9 show that the finitary radical of G contains L. If we suppose, for a contradiction, that L has finite central dimension, then since $\mathbf{Fin}(G)/L$ is finitely generated Lemma 4.1 implies that $\mathbf{Fin}(G)$ has finite central dimension, contrary to our hypothesis. It follows that L has infinite central dimension. Finally, let $g \notin \mathbf{Fin}(G)$. Using Corollary 5.48 we see that L satisfies the maximal condition on $\langle g\rangle$-invariant subgroups as required. □

As with the condition min-icd we give examples of soluble groups satisfying the maximal condition on subgroups but which have infinite central dimension. The examples constructed earlier in this chapter illustrate Theorem 5.60 which is the basic theorem in the case of groups which are not finitely generated. The following example illustrates Theorem 5.63.

Let F be a field whose multiplicative group is not periodic and let A be a vector space over F having the countably infinite basis $\{a_n | n \in \mathbb{N}\}$. In the multiplicative group of F choose an element γ of infinite order and consider the infinite matrix $M(\gamma) = [\alpha_{ij}]_{i,j\in\mathbb{N}}$ where

$$\alpha_{j,j} = \gamma^j \text{ for all } j \in \mathbb{N}$$
$$\alpha_{i,j} = 0 \text{ for all } i \neq j.$$

Thus

$$M(\gamma) = \begin{pmatrix} \gamma & 0 & 0 & 0\dots & 0 & 0 & 0\dots \\ 0 & \gamma^2 & 0 & \dots & 0 & 0 & 0\dots \\ 0 & 0 & \gamma^3 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0\dots & \gamma^n & 0 & \dots & \dots \\ 0 & 0 & 0 & 0\dots & \gamma^{n+1} & 0 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \end{pmatrix}.$$

Now let Σ denote the set of all matrices having the form

$$\begin{pmatrix} \alpha_1 & \beta & 0 & 0\dots & 0 & 0 & 0\dots \\ 0 & \alpha_2 & 0 & \dots & 0 & 0 & 0\dots \\ 0 & 0 & \alpha_3 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0\dots & \alpha_n & 0 & \dots & \dots \\ 0 & 0 & 0 & 0\dots & \alpha_{n+1} & 0 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \end{pmatrix},$$

where $0 \neq \alpha_i \in F$ for all i. In other words, if we let

$$D(\alpha_1, \dots, \alpha_n, \dots) = D(\alpha_n | n \in \mathbb{N})$$

be the diagonal matrix with main diagonal entries $\alpha_1, \dots, \alpha_n \dots$, then the set Σ consists of those matrices of the form $\beta E_{12} + D(\alpha_n | n \in \mathbb{N})$. We have

$$\begin{aligned}&(\beta E_{12} + D(\alpha_n | n \in \mathbb{N}))(\mu E_{12} + D(\nu_n | n \in \mathbb{N}))\\&= (\alpha_1 \mu + \beta \nu_2) E_{12} + D(\alpha_n \nu_n | n \in \mathbb{N}).\end{aligned}$$

Hence the subset Σ is closed under multiplication. In addition, we see that $\beta E_{12} + D(\alpha_n | n \in \mathbb{N})$ is nonsingular and

$$(\beta E_{12} + D(\alpha_n | n \in \mathbb{N}))^{-1} = -\alpha_1^{-1} \beta \alpha_2^{-1} E_{12} + D(\alpha_n^{-1} | n \in \mathbb{N}).$$

Hence the subset Σ_1 of Σ consisting of nonsingular matrices is a subgroup. As we saw in the earlier example, the set $T_{12} = \{t_{12}(\alpha) | \alpha \in F\}$ is a subgroup under multiplication and this subgroup is isomorphic to the additive group of the field F. Furthermore,

$$\begin{aligned}&(\beta E_{12} + D(\alpha_n | n \in \mathbb{N}))^{-1} t_{12}(\kappa)(\beta E_{12} + D(\alpha_n | n \in \mathbb{N}))\\&= t_{12}(\alpha_1^{-1} \kappa \alpha_2),\end{aligned}$$

which shows that T_{12} is a normal subgroup of Σ_1. In particular,

$$M(\gamma)^{-1} t_{12}(\kappa) M(\gamma) = t_{12}(\kappa\gamma).$$

Now let $\Gamma = \langle M(\gamma), t_{12}(1) \rangle$. Then $\Gamma = \Theta \langle M(\gamma) \rangle$ where Θ is isomorphic to a subgroup of the multiplicative group of the field F, generated by the elements γ^n for $n \in \mathbb{N}$. Clearly the subgroup Θ has finite central dimension and the subgroup $\langle M(\gamma) \rangle$ has infinite central dimension. Let H be an element of Γ. Suppose that $H = \beta E_{12} + D(\gamma^{kn} | n \in \mathbb{N}) \notin \Theta$ for some non-zero integer k. We see that the subgroup $\langle H \rangle$ has infinite central dimension and it follows that the finitary radical coincides with Θ. Let

$$\Lambda_1 \le \Lambda_2 \le \dots \le \Lambda_n \le \Lambda_{n+1} \le \dots \tag{5.2}$$

be an ascending chain of subgroups of Γ each having infinite central dimension. Since Θ has finite central dimension Θ contains no Λ_j for any $j \in \mathbb{N}$. If $\Lambda_j \cap \Theta = 1$ for all $j \in \mathbb{N}$, then $(\bigcup_{n \in \mathbb{N}} \Lambda_n) \cap \Theta = 1$ and it follows that $\bigcup_{n \in \mathbb{N}} \Lambda_n$ is an infinite cyclic group. In this case the ascending chain (5.2) must terminate. On the other hand suppose that there is a natural number m such that $\Theta_m = \Lambda_m \cap \Theta \ne 1$. Then $\Lambda_m = \Theta_m \langle H \rangle$ where $H = t_{12}(\tau) M(\gamma)^k$ for some $\tau \in F$ and some non-zero integer k. Let $X = M(\gamma)^k$. We can think of Θ as a cyclic $\mathbb{Z}\langle M(\gamma) \rangle$-module. Since the index $|M(\gamma) : M(\gamma)^k|$ is finite Θ is a finitely generated $\mathbb{Z}\langle X \rangle$-module. Since the group ring of an infinite cyclic group over the ring of integers is Noetherian it follows that the $\mathbb{Z}\langle X \rangle$-module Θ is also Noetherian. Note that every $\langle X \rangle$-invariant subgroup of Θ is also $\langle H \rangle$-invariant. In other words, the subgroup Θ satisfies the maximal condition on $\langle H \rangle$-invariant subgroups. It follows that the ascending chain

$$(\Theta \cap \Lambda_1)\langle H \rangle \le (\Theta \cap \Lambda_2)\langle H \rangle \le \dots \le (\Theta \cap \Lambda_n)\langle H \rangle \le (\Theta \cap \Lambda_1)\langle H \rangle \le \dots$$

cannot be infinite and hence the ascending chain (5.2) must terminate in finitely many steps. This means that the subgroup Γ satisfies the maximal condition on subgroups and has infinite central dimension giving the required example.

The structure of finite dimensional soluble linear groups is often defined by the structure of the multiplicative group of the field over which such subgroups are considered. More precisely, it is defined by not only the structure of this multiplicative group, but also by the structure of the multiplicative groups of finite extensions of the base field. So it is reasonable to expect that the same dependence takes place for infinite dimensional linear groups satisfying max-icd. Theorem 5.60 logically leads us to the consideration of fields satisfying the condition (RE) discussed earlier in this chapter.

Lemma 5.64. *Let A be a vector space over the field F which satisfies the condition (RE) and let G be a soluble subgroup of $GL(F, A)$. If G satisfies the condition max-icd, then either G has finite central dimension or the factor group G/G' is finitely generated.*

Proof. We may assume that G has infinite central dimension. Suppose the contrary and that the factor group G/G' is not finitely generated. Then Theorem 5.60 implies that the multiplicative group of some finite field extension of F contains a Prüfer q-subgroup for some prime q and this gives us our contradiction. □

Proposition 5.65. *Let A be a vector space over the field F which satisfies the condition (RE) and let G be a soluble subgroup of $GL(F, A)$. Suppose that G satisfies the condition max-icd.*

(i) If G is not finitely generated, then G has finite central dimension.

(ii) If G is finitely generated, then its finitary radical has finite central dimension.

Proof. Let

$$1 = D_0 \leq D_1 \leq \cdots \leq D_k = G$$

be the derived series of the group G. If every factor of this series is finitely generated, then G is polycyclic. In particular, G is finitely generated and its finitary radical is also finitely generated. Lemma 4.1 shows that in this case the finitary radical of G has finite central dimension.

Therefore we assume that there is a natural number t such that the factor group G/D_t is polycyclic, but the factor group D_t/D_{t-1} is not finitely generated. In particular $G = \langle D_t, S\rangle$ for some finite subset S. By Lemma 5.64 the subgroup D_t has finite central dimension. If G is not finitely generated, then Lemma 5.49 shows that G is a finitary linear group. Since $G = \langle D_t, S\rangle$ Lemma 4.1 then implies that G has finite central dimension. If G is finitely generated, then the finitary radical of G contains the subgroup D_t. Since the factor $\mathbf{Fin}(G)/D_t$ is finitely generated, Lemma 4.1 shows that $\mathbf{Fin}(G)$ has finite central dimension. □

We now consider one particular case of Theorem 5.60, but first we give the following result. As we have seen, the group G considered in Theorem 5.60 has a normal unipotent subgroup K such that G/K is abelian-by-finite and (finitely generated)-by-Prüfer. We consider now the particular case when G/K is Prüfer-by-finitely generated.

Lemma 5.66. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ satisfying the condition max-icd. Suppose that G contains subgroups H, V satisfying the following conditions:*

(i) H is a normal subgroup of V;

(ii) H is a nilpotent bounded p-subgroup for some prime p;

(iii) V/H is a Prüfer q-group for some prime $q \neq p$;

(iv) V has infinite central dimension.

Then H has a finite V-composition series.

Proof. The subgroup V contains a q-subgroup Q such that $V = HQ$ (see [100, Lemma 1.D.4], for example). Let

$$Q = \langle z_n | z_1^p = 1, z_{n+1}^p = z_n, n \in \mathbb{N}\rangle.$$

If the subgroup H is finite then the result follows immediately so we suppose that H is infinite. The fact that H is nilpotent implies that H/H' is infinite (see [183, Corollary of Theorem 2.26], for example)and hence not finitely generated. Corollary 5.44 shows that the subgroup H has finite central dimension. Since $V = HQ$, Lemma 4.1 and the hypotheses imply that the subgroup Q has infinite central dimension.

Being nilpotent and bounded, H has a finite series

$$1 = D_0 \leq D_1 \leq \cdots \leq D_{k-1} \leq D_k = H$$

of V-invariant subgroups whose factors are elementary abelian p-groups. Let $B = D_j/D_{j-1}$ be an arbitrary factor of this series. If z is an element of Q, then $B = C_B(z) \times [B, z]$ (see [116, Corollary 5.16], for example), Since Q is abelian, $C_B(z)$ and $[B, z]$ are Q-invariant subgroups of B. If y is also an element of Q such that $\langle z\rangle \leq \langle y\rangle$, then $C_B(y) \leq C_B(z)$ and $[B, z] \leq [B, y]$. Thus we have an ascending series

$$[B, z_1] \leq [B, z_2] \leq \cdots \leq [B, z_n] \leq \dots$$

of Q-invariant subgroups. By Corollary 5.48 there is a natural number m such that $[B, z_n] = [B, z_m]$ for all $n \geq m$. The equation $B = C_B(z_n) \times [B, z_n]$ implies that $C_B(z_n) = C_B(z_m)$ for all $n \geq m$. In other words B satisfies the minimal condition on centralizers. In this case B is a semisimple $\mathbb{F}_pQ$-module by [77, Theorem A]. By Corollary 5.48 B satisfies the maximal condition on

Q-invariant subgroups. Thus B is the direct product of finitely many minimal Q-invariant subgroups. Since this is true for each factor D_j/D_{j-1} for $1 \le j \le k$, H has a finite Q-composition series. Then H has a finite V-composition series. □

Theorem 5.67. *Let A be a vector space over the field F of prime characteristic p and let G be a soluble subgroup of $GL(F, A)$ satisfying the condition max-icd. Suppose that G has infinite central dimension and is not finitely generated. If the factor group $G/\boldsymbol{Tor}(G)$ is polycyclic, then G contains normal subgroups K, L satisfying the following conditions:*

(i) K is a bounded nilpotent p-subgroup;

(ii) K has finite central dimension and A is K-nilpotent;

(iii) K has a finite L-composition series;

(iv) $L = KQ$, where Q is a Prüfer q-group for some prime $q \neq p$ and Q has infinite central dimension;

(v) G/L is polycyclic and metabelian-by-finite.

Proof. By Lemma 5.49 G is a finitary linear group. Taking account of Theorem 5.61 we see that G contains normal subgroups K, L, W such that $K \le L \le W$, G/W is a finitely generated abelian-by-finite group, W/L is a finitely generated abelian group, L/K is a Prüfer q-group for some prime $q \neq p$, K is a bounded nilpotent p-subgroup and the vector space A is K-nilpotent. Then, since G/L is finitely generated we deduce that L has infinite central dimension. The subgroup L contains a q-subgroup Q such that $L = KQ$ (see [100, Lemma 1.D.4], for example). Clearly Q has infinite central dimension, since K has finite central dimension. Finally, Lemma 5.66 implies that K has a finite L-composition series. □

As a corollary we obtain a description of the periodic soluble groups satisfying the maximal condition on subgroups of infinite central dimension.

Theorem 5.68. *Let A be a vector space over the field F of prime characteristic p and let G be an infinite periodic soluble subgroup of $GL(F, A)$ satisfying the condition max-icd. Suppose that G has infinite central dimension. Then G contains a normal subgroup L satisfying the following conditions:*

(i) L has finite index in G;

(ii) $L = KQ$, where K is a normal bounded nilpotent p-subgroup and Q is a Prüfer q-group for some prime $q \neq p$;

(iii) K has finite central dimension;

(iv) A is K-nilpotent;

(v) K has a finite G-composition series;

(vi) Q has infinite central dimension.

Proof. Clearly G is a locally finite group and since it is infinite it cannot be finitely generated. Furthermore $G = \mathbf{Tor}(G)$ in this case, so we may apply Theorem 5.67.

□

Theorem 5.69. *Let A be a vector space over the field F of characteristic 0 and let G be an infinite periodic soluble subgroup of $GL(F, A)$ satisfying the condition max-icd. Suppose that G has infinite central dimension. Then G is a Prüfer-by-finite group.*

Proof. Since G is clearly locally finite it is not finitely generated. Then Theorems 5.60 and 5.61 give the result that G is Prüfer-by-finite. □

We note that the description given above of soluble periodic linear groups satisfying max-icd is not included in [127].

The question arises as to when the conditions given in Lemma 5.66 concerning the group G can actually arise in practice. In particular what types of field allow us to deduce that Theorem 5.67 follows? We explore one such possibility now. We let $U(E)$ denote the multiplicative group of a field E.

Accordingly, we shall say that a field F has the property (FAE) if it satisfies the following condition:

(FAE) for each finite field extension E of F the factor group $U(E)/\mathbf{Tor}(U(E))$ is free abelian.

In particular this property is required to hold for the field F itself of course.

Proposition 5.70. *Let A be a vector space over the field F of prime characteristic p and let G be a soluble subgroup of $GL(F, A)$ satisfying the condition max-icd. Suppose that G has infinite central dimension and that G is not finitely generated. If the field F satisfies the condition (FAE), then G is a group satisfying all the conditions of Theorem 5.67.*

Proof. Suppose first that G/G' is not finitely generated. By Lemma 5.56 G contains a normal subgroup S such that $G' \le S$, the factor S/G' is finitely generated and G/S is a Prüfer q-group for some prime q. Lemma 5.42 shows that the subgroup S has finite central dimension. Let $C = C_A(S)$ and note that the subspace C has finite codimension in A. Since S is a normal subgroup of G, the subspace C is G-invariant. Let $V = C_G(C)$ so that V is normal in G and $S \le V$. If we assume that $V = G$, then G has finite central dimension, contrary to the hypotheses. Hence V is a proper subgroup of G. It follows that the factor group V/S is finite and that G/V is a Prüfer q-group. The finiteness of $\mathbf{dim}_F(A/C)$ implies that the series $C \le A$ can be refined to a series

$$C = C_1 \le C_2 \le \cdots \le C_n = A$$

whose factors are G-chief for $1 \leq j \leq n-1$. Using Proposition 5.34 we deduce that the factor $G/C_G(C_{j+1}/C_j)$ contains an abelian normal subgroup $G_{j+1}/C_G(C_{j+1}/C_j)$ such that G/G_{j+1} is finite and

$$(G_{j+1}/C_G(C_{j+1}/C_j))/\mathbf{Tor}(G_{j+1}/C_G(C_{j+1}/C_j))$$

is free abelian for $1 \leq j \leq n-1$. Let

$$T_{j+1}/C_G(C_{j+1}/C_j) = \mathbf{Tor}(G_{j+1}/C_G(C_{j+1}/C_j))$$

for $1 \leq j \leq n-1$ and let

$$\begin{aligned} D &= G_2 \cap G_3 \cap \cdots \cap G_n, \\ T &= T_2 \cap T_3 \cap \cdots \cap T_n, \\ H &= C_G(C_2/C_1) \cap C_G(C_3/C_2) \cap \cdots \cap C_G(C_n/C_{n-1}). \end{aligned}$$

Then several applications of Remak's theorem yield embeddings as follows:

$$\begin{aligned} G/D &\leq \operatorname*{Dr}_{1\leq j\leq n-1} G/G_{j+1}, \\ D/T &\leq \operatorname*{Dr}_{1\leq j\leq n-1} G_{j+1}/T_{j+1}, \\ T/H &\leq \operatorname*{Dr}_{1\leq j\leq n-1} T_{j+1}/C_G(C_{j+1}/C_j) \end{aligned}$$

which in turn show that G/D is finite, D/T is free abelian and T/H is periodic. Let $K = H \cap V$. Again using Remak's theorem we obtain an embedding of T/K in the direct product $T/(V \cap T) \times T/H$ and this implies that T/K is periodic. As in the proof of Theorem 5.60 we can show that H is a p-group. It follows that T is likewise periodic. Then $G/\mathbf{Tor}(G)$ is (free abelian)-by-finite. In general we can apply the same reduction that was used in the proof of Theorem 5.61 and obtain the same result. Since $G/\mathbf{Tor}(G)$ has finite 0-rank it must be polycyclic, so G satisfies the conditions of Theorem 5.67. □

We next give some examples of fields satisfying the property (FAE) that were kindly provided to the authors by R. Guralnick, to whom we are indebted.

Proposition 5.71. *Let F be a finitely generated extension of a locally finite field L. Then $U(F)/\mathbf{Tor}(U(F))$ is free abelian.*

Proof. Note that the property (FAE) holds for any subfield. Thus there is no loss of generality if we replace L by K, the algebraic closure of L, and F by $P = FK$. We note that the field K is still locally finite and that the hypotheses still hold for P. In this case P is a finite extension of $K(x_1, \ldots x_r)$ where the x_j are algebraically independent for $1 \leq j \leq r$. We apply Noether's Lemma and see that we may choose the x_j so that P is the quotient field of the integral closure D of $K[x_1, \ldots, x_r]$.

Let V be the set of discrete valuations corresponding to the places of D. Then we have the map

$$1 \longrightarrow U(D) \longrightarrow U(P) \longrightarrow G$$

where $G \cong \bigoplus_{v\in V} Y_v$ and $Y_v \cong \mathbb{Z}$ for $v \in V$. Furthermore V is the set of valuations of P (given by $u \longmapsto \sum v(u)$). Now it is well-known that $U(D)/U(K)$ is a finitely generated abelian group (see [33]) and the image of the map is contained in the free abelian group G, and so is also free abelian. Thus the map splits and we see that $U(P)/U(K) \cong A \oplus B$ where A is finitely generated and B is free abelian. Since $U(K)$ is the torsion subgroup of $U(P)$ the result follows. □

Proposition 5.72. *Let F be a field and let $E = F(X_\lambda | \lambda \in \Lambda)$ be a field extension of F such that the elements $X_\lambda, \lambda \in \Lambda$ are algebraically independent. Then $U(E) \cong U(F) \times B$ where B is free abelian.*

Proof. We have the obvious F-valuations on E (corresponding to irreducible polynomials). This gives a map

$$1 \longrightarrow U(F) \longrightarrow U(E) \longrightarrow A$$

where A is free abelian. Thus $U(E) \cong U(F) \times B$ where B is the image of the map and so is free. □

Proposition 5.73. *Let L be a rational function field over a finite extension of the field $\mathbb{Q}$ of rational numbers and let F be a finite extension of L. Then $U(L)/T$ is free abelian, where T is the torsion subgroup of $U(L)$. Moreover, T is finite.*

Proof. We have $F = L(a)$ for some element a. The minimal polynomial of a involves only finitely many elements of the transcendence basis for L. So we see that F is also a rational function field over a field finitely generated over $\mathbb{Q}$ and so it suffices to prove the result for L finitely generated over $\mathbb{Q}$. However, the result is well known in this case, since there are enough discrete valuations. □

Antifinitary Linear Groups

As we remarked above, taking into account Lemma 5.1, a linear group G is finitary if and only if every finitely generated subgroup of G has finite central dimension. In other words a finitary linear group has a local family consisting of subgroups having finite central dimension.

Thus finitary groups give us one example of using finiteness conditions to study infinite dimensional linear groups, which for finitary linear groups has been very successful. A large array of articles have been devoted to finitary linear groups, establishing many interesting results and techniques. One very nice article detailing some of the stages of the development of the theory of finitary linear groups is the survey article [175] of R. E. Phillips. Some results on finitary linear groups have already been cited and used above. However, finitary groups have not really been a subject of special consideration in this book; indeed, the theory of finitary linear groups deserves a separate book of its own.

A local family of infinite groups, consisting of finitely generated subgroups, cannot really be described as "quite large" since the cardinality of such a local family will typically be the cardinality of the group, while in many infinite groups the cardinality of the family of all subgroups is larger than the cardinality of the group. These considerations naturally lead us to the following type of linear group.

Let G be a subgroup of $GL(F, A)$. Then G is called *antifinitary* if G has infinite central dimension, but every proper subgroup of G which is not finitely generated has finite central dimension. Our immediate goal will be the description of antifinitary linear groups. In the paper [107] of L. A. Kurdachenko, J. M. Muñoz-Escolano and J. Otal there is an alternative definition of antifinitary linear groups based on the augmentation dimension rather than the central dimension. As we saw in Chapter 4, these two dimensions of a linear group need not coincide. However, as we shall see later, the proofs that we obtain for our variation of antifinitary groups are virtually the same as the proofs given in [107] and the final results are the same. We also note here the survey paper of L. A. Kurdachenko [105].

One of the first questions that arises here is the question of the structure of groups all of whose proper subgroups are finitely generated. This question can be viewed as an extension of Schmidt's problem. At the beginning of this section we describe the structure of such groups with the additional condition that they are locally generalized radical.

We begin with several standard group theoretic results concerned with finitely generated groups.

Lemma 5.74. *Let G be a group whose proper subgroups are finitely generated. If G is not finitely generated, then G is directly indecomposable.*

Proof. If $G = A \times B$ for some proper nontrivial subgroups A, B, then A, B are finitely generated and hence G is also finitely generated. This is a contradiction which proves the result. □

Lemma 5.75. *Let G be a group whose proper subgroups are finitely generated. If G is abelian and not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. Let $T = \mathbf{Tor}(G)$, the torsion subgroup of G. If $T = G$, then each proper subgroup of G must be finite and G is then a Prüfer p-group for some prime p.

Hence we may suppose that T is a proper subgroup of G, so that T is finite. If T is nontrivial, then we may use [58, Theorem 27.5] (for example) to deduce that $G = T \times D$ for some subgroup D. This contradicts Lemma 5.74 and hence G must be torsion-free.

In the group G choose a maximal $\mathbb{Z}$-independent subset $\{a_\lambda | \lambda \in \Lambda\}$ and let $A = \langle a_\lambda | \lambda \in \Lambda \rangle$. Then $A = \underset{\lambda\in\Lambda}{\mathrm{Dr}} \langle a_\lambda \rangle$ and Lemma 5.74 shows that A is a proper subgroup of G. Hence A is finitely generated, so that G has finite 0-rank. The factor group G/A is periodic and the hypotheses on G imply that all proper subgroups of G/A are finite. This implies that G/A is a Prüfer group for some prime p. Let q be a prime different from p. Then $A^q \neq A$ and the factor A/A^q is finite. Furthermore, A/A^q is the q-component of the periodic abelian group G/A^q, so that $G/A^q = A/A^q \times P/A^q$ where P/A^q is the q' component of G/A^q. However all proper subgroups of G/A^q are finitely generated, whereas P/A^q is a Prüfer p-group which yields a contradiction. It follows that the case when G is torsion-free does not occur. The result follows. □

Lemma 5.76. *Let G be a group whose proper subgroups are finitely generated. If G is not finitely generated, then for every proper normal subgroup H the factor group G/H is not finitely generated. In particular G is an $\mathfrak{F}$-perfect group.*

Proof. Since H is a proper subgroup it is finitely generated. If G/H is finitely generated, then we obtain the contradiction that G too is finitely generated, since it is well-known that an extension of one finitely generated group by another such group is finitely generated. It is then clear that G has no proper subgroups of finite index. □

Lemma 5.77. *Let G be a group whose proper subgroups are finitely generated. If G is locally nilpotent and not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. Let $T = \mathbf{Tor}(G)$. Since G is locally nilpotent T is the set of all elements of G of finite order and G/T is torsion-free (see [52, Proposition 1.2.11], for example). If $T = G$, then each proper subgroup of G is finite and as we noted earlier in this chapter in this case G is a Prüfer p-group for some prime p.

Hence we may suppose that T is a proper subgroup of G. In this case, since T is a finitely generated periodic nilpotent group it is finite. The factor group G/T is not finitely generated and its proper subgroups are all finitely generated. For now we suppose that T is trivial and hence assume that G is torsion-free. Since G is locally nilpotent, it has a normal Kurosh-Chernikov system $\mathcal{S}$ whose terms are isolated and whose factors are central (see [62, Theorem 7], for example). If we suppose that $\mathcal{S} = \{\langle 1 \rangle, G\}$, then since $G/\langle 1 \rangle$ is central it follows that G is abelian and we obtain a contradiction to Lemma 5.75. Thus we may suppose that $\mathcal{S}$ contains some proper nontrivial terms and in particular we let $1 \neq H \in \mathcal{S}$ be such. Then H is a finitely generated nilpotent group and in particular H has finite 0-rank. Then, by [52, Theorem 2.3.1] (for example), it follows that for some natural number k some term $\zeta_k(G)$ of the upper central series of G contains H. In particular it follows that $Z = \zeta(G) \neq 1$. Using the previous arguments we deduce that the factor group G/Z either is abelian or itself has nontrivial centre C/Z. In the former case it follows that G is abelian and we again obtain a contradiction using Lemma 5.75. In the latter case we choose an element $g \in C \setminus Z$ so that $[g, G] \neq 1$. The inclusion $[g, G] \leq Z$ implies that $[g.G]$ is a finitely generated torsion-free abelian group. On the other hand we have $G/C_G(g) \cong [g, G]$ (see [52, Lemma 3.2.5], for example) and hence the factor group $G/C_G(g)$ is likewise a finitely generated torsion-free abelian group which contradicts Lemma 5.76. The result now follows in the general case also. □

Lemma 5.78. *Let G be a group whose proper subgroups are finitely generated. If G is radical and not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. Let R be the Hirsch-Plotkin radical of G. If $R = G$, then the result follows from Lemma 5.77 so we may suppose that R is a proper subgroup of G. Then the hypotheses imply that R is a finitely generated nilpotent subgroup of G and it follows that the torsion subgroup $\mathbf{Tor}(R)$ is finite. If $\mathbf{Tor}(R) \neq 1$, then the factor group $G/C_G(\mathbf{Tor}(R))$ is finite and Lemma 5.76 implies that $G = C_G(\mathbf{Tor}(R))$, so that $\mathbf{Tor}(R) \leq \zeta(G)$. In particular the subgroup $\zeta(G)$ is nontrivial.

Suppose next that $\mathbf{Tor}(R) = 1$, so that R is torsion-free. Then its centre Z is a free abelian group and has finite 0-rank. This implies that $Z \neq Z^p$ for each prime p and of course Z/Z^p is finite. Hence the factor group $G/C_G(Z/Z^p)$ is also finite and Lemma 5.76 again implies that $G = C_G(Z/Z^p)$. Hence in this case we have $[g, z] \in Z^p$ for all elements $g \in G, z \in Z$. This is true for each prime p and hence $[g, z] \in \bigcap_{p\in\mathbb{P}} Z^p = 1$. Hence we have $Z \leq \zeta(G)$, so that once again the centre of G is nontrivial.

It follows from these arguments and a simple mathematical induction that G is a hypercentral group and the result now follows from Lemma 5.77. □

Corollary 5.79. *Let G be a group whose proper subgroups are finitely generated. If G is locally soluble and not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. Suppose that G is non-abelian. Since G is locally soluble, G has a normal Kurosh-Chernikov system $\mathcal{S}$ whose factors are abelian (see [132, §6], for example). If we suppose that $\mathcal{S} = \{\langle 1\rangle, G\}$, then the factor $G/\langle 1\rangle$ is abelian, contrary to our assumption. Hence the system $\mathcal{S}$ contains a proper nontrivial subgroup H. Since $H \neq G$ it is a finitely generated subgroup of the locally soluble subgroup G, so H is soluble. It then follows that H contains a characteristic abelian subgroup A and since H is normal in G we see that A is normal in G. An easy mathematical induction shows that G must be hyperabelian and hence G is a radical group. The result now follows from Lemma 5.78. □

Proposition 5.80. *Let G be a group whose proper subgroups are finitely generated. If G is a locally generalized radical group and not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. It is not hard to see that G must be countable. Then G has an ascending chain of finitely generated subgroups

$$G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \leq \ldots$$

such that $G = \bigcup_{n \in \mathbb{N}} G_n$. Each of the subgroups G_n is finitely generated and furthermore, all subgroups of G_n are also finitely generated. It follows that G_n has the maximal condition on subgroups and hence is polycyclic-by-finite. Hence G_n has a greatest soluble normal subgroup S_n of finite index in G_n. Since S_2 is normal in G_2 it follows that S_1S_2 is also soluble and using mathematical induction it is then easy to see that $S_1S_2 \ldots S_n$ is soluble. Hence the union $S = \bigcup_{n \in \mathbb{N}} S_1S_2 \ldots S_n$ is a locally soluble subgroup of G. If $S = G$, then the result follows from Corollary 5.79. We may therefore suppose that S is a proper subgroup of G, so that S is finitely generated and all its subgroups are finitely generated. Hence S has the maximal condition and it follows that S is polycyclic. In particular S has finite 0-rank. Let H be an arbitrary proper subgroup of G. Then H is finitely generated, so $H \leq G_m$ for some natural number m. Since G_m/S_m is finite it follows that $H/(H \cap S_m)$ is finite and hence H has finite 0-rank at most r. An application of [52, Theorem 2.5.7] implies that G has finite 0-rank at most some natural number r_1. Then $G/\mathbf{Tor}(G)$ is soluble-by-finite (see [52, Theorem 2.4.13], for example). However, by Lemma 5.76, G is $\mathfrak{F}$-perfect so that $G/\mathbf{Tor}(G)$ is soluble. Corollary 5.79 shows that G is periodic and as we saw earlier in the chapter this means G is a Prüfer p-group, as required. □

We note that in the general case the conclusion of Proposition 5.80 no longer holds. The work of A. Yu. Olshanskii allows us to construct a corresponding counterexample. For this we need the following remarkable theorem of Olshanskii [162, Theorem 35.1]

Theorem 5.81. *Let $\{G_\lambda | \lambda \in \Lambda\}$ be a finite or countable set of nontrivial finite or countably infinite groups without involutions. Suppose $|\Lambda| \geq 2$ and that n is a sufficiently large odd number (at least 10^{75}). Suppose that if $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$, then $G_\lambda \cap G_\mu = 1$. Then there is a countable simple group $G = OG(G_\lambda | \lambda \in \Lambda)$ containing a copy of G_λ for all $\lambda \in \Lambda$ with the following properties:*

- *(i) If $x, y \in G$ and $x \in G_\lambda \setminus \{1\}, y \notin G_\lambda$, for some $\lambda \in \Lambda$, then G is generated by x and y;*
- *(ii) Every proper subgroup of G is either a cyclic group of order dividing n or is contained in some subgroup conjugate to some G_λ.*

Hence these groups of Olshanskii are 2-generator and have subgroups which are restricted by the choice of the constituent groups G_λ. An application of this theorem yields the following examples.

Proposition 5.82. *There is a non-periodic group G which is not finitely generated, but each proper subgroup of G is finitely generated.*

Proof. For each natural number n let H_n be a free abelian group having finite 0-rank n. Let $G_1 = H_1$, let $G_2 = OG(G_1, H_2)$ and, for $n \geq 2$, let $G_{n+1} = OG(G_n, H_{n+1})$. Note that

$$H_1 \lneqq H_2 \lneqq \cdots \lneqq H_n \lneqq \ldots$$

and let

$$G = \bigcup_{n \in \mathbb{N}} G_n.$$

We show that each G_n and each of it subgroups are finitely generated and to do this we use induction on n. Let L be a subgroup of G_2. If L is a proper subgroup of G_2, then Theorem 5.81 implies that either L is finite cyclic or that L is isomorphic to a subgroup of G_1 or H_2. In each case it is clearly true that L is finitely generated. If $L = G_2$, then L is finitely generated by construction. Hence the induction starts.

Assume now that $n \geq 2$ and that we have already proved that G_n is finitely generated. Let L be a subgroup of G_{n+1}. If L is a proper subgroup of G_{n+1}, then Theorem 5.81 shows that either L is finite cyclic or L is isomorphic to a subgroup of G_n or H_{n+1}, so L is finitely generated. If $L = G_{n+1}$, then L is finitely generated by construction.

Let $1 \neq K$ be a subgroup of G. Then the equality $G = \bigcup_{n \in \mathbb{N}} G_n$ implies that there exists an integer j such that $G_j \cap K \neq 1$. Let x be a nontrivial element of $G_j \cap K$. If $G_n \cap K \leq G_j$ for all $n \in \mathbb{N}$, then $K \leq G_j$ and our work above shows that K is finitely generated. Hence we may assume that there is a natural number $m > j$ such that $G_j \cap K \neq G_m \cap K$. Hence there is an element $y \in G_m \cap K$ such that $y \notin G_j \cap K$. By Theorem 5.81 it follows that $G_m = \langle x, y \rangle$. On the other hand $\langle x, y \rangle \leq K$, so we obtain $G_m \leq K$. If $G_n \cap K \leq G_m$ for all $m \in \mathbb{N}$, then $K = G_m$, but otherwise, as above, there is a natural number $s > m$ such that $G_m \cap K \neq G_s \cap K$ and we then deduce that $K = G_s$. Continuing this line of argument, we deduce that if K is a proper subgroup of G, then there is a natural number t such that $K \leq G_t$ and then K is finitely generated.

By construction $G_n \neq G_{n+1}$ for each natural number n. In particular it follows that the group G is not finitely generated. This completes the proof. □

With these preliminaries out of the way, we now proceed to the study of antifinitary linear groups. As usual we start with the simplest properties that such groups possess.

Lemma 5.83. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. If K, L are proper subgroups of G which are not finitely generated, then the subgroup $\langle K, L \rangle$ has finite central dimension.*

Proof. By definition, K, L have finite central dimension, so we can apply Lemma 4.1 to deduce the result. □

Lemma 5.84. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. If K is a proper subgroup of G such that the finitary radical of G does not contain K, then K is finitely generated.*

Proof. If K is not finitely generated, then as G is antifinitary, it follows that K has finite central dimension. Then $K \leq \mathbf{Fin}(G)$ and we obtain a contradiction to the hypotheses. □

Corollary 5.85. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Then every proper subgroup of $G/\boldsymbol{Fin}(G)$ is finitely generated.*

Proof. Let $D = \mathbf{Fin}(G)$ and let H/D be a proper nontrivial subgroup of G/D. Then $H \nleq D$, so H is finitely generated by Lemma 5.84 and hence its homomorphic image H/D is also finitely generated, as required. □

Corollary 5.86. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. If G is a locally generalized radical group, then $G/\boldsymbol{Fin}(G)$ is either a polycyclic-by-finite group or a Prüfer p-group for some prime p.*

Proof. Once again let $D = \mathbf{Fin}(G)$. By Corollary 5.85 each proper subgroup of G/D is finitely generated. If G/D is finitely generated, then since it satisfies the maximal condition it follows that G/D is polycyclic-by-finite. If G/D is not finitely generated, then Proposition 5.80 implies that this factor group must be a Prüfer p-group for some prime p. This completes the proof. □

Lemma 5.87. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. If G is soluble and the factor group $G/\boldsymbol{Fin}(G)$ is not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. Let $D = \mathbf{Fin}(G)$. Clearly we have $D \neq G$ and Corollary 5.86 implies that G/D is a Prüfer p-group for some prime p. If y is an arbitrary element of G such that $y \notin D$, then Lemma 5.84 implies that the subgroup $\langle y, D\rangle$ is finitely generated. Since G/D is a Prüfer p-group it follows that $\langle y, D\rangle/D$ is finite. Hence the subgroup D is finitely generated (see [52, Proposition 1.2.13], for example). Let

$$1 = D_0 \leq D_1 \leq \cdots \leq D_{n-1} \leq D_n = D$$

be the derived series of the subgroup D. The proof proceeds by induction on n. Suppose first that $n = 1$, so that the subgroup D is abelian and finitely generated. Then every subgroup of D is finitely generated. If K is a proper subgroup of G and D does not contain K, then Lemma 5.84 implies that K is finitely generated. Thus every proper subgroup of G is finitely generated. Using Proposition 5.80 we deduce that G is a Prüfer p-group.

Suppose now that $n > 1$ and that we have already proved that the factor group G/D_1 is a Prüfer p-group. In this case the factor D/D_1 is finite. Since D is finitely generated it follows that D_1 is finitely generated (again by [52, Proposition 1.2.13]). Then since D_1 is abelian all subgroups of D_1 are finitely generated. If K is a proper subgroup of G such that D does not contain K, then our arguments above imply that K is finitely generated. Thus every proper subgroup of G is finitely generated and again Proposition 5.80 enables us to deduce that G is a Prüfer p-group, as required. □

Theorem 5.88. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. If G is a locally generalized radical group and the factor group $G/\boldsymbol{Fin}(G)$ is not finitely generated, then G is a Prüfer p-group for some prime p.*

Proof. We let $D = \mathbf{Fin}(G)$. Clearly $D \neq G$ and we note that by Corollary 5.86 G/D is a Prüfer p-group for some prime p. If y is an arbitrary element of G such that $y \notin D$, then Lemma 5.84 implies that the subgroup $\langle y, D\rangle$ is finitely generated. Since G/D is a Prüfer p-group it follows that $\langle y, D\rangle/D$ is finite. Hence the subgroup D is finitely generated (see [52, Proposition 1.2.13], for example). It follows that D has finite central dimension. By Lemma 4.1 D contains a normal nilpotent subgroup L such that the factor group D/L is isomorphic to some subgroup of $GL_m(F)$, where $m = \mathbf{centdim}_F(D)$. Since D/L is generalized radical it contains no nonabelian free subgroups, so D/L is soluble-by-finite (see [202, Theorem 10.16], for example). Suppose, for a contradiction, that D/L is not soluble and let S/L be the soluble radical of D/L. Then G/S is not soluble and D/S is finite. In particular every subgroup of D/S is finitely generated. If K is a proper subgroup of G containing S and K is not contained in D, then Lemma 5.84 implies that K is finitely generated. Then K/S is finitely generated also and this shows that every subgroup of G/S is finitely generated. Using Proposition 5.80 we see that G/S is a Prüfer p-group, so that G and hence D is soluble. With this contradiction we see that D/L is soluble. In turn, it follows that D is soluble and hence so is G. Lemma 5.87 may now be applied to deduce the result. □

We now turn to the case when $G/\mathbf{Fin}(G)$ is finitely generated, but G is not finitely generated.

Lemma 5.89. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that $G \neq \boldsymbol{Fin}(G)$ and the factor group $G/\boldsymbol{Fin}(G)$ is finitely generated. If G is not finitely generated, then the factor group $G/\boldsymbol{Fin}(G)$ is finite and has prime order p.*

Proof. Let $D = \mathbf{Fin}(G)$. Since the factor group G/D is finitely generated, there is a finite subset X of G such that $G = \langle X\rangle D$. Let

$$\mathcal{S} = \{Y | Y \text{ is a finite subset of } G \text{ such that } G = \langle Y\rangle D\}$$

and in the family $\mathcal{S}$ choose a subset M which has the smallest size.

Suppose that $|M| \geq 2$ so that M has two non-empty subsets X, Y such that $M = X \cup Y$ and $X \cap Y = \emptyset$. The choice of M implies that $\langle X \rangle D$ and $\langle Y \rangle D$ are proper subgroups of G. If $\langle X \rangle D = D$, then we have $\langle M \rangle D = \langle Y \cup X \rangle D = \langle Y \rangle D$, contrary to the choice of M. This contradiction implies that $\langle X \rangle D \neq D$ and by similar reasoning we also have $\langle Y \rangle D \neq D$. Lemma 5.84 then implies that the subgroups $\langle X, D \rangle$ and $\langle Y, D \rangle$ are finitely generated. Then, since $M = X \cup Y$ we deduce that $G = \langle M, D \rangle = \langle X, Y, D \rangle$ is finitely generated, contrary to the hypotheses on G. This contradiction shows that M contains just one element and hence the factor group G/D is cyclic.

If G/D is infinite cyclic or G/D is finite cyclic, with non-prime order, then G/D contains a proper nontrivial subgroup K/D. Again using Lemma 5.84 we see that K is finitely generated. However in either case the index $|G : K|$ is finite which implies that G is also finitely generated, again giving a contradiction. Thus the factor group G/D has prime order and the proof is complete. $\square$

Lemma 5.90. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and $K/H = \operatorname{Dr}_{\lambda \in \Lambda} K_\lambda/H$, where $K_\lambda \neq H$ for each λ in the index set Λ. If Λ is infinite, then K has finite central dimension.*

Proof. The fact that Λ is infinite ensures the existence of infinite subsets Γ and Δ of Λ such that

$$\Gamma \cup \Delta = \Lambda \text{ and } \Gamma \cap \Delta = \emptyset.$$

It follows that the subgroups

$$K_\Gamma/H = \operatorname*{Dr}_{\lambda \in \Gamma} K_\lambda/H \text{ and } K_\Delta/H = \operatorname*{Dr}_{\lambda \in \Delta} K_\lambda/H$$

are not finitely generated. Therefore K_Γ and K_Δ are not finitely generated either. However, K_Γ, K_Δ are proper subgroups of G so they must then have finite central dimension. Then the equality $K = K_\Gamma K_\Delta$ and Lemma 5.83 together imply that K has finite central dimension, as required. $\square$

Corollary 5.91. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and $K/H = \operatorname{Dr}_{\lambda \in \Lambda} K_\lambda/H$, where $K_\lambda \neq H$ for each λ in the infinite index set Λ. If g is an element of G such that K_λ is $\langle g \rangle$-invariant for each $\lambda \in \Lambda$, then $g \in \boldsymbol{Fin}(G)$.*

Proof. The fact that K_λ is $\langle g \rangle$-invariant for each $\lambda \in \Lambda$ implies that the subgroups H, K are also $\langle g \rangle$-invariant. Since the index set Λ is infinite it contains a subset Γ such that $\Lambda \setminus \Gamma$ is finite and $\langle gH \rangle \cap \operatorname{Dr}_{\lambda \in \Gamma} K_\lambda/H = 1$. Furthermore Γ contains an infinite subset Δ such that the subgroup $L/H = (\operatorname{Dr}_{\lambda \in \Delta} K_\lambda/H) \rtimes \langle gH \rangle$ is proper. Then Lemma 5.90 implies that the subgroup

L has finite central dimension. Since $\langle g\rangle$ is a subgroup of L it also has finite central dimension and hence $g \in \mathbf{Fin}(G)$ as required. □

The next result is then very easy to deduce.

Corollary 5.92. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and $K/H = \underset{\lambda\in\Lambda}{Dr}\, K_\lambda/H$, where $K_\lambda \neq H$ for each λ in the infinite index set Λ. If K_λ is a G-invariant subgroup for each $\lambda \in \Lambda$, then the group G is finitary.*

Lemma 5.93. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and K/H is an infinite elementary abelian p-group for some prime p. If g is an element of G such that H, K are both $\langle g\rangle$-invariant and $g^k \in C_G(K/H)$ for some natural number k, then $g \in \boldsymbol{Fin}(G)$.*

Proof. It follows from Lemma 5.12 that K contains a family of $\langle g\rangle$-invariant subgroups

$$\{B_j | H \leq B_j \text{ and } B_j/H \text{ is finite for each } j \in \mathbb{N}\}$$

such that $\langle B_j/H | j \in \mathbb{N}\rangle = \underset{j\in\mathbb{N}}{\mathrm{Dr}}\, B_j/H$. Then Corollary 5.91 implies that $g \in \mathbf{Fin}(G)$. □

Corollary 5.94. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and K/H is a periodic (locally soluble)-by-finite group. If K/H is not a Chernikov group, then $K \leq \boldsymbol{Fin}(G)$.*

Proof. Let L/H be a normal locally soluble subgroup of K/H which has finite index in K/H. Since the section K/H is not Chernikov, neither is L/H. Let g be an element of K. Then a result of D. I. Zaitsev [224] shows that the section L/H contains a $\langle g\rangle$-invariant abelian subgroup A/H which is also not Chernikov. If the set $\Pi(A/H)$ is infinite, then Corollary 5.91 implies that $g \in \mathbf{Fin}(G)$. Suppose that the set $\Pi(A/H)$ is finite. Then there is a prime p such that the p-component P/H of A/H is not Chernikov. Then the lower layer of P/H, $\boldsymbol{\Omega}_1(P/H)$, is an infinite elementary abelian p-group. Clearly $\boldsymbol{\Omega}_1(P/H)$ is $\langle g\rangle$-invariant and Lemma 5.93 implies that $g \in \mathbf{Fin}(G)$. □

Corollary 5.95. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that H, K are subgroups of G such that H is normal in K and K/H is a locally finite group. If K/H is not a Chernikov group, then $K \leq \boldsymbol{Fin}(G)$.*

Proof. Let g be an arbitrary element of K and consider the subgroup $C/H = C_{K/H}(gH)$. If C/H is not a Chernikov group, then it contains an abelian subgroup A/H which is also not Chernikov (see [100, Theorem 5.8], for example). It easily follows that A/H contains a subgroup D/H which is a direct product of infinitely many cyclic subgroups and an application of Corollary 5.91 shows that $g \in \mathbf{Fin}(G)$ in this case.

Suppose now that the section C/H is Chernikov. Then by a result of B. Hartley [79] K/H is a (locally soluble)-by-finite group and since it is periodic we may apply Corollary 5.94. This again implies that $g \in \mathbf{Fin}(G)$. □

Immediately we may deduce the following interesting result.

Corollary 5.96. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite locally finite group. If G is not a Chernikov group, then G is a finitary linear group.*

We now need some further terminology. Let G be a group and let A be a periodic abelian normal subgroup of G. Then A is said to be *G-quasifinite* if A is infinite but every proper G-invariant subgroup of A is finite.

It is not hard to see that a G-quasifinite subgroup A must be a p-group for some prime p. Then either A^p is a proper subgroup of A or $A = A^p$. In the first case A^p must be finite and it follows that A must be a bounded p-group. It then follows that A is a direct product of cyclic p-subgroups (see the book of L. Fuchs [58, Theorem 17.2], for example). In this case it also then follows that $|A| = |\mathbf{\Omega}_1(A)|$, so that $\mathbf{\Omega}_1(A)$ is infinite. This means that $\mathbf{\Omega}_1(A) = A$ and hence A is an infinite elementary abelian p-group.

If $A = A^p$, then the subgroup A must be divisible and in this case the subgroup $\mathbf{\Omega}_1(A)$ is finite. Then A is a Chernikov p-subgroup (see the book [52, Corollary 3.1.5], for example).

It is now possible to begin a deeper investigation into the structure of antifinitary linear groups. When G is infinite and locally finite, but $G/\mathbf{Fin}(G)$ is finitely generated, then of course $G/\mathbf{Fin}(G)$ is finite.

Proposition 5.97. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite locally finite group and that the factor group $G/\mathbf{Fin}(G)$ is a nontrivial finite group. Then G satisfies the following conditions:*

(i) *$G/\mathbf{Fin}(G)$ has prime order p;*

(ii) *$G = K\langle g\rangle$ where K is a normal subgroup of G and g is a p-element;*

(iii) *K is a q-subgroup for some prime q; moreover $K = \underset{1\le j\le m}{Dr}\, C_j$ where C_j is a Prüfer q-subgroup for $1 \le j \le m$;*

(iv) *K is G-quasifinite;*

(v) *if $q = p$, then $m = p^k(p-1)$, where $p^k = |\langle g\rangle/C_{\langle g\rangle}(K)|$;*

(vi) *if $q \ne p$, then m is the order of q modulo p^k and $p^k = |\langle g\rangle/C_{\langle g\rangle}(K)|$.*

Proof. Corollary 5.96 implies that G must be a Chernikov group, since $G/\mathbf{Fin}(G)$ is nontrivial. Using Lemma 5.89 we see that the factor group $G/\mathbf{Fin}(G)$ has prime order p. Hence there is a p-element g such that $G = \langle g\rangle\mathbf{Fin}(G)$. This proves (i) and (ii). Let K denote the divisible part of the group G. Since K is an infinite Chernikov group it is not finitely generated and the subgroup $\langle g^p\rangle K$ is a proper subgroup of G that is also not finitely generated. Hence, since G is antifinitary, $\langle g^p\rangle K$ has finite central dimension so $\mathbf{Fin}(G) = \langle g^p\rangle K$.

Suppose that K contains a proper infinite G-invariant subgroup L. Then $\langle g\rangle L$ is a proper subgroup of G and since it is not finitely generated it follows that $\langle g\rangle L$ has finite central dimension. In this case however Lemma 5.1 implies that $\langle g\rangle$ has finite central dimension also. Then the equality $G = \langle g\rangle K$ and Lemma 4.1 together imply that G has finite central dimension, contrary to the hypotheses on G. This contradiction shows that the subgroup K is G-quasifinite which proves (iv). In particular, it follows that K is a q-subgroup for some prime q which proves (iii). Finally assertion (v) follows from the results of [220, Section 3] while (vi) follows from [78, Theorem 3.4]. □

In the last result the group $G/\mathbf{Fin}(G)$ was finite. Now we assume that G is not finitely generated, but that $G/\mathbf{Fin}(G)$ is.

Lemma 5.98. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that the group G is not finitely generated and that $D = \boldsymbol{Fin}(G) \neq G$. Suppose also that the factor group G/D is finitely generated. Then D/D' is a Chernikov group.*

Proof. Let $K = D'$, the derived subgroup of D. Clearly we may suppose that $D \neq K$. It follows from Lemma 5.89 that the factor group G/D has prime order p. Then we can choose an element g such that $G = \langle g\rangle D$. Let $T/K = \mathbf{Tor}(D/K)$ and suppose, for a contradiction, that $D \neq T$.

First we suppose that the factor group D/T is finitely generated, so that there exists a finite subset S such that $D = \langle S, T\rangle$. If we now suppose that the subgroup $\langle g, T\rangle$ is finitely generated, then we deduce that $G = \langle g, D\rangle = \langle g, S, T\rangle$ is finitely generated which contradicts the hypotheses on G. Hence $\langle g, T\rangle$ is not finitely generated. Furthermore, if we suppose that $\langle g, T\rangle$ is a proper subgroup of G, then it has finite central dimension, since G is antifinitary. Then Lemma 5.1 implies that $g \in D = \mathbf{Fin}(G)$, again a contradiction. Therefore $G = \langle g, T\rangle$. Since $g^p \in D$ it follows that $D = \langle g^p\rangle T$. If now we suppose that g has finite order, then the element g^p has finite order also and hence $g^p \in T$. This implies that $D = T$, contradicting our supposition that $D \neq T$. Hence the element g has infinite order. Since the factor group D/T is torsion-free it is clear that $\langle g\rangle \cap T = 1$. Let q be a prime different from p. Then the subgroup $\langle g^q\rangle T$ is a proper subgroup of D. It cannot be finitely generated otherwise G would also be finitely generated. Thus since G is antifinitary the subgroup $\langle g^q\rangle T$ has finite central dimension and Lemma 5.1 implies that $g^q \in D = \mathbf{Fin}(G)$. The choice of q implies that $\langle g\rangle = \langle g^q\rangle\langle g^p\rangle$ so that $g \in \mathbf{Fin}(G)$, yet another contradiction.

It follows that the factor D/T is not finitely generated. Suppose, for a further contradiction, that D/T has infinite 0-rank. Then there is a subgroup B/T of D/T such that $\langle gT\rangle \cap B/T = 1$ and the section D/B is a torsion-free group of finite 0-rank. Since $g^p \in D$, the family $\{B^x/T | x \in \langle g\rangle\}$ is finite and we let

$$\{B^x/T | x \in \langle g\rangle\} = \{B_1/T, \ldots, B_n/T\}.$$

For each $x \in \langle g\rangle$ the isomorphism $D/B^x = D^x/B^x \cong D/B$ implies that every section D/B_j is a torsion-free abelian group having finite 0-rank for $1 \leq j \leq n$. Let

$$C/T = \cap_{x\in\langle g\rangle} B^x/T = B_1/T \cap \cdots \cap B_n/T.$$

The subgroup C is $\langle g\rangle$-invariant and Remak's theorem gives an embedding from D/C into the direct product $D/B_1 \times \cdots \times D/B_n$. This shows that D/C is a torsion-free abelian group of finite 0-rank and hence the subgroup C/T has infinite 0-rank. The fact that $g^p \in D$ implies that the subgroup $\langle c^x T | x \in \langle g\rangle\rangle$ is finitely generated for each element $c \in C$. Hence the subgroup $\langle gT, C/T\rangle$ is not finitely generated and it follows that its preimage $\langle g, C\rangle$ also is not finitely generated. The choice of C shows that $\langle g, C\rangle$ is a proper subgroup of G and since G is antifinitary we deduce that $\langle g, C\rangle$ has finite central dimension. Lemma 5.1 implies that $g \in \mathbf{Fin}(G)$ which again yields a contradiction. This contradiction shows that the factor D/T has finite 0-rank.

Next we choose a subgroup U of D such that $T \leq U$, U/T is a free abelian group of finite 0-rank and D/U is periodic. Let $V/T = (U/T)^{G/T}$. Since G/D is finite, the factor V/T is finitely generated, so Corollary 5.94 implies that G/V is a Chernikov group. It follows that the factor group D/T is minimax.

Suppose that $\langle gT\rangle \cap D/T = 1$. Since D/T is minimax, there is a prime number r such that $D_1/T = (D/T)^r \neq D/T$. Then

$$\langle gT, D_1/T\rangle \neq \langle gT, D/T\rangle = G/T,$$

so that the subgroup $\langle g, D_1\rangle$ is a proper subgroup of G. The fact that D/T is minimax implies that the factor D/D_1 is finite and hence $\langle g, D_1\rangle$ has finite index in G. Hence $\langle g_1, T\rangle$ is not finitely generated so must have finite central dimension, since G is antifinitary. Thus the subgroup $\langle g\rangle$ has finite central dimension by Lemma 5.1 and this is yet another contradiction.

Hence the intersection $\langle gT\rangle \cap D/T$ is nontrivial. In particular, the element gT has infinite order and $C_{D/T}(gT)$ is nontrivial. Since D/T is torsion-free abelian $Z/T = C_{D/T}(gT)$ is a pure subgroup of D/T. Suppose that Z/T is finitely generated so that the factor D/Z is torsion-free and not finitely generated. Since $\langle g\rangle Z \cap D/Z = 1$ we may repeat the arguments above and obtain a contradiction. This contradicition shows that Z/T is not finitely generated.

In this case the section $\langle g, Z\rangle/T$ is also not finitely generated and we then deduce that the subgroup $\langle g, Z\rangle$ is likewise not finitely generated. If we suppose that $\langle g, Z\rangle$ is a proper subgroup of G, then as above we obtain a contradiction which shows that $G = \langle g, Z\rangle$. This shows, in other words, that the factor group G/T is a torsion-free abelian minimax group.

Let
$$P/\langle g, T\rangle = \mathbf{Tor}(G/\langle g, T\rangle)$$
and choose primes t, s such that $s, t \notin \Pi(G/\langle g, T\rangle)$ and $t \neq s$.Then $\langle g, T\rangle/\langle g^s, T\rangle$ is the finite s-component of $G/\langle g^s, T\rangle$ and $\langle g, T\rangle/\langle g^t, T\rangle$ is the finite t-component of $G/\langle g^t, T\rangle$. Then
$$G/\langle g^s, T\rangle = \langle g, T\rangle/\langle g^s, T\rangle \times U/\langle g^s, T\rangle \text{ and}$$
$$G/\langle g^t, T\rangle = \langle g, T\rangle/\langle g^t, T\rangle \times V/\langle g^t, T\rangle$$
for certain subgroups U, V of G (using [58, Theorem 27.5], for example). Clearly the subgroups U and V have finite index in G, so neither of them are finitely generated. Since G is antifinitary it follows that U, V have finite central dimension and Lemma 5.1 implies that $g^s, g^t \in \mathbf{Fin}(G)$. However, our choice of s, t implies that $\langle g\rangle = \langle g^s\rangle\langle g^t\rangle$ from which it follows that $g \in \mathbf{Fin}(G)$ also, contrary to the choice of g. This final contradiction implies that D/D' must be periodic, Then an application of Corollary 5.94 shows that D/D' must be Chernikov. □

Corollary 5.99. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that the group G is not finitely generated and that $\boldsymbol{Fin}(G) \neq G$. Suppose also that the factor group $G/\boldsymbol{Fin}(G)$ is finitely generated. If R is the finite residual of G, then $G \neq R$ and G/R is a finite cyclic p-group for some prime p. In particular, R is an $\mathfrak{F}$-perfect group.*

Proof. Let $D = \mathbf{Fin}(G)$. Using Lemma 5.89 we see that the factor group G/D is finite and its order is a prime p. Since $G \neq D$, the finiteness of G/D implies that $G \neq R$. Let g be an element of G such that $G = \langle g\rangle D$. If H is a normal subgroup of G of finite index, then the subgroup $\langle g\rangle H$ also has finite index and hence $\langle g\rangle H$ is not finitely generated. Since $g \notin D$ and G is antifinitary we have that $G = \langle g\rangle H$ and hence the factor group G/H is abelian. Using Remak's theorem we deduce that the factor group G/R is also abelian. Since G/D is finite we also have $R \leq D$. If $R = D$, then the result follows. If $R \neq D$, then $D' \leq G' \leq R$ and Lemma 5.98 implies that G/D' is a Chernikov group. However G/R is residually finite and also Chernikov, so G/R must be finite. Then $\langle g\rangle R$ is not finitely generated and as above $G = \langle g\rangle R$. It is easy to see that the factor group G/R must be a p-group. This completes the proof. □

Corollary 5.100. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that the group G is not finitely generated and that $\boldsymbol{Fin}(G) \neq G$. Suppose also that the factor group $G/\boldsymbol{Fin}(G)$ is finitely generated. If R is the finite residual of G and $R \neq R'$, then G/R' is an infinite Chernikov group.*

Proof. The proof of this result simply repeats the arguments given in the proof of Lemma 5.98. □

Proposition 5.101. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that the group G is not finitely generated and that $\boldsymbol{Fin}(G) \neq G$. Suppose also that the factor group $G/\boldsymbol{Fin}(G)$ is finitely generated. If G is a locally generalized radical group, then G is locally finite.*

Proof. Let R be the finite residual of the group G. By Corollary 5.99 the factor group G/R is finite cyclic and has order p^n for some prime p. Hence we can choose an element g such that $G = \langle g \rangle R$. Since G/R is finite and G is not finitely generated, R isn't finitely generated either. Since G is antifinitary it follows that R has finite central dimension and, since $G \neq \mathbf{Fin}(G)$, Lemma 4.1 implies that $g \notin \mathbf{Fin}(G)$.

Let $C = C_A(R)$ so that C has finite codimension in A. Let $n = \mathbf{dim}_F(A/C)$ and let $H = C_R(A/C)$. Then we may think of R/H as a locally generalized radical subgroup of $GL_n(F)$. Since a locally generalized radical group contains no nonabelian free subgroups it follows using the Tits alternative that R/H contains a normal soluble subroup K/H such that R/K is locally finite (see [202, Corollary 10.17], for example). Lemma 4.1 shows that the subgroup H is abelian and hence K is also soluble. Since G/R is finite $L = K^G$ is a product of finitely many soluble normal subgroups of R. Hence L is a soluble G-invariant subgroup and the factor G/L is locally finite.

Let

$$1 = L_0 \leq L_1 \leq \cdots \leq L_{k-1} \leq L_k = L$$

be a series of G invariant subgroups of L whose factors are abelian and without loss of generality we may suppose that each factor in this series either is periodic or is torsion-free.

Assume that L is not periodic. Then there is a natural number m such that the factor L_{m+1}/L_m is torsion-free but the factor L/L_{m+1} is periodic, so that R/L_{m+1} is then locally finite. By Corollary 5.95 the factor group G/L_{m+1} must be Chernikov. Corollary 5.99 implies that R is an $\mathfrak{F}$-perfect group which means that R/L_{m+1} is a divisible Chernikov group. Since the factor L_{m+1}/L_m is torsion-free abelian, Corollary 5.100 shows that $R \neq L_{m+1}$ and in turn this implies that the subgroup $\langle g, L_{m+1} \rangle$ is proper. Since $g \notin \mathbf{Fin}(G)$, Lemma 5.84 and the fact that G is antifinitary together imply that $\langle g, L_{m+1} \rangle$ is a finitely generated subgroup of G. Hence the section $U/L_m = \langle g, L_{m+1} \rangle / L_m$ is finitely generated. Then U/L_{m+1} is finitely generated and also locally finite so U/L_{m+1} is finite. Then L_{m+1}/L_m is finitely generated (see [52, Proposition 1.2.13], for example). Since L_{m+1}/L_m is torsion-free abelian we can identify $G/C_G(L_{m+1}/L_m)$ with a subgroup of $GL_r(\mathbb{Z})$ where r is the 0-rank of L_{m+1}/L_m. It is well-known that the periodic subgroups of $GL_r(\mathbb{Z})$ are finite (see [202, Theorem 9.33], for example). We already know that the factor group G/L_{m+1} is locally finite so the factor group $G/C_G(L_{m+1}/L_m)$ is finite. It follows that $R \leq C_G(L_{m+1}/L_m)$. In other words the factor L_{m+1}/L_m is central in R. Then $(R/L_m)'$ is periodic (see [52, Corollary 1.5.17], for example). It follows that R has a non-periodic abelian factor group, yielding a contradiction with Corollary 5.100. This contradiction shows that the subgroup L is periodic and hence G must also be periodic, so locally finite. □

We summarize the preceding results in the following theorem.

Theorem 5.102. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite locally generalized radical group such that $\boldsymbol{Fin}(G) \neq G$.*

(a) If the factor group $G/\boldsymbol{Fin}(G)$ is not finitely generated, then G is a Prüfer p-group for some prime p.

(b) If $G/\boldsymbol{Fin}(G)$ is finitely generated, but G is not finitely generated, then G satisfies the following conditions:

(i) $G/\boldsymbol{Fin}(G)$ has prime order p, for some prime p;

(ii) $G = K\langle g\rangle$ for some normal subgroup K of G and some p-element g;

(iii) there is a prime q, a natural number m and Prüfer q-subgroups C_j, for $j = 1, \ldots, m$ such that $K = \underset{1\leq j\leq m}{Dr} C_j$;

iv) K is G-quasifinite;

((v) if $q = p$, then $m = p^k(p-1)$ where $p^k = |\langle g\rangle/C_{\langle g\rangle}(K)|$;

(vi) if $q \neq p$, then m is the order of q modulo p^k and $p^k = |\langle g\rangle/C_{\langle g\rangle}(K)|$.

Proof. (a) If the factor group $G/\mathbf{Fin}(G)$ is not finitely generated, then Theorem 5.88 implies that G is a Prüfer p-group for some prime p.

(b) If $G/\mathbf{Fin}(G)$ is finitely generated, then Proposition 5.101 implies that the group G is locally finite and Proposition 5.97 can then be applied to give the result. □

The final set of results in this section is concerned with the finitely generated antifinitary linear groups. We start with the following result.

Lemma 5.103. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite finitely generated radical group. Then the finitary radical of G has finite central dimension and G is soluble.*

Proof. Since G is finitely generated, it cannot be a finitary linear group so $G \neq \mathbf{Fin}(G)$. Then Corollary 5.86 implies that the factor group $G/\mathbf{Fin}(G)$ is polycyclic. If $\mathbf{Fin}(G)$ is not finitely generated it follows that $\mathbf{Fin}(G)$ has finite central dimension, as G is antifinitary. If $\mathbf{Fin}(G)$ is finitely generated, then it has finite central dimension by Lemma 5.1. The fact that G is soluble follows from Lemma 5.15. □

Lemma 5.104. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite finitely generated soluble group. Suppose that U, V are normal subgroups of G such that $U \leq V$ and the factor V/U is abelian. If $g \notin \boldsymbol{Fin}(G)$, the $\mathbb{Z}\langle g\rangle$-module V/U is finitely generated.*

Proof. Suppose the contrary and let V/U be a $\mathbb{Z}\langle g\rangle$-module that is not finitely generated. Then there is an infinite subset of elements $\{v_n | n \in \mathbb{N}\}$ of V such that

$$\langle v_1U, \dots, v_nU\rangle^{\langle g\rangle} \neq \langle v_1U, \dots, v_nU, v_{n+1}U\rangle^{\langle g\rangle}$$

for each natural number n. Let

$$V_n = \langle v_1U, \dots, v_nU\rangle^{\langle g\rangle} \text{ for each } n \in \mathbb{N} \text{ and } W = \cup_{n\in\mathbb{N}} V_n.$$

Suppose, for a contradiction, that $\langle W/U, gU\rangle$ is finitely generated. In this case there exist elements $w_1, \dots, w_m \in W$ such that $W/U = \langle w_1U, \dots, w_mU\rangle^{\langle g\rangle}$ by [5]. By the construction of W it follows that there is a natural number t such that the subgroup V_t contains all the elements $w_1, \dots, w_m$. Since the subgroups V_n are $\langle g\rangle$-invariant we deduce that

$$\langle w_1U, \dots, w_mU\rangle^{\langle g\rangle} \leq V_t/U \neq V_{t+1}/U.$$

On the other hand,

$$V_{t+1}/U \leq W/U = \langle w_1U, \dots, w_mU\rangle^{\langle g\rangle}$$

which gives the contradiction sought. It follows that $\langle W/U, gU\rangle$ is not finitely generated. Hence the subgroup $\langle W, u\rangle$ isn't finitely generated either and since G is finitely generated we see that $\langle W, g\rangle$ is a proper subgroup of G. But G is antifinitary so $\langle W, g\rangle$ has finite central dimension and hence Lemma 5.1 implies that $\langle g\rangle$ also has finite central dimension. Thus $g \in \mathbf{Fin}(G)$, contrary to the choice of g. This contradiction proves the result. □

Our final result in this chapter gives some details concerning the structure of a finitely generated antifinitary group.

Theorem 5.105. *Let A be a vector space over the field F and let G be an antifinitary subgroup of $GL(F, A)$. Suppose that G is an infinite finitely generated radical group such that $G \neq \boldsymbol{Fin}(G)$ Then the group G satisfies the following conditions.*

(i) *$\boldsymbol{Fin}(G)$ has finite central dimension;*

(ii) *G contains a normal subgroup U such that G/U is polycyclic;*

(iii) *there is a natural number m such that $(x - 1)^m = 0$ for each element $x \in U$, so that every element of U is unipotent and the subgroup U is nilpotent;*

(iv) *if* $\textbf{\textit{char}}(F) = 0$, *then* U *is torsion-free and if* $\textbf{\textit{char}}(F)$ *is a prime* p, *then* U *is a bounded subgroup;*

(v) U *satisfies the maximal condition on* $\langle g\rangle$*-invariant subgroups for each element* $g \notin \textbf{\textit{Fin}}(G)$.

Proof. Lemma 5.103 implies that G is soluble and that the finitary radical $\mathbf{Fin}(G)$ of G has finite central dimension. Corollary 5.86 implies that the factor group $G/\mathbf{Fin}(G)$ is polycyclic. Let $C = C_A(\mathbf{Fin}(G))$ so that C has finite codimension in A. Then the vector space A has a finite series of G-invariant subspaces

$$0 = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_m = A$$

such that the factors $C_2/C_1, \ldots C_m/C_{m-1}$ are finite dimensional and G-chief. Applying Maltsev's theorem (see [202, Lemma 3.5], for example) we see that the factors $G/C_G(C_2/C_1), \ldots, G/C_G(C_m/C_{m-1})$ are abelian-by-finite groups. Since G is finitely generated these factors are polycyclic. Let

$$U = C_G(C_1) \cap C_G(C_2/C_1) \cap \cdots \cap C_G(C_m/C_{m-1}).$$

The inclusion $\mathbf{Fin}(G) \leq C_G(C_1)$ shows that the factor group $G/C_G(C_1)$ is also polycyclic and by Remak's theorem there is an embedding

$$G/U \longrightarrow G/C_G(C_1) \cap G/C_G(C_2/C_1) \cap \cdots \cap G/C_G(C_m/C_{m-1}),$$

which proves that the factor group G/U is also polycyclic. Each element of U acts trivially on each of the factors C_{j+1}/C_j for $0 \leq j \leq m-1$ so we can apply Theorem 1.2 to obtain (iii) and (iv).

Finally suppose that $g \notin \mathbf{Fin}(G)$. Then the cyclic subgroup $\langle g\rangle$ has infinite central dimension. Because U is nilpotent it has a finite upper central series whose terms are G-invariant subgroups of G and whose factors are abelian. Lemma 5.104 implies that each factor of this series is finitely generated as a $\mathbb{Z}\langle g\rangle$-module. Since the group ring $\mathbb{Z}\langle g\rangle$ is clearly Noetherian it follows that each factor of the series is a Noetherian $\mathbb{Z}\langle g\rangle$-module. Since the upper central series of U is finite it is then an easy matter to prove that U satisfies the maximal condition on $\langle g\rangle$-invariant subgroups. □

In conclusion we observe that the study of infinite dimensional linear groups in which the family of subgroups having infinite central dimension and satisfying some classical finiteness condition has been carried out in various other cases. Our goal in this chapter was to give the general flavour of some of the results that can be obtained when certain finiteness conditions hold, so we presented only three key situations. The interested reader can also find a number of papers related to the results in this chapter, a list which includes, but is not limited to, the following papers: O. Yu Dashkova, M. R. Dixon and

L. A. Kurdachenko [38, 39], M. R. Dixon and L. A. Kurdachenko [44], M. R. Dixon, L. A. Kurdachenko, J. M. Muñoz-Escolano and J. Otal [47], L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal [108, 109], L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal and N. N. Semko [110, 111, 112] and J. M. Muñoz-Escolano, J. Otal and N. N. Semko [151].

Chapter 6

Covering by Finite Dimensional Subspaces

Linear Actions with Finite Orbits

As usual we let G denote a linear group, so that G is a subgroup of $GL(F, A)$, for some (usually infinite dimensional) vector space A over a field F. As we already saw in the earlier chapters, the family of G-invariant subspaces, the properties of the group's elements, the size, location and saturation of the family of all subspaces, G-invariant subspaces and so on, have a significant effect on the structure of the linear group G. The case when the family of G-invariant subspaces consists of just two subspaces, namely the space itself and the zero subspace was considered in Chapter 2. In the current chapter we consider another extreme, which is really the exact opposite of that in Chapter 2.

Suppose, for example, that every subspace of A is G-invariant, akin to the situation in abstract group theory, when every subgroup is a normal subgroup (such groups are usually called *Dedekind groups*). The structure of groups when every subgroup is normal is well-known (due to R. Dedekind and R. Baer, see the book of D. J. S. Robinson [184, 5.3.7], for example). In the situation when all the subspaces are G-invariant it follows that for each element $a \in A$ the subspace Fa is G-invariant and hence if f, g are arbitrary elements of the linear group G, then $f(a) = \alpha a, g(a) = \beta a$, for certain elements α, β of the field F. We then have

$$(fg)(a) = f(g(a)) = f(\beta a) = \beta f(a) = \beta(\alpha a) = \beta\alpha a$$

and similarly

$$(gf)(a) = \beta\alpha a$$

Since $\alpha\beta = \beta\alpha$ in the field F it follows that $(fg)(a) = (gf)(a)$ is valid for each element a of A and hence $fg = gf$. Thus for linear groups the situation is somewhat simpler, for in this case the group G must be abelian.

This simple example nevertheless gives a good illustration of the influence that a family of G-invariant subspaces can have on the structure of the group

G. Figuratively speaking, if the family of G-invariant subspaces is quite large, then the group will, to a greater or lesser extent, be close to being abelian. As before, in this chapter we shall consider linear groups that are in one sense or another close to finite dimensional.

In Chapter 1 for a linear group G we considered the G-invariant subspace

$$\begin{aligned}\mathbf{FO}_G(A) &= \{a \in A | a^G \text{ is finite } \} \\ &= \{a \in A | C_G(a) \text{ has finite index in } G\}.\end{aligned}$$

Now we will consider the situation when $A = \mathbf{FO}_G(A)$. We shall say that the linear group G *has finite orbits on* A if the orbits a^G are finite for every element a of the vector space A. As we saw in Chapter 1 if the G-orbit of an element a is finite, then $|a^G| = |G : C_G(a)|$.

For the group G, one of the natural actions of G on itself is the action of conjugation by a given element. At the same time for an abstract group the orbit of an element $g \in G$ is the conjugacy class $\{x^{-1}gx = g^x | x \in G\}$. Hence we may consider the class of linear groups having finite orbits on A as an analogue of the class of FC-groups, which has been studied extensively (see the book of M. J. Tomkinson [199], for example).

To begin, we note some properties of linear groups having finite orbits. We recall that if B is a G-invariant subspace of A, then there is a natural action of G on both the vector spaces B and A/B.

Proposition 6.1. *Let F be a field and let A be a vector space over F.*

(i) If G is a subgroup of $GL(F, A)$ having finite orbits on A and if K is a subgroup of G, then K has finite orbits on A.

(ii) If G is a subgroup of $GL(F, A)$ having finite orbits on A and if B is a G-invariant subspace of A, then G has finite orbits on B.

(iii) If G is a subgroup of $GL(F, A)$ having finite orbits on A and if B is a G-invariant subspace of A, then G has finite orbits on A/B.

(iv) If $G$is a subgroup of $GL(F, A)$ having finite orbits on A, then G is residually finite;

(v) If $\{G_\lambda | \lambda \in \Lambda\}$ is a family of finite groups and $G = \underset{\lambda \in \Lambda}{Cr}\, G_\lambda$, then for each field F there is a vector space V over F such that G has finite orbits on V.

(vi) If G is a residually finite group, then for each field F there exists a vector space V over F such that G has finite orbits on V.

Proof. The proofs of parts (i)-(iii) are very easy and are omitted.

(iv) In Chapter 1 we noted that the centralizer of each G-orbit is a normal subgroup of G. We have also noted that $|a^G| = |G : C_G(a)|$ for every element

$a \in A$. Let $a^G = \{a_1, a_2, \ldots, a_n\}$. Then from the equality $C_G(a^G) = C_G(a_1) \cap C_G(a_2) \cap \cdots \cap C_G(a_n)$ we deduce that $C_G(a^G)$ has finite index in G. However A is the union of all the G-orbits, so $\bigcap_{a \in A} C_G(a^G) = C_G(A) = 1$ and it now follows that G is residually finite.

(v) First we suppose that G is an arbitrary group and we let F be an arbitrary field. We choose a vector space $W(G)$ having a basis $\mathcal{B}$ such that the cardinalities of $\mathcal{B}$ and G are the same. In this case we can index the elements of the basis $\mathcal{B}$ using the elements of the group G so we write $\mathcal{B} = \{b_g | g \in G\}$. If $g, x \in G$ we now define an action of G on the basis $\mathcal{B}$ by setting $gb_x = b_{gx}$ and then extend this natural action in the obvious way to the whole space $W(G)$. It is clear that $C_G(W(G)) = 1$, so we can think of G as being a subgroup of $GL(F, W(G))$.

For every group G_λ in the given family we now construct the corresponding vector space $A_\lambda = W(G_\lambda)$. We set $V = \bigoplus_{\lambda \in \Lambda} A_\lambda$, the direct sum of the F-vector spaces A_λ and let G be the Cartesian product of the groups G_λ, for $\lambda \in \Lambda$. Thus $G = \underset{\lambda \in \Lambda}{\mathrm{Cr}}\, G_\lambda$. Every element of V can be written in the form $\sum_{\lambda \in \Lambda} a_\lambda$, where $a_\lambda \in A_\lambda$ and $a_\lambda = 0$ for all but finitely many of the indices λ. For each $\lambda \in \Lambda$, let $g_\lambda \in G_\lambda$ and $a_\lambda \in A_\lambda$. We define an action of G on V by

$$(g_\lambda)_{\lambda \in \Lambda} \left(\sum_{\lambda \in \Lambda} a_\lambda \right) = \sum_{\lambda \in \Lambda} g_\lambda a_\lambda.$$

We see at once, from this definition, that

$$\underset{\mu \in \Lambda, \mu \neq \lambda}{\mathrm{Cr}} G_\mu = C_G(A_\lambda),$$

which implies the equation

$$C_G(V) = \bigcap_{\lambda \in \Lambda} C_G(A_\lambda) = \bigcap_{\lambda \in \Lambda} \underset{\mu \in \Lambda, \mu \neq \lambda}{\mathrm{Cr}} G_\mu = 1.$$

This shows that we can think of G as a subgroup of $GL(F, V)$. To show that G has finite orbits on V, let a be an arbitrary element of V, so that $a = \sum_{\lambda \in \Lambda} a_\lambda$ for certain elements $a_\lambda \in A_\lambda$ and let Γ be the support of a, so $\Gamma = \{\lambda \in \Lambda | a_\lambda \neq 0\}$. Then we have $a = \sum_{\lambda \in \Gamma} a_\lambda$ and it is then clear that $\underset{\lambda \in \Lambda \setminus \Gamma}{\mathrm{Cr}} G_\lambda \leq C_G(a)$. Since the groups G_λ are finite, it follows that $C_G(a)$ has finite index in G. Of course we know that $|a^G| = |G : C_G(a)|$ for all $a \in A$, so we deduce that the G-orbit of every element $a \in A$ is finite.

(vi) Now let G be an arbitrary residually finite group. Then G has a family of normal subgroups $\{K_\lambda | \lambda \in \Lambda\}$ such that $G_\lambda = G/K_\lambda$ is finite for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} K_\lambda = 1$. We define the mapping $\Psi : G \longrightarrow \underset{\lambda \in \Lambda}{\mathrm{Cr}}\, G_\lambda$ by setting $\Psi(g) = (gK_\lambda)_{\lambda \in \Lambda}$, for each $g \in G$. By the classical theorem due to Remak, it follows that Ψ is an embedding of G into $\underset{\lambda \in \Lambda}{\mathrm{Cr}}\, G_\lambda$, because $\mathbf{ker}(\Psi) = \bigcap_{\lambda \in \Lambda} K_\lambda = 1$. We can identify G with this image in $\underset{\lambda \in \Lambda}{\mathrm{Cr}}\, G_\lambda$.

Now let F be an arbitrary field. By (v) there is a vector space V over F such that $\underset{\lambda\in\Lambda}{\mathrm{Cr}}\, G_\lambda$ is a subgroup of $GL(F,V)$ and such that $\underset{\lambda\in\Lambda}{\mathrm{Cr}}\, G_\lambda$ has finite orbits. By (i) G also has finite G-orbits on V as required. □

As we see from this result the fact that the orbits of all elements are finite is equivalent to the finite approximability (residual finiteness) of the group. The theory of residually finite groups is quite well established and has many interesting results, but this theory is not really near completion. Thus to proceed further we shall need to impose some further restrictions. Returning to our analogy between linear groups having finite orbits and FC-groups, the same can also be said concerning the theory of FC-groups. In the theory of FC-groups some subclasses of groups emerged naturally and occupied a special place in this theory. One such subclass appeared in the paper [155], for example. In this paper B. H. Neumann discussed the class of groups in which the conjugacy classes are finite but of bounded size. Neumann called such groups BFC-groups and proved that such groups are finite-by-abelian. Thus in this case the derived subgroup G' of the BFC-group G is finite. There is a natural analogue of such groups when we discuss linear groups.

As usual, we let A be a vector space over the field F and let G denote a subgroup of $GL(F,A)$. We say that G has *boundedly finite orbits on A* if there is a natural number b such that $|a^G| \leq b$, for each element a of A.

We let $\mathbf{Io}_A(G) = \max\{|a^G| | a \in A\}$ and first discuss some simple situations when the values of $\mathbf{Io}_A(G)$ are small.

If $\mathbf{Io}_A(G) = 1$, then $a^G = \{a\}$ for each element $a \in A$. It follows that $ga = a$ for all $g \in G$. Thus in this case G is the trivial group.

Suppose that $\mathbf{Io}_A(G) = 2$ and let a be an element of A such that $a^G \neq \{a\}$. In this case the group G contains an element g such that $ga \neq a$. For the element $g^2a = g(g(a))$ we have two choices, namely $g^2a = a$ or $g^2a = ga$. However, in the latter case we immediately deduce that $ga = a$, contradicting the choice of g. Thus $g^2a = a$. If d is a further element of A such that $gd \neq d$, then the above argument can be repeated to deduce that $g^2d = d$. Consequently, for each arbitrary element c of the space A we have $gc = c$ or $g^2c = c$. It follows that g has order 2, so that in this case the group G is an elementary abelian 2-group. Of course, conversely, if G is an elementary abelian 2-group, then $\mathbf{Io}_A(G) = 2$.

We now give an example of a group with boundedly finite orbits.

Example 6.2. *Let F be an arbitrary field and let A be a vector space over F having the countable basis $\{c, a_n | n \in \mathbb{N}\}$. For each natural number n, we define a linear transformation $g_n : A \longrightarrow A$ as follows:*

$$g_n(a) = \begin{cases} a & \text{if } a = c \\ a + c & \text{if } a = a_n \\ a & \text{if } a = a_k \text{ whenever } k \neq n \end{cases}$$

This mapping is then extended linearly in the usual manner. Let $n \neq t$ be natural numbers. We have

$$\begin{aligned} g_n g_t(c) &= g_n(g_t(c)) = g_n(c) = c \\ g_n g_t(a_n) &= g_n(g_t(a_n)) = g_n(a_n) = a_n + c \\ g_n g_t(a_t) &= g_n(g_t(a_t)) = g_n(a_t + c) = g_n(a_t) + g_n(c) = a_t + c \\ g_n g_t(a_k) &= g_n(g_t(a_k)) = g_n(a_k) = a_k \text{ whenever } k \neq n, t \end{aligned}$$

We can see from the above equalities that $g_n g_t = g_t g_n$ for all $n, t \in \mathbb{N}$ and it follows that the group $G = \langle g_n | n \in \mathbb{N} \rangle$ is abelian. Furthermore,

$$g_n^m(c) = c, g_n^m(a_n) = a_n + mc \text{ and } g_n^m(a_k) = a_k \text{ whenever } k \neq n.$$

It follows that if F has characteristic the prime p, then the linear transformation g_n has prime order p, so that G is an elementary abelian p-group; if F has characteristic 0, then g_n has infinite order and G is a free abelian group. Thus in the case when $\boldsymbol{char}(F) = p$ we have $a^G = \{a_n, a_n + c, a_n + 2c, \ldots, a_n + (p-1)c\}$. It follows that $[G, a_n] \leq Fc$ for each $n \in \mathbb{N}$. In turn this implies that $[G, A] \leq Fc$. Since $a^G \subseteq a + [G, A]$, this means that $|a^G| \leq p$ for each element $a \in A$. Thus G has boundedly finite orbits on A and in this case $\boldsymbol{Io}_A(G) = p$.

Taking into account the above-mentioned result of B. H. Neumann concerning the structure of BFC-groups and the natural similarity between the derived subgroup and the G-commutator subspace of a vector space we pose the following very natural question.

- Let F be a field, let A be a vector space over F and let G be a subgroup of $GL(F, A)$. Suppose that G has boundedly finite orbits on A. Is it then the case that the G-commutator subspace $[G, A]$ has finite dimension over F?

We note at once however that the answer to this question is negative as the following example shows.

Example 6.3. *Let F be an arbitrary field of characteristic 0 and suppose that F is* not *a finite extension of the field $\mathbb{Q}$ of rational numbers. For example, F could be the field of real numbers. As usual we can consider F as a vector space over $\mathbb{Q}$. The function $h : F \longrightarrow F$ defined by $h(x) = -x$ for each $x \in F$ is clearly linear and the definition of h implies that $[h, x] = -2x$ for all elements $x \in F$, from which it follows that $[h, F] = F$. Thus $\boldsymbol{dim}_{\mathbb{Q}}([h, F])$ is infinite. However, for each element $x \in F$ we have $x^{\langle h \rangle} = \{x, -x\}$ which shows that the linear group $\langle h \rangle$ has boundedly finite orbits on A. Indeed $\boldsymbol{Io}_A(\langle h \rangle) = 2$ in this case.*

For the prime characteristic case we have the following example (we note that the example above can be modified slightly to also give examples in all cases but those of characteristic 2).

Example 6.4. *Let F be a field of prime characteristic p and let A be a vector space over F having a countable basis $\{a_n | n \in \mathbb{N}\}$. Define a linear transformation $f : A \longrightarrow A$ by*

$$f(a_{2n}) = a_{2n}, f(a_{2n+1}) = a_{2n+1} + a_{2n} \text{ for all natural numbers } n.$$

It is easy to show that f is an F-automorphism of A. We have

$$[f, a_{2n}] = 0, [f, a_{2n+1}] = a_{2n+1} + a_{2n} - a_{2n+1} = a_{2n}, \text{ for all } n \in \mathbb{N},$$

from which it follows that $[f, A] = \bigoplus_{n\in\mathbb{N}} Fa_{2n}$ and hence $[F, A]$ has infinite dimension over F. Furthermore,

$$\begin{aligned} a_{2n}^{\langle f\rangle} &= \{a_{2n}\} \text{ and} \\ a_{2n+1}^{\langle f\rangle} &= \{a_{2n+1}, a_{2n+1} + a_{2n}, a_{2n+1} + 2a_{2n}, \dots a_{2n+1} + (p-1)a_{2n}\}, \end{aligned}$$

which implies that $|a^{\langle f\rangle}| \leq p$ for each element of the vector space A. Thus in this instance $\boldsymbol{Io}_A(\langle f\rangle) = p$.

The examples above appear to show that the situation of a linear group, having boundedly finite orbits, looks more complicated than the situation for BFC-groups.

The next few results in this chapter were obtained in the paper [48] of M. R. Dixon, L. A. Kurdachenko and J. Otal where the more general situation of modules over group rings was considered, but here we shall limit ourselves to considering only the case of linear groups.

We begin our consideration with elementary properties of linear groups having boundedly finite orbits.

Proposition 6.5. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A.*

(i) If H is a subgroup of G, then H has boundedly finite orbits on A; moreover $\boldsymbol{Io}_A(H) \leq \boldsymbol{Io}_A(G)$;

(ii) if B is a G-invariant subspace of A, then G has boundedly finite orbits on B; moreover $\boldsymbol{Io}_B(G) \leq \boldsymbol{Io}_A(G)$;

(iii) if B is a G-invariant subspace of A, then G has boundedly finite orbits on A/B; moreover $\boldsymbol{Io}_{A/B}(G) \leq \boldsymbol{Io}_A(G)$;

(iv) if H is a subgroup of G and if B, C are G-invariant subspaces of A such that $C \leq B$, then H has boundedly finite orbits on B/C; moreover $\boldsymbol{Io}_{B/C}(H) \leq \boldsymbol{Io}_A(G)$;

(v) *G can be embedded in a Cartesian product of finite groups whose orders divide $\boldsymbol{Io}_A(G)!$. In particular G is a locally finite group of exponent at most $\boldsymbol{Io}_A(G)!$.*

Proof. The assertions (i)-(iv) are very easy to prove.

(v) In Chapter 1 we noted that the centralizer of each G-orbit is a normal subgroup of G. We have also noted that $|a^G| = |G : C_G(a)|$ for each element a of A. Let $t = \mathbf{Io}_A(G)$ and let $a^G = \{a_1, a_2, \ldots, a_n\}$ where $n \leq t$ so that $|G : C_G(a)| \leq t$. It follows that the index of $\mathbf{core}_G C_G(a)$ in G is finite and at most $t!$. As we saw in Chapter 1, the subgroups $C_G(a_k)$ and $C_G(a_j)$ are conjugate in G which implies that $\mathbf{core}_G C_G(a_k) = \mathbf{core}_G C_G(a_j)$ for $j, k \in \{1, 2, \ldots, n\}$. From the equality $C_G(a^G) = C_G(a_1) \cap \cdots \cap C_G(a_n)$ we deduce that $C_G(a^G) = \mathbf{core}_G C_G(a)$ and hence $C_G(a^G)$ has finite index at most $t!$ in G. Since A is the union of all the G-orbits we have $\bigcap_{a \in A} C_G(a^G) = C_G(A) = 1$.

Let $G_a = G/C_G(a^G)$ for each $a \in A$ and define the mapping $\Psi : G \longrightarrow \underset{a \in A}{\mathrm{Cr}}\, G_a$ by setting $\Psi(g) = (gC_G(a^G))_{a \in A}$ for every $g \in G$. By Remak's theorem Ψ is an embedding of G into $\underset{a \in A}{\mathrm{Cr}}\, G_a$, because $\mathbf{ker}(\Psi) = \bigcap_{a \in A} C_G(a^G) = 1$. Since $|G_a| \leq t!$ the group $\underset{a \in A}{\mathrm{Cr}}\, G_a$ has finite exponent at most $t!$. However the group $\underset{a \in A}{\mathrm{Cr}}\, G_a$ is locally finite (see [52, Proposition 2.5.4], for example) and hence G is also locally finite, of exponent at most $t!$. □

The next result will play a prominent role in the results to follow. In particular it makes certain induction proofs possible.

Proposition 6.6. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b > 1$ and let d be an element of A such that $|d^G| = b$. If D is the subspace of A generated by the G-orbit of the element d and if $K = C_G(D)$, then the K-orbits of the quotient space A/D have at most $b - 1$ elements.*

Proof. It is clear from Proposition 6.5 that the K-orbits of A/D have at most b elements. We suppose, for a contradiction, that the vector space A contains an element c such that the K-orbit of $c+D$ in the quotient space A/D contains exactly b elements. Proposition 6.5 shows that $|c^G| = b$. Let

$$(c + D)^K = \{c + D = y_1c + D, y_2c + D, \ldots y_bc + D\}$$

for certain elements $y_1 = 1, y_2, \ldots, y_k$ of K. It is clear that the set $\{y_1c, \ldots, y_bc\}$ contains b distinct elements and since $\mathbf{Io}_A(G) = b$ it follows that $c^G = \{y_1c, \ldots, y_bc\}$. Suppose that g, x are elements of G such that $gc+D = xc+D$ and assume that $gc \neq xc$. We have $gc = y_jc, xc = y_mc$ for some $j, m \in \{1, \ldots b\}$ such that $j \neq m$. Then $y_jc+D = gc+D = xc+D = y_mc+D$ which gives us a contradiction. Thus $gc + D = xc + D$ always implies that $gc = xc$.

Let $z \in C_G(c + d)$ where d is as in the statement of the proposition. Then $c + d = z(c + d) = zc + zd$ and since D is G-invariant it follows that

$zc + D = c + D$. From what we proved above, this implies that $zc = c$. However this means that $zd = d + c - zc = d + c - c = d$ and hence $z \in C_G(c) \cap C_G(d)$. Consequently $C_G(c+d) \leq C_G(c) \cap C_G(d)$ and since it is clear that $C_G(c) \cap C_G(d) \leq C_G(c+d)$ it follows that

$$C_G(c) \cap C_G(d) = C_G(c+d).$$

We then deduce that

$$|G : C_G(d)| \leq |G : C_G(c+d)| = |(c+d)^G|.$$

The fact that $\mathbf{Io}_A(G) = b = |d^G|$ shows that $|(c+d)^G| = b$. From these arguments we see that $C_G(c+d) = C_G(d)$, since we also know that $C_G(c+d) \leq C_G(d)$. In turn, since $C_G(c) \cap C_G(d) = C_G(c+d) = C_G(d)$ we deduce that $C_G(d) \leq C_G(c)$. It then follows that $K = C_G(F(d^G)) \leq C_G(d) \leq C_G(c)$. However, in this case, for every element $y \in K$ we see that $y(c + D) = yc + D = c + D$, which shows that $(c + D)^K = \{c + D\}$. It follows that $b = |(c + D)^K| = 1$, which gives the contradiction sought. This contradiction proves the result. □

Lemma 6.7. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Then G contains a subnormal subgroup H and A contains an H-invariant subspace B satisfying the following conditions:*

(i) H has finite index in G;

(ii) B has finite dimension;

(iii) $[H, A] \leq B$.

Moreover, there are functions β_1, β_2 such that $|G : H| \leq \beta_1(b)$ and $\boldsymbol{dim}_F(B) \leq \beta_2(b)$.

Proof. To prove this result we use induction on the natural number b. If $b = 1$, then as we saw above, the group G just consists of the identity linear transformation, so the result follows trivially on setting $H = G$ and $B = 0$. Suppose now that $b > 1$ and that the result is true for all linear groups having boundedly finite orbits strictly smaller than b. Choose an element d in the vector space A such that $|d^G| = b$ and let D be the subspace of A generated by d^G. Then D is clearly G-invariant and since d^G is finite it follows that D is finite dimensional. Furthermore $C_G(D)$ has finite index in G since $C_G(D) = C_G(d^G)$. Indeed, as we have seen before, we have $|G : C_G(d^G)| \leq b!$. Let $K = C_G(D)$ and note that by Proposition 6.6 we have $\mathbf{Io}_{A/D}(K) \leq b-1$, so we may apply the induction hypothesis to the linear group K acting on A/D. Thus the normal subgroup K contains a subnormal subgroup H of finite index at most $\beta_1(b-1)$ and A/D has an H-invariant subspace B/D having finite dimension at most $\beta_2(b-1)$ such that $[H, A/D] \leq B/D$. Then

H is subnormal in G and $[H, A] \leq B$ so that (i) and (iii) follow. Furthermore $\mathbf{dim}_F(B) \leq b + \beta_2(b-1) = \beta_2(b)$ and also $|G : H| = |G : K| \cdot |K : H| \leq b!\beta_1(b-1) = \beta_1(b)$. The result (ii) now follows. $\square$

It is easy to see from the proof of this result that the functions β_1 and β_2 can actually defined as follows:

$$\beta_1(b) = 1!2! \ldots b! \text{ and } \beta_2(b) = 1 + 2 + \cdots + b = b(b+1)/2.$$

Naturally, it is unlikely that these functions are best possible.

We may now obtain a result giving the basic details of the structure of linear groups having boundedly finite orbits.

Theorem 6.8. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Then G contains a normal subgroup L and A contains a G-invariant subspace C satisfying the following conditions:*

(i) L has finite index in G;

(ii) C has finite dimension;

(iii) $[L, A] \leq C$ and $[L, C] = 0$;

(iv) if $\boldsymbol{char}(F) = p$, a prime, then L is an elementary abelian p-group; if $\boldsymbol{char}(F) = 0$, then $L = 1$ and G is finite.

Moreover, there are functions β_3 and β_4 such that $|G : L| \leq \beta_3(b)$ and $\boldsymbol{dim}_F(C) \leq \beta_4(b)$.

Proof. By Lemma 6.7 the group G contains a subnormal subgroup H having finite index at most $\beta_1(b)$ and the vector space A contains an H-invariant subspace B of finite dimension at most $\beta_2(b)$ with the property that $[H, A] \leq B$.

We first construct a G-invariant subspace satisfying condition (ii) and, to this end, let C be the G-invariant subspace generated by B. If a is an arbitrary element of A, then $|a^G| \leq b$, so the G-invariant subspace of A generated by a has finite dimension at most b. It follows that the subspace C has finite dimension at most $b\beta_2(b) = \beta_4(b)$. This proves (ii). We have noted before that $|G : C_G(a^G)| \leq b!$ for each element a of the space A and the fact that $\mathbf{dim}_F(C) \leq b\beta_2(b)$ implies that $C_G(C)$ has finite index at most $(b!)^{b\beta_2(b)}$ in G.

The inclusions $[H, A] \leq B$ and $B \leq C$ together imply that $[H, A] \leq C$. Let $K = \mathbf{core}_G H$ so that K has finite index in G and indeed $|G : K| \leq \beta_1(b)!$. However $K \leq H$ so we have $[K, A] \leq C$ and it follows that $K \leq C_G(A/C)$. In particular, $C_G(A/C)$ has finite index at most $\beta_1(b)!$.

Let $L = C_G(C) \cap C_G(A/C)$. By Theorem 1.2 the subgroup L is abelian and because the indices $|G : C_G(C)|$ and $|G : C_G(A/C)|$ are finite it follows that L has finite index in G, so (i) follows. Furthermore,

$$|G : L| \leq |G : C_G(C)| \cdot |G : C_G(A/C)| \leq (b!)^{b\beta_2(b)}\beta_1(b)! = \beta_3(b).$$

Clearly also (iii) now follows. To prove (iv) we note that if $\mathbf{char}\,(F) = p$, a prime, then again applying Theorem 1.2 we obtain that L is an elementary abelian p-group. Suppose then that the characteristic of F is 0 and that L is nontrivial. Then for each nontrivial linear transformation $h \in L$, there is an element $u \in A$ such that $hu \neq u$, so that in this case we have $hu = u + c$ for some non-zero element $c \in C$. We have

$$h^2u = h(h(u)) = h(u + c) = h(u) + h(c) = u + c + c = u + 2c$$

and a straightforward induction argument implies that $h^n u = u + nc$, for each natural number n. Since $\mathbf{char}\,(F) = 0$ we have $nc \neq mc$, whenever $m \neq n$ and $m, n \in \mathbb{N}$. It follows that the $\langle h\rangle$-orbit of u is infinite, contradicting the fact that the group has boundedly finite orbits. Hence in this case the subgroup L is trivial and G is finite. □

The theorem just proved shows that in the presence of boundedly finite orbits the structure of the linear group G is very restricted and indeed, for fields of characteristic 0, it implies the finiteness of G. We now consider some other situations where the condition that there are boundedly finite orbits implies the finiteness of the group G.

The following lemma, despite its simplicity, turns out to be very useful in this regard.

Lemma 6.9. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Let Λ be an index set and suppose that $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ where each subspace A_λ is G-invariant and the factor groups $G/C_G(A_\lambda)$ are finite for all $\lambda \in \Lambda$. Then*

(i) G is finite;

(ii) if $G/C_G(A_\lambda) \leq k$ for all $\lambda \in \Lambda$, then $|G| \leq k^b$;

(iii) if there exists a set of primes π such that $G/C_G(A)$ is a π-group for every $\lambda \in \Lambda$, then G is also a π-group.

Proof. (i) Suppose, for a contradiction, that G is infinite. This assumption and the hypotheses imply that $C_G(A_\lambda) \neq 1$, for all indices λ. Choose an arbitrary subspace $D_1 = A_{\lambda(1)}$. The condition $C_G(D_1) \neq 1$ implies that there are elements $x_1 \in G$ and $d_1 \in D_1$ such that $x_1 d_1 \neq d_1$. The fact that G is infinite again implies that there is an element $\lambda(2) \in \Lambda$ such that $C_G(D_1 \oplus A_{\lambda(2)}) \neq C_G(D_1)$ and we let $D_2 = A_{\lambda(2)}$. This time we choose an element

$x_2 \in C_G(D_1) \setminus C_G(D_1 \oplus D_2)$. Then $x_2 \notin C_G(D_2)$, so there is an element $d_2 \in D_2$ such that $x_2 d_2 \neq d_2$ and we note that $x_2 d_1 = d_1$. Again using the fact that G is infinite, we obtain the existence of an index $\lambda(3)$ such that $C_G(D_1 \oplus D_2 \oplus A_{\lambda(3)}) \neq C_G(D_1 \oplus D_2)$. We set $D_3 = A_{\lambda(3)}$ and choose an element $x_3 \in C_G(D_1 \oplus D_2) \setminus C_G(D_1 \oplus D_2 \oplus D_3)$. With this choice of $x_3 \notin C_G(D_1 \oplus D_2 \oplus D_3)$ it follows that there is an element $d_3 \in D_3$ such that $x_3 d_3 \neq d_3$ and yet $x_3 d_1 = d_1, x_3 d_2 = d_2$. We repeat this argument m times, where $m > b$. Using the fact that G is infinite we can construct G-invariant subspaces $D_1, D_2, \dots, D_m$ and elements $d_i \in D_i, x_i \in G$ for $1 \leq i \leq m$ such that

$$D_1 + D_2 + \cdots + D_m = D_1 \oplus D_2 \oplus \cdots \oplus D_m.$$

Furthermore,

$$\begin{aligned}
&x_1 d_1 \neq d_1 \\
&x_r d_i = d_i \text{ for all } 1 < r \leq m \text{ and for all } i < r \\
&x_r d_r \neq d_r \text{ for all } 1 < r \leq m.
\end{aligned}$$

We let $d = d_1 + d_2 + \dots d_m$ and note that

$$\begin{aligned}
x_1 d &= x_1 d_1 + \dots x_1 d_m \\
x_2 d &= x_2 d_1 + \cdots + x_2 d_m = d_1 + x_2 d_2 + \dots x_2 d_m \\
x_3 d &= x_3 d_1 + \cdots + x_3 d_m = d_1 + d_2 + x_3 d_3 \dots x_3 d_m \\
&\vdots \ddots \\
x_m d &= x_m d_1 + \cdots + x_m d_m = d_1 + d_2 + \cdots + d_{m-1} + x_m d_m
\end{aligned}$$

It is easy to see from these equations, using the fact that the subspaces A_λ are G-invariant, that the elements $x_i d$ for $i = 1, \dots, m$ are pairwise distinct and therefore the G-orbit of the element d contains at least $m > b$ elements, which gives us a contradiction. This contradiction shows that the group G must be finite.

(ii) and (iii). Repeating the arguments above, we can now find G-invariant subspaces $A_{\mu(1)}, A_{\mu(2)}, \dots A_{\mu(k)}$ such that $k \leq b$ and

$$1 = C_G(A) = C_G(A_{\mu(1)} \oplus A_{\mu(2)} \cdots \oplus A_{\mu(k)}).$$

It follows that $\bigcap_{1 \leq i \leq k} C_G(A_{\mu(i)}) = 1$. Using the classical theorem of Remak, we obtain an embedding of G into $\mathrm{Dr}_{\,1 \leq i \leq k} G/C_G(A_{\mu(i)})$ and the results (ii) and (iii) then follow immediately. □

Theorem 6.10. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Suppose also that G is a periodic p'-subgroup whenever F has prime characteristic p. Then G is finite of order at most $(b!)^b$.*

Proof. We note immediately that Theorem 6.8 shows that G is finite. Let a be an arbitrary element of A. Since $|a^G| \leq b$, the subspace A_1 generated by

a^G has finite dimension at most b. It is clear that A_1 is G-invariant and that $G/C_G(A_1)$ is finite. Since $p \notin \Pi(G/C_G(A_1))$ it follows by Maschke's theorem that A_1 is a direct sum of simple FG-submodules (see [116, Corollary 5.15], for example). Since a is an arbitrary element of A we deduce that A is a sum of simple FG-submodules and Lemma 2.5 then shows that in this case there is an index set Λ and a collection of simple submodules A_λ for $\lambda \in \Lambda$ such that

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda$$

Each of the subspaces A_λ is generated by some G-orbit and hence, as above, $|G/C_G(A_\lambda)| \leq b!$. We now apply Lemma 6.9 to deduce that G is a finite group of order at most $(b!)^b$, as required. □

Two simple corollaries follow which are worth mentioning explicitly. In the first the finiteness of G was established in Theorem 6.8, where the bound on the order of G was given as $\beta_3(b)$. However it is apparent from looking at the proofs that the estimate given in Theorem 6.10 is somewhat sharper.

Corollary 6.11. *Let A be a vector space over a field F of characteristic 0 and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Then G is finite and $|G| \leq (b!)^b$.*

The second of these corollaries is also now immediate from Theorem 6.10

Corollary 6.12. *Let A be a vector space over a field F of prime characteristic p and let G be a subgroup of $GL(F, A)$ having boundedly finite orbits on A. Suppose that $\boldsymbol{Io}_A(G) = b$. Then every p'-subgroup of G is finite of order at most $(b!)^b$.*

Linear Actions with Finite Dimensional Orbits

As we saw above, the condition that the G-orbits of the linear group G are finite is a rather strong one. If the orbit a^G of an element $a \in A$ is finite, then the dimension of the subspace of A generated by a^G is also finite. However it is clear that the converse of this is not true, because there are finite dimensional vector spaces having infinite G-orbits. In Chapter 1 for the linear group G we defined a G-invariant subspace

$$\mathbf{FDO}_G(A) = \{a \in A | Fa^G \text{ is finite dimensional}\}.$$

We now consider the situation in which $A = \mathbf{FDO}_G(A)$, that is the case in which all subspaces Fa^G are finite dimensional for all $a \in A$.

We say that a linear group has *finite dimensional orbits (on A)* if the G-orbit a^G of every element $a \in A$ generates a finite dimensional subspace.

Here we also have an analogy with the class of FC-groups, although now the analogy is much weaker.

We next note some of the properties of linear groups having finite dimensional orbits.

Proposition 6.13. *Let F be a field and let A be a vector space over F.*

(i) If G is a subgroup of $GL(F, A)$ having finite dimensional orbits on A and if K is a subgroup of G, then K has finite dimensional orbits on A;

(ii) if G is a subgroup of $GL(F, A)$ having finite dimensional orbits on A and if B is a G-invariant subspace of A, then G has finite dimensional orbits on B;

(iii) if G is a subgroup of $GL(F, A)$ having finite dimensional orbits on A and if B is a G-invariant subspace of A, then G has finite dimensional orbits on A/B;

(iv) if G is a subgroup of $GL(F, A)$ having finite dimensional orbits on A, then G has a family of normal subgroups $\{H_\lambda | \lambda \in \Lambda\}$ and there exist natural numbers $n(\lambda)$ such that G/H_λ is a subgroup of $GL_{n(\lambda)}(F)$ where $\bigcap_{\lambda\in\Lambda} H_\lambda = 1$.

(v) Let F be an arbitrary field. If $\{G_\lambda | \lambda \in \Lambda\}$ is a family of groups such that, for each $\lambda \in \Lambda$, G_λ is a subgroup of $GL_{n(\lambda)}(F)$ for some natural number $n(\lambda)$ and if $G = \underset{\lambda\in\Lambda}{Cr}\, G_\lambda$, then there is a vector space V over F such that G has finite dimensional orbits on V.

(vi) Let F be an arbitrary field. If G has a family of normal subgroups $\{H_\lambda | \lambda \in \Lambda\}$ such that, for each $\lambda \in \Lambda$, G/H_λ is a subgroup of $GL_{n(\lambda)}(F)$ for some natural number $n(\lambda)$ and $\bigcap_{\lambda\in\Lambda} H_\lambda = 1$, then for each field F there exists a vector space V over F such that G has finite dimensional orbits on V.

The proof of this result is almost word for word the same as the proof of Proposition 6.1, so is omitted.

As we can see from this result, the fact that the orbits of all the elements of a vector space are finite dimensional is equivalent to saying that a linear group acting in this way is residually (finite dimensional). The class of such groups is very much broader than the class of residually finite groups.

If we turn again to the analogy with FC-groups, we can see that another natural analogue of BFC-groups will be that of linear groups G acting on the vector space A in such a way that each G-orbit generates a finite dimensional subspace having dimension bounded by some natural number b. Such linear groups will be discussed in this section and were first studied in the paper [49] of M. R. Dixon, L. A. Kurdachenko and J. Otal. We shall present the main results of this paper here. The results for groups having boundedly finite orbits given earlier once again show that here we cannot expect a complete analogue of the theorem of B. H. Neumann. Naturally the situation here is more complicated and, unlike with boundedly finite orbits, the case of fields of characteristic 0 plays a more interesting role.

As usual, let F be a field, let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. We shall say that G has *boundedly finite dimensional orbits on A* if there is a natural number b such that $\mathbf{dim}_F(Fa^G) \leq b$ for each element a of the vector space A.

In this case we let $\mathbf{do}_A(G) = \max\{\mathbf{dim}_F(Fa^G)|a \in A\}$.

We first consider the case when $\mathbf{do}_A(G) = 1$, so that in this case each of the subspaces Fa^G is one dimensional. Naturally, in this case, Fa is a subspace of Fa^G and it follows that $Fa^G = Fa$. Thus in this case every subspace Fa is G-invariant.

In this case we saw in Lemma 3.57 that for each element $g \in G$ there exists an element $\tau_g \in F$ such that $ga = \tau_g a$ for all elements $a \in A$. Furthermore we saw in Corollary 3.58 that this implies the existence of a homomorphism $\vartheta : G \longrightarrow U(F)$ defined by $\vartheta(g) = \tau_g$ for all elements $g \in G$ so that then G is isomorphic to a subgroup of the multiplicative group of the field F. In particular, as we noted earlier, the group G is then abelian.

We note the following elementary properties concerning boundedly finite dimensional orbits. The proofs are analogous to ones given earlier in Proposition 6.5 so are omitted.

Lemma 6.14. *Let F be a field and let A be a vector space over F. Let G be a subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A.*

(i) If K is a subgroup of G, then K has boundedly finite dimensional orbits on A. Furthermore $\boldsymbol{do}_A(K) \leq \boldsymbol{do}_A(G)$.

(ii) If B is a G-invariant subspace of A, then G has boundedly finite dimensional orbits on B. Furthermore $\boldsymbol{do}_B(G) \leq \boldsymbol{do}_A(G)$.

(iii) If B is a G-invariant subspace of A, then G has boundedly finite dimensional orbits on A/B. Furthermore $\boldsymbol{do}_{A/B}(G) \leq \boldsymbol{do}_A(G)$.

(iv) If K is a subgroup of G and if B, C are G-invariant subspaces of A such that $C \leq B$, then K has boundedly finite dimensional orbits on B/C.Furthermore $\boldsymbol{do}_{B/C}(K) \leq \boldsymbol{do}_A(G)$.

The next result will play a significant role in the description of linear groups with boundedly finite dimensional orbits.

Proposition 6.15. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A. Suppose also that $\boldsymbol{do}_A(G) = b > 1$. Let d be an element of A such that $\boldsymbol{dim}_F(Fd^G) = b$. If D is the subspace of A generated by the G-orbit of d and if $K = C_G(D)$, then $\boldsymbol{dim}_F(F(a + D)^K) \leq b - 1$ for all $a \in A$.*

Proof. Lemma 6.14 implies that $\mathbf{dim}_F(F(a+D)^K) \leq b$. Suppose, for a contradiction, that there is an element $c \in A$ such that the F-subspace of A/D, generated by the element $c+D$ has dimension b. We have that $b = \mathbf{dim}_F(F(c+D)^K) \leq \mathbf{dim}_F(F(c+D)^G)$ and Lemma 6.14 implies that $\mathbf{dim}_F(F(c+D)^G) \leq \mathbf{dim}_F(Fc^G) \leq b$ so it follows that

$$F(c+D)^K = F(c+D)^G \text{ and } \mathbf{dim}_F(F(c+D)^G) = \mathbf{dim}_F(Fc^G) = b. \quad (6.1)$$

Let $C = Fc^G$. Since

$$F(c+D)^G = (Fc^G + D)/D = (C+D)/D,$$

(6.1) implies that $\mathbf{dim}_F(C+D) = 2b$. On the other hand

$$\mathbf{dim}_F(C+D) = \mathbf{dim}_F(C) + \mathbf{dim}_F(D) - \mathbf{dim}_F(C \cap D),$$

so that $\mathbf{dim}_F(C \cap D) = 0$ and hence $C \cap D = 0$.

Let $x \in C_G(c+D)$. Then $c+D = x(c+D) = xc+D$ and hence $(x-1)c \in D$. On the other hand $c \in C$, a G-invariant subspace, so $(x-1)c \in C$. It follows that $(x-1)c \in C \cap D = 0$ and hence $(x-1)c = 0$. Using the obvious inclusion $C_G(c) \leq C_G(c+D)$ we deduce that $C_G(c+D) = C_G(c)$.

Let $d \in D$ and let $z \in C_G(d+c)$. Then $d+c = z(d+c) = zd+zc$. Since $zd \in D$ we have $c+D = zc+D$, so $z \in C_G(c+D)$. By what we showed above it follows that $zc = c$ and hence $C_G(d+c) \leq C_G(c)$. Furthermore, $zd = d+c-zc = d$, so we deduce that $z \in C_G(d) \cap C_G(c)$. Hence $C_G(d+c) \leq C_G(d) \cap C_G(c)$.

Let $e = c+d$ and $E = Fe^G$. Then

$$\begin{aligned}(E+C)/C = F(e+C)^G &= F(c+d+C)^G = F(d+C)^G \\ &= (Fd^G + C)/C = (D+C)/C.\end{aligned}$$

As we saw above $D \cap C = 0$. It follows that

$$(E+C)/C = (D+C)/C \cong D.$$

In turn, this implies that $\mathbf{dim}_F((E+C)/C) = \mathbf{dim}_F(D) = b$ and hence $\mathbf{dim}_F(E) \geq \mathbf{dim}_F((E+C)/C) = b$ which is possible only if $E \cap C = 0$. The arguments above show that $C_G(e+C) = C_G(e)$. However

$$e+C = c+d+C = d+C.$$

The equation $D \cap C = 0$ implies that $C_G(d+C) = C_G(d)$. All these equations imply that $C_G(e) = C_G(d+c) = C_G(d)$. But $C_G(d+c) \leq C_G(d) \cap C_G(c)$ and hence $C_G(d) \leq C_G(c)$. The obvious inclusion $C_G(D) \leq C_G(d)$ implies that $C_G(D) \leq C_G(c)$. Hence we have $c^{C_G(D)} = c^K = c$ and therefore $(c+D)^K = c^K + D = \{c+D\}$ which contradicts the fact that $\mathbf{dim}_F(F(c+D)^K) = b > 1$. This contradiction gives the required result. $\square$

Now we can obtain the following description of linear groups with boundedly finite dimensional orbits analogous to Theorem 6.8 in which the orbits were finite. The beauty of our previous result is that it allows us to use induction arguments.

Theorem 6.16. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A. Suppose also that $\boldsymbol{do}_A(G) = b > 1$. Then*

(i) A contains a G-invariant subspace D of finite dimension such that every subspace of A/D is K-invariant, where $K = C_G(D)$. Moreover, there is a function β_5 such that $\boldsymbol{dim}_F(D) \leq \beta_5(b)$;

(ii) K contains a G-invariant subgroup L such that $[L, A] \leq D$ and K/L is isomorphic to a subgroup of the multiplicative group of the field F;

(iii) if F has prime characteristic p, then L is an elementary abelian p-group and if F has characteristic 0, then L is an abelian torsion-free subgroup.

Proof. The proof is carried out using induction on b. If $b = 1$, then as we noted above the subspace Fa is G-invariant for every element a of A. Corollary 3.58 shows that in this case G is isomorphic to some subgroup of the multiplicative group of the field F. Hence our assertion is true in the case when $b = 1$.

Suppose now that $b > 1$ and that our assertion is true for all natural numbers c such that $c < b$. In the vector space A we choose an element $d_1 \in A$ such that $\mathbf{dim}_F(Fd_1^G) = b$. Let $D_1 = Fd_1^G$ and $H = C_G(D_1)$. Proposition 6.15 shows that the H-orbit of every element of the quotient space A/D generates a subspace of finite dimension at most $b - 1$. Choose an element $d_2 \in A$ such that $\mathbf{dim}_F(F(d_2 + D_1)^H) = k$ is maximal. Then as we saw above $k \leq b - 1$ and we set $B/D_1 = F(d_2 + D_1)^H$. Let $\{e_1 + D_1, \dots, e_k + D_1\}$ be a basis of B/D_1 and let D_2/D_1 be the G-invariant subspace of A/D_1 generated by B/D_1. Then

$$D_2/D_1 = F(e_1 + D_1)^G + \cdots + F(e_k + D_1)^G.$$

By Lemma 6.14 $\mathbf{dim}_F(F(e_j + D_1)^G) \leq b$ for every j such that $1 \leq j \leq k$. Therefore we have that $\mathbf{dim}_F(D_2/D_1) \leq kb \leq (b - 1)b$. It follows that

$$\mathbf{dim}_F(D_2) \leq b + (b - 1)b.$$

Let $U = C_H(B/D_1)$. By Proposition 6.15 $\mathbf{dim}_F(F(a + B)^U) \leq b - 2$ for each element a of A. Since $B \leq D_2$ we have $\mathbf{dim}_F(F(a + D_2)^U) \leq b - 2$. Let $R = C_G(D_2)$. By construction D_2 contains D_1 so that $R = C_G(D_2) \leq C_G(D_1) = H$. Since $B \leq D_2$ we have $R \leq C_H(B/D_1) = U$. It follows that $\mathbf{dim}_F(F(a + D_2)^R) \leq b - 2$ for each element a of A. We then repeat this argument; after finitely many steps we obtain an FG-submodule D such that $\mathbf{dim}_F(D) \leq \beta_5(b)$ and if $K = C_G(D)$, then $\mathbf{dim}_F(F(a + D)^K) = 1$ for each element a of A. This means that $K \leq \mathbf{Norm}_G(A/D)$. Furthermore, from the above argument, we deduce that it is possible to choose the function β_5 as

$$\begin{aligned}\beta_5(b) &= b + (b-1)b + (b-2)b + \cdots + 2b \\ &= b((b-1) + (b-2) + \cdots + 2 + 1) \\ &= b(b-1)b/2 = b^2(b-1)/2.\end{aligned}$$

This completes the proof of (i).

Let $L = C_K(A/D)$. Clearly L is G-invariant and $[L, A] \leq D$. Furthermore, Corollary 3.58 implies that K/L is isomorphic to a subgroup of the multiplicative group of the field F. This proves (ii). By Theorem 1.2 we deduce that the subgroup L is abelian. Furthermore, if F has prime characteristic p, then L is an elementary abelian p-group and if F has characteristic 0, then L is a torsion-free subgroup. Hence (iii) follows □

In the corollaries below we use the notation introduced during the proof of the preceding theorem.

Corollary 6.17. *Let A be a vector space over a field F and let G be a periodic subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A. Then G is locally finite.*

Proof. Since K/L and L are abelian it follows that K is a periodic metabelian group so is locally finite. Furthermore G/K is isomorphic to a subgroup of $GL_m(F)$, where $m = \mathbf{dim}_F(D)$. However a finite dimensional periodic linear group is locally finite (by [202, Corollary 4.8], so G/K is also locally finite. Since an extension of a locally finite group by a locally finite group is locally finite, the result follows. □

Next we prove a result reminiscent of the Tits Alternative (see [202, Corollary 10.7], for example).

Corollary 6.18. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A. If G contains no non-cyclic free subgroups then G is soluble-by-locally finite. Furthermore, if F has characteristic* 0*, then G is soluble-by-finite.*

Proof. As in the proof of Corollary 6.17 K is metabelian and G/K is isomorphic to a subgroup of $GL_m(F)$ where $m = \mathbf{dim}_F(D)$. Suppose that G/K contains a non-cyclic free subgroup S/K. Then S contains a subgroup S_1 such that $S = KS_1$ and $K \cap S_1 = 1$ (see [130, §52], for example). Then the isomorphism $S_1 \cong S/K$ implies that S_1 is a non-cyclic free subgroup of G contradicting the hypotheses on G. Thus G/K contains no non-cyclic free subgroups and the Tits Alternative (see [202, Corollary 10.7], for example) shows that G/K is soluble-by-locally finite. Furthermore, if F has characteristic 0, then G/K is soluble-by-finite. Since K is metabelian the result follows. □

Corollary 6.19. *Let A be a vector space over a field F and let G be a periodic subgroup of $GL(F, A)$ having boundedly finite dimensional orbits on A. Suppose that if F has prime characteristic p, then G is a p'-group. Then the center of G contains a locally cyclic subgroup K such that G/K is abelian-by-finite.*

Proof. Let D be the G-invariant subspace of A and let K, L be the subgroups of G which were defined during the statement and proof of Theorem 6.16. Conditions (ii) and (iii) of that theorem show that $L = 1$, so that K is isomorphic to some subgroup of the multiplicative group of the field F by Theorem 6.16(ii). Since K is periodic it is locally cyclic. Corollary 3.63 implies that $[g, x] \in C_G(A/D)$ for each element $x \in K$ and each element $g \in G$. Since $K = C_G(D)$ is a normal subgroup of G we have $[g, x] \in C_G(D)$, so that $[g, x] \in C_G(A/D) \cap C_G(D) = L = 1$. Thus $K \leq \zeta(G)$. Let $m = \mathbf{dim}_F(D)$. Then $H = G/C_G(D)$ is isomorphic to some subgroup of $GL_m(F)$. If F has characteristic 0, then as H is periodic it is abelian-by-finite (see [202, Corollary 9.4], for example). Suppose now that $\mathbf{char}\,(F) = p$ is a prime and let $q \in \Pi(H)$. Since, by hypothesis, $q \neq p$, Corollary 4.14 shows that the Sylow q-subgroups of H have finite special rank at most $(5m^2 + m)/2$. Let V be an arbitrary finite subgroup of H. As we noted above, for every prime $q \in \Pi(V)$, the Sylow q-subgroups of V have finite special rank at most $(5m^2 + m)/2$. In particular, every Sylow q-subgroup of V is generated by at most $(5m^2 + m)/2$ elements. Then V is generated by at most $(5m^2 + m)/2 + 1$ elements (see [52, Corollary 6.3.15], for example). Since this is true for each finite subgroup V of H, the group H has finite special rank at most $(5m^2 + m)/2 + 1$. Then H is abelian-by-finite (see [202, Theorem 10.9], for example). This completes the proof. $\square$

Linear Actions with Finite Orbits of Subspaces

Let A be a vector space over a field F, let B be a subspace of A and let G be a subgroup $GL(F, A)$. For every element $g \in G$, we let

$$gB = g(B) = \{gb | b \in B\}.$$

Clearly gB is a subspace of A. Just as we defined the G-orbits for the elements of A we can do the same for subspaces.

Indeed, we define the *G-orbits of a subspace* B of A to be the family $B^G = \{gB | g \in G\}$.

Let $C \in B^G$. Then $C = gB$ for some element $g \in G$. Let $x \in \mathbf{Inv}_G(B)$ and consider the element gxg^{-1}. We have

$$gxg^{-1}(C) = gx(g^{-1}(C)) = gx(B) = g(B) = C$$

which shows that $g\mathbf{Inv}_G(B)g^{-1} \leq \mathbf{Inv}_G(C)$. On the other hand, let $y \in \mathbf{Inv}_G(C)$. Then $y(C) = C$, so $y(g(B)) = g(B)$ and it follows that $g^{-1}(y(g(B))) = g^{-1}((gB)) = B$. Hence $g^{-1}yg \in \mathbf{Inv}_G(B)$, so $g^{-1}\mathbf{Inv}_G(C)g \leq \mathbf{Inv}_G(B)$ and $\mathbf{Inv}_G(C) \leq g\mathbf{Inv}_G(B)g^{-1}$. We deduce that $\mathbf{Inv}_G(gB) = g\mathbf{Inv}_G(B)g^{-1}$.

If the G-orbit of a subspace B is finite, then as was shown with the G-orbit of an element, we can show that $|B^G| = |G : \mathbf{Inv}_G(B)|$.

Returning again to the analogy with FC-groups, another classic result of B. H. Neumann immediately comes to mind. In the paper [156] Neumann proved that if the subgroups of an abstract group G have a finite number of conjugates (so have finite G-orbits under conjugation) then the group is center-by-finite. Of course center-by-finite groups have the property that all subgroups have finitely many conjugates. Thus we arrive at the problem of describing the structure of linear groups having finite G-orbits of subspaces.

Suppose that the G-orbit of every subspace B of a vector space A is finite. Then the G-orbit of every one dimensional subspace Fa is finite for each $a \in A$. From what we noted above it then follows that the index $|G : \mathbf{Inv}_G(Fa)|$ is finite. Let T be a transversal to $\mathbf{Inv}_G(Fa)$ in the group G. Then $G(Fa) = Fa^G = \sum_{g\in T} g(Fa)$ and the finiteness of the set T implies that the G-invariant subspace $G(Fa)$ has finite dimension. Furthermore $\mathbf{dim}_F(G(Fa)) \leq |T|$. If we now suppose that there is a natural number b such that $|B^G| \leq b$ for each subspace B, then we deduce that the group G has boundedly finite dimensional orbits of elements and we arrive at the situation that has already been discussed previously. Consequently, in what follows below we shall consider the situation when the orbits of subspaces are finite without assuming, however, that the sizes of these orbits are bounded. On the other hand, the theorem of B. H. Neumann mentioned above implies that if the sizes of the conjugacy classes of subgroups of a group are finite, then their sizes are bounded. This prompts the following very natural question:

- Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the G-orbit of every subspace B of A is finite. In this case, will the sizes of these orbits be bounded?

We begin the study of this question with the following result.

Lemma 6.20. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If a is a non-zero element of A, then the quotient group $\boldsymbol{Inv}_G(Fa)/C_G(Fa)$ is isomorphic to some subgroup of the multiplicative group of the field F.*

Proof. Let g be an arbitary element of $\mathbf{Inv}_G(Fa)$. Then $ga \in Fa$, so $ga = \tau_g a$ for some element $\tau_g \in F$. Each element c of the subspace Fa has the form $c = \gamma a$, for some element $\gamma \in F$ and this representation is unique. Then

$$gc = g(\gamma a) = \gamma(ga) = \gamma(\tau_g a) = (\gamma\tau_g)a = (\tau_g\gamma)a = \tau_g c.$$

Thus the linear transformation g is completely determined by the element τ_g of the field F. We may therefore define a mapping

$$\Phi : \mathbf{Inv}_G(Fa) \longrightarrow U(F) \text{ by } \Phi(g) = \tau_g,$$

for each $g \in \mathbf{Inv}_G(Fa)$. We note that if $g, h \in \mathbf{Inv}_G(Fa)$, then we have

$$\tau_{gh}a = (gh)a = g(ha) = g(\tau_h a) = \tau_h(ga) = \tau_h(\tau_g a) = (\tau_g \tau_h)a,$$

from which it follows that $\Phi(gh) = \tau_{gh} = \tau_g\tau_h = \Phi(g)\Phi(h)$. Hence the function Φ is a homomorphism from the group $\mathbf{Inv}_G(Fa)$ to the multiplicative group of the field F. Furthermore it is clear that $\mathbf{ker}(\Phi) = C_G(Fa)$ and this proves the result. □

Lemma 6.21. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. If d is an element of A such that the G-orbit of the subspace Fd is finite, then the G-invariant subspace $G(Fd)$ has finite dimension and the factor group $G/C_G(G(Fd))$ is abelian-by-finite.*

Proof. Let $D = Fd$ and note that the family $\{gD|g \in G\}$ is finite by hypothesis. Let

$$D^G = \{gD|g \in G\} = \{D_1, \dots, D_m\}.$$

Then $GD = D_1 + \cdots + D_m$, which shows that the subspace GD has finite dimension. The fact that the G-orbit of the subspace D is finite implies that $|D^G| = |G : \mathbf{Inv}_G(D)|$ and in particular this last index is finite. The equality $\mathbf{Inv}_G(gD) = g\mathbf{Inv}_G(D)g^{-1}$ proved earlier implies that $\{\mathbf{Inv}_G(gD)|g \in G\} = \{g\mathbf{Inv}_G(D)g^{-1}|g \in G\}$. It follows that

$$L = \mathbf{core}_G\,(\mathbf{Inv}_G(D)) = \bigcap_{g\in G} g\mathbf{Inv}_G(D)g^{-1} = \bigcap_{g \in G} \mathbf{Inv}_G(gD)$$
$$= \bigcap_{1\le j\le m} \mathbf{Inv}_G(D_j)$$

is a normal subgroup of finite index in G. We note that $C_G(D_j)$ is a normal subgroup of $\mathbf{Inv}_G(D_j)$, so that $C_G(D_j)$ is normal in L for $1 \le j \le m$. Then

$$L/(L \cap C_G(D_j)) \cong LC_G(D_j)/C_G(D_j) \le \mathbf{Inv}_G(D_j)/C_G(D_j)$$

for $1 \le j \le m$.

We note that $C_G(gD) = gC_G(D)g^{-1}$ for every element $g \in G$ which shows that

$$\{C_G(gD)|g \in G\} = \{gC_G(D)g^{-1}|g \in G\} = \{C_G(D_j)|1 \le j \le m\}.$$

Furthermore, if $D_j = gD$ for some $g \in G$, then we have

$$\mathbf{Inv}_G(D_j)/C_G(D_j) = g\mathbf{Inv}_G(D)g^{-1}/gC_G(D)g^{-1} \cong \mathbf{Inv}_G(D)/C_G(D).$$

This holds for $1 \le j \le m$.

By Lemma 6.20 the factor group $\mathbf{Inv}_G(D)/C_G(D)$ is abelian, so that $\mathbf{Inv}_G(D_j)/C_G(D_j)$ is likewise abelian, for $1 \leq j \leq m$. In turn, this means that the factor group $L/(L \cap C_G(D_j))$ is also abelian, for $1 \leq j \leq m$.

Furthermore, $\bigcap_{1 \leq j \leq m} C_G(D_j)$ is a normal subgroup of G and

$$\bigcap_{1 \leq j \leq m} C_G(D_j) \leq C_G(GD).$$

Using Remak's theorem we obtain the embedding

$$L/C_G(GD) \longrightarrow L/(L \cap C_G(D_1)) \times \cdots \times L/(L \cap C_G(D_m))$$

which shows that $L/C_G(GD)$ is abelian. Since G/L is finite the result follows. □

Proposition 6.22. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the G-orbit of every subspace of A is finite. Then A contains a G-invariant subspace D of finite codimension, such that the index $|G : \boldsymbol{Norm}_G(D)|$ is finite.*

Proof. If $G = \mathbf{Norm}_G(A)$, then the result is clear upon setting $D = A$. Therefore we may assume that the vector space A contains an element u_1 such that the subspace $U_1 = Fu_1$ is not G-invariant, which means that the group G contains an element g_1 such that $U_1 \neq g_1U_1$. Since the G-orbit of the subspace U_1 is finite it follows that the index $|G : \mathbf{Inv}_G(U_1)|$ is finite. Let $G_1 = \mathbf{Inv}_G(U_1)$ and $V_1 = GU_1$. By Lemma 6.21 we deduce that the G-invariant subspace V_1 has finite dimension and we note that by construction $g_1U_1 \leq V_1$.

The F-subspace V_1 has an F-complement W_1 in the vector space A so that $A = V_1 \oplus W_1$. As F-vector spaces we have $A/W_1 \cong V_1$, so the subspace W_1 has finite codimension in A. Then the F-isomorphism $A/gW_1 = gA/gW_1 \cong A/W_1$ shows that the subspace gW_1 has finite codimension in A. This is true for each element $g \in G$. The G-orbit of W_1 is finite and we let

$$\{gW_1 | g \in G\} = \{B_1 = W_1, B_2, \ldots, B_m\}.$$

Let $Y_1 = B_1 \cap B_2 \cap \cdots \cap B_m$. Then Y_1 is G-invariant and since $\mathbf{dim}_F(A/B_j)$ is finite for $1 \leq j \leq m$ it follows that the subspace Y_1 has finite codimension in A. If $G_1 = \mathbf{Inv}_G(U_1) = \mathbf{Norm}_G(Y_1)$, then set $D = Y_1$. Since $|G : G_1|$ is finite, the result follows in this case. Therefore suppose that $G_1 \neq \mathbf{Norm}_G(Y_1)$.

In this case the subspace Y_1 contains an element u_2 such that the subspace $U_2 = Fu_2$ is not G_1-invariant, so there exists an element $g_2 \in G_1$ such that $g_2U_2 \neq U_2$. Since Y_1 is G-invariant it follows that $g_2U_2 \leq Y_1$ and since $g_2 \in G_1 = \mathbf{Inv}_G(U_1)$ it follows that $g_2U_1 = U_1$.

The subspace $U_1 \oplus U_2$ has a finite G-orbit and this implies that the index $|G : \mathbf{Inv}_G(U_1 \oplus U_2)|$ is finite. Let $G_2 = \mathbf{Inv}_G(U_1 \oplus U_2)$ and write $V_2 = GU_2$. We can again use Lemma 6.21 which this time shows that the G-invariant subspace

V_2 has finite dimension. The choice of V_2 shows that $g_2U_2 \le V_2$ and using similar arguments to those used above we can find a G-invariant subspace Y_2 of finite codimension in A such that $Y_2 \cap (V_1 \oplus V_2) = 0$. If $G_2 = \mathbf{Norm}_G(Y_2)$, then we set $D = Y_2$ and the result follows as above. Therefore we may suppose that $G_2 \ne \mathbf{Norm}_G(Y_2)$ and then continue this process.

Suppose, for a contradiction, that this process does not terminate. This means that the vector space A contains an infinite subset $\{u_n | n \in \mathbb{N}\}$ and the group G contains an infinite subset $\{g_n | n \in \mathbb{N}\}$ such that the following conditions hold:

(i) $g_nU_n \ne U_n$ where $U_n = Fu_n$;

(ii) $g_{n+k}U_n = U_n$ for all $k \in \mathbb{N}$;

(iii) $GU_{n+1} \cap (GU_1 + \cdots + GU_n) = 0$ for all $n \in \mathbb{N}$.

The condition (iii) implies that $\sum_{n\in\mathbb{N}} GU_n = \bigoplus_{n\in\mathbb{N}} GU_n$ and hence $\sum_{n\in\mathbb{N}} U_n = \bigoplus_{n\in\mathbb{N}} U_n$. We have

$$g_k(\oplus_{n\in\mathbb{N}}U_n) = \oplus_{n\in\mathbb{N}}g_kU_n = g_kU_1 \oplus \cdots \oplus g_kU_k \oplus (\oplus_{n>k}U_n)$$

Since $g_nU_n \ne U_n$ we deduce that $g_k(\bigoplus_{n\in\mathbb{N}} U_n) \ne g_m(\bigoplus_{n\in\mathbb{N}} U_n)$ whenever $k \ne m$. It follows that the G-orbit of the subspace $\bigoplus_{n\in\mathbb{N}} U_n$ is infinite, which gives us the required contradiction. This contradiction shows that the process described above terminates in finitely many steps and this in turn proves the proposition. □

We now gather some further information.

Proposition 6.23. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the G-orbit of every subspace of A is finite. Then G contains a subgroup H of finite index and A contains a G-invariant subspace D together with elements $a_1, \dots, a_n$ satisfying the following conditions:*

(i) $A = Fa_1 \oplus \cdots \oplus Fa_n \oplus D$;

(ii) for each element h of the subgroup H there exists an element $\tau_h \in F$ such that $hd = \tau_h d$ for all elements $d \in D$;

(iii) for each element h of the subgroup H there are elements $\sigma_{h,j} \in F$ such that $ha_j = \sigma_{h,j}a_j$, for $1 \le j \le n$;

(iv) there exists a natural number m such that $\sigma_{h,j} = \theta_{h,j}\tau_{h,j}$ where $\theta_{h,j}^m = 1$ for all $h \in H$.

Proof. Proposition 6.22 implies that the vector space A contains a G-invariant subspace D of finite codimension and such that the index $|G : \mathbf{Norm}_G(D)|$ is finite. In the quotient space A/D we choose an arbitrary basis

$\{a_1 + D, \ldots, a_n + D\}$. It then follows easily that $A = Fa_1 \oplus \cdots \oplus Fa_n \oplus D$ so (i) holds. Since the G-orbit of every subspace Fa_j is finite, the index $|G : \mathbf{Inv}_G(Fa_j)|$ is finite for $1 \leq j \leq n$ and it follows that the index of

$$H = \mathbf{Norm}_G(D) \cap \mathbf{Inv}_G(Fa_1) \cap \cdots \cap \mathbf{Inv}_G(Fa_n)$$

in G is finite. Since $H \leq \mathbf{Norm}_G(D)$ Lemma 6.20 implies that for each element h of the subgroup H there is an element $\tau_h \in F$ such that $hd = \tau_h d$ for all elements $d \in D$ so (ii) holds. The inclusion $H \leq \mathbf{Inv}_G(Fa_j)$ shows that the subspace Fa_j is H-invariant. In turn this implies that for every element h of the subgroup H there is an element $\sigma_{h,j} \in F$ such that $ha_j = \sigma_{h,j} a_j$ for $1 \leq j \leq n$. Hence (iii) holds. In order to prove (iv) we let d be an arbitrary non-zero element of the subspace D and set $a = a_1 + \cdots + a_n + d$. Then

$$\begin{aligned} ha &= h(a_1 + \cdots + a_n + d) = ha_1 + \ldots ha_n + hd \\ &= \sigma_{h,1} a_1 + \cdots + \sigma_{h,n} a_n + \tau_h d. \end{aligned}$$

By hypothesis, the subspace Fa has finite G-orbit, so it follows that the index $|G : \mathbf{Inv}_G(Fa)|$ is finite. We let $|G : \mathbf{Inv}_G(Fa)| = m$. Then $y = h^m \in \mathbf{Inv}_G(Fa)$. Since the subspace Fa is $\langle y \rangle$-invariant it follows that there is an element $\eta \in F$ such that $ya = \eta a$. Then

$$ya = ya_1 + \cdots + ya_n + yd = \eta a_1 + \cdots + \eta a_n + \eta d.$$

However, we also have

$$ya = h^m(a_1 + \cdots + a_n + d) = \sigma_{h,1}^m a_1 + \cdots + \sigma_{h,n}^m a_n + \tau_h^m d.$$

Since the elements $a_1, \ldots, a_n, d$ are linearly independent, we deduce that

$$\eta = \sigma_{h,1}^m = \cdots = \sigma_{h,n}^m = \tau_h^m.$$

In particular we have $\sigma_{h,j}^m \tau_h^{-m} = (\sigma_{h,j} \tau_h^{-1})^m = 1$, for $1 \leq j \leq n$. Set $\sigma_{h,j} \tau_h^{-1} = \theta_{h,j}$. Then $\theta_{h,j}^m = 1$ and $\sigma_{h,j} = \theta_{h,j} \tau_h$ for $1 \leq j \leq n$. This completes the proof. □

Corollary 6.24. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the G-orbit of every subspace of A is finite. Then G contains a subgroup H of finite index and there exists an embedding*

$$\nu : H \longrightarrow V = U_1 \times \cdots \times U_{n+1},$$

where U_j is a multiplicative subgroup of the field F for $1 \leq j \leq n+1$. Furthermore,

$$\boldsymbol{Im}(\nu) \leq (W_1 \times \ldots W_n)Y$$

where $W_j = \{\alpha | \alpha \in F$ and $\alpha^m = 1\}$ and Y is the diagonal subgroup of V.

Proof. We use the notation from the proof of Proposition 6.23. Thus

$$\begin{aligned} A &= Fa_1 \oplus \cdots \oplus Fa_n \oplus D \\ H &= \mathbf{Norm}_G(D) \cap \mathbf{Inv}_G(Fa_1) \cap \cdots \cap \mathbf{Inv}_G(Fa_n) \text{ and} \\ |G : H| &= m \end{aligned}$$

We define the mapping ν as follows. For every element $h \in H$ let

$$\nu(h) = (\theta_{h,1}\tau_h, \ldots, \theta_{h,n}\tau_h, \tau_h).$$

It is now quite easy to show that ν is a group homomorphism from the subgroup H into V, that $\mathbf{ker}(\nu) = C_H(A) = 1$ and that $\mathbf{Im}(\nu) \leq (W_1 \times \ldots W_n)Y$, as required. □

We may now obtain the description of linear groups having finite orbits of subspaces. This gives us an analogue, for infinite dimensional linear groups, of the work of B. H. Neumann [156] mentioned earlier.

Theorem 6.25. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that the G-orbit of every subspace of A is finite. Then the following conditions hold:*

(i) G contains a normal subgroup K of finite index such that every subspace of A is K-invariant;

(ii) G is center-by-finite;

(iii) there is a natural number k such that the G-orbit of every subspace of A contains at most k subspaces.

In particular G has boundedly finite orbits of subspaces.

Proof. We use the notation introduced in Corollary 6.24 and Proposition 6.23. Let H be the subgroup of G of finite index defined in the corollary. Since every bounded subgroup of the multiplicative group of a field is cyclic, the subgroup W_j has order m for $1 \leq j \leq n$ so that

$$|(W_1 \times \cdots \times W_n)Y : Y| \leq m^n.$$

Let K denote the preimage of Y in H under the mapping ν. Clearly the subgroup $H \cap \mathbf{Norm}_G(A)$ contains K. This choice of K implies that K is a normal subgroup of finite index in G and clearly every subspace of A is K-invariant. Hence (i) follows. Corollary 3.63 shows that G is center-by-finite, so (ii) also holds.

Finally, to prove (iii), let $|G : K| = k$. If B is an arbitrary subspace of A, then $K \leq \mathbf{Inv}_G(B)$. It follows that $|G : \mathbf{Inv}_G(B)| \leq k$. The equality $|G : \mathbf{Inv}_G(B)| = |B^G|$ implies that the G-orbit of every subspace of B has at most k subspaces and the theorem follows. □

A Measure of Non-G-Invariance

As usual we let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. As we have seen, with each subspace B of the space A we may associate two natural G-invariant subspaces:

$$GB, \text{ the subspace of } A \text{ generated by all subspaces } gB \text{ for } g \in G$$

and

$$\mathbf{core}_G\, B = \bigcap_{g\in G} gB.$$

As we noted in Chapter 3, GB is the smallest G-invariant subspace containing B and $\mathbf{core}_G\, B$ is the largest G-invariant subspace contained in B.

Thus a subspace B is G-invariant if and only if $GB = B = \mathbf{core}_G\, B$. Thus the values of $\mathbf{dim}_F(GB/B)$ and $\mathbf{dim}_F(B/\mathbf{core}_G\, B)$ give some indication about how close B is to being G-invariant or how far B is from being G-invariant.

For a subspace B of the vector space A we shall call $\mathbf{dim}_F(GB/B)$ the *upper measure of non-G-invariance of* B and we call $\mathbf{dim}_F(B/\mathbf{core}_G\, B)$ the *lower measure of non-G-invariance of* B.

Here it is also appropriate to recall the corresponding group-theoretic counterparts of these ideas. Groups in which the index $|H^G : H|$ is finite for all subgroups H of a group G were studied by B. H. Neumann, again in the paper [156]. In this case such groups are finite-by-abelian. The dual situation, the case in which the indices $|H : \mathbf{core}_G\, H|$ are finite for each subgroup H has been studied in the papers [27, 196] of J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith and J. Wiegold and H. Smith and J. Wiegold respectively. In this latter case (which turned out to be rather more complicated) the groups turned out to be abelian-by-finite. In both cases though it is evident that the groups turn out to be quite close to being abelian. As we saw earlier, if every subspace of A is G-invariant the group G is abelian. This leads one to suspect that a group G acting on a vector space in such a way that the measure of non-G-invariance of every subspace is bounded should lead in one sense or another to a group which is close to being abelian. For the upper measure of non G-invariance this is seen at once. Indeed, suppose that there exists a natural number b such that $\mathbf{dim}_F(GB/B) \leq b$ for each subspace B of our vector space A. Then $\mathbf{dim}_F(G(Fa)) \leq b + 1$ for every element $a \in A$. This situation was considered in the second section of this chapter. In the current section, the second case will be considered; that is we study the structure of linear groups acting on a vector space A when the lower measure of non-G-invariance is finite and bounded. A description of such groups was obtained in the paper [106] of L. A. Kurdachenko, J. M. Muñoz-Escolano and J. Otal and here we present the main results of that paper.

Lemma 6.26. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that D is a subspace of A satisfying the following conditions:*

(i) $D = \bigoplus_{\lambda\in\Lambda} D_\lambda$ *for some index set Λ, where D_λ is a non-zero G-invariant subspace of A for all $\lambda \in \Lambda$;*

(ii) $\boldsymbol{dim}_F(D_\lambda) > 1$ *for all $\lambda \in \Lambda$;*

(iii) D_λ *contains an element c_λ such that the subspace Fc_λ is not G-invariant for $\lambda \in \Lambda$.*

If there is a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b, then $|\Lambda| \leq b$.

Proof. For every index $\mu \in \Lambda$ we define

$$E_\mu = \oplus_{\lambda\in\Lambda,\lambda\neq\mu} D_\lambda.$$

Then $A/E_\mu \cong D_\mu$ for all $\mu \in \Lambda$. We let $C = \bigoplus_{\lambda\in\Lambda} Fc_\lambda$. Then clearly $C+E_\mu = Fc_\mu + E_\mu$ so that $(C + E_\mu)/E_\mu = (Fc_\mu + E_\mu)/E_\mu$. Since $Fc_\mu \cap E_\mu = 0$ we have

$$(Fc_\mu + E_\mu)/E_\mu \cong_F Fc_\mu/(Fc_\mu \cap E_\mu) \cong_F Fc_\mu.$$

Thus $(C + E_\mu)/E_\mu$ is a one-dimensional subspace of the quotient space A/E_μ which is not G-invariant. It follows that $\mathbf{core}_G\,(C + E_\mu)/E_\mu = 0$. On the other hand, $\mathbf{core}_G\,(C + E_\mu)/E_\mu = (\mathbf{core}_G\,C + E_\mu)/E_\mu$, so we deduce that $\mathbf{core}_G\,C \leq E_\mu$. This containment is true for all indices $\mu \in \Lambda$ and hence we have $\mathbf{core}_G\,C \leq \bigcap_{\mu\in\Lambda} E_\mu = 0$. The hypotheses imply that $C/\mathbf{core}_G\,C = \bigoplus_{\lambda\in\Lambda} Fc_\lambda/\mathbf{core}_G\,C \cong \bigoplus_{\lambda\in\Lambda} Fc_\lambda$ has finite dimension at most b and hence it follows that the index set Λ is finite. Indeed, it is clear that $|\Lambda| \leq b$, as required. □

Corollary 6.27. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that D is a subspace of A satisfying the following conditions:*

(i) $D = \bigoplus_{\lambda\in\Lambda} D_\lambda$ *for some index set Λ, where D_λ is a minimal non-zero G-invariant subspace of A for all $\lambda \in \Lambda$;*

(ii) $\boldsymbol{dim}_F(D_\lambda) > 1$ *for all $\lambda \in \Lambda$.*

If there is a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b, then $|\Lambda| \leq b$ and $\boldsymbol{dim}_F(D) \leq 2b$.

Proof. Since D_λ is a minimal G-invariant subspace of A of dimension strictly greater than 1, it follows that for each element $d \in D_\lambda$ and each $\lambda \in \Lambda$ the subspace Fd cannot be G-invariant. Using Lemma 6.26 we deduce that $|\Lambda| \leq b$. We may then assume that $\Lambda = \{1, \dots, k\}$ where $k \leq b$ and in this case we have $D = D_1 \oplus \cdots \oplus D_k$. In each subspace D_j we choose a non-zero element f_j, for

$1 \le j \le k$. There is a subspace B_j of D such that $D = Fd_j \oplus B_j$ for $1 \le j \le k$. Since D_j is a minimal G-invariant subspace of A we have $\mathbf{core}_G\, B_j = 0$ for $1 \le j \le k$. Set $E_j = \bigoplus_{1 \le m \le k, m \ne j} D_m$. Then $D/E_j \cong_{FG} D_j$ so that D/E_j is a minimal non-zero G-invariant subspace of A/E_j for $1 \le j \le k$. It follows that $\mathbf{core}_G\,(B_j + E_j)/E_j = 0$ for $1 \le j \le k$. Let $B = \bigoplus_{j=1}^k B_j$. Clearly $B_j + E_j = B + E_j$ from which it follows that $\mathbf{core}_G\,(B + E_j)/E_j = 0$. Since

$$\mathbf{core}_G\,(B + E_j)/E_j = (\mathbf{core}_G\, B + E_j)/E_j$$

it follows that $(\mathbf{core}_G\, B + E_j)/E_j = 0$. In other words, we have that $\mathbf{core}_G\, B \le E_j$ and this is true for all j with $1 \le j \le k$. We deduce that $\mathbf{core}_G\, B \le \bigcap_{1 \le j \le k} E_j = 0$. Thus $B/\mathbf{core}_G\, B \cong_F B$ and hence $\mathbf{dim}_F(B/\mathbf{core}_G\, B) = \mathbf{dim}_F(B)$. Our hypotheses imply that $\mathbf{dim}_F(B) \le b$ and by construction

$$\begin{aligned} D &= \oplus_{1 \le j \le k}(Fd_j \oplus B_j) = (\oplus_{1 \le j \le k} Fd_j) \oplus (\oplus_{1 \le j \le k} B_j) \\ &= \oplus_{1 \le j \le k} Fd_j \oplus B. \end{aligned}$$

From this it follows that

$$\mathbf{dim}_F(D) = k + \mathbf{dim}_F(B) \le b + b = 2b,$$

as required. □

Corollary 6.28. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that D is a subspace of A satisfying the following conditions:*

(i) $D = \bigoplus_{\lambda \in \Lambda} D_\lambda$ for some index set Λ, where D_λ is a minimal non-zero G-invariant subspace of A for all $\lambda \in \Lambda$;

(ii) $\mathbf{dim}_F(D_\lambda) = 1$ for all $\lambda \in \Lambda$.

If there is a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b, then D contains a G-invariant subspace C such that $\mathbf{dim}_F(D/C) \le 2b$ and every subspace of C is G-invariant.

Proof. For each $\lambda \in \Lambda$ choose a non-zero element $d_\lambda \in D_\lambda$. Then the hypotheses imply that $D_\lambda = Fd_\lambda$. Since D_λ is G-invariant, d_λ is an eigenvector of each element g of the linear group G. In other words, the field F contains an element $\tau_{\lambda,g}$ such that $gd_\lambda = \tau_{\lambda,g} d_\lambda$. It follows easily that in this case we have $gc = \tau_{\lambda,g} c$ for each element $c \in D_\lambda$.

Let $\lambda_1 \in \Lambda$ and let g_1 be an element of the group G such that $g_1 d_{\lambda_1} = \tau_{\lambda_1,g_1} d_{\lambda_1}$. Suppose that there is an index $\mu_1 \in \Lambda$ such that $g_1 d_{\mu_1} = \tau_{\mu_1,g_1} d_{\mu_1}$ and $\tau_{\lambda_1,g_1} \ne \tau_{\mu_1,g_1}$. In this case we obtain

$$g_1(d_{\lambda_1} + d_{\mu_1}) = g_1 d_{\lambda_1} + g_1 d_{\mu_1} = \tau_{\lambda_1,g_1} d_{\lambda_1} + \tau_{\mu_1,g_1} d_{\mu_1} \notin F(d_{\lambda_1} + d_{\mu_1}),$$

using the fact that d_{λ_1} and d_{μ_1} are linearly independent. This shows that the subspace $F(d_{\lambda_1} + d_{\mu_1})$ cannot be G-invariant. Set $E_1 = Fd_{\lambda_1} + Fd_{\mu_1}$, so that E_1 is G-invariant, yet it contains the one-dimensional subspace $F(d_{\lambda_1} + d_{\mu_1})$ which is not G-invariant.

Suppose that there are indices $\lambda_2, \mu_2 \in \Lambda \setminus \{\lambda_1, \mu_1\}$ and an element $g_2 \in G$ such that

$$g_2 d_{\lambda_2} = \tau_{\lambda_2,g_2} d_{\lambda_2}, g_2 d_{\mu_2} = \tau_{\mu_2,g_2} d_{\mu_2} \text{ and } \tau_{\lambda_2,g_2} \neq \tau_{\mu_2,g_2}.$$

As above, we see that the subspace $F(d_{\lambda_2} + d_{\mu_2})$ is not G-invariant. Set $E_2 = Fd_{\lambda_2} + Fd_{\mu_2}$ and note that the choice of these elements implies that $E_1 \cap E_2 = 0$.

Using similar reasoning we choose pairwise different indices

$$\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n$$

and elements $g_1, \ldots g_n$ in the group G such that

$$g_j d_{\lambda_j} = \tau_{\lambda_j,g_j} d_{\lambda_j}, g_j d_{\mu_j} = \tau_{\mu_j,g_j} d_{\mu_j} \text{ and } \tau_{\lambda_j,g_j} \neq \tau_{\mu_j,g_j},$$

for $1 \leq j \leq n$. As above, in each case the subspace $F(d_{\lambda_j} + d_{\mu_j})$ is not G-invariant for $1 \leq j \leq n$ and for each j we let $E_j = Fd_{\lambda_j} + Fd_{\mu_j}$. Then each subspace E_j is G-invariant and $\mathbf{dim}_F(E_j) = 2$, for $1 \leq j \leq n$, so we may apply Lemma 6.26. According to this result we have $n \leq b$. This shows that the process of obtaining the indices $\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n$ and the elements $g_1, \ldots, g_n$ must terminate in finitely many steps. Let this process stop at $n = t \leq b$ and let

$$\Lambda_t = \Lambda \setminus \{\lambda_1, \mu_1, \ldots, \lambda_t, \mu_t\}.$$

If g is an arbitrary element of the group G and if $\lambda, \mu \in \Lambda_t$, then we have

$$gd_\lambda = \tau_{\lambda,g} d_\lambda, gd_\mu = \tau_{\mu,g} d_\mu \text{ and } \tau_{\lambda,g} = \tau_{\mu,g}.$$

This means that the elements $\tau_{\mu,g}$ are independent of λ and we simply denote the common value by τ_g. Thus

$$gd_\lambda = \tau_g d_\lambda \text{ for all } \lambda \in \Lambda_t.$$

Let $C = \bigoplus_{\lambda \in \Lambda_t} Fd_\lambda$. We note that the subspace C is G-invariant and $\mathbf{codim}_F(C) = 2t \leq 2b$. If c is an arbitrary element of C, then $c = \sum_{\lambda \in \Gamma} \nu_\lambda d_\lambda$ for some finite subset Γ of Λ_t and $\nu_\lambda \in F$ for all $\lambda \in \Lambda_t$. We have

$$gc = g\left(\sum_{\lambda \in \Gamma} \nu_\lambda d_\lambda\right) = \sum_{\lambda \in \Gamma} \nu_\lambda g d_\lambda = \sum_{\lambda \in \Gamma} \nu_\lambda \tau_g d_\lambda = \tau_g \left(\sum_{\lambda \in \Gamma} \nu_\lambda d_\lambda\right) = \tau_g c.$$

Thus every one-dimensional subspace of C is $\langle g \rangle$-invariant. Since this is true for every element $g \in G$, every one-dimensional subspace of C is G-invariant. In turn, this means that every subspace of C is G-invariant, which completes the proof. $\square$

Corollary 6.29. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Suppose that D is a subspace of A such that $D = \bigoplus_{\lambda \in \Lambda} D_\lambda$ for some index set Λ, where D_λ is a minimal non-zero G-invariant subspace of A for all $\lambda \in \Lambda$. If there exists a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b, then D contains a G-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(D/C) \leq 4b$ *and*

(ii) every subspace of C is G-invariant.

Proof. Subdivide the set of indices Λ into two disjoint subsets Γ and Δ as follows:

$$\Gamma = \{\lambda \in \Lambda |\, \dim_F(D_\lambda) > 1\}, \Delta = \{\lambda \in \Lambda |\, \dim_F(D_\lambda) = 1\}.$$

Then let $U = \bigoplus_{\lambda \in \Gamma} D_\lambda$ and $V = \bigoplus_{\lambda \in \Delta} D_\lambda$. Clearly both the subspaces U and V are G-invariant and $D = U \oplus V$. Corollary 6.27 shows that $\mathbf{dim}_F(U) \leq 2b$. Using Corollary 6.28 we deduce that the subspace V contains a G-invariant subspace C such that $\mathbf{dim}_F(V/C) \leq 2b$ and every subspace of C is G-invariant. We clearly have $\mathbf{dim}_F(D/C) \leq 4b$ which completes the proof. □

Let G be a subgroup of $GL(F, A)$ and suppose that the lower measure of non G-invariance of all subspaces of A is bounded. If H is a subgroup of G, then $\mathbf{core}_G\, B \leq \mathbf{core}_H\, B$ for every subspace B of A. It follows that the lower measure of non H-invariance of all subspaces of A is also bounded. This suggests that a natural first step here is to study the case when the lower measure of non $\langle g \rangle$-invariance of all subspaces is bounded for each cyclic subgroup $\langle g \rangle$ of the group G. Clearly such an investigation subdivides naturally into two cases depending on whether an element g of G has finite or infinite order. If the element g has finite order, then the general case can be reduced to the case when the element g has order q^n for some prime q and natural number n. There are then two further cases depending on whether the characteristic of the field F is precisely the prime q or whether q is relatively prime to $\mathbf{char}\,(F)$.

Lemma 6.30. *Let A be a vector space over a field F and let G be a subgroup of $GL(F, A)$. Let g be a unipotent element of G and suppose that there exists a natural number b such that the lower measure of non $\langle g \rangle$-invariance of every subspace of A is at most b. If D is a subspace of A such that $D \cap C_A(g) = 0$, then $\boldsymbol{dim}_F(D) \leq b$.*

Proof. Since g is unipotent it follows that the vector space A is $\langle g \rangle$-nilpotent. If B is a $\langle g \rangle$-invariant subspace of A, then Lemma 1.6 implies that $B \cap C_A(g) \neq 0$. Let $E = \mathbf{core}_{\langle g \rangle}\, D$. Since E is contained in D and $D \cap C_A(g) = 0$ it follows that $E \cap C_A(g) = 0$ also. If we suppose that $E \neq 0$, then since E is $\langle g \rangle$-invariant it follows, from our remark above, that $E \cap C_A(g) \neq 0$ and we obtain a contradiction. This contradiction shows that $E = 0$. We therefore deduce that $\mathbf{dim}_F(D) = \mathbf{dim}_F(D/\mathbf{core}_{\langle g \rangle}\, D) \leq b$. □

Proposition 6.31. *Let F be a field of prime characteristic p. Let A be a vector space over F and let G be a subgroup of $GL(F, A)$. Let g be a p-element of G and suppose that there exists a natural number b such that the lower measure of non $\langle g\rangle$-invariance of every subspace of A is at most b. Then $\boldsymbol{dim}_F(A/C_A(g)) \leq b$.*

Proof. Lemma 1.28 shows that the element g is unipotent. In other words, there exists a natural number n such that $(g-1)^n = 0$. We can think of A as an $F\langle x\rangle$-module, where $\langle x\rangle$ is an infinite cyclic group and the action of x on A is defined via the rule $xa = ga$ for each element $a \in A$. It follows from this definition that $(x-1)^n a = 0$ for each element $a \in A$. Therefore $(x-1)^n \in \mathbf{Ann}_{F\langle x\rangle}(A)$ and in particular we note that $\mathbf{Ann}_{F\langle x\rangle}(A) \neq 0$. Then A can be decomposed into a direct sum of cyclic $F\langle x\rangle$-submodules according to [119, Corollary 6.5], for example. Hence

$$A = \oplus_{\lambda\in\Lambda} F\langle x\rangle a_\lambda$$

for certain elements $a_\lambda \in A$ and for some index set Λ.

The fact that g is a unipotent element implies that there are elements $a_{\lambda,j} \in A$ such that

$$F\langle x\rangle a_\lambda = Fa_{\lambda,1} \oplus Fa_{\lambda,2} \oplus \cdots \oplus Fa_{\lambda,k(\lambda)} \text{ where } k(\lambda) \leq n.$$

Here we define the $a_{\lambda,j}$ by

$$(g-1)a_{\lambda,1} = 0, (g-1)a_{\lambda,2} = a_{\lambda,1}, \ldots, (g-1)a_{\lambda,k(\lambda)} = a_{\lambda,k(\lambda)-1}.$$

We now set

$$\begin{aligned} D_j &= \oplus_{\lambda\in\Lambda} Fa_{\lambda,j} \text{ for } 1 \leq j \leq n \text{ and} \\ D &= \oplus_{2\leq j\leq n} D_j. \end{aligned}$$

Then $A = D_1 \oplus D_2 \oplus \cdots \oplus D_n = D_1 \oplus D$. We have $D_1 = C_A(g)$ so $D \cap C_A(g) = 0$. Using Lemma 6.30 we deduce that $\mathbf{dim}_F(D) \leq b$. It remains to now notice that $\mathbf{dim}_F(D) = \mathbf{dim}_F(A/C_A(g))$ and the result follows. □

Proposition 6.32. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G of finite order and suppose that if $\boldsymbol{char}(F) = p$ a prime, then g is a p'-element. Suppose that there exists a natural number b such that the lower measure of non $\langle g\rangle$-invariance of every subspace of A is at most b. Then A contains a $\langle g\rangle$-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(A/C) \leq 4b$ and

(ii) every subspace of C is $\langle g\rangle$-invariant.

Proof. We may again think of A as an $F\langle x\rangle$-module, where $\langle x\rangle$ is an infinite cyclic group and the action of x on A is defined by the rule $xa = ga$ for each element $a \in A$. Our hypotheses imply that A is a semisimple $F\langle x\rangle$-module (see [116, Corollary 5.15], for example) and hence $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ where each of the direct summands A_λ is a non-zero minimal $\langle g\rangle$-invariant subspace for each $\lambda \in \Lambda$. We may now apply Corollary 6.29 to deduce the result. □

Corollary 6.33. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G of finite order and suppose that there exists a natural number b such that the lower measure of non $\langle g\rangle$-invariance of every subspace of A is at most b. Then A contains a $\langle g\rangle$-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(A/C) \le 5b$ and

(ii) every subspace of C is $\langle g\rangle$-invariant.

Proof. If the field F has characteristic 0, then we can apply Proposition 6.32 to deduce the result. Therefore we may assume that the field F has prime characteristic p. In this case, since g has finite order, we may express the element g as a product $g = xy$ in which x is a p-element, y is a p'-element and $xy = yx$. Let $D = C_A(x)$. Clearly the lower measure of non-$\langle x\rangle$-invariance of every subspace of A is at most b and by Proposition 6.31 we deduce that $\mathbf{dim}_F(A/D) \le b$. For every element $d \in D$ we have

$$gd = (yx)d = y(xd) = yd. \tag{6.2}$$

We apply Proposition 6.32 to the subspace D to see that D contains a $\langle y\rangle$-invariant subspace C such that $\mathbf{dim}_F(D/C) \le 4b$; furthermore every subspace of C is $\langle y\rangle$-invariant. The equation (6.2) implies that every subspace of D that is $\langle y\rangle$-invariant is also $\langle g\rangle$-invariant and it follows that every subspace of C is actually $\langle g\rangle$-invariant. Finally it is clear that $\mathbf{dim}_F(A/C) \le 5b$ which completes the proof. □

The next natural consideration concerns the case when an element has infinite order. Here we need to apply some concepts and results from the theory of modules over an integral domain.

Lemma 6.34. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G of infinite order and suppose that there exists a natural number b such that the lower measure of non $\langle g\rangle$-invariance of every subspace of A is at most b. Then A is periodic as an $F\langle g\rangle$-module.*

Proof. We suppose, for a contradiction, that A contains a non-zero element a such that $\mathbf{Ann}_{F\langle g\rangle}(a) = 0$. It follows that the $\langle g\rangle$-invariant subspace $\langle g\rangle(Fa)$ generated by a (which is precisely the same as the $F\langle g\rangle$-submodule generated by a) is isomorphic to the group ring $F\langle g\rangle$, when thought of as a module over itself. Since $F\langle g\rangle = \bigoplus_{n\in\mathbb{Z}} Fg^n$ we obtain the direct decomposition

$$\langle g\rangle(Fa) = \oplus_{n\in\mathbb{Z}} g^n(Fa).$$

Let $x = g^2$ and $D = \langle x \rangle (Fa)$. We observe at once that

$$D \cong \oplus_{n \in \mathbb{Z}} F x^n = \oplus_{n \in \mathbb{Z}} F g^{2n}.$$

It follows that D has infinite F-dimension and that $\langle g \rangle (Fa) = D \oplus gD$. Let $C = \mathbf{core}_{\langle g \rangle} D$. Our hypotheses imply that $\mathbf{dim}_F(D/C) \leq b$ which shows that C also has infinite F-dimension. In particular, the subspace C is non-zero. On the other hand, let c be an arbitrary non-zero element of C. Since C is $\langle g \rangle$-invariant we have $gc \in C$, so that $gc \in D$. However $gc \in gD$, so that $gc \in D \cap gD = 0$. It follows that $gc = 0$ and from this we deduce the contradiction that $c = 0$. This proves the result. $\square$

In the next result we let

$$A[1] = \bigoplus_{P \in \pi} \Omega_{P,1}(A_P).$$

Lemma 6.35. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G of infinite order and suppose that there exists a natural number b such that the lower measure of non $\langle g \rangle$-invariance of every subspace of A is at most b. If C is a subspace of A such that $C \cap A[1] = 0$, then $\boldsymbol{dim}_F(C) \leq b$.*

Proof. By Lemma 6.34 A is periodic as an $F\langle g \rangle$-module. Proposition 3.11 shows that in this case every non-zero $\langle g \rangle$-invariant subspace has non-zero intersection with $A[1]$. Let $E = \mathbf{core}_{\langle g \rangle} C$ so that $\mathbf{dim}_F(C/E) \leq b$. Since E is $\langle g \rangle$-invariant, if we assume that $E \neq 0$, then our previous remark shows that $E \cap A[1] \neq 0$. On the other hand $E \leq C$ and $C \cap A[1] = 0$ by hypothesis. This contradiction shows that $E = 0$ and hence $\mathbf{dim}_F(C) \leq b$, as required. $\square$

Proposition 6.36. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G of infinite order and suppose that there exists a natural number b such that the lower measure of non $\langle g \rangle$-invariance of every subspace of A is at most b. Then A contains a $\langle g \rangle$-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(A/C) \leq 5b$ *and*

(ii) every subspace of C is $\langle g \rangle$-invariant.

Proof. Again using Lemma 6.34 we see that A is periodic as an $F\langle g \rangle$-module. We let D denote the F-complement to the subspace $A[1]$, so that $A = A[1] \oplus D$. Then Lemma 6.35 shows that the F-dimension of D is at most b, so that $\mathbf{dim}_F(A/A[1]) \leq b$ also.

Let $\pi = \mathbf{Ass}(F\langle g \rangle)$. Then we have $A[1] = \bigoplus_{P \in \pi} \Omega_{P,1}(A_P)$. Furthermore, $P = \mathbf{Ann}_{F\langle g \rangle}(A_P)$ is a prime ideal of the group ring $F\langle g \rangle$. Since $F\langle g \rangle$ is a principal ideal domain it follows that P is a maximal ideal of $F\langle g \rangle$ so that in this case the quotient ring $F_P = F\langle g \rangle / P$ is a field. Hence we can think of

$\Omega_{P,1}(A_P)$ as a vector space over F_P. Let $\{a_\lambda | \lambda \in \Lambda_P\}$ be a basis for this vector space, for some index set Λ_P. Then $\Omega_{P,1}(A_P) = \bigoplus_{\lambda \in \Lambda_P} F_P a_\lambda$. As we saw in Chapter 2 it is then the case that $F_P a_\lambda = A_\lambda$ is a simple module over $F\langle g\rangle$ for $\lambda \in \Lambda_P$. Let $\Lambda = \bigcup_{P \in \pi} \Lambda_P$, so that $A[1] = \bigoplus_{\lambda \in \Lambda} A_\lambda$ and each A_λ is a simple $F\langle g\rangle$-module for all $\lambda \in \Lambda$. Corollary 6.29 then implies that $A[1]$ contains a $\langle g\rangle$-invariant subspace C such that $\mathbf{dim}_F(A[1]/C) \leq 4b$ and every subspace of C is $\langle g\rangle$-invariant. Since $\mathbf{dim}_F(A/A[1]) \leq b$ we deduce that $\mathbf{dim}_F(A/C) \leq 5b$ which proves the proposition. □

Corollary 6.37. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Let g be an element of G and suppose that there exists a natural number b such that the lower measure of non $\langle g\rangle$-invariance of every subspace of A is at most b. Then A contains a $\langle g\rangle$-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(A/C) \leq 5b$ and

(ii) every subspace of C is $\langle g\rangle$-invariant.

Proof. If the element g has finite order, then this result follows from Corollary 6.33 and if the element g has infinite order, then we can apply Proposition 6.36. □

We can now prove the main results of this section.

Theorem 6.38. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that there exists a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b. Then A contains a G-invariant subspace C such that*

(i) $\boldsymbol{dim}_F(A/C) \leq 6b^2 + b$ and

(ii) every subspace of C is G-invariant.

Proof. If every one-dimensional subspace of A is G-invariant, then every subspace of A is G-invariant and the result follows with $C = A$ in this case. Hence we may suppose that there exists at least one one-dimensional subspace that is not G-invariant. This means that there is an element $g_1 \in G$ and an element $a_1 \in A$, such that $g_1 a_1 \notin F a_1$ Let $d_1 = (g_1 - 1)a_1$. If we suppose that the element $d_1 \in F a_1$, then $d_1 = \nu a_1$ for some element $\nu \in F$. We then have $\nu a_1 = g_1 a_1 - a_1$, so that $g_1 a_1 = (\nu + 1)a_1$, which contradicts the fact that $g_1 a_1 \notin F a_1$. Hence $d_1 \notin F a_1$ and hence the subspace $F a_1 + F d_1$ has dimension 2. Using Corollary 6.37 we deduce that A contains a $\langle g_1\rangle$-invariant subspace D_1 such that $\mathbf{dim}_F(A/D_1) \leq 5b$ and every subspace of D_1 is $\langle g_1\rangle$-invariant. Since $F a_1$ is not $\langle g_1\rangle$-invariant this means that the intersection $(F a_1 + F d_1) \cap D_1$ is either 0 or has dimension 1. In the former case we let $E_1 = D_1$. On the other hand, when $\mathbf{dim}_F((F a_1 + F d_1) \cap D_1) = 1$ there exist $\lambda, \mu \in F$ such that $0 \neq \lambda a_1 + \mu d_1 \in D_1$. In this case we let E_1 be an F-complement to

$(Fa_1 + Fd_1) \cap D_1$ in the subspace D_1. In either case E_1 is a $\langle g_1 \rangle$-invariant subspace of D_1 and every subspace of E_1 is likewise $\langle g_1 \rangle$-invariant. If $\lambda = 0$, then $Fd_1 = (Fa_1 + Fd_1) \cap D_1$, so E_1 is actually a complement to Fd_1 and hence $d_1 \notin E_1$. If $\lambda \neq 0$ and if $d_1 \in D_1$, then Fa_1 is a $\langle g_1 \rangle$-invariant subspace of D_1, a contradiction. Hence if $\lambda \neq 0$, then $d_1 \notin D_1$ so in any case $d_1 \notin E_1$. Hence we have

$$\mathbf{dim}_F(A/E_1) = \mathbf{dim}_F(A/D_1) + \mathbf{dim}_F(D_1/E_1) \leq 5b + 1.$$

Furthermore, $(Fa_1 + Fd_1) \cap E_1 = 0$. Let $L_1 = \mathbf{core}_G\, E_1$, so that L_1 is a G-invariant subspace. Then we have, by hypothesis,

$$\begin{aligned}\mathbf{dim}_F(A/L_1) &= \mathbf{dim}_F(A/E_1) + \mathbf{dim}_F(E_1/L_1)\\ &\leq 5b + 1 + b = 6b + 1\end{aligned}$$

and every subspace of L_1 is $\langle g_1 \rangle$-invariant.

If every one-dimensional subspace of L_1 is G-invariant, then we set $C = L_1$ to complete the proof. Hence we may assume that there is an element $g_2 \in G$ and an element $a_2 \in L_1$ such that $g_2 a_2 \notin Fa_2$. Let $d_2 = (g_2 - 1)a_2$. Note that, as before, we can show that $d_2 \notin Fa_2$ and therefore the subspace $Fa_2 + Fd_2$ has dimension 2. Using the same arguments as those above, we can select a subspace L_2 of L_1 satisfying the following conditions:

(a) $\mathbf{dim}_F(L_1/L_2) \leq 6b + 1$;

(b) there is a $\langle g_2 \rangle$-invariant subspace E_2 such that $(Fa_2 + Fd_2) \cap E_2 = 0$;

(c) Every subspace of L_2 is $\langle g_2 \rangle$-invariant.

Then, in particular, $\mathbf{dim}_F(A/L_2) \leq 2(6b+1)$. Furthermore every subspace of L_2 is $\langle g_1, g_2 \rangle$-invariant.

If L_2 contains a subspace which is not G-invariant, then we continue the process of choosing elements $g_j \in G, a_j \in A$. Assume that we have already chosen elements $g_1, \ldots, g_n$ of the group G and elements $a_1, \ldots, a_n$ in the vector space A satisfying the conditions

(a) $a_j(g_j - 1) = d_j$, for every $1 \leq j \leq n$;

(b) $\mathbf{dim}_F(a_j F + d_j F) = 2, g_j a_j \notin Fa_j$ and $Fd_j \cap Fa_j = 0$;

(c) $(Fa_j + Fd_j) \cap \bigoplus_{k \neq j}(Fa_k + Fd_k) = 0$ for all j such that $1 \leq j \leq n$;

(d) the subspaces Fa_j and Fd_j are $\langle g_k \rangle$-invariant for all $k < j$, for $1 \leq j \leq n$.

Suppose that $n > b$ and set

$$\begin{aligned}B &= Fa_1 \oplus Fa_2 \oplus \cdots \oplus Fa_n,\\ D &= Fd_1 \oplus Fd_2 \oplus \cdots \oplus Fd_n.\end{aligned}$$

Then

$$B \cap D = 0 \text{ and } \mathbf{dim}_F(B) = \mathbf{dim}_F(D) = n.$$

Let $Y = \mathbf{core}_G B$. Our hypotheses imply that $\mathbf{dim}_F(B/Y) \leq b$. Then since $n > b$ we deduce that Y is a non-zero subspace. Let y be an arbitrary element of the subspace Y. Since $Y \leq B$ we have that $y = \sigma_1 a_1 + \cdots + \sigma_n a_n$, for certain elements $\sigma_1, \ldots, \sigma_n \in F$. Let m be the first index j for which $\sigma_j \neq 0$. Then actually we have $y = \sigma_m a_m + \cdots + \sigma_n a_n$. However,

$$\begin{aligned}(g_m - 1)y &= (g_m - 1)(\sigma_m a_m + \cdots + \sigma_n a_n)\\ &= (g_m - 1)\sigma_m a_m + \cdots + (g_m - 1)\sigma_n a_n\\ &= \sigma_m d_m + \tau_{m+1} a_{m+1} + \cdots + \tau_n a_n\end{aligned}$$

for certain elements $\tau_{m+1}, \ldots, \tau_n \in F$. The fact that Y is G-invariant implies that $(g_m - 1)y \in Y \leq B$. Clearly, B contains the element $\tau_{m+1}a_{m+1} + \cdots + \tau_n a_n$ so that

$$\sigma_m d_m = (g_m - 1)y - (\tau_{m+1} a_m + \cdots + \tau_n a_n) \in B$$

which contradicts the fact that $B \cap D = 0$. This contradiction shows that the process of choosing the elements g_j, a_j must terminate in at most b steps. This means that there is a G-invariant subspace C such that $\mathbf{dim}_F(A/C) \leq b(6b+1)$ and every subspace of C is G-invariant. This completes the proof. □

The following theorem gives us information concerning the structure of linear groups G in which the lower measures of non G-invariance of every subspace of the associated vector space are bounded.

Theorem 6.39. *Let A be a vector space over the field F and let G be a subgroup of $GL(F, A)$. Suppose that there exists a natural number b such that the lower measure of non G-invariance of every subspace of A is at most b. Then G contains a normal subgroup K satisfying the following conditions:*

(i) G/K is isomorphic to some subgroup of a direct product $U \times V$, where U is the multiplicative group of the field F and $V = GL_k(F)$ where $k = 6b^2 + b$;

(ii) if F has characteristic 0, then K is a torsion-free abelian group;

(iii) if F has characteristic the prime p, then K is an elementary abelian p-group.

Proof. Theorem 6.38 shows that the vector space A contains a G-invariant subspace C having codimension at most $k = 6b^2 + b$ such that the subspaces of C are all G-invariant. Let $G_1 = C_G(C)$. Then Corollary 3.58 shows that G/G_1 is isomorphic to some subgroup of the multiplicative group $U = U(F)$ of the field F.

Let $G_2 = C_G(A/C)$. Since A/C has finite dimension at most k it follows that G/G_2 is isomorphic to some subgroup of the finite dimensional linear group $V = GL_k(F)$. Let $K = G_1 \cap G_2$ and observe that K is a normal subgroup of G. Using Remak's theorem we obtain an embedding of G/K in the direct product $G/G_1 \times G/G_2$. Furthermore, the series $0 \leq C \leq A$ is K-central and an application of Theorem 1.2 implies that K is abelian; indeed, when $\mathbf{char}\,(F) = 0$, K is torsion-free and when $\mathbf{char}\,(F) = p$, a prime, then K is an elementary abelian p-group. This completes the proof. $\square$

We remark that the study of linear groups associated with measures of non G-invariance of subspaces was continued in the papers [117] and [118] of L. A. Kurdachenko, A. V. Sadovnichenko and I. Ya. Subbotin, but we shall not investigate the results of those papers here.

In this book we have attempted to give an account concerning some of the theory of infinite dimensional linear groups as that theory now stands. Of necessity, due to a lack of space, there are many papers and results which we have not discussed, or only briefly so. It is hoped that on reading this book researchers will investigate the algebraic properties of infinite dimensional linear groups further. Certainly much more research can and should be done concerning this class of groups.

Bibliography

F. W. Anderson and K. R. Fuller.

[1] *Rings and Categories of Modules.* Springer-Verlag, New York-Heidelberg, 1974. Graduate Texts in Mathematics, Vol. 13.

M. F. Atiyah and I. G. Macdonald.

[2] *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

R. Baer.

[3] Finiteness properties of groups. *Duke Math. J.*, 15:1021–1032, 1948.

[4] Endlichkeitskriterien für Kommutatorgruppen. *Math. Ann.*, 124:161–177, 1952.

[5] Noethersche Gruppen. *Math. Z.*, 66:269–288, 1956.

G. Baumslag.

[6] *Lecture Notes on Nilpotent Groups.* American Mathematical Society, Providence, R.I., 1971.

V. V. Belyaev.

[7] Locally finite groups with Chernikov Sylow p-subgroups. *Algebra i Logika*, 20:605–619, 1981. English transl. in Algebra and Logic, **20** (1981) 393-402.

[8] Semisimple periodic groups of finitary transformations. *Algebra i Logika*, 32:17–33, 1993. English transl. in Algebra and Logic, **32** (1993) 8-16.

[9] Irreducible periodic groups of finitary transformation. *Algebra i Logika*, 33:109–134, 1994. English transl. in Algebra and Logic, **33** (1994) 65-77.

[10] Structure of periodic finitary transformation groups. *Algebra i Logika*, 33:347–366, 1994. English transl. in Algebra and Logic, **33** (1994) 195-204.

M. Bernkopf.

[11] A history of infinite matrices: A study of denumerably infinite linear systems as the first step in the history of operators defined on function spaces. *Arch. History Exact Sci.*, 4(4):308–358, 1968.

A. Bier.

[12] Commutators and powers of infinite unitriangular matrices. *Linear Algebra Appl.*, 457:162–178, 2014.

[13] On lattices of closed subgroups in the group of infinite triangular matrices over a field. *Linear Algebra Appl.*, 485:132–152, 2015.

A. Bier and W. Hołubowski.

[14] A note on commutators in the group of infinite triangular matrices over a ring. *Linear Multilinear Algebra*, 63(11):2301–2310, 2015.

S. Black.

[15] A finitary Tits' alternative. *Arch. Math. (Basel)*, 72(2):86–91, 1999.

A. I. Borevich and I. R. Shafarevich.

[16] *Number Theory.* Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20. Academic Press, New York-London, 1966.

N. Bourbaki.

[17] *Éléments de mathématique. XI. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre IV: Polynomes et fractions rationnelles. Chapitre V: Corps commutatifs.* Actualités Sci. Ind., no. 1102. Hermann et Cie., Paris, 1950.

[18] *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation.* Actualités Scientifiques et Industrielles, No. 1290. Herman, Paris, 1961.

[19] *Éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.* Actualités Scientifiques et Industrielles, No. 1293. Hermann, Paris, 1961.

[20] *Éléments de mathématique. Première partie. Fascicule VI. Livre II: Algèbre. Chapitre 2: Algèbre linéaire.* Troisième édition, entièrement refondue. Actualités Sci. Indust., No. 1236. Hermann, Paris, 1962.

[21] *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations.* Actualités Scientifiques et Industrielles, No. 1308. Hermann, Paris, 1964.

[22] *Éléments de mathématique. Fasc. XXXI. Algèbre commutative. Chapitre 7: Diviseurs.* Actualités Scientifiques et Industrielles, No. 1314. Hermann, Paris, 1965.

A. A. Bovdi.

[23] The structure of the multiplicative group of an integral group ring. *Dokl. Akad. Nauk SSSR*, 301(6):1295–1297, 1988.

B. Bruno, M. Dalle Molle, and F. Napolitani.

[24] On finitary linear groups with a locally nilpotent maximal subgroup. *Arch. Math. (Basel)*, 76(6):401–405, 2001.

B. Bruno and R. E. Phillips.

[25] Residual properties of finitary linear groups. *J. Algebra*, 166(2):379–392, 1994.

[26] A note on groups with nilpotent-by-finite proper subgroups. *Arch. Math.*, 65:369–374, 1995.

J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith, and J. Wiegold.

[27] Groups with all subgroups normal-by-finite. *J. Austral. Math. Soc. Ser. A*, 59(3):384–398, 1995.

V. S. Charin and D. I. Zaitsev.

[28] Groups with finiteness conditions and other restrictions for subgroups. *Ukrain. Mat. Zh.*, 40(3):277–287, 405, 1988.

A. W. Chatters and C. R. Hajarnavis.

[29] *Rings with Chain Conditions*, volume 44 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.

S. N. Chernikov.

[30] Finiteness conditions in the general theory of groups. *Uspehi Mat. Nauk*, 14(5 (89)):45–96, 1959.

[31] *Gruppy s zadannymi svoistvami sistemy podgrupp (Groups with Prescribed Properties of Systems of Subgroups).* "Nauka", Moscow, 1980. Sovremennaya Algebra. [Modern Algebra].

A. H. Clifford.

[32] Representations induced in an invariant subgroup. *Ann. of Math. (2)*, 38(3):533–550, 1937.

J.-L. Colliot-Thélène, R. M. Guralnick, and R. Wiegand.

[33] Multiplicative groups of fields modulo products of subfields. *J. Pure Appl. Algebra*, 106(3):233–262, 1996.

R. G. Cooke.

[34] *Infinite Matrices and Sequence Spaces.* Dover Publications, Inc., New York, 1965.

C. W. Curtis and I. Reiner.

[35] *Representation Theory of Finite Groups and Associative Algebras.* Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[36] *Methods of Representation Theory, with Applications to Finite Groups and Orders. Vol. I.* Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1981. A Wiley-Interscience Publication.

[37] *Methods of Representation Theory, with Applications to Finite Groups and Orders. Vol. II.* Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1987. A Wiley-Interscience Publication.

O. Yu Dashkova, M. R. Dixon, and L. A. Kurdachenko.

[38] Infinite dimensional linear groups with the restrictions on subgroups of infinite rank. *Proceedings of Gomel University*, 3:109–123, 2006.

[39] Linear groups with rank restrictions on the subgroups of infinite central dimension. *J. Pure Appl. Algebra*, 208(3):785–795, 2007.

R. Dedekind.

[40] Über Gruppen deren sämtliche Teiler Normalteiler sind. *Math. Ann.*, 48:548–561, 1897.

M. R. Dikson, M. 'Evans, and L. A. Kurdachenko.

[41] Linear groups with the minimality condition for some infinite-dimensional subgroups. *Ukraïn. Mat. Zh.*, 57(11):1476–1489, 2005.

J. D. Dixon.

[42] *Linear Groups.* Van Nostrand Reinhold Company, London, 1971.

M. R. Dixon, M. J. Evans, and L. A. Kurdachenko.

[43] Linear groups with the minimal condition on subgroups of infinite central dimension. *J. Algebra*, 277:172–186, 2004.

M. R. Dixon and Y. Karatas.

[44] Groups with all subgroups permutable or of finite rank. *Cent. Eur. J. Math.*, 10(3):950–957, 2012.

M. R. Dixon and L. A. Kurdachenko.

[45] Linear groups with infinite central dimension. In *Groups St. Andrews 2005. Vol. 1*, volume 339 of London Math. Soc. Lecture Note Ser., pages 306–312. Cambridge Univ. Press, Cambridge, 2007.

[46] Abstract and linear groups with some specific restrictions. In *Meeting on Group Theory and Its Applications*, pages 87–106. Madrid, Zaragoza 2011, 2012.

M. R. Dixon, L. A. Kurdachenko, J. M. Muñoz-Escolano, and J. Otal.

[47] Trends in infinite dimensional linear groups. In *Groups St Andrews 2009 in Bath. Volume 1*, volume 387 of London Math. Soc. Lecture Note Ser., pages 271–282. Cambridge Univ. Press, Cambridge, 2011.

M. R. Dixon, L. A. Kurdachenko, and J. Otal.

[48] Linear groups with bounded action. *Algebra Colloq.*, 18(3):487–498, 2011.

[49] "Linear Groups with Finite Dimensional Orbits." In *Ischia Group Theory 2010*, pages 131–145. World Sci. Publ., Hackensack, NJ, 2012.

[50] Linear analogues of theorems of Schur, Baer and Hall. *Int. J. Group Theory*, 2(1):79–89, 2013.

[51] On the structure of some infinite dimensional linear groups. *Comm. Algebra*, 45(1):234–246, 2017.

M. R. Dixon, L. A. Kurdachenko, and I. Ya. Subbotin.

[52] *Ranks of Groups: The Tools, Characteristics, and Restrictions.* John Wiley & Sons, Inc., Hoboken, NJ, 2017.

K. Doerk and T. Hawkes.

[53] *Finite Soluble Groups*, volume 4 of De Gruyter Expositions in Mathematics. Walter de Gruyter & Co., Berlin, 1992.

C. Faith.

[54] *Algebra: Rings, Modules and Categories. I.* Springer-Verlag, New York-Heidelberg, 1973. Die Grundlehren der Mathematischen Wissenschaften, Band 190.

[55] *Algebra. II.* Springer-Verlag, Berlin-New York, 1976. Ring Theory, Grundlehren der Mathematischen Wissenschaften, No. 191.

W. Feit.

[56] *The Representation Theory of Finite Groups*, volume 25 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York, 1982.

S. Franciosi, F. de Giovanni, and L. A. Kurdachenko.

[57] Groups whose proper quotients are FC-groups. *J. Algebra*, 186(2):544–577, 1996.

L. Fuchs.

[58] *Infinite Abelian Groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.

[59] *Infinite Abelian Groups. Vol. II.* Pure and Applied Mathematics. Vol. 36-II. Academic Press, New York, 1973.

L. Fuchs and L. Salce.

[60] *Modules Over Valuation Domains*, volume 97 of Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1985.

R. Gilmer.

[61] *Multiplicative Ideal Theory.* Marcel Dekker, Inc., New York, 1972. Pure and Applied Mathematics, No. 12.

V. M. Glushkov.

[62] On some questions of the theory of nilpotent and locally nilpotent groups without torsion. *Mat. Sbornik N.S.*, 30(72):79–104, 1952.

J. S. Golan. and T. Head.

[63] *Modules and the Structure of Rings*, volume 147 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1991. A primer.

Yu. M. Gorchakov.

[64] *Gruppy s konechnymi klassami sopryazhennykh elementov [Groups with Finite Classes of Conjugate Elements].* “Nauka”, Moscow, 1978. Sovremennaya Algebra. [Modern Algebra Series].

D. Gorenstein.

[65] *Finite Groups.* Harper & Row Publishers, New York, 1968.

[66] *Finite Simple Groups, an Introduction to Their Classification.* University Series in Mathematics. Plenum Publishing Corp., New York, 1982.

R. Grigorchuk.

[67] On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.

R. Grigorchuk, Y. Leonov, V. Nekrashevych, and V. Sushchansky.

[68] Self-similar groups, automatic sequences, and unitriangular representations. *Bull. Math. Sci.*, 6(2):231–285, 2016.

W. H. Gustafson.

[69] Book Review: Matrix groups. *Bull. Amer. Math. Soc.*, 83(3):318–322, 1977.

J. I. Hall.

[70] Locally finite simple groups of finitary linear transformations. In B. Hartley, G. M. Seitz, A. V. Borovik, and R. M. Bryant, editors, *Finite and Locally Finite Groups*, volume 471 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pages 147–188, Istanbul 1994, 1995. Kluwer Acad. Publ., Dordrecht.

[71] Periodic simple groups of finitary linear transformations. *Ann. of Math. (2)*, 163(2):445–498, 2006.

J. I. Hall and B. Hartley.

[72] A group theoretical characterization of simple, locally finite, finitary linear groups. *Arch. Math. (Basel)*, 60(2):108–114, 1993.

P. Hall.

[73] Finite-by-nilpotent groups. *Proc. Cambridge Philos. Soc.*, 52:611–616, 1956.

[74] "Nilpotent Groups." Notes of lectures given at the Canadian Mathematical Congress, University of Alberta, 1957.

[75] *The Edmonton Notes on Nilpotent Groups*. Queen Mary College Mathematics Notes, London, 1969.

P. Hall and C. R. Kulatilaka.

[76] A property of locally finite groups. *J. London Math. Soc.*, 39:235–239, 1964.

B. Hartley.

[77] A class of modules over a locally finite group. II. *J. Austral. Math. Soc.*, 19(4):437–469, 1975.

[78] A dual approach to Chernikov modules. *Math. Proc. Camb. Phil. Soc.*, 82:215–239, 1977.

[79] Fixed points of automorphisms of certain locally finite groups and Chevalley groups. *J. London Math. Soc.*, 37:421–436, 1988.

B. Hartley and T. O. Hawkes.

[80] *Rings, Modules and Linear Algebra. A Further Course in Algebra Describing the Structure of Abelian Groups and Canonical Forms of Matrices Through the Study of Rings and Modules.* Chapman and Hall, Ltd., London, 1970.

W. Holubowski.

[81] Free subgroups of the group of infinite unitriangular matrices. *Internat. J. Algebra Comput.*, 13(1):81–86, 2003.

[82] *Algebraic Properties of Groups of Infinite Matrices*, volume 671 of Monografia (Gliwice) [Monograph (Gliwice)]. Wydawnictwo Politechniki Śląskiej, Gliwice, 2017. in Russian.

W. Holubowski, I. Kashuba, and S. Żurek.

[83] Derivations of the Lie algebra of infinite strictly upper triangular matrices over a commutative ring. *Comm. Algebra*, 45(11):4679–4685, 2017.

W. Holubowski and R. Slowik.

[84] Parabolic subgroups of groups of column-finite infinite matrices. *Linear Algebra Appl.*, 437(2):519–524, 2012.

X. Hou.

[85] Decomposition of infinite matrices into products of commutators of involutions. *Linear Algebra Appl.*, 563:231–239, 2019.

T. W. Hungerford.

[86] *Algebra*, volume 73 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1980. Reprint of the 1974 original.

B. Huppert.

[87] *Endliche Gruppen. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, 1967.

B. Huppert and N. Blackburn.

[88] *Finite groups. II*, volume 242 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin-New York, 1982. AMD, 44.

[89] *Finite groups. III*, volume 243 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin-New York, 1982.

K. Iwasawa.

[90] Einege sätze über freie gruppen. *Proc. Imp. Acad.Tokyo*, 19:272–274, 1943.

A. V. Jategaonkar.

[91] *Localization in Noetherian rings*, volume 98 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1986.

I. Kaplansky.

[92] *Infinite Abelian Groups.* Revised edition. The University of Michigan Press, Ann Arbor, Mich., 1969.

[93] "Topics in Commutative Ring Theory." Department of Mathematics, University of Chicago, Chicago, Ill., 1974. Lecture notes.

M. I. Kargapolov.

[94] On a problem of O. Yu. Schmidt. *Sibirsk. Mat. Ž.*, 4:232–235, 1963.

M. I. Kargapolov and Yu. I. Merzljakov.

[95] *Fundamentals of the Theory of Groups*, volume 62 of Graduate Texts in Mathematics. Springer Verlag, Berlin, Heidelberg, New York, 1979.

[96] *Osnovy teorii grupp (Foundations of the Theory of Groups).* "Nauka", Moscow, third edition, 1982.

G. Karpilovsky.

[97] *Commutative Group Algebras*, volume 78 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1983.

[98] *Field Theory, Classical Foundations and Multiplicative Groups*, volume 120 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1988.

L. S. Kazarin and L. A. Kurdachenko.

[99] Finiteness conditions and factorizations in infinite groups. *Uspekhi Mat. Nauk*, 47:3:75–114, 1992. English transl. in Math. Surveys, **47:3** (1992), 81-126.

O. H. Kegel and B. A. F. Wehrfritz.

[100] *Locally Finite Groups.* North-Holland Mathematical Library. North-Holland, Amsterdam, London, 1973. Volume 3.

A. Kertész.

[101] *Lectures on Artinian Rings*, volume 14 of Disquisitiones Mathematicae Hungaricae [Hungarian Mathematics Investigations]. Akadémiai Kiadó, Budapest, 1987. With chapters by Gerhard Betsch, Alfred Widiger and Richard Wiegandt, Translated from the German by Manfred Stern, Translation edited and with a preface by Wiegandt.

L. Kurdachenko and I. Subbotin.

[102] On some infinite dimensional linear groups. In *Groups St. Andrews 2001 in Oxford. Vol. II*, volume 305 of London Math. Soc. Lecture Note Ser., pages 377–384. Cambridge Univ. Press, Cambridge, 2003.

L. A. Kurdachenko.

[103] Nonperiodic FC-groups and related classes of locally normal groups and abelian torsion-free groups. *Sibirsk. Mat. Zh.*, 27(2):104–116, 222, 1986. English Translation in Siberian Math. J. **27** (1986), no. 2, 227–236.

[104] Modules over group rings with some finiteness conditions. *Ukraïn. Mat. Zh.*, 54(7):931–940, 2002.

[105] On some infinite dimensional linear groups. *Note Mat.*, 30(suppl. 1):21–36, 2010.

L. A. Kurdachenko, J. M. Muñoz-Escolano, and J. Otal.

[106] Groups acting on vector spaces with a large family of invariant subspaces. *Linear Multilinear Algebra*, 60(4):487–498, 2012.

[107] Antifinitary linear groups. *Forum Math.*, 20(1):27–44, 2008.

[108] Locally nilpotent linear groups with the weak chain conditions on subgroups of infinite central dimension. *Publ. Mat.*, 52(1):151–169, 2008.

[109] Soluble linear groups with some restrictions on subgroups of infinite central dimension. In *Ischia Group Theory 2008*, pages 156–173. World Sci. Publ., Hackensack, NJ, 2009.

L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, and N. N. Semko.

[110] Linear groups admitting an infinite dimensional deviation. In *Abstracts, First French-Spanish Congress of Mathematics*, pages 11–12. Zaragoza, 2007.

[111] Locally nilpotent linear groups with restrictions on their subgroups of infinite central dimension. *Geom. Dedicata*, 138:69–81, 2009.

[112] Locally nilpotent linear groups with some restrictions on subgroups of infinite central dimension. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, (3):25–28, 2009.

L. A. Kurdachenko and J. Otal.

[113] Simple modules over CC-groups and monolithic just non-CC-groups. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 4(2):381–390, 2001.

[114] The rank of the factor-group modulo the hypercenter and the rank of the some hypocenter of a group. *Cent. Eur. J. Math.*, 11(10):1732–1741, 2013.

L. A. Kurdachenko, J. Otal, and I. Ya. Subbotin.

[115] *Groups with Prescribed Quotient Groups and Associated Module Theory.* Series in Algebra. World Scientific, Singapore, 2002. Volume 8.

[116] *Artinian Modules over Group Rings.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

L. A. Kurdachenko, A. V. Sadovnichenko, and I. Ya. Subbotin.

[117] On some infinite dimensional linear groups. *Cent. Eur. J. Math.*, 7(2):176–185, 2009.

[118] Infinite dimensional linear groups with a large family of G-invariant subspaces. *Comment. Math. Univ. Carolin.*, 51(4):551–558, 2010.

L. A. Kurdachenko, N. N. Semko, and I. Ya. Subbotin.

[119] *Insight into Modules over Integral Domains*, volume 75 of Mathematics and its Applications. Proceedings of Institute of Mathematics, Kiev, Ukraine, Kiev, 2008.

L. A. Kurdachenko and H. Smith.

[120] Groups with the weak maximal condition for non-subnormal subgroups. *Ricerche Mat.*, 47(1):29–49, 1998.

[121] Groups in which all subgroups of infinite rank are subnormal. *Glasg. Math. J.*, 46(1):83–89, 2004.

L. A. Kurdachenko and I. Ya. Subbotin.

[122] Groups whose proper quotients are hypercentral. *J. Austral. Math. Soc. Ser. A*, 65(2):224–237, 1998.

[123] On Artinian modules over hyperfinite groups. In *Algebra and Its Applications (Athens, OH, 1999)*, volume 259 of Contemp. Math., pages 323–331. Amer. Math. Soc., Providence, RI, 2000.

[124] On minimal Artinian modules and minimal Artinian linear groups. *Int. J. Math. Math. Sci.*, 27(12):707–714, 2001.

[125] On some infinite-dimensional linear groups. *Comm. Algebra*, 29(2):519–527, 2001.

[126] On some infinite dimensional linear groups. *Southeast Asian Bull. Math.*, 26(5):773–787, 2003.

[127] Linear groups with the maximal condition on subgroups of infinite central dimension. *Publ. Mat.*, 50(1):103–131, 2006.

[128] A brief history of an important classical theorem. *Adv. Group Theory Appl.*, 2:121–124, 2016.

L. A. Kurdachenko, I. Ya. Subbotin, and T. Velichko.

[129] On some groups with only two types of subgroups. *Asian-Eur. J. Math.*, 7(4):1450057, 15, 2014.

A. G. Kurosh.

[130] *Teoriya Grupp.* Izdat. "Nauka", Moscow, augmented edition, 1967.

[131] *General Algebra.* Izdat. "Nauka", Moscow, 1974. Lectures for the academic year 1969–1970, Edited by T. M. Baranovič, in Russian.

A. G. Kurosh and S. N. Chernikov.

[132] Solvable and nilpotent groups. *Uspehi Matem. Nauk (N.S.)*, 2(3(19)):18–59, 1947.

T. Y. Lam.

[133] *A First Course in Noncommutative Rings*, volume 131 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.

S. Lang.

[134] *Algebra.* Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.

F. Leinen.

[135] Absolute irreducibility for finitary linear groups. *Rend. Sem. Mat. Univ. Padova*, 92:59–61, 1994.

[136] Hypercentral unipotent finitary skew linear groups. *Comm. Algebra*, 22(3):929–949, 1994.

[137] Irreducible representations of periodic finitary linear groups. *J. Algebra*, 180(2):517–529, 1996.

F. Leinen and O. Puglisi.

[138] Unipotent finitary linear groups. *J. London Math. Soc. (2)*, 48(1):59–76, 1993.

[139] Countable recognizability of primitive periodic finitary linear groups. *Math. Proc. Cambridge Philos. Soc.*, 121(3):425–435, 1997.

[140] Confined subgroups in periodic simple finitary linear groups. *Israel J. Math.*, 128:285–324, 2002.

M. S. Lucido.

[141] Changing the field characteristic on finitary linear groups. *Arch. Math. (Basel)*, 70(2):97–103, 1998.

A. I. Maltsev.

[142] On groups of finite rank. *Mat. Sbornik*, 22:351–352, 1948.

[143] On certain classes of infinite soluble groups. *Mat. Sbornik*, 28:367–388, 1951. English transl. Amer. Math. Soc. Translations.,**2** (1956), 1-21.

A. Marcoci, L. Marcoci, L. E. Persson, and N. Popa.

[144] Schur multiplier characterization of a class of infinite matrices. *Czechoslovak Math. J.*, 60(135)(1):183–193, 2010.

E. Matlis.

[145] *Torsion-free Modules*. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago-London, 1972.

J. C. McConnell and J. C. Robson.

[146] *Noncommutative Noetherian rings*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1987. With the cooperation of L. W. Small, A Wiley-Interscience Publication.

U. Meierfrankenfeld.

[147] Ascending subgroups of irreducible finitary linear group. *J. London Math. Soc. (2)*, 51(1):75–92, 1995.

U. Meierfrankenfeld, R. E. Phillips, and O. Puglisi.

[148] Locally solvable finitary linear groups. *J. London Math. Soc. (2)*, 47:31–40, 1993.

Yu I. Merzljakov.

[149] Locally solvable groups of finite rank. *Algebra i Logika Sem.*, 3(2):5–16, 1964.

G. A. Miller and H. Moreno.

[150] Non-abelian groups in which every subgroup is abelian. *Trans. Amer. Math. Soc.*, 4:398–404, 1903.

J. M. Muñoz-Escolano, J. Otal, and N. N. Semko.

[151] Periodic linear groups with the weak chain conditions on subgroups of infinite central dimension. *Comm. Algebra*, 36(2):749–763, 2008.

M. M. Murach.

[152] Certain generalized FC-groups of matrices. *Ukrain. Mat. Ž.*, 28(1):92–97, 143, 1976.

W. Narkiewicz.

[153] *Elementary and Analytic Theory of Algebraic Numbers.* Springer-Verlag, Berlin; PWN—Polish Scientific Publishers, Warsaw, second edition, 1990.

B. H. Neumann.

[154] Groups with finite classes of conjugate elements. *Proc. London Math. Soc. (3)*, 1:178–187, 1951.

[155] Groups covered by permutable subsets. *J. London Math. Soc.*, 29:236–248, 1954.

[156] Groups with finite classes of conjugate subgroups. *Math. Z.*, 63:76–96, 1955.

D. G. Northcott.

[157] *Lessons on Rings, Modules and Multiplicities.* Cambridge University Press, London, 1968.

A. S. Olijnyk and V. I. Sushchansky.

[158] A free group of infinite unitriangular matrices. *Mat. Zametki*, 67(3):382–386, 2000. English transl. in Math. Notes **67** (2000), 320-324.

A. Yu. Olshanskii.

[159] An infinite simple torsion-free Noetherian group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(6):1328–1393, 1979.

[160] An infinite group with subgroups of prime orders. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(2):309–321, 479, 1980.

[161] Groups of bounded exponent with subgroups of prime order. *Algebra i Logika*, 21:553–618, 1982. English transl. in Algebra and Logic, **21** (1982), 369-418.

[162] *Geometry of Defining Relations in Groups*, volume 70 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 1991.

J. Otal and N. N. Semko.

[163] Groups with small cocentralizers. *Algebra Discrete Math.*, (4):135–157, 2009.

R. Słowik.

[164] Epimorphisms of infinite triangular and unitriangular matrices. *Linear Algebra Appl.*, 462:186–203, 2014.

[165] Derivations of rings of infinite matrices. *Comm. Algebra*, 43(8):3433–3441, 2015.

[166] Epimorphisms of the ring of infinite triangular matrices. *Linear Multilinear Algebra*, 63(7):1372–1378, 2015.

[167] Linear minimal rank preservers on infinite triangular matrices. *Kyushu J. Math.*, 69(1):63–68, 2015.

[168] Some facts about zero divisors of triangular infinite matrices. *Math. Commun.*, 20(2):175–183, 2015.

[169] Linear rank preservers on infinite triangular matrices. *J. Korean Math. Soc.*, 53(1):73–88, 2016.

[170] Sums of square-zero infinite matrices revisited. *Bull. Iranian Math. Soc.*, 45(3):911–916, 2019.

D. S. Passman.

[171] *Infinite Group Rings.* Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 6, 1971.

[172] *The Algebraic Structure of Group Rings.* Pure and Applied Mathematics. J. Wiley and Sons, New York, London, Sydney, Toronto, 1977.

[173] “Group Rings of Polycyclic Groups.” In *Group Theory*, pages 207–256. Academic Press, London, 1984.

[174] *A Course in Ring Theory*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

R. E. Phillips.

[175] "Finitary Linear Groups: a Survey." In B. Hartley, G. M. Seitz, A. V. Borovik, and R. M. Bryant, editors, *Finite and Locally Finite Groups*, volume 471 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pages 111–146, Istanbul 1994, 1995. Kluwer Acad. Publ., Dordrecht.

R. S. Pierce.

[176] *Associative Algebras*, volume 88 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

B. I. Plotkin.

[177] *Groups of Automorphisms of Algebraic Systems*. Izdat. "Nauka", Moscow, 1966. in Russian.

Ja. D. Polovicky.

[178] Groups with extremal classes of conjugate elements. *Sibirsk. Mat. Ž.*, 5:891–895, 1964.

O. Puglisi.

[179] Homomorphic images of finitary linear groups. *Arch. Math. (Basel)*, 60(6):497–504, 1993.

[180] Free products of finitary linear groups. *Proc. Amer. Math. Soc.*, 124(4):1027–1033, 1996.

[181] Maximal unipotent subgroups of finitary linear groups. *J. Algebra*, 181:628–658, 1996.

[182] Sylow subgroups of finitary linear groups. *Geom. Dedicata*, 63(1):95–112, 1996.

D. J. S. Robinson.

[183] *Finiteness Conditions and Generalized Soluble Groups vols. 1 and 2.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, Heidelberg, New York, 1972. Band 62 and 63.

[184] *A Course in the Theory of Groups*, volume 80 of Graduate Texts in Mathematics. Springer Verlag, Berlin, Heidelberg, New York, 1996.

J. S. Rose.

[185] *A Course on Group Theory.* Cambridge University Press, Cambridge-New York-Melbourne, 1978.

O. F. G. Schilling.

[186] *The Theory of Valuations.* Mathematical Surveys, No. 4. American Mathematical Society, New York, N. Y., 1950

O. J. Schmidt.

[187] Groups all of whose subgroups are nilpotent. *Mat. Sb.*, 31:366–372, 1924.

R. Schmidt.

[188] *Subgroup Lattices of Groups*, volume 14 of de Gruyter Expositions in Mathematics. Walter de Gruyter & Co., Berlin, 1994.

M. P. Sedneva.

[189] Some remarks on infinite-dimensional linear groups. *Math. Proceedings of Acadamy of Sciences of Latviya, Series of Physical and Technical Science*, 1965(6):59–62, 1965.

D. Segal.

[190] *Polycyclic groups*, volume 82 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1983.

S. K. Sehgal.

[191] *Topics in Group Rings*, volume 50 of Monographs and Textbooks in Pure and Applied Math. Marcel Dekker, Inc., New York, 1978.

R. Y. Sharp.

[192] *Steps in Commutative Algebra*, volume 19 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1990.

D. W. Sharpe and P. Vámos.

[193] *Injective Modules.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 62. Cambridge University Press, London-New York, 1972.

P. N. Shivakumar and K. C. Sivakumar.

[194] A review of infinite matrices and their applications. *Linear Algebra Appl.*, 430(4):976–998, 2009.

P. N. Shivakumar, K. C. Sivakumar, and Y. Zhang

[195] *Infinite Matrices and Their Recent Applications.* Springer, [Cham], 2016.

H. Smith and J. Wiegold.

[196] Locally graded groups with all subgroups normal-by-finite. *J. Austral. Math. Soc. Ser. A*, 60(2):222–227, 1996.

D. A. Suprunenko.

[197] *Matrix Groups.* Translated from the Russian, Translation edited by K. A. Hirsch, Translations of Mathematical Monographs, Vol. 45. American Mathematical Society, Providence, R.I., 1976.

F. Szechtman.

[198] Groups having a faithful irreducible representation. *J. Algebra*, 454:292–307, 2016.

M. J. Tomkinson.

[199] *FC-Groups.* Pitman Publishing Limited, Boston, London, Melbourne, 1984.

[200] FC-groups: recent progress. In *Infinite Groups 1994 (Ravello)*, pages 271–285. de Gruyter, Berlin, 1996.

A. V. Tushev.

[201] On the irreducible representations of soluble groups of finite rank. *Asian-Eur. J. Math.*, 5(4):1250061, 12, 2012.

B. A. F. Wehrfritz.

[202] *Infinite Linear Groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, New York, Heidelberg, Berlin, 1973. Band 76.

[203] Nilpotence in finitary skew linear groups. *J. Pure Appl. Algebra*, 83(1):27–41, 1992.

[204] Algebras generated by locally nilpotent finitary skew linear groups. *J. Pure Appl. Algebra*, 88(1-3):305–316, 1993.

[205] Locally soluble finitary skew linear groups. *J. Algebra*, 160(1):226–241, 1993.

[206] Nilpotence in finitary linear groups. *Michigan Math. J.*, 40(3):419–432, 1993.

[207] Generalized soluble primitive finitary skew linear groups. *Algebra Colloq.*, 1(4):323–334, 1994.

[208] Locally nilpotent finitary skew linear groups. *J. London Math. Soc. (2)*, 50(2):323–340, 1994.

[209] Primitive finitary skew linear groups. *Arch. Math. (Basel)*, 62(5):393–400, 1994.

[210] Irreducible locally nilpotent finitary skew linear groups. *Proc. Edinburgh Math. Soc. (2)*, 38(1):63–76, 1995.

[211] Locally soluble primitive finitary skew linear groups. *Comm. Algebra*, 23(3):803–817, 1995.

[212] The complete reducibility of locally completely reducible finitary linear groups. *Bull. London Math. Soc.*, 29(2):173–176, 1997.

[213] Free products of finitary skew linear groups. *Arch. Math. (Basel)*, 68(3):177–183, 1997.

[214] Finitary linear images of finitary linear groups. *Math. Pannon.*, 10(2):277–290, 1999.

[215] *Finite Groups: A Second Course on Group Theory.* World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

[216] Absolutely irreducible finitary skew linear groups. *Q. J. Math.*, 51(2):267–274, 2000.

[217] On nonmodular periodic finitary linear groups. *J. Algebra*, 223(1):295–306, 2000.

[218] Locally finite finitary skew linear groups. *Proc. London Math. Soc. (3)*, 83(1):71–92, 2001.

[219] *Group and Ring Theoretic Properties of Polycyclic Groups*, volume 10 of Algebra and Applications. Springer-Verlag London, Ltd., London, 2009.

D. I. Zaitsev.

[220] Complementation of subgroups of extremal groups. *Inst. Mat., Akad Nauk Ukrain, Kiev*, pages 72–130, 1974.

[221] On solvable subgroups of locally solvable groups. *Dokl. Akad. Nauk SSSR*, 214:1250–1253, 1974. English transl. in Soviet Math. Dokl., **15** (1974), 342-345.

[222] "The Existence of Direct Complements in Groups with Operators." In *Studies in Group Theory (Russian)*, pages 26–44, 168. Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1976.

[223] "Infinitely Irreducible Normal Subgroups." In *Structures of Groups and Properties of Their Subgroups (Russian)*, pages 17–38, 150–151. Inst. Kibernet., Akad. Nauk Ukrain. SSR, Kiev, 1978.

[224] Locally solvable groups of finite rank. *Dokl. Akad. Nauk SSSR*, 240(2):257–260, 1978. English transl. in Soviet Math Dokl. **19** (1978), 571-574.

[225] "Hypercyclic Extensions of Abelian Groups." In *Groups Defined by Properties of a System of Subgroups (Russian)*, pages 16–37, 152. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979.

D. I. Zaitsev., M. I. Kargapolov, and V. S. Charin.

[226] Infinite groups with prescribed properties of subgroups. *Ukrain. Mat. Zh.*, 24:618–633, 716, 1972.

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Symbol Index

$A \rtimes B$: semidirect product of A and B, 17

$A \wr B$: wreath product of A and B, 17

$A^{\mathfrak{N}}$: G-nilpotent residual of A, 157

$A^{\mathbf{L}\mathfrak{N}}$: G-locally nilpotent residual of A, 157

A_U: $\{a \in A | aU^n = 0, n \in \mathbb{N}\}$, 71

B^G: $\{gB | g \in G\}$ for subspace B, 264

$C_A(H)$: centralizer of subgroup H in A, 5

$C_G(M)$: centralizer of subset M in G, 5

$C_G(V/U)$: $\{g \in G | g(v) \in v + U$ for every $v \in V\}$, 6

$C_G(a)$: centralizer of element a, 4

C_{P^∞}: Prüfer P-module, 75

$FGL(F, A)$: group of finitary linear transformations, 32

FM: F-subspace spanned by M, 4

$F[H]$: subring generated by H over F, 51

$GL(F, A)$: general linear group, 1

GM: least G-invariant subspace containing M, 6

$G^{\mathfrak{X}}$: $\mathfrak{X}$-residual of the group G, 15

$M_{\mathcal{B}}(f)$: matrix of linear transformation f relative to basis $\mathcal{B}$, 2

RG: group ring of G over R, 7

$T_{\mathbb{N}}(F)$: group of upper triangular matrices, 19

$U(R)$: group of units of ring R, 41

$UT_n(F)$: group of unitriangular $n \times n$ matrices, 10

$UT_{\mathbb{N}}(F)$: group of unitriangular matrices, 19

$[G, A]$: G-commutator subspace, 7

$[g, a]$: $(g-1)a$, 7

$[g_n, g_{n-1}, \dots, g_1, a]$: $[g_n, [g_{n-1}, [\dots [g_1, a]] \dots]$, 7

$\underset{\lambda \in \Lambda}{\mathrm{Cr}}\, A_\lambda$: Cartesian product of subspaces, 12

$\Omega_{U,k}(A)$: $\{a \in A | U^k a = 0\}$, 71

$\Pi(G)$: $\{p \in \mathbb{P} | p \in \Pi(g)$ for some $g \in G\}$, 24

$\Pi(g)$: $\{p \in \mathbb{P} | p \mid |g|\}$, 24

$\mathbf{Alg}\,(G)$: the set of algebraic elements of G, 44

$\mathbf{Ann}_A(Y)$: $\{a \in A | ya = 0, \forall y \in Y\}$, 7

$\mathbf{Ann}_{FG}(V)$: $\{x \in FG | xv = 0, \forall x \in V\}$, 7

$\mathbf{Ass}_R(A)$: {prime ideals $P | \mathbf{Ann}_A(P) \neq 0\}$, 71

$\mathbf{augdim}_F(G)$: augmentation dimension of G, 141

$\mathbf{Aut}\,(A)$: automorphism group of A, 20

$\mathbb{Q}$: field of rational numbers, 20

$\mathbf{do}_A(G)$: $\max\{\mathbf{dim}_F(Fa^G) | a \in A\}$, 260

$\mathbf{FC}(G)$: $\{x \in G | x^G$ is finite$\}$, 54

$\mathbf{O}_\pi(G)$: largest normal π-subgroup of G, 69

$\bigcap \mathcal{M}$: $\bigcap\{M : M \in \mathcal{M}\}$, 15

$\bigcup\mathcal{M}$: $\bigcup\{M : M \in \mathcal{M}\}$, 15
$\bigoplus_{\lambda\in\Lambda} A_\lambda$: direct sum of subspaces A_λ, 12
$\boldsymbol{\Omega}_n(P)$: $\{a \in P | a^{p^n} = 1\}$, 47
$\mathbf{centdim}_F(G)$: central dimension of G, 139
$\mathbf{char}\,(F)$: characteristic of ring F, 40
$\mathbf{chl}_G(A)$: length of finite G-chief series, 160
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$\mathbf{R}\mathfrak{X}$: class of residually $\mathfrak{X}$-groups, 16
$\mathbf{dim}_F(A)$: F-dimension of A, 2
$\underset{\lambda\in\Lambda}{\mathrm{Dr}}\,\langle G_\lambda\rangle$: direct product of groups, 46
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$\mathbf{Im}(\Phi)$: image of homomorphism Φ, 40
$\mathbf{Inv}_G(V)$: $\{g \in G | g(v) \in V$ for every $v \in V\}$, 6
$\mathbb{F}_p$: field of order p, 47
$\mathbb{N}$: set of natural numbers, 4
$\mathcal{L}_{icd}(G)$: family of subgroups of G with infinite central dimension, 174
$\mathfrak{F}_p$: class of finite p-groups, 16
$\mathbf{mon}_G(A)$: FG-monolith, 130
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$\mathbf{Norm}_A(G)$: $\{a \in A | g(a) \in Fa$ for each element $g \in G\}$, 114
$\omega(FG)$: augmentation ideal of FG, 7
$\mathbf{QSoc}(G)$: quasi-socle of G, 58
$\mathbf{r}_p(G)$: p-rank of abelian G, 47
$\mathbf{r}_\mathbb{Z}(G)$: $\mathbb{Z}$-rank of abelian G, 47
$\mathbf{Soc}(G)$: socle of group G, 58
$\mathbf{Soc}_G(A)$: G-socle of A, 66
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$\mathbf{r}(G)$: special rank of G, 145
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$\mathbf{FDO}_G(A)$: $\{a \in A | Fa^G$ has finite dimension$\}$, 6
$\mathbf{FO}_G(A)$: $\{a \in A | a^G$ is finite $\}$, 6
$\mathbf{Out}(G)$:outer automorphism group of G, 196
$\mathbf{Res}_\mathfrak{X}G$: $\{N \lhd G | G/N \in \mathfrak{X}\}$, 16
$\mathbf{r}_0(G)$: 0-rank of group G, 58
$\mathbf{Tor}(G)$: {maximal normal torsion subgroup}, 24
$\mathbf{Tor}_R(A)$: R-periodic part of module A, 71
$\mathbf{Tor}_p(G)$: {elements of p-power order in abelian G}, 47
$\zeta_G(A)$: G-center of A, 6
$\zeta_{G,\alpha}(A)$: term of upper G-central series, 8
$\zeta_{G,\infty}(A)$: upper G-hypercenter, 8
$\mathbf{zl}_G(A)$: G-central length, 8
a^G: G-orbit of a, 4
x^G: $\{x^g = g^{-1}xg | g \in G\}$, 54

$\mathbf{Fin}(G)$:set of elements of finite central dimension, 178
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$UT_m(\mathbb{Z})$: unitriangular matrix group with coefficients in $\mathbb{Z}$, 20

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