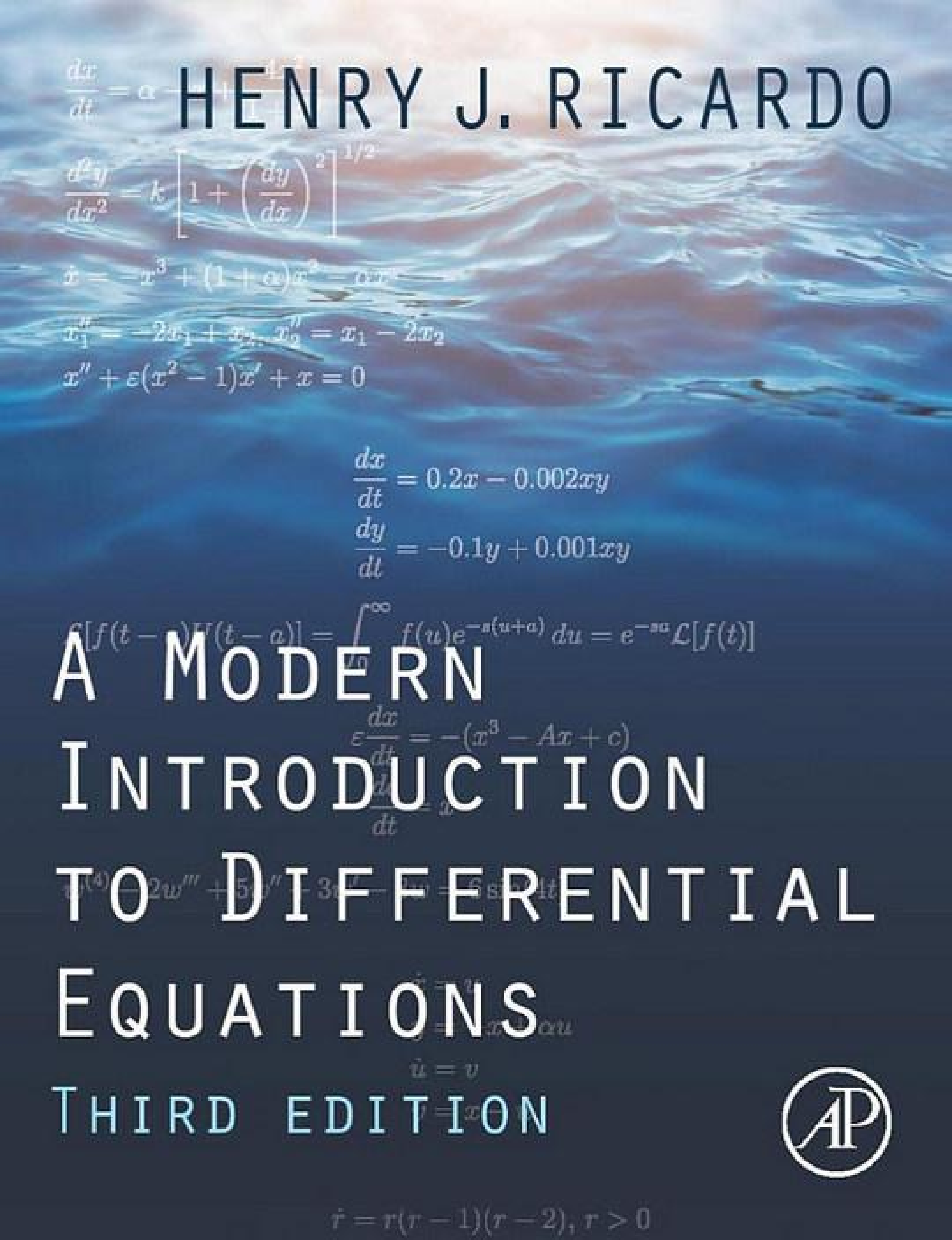
HENRY J. RICARDO
$x'' + \varepsilon(x^2 - 1)x' + x = 0$
$\frac{dx}{dt} = 0.2x - 0.002xy$
$\frac{dy}{dt} = -0.1y + 0.001xy$
A MODERN INTRODUCTION TO DIFFERENTIAL EQUATIONS
THIRD EDITION
AP
$\dot{r} = r(r-1)(r-2),\ r > 0$

A Modern Introduction to Differential Equations

A Modern Introduction to Differential Equations

Third Edition

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For Catherine, the sole stable equilibrium point in my life, and for all the second derivatives:

Tomás and Nicholas Ricardo

Christopher Corcoran

Joshua and Elizabeth Gritmon

Contents

[1] $*$ Denotes an optional section.

Preface

Philosophy

The evolution of the differential equations course I described in the prefaces to the first two editions of this book has progressed nicely. In particular, the quantitative, graphical, and qualitative aspects of the subject have been receiving increased attention, due in large part to the availability of technology in the classroom and at home.

As did the previous editions, this new edition presents a solid yet highly accessible introduction to differential equations, developing many concepts from the perspective of dynamical systems and employing technology to treat topics graphically, numerically, and analytically. In particular, the book acknowledges that most differential equations cannot be solved in closed form and makes extensive use of qualitative and numerical methods to analyze solutions.

The text includes discussions of many significant mathematical models, although there is no systematic attempt to teach the art of modeling. Similarly, the text introduces only the minimum amount of linear algebra sufficient for an analysis of systems of equations.

This book is intended to be the text for a one-semester ordinary differential equations course that is typically offered at the sophomore or junior level, but with some differences. The prerequisite for the course is two semesters of calculus. No prior knowledge of multivariable calculus and linear algebra is needed because basic concepts from these subjects are developed within the text itself. This book is aimed primarily at students majoring in mathematics, the natural sciences, and engineering. However, students in economics, business, and the social sciences who have the necessary background should also benefit from the material presented in this book.

Use of technology

This text assumes that the student has access to a computer algebra system (CAS) or perhaps some specialized software that will enable him or her to construct the required graphs (solution curves, slope fields, phase portraits, etc.) and numerical approximations. For example, a spreadsheet program can be used effectively to implement Euler's method of approximating solutions. Although I have used *Maple*® in my own teaching, no specific software or hardware platform is assumed for this book. To a large extent, even a graphing calculator will suffice.

Pedagogical features and writing style

This book is truly meant to be *read* by the students. The style is accessible without excessive mathematical formality and extraneous material, although it does provide a solid foundation upon which individual teachers can build according to their taste and the students' needs. (*Feedback from users of the first two editions suggests that students find the book easy to read*.) Every chapter has an informal *Introduction* that sets the tone and motivates the material to come. I have tried to motivate the introduction of new concepts in various ways, including references to earlier, more elementary mathematics courses taken by the students. Each chapter concludes with a narrative *Summary* reminding the readers of the important concepts in the chapter. Within the sections there are figures and tables to help the students visualize or summarize the concepts. There are many worked-out examples and exercises taken from biology, chemistry, and economics, and from traditional pure mathematics, physics, and engineering. In the text itself I lead the students through qualitative and numerical analyses of problems that would have been difficult to handle before the ubiquitous presence of graphing calculators and computers. The exercises that appear at the end of each content section are divided into A, B, and C problems to indicate a range from the routine to the challenging, the later problems often requiring some sophisticated exploration and/or theoretical justification. Some exercises introduce students to supplementary concepts. I have provided answers to the odd-numbered problems at the back of the book, with detailed solutions to these problems in the separate *Student Solutions Manual.*

I wrote the book in the same way that I have taught the course, using a colloquial and interactive style. The student is frequently urged to "Think about this," "Check this," or "Make sure you understand." In general there are no proofs of theorems except for those mathematical statements that can be justified by a sequence of fairly obvious calculations or algebraic manipulations. In fact, there is no general labeling of facts as theorems, although some definitions are stated formally and key results are italicized within the text or emphasized in other ways. Also, brief historical remarks related to a particular concept or result are placed throughout the text without obstructing the flow. This is not a mathematical treatise, but a friendly, informative, modern introduction to tools needed by students in many disciplines. I have enjoyed teaching such a course, and I believe my students have benefited from the experience. I sincerely hope that the users of this book also gain some insight into the modern theory and applications of differential equations.

Key content features

Chapters 1–3 introduce the *basic concepts* of differential equations and focus on the analytical, graphical, and numerical aspects of first-order equations, including *slope fields*, *phase lines*, and *bifurcations*. In the later chapters these aspects (including the *Superposition Principle*) are generalized in natural ways to higher-order equations and systems of equations. The numerical approximation of solutions is explored via variants of Euler's method and more sophisticated techniques such as the Runge–Kutta–Fehlberg methods.

Chapter 4 starts with methods of solving important *second-order homogeneous and nonhomogeneous linear equations with constant coefficients* and introduces applications to electrical circuits and spring-mass problems. In this chapter the standard methods of undetermined coefficients and variation of parameters are explained and applied to second-, third-, and fourth-order equations.

Chapter 5 presents the *Laplace transform* and its applications to the solution of differential equations and systems of differential equations. This is one of the more traditional topics in the book; it is included because of its usefulness in many applied areas. In particular, students can deal with *nonhomogeneous* linear equations and systems more easily and handle *discontinuous driving forces*. The Laplace transform is applied to electric circuit problems, the deflection of beams (a boundary-value problem), and spring-mass systems. There is a new section on the application of the Laplace transform to *linear equations with variable coefficients*.

Chapter 6 begins with the important demonstration that any higher-order differential equation is equivalent to a system of first-order equations. This is followed by an existence and uniqueness result for systems and the extension of first-order equation numerical methods to systems. *Phase portraits* of planar systems are introduced as an entree to the qualitative analysis of systems. There is a brief introduction to the *matrix algebra* concepts needed for the systematic exposition of *two-dimensional systems of autonomous linear equations.* (This treatment is supplemented by Appendix B.) The importance of linearity is emphasized, and the *Superposition Principle* is discussed again. The *stability* of these systems is completely characterized by means of the *eigenvalues* of the matrix of coefficients. *Spring-mass* systems are discussed in terms of their eigenvalues. There is also a brief introduction to the complexities of *nonhomogeneous* systems. Finally, via 3×3 and 4×4 examples, the student is shown how the ideas previously developed can be extended to *nth-order equations* and their equivalent systems. Among the examples treated in this chapter are *predator-prey* systems, an *arms race* illustration, and *spring-mass* systems (including one showing *resonance*).

Chapter 7 provides an introduction to systems of *nonlinear* equations. The *stability* of nonlinear systems is analyzed. The important notion of a *linear approximation* to a nonlinear equation or system is developed, including the use of a qualitative result due to Hartman and Grobman. Some important examples of nonlinear systems are treated in detail, including the *Lotka–Volterra equations*, the *undamped pendulum*, and the *van der Pol oscillator*. There is a new section on *bifurcations* in linear and nonlinear systems. The book concludes with a discussion of *limit cycles* and the *Hopf bifurcation*.

Appendices A–C present important prerequisite or corequisite material from *calculus* (*single-variable and multivariable*), *vector-matrix algebra*, and *complex numbers*, respectively. **Appendix D** supplements the text by introducing the *series solutions of first- and second-order differential equations*.

New to the Third Edition

- Enhanced treatment of *bifurcations* for first-order equations.
- A brief discussion of *bistability* and *hysteresis*.
- A treatment of the *backward Euler method*.
- A sketch of the Picard iteration method and a *proof* of a uniqueness theorem for solutions of first-order equations.
- Improved flow of Chapters 4–6 (second-order and higher-order equations, Laplace transforms, systems of linear equations) via reorganization of the material. *Laplace transforms* have been placed earlier in the text, and there is a new section on Laplace transforms of linear equations with *variable* coefficients.
- Expanded treatment of the *Hartman–Grobman theorem* (referred to as the Poincaré–Lyapunov theorem in the Second Edition).
- New material on *bifurcations* in linear and nonlinear systems, including a treatment of the *Hopf bifurcation*.
- New examples, figures, and exercises. The text now has more than 160 examples, 130 figures (some with multiple graphs), 16 tables, and 940 exercises (many with multiple parts).
- Improved numbering of exercises.
- Updated and more detailed *Index*.

Supplements

- **Instructor's Resource Manual**—http://textbooks.elsevier.com/web/Manuals.aspx?isbn=9780128182178—Contains solutions to all exercises in the text, section-by-section teaching comments and suggestions, additional examples and problems, references to books and journal articles, and links (including hyperlinks) to Internet resources for differential equations. *A link to a web page containing errata (if any) and updated supplementary material is provided*. This manual is available free to instructors who adopt the text.
- **Student Solutions Manual**—https://www.elsevier.com/books-and-journals/book-companion/9780128182178—Provides complete solutions to the odd-numbered exercises in the text.

Acknowledgments

The seminal influences on the philosophy and content of this book were (1) The Boston University Ordinary Differential Equations Project; (2) The Consortium for Ordinary Differential Equations Experiments (C• ODE• E); (3) The Special Issue on Differential Equations: *College Mathematics Journal*, Vol. 25, No. 5 (November 1994); (4) David Sánchez's review of ordinary differential equation (ODE) texts in the April 1998 issue of the *American Mathematical Monthly* (pp. 377–383); and *Revolutions in Differential Equations: Exploring ODEs with Modern Technology*, edited by Michael J. Kallaher (MAA Notes #50, 1999).

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Above all, I am (still) grateful to my wife, Catherine, for her love, steadfast support, and patience during the writing of this book and at all other times. Her encouragement and active assistance in proofreading and critiquing the manuscript were invaluable.

Henry J. Ricardo

CHAPTER

Introduction to differential equations

1

Introduction

What do the following situations have in common?

- An arms race between nations
- The rate at which HIV-positive patients come to exhibit AIDS
- The dynamics of supply and demand in an economy
- The interaction between two or more species of animals on an island

The answer is that each of these areas of investigation can be modeled with differential equations. This means that the essential features of these problems can be represented using one or several differential equations, and the solutions of the mathematical problems provide insights into the future behavior of the systems being studied.

This book deals with *change*, with *flux*, with *flow*, and, in particular, with the *rate* at which change takes place. Every living thing changes. The tides ebb and flow over the course of a day. Countries increase and diminish their stockpiles of weapons. The price of oil rises and falls. The proper framework of this course is **dynamics**—the study of systems that evolve over time.

The origin of dynamics/dynamical systems (originally an area of physics) and of differential equations lies in the earliest work by the English scientist and mathematician Sir Isaac Newton (1642–1727) and the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716) in developing the new science of calculus in the 17th century. Newton in particular was concerned with determining the laws governing motion, whether of an apple falling from a tree or of the planets moving in their orbits. He was concerned with *rates of change*. In the late 19th century Henri Poincaré and others analyzed the positions, motion, and stability of the planets using a powerful geometric approach to the analysis of dynamical systems. The development of these methods, aided by technology, continues to this day. However, you mustn't think that the subject of differential equations is all about physics. The same types of equations and the same kind of analysis of dynamical systems can be used to model and understand situations in biology, chemistry, engineering, economics, sociology, and medicine, for example. Applications to various areas of the physical, biological, and social sciences will be found throughout this book.

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In the words of the great Norwegian mathematician Sophus Lie (1842–99),

Among all of the mathematical disciplines, the theory of differential equations is the most important... It furnishes the explanation of all those elementary manifestations of nature which involve time.

1.1 Basic terminology

1.1.1 Ordinary and partial differential equations

Ordinary differential equations

Definition 1.1.1

An **ordinary differential equation** (**ODE**) is an equation that involves an unknown function of a single variable, its independent variable, and one or more of its derivatives.

Example 1.1.1 An Ordinary Differential Equation

Here's a typical elementary ODE, with some of its components indicated:

unknown function, y ↓

$$3\frac{dy}{dt} = y$$

independent variable, t ↑

This equation describes an unknown function of t that is equal to three times its own derivative. Expressed another way, the differential equation describes a function whose rate of change is proportional to its size (value) at any given time, with constant of proportionality one-third.

The Leibniz notation for a derivative, $\frac{d()}{d()}$, is helpful because the independent variable (the fundamental quantity whose change causes other changes) appears in the denominator, the dependent variable in the numerator. The three equations

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

$$x''(t) - 5x'(t) + 6x(t) = 0$$

$$\frac{dx}{dt} = \frac{3t^2 + 4t + 2}{2(x-1)}$$

leave no doubt about the relationship between the independent and dependent variables. But in an equation such as $(w')^2 + 2t^3w' - 4t^2w = 0$, using Lagrange's notation (prime marks), we must *infer* that the unknown function w is really $w(t)$, a function of the independent variable t.

In many dynamical applications the independent variable is time, represented by t, and we denote the function's derivative using Newton's dot notation,[1] as in the equation $\ddot{x} + 3t\dot{x} + 2x = \sin(\omega t)$. You should be able to recognize a differential equation no matter what letters are used for the independent and dependent variables and no matter what derivative notation is employed. The context will determine what the various letters mean, and it's the *form* of the equation that should be recognized. For example, you should be able to see that the two ordinary differential equations

$$\text{(A)}\ \frac{d^2u}{dt^2} - 3\frac{du}{dt} + 7u = 0 \quad \text{and} \quad \text{(B)}\ \frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 7y$$

are the same—that is, they describe the same mathematical or physical behavior. In Eq. (A) the unknown function u depends on t, whereas in Eq. (B) the function y is a function of the independent variable x, but both equations describe the same relationship that involves the unknown function, its derivatives, and the independent variable. Each equation is describing a function whose second derivative equals three times its first derivative minus seven times itself.

Partial differential equations

If we are dealing with functions of *several* variables and the derivatives involved are *partial* derivatives, then we have a **partial differential equation** (**PDE**) (see Section A.7 if you are not familiar with partial derivatives). For example the partial differential equation $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$, which is called the *wave equation*, is of fundamental importance in many areas of physics and engineering. In this equation we are assuming that $u = u(x, t)$, a function of the two variables x and t. However, in this text, when we use the term *differential equation*, we mean an *ordinary* differential equation. Often we just write *equation* if the context makes it clear that an ordinary differential equation is intended.

The order of an ordinary differential equation

One way to classify differential equations is by their **order**.

> **Definition 1.1.2**
>
> An ordinary differential equation is of **order n**, or is an **nth-order equation**, if the highest derivative of the unknown function in the equation is the nth derivative.

The equations

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

[1] In this notation, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, and $\dddot{x} = d^3x/dt^3$.

$$(w')^2 + 2t^3 w' - 4t^2 w = 0$$

$$\frac{dx}{dt} = \frac{3t^2 + 4t + 2}{2(x-1)}$$

are all *first-order* differential equations because the highest derivative in each equation is the first derivative. The equations

$$x''(t) - 5x'(t) + 6x(t) = 0$$

and

$$\ddot{x} + 3t\,\dot{x} + 2x = \sin(\omega t)$$

are second-order equations, and $e^{-x} y^{(5)} + (\sin x) y''' = 3e^x$ is of order 5.

A general form for an ordinary differential equation

If y is the unknown function with a single independent variable x, and $y^{(k)}$ denotes the kth derivative of y, we can express an nth-order differential equation in a concise mathematical form as the relation

$$F\left(x, y, y', y'', y''', \ldots, y^{(n-1)}, y^{(n)}\right) = 0,$$

where F is a real-valued function of the $n+2$ variables x, $y(x)$, $y'(x), \ldots, y^{(n)}(x)$.

The **normal form** of an nth-order differential equation involves solving for the highest derivative and placing all the other terms on the other side of the equation:

$$y^{(n)} = G\left(x, y, y', y'', y''', \ldots, y^{(n-1)}\right).$$

The next example shows what these forms look like in practice. However, it may not be easy (or even possible) to solve a general form explicitly for the highest derivative. *We will deal only with equations that can be expressed in normal form.*

Example 1.1.2 General Form for a Second-Order ODE

If y is an unknown function of x, then the second-order ordinary differential equation $2\frac{d^2y}{dx^2} + e^x \frac{dy}{dx} = y + \sin x$ can be written as $2\frac{d^2y}{dx^2} + e^x \frac{dy}{dx} - y - \sin x = 0$ or as

$$\underbrace{2y'' + e^x y' - y - \sin x}_{F(x,y,y',y'')} = 0.$$

Note that F denotes a mathematical expression involving the independent variable x, the unknown function y, and the first and second derivatives of y.

Alternatively, in this last example we could use ordinary algebra to solve the original differential equation for its highest derivative and write the equation as $y'' = \underbrace{\frac{1}{2}\sin x + \frac{1}{2}y - \frac{1}{2}e^x y'}_{G(x,y,y')}$.

Linear and nonlinear ordinary differential equations

Another important way to categorize differential equations is in terms of whether they are *linear* or *nonlinear*.

Definition 1.1.3

If y is a function of x, then the general form of a **linear ordinary differential equation of order *n*** is

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x). \tag{1.1.1}$$

What is important here is that each coefficient function a_i, as well as f, depends on the independent variable x alone and doesn't have the dependent variable y or any of its derivatives in it. In particular, Eq. (1.1.1) involves no products or quotients of y and/or its derivatives.

Example 1.1.3 A Second-Order Linear Equation

The equation $x'' + 3tx' + 2x = \sin(\omega t)$, where ω is a constant, is linear. We can see the form of this equation is as follows:

$$\overbrace{1}^{a_2(t)} \cdot x'' + \overbrace{3t}^{a_1(t)} \cdot x' + \overbrace{2}^{a_0(t)} \cdot x = \overbrace{\sin(\omega t)}^{f(t)}.$$

The coefficients of the various derivatives of the unknown function x are functions (sometimes constant) of the independent variable t alone.

The next example shows that not all first-order equations are linear.

Example 1.1.4 A First-Order Nonlinear Equation (an HIV Infection Model)

The equation $\frac{dT}{dt} = s + rT\left(1 - \frac{T}{T_{\max}}\right) - \mu T$ models the growth and death of T cells, which are important components of the immune system.[2] Here $T(t)$ is the number of T cells present at time t. If we rewrite the equation by removing the parentheses we get $\frac{dT}{dt} = s + rT - \left(\frac{r}{T_{\max}}\right)T^2 - \mu T$, and we see that there is a term involving the square of the unknown function. Therefore the equation is not linear.

In general, there are more systematic ways to analyze linear equations than to analyze nonlinear equations, and we will look at some of these linear methods in Chapters 2, 4, 5, and 6. However, nonlinear equations are important and appear throughout this book. In particular, Chapter 7 is devoted to the analysis of nonlinear systems of equations.

[2] E.K. Yeargers, R.W. Shonkwiler, and J.V. Herod, *An Introduction to the Mathematics of Biology: With Computer Algebra Models* (Boston: Birkhäuser, 1996): 341.

1.1.2 Systems of ordinary differential equations

In earlier mathematics courses, you had to deal with *systems* of algebraic equations, such as

$$\begin{aligned} 3x - 4y &= -2 \\ -5x + 2y &= 7. \end{aligned}$$

Similarly, in working with differential equations, you may have found yourself confronting **systems of differential equations**, such as

$$\begin{aligned} \frac{dx}{dt} &= -3x + y \\ \frac{dy}{dt} &= x - 3y \end{aligned}$$

or

$$\begin{aligned} \dot{x} &= -sx + sy \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned}$$

where b, r, and s are constants. (Recall that $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, and $\dot{z} = \frac{dz}{dt}$.) The second system arose in a famous study of meteorological conditions.

Note that each of the two systems of differential equations has a different number of equations and that each equation in the first system is *linear*, whereas the last two equations in the second system are *nonlinear* because they contain products—xz in the second equation and xy in the third—of some of the unknown functions. Naturally, we call a system in which all equations are linear a **linear system**, and we refer to a system with at least one nonlinear equation as a **nonlinear system**. In Chapters 5, 6, and 7, we'll see how systems of differential equations arise and learn how to analyze them. For now, just try to get used to the *idea* of a system of differential equations.

Exercises 1.1

A

In Problems 1–12, (a) identify the independent variable and the dependent variable of each equation; (b) give the order of each differential equation; and (c) state whether the equation is linear or nonlinear. If your answer to (c) is nonlinear, explain why this is true.

1. $y' = y - x^2$
2. $xy' = 2y$
3. $x'' + 5x = e^{-x}$

4. $(y')^2 + x = 3y$

5. $xy'(xy' + y) = 2y^2$

6. $\frac{d^2r}{dt^2} = 3\frac{dr}{dt} + \sin t$

7. $y^{(4)} + xy''' + e^x = 0$

8. $y'' + ky'(y^2 - 1) + 3y = -2\cos t$

9. $\dddot{x} - 2\ddot{x} + 4t\dot{x} - e^t x = t + 1$

10. $x^{(7)} + t^2 x^{(5)} = xe^t$

11. $e^{y'} + 3xy = 0$

12. $t^2 R''' - 4tR'' + R' + 3R = e^t$

13. Classify each of the following systems as linear or nonlinear:

a. $\frac{dy}{dt} = x - 4xy$

$\frac{dx}{dt} = -3x + y$

b. $Q' = tQ - 3t^2 R$

$R' = 3Q + 5R$

c. $\dot{x} = x - xy + z$

$\dot{y} = -2x + y - yz$

$\dot{z} = 3x - y + z$

d. $\dot{x} = 2x - ty + t^2 z$

$\dot{y} = -2tx + y - z$

$\dot{z} = 3x - t^3 y + z$

14. If $y(x) = \int_1^x \sin t\, dt$, calculate $y'''(x) + y'(x)$. (See Section A.4 of Appendix A.)

B

15. For what value(s) of the constant a is the differential equation

$$\frac{d^2x}{dt^2} + (a^2 - a)x\frac{dx}{dt} = te^{(a-1)x}$$

a linear equation?

16. Rewrite the following equations as linear equations, if possible.

a. $\frac{dx}{dt} = \ln(2^x)$

b. $x' = \begin{cases} \frac{x^2-1}{x-1} & \text{for} \quad x \neq 1 \\ 2 & \text{for} \quad x = 1 \end{cases}$

c. $x' = \begin{cases} \frac{x^4-1}{x^2-1} & \text{for} \quad x \neq 1 \\ 2 & \text{for} \quad x = 1. \end{cases}$

C

17. If f is a function whose arc length over an interval $[a, x]$ is equal to the area under the curve $y = f(t)$ and above the horizontal axis on the same interval, show that $[f'(x)]^2 = f^2(x) - 1$ for all values of x. (See the end of Appendix A.4.)

1.2 Solutions of differential equations

1.2.1 Basic notions

In past mathematics courses, whenever you encountered an equation you were probably asked to *solve* it, or find a *solution*. Simply put, a *solution* of a differential equation is a function that *satisfies* the equation. When you substitute this function into the differential equation, you get a true mathematical statement—an *identity*.

Definition 1.2.1

A **solution** of an nth-order differential equation $F(x, y, y', y'', y''', \ldots, y^{(n-1)}, y^{(n)}) = 0$, or $y^{(n)} = G(x, y, y', y'', y''', \ldots, y^{(n-1)})$, on an interval (a, b) is a real-valued function $y = y(x)$ such that all the necessary derivatives of $y(x)$ exist on the interval, and $y(x)$ satisfies the equation for every value of x in the interval. **Solving** a differential equation means finding all the possible solutions of a given equation.

A subtlety to be aware of is that solutions of differential equations, including their domains, can be different according to which form of the differential equation—general or normal—is used (see Section 1.1). For example, the differential equations $dy/dx = (y + 1)/x$ and $x(dy/dx) = y + 1$ are different equations! No solution of the first equation can contain $x = 0$ in its domain, but the solution $y = -2x - 1$ *does* satisfy the equation $x(dy/dx) = y + 1$ for all real values of x. This is analogous to the fact that the algebraic equations $(x^2 - 1)/(x - 1) = x$ and $x^2 - 1 = x(x - 1)$ are not the same. The first equation has no solution, while the second has $x = 1$ as a solution. Even before we begin learning formal solution methods in Chapter 2, we can *guess* the solutions of some simple differential equations. The next example shows how to guess intelligently.

Example 1.2.1 Guessing and Verifying a Solution to an ODE

The first-order linear differential equation $\frac{dB}{dt} = kB$, where k is a given positive constant, is a simple model of a bank balance, $B(t)$, under continuous compounding t years after the initial deposit. The rate of change of B at any instant is proportional to the *size* of B at that instant, with k as the constant of proportionality—a rate of growth, an interest rate. This equation expresses the fact that the larger the bank balance at any time t, the faster it will grow.

You can guess what kind of function describes $B(t)$ if you think about the elementary functions you know and their derivatives. What kind of function has a derivative that is a constant multiple of itself? You should be able to see why $B(t)$ must be an *exponential* function of the form ae^{kt}, where a is any constant. By substituting $B(t) = ae^{kt}$ into the original differential equation, you can verify that you have guessed correctly. The left-hand side of the equation becomes $\frac{d(ae^{kt})}{dt}$, which equals kae^{kt}, and the right-hand side of the equation is $k(ae^{kt})$. The left-hand side equals the right-hand side for all values of t, giving us an identity.

Anticipating an idea that we'll discuss later in this section, we can let $t = 0$ in our solution function to conclude that $B(0) = ae^{k(0)} = a$—that is, the constant a must equal the initial deposit. Finally, we can express the solution as $B(t) = B(0)e^{kt}$.

Note that in Definition 1.2.1 we say "a" solution rather than "the" solution. A differential equation, if it has a solution at all, usually has more than one solution. Also, we should pay attention to the interval on which the solution may be defined. Later in this section and in Section 2.8, we will discuss in more detail the question of the existence and uniqueness of solutions. For now, let's just learn to recognize when a function is a solution of a differential equation and determine what the domain of a solution is.

Example 1.2.2 Verifying a Solution and Its Domain

For each constant c, the function $x = ce^{1/t}$ is a solution of the differential equation $t^2 \frac{dx}{dt} + x = 0$. We see that

$$\frac{dx}{dt} = ce^{1/t} \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{c}{t^2}e^{1/t},\ t^2\frac{dx}{dt} = -ce^{1/t},\ t^2\frac{dx}{dt} + x = -ce^{1/t} = 0.$$

Now we determine the (maximum) domain of the solution.

First note that the constant function $x \equiv 0$ is a solution (corresponding to $c = 0$) for all real values of t. But the right side of the equation in the form $dx/dt = -x/t^2$ is *undefined* at the origin, so we must restrict the solution in this normal form to either $-\infty < t < 0$ or $0 < t < \infty$. If $c \neq 0$, we can't have $t = 0$ in our domain. This means that we can choose *either* of the two intervals $(-\infty, 0)$ and $(0, \infty)$.

Example 1.2.3 Intervals of Validity

Suppose we want the solution of the equation $x' = 2tx^2$ that satisfies the additional condition $x(0) = 1$. This suggests that there may be many solutions of the equation, but we want to find the unique solution (we hope) passing through the point $(t, x(t)) = (0, 1)$.

We can easily verify that $x(t) = 1/(1 - t^2)$ is a solution of $x' = 2tx^2$ such that $x(0) = 1$. Since the solution is not defined for $t = \pm 1$ and since we want $t = 0$ to be in the domain of the solution, the only interval possible is $(-1, 1)$. On the other hand, if we want a solution of the equation such that $x(0) = -1$, then $x(t) = -1/(1 + t^2)$ is such a solution (*Check this*), and this solution has the whole real number line as its domain.

Example 1.2.4 Verifying a Solution of a Second-Order Equation

Suppose that someone claims that $x(t) = 5e^{3t} - 7e^{2t}$ is a solution of the second-order linear equation $x'' - 5x' + 6x = 0$ on the whole real line—that is, for all values of t in the interval $(-\infty, \infty)$. You can prove that this claim is correct by calculating $x'(t) = 15e^{3t} - 14e^{2t}$ and $x''(t) = 45e^{3t} - 28e^{2t}$ and then substituting these expressions into the original equation:

$$
\begin{aligned}
x''(t) - 5x'(t) + 6x(t) &= \overbrace{(45e^{3t} - 28e^{2t})}^{x''(t)} - 5\,\overbrace{(15e^{3t} - 14e^{2t})}^{x'(t)} + 6\,\overbrace{(5e^{3t} - 7e^{2t})}^{x(t)} \\
&= 45e^{3t} - 28e^{2t} - 75e^{3t} + 70e^{2t} + 30e^{3t} - 42e^{2t} \\
&= -30e^{3t} + 42e^{2t} + 30e^{3t} - 42e^{2t} = 0.
\end{aligned}
$$

Because $x(t) = 5e^{3t} - 7e^{2t}$ satisfies the original equation, we see that $x(t)$ is a solution. But this is not the only solution of the given differential equation. For example, you can check that $x_2(t) = -\pi e^{3t} + \frac{2}{3}e^{2t}$ is also a solution. We'll discuss this kind of situation in more detail a little later.

Implicit solutions

Think back to the concept of *implicit function* in calculus. The idea here is that sometimes functions are not defined cleanly (explicitly) by a formula in which the dependent variable (on one side) is expressed in terms of the independent variable and some constants (on the other side), as in the solution $x = x(t) = 5e^{3t} - 7e^{2t}$ of Example 1.2.4. For instance, you may be given the *relation* $x^2 + y^2 = 5$, which can be written in the form $G(x, y) = 0$, where $G(x, y) = x^2 + y^2 - 5$. The graph of this relation is a circle of radius $\sqrt{5}$ centered at the origin, and this graph does not represent a function. (*Why?*) However, this relation *does* define two functions *implicitly*: $y_1(x) = \sqrt{5 - x^2}$ and $y_2(x) = -\sqrt{5 - x^2}$, both having domains $[-\sqrt{5}, \sqrt{5}]$.

Definition 1.2.2

A relation $F(x, y) = 0$ is said to be an **implicit solution** of a differential equation involving x, y, and derivatives of y with respect to x if $F(x, y) = 0$ *defines* one or more explicit solutions of the differential equation.

More advanced courses in analysis discuss when a relation actually defines one or more implicit functions. This involves a result called the **Implicit Function Theorem**. For now, just remember that even if you can't untangle a relation to get an explicit formula for a function, you can use implicit differentiation to find derivatives of any differentiable functions that may be buried in the relation.

When trying to solve differential equations, often we can't find an explicit solution and must be content with a solution defined implicitly.

Example 1.2.5 Verifying an Implicit Solution

We want to show that any function y that satisfies the relation $G(x, y) = x^2 + y^2 - 5 = 0$ is a solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

First, we differentiate the relation implicitly, treating y as $y(x)$, an implicitly defined function of the independent variable x:

$$(1)\ \frac{d}{dx}G(x, y) = \frac{d}{dx}(x^2 + y^2 - 5) = \frac{d}{dx}(0) = 0$$

$$(2)\ 2x + \overbrace{2y\frac{dy}{dx}}^{\text{Chain Rule}} - \frac{d}{dx}(5) = 0$$

$$(3)\ 2x + 2y\frac{dy}{dx} = 0.$$

Now, assuming that $y \neq 0$, we solve Eq. (3) for $\frac{dy}{dx}$, getting $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$ and proving that any function defined implicitly by the relation above is a solution of our differential equation.

1.2.2 Families of solutions I

Next, we discuss how many solutions a differential equation could have. For example, the equation $(y')^2 + 1 = 0$ has *no* real-valued solution (*Think about this*), whereas the equation $|y'| + |y| = 0$ has exactly one solution, the function $y \equiv 0$. (*Why*?) As we saw in Example 1.2.4, the differential equation $x'' - 5x' + 6x = 0$ has at least two solutions.

The situation gets more complicated, as the next example shows.

Example 1.2.6 An Infinite Family of Solutions

Suppose two students, Joshua and Ellie, look at the simple first-order differential equation $\frac{dy}{dx} = f(x) = x^2 - 2x + 7$. A solution of this equation is a function of x whose first derivative equals $x^2 - 2x + 7$. Joshua thinks the solution is $\frac{x^3}{3} - x^2 + 7x$, and Ellie thinks the solution is $\frac{x^3}{3} - x^2 + 7x - 10$. Both answers seem to be correct.

Solving this problem is simply a matter of integrating both sides of the differential equation:

$$y = \int dy = \int \frac{dy}{dx}\,dx = \int x^2 - 2x + 7\,dx.$$

Because we are using an *indefinite* integral, there is always a constant of integration that we mustn't forget. The solution to our problem is actually an *infinite family of solutions*, $y(x) = \frac{x^3}{3} - x^2 + 7x + C$, where C is any real constant. Every particular value of C gives us another member of the family. We have just solved our first differential equation in this course without guessing! Every time we performed an indefinite integration (found an antiderivative) in calculus class, we were solving a simple differential equation.

When describing the set of solutions of a first-order differential equation such as the one in the previous example, we usually refer to it as a **one-parameter family of solutions**. The *parameter* is the constant C. Each definite value of C gives us what is called a **particular solution** of the differential equation. In the preceding example Joshua and Ellie produced particular solutions, one with $C = 0$ and the other with $C = -10$. A particular solution is sometimes called an **integral** of the equation, and its graph is called an **integral curve** or a **solution curve**.

Fig. 1.1 shows three of the integral curves of the equation $\frac{dy}{dx} = x^2 - 2x + 7$, where $C = 15$, 0, and -10 (from top to bottom).

The curve passing through the origin is Joshua's particular solution; the solution curve passing through the point $(0, -10)$ is Ellie's.

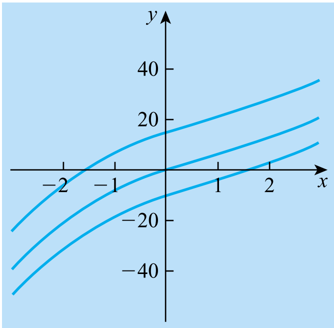


FIGURE 1.1

Integral curves of $\frac{dy}{dx} = x^2 - 2x + 7$ with parameters 15, 0, and -10

Exercises 1.2

In Problems 1–10, verify that the indicated function is a solution of the given differential equation. The letters a, b, c, and d denote constants.

A

1. $y'' + y = 0$; $y = \sin x$
2. $x'' - 5x' + 6x = 0$; $x = -\pi e^{3t} + \frac{2}{3}e^{2t}$
3. $\frac{1}{4}\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$; $y = x^2$
4. $t\frac{dR}{dt} - R = t^2 \sin t$; $R = t(c - \cos t)$
5. $\frac{d^4y}{dt^4} = 0$; $y = at^3 + bt^2 + ct + d$
6. $\frac{dr}{dt} = at + br$; $r = ce^{bt} - \frac{a}{b}t - \frac{a}{b^2}$
7. $xy' - 2 = 0$; $y = \ln(x^2)$
8. $y'' = a\sqrt{1 + (y')^2}$; $y = \frac{e^{ax} + e^{-ax}}{2a}$
9. $xy' - \sin x = 0$; $y = \int_1^x \frac{\sin t}{t}\,dt$ [*Hint*: Think of the *Fundamental Theorem of Calculus*. See Section A.4]
10. $y'' + 2xy' = 0$; $y = \int_3^x e^{-t^2}\,dt$ [*Hint*: Think of the *Fundamental Theorem of Calculus*. See Section A.4]
11. For each function, find a differential equation satisfied by that function:
    - **a.** $y = c + \frac{x}{c}$, where c is a constant
    - **b.** $y = e^{ax} \sin bx$, where a and b are constants
    - **c.** $y = (A + Bt)e^t$, where A and B are constants
    - **d.** $y(t) = e^{-3t} + \int_1^t uy(u)\,du$

In Problems 12–15, assume that the function y is defined implicitly as a function of x by the given equation, where C is a constant. In each case, use the technique of implicit differentiation to find a differential equation for which y is a solution.

12. $xy - \ln y = C$

13. $y + \arctan y = x + \arctan x + C$

14. $y^3 - 3x + 3y = 5$

15. $1 + x^2 y + 4y = 0$

16. Is a function y satisfying $x^2 + y^2 - 6x + 10y + 34 = 0$ a solution of the differential equation $\frac{dy}{dx} = \frac{3-x}{y+5}$? Explain your answer.

B

17. Verify that $y = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \ln\sqrt{x + \sqrt{x^2+1}}$ is a solution of the equation $2y = xy' + \ln(y')$.

18. Write a paragraph explaining why $B(t)$ in Example 1.2.1—a solution of the differential equation $\frac{dB}{dt} = kB$—can't be a polynomial, trigonometric, or logarithmic function.

19. **a.** Why does the equation $(y')^2 + 1 = 0$ have no real-valued solution?

b. Why does the equation $|y'| + |y| = 0$ have only one solution? What is the solution?

20. Explain why the equation $\frac{dx}{dt} = \sqrt{-|x-t|}$ has *no* real-valued solution.

21. If c is a positive constant, show that the two functions $y = \sqrt{c^2 - x^2}$ and $y = -\sqrt{c^2 - x^2}$ are both solutions of the nonlinear equation $y\frac{dy}{dx} + x = 0$ on the interval $-c < x < c$. Explain why the solutions are not valid outside the open interval $(-c, c)$.

22. **a.** Verify that the function $y = \ln(|C_1 x|) + C_2$ is a solution of the differential equation $y' = \frac{1}{x}$ for each value of the parameters C_1 and C_2 and every x in the interval $(0, \infty)$.

b. Show that there is only one genuine parameter needed for y. In other words, write $y = \ln(|C_1 x|) + C_2$ using only one parameter C.

23. Find a solution of $\frac{dy}{dx} + y = \sin x$ of the form $y(x) = c_1 \sin x + c_2 \cos x$, where c_1 and c_2 are constants.

24. Find a second-degree polynomial $y(x)$ that is a particular solution of the linear differential equation $2y' - y = 3x^2 - 13x + 7$.

25. Show that the first-order nonlinear equation $(xy' - y)^2 - (y')^2 - 1 = 0$ has a one-parameter family of solutions given by $y = Cx \pm \sqrt{C^2+1}$, but that any function y defined implicitly by the relation $x^2 + y^2 = 1$ is also a solution—one that does not correspond to a particular value of C in the one-parameter solution formula.

26. Consider the equation $xy'' - (x+n)y' + ny = 0$, where n is a nonnegative integer.

a. Show that $y = e^x$ is a solution.

b. Show that $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ is a solution.

C

27. Find a differential equation satisfied by the function

$$y(t) = \cos t + \int_0^t (t-u)y(u)\,du.$$

28. Show that the function y defined by $y(x) = \int_0^\pi \sin(x\cos t)\,dt$ satisfies the linear ordinary differential equation of order two

$$xy'' + y' + xy = 0.$$

[*Hint*: Use *Leibniz's Rule*. See Appendix A.7.]

1.3 Initial-value problems and boundary-value problems

Now suppose that we want to solve a first-order differential equation for y, a function of the independent variable t, and we specify that one of its integral curves must pass through a particular point (t_0, y_0) in the plane. We impose the condition $y(t_0) = y_0$, which is called an **initial condition**, and the problem is thus called an **initial-value problem** (**IVP**). Note that we are trying to pin down a particular solution this way. We find this solution by choosing a specific value of the constant of integration (the parameter).

Example 1.3.1 A First-Order Initial-Value Problem

Suppose that an object is moving along the x-axis in such a way that its instantaneous velocity at time t is given by $v(t) = 12 - t^2$. First, we find the *position* x of the object measured from the origin at any time $t > 0$.

Because the velocity function is the derivative of the position function, we can set up the first-order differential equation $\frac{dx}{dt} = v(t) = 12 - t^2$ to describe our problem.

A simple integration of both sides yields

$$x(t) = \int dx = \int \frac{dx}{dt}\,dt = \int 12 - t^2\,dt = 12t - \frac{t^3}{3} + C.$$

This last result tells us that the position of the object at an arbitrary time $t > 0$ can be described by any member of the one-parameter family $12t - \frac{t^3}{3} + C$, which is not a very satisfactory conclusion. But if we have some additional information, we can find a definite value for C and end the uncertainty.

Suppose we know, for example, that the object is located at $x = -5$ when $t = 1$. Then we can use this *initial condition* to get

$$-5 = x(1) = 12(1) - \frac{1^3}{3} + C, \text{ or } -5 = \frac{35}{3} + C.$$

This last equation implies that $C = \frac{-50}{3}$, so the position of the object at time t is given by the particular function $x(t) = 12t - \frac{t^3}{3} - \frac{50}{3} = 12t - \frac{(t^3+50)}{3}$.

We selected the initial condition $x(1) = -5$ randomly. Any other choice of $x(t_0) = x_0$ would have led to a definite value for C and a particular solution of the problem.

1.3.1 An integral form of an IVP solution

If a first-order equation can be written in the form $y' = f(x)$—that is, if the right-hand side is a continuous (or piecewise continuous) function of the independent variable alone—then we can always express the solution to the IVP $y' = f(x)$, $y(x_0) = y_0$ on an interval (a, b) as

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0 \tag{1.3.1}$$

for x in (a, b). Note that we use the x value of the initial condition as the lower limit of integration and the y value of the initial condition as a particular constant of integration. We use t as a *dummy variable*. Given Eq. (1.3.1), the *Fundamental Theorem of Calculus* (*FTC*) (Section A.4) implies that $y' = f(x)$, and we see that $y(x_0) = \int_{x_0}^{x_0} f(t)\,dt + y_0 = 0 + y_0 = y_0$, which is what we want. This way of handling certain types of IVPs is common in physics and engineering texts. In Example 1.2.6 the solution of the equation with $y(-1) = 2$, for example, is

$$\begin{aligned} y(x) &= \int_{-1}^{x} t^2 - 2t + 7\,dt + 2 \\ &= \left(\frac{t^3}{3} - t^2 + 7t\right)\Bigg|_{t=x} - \left(\frac{t^3}{3} - t^2 + 7t\right)\Bigg|_{t=-1} + 2 \\ &= \left(\frac{x^3}{3} - x^2 + 7x\right) - \left(\frac{-25}{3}\right) + 2 = \frac{x^3}{3} - x^2 + 7x + \frac{31}{3}. \end{aligned}$$

You could also solve this problem the way we did in Example 1.3.1—that is, without using a definite integral formula.

1.3.2 Families of solutions II

Although we have seen examples of first-order equations that have no solution or only one solution, in general we should expect a first-order differential equation to have an infinite set of solutions, described by a single parameter.

Extending the discussion in Section 1.2, we state that an nth-order differential equation may have an ***n*-parameter family of solutions**, involving n arbitrary constants $C_1, C_2, C_3, \ldots, C_n$ (the parameters). For example, a solution of a second-order equation $y'' = g(t, y, y')$ may have *two* arbitrary constants. By prescribing the **initial conditions** $y(t_0) = y_0$ *and* $y'(t_0) = y_1$, we can determine specific values for these two constants and obtain a *particular* solution. Note that we use the same value, t_0, of the independent variable for each condition.

The next example shows how to deal with a second-order IVP.

Example 1.3.2 A Second-Order IVP

We will show in Section 4.1 that any solution of the second-order linear equation $y'' + y = 0$ has the form $y(t) = A\cos t + B\sin t$ for arbitrary constants A and B. (You should verify that any function having the form indicated in the preceding sentence is a solution of the differential equation.) If a solution of this equation represents the *position* of a moving object relative to some fixed location, then the derivative of the solution represents the *velocity* of the particle at time t. If we specify, for example, the initial conditions $y(0) = 1$ and $y'(0) = 0$, we are saying that we want the position of the particle when we begin our study to be 1 unit in a positive direction from the fixed location and we want the velocity to be 0. In other words, our particle starts out at rest 1 unit (in a positive direction) from the fixed location.

We can use these initial conditions to find a particular solution of the original differential equation:

1. $y(0) = 1$ implies that $1 = y(0) = A\cos(0) + B\sin(0) = A$;
2. $y'(0) = 0$ implies that $0 = y'(0) = -A\sin(0) + B\cos(0) = B$.

Combining the results of (1) and (2), we find the particular solution $y(t) = \cos t$.

Definition 1.3.1

Finding the particular solution of the nth degree equation

$$F\left(t, y, y', y'', y''', \ldots, y^{(n-1)}, y^{(n)}\right) = 0$$

such that $y(t_0) = y_0$, $y'(t_0) = y_1$, $y''(t_0) = y_2$, $\ldots$, and $y^{(n-1)}(t_0) = y_{n-1}$, where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary real constants, is called solving an **initial-value problem** (**IVP**). The n specified values $y(t_0) = y_0$, $y'(t_0) = y_1$, $y''(t_0) = y_2, \ldots$, and $y^{(n-1)}(t_0) = y_{n-1}$ are called **initial conditions**.

Right now we can't be sure of the circumstances under which we can solve such an initial-value problem. We will discuss the *existence* and *uniqueness* of solutions of single equations in Section 2.8. Then in Section 6.2 we will consider IVPs for *systems* of differential equations.

Boundary-value problems

For differential equations of second order and higher, we can also determine a particular solution by specifying what are called **boundary conditions**. The idea here is to give conditions that must be satisfied by the solution function and/or its derivatives at *two different points* of the domain of the solution.

Definition 1.3.2

A **boundary-value problem** (**BVP**) is a problem of determining a solution to a differential equation subject to conditions on the unknown function specified at *two or more* values of the independent variable. These conditions are called **boundary conditions**.

The points chosen depend on the nature of the problem you are trying to solve and on the data you are given about the problem. For example, if you are analyzing the stresses on a steel girder of length L whose ends are embedded in concrete, you may want to find $y(x)$, the bend or "give" at a point x units from one end if a load is placed somewhere on the beam (Fig. 1.2). Note that the domain of y is $[0, L]$. In this problem it is natural to specify $y(0) = 0$ and $y(L) = 0$, reasonable values at the endpoints, or *boundaries*, of the solution interval. Graphically, we require the solution y to pass through the points $(0, 0)$ and $(L, 0)$. (See Problem 20 in Exercises 1.3 for an applied problem of this type.)

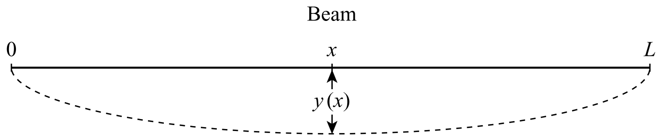


FIGURE 1.2

A solution $y(x)$ satisfying the boundary conditions $y(0) = 0$ and $y(L) = 0$

The next example shows, just as in the case of an initial-value problem, that without further analysis we can't be sure whether there are solutions of a particular BVP or whether any solution we find is unique. In general, BVPs are harder to solve than IVPs. Although BVPs will appear in this book from time to time, we focus most of our attention on initial-value problems.

As the next example shows, some boundary-value problems have no solution, others have one solution, and some have infinitely many solutions.

Example 1.3.3 A BVP Can Have Many, One, or No Solutions

We'll use the second-order differential equation from Example 1.3.2, $y'' + y = 0$, which has the two-parameter family of solutions $y(t) = c_1 \cos t + c_2 \sin t$.

Now let's see what happens if we impose the boundary conditions $y(0) = 1$, $y(\pi) = 1$. The first condition implies that $1 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$, and the second condition tells us that $1 = y(\pi) = c_1 \cos(\pi) + c_2 \sin(\pi) = -c_1$. Because we can't have c_1 equaling 1 and -1 at the same time, this contradiction says that the boundary-value problem has *no solution*.

On the other hand, the boundary conditions $y(0) = 1$, $y(2\pi) = 1$ lead to a different conclusion. If we use the first condition, we get $1 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$. The second condition yields the result $1 = y(2\pi) = c_1 \cos(2\pi) + c_2 \sin(2\pi) = c_1$. The fact that we can't pin down the value of c_2 tells us that *any* value is all right. In other words, the BVP has *infinitely many solutions* of the form $y(t) = \cos t + c_2 \sin t$.

Finally, if we demand that $y(0) = 1$ and $y(\pi/4) = 1$, we find that $1 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$ and

$$
\begin{aligned}
1 = y\left(\frac{\pi}{4}\right) &= c_1 \cos(\pi/4) + c_2 \sin\left(\frac{\pi}{4}\right) = c_1\left(\frac{\sqrt{2}}{2}\right) + c_2\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}}{2} + c_2\left(\frac{\sqrt{2}}{2}\right),
\end{aligned}
$$

which implies that $c_2 = \sqrt{2} - 1$. Therefore, this BVP has the *unique solution* $y(t) = \cos t + (\sqrt{2} - 1) \sin t$.

You should realize that for a general nth-order equation (or for a system of equations), there are many possible ways to specify boundary conditions, not always at the endpoints of solution intervals. The idea is to have a number of conditions that will enable us to solve for (or specify) the appropriate number of arbitrary constants.

The following example shows how boundary conditions occur naturally in the solution of an interesting problem.

Example 1.3.4 A Cool BVP

The *Car and Driver* magazine website (August 30, 2018) reports that the 2018 Lamborghini Huracán Performante Spyder (base price $314,654) goes from 0 to 60 mph in 2.4 seconds. Assuming constant acceleration, we ask how far the car travels before it reaches 60 mph.

If $s(t)$ denotes the position of the car after t seconds, then we must calculate $s(2.4) - s(0)$, the total distance covered by the car in the 2.4-second interval. We know the acceleration can be described as $a(t) = \frac{d^2s}{dt^2}$, which in this problem equals some constant C; and we know that $s(0) = s'(0) = 0$; that is, our initial position is considered 0, and the velocity when we first put our foot on the gas pedal is also 0. The last bit of information we have is that $s'(2.4)$, the velocity at the end of 2.4 seconds, is 60 mph. Thus, we have a second-order differential equation $\frac{d^2s}{dt^2} = C$, initial conditions, and some boundary conditions, and we must solve for the unknown function $s(t)$.

Now the basic rules of integral calculus tell us that when we find the antiderivative of each side of the differential equation in the last paragraph, we get

$$\int \frac{d^2s}{dt^2}\,dt = \int C\,dt = Ct + C_1,$$

where C_1 is a constant of integration. But $\int \frac{d^2s}{dt^2}\,dt = \frac{ds}{dt}$, so $\frac{ds}{dt} = Ct + C_1$. Integrating each side of this last equation gives us

$$s(t) = \frac{Ct^2}{2} + C_1 t + C_2.$$

Thus, we have an expression for $s(t)$, but it contains three arbitrary constants. Now we can use the condition $s(0) = 0$ to write

$$0 = s(0) = \frac{C(0)^2}{2} + C_1(0) + C_2,$$

which boils down to $0 = C_2$, so we can say

$$s(t) = \frac{Ct^2}{2} + C_1 t.$$

Because $s'(0) = 0$, we can see that $0 = s'(0) = (Ct + C_1)|_{t=0} = C_1$, and thus $s(t) = \frac{Ct^2}{2}$.

We still have one unknown constant, C, in our formula, but we know that at the end of 2.4 seconds the velocity is 60 miles per hour. *We have to be careful of our units here*. We don't want to mix seconds and hours. To make all our units consistent, we convert 2.4 seconds to 1/1500 (= 2.4/3600) of an hour. Then we can claim that $60 = s'(1/1500) = C \cdot (1/1500)$, so we have

$$C = \frac{60(1500)}{7} = 90{,}000 \text{ (miles/hr}^2\text{)},$$

$$s(t) = \frac{Ct^2}{2} = 45{,}000t^2,$$

and

$$s\left(\frac{1}{1500}\right) = 45{,}000\left(\frac{1}{1500}\right)^2 = 0.02\ldots \text{ mile} \approx 106 \text{ feet.}$$

We have shown that in going from 0 to 60, the Lamborghini will travel approximately 106 feet.

General solutions

If *every* solution of an nth-order differential equation $F\big(x, y, y', y'', y''', \ldots, y^{(n-1)}, y^{(n)}\big) = 0$ on an interval (a, b) can be obtained from an n-parameter family by choosing appropriate values for the n constants, we say that the family is the **general solution** of the differential equation. In this case, we will need n initial conditions or n boundary conditions (or a combination of n conditions) to determine the constants.

A **particular solution** of an nth-order differential equation is any solution that does not contain an arbitrary constant. For example, this solution may be the result of choosing specific values for all the parameters of a general solution.

Sometimes, however, we can't find *every* solution among the members of an n-parameter family. For example, you should verify that the first-order nonlinear differential equation $2xy' + y^2 = 1$ has a one-parameter family of solutions given by $y = \frac{Cx-1}{Cx+1}$. However, for all values of x the constant function $y \equiv 1$ is also a solution, but it can't be obtained from the family by choosing a particular value of the parameter C. Suppose we *could* find a value of C such that $\frac{Cx-1}{Cx+1} = 1$. Cross-multiplication gives us $Cx - 1 = Cx + 1$, so that $-1 = 1$!

Also, $y(x) = kx^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$ for any constant k and for all values of x, but $y(x) = x^2 \ln|x|$ is also a solution for all x. (*Check these claims.*) Of course, because the equation is second-order, we should realize that a one-parameter family can't be the general solution.

A particular solution of an nth-order differential equation that can't be obtained by picking specific values of the parameters in an n-parameter family of solutions is called a **singular solution**. We'll see in Chapter 2 that some of these singular solutions are created when we perform certain algebraic manipulations on differential equations.

1.3.3 Solutions of systems of ODEs

For a *system* of two equations with unknown functions $x(t)$ and $y(t)$, a solution on an interval (a, b) consists of a *pair* of differentiable functions $x(t)$, $y(t)$ satisfying both equations that make up the system at all points of the interval. Initial conditions are given as $x(t_0) = x_0$ and $y(t_0) = y_0$.

Example 1.3.5 A System IVP

Later, when we study systems in more depth, we will see why the only solution of the linear system

$$\frac{dx}{dt} = -3x + y$$

$$\frac{dy}{dt} = x - 3y$$

satisfying the conditions $x(0) = 0$ and $y(0) = 7$ is $\left\{x(t) = \frac{7}{2}e^{-2t} - \frac{7}{2}e^{-4t},\ y(t) = \frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t}\right\}$. We can verify that these functions constitute a solution and accept the uniqueness as a fact for now.

We calculate

$$\frac{dx}{dt} = -7e^{-2t} + 14e^{-4t} \quad \text{and} \quad \frac{dy}{dt} = -7e^{-2t} - 14e^{-4t}.$$

Then $-3x + y = -3\left(\frac{7}{2}e^{-2t} - \frac{7}{2}e^{-4t}\right) + \left(\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t}\right) = -7e^{-2t} + 14e^{-4t} = \frac{dx}{dt}$ and $x - 3y = \left(\frac{7}{2}e^{-2t} - \frac{7}{2}e^{-4t}\right) - \left(\frac{21}{2}e^{-2t} + \frac{21}{2}e^{-4t}\right) = -7e^{-2t} - 14e^{-4t} = \frac{dy}{dt}$.

You can think of the solution pair in the preceding example as coordinates of a point $(x(t), y(t))$ in two-dimensional space, R^2. As the independent variable t changes, the points trace out a curve in the x-y plane called a **trajectory**. The *positive* direction of the curve is the direction it takes as t *increases*. Fig. 1.3a shows the curve in the x-y plane corresponding to the system solution in Example 1.3.5, together with arrows indicating its direction. The initial point $(x(0), y(0)) = (0, 7)$ is indicated. Looking at the solution formulas for $x(t)$ and $y(t)$, we see that $\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} y(t)$, so that the curve tends toward the origin as t increases.

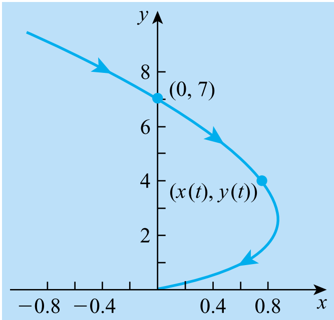


FIGURE 1.3a

Plot of $(x(t), y(t)) = \left(\frac{7}{2}e^{-2t} - \frac{7}{2}e^{-4t}, \frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t}\right)$ in the x-y plane, $-0.1 \le t \le 4$

Fig. 1.3b shows x plotted against t, and Fig. 1.3c shows y plotted against t.

A very important, *dynamical* way of looking at the situation in the preceding example is to think of the curve in Fig. 1.3a as the path (or trajectory) of an object or quantity whose motion or change is governed by the system of differential equations. Initial conditions specify the behavior (the value, rate of change, and so on) at a single point on the path of the moving object or changing quantity. The proper graph of the solution of the system in Example 1.3.5 is a **space curve**, the set of points $(t, x(t), y(t))$. Boundary conditions also determine certain aspects of the path of the phenomenon under study.

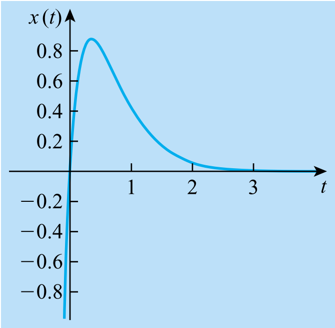


FIGURE 1.3b

Plot of $x(t) = \frac{7}{2}e^{-2t} - \frac{7}{2}e^{-4t}$, $-0.1 \leq t \leq 4$

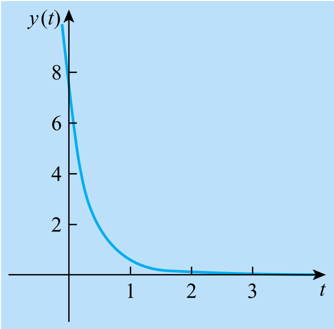


FIGURE 1.3c

Plot of $y(t) = \frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t}$, $-0.1 \leq t \leq 4$

Similarly, each solution of the nonlinear system

$$\begin{aligned}\dot{x} &= -sx + sy \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz\end{aligned}$$

where b, r, and s are constants, is an ordered triple $(x(t), y(t), z(t))$, and initial conditions have the form $x(t_0) = x_0$, $y(t_0) = y_0$, and $z(t_0) = z_0$. Boundary conditions in this situation can take various forms. The trajectory in this case is a *space curve*, a path in three-dimensional space. The true graph of the solution is the set of points $(t, x(t), y(t), z(t))$ in *four-dimensional space*. These geometrical points of view, especially the idea of a trajectory, are very useful, and we'll follow up on these concepts in Chapters 6 and 7.

Exercises 1.3

A

1. Consider the equation and solution in Problem 4 in Exercises 1.2. Find the particular solution that satisfies the initial condition $R(\pi) = 0$.
2. Consider the equation and solution in Problem 5 in Exercises 1.2. Find the particular solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, and $y'''(0) = 6$. [*Hint*: Use the initial conditions one at a time, beginning from the left.]
3. Consider the equation and solution in Problem 6 in Exercises 1.2. Find the particular solution that satisfies the initial condition $r(0) = 0$. (Your answer should involve only the constants a and b.)
4. Consider the equation and solution in Problem 8 in Exercises 1.2. Find the particular solution that satisfies the initial conditions $y(0) = 2$, $y'(0) = 0$.
5. Find constants A, B, and C such that $\frac{1}{8} - \frac{1}{4}x + \frac{11}{296}e^{6x} + Ax^2 + B\sin x + C\cos x$ is a solution of the IVP $y''' - 6y'' = 3 - \cos x$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.
6. A class of functions is described by the formula $y = Ct/\sqrt{C^2 - t^2}$ for each real number. Is there a function in this family such that $y = 2$ when $t = 1$? Such that $y = 1$ when $t = 2$?

B

7. A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 1/(t^2 + 1)$. Assuming that the initial position of the particle is the origin, show that it will never get past $x = \pi/2$.
8. Show that the functions $y_1(x) \equiv 0$ and $y_2(x) = (x - x_0)^3$, each defined for $-\infty < x < \infty$, are both solutions of the initial-value problem
$$\frac{dy}{dx} = 3y^{2/3}, \quad y(x_0) = 0.$$
9. Show that $y = e^{x^2} \int_1^x e^{-t^2}\, dt$ is a solution of the IVP $y' = 1 + 2xy$, $y(1) = 0$.
10. The differential equation of a family of curves in the x-y plane is given by
$$y''' = -24\cos(\pi x/2).$$
    a. Find an equation for the family and give the number of parameters involved.
    b. Find a member of the family that passes through the points $(0, -4)$ and $(1, 0)$ and that has a slope of 6 at the point where $x = 1$.
11. Is it possible for the differential equation corresponding to a three-parameter family of solutions to be of order four? Explain.
12. Given the first-order nonlinear equation $(y')^2 + xy' = y$, verify the following statements.

- **a.** Each member of the one-parameter family of functions $y(x) = cx + c^2$, where c is a real constant, is a solution of the differential equation.
- **b.** The function $y(x) = -\frac{1}{4}x^2$ can't be obtained from $y(x) = cx + c^2$ by any choice of c, yet it satisfies the given differential equation. (Thus $y = -\frac{1}{4}x^2$ is a singular solution.)

13. A 727 jet needs to be flying 200 mph to take off. If the plane can accelerate from 0 to 200 mph in 30 seconds, how long must the runway be, assuming constant acceleration?

14. An automobile website reported that a 2008 Mercedes-Benz SLR McLaren went from 0 to 62 mph in 3.8 seconds.

- **a.** Assuming constant acceleration, how far did the car travel before it reached 60 mph?
- **b.** The car's "carbon ceramic" brakes were applied when the car was going 62 mph. Assuming constant deceleration, how long did it take the car to stop if it stopped (according to the report) in 114 feet?

15. **a.** Show that the functions $x(t) = (A + Bt)e^{3t}$ and $y(t) = (3A + B + 3Bt)e^{3t}$ are solutions of the system

$$
\begin{aligned}
x' &= y \\
y' &= -9x + 6y
\end{aligned}
$$

for all values of the parameters A and B.

- **b.** Find the solution to the system in part (a) with $x(0) = 1$ and $y(0) = 0$.

16. Show that the functions $x(t) = e^{-t/10}\sin t$ and $y(t) = \frac{1}{10}e^{-t/10}(-10\cos t + \sin t)$ are solutions of the initial value problem

$$
\begin{aligned}
\frac{dx}{dt} &= -y \\
\frac{dy}{dt} &= (1.01)x - (0.2)y; \;\; x(0) = 0, \, y(0) = -1.
\end{aligned}
$$

17. A mathematical model of an idealized company consists of the equations

$$
\begin{aligned}
\frac{du}{dt} &= kau, \quad u(0) = A \\
\frac{dw}{dt} &= a(1 - k)u, \quad w(0) = 0.
\end{aligned}
$$

Here $u(t)$ represents the capital invested in the company at time t, $w(t)$ denotes the total dividend paid to shareholders in the period $[0, t]$, and a and k are constants with $0 \le k \le 1$.

- **a.** Solve the first equation for $u(t)$. (See Example 1.2.1.)
- **b.** Substitute your answer for part (a) in the differential equation for w and integrate to find $w(t)$. (Distinguish between $w(t)$ for $0 < k \le 1$ and for $k = 0$.)

18. The general solution of the differential equation $x'' = -x$ is $x(t) = C_1 \sin t + C_2 \cos t$. Find the solution that satisfies the initial conditions $x(0) = 2$, $x'(0) = -1$.

C

19. Let $W = W(t)$ denote your weight on day t of a diet. If you eat C calories per day and your body burns EW calories per day, where E represents calories per pound, then the equation $\frac{dW}{dt} = k(C - EW)$ models your change in weight.[3] (This equation says that your change in weight is proportional to the difference between calories eaten and calories burnt, with constant of proportionality k.)

a. Show that $W = \frac{C}{E} + \left(W_0 - \frac{C}{E}\right) e^{-kEt}$ is a solution of the equation, where $W_0 = W(0)$, your weight at the beginning of the diet.

b. Given the solution in part (a), what happens to $W(t)$ as $t \to \infty$?

c. If $W_0 = 180$ lb, $E = 20$ cal/lb, and $k = 1/3500$ lb/cal, then how long will it take to lose 20 lb? How long for 30 lb? 35 lb? What do your answers seem to say about the process of weight loss?

20. Solve the equation $EI\frac{d^4y}{dx^4} = -\frac{W}{L}$, with the boundary conditions $y(0) = 0$, $y'(0) = 0$; $y(L) = 0$, $y'(L) = 0$. (This problem arises in the analysis of the stresses on a uniform beam of length L and weight W, both of whose ends are fixed in concrete. The solution y describes the shape of the beam when a certain type of load is placed on it. Here, E and I are constants, and the product EI is a constant called the *flexural rigidity* of the beam.) [*Hint*: Integrate successively, introducing a constant of integration at each stage. Then use the boundary conditions to evaluate these constants of integration.]

21. The *logistic equation* $\frac{dy}{dt} = ky(t)\left(1 - \frac{y(t)}{M}\right)$ is used to describe the growth of certain kinds of human and animal populations. Here, k and M denote constants describing characteristics of the population being modeled.

a. Show that the function $y(t) = \frac{M}{1+Ae^{-kt}}$ satisfies the logistic equation with $y(0) = \frac{M}{1+A}$.

b. Analysis of U.S. population data for the years 1800, 1900, and 2000 indicates that the solution given in part (a) provides a good fit if $M = 333.2361$, $A = 61.7743$, and $k = 0.0290784$. Using technology, plot the graph of $y(t)$ using these values of M, A, and k. (Here, t denotes the time in years since 1800, the year of the second U.S. census.)

c. In 1800 the U.S. population was 5,308,483. In 1980 the figure was 226,542,199, while for 2000 the population was 281,421,906. By evaluating the function plotted in part (b) at $t = 0$, 180, and 200, compare the values (in millions) given by $y(t)$ to the actual populations. (Check the official website of the Bureau of the Census for additional information: www.census.gov.)

[3] A.C. Segal, "A Linear Diet Model," *College Mathematics Journal* **18** (1987): 44–45.

d. According to the model with parameters as given in part (b), what happens to the population of the U.S. as $t \to \infty$?

22. The equations

$$\frac{dT^*}{dt} = kV_1T_0 - \delta T^*$$
$$\frac{dV_1}{dt} = -cV_1$$

are used in modeling HIV-1 infections.[4] Here, $T^* = T^*(t)$ denotes the number of infected cells, $T_0 = T(0)$ is the number of potentially infected cells at the time therapy is begun, $V_1 = V_1(t)$ is the concentration of viral particles in plasma, k is the rate of infection, c is the rate constant for viral particle clearance, and δ is the rate of loss of virus-producing cells.

a. Imitate the analysis shown in Example 1.2.1 and solve the second equation for $V_1(t)$, expressing your solution in terms of $V_0 = V_1(0)$.

b. Using the solution found in part (a), show that the solution of the differential equation for T^* can be written as

$$T^*(t) = T^*(0)e^{-\delta t} + \frac{kT_0V_0}{c-\delta}(e^{-ct} - e^{-\delta t}).$$

c. What does the solution in part (a) say about the number of infected cells as $t \to \infty$?

23. Consider the linear equation $x^2y'' + xy' - 4y = x^3$ (*). Let y_{GR} be the *general* solution of the "reduced" (or "complementary") equation $x^2y'' + xy' - 4y = 0$ and let y_P be a *particular* solution of (*). Show that $y_{GR} + y_P$ is the *general* solution of (*). (For this problem, define the general solution of a second-order ODE as a solution having two arbitrary constants. A particular solution, of course, has *no* arbitrary constants.)

Summary

The study of differential equations is as old as the development of calculus by Newton and Leibniz in the late 17th century. Originally, motivation was provided by important questions about change and motion on earth and in the heavens.

An **ordinary differential equation** (**ODE**) is an equation that involves an unknown function, its independent variable, and one or more of its derivatives:

$$F\left(x, y, y', y'', y''', \ldots, y^{(n-1)}, y^{(n)}\right) = 0.$$

[4] A.S. Perelson, A.U. Neumann, M. Markowitz, J.M. Leonard, and D.D. Ho, "HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time," *Science* **271** (1996): 1582–1586.

Such an equation can be described in terms of its **order**, the order of the highest derivative of the unknown function in the equation.

Differential equations can also be classified as either **linear** or **nonlinear**. **Linear equations** can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x),$$

where each coefficient function $a_i(x)$ depends on x alone and doesn't involve y or any of its derivatives. **Nonlinear equations** usually contain products, quotients, or more elaborate combinations of the unknown function and its derivatives.

A **solution** of an ODE is a real-valued function that, when substituted in the equation, makes the equation valid on some interval. A given nth-order ODE may have *no* solutions, *only one* solution, or *infinitely many solutions*. An *infinite family of solutions* may be characterized by n constants (parameters). These arbitrary constants, if present, may be evaluated by imposing appropriate **initial conditions** (usually n of them, involving the behavior of the solution at a single point of its domain) or **boundary conditions** (at two or more points). Solving a differential equation with initial conditions is referred to as solving an **initial-value problem** (**IVP**). Solving a differential equation with boundary conditions is referred to as solving a **boundary-value problem** (**BVP**). In general, BVPs are harder to solve than IVPs. The result of solving either an IVP or a BVP is called a **particular solution** of the equation or an **integral** of the equation. The graph of a particular solution is called an **integral curve** or **solution curve**. In later chapters, we'll discuss the question of *existence and uniqueness* for IVPs: Does the equation or system have a solution that satisfies the initial conditions? If so, is there only one solution?

If *every* solution of an nth-order ODE on an interval can be obtained from an n-parameter family by choosing appropriate values for the n constants, then we say that the family is the **general solution** of the differential equation. In this case we need n initial conditions or n boundary conditions to determine the constants. A **particular solution** is one containing no arbitrary constant. However, sometimes there are **singular solutions** that can't be found just by choosing particular values of the constants.

Just as high school or college algebra introduces systems of algebraic equations, the study of certain problems often leads to **systems of differential equations**. These systems, in turn, can be classified as either **linear systems** or **nonlinear systems**. We can specify initial or boundary conditions for systems. Whether we're considering single equations or systems of equations we are dealing with dynamical situations—situations in which objects and quantities are moving and changing. For certain systems of two equations involving the unknown functions x and y, it is often useful to look at a **trajectory**, the set of points $(x(t), y(t))$ where x and y are solutions of the system.

CHAPTER

First-order differential equations

2

Introduction

The various examples in the preceding chapter should have convinced you that there are different possible answers to the question of what the solution or solutions to a differential equation look like. In this chapter, we'll examine first-order differential equations from the analytic and the qualitative points of view.

First, we'll learn *analytic* solution techniques for two important types of first-order equations, separable and linear. For these kinds of equations, we'll come up with explicit or implicit formulas for their solution curves. This method of solution is often referred to as *integrating* a differential equation.

Next, there is a *qualitative* way of viewing differential equations. This is a neat geometrical way of studying the behavior of solutions without actually solving the differential equation. The idea is to examine certain pictures or graphs derived from differential equations. Although we can do some of this work by hand, computer algebra systems, specialized programs, and many calculators can produce these graphs, and you'll be expected to use technology when appropriate. (Follow your instructor's guidance in using technology.)

After the analytic and qualitative treatments in this chapter, Chapter 3 will focus on some numerical solution methods concerned with approximating values of solutions. As we'll see, qualitative and numerical methods are both necessary because it is often impossible to represent the solutions of differential equations—even first-order equations—by formulas involving elementary functions.[1]

2.1 Separable equations

The simplest type of differential equation to solve is one in which the variables are *separable*. Formally, a first-order differential equation $\frac{dy}{dx} = F(x, y)$ is called **separable** if it can be written in the form $\frac{dy}{dx} = f(x)g(y)$, where f denotes a function of

[1] In general, "elementary functions" are finite combinations of integer powers of the independent variable, roots, exponential functions, logarithmic functions, trigonometric functions, and inverse trigonometric functions.

A Modern Introduction to Differential Equations. https://doi.org/10.1016/B978-0-12-818217-8.00009-9

the independent variable x alone and g denotes a function of the dependent variable y alone. For example, the equation $\frac{dy}{dx} = e^x y^2$ is separable.

If $y(x)$ is a nonconstant solution of the equation $\frac{dy}{dx} = f(x)g(y)$ and $g(y(x))$ is nonzero on an interval (a, b), we can divide both sides of the equation by $g(y(x))$ (a process called **separation of variables**) to get

$$\frac{1}{g(y(x))}\frac{dy}{dx} = f(x).$$

If $F(x)$ is an antiderivative of $f(x)$ and $G(y)$ is an antiderivative of $\frac{1}{g(y)}$, we can integrate both sides of the preceding equation with respect to the independent variable x to obtain

$$\int \frac{1}{g(y(x))}\frac{dy}{dx}\,dx = \int \frac{1}{g(y)}\,dy = \int f(x)\,dx,$$

or $G(y(x)) = F(x) + C$, where the constants of integration associated with G and F have been combined in the single constant C. Note how we use the *Chain Rule* in working with $G(y(x))$:

$$\frac{d}{dx}G(y(x)) = G'(y(x))\frac{dy}{dx} = \frac{1}{g(y(x))}\frac{dy}{dx}.$$

There are three things to be careful about: (1) not every first-order differential equation is separable; (2) even after you have separated the variables and integrated, it may not be possible to solve for one variable (say y) in terms of the other (say x)—you may have to express your answer *implicitly*; (3) you may not be able to carry out the integration(s) in terms of elementary functions. We'll see examples of these situations later on.

Also, note that in a separable equation, for example $y' = f(x)g(y)$, a solution of $g(y) \equiv 0$ is also a solution of the differential equation—possibly a *singular* solution (see Section 1.3). If $g(y) = 0$, then $y' = 0$, implying that y is a constant. Conversely, if $y(x) = c$ is a constant solution, then $y' = 0$, which implies that $g(y) = 0$ because $f(x) \equiv 0$ is unlikely in a physical problem. This says that the zeros of g are constant solutions and, in general, they are the only constant solutions.

The preceding analysis can be refined by considering three cases for a separable differential equation $\frac{dy}{dx} = f(x)\,g(y)$: (1) $g(y) \equiv 1$; (2) $f(x) \equiv 1$; (3) neither (1) nor (2). In case 1 the equation takes the simple form $\frac{dy}{dx} = f(x)$. If $f(x)$ is continuous on some interval $a < x < b$, then the initial-value problem (IVP) $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$ has a unique solution on (a, b) given by $y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$. [See Eq. (1.3.1). The uniqueness of this solution will be discussed in Section 2.8.]

Example 2.1.1 A Separable Equation: Case 1

The IVP $\frac{dy}{dx} = -x^3 + \cos x$, $y(1) = 1$ is a case 1 situation with $f(x) = -x^3 + \cos x$, a continuous function for all real values of x. We can solve the IVP by integrating both sides of the equation to obtain the general solution and then substituting the initial condition. Alternatively, we can use the simple solution formula

$$y(x) = 1 + \int_1^x (-t^3 + \cos t)\,dt = -\frac{x^4}{4} + \sin x + \left(\frac{5}{4} - \sin 1\right).$$

In case 2, we have $\frac{dy}{dx} = g(y)$. We can rewrite the equation as $\frac{1}{g(y)}\frac{dy}{dx} = 1$, or more accurately as $\frac{1}{g(y(x))}\frac{dy}{dx} = 1$. Integrating both sides with respect to x, we get $\int \frac{1}{g(y(x))}\frac{dy}{dx}\,dx = \int 1\,dx$. If we make the substitution $y = y(x)$, then $dy = y'(x)\,dx = \frac{dy}{dx}\,dx$, and we have $\int \frac{1}{g(y)}\,dy = \int 1\,dx = x + C$. This gives us a solution (possibly implicit) to our case 2 ordinary differential equation (ODE). Letting $G(y) = \int \frac{1}{g(y)}\,dy$, we can express the solution of the ODE in the form $G(y) = x + C$. If we are given the initial condition $y(x_0) = y_0$, we choose the constant of integration C such that $G(y_0) = x_0 + C$. Finally, if G has an inverse, we can write $y(x) = G^{-1}(x + C) = G^{-1}(x + G(y_0) - x_0)$.

The next example, illustrating case 2, showcases a simple differential equation with important applications to problems involving growth and decline (decay). In Example 1.2.1, we *guessed* at the solution and then verified that our guess was correct. In this example we use the fact that the logarithmic and exponential functions are inverses of each other.

Example 2.1.2 Solving a Separable Equation, Case 2

The principle here is that *the rate of growth (or decline) of a quantity Q is directly proportional to the quantity at a given time*: $\dot{Q}(t) = \frac{dQ}{dt} = kQ$, where k is a positive constant in a growth situation and k is negative when the quantity is decreasing over time.

Separating the variables, we can write $\frac{dQ}{Q} = k\,dt$, so that $\int \frac{dQ}{Q} = \int k\,dt$ and $\ln|Q| = kt + C$. Then we exponentiate: $e^{\ln|Q|} = e^{kt+C} = e^{kt}e^{C}$, or $|Q| = Me^{kt}$, where $M = e^{C}$, a positive constant. Since we can reasonably assume that Q is positive, we can just write $Q(t) = Me^{kt}$, with $M > 0$. Finally, we see that $Q(0) = Me^0 = M$, so that our final solution is $Q = Q(t) = Q(0)e^{kt}$.

Case 3, $\frac{dy}{dx} = f(x)g(y)$, where neither f nor g is a constant function, is the most interesting case. Now we can separate the variables to write

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x), \quad \int \frac{1}{g(y(x))}\frac{dy}{dx}\,dx = \int f(x)dx, \text{ and } \int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

The case 3 example that follows adds the use of technology.

Example 2.1.3 A Separable Equation and the Graph of a Solution

Suppose that an insect population P shows seasonal growth modeled by the differential equation $\frac{dP}{dt} = kP\cos(\omega t)$, where k and ω are positive constants. (The cosine factor suggests periodic fluctuation.)

We see that the equation is separable: $\frac{dP}{dt} = f(P)g(t)$, where $f(P) = P$ and $g(t) = k\cos(\omega t)$. (We could have left the constant k with the factor P, but if we think ahead, we'll realize that there's one less algebraic step if we keep the constant with the cosine term.) Separating the variables, we get $\frac{dP}{P} = k\cos(\omega t)\,dt$, so that $\int \frac{dP}{P} = k\int \cos(\omega t)\,dt$, or $\ln|P| = \frac{k}{\omega}\sin(\omega t) + C$.

Exponentiating, we see that $P(t) = Re^{\frac{k}{\omega}\sin(\omega t)}$, where $R > 0$. (This is a population problem, so $R > 0$ is a realistic assumption.) Letting $P_0 = P(0)$ denote the initial insect population, we have $P(t) = P_0 e^{\frac{k}{\omega}\sin(\omega t)}$ as the solution.

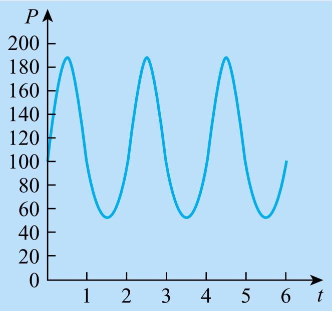


FIGURE 2.1

The solution of the IVP $\frac{dP}{dt} = 2P\cos(\pi t);\ P(0) = 100$

Graphing the solution curve for $P_0 = 100$, $k = 2$, and $\omega = \pi$ (Fig. 2.1), we can see that the population varies periodically, fluctuating from a minimum value of $100e^{-\frac{2}{\pi}}$ (approximately 53) to a maximum value of $100e^{\frac{2}{\pi}}$ (approximately 189).

Sometimes, as the next example shows, it may not be easy to find an explicit solution for a separable differential equation.

Example 2.1.4 A Separable Equation with Implicit Solutions

The equation $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ can be written as $\frac{dy}{dx} = f(x)g(y)$, where $f(x) = x^2$ and $g(y) = \frac{1}{1+y^2}$. Separating the variables, we get $(1 + y^2)\,dy = x^2\,dx$.

Integrating both sides, we find that $y + \frac{y^3}{3} = \frac{x^3}{3} + C$, or $\frac{x^3}{3} - \left(y + \frac{y^3}{3}\right) = C$. This gives the solution *implicitly*. To get an *explicit* solution, we must solve this last equation for y in terms of x or for x in terms of y. Either way is acceptable, although solving for x as a function of y is easier algebraically. But even if we don't find an explicit solution, we can plot solution curves for different values of the constant C. (This may be a good time to find out how to graph implicit functions using your available technology.) In Fig. 2.2 we use (from top to bottom) $C = -7, -5, -3, 0, 3, 5,$ and 7.

The third concern we mentioned earlier is that you may not be able to integrate one or both of the sides after you have separated the variables. We will address this problem next.

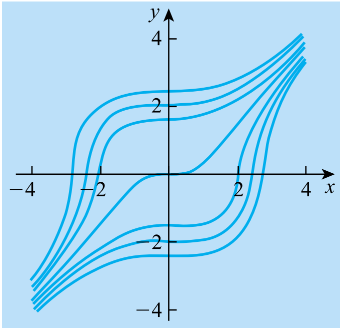


FIGURE 2.2

Implicit solutions of $\frac{dy}{dx} = \frac{x^2}{1+y^2}$: the curves $\frac{x^3}{3} - \left(y + \frac{y^3}{3}\right) = C$; $C = -7, -5, -3, 0, 3, 5,$ and 7; $-4 \le x \le 4$, $-4 \le y \le 4$

Example 2.1.5 A Function with No Explicit Integral

The differential equation $\frac{dy}{dt} = e^{y^2} t$ is clearly separable—we can write $e^{-y^2}\,dy = t\,dt$. However, we can't carry out the integration $\int e^{-y^2}\,dy$ on the left-hand side because there is no combination of elementary functions whose derivative is e^{-y^2}. Consequently, we are forced to write the family of solutions as

$$\int e^{-y^2}\,dy = \frac{t^2}{2} + C, \quad \text{or} \quad 2\int e^{-y^2}\,dy = t^2 + K,$$

where $K = 2C$.

Integrals of the form $\int_a^b e^{-y^2}\,dy$ have many applications in mathematics and science, especially in problems dealing with probability and statistics. For instance, the **error function** $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy$ appears in many applied problems and can be evaluated easily by any computer algebra system (CAS).

Example 2.1.6 An Equation with Infinitely Many Singular Solutions

The differential equation $\frac{dy}{dx} = -\frac{x \cot y}{\sqrt{x+1}}$ is separable and we can write $\tan y\,dy = -\frac{x}{\sqrt{x+1}}\,dx$, or $\int \tan y\,dy = -\int \frac{x}{\sqrt{x+1}}\,dx$. Writing $\int \tan y\,dy$ as $\int \frac{\sin y}{\cos y}\,dy$ or, transferring the negative sign to the left side of the equation, $\int -\frac{\sin y}{\cos y}\,dy$, suggests the logarithmic nature of the integral: $\int \frac{-\sin y}{\cos y}\,dy = \ln|\cos y|$. The integral $\int \frac{x}{\sqrt{x+1}}\,dx$ suggests a substitution to eliminate the radical sign. If we let $x + 1 = u^2$ for $x > -1$, we have $dx = 2u\,du$. This gives us the equation $\ln|\cos y| = \int \frac{(u^2-1)}{u} \cdot 2u\,du = 2\int u^2 - 1\,du = 2\left(\frac{u^3}{3} - u\right) + C = 2\left(\frac{(x+1)\sqrt{x+1}}{3} - \sqrt{x+1}\right) - \frac{2}{3}(x-2)\sqrt{x+1} + C.$ The implicit solution to the original differential equation is given by the one-parameter family of solutions $\ln|\cos y| - \frac{2(x-2)}{3}\sqrt{x+1} = C$ for $x > -1$ and $y \ne \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. (We avoid these values of y because $\cos y = 0$ and $\ln 0$ is not defined. However, the constant functions

$y = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ are also solutions of the differential equation (*Check this.*) that cannot be obtained from the one-parameter family.

Dealing with separable equations often requires some algebraic skills and some integration intuition, although technology can help in tough situations. The next example introduces a common algebraic challenge.

Example 2.1.7 Using Partial Fractions

The equation $\frac{dz}{dt} + 1 = z^2$ looks simple enough, but it requires some algebraic manipulation to get a neat solution. Separating the variables, we get $\frac{dz}{z^2-1} = dt$. Using the method of partial fractions (see Section A.5), we can write $\frac{1}{z^2-1}$ as $\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$, so integration gives us $\int \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) dz = \int 1\, dt$, or $\frac{1}{2}(\ln|z-1| - \ln|z+1|) = t + C_1$. Multiplying both sides of this last equation by 2 and then simplifying the logarithmic expression, we get $\ln\left|\frac{z-1}{z+1}\right| = 2t + C_2$. Exponentiating, we find that $\frac{z-1}{z+1} = Ke^{2t}$. Finally, solving this last equation for z (*Try this—it's a bit tricky.*), we conclude that $z = \frac{1+Ke^{2t}}{1-Ke^{2t}}$, a one-parameter family of solutions.

Note that in going through the process of separating the variables, we divided by $z^2 - 1$, implicitly assuming that this expression was not zero. Coming back to this, we see that the constant function $z \equiv 1$ corresponds to $K = 0$ in our one-parameter family, whereas $z \equiv -1$ is a singular solution. (*Why?*)

As simple as separable equations may seem, they have some very important applications. The calculations and manipulations involved in the next example may seem tedious, but they should remind you of things you have seen in previous classes. The analysis at the end of the example should convince you that a graphical approach can be enlightening.

Example 2.1.8 A Model of a Bimolecular Chemical Reaction

Most chemical reactions can be viewed as interactions between two molecules that undergo a change and result in a new product. The rate of a reaction therefore depends on the number of interactions or collisions, which in turn depends on the concentrations (in moles per liter) of the two types of molecules. Consider the simple (*bimolecular*) reaction $A + B \to X$ in which molecules of substance A collide with molecules of substance B to create substance X.

Let's designate the concentrations at time 0 of A and B by α and β, respectively. We'll assume that the concentration of X at the beginning is 0 and that at time t it is $x = x(t)$. The concentrations of A and B at time t are, correspondingly, $\alpha - x$ and $\beta - x$. Note that $\alpha - x > 0$ and $\beta - x > 0$ (*Why?*). The rate of formation (the *velocity of reaction* or *reaction rate*) is given by the differential equation $\frac{dx}{dt} = k(\alpha - x)(\beta - x)$, where k is a positive number called the *velocity constant*. The product on the right-hand side of the equation reflects the interactions or collisions between the two kinds of molecules. We want to determine $x(t)$.

Separating the variables and integrating, we get

$$\int \frac{dx}{(\alpha - x)(\beta - x)} = \int k\, dt.$$

To simplify the integrand $\frac{1}{(\alpha-x)(\beta-x)}$, we use the technique of partial fractions so that we can write

$$\int \frac{dx}{(\alpha-x)(\beta-x)} = \frac{1}{\beta-\alpha}\int \frac{dx}{\alpha-x} + \frac{1}{\alpha-\beta}\int \frac{dx}{\beta-x} = \int k\,dt,$$

or

$$-\frac{1}{\beta-\alpha}\ln(\alpha-x) - \frac{1}{\alpha-\beta}\ln(\beta-x) = kt + C,$$

which simplifies to

$$\frac{1}{\alpha-\beta}\ln\left(\frac{\alpha-x}{\beta-x}\right) = kt + C.$$

The initial condition $x(0) = 0$ leads us to conclude that

$$C = \frac{1}{\alpha-\beta}\ln\left(\frac{\alpha}{\beta}\right).$$

Then

$$\frac{1}{\alpha-\beta}\ln\left(\frac{\alpha-x}{\beta-x}\right) = kt + \frac{1}{\alpha-\beta}\ln\left(\frac{\alpha}{\beta}\right),$$

so

$$\ln\left(\frac{\alpha-x}{\beta-x}\right) = (\alpha-\beta)kt + \ln\left(\frac{\alpha}{\beta}\right),$$

or

$$\frac{\alpha-x}{\beta-x} = \frac{\alpha}{\beta}e^{(\alpha-\beta)kt}.$$

A few more algebraic manipulations lead to the solution

$$x = x(t) = \frac{\alpha\beta\left(1 - e^{(\alpha-\beta)kt}\right)}{\beta - \alpha e^{(\alpha-\beta)kt}}. \tag{2.1.1}$$

Eq. (2.1.1) does not seem very informative as far as understanding the nature of the chemical reaction goes, but Problem 31 of Exercises 2.1 suggests some useful ways of analyzing the solution. A CAS generated graph of a solution (Fig. 2.3) of the equation with $\alpha = 250$, $\beta = 40$, and $k = 0.0006$ is more informative, and shows the steady rise in the concentration of molecule X to what is called an *equilibrium value* of 40. (We'll explore the idea of an equilibrium value in Section 2.6.) The particular solution shown corresponds to $x(0) = 0$.

Of course, as we noted previously, the right-hand side of the original differential equation is positive, so we know ahead of time that the concentration function is increasing. Also, you can calculate $\frac{d^2x}{dt^2}$ from the original differential equation to see why the graph of x is concave down. Remember that $k > 0$ and $0 \le x < \alpha$, $0 \le x < \beta$.

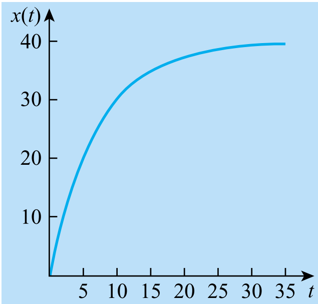


FIGURE 2.3

Solution of the IVP $\frac{dx}{dt} = 0.0006(250 - x)(40 - x);\ x(0) = 0,\ 0 \le t \le 35,\ 0 \le x \le 40$

Exercises 2.1

A

Solve the equations or IVPs in Problems 1–9 by separating the variables. Be sure to describe any singular solutions where appropriate.

1. $\frac{dy}{dx} = \frac{A-2y}{x}$, where A is a constant
2. $\frac{dy}{dx} = \frac{-xy}{x+1}$
3. $y' = 3\sqrt[3]{y^2};\ y(2) = 0$
4. $\frac{dy}{dx} = \frac{(y-1)(y-2)}{x}$
5. $(\cot x)y' + y = 2;\ y(0) = -1$
6. $x' = -\frac{\sin t \cos^2 x}{\cos^2 t};\ x(0) = 0$
7. $x^2y^2y' + 1 = y$
8. $xy' + y = y^2;\ y(1) = 0.5$
9. $z' = 10^{x+z}$
10. Solve the equation $y' = 1 + x + y^2 + xy^2$. [*Hint*: Factor cleverly.]
11. Solve the equation $(y')^2 + (x + y)y' + xy = 0$. [*Hint*: Solve this quadratic equation for y' by factoring or by using the quadratic formula, and then solve the two resulting differential equations separately.]

An equation of the form $dy/dx = f(ax + by)$ can be transformed into an equation with separable variables by making the substitution $z = ax + by$ or $z = ax + by + c$, where c is an arbitrary constant. For example, the equation $y' = (y - x)^2$ is not separable, but the substitution $z = y - x$ leads to the separable equation $z' + 1 = z^2$, which was solved as Example 2.1.7. Then substitute the original variables for z. Use this technique to solve the equations in Problems 12–14.

12. $y' - y = 2x - 3$

13. $(x + 2y)y' = 1;\ y(0) = -1$

14. $y' = \sqrt{4x + 2y - 1}$

A **homogeneous** equation has the form $dy/dx = f(x, y)$, where $f(x, y)$ can be expressed in the form $g(y/x)$ or $g(x/y)$—that is, as a function of the quotient y/x or the quotient x/y alone. For example, by dividing numerator and denominator by x^2, we can write the equation $\frac{dy}{dx} = \frac{2x^2-y^2}{3xy}$ in the form $\frac{dy}{dx} = \frac{2-(y/x)^2}{3(y/x)} = g\left(\frac{y}{x}\right)$. Any such equation can be changed into a separable equation by making the substitution $z = y/x$ (or $z = x/y$). Making the substitution $z = y/x$ in our example, we have $\frac{dy}{dx} = \frac{d}{dx}(x\,z) = 1 \cdot z + x\left(\frac{dz}{dx}\right)$, so that our equation becomes $x\frac{dz}{dx} + z = \frac{2-z^2}{3z}$ or, separating the variables, $\left(\frac{3z}{2-4z^2}\right)dz = \frac{1}{x}dx$. After integrating, remember to replace z by y/x (or x/y). Use this technique to solve the equations in Problems 15–18. [This use of the term "homogeneous" differs from that found in Section 2.2.]

15. $y' = \frac{x+y}{x-y}$

16. $\dot{x} = \frac{t-3x}{3t+x}$

17. $y' = \frac{x}{y} + \frac{y}{x}$

18. $\frac{dy}{dx} = \frac{y^2+2xy-x^2}{x^2+2xy-y^2}$

19. Under continuous compounding the balance B of a bank account demonstrates the "snowball" effect: The larger the balance at a given time, the greater the rate of growth: $\frac{dB}{dt} = rB$, where r is the annual interest rate (expressed as a decimal). If you invest one thousand dollars at 4% interest compounded continuously for six years, how much will be in your account?

20. The **Malthusian growth model** is named after Thomas Malthus (1766–1834), who wrote one of the earliest and most influential books on population. This model postulates that the rate of growth is proportional to the actual population at any time t: $\frac{dP}{dt} = rP$, where $P = P(t)$ denotes the population at time t and r is the population growth rate (usually the net result of birth rate minus death rate), which could be negative. If the growth rate of a population is 0.0125 per year and the initial population ($t = 0$) is six million, what is the projected population in ten years?

21. The phenomenon of radioactive decay was studied at the turn of the 20th century by many scientists, including Marie and Pierre Curie. As a result of this research and their own experiments, Frederick Soddy and Ernest Rutherford formulated the **radioactive decay law** given by the equation $\frac{dm}{dt} = -km$, where $m(t)$ denotes the mass of a radioactive substance at time t and k is a positive constant (called the *decay constant* of the substance). If $k = 0.0256$

and the time is in years, what is the mass of the radioactive substance after two years?

22. **Newton's law of cooling** states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium*. Using Newton's law of cooling, derive a differential equation for the cooling of a hot body surrounded by a cool medium.

B

23. Suppose that f is a function such that $f(x) = \int_0^x f(t)\,dt$ for all real numbers x. Show that $f(x) \equiv 0$. [*Hint*: Use the Fundamental Theorem of Calculus to get a differential equation. Then think of an appropriate initial condition.]

24. Consider the equation $\dot{x} = \frac{x^2+x}{t}$.
a. Find a one-parameter family of solutions.
b. Can you find a solution satisfying the initial condition $x(0) = -1$? If so, give it. If not, give a reason.
c. Find a singular solution.

25. **a.** Solve the IVP $\dot{x} = x^2$, $x(1) = 1$.
b. If the solution in part (a) is valid over an interval I, how large can I be?
c. Use technology to draw the graph of the solution $x(t)$ found in part (a).
d. Solve the IVP $\dot{x} = x^2$, $x(0) = 0$.

26. The equation

$$\frac{dQ}{dP} = -\frac{cQ}{1+cP}$$

is one model used to estimate the cost of national health insurance,[2] where $Q(P)$ represents the quantity of health services performed at price P, P represents the proportion of the total cost of health services that an individual pays directly ("out of pocket expenses," or coinsurance), and c is a constant.
a. Solve the equation for Q.
b. If $Q(0)/Q(1)$ is approximately 2, what is the value of c?
c. Using the value of c found in part (b), determine $Q(0.20)$ in terms of $Q(0)$. What does your answer tell you about the effect of a 20% coinsurance (versus no coinsurance)?

27. A quantity y varies in such a way that $\frac{dy}{dt} = -\frac{\ln 2}{30}(y - 20)$. If $y = 60$ when $t = 30$, find the value of t for which $y = 40$.

28. In analyzing the change in the percentage of red blood cells in a hospital patient undergoing surgery, the following equation has been used[3]

[2] A.J. Kroopnick, "Estimating the Cost of National Health Insurance Using Three Simple Models," *Math. and Comp. Ed.* **30** (1996): 267–271.

[3] M.E. Brecher and M. Rosenfeld, "Mathematical and Computer Modeling of Acute Normovolemic Hemodilution," *Transfusion* **34** (1994): 176–179. See also "Calculus in the Operating Room" by P. Toy and S. Wagon, *Amer. Math. Monthly* **102** (1995): 101.

$$\frac{dH}{dV_L} = -\frac{H}{EBV},$$

where H denotes the *hematocrit* (percentage of red blood cells in the total volume of blood), V_L represents the volume of blood loss, and *EBV* is the patient's estimated total blood volume.

a. Solve the differential equation for H.

b. If the patient's total blood volume of 5 liters is maintained throughout surgery via the injection of saline solution, and the initial value of H is 0.40, what is the patient's volume of red blood cells at the end of the operation?

29. The volume V of water in a particular container is related to the depth h of the water by the equation $\frac{dV}{dh} = 16\sqrt{4-(h-2)^2}$. If $V = 0$ when $h = 0$, find V when $h = 4$.

30. The slope m of a curve is 0 where the curve crosses the y-axis, and $\frac{dm}{dx} = \sqrt{1+m^2}$. Find m as a function of x.

31. Consider Eq. (2.1.1), the solution to Example 2.1.8.

a. If $\alpha > \beta$, factor $e^{(\alpha-\beta)kt}$ from the numerator and denominator and show that $x(t) \to \beta$ as $t \to \infty$.

b. If $\alpha < \beta$, explain what happens to $e^{(\alpha-\beta)kt}$ as $t \to \infty$ and show that $x(t) \to \alpha$ as $t \to \infty$.

32. Solve the IVP $\frac{dQ}{dt} = \frac{Q^3+2Q}{t^2+3t}$, $Q(1) = 1$ explicitly for $Q(t)$, and state the interval for which the solution is valid.

33. The number of bacteria in a Petri dish is observed to grow at a rate proportional to the number of cells present. At the beginning of an experiment, there are 10,000 cells, and after three hours there are 500,000.

a. How many bacteria will there be after one day of growth if this unlimited growth continues? (Be consistent with your units!)

b. What is the *doubling time* of the bacteria—that is, how many hours will it take for the initial population to double?

34. A bacteria culture is known to grow at a rate proportional to the amount present. Find an expression for the approximate number of bacteria in such a culture if the initial number is 300 and if it is observed that the population has increased by 20% after two hours.

35. A certain radioactive material is known to decay at a rate proportional to the amount present. If after one hour it is observed that 10% of the material has decayed, find the *half-life* of the material—that is, the time it takes for one-half of the material to decay.

36. After two days, 10 grams of a radioactive chemical is present. Three days later 5 grams is present. How much of the chemical was present initially, assuming that the rate of disintegration is proportional to the amount present?

C

37. A police department forensics expert checks a gun by firing a bullet into a bale of cotton. The friction force resulting from the passage of the bullet through the cotton causes the bullet to slow down at a rate proportional to the square root of its velocity. It stopped in 0.1 second and penetrated 10 feet into the bale of cotton. How fast was the bullet going when it hit the bale?

38. The relationship between the velocity v of a rifle bullet and the distance L traveled by it in the barrel of the gun is established in ballistics by the equation $v = \frac{aL^n}{b+L^n}$, where $v = \frac{dL}{dt}$ and $n < 1$. Find the relationship between the time t during which the bullet moves in the barrel and the distance L covered.

39. In trying to determine the shape of a flexible nonstretching cable suspended between two points A and B of equal height, we can analyze the forces acting on the cable and get the differential equation

$$\frac{d^2y}{dx^2} = k\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2},$$

where $k > 0$ is a constant.

a. Use the substitution $p(x) = dy/dx$ to reduce the second-order equation to a separable first-order equation.

b. Express the general solution of the equation in terms of exponential functions. (You may need a table of integrals here. Your CAS may evaluate the more difficult integral in an awkward way.)

40. When the drug theophyllin is administered for asthma, a concentration in the blood below 5 mg/liter of blood has little effect, while undesirable side effects appear if the concentration exceeds 20 mg/liter. Suppose a dose corresponding to 14 mg/liter of blood is administered initially. The concentration satisfies the differential equation $\frac{dC}{dt} = -\frac{C}{6}$, where the time t is measured in hours.

a. Find the concentration at time t.

b. Show that a second injection will need to be given after about 6 hours to prevent the concentration becoming ineffective.

c. Given that the second injection also increases the concentration by 14 mg/liter, how long is it before another injection is necessary?

d. What is the shortest safe time that a second injection may be given so that side effects do not occur?

e. Sketch graphs of the situations in parts (b), (c), and (d).

41. One method of administering a drug is to feed it continuously into the bloodstream by a process called *intravenous infusion*. This process can be modeled by the separable (and linear) differential equation $\frac{dC}{dt} = -\mu C + D$, where C is the concentration in the blood at time t, μ is a positive constant, and D is also a positive constant, the rate at which the drug is administered.

- **a.** Find the equilibrium solution of the differential equation such that $\frac{dC}{dt} = 0$.
- **b.** Given $C = C_0$ when $t = 0$, find the concentration at time t. What limit does the concentration approach as $t \to \infty$? Compare with your answer to part (a).
- **c.** Sketch the graph of a typical solution.

42. Let $\frac{dP}{dt} = P(1 - P)$.

- **a.** Find all solutions by separating the variables. (You will have to integrate by using partial fractions.)
- **b.** Let $P(0) = P_0$. Suppose $0 < P_0 < 1$. What happens to $P(t)$ as $t \to \infty$?
- **c.** Let $P(0) = P_0$. Suppose $P_0 > 1$. What happens to $P(t)$ as $t \to \infty$?

43. Consider the following model for the growth of a city. The shape of the city remains roughly circular. The maximum travel time is proportional to the diameter of the city. The population is proportional to the area of the city. The rate of increase of the city's population is inversely proportional to the maximum travel time. If the population of the city was 5000 in 1945 and 20,000 in 1985, what is the predicted population for 2025?

44. The equation $\frac{dy}{dx} = \frac{6x^2-5xy-2y^2}{6x^2-8xy+y^2}$ is not separable, but it can be solved by making the substitution $v = y/x$.

- **a.** Make the suggested substitution to get a separable equation using the variables x and v.
- **b.** Solve the equation obtained in part (a) and change the variables to x and y.
- **c.** Letting any constant found in part (b) equal 1, graph the implicit solution found in part (b).

2.2 Linear equations

We introduced the idea of a linear differential equation in Section 1.1. Now let's see what we can do when the order of the differential equation is 1.

Definition 2.2.1

A **linear first-order differential equation** is an equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

where a_1, a_0, and f are functions of the independent variable alone.

After dividing through by $a_1(x)$—being careful to note where this function is zero—we can write the equation in the *standard form*

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2.2.1}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = f(x)/a_1(x)$ are functions of x alone. In this standard form, if the function $Q(x)$ is the zero function, we call Eq. (2.2.1) **homogeneous**. Otherwise, we say that the equation is **nonhomogeneous**. (Don't confuse this terminology with the use of the term *homogeneous* as explained just before Problem 15 in Exercises 2.1.) In certain applied problems, $Q(x) \not\equiv 0$ may be referred to as the **forcing term**, the **driving term**, or the **input**. The solution y can be called the **output**.

For example, $\frac{dy}{dx} + \sin(x)y = e^{-x}$ is linear with $P(x) = \sin x$ and $Q(x) = e^{-x}$. The equation $x\frac{dy}{dx} + y^2 = 0$ is not linear, because even when we divide by x (assuming that x is nonzero), we get $\frac{dy}{dx} + \left(\frac{y}{x}\right)y = 0$. The function $Q(x)$ can be taken as $Q(x) \equiv 0$, but the coefficient of y, $\frac{y}{x}$, is not a function of x alone.

However, even the equation $2tz^3 + 3t^2z^2\frac{dz}{dt} = t^5z^2$, which looks complicated, can be made linear. Just divide by $3t^2z^2$ to get $\frac{dz}{dt} + \left(\frac{2}{3t}\right)z = \frac{1}{3}t^3$, so that $P(t) = \frac{2}{3t}$ and $Q(t) = \frac{1}{3}t^3$. Of course, we must consider the cases $t = 0$ and $z \equiv 0$ separately.

2.2.1 The superposition principle

In some applications it is useful to think of a linear first-order equation in terms of an **operator**, or **transformation**, L, that changes a differentiable function y into the left-hand side of Eq. (2.2.1): $L(y) = \frac{dy}{dx} + P(x)y$. Then Eq. (2.2.1) can be expressed simply as $L(y) = Q(x)$. For example, if the nonhomogeneous linear equation in standard form is $\frac{dy}{dx} - y = x$, then we have the operator L defined as $L(y) = \frac{dy}{dx} - y$. If $y(x) = x^2$, for instance, then $L(y) = 2x - x^2$. A solution y of the differential equation $\frac{dy}{dx} - y = x$ would have to satisfy $L(y) = x$. (We can see that $y = x^2$ is not a solution.)

In this general context, suppose y_1 is a solution of $L(y) = Q_1(x)$, y_2 is a solution of $L(y) = Q_2(x)$, and c_1, c_2 are arbitrary constants. Then $c_1y_1 + c_2y_2$ is called a **linear combination** of y_1 and y_2, and we have

$$\begin{aligned}
L(c_1y_1 + c_2y_2) &= \frac{d}{dx}(c_1y_1 + c_2y_2) + P(x)\,(c_1y_1 + c_2y_2) \\
&= c_1\frac{d}{dx}y_1 + c_2\frac{d}{dx}y_2 + c_1P(x)y_1 + c_2P(x)y_2 \\
&= c_1\left(\frac{d}{dx}y_1 + P(x)y_1\right) + c_2\left(\frac{d}{dx}y_2 + P(x)y_2\right) \\
&= c_1Q_1 + c_2Q_2 = c_1L(y_1) + c_2L(y_2).
\end{aligned}$$

Any operator that satisfies the condition $L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2)$ is called a **linear operator**. Otherwise, it is called **nonlinear**. We can describe this situation by saying that adding two inputs (Q_1 and Q_2) of a linear equation gives us an output that is the sum $(y_1 + y_2)$ of the individual outputs.

The general form of this last observation is called the **Superposition Principle** and, as we will see later, it applies to linear equations of any order. In particular, when

$f(x) \equiv 0$, we see that *any linear combination of solutions to a homogeneous linear equation is also a solution.*

Superposition Principle for Homogeneous Equations

Suppose y_1 and y_2 are solutions of the homogeneous linear first-order differential equation

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval (a, b). Then the linear combination $c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are arbitrary constants, is also a solution on this interval.

The next two examples should clarify the difference between linear and nonlinear operators.

Example 2.2.1 A Linear Operator

We can check that $y_1 = e^{-x}$ is a solution of the homogeneous linear equation $L(y) = y' + y = 0 = Q_1$ and that $y_2 = \sin x$ is a solution of $L(y) = y' + y = \cos x + \sin x = Q_2$. Note that we have the same left-hand side, but different right-hand sides. You should see that $y_1 + y_2 = e^{-x} + \sin x$ is a solution of the equation $y' + y = Q_1 + Q_2 = 0 + \cos x + \sin x = \cos x + \sin x$—that is, $L(y_1 + y_2) = Q_1 + Q_2 = L(y_1) + L(y_2)$.

However, not every operator defined by a first-order equation is linear.

Example 2.2.2 A Nonlinear Operator

Now consider the operator defined as $T(y) = xy' + y^2$ and suppose that $T(y_1) = 0$ and $T(y_2) = 0$. Then

$$\begin{aligned} T(y_1 + y_2) &= x(y_1 + y_2)' + (y_1 + y_2)^2 \\ &= xy_1' + xy_2' + y_1^2 + y_2^2 + 2y_1 y_2 \\ &= \overbrace{\left(xy_1' + y_1^2\right)}^{T(y_1)} + \overbrace{\left(xy_2' + y_2^2\right)}^{T(y_2)} + 2y_1 y_2 \\ &= 0 + 0 + 2y_1 y_2 = 2y_1 y_2 \neq T(y_1) + T(y_2). \end{aligned}$$

The equation $T(y) = xy' + y^2 = 0$ is nonlinear, and the operator T is not a linear operator.

2.2.2 Variation of parameters and the integrating factor

As a first step in finding the general solution of Eq. (2.2.1), let's consider the case $Q(x) \equiv 0$: $\frac{dy}{dx} + P(x)y = 0$. This homogeneous equation is separable, so we can see that

$$\frac{dy}{dx} = -P(x)y, \quad \frac{dy}{y} = -P(x)\,dx, \quad \ln|y| = -\int P(x)\,dx, \quad y = Ce^{-\int P(x)\,dx}.$$

Recognizing $y = Ce^{-\int P(x)\,dx}$ as the general solution of the homogeneous linear equation, several important figures in the early history of calculus—notably Leibniz, Euler, and Lagrange—are credited with devising a clever way to solve the nonhomogeneous Eq. (2.2.1), the method of *variation of constants*, or *variation of parameters*. The idea is to look for a solution similar to $y = Ce^{-\int P(x)\,dx}$ in form. Clearly something has to change since the differential equation has changed. The key is to replace the constant C in $Ce^{-\int P(x)\,dx}$ with a *function* $C(x)$, so that we have $y = C(x)e^{-\int P(x)\,dx}$. We assume that a solution of the nonhomogeneous equation has this form, and we try to determine what $C(x)$ is.

Differentiating using the product and chain rules, we find that

$$\frac{dy}{dx} = \frac{dC(x)}{dx}e^{-\int P(x)\,dx} - C(x)P(x)e^{-\int P(x)\,dx}$$

and

$$\frac{dy}{dx} + P(x)y = \frac{dC(x)}{dx}e^{-\int P(x)\,dx}.$$

Consequently, our nonhomogeneous equation becomes

$$\frac{dC(x)}{dx}e^{-\int P(x)\,dx} = Q(x), \text{ or } \quad \frac{dC(x)}{dx} = Q(x)e^{\int P(x)\,dx}.$$

Integrating this last equation, we find that

$$C(x) = \int Q(x)e^{\int P(x)\,dx}\,dx + K.$$

Putting the pieces together, we see that the general solution of Eq. (2.2.1) is given by

$$\begin{aligned} y(x) &= e^{-\int P(x)\,dx} \cdot \left(\int Q(x)e^{\int P(x)\,dx}\,dx + K\right) \\ &= e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx + Ke^{-\int P(x)\,dx}. \end{aligned}$$

This is an explicit formula for the general solution of any first-order linear differential equation in standard form. Even if the integrals involved can't be evaluated in closed form, they can still be approximated by numerical methods usually learned in a calculus course. (*Do not bother memorizing this formula*. Just remember that *any linear first-order equation has an explicit general solution* and understand how to use variation of parameters to find it.)

There is a closely related method using something called an **integrating factor**—a special multiplier function that has been used to solve first-order linear equations since the late 1600s. The idea is that if we take Eq. (2.2.1) and multiply through by a nonzero function $\mu(x)$ to get $\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$, the left-hand side looks as if it could be a derivative obtained via the product rule:

$$\frac{d}{dx}(\mu(x)y) = \mu(x)\frac{dy}{dx} + \frac{d\mu(x)}{dx}y. \tag{2.2.2}$$

If we can find a function μ such that $\frac{d\mu(x)}{dx} = \mu(x)P(x)$ in Eq. (2.2.2), this leads to the relation

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x),$$

and we can integrate both sides to get

$$\mu(x)y = \int \mu(x)Q(x)\,dx, \quad \text{or} \quad y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx.$$

It is easy to see that the equation $\frac{d\mu(x)}{dx} = \mu(x)P(x)$ is separable with solution $\mu(x) = Ce^{\int P(x)\,dx}$, where C is an arbitrary constant. (*Check this*.) For convenience, we take $C = 1$, but you should see that any nonzero constant produces the same result ultimately. The general solution of Eq. (2.2.1) is

$$\begin{aligned} y &= \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}Q(x)\,dx + K\right) \\ &= e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx + Ke^{-\int P(x)\,dx}. \end{aligned}$$

Some people find it easier to memorize the formula for the integrating factor μ than to use the method of variation of parameters. Ultimately, these methods use the same calculations, but the variation of parameters is a powerful technique that will prove its utility in Chapter 4, when it is applied to the solutions of higher-order differential equations. For now, we will demonstrate both methods.

Example 2.2.3 Solving a Linear Differential Equation

Suppose we want to solve the linear nonhomogeneous equation $y' + xy = 2x$, which is already in standard form with $P(x) = x$, $Q(x) = 2x$. The solution of the homogeneous (separable) equation is $y = Ce^{-\int x\,dx} = Ce^{-x^2/2}$. Replacing the constant C by a function $C(x)$, we have $y = C(x)e^{-x^2/2}$ and calculate

$$y' = C'(x)e^{-x^2/2} - xC(x)e^{-x^2/2}$$

and

$$y' + xy = C'(x)e^{-x^2/2} = 2x.$$

This gives us $C'(x)e^{-x^2/2} = 2x$, from which we derive $C'(x) = 2xe^{x^2/2}$ and $C(x) = 2e^{x^2/2} + K$. Finally, variation of parameters gives us $y = C(x)e^{-x^2/2} = \left(2e^{x^2/2} + K\right) = 2 + Ke^{-x^2/2}$.

Alternatively, we can reach into our sleeve and pluck out the magic integrating factor $\mu(x) = e^{\int P(x)\,dx} = e^{x^2/2}$ for this equation. Now we get an equivalent differential equation by multiplying each side of the original equation by $\mu(x)$:

$$e^{x^2/2}y' + xe^{x^2/2}y = 2xe^{x^2/2}.$$

Note that if we assume that $y = y(x)$, an implicit function of x, the Product Rule gives us $\left(e^{x^2/2}y\right)' = xe^{x^2/2}y + e^{x^2/2}y'$, the left-hand side of our new differential equation. This observation tells us that the left side is an *exact derivative* and enables us to write the differential equation in a more compact form: $\left(e^{x^2/2}y\right)' = 2xe^{x^2/2}$. (*Be sure that you see this.*) Now we can integrate each side with respect to x to get $e^{x^2/2}y = \int 2xe^{x^2/2}\,dx = 2e^{x^2/2} + C$. Solving for y by multiplying each side of this last equation by $e^{-x^2/2}$, we get $y(x) = 2 + Ce^{-x^2/2}$, valid for $-\infty < x < \infty$.

We see from either closed-form solution that all solutions approach 2 as $x \to \pm\infty$: If $y(x)$ is any solution of the differential equation $y' + xy = 2x$, then $\lim_{x\to\infty} y(x) = 2 = \lim_{x\to-\infty} y(x)$. Fig. 2.4 shows five solutions of this linear equation.

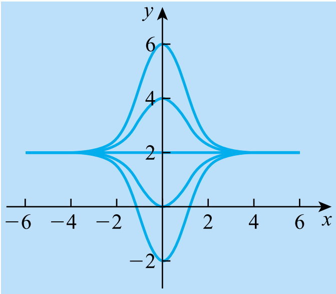


FIGURE 2.4

Solutions of the IVP $y' + xy = 2x$; $y(0) = -2$, 0, 2, 4, and 6; $-6 \le x \le 6$, $-2 \le y \le 6$

From top to bottom, the five particular solutions plotted correspond to $C = 4, 2, 0, -2$, and -4. The choice $C = 0$ gives us the asymptotic solution $y \equiv 2$.

Finally, you may have recognized that our original equation is actually a separable equation. You should solve by separating the variables and then compare your solution to the one we gave.

Let's see another example of these two techniques.

Example 2.2.4 Solving a Linear Equation

The linear equation $x\frac{dy}{dx} - 2y = x^3e^{-2x}$ can be written in standard form as $\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x^2e^{-2x}$. The related homogeneous equation has the solution $y = Cx^2$.

Now let $y = C(x)x^2$ be our experimental solution. Then

$$\frac{dy}{dx} = \frac{dC(x)}{dx} \cdot x^2 + C(x)(2x).$$

The equation

$$\frac{dy}{dx} - 2y = \frac{dC(x)}{dx} \cdot x^2 = x^2e^{-2x}$$

implies $\frac{dC(x)}{dx} = e^{-2x}$ and $C(x) = -\frac{1}{2}e^{-2x} + K$. Finally, we have $y(x) = C(x)x^2 = \left(-\frac{1}{2}x^2e^{-2x} + K\right)x^2 = -\frac{1}{2}x^2e^{-2x} + Kx^2$ as our general solution.

Now let's try the integrating factor technique on the linear equation $\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x^2e^{-2x}$. Our integrating factor is $\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|} = x^{-2}$.

Multiplying both sides of the equation by this factor, we get $x^{-2}\frac{dy}{dx} - 2x^{-3}y = e^{-2x}$. Recognizing the left side as the derivative of the product $\mu(x)y = x^{-2}y$, we can write the differential equation as $\frac{d}{dx}\left(x^{-2}y\right) = e^{-2x}$.

Integrating both sides, we find that $x^{-2}y = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$. Now we solve for y and see that $y = -\frac{1}{2}x^2e^{-2x} + Cx^2$.

The next example, an important application of linear differential equations to electrical network theory, shows that the details of using an integrating factor may get messy.

Example 2.2.5 A Circuit Problem

As a consequence of one of *Kirchhoff's laws* in physics, suppose we know that the current I flowing in a particular electrical circuit satisfies the first-order linear differential equation $L\frac{dI}{dt} + RI = v_0\sin(\omega t)$, where L, R, v_0, and ω are positive constants that give information about the circuit. Let's try to find the current $I(t)$ at time t, for $t > 0$, given that $I(0) = 0$. This initial condition says that at the beginning of our analysis ($t = 0$), there is no current flowing in the circuit.

First, we divide both sides of the differential equation by L to put our equation into standard form: $\frac{dI}{dt} + \left(\frac{R}{L}\right)I = \left(\frac{v_0}{L}\right)\sin(\omega t)$. Now, in terms of the standard form [Eq. (2.2.1)], we make the identifications $P(t) \equiv R/L$, a constant function, and $Q(t) = (v_0/L)\sin(\omega t)$. In this problem the forcing term $Q(t)$ represents an (alternating) electromotive force supplied by a generator. Next, we compute the integrating factor

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{R}{L}t}.$$

Multiplying each side of the equation in standard form by $\mu(t)$, we get

$$e^{\frac{R}{L}t}\frac{dI}{dt} + e^{\frac{R}{L}t}\left(\frac{R}{L}\right)I = \left(\frac{v_0}{L}\right)e^{\frac{R}{L}t}\sin(\omega t), \quad \text{or} \quad \frac{d}{dt}\left(e^{\frac{R}{L}t}I\right) = \left(\frac{v_0}{L}\right)e^{\frac{R}{L}t}\sin(\omega t).$$

Integrating each side yields $e^{\frac{R}{L}t}I = \left(\frac{v_0}{L}\right)\int e^{\frac{R}{L}t}\sin(\omega t)\,dt$.

To evaluate this last integral, we have three choices: (1) integrate by parts twice, (2) use a table of integrals, or (3) submit the integral to a computer algebra system capable of integration. In all three cases we get

$$\begin{aligned} e^{\frac{R}{L}t}I &= \left(\frac{v_0}{L}\right)\left[\frac{Re^{\frac{R}{L}t}\sin(\omega t)}{L\left(\frac{R^2}{L^2}+\omega^2\right)} - \frac{\omega e^{\frac{R}{L}t}\cos(\omega t)}{\left(\frac{R^2}{L^2}+\omega^2\right)}\right] + C \\ &= \left(\frac{v_0}{L}\right)\frac{e^{\frac{R}{L}t}}{\left(\frac{R^2}{L^2}+\omega^2\right)}\cdot\left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right] + C. \end{aligned}$$

To find the general solution, we multiply each side of this last equation by $e^{-\frac{R}{L}t}$ to get

$$I(t) = \frac{\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2}+\omega^2\right)}\cdot\left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right] + Ce^{-\frac{R}{L}t}.$$

Now we use the initial condition $I(0) = 0$:

$$0 = I(0) = \frac{\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2} + \omega^2\right)} \cdot \left[\frac{R}{L}\sin(\omega \cdot 0) - \omega\cos(\omega \cdot 0)\right] + Ce^0$$
$$= \frac{-\omega\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2} + \omega^2\right)} + C,$$

so that $C = \frac{\omega\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2}+\omega^2\right)}$, and we have (finally!)

$$I(t) = \frac{\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2} + \omega^2\right)} \cdot \left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right] + \frac{\omega\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2} + \omega^2\right)}e^{-\frac{R}{L}t}$$
$$= \frac{\left(\frac{v_0}{L}\right)}{\left(\frac{R^2}{L^2} + \omega^2\right)} \cdot \left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t) + \omega e^{-\frac{R}{L}t}\right].$$

In the preceding example, we call the term $\omega e^{-\frac{R}{L}t}$ (or its constant multiple) in the formula for $I(t)$ a **transient** term because it eventually goes to 0. "Eventually" means as $t \to \infty$. The trigonometric terms make up the **steady-state** part of the solution and have the same period as the original forcing term. (*Can you see that this last claim is true?*)

Exercises 2.2

A

For Problems 1–18, solve each equation or initial-value problem using both variation of parameters and integrating factors.

1. $y' + 2y = 4x$

2. $y' + 2xy = xe^{-x^2}$

3. $\dot{x} + 2tx = t^3$

4. $y' + y = \cos x$

5. $ty' = -3y + t^3 - t^2$

6. $\frac{dx}{ds} = \frac{x}{s} - s^2$

7. $y = x(y' - x\cos x)$

8. $(1 + x^2)y' - 2xy = (1 + x^2)^2$

9. $t(x' - x) = (1 + t^2)e^t$

10. $Q' - (\tan t)Q = \sec t;\ Q(0) = 0$

11. $xy' + y - e^x = 0;\ y(a) = b$ [a and b are constants.]

12. $(xy' - 1)\ln x = 2y$

13. $y' + ay = e^{mx}$ [Consider two cases: $m \neq -a$ and $m = -a$.]

14. $y' + \left(\frac{1-2x}{x^2}\right) y = 1$

15. $tx' - \left(\frac{x}{t+1}\right) = t;\ x(1) = 0$

16. $y = (2x + y^3)y'$ [*Hint*: Think of y as the independent variable, x as the dependent variable, and rewrite the equation in terms of dx/dy.]

17. $x(e^y - y') = 2$ [Use the hint from Problem 16.]

18. $y(x) = \int_0^x y(t)\,dt + x + 1$ [Use the Fundamental Theorem of Calculus to get an ODE.]

B

An equation of the form $y' + a(x)y = b(x)y^n$ is called a **Bernoulli equation**, named for Jakob Bernoulli (1654–1705), one of a family of noted Swiss scientists and mathematicians. Note that if $n = 0$ or $n = 1$, we have a linear equation. Now if n is not equal to 0 or 1 and we divide both sides of the equation by y^n, we can let $z = y^{1-n}$ and get a linear equation in the variable z. We solve the linear equation for z in terms of x and then return to the original variables x and y. This substitution method was found by Leibniz in 1696. For example, $y' - y = xy^2$ is a Bernoulli equation with $a(x) \equiv -1$, $b(x) = x$, and $n = 2$. Divide by y^2 and the equation becomes $y^{-2}\,y' - y^{-1} = x$. Letting $z = y^{-1}$, we get the linear equation $-z' - z = x$, or $z' + z = -x$. Solving for z, we find that $z = 1 - x + ce^{-x}$. Since $z = y^{-1}$, we conclude that $y = \left(1 - x + ce^{-x}\right)^{-1}$. Note that we divided by y^2 and that $y \equiv 0$ is a singular solution. Find all the solutions for each Bernoulli equation in Problems 19–24.

19. $y' = \frac{4}{t}y - 6ty^2$

20. $\dot{x} = \frac{1}{t}x + \sqrt{x}$

21. $\frac{dy}{dx} + y = xy^3$

22. $y' + xy = \sqrt{y}$

23. $y' = 2ty + ty^2$

24. $y' = x^3y^2 + xy$

25. In trying to regulate fishing in the oceans, international commissions have been set up to implement controls. To understand the effect of these controls, mathematical models of fish populations have been constructed. One stage in this modeling effort involves predicting the growth of an individual fish. The **von Bertalanffy growth model** is reflected in the Bernoulli equation (see above):

$$\frac{dW}{dt} = \alpha W^{2/3} - \beta W,$$

where $W = W(t)$ denotes the weight of a fish and α, β are positive constants.

a. Find the general solution of the equation.
b. Calculate $W_\infty = \lim_{t\to\infty} W(t)$, the limiting weight of the fish.
c. Using the answer to part (b) and the initial condition $W(0) = 0$, write the formula for $W(t)$ free of any arbitrary constants.
d. Sketch a graph of W against t.

26. Show that if a linear first-order differential equation is homogeneous, then the equation is separable.

27. When a switch is closed in a circuit containing a resistance R, an inductance L, and a battery that supplies a constant voltage E, the current I builds up at a rate described by the equation $L\frac{dI}{dt} + RI = E$. [In Example 2.2.5 the electromotive force on the right-hand side of the equation is not constant. Instead of a battery, there is a generator supplying an alternating voltage equal to $(v_0/L)\sin(\omega t)$.]
a. Find the current I as a function of time.
b. Evaluate $\lim_{t\to\infty} I(t)$.
c. How long will I take to reach one-half its final value?
d. Find I if $I_0 = I(0) = E/R$.

28. In an electrical circuit, when a capacitor of capacitance C is being charged through a resistance R by a battery that supplies a constant voltage E, the instantaneous charge Q on the capacitor satisfies the differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = E.$$

a. Find Q as a function of time if the capacitor is initially uncharged—that is, if $Q_0 = Q(0) = 0$.
b. How long will it be before the charge on the capacitor is one-half its final value?

29. In Problem 28, determine Q if $Q_0 = 0$ and if the battery is replaced by a generator that supplies an alternating voltage equal to $E_0 \sin(\omega t)$.

30. In analyzing the effect of advertising on the sales of a product, we can extract the following model from work done by the economists Vidale and Wolfe[4]:

$$\frac{dS}{dt} + \left(\frac{rA}{M} + \lambda\right) S = rA.$$

Here, $S = S(t)$ denotes sales, $A = A(t)$ indicates the amount of advertising, M is the saturation level of the product (the practical limit of sales that can be generated), and r and λ are positive constants. Clearly, the solution of this linear equation depends on the form of the advertising function A.

[4] M.L. Vidale and H.B. Wolfe, "Response of Sales to Advertising," in *Mathematical Models in Marketing*, ed. Robert G. Murdick (Scranton, PA: Intext Educational Publishers, 1971): 249–256.

a. Solve the equation if A is constant over a particular time interval and zero after this:

$$A(t) = \begin{cases} \overline{A} & \text{for} \quad 0 < t < T \\ 0 & \text{for} \quad t > T. \end{cases}$$

(You will have to solve two equations and then combine the solutions appropriately.)

b. Sketch a typical graph of S against t. (Choose reasonable values for any constants in your solution.)

31. In the study of population genetics, biological units called *genes* determine what characteristics living things inherit from their parents. Suppose we look at a gene with two "flavors" A and a that occur in the proportions $p(t)$ and $q(t) = 1 - p(t)$, respectively, at time t in a particular population. Suppose that we have the relation $\frac{dp}{dt} = \nu - (\mu + \nu)p$, where μ is a constant describing a "forward mutation rate" and ν is another constant representing the "backward mutation rate."

a. Determine $p(t)$ and $q(t)$ in terms of $p(0)$, $q(0)$, μ, and ν.

b. Show that $\lim_{t\to\infty} p(t) = \nu/(\mu + \nu)$ and $\lim_{t\to\infty} q(t) = \mu/(\mu + \nu)$. These are called the *equilibrium* gene frequencies.

32. Prove that any Bernoulli differential equation $y' + a(x)y = b(x)y^n$, where $n \neq 0, 1$, can be converted into a linear equation by the special substitution $y = u^{1/(1-n)}$. (See the paragraph at the beginning of the B exercises.)

C

33. If $V = V(t)$ represents the value of a bond at time t, $r(t)$ is the interest rate, and $K(t)$ is the coupon payment, then $\frac{dV}{dt} + K(t) = r(t)V$ describes the value of the bond at a time before maturity.

a. If T is the time to the bond's maturity and $V(T) = Z$, show that

$$V(t) = e^{-\int_t^T r(x)\,dx}\left(Z + \int_t^T K(u)e^{\int_u^T r(x)\,dx}\,du\right).$$

b. What does $V(t)$ look like if you have a zero-coupon bond—that is, if $K(t) \equiv 0$?

34. A disease has spread throughout a community. The number of infected individuals I in the community at any time $t > 0$ is given by $\frac{dI}{dt} - k(P_0 + rt)I = -kI^2$, where $I(0) = I_0$, P_0 is the population of the community at $t = 0$, r is the constant growth rate of the community, and k is a constant. Show that

$$I = e^{kP_0t+(1/2)krt^2}\left[\frac{1}{I_0} + k\int_0^t e^{kP_0u+(1/2)kru^2}\,du\right]^{-1}.$$

[*Hint*: The differential equation is a Bernoulli equation, as discussed in Problem 32.]

35. Suppose you have a linear first-order differential equation in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$, where $Q(x)$ is not the zero function.

a. Looking at the general solution given by Eq. (2.2.2), show that the term $Ce^{-\int P(x)\,dx}$ is the general solution, y_{GH}, of the homogeneous equation you get by setting $Q(x) \equiv 0$.

b. Show that the term $e^{-\int P(x)\,dx} \cdot \int e^{\int P(x)\,dx} Q(x)\,dx$ is a particular solution, y_{PNH}, of the original nonhomogeneous equation. (Thus, we can express the general solution, y_{GNH}, of the nonhomogeneous equation as follows: $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$.) (See Problem 23 of Exercises 1.3.)

c. Examine the result of part (b) in light of the Superposition Principle.

36. If $x \neq 0$, $x \neq 1$, $y > 0$, and $y \neq 1$, find y as a function of x provided that

$$y' + y(\ln y)^2 - \frac{1}{x(x-1)} y \ln y = 0.$$

(*Hint*: Let $y = e^{1/z}$, where $z = z(x)$ is nonzero and differentiable for all real values of x.)

2.3 Compartment problems

In analyzing certain systems in biology and chemical engineering, researchers encounter a class of problems called **mixing problems** or **compartment problems**.

Suppose we have a single container, or compartment, containing some substance. Now think of some other substance entering the compartment at a certain rate, and imagine that a mixture of the two substances leaves the compartment at another rate. For example, we could be talking about a tank of water into which some chemical is introduced via a pipe. What emerges from the tank through another pipe will be a mixture of the water and the chemical. In biology and physiology the compartment may be the bloodstream or a particular organ, such as the kidneys. In fact, mathematical analyses in these research fields often regard the organism under study as a whole collection of individual components (compartments). In the human body these could be different organs or groups of cells, for example.

To get our bearings, we can start with a simple one-compartment model (Fig. 2.5).

We have a single tank with a certain amount of material in it. The amount of substance (or the concentration of the substance) that is added to the tank is called the **inflow**, and the amount (or concentration) of substance leaving the tank is called the **outflow**. We assume that there is a thorough mixing process taking place in the tank—an almost instantaneous uniform blending of the two substances. To model this process using a differential equation, it is important to focus on three different rates associated with this situation: (1) the rate of inflow, (2) the net rate at which

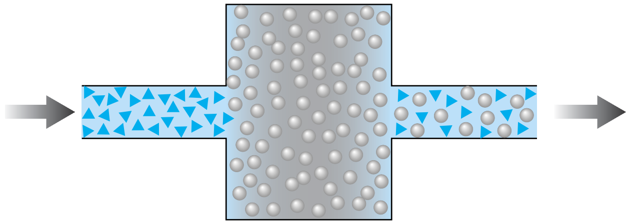


FIGURE 2.5

A one-compartment model

some aspect of the mixture in the tank is changing, and (3) the rate of outflow of the mixture.

The basic principle we will use in solving compartment problems is the **Balance Law** (or **Balance Equation**):

$$\boxed{\textbf{Net Rate of Change} = \textbf{Rate of Inflow} - \textbf{Rate of Outflow.}} \qquad (2.3.1)$$

As our first example, let's look at a simple model of medicine in the bloodstream.

Example 2.3.1 Medicine in the Bloodstream

Intravenous infusion is the process of administering a substance into the veins at a steady rate (see Problem 41 of Exercises 2.1). Suppose a patient in a hospital is receiving medication through an intravenous tube that drips the substance into the bloodstream at a constant rate of I milligrams (mg) per minute. Also suppose that the medication is dispersed through the body and is eliminated at a rate proportional to the concentration of the medication at the time. In this problem, concentration is defined as

$$\frac{\text{Quantity of medication}}{\text{Volume of blood plus medication}},$$

where we assume that the volume V of blood plus medication remains constant. The problem is to find the concentration of the medication in the body at any time t. To do this, we can consider the bloodstream as a single compartment and examine a differential equation that models the process.

If we let $C = C(t)$ denote the concentration of the medication at time t (in mg/cm^3), then the Balance Law (2.3.1) leads us to the relation

$$V\frac{dC}{dt} = I - kC,$$

where k is a positive constant of proportionality that depends on the specific medication and the physiological characteristics of the patient.

Note that the left-hand side of the differential equation is in units of cm^3 $\times$ (mg/cm^3) $\times$ (1/min) $=$ mg/min and that the right-hand term I is also in mg/min. Because C is expressed in units of $\frac{\text{mg}}{\text{cm}^3}$, we see that the appropriate unit for k, representing a removal rate, must be cm^3/min. (Be sure that you understand this "dimensional analysis.")

This is a linear equation that we can write in the standard form

$$\frac{dC}{dt} + \left(\frac{k}{V}\right)C = \frac{I}{V}.$$

An integrating factor for this equation is $\mu = e^{\int \frac{k t}{V} dt} = e^{\frac{k t}{V}}$. Multiplying each side of this last differential equation by μ gives us

$$e^{\frac{k t}{V}} \frac{dC}{dt} + \frac{k}{V} e^{\frac{k t}{V}} C = \left(\frac{I}{V}\right) e^{\frac{k t}{V}},$$

or

$$\frac{d}{dt}\left(e^{\frac{k t}{V}} C\right) = \left(\frac{I}{V}\right) e^{\frac{k t}{V}},$$

so that integrating each side gives us

$$e^{\frac{k t}{V}} C = \int \left(\frac{I}{V}\right) e^{\frac{k t}{V}}\, dt,$$

yielding

$$C(t) = e^{-\frac{k t}{V}} \int \left(\frac{I}{V}\right) e^{\frac{k t}{V}}\, dt = e^{-\frac{k t}{V}} \left[\frac{V}{k}\left(\frac{I}{V}\right) e^{\frac{k t}{V}} + \alpha\right] = \frac{I}{k} + \alpha e^{-\frac{k t}{V}}.$$

Using the implied initial condition $C(0) = 0$, we find that $\alpha = -\frac{I}{k}$, so that we can write our solution as

$$C(t) = \frac{I}{k} - \frac{I}{k} e^{-\frac{k t}{V}} = \frac{I}{k}\left(1 - e^{-\frac{k t}{V}}\right).$$

Note what happens as time goes by. Analytically, $\lim_{t\to\infty} C(t) = \frac{I}{k}$. This says that the concentration of medication in the patient's body reaches a *threshold*, or *saturation level*, of $\frac{I}{k}$. Fig. 2.6 is a graph of the concentration when $I = 4$, $V = 1$, and $k = 0.2$, showing a saturation level of 20 mg/cm^3.

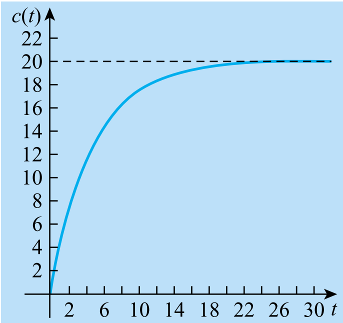


FIGURE 2.6

$C(t) = 20\left(1 - e^{-0.2t}\right)$, $0 \le t \le 30$, $0 \le C \le 20$

In compartment model problems, it is often important to determine *how long* it may take for a certain result to occur.

Example 2.3.2 Air Pollution

By 10:00 P.M. on a lively Friday night, a club of dimensions 30 feet by 50 feet by 10 feet is full of customers. Sadly, many of these customers are smokers, so cigarette smoke containing 4% carbon monoxide is introduced into the room at a rate of 0.15 cubic feet per minute. Suppose that this rate does not vary significantly during the evening. Before 10:00 there is no trace of carbon monoxide in the club; fortunately, this club is equipped with good ventilators. These ventilators allow the formation of a uniform smoke-air mixture in the room, and they provide for the ejection of this mixture to the outside at the rate of 1.5 cubic feet per minute—that is, at a rate 10 times greater than that of the arrival of pollutants.

You want to dance and socialize, but you also want to preserve your health. A prolonged exposure to a concentration of carbon monoxide greater than or equal to 0.012% is considered dangerous by the Health Department. Knowing that the club closes its doors at 3 A.M., will you allow yourself to stay until the end? To be more precise, you want to find the time when the concentration of carbon monoxide reaches the critical concentration of 0.012%.

The key to this type of single-compartment problem is the fundamental relation we saw used in the preceding example:

Net rate of change = rate of inflow − rate of outflow.

Let $C(t)$ be the concentration of carbon monoxide in the club (the grams of carbon monoxide per cubic foot of air, abbreviated g/ft^3) at any time t, where $t = 0$ represents 10 P.M. Then $Q(t)$, the *amount* of pollutant in the room at time t, is described by the equation $Q(t) = (\text{volume of room}) \times C(t)$. Because the room is $30 \times 50 \times 10 = 15{,}000$ cubic feet, this expression for the amount of carbon monoxide in the room at time t becomes $Q(t) = 15{,}000\,C(t)$.

Now the *rate* at which carbon monoxide is *entering* the room is given by

$$\left(0.15\,\frac{\text{ft}^3}{\text{min}}\right)\left(0.04\,\frac{\text{g}}{\text{ft}^3}\right) = 0.006\,\frac{\text{g}}{\text{min}}.$$

Similarly, the *rate* at which carbon monoxide is *leaving* the room (via the ventilators) is $\left(1.5\,\frac{\text{ft}^3}{\text{min}}\right) \cdot C(t)$.

The Balance Law (2.3.1) tells us that the rate of change of the amount of carbon monoxide in the room is equal to the rate at which the pollutant is introduced minus the rate at which it leaves:

$$\begin{aligned}\frac{dQ(t)}{dt} &= \frac{d}{dt}\{15{,}000\,C(t)\} = \text{rate of inflow} - \text{rate of outflow} \\ &= \left(0.15\,\frac{\text{ft}^3}{\text{min}}\right)\left(0.04\,\frac{\text{g}}{\text{ft}^3}\right) - \left(1.5\,\frac{\text{ft}^3}{\text{min}}\right)C(t) \\ &= 0.006 - 1.5C(t)\ \text{g/min}\end{aligned}$$

and we have the differential equation

$$15{,}000\,\frac{d}{dt}\,C(t) = 0.006 - 1.5\,C(t).$$

This is a linear equation, and we can write it in the form

$$\frac{d}{dt}C(t) + (0.0001)C(t) = \left(4 \times 10^{-7}\right).$$

An integrating factor is $\mu(t) = e^{\int (0.0001)\,dt} = e^{0.0001t}$, so the last equation has the form

$$\frac{d}{dt}\left\{e^{0.0001t}C(t)\right\} = \left(4\times 10^{-7}\right)e^{0.0001t}.$$

Integrating, we find that

$$\begin{aligned} C(t) &= \left(4\times 10^{-7}\right)e^{-0.0001t}\int e^{0.0001t}\,dt \\ &= \left(4\times 10^{-7}\right)e^{-0.0001t}\left(\frac{e^{0.0001t}}{0.0001}+k\right) \\ &= 0.004 + \alpha\cdot e^{-0.0001t}, \end{aligned}$$

where $\alpha = \left(4\times 10^{-7}\right)k$.

Because we are told that $C(0) = 0$, we have $0 = C(0) = 0.004 + \alpha$, which gives us the information that $\alpha = -0.004$. Therefore, we can write the solution of our differential equation as

$$C(t) = 0.004\left(1 - e^{-0.0001t}\right).$$

Because we want to know the time t at which the concentration equals 0.012%, we must solve the equation $C(t) = 0.00012$ for t. Hence, we must have

$$\begin{aligned} 0.00012 &= 0.004\left(1 - e^{-0.0001t}\right) \\ 0.03 &= 1 - e^{-0.0001t} \\ e^{-0.0001t} &= 1 - 0.03 = 0.97 \\ -0.0001t &= \ln(0.97) \\ t &= \frac{\ln(0.97)}{(-0.0001)} \end{aligned}$$

so $t = 304.59$ minutes ≈ 5.08 hours ≈ 5 hours, 5 minutes. Therefore, the critical concentration of carbon monoxide is reached at 3:05 A.M. *That's cutting it too close!*

The next example shows a different sort of compartment and alerts us to the fact that not all compartments have constant "volumes."

Example 2.3.3 Fairness in Employment

Suppose that a government agency has a current staff of 6000, of whom 25% are women. Employees are quitting randomly at the rate of 100 per week. If we know that replacements are being hired at the rate of 50 per week, with the requirement that half be women, what is the size of the agency staff in 40 weeks, and what percentage is then female?

This is a compartment problem, with the agency as the compartment. We note that, in contrast to the previous examples, our compartment size (agency staff size) varies with time. Let $W(t)$ be the number of women at time t, with $W(0) = 25\%$ of $6000 = 1500$. Now the net change in total staff is $50 - 100 = -50$ people/week, so that the staff size at time t is $6000 - 50t$ people. Summarizing this information, we have

$$\underbrace{\frac{dW}{dt}}_{\substack{\text{rate of change}\\ \text{in no. of women}}} = \underbrace{25 \text{ women/week}}_{\substack{\text{rate of inflow of women}\\ =50\% \text{ of all replacements}}} - \underbrace{\left(\underbrace{100 \text{ people/week}}_{\text{rate of people leaving}} \cdot \underbrace{\frac{W(t)}{6000-50t} \text{women/people}}_{\substack{\text{proportion of women on staff}\\ \text{at time of leaving}}}\right)}_{\text{rate of women leaving}}$$

or

$$\frac{dW}{dt} + \left(\frac{100}{6000-50t}\right) W = 25.$$

The integrating factor is $\mu(t) = (6000-50t)^{-2}$, so we get $W(t) = \frac{1}{2}(6000-50t) + C(6000-50t)^2$. Because $W(0) = 1500$, we find that $C = -1/24{,}000$. Thus, $W(t) = \frac{1}{2}(6000-50t) - \frac{1}{24{,}000}(6000-50t)^2$, and when $t = 40$ we get a staff total equal to $6000 - 50(40) = 4000$ and $W(40) = 2000 - 2000/3$, so that the staff is about $(2000 - 2000/3)/4000 = 1/3$ or $\mathbf{33\frac{1}{3}}$% female.

After we've treated systems of equations in Chapter 6, we'll be able to solve *multicompartment* problems.

Exercises 2.3

A

1. Suppose a population has a constant per capita birth rate $b > 0$ and a constant per capita death rate $d > 0$. Using the Balance Law (2.3.1), write a differential equation for the population $p(t)$ at time t. (Do not solve this equation.)
2. In Problem 1, suppose the per capita death rate d is not constant, but is instead proportional to the population $p(t)$. Write (but do not solve) a differential equation for the population $p(t)$.
3. In Problem 1, suppose the per capita birth rate b is not constant, but is instead proportional to the population $p(t)$. Write (but do not solve) a differential equation for the population $p(t)$.
4. Suppose a country with constant per capita birth and death rates b and d, respectively, has an influx of immigrants at a constant rate I (not a per capita rate, but a constant rate). Write a differential equation for the size of the population $p(t)$.

B

5. A study of the population of Botswana from 1975 to 1990 leads to a model for the country's growth rate, $\frac{dP}{dt} = kP - \alpha t$, where t denotes time in years with 1990 corresponding to $t = 0$, $P(0) = 1.285$ (million), $k = 0.0355$, and $\alpha = 1.60625 \times 10^{-3}$. (The term kP reflects births and immigration, while the term αt captures deaths and emigration.)
   a. Find a formula for $P(t)$.
   b. Estimate Botswana's population in the year 2025.

6. A tank with a capacity of 100 gallons is half full of fresh water. A pipe is opened which lets treated sewage enter the tank at the rate of 4 gal/min. At the same time, a drain is opened to allow 3 gal/min of the mixture to leave the tank. If the treated sewage contains 10 grams per gallon of usable potassium, what is the *concentration* of potassium in the tank when it is full? (Be careful of your units!)

7. A tank with a capacity of 100 gallons is initially full of water. Pure water is allowed to run into the tank at the rate of 1 gallon per minute. At the same time, brine (a mixture of salt and water) containing 1/4 pound of salt per gallon flows into the tank at the rate of 1 gallon per minute. (Assume that there is perfect mixing.) The mixture flows out at the rate of 2 gallons per minute. Find the *amount* of salt in the tank after t minutes.

8. Suppose you have a 200-gallon tank full of fresh water. A drain is opened that removes 3 gal/s from the tank and, at the same moment, a valve is opened that lets in a 1% solution (a 1% concentration) of chlorine at 2 gal/s.

a. When is the tank half full and what is the concentration of chlorine then?

b. If the drain is closed when the tank is half full and the tank is allowed to fill, what will be the final concentration of chlorine in the tank?

9. A tank contains 50 gallons of fresh water. Brine (see Problem 7) at a concentration of 2 lbs/gal (i.e., 2 lbs of salt per gallon) starts to run into the tank at 3 gal/min. At the same time the mixture of fresh water and brine runs out at 2 gal/min.

a. How much liquid is there in the tank after 50 minutes?

b. How many pounds of dissolved salt is in the tank after 50 minutes?

10. In a large tank are 100 gallons of brine containing 75 pounds of dissolved salt. Water runs into the tank at the rate of 3 gal/min, and the mixture runs out at the rate of 2 gal/min. The concentration is kept uniform by stirring. How much salt is there in the solution after 1.5 hours?

11. A swimming pool holds 10,000 gallons of water. It can be filled at the rate of 100 gal/min and emptied at the same rate. Right now the pool is filled, but there are 20 pounds of an impurity dissolved in the water. For the safety of the swimmers, this must be reduced to less than 1 pound. It would take 200 minutes to empty the pool completely and refill it, but during part of this time the pool could not be used. How long will it take to restore the pool to a safe condition if at all times the pool must be at least half full?

12. Assuming the information in Example 2.3.3, what would be the percentage of female staff members after 40 weeks if *all* the new employees were required to be women?

C

13. Suppose that the maximum concentration of a drug present in a given organ of constant volume V must be $c_{\max}$. Assuming that the organ does not contain the drug initially, that the liquid carrying the drug into the organ has constant concentration $c > c_{\max}$, and that the inflow and outflow rates are both equal to r, show that the liquid must not be allowed to enter for a time longer than

$$\frac{V}{r}\ln\left(\frac{c}{c - c_{\max}}\right).$$

14. A tank that holds 100 gallons is half full of a brine solution with a concentration of $\frac{1}{3}$ pound of salt per gallon. Two pipes lead into it at the top, one supplying a brine solution of $\frac{1}{2}$ lb/gal and the other pure water. Each pipe has a flow of 4 gal/min. One pipe leads out at the bottom and removes the mixture at 3 gal/min. What is the *concentration* of the mixture that first flows out the overflow pipe at the top? Assume uniform mixing.

15. Using the data in Example 2.3.2, determine the rate at which the ventilators should operate if the concentration of carbon monoxide is *never* to reach the lethal level. Explain your reasoning.

2.4 Slope fields

Now that we have become familiar with the basic concepts of ordinary differential equations (ODEs) and have learned how to solve separable and linear equations, we can consider a *qualitative* approach to understanding solutions of first-order equations. This is a graphical approach to an equation that provides insights into the behavior of solutions, even when we may not know the techniques for solving the equation.

Let's look at first-order equations in normal form:

$$\frac{dy}{dx} = y' = f(x, y).$$

For example, we could have $\frac{dy}{dx} = f(x, y) = 3y - 4x$, $y' = g(x, y) = \sqrt{xy}$, $y' = F(x) = 2x^3 - 1$, or $y' = G(y) = 2 - y^2$. Now remember what a first derivative tells us. One interpretation of a derivative is as the slope of the tangent line drawn to a curve at a particular point. The equation $y' = f(x, y)$ means that at the point (x, y) of any solution curve of the differential equation, the slope of the tangent line is given by the value of the function f at that point—that is, the slope is given by $f(x, y)$. Remember that there may be a whole family of solution curves in addition to the singular solutions.

For a first-order differential equation, a set of possible tangent line segments (sometimes called **lineal elements**), whose slopes at (x, y) are given by $f(x, y)$, is called a **slope field** (or **direction field**) of the equation. Visually, this establishes a flow pattern for solutions of the equation. A slope field includes tangent line segments for many solutions of the equation, but the general shapes of the integral curves should be clear. You can think of these outlines as the "ghosts" of solution curves, and they may reveal certain *qualitative* aspects of the solutions, even if a closed form solution is difficult or impossible to find.

Our first example will indicate how to generate a slope field.

Example 2.4.1 A Slope Field

To get a feeling for these ideas, let's get a piece of graph paper and plot some tangent line segments for the first-order linear equation $y' - y = x$, which we can write as $y' = f(x, y) = x + y$. To make things a bit easier, we can construct a table (Table 2.1).

Table 2.1 Slopes at Points (x, y) for $y' = x + y$

Point		$y' = x + y$	
x	y	(x, y)	Slope at (x, y)
−3	3	(−3, 3)	0
1	−1	(1, −1)	0
0	0	(0, 0)	0
0	1	(0, 1)	1
1	0	(1, 0)	1
2	−1	(2, −1)	1
−1	2	(−1, 2)	1
0	−1	(0, −1)	−1
−1	0	(−1, 0)	−1
2	−3	(2, −3)	−1
⋮	⋮	⋮	⋮

We've made things even easier for ourselves by choosing points at which the slopes are 0, 1, and −1. Now we can draw some tangent line segments corresponding to these slopes (Fig. 2.7a).

Note that we have drawn the little tangent line segments so that the midpoint of each segment is the point (x, y). We have used portions of the slope field given by $f(x, y) = 0$ and $f(x, y) = \pm 1$. Fig. 2.7b is a computer-drawn direction field for the same ODE, with some solution curves superimposed on the slope field.

Note that as $x \to -\infty$, the solution curves seem to be approaching a straight line as an asymptote. The solution curves seem to be veering *away* from this line as $x \to +\infty$. If you look very closely, you may be able to guess that the straight line is $x + y = -1$, or $y = -1 - x$. In Section 2.2 we learned how to find the general solution, $y = -x - 1 + Ce^x$, for this linear equation. The straight line $y = -1 - x$ is the particular solution of the ODE corresponding to $C = 0$, a solution of the IVP $y' - y = x$, $y(0) = -1$. Also note that if y is the general solution and $C \neq 0$, then $\lim_{x\to+\infty} y(x) = \infty$ if $C > 0$ and $\lim_{x\to+\infty} y(x) = -\infty$ if $C < 0$.

Although the slope field suggests some features of the solution curves, we have to be careful not to read too much into this picture. In the preceding example, without the analytic form of the general solution or some sound numerical evidence, we can't be sure that y doesn't have vertical asymptotes, so that $y \to \pm\infty$ as x approaches some *finite* value x_0.

Note that in Example 2.4.1 we used portions of the slope field given by $f(x, y) = 0$ and $(x, y) = \pm 1$. For any first-order equation $y' = f(x, y)$, if we look at

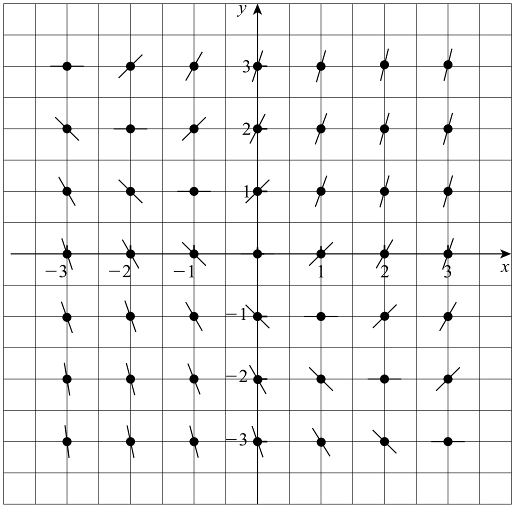


FIGURE 2.7a

Some lineal elements for $y' - x + y$

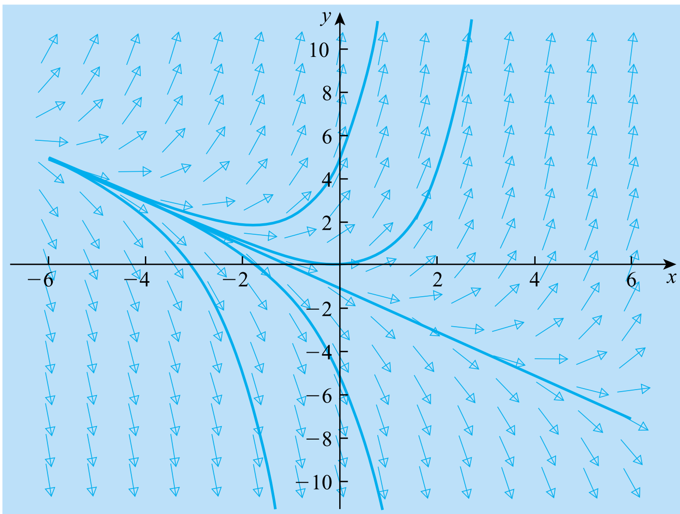


FIGURE 2.7b

Slope field for $y' = x + y$, $-6 \leq x \leq 6$, $-10 \leq y \leq 10$, and five computer-generated solution curves

the set of points (x, y) such that $f(x, y) = C$, a constant, we get an **isocline**—a curve along which the slopes of the tangent lines are all the same. (The word *isocline* is made up of parts that mean "equal" and "inclination" or "slope.") Isoclines are used to simplify the construction of a slope field; once you draw the isoclines, you can quickly and easily draw, for each C, a series of parallel line segments of slope C, all having their midpoints on the curve $f(x, y) = C$. In Example 2.4.1 the isoclines are the curves $x + y = C$, which are straight lines through $(0, C)$ and $(C, 0)$ with a slope of -1.

It is important to realize that *an isocline is usually not a solution curve*, but that through any point on an isocline a solution to the differential equation passes with slope C. However, as we'll see in Section 2.5, isoclines corresponding to $C = 0$—called **nullclines**—turn out to be important solutions (*equilibrium solutions*) of equations in which the independent variable does not appear explicitly—that is, equations of the form $y' = f(y)$.

The next example has something important to say about the difference between equations of the general form $y' = f(x, y)$ and equations of the special form $y' = f(y)$.

Example 2.4.2 A Special Slope Field

The slope field (Fig. 2.8) corresponding to the equation $x' = f(x) = -2x$ reveals something interesting about certain kinds of equations and their corresponding slope fields. (*Don't be confused by the labeling of the axes*.) Here, we are assuming that t is the independent variable and x is the dependent variable: $x = x(t)$.) First of all, note that algebraically we can write the equation in the form $F(x, x') = x' + 2x = 0$, or $x' = f(x) = -2x$. In other words, we have a first-order equation in which the independent variable t does not appear explicitly. This says that the slopes of the tangent line segments making up the slope field of this equation depend only on the values of x.

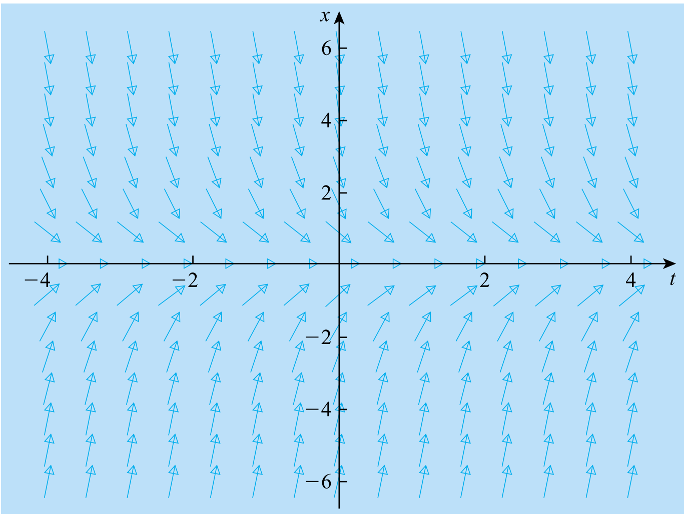


FIGURE 2.8

Slope field for $x' = -2x$, $-4 \le t \le 4$, $-6 \le x \le 6$

In the slope field plot given in Fig. 2.8, if you fix the value of x by drawing a horizontal line $x = C$ for any constant C, you will see that all the tangent line segments along this line have the same slope, no matter what the value of t. Another way to look at this is to realize that you can generate infinitely many solutions by taking any one solution and translating (shifting) its graph left or right. (See Problem 33 of Exercises 2.4.)

2.4.1 Autonomous and nonautonomous equations

A differential equation, such as the one in the preceding example, in which the independent variable does not appear explicitly is called an **autonomous** equation. If the independent variable does appear, the equation is called **nonautonomous**. This definition is valid for an equation of any order.

For example, $y' = y^2 - t^2$ is nonautonomous because the independent variable t appears explicitly, whereas $y' = 3y^4 + 2\sin(y)$ is autonomous because the independent variable (t, x, or whatever) is missing. Note that the independent variable is always present *implicitly* (in the background), but if you don't see it "up front," the equation is autonomous. Example 2.4.1 discusses a nonautonomous equation. If we look carefully at its slope field (Fig. 2.7b), we see that the slopes change as we move along any horizontal line.

Autonomous equations arise frequently in physical problems because the physical components generally depend on the *state* of the system, but not on the actual time. We can define the **state** of a system loosely as the set of values of the dependent variables in the system. For example, according to *Newton's Second Law of Motion*, an object of mass m falling under the influence of gravity satisfies the autonomous equation $\ddot{x} = -g$, where $x(t)$ is the position of the mass measured from the Earth's surface and g is the acceleration due to gravity. Gravity is considered time-independent because the mass follows the same path no matter when the mass is dropped.

Now let's see how to recognize the correspondence between first-order differential equations and their slope fields.

Example 2.4.3 Matching Equations and Slope Fields

$$\text{(A)}\ \frac{dx}{dt} = x^2 - t^2 \quad \text{and} \quad \text{(B)}\ \frac{dx}{dt} = x^2 - 1$$

Looking at the two differential equations and the accompanying slope fields 1–4, let's try to match each equation with exactly one of the slope fields (Figs. 2.9a–d).

We can start with Eq. (A) and note that it is a *nonautonomous* equation. This tells us that we should not expect equal slopes along horizontal lines. As we move horizontally—that is, if we fix the value of x and vary the value of t—the value of the slope changes according to the formula $x^2 - t^2$. This analysis eliminates slope fields 2 and 3 because the inclinations of the tangent line segments clearly remain constant along horizontal lines. Now if we write Eq. (A) in factored form, $\frac{dx}{dt} = (x + t)(x - t)$, we can see that the tangent line segments must be horizontal where $x = t$ or $x = -t$ because that's where the slope $\frac{dx}{dt}$ equals 0. (These are the *nullclines*—isoclines corresponding to $C = 0$.) Looking carefully at slope fields 1 and 4, we see that field 4 exhibits a series of horizontal "steps" forming an X through the origin. If we look closely, it seems that these horizontal line segments lie on the lines $x = t$ and $x = -t$, so we conclude that Eq. (A) corresponds to slope field 4.

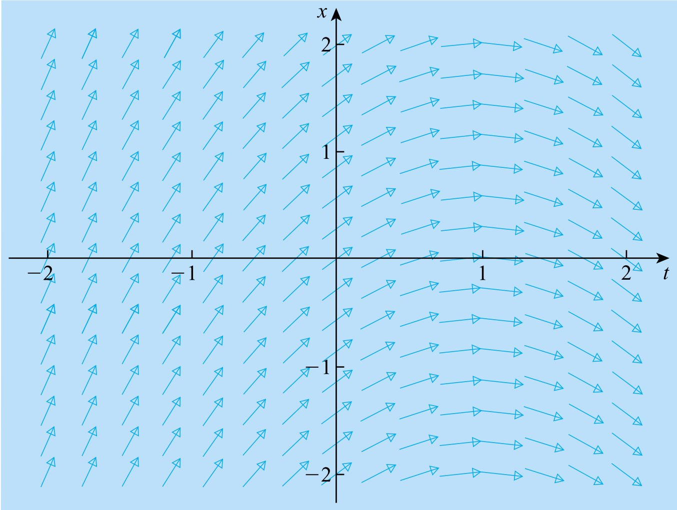


FIGURE 2.9a

Slope field 1

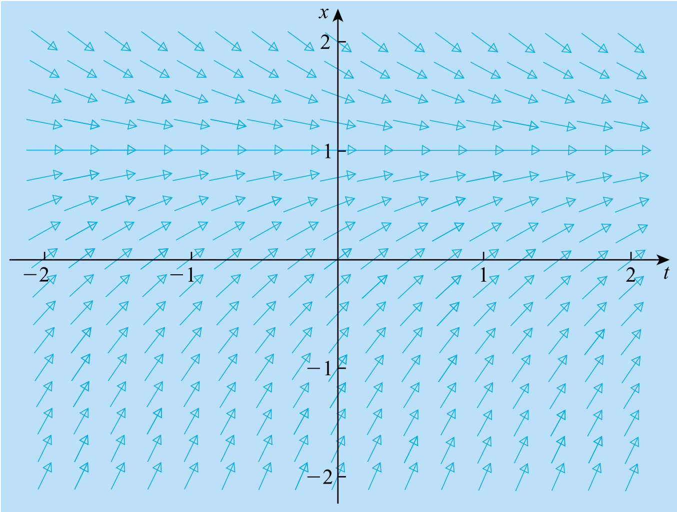


FIGURE 2.9b

Slope field 2

Eq. (B) is *autonomous* because the independent variable t does not appear explicitly. The corresponding slope field must show equal slopes along any horizontal line. Only fields 2 and 3 exhibit this behavior. What else can we look for? Well, if we factor Eq.(B) to get $\frac{dx}{dt} = (x+1)(x-1)$, we realize that the slope field must show horizontal line segments when $\frac{dx}{dt}$ equals 0—that is, where

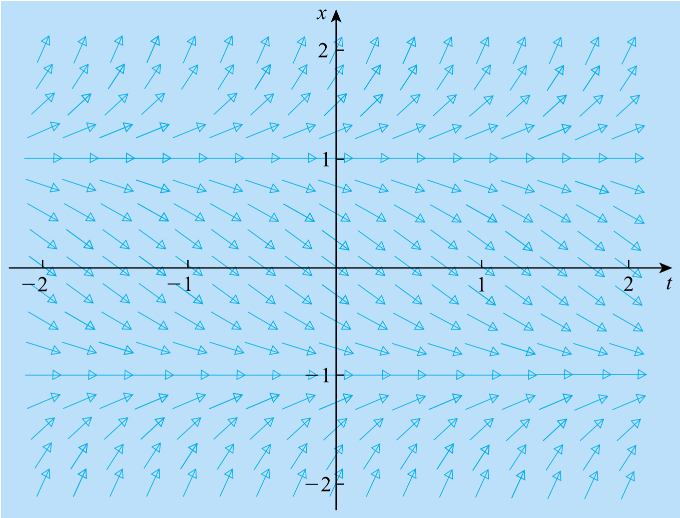


FIGURE 2.9c

Slope field 3

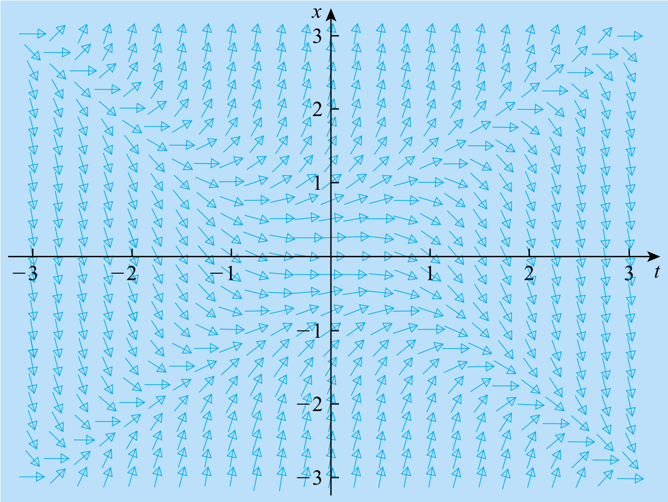


FIGURE 2.9d

Slope field 4

$x = 1$ or $x = -1$. Slope field 2 has horizontal tangents at $x = 1$ but doesn't have them at $x = -1$. Only slope field 3 shows zero slopes along both horizontal lines $x = 1$ and $x = -1$, so we conclude that Eq. (B) must match up with slope field 3.

The next example shows an advantage—and a possible drawback—of using slope fields.

Example 2.4.4 A Slope Field for an Autonomous Equation

The first-order nonlinear autonomous equation $y' = y^4 + 1$ looks innocent, but (*Surprise!*) it has the one-parameter family of implicit solutions:

$$\frac{\sqrt{2}}{8}\ln\left(\frac{y^2+\sqrt{2}y+1}{y^2-\sqrt{2}y+1}\right)+\frac{\sqrt{2}}{4}\left\{\arctan\left(y\sqrt{2}+1\right)+\arctan\left(y\sqrt{2}-1\right)\right\}=t+C.$$

(The equation is separable, but the integration required to solve it is tricky. Use your CAS to evaluate the integral, but don't be surprised if your answer doesn't look exactly like the one given here.) Without looking at the solution formula, you can see immediately that the differential equation has no constant function as a solution: If y is constant, then $y' = 0$; but the right-hand side of the differential equation, $y^4 + 1$, can never be zero. This simple analysis shows that any solution y must be an *increasing* function. (*Why?*)

The fearsome formula describing a family of implicit solutions gives little useful information. However, let's take a look at the equation's slope field (Fig. 2.10). First of all, the autonomous nature of the equation is clear because along any horizontal line the inclinations of the tangent line segments are equal.

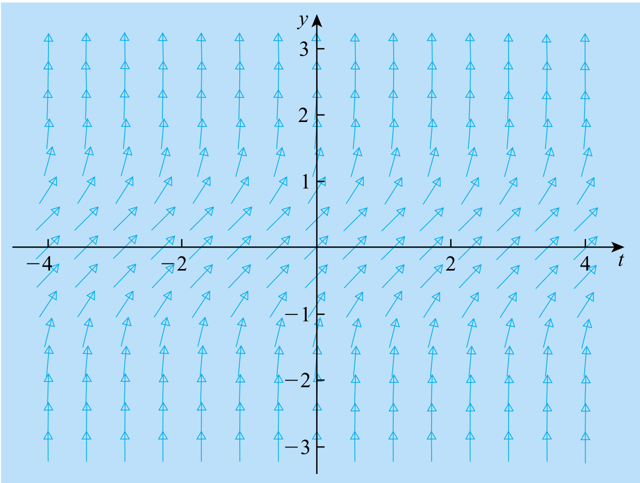


FIGURE 2.10

Slope field for $y' = y^4 + 1$, $-4 \le t \le 4$, $-3 \le y \le 3$

Furthermore, it should be evident that any solution curve is increasing. In fact, any solution curve has vertical asymptotes bounding it on the left and on the right. Of course, we can't tell whether this last statement is true by merely looking at the slope field. A purely graphical analysis can't reveal this. But the slope field does give us an idea of what to expect when we try to solve the equation analytically or to approximate a solution numerically.

As we will see in later chapters, a type of slope field can help us analyze certain *systems* of differential equations as well.

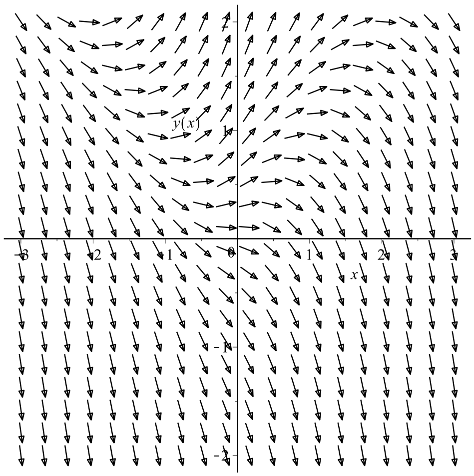
Exercises 2.4

A

In Problems 1–14, first sketch the slope field for the given equation by hand, and then try using a computer or graphing calculator to generate the slope field. Sketch several possible solution curves for each equation.

1. $y' = x$
2. $\frac{dx}{dt} = t$
3. $\frac{dr}{dt} = t - 2r$
4. $\frac{dx}{dt} = 1 - 0.01x$
5. $Q' = |Q|$
6. $y' = y - x$
7. $r\frac{dr}{dt} = -t$
8. $\frac{dy}{dx} = \frac{1}{y}$
9. $\frac{dy}{dx} = \frac{2y}{x}$
10. $y' = \max(x, y)$, the larger of the two values x and y
11. $y' = x^2 + y^2$
12. $x' = 1 - tx$
13. $\frac{dy}{dt} = \frac{ty}{t^2-1}$
14. $\frac{dP}{dt} = 2P(1 - P)$
15. Use technology to determine the slope field for the equation $y' = \frac{\cos x}{\cos y}$. Describe the nullclines of the equation.
16. In any way your instructor tells you, manually or using technology, sketch the slope field for each of the following equations and then sketch the solution curve that passes through the given point (x_0, y_0).
    a. $\frac{dy}{dx} = x^2$; $(x_0, y_0) = (0, -2)$
    b. $\frac{dy}{dx} = -xy$; $(x_0, y_0) = (0, 3)$
17. The German physiologist Gustav Fechner (1801–87) devised the model expressed as $\frac{dR}{dS} = \frac{k}{S}$, where k is a constant, to describe the response, R, to a stimulus, S. Use technology to sketch the slope field for $k = 0.1$.
18. Describe the isoclines of the equation $\frac{dy}{dt} = \frac{y+t}{y-t}$.
19. Which of the equations in Problems 1–15 are autonomous? If you have done some of these problems, look at their slope fields to confirm your answers.
20. Which of the following differential equations would produce the slope field shown below?

A. $\frac{dy}{dx} = |y - x|$
B. $\frac{dy}{dx} = |y| - x$
C. $\frac{dy}{dx} = y - |x|$
D. $\frac{dy}{dx} = |y + x|$
E. $\frac{dy}{dx} = |y| - |x|$



Slope field for Exercise 20

B

21. A **bimolecular chemical reaction** is one in which molecules of substance A collide with molecules of substance B to create substance X. The rate of formation (the *velocity of reaction*) is given by a differential equation of the form $\frac{dx}{dt} = k(\alpha - x)(\beta - x)$, where α and β represent the initial amounts of substances A and B, respectively, and $x(t)$ denotes the amount of substance X present at time t. (See Example 2.1.8.)

a. Use technology to plot the slope field when $\alpha = 250$, $\beta = 40$, and $k = 0.0006$.

b. If $x(0) = 0$, what seems to be the behavior of x as $t \to \infty$?

22. The one-parameter family $y = \frac{c}{t}$ represents a solution of $\frac{dy}{dt} = f(t, y)$. Sketch (by hand) the slope field of the differential equation.

23. Describe the isoclines of the equation $y' = \frac{1}{\sqrt{1+t^2+y^2}}$.

24. Describe the nullclines of the equation $xy\frac{dy}{dx} = y^2 - x^2$.

25. Describe the nullclines of the equation in Problem 21.

26. In your own words, explain the important differences in the slope fields for the following forms of first-order differential equations:

a. $y' = f(t, y)$

b. $y' = f(t)$

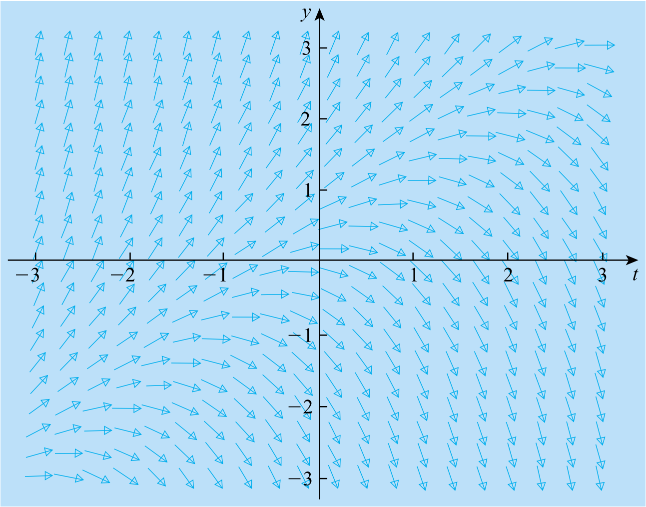
c. $y' = f(y)$

27. Match each of the equations a–c with one of the accompanying slope fields.

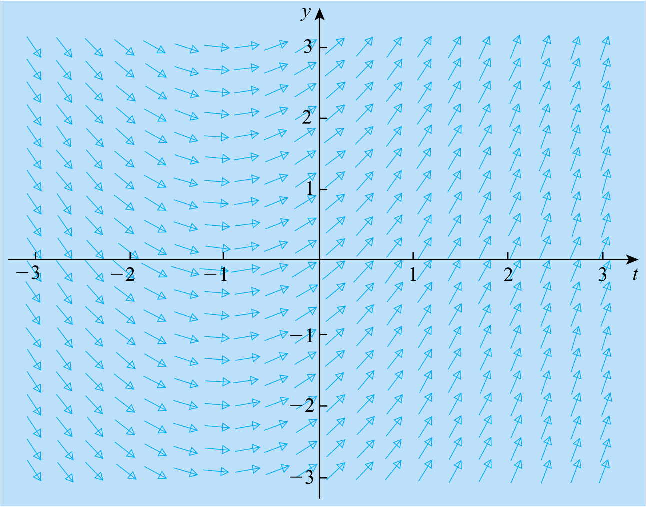
a. $\frac{dy}{dt} = y + 1$

b. $\frac{dy}{dt} = y - t$

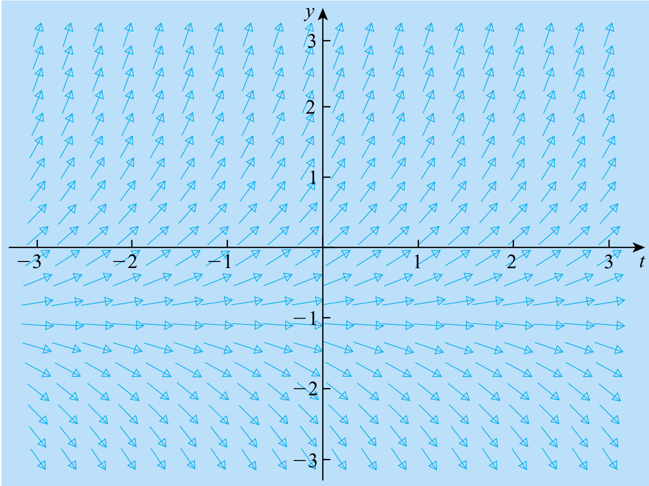
c. $\frac{dy}{dt} = t + 1$



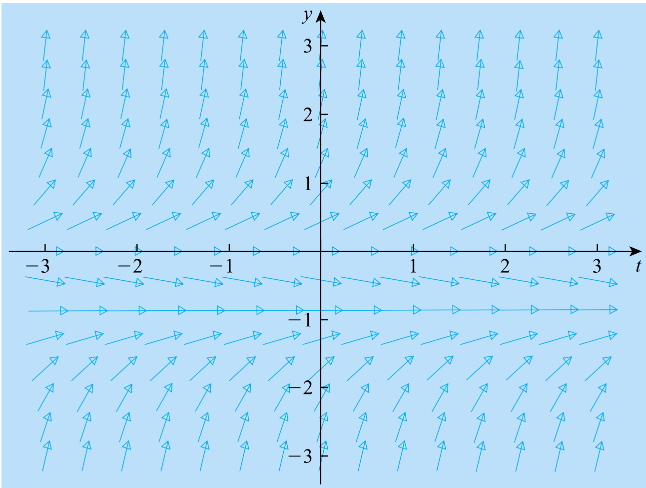
Slope field 1 for Exercise 27



Slope field 2 for Exercise 27



Slope field 3 for Exercise 27



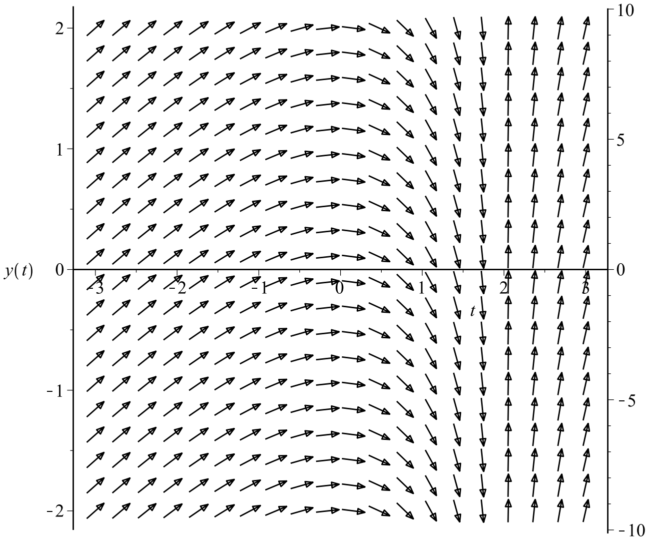
Slope field 4 for Exercise 27

28. Match each of the equations (A)–(C) with one of the accompanying slope fields.

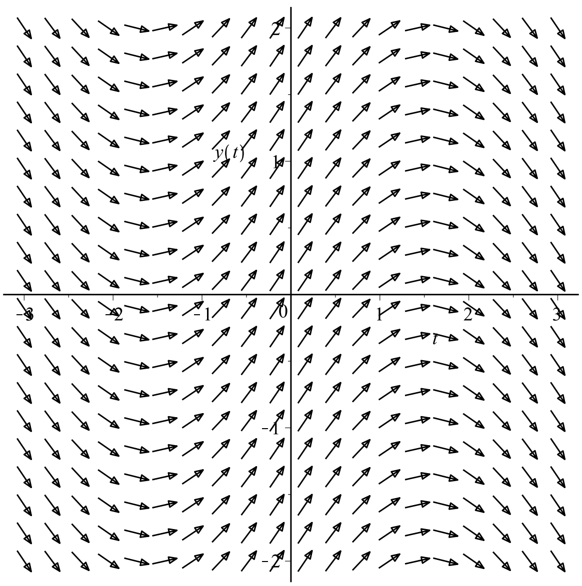
A. $\frac{dy}{dt} = \ln|t-1|$

B. $\frac{dy}{dt} = (y-1)(y+3)$

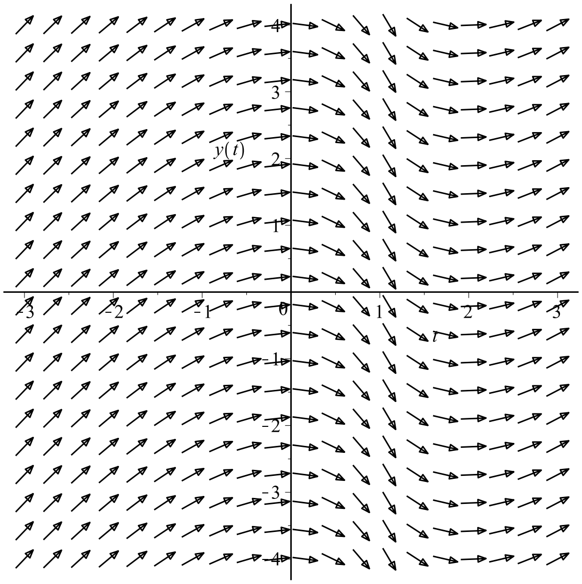
C. $\frac{dy}{dt} = t/(t-2)$



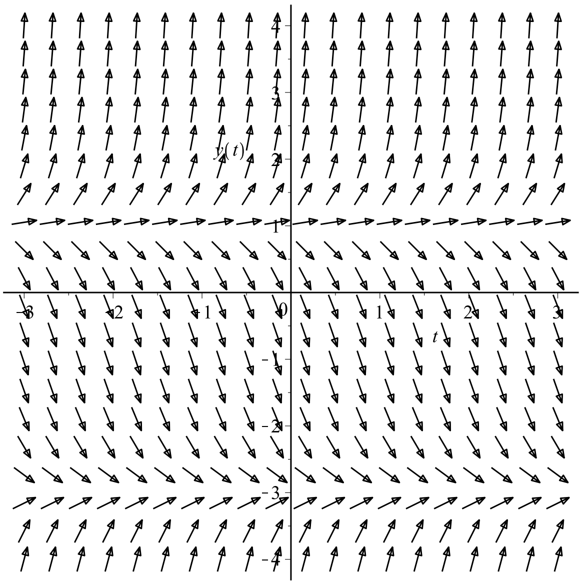
Slope field 1 for Exercise 28



Slope field 2 for Exercise 28



Slope field 3 for Exercise 28



Slope field 4 for Exercise 28

29. By looking at the slope field for each of the following equations, describe the behavior of the solutions of each equation as $t \to \infty$. How do your answers seem to depend on the initial conditions in each case?

a. $y' = 3y$

b. $\frac{dP}{dt} = P(1 - P)$

c. $y' = e^{-t} + y$

d. $y' = 3\sin t + 1 + y$

30. Examine the slope field for the first-order nonlinear equation $\frac{dy}{dx} = e^{-2xy}$. Based on this examination, what can you say about the solutions to this equation? (You may want to look at parts of the various quadrants more closely.) In particular, what can you say about the behavior of the solutions as $x \to \pm\infty$? (Be careful: Some solutions become infinite as x approaches finite values.)

31. Consider the autonomous equation $\frac{dx}{dt} = x(1 - x)(x + 1)$.

a. Use technology to determine the slope field of this equation.

b. Describe the behavior of a solution satisfying $x(0) = 0.7$.

c. Describe the behavior of a solution satisfying $x(0) = 1.3$.

d. Describe the behavior of a solution satisfying $x(0) = -0.7$.

e. Describe the behavior of a solution satisfying $x(0) = -1.3$.

32. Consider the equation $\frac{dy}{dx} = 2y - 4x$.

a. The slope field for the differential equation is shown below. Make a copy (or recreate the slope field using a CAS) and sketch the solution curve that goes through the point $(0, -1)$.

b. There is a value of b for which $y = 2x + b$ is a solution of the differential equation. Find this value of b and justify your answer.

c. Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. It appears from the slope field that g has a local maximum at the point $(0, 0)$. Using the differential equation, prove analytically that this is so.

C

33. a. If $\varphi(t)$ is a solution of an autonomous differential equation $x' = f(x)$ and k is any real number, show that $\varphi(t + k)$ is also a solution. [*Hint*: Use the Chain Rule.]

b. Use the result of (a) to show that if $\sin t$ is a solution of an autonomous differential equation $x' = f(x)$, then $\cos t$ is also a solution. [*Hint*: How are the graphs of sine and cosine related?]

c. Let $\phi(t)$ be a nonzero solution of the nonautonomous differential equation $x' = tx$. Is $\phi(t + c)$ also a solution for $c \neq 0$? Prove your answer.

2.5 Phase lines and phase portraits

2.5.1 The logistic equation

The slope fields introduced in the last section are useful for viewing two-dimensional information. When we deal with an *autonomous* first-order equation, say $\dot{x} = f(x)$, there is no explicit dependence on the variable t, and so we find ourselves in a situation where the relevant data is one-dimensional. Thus, we can hope to use a simpler graphical representation of the information. As we'll see, this new method of qualitative analysis can be used quite effectively to provide useful information about solution curves.

We'll begin to examine this new analysis technique by using an important population growth model, first studied by the Belgian mathematician Pierre Verhulst in 1838 and later rediscovered independently by the American scientists Raymond Pearl and Lowell Reed in the 1920s.

Example 2.5.1 The Qualitative Analysis of the Logistic Equation

The autonomous differential equation $\frac{dP}{dt} = P(1 - P)$, a particular example of something called a **logistic equation**, is useful, for instance, in analyzing such phenomena as epidemics. (We dealt with this equation in another way in part (b) of Problem 29 in Exercises 2.4.) In an epidemiological situation, P could represent the infected population (or the *percentage* of the total population that is infected) as a function of time. We'll work more with this kind of model later, but for now let's ignore the fact that this is a separable equation that we can solve explicitly and see what basic calculus can tell us.

First of all, the right-hand side represents a derivative, the instantaneous rate at which P is changing with respect to time. From calculus we know that if the derivative is *positive*, then P is *increasing*, and if the derivative is *negative*, then P is *decreasing*. Now when is dP/dt positive? The answer is when $P(1 - P)$ is greater than zero. Similarly, dP/dt is negative when $P(1 - P)$ is less than zero. Finally, we see that $dP/dt = 0$ when $P(1 - P) = 0$—that is, when $P = 0$ or $P = 1$. These two critical points split the P-axis into three pieces (Fig. 2.11): $-\infty < P < 0$, $0 < P < 1$, and $1 < P < \infty$.

What is the sign of dP/dt when P satisfies $-\infty < P < 0$? Well, for these values of P, P is negative and $1 - P$ is positive, making the product $dP/dt = P(1 - P)$ *negative*. This means that P is *decreasing*. When P is between 0 and 1, we see that P is positive and $1 - P$ is positive, so dP/dt is *positive* and P is *increasing*. Finally, when P is greater than 1, we see that P is positive and $1 - P$ is negative, so dP/dt is *negative* and P is *decreasing*.

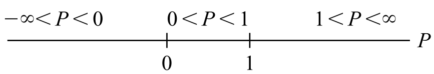


FIGURE 2.11

P-axis divided by critical points

We can redraw Fig. 2.11 with arrows indicating whether P is increasing or decreasing on a particular interval for P. The direction of any arrow shows the algebraic sign of dP/dt in a subinterval and so indicates whether P is increasing or decreasing: $\rightarrow$ means "positive derivative/increasing P" and $\leftarrow$ means "negative derivative/decreasing P." Fig. 2.12 is called the (**one-dimensional**) **phase**

portrait of the differential equation $\frac{dP}{dt} = P(1-P)$. The horizontal line itself is called the **phase line**.

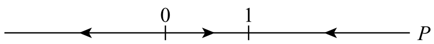


FIGURE 2.12

Phase portrait of $\frac{dP}{dt} = P(1-P)$

We can actually do a little more in this situation. If we differentiate each side of our original differential equation with respect to t, we get

$$\frac{d^2P}{dt^2} = \frac{dP}{dt} \cdot (1-2P) = P(1-P)(1-2P),$$

where we have replaced dP/dt by the right-hand side of the original differential equation. (*Check all this!*) Remember that the second derivative of a function tells us about the *concavity* of the function: P is concave *up* when $d^2P/dt^2 > 0$ and P is concave *down* when $d^2P/dt^2 < 0$. Using the critical points 0, $\frac{1}{2}$, and 1 of $\frac{d^2P}{dt^2}$ as a guide, we can construct the table of signs (Table 2.2).

Table 2.2 Table of Signs

P **Interval**	P	$1-P$	$1-2P$	$P'' = P(1-P)(1-2P)$	**Concavity**
$(-\infty, 0)$	$-$	$+$	$+$	$-$	Down
$\left(0, \frac{1}{2}\right)$	$+$	$+$	$+$	$+$	Up
$\left(\frac{1}{2}, 1\right)$	$+$	$+$	$-$	$-$	Down
$(1, \infty)$	$+$	$-$	$-$	$+$	Up

We have to remember that t is the independent variable in this problem and P is the dependent variable. It's easy to lose sight of this because the (autonomous) form of this differential equation makes us focus on P alone.

On the basis of our analysis of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$, let's take a look at what the graph of P could look like in the $t-P$ coordinate plane (Fig. 2.13). We'll focus on the first quadrant because $t \geq 0$ and $P \geq 0$ are realistic assumptions when dealing with a population growth model. Note that the phase line (representing the P-axis) is now drawn vertically and placed next to the graph and that we've marked the important values from our previous investigation of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

The graph indicates the change of concavity at $P = \frac{1}{2}$. Notice how each of the three solutions we have sketched seems to approach $P = 1$ as an asymptote as t increases. In terms of a realistic scenario, this says that if the initial population is below 1 (the unit could be thousands or millions), the population will increase to 1 asymptotically. On the other hand, any population starting above 1 will eventually decrease toward 1. If we had drawn the rest of the phase line (for $P < 0$) and solutions in the fourth quadrant ($t \geq 0$, $P < 0$), we would have seen these solutions moving away from the t-axis. We'll say more about this phenomenon in the next section.

The **logistic equation**, which is commonly used to model population growth when resources (such as food) are limited, is usually written as $\frac{dP}{dt} = rP(1 - \frac{P}{k})$, where r

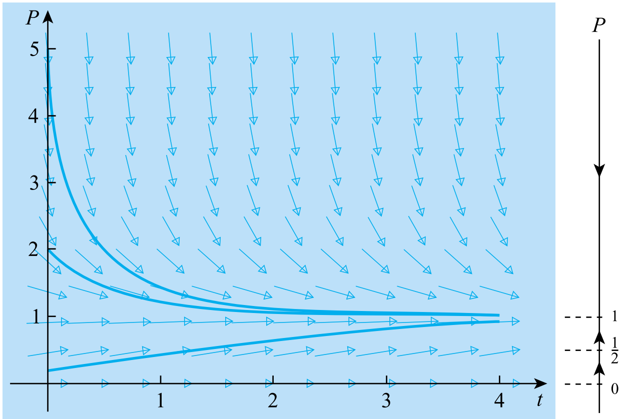


FIGURE 2.13

Sketch of three solutions of $\frac{dP}{dt} = P(1 - P)$, based on the phase portrait and concavity. Initial conditions are $P(0) = 0.2$, 2, and 5; $0 \leq t \leq 4$, $0 \leq P \leq 5$

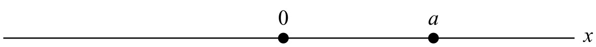
is a per capita growth rate balancing births and deaths and k represents the theoretical maximum population that a given environment (forest, Petri dish, etc.) can sustain. The value k is called the **carrying capacity**. This model will reappear from time to time in this text.

Sometimes an autonomous differential equation will contain a parameter whose possible values affect the behavior of solutions.

Example 2.5.2 An Equation with a Parameter

Consider the equation $x' = x^2 - ax = x(x - a)$, where a is a constant. There are two apparent critical points: $x = 0$ and $x = a$. The first challenge is to position the critical points properly on the x-axis.

First, let's assume that $a > 0$. Then the phase line is



If $x < 0$, then $x - a < 0$ and $x' = x(x - a) > 0$. If $0 < x < a$, then $x > 0$ and $x - a < 0$, so $x(x - a) < 0$. Finally, if $x > a$, we have $x > 0$ and $x - a > 0$, so $x(x - a) > 0$. Thus, if $a > 0$, the phase portrait is as shown in Fig. 2.14.

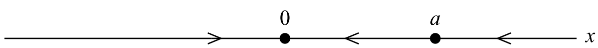
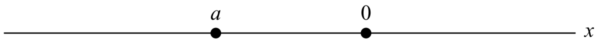


FIGURE 2.14

Phase portrait of $x' = x^2 - ax$, $a > 0$

Now suppose $a < 0$. The phase line is



If $x < a$, then $x < 0$ and $x - a < 0$, so $x' = x(x - a) > 0$. When $a < x < 0$, we have $x < 0$ and $x - a > 0$, so $x(x - a) < 0$. If $x > 0$, then $x - a > 0$, so $x(x - a) > 0$. For $a < 0$ the phase portrait is as shown in Fig. 2.15.

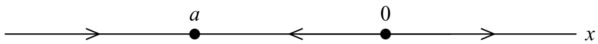


FIGURE 2.15

Phase portrait of $x' = x^2 - ax$, $a < 0$

There is one last case: $a = 0$. Now there is only one critical point, and it is easy to see that the phase portrait is as shown in Fig. 2.16.

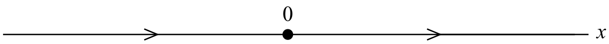


FIGURE 2.16

Phase portrait of $x' = x^2 - ax$, $a = 0$

In summary, we see that the presence of parameters in a differential equation may affect the behavior of its solutions. Sketches of typical solution curves corresponding to $a > 0$, $a < 0$, and $a = 0$ are left as an exercise (see Problem 16 of Exercises 2.5).

Since differential equations with parameters occur in many important applications, we give an additional, slightly more complicated example.

Example 2.5.3 Another Equation with a Parameter

If we analyze the nonlinear differential equation $\dot{x} = \mu x - x^3 = x(\mu - x^2)$, where μ is a real parameter, we see how the behavior of solutions varies depending on the value of μ.

First of all, we note that $x = 0$ is a critical point ($\dot{x} = 0$) no matter what the value of μ is. On the other hand, the two symmetric critical points $x = \pm\sqrt{\mu}$ exist only for $\mu > 0$.

Assume that $\mu > 0$. If $x < -\sqrt{\mu}$, then $\mu - x^2 < 0$ (*Check this.*) and $x(\mu - x^2) > 0$. If $-\sqrt{\mu} < x < 0$, then $\mu - x^2 > 0$ and $x(\mu - x^2) < 0$. If $0 < x < \sqrt{\mu}$, we have $x^2 < \mu$, so $\mu - x^2 > 0$ and $x(\mu - x^2) > 0$. Finally, if $x > \sqrt{\mu}$, we see that $\mu - x^2 < 0$ and $x(\mu - x^2) < 0$. Thus, for $\mu > 0$ the phase portrait is shown in Fig. 2.17.

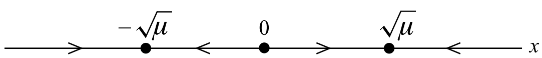


FIGURE 2.17

Phase portrait of $\dot{x} = \mu x - x^3$, $\mu > 0$

If $\mu \le 0$, there is only one critical point, $x = 0$. It is easy to see that the phase portrait is shown in Fig. 2.18.

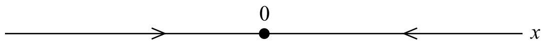


FIGURE 2.18

Phase portrait of $\dot{x} = \mu x - x^3$, $\mu \le 0$

In the next section, we will refine the phase line and phase portrait analyses.

Exercises 2.5

A

Draw phase portraits for each of the equations in Problems 1–13.

1. $\frac{dy}{dt} = y^2 - 1$
2. $y' = y^2(1-y)^2$
3. $x' = (x+1)(x-3)$
4. $\dot{x} = \cos x$
5. $y' = e^y - 1$
6. $y' = y(1-y)(2-y)$
7. $\dot{y} = \sin y$
8. $x' = 1 - \frac{x}{1+x}$
9. $y' = ye^{y-1}$
10. $\dot{y} = \sin y \cos y$
11. $x' = -3x(1-x)(3-x)$
12. $y' = y(y^2 - 4)$
13. $x' = x(1 - e^x)$
14. The equation in Problem 11 could represent a model of a population that can become extinct if it drops below a particular critical value. What is this critical value?
15. Consider the equation $\dot{x} = 1 + \frac{1}{2}\cos x$.
    - **a.** Draw the phase portrait of this equation.
    - **b.** What does your qualitative analysis tell you about solutions of this equation?

16. Consider the equation $x' = x^2 - ax$ of Example 2.5.2. For the cases $a > 0$, $a < 0$, and $a = 0$, sketch solution curves that illustrate Fig. 2.14, Fig. 2.15, and Fig. 2.16, respectively.

B

17. Using one of *Kirchhoff's laws* in physics, we find that the current, I, flowing in a particular electric circuit satisfies the equation $0.5\frac{dI}{dt} + 10I = 12$. (The resistance is 10 ohms, the inductance is 0.5 henry, and there is a 12 volt battery.)

a. Sketch the phase portrait of the equation.

b. If the initial current, $I(0)$, is 3 amps, use part (a) to describe the behavior of I for large values of t.

18. Example 2.1.8 and Problem 21 of Exercises 2.4 indicated that a type of chemical reaction can be modeled by the equation $\frac{dx}{dt} = k(\alpha - x)(\beta - x)$.

a. If $\alpha = 250$, $\beta = 40$, and k is a positive constant, produce the phase portrait of the equation.

b. If $x(0) = 0$, how does x behave as $t \to \infty$?

19. Consider the equation $y' = y^3 - 4y$.

a. If $y(0) = -1$, what happens to $y(t)$ as t increases?

b. Describe the behavior of $y(t)$ as $t \to \infty$ if $y(0) = \pi$.

20. Consider the equation $\frac{dy}{dt} = (1 + y)^2$.

a. What happens to solutions with initial conditions $y(0) > -1$ as t increases?

b. Describe the behavior of solutions with initial conditions $y(0) < -1$ as t increases.

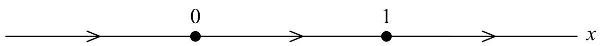
21. Consider the equation $\frac{dP}{dt} = \left(1 - \frac{P}{15}\right)^3 \left(\frac{P}{7} - 1\right) P^5$, with $P(0) = 3$.

a. Use the phase portrait for this equation to give a rough sketch of the solution $P(t)$.

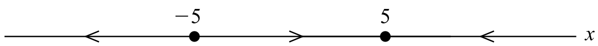
b. What happens to $P(t)$ as t becomes very large?

22. For each of the phase portraits shown below, write down a corresponding first-order equation of the form $x' = f(x)$.

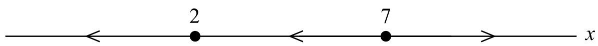
a.



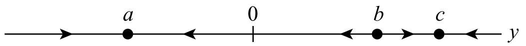
b.



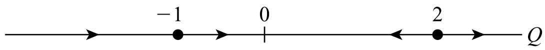
c.



23. Given the following phase portrait for $\frac{dy}{dt} = f(y)$, make a rough sketch of the graph of $f(y)$ assuming that $y = 0$ is in the center of the phase line.



24. Given the following phase portrait, find a first-order ODE that is consistent with this phase portrait.



C

25. Consider the equation $x' = (2\alpha - 1)x + 1$, where α is a parameter. Describe the behavior of solutions $x(t)$ of this equation as $t \to \infty$, noting how this behavior depends on the value of α.

26. Consider the equation $x' = (\alpha^2 - 1)x + 1 + \alpha$, where α is a parameter. Describe the behavior of solutions $x(t)$ of this equation as $t \to \infty$, noting how this behavior depends on the value of α.

27. The **Landau equation** arises in the analysis of the dynamics of fluid flow. It is $\frac{dx}{dt} = ax - bx^3$, where a and b are positive real constants.

a. Draw the phase portrait of Landau's equation.

b. What happens to x as t increases if $x(0) = \sqrt{\frac{a}{b}} + \varepsilon$, where ε is a small positive quantity?

c. What happens to x as t increases if $x(0) = 0$?

d. How does x behave as t increases if $x(0) = \sqrt{\frac{a}{b}} - \varepsilon$?

28. If we consider Problem 7 in a different way, we can see the difference between a traditional treatment of a differential equation and a qualitative analysis.

a. Without using technology, solve the differential equation in Problem 7 with initial condition $y(0) = y_0$.

b. Suppose that $y_0 = \pi/2$. Using the solution found in part (a), can you describe what happens as $t \to \infty$?

c. For an arbitrary initial condition y_0, what is the behavior of $y(t)$ as $t \to \infty$?

d. Comment on the contrast between the insight obtained from the phase portrait (Problem 7) and the information obtained from the solution found in part (a).

2.6 Equilibrium points: sinks, sources, and nodes

Let's take another look at Fig. 2.13 and focus on the **critical points**—the places where $\frac{dP}{dt} = 0$. Geometrically, these are the horizontal lines $P = 0$ and $P = 1$, which represent the functions $P(t) \equiv 0$ and $P(t) \equiv 1$, constant solutions of the differential equation $\frac{dP}{dt} = P(1-P)$. These values of P are called **equilibrium points** or **stationary points** of the autonomous differential equation. We also say that $P(t) \equiv 0$ and $P(t) \equiv 1$ are **equilibrium solutions** or **stationary solutions** of the equation. Assuming that the solutions of an autonomous differential equation describe some physical, economic, or biological system, we can conclude that if the system actually reaches an equilibrium point P, it must always have been at P—and will always remain at P. (*Think about this. You have* $\frac{dP}{dt} = 0$ *at an equilibrium point*.)

We can go further in this analysis and classify equilibrium points for autonomous first-order differential equations. There are three basic kinds of equilibrium points: **sinks**, **sources**, and **nodes**.

If we look at the equilibrium point $P = 1$ in Example 2.5.1, we see from Fig. 2.13 that the solution curves near the line $P = 1$ seem to swarm into (or converge to) the horizontal line. We call $P = 1$ a **sink**. Sinks are also called **attractors** or **asymptotically stable solutions**. A little more accurately, an equilibrium solution $P \equiv k$ is a **sink** if solutions with initial conditions sufficiently close to $P \equiv k$ are asymptotic to $P \equiv k$ as $t \to \infty$. This idea of being "sufficiently close" can be made mathematically precise: Suppose x^* is an equilibrium solution of the autonomous equation $\dot{x} = f(x)$. Then x^* is called **asymptotically stable** if there exists a value for $\delta > 0$ such that if a solution $x = \varphi(t)$ satisfies $|\varphi(t_0) - x^*| < \delta$, then $\lim_{t\to\infty} \varphi(t) = x^*$. However, we will just consider the situation intuitively. An asymptotically stable equilibrium solution has long-term behavior that is insensitive to slight (or sometimes large) variations in its initial condition. The term *sink* is meant to suggest the drain of a bathroom or kitchen sink: Along the sides, water that is close enough will flow into the drain.

On the other hand, we see different behavior near $P = 0$. As we look along solution curves from left to right, they seem to be moving *away* from the line $P = 0$. The equilibrium point $P = 0$ is called a **source**. In other words, an equilibrium solution $P \equiv k$ is a **source** if solutions with initial conditions sufficiently close to $P \equiv k$ are asymptotic to $P \equiv k$ as $t \to -\infty$ (i.e., as we go backward in time). A source is also called a **repeller** or an **unstable equilibrium solution**. Such a solution is extremely sensitive to even the slightest variations in its initial condition. Here, we can think of a faucet discharging water or a hand-held hair dryer putting out streams of hot air.

If we have another equilibrium point such that nearby solutions show any other kind of behavior—perhaps somewhat like a sink and somewhat like a source at the same time—we call that equilibrium point a **node**. (See Example 2.6.2 and Fig. 2.16 for instance.) More technically, we can refer to a node as a **semistable equilibrium solution**.

2.6.1 A test for equilibrium points

We can test equilibrium points/solutions for autonomous first-order equations by using a criterion that is reminiscent of a basic calculus result for determining maximum and minimum values of a function.

The First Derivative Test

Suppose that f and f' are continuous. If x^* is an equilibrium point for the autonomous equation $\frac{dx}{dt} = f(x)$, it is true that (1) if $f'\left(x^*\right) > 0$, then x^* is a source; (2) if $f'\left(x^*\right) < 0$, then x^* is a sink; and (3) if $f'(x^*) = 0$, the test fails—that is, we can't tell what sort of equilibrium point x^* may be without further investigation.

In Example 2.5.1, we saw that $P = 0$ and $P = 1$ were equilibrium points. Because $f(P) = P(1 - P)$, we have $f'(P) = 1 - 2P$, so $f'(0) = 1 > 0$ indicates that $P = 0$ is a *source* and $f'(1) = -1 < 0$ shows that $P = 1$ is a *sink*.

We can understand why the First Derivative Test works by using the concept of *local linearity*: Near an equilibrium solution x^*, we can approximate $f(x)$ by the equation of its *tangent line* at x^*. (See Section A.1 if necessary.) Therefore, if x is close enough to x^*, we can write

$$\frac{dx}{dt} = f(x) \approx f(x^*) + f'(x^*)(x - x^*) = f'(x^*)(x - x^*)$$

because $f(x^*) = 0$ when x^* is an equilibrium solution. Note that if we form the approximate differential equation $\frac{dx}{dt} \approx f'(x^*)(x - x^*)$, we can separate the variables and integrate to see that

$$|x - x^*| \approx K e^{f'(x^*)t},$$

where K is a positive constant. Thus, if $f'(x^*) > 0$ the distance between x^* and x increases exponentially as $t \to \infty$, whereas if $f'(x^*) < 0$ the difference $|x - x^*|$ decays.

We can also use Table 2.3 to compare the signs of $f'(x^*)$ and $(x - x^*)$.

Table 2.3 Signs of $\frac{dx}{dt}$

$(x - x^*)$	$f'(x^*)$	$\frac{dx}{dt}$
$+$	$+$	$+$
$+$	$-$	$-$
$-$	$+$	$-$
$-$	$-$	$+$

The first row of signs in Table 2.3, for example, tells us that if $(x - x^*)$ is positive, such that a solution x is slightly *above* the equilibrium solution, and $f'(x^*) > 0$, then $\frac{dx}{dt} > 0$, which means that the solution x is moving *away* from x^*. This last statement says that x^* must be a *source*. Similarly, the third row of signs indicates that if x starts out *below* x^* and $f'(x^*) > 0$, then x falls away from x^* as t increases, so x^* is a source. The remaining two rows describe a sink.

The next two examples show us the power and the limitations of the First Derivative Test.

Example 2.6.1 Using the First Derivative Test

If we examine the autonomous equation $\frac{dx}{dt} = x - x^3 = x(1 - x^2)$, we see that the equilibrium points are $x = 0$, $x = -1$, and $x = 1$. Can we determine what kinds of equilibrium points these are without any kind of graph?

Yes, we just apply the First Derivative Test. First of all, we have $\frac{dx}{dt} = f(x)$, where $f(x) = x - x^3$, so $f'(x) = 1 - 3x^2$. Because $f'(0) = 1 > 0$, we know that $x = 0$ is a *source*. The fact that $f'(-1) = -2 < 0$ tells us that $x = -1$ is a *sink*. Finally, because $f'(1) = -2 < 0$, we see that $x = 1$ is another *sink*.

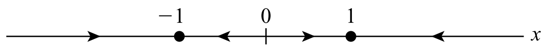


FIGURE 2.19

Phase portrait of $\frac{dx}{dt} = x - x^3$

The phase portrait shown in Fig. 2.19 reflects this information. Finally, the slope field (Fig. 2.20) confirms our analysis.

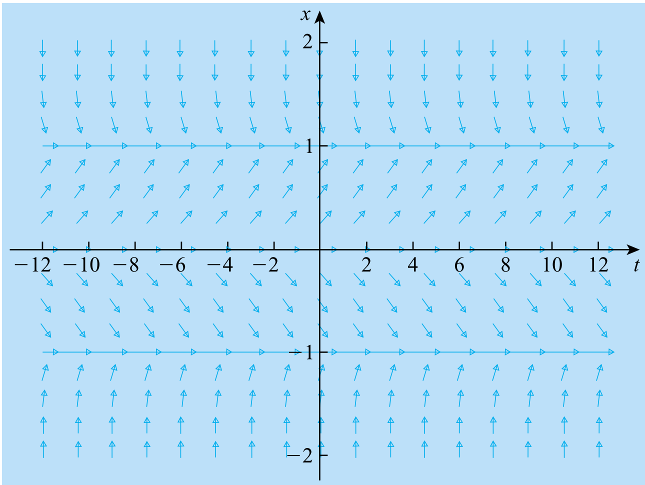


FIGURE 2.20

Slope field for $\frac{dx}{dt} = x - x^3$

Example 2.6.2 Failure of the First Derivative Test

Let's look at the first-order nonlinear equation $\frac{dx}{dt} = f(x) = (1-x)^2$. The only equilibrium solution is $x \equiv 1$, and we have $f'(x) = 2(1 - x)(-1) = 2(x - 1)$. Because $f'(1) = 0$, the First Derivative Test doesn't allow us to draw any conclusions. However, we can examine the behavior of $f'(x)$ *near* $x = 1$ to get an idea of what's going on.

We can see that $f'(x)$ is greater than zero for values of x greater than 1, so $x \equiv 1$ looks like a source, but values of x just below 1 give us *negative* values of the derivative, so $x \equiv 1$ looks like a sink. This ambivalent behavior leads us to conclude that $x \equiv 1$ is a *node*. Fig. 2.21 shows the phase portrait of this equation. Fig. 2.22 shows the slope field with some particular solutions superimposed.

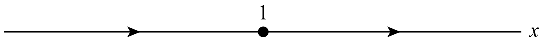


FIGURE 2.21

Phase portrait of $\frac{dx}{dt} = (1 - x)^2$

Notice that solution curves starting *above* the line $x = 1$ seem to flow *away* from the line $x = 1$, whereas those starting *below* the equilibrium solution flow *toward* the line $x = 1$. In other words, the point $x = 1$ is neither a sink nor a source. It is a *node*.

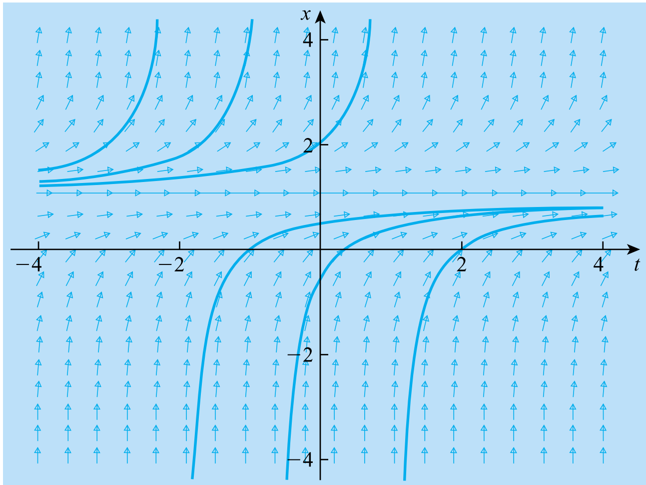


FIGURE 2.22

Solutions of $\frac{dx}{dt} = (1 - x)^2$: $x(0) = -\frac{1}{2}$, $\frac{1}{2}$, and 2

In the event that $f'(x^*) = 0$, where x^* is an equilibrium solution of an equation $\frac{dx}{dt} = f(x)$, we can sometimes determine the nature of the equilibrium by using these criteria:

a. The equation $\dot{x} = f(x)$ has a *sink* at $x = x^*$ if $f(x)$ changes sign from positive to negative at $x = x^*$.

b. The equation $\dot{x} = f(x)$ has a *source* at $x = x^*$ if $f(x)$ changes sign from negative to positive at $x = x^*$.

Note that in Example 2.6.2 the function $f(x)$ does not change sign at $x = 1$.

Example 2.6.3 Another Example of the First Derivative Test

Let's find and classify the equilibrium solutions of $\dot{y} = \frac{1}{2}y(y-2)^2(y-4)$.

It is easy to see that the equilibrium solutions are $y = 0$, $y = 2$, and $y = 4$, and we have (after some simplification) $f'(y) = 2(y-2)(y^2 - 4y + 2)$. Now we apply the First Derivative Test to each of the equilibrium points.

We see that $f'(0) = -8 < 0$, so we conclude that $y = 0$ is a *sink*. Then $f'(4) = 8 > 0$, so $y = 4$ is a *source*. We've saved the solution $y = 2$ for last because the First Derivative Test fails: $f'(2) = 0$. However, we can use criteria a and b above to examine changes in sign of $f(y)$ near $x = 2$.

For values of y a little less than 2 (try $y = 1$, for example), we have $f'(y) > 0$ (source-like behavior), while for values of y slightly greater than $y = 2$, we see that $f'(y) < 0$ (sink-like behavior). We thus conclude that $y = 2$ is a *node*, and we can construct the phase portrait (Fig. 2.23).

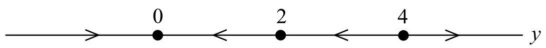


FIGURE 2.23

Phase portrait of $\dot{y} = \frac{1}{2}y(y-2)^2(y-4)$

In analyzing an autonomous first-order differential equation $\frac{dx}{dt} = f(x)$, it is useful to sketch the phase line using the graph of $f(x)$. First of all, equilibrium points are the zeros of f. Furthermore, regions where f is positive and regions where f is negative correspond to parts of the phase line where the arrows point to the right and to the left, respectively.

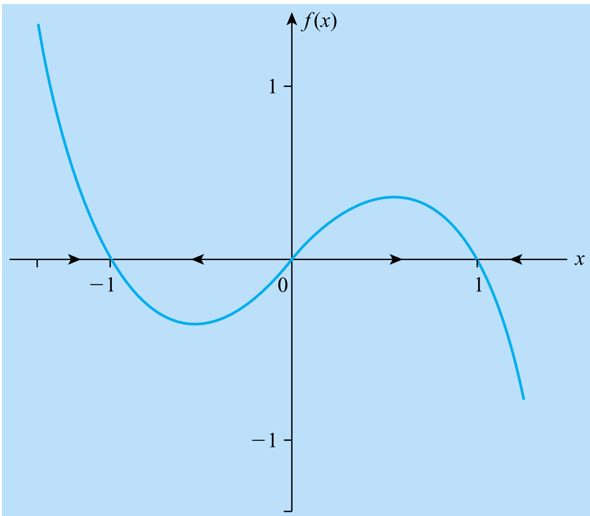


FIGURE 2.24

$f(x) = x - x^3$ compared to the phase line of $\frac{dx}{dt} = x - x^3$

Example 2.6.4 Using the Graph of $f(x)$ to Sketch the Phase Portrait

Let's return to the equation in Example 2.6.1, $\frac{dx}{dt} = x - x^3$, this time focusing on what the graph of $f(x) = x - x^3$ reveals. Fig. 2.24 shows the graph of $f(x)$ aligned with the phase portrait of the differential equation.

Note the equivalence between equilibrium points and the zeros of f and the correspondence between regions of positivity and negativity for $f(x)$ and the directions of the arrows on the phase line.

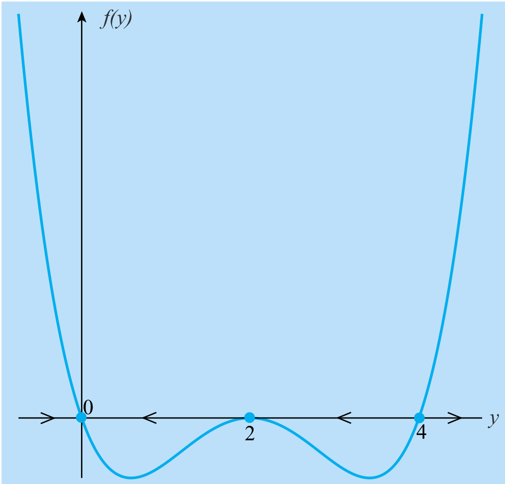


FIGURE 2.25

$f(y) = \frac{1}{2}y(y-2)^2(y-4)$ compared to the phase line of $\dot{y} = \frac{1}{2}y(y-2)^2(y-4)$

Similarly, we can show the graph of $f(y)$ from Example 2.6.3 aligned with the phase portrait (Fig. 2.25) of the differential equation.

Equilibrium solutions and their nature will be particularly useful when we discuss qualitative aspects of systems of linear and nonlinear equations in Chapters 6 and 7.

Exercises 2.6

A

For Problems 1–15, find the equilibrium point(s) of each equation and classify it as **sinks**, **sources**, or **nodes**.

1. $y' = y^2(1-y)^2$
2. $\dot{x} = \cos x$
3. $y' = e^y - 1$
4. $y' = y^2(y^2 - 1)$
5. $\dot{x} = ax + bx^2$, $a > 0$, $b > 0$

6. $\dot{x} = x^3 - 1$

7. $\dot{x} = x^2 - x^3$

8. $\dot{y} = 10 + 3y - y^2$

9. $\dot{x} = x(2 - x)(4 - x)$

10. $\dot{x} = -x^3$

11. $\dot{x} = x^3$

12. $\dot{y} = y \ln(y + 2)$

13. $\dot{x} = x - \cos x$ [*Hint*: Either use technology to find the equilibrium point explicitly (via a CAS **solve** command) or graph $y = x$ and $y = \cos x$ together on the same set of axes to estimate the graphs' point of intersection.]

14. $\dot{x} = x - e^{-x}$ (Use technology—see the preceding exercise.)

15. $x' = x(x + 1)(x - 0.5)^6$

16. A lake has two rivers flowing into it, one discharging a certain amount of water containing a concentration of pollutant and the other discharging an amount of clean water per day. Assuming that the lake volume is constant, the total amount of pollution in the lake, $Q(t)$, can be modeled by the balance equation $\frac{dQ}{dt} = D(Q^* - Q)$, where D is a positive constant involving the two rates of flow into the lake and the lake's volume and Q^* is a positive constant involving volume, rates of flow, and the pollutant concentration.

a. What is the equilibrium solution of this equation?

b. Is the solution found in part (a) stable or unstable? (For example, a *sink* would indicate that the clean river input *reduces* the long-term amount of pollution in the lake.)

17. Give an example of an autonomous differential equation having *no* real-valued equilibrium solution.

18. Give an example of an autonomous differential equation having exactly n equilibrium solutions ($n \geq 1$).

B

19. The following equation has been proposed for determining the speed of a rowboat[5]: $M\frac{du}{dt} = \frac{8P}{u} - bSu^2$, where $u(t)$ denotes the speed of the boat at time t; M its mass; and P, S, and b are positive constants describing various other aspects of the boat and the person rowing it.

a. Determine the equilibrium speed of the boat.

b. Determine whether the speed found in part (a) is a sink or a source.

c. Interpret the result of part (b) physically.

20. Given $\frac{dx}{dt} = f(x)$ and the following graph of $f(x)$,

a. Sketch the phase portrait of the equation.

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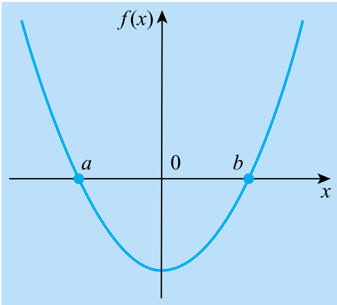
[5] M. Mesterton-Gibbons, *A Concrete Approach to Mathematical Modelling* (New York: John Wiley & Sons, 1995): 32–34; 53–56; 130–132.

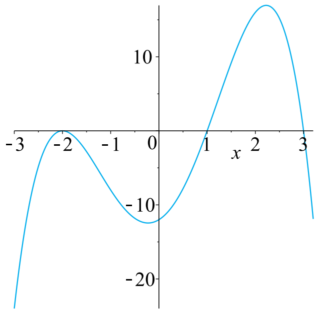
b. Identify all equilibrium points and classify each as a sink, a source, or a node.



21. Given $\frac{dx}{dt} = f(x)$ and the following graph of $f(x)$,

a. Sketch the phase portrait of the equation.

b. Identify all equilibrium points and classify each as a sink, a source, or a node.



22. The velocity $v(t)$ of a skydiver falling to the ground is given by $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance. Assume $v(0) = 0$. Without solving the equation, determine the limit of $v(t)$ as $t \to \infty$. (This limit is called the *terminal velocity*.)

23. A population growth model that is fairly simple yet amazingly accurate in predicting tumor growth is described by the **Gompertz equation**, $\frac{dN}{dt} = -aN \ln(bN)$, where $N(t) > 0$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters that are determined experimentally. [Benjamin Gompertz (1779–1865) was an English mathematician and actuary.]

a. Sketch the phase portrait for this equation.

b. Sketch the graph of $f(N)$ against N.

c. Find and classify all equilibrium points for this equation.

d. For $0 < N \leq 1$, determine where the graph of $N(t)$ against t is concave up and where it is concave down. (You may want to review Example 2.5.1.)

e. Sketch $N(t)$.

24. Find an equation $\dot{x} = f(x)$ with the property that there are exactly three equilibrium points and all of them are *sinks*.

C

25. A population of animals that follows a logistic growth pattern is harvested at a constant rate—that is, as long as the population size, P, is positive, a fixed number, h, of animals is removed per unit of time. The equation modeling the dynamics of this situation is

$$\frac{dP}{dt} = rP(1 - \frac{P}{k}) - h$$

for $P > 0$.

a. Show that if $h < \frac{rk}{4}$, there are two nonzero equilibrium solutions.

b. Show that the smaller of the equilibrium solutions in part (a) is a source, whereas the larger of the two is a sink.

26. Consider the equation $\dot{x} = -x^3 + (1 + \alpha)x^2 - \alpha x$, where α is a constant.

a. If $\alpha < 0$, find all equilibrium solutions of this equation and classify them.

b. If $0 < \alpha < 1$, find all equilibrium solutions and classify them.

c. If $\alpha > 1$, find all equilibrium solutions and classify them.

d. Describe the equilibrium solutions if $\alpha = 0$.

e. Describe the equilibrium solutions if $\alpha = 1$.

2.7 Bifurcations

2.7.1 Basic concepts

To get an idea of what this topic is all about, let's go back to elementary algebra and look at the quadratic function $f(x) = x^2 + x + c$, where c is a constant. We should realize that the zeros of this function depend on the parameter c. To see this, let's write

$$x^2 + x + c = \left(x + \frac{1}{2}\right)^2 + \left(c - \frac{1}{4}\right). \tag{2.7.1}$$

Clearly, the term $\left(x + \frac{1}{2}\right)^2$ is always nonnegative, so that if $c > \frac{1}{4}$ the expression (2.7.1) is always *strictly greater than zero*, and the quadratic equation $x^2 + x + c = 0$ has *no* real solutions. If $c = \frac{1}{4}$, then the equation has $x = -\frac{1}{2}$ as its *only* root, a repeated root. Finally, if $c < \frac{1}{4}$ we have *two* solutions, $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$. (*Verify all the assertions in this paragraph*). Fig. 2.26 shows the graph of $y = x^2 + x + c$ for three values of c.

The important point in this example is that $\frac{1}{4}$ is the value of the parameter c at which the nature of the solutions of the quadratic equation changes. We say that

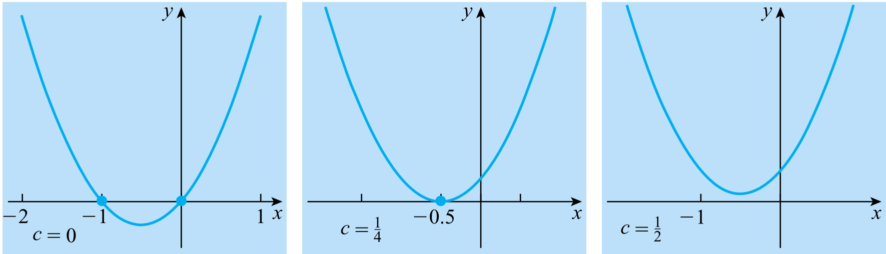


FIGURE 2.26

Graphs of $y = x^2 + x + c$

$c = \frac{1}{4}$ is a **bifurcation point** because as c decreases through $\frac{1}{4}$ the solution $x = 0$ splits into two solutions. (The word *bifurcation* refers to a splitting or branching.)

We can see the effect of the bifurcation most clearly by plotting the solution x against the parameter c in our example, showing the graph of the relationship $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ (Fig. 2.27). This graph showing the dependence of a solution on a parameter is called the **bifurcation diagram** for the equation $x^2 + x + c = 0$. Be sure you understand what this diagram tells you. Note, in particular, what happens as c passes through the value $\frac{1}{4}$.

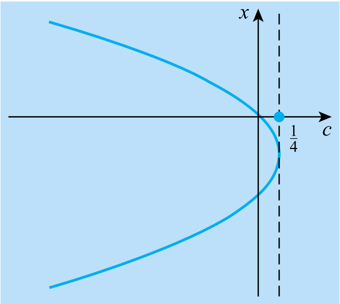


FIGURE 2.27

Bifurcation diagram for $x^2 + x + c = 0$

2.7.2 Application to differential equations

Most continuous dynamical systems involving differential equations contain parameters. In analyzing such a system, you may observe that a slight change in a parameter value can have a significant impact on the behavior of the system. A parameter, for example, may be a function of time: the mass of a rocket changes as it burns fuel, the load of an electricity power grid fluctuates during the day, and so forth. Bifurcation analysis has a large number of applications in science and engineering.

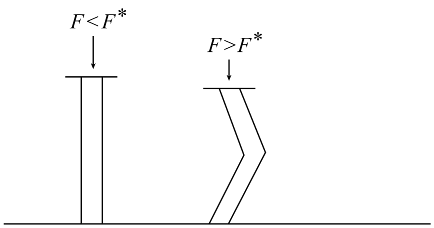
This sort of qualitative change caused by a change in parameter value is particularly interesting when we observe it in an autonomous differential equation. *What changes for such an ODE at a bifurcation point is the number and/or nature of the equilibrium solutions.*

Definition 2.7.1

Given a family of autonomous differential equations $\frac{dx}{dt} = f(x, \lambda)$ containing a parameter λ, a **bifurcation point** (or **bifurcation value**) is a value λ_0 of the parameter for which the qualitative nature of the equilibrium solutions changes as λ passes through λ_0. The actual change in the equilibrium solutions is called a **bifurcation**.

For a family of autonomous differential equations $\frac{dx}{dt} = f(x, \lambda)$, a **bifurcation diagram** is a diagram that summarizes the change in qualitative behavior of the equation as the parameter λ is varied. We construct the bifurcation diagram by plotting the parameter value λ against all corresponding equilibrium values y^*. Typically, λ is plotted on the horizontal axis and y^* on the vertical axis. A "curve" of sinks is indicated by a solid line and a curve of sources is indicated by a dashed line. The examples in this section should clarify the process.

The real significance of bifurcations was first revealed by Leonhard Euler (1707–83), the great Swiss mathematician who has been called "the Shakespeare of mathematics," in his work published in 1744 on the buckling of an elastic straight beam or a column under a compressive force. The normal upright position represents an equilibrium position. The parameter here is the force F exerted on the top of the column. For certain values of F, say $F < F^*$, the column maintains its vertical position, but if the force is too great, say $F > F^*$, the vertical equilibrium position becomes unstable, and the column may buckle. The critical force F^* is the bifurcation point. The equilibrium situation changes as the size of the force passes through the value F^*.



There are many types of bifurcations (some of whose names haven't been standardized yet), but the examples discussed in this section will introduce us to three common types for autonomous first-order equations:

1. **Transcritical Bifurcation**: In this scenario, there will be an equilibrium solution for all values of the parameter and such a solution never disappears. Both

before and after the bifurcation point, there will be an equilibrium solution—one stable and one unstable—and their stability is *exchanged* from stable to unstable or vice versa as the parameter passes through the bifurcation point.

2. **Saddle-Node Bifurcation**: In this situation, two equilibrium solutions collide and annihilate each other. One of the solutions is unstable (the *saddle*), while the other (the *node*) is stable.
3. **Pitchfork Bifurcation**: This kind of bifurcation is characterized by the system transitioning from one equilibrium solution to three equilibrium solutions.

These descriptions will make sense once we have studied the following examples. The first example reveals the bifurcation point for a simple first-order equation that we discussed earlier.

Example 2.7.1 A Transcritical Bifurcation

The equation $\frac{dy}{dt} = ay$ expresses the fact that at any time t, some quantity y grows at a rate proportional to its size at time t. The parameter a is the constant of proportionality that captures some growth characteristic of the quantity.

Setting $\frac{dy}{dt} = 0$, we find that the equilibrium solutions are described by $ay = 0$. If $a = 0$, then *every* value of y is an equilibrium point. If $a \neq 0$, then $y \equiv 0$ is the only equilibrium point. For the equation $\frac{dy}{dt} = ay = f(y)$, we have $f'(y) \equiv a$; we use the First Derivative Test from Section 2.6 to conclude that if $a > 0$, then $y \equiv 0$ is a *source*, and if $a < 0$, then $y \equiv 0$ is a *sink*. Clearly, $a = 0$ is a bifurcation point because the number and nature of the equilibrium solutions change as a passes through 0. Fig. 2.28 shows graphs of $f(y)$ against y for the three possibilities for a and the corresponding phase portraits.

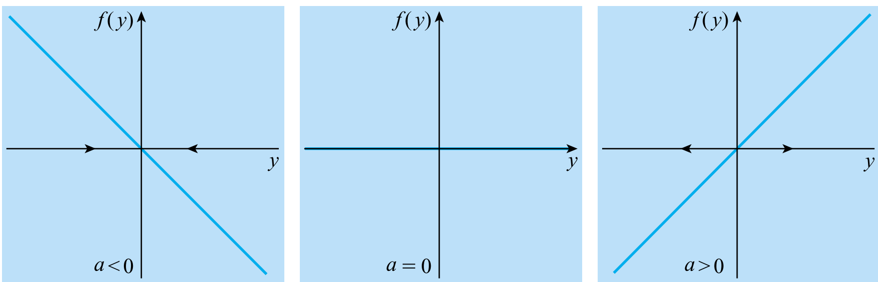


FIGURE 2.28

$\frac{dy}{dx} = f(y) = ay$ vs. y

Note how the conditions for a transcritical bifurcation are satisfied. Equilibrium solutions exist for all values of a and the stability of the equilibrium solution $y \equiv 0$ changes when the parameter a crosses the origin.

We can show the dependence of the equilibrium points on a by drawing a bifurcation diagram, plotting $y(t)$ against a (Fig. 2.29). The y-axis itself represents all the solutions $y = C$, where C is any constant for $a = 0$. It is usual in bifurcation diagrams to use solid curves to indicate stable equilibrium solutions (sinks) and dashed lines to denote unstable solutions (sources). Arrows indicate the directions of change of some solutions with time.

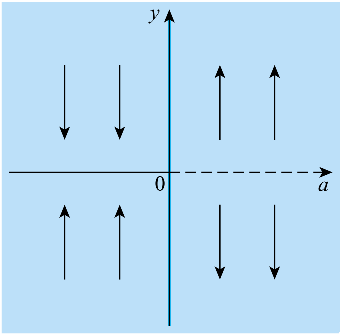


FIGURE 2.29

Bifurcation diagram for $\frac{dy}{dt} = ay$

Example 2.7.2 A Pitchfork Bifurcation

Let's look at the first-order nonlinear equation $\frac{dy}{dx} = \alpha y - y^3 = f(y)$. This is the **Landau equation**, a variant of which appeared in Problem 27 of Exercises 2.5; it arises in the study of one-dimensional patterns in fluid systems. Here $y = y(t)$ gives the amplitude of the patterns, and α is a small, dimensionless parameter that measures the distance from the bifurcation. [L.D. Landau (1908–1968) was a Russian physicist who won the Nobel Prize in 1962.]

We see that $\frac{dy}{dx} = 0$ implies that $\alpha y - y^3 = y(\alpha - y^2) = 0$, so $y = 0$, $y = \sqrt{\alpha}$, and $y = -\sqrt{\alpha}$ are the only equilibrium points. Looking at these points, we can see (because of the radical sign) that we have three cases to consider: (1) $\alpha = 0$, (2) $\alpha > 0$, and (3) $\alpha < 0$.

When $\alpha = 0$, there is the single equilibrium point $y = 0$. Then we have $f'(y) = \alpha - 3y^2 = -3y^2$, so $f'(0) = 0$ and we can't determine the nature of the equilibrium solution $y = 0$ from the First Derivative Test. However, we can see that when $\alpha = 0$ the differential equation is $\frac{dy}{dx} = -y^3$, a separable equation whose solution tends to zero as x becomes infinite in the positive direction. (*Solve the equation to see for yourself.*) Thus, $y = 0$ is a *sink*.

If α is less than zero, then $y = 0$ is the only equilibrium point because $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ are imaginary numbers. For this case, we see that $f'(0) = \alpha < 0$, so $y = 0$ is a *sink*.

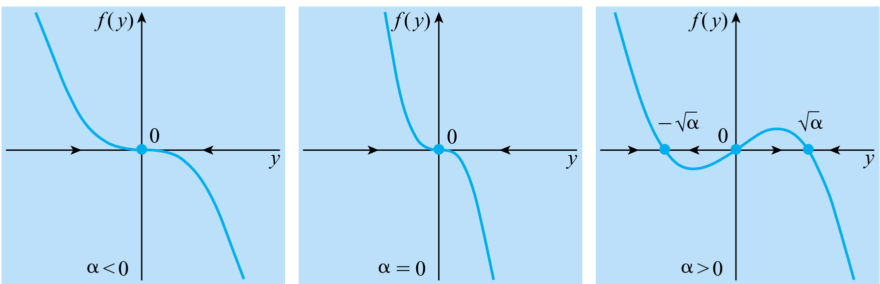


FIGURE 2.30

$f(y) = \alpha y - y^3$ vs. y

However, if α is greater than zero, then the equation has three distinct equilibrium points: $y = 0$, $y = \sqrt{\alpha}$, and $y = -\sqrt{\alpha}$. We see that $f'(0) = \alpha > 0$, so $y = 0$ is a *source*; $f'(\sqrt{\alpha}) = -2\alpha < 0$, so $y = \sqrt{\alpha}$ is a *sink*; and $f'(-\sqrt{\alpha}) = -2\alpha < 0$, so $y = -\sqrt{\alpha}$ is also a *sink*. This behavior clearly describes a pitchfork bifurcation, based on the characterization given above.

Note how the value of α determines the number and the nature of the equilibrium solutions of our equation. Clearly, $\alpha = 0$ is the only bifurcation point for our original equation. Fig. 2.30 shows two representations for each set of values of α: graphs of $f(y)$ against y for the three descriptions of α considered above and corresponding phase portraits.

We can show this dependence of the equilibrium points on α also by means of a bifurcation diagram, in which we plot y against α (Fig. 2.31). This diagram shows that we have a pitchfork bifurcation in an obvious geometric way.

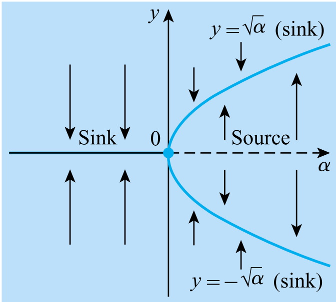


FIGURE 2.31

Bifurcation diagram for $\frac{dy}{dx} = \alpha y - y^3$

A **laser**—the word stands for **l**ight **a**mplification by **s**timulated **e**mission of **r**adiation—is a marvelous device that produces a beam of intense, concentrated pure light that can be used to cut diamonds, destroy cancerous cells, perform eye surgery, and enhance telecommunications when used in fiber optics. Basically, an external energy source is used to excite atoms and produce photons (light particles) that have the same frequency and phase. A. Schawlow and C. Townes received a patent for the invention of the laser in 1960, and the first laser was built by the American physicist T.H. Maiman in the same year. The mathematical model of a laser that follows is an important scientific example that illustrates another type of bifurcation. It is more complicated than the previous two examples because the bifurcation behavior depends on the values of *two* parameters.

Example 2.7.3 A Transcritical Bifurcation in a Laser Model

A simplified model of the basic physics behind a laser is given by the equation

$$\dot{n} = f(n) = Gn(N_0 - n) - kn = (GN_0 - k)n - Gn^2.$$

In this equation, $n = n(t)$ represents the number of photons at time t, N_0 is the (constant) number of “excited” atoms (in the absence of laser action), and G and k are positive parameters related to the gain and loss, respectively, of photons that have the same frequency and phase. We emphasize that we have *two* parameters in our equation, and we will see that our bifurcation analysis depends on the value of N_0 in relationship to them.

We can write the equation as $\dot{n} = n(GN_0 - k - Gn)$, so setting $\dot{n}$ equal to zero gives us $n \equiv 0$ or $GN_0 - k - Gn \equiv 0$. This tells us that the equilibrium solutions are $n \equiv 0$ and $n = (GN_0 - k)/G = N_0 - (k/G)$, where $N_0 \neq k/G$.

Looking at the first equilibrium solution, $n \equiv 0$, we see that this equilibrium solution is a sink when $N_0 < k/G$—that is, when $GN_0 - k < 0$. From the original equation, we have $f(n) = (GN_0 - k)n - Gn^2$, so $f'(n) = (GN_0 - k) - 2Gn$. Then $f'(0) = GN_0 - k < 0$, so $n \equiv 0$ is indeed a sink by the First Derivative Test. Physically, this means that there is no stimulated emission and no photons are produced that have the same frequency and phase. The laser device functions like a light bulb. Similarly, we can determine that $n \equiv 0$ is a source when $N_0 > k/G$.

Focusing on the second equilibrium point, $n = N_0 - (k/G)$, where $N_0 \neq k/G$, we see that

$$f'\left(N_0 - \frac{k}{G}\right) = (GN_0 - k) - 2G\left(N_0 - \frac{k}{G}\right) = -GN_0 + k.$$

If $N_0 < k/G$, then $-GN_0 + k > 0$ and therefore $n = N_0 - (k/G)$ is a source. If $N_0 > k/G$, then $-GN_0 + k < 0$ and therefore $n = N_0 - (k/G)$ is a sink. The physical interpretation of this last fact is that the external energy source has excited the atoms enough so that some atoms produce photons that have the same frequency and phase. The device is now producing coherent light.

Finally, if $N_0 = k/G$, then our original equation reduces to $\dot{n} = -Gn^2$, so we get only one equilibrium solution, $n \equiv 0$, which is a sink if we consider only positive values of n. Because of this change in the nature and number of equilibrium solutions, we can interpret $N_0 = k/G$ as our bifurcation point (called the *laser threshold*). The bifurcation diagram (Fig. 2.32) summarizes this model.

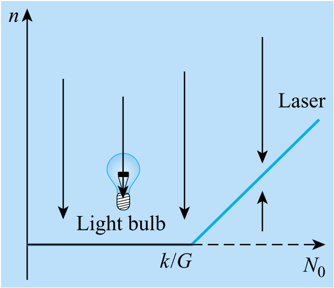


FIGURE 2.32

Bifurcation diagram for the laser model[6]

The physical interpretation is that when the amount of energy supplied to the laser exceeds a certain threshold—that is, when $N_0 > k/G$—the light bulb turns into a laser. Notice that at the bifurcation value $N_0 = k/G$ the two equilibrium solutions merge, and when they split apart, they have interchanged stability. For $N_0 > k/G$ the equilibrium solution $n \equiv 0$ becomes unstable by transferring its stability to another equilibrium solution, $n = N_0 - (k/G)$, the straight line with a slope of 1 in Fig. 2.32. This is the defining characteristic of a transcritical bifurcation.

[6] Adapted from Fig. 3.3.3 in S.H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Second Edition) (Boulder, CO: Westview Press, 2015): 56.

The next example demonstrates the behavior of a saddle-node bifurcation, and the analysis resembles that of the quadratic function at the beginning of Section 2.7.1.

Example 2.7.4 A Saddle-Node Bifurcation

Consider the nonlinear autonomous differential equation $\dot{x} = r + x^2$, where r is a real parameter. The equilibrium solutions satisfy the equation $r + x^2 = 0$, so we have $x = \pm\sqrt{-r}$.

If $r < 0$, then $-r > 0$ and there are *two* equilibrium solutions, $x = \sqrt{-r}$ and $x = -\sqrt{-r}$. If $r = 0$, we have $\dot{x} = x^2$, so that $x = 0$ is the only equilibrium solution. With $f(x) = x^2$, we see that $f'(0) = 2(0) = 0$, so the First Derivative Test fails to reveal the nature of this equilibrium point. However, for values of x just below 0, we see that $f'(x) < 0$, suggesting that $x = 0$ could be a sink, but for values of x above 0, $f'(x) > 0$, so that $x = 0$ looks like a source. We conclude that $x = 0$ is a *node*.

If $r > 0$, then $-r < 0$, and there are *no* equilibrium solutions.

Clearly $r = 0$ is a bifurcation point because the number and nature of the equilibrium solutions change as r passes through zero.

Fig. 2.33 shows the graphs of $f(x)$ against x and the corresponding phase portraits.

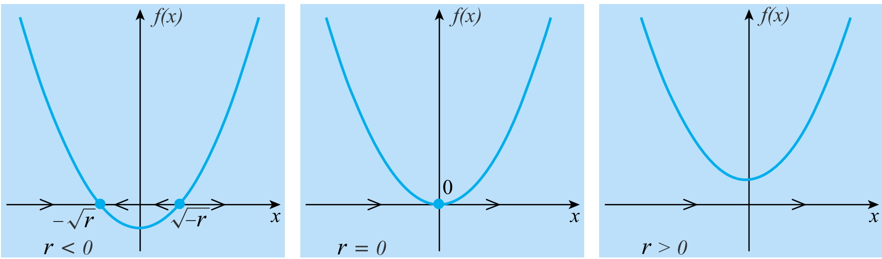


FIGURE 2.33

$f(x) = r + x^2$ vs. x

What happens here is that as r passes from less than zero to greater than zero, the two equilibrium solutions $-\sqrt{-r}$ and $\sqrt{-r}$ collide and cancel each other out. (Picture the parabola rising rigidly, forcing the x-intercepts to fuse into one point, the origin.) As the parabola continues to rise, it eventually avoids any contact with the x-axis—that is, there are no longer any equilibrium points.

The bifurcation diagram is shown in Fig. 2.34.

The bifurcation examples we have seen so far have involved a change in stability in equilibrium solutions or a disappearance of solutions as the parameter crosses a bifurcation point. Stranger behavior is possible.

Example 2.7.5 Bistability

Let's look at the equation $\dot{x} = f(x) = \alpha x + x^3 - x^5$, where α is a real parameter. First we find the equilibrium solutions and determine their stability.

The solutions of $\alpha x + x^3 - x^5 = x(\alpha + x^2 - x^4) = 0$ are $x = 0$ and x equal to the solutions of the polynomial equation $\alpha + x^2 - x^4 = 0$. Letting $y = x^2$, we determine the solutions of $y^2 - y - \alpha = 0$, which are $y_1 = (1 + \sqrt{1 + 4\alpha})/2$ and $y_2 = (1 - \sqrt{1 + 4\alpha})/2$, provided that $1 + 4\alpha \geq 0$—that is, $\alpha \geq -1/4$. Since we have $y = x^2$, we can only use y_1 and y_2 when they are nonnegative.

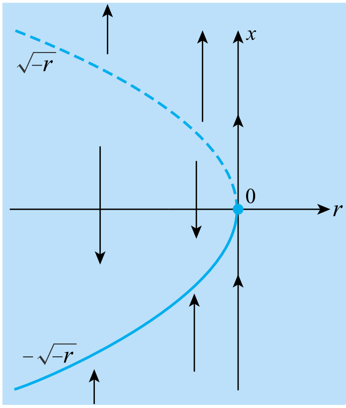


FIGURE 2.34

Bifurcation diagram for $\dot{x} = r + x^2$

Now $y_1 = (1 + \sqrt{1 + 4\alpha})/2$ is always positive when $\alpha \geq -1/4$. But $y_2 = (1 - \sqrt{1 + 4\alpha})/2$ is nonnegative only when $1 \geq \sqrt{1 + 4\alpha}$—that is, when $-1/4 \leq \alpha \leq 0$.

In summary, when $\alpha < -1/4$, there is only one equilibrium solution: $x = 0$. Since $\frac{d}{dx} f(x) = \alpha + 3x^2 - 5x^4 = \alpha < 0$ at $x = 0$, the First Derivative Test shows us that $x = 0$ is a *sink*. When $-1/4 \leq \alpha \leq 0$, there are *five* equilibrium solutions. The solution $x = 0$ is valid and is still a sink; and the four other equilibrium solutions are $\pm\sqrt{y_1}$ and $\pm\sqrt{y_2}$. The First Derivative Test shows that the two equilibrium solutions $\pm\sqrt{(1 - \sqrt{1 + 4\alpha})/2}$ are *sources*, while $\pm\sqrt{(1 + \sqrt{1 + 4\alpha})/2}$ are *sinks*. (*Verify that* $f'\left(\pm\sqrt{(1 - \sqrt{1 + 4\alpha})/2}\right) = -4\alpha - 1 + \sqrt{1 + 4\alpha} > 0$ for $-1/4 < \alpha < 0$ and $f'\left(\pm\sqrt{(1 + \sqrt{1 + 4\alpha})/2}\right) = -4\alpha - 1 - \sqrt{1 + 4\alpha} < 0$ for $-1/4 < \alpha < 0$.)

For $\alpha > 0$, only three equilibrium solutions remain because the solutions $\pm\sqrt{(1 - \sqrt{1 + 4\alpha})/2})$ no longer exist: $\alpha > 0$ implies that $1 + 4\alpha > 1$, giving us $1 - \sqrt{1 + 4\alpha} < 0$, so that we have a square root of a negative number. The equilibrium solution $x = 0$ is now a *source* since $\alpha > 0$, while the solutions $\pm\sqrt{(1 + \sqrt{1 + 4\alpha})/2}$ are still sinks.

In the interval $[-1/4, 0]$, several stable solutions (sinks) co-exist. We say that there is **bistability** between the solutions. The choice between $x = 0$ and one or the other of the symmetric stable solutions (sinks) depends on the history of the system the differential equation describes, for example initial conditions. The bifurcation diagram for this example is given in Fig. 2.35.

Let's step back and examine a modified version (Fig. 2.36) of the last example's bifurcation diagram of $\dot{x} = \alpha x + x^3 - x^5$. We have seen that for $\alpha < -1/4$ the equilibrium solution $x = 0$ is stable (a sink). The system will remain in this state as α moves slowly toward 0. As α is increased through 0, the system loses its stability by jumping to one of the stable equilibrium branches near ± 1. Which one of these symmetric states occurs is not determined by the system, but will depend on external asymmetries such as noise or imperfections in the system. As α is increased further the equilibrium solution remains on the same stable equilibrium branch. However, if

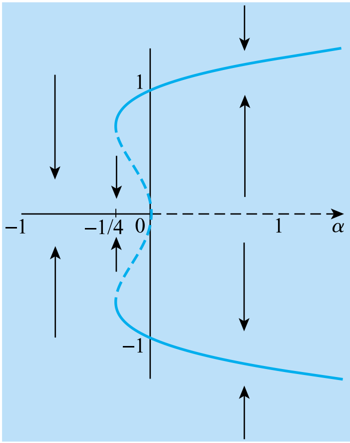


FIGURE 2.35

$f(x) = r + x^2$ vs. x

α is decreased from large values the solution will not jump back to 0 at $\alpha = 0$. Instead, it will remain on the branch until $\alpha = -1/4$, at which point the system jumps back to $x = 0$.

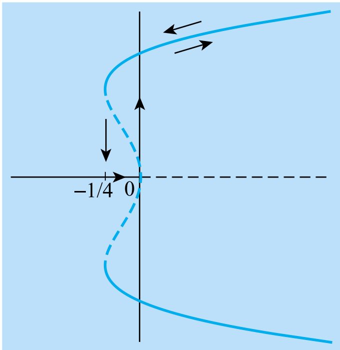


FIGURE 2.36

$f(x) = r + x^2$ vs. x

This phenomenon in the last example, involving a jump and irreversible behavior, is referred to as **hysteresis** in various applications. The arrows in Fig. 2.36 indicate a *hysteresis loop*. The term was introduced by the physicist J.A. Ewing in 1881 and refers to a phenomenon in which a temporary change in one factor causes a permanent change in another. Applications exhibiting hysteresis (from the Greek word for "remaining") have traditionally been found in physics—for example, ferromagnetism—but this word also conveys the idea that the choice of the stable equilibrium solution by a system depends on the *past history* of the system. When an external magnetic field is applied to a ferromagnetic material such as iron the material is magnetized. But even when the imposed magnetizing field is removed, part of the magnetization will be retained. This property of ferromagnetic materials is useful as a magnetic "memory," and hysteresis forms the basis for the element of memory in a tape recorder, a computer hard disk drive, and a credit card.

In recent years the effects of hysteresis have been studied in various areas ranging from physics to biology, from material science to mechanics, and from electronics to economics. (For a very readable (non-mathematical) discussion of hysteresis, see Section 6.4 of *Six Sources of Collapse* by Charles R. Hadlock (Washington, D.C.: Mathematical Association of America, 2012).)

Exercises 2.7

A

For each of the following equations in Problems 1–6, (1) sketch all the qualitatively different graphs of $f(x)$ against x as the parameter c is varied; (2) determine the bifurcation point(s); and (3) sketch the bifurcation diagram of equilibrium solutions against c.

1. $\frac{dx}{dt} = x^2 - c$
2. $\frac{dx}{dt} = 1 + cx + x^2$
3. $\frac{dx}{dt} = x - cx(1 - x)$
4. $\frac{dx}{dt} = x^2 - 2x + c$
5. $\frac{dx}{dt} = x(x - c)$
6. $\frac{dx}{dt} = cx - x^2$

B

7. Consider the logistic equation (see Example 2.5.1) with a constant rate of harvesting (hunting, fishing, reaping, etc.) h : $\frac{dP}{dt} = P(5 - P) - h$. Is there a maximum harvest rate h^* beyond which the population will become extinct for every initial population $P_0 = P(0)$?
8. Construct the bifurcation diagram for the equation $x' = \alpha - e^{-x^2}$, where $\alpha > 0$.
9. Construct the bifurcation diagram for $\frac{dx}{dt} = x(c - x^2)$, where c is a parameter.
10. Construct the bifurcation diagram for $\frac{dx}{dt} = x(x^2 - 1 - \alpha)$, $-\infty < \alpha < \infty$.
11. Construct the bifurcation diagram for the equation $x' = 3x - x^3 - \alpha$, where α is a parameter.
12. Determine what kind of bifurcation the system represented by the equation $\dot{x} = (2 + \mu)x - 5x^2$ undergoes by finding the bifurcation value and drawing the bifurcation diagram, complete with arrows indicating the nature of the equilibrium solutions.

C

13. The **Landau equation** in the form $\dot{x} = (R - R_c)x - kx^3$, where k and R_c are positive constants and R is a parameter that may take on various values, is important in the field of fluid mechanics.

a. If $R < R_c$, show that there is only the equilibrium solution $x = 0$, and that it is a sink.

b. If $R > R_c$, show that there are three equilibrium solutions, $x = 0$, $x = \sqrt{(R - R_c)/k}$, and $x = \sqrt{(R - R_c)/k}$, and that the first solution is a source while the other two are sinks.

c. Sketch a graph in the R-x plane showing all equilibrium solutions and label each one as a sink or a source. How would you describe the bifurcation point $R = R_c$?

14. The following equation occurs in the study of *gene activation*:

$$\frac{dx}{dt} = \alpha - x + \frac{4x^2}{1 + x^2}.$$

Here $x(t)$ is the concentration of gene product at time t.

a. Sketch the phase portrait for $\alpha = 1$.

b. There is a small value of α, say α_0, where a bifurcation occurs. Estimate α_0 and sketch the phase portrait for some α in the open interval $(0, \alpha_0)$.

c. Draw the bifurcation diagram for this differential equation.

*2.8 Existence and uniqueness of solutions[7]

This is the time to acknowledge that we have been avoiding a very important question. When we're trying to solve a differential equation, how do we know that there *is* a solution? We could be looking for something that doesn't exist—a waste of time, effort, and (these days) computer resources. As strange as it may seem when we think about it, those who investigated differential equations, from Newton's time through the late nineteenth century, were concerned for the most part about solution techniques, not the general question of existence and uniqueness of solutions.

We already noted in Section 1.2 that the equation $(y')^2 + 1 = 0$ has no real-valued solution. You can easily check that the IVP $y' = \frac{3}{2}y^{1/3}$, $y(0) = 0$ has three distinct solutions: $y \equiv 0$, $y = -x^{3/2}$, and $y = x^{3/2}$.

Calculators and computers can mislead. They may present us with a solution where there is none. If there are several possible solutions, our user-friendly device may make its own selection, whether or not it is the one that we want for our problem. A skeptical attitude and a knowledge of mathematical theory will protect us against inappropriate answers.

[7] * Denotes an optional section.

First, let's look at what can happen when we try to solve first-order IVPs. Then we'll discuss an important result guaranteeing when such IVPs have one and only one solution.

Example 2.8.1 An IVP with a Unique Solution on a Restricted Domain

We'll see that for each value of x_0 the IVP $x' = 1 + x^2$, $x(0) = x_0$ has a unique solution, but that this solution does not exist for all values of the independent variable t. The slope field for this equation (Fig. 2.37) gives us some clues.

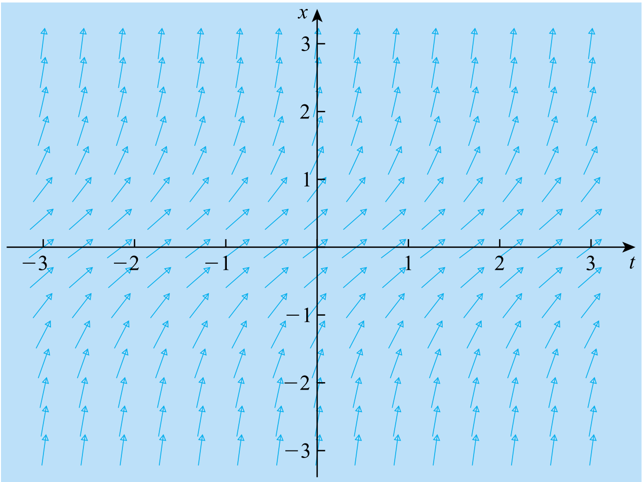


FIGURE 2.37

Slope field for $x' = 1 + x^2$; $-3 \le t \le 3$, $-3 \le x \le 3$

To see things clearly, we can focus on the initial condition $x(0) = 0$. There seems to be only one solution satisfying this condition, but the direction field suggests that the solution curve may have vertical asymptotes. Separating the variables, we see that $\int \frac{dx}{1+x^2} = \int 1\,dt$, which gives us $\arctan x = t + C$, or $x(t) = \tan(t + C)$. The initial condition $x(0) = 0$ implies that $C = 0$, so that the solution of the IVP is $x(t) = \tan t$. But this solution's domain is the open interval $(-\pi/2, \pi/2)$. Recall that the function approaches $\pm\infty$ as $t \to \pm\pi/2$ (we say the function "blows up in finite time"); therefore, the unique solution of our IVP doesn't exist outside the (time) interval $(-\pi/2, \pi/2)$.

Now even if we have determined that a given equation *has* a solution, a second important concern is whether there is *only one* solution. This question is usually asked about solutions to IVPs.

Example 2.8.2 An IVP with Infinitely Many Solutions

The nonlinear separable differential equation $x' = x^{2/3}$ has *infinitely many* solutions satisfying $x(0) = 0$ on every interval $[0, \beta]$. To prove this claim, we actually construct the family of solutions of the IVP.

For each number c such that $0 < c < \beta$, we can define the function

$$x_c(t) = \begin{cases} 0 & \text{for } 0 \le t \le c \\ \frac{1}{27}(t-c)^3 & c \le t \le \beta. \end{cases}$$

You should verify that each such function satisfies the differential equation with $x(0) = 0$. You should even be able to show that such a function is differentiable at the break point c. Because there are infinitely many values of the parameter c, our IVP has infinitely many solutions. Fig. 2.38 shows a few of these solutions with $\beta = 7$.

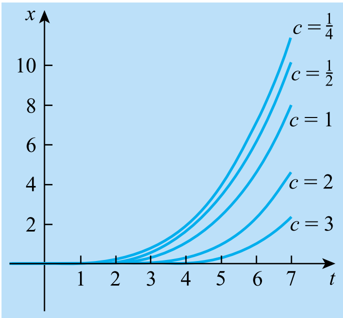


FIGURE 2.38

Solutions of the IVP $x' = x^{2/3}$, $x(0) = 0$. The functions $x_c(t)$ for $c = \frac{1}{2}, \frac{1}{4}, 1, 2$, and 3; $\beta = 7$; $-1 \le t \le 7$

2.8.1 An Existence and Uniqueness Theorem

For first-order differential equations the answers to the existence and uniqueness questions we have just posed are fairly easy. We have an **Existence and Uniqueness Theorem**—simple conditions that guarantee one and only one solution of an IVP.

Existence and Uniqueness Theorem

Let R be a rectangular region in the x-y plane described by the two inequalities $a \le x \le b$ and $c \le y \le d$. Suppose that the point (x_0, y_0) is inside R. Then if $f(x, y)$ and the partial derivative $\frac{\partial f}{\partial y}(x, y)$ are continuous functions on R, there is an interval I centered at $x = x_0$ and a unique function $y(x)$ defined on I such that y is a solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$.

This statement may seem a bit abstract, but it is the simplest and probably the most widely used result that guarantees the existence and uniqueness of a solution of a first-order IVP. Using this theorem is simple. Take your IVP, write it in the form $y' = f(x, y)$, $y(x_0) = y_0$, and then examine the functions $f(x, y)$ and $\frac{\partial f}{\partial y}$, the *partial derivative* of f with respect to the dependent variable y. (Section A.7 has a quick review of partial differentiation.)

Fig. 2.39 gives an idea of what such a region R and interval I in the Existence and Uniqueness Theorem may look like.

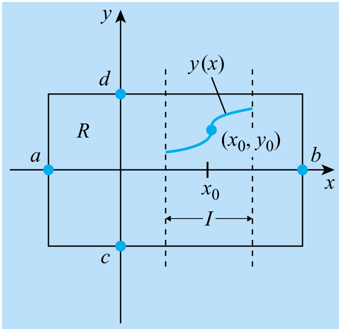


FIGURE 2.39

Region of existence and uniqueness: $R = \{(x, y) | a \le x \le b, c \le y \le d\}$

It's important to note the following comments about this fundamental theorem:

1. If the conditions of our result are satisfied, then solution curves for the IVP can never intersect. (*Do you see why?*)
2. If $f(x, y)$ and $\partial f/\partial y$ happen to be continuous for *all* values of x and y, our result does *not* say that the unique solution must be valid for *all* values of x and y.
3. The continuity of $f(x, y)$ and $\partial f/\partial y$ are *sufficient* for the existence of solutions, but they may not be *necessary* to guarantee their existence. This means that you may have solutions even if the continuity condition is not satisfied.
4. This is an *existence theorem*, which means that if the right conditions are satisfied, you can find a solution, but you are not told how to find it. In particular, you may not be able to describe the interval I without actually solving the differential equation.

The significance of these remarks will be explored in some of the following examples and in some of the problems in Exercises 2.8. First, let's apply the Existence and Uniqueness Theorem to IVPs involving first-order linear ODEs. In the last section of this chapter we'll sketch a proof of this important result.

Example 2.8.3 Any "Nice" Linear IVP Has a Unique Solution

Because linear equations model many important physical situations, it's important to know when these equations have unique solutions. We show that if $P(x)$ and $Q(x)$ are continuous ("nice") on an interval (a, b) containing x_0, then any IVP of the form $\frac{dy}{dx} + P(x)y = Q(x)$, $y(x_0) = y_0$, has one and only one solution on (a, b).

In terms of the Existence and Uniqueness Theorem, we have

$$f(x, y) = -P(x)y + Q(x) \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = -P(x).$$

But both $P(x)$ and $Q(x)$ are assumed continuous on the rectangle $R = \{(x, y) | a \le x \le b, c \le y \le d\}$ for any values of c and d, and $f(x, y)$ is a combination of continuous functions. (There are no values of x and y that give us division by zero or an even root of a negative number, for example.) The conditions of the theorem are satisfied, and so any IVP of the form described previously has a unique solution.

In Section 2.2 we showed how to find a solution of a linear differential equation explicitly. Now we see, given an appropriate initial condition, that we have learned how to find the *unique* solution.

Now let's go back and re-examine some examples we discussed earlier.

Example 2.8.4 Example 2.8.1 Revisited

Assume that x is a function of the independent variable t. If we look at the IVP $x' = 1 + x^2$, $x(0) = x_0$, in light of the Existence and Uniqueness Theorem, we see that $f(t, x) = 1 + x^2$, a function of x alone that is clearly continuous at all points (t, x), and $\frac{\partial f}{\partial x} = 2x$, also continuous for all (t, x).

The conditions of the theorem are satisfied, and so the IVP has a unique solution. But even though both $f(t, x)$ and $\frac{\partial f}{\partial x}$ are continuous for *all* values of t and x, we know that any unique solution is limited to an interval

$$\left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right), \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

separating the consecutive vertical asymptotes of the tangent function. (Go back and look at the one-parameter family of solutions for the equation, and see comment 2 that follows the statement of the Existence and Uniqueness Theorem.)

Next, we reexamine Example 2.8.2 in light of the Existence and Uniqueness Theorem.

Example 2.8.5 Example 2.8.2 Revisited

Here, we have the form $x' = x^{2/3} = f(x)$, with $x(0) = 0$, so we must look at $f(x)$ and $\frac{\partial f}{\partial x}$. But $\frac{\partial f}{\partial x} = f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which is not continuous in any rectangle in the t-x plane that includes $x = 0$ (that is, any part of the t-axis). Therefore, we shouldn't expect to have both existence and uniqueness on an interval of the form $[0, \beta]$—and in fact we don't have uniqueness, as we have seen.

However, if we avoid the t-axis—that is, if we choose an initial condition $x(t_0) = x_0 \neq 0$—then the Existence and Uniqueness Theorem guarantees that there will be a unique solution for the IVP. Fig. 2.40a shows the slope field for the autonomous equation $x' = x^{2/3}$ in the rectangle $-1 \le t \le 5$, $0 \le x \le 3$. This rectangle includes part of the t-axis, and it is easy to visualize many solutions starting at the origin, gliding along the t-axis for a little while, and then taking off. Fig. 2.38 shows some of these solution curves.

Fig. 2.40b, on the other hand, shows what happens if we choose a rectangle that avoids the t-axis. It should be clear that if we pick any point (t_0, x_0) in this rectangle, there will be one and only one solution of the equation that passes through this point.

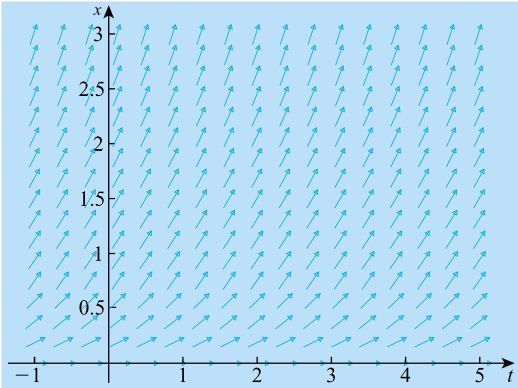


FIGURE 2.40a

Slope field for $x' = x^{2/3}$, $-1 \le t \le 5$, $0 \le x \le 3$

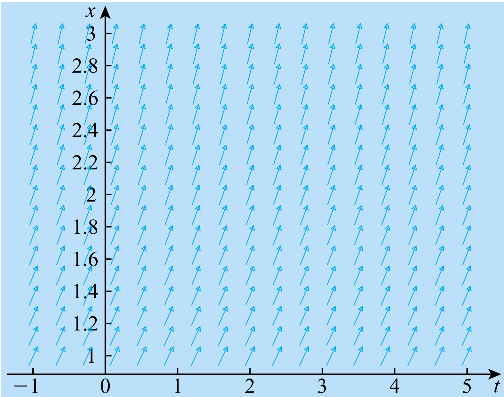


FIGURE 2.40b

Slope field for $x' = x^{2/3}$, $-1 \le t \le 5$, $1 \le x \le 3$

2.8.2 A sketch of a proof of the Existence and Uniqueness Theorem

The proof of the Existence and Uniqueness Theorem is due to Émile Picard (1856-1941), who used an *iteration scheme* that guarantees a solution under the conditions specified.

We begin by recalling that any solution to the IVP $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ must also satisfy the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{I}$$

The converse is also true: If $y(x)$ satisfies the integral equation, then $\frac{dy}{dx} = f(x, y(x))$ and $y(x_0) = y_0$. This means that the IVP may be replaced by Eq. (I) and we can implement Picard's method.

In the integrand in Eq. (I), replace $y(t)$ by the constant y_0, then integrate and call the resulting right-hand side of Eq. (I) $y_1(x)$:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt. \tag{II}$$

This starts the process (also called *successive approximation*). To keep it going, we use the iterative formulas

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt. \tag{III}$$

A rigorous proof of Picard's theorem consists of showing that *this process produces a sequence of functions* $\{y_n(x)\}$ *that converges to a function* $y(x)$ *that satisfies the IVP* $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ *for values of* x *sufficiently near* x_0. The convergence follows from the continuity assumptions in the Existence and Uniqueness Theorem. Another technicality in the proof involves ensuring that the graph of each function $y = y_n(x)$ lies within the rectangle R of the theorem for appropriate values of x.

Let's look at a simple example of how Picard's method works. In many situations the actual computations are rather tedious.

Example 2.8.6 Picard's Iteration Method

Suppose we have the very simple IVP $\frac{dy}{dx} = y$, $y(0) = 1$, or $y(x) = 1 + \int_0^x y(t)\, dt$. Then $f(x, y) = y$, and Eq. (II) becomes

$$y_1(x) = 1 + \int_0^x 1\, dt = 1 + x.$$

Using Eq. (III) with $n = 1$, we get

$$y_2(x) = 1 + \int_0^x f(t, y_1(t))\, dt = 1 + \int_0^x (1 + t)\, dt = 1 + x + \frac{x^2}{2!}.$$

The next iteration, with $n = 2$, gives

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Spoiler alert: $y_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$, $y_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}, \ldots, y_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. At this point, you should recognize that the limit of the sequence $\{y_n(x)\}$ as $n \to \infty$ is the infinite Taylor series for e^x: $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. This, of course, is a solution of the IVP. (It would be interesting to graph e^x, $y_1(x)$, and $y_2(x)$, for example, on the same set of axes to get a sense of the approximation.)

A rigorous proof that Picard's iterates converge to a solution of an IVP also shows that $y(x) = \lim_{n\to\infty} y_n(x)$ is uniquely determined, but we can sketch an independent proof of uniqueness.

Suppose that the hypotheses of the Existence and Uniqueness Theorem hold and that $Y_1(x)$ and $Y_2(x)$ are two solutions of the IVP $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. We want to show that these solutions are equal—that is, $|Y_1(x) - Y_2(x)| = 0$. Now we have

$$\frac{d}{dx}[Y_1(x) - Y_2(x)] = f(x, Y_1(x)) - f(x, Y_2(x)), \quad x_0 < x < x_0 + a. \qquad \text{(IV)}$$

Since the function $Y_1(x) - Y_2(x)$ is continuous on I and has a value of zero at $x = x_0$, we can integrate each side of Eq. (IV) from x_0 to some $x \in I$ to see that

$$Y_1(x) - Y_2(x) = \int_{x_0}^{x} \left[f(t, Y_1(t)) - f(t, Y_2(t))\right] dt, \ x \in I$$

If we fix the value of x, the function f can be regarded as a function of y alone. Since $\frac{\partial f}{\partial y}$ is continuous in the rectangle R, the Mean Value Theorem and the Extreme Value (Maximum/Minimum) Theorem for continuous functions of a single variable imply that there is a positive constant M such that

$$|f(x, Y_1) - f(x, Y_2)| \le M|Y_1 - Y_2| \quad \text{for } (x, Y_1(x)), (x, Y_2(x)) \in R. \qquad \text{(V)}$$

Taking absolute values of each side of inequality (V), denoting $|Y_1(x) - Y_2(x)|$ by $Z(x)$, estimating the integral, and using inequality (V), we get the integral inequality

$$0 \le Z(x) \le \int_{x_0}^{x} |f(t, Y_1(t)) - f(t, Y_2(t))|\, dt \le M \int_{x_0}^{x} Z(t)\, dt, \ t \in I. \qquad \text{(VI)}$$

Now we show that the only nonnegative solution $Z(x)$ of inequality (VI) is the zero function, $Z(x) = 0$ for all values of x.

Let

$$W(x) = \int_{x_0}^{x} Z(t)\, dt \quad \text{for all } x \in I.$$

Since $W'(x) = Z(x)$ on I by the Fundamental Theorem of Calculus (Appendix A.4), we can rewrite Eq. (VI) as the inequality $W'(x) - MW(x) \le 0$, all $x \in I$. But the left-hand side of this inequality resembles a first-order linear differential equation. Multiplying this inequality by the integrating factor e^{-Mx}, we get

$$\frac{d}{dx}\left[W(x)e^{-Mx}\right] \le 0, \quad x_0 \le x \le x_0 + a.$$

This means that $W(x)e^{-Mx}$ is a nonincreasing function of x for $x_0 \le x \le x_0 + a$. This implies that

$$W(x)e^{-Mx} \le W(x_0)e^{-Mx}, \quad x_0 \le x \le x_0 + a.$$

But $W(x_0) = \int_{x_0}^{x_0} Z(t)\,dt = 0$ and so $W(x) \le 0$ for all $x \in I$. On the other hand, since $W(x) = \int_{x_0}^{x} Z(t)\,dt$ for $x > x_0$ and $Z(x) \ge 0$, we know that $W(x) \ge 0$ for all $x \in I$. Since $W(x) \le 0 \le W(x)$, we see that $W(x) = 0$ for all x, so $W'(x) = Z(x) = 0$ for all x as well—and we have shown that $Y_1(x) = Y_2(x)$ for all $x \in I$. Therefore, our IVP can't have more than one solution on I.

Exercises 2.8

A

For each of the following IVPs 1–8, determine a rectangle R in the appropriate plane (x-y, t-x, etc.) for which the given differential equation would have a unique solution through a point in the rectangle. **Do not solve the equations**.

1. $\frac{dx}{dt} = \frac{1}{x}$, $x(0) = 3$
2. $\frac{dy}{dt} = \frac{5}{4}y^{1/5}$, $y(0) = 0$
3. $t\frac{dx}{dt} = x$, $x(0) = 0$
4. $y' = -\frac{t}{y}$, $y(0) = 0.2$
5. $y' = \frac{t}{1+t+y}$, $y(-2) = 1$
6. $x' = \tan x$, $x(0) = \frac{\pi}{2}$
7. $(1+t)\frac{dy}{dt} = 1 - y$
8. $y' = v\frac{x+y}{x-y}$
9. What is the length of the largest interval I on which the IVP $y' = 1 + y^2$, $y(0) = 0$ has a solution?
10. Show that $y \equiv -1$ is the only solution of the IVP $y' = t(1+y)$, $y(0) = -1$.
11. What is the solution to the IVP $\frac{dx}{dt} = x^{2/3}$, $x(0) = x_0$ if $x_0 < 0$? Compare your answer to the answer(s) in Example 2.8.2. What has changed?
12. What is the largest interval I on which the IVP $\frac{dy}{dx} = e^y$, $y(0) = c$ (a constant) has a solution?

B

13. Look at the IVP $Q' = |Q - 1|$, $Q(0) = 1$.
    a. Explain why the conditions of the Existence and Uniqueness Theorem do not hold for this equation.
    b. Guess at a solution of this IVP.
    c. Explain why the solution you found in part (b) is unique.
14. Consider the equation $\dot{y} = \sqrt{|y|} + k$, where k is a positive constant.
    a. Solve the equation. (You will get an implicit solution.)
    b. For what initial values $(t_0,\ y_0)$ does the equation have a unique solution?
    c. For what values of $k \le 0$ does the equation have unique solutions?

15. The parabola $y = x^2$ and the line $y = 2x - 1$ are both solutions of the equation $y' = 2x - 2\sqrt{x^2 - y}$ and satisfy the initial condition $y(1) = 1$. Does this contradict the Existence and Uniqueness Theorem?

16. Consider the equation $\frac{dy}{dx} + x^2y^3 = \cos x$.

a. Does this equation have a unique solution passing through any point (x_0, y_0)?

b. Try to solve the equation using the ODE solver in your CAS. Comment on the result.

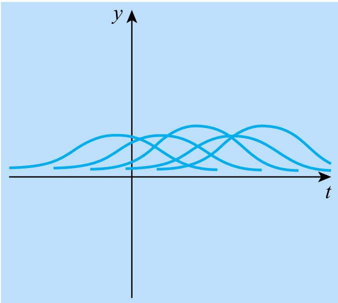
17. Consider the IVP $\frac{dy}{dx} = P(x)y^2 + Q(x)y$, $y(2) = 5$, where $P(x)$ and $Q(x)$ are third-degree polynomials in x. Does this problem have a unique solution on some interval $|x - 2| \le h$ around $x = 2$? Explain why or why not.

18. Consider the nonlinear equation $\frac{dx}{dt} = (\alpha - x)(\beta - x)$, where α and β are positive constants. (See Example 2.1.8.) Without solving the equation, show that the solution of any IVP involving this equation is unique.

C

19. Consider the equilibrium solution $P \equiv b$ of the logistic equation (Section 2.5) $\frac{dP}{dt} = kP(b - P)$, where k and b are positive constants. Is it possible for a solution near $P \equiv b$ to reach (i.e., equal) this solution for a finite value of x? [*Hint*: Use the uniqueness part of the Existence and Uniqueness Theorem.]

20. Why can't the family of curves shown in the following be the solution curves for the differential equation $y' = f(t, y)$, where f is a polynomial in t and y?



21. Example 2.8.3 indicates that the linear IVP $y' = y$, $y(0) = 1$ has a unique solution. Ignoring the fact that you can actually solve this equation, prove the following properties of $y(t)$:

a. For all real values of t, $y(t) \cdot y(-t) = 1$.

b. $y(t) > 0$ for all real numbers t.

c. For all real numbers t_1 and t_2, $y(t_1 + t_2) = y(t_1) \cdot y(t_2)$.

22. Suppose that a differential equation is a model for a certain type of chemical reaction. Could the fact that the equation does *not* have a solution indicate that the reaction *cannot* take place? Would the fact that the equation *has* a solution guarantee that the reaction *does* take place?

Summary

An easy type of first-order ODE to solve is a **separable equation**, one that can be written in the form $\frac{dy}{dx} = f(x)g(y)$, where f denotes a function of x alone and g denotes a function of y alone. "Separating the variables" leads to the equation $\int \frac{dy}{g(y)} = \int f(x)\,dx$. It is possible that you cannot carry out one of the integrations in terms of elementary functions or you may wind up with an *implicit* solution. Furthermore, the process of separation of variables may introduce **singular solutions**.

Another important type of first-order ODE is a **linear equation**, one that can be written in the form $a_1(x)y' + a_0(x)y = f(x)$, where $a_1(x)$, $a_0(x)$, and $f(x)$ are functions of the independent variable x alone. The standard form of such an equation is $\frac{dy}{dx} + P(x)y = Q(x)$. The equation is called **homogeneous** if $Q(x) \equiv 0$ and **nonhomogeneous** otherwise. Any homogeneous linear equation is separable.

After writing a first-order linear equation in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$, we can solve it by the method of **variation of parameters** or by introducing an **integrating** factor, $\mu(x) = e^{\int P(x)\,dx}$.

A typical first-order differential equation can be written in the form $\frac{dy}{dx} = f(x, y)$. Graphically, this tells us that at any point (x, y) on a solution curve of the equation, the slope of the tangent line is given by the value of the function f at that point. We can outline the solution curves by using possible tangent line segments. Such a collection of tangent line segments is called a **direction field** or **slope field** of the equation. The set of points (x, y) such that $f(x, y) = C$, a constant, defines an **isocline**, a curve along which the slopes of the tangent lines are all the same (namely, C). In particular, the **nullcline** (or **zero isocline**) is a curve consisting of points at which the slopes of solution curves are zero. A differential equation in which the independent variable does not appear explicitly is called an **autonomous** equation. If the independent variable *does* appear, the equation is called **nonautonomous**. For an autonomous equation the slopes of the tangent line segments that make up the slope field depend only on the values of the dependent variable. Graphically, if we fix the value of the dependent variable, say x, by drawing a horizontal line $x = C$ for any constant C, we see that all the tangent line segments along this line have the same slope, no matter what the value of the independent variable, say t. Another way to look at this is to realize that we can generate infinitely many solutions by taking any one solution and translating (shifting) its graph left or right. Even when we can't solve an equation, an analysis of its slope field can be very instructive. However, such a graphical analysis may miss certain important features of the integral curves, such as vertical asymptotes.

An *autonomous* first-order equation can be analyzed qualitatively by using a **phase line** or **phase portrait**. For an autonomous equation the points x such that $\frac{dx}{dt} = f(x) = 0$ are called **critical points**. We also use the terms **equilibrium points**,

equilibrium solutions, and **stationary points** to describe these key values. There are three kinds of equilibrium points for an autonomous first-order equation: **sinks**, **sources**, and **nodes**. An equilibrium solution y is a **sink** (or **asymptotically stable solution**) if solutions with initial conditions "sufficiently close" to y approach y as the independent variable tends to infinity. On the other hand, if solutions "sufficiently close" to an equilibrium solution y are asymptotic to y as the independent variable tends to negative infinity, then we call y a **source** (or **unstable equilibrium solution**). An equilibrium solution that shows any other kind of behavior is called a **node** (or **semistable equilibrium solution**). The **First Derivative Test** is a simple (but not always conclusive) test to determine the nature of equilibrium points.

Suppose that we have an autonomous differential equation with a parameter α. A **bifurcation point** α_0 is a value of the parameter that causes a change in the nature of the equation's equilibrium solutions as α passes through the value α_0. There are three main types of bifurcation for a first-order equation: (1) **pitchfork bifurcation**; (2) **saddle-node bifurcation**; and (3) **transcritical bifurcation**.

When we are trying to solve a differential equation, especially an IVP, it is important to understand whether the problem *has* a solution and whether any solution is *unique*. The **Existence and Uniqueness Theorem** provides simple sufficient conditions that guarantee that there is one and only one solution of an IVP. A standard proof of this result involves *successive approximations*, or *Picard iterations*.

CHAPTER

3 The numerical approximation of solutions

Introduction

Historically, numerical methods of working with differential equations were developed when some equations could not be solved analytically—that is, with their solutions expressed in terms of elementary functions. Over the past 300 years, mathematicians and scientists have learned to solve an increasing variety of differential equations. However, today there are still equations that are impossible to solve in closed form (for instance, Example 2.1.5). In fact, very few differential equations that arise in applications can be solved exactly and, perhaps more importantly, even solution formulas often express the solutions *implicitly* via complicated combinations of the solution and the independent variable that are difficult to work with. Take a look back at the solution in Example 2.4.4, for instance. In this chapter we will describe some ways of getting an *approximate numerical solution* of a first-order initial-value problem (IVP) $y' = f(x, y)$, $y(x_0) = y_0$. This means we will be able to calculate approximate values of the solution function y by some process requiring a finite number of steps, so that at the end of this step-by-step process we are reasonably close to the "true" answer. Graphically, we want to approximate the solution curve with a simpler curve, usually a curve made up of straight line segments.

The very nature of what we will be trying to do contains the notion of *error*, the discrepancy between a true value and its approximate value. Error is what stands between reality and perfection. It is the static in our telephone line, the wobble in a kitchen chair, a slip of the tongue. Although there are various ways to measure error, we will focus on **absolute error**, which is defined by the quantity |true value − approximation|, the absolute value of the difference between the exact value and the approximate value. We'll have more to say about error in the following sections.

Let's see how all this applies to a first-order IVP $y' = f(x, y)$, $y(x_0) = y_0$.

3.1 Euler's method

One of the easiest methods of obtaining an approximation to a solution curve is attributed to the mathematician Euler. He used this approach to solve differential equations around 1768. A modern way of expressing his idea is to say that he used

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local linearity. Geometrically, this simply means that if a function F is differentiable at $x = x_0$ and we "zoom in" on the point $(x_0, F(x_0))$ lying on the curve $y = F(x)$, then we will think we're looking at a straight line segment. Numerically, we're saying that if we have a straight line tangent to a curve $y = F(x)$ at a point $(x_0, F(x_0)) = (x_0, y_0)$, then for a value of x close to x_0, the corresponding value on the tangent line is approximately equal to the value on the curve. In other words, we can avoid the complexity of dealing with values on what may be a complicated curve by dealing with values on a straight line. (See Section A.1 for more details on this topic.)

Using the familiar point-slope formula for the equation of a straight line, we can derive the equation of the line T tangent to the curve $y = F(x)$ at the point (x_0, y_0)

$$T(x) = F'(x_0)(x - x_0) + F(x_0) = y'(x_0)(x - x_0) + y_0. \qquad (3.1.1)$$

Now we can express the idea of local linearity by writing

$$\underbrace{y(x)}_{\text{value on the curve}} \approx \underbrace{y'(x_0)(x - x_0) + y_0}_{\text{value on the tangent line}},$$

where the symbol $\approx$ means "is approximately equal to." Fig. 3.1 shows what we are saying.

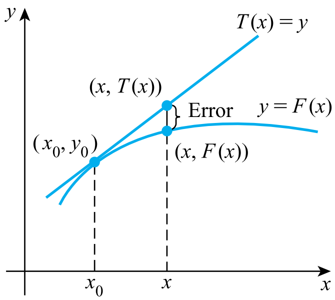

FIGURE 3.1

Local linearity

Note that because the curve we're using as an illustration is concave down near the point (x_0, y_0), the tangent line lies *above* the curve here, so the value $T(x)$ given by the tangent line is *greater than* the true value $y(x)$ for x near x_0.

Now let's look at an IVP $y' = f(x, y)$, $y(x_0) = y_0$ so that we can write $y_0' = y'(x_0) = f(x_0, y_0)$. **In what follows, we assume that there is a unique solution $\boldsymbol{\varphi}$ in some interval containing $\boldsymbol{x_0}$.** Suppose we want to know the height of the solution curve corresponding to a value x_1 that is close to x_0, but to the *right* of x_0. We can describe this new value of the independent variable as $x_1 = x_0 + h$, where $h > 0$ is the size of a small "step." Now let's try to approximate $\varphi(x_1)$, a value on the actual solution curve, by some value y_1 on the tangent line to $y = \varphi(x)$ at x_0:

$$\begin{aligned}\varphi(x_1) \approx y_1 &= \varphi'(x_0)(x_1 - x_0) + y_0\\ &= f(x_0, y_0)(x_0 + h - x_0) + y_0\\ &= f(x_0, y_0)\cdot h + y_0.\end{aligned}$$

Therefore, we can write

$$\varphi(x_1) \approx y_1 = f(x_0, y_0)\cdot h + y_0$$

and we have a good *local linear approximation* of $\varphi(x)$ at $x = x_1$ if we choose a value of h that is small enough. Fig. 3.2 illustrates what's going on.

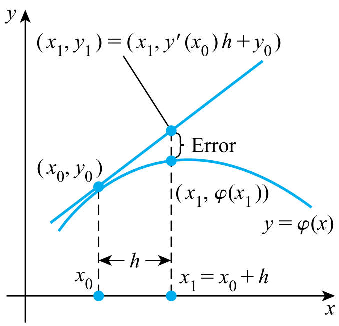

FIGURE 3.2

A local linear approximation of a solution

We can repeat the process using (x_1, y_1) as our jumping-off point, realizing that the value y_1 is only an approximation. Using Eq. (3.1.1) again, with (x_0, y_0) replaced by (x_1, y_1), we see that the line through (x_1, y_1) with slope equal to $f(x_1, y_1)$ has y values given by

$$f(x_1, y_1)(x - x_1) + y_1. \tag{3.1.2}$$

We should realize that the point (x_1, y_1) is not expected to be on the actual solution curve, so in general $f(x_1, y_1) \neq f(x_1, \varphi(x_1))$, the slope of the actual solution at x_1.

For convenience, suppose that we want to approximate the height of a solution curve corresponding to a value x_2 that is the same distance from x_1 as x_1 is from x_0—that is, we take a step to the right of size $h : x_2 = x_1 + h = (x_0 + h) + h = x_0 + 2h$. We can approximate $\varphi(x_2)$, the actual value of the solution function at $x = x_2$, by using Eq. (3.1.2):

$$\varphi(x_2) \approx y_2 = f(x_1, y_1)\cdot h + y_1.$$

Similarly, for $x_3 = x_2 + h = (x_0 + 2h) + h = x_0 + 3h$ we approximate $\varphi(x_3)$ as follows:

$$\varphi(x_3) \approx y_3 = f(x_2, y_2) \cdot h + y_2.$$

Fig. 3.3 shows what we are doing.

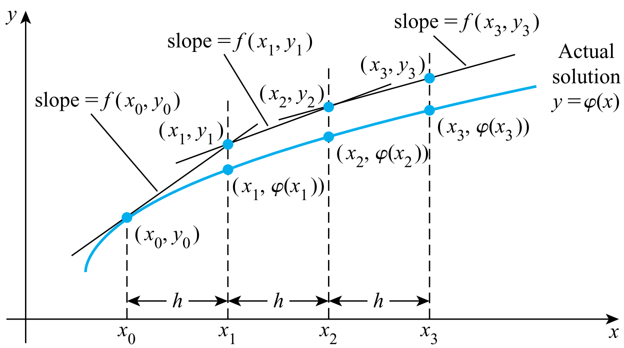

FIGURE 3.3

A three-step linear approximation

Continuing in this way, we generate a sequence of approximate values $y_1, y_2, y_3, \ldots, y_n$ for the solution function φ at various equally spaced points $x_1, x_2, x_3, \ldots, x_n$:

$$\underbrace{y_{k+1}}_{\text{new approx. value}} = \underbrace{y_k}_{\text{old approx. value}} + \underbrace{h}_{\text{step size}} \cdot f(x_k, y_k), \tag{3.1.3}$$

where $x_k = x_0 + kh$, $k = 0, 1, \ldots, n$. If you go back through the derivation, you'll realize that Eq. (3.1.3) is also valid for $h < 0$. Note that if the points x_k are equally spaced with step size h and we want to get from (x_0, y_0) to (x^*, y^*) along the approximating polygonal curve, then we must have $n = \frac{x^* - x_0}{h}$ steps. For example, if we start at $x_0 = 2$ and want to approximate $\varphi(2.7)$ using steps of size $h = 0.1$, we can reach $x^* = 2.7$ by taking $n = \frac{2.7-2}{0.1} = 7$ steps. In practice, once we have chosen the step size h, the number of steps needed, n, will be obvious.

If we stand back from all these equations and look at Figs. 3.2 and 3.3 again, we can see that what we are doing is using the slope field for our IVP as a set of stepping stones. We "walk" on a tangent line segment for a short distance, stop to look forward for the next step, jump to that step, and so on. We are approximating the flow of the solution curve by using flat rocks set into the "stream." If you play "connect the dots" with the points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, you see a polygonal line (called the **Euler polygon** or the **Cauchy–Euler polygon**) that approximates the actual solution curve. Fig. 3.4 shows this for the IVP $y' = x^2 + y$, $y(0) = 1$, where we try to approximate $y(1)$ with different step sizes $h = 1$, 0.5, and 0.25.

We can look at this approximation process, Eq. (3.1.3), in another geometrical way. Suppose we have the differential equation $y' = f(x, y)$. Then the Fundamental

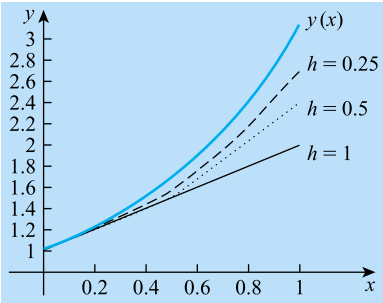

FIGURE 3.4

The actual solution of the IVP $y' = x^2 + y$, $y(0) = 1$, and three Euler approximations ($h = 1$, 0.5, and 0.25) on the interval $[0, 1]$

Theorem of Calculus tells us that

$$y_{k+1} - y_k \approx y(x_{k+1}) - y(x_k) = \int_{x_k}^{x_{k+1}} y'(x)\,dx = \int_{x_k}^{x_{k+1}} f(x, y)\,dx.$$

But Eq. (3.1.3) requires us to replace $y_{k+1} - y_k$ by $h \cdot f(x_k, y_k)$. This means that we are approximating $\int_{x_k}^{x_{k+1}} f(x, y)\,dx$ with $h \cdot f(x_k, y_k)$. Fig. 3.5 shows the geometry of the situation in the interval $[x_k, x_{k+1}]$.

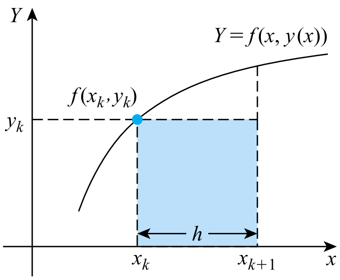

FIGURE 3.5

Approximation of an integral using a rectangular area

We have approximated the area under the curve $Y = f(x, y(x))$ with the area of the shaded rectangle—a rectangle formed by using the height of the curve at the left-hand endpoint of the interval. Thus, Euler's method amounts to using a left-hand *Riemann sum* approximation of the area under a curve.

If y is the solution of the equation $y' = f(x, y)$, we can view Euler's method in yet another way by considering the Taylor expansion of $y(x)$ about $x = x_k$ (see

Section A.3):

$$y(x_{k+1}) = y(x_k) + y'(x_k)h + y''(\xi_k)\frac{h^2}{2} = \underbrace{y(x_k) + f(x_k, y(x_k))h}_{y_{k+1}} + y''(\xi_k)\frac{h^2}{2}$$

with $x_k < \xi_k < x_{k+1}$. Assuming that $y(x)$ has a bounded second derivative and realizing that $h^2 < h$ for small values of h, we see that Euler's method is essentially using a first-degree Taylor polynomial to approximate the solution curve:

$$y(x_{k+1}) \approx y(x_k) + f(x_k, y(x_k))h.$$

The approximation processes we have been describing (and will describe later in this chapter) are subject to two basic kinds of error: **truncation error**, which occurs when we stop (or truncate) an approximation process after a certain number of steps, and **propagated error**, the accumulated error resulting from many calculations with rounded values. (See Section A.3 for additional remarks about these types of errors.) We must be aware that there is usually a trade-off in dealing with error. If we try to reduce the *truncation* error and increase the accuracy of our approximation by carrying out more steps (for example, by taking more terms of a Taylor series or more steps in the Euler method), we are increasing our calculation load and consequently running the risk of increasing *propagated* error. Fig. 3.6 shows the trade-off in general terms.

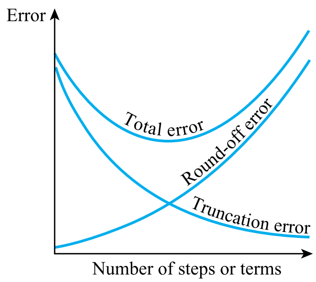

FIGURE 3.6

Total error = round-off error + truncation error

Clearly, at each stage of Euler's method, we choose to round off the entries in a certain way. Even if we assume for the sake of simplicity that the round-off error is negligible, our use of local linearity—using straight lines to approximate curves—introduces truncation error. Now suppose that we are given the value, $y(x_0)$, of a solution at an initial point and want to approximate the value, $y(b)$, at some later point $b = x_0 + nh$. First, there is a **local truncation error** at each step, defined as $y(x_{k+1}) - y_{k+1}$ for each k ($k = 0, 1, 2, \ldots, n-1$). This is the error introduced when computing the value y_{k+1} from the value y_k, *assuming that* y_k *is exact*. Then we have the **cumulative truncation error**, defined as $y(b) - y_n = y(x_n) - y_n$, which is the total actual error in the value of $y(x_0 + nh)$, or $y(b)$, caused by all the previous approximations—that is, by the cumulative effect after n steps of the local errors from previous steps. (This is not just the sum of all the local truncation errors. *Life isn't that simple*.)

In any case, a mathematically rigorous analysis of the errors produced shows that *the local truncation error at any step of Euler's method behaves like a constant multiple of* h^2, which is smaller than h when h is small:

$$|\text{local truncation error at step } k| = |y(x_{k+1}) - y_{k+1}| \leq \frac{M}{2}h^2,$$

where $M = \max_{x_k < x < x_{k+1}} |y''(x)|$. This follows easily from the Taylor series expansion

$$y(x_{k+1}) = \underbrace{y(x_k) + f(x_k, y(x_k))h}_{y_{k+1}} + y''(\xi_k)\frac{h^2}{2}$$

given previously.

It is also true that for Euler's method, the cumulative truncation error is no greater than a constant multiple of the step size h:

$$|\text{true value} - \text{approximation}| = |y(b) - y_n| \leq K \cdot h,$$

where K is independent of h, but depends on $|y''(x)|$ and the interval $[x_0, b]$. Because the cumulative error is bounded by a constant multiple of the *first* power of the step size h, we say that Euler's method is a **first-order method**. (The number K is a *maximum* bound. In practice, the actual error incurred in a problem will usually be less than this bound.) Intuitively, we can reason as follows: There are $n = \frac{b-x_0}{h}$ steps in the Euler method approximation, each having a local error that is less than or equal to some multiple of h^2. If K^* is the largest of the multipliers, then the cumulative error is less than or equal to $\frac{b-x_0}{h} \cdot K^* h^2 = Kh$.

Therefore, if we ignore the round-off error as essentially a statistical problem outside our range of interest right now, we can make the total error as small as we wish by making h sufficiently small—that is, by making the *number* of steps sufficiently large. This is not very satisfactory because a larger number of steps *does* require more calculating time by hand or by computer, and in real-life problems the larger number of steps often leads to a "snowball effect" of round-off errors. Take another look at Fig. 3.6.

If you want to understand and improve the accuracy of your approximations, here are two rules of thumb you can use: (1) Start your calculations with many more decimal places than you need. (2) Keep on redoing your calculations with a step size h equal to one-half its previous value. If you reach a stage at which the new result agrees with the previous one to d decimal places after appropriate rounding, then you can assume that you have d decimal place accuracy. (Look at Example 3.1.4 for a slight variation of this rule.)

Euler's method is not very accurate and is not used widely in practice. But the method is simple and displays the essential characteristics of more sophisticated methods. In the next section we discuss an improved method, one that uses Euler's basic idea in a more efficient way.

This is enough theory for now. Let's see how Euler's method of using tangent lines works with a simple IVP.

Example 3.1.1 Euler's Method with Error Analysis

Suppose we're given the IVP $\frac{dx}{dt} = t^2 + x$, $x(1) = 3$. We want to use Euler's method to approximate $x(1.5)$.

This is a first-order linear equation whose particular solution for the initial condition $x(1) = 3$ is $x(t) = -t^2 - 2t - 2 + 8e^{t-1}$. (*Verify this.*) Thus the actual value of $x(1.5)$ is $-(1.5)^2 - 2(1.5) - 2 + 8e^{(1.5)-1} = 5.939770\ldots$. We'll use the actual value to see how good an approximation Euler's method gives us.

In our problem $f(t, x) = t^2 + x$, so Euler's formula (3.1.3) becomes

$$x_{k+1} = x_k + h \cdot (t_k^2 + x_k), \tag{3.1.4}$$

where $t_k = t_0 + kh$, $k = 0, \ldots, n$, $t_0 = 1$, and $x_0 = 3$. (By now you should be comfortable with the switch from the traditional x-y coordinates to t-x coordinates.) Suppose we take $h = 0.1$—that is, our step size is one-tenth of a unit. Because our target $t = 1.5$ is 0.5 units away from our initial point $t = 1$, we'll need $n = 5$ steps of size $h = 0.1$ to reach this with Euler's process (Fig. 3.7).

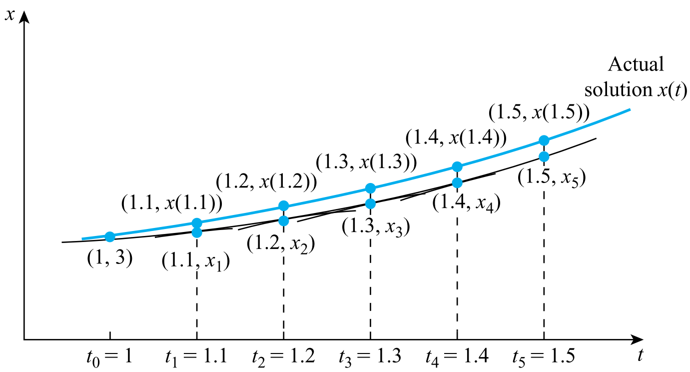

FIGURE 3.7

A five-step approximation

Using formula (3.1.4), let's generate our approximate values, stepping from $t = 1$ to $t = 1.5$:

$$x_1 = x_0 + (0.1)(t_0^2 + x_0) = 3 + (0.1)(1^2 + 3) = 3.40$$
$$x_2 = x_1 + (0.1)(t_1^2 + x_1) = 3.40 + (0.1)(1.1^2 + 3.4) = 3.861$$
$$x_3 = x_2 + (0.1)(t_2^2 + x_2) = 3.861 + (0.1)(1.2^2 + 3.861) = 4.3911$$

$$x_4 = x_3 + (0.1)(t_3^2 + x_3) = 4.3911 + (0.1)(1.3^2 + 4.3911) = 4.99921$$
$$x_5 = x_4 + (0.1)(t_4^2 + x_4) = 4.99921 + (0.1)(1.4^2 + 4.99921) = 5.695131.$$

Thus, Euler's method gives the approximation 5.695131 for the value $x(1.5)$. In this example, the *absolute error* is |true value − approximation| = |5.939770 − 5.695131| = 0.244639.

If we try again, using a step size only *half* the size of the step we used before—that is, using a step size $h = 0.05$—it will take *twice* as many steps to bridge the gap between the initial value $t = 1$ and the final value $t = 1.5$. Table 3.1 shows the result of a spreadsheet calculation of Euler's method for this new sequence of steps. If you have access to a spreadsheet program, you'll find it fairly easy to use it to set up Euler's method.

Table 3.1 Euler's Method for $\frac{dx}{dt} = t^2 + x$, $x(1) = 3$, with $h = 0.05$

k	t_k	x_k	True Value	Absolute Error
0	1	3.000000	3.000000	0.00000
1	1.05	3.200000	3.207669	0.00767
2	1.1	3.415125	3.431367	0.01624
3	1.15	3.646381	3.672174	0.02579
4	1.2	3.894825	3.931222	0.03640
5	1.25	4.161567	4.209703	0.04814
6	1.3	4.447770	4.508870	0.06110
7	1.35	4.754658	4.830040	0.07538
8	1.4	5.083516	5.174598	0.09108
9	1.45	5.435692	5.543997	0.10831
10	1.5	5.812602	5.939770	0.12717

Note that the number of calculations (steps) has doubled, but the absolute error has been cut almost in half. Also note the cumulative growth of the error in the last column.

If we cut the step size in half again, working with $h = 0.025$ this time (Table 3.2), we can see a pattern emerging.

Note that the absolute error increases as a function of k, the number of steps. Furthermore, the differences between successive errors are increasing slightly. For example, if you subtract the error for $k = 9$ from the error corresponding to $k = 10$, you get 0.0030, whereas subtracting the $k = 10$ error from the $k = 11$ error yields 0.0032.

If this error pattern seems vaguely familiar, it may be because you have seen error analysis applied to left- and right-hand Riemann sum approximations in calculus.

Let's try another problem. Practice makes perfect—or at least we can approximate perfection.

Example 3.1.2 Euler's Method with Error Analysis

Consider the IVP $\frac{dy}{dt} = \frac{1}{t}$, $y(1) = 0$, and suppose we want to approximate $y(2)$.

You should recognize the solution of this IVP as $y = \ln t$, so we're really trying to approximate $\ln 2 = 0.69314718056\ldots$. (*Hint*: If you check this "exact" answer on your calculator or

Table 3.2 Euler's Method for $\frac{dx}{dt} = t^2 + x$, $x(1) = 3$, with $h = 0.025$

k	t_k	x_k	True Value	Absolute Error
0	1	3.000000	3.000000	0.0000
1	1.025	3.100000	3.101896	0.0019
2	1.05	3.203766	3.207669	0.0039
3	1.075	3.311422	3.317448	0.0060
4	1.1	3.423098	3.431367	0.0083
5	1.125	3.538926	3.549563	0.0106
6	1.15	3.65904	3.672174	0.0131
7	1.175	3.783578	3.799345	0.0158
8	1.2	3.912683	3.931222	0.0185
9	1.225	4.046500	4.067957	0.0215
10	1.25	4.185178	4.209703	0.0245
11	1.275	4.32887	4.356620	0.0277
12	1.3	4.477733	4.508870	0.0311
13	1.325	4.631926	4.666620	0.0347
14	1.35	4.791615	4.830040	0.0384
15	1.375	4.956968	4.999306	0.0423
16	1.4	5.128158	5.174598	0.0464
17	1.425	5.305362	5.356098	0.0507
18	1.45	5.488761	5.543997	0.0552
19	1.475	5.678543	5.738489	0.0599
20	1.5	5.874897	5.939770	0.0649

computer algebra system (CAS), realize that these devices use very sophisticated approximation methods themselves!)

If we take $h = 0.05$, we'll need 20 steps to stretch from $t = 1$ to $t = 2$. In our example, Euler's method gives us the formula

$$y_{k+1} = y_k + \frac{0.05}{t_k}$$

for $t_k = 1 + 0.05k$ $(k = 0, \ldots, 20)$. Table 3.3 gives the results.

The solution curve $y = \ln t$ is concave down, so the approximating tangent lines all lie *above* the solution curve, leading to an approximation of $\ln 2$ that's too large. Just as in Example 3.1.1, the errors increase with the value of k, but this time, if you subtract successive errors (corresponding to successive values of k), you'll see that the differences are *decreasing*. (To gain some insight into this phenomenon, compare $f(x, y)$ in Examples 3.1.1 and 3.1.2.)

Changing to $h = 0.025$ and $n = 40$ yields the approximate value 0.699436, whereas setting $h = 0.01$ and $n = 100$ gives us an approximation of 0.695653. Of course, technology (a spreadsheet) was used to obtain the last two approximations.

Next, we'll see what happens when we are given an equation whose solution we don't know.

Table 3.3 Euler's Method for $\frac{dy}{dt} = \frac{1}{t}$, $y(1) = 0$, with $h = 0.05$

k	t_k	y_k	True Value	Absolute Error
0	1	0.000000	0.000000	0.00000
1	1.05	0.050000	0.048790	0.00121
2	1.1	0.097619	0.095310	0.00231
3	1.15	0.143074	0.139762	0.00331
4	1.2	0.186552	0.182322	0.00423
5	1.25	0.228219	0.223144	0.00507
6	1.3	0.268219	0.262364	0.00585
7	1.35	0.306680	0.300105	0.00658
8	1.4	0.343717	0.336472	0.00724
9	1.45	0.379431	0.371564	0.00787
10	1.5	0.413914	0.405465	0.00845
11	1.55	0.447247	0.438255	0.00899
12	1.6	0.479506	0.470004	0.00950
13	1.65	0.510756	0.500775	0.00998
14	1.7	0.541059	0.530628	0.01043
15	1.75	0.570470	0.559616	0.01085
16	1.8	0.599042	0.587787	0.01126
17	1.85	0.626820	0.615186	0.01163
18	1.9	0.653847	0.641854	0.01199
19	1.95	0.680162	0.667829	0.01233
20	2	0.705803	0.693147	0.01266

Example 3.1.3 Euler's Method—Unknown Exact Solution

Suppose we're given the IVP $y' = \sqrt{x+y}$, $y(5) = 4$, and we want to find $y(4)$.

The first thought that should occur to us is that the equation is neither separable nor linear. Are we in trouble here? *No*, not if we understand Euler's method.

In our problem $f(x, y) = \sqrt{x+y}$, so Eq. (3.1.3) takes the form

$$y_{k+1} = y_k + h\sqrt{x_k + y_k},$$

where $x_k = 5 + kh$, $k = 0, 1, \ldots, n$. As usual, n denotes the number of steps we choose.

Let's start off by choosing five steps to get us from the initial point $x = 5$ to our destination $x = 4$. Each step has to have length 0.2, and because we are moving *backward* from the initial point, we must take $h = -0.2$ in the formula. We'll carry out this first attempt at approximation by hand and then use a spreadsheet when the calculations become more numerous (and more tedious).

The formula gives us

$$y_1 = y_0 + h\sqrt{x_0 + y_0} = 4 + (-0.2)\sqrt{5+4} = 3.4$$
$$y_2 = y_1 + h\sqrt{x_1 + y_1} = 3.4 + (-0.2)\sqrt{4.8 + 3.4} = 2.82728716$$
$$y_3 = y_2 + h\sqrt{x_2 + y_2} = 2.82728716 + (-0.2)\sqrt{4.6 + 2.82728716} = 2.28222616$$

$$y_4 = y_3 + h\sqrt{x_3 + y_3} = 2.28222616 + (-0.2)\sqrt{4.4 + 2.28222616} = 1.76522612$$
$$y_5 = y_4 + h\sqrt{x_4 + y_4} = 1.76522612 + (-0.2)\sqrt{4.2 + 1.76522612} = 1.27674987.$$

These calculations tell us that $y(4) \approx 1.27674987$. The *true answer* is **1.34042895566892**. . . .[1] Therefore, when we round the "true" answer to eight places, the absolute error is $|1.34042896 - 1.27674987| = 0.06367909$. If we choose $h = -0.01$ and use 100 steps, a spreadsheet calculation gives us an approximation of 1.337296, with an absolute error of 0.0031.

So far we've been cheating a bit, discussing the numerical solutions of equations for which we were able to find an analytic solution (even if implicit). Knowing the exact solution allowed us to analyze the error—the gap between the true solution value and the approximate value of a solution at a point. However, it's time to consider a more typical example.

Example 3.1.4 Euler's Method—A Completely Unknown Solution

The IVP $\frac{dy}{dt} = y^2 - t^2$, $y(0) = \frac{1}{2}$, cannot be solved by any of the methods we have discussed so far, although this special type of *Riccati equation*[2] does have a *series solution* in terms of *Bessel functions* (see Section D.3). Nevertheless, we can approximate the solution at $t = 1$ (for example) so that it is accurate to, say, three decimal places.

"Without the exact answer as a guide, how do we know that these three decimal places are accurate?" you may be asking. Let's skip the detailed formula and see what happens for different step sizes (Table 3.4). We have rounded the approximations in the last column to six decimal places.

Table 3.4 The IVP $\frac{dy}{dt} = y^2 - t^2$, $y(0) = \frac{1}{2}$: Approximate Values of $y(1)$ for Various Step Sizes

Step Size	Number of Steps	Approximate Value
1/100	100	0.512113
1/1000	1000	0.506106
1/2000	2000	0.505769
1/4000	4000	0.505600
1/8000	8000	0.505515
1/16,000	16,000	0.505473
1/20,000	20,000	0.505464

[1] This answer is obtained by making a substitution to transform the given equation into a separable equation (see the explanation that precedes Problem 12 of Exercises 2.1), solving the equation to get an implicitly defined solution to the IVP, and then solving for the value of y when $x = 4$. Solving the implicit relation for y requires a calculator or CAS with a "solve" function for general equations. Even this "true" answer is only an approximation (although presumably a very accurate one) because the algebraic equation can't be solved exactly.

[2] See Chapter 2, Section 5 of *Ordinary Differential Equations: A Brief Eclectic Tour* by David Sánchez (Washington, D.C.: Mathematical Association of America, 2002) for an informative discussion of this important class of differential equations.

We have reached a stage where the first three digits of the approximate values do not seem to be changing. The last approximate value agrees with the previous one to three decimal places after appropriate rounding, so we can assume that the approximation is 0.505, accurate to three significant digits. The idea—a rule of thumb based on mathematical analysis—is to keep on using smaller step sizes until there are changes only *past* the decimal place we are interested in. Then we can be sure of those decimal places that do *not* change.

*3.1.1 Stiff differential equations[3]

We can encounter difficulty applying Euler's method (and some other methods) to approximate solutions when these solutions have components whose timescales differ widely. Another way of saying this is that stiffness occurs when some components of the solution decay much more rapidly than others. For instance, the solution of the circuit problem in Example 2.2.5 had two components: (1) a *transient term* of the form Ce^{-at}, with $C > 0$ and $a > 0$, that decreased rapidly to zero as $t \to \infty$ and (2) a *steady-state term* of the form $A\sin(\omega t) + B\cos(\omega t)$ that oscillated with time. Instead of approximating the steady-state part of the solution, Euler's method may allow the error associated with the transient part to dominate, producing meaningless results.

Equations exhibiting this characteristic behavior include many that arise in electrical circuit theory and in the study of chemical reactions. The term *stiff* is used because these numerical difficulties occur when analyzing the motion of spring-mass systems with large spring constants—that is, systems with "stiff" springs. (The stiffness of a spring depends on the materials it is made of and on the specific manufacturing processes used.) In Chapter 6, we'll discuss spring-mass problems in greater detail.

For now, let's look at an example that highlights the difficulty.

Example 3.1.5 A Stiff Differential Equation

Suppose we look at the IVP $\frac{dI}{dt} + 50I = \sin(\pi t)$, $I(0) = 0$, which is just the equation in Example 2.2.5 with $L = 1$, $R = 50$, $v_0 = 1$, and $\omega = \pi$. According to that example, the solution is

$$I(t) = \frac{1}{(2500 + \pi^2)}\left\{50\sin(\pi t) - \pi\cos(\pi t) + \pi e^{-50t}\right\}.$$

(*Check this for yourself.*) Note both the transient component and the steady-state part.

Now suppose that we want to approximate the solution at $t = 2$. To understand the accuracy of the approximation, we can first use the solution formula to find the exact answer,

$$I(2) = \frac{-\pi}{2500 + \pi^2}\left(1 - \frac{1}{e^{100}}\right) = -0.001251695566\ldots.$$

For further comparison with approximations, here are some actual values of I at intermediate points between 0 and 2:

---

[3] * Denotes an optional section.

$$I(0.5) = 0.01992135365\ldots \qquad I(1.0) = 0.001251695566\ldots$$

$$I(1.5) = -0.1992135365\ldots.$$

Euler's method yields the formula $i_{k+1} = i_k + h(\sin(\pi t_k) - 50 i_k)$. Table 3.5a displays the results of using Euler's method with $h = 0.1$, Table 3.5b shows the results when $h = 0.05$, and Table 3.5c shows what happens when $h = 0.01$. We omit the error column in each table because the discrepancies or agreements between the actual values of I and the approximate values are fairly obvious in each table.

Table 3.5a $h = 0.1$

k	t_k	i_k
0	0	0
5	0.5	−1.26575
10	1.0	1316.73504
15	1.5	-0.13483×10^7
20	2.0	0.13807×10^{10}

Table 3.5b $h = 0.05$

k	t_k	i_k
0	0	0
10	0.5	0.09262
20	1.0	4.18748
30	1.5	241.37834
40	2.0	13,920.24471

Table 3.5c $h = 0.01$

k	t_k	i_k
0	0	0
50	0.5	0.01994
100	1.0	0.00125396
150	1.5	−0.01994
200	2.0	−0.00125396

You can see that the errors in approximating $I(2)$ are unacceptably large for $h = 0.1$ and $h = 0.05$, whereas there is very little error when we have reduced h to 0.01. Comparing the graph of the actual solution curve for $I(t)$ with the graphs of the approximation curves given by Euler's method for these three values of h is a real eye-opener (see Figs. 3.8a, 3.8b, and 3.8c). In each graph, the blue line is the actual solution curve, and the black line is the approximation. Note that the scales are different from graph to graph.

Fig. 3.8c shows that you can hardly distinguish between the actual solution curve and its Euler method approximation when $h = 0.01$. The choice of the interval [0, 0.35] for t was made after some experimentation. Using the technology available to you, you should look at the graphs of the approximations on larger intervals. For larger values of t, beginning around $t = 1$, you will find that the approximation curves are rather alarming distortions of the steady-state solution.

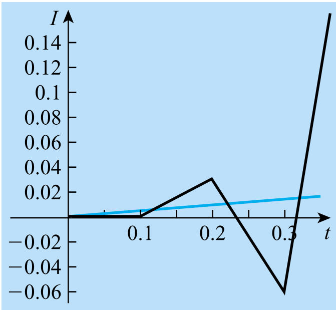

FIGURE 3.8a

$h = 0.1$

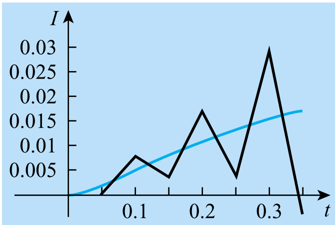

FIGURE 3.8b

$h = 0.05$

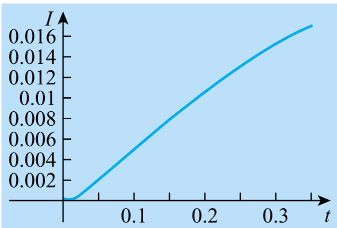

FIGURE 3.8c

$h = 0.01$

In general, the solution of a stiff differential equation is impractical with numerical methods not designed specifically for such problems. As we have seen, Euler's method (also called the *forward* Euler method) can fail to approximate the solution of a stiff equation.

Now let's look at a simple IVP, $\frac{dy}{dt} = ay$, $y(0) = y_0$, that is often used as a test case for numerical methods. This can be considered a problem with a transient term and a zero steady-state term. Given $t_0 = 0$ and $y(0) = y_0$, we can apply Euler's algorithm with step size h,

$$y_{n+1} = y_n + hf(t_n, y_n),$$

so

$$\begin{aligned}
y_1 &= y_0 + h(ay_0) = (1+ha)y_0, \\
y_2 &= y_1 + h(ay_1) = (1+ha)y_1 = (1+ha)^2 y_0, \\
&\quad\vdots \\
y_n &= y_{n-1} + h(ay_{n-1}) = \cdots = (1+ha)^n y_0.
\end{aligned}$$

If we're trying to approximate $y(T)$, where T is some value of t other than 0, then $h = T/n$ since we are dividing the interval $[0, T]$ into n equal subintervals. Thus we can write

$$y_n = y_0 \left(1 + \frac{aT}{n}\right)^n.$$

Recalling that $\lim_{n\to\infty}(1+1/n)^n = e$, we can let $m = n/aT$ to see that

$$\lim_{n\to\infty} y_0 \left(1+\frac{aT}{n}\right)^n = y_0 \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{maT} = y_0 \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^{aT} = y_0 e^{aT}.$$

Going back to our original notation, this says that $y_n = (1+ha)^n y_0$ converges to $y_0 e^{aT}$ as $h \to 0$. Since Euler's method is a first-order method, the error decreases as a multiple of h: $|e^{aT} - (1+ha)^n| \le Mh$ for some constant M. HOWEVER, if $a < 0$, then not only will the error be large if the value of h is too large, but the numerical approximation will grow exponentially instead of decaying exponentially like the true solution.

For example, consider $a = -500$, which should lead to a rapidly decaying solution. Then if we take $h = 0.01$, we see that $y_n = (1 - 500/100)^n y_0 = (-4)^n y_0$, which is a rapidly growing approximation oscillating between positive and negative values. In order to guarantee a decaying solution, h must satisfy $|1 + ha| < 1$ (so that $(1+ha)^n \to 0$), or $0 < h < -2/a$. If, for example, $a = -500$, we must have $h < -2/(-500) = 1/250$.

Euler's procedure is called an *explicit* method because in equation (3.1.3) the value y_{k+1} is given explicitly in terms of known quantities such as y_k and $f(x_k, y_k)$. In contrast, the **backward Euler method** is an *implicit* scheme that finds the solution by solving an equation involving the current state of a system and the later one. More precisely, we have the new algorithm

$$y_{k+1} = y_k + hf(x_{k+1}, y_{k+1}). \tag{3.1.5}$$

This is not a typo. The new approximation y_{k+1} appears on both sides of equation (3.1.5) and must be solved for y_{k+1}—that is, expression (3.1.5) is not a *formula* for

y_{k+1}, but an *equation* for y_{k+1}. It expresses y_{k+1} *implicitly*. In some applications of this method we may have to solve nonlinear equations to find y_{k+1}. This can be done, for example, by using an algorithm such as Newton's method (the Newton–Raphson method) or simply by using a CAS *solve* command. Like Euler's forward procedure, Euler's backward method is a first-order method. Its cumulative error is bounded by a constant multiple of the first power of the step size h.

The reason for using a method with such a high computational cost is that implicit techniques are *stable*. Loosely speaking, errors introduced in the course of such a procedure remain bounded (possibly even die out) and the numerical approximation of a solution will mimic the actual solution. Another aspect of stability is that solutions resulting from changes ("perturbations") in initial value remain close to the original solution.

Let's try this new algorithm on our last IVP, contrasting the accuracy and stability of Euler's forward and backward methods.

Example 3.1.6 Euler's Backward Method

Let's tackle the IVP $\frac{dy}{dt} = -10y$, $y(0) = 1$ and use both Euler methods to approximate $y(2)$.

Using Euler's forward process with $h = 0.5$, which is larger than the value of h recommended in our previous analysis ($h < -2/(-10) = 0.2$), we get Table 3.6a.

Table 3.6a $h = 0.5$

k	t_k	y_k
0	0	1
1	0.5	−4
2	1.0	16
3	1.5	−64
4	2.0	256

We can't help noticing the chaotic behavior since $y_n = (1+hk)^n = (1+(0.5)(-10))^n = (-4)^n$. Now let's see what Euler's backward method gives us.

We have

$$y_1 = y_0 + hf(t_1, y_1) = 1 + (0.5)(-10)y_1, \text{ or } y_1 = \frac{1}{6}$$

$$y_2 = y_1 + hf(t_2, y_2) = \frac{1}{6} - 5y_2, \text{ or } y_2 = \frac{1}{36}$$

$$y_3 = y_2 + hf(t_3, y_3) = \frac{1}{6} - 5y_3, \text{ or } y_3 = \frac{1}{216}$$

$$y_4 = y_3 + hf(t_4, y_4) = \frac{1}{6} - 5y_4, \text{ or } y_4 = \frac{1}{1296}.$$

Table 3.6b shows the values for the backward Euler method.

First of all, we notice that the approximate values y_n, $n = 1, 2, 3, 4$, are decreasing rapidly, as they should. Next, the exact value of $y(2)$ is $e^{(-10)(2)} = e^{-20} \approx 2.061153622 \times 10^{-9}$, so Euler's backward method gives a better (but not superb) approximation. With a better choice of h, we would expect Euler's backward method to yield a much more accurate approximation.

Table 3.6b $h = 0.5$

k	t_k	y_k
0	0	1
1	0.5	0.1667
2	1.0	0.0278
3	1.5	0.0046
4	2.0	0.0008

See Section A.3 for a further discussion of approximation error. In the subsequent sections of this chapter, we will investigate improved algorithms.

Exercises 3.1

In the following problems, being asked to do a problem "by hand" or "manually" means that each step should be written out and although calculators may be used to do arithmetic, no calculator routine or CAS program for Euler's method should be used. This is the opposite of being allowed to "use technology."

A

For Problems 1–3, use Euler's method by hand with the given step sizes to approximate the solution to the given IVP over the specified interval. Include a table of values, and give a sketch of the approximate solution by plotting the values you have calculated.

1. $\frac{dy}{dt} = t^2 - y^2$, $y(0) = 1$; $0 \le t \le 1$, $h = 0.25$
2. $\frac{dy}{dt} = e^{(2/y)}$, $y(0) = 2$; $0 \le t \le 2$, $h = 0.5$
3. $\frac{dy}{dt} = e^{(2/y)}$, $y(1) = 2$; $1 \le t \le 3$, $h = 0.5$
4. Compare your answers to Problems 2 and 3 and explain what you see.
5. If y is the solution of the IVP $\frac{dy}{dt} = \cos t$, $y(0) = 0$, use Euler's method manually with $h = \pi/10$ to approximate $y(\pi/2)$. What is the absolute error?
6. Approximate $y(1.4)$ by hand if y is the solution to the IVP $\frac{dy}{dx} = x^3$, $y(1) = 1$. Use $h = 0.1$.
7. Given $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 1$, use $h = 0.1$ to approximate $y(1)$ manually.
8. Given the IVP $y' = y \sin 3t$, $y(0) = 1$, use technology to approximate $y(4)$ using 20 steps.
9. Given $y' = 1/(1 + x^2)$, $y(0) = 0$, use $h = 0.1$ to approximate $y(1)$ manually. How can you use your result to compute π?
10. Consider the IVP $y' = x^2 + y$, $y(0) = 1$. By hand, approximate $y(0.1)$, $y(0.2)$, and $y(0.3)$ using both $h = 0.1$ and $h = 0.05$ for each approximation.
11. Consider the IVP $y' = y^2$, $y(0) = 1$.
    a. Using $h = 0.2$, approximate the solution y over the interval [0, 1.2] by hand.

b. Show that the exact solution is given by $y = \frac{1}{1-t}$.

c. Compare the values found in part (a) with values given by the formula in part (b). Explain any strange numerical behavior. [*Hint*: A slope field or solution graph may help.]

12. Consider the IVP $\frac{dy}{dt} = \frac{\sin y}{t}$, $y(2) = 1$. Using $h = 0.1$, approximate the solution y over the interval $[2, 3]$ by hand.

B

13. In Problem 5 of Exercises 2.3, you were given the following model for the population of Botswana: $\frac{dP}{dt} = 0.0355\,P - 0.00160625\,t$, with $P(0) = 1.285$ (million). The value $t = 0$ corresponds to 1990.

a. Use technology and Euler's method with $h = 0.01$ to approximate $P(1)$, the population in 1991.

b. Using the approximation for $P(1)$ found in part (a) as your starting point and $h = -0.01$, approximate $P(0)$.

14. In the area of pharmacokinetics, the *Michaelis–Menten equation*, $\frac{dx}{dt} = \frac{-Kx}{A+x}$, describes the rate at which a body processes a drug. Here $x(t)$ is the concentration of the drug in the body at time t, and K and A are positive constants. [The equation was developed by the biochemical/medical researchers Leonor Michaelis (1875–1949) and Maud Menten (1879–1960).]

a. For a particular controlled substance, let $A = 6$, $K = 1$, and $x(0) = 0.0025$. Use technology and Euler's method with $h = 0.1$ to evaluate x for $t = 1, 2, 3, 10$, and 20. Estimate how long it takes for the concentration to be half of its initial value.

b. For alcohol, let $A = 0.005$, $K = 1$, and $x(0) = 0.025$. Use technology and Euler's method with $h = 0.01$ to evaluate x for $t = 0.01, 0.02, 0.03, 0.04$, and 0.05. Estimate how long it takes for the concentration to be half its initial value.

15. In modeling aircraft speed and altitude loss in a pull-up from a dive, the basic laws of physics yield the differential equation

$$\frac{dV}{d\theta} = \frac{-gV\sin\theta}{kV^2 - g\cos\theta},$$

where θ denotes the dive angle (in radians), $V = V(\theta)$ is the speed of the plane, $g = 9.8$ m/s^2 is the acceleration constant, and k is a constant related to the wing surface area. For a particular plane, $k = 0.00145$, $\theta_0 = -0.786$, and $V(\theta_0) = V_0 = 150$ m/s. Use $h = 0.006$ (which divides θ_0 evenly) and $n = 131$ to estimate $V(0)$, the plane's speed at the completion of its pull-up—that is, when it levels out to $\theta = 0$. (Use technology, of course!)

16. Consider the IVP $y' = 1 - t + 4y$, $y(0) = 1$. Using technology and $h = 0.1$, approximate the solution on the interval $0 \le t \le 1$. What error is made at $t = 1/2$ and $t = 1$?

17. Use Euler's method manually with both $h = 0.5$ and $h = 0.25$ to approximate $x(2)$, where $x(t)$ is the solution of the IVP $\frac{dx}{dt} = \frac{3t^2}{2x}$, $x(0) = 1$. Solve the equation exactly and compare the absolute errors you get with the different values of h.

18. Consider the IVP $y' = y(1 - y^2)$, $y(0) = 0.1$. Note that the equation has three equilibrium solutions.

a. Use a phase portrait analysis or a direction field to predict what *should* happen to the solution.

b. Use technology and Euler's method with $h = 0.1$ to step out to $x = 3$. What happens to the numerical solution?

19. Suppose $x' = x^3$.

a. Find an expression for x'' in terms of x, assuming that x is a function of t.

b. Suppose $x(0) = 1$. Is the solution curve concave up or concave down? Use the result in part (a) to justify your answer.

c. Does Euler's method overestimate or underestimate the true value of the solution at $t = 0.1$? Explain. (Don't actually carry out Euler's method.)

20. Consider the IVP $y' = y^\alpha$, $\alpha < 1$, $y(0) = 0$.

a. Find the *exact* solution of the IVP.

b. Show that Euler's method fails to determine an approximate solution to the IVP.

c. Show what happens if the initial condition is changed to $y(0) = 0.01$.

21. **a.** Sketch the direction field for $y' = \sqrt{1 - y^2}$.

b. Verify that a solution for this equation satisfying the initial condition $y(0) = 0$ is given by

$$y = \begin{cases} \sin t & 0 \le t < \frac{1}{2}\pi \\ 1 & \frac{1}{2}\pi \le t \end{cases}.$$

c. Describe the behavior of Euler's method when $h = 0.4$. Could you have predicted this behavior without any calculations?

22. Consider the IVP $\frac{dy}{dx} = x - y$, $y(0) = 1$.

a. Find the exact solution of this IVP.

b. Use Euler's forward method with $h = 0.5$ to approximate $y(2)$.

c. Approximate $y(2)$ using the following algorithm (*Euler's backward method*) with $h = 0.5$: $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$. [Note that this defines y_{n+1} *implicitly*, so you have to solve for y_{n+1} at each stage.]

C

23. Describe a class of differential equations for which Euler's method gives a *completely accurate* numerical solution—that is, for which y_k exactly equals the true solution $\varphi(x_k)$ for every k. [*Hint*: Try to think of differential equations for which all solution curves coincide with the tangent line segments.]

24. Consider the stiff differential equation $\frac{dy}{dt} = -100y + 1$, with $y(0) = 1$.

a. Solve this IVP and calculate the exact value of $y(1)$.

b. Use technology and Euler's method to approximate $y(1)$ with $h = 0.1$, 0.05, and 0.01.

c. Use technology to plot the exact solution and an approximate solution of the equation over the interval [0, 0.03] on the same set of axes. Do this for each of the three values of h mentioned in this problem.

25. The equation $y' = -50(y - \cos x)$ is stiff.

a. Use software to solve the equation with the initial condition $y(0) = 0$. Then calculate the exact value $y(0.2)$.

b. Use Euler's method and technology to approximate $y(0.2)$ with step size $h = 1.974/50$. What is the absolute error?

c. Use Euler's method and technology to approximate $y(0.2)$ with step size $h = 1.875/50$. What is the absolute error?

d. Use Euler's method and technology to approximate $y(0.2)$ with step size $h = 2.1/50$. What is the absolute error now?

e. Using technology, plot the three approximation curves found in parts (b), (c), and (d) on the same axes. Use the interval [0, 1]. Would you call the Euler method solution of the equation "sensitive to step size"?

26. The second-order IVP $y'' = F(x, y, y')$, $y(a) = c_1$, $y'(a) = c_2$ can be written as two simultaneous first-order equations: $y' = u$, $u' = F(x, y, u)$, where $y(a) = c_1$, $u(a) = c_2$.

a. Devise a procedure for approximating y and y' when $x = a + h$.

b. Use the method found in part (a) to approximate the solution of the IVP $y'' = x + y$, $y(0) = y'(0) = 0$ at $x = 1$.

c. Given that the exact solution of the IVP in part (b) is $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x} - x$, compare the approximate value of $y(1)$ found in part (b) to the exact value.

3.2 The improved Euler method

In Euler's original method, the slope $f(x, y)$ over any interval $x_k \le x \le x_{k+1}$ of length h is replaced by $f(x_k, y_k)$, so that x always takes the value of the left endpoint of the interval. (If $y' = f(x)$, a function of x alone, then Euler's method is equivalent to using a left-hand Riemann sum to approximate a definite integral.)

Now instead of always using the slope at the *left* endpoint of the interval $[x_k, x_{k+1}]$, we can think of using an *average* derivative value over the interval. The **improved Euler method** involves two stages that will be combined into one approximation formula. The first stage involves moving tentatively across the interval $[x_k, x_{k+1}]$ using Euler's original method, thereby producing a guess, or trial value, $\hat{y}_{k+1} = y_k + h \cdot f(x_k, y_k)$. Note that the values $f(x_k, y_k)$ and $f(x_{k+1}, \hat{y}_{k+1})$ approximate the slopes of the solution curve at $(x_k, y(x_k))$ and $(x_{k+1}, y(x_{k+1}))$, respectively. Now the second stage looks at the *average* of the derivative $f(x_k, y_k)$ and the guess $f(x_{k+1}, \hat{y}_{k+1}) = f(x_{k+1}, y_k + hf(x_k, y_k))$ and uses this average to take the *real* step across the interval.

$$\textbf{Guess (tentative step): } \hat{y}_{k+1} = y_k + h \cdot f(x_k, y_k)$$

$$\begin{aligned} \textbf{Real step: } y_{k+1} &= y_k + h\left\{\frac{f(x_k, y_k) + f(x_{k+1}, \hat{y}_{k+1})}{2}\right\} \\ &= y_k + h\left\{\frac{f(x_k, y_k) + f(x_{k+1}, y_k + h \cdot f(x_k, y_k))}{2}\right\} \\ &= y_k + \frac{h}{2}\{f(x_k, y_k) + f(x_{k+1}, y_k + h \cdot f(x_k, y_k))\}. \end{aligned} \tag{3.2.1}$$

Formula (3.2.1) describes the **improved Euler method** (or **Heun's method**, named for Karl Heun (1859–1929), a German applied mathematician who devised this scheme around 1900). It is an example of a **predictor-corrector method:** We use $\hat{y}_{k+1}$ (via Euler's method) to *predict* a value of $y(x_{k+1})$ and then use y_{k+1} to *correct* this value by averaging.

Look carefully at Eq. (3.2.1). If $f(x, y)$ is really just $f(x)$, a function of x alone, then solving the IVP $y' = f(x, y)$, $y(x_0) = x_0$ amounts to solving the equation $y' = f(x)$, which is a matter of simple integration. In Section 1.3 [Eq. (1.3.1)] we saw that we can write the solution as

$$y(x) = \int_{x_0}^{x} f(t)\,dt + y_0 = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(t)\,dt + y_0,$$

where $x_n = x$. In this case formula (3.2.1) reduces to

$$y_{k+1} = y_k + \frac{h}{2}\{f(x_k) + f(x_{k+1})\}$$

and the Fundamental Theorem of Calculus tells us on the interval $[x_k, x_{k+1}]$ that

$$\int_{x_k}^{x_{k+1}} \overbrace{f(t)}^{y'}\,dt = y(x_{k+1}) - y(x_k) \approx y_{k+1} - y_k = \frac{h}{2}\{f(x_k) + f(x_{k+1})\}.$$

In other words, in this situation we are using the *Trapezoid Rule* from calculus to approximate each integral on $[x_k, x_{k+1}]$.

Next, we see some illustrations of why this method is called "improved."

Example 3.2.1 The Improved Euler Method

Let's use the improved Euler formula (3.2.1), to calculate an approximate value of the solution of the IVP $y' = y$, $y(0) = 1$ at $x = 1$. (*Of course you realize that this is just a roundabout way of asking for an approximation of that important mathematical constant e, right?*)

Let's start with $h = 0.1$, so we'll need 10 steps to reach $x = 1$ from the initial point $x = 0$. Thus, in formula (3.2.1) we have $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_k = 0 + kh = kh$ $(k = 0, \ldots, 10)$, and $f(x, y) = y$. When we put all this information together, we see that the formula takes a simplified form:

$$\begin{aligned} y_{k+1} &= y_k + \frac{h}{2}\{y_k + (y_k + hy_k)\} \\ &= y_k + \frac{h}{2}\{(2+h)y_k\} = y_k + (0.05)(2.1)y_k = 1.105\, y_k. \end{aligned}$$

Therefore the calculations are

$$\begin{aligned} y_1 &= 1.105 y_0 = 1.105(1) = 1.105 \\ y_2 &= 1.105 y_1 = (1.105)^2 = 1.221025 \\ y_3 &= 1.105 y_2 = (1.105)^3 = 1.349232625 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ y_{10} &= 1.105 y_9 = (1.105)^{10} = 2.71408084661. \end{aligned}$$

Comparing this approximation to the actual value 2.71828182846 (rounded to 11 decimal places), we find that the absolute error is 0.00420098185. (In Problem 4 of Exercises 3.2 you'll be asked to try this with the original Euler method.)

Using 20 steps, a CAS gives the approximate value 2.71719105435, so the absolute error is now 0.00109077410. Note that when we doubled the number of steps from 10 to 20, the result was that the absolute error was roughly one-fourth what it was before.

Now let's revisit Example 3.1.1 to see how the improved method compares with the original process of approximation.

Example 3.2.2 The Improved Euler Method—Example 3.1.1 Revisited

We want to approximate $x(1.5)$, given the IVP $\frac{dx}{dt} = t^2 + x$, $x(1) = 3$. The actual value is $5.939770\ldots$. We'll start with $h = 0.1$, so we'll need five steps to stretch between $t = 1$ and $t = 1.5$.

For this problem, the improved Euler formula is

$$\begin{aligned} x_{k+1} &= x_k + \frac{h}{2}\left\{\left(t_k^2 + x_k\right) + t_{k+1}^2 + x_k + h\left(t_k^2 + x_k\right)\right\} \\ &= x_k + \frac{h}{2}\left\{t_{k+1}^2 + (1+h)t_k^2 + (2+h)x_k\right\} \\ &= x_k + (0.05)\left\{t_{k+1}^2 + 1.1t_k^2 + 2.1x_k\right\}, \end{aligned}$$

where $t_0 = 1$, $t_1 = 1.1$, $t_2 = 1.2$, $t_3 = 1.3$, $t_4 = 1.4$, and $t_5 = 1.5$. Therefore,

$$\begin{aligned} x_1 &= 3 + (0.05)\{(1.1)^2 + 1.1(1)^2 + 2.1(3)\} = 3.4305 \\ x_2 &= 3.4305 + (0.05)\{(1.2)^2 + 1.1(1.1)^2 + 2.1(3.4305)\} = 3.9292525 \\ x_3 &= 3.9292525 + (0.05)\{(1.3)^2 + 1.1(1.2)^2 + 2.1(3.9292525)\} \\ &= 4.5055240125 \\ x_4 &= 4.5055240125 + (0.05)\{(1.4)^2 + 1.1(1.3)^2 + 2.1(4.5055240125)\} \\ &= 5.16955403381 \end{aligned}$$

$$x_5 = 5.16955403381 + (0.05)\{(1.5)^2 + 1.1(1.4)^2 + 2.1(5.16955403381)\}$$
$$= 5.93265720736.$$

To five decimal places we have $x(1.5) \approx 5.93266$. The absolute error is 0.00711. When we employed Euler's method in Example 3.1.1, the error was 0.244639.

If we use 10 steps in the improved Euler method, then we get $x(1.5) \approx 5.943455$, with an absolute error of 0.00369, compared to the Euler method's error of 0.12717.

Now let's go back and redo another earlier example with the new method.

Example 3.2.3 Improved Euler Method—Example 3.1.3 Revisited

In Example 3.1.3 we discussed the IVP $y' = \sqrt{x+y}$, $y(5) = 4$ with the goal of approximating $y(4)$. If we apply the improved method to the problem, with five backward steps each of length 0.2—that is, with $h = -0.2$—we get the values shown in Table 3.7.

Table 3.7 Improved Euler Method with $h = -0.2$

k	x_k	y_k	True Value	Absolute Error
0	5.0	4.000000	4.000000	0.000000
1	4.8	3.413644	3.413384	0.000260
2	4.6	2.854277	2.853750	0.000527
3	4.4	2.322249	2.321444	0.000805
4	4.2	1.817952	1.816857	0.001100
5	4.0	1.341827	1.34043	0.00140

Thus, $y(4) \approx 1.341827$ by the improved method, compared to the "true" answer 1.34042895566892 and the original Euler method approximate value 1.27674987.

An analysis of error shows that the *local* truncation error at any stage of the improved Euler method behaves like a constant multiple of h^3 and that *the cumulative truncation error is no greater than a constant multiple of the square of the step size* h: $|\text{true value} - \text{approximation}| \le K \cdot h^2$, where K is a constant that depends on the function $f(x, y)$, on its partial derivatives, and on the interval involved, but does not depend on h. We say that the improved Euler method is a **second-order method**.

In the next section, we'll look at a fourth-order method and a powerful combination of fourth- and fifth-order techniques.

Exercises 3.2

A

Use the following table to enter the data from Problems 1 and 2.

	TRUE VALUE	Euler's Method	Absolute Error	Improved Euler Method	Absolute Error
$h = 0.1$					
$h = 0.05$					
$h = 0.025$					

1. Use the improved Euler method to redo Example 3.1.1 with $h = 0.1$, 0.05, and 0.025.
2. Use the improved Euler method to redo Example 3.1.2 with $h = 0.1$, 0.05, and 0.025. (You'll also have to use Euler's method for $h = 0.1$.)
3. a. Find the exact solution to the IVP $\frac{dx}{dt} = t + x$, $x(0) = 1$.
   b. Apply the improved Euler method with step size $h = 0.1$ to approximate the value $x(1)$.
   c. Calculate the absolute error at each step of part (b).
4. Using technology, redo Example 3.2.1 with both Euler's method and the improved Euler method, using a step size of $h = 0.01$—that is, using 100 steps. For each method calculate the absolute errors incurred in approximating $y(0.01), y(0.02), \ldots, y(0.99), y(1.0)$. (A spreadsheet program can be particularly useful here.)

B

5. Redo Problem 13 in Exercises 3.1 using the improved Euler method.
6. Redo Problem 15 in Exercises 3.1 using the improved Euler method.
7. Redo Problem 20 in Exercises 3.1 using the improved Euler method.
8. Consider the stiff IVP (see Section 3.1.1) $\frac{dy}{dx} = -50(y - \cos x)$, $y(0) = 0$.
   a. Use a CAS and apply the improved Euler method (Heun's method) with $h = 0.5$, 0.1, and 0.01 to approximate the value of $y(1)$.
   b. Find the exact solution of the (linear) IVP by hand and use this to calculate $y(1)$.
   c. Plot the solution found in part (b) from $x = -0.01$ to $x = 7$.

C

9. Redo Problem 24 in Exercises 3.1 using the improved Euler method.

3.3 More sophisticated numerical methods: Runge–Kutta and others

Modern computers (and even hand-held calculators) have many algorithms for solving differential equations numerically. Some of these are highly specialized and are meant to handle very particular types of ordinary differential equations (ODEs) (such as stiff equations; see Section 3.1.1) and systems of ODEs. Euler's method and its improved version are useful for illustrating the idea behind numerical approximation, but they are not very efficient in terms of approximating a solution of an IVP very accurately and with a minimum number of steps.

A very good method, implemented in many CASs and in calculator firmware, is the **fourth-order Runge–Kutta method (RK4),** which was developed in an 1895 paper by Carl Runge (1856–1927), a German mathematician, and was generalized to *systems* of ODEs in 1901 by M. Wilhelm Kutta (1867–1944), a German mathematician and aerodynamicist. As the description indicates, in this method the total accumulated error is proportional to h^4, so reducing the step size by a factor of $1/10$ produces four more digits of accuracy—for example, reducing the step size from $h = 0.1$ to $h = 0.01$ generally decreases the total error by a factor of 0.0001. (The *local* truncation error behaves as h^5.) There are also second- and third-order Runge–Kutta methods. (Euler's method can be called a first-order Runge–Kutta method.)

Now suppose we have an IVP $y' = f(x, y)$, $y(x_0) = y_0$. The RK4 formula looks a bit strange, but not if we realize that it is approximating the value $y(x_{k+1})$ by a *weighted average*, y_{k+1}, of values of $f(x, y)$ calculated at different points in the interval $[x_k, x_{k+1}]$. For each interval $[x_k, x_{k+1}]$, we calculate the following slopes in the order given:

$$\begin{aligned} m_1 &= f(x_k, y_k) \\ m_2 &= f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}m_1\right) \\ m_3 &= f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}m_2\right) \\ m_4 &= f(x_k + h, y_k + hm_3) = f(x_{k+1}, y_k + hm_3). \end{aligned} \tag{3.3.1}$$

Then the classical RK4 formula is

$$y_{k+1} = y_k + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4), \tag{3.3.2}$$

where the sum $(m_1 + 2m_2 + 2m_3 + m_4)/6$ is a weighted average of slopes.

The value m_1 is the slope at x_k calculated by Euler's method. Then m_2 is an estimate of the slope at the midpoint of the interval $[x_k, x_{k+1}]$, where Euler's method has been used to estimate the y value there. Now m_3 is a value for the slope at the midpoint of $[x_k, x_{k+1}]$ using the improved Euler's method. Finally, m_4 is the slope at x_{k+1} calculated by Euler's method, using the improved slope m_3 at the midpoint to step to x_{k+1}.

Perhaps this formula won't be so alarming if we look at the simplified situation when $f(x, y)$ is independent of y in the equation $y' = f(x, y)$. If $f(x, y) = g(x)$, then the set of equations (3.3.1) for m_1, m_2, m_3, and m_4 reduces to

$$\begin{aligned} m_1 &= g(x_k) \\ m_2 &= g\left(x_k + \frac{h}{2}\right) \\ m_3 &= g\left(x_k + \frac{h}{2}\right) \\ m_4 &= g(x_k + h) = g(x_{k+1}) \end{aligned}$$

so formula (3.3.2) becomes

$$y_{k+1} = y_k + \frac{h}{6}\left\{g(x_k) + 2g\left(x_k + \frac{h}{2}\right) + 2g\left(x_k + \frac{h}{2}\right) + g(x_{k+1})\right\}$$
$$= y_k + \frac{h}{6}\left\{g(x_k) + 4g\left(x_k + \frac{h}{2}\right) + g(x_{k+1})\right\}$$

and you may recognize the expression $\frac{h}{6}\left\{g(x_k) + 4g\left(x_k + \frac{h}{2}\right) + g(x_{k+1})\right\}$ as a form of *Simpson's Rule* for approximating $\int_{x_k}^{x_{k+1}} g(x)\,dx$. (Note that $x_k + \frac{h}{2}$ in the expression is the *midpoint* of the interval $[x_k, x_{k+1}]$ because $h = x_{k+1} - x_k$.)

To get a feel for the calculations, let's choose an example that we've seen before.

Example 3.3.1 RK4—Example 3.1.1 Yet Again

Let's approximate $x(1.5)$ by the Runge–Kutta method if we are given the IVP $\frac{dx}{dt} = t^2 + x$, $x(1) = 3$. We'll use $h = 0.1$, so we need five steps.

Just to get the idea, let's focus on the interval $[t_0, t_1] = [1, 1.1]$. We calculate

$$m_1 = f(t_0, x_0) = f(1, 3) = (1^2 + 3) = 4$$
$$m_2 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}m_1\right) = f(1 + 0.05, 3 + 0.05(4))$$
$$= 1.05^2 + 3.2 = 4.3025$$
$$m_3 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}m_2\right) = f(1 + 0.05, 3 + 0.4317625)$$
$$= 1.05^2 + 3.215125 = 4.317625$$
$$m_4 = f(t_0 + h, x_0 + hm_3) = f(t_1, x_0 + hm_3) = f(1.1, 3 + 0.4317625)$$
$$= 1.1^2 + 3.4317625 = 4.6417625$$

so

$$x(1.1) \approx x_1 = 3 + \frac{0.1}{6}(4 + 2(4.3025) + 2(4.317625) + 4.6417625)$$
$$= 3 + \frac{0.1}{6}(25.8820125) = 3.431366875.$$

The actual value of $x(1.1)$ (from the solution formula $x(t) = -t^2 - 2t - 2 + 8e^{t-1}$) is 3.4313673446.... Here, the absolute error is 0.0000004696. (*This is an amazingly close approximation!*)

Table 3.8 shows, for the same value $h = 0.1$, the exact values and the approximate values given for this problem by Euler's method, the improved Euler method, and the Runge–Kutta method. We can see how accurate the Runge–Kutta method is at each step.

As accurate as the classical Runge–Kutta method is, improvements are possible. For example a very popular method, the **Runge–Kutta–Fehlberg algorithm**, combines fourth-order and fifth-order methods in a clever way announced by E. Fehlberg in 1969. The **rkf45** method, as its computer implementation is known, uses *variable* step sizes, choosing the step size at each stage to try to achieve a predetermined degree of accuracy. Such clever numerical techniques are called **adaptive methods**. They may be useful, for example, in handling stiff differential equations (Subsection 3.1.1).

Table 3.8 Comparison of Methods with $h = 0.1$

t_k	True Value of $x(t_k)$	Euler's Method	Improved Euler Method	Runge–Kutta Method
1	3.00000	3.00000	3.00000	3.00000
1.1	3.43137	3.40000	3.43050	3.43137
1.2	3.93122	3.86100	3.92925	3.93122
1.3	4.50887	4.39110	4.50552	4.50887
1.4	5.17460	4.99921	5.16955	5.17460
1.5	5.93977	5.69513	5.93266	5.93977

A more dramatic problem that can be handled by adaptive methods is that of calculating a flight path from the Earth to the Moon and back,[4] which involves solving a system of differential equations numerically. In deep space, the driving force (the gravitational potential gradients of the Earth, Venus, and the Sun) varies slowly, so a relatively large step size can be used in solving the system. However, near the Earth or the Moon a relatively small step size is needed to achieve the same accuracy. (If this small step size is used for the entire flight, the calculation will be unnecessarily long, but if the spacecraft gets too close to the earth and the step size is too large, the calculation gives the craft too much kinetic energy and the craft zips out of the Earth–Moon system at the speed of light.)

Exercises 3.3

In the problems that follow, it is assumed that you have versions of the Runge–Kutta fourth-order method (RK4) and Runge–Kutta–Fehlberg (rkf45) method available to you. Use the following table to enter the data from Problems 1 and 2. You may go back to earlier examples to find the needed values.

A

	TRUE VALUE	Euler's Method	Improved Euler Method	RK4 Method
$h = 0.1$				
$h = 0.05$				
$h = 0.025$				

1. Use the RK4 method to redo Example 3.2.1 with $h = 0.1$, 0.05, and 0.025. (Using the improved Euler method, the cases $h = 0.1$ and $h = 0.05$ were done for you in the example.)

[4] The astronauts of Apollo 13, whose engine failed on the flight to the moon, had to follow such a flight path. The spacecraft had to be swung around by the gravity of the moon and returned to the vicinity of the Earth without further thrusts from its rocket motors.

2. Use the RK4 method to redo Example 3.2.2 with $h = 0.1$, 0.05, and 0.025.

3. Use the rkf45 method to approximate the solution of $y' = y$, $y(0) = 1$, at $t = 1$, with $h = 0.1$. (That is, approximate the value of the constant e. See Example 3.2.1.)

4. **a.** Find the exact solution of the IVP $\frac{dx}{dt} = t + x$, $x(0) = 1$.

b. Apply the rkf45 method with step size $h = 0.1$ to approximate $x(1)$, calculating the absolute error at each step.

5. **a.** Find the closed-form solution of the equation $\frac{dx}{dt} = -tx^2$.

b. Using the rkf45 method with $h = 0.1$, approximate the value $x(1)$ if x is the solution of the IVP $\frac{dx}{dt} = -tx^2$, $x(0) = 2$.

6. Approximate $y(0.8)$ using the rkf45 method with $h = 0.01$ if y is the solution of the IVP $\frac{dy}{dx} = \sin(xy)$, $y(0) = 0$.

B

7. A daredevil named Ellie goes skydiving and jumps from a plane at an initial altitude of 10,000 feet. At time t her velocity $v(t)$ satisfies the IVP $\frac{dv}{dt} = f(v)$, $v(0) = 0$, where

$$f(v) = 32 - (0.000025) \cdot (100v + 10v^2 + v^3).$$

If she does not open her parachute, she will reach *terminal velocity* when the forces of gravity and air resistance balance.

a. Use the rkf45 method to approximate her velocity at times $t = 5$, 10, 15, 16, 17, 18, 19, and 20, and so guess her terminal velocity (accurate to three decimal places).

b. Use technology to graph Ellie's velocity over the interval [0, 30].

8. In 1927, British scientists Kermack and McKendrick laid the foundations for the theory of epidemiology by presenting data on the number of deaths resulting from a plague spread by rats in Bombay (now Mumbai) during the period from December 1905 to July 1906. They gave the following equation for the total number of deaths as of week t:

$$\frac{dR}{dt} = 890\,\text{sech}^2(0.20t - 3.4),$$

where $\text{sech}\, t$ denotes the *hyperbolic secant function*.

a. Assuming that $R(0) = 0$, use the rkf45 method to fill in the following table.

t (weeks)	Actual Deaths	Predicted Deaths
1	4	
5	68	
10	425	
20	6339	
30	9010	

b. According to the Kermack–McKendrick model, what is the asymptotic value of R—that is, $\lim_{t\to\infty} R(t)$?

9. When using the Runge–Kutta methods (including Euler's method and Heun's method), it is important to realize that the error depends on the *form* of the equation and on the solution itself. To see an example of this, note that $y(x) = (x+1)^2$ is the solution to each of the two problems

$$y' = 2(x+1), \quad y(0) = 1$$
$$y' = 2y/(x+1), \quad y(0) = 1.$$

a. Show that Heun's method is exact for the first equation.

b. Show that the method is *not* exact when applied to the second equation even though it has the same solution as the first equation.

10. Consider the IVP $\frac{dx}{dt} = x^2$, $x(0) = 2$.

a. Use Euler's method with $h = 0.1$ to approximate $x(1)$. Does your answer seem strange?

b. Use the rkf45 method with $h = 0.1$ to approximate $x(1)$. Compare your answer to the answer in part (a) if your calculator or CAS gives you a meaningful answer in both cases.

c. To help explain your difficulties in parts (a) and (b), find the closed-form solution of the IVP.

d. Use your answer to part (c) to explain why your answers to parts (a) and (b) are both wrong.

e. How do you think you may be able to avoid the difficulty uncovered in part (d)? Maybe by changing step size? Try to solve the problem again using the rkf45 method.

C

11. Consider the generalized logistic equation

$$\frac{dP}{dt} = kP^{\alpha}\left(1 - \frac{P^{\beta}}{M}\right).$$

a. Let $k = 1$, $M = 5$, and $P(0) = 1$. Find numerical approximations to the solution in the range $0 \le t \le 10$ for the parameter pairs $(\alpha, \beta) = (0.5, 1), (0.5, 2), (1.5, 1), (1.5, 2), (2, 2)$.

b. Estimate a parameter pair (r, q) that approximately yields the values $P(0) = 1$, $P(2) = 2.4$, $P(4) = 2.9$.

Summary

Even if we can solve a first-order differential equation, we may not be able to find a closed-form solution. This difficulty has led to the development of numerical methods to *approximate* a solution to any degree of accuracy. Leaving aside *input error*, there are two main sources of error in numerical calculations done by hand, calculator, or computer: *round-off error* and *truncation error*. **Round-off error** is the kind of inaccuracy we get by taking a certain number of decimal places instead of taking the entire number. In particular, remember that a calculator or computer is limited in the number of decimal places it can handle. **Truncation error** occurs when we stop (or truncate) an approximation process after a certain number of steps. Finally, we must be aware that there is usually a trade-off when dealing with error. If we try to reduce the truncation error and increase the accuracy of our approximation by carrying out more steps (for example, by taking more terms of a Taylor series), we increase the calculation load and consequently run the risk of increasing *propagated* (cumulative) *error*. (See Section A.3.)

Euler's method uses the idea that values near a point on a curve can be approximated by values on the tangent line drawn to that point. If we want to approximate the solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$ on an interval $[a, b]$, we first partition $[a, b]$ by using $n + 1$ equally spaced points x_i, where $x_{i+1} - x_i = \frac{b-a}{n} = h$ for $i = 0, 1, \ldots, n - 1$. Then if y_i is an approximate value for $y(x_i)$, we can define the sequence of approximate solution values as $y_{k+1} = y_k + hf(x_k, y_k)$. Generally speaking, we can increase the accuracy of the approximation (reduce the error) by making the step size h smaller—that is, by making the *number* of steps n larger. For Euler's method, a **first-order method**, the cumulative truncation error is bounded by a constant multiple of the step size: $|\text{true value} - \text{approximation}| \leq K \cdot h$, where K is independent of h but depends on $|y''(x)|$ and the interval $[x_0, b]$. In practice, the actual error incurred in a problem will usually be less than this bound.

Euler's method (sometimes called the *forward Euler method*) does not work well for **stiff** differential equations. Stiffness occurs when some components of the solution decay much more rapidly than others. The **Euler backward method** does a better job in these cases, but is still not ideal for stiff equations. The sequence of approximate solution values is defined by the equation $y_{k+1} = y_k + hf(x_{k+1}, y_{k+1})$. Since you must solve for y_{k+1} each time you use this equation, it is called an *implicit* method.

An improvement on Euler's method called **Heun's method** guesses a value of $y(x_k)$ and then uses y_k to correct this guess by an averaging process. The algorithm can be expressed as follows:

Guess (tentative step): $\hat{y}_{k+1} = y_k + h \cdot f(x_k, y_k)$

Real Step:
$$
\begin{aligned}
y_{k+1} &= y_k + h\left\{\frac{f(x_k, y_k) + f(x_{k+1}, \hat{y}_{k+1})}{2}\right\} \\
&= y_k + h\left\{\frac{f(x_k, y_k) + f(x_{k+1}, y_k + h \cdot f(x_k, y_k))}{2}\right\} \\
&= y_k + \frac{h}{2}\left\{f(x_k, y_k) + f(x_{k+1}, y_k + h \cdot f(x_k, y_k))\right\}.
\end{aligned}
$$

For the improved Euler method, the cumulative truncation error is no greater than a constant multiple of the square of the step size h: |true value − approximation| $\leq K \cdot h^2$, where K is a constant that depends on the function $f(x, y)$, on its partial derivatives, and on the interval involved, but not on h. We say that the improved Euler method is a **second-order method**.

There are many more sophisticated algorithms for solving differential equations numerically. Two very effective methods implemented in many CASs and even some calculators are the **fourth-order Runge–Kutta method** and the **Runge–Kutta–Fehlberg algorithm**. The *rkf45* method, as the computer implementation of this last algorithm is known, uses *variable* step sizes and chooses the best step size at each stage to achieve a predetermined degree of accuracy. Such clever numerical techniques are called **adaptive methods**.

CHAPTER

Second- and higher-order equations

4

Introduction

In Chapters 2 and 3, we analyzed first-order equations graphically, numerically, and analytically and introduced qualitative concepts that will be useful in later chapters.

In this chapter, we will make the jump from first-order equations to higher-order equations, especially second- and third-order equations. We'll start by investigating types of second-order linear equations that occur frequently in science and engineering applications. These equations have a fully developed theory that generalizes to higher-order equations of the same type.

This chapter will close with a discussion of existence and uniqueness of solutions for higher-order differential equations.

4.1 Homogeneous second-order linear equations with constant coefficients

A very important application of differential equations is the analysis of an *RLC circuit* containing a resistance R, an inductance L, and a capacitance C. (We have already seen some first-order examples in Chapter 2.) In electrical circuit theory, if $I = I(t)$ represents the current, *Kirchhoff's Voltage Law* leads to the equation $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$ when the voltage applied to the circuit is constant (for example, when a battery is used). We describe any equation of the form $ay'' + by' + cy = 0$, where a, b, and c are constants and $a \neq 0$, as a **homogeneous second-order linear equation with constant coefficients**. In this section, we will develop a technique for solving any equation of this type.

Extending the way we considered a first-order linear equation in Section 2.2, we see that a linear second-order equation with constant coefficients can be viewed in terms of an *operator* L transforming functions that have two derivatives: $L(y) = ay'' + by' + cy$. To solve a homogeneous equation, we must find a function y such that $L(y) = 0$. There is a natural extension of the *Superposition Principle* (see Section 2.2) for homogeneous equations:

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Superposition Principle for Homogeneous Equations

Suppose y_1 and y_2 are solutions of the homogeneous second-order linear differential equation

$$ay'' + by' + cy = 0$$

on an interval I. Then the linear combination $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants, is also a solution on this interval.

(You'll be asked to prove this in Problem 15 of Exercises 4.1.)

If we consider a homogeneous *first*-order linear equation with constant coefficients, $ay' + by = 0$, where $a \neq 0$, we know that the general solution is $y = Ce^{-\frac{b}{a}t}$. In 1739, aware of this solution, Euler[1] proposed solving an nth-order homogeneous linear equation with constant coefficients by looking for solutions of the form $y = e^{\lambda t}$, where λ is a constant to be determined. Let's see how this works for the equation

$$ay'' + by' + cy = 0, \tag{4.1.1}$$

where a, b, and c are constants.

If we assume that $y = e^{\lambda t}$ is a solution of Eq. (4.1.1), then $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$. Substituting these derivatives into (4.1.1), we get $a\left(\lambda^2 e^{\lambda t}\right) + b\left(\lambda e^{\lambda t}\right) + c\left(e^{\lambda t}\right) = 0$, which simplifies to $\left(a\lambda^2 + b\lambda + c\right)e^{\lambda t} = 0$. Because the exponential factor is never zero, we must have $a\lambda^2 + b\lambda + c = 0$.

4.1.1 The characteristic equation and eigenvalues

We have just concluded that if $y = e^{\lambda t}$ is a solution of Eq. (4.1.1), then λ must satisfy the equation $a\lambda^2 + b\lambda + c = 0$, which is called the **characteristic equation** (or **auxiliary equation**) of the differential Eq. (4.1.1). The roots of this characteristic equation will reveal to us the nature of the solution(s) of (4.1.1). Note that we can go straight from the ordinary differential equation (ODE) to the characteristic equation:

$$\underbrace{ay''}_{a\cdot\text{2nd derivative}} + \underbrace{by'}_{b\cdot\text{1st derivative}} + \underbrace{cy}_{c\cdot\text{0th derivative}} = 0$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\overbrace{a\lambda^2}^{a\cdot\text{2nd-degree term}} + \overbrace{b\lambda}^{b\cdot\text{1st-degree term}} + \overbrace{c}^{c\cdot\text{0th-degree term}} = 0$$

Because the characteristic equation of our second-order ODE is a quadratic equation, we know that there are two roots, say λ_1 and λ_2.

[1] In a letter to John (Johannes) Bernoulli, who first solved the important type of differential equation devised by his brother Jakob. See the start of the "B" problems in Exercises 2.2.

There are only three possibilities for these roots:

1. The roots are both real numbers with $\lambda_1 \neq \lambda_2$
2. The roots are real numbers with $\lambda_1 = \lambda_2$
3. The roots are complex numbers: $\lambda_1 = p + qi$ and $\lambda_2 = p - qi$, where p and q are real numbers (called the *real part* and the *imaginary part*, respectively) and $i = \sqrt{-1}$.

In the third case, we say that λ_1 and λ_2 are *complex conjugates* of each other. (See Appendix C, especially Section C.3, for more information about complex numbers.)

4.1.2 Real but unequal roots

In case (1), where λ_1 and λ_2 are unequal real numbers, then both $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are solutions of Eq. (4.1.1). By the extension of the Superposition Principle given earlier in this section, any *linear combination* of the form $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ is also a solution, where c_1 and c_2 are arbitrary constants. It can be shown (see Section 4.2 for the details) that this is the *general* solution of Eq. (4.1.1)—that is, if the roots of the characteristic equation (4.1.1) are real and distinct, then *any* solution of (4.1.1) must have the form $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ for some constants c_1 and c_2. The next example shows how to solve equations of the form (4.1.1) using these roots.

Example 4.1.1 The Characteristic Equation—Unequal Roots

Let's solve the homogeneous linear second-order equation with constant coefficients $6y'' + 13y' - 5y = 0$. We find that the characteristic equation of this ODE is $6\lambda^2 + 13\lambda - 5 = 0$:

$$\begin{array}{ccccccc}
\underbrace{6y''}_{6\cdot\text{2nd derivative}} & + & \underbrace{13y'}_{13\cdot\text{1st derivative}} & + & \underbrace{-5y}_{-5\cdot\text{0th derivative}} & = 0 \\
\updownarrow & & \updownarrow & & \updownarrow & \\
\overbrace{6\lambda^2}^{6\cdot\text{2nd-degree term}} & + & \overbrace{13\lambda}^{13\cdot\text{1st-degree term}} & + & \overbrace{(-5)}^{-5\cdot\text{0th-degree term}} & = 0
\end{array}$$

Using the quadratic formula, we find

$$\lambda = \frac{-13 \pm \sqrt{13^2 - 4(6)(-5)}}{2(6)} = \frac{-13 \pm \sqrt{289}}{12} = \frac{-13 \pm 17}{12} = \frac{1}{3} \text{ and } -\frac{5}{2},$$

so that we have two distinct real roots, $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = -\frac{5}{2}$, and we can write the general solution of our equation as $y(t) = c_1 e^{\frac{t}{3}} + c_2 e^{-\frac{5t}{2}}$.

4.1.3 Real but equal roots

Next, we consider possibility (2), that the roots of the characteristic equation are real numbers with $\lambda_1 = \lambda_2$. In this situation, we get only one solution, $y = e^{\lambda t}$, where λ is the value of the repeated root. To obtain the general solution in this case, we have to find another solution that is not merely a constant multiple of $e^{\lambda t}$ (or else the "two" solutions can be merged into a single solution requiring only one arbitrary constant). Again, Euler comes to the rescue (this time in 1743). He suggested that an independent[2] second solution might be found by considering functions of the form $y_2(t) = u(t)e^{\lambda t}$, where $u(t)$ is an unknown function that must be determined. This may remind you of the method of *variation of parameters* discussed in Section 2.2.2.

Rather than deriving the consequences of Euler's assumption in the general case (see Problem 25 in Exercises 4.1), we'll illustrate his ingenious technique with an example.

Example 4.1.2 The Characteristic Equation—Equal Roots

The equation $y'' - 4y' + 4y = 0$ has the characteristic equation $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$, so $\lambda = 2$ is a repeated root. We know that $y_1 = e^{2t}$ is one solution of the differential equation. Taking Euler's suggestion, we consider $y_2(t) = u(t)e^{2t}$.

Now the Product Rule and the Chain Rule yield $y_2' = 2ue^{2t} + u'e^{2t}$ and $y_2'' = 4ue^{2t} + 4u'e^{2t} + u''e^{2t}$. Substituting y_2 and its derivatives into our original differential equation, we obtain

$$\begin{aligned} y_2'' - 4y_2' + 4y_2 &= \left(4ue^{2t} + 4u'e^{2t} + u''e^{2t}\right) - 4\left(2ue^{2t} + u'e^{2t}\right) + 4\left(ue^{2t}\right) \\ &= u''e^{2t} = 0. \end{aligned}$$

Therefore, we must have $u''(t) = 0$, and two successive integrations give us $u'(t) = A$ and $u(t) = At + B$, where A and B are arbitrary constants. Our conclusion is that $y_2(t) = u(t)e^{2t} = (At + B)e^{2t}$ is a solution of the original ODE that is *not* a constant multiple of $y_1 = e^{2t}$. The Superposition Principle tells us that the general solution is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{2t} + c_2(At + B)e^{2t} = (C_1t + C_2)\,e^{2t},$$

where $C_1 = c_2A$ and $C_2 = c_1 + c_2B$ are arbitrary constants.

4.1.4 Complex conjugate roots

When the roots are complex numbers—$\lambda_1 = p + qi$ and $\lambda_2 = p - qi$, where p and q are real numbers—the two corresponding solutions of the differential equation $ay'' + by' + cy = 0$ are $y_1(t) = e^{(p+qi)t}$ and $y_2(t) = e^{(p-qi)t}$. At this point, a crucial fact to know is **Euler's formula**,[3] which defines the exponential function for complex values of the argument (exponent):

$$e^{p+qi} = e^p\,(\cos q + i\,\sin q)\,.$$

[2] Two functions f_1 and f_2 are called **(linearly) independent** on an interval I if one is not a constant multiple of the other. Equivalently, if c_1 and c_2 are constants, then the only way for $c_1 f_1 + c_2 f_2$ to be the zero function on I is if $c_1 = c_2 = 0$. It can be shown that Euler's technique produces a new solution independent of the first.

[3] Euler discovered this formula in 1740 while investigating solutions of the equation $y'' + y = 0$. For a marvelous account of this formula and its consequences, see *Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills* by Paul J. Nahin (Princeton: Princeton University Press, 2006).

(If we let $p = 0$ and $q = \pi$, we get a particularly elegant formula connecting five of the most significant constants in all of mathematics: $e^{\pi i} + 1 = 0$. See also Appendix C.4.)

Using Euler's formula, we can write the solutions as

$$y_1(t) = e^{(p+qi)t} = e^{pt} e^{(qt)i} = e^{pt}\left(\cos(qt) + i\,\sin(qt)\right)$$

and

$$\begin{aligned} y_2(t) &= e^{(p-qi)t} = e^{pt} e^{-(qt)i} = e^{pt}\left(\cos(-qt) + i\,\sin(-qt)\right) \\ &= e^{pt}\left(\cos(qt) - i\,\sin(qt)\right), \end{aligned}$$

where we have simplified $y_2(t)$ by recognizing that the cosine is an even function and the sine is an odd function: $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$. If we combine these complex-valued solutions carefully (see Problem 23 in Exercises 4.1), we find that

$$y(t) = e^{pt}\left(C_1 \cos(qt) + C_2 \sin(qt)\right),$$

a real-valued function, is a solution of $ay'' + by' + cy = 0$ for all constants C_1 and C_2. **In fact, $y(t) = e^{pt}\left(C_1 \cos(qt) + C_2 \sin(qt)\right)$ is the general solution of the homogeneous equation when the characteristic equation has complex conjugate roots $p \pm qi$.**

Now let's practice with complex roots of a characteristic equation.

Example 4.1.3 Complex Conjugate Roots

The equation $\ddot{x} + 8\dot{x} + 25x = 0$ models the motion of a steel ball suspended from a spring, where $x(t)$ is the ball's distance (in feet) from its rest (equilibrium) position at time t seconds. Distance below the rest position is considered positive, and distance above is considered negative. We want to describe the motion (position) of the ball by finding a formula for $x(t)$.

The characteristic equation is $\lambda^2 + 8\lambda + 25 = 0$. The quadratic formula gives us

$$\lambda = \frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i,$$

so the roots are $\lambda_1 = -4 + 3i$ and $\lambda_2 = -4 - 3i$. Using the solution formula derived previously, with $p = -4$ and $q = 3$, we see that $x(t) = e^{-4t}\left(C_1 \cos(3t) + C_2 \sin(3t)\right)$ for arbitrary constants C_1 and C_2.

Suppose that we specify initial conditions, say $x(0) = 0$ and $\dot{x}(0) = 4$. These conditions tell us that the ball is at its equilibrium position at the beginning of our investigation and that the ball is started in motion from its equilibrium position with an initial velocity of 4 ft/s in the downward direction. Applying these conditions, we have

$$x(0) = e^{-4(0)}\left(C_1 \cos(0) + C_2 \sin(0)\right) = C_1 = 0$$

and

$$\begin{aligned} \dot{x}(0) &= e^{-4(0)}\left(-3C_1 \sin(0) + 3C_2 \cos(0)\right) - 4e^{-4(0)}\left(C_1 \cos(0) + C_2 \sin(0)\right) \\ &= 3C_2 - 4C_1 = 4. \end{aligned}$$

Therefore, $C_1 = 0$, $C_2 = \frac{4}{3}$, and the solution of our initial-value problem (IVP) is $x(t) = \frac{4}{3}e^{-4t}\sin(3t)$. The graph of this solution (Fig. 4.1) shows that the motion is dying out as time passes—that is, $x \to 0$ as $t \to \infty$. Another way of saying this is that the ball eventually returns to its equilibrium position.

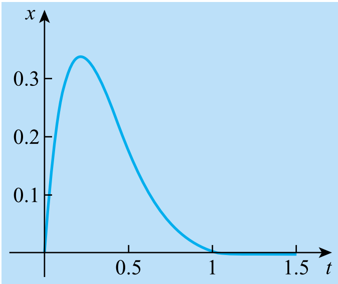

FIGURE 4.1

Graph of $\frac{4}{3}e^{-4t}\sin(3t)$, the solution of the IVP $\ddot{x} + 8\dot{x} + 25x = 0$; $x(0) = 0$, $\dot{x}(0) = 4$; $0 \le t \le 1.5$

As we'll see in Chapter 6, the differential equation has a term in it that represents air resistance, and this results in what is called *damped* motion.

4.1.5 The amplitude-phase angle form of a solution

In working with a differential equation part of whose solution is a linear combination of $\cos\omega t$ and $\sin\omega t$, where ω is a parameter, we can use a basic identity to write the trigonometric part of the solution as a single trigonometric function. This will enable us to "read" the solution more clearly and visualize its graph more easily.

Specifically, we show that

$$c_1\cos\omega t + c_2\sin\omega t = M\cos(\omega t - \phi), \tag{4.1.2}$$

where M is a constant depending on c_1 and c_2 and ϕ is an angle depending on c_1 and c_2, where not both c_1 and c_2 are zero. To see this, we write

$$c_1\cos\omega t + c_2\sin\omega t = \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega t\right]. \tag{4.1.3}$$

We note three things about the right-hand side of (4.1.3): (1) $\left|\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right| \le 1$ and $\left|\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right| \le 1$; (2) $\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)^2 + \left(\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)^2 = 1$; and (3) the right-hand side resembles the trigonometric identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$, where $B = \omega t$. To see the similarity indicated in (3) more clearly, we choose an angle ϕ (measured in radians) such that $\cos\phi = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\sin\phi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$. We may be tempted to define $\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right)$, but as we'll see in Example 4.1.4(b) the value of the quotient c_2/c_1 does not determine the quadrant in which ϕ lies. The range of the inverse tangent is assumed to be the open interval $(-\pi/2, \pi/2)$, which may not be consistent with the true location of ϕ. The necessary adjustments are indicated in the summary later in this section.

The angle ϕ, called the **phase angle**, always exists because of properties (1) and (2). The quantity $M = \sqrt{C_1^2 + C_2^2}$ is the **amplitude**. The expression $M\cos(\omega t - \phi)$ is called the **amplitude-phase angle** form of the solution $C_1\cos\omega t + C_2\sin\omega t$. We see that $M\cos(\omega t - \phi) = M\cos\left(\omega\left[t - \frac{\phi}{\omega}\right]\right)$ has period $2\pi/\omega$ and that the graph of $M\cos(\omega t - \phi)$ is the graph of $M\cos(\omega t)$ shifted ϕ/ω units to the right.

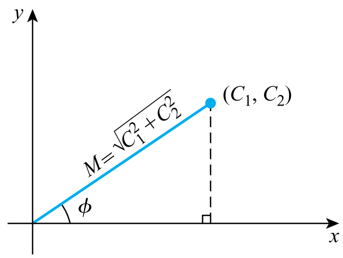

FIGURE 4.2

Amplitude-phase angle

Fig. 4.2 illustrates the situation for a first-quadrant point (C_1, C_2).

Example 4.1.4 The Amplitude-Phase Angle Forms of Solutions

(a) The IVP $\ddot{x} + 256x = 0$, $x(0) = 1/4$, $x'(0) = 1$ has the solution $x(t) = \frac{1}{4}\cos 16t + \frac{1}{16}\sin 16t$. To write this solution in the amplitude-phase angle form $M\cos(16t - \phi)$, we first calculate $M = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{16}\right)^2} = \frac{\sqrt{17}}{16}$. Next we note that $(c_1, c_2) = (1/4, 1/16)$ is in the first quadrant, and we determine the phase angle ϕ by the equations $\cos\phi = \frac{1/4}{\sqrt{17}/16} = \frac{4\sqrt{17}}{17}$ and $\sin\phi = \frac{1/16}{\sqrt{17}/16} = \frac{\sqrt{17}}{17}$, which indicate that $\phi = 0.2450$ radian. Therefore, we can write the solution as $\frac{\sqrt{17}}{16}\cos(16t - 0.2450)$.

(b) The solution of the IVP $y'' + 9y = 0$, $y(0) = -0.3$, $y'(0) = 1.2$ is $y(t) = -0.3\cos 3t + 0.4\sin 3t$. The amplitude of this solution is $M = \sqrt{c_1^2 + c_2^2} = \sqrt{(-0.3)^2 + (0.4)^2} = 0.5$. The point $(c_1, c_2) = (-0.3, 0.4)$ lies in the second quadrant, and the unique solution of the equations $\cos\phi = \frac{-0.3}{0.5} = -0.6$ and $\sin\phi = \frac{0.4}{0.5} = 0.8$ is approximately 2.2143 radians ($= \pi - \sin^{-1}(0.8)$). Thus, $y(t) = 0.5\cos(3t - 0.2143)$. Note that in this example $\tan^{-1}(c_2/c_1) = \tan^{-1}(0.4/-0.3) = -0.9273$, an incorrect answer that places ϕ in the fourth quadrant. However, ϕ can be calculated correctly as $\tan^{-1}(0.4/-0.3) + \pi$.

We can summarize the process of expressing a solution in amplitude-phase angle form as follows:

Amplitude-Phase Angle Form

$$c_1 \cos \omega t + c_2 \sin \omega t = M \cos(\omega t - \phi)$$

$$M = \sqrt{c_1^2 + c_2^2}$$

$$\tan \phi = \frac{c_2}{c_1}$$

$$\phi = \begin{cases} \tan^{-1}(c_2/c_1) & \text{if} \quad c_1 \geq 0 \\ \tan^{-1}(c_2/c_1) + \pi & \text{if} \quad c_1 < 0 \end{cases}$$

Using this technique, you can express a linear combination of sines and cosines as a single sine term as well: $c_1 \cos \omega t + c_2 \sin \omega t = R \sin(\omega t - \phi)$.

4.1.6 Summary

We can summarize the situation for homogeneous linear second-order equations with constant coefficients as follows:

Suppose that we have the equation $ax'' + bx' + cx = 0$, where a, b, and c are constants, $a \neq 0$, and λ_1 and λ_2 are the roots of the characteristic equation $a\lambda^2 + b\lambda + c = 0$. Then

1. If there are two distinct real roots—λ_1, λ_2, with $\lambda_1 \neq \lambda_2$—corresponding to our equation, the general solution is
$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$
2. If there is a repeated real root λ, the general solution has the form
$$x(t) = c_1 e^{\lambda t} + c_2 t\, e^{\lambda t} = (c_1 + c_2 t)\, e^{\lambda t}.$$
3. If the roots form a complex conjugate pair $p \pm qi$, then Euler's formula can be used to show that the real-valued general solution has the form
$$x(t) = e^{pt}\left(c_1 \cos(qt) + c_2 \sin(qt)\right).$$

Exercises 4.1

A

Find the general solution of each of the equations in Problems 1–10.

1. $y'' - 4y' + 4y = 0$
2. $\ddot{x} + 4\dot{x} - 5x = 0$
3. $x'' - 2x' + 2x = 0$

4. $x'' + 5x' + 6x = 0$
5. $\ddot{x} + 2\dot{x} = 0$
6. $\ddot{x} - x = 0$
7. $y'' + 4y = 0$
8. $6\ddot{x} - 11\dot{x} + 4x = 0$
9. $\ddot{r} - 4\dot{r} + 20r = 0$
10. $y'' + 4ky' - 12k^2 y = 0$ (k is a parameter)

11. Solve the IVP $\ddot{x} - 3\dot{x} + 2x = 0$; $x(0) = 1$, $\dot{x}(0) = 0$.
12. Solve the IVP $y'' - 2y' + y = 0$; $y(0) = 0$, $y'(0) = 0$.
13. Solve the IVP $y'' - 4y' + 20y = 0$; $y(\pi/2) = 0$, $y'(\pi/2) = 1$.
14. Write each of the following functions in amplitude-phase angle form:
    a. $x(t) = 3\cos 5t - 7\sin 5t$
    b. $y(t) = \sqrt{3}\cos 14t + \sin 14t$
    c. $x(t) = -6\cos 5t + 6\sin 5t$
    d. $y(t) = \sqrt{3}\cos 6t - \sin 6t$

B

15. a. Show that if a, b, and c are constants and y is any function having at least two derivatives, then the *differential operator* L defined by the relation $L(y) = ay'' + by' + cy$ is *linear*: $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$ for any twice-differentiable functions y_1 and y_2 and any constants c_1 and c_2.
    b. Show that if y_1 and y_2 are two solutions of $L(y) = 0$, then the function $c_1 y_1 + c_2 y_2$ is also a solution of $L(y) = 0$.

As we noted at the beginning of Section 4.1, if $I = I(t)$ represents the current in an electrical circuit, then *Kirchhoff's Voltage Law* gives us the equation $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$ when the voltage applied to the circuit is constant. In this equation, L is the inductance, R is the resistance, and C is the capacitance. Use this equation in Problems 16 and 17.

16. An *RLC* circuit with $R = 10$ ohm, $L = 0.5$ henry, and $C = 0.01$ farad has a constant voltage of 12 volt. Assume no initial current and that $\frac{dI}{dt} = 60$ when the voltage is first applied. Find an expression for the current in the circuit at time $t > 0$.
17. An *RLC* circuit with $R = 6$ ohm, $L = 0.1$ henry, and $C = 0.02$ farad has a constant voltage of 6 volt. Assume no initial current and that $\frac{dI}{dt} = 60$ when the voltage is first applied.
    a. Find an expression for the current in the circuit at time $t > 0$.
    b. Use technology to graph the answer found in part (a) for $0 \le t \le 0.5$.

c. From the graph in part (b) estimate the maximum value of I, and find the exact value by calculus techniques applied to the expression found in part (a).

d. At what time is the maximum value found in part (c) achieved? (You can use a calculator or a CAS for this.)

According to *Newton's Second Law of Motion*, if an object with mass m is suspended from a spring attached to the ceiling, then the motion of the object is governed by the equation $m\ddot{x} + a\dot{x} + kx = 0$. In this equation, $x(t)$ is the object's distance from its rest (or equilibrium) position at time t seconds. Distance below the rest position is considered positive, while distance above is considered negative. Also, a is a constant representing the air resistance and/or friction present in the system and k is the spring constant describing the "give" in the spring. (Recall that mass = weight/g, where g is the gravitational constant: 32 ft/s^2, or 9.8 m/s^2.) Use this equation to do Problems 18–21.

18. An object of mass 4 slugs (= 128 lb/32 ft/s^2) is suspended from a spring having spring constant 64 lb/ft. The object is started in motion, with no initial velocity, by pulling it 6 inches (*Be careful of the units!*) below the equilibrium position and then releasing it. If there is no air resistance, find a formula for the position of the object at any time $t > 0$. (Note that the problem statement contains two initial conditions.)

19. A 20 g mass hangs from the end of a spring having a spring constant of 2880 dyn/cm and is allowed to come to rest. It is then set in motion by stretching the spring 3 cm from its equilibrium position and releasing the mass with an initial velocity of 10 cm/s in the downward (positive) direction. Find the position of the mass at time $t > 0$ if there is no air resistance.

20. A $\frac{1}{2}$ kg mass is attached to a spring having a spring constant of 6 lb/ft. The mass is set in motion by displacing it 6 inches below its equilibrium position with no initial velocity. Find the subsequent motion of the mass if a, the constant representing air resistance, is 4 lb.

21. A $\frac{1}{2}$ kg mass is attached to a spring having a spring constant of 8 N/m (newton per meter). The mass is set in motion by displacing it 10 cm above its equilibrium position with an initial velocity of 2 m/s in the upward direction.

a. Find the subsequent motion of the mass if the constant representing air resistance is 2 newton.

b. Graph the function $x(t)$ found in part (a) for $0 \le t \le 3$, $2 \le t \le 3$, and $3 \le t \le 4$. Describe the motion of the mass in your own words.

c. Estimate the greatest distance of the mass above its equilibrium position.

22. The equation $\theta'' = -4\theta - 5\theta'$ represents the angle $\theta(t)$ made by a swinging door, where θ is measured from the equilibrium position of the door, which is the closed position. The initial conditions are $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 0$.

a. Determine the angle $\theta(t)$ as a function of time ($t > 0$).

b. What does your solution tell you is going to happen as t increases?

c. Use technology to graph the solution $\theta(t)$ on the interval $[0, 5]$.

C

23. We know that $y_1(t) = e^{pt}(\cos(qt) + i\sin(qt))$ and $y_2(t) = e^{pt}(\cos(qt) - i\sin(qt))$ are complex-valued solutions of Eq. (4.1.1) when the roots of the characteristic equation are complex conjugate numbers $p \pm qi$. In what follows, you may assume that complex constants are valid in the Superposition Principle.

a. Calculate $Y_1 = \frac{y_1+y_2}{2}$ and show that Y_1 is a real-valued solution of (4.1.1).

b. Calculate $Y_2 = \frac{y_1-y_2}{2i}$ and show that Y_2 is a real-valued solution of (4.1.1).

c. Calculate $Y = c_1Y_1 + c_2Y_2$ and conclude that Y is a real-valued solution of (4.1.1) for arbitrary real constants c_1 and c_2.

24. Given the equation $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$,

a. For what values of R will the current subside to zero without oscillating? [*Hint*: Oscillation is equivalent to having trigonometric functions in the solution.]

b. For what values of R will the current oscillate before subsiding to zero?

25. Suppose that we have a constant-coefficient equation $ay'' + by' + cy = 0$ whose characteristic equation has a repeated root r. Then we know that $y_1(t) = e^{rt}$ is a solution of the equation. If we form the new function $y_2(t) = u(t)e^{rt}$, where $u(t)$ is unknown, we want to determine $u(t)$ so that y_2 is a solution of the differential equation, but is not a constant multiple of y_1.

a. Show that any constant-coefficient equation $ay'' + by' + cy = 0$ whose characteristic equation has a double root r must have the form $y'' - 2ry' + r^2y = 0$.

b. Find y_2' and y_2'' and then substitute y_2 and these derivatives into the equation $y'' - 2ry' + r^2y = 0$. Simplify the result.

c. Solve the equation you get in part (b) for $u(t)$.

26. Consider the equation $ay'' + by' + cy = 0$. Another approach to the situation in which the characteristic equation has a double real root λ^* was developed by the French mathematician d'Alembert (1717–83). He proposed (in c. 1748) splitting this root into two "neighboring" roots λ^* and $\lambda^* + \varepsilon$, where ε is small (but not zero).

a. Show that both $e^{\lambda^* t}$, $e^{(\lambda^*+\varepsilon)t}$ and the combination $y_\varepsilon(t) = \frac{e^{(\lambda^*+\varepsilon)t} - e^{\lambda^* t}}{\varepsilon}$ are solutions of the "perturbed" equation $ay'' + (b - \varepsilon a)\,y' + (c + \varepsilon a\lambda^*)\,y = 0$. [Use the result of Problem 25a.]

b. Show that as $\varepsilon \to 0$, $y_\varepsilon(t)$ becomes the function $te^{\lambda^* t}$, which is a solution of the original equation.

4.2 Nonhomogeneous second-order linear equations with constant coefficients

4.2.1 The structure of solutions

If we take the same *RLC* circuit that we considered at the beginning of the preceding section and hook up a generator supplying alternating current to it, Kirchhoff's Voltage Law will now take the form $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$, where E is the applied nonconstant voltage. An equation of this kind is called a **nonhomogeneous second-order linear equation with constant coefficients**. (The nonzero right-hand side of such an equation is often called the **forcing function** or the **input**. The solution of the equation is the **output**. See Section 2.2.)

To get a handle on solving a nonhomogeneous linear equation with constant coefficients, let's think a bit about the difference between a nonhomogeneous equation

$$ay'' + by' + cy = f(t) \tag{4.2.1}$$

and its **associated homogeneous equation** $ay'' + by' + cy = 0$. If y is the general solution of the *homogeneous* system, then y doesn't quite "reach" all the way to $f(t)$ under the transformation $L(y) = ay'' + by' + cy$. It stops short at 0. Perhaps we could enhance the solution y in some way so that operating on this new function *does* give us all of f. We have to be able to capture the "leftover" term $f(t)$.

For nonhomogeneous second-order equations with constant coefficients, the proper form of the Superposition Principle is the following:

If y_1 is a solution of $ay'' + by' + cy = f_1(t)$ and y_2 is a solution of $ay'' + by' + cy = f_2(t)$, then $y = c_1y_1 + c_2y_2$ is a solution of $ay'' + by' + cy = c_1f_1(t) + c_2f_2(t)$ for any constants c_1 and c_2.

From the Superposition Principle follows **a fundamental truth about the solutions of linear equations with constant coefficients**:

> **The general solution, y_{GNH}, of a linear nonhomogeneous equation $ay'' + by' + cy = f(t)$ is obtained by finding a *particular* solution, y_{PNH}, of the nonhomogeneous equation and adding it to the *general* solution, y_{GH}, of the associated homogeneous equation: $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$.**

We can prove this easily using operator notation, in which $L(y) = ay'' + by' + cy$:

1. First note that $L(y_{\text{GH}}) = 0$ and $L(y_{\text{PNH}}) = f(t)$ by definition.
2. Then if $y = y_{\text{GH}} + y_{\text{PNH}}$, we have $L(y) = L(y_{\text{GH}} + y_{\text{PNH}}) = L(y_{\text{GH}}) + L(y_{\text{PNH}}) = 0 + f(t) = f(t)$, so y is a solution of Eq. (4.2.1).
3. Now we must show that *every* solution of Eq. (4.2.1) has the form $y = y_{\text{GH}} + y_{\text{PNH}}$. To do this, we assume that y^* is an arbitrary solution of $L(y) = f(t)$ and y_{PNH} is a particular solution of $L(y) = f(t)$. If we let $z = y^* - y_{\text{PNH}}$, then

$$L(z) = L\left(y^* - y_p\right) = L\left(y^*\right) - L\left(y_p\right) = f(t) - f(t) = 0,$$

which shows that z is a solution to the homogeneous equation $L(y) = 0$. Because $z = y^* - y_{\text{PNH}}$, it follows that $y^* = z + y_{\text{PNH}}$. Since y^* is an arbitrary solution of the nonhomogeneous equation, the expression $z + y_{\text{PNH}}$ includes *all* solutions. (See Problem 23 of Exercises 1.3 and Problem 35 of Exercises 2.2 for related results.)

Let's go through a few simple examples to develop some intuition for the solutions of nonhomogeneous equations.

Example 4.2.1 Solving a Nonhomogeneous Equation

If we are given the nonhomogeneous equation $y'' + 4y' + 5y = 10e^{-2x}\cos x$, the general solution will be made up of the general solution of the associated homogeneous equation and a particular solution of the nonhomogeneous equation: $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$. The characteristic equation $\lambda^2 + 4\lambda + 5 = 0$ has roots $-2 \pm i$, so we know that $y_{\text{GH}} = e^{-2x}(c_1 \cos x + c_2 \sin x)$. We can verify that a particular solution of the nonhomogeneous equation is $5xe^{-2x}\sin x$. Therefore, the general solution of the nonhomogeneous equation is $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + 5xe^{-2x}\sin x$.

In the preceding example, a particular solution appeared magically. The next example hints at how we may find y_{PNH} by examining the forcing function on the right-hand side of the equation. Sections 4.3 and 4.4 will provide systematic procedures for determining a particular solution of a nonhomogeneous equation.

Example 4.2.2 Solving a Nonhomogeneous Equation

Suppose we want to find the general solution of $y'' + 3y' + 2y = 12e^t$. Because the characteristic equation of the associated homogeneous equation is $\lambda^2 + 3\lambda + 2 = 0$, with roots -1 and -2, we know that the general solution of the homogeneous equation is $y_{\text{GH}} = c_1 e^{-t} + c_2 e^{-2t}$.

Now we look carefully at the form of the nonhomogeneous equation. In looking for a particular solution y_{PNH}, we can ignore any terms of the form e^{-t} or e^{-2t} because they are part of the homogeneous solution and won't contribute anything new. But somehow, after differentiations and additions, we have to wind up with the term $12e^t$. We guess that $y = ce^t$ for some undetermined constant c. Substituting this expression into the left-hand side of the nonhomogeneous equation, we get $(ce^t) + 3(ce^t) + 2(ce^t) = 6ce^t$. If we choose $c = 2$, then $y_{\text{PNH}} = 2e^t$ is a particular solution of the nonhomogeneous equation.

Putting these two components together, we can write the general solution of the nonhomogeneous equation as $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}} = c_1 e^{-t} + c_2 e^{-2t} + 2e^t$.

The intelligent guessing used in the preceding example can be formalized into the **method of undetermined coefficients**, which will be discussed in the next section. But, as we'll see, this method is effective only when the forcing function $f(t)$ in the equation $ay'' + by' + cy = f(t)$ is of a special type.

Exercises 4.2

A

For each of the nonhomogeneous differential equations in Problems 1–5, verify that the given function y_p is a particular solution.

1. $y'' + 3y' + 4y = 3x + 2$; $y_p = \frac{3}{4}x - \frac{1}{16}$
2. $y'' - 4y = 2e^{3x}$; $y_p = \frac{3}{5}e^{3x}$
3. $3y'' + y' - 2y = 2\cos x$; $y_p = -\frac{5}{13}\cos x + \frac{1}{13}\sin x$
4. $y'' + 5y' + 6y = x^2 + 2x$; $y_p = \frac{1}{6}x^2 + \frac{1}{18}x - \frac{11}{108}$
5. $y'' + y = \sin x$; $y_p = -\frac{1}{2}x\cos x$
6. If $x_1(t) = \frac{1}{2}e^t$ is a solution of $\ddot{x} + \dot{x} = e^t$ and $x_2(t) = -te^{-t}$ is a solution of $\ddot{x} + \dot{x} = e^{-t}$, find a particular solution of $\ddot{x} + \dot{x} = e^t + e^{-t}$ and verify that your solution is correct.
7. Given that $y_p = x^2$ is a solution of $y'' + y' - 2y = 2\left(1 + x - x^2\right)$, use the Superposition Principle to find a particular solution of $y'' + y' - 2y = 6\left(1 + x - x^2\right)$ and verify that your solution is correct.
8. If $y_1 = 1 + x$ is a solution of $y'' - y' + y = x$ and $y_2 = e^{2x}$ is a solution of $y'' - y' + y = 3e^{2x}$, find a particular solution of $y'' - y' + y = -2x + 4e^{2x}$. Verify that your solution is correct.

B

9. Find the general solution of the equation given in Problem 1.
10. Find the general solution of the equation given in Problem 2.
11. Find the general solution of the equation given in Problem 3.
12. Find the general solution of the equation given in Problem 4.
13. Find the general solution of the equation given in Problem 5.
14. Find the general solution of the equation $y'' + y' - 2y = 6\left(1 + x - x^2\right)$ given in Problem 7.
15. Find the general solution of the equation $\ddot{x} + \dot{x} = e^t + e^{-t}$ given in Problem 6.
16. Find the form of a particular solution of $y'' - y = x$ by intelligent guessing and use this information to solve the IVP $y'' - y = x$, $y(0) = y'(0) = 0$.

C

17. Suppose $x(t)$ satisfies the IVP

$$\ddot{x} + \pi^2 x = f(t) = \begin{cases} \pi^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

with $x(0) = 1$ and $\dot{x}(0) = 0$. Determine the continuously differentiable solution for $t \ge 0$. (This means that the solution has a continuous derivative function. Note that it will have a discontinuous second derivative at $t = 1$.)

4.3 The method of undetermined coefficients

As we saw in Section 4.2, to find the general solution of a linear nonhomogeneous equation with constant coefficients $ay'' + by' + cy = f(t)$, we must find a particular solution y_{PNH} of this equation and add it to the general solution y_{GH} of the associated homogeneous equation $ay'' + by' + cy = 0$.

The **method of undetermined coefficients** is the systematic version of the "intelligent guessing" discussed in the preceding section. It was developed by Euler in his 1753 study of the motion of the Moon. This technique uses the forcing function $f(t)$ on the right-hand side of the differential equation to suggest a form for y_{PNH}. This trial solution (guess) will contain undetermined constants that can be evaluated by substituting the suggested function y_{PNH} into the nonhomogeneous equation. (Look back at Example 4.2.1 and Example 4.2.2.)

For example, if the forcing function $f(t)$ is a polynomial of degree n, it is reasonable to suspect that y_{PNH} is also an nth degree polynomial $a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ whose coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ we must determine. As we'll see, sometimes it's more intelligent to assume the degree of the trial solution is $n+1$ or $n+2$.

Example 4.3.1 Finding y_{PNH} by the Method of Undetermined Coefficients

Consider the equation $y'' - y' - 2y = 3x^2 - 2x + 1$. The characteristic equation of the associated homogeneous equation is $\lambda^2 - \lambda - 2 = 0$, with roots $\lambda_1 = -1$ and $\lambda_2 = 2$, so y_{GH} is $c_1 e^{-x} + c_2 e^{2x}$. Because the forcing function is $3x^2 - 2x + 1$, we guess that

$$y_{\text{PNH}} = a_2 x^2 + a_1 x + a_0.$$

Then we calculate $y'_{\text{PNH}} = 2a_2 x + a_1$ and $y''_{\text{PNH}} = 2a_2$. Substituting these derivatives into the differential equation, we get

$$2a_2 - (2a_2 x + a_1) - 2\left(a_2 x^2 + a_1 x + a_0\right) = 3x^2 - 2x + 1,$$

or, after collecting like terms,

$$-2a_2 x^2 + (-2a_2 - 2a_1)\,x + (2a_2 - a_1 - 2a_0) = 3x^2 - 2x + 1.$$

Equating the coefficients of like powers of x, we find

$$-2a_2 = 3,\ \ -2a_2 - 2a_1 = -2,\ \ 2a_2 - a_1 - 2a_0 = 1.$$

The first equation yields $a_2 = -3/2$. Substituting this value for a_2 into the second equation gives $a_1 = 5/2$. Finally, we solve the third equation for a_0 in terms of a_1 and a_2 to find $a_0 = (1 + a_1 - 2a_2)/(-2) = -13/4$.

Now that we have determined the coefficients of our polynomial y_{PNH}, we see that $y_{\text{PNH}} = -\frac{3}{2}x^2 + \frac{5}{2}x - \frac{13}{4}$. Thus, the general solution of the nonhomogeneous equation is

$$y_{\text{GH}} + y_{\text{PNH}} = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{2}x^2 + \frac{5}{2}x - \frac{13}{4}.$$

The guesswork involved in the preceding example was a bit misleading. We assumed that the degree of the undetermined polynomial was less than or equal to the degree of the polynomial on the right-hand side of the equation. (It is possible that some of the coefficients were zero.) This assumption was valid in this example because the left-hand side of the differential equation $y'' - y' - 2y = 3x^2 - 2x + 1$ contained a term of the form cy. If we suppose that y is an nth degree polynomial, then the calculation of y'' reduces the degree of y by 2, but the term $-2y$ restores a polynomial of degree n to the result. However, if this constant multiple of y is missing (that is, if $c = 0$), we should guess that y is a polynomial of degree $n + 1$. If the y' term is also missing, then the degree of our trial solution should be $n + 2$.

If the equation is $ay'' + by' + cy = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, assuming that $a \neq 0$, Table 4.1 summarizes our choice of a polynomial trial solution.

Table 4.1 Degree of Polynomial Trial Solutions

	Degree of Trial Solution
$b \neq 0,\ c \neq 0$	n
$b = 0,\ c \neq 0$	n
$b \neq 0,\ c = 0$	$n + 1$
$b = 0,\ c = 0$	$n + 2$

Please don't memorize this table. It's just an aid to help develop your intuition.

Example 4.3.2 A Particular Solution of $ay'' + by' + cy = f(t)$ When $c = 0$

If we want to find a particular solution of the equation $y'' - 7y' = t^2 + 2t + 3$, we realize that choosing a trial solution of the form $a_2 t^2 + a_1 t + a_0$ won't work. Substituting a second-degree polynomial into the left-hand side of the equation yields a *linear* result because of the differentiations. Therefore, we guess (see Table 4.1) that $y_{\text{PNH}} = a_3 t^3 + a_2 t^2 + a_1 t + a_0$. Then $y'_{\text{PNH}} = 3a_3 t^2 + 2a_2 t + a_1$ and $y''_{\text{PNH}} = 6a_3 t + 2a_2$. Substituting y_{PNH} into the equation and collecting terms, we find

$$-21a_3 t^2 + (6a_3 - 14a_2)\,t + (2a_2 - 7a_1) = t^2 + 2t + 3.$$

Equating coefficients of equal powers of t, we have

$$-21a_3 = 1, \quad 6a_3 - 14a_2 = 2, \quad 2a_2 - 7a_1 = 3.$$

Solving this system of equations, we conclude that $a_3 = -1/21$, $a_2 = -8/49$, and $a_1 = -163/343$, so $y_{\text{PNH}} = -\frac{1}{21}t^3 - \frac{8}{49}t^2 - \frac{163}{343}t$. We can take a_0 to be zero because it is a "free" variable and cannot be determined. (Also, noting that $y_{\text{GH}} = c_1 e^{7t} + c_2$, we see that any nonzero value of a_0 would be absorbed by c_2 in assembling y_{GNH}.)

In general, the key idea behind the method of undetermined coefficients is that all the derivatives of the forcing function $f(t)$ should have the same form as $f(t)$ itself. If this is true, the method will work. If this is not true, we should not use the method.

If we think about functions whose derivatives have the same forms as themselves, we realize that we are limited to polynomials, exponential functions, linear combinations of sines and cosines, or combinations of the sums and products of these functions. For example, the method of undetermined coefficients applies to equations whose forcing terms are

$$
\begin{gathered}
-3, \\
-2t^5 - 6t^3 + 4, \\
2\cos 4t - \frac{7}{3}\sin 4t, \\
25e^t \sin t, \\
t^2 e^{3t} + (1 - t^3)\cos 5t.
\end{gathered}
$$

To state this more concisely, the forcing function $f(t)$ must be a sum of terms of the general form $P(t)e^{kt}\cos(mt)$ or $P(t)e^{kt}\sin(mt)$, where $P(t)$ is a polynomial and k, m are constants. We should be aware that these possible functions f include $\sinh x$ (the hyperbolic sine) and $\cos^3 x$ due to identities from algebra and trigonometry. Specifically *excluded* are functions such as $\ln|x|$, $|x|$, e^{x^2}, and rational functions such as $x/(1 + x^2)$. Table 4.2 provides suggestions for the forms of the particular solutions.

Table 4.2 Trial Particular Solutions for Nonhomogeneous Equations

$f(t)$	**Form of Trial Solution**
$c \neq 0$, a constant	K, a constant
$P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$	$Q_m(t) = c_m t^m + c_{m-1}t^{m-1} + \cdots + c_1 t + c_0$ (See Table 4.1)
ce^{at}	$K e^{at}$
$a\cos rt + b\sin rt$	$\alpha\cos rt + \beta\sin rt$
$e^{Rt}(a\cos rt + b\sin rt)$	$e^{Rt}(\alpha\cos rt + \beta\sin rt)$
$P_n(t)e^{at}$	$Q_m(t)e^{at}$

By the Superposition Principle (see Section 4.2), the forcing functions that are linear combinations of the forms on the left side of Table 4.2 require the same linear combinations of the corresponding trial solutions. **If forcing function $f_1(t)$ suggests a trial solution of form $F_1(t)$, and forcing function $f_2(t)$ suggests a trial solution of form $F_2(t)$, then a forcing function of the form $c_1 f_1(t) + c_2 f_2(t)$ requires a trial solution of the form $c_1 F_1(t) + c_2 F_2(t)$**. This result generalizes to any finite linear combination $c_1 f_1(t) + c_2 f_2(t) + \cdots + c_k f_k(t)$ of forcing functions. **However, as we shall see after Example 4.3.5, we must modify this neat scheme if the forcing function f is a solution of the homogeneous equation**.

Example 4.3.3 Undetermined Coefficients with an Exponential Forcing Function

We can easily verify that $x_{GH} = c_1\cos 2t + c_2\sin 2t$ for the equation $\ddot{x} + 4x = 3e^{2t}$. To find a particular solution of the nonhomogeneous equation, we choose a function of the form $x = Ke^{2t}$, where K is an undetermined constant. We calculate $\dot{x} = 2Ke^{2t}$, $\ddot{x} = 4Ke^{2t}$; a substitution into the nonhomogeneous equation gives us $4Ke^{2t} + 4Ke^{2t} = 4e^{2t}$, or $8Ke^{2t} = 4e^{2t}$, so $K = 1/2$.

Thus, a particular solution is $x_p(t) = \frac{1}{2}e^{2t}$, and the general solution of the nonhomogeneous equation is $x(t) = c_1\cos 2t + c_2\sin 2t + \frac{1}{2}e^{2t}$.

Example 4.3.4 Undetermined Coefficients with a Trigonometric Forcing Function

A 2560 lb car supported by a *MacPherson strut* (a particular type of suspension system for shock absorption) is traveling over a bumpy road at a constant velocity v. The equation modeling the motion is

$$80\ddot{x} + 10{,}000x = 2500\cos\left(\frac{\pi v t}{6}\right),$$

where x represents the vertical position of the car's axle relative to its equilibrium position, and the basic units of measurement are feet and feet per second where appropriate. (Note that the coefficient of $\ddot{x}$ is $2560/g = 2560/32 = 80$, the *mass* of the car.) We want to determine how the velocity affects the way the car vibrates.

The general solution of the associated homogeneous equation is $x_{GH} = c_1\sin\left(5\sqrt{5}t\right) + c_2\cos\left(5\sqrt{5}t\right)$. Table 4.2 suggests we choose a trial solution of the form $x_P = A\sin\left(\frac{\pi v t}{6}\right) + B\cos\left(\frac{\pi v t}{6}\right)$. If we examine the left-hand side of the differential equation, we can simplify our work. We notice that if the trial solution x contains a sine term, then $\ddot{x}$ yields a sine term and $10{,}000x$ contributes another sine. Because there is no multiple of $\sin\left(\frac{\pi v t}{6}\right)$ in the forcing function, we conclude that $A = 0$, giving the trial solution the simpler form $B\cos\left(\frac{\pi v t}{6}\right)$.

Now $\dot{x}_P = -\frac{\pi v}{6}B\sin\left(\frac{\pi v t}{6}\right)$ and $\ddot{x}_P = -\left(\frac{\pi v}{6}\right)^2 B\cos\left(\frac{\pi v t}{6}\right)$. Substituting these derivatives into the nonhomogeneous equations and collecting terms, we find

$$\left[10{,}000 - 80\left(\frac{\pi v}{6}\right)^2\right]B\cos\left(\frac{\pi v t}{6}\right) = 2500\cos\left(\frac{\pi v t}{6}\right),$$

so $B = 2500/\left[10{,}000 - 80\left(\frac{\pi v}{6}\right)^2\right]$. Therefore, $x_{PNH} = \frac{2500}{10{,}000-80\left(\frac{\pi v}{6}\right)^2}\cos\left(\frac{\pi v t}{6}\right)$ and the general solution of our equation of motion is

$$x(t) = c_1\sin\left(5\sqrt{5}t\right) + c_2\cos\left(5\sqrt{5}t\right) + \frac{2500}{10{,}000 - 80\left(\frac{\pi v}{6}\right)^2}\cos\left(\frac{\pi v t}{6}\right).$$

Clearly, the first two trigonometric terms help describe the bumpy ride, but they have fixed amplitudes $|c_1|$ and $|c_2|$, so the ride can't get alarmingly bumpy. However, the amplitude of the last term is given by $\frac{2500}{10{,}000-80\left(\frac{\pi v}{6}\right)^2}$, which grows larger and larger as the denominator expression $10{,}000 - 80\left(\frac{\pi v}{6}\right)^2$ gets closer and closer to zero. Thus, we get *unbounded* vibrations when $10{,}000 - 80\left(\frac{\pi v}{6}\right)^2 = 0$—that is, when $v = \sqrt{\frac{4500}{\pi^2}} \approx 21.35$ ft/s. The dimensional equation $\frac{\text{mile}}{\text{h}} = \frac{\text{mile}}{\text{ft}} \cdot \frac{\text{ft}}{s} \cdot \frac{s}{\text{h}}$ allows us to express our answer as $\frac{1}{5280} \cdot \frac{21.35}{1} \cdot \frac{3600}{1} \approx 14.56$ miles per hour.

Assuming the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, Fig. 4.3 shows two graphs of $x(t)$ against t, the solid graph having $v = 15$ ft/s and the dashed line graph using $v = 21$ ft/s. You can see that the car's vibrations become wilder over time for a speed close to 21.35 feet per second.

The preceding example illustrates the phenomenon of **resonance**, the presence of oscillations of unbounded amplitude. We will encounter resonance again and discuss it further in Chapter 6.

As indicated earlier, the Superposition Principle allows us to handle more complicated combinations of basic forcing functions.

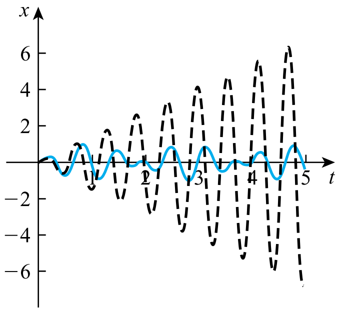

FIGURE 4.3

$x(t) = c_1 \sin\left(5\sqrt{5t}\right) + c_2 \cos\left(5\sqrt{5t}\right) + \frac{2500}{10{,}000 - 80\left(\frac{\pi v}{6}\right)^2} \cos\left(\frac{\pi v t}{6}\right)$; $x(0) = 0$, $\dot{x}(0) = 0$;
$0 \le t \le 5$, $v = 15$ (solid curve); $v = 21$ (dashed curve)

Example 4.3.5 Undetermined Coefficients with a Linear Combination of Forcing Functions

Suppose we have the equation $y'' + y' - 2y = x^2 + 2\sin x - \cos x + e^{3x}$. In this case, the forcing function is a linear combination of familiar terms from the left-hand side of Table 4.2, so we choose a trial solution y_P that is a combination of the second, third, and fourth entries of Table 4.2 (second column):

$$y_P = (Ax^2 + Bx + C) + (D\sin x + E\cos x) + Fe^{3x}.$$

Consequently, $y_P' = 2Ax + B + D\cos x - E\sin x + 3Fe^{3x}$ and $y_P'' = 2A + B - D\sin x - E\cos x + 9Fe^{3x}$. When we substitute these derivatives in the nonhomogeneous equation and collect terms, we get the equation

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) + (-3D - E)\sin x + (-3E + D)\cos x + 10Fe^{3x}$$
$$= x^2 + 2\sin x - \cos x + e^{3x}.$$

Matching the coefficients of like terms on each side, we get the system

1. $-2A = 1$ [The coefficients of x^2 must be equal.]
2. $2(A - B) = 0$ [The coefficients of x must be equal.]
3. $2A + B - 2C = 0$ [The constant terms must be equal.]
4. $-3D - E = 2$ [The coefficients of $\sin x$ must be equal.]
5. $-3E + D = -1$ [The coefficients of $\cos x$ must be equal.]
6. $10F = 1$ [The coefficients of e^{3x} must be equal.]

Working from the top down, we find $A = -1/2$, $B = A = -1/2$, $C = -3/4$, $D = -7/10$, $E = 1/10$, and $F = 1/10$. Therefore,

$$y_{PNH} = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} - \frac{7}{10}\sin x + \frac{1}{10}\cos x + \frac{1}{10}e^{3x}.$$

The general solution of the associated homogeneous equation is $c_1 e^x + c_2 e^{3x}$, so the general solution of the nonhomogeneous equation is

$$y(x) = c_1 e^x + c_2 e^{3x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4} - \frac{7}{10}\sin x + \frac{1}{10}\cos x + \frac{1}{10}e^{3x}.$$

There is an exception to the neatness of Table 4.2, however. If any term of the initial trial solution is also a term (or a multiple of a term) of $y_{GH}(t)$, then the trial solution must be modified by multiplying it by t^m, where m is the smallest positive integer such that the product of t^m and the trial solution has no terms in common with $y_{GH}(t)$. The next example illustrates this change of strategy.

Example 4.3.6 Undetermined Coefficients—An Exception to the General Rule

Let's solve the equation $y'' + 4y = 3\cos 2x + 2x^2$. The general solution of the homogeneous equation $y'' + 4y = 0$ is $c_1 \sin 2x + c_2 \cos 2x$. Ordinarily we would choose a trial solution of the form $A\sin 2x + B\cos 2x + Cx^2 + Dx + E$. However, the expression $A\sin 2x + B\cos 2x$ duplicates the terms of y_{GH}. To deal with this, we choose a modified trial solution of the form

$$y_P = x\,(A\sin 2x + B\cos 2x) + Cx^2 + Dx + E,$$

noting that the second-degree polynomial does not have to be changed. Then $y_P' = x(2A\cos 2x - 2B\sin 2x) + (A\sin 2x + B\cos 2x) + 2Cx + D$ and

$$\begin{aligned} y_P'' =& x\,(-4A\sin 2x - 4B\cos 2x) + (2A\cos 2x - 2B\sin 2x) \\ &+ (2A\cos 2x - 2B\sin 2x) + 2C. \end{aligned}$$

Substituting these derivatives in the nonhomogeneous equation and collecting terms, we get

$$-4B\sin 2x + 4A\cos 2x + 4Cx^2 + 4Dx + (4E + 2C) = 3\cos 2x + 2x^2.$$

Equating coefficients of equal terms gives us the equations $-4B = 0$, $4A = 3$, $4C = 2$, $4D = 0$, and $4E + 2C = 0$, with solutions $A = 3/4$, $B = 0$, $C = 1/2$, $D = 0$, and $E = -1/4$. Therefore, after some simplification, $y_{PNH} = \frac{3}{4}x\sin 2x + \frac{1}{2}x^2 - \frac{1}{4}$ and $y_{GNH} = c_1 \sin 2x + c_2\cos 2x + \frac{3}{4}x\sin 2x + \frac{1}{2}x^2 - \frac{1}{4}$.

Exercises 4.3

A

For each of the equations in Problems 1–10, find y_{GH} and the expression in terms of undetermined coefficients that you would use to find y_{PNH}. **Do not actually determine the coefficients**.

1. $y'' + 3y' = 3$
2. $y'' - 7y' = (x-1)^2$
3. $y'' + 7y' = e^{-7x}$
4. $y'' - 8y' + 16y = (1-x)e^{4x}$
5. $y'' + 25y = \cos 5x$
6. $y'' + y = xe^{-x}$
7. $y'' + 6y' + 13y = e^{-3x}\cos 2x$
8. $y'' - 4y' + 3y = 3e^{x} + 2e^{-x} + x^3e^{-x}$

9. $y'' + k^2 y = k$, where k is a parameter

10. $4y'' + 8y' = x \sin x$

Find the general solution of each of the equations in Problems 11–20 by using the method of undetermined coefficients.

11. $y'' - 2y' - 3y = e^{4t}$

12. $\ddot{x} - 3\dot{x} + 2x = \sin t$

13. $x'' - 2x' + 2x = e^t + t \cos t$

14. $x'' + x' = 4t^2 e^t$

15. $\ddot{x} + \dot{x} = 4 \sin t$

16. $\ddot{x} - x = 2e^t - t^2$

17. $y'' + 10y' + 25y = 4e^{-5x}$

18. $6\ddot{x} - 11\dot{x} + 4x = t$

19. $\ddot{x} + 3\dot{x} + 2x = t \sin t$

20. $y'' + 5y' + 6y = 10(1 - x)e^{-2x}$

B

21. Solve the IVP $y'' - 3y' - 4y = 3e^{4x}$; $y(0) = 0$, $y'(0) = 0$.

22. Solve the IVP $y'' + \omega^2 y = t\,(\sin \omega t + \cos \omega t)$; $y(0) = 0$, $y'(0) = 0$.

23. Solve the IVP $y'' + y' + y = t^2 e^{-t} \cos t$; $y(0) = 1$, $y'(0) = 0$ using a CAS. [*Warning*: Serious mental injury may result from attempting to do this manually.]

As mentioned at the beginning of Section 4.2, if $I = I(t)$ represents the current in an electrical circuit, then *Kirchhoff's Voltage Law* gives us the nonhomogeneous equation $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$, where E is the applied nonconstant voltage. In this equation, L is the inductance, R is the resistance, and C is the capacitance. Use this equation in Problems 24–26.

24. An *RLC* circuit has a resistance of 5 ohm, an inductance of 0.05 henry, a capacitor of 0.0004 farad, and an applied alternating voltage of $200\cos(100\,t)$ volt.

  a. Without using technology, find an expression for the current flowing through this circuit if the initial current is zero and $\frac{dI}{dt}(0)$ is 4000.

  b. Check your answer to part (a) by using technology.

25. Look for a particular solution of $y'' + 0.2y' + y = \sin(\omega x)$ and investigate its amplitude as a function of ω. Use technology to graph the particular solution for values of ω that seem significant to you and describe the behavior of this solution.

26. In her dorm room, a student attaches a weight to a spring hanging from the ceiling. She starts the mass in motion from the equilibrium position with an initial velocity in the upward direction. But during this experiment, there is rhythmic stomping (dancing or pest control?) from the student upstairs that causes the ceiling and the entire spring-mass system to vibrate. Taking into account air resistance and this "external force," she determines that the equation of motion is $\ddot{x} + 9\dot{x} + 14x = \frac{1}{2}\sin t$, with $x(0) = 0$ and $\dot{x}(0) = -1$.

a. Solve this equation for $x(t)$, the position of the weight relative to its rest position.

b. Use technology to graph $x(t)$ for $0 \le t \le 10$.

27. Consider the equation

$$y'' + y = F(t), \quad \text{where } F(t) = \begin{cases} t & 0 \le t \le \pi, \\ 0 & t > \pi \end{cases}$$

$y(0) = y'(0) = 0$, and y and y' are continuous at π.

a. Plot the forcing function against t.

b. Solve the IVP for $0 \le t \le \pi$.

c. Solve the IVP for $t > \pi$, and determine the constants from the continuity conditions at $t = \pi$.

d. Combine your answers to parts (b) and (c) into a single solution Y to the original IVP and plot Y against t.

28. Find the solution of the IVP $y'' - 3y' + 2y = 6e^{-x}$; $y(0) = 1$, $y'(2) = 2$.

C

29. Find the general solution of $y'' - 3y' + 2y = 4\sin^3 3x$. [*Hint*: Use trigonometric identities to reduce the forcing function to a linear combination of functions in Table 4.2.]

30. Find the general solution of $y'' + 4y = \sin^4 x$. (See the hint for the preceding problem.)

31. Find the general solution of $y'' + 4y = \cos x \cos 2x \cos 3x$. (See the hint for Problem 29.)

32. Find the general solution of $y'' + \lambda^2 y = \sum_{k=1}^{N} a_k \sin k\pi t$, where $\lambda > 0$ and $\lambda \ne k\pi$ for $k = 1, 2, \ldots, N$.

33. Consider the equation $ay'' + by' + cy = g(t)$, where a, b, and c are positive constants.

a. If $Y_1(t)$ and $Y_2(t)$ are solutions of the given equation, show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. [*Hint*: Show that $\sqrt{b^2 - 4ac} \le b$.]

b. Is the result of part (a) true if $b = 0$?

c. If $g(t) = K$, a constant, show that every solution of the given equation approaches K/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ as well?

A special type of second-order differential equation with variable coefficients is the **Cauchy–Euler equation** (or **Euler's differential equation**): $x^2y'' + axy' + by = 0$, where a and b are real constants. The substitution $x = e^z$ reduces the equation to a second-order linear ODE with constant coefficients. Use this method to transform each of the following equations and then solve the resulting constant-coefficient equation.

34. $x^2y'' + xy' + 4y = 1$

35. $x^2y'' + 5xy' - 6y = \frac{4}{x^2} - 12$

36. $x^2y'' - 2xy' + 2y = x^2 + 4$

37. $x^2y'' - xy' + 2y = x \ln x$

4.4 Variation of parameters

There are various techniques for finding a particular solution of the nonhomogeneous equation. The **method of variation of parameters** (or **variation of constants**) was developed by the French-Italian mathematician Joseph Louis Lagrange (1736–1813) in 1775. We've already seen this technique for first-order linear equations (Section 2.2.2).

Let's look at the nonhomogeneous equation $ay'' + by' + cy = f(t)$ and assume that $y_1(t)$ and $y_2(t)$ are two known solutions of the homogeneous equation $ay'' + by' + cy = 0$ that are independent of each other. (Footnote 2 in Chapter 4 defines the notion of 'independence.') Then we know that $c_1y_1(t) + c_2y_2(t)$ is also a solution of the homogeneous equation for any constants c_1 and c_2. Lagrange's idea was to look for a particular solution of the *nonhomogeneous* equation of the form $c_1(t)y_1(t) + c_2(t)y_2(t)$, where $c_1(t)$ and $c_2(t)$ are unknown *functions* that must be determined. By substituting this trial solution into the original nonhomogeneous differential equation, we will obtain one equation that must be satisfied by $c_1(t)$ and $c_2(t)$; however, because we have *two* unknown functions, this single equation is not enough. We need two equations to determine these functions completely, and Lagrange's method imposes an additional condition to simplify the calculations involved. If you carefully look at Eq. (*) in each of the three examples below, you'll realize that we have chosen C_1 and C_2 such that the first derivative of y will be the same, as if C_1 and C_2 were *constants* (although they are *functions*).

Rather than go through this method in complete generality, we'll illustrate the technique via specific examples.

Example 4.4.1 Using Variation of Parameters

Suppose we want to solve $y'' + 3y' + 2y = 3e^{-2x} + x$. The characteristic equation of the associated homogeneous equation is $\lambda^2 + 3\lambda + 2 = 0$, with roots -2 and -1. Then $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-x}$ are two independent solutions of the homogeneous equation, and the general solution of the homogeneous equation is $y_{GH} = c_1e^{-2x} + c_2e^{-x}$, where c_1 and c_2 are arbitrary constants. Now we assume that $y = C_1y_1 + C_2y_2 = C_1e^{-2x} + C_2e^{-x}$ is a particular solution of the nonhomogeneous equation, where $C_1 = C_1(x)$ and $C_2 = C_2(x)$ are unknown *functions*. Differentiating, we obtain

$$\begin{aligned} y' &= -2C_1e^{-2x} + C_1'e^{-2x} - C_2e^{-x} + C_2'e^{-x} \\ &= \left(-2C_1e^{-2x} - C_2e^{-x}\right) + \left(C_1'e^{-2x} + C_2'e^{-x}\right). \end{aligned}$$

To avoid messy higher derivatives (and being aware that we're looking for a *particular* solution), Lagrange's method requires that we impose the condition

$$C_1'e^{-2x} + C_2'e^{-x} = 0. \qquad (*)$$

Accepting this condition, we have $y' = -2C_1e^{-2x} - C_2e^{-x}$, from which we calculate $y'' = 4C_1e^{-2x} - 2C_1'e^{-2x} + C_2e^{-x} - C_2'e^{-x}$. Substituting these expressions for y, y', and y'' into the equation $y'' + 3y' + 2y = 3e^{-2x} + x$, we find that

$$\begin{aligned} &\left(4C_1e^{-2x} - 2C_1'e^{-2x} + C_2e^{-x} - C_2'e^{-x}\right) + 3\left(-2C_1e^{-2x} - C_2e^{-x}\right) + 2\left(C_1e^{-2x} + C_2e^{-x}\right) \\ &\quad = 3e^{-2x} + x, \end{aligned}$$

or

$$-2C_1'e^{-2x} - C_2'e^{-x} = 3e^{-2x} + x. \qquad (**)$$

Eqs. (*) and (**) form a system of equations that we must solve for C_1' and C_2':

$$\begin{aligned} C_1'e^{-2x} + C_2'e^{-x} &= 0 & (*) \\ -2C_1'e^{-2x} - C_2'e^{-x} &= 3e^{-2x} + x. & (**) \end{aligned}$$

Adding (*) and (**) gives us $-C_1'e^{-2x} = 3e^{-2x} + x$, so that $C_1' = -3 - xe^{2x}$.

Integrating (manually or using a CAS) yields $C_1(x) = -3x - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}$. When using variation of parameters, we make all constants of integration 0 because we want only a *particular* solution. Next we use (*) to find that $C_2' = e^x\left(-C_1'e^{-2x}\right) = e^x\left(3e^{-2x} + x\right) = 3e^{-x} + xe^x$. Integrating gives us $C_2(x) = -3e^{-x} + xe^x - e^x$.

Finally,

$$\begin{aligned} y_{\text{PNH}} = C_1y_1 + C_2y_2 &= \left(-3x - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}\right)\left(e^{-2x}\right) + \left(-3e^{-x} + xe^x - e^x\right)\left(e^{-x}\right) \\ &= -3xe^{-2x} - \frac{x}{2} + \frac{1}{4} - 3e^{-2x} + x - 1 = -3xe^{-2x} - 3e^{-2x} + \frac{x}{2} - \frac{3}{4}, \end{aligned}$$

so that $y_{\text{GNH}} = c_1e^{-2x} + c_2e^{-x} + \frac{x}{2} - \frac{3}{4} - 3xe^{-2x}$ is the general solution of the original nonhomogeneous equation. (Note that the term $-3e^{-2x}$ in y_{PNH} has been absorbed by the term c_1e^{-2x} in y_{GH}.)

Example 4.4.2 Using Variation of Parameters

The equation $\ddot{x} + x = \tan t$ cannot be solved by using the method of undetermined coefficients because the forcing function $f(t) = \tan t$ cannot be expressed as a linear combination of one of the basic forms given in Table 4.2. However, Lagrange's method works.

The general solution of the associated homogeneous equation $\ddot{x} + x = 0$ is $y_{\text{GH}}(t) = c_1 \sin t + c_2 \cos t$, where c_1 and c_2 are arbitrary constants. Therefore, we try to find functions $C_1 = C_1(t)$ and

$C_2 = C_2(t)$ such that $x_P(t) = C_1(t)\sin t + C_2(t)\cos t$ is a particular solution of the nonhomogeneous equation. To do this, we must first calculate

$$\begin{aligned}\dot{x} &= C_1 \cos t + \dot{C}_1 \sin t - C_2 \sin t + \dot{C}_2 \cos t \\ &= (C_1 \cos t - C_2 \sin t) + \left(\dot{C}_1 \sin t + \dot{C}_2 \cos t\right).\end{aligned}$$

Before calculating $\ddot{x}$ and increasing the complexity of the expressions we have to use, we impose the condition

$$\dot{C}_1 \sin t + \dot{C}_2 \cos t = 0. \qquad (*)$$

In particular, this condition ensures that $\ddot{x}$ will contain no second derivatives of C_1 or C_2.

Now differentiating the simplified expression for $\dot{x}$ gives us $\ddot{x} = -C_1 \sin t + \dot{C}_1 \cos t - C_2 \cos t - \dot{C}_2 \sin t$. Substituting for x and $\ddot{x}$ in our original equation yields

$$\ddot{x} + x = \left(-C_1 \sin t + \dot{C}_1 \cos t - C_2 \cos t - \dot{C}_2 \sin t\right) + (C_1 \sin t + C_2 \cos t) = \tan t,$$

or

$$\dot{C}_1 \cos t - \dot{C}_2 \sin t = \tan t. \qquad (**)$$

Now (*) and (**) give us a system of simultaneous equations for $\dot{C}_1$ and $\dot{C}_2$:

$$\left\{\begin{array}{l}\dot{C}_1 \sin t + \dot{C}_2 \cos t = 0 \\ \dot{C}_1 \cos t - \dot{C}_2 \sin t = \tan t\end{array}\right\}.$$

Multiplying the first equation by $\sin t$ and the second by $\cos t$ and then adding, we find $\dot{C}_1\left(\sin^2 t + \cos^2 t\right) = \tan t \cos t$, or $\dot{C}_1 = \sin t$. Thus, because we need only a particular solution, we can take the constant of integration to be zero and get $C_1 = -\cos t$. Using Eq. (*), we derive $\dot{C}_2 = -\dot{C}_1 \sin t/\cos t = -\sin^2 t/\cos t = \left(\cos^2 t - 1\right)/\cos t = \cos t - 1/\cos t$. Then $C_2 = \int (\cos t - 1/\cos t)\,dt = \sin t - \int \sec t\,dt = \sin t - \ln|\sec t + \tan t|$.

Therefore, a particular solution of our original nonhomogeneous equation is given by $x_{\text{PNH}} = C_1(t)\sin t + C_2(t)\cos t = -\cos t \sin t + (\sin t - \ln|\sec t + \tan t|)\cos t = -\cos t \sin t + (\sin t - \ln|\sec t + \tan t|)\cos t = -\ln|\sec t + \tan t|\cos t$, and $y_{\text{GNH}}(t) = c_1 \sin t + c_2 \cos t - \ln|\sec t + \tan t|\cos t$. (Note that because the last term is not a *constant* multiple of $\cos t$, it doesn't get absorbed by the term $c_2 \cos t$.)

Example 4.4.3 Using Variation of Parameters

We cannot use the method of undetermined coefficients to solve the equation $y'' - 3y' + 2y = \sin\left(e^{-x}\right)$ because the forcing function $f(x) = \sin\left(e^{-x}\right)$ does not fit any of the patterns in Table 4.2. So we try variation of parameters.

We find that $y_{\text{GH}} = c_1 e^x + c_2 e^{2x}$, so we assume that a particular solution of the nonhomogeneous equation has the form $y_P = C_1 e^x + C_2 e^{2x}$, where C_1 and C_2 are undetermined functions of x. Then $y_{\text{P}}' = \left(C_1 e^x + 2C_2 e^{2x}\right) + \left(C_1' e^x + C_2' e^{2x}\right)$ and, assuming

$$C_1' e^x + C_2' e^{2x} = 0, \qquad (*)$$

$y_{\text{P}}'' = C_1 e^x + C_1' e^x + 4C_2 e^{2x} + 2C_2' e^{2x}$. Substituting in the nonhomogeneous equation, we get

$$\left[\left(C_1 + C_1'\right)e^x + \left(4C_2 + 2C_2'\right)e^{2x}\right] - 3\left[C_1 e^x + 2C_2 e^{2x}\right]$$

$$+2\left[C_1e^x + C_2e^{2x}\right] = \sin\left(e^{-x}\right)$$

or, simplifying,

$$e^xC_1' + 2e^{2x}C_2' = \sin\left(e^{-x}\right). \qquad (**)$$

Subtracting Eq. (*) from (**) gives us $e^{2x}C_2' = \sin\left(e^{-x}\right)$, or $C_2' = e^{-2x}\sin\left(e^{-x}\right)$. Then, from Eq. (*), $C_1' = e^{-x}\left(-e^{2x}C_2'\right) = -e^x\left(e^{-2x}\sin\left(e^{-x}\right)\right) = -e^{-x}\sin\left(e^{-x}\right)$. Therefore, $C_1 = \int \sin\left(e^{-x}\right)\left(-e^{-x}\right)dx$ and $C_2 = -\int e^{-x}\sin\left(e^{-x}\right)\left(-e^{-x}\right)dx$. Making the substitutions $u = e^{-x}$, $du = -e^{-x}\,dx$ in each integral, we find $C_1 = -\cos\left(e^{-x}\right)$ and $C_2 = -\sin\left(e^{-x}\right) + e^{-x}\cos\left(e^{-x}\right)$. Therefore, $y_{\text{PNH}} = C_1e^x + C_2e^{2x} = \left[-\cos\left(e^{-x}\right)\right]e^x + \left[-\sin\left(e^{-x}\right) + e^{-x}\cos\left(e^{-x}\right)\right]e^{2x} = -e^{2x}\sin\left(e^{-x}\right)$, and the general solution is

$$y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}} = c_1e^x + c_2e^{2x} - e^{2x}\sin\left(e^{-x}\right).$$

The preceding three examples involved quite a bit of algebra and calculus, but the method of variation of parameters is guaranteed to work. Even if the integrations of $C_1'(x)$ and $C_2'(x)$ can't be done in closed form, we can still use numerical methods such as Simpson's Rule to approximate the solution. The next example illustrates this kind of integration difficulty.

Example 4.4.4 Variation of Parameters—No Closed Form Solution

Let's solve the equation $y'' + y' - 2y = \ln x$. The characteristic equation of the associated homogeneous equation is $\lambda^2 + \lambda - 2 = 0$, with roots -2 and 1, so we know that $y_{\text{GH}} = c_1e^{-2x} + c_2e^x$, where c_1 and c_2 are constants.

Next, we consider $y = C_1e^{-2x} + C_2e^x$, where C_1 and C_2 are unknown functions of x. Differentiating, we get

$$y' = \left(-2C_1e^{-2x} + C_2e^x\right) + \left(C_1'e^{-2x} + C_2'e^x\right) = -2C_1e^{-2x} + C_2e^x$$

because we must assume that

$$C_1'e^{-2x} + C_2'e^x = 0. \qquad \text{(A)}$$

Then $y'' = 4C_1e^{-2x} - 2C_1'e^{-2x} + C_2e^x + C_2'e^x$.

After substituting these expressions for y, y', and y'' into the equation $y'' + y' - 2y = \ln x$, we get

$$-2C_1'e^{-2x} + C_2'e^x = \ln x. \qquad \text{(B)}$$

Now we must solve the following system for C_1' and C_2':

$$C_1'e^{-2x} + C_2'e^x = 0 \qquad \text{(A)}$$
$$-2C_1'e^{-2x} + C_2'e^x = \ln x. \qquad \text{(B)}$$

Subtracting (B) from (A) gives us $3C_1'e^{-2x} = -\ln x$, so that $C_1' = -\frac{1}{3}e^{2x}\ln x$ and $C_1(x) = -\frac{1}{3}\int e^{2x}\ln x\,dx = -\frac{1}{6}e^{2x}\ln x + \frac{1}{6}\int \frac{e^{2x}}{x}\,dx$. This integration was done manually (integration by parts: $u = \ln x$, $dv = e^{2x}\,dx$, etc.). A CAS might give an answer in terms of the "exponential integral," which you may not recognize. In any case, the integral $\int \frac{e^{2x}}{x}\,dx$ cannot be expressed in closed form.

Eq. (A) tells us that $C_2' = e^{-x}\left(-C_1' e^{-2x}\right) = -e^{-3x}\left(-\frac{1}{3}e^{2x}\ln x\right) = \frac{1}{3}e^{-x}\ln x$, and an integration by parts leads to the conclusion that $C_2(x) = -\frac{1}{3}e^{-x}\ln x + \frac{1}{3}\int \frac{e^{-x}}{x}\,dx$.

The next to last step is to calculate

$$\begin{aligned} y_{\text{PNH}} = c_1 y_1 + c_2 y_2 &= \left(-\frac{1}{6}e^{2x}\ln x + \frac{1}{6}\int \frac{e^{2x}}{x}\,dx\right)\left(e^{-2x}\right) \\ &\quad + \left(-\frac{1}{3}e^{-x}\ln x + \frac{1}{3}\int \frac{e^{-x}}{x}\,dx\right)\left(e^{x}\right) \\ &= -\frac{\ln x}{2} + \frac{e^{-2x}}{6}\int \frac{e^{2x}}{x}\,dx + \frac{e^{x}}{3}\int \frac{e^{-x}}{x}\,dx. \end{aligned}$$

Finally, the general solution is given by the formula

$$y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}} = c_1 e^{-2x} + c_2 e^{x} - \frac{\ln x}{2} + \frac{e^{-2x}}{6}\int \frac{e^{2x}}{x}\,dx + \frac{e^{x}}{3}\int \frac{e^{-x}}{x}\,dx.$$

Another important feature of variation of parameters is that the method remains valid for a linear equation whose coefficients are continuous functions of the independent variable.

Suppose $p(t)$ and $q(t)$ are continuous functions and $f(t)$ is continuous or piecewise continuous. If $y_{\text{GH}} = c_1 y_1(t) + c_2 y_2(t)$ is the general solution of $y'' + p(t)y' + q(t)y = 0$, then we can express the general solution of $y'' + p(t)y' + q(t)y = f(t)$ as $y = y_{\text{GH}} + C_1(t)y_1(t) + C_2(t)y_2(t)$, where $C_1(t)$ and $C_2(t)$ can be found by the method of variation of parameters.

The last paragraph extends the formula $\boldsymbol{y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}}$ of Section 4.2 to the situation where the coefficients of y and y' are continuous functions rather than constants. Of course, in the context of an equation with nonconstant coefficients, finding linearly independent functions such that $y_{\text{GH}} = c_1 y_1(t) + c_2 y_2(t)$ can be difficult. Rather than focus on specialized techniques, we will be content to illustrate what to do once we have y_{GH}.

Example 4.4.5 Variation of Parameters—Nonconstant Coefficients

It is easy to see that $y_1(x) = x$ and $y_2(x) = 1/x$ are linearly independent solutions of the differential equation $x^2 y'' + xy' - y = 0$. If we suppose the functions are *not* independent, then $x = c/x$ for some constant c and every $x \neq 0$. Now let $x = 1$ and $x = 2$, forcing the contradiction that c must be equal to both 1 and 4. Thus, the functions are independent and $y_{\text{GH}} = c_1 x + c_2/x$.

Now we use the method of variation of parameters to find the general solution of $x^2 y'' + xy' - y = x$, $x \neq 0$.

We start with a trial solution $y_{\text{P}} = C_1 x + C_2/x$, where $C_1 = C_1(x)$ and $C_2 = C_2(x)$. Then $y_{\text{P}}' = C_1 + C_1' x - C_2/x^2 + C_2'/x = \left(C_1 - C_2/x^2\right) + \left(C_1' x + C_2'/x\right)$, where we impose the condition

$$C_1' x + \frac{C_2'}{x} = 0 \qquad (*)$$

before calculating the next derivative.

Now $y_P'' = C_1' + 2C_2/x^3 - C_2'/x^2$. Substituting in the nonhomogeneous equation, we get

$$x^2\left[C_1' + 2C_2/x^3 - C_2'/x^2\right] + x\left[C_1 - C_2/x^2\right] - \left[C_1 x + C_2/x\right] = x,$$

or

$$x^2 C_1' - C_2' = x. \tag{**}$$

Eqs. (*) and (**) must be solved for C_1' and C_2'. Multiplying (*) by x gives us $x^2C_1' + C_2' = 0$, and adding this result to (**) yields $2x^2C_1' = x$, from which it follows that $C_1(x) = \frac{1}{2}\ln|x|$. Then Eq. (**) yields $C_2' = x^2C_1' - x$, or $C_2' = x^2(1/(2x)) - x = -x/2$, so $C_2 = -x^2/4$.

Therefore, $y_{\text{PNH}} = C_1x + C_2/x = (x/2)\ln|x| - x/4$ and $y_{\text{GNH}} = c_1x + c_2/x + (x/2)\ln|x|$, where the term $-x/4$ of y_{PNH} has been absorbed into the term c_1x from y_{GH}.

In summary, **the method of variation of parameters works for all second-order linear differential equations provided that the coefficients are continuous functions of the independent variable. In those limited situations in which the method of undetermined coefficients (Section 4.3) can be used, that method is usually easier than variation of parameters.**

Exercises 4.4

A

Find the general solution of each of the equations in Problems 1–10 by using the method of variation of parameters.

1. $x'' - 2x' + x = \frac{e^t}{t}$
2. $y'' + 4y = 2\tan x$
3. $\ddot{r} + r = \frac{1}{\sin t}$
4. $y'' + 2y' + y = \frac{e^{-x}}{x}$
5. $y'' + 4y' + 4y = 3xe^{-2x}$
6. $y'' + y = \sec x$
7. $y'' + 2y' + y = e^{-x}\ln x$
8. $y'' - y = \sin^2 x$
9. $y'' - 3y' + 2y = \cos\left(e^{-x}\right)$
10. $y'' + 3y' + 2y = \frac{1}{1+e^x}$

B

Find the general solution of each of the equations in Problems 11–16. Linearly independent solutions for the associated homogeneous equation are shown next to each nonhomogeneous equation.

11. $x^2y'' - xy' + y = x;\ y_1 = x,\ y_2 = x\ln x$

12. $2x^2y'' + 3xy' - y = \frac{1}{x};\ y_1 = \sqrt{x},\ y_2 = \frac{1}{x}$

13. $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x\ln x;\ y_1 = x,\ y_2 = x^2$

14. $y'' + \frac{2}{x}y' + y = \frac{1}{x},\ x \neq 0;\ y_1 = \frac{\sin x}{x},\ y_2 = \frac{\cos x}{x}$

15. $y'' - 2(\tan x)y' = 1;\ y_1 = C$ (a constant), $y_2 = \tan x$

16. $x^2y'' + xy' - y = x,\ x \neq 0;\ y_1 = x,\ y_2 = 1/x$

17. Consider the equation $x^2y'' - 4xy' + 6y = 0$.

a. Show that the general solution of this equation is $y = c_1x^3 + c_2x^2$.

b. Use the result of part (a) to find the general solution of

$$x^2y'' - 4xy' + 6y = x^4.$$

18. Find the general solution of $x^2y'' - 4xy' + 6y = x^4$ [part (b) of the previous problem] by reducing this *Cauchy–Euler equation* to one with constant coefficients via the substitution $x = e^z$. (See the paragraph between Problems 33 and 34 of Exercises 4.3.)

C

19. Show that the solution of the IVP $y'' + a^2y = F(x),\ a \neq 0,\ y(0) = y'(0) = 0$, is

$$y = \frac{1}{a}\int_0^x F(u)\sin a(x-u)\,du.$$

20. Suppose $y'' + a(x)y' + b(x)y = 0$ has continuous coefficient functions $a(x)$ and $b(x)$ on an interval I. Assume $y_1(x)$ is a solution of the equation and is nonzero on a subinterval J of I.

a. Let $y_2(x) = y_1(x)u(x)$, where $u(x)$ is a nonconstant function. Assuming that $y_2(x)$ is a solution of the differential equation, show that

$$u(x) = c_1\int \frac{e^{-\int a(x)\,dx}}{y_1^2(x)}\,dx + c_2,$$

where c_1 and c_2 are arbitrary constants.

b. Show that $y_2(x) = y_1(x)\int \frac{e^{-\int a(x)\,dx}}{y_1^2(x)}\,dx$ defines a solution on J that is independent of $y_1(x)$.

21. If the solution of $y'' + p(x)y' + q(x)y = 0$ is $\alpha y_1(x) + \beta y_2(x)$, show that the general solution of $y'' + p(x)y' + q(x)y = r(x)$ is

$$y = c_1y_1(x) + c_2y_2(x) + y_2(x)\int \frac{r(x)y_1(x)}{W(y_1, y_2)}\,dx - y_1(x)\int \frac{r(x)y_2(x)}{W(y_1, y_2)}\,dx,$$

where $W(y_1, y_2) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is called the *Wronskian* of $y_1(x)$ and $y_2(x)$ and is not the zero function. Discuss the case in which $W(y_1, y_2)$ is identically equal to zero.

22. An alternative method of solving a *Cauchy–Euler differential equation* $x^2y'' + axy' + by = 0$ (see Section C of Exercises 4.3), where a and b are real constants, requires the substitution $y = x^r$, where $x > 0$.

a. If $y = x^r$, calculate y' and y'' and show that for y to be a solution of the Cauchy–Euler equation, r must satisfy the *indicial equation*

$$r^2 + (a-1)r + b = 0.$$

b. Show that if the indicial equation has real roots $r_1 \neq r_2$, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ are linearly independent solutions of the differential equation.

c. Show that if the indicial equation has complex conjugate roots $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, then $y_1 = x^{\alpha}\cos(\beta \ln x)$ and $y_2 = x^{\alpha}\sin(\beta \ln x)$ are solutions. [*Note*: $x^{\alpha \pm \beta i} = x^{\alpha}e^{\pm i\beta \ln x}$ for $x > 0$.]

4.5 Higher-order linear equations with constant coefficients

Linearity is such a marvelous property that we can generalize our work in the preceding few sections in a very natural way. The details may get a bit complicated, but the theory is crisp and clear.

If y is a function that is n-times differentiable and $a_0, a_1, a_2, \ldots, a_n$ are constants, $a_n \neq 0$, then we can define the *nth-order linear operator L* as follows:

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y.$$

Any nth-order linear differential equation with constant coefficients can be expressed concisely as $L(y) = f(t)$. If $f(t) \equiv 0$, then the equation is called a **homogeneous *n*th-order linear equation with constant coefficients**. If $f(t)$ is not the zero function, then we have a **nonhomogeneous *n* th-order linear equation with constant coefficients**.

An important property of such nth-order equations is the extended Superposition Principle:

Superposition Principle

If y_j is a solution of $L(y) = f_j$ for $j = 1, 2, \ldots, n$, and $c_1, c_2, \ldots, c_n$ are arbitrary constants, then $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution of $L(y) = c_1f_1 + c_2f_2 + \cdots + c_nf_n$—that is,

$$\begin{aligned} L(c_1y_1 + c_2y_2 + \cdots + c_ny_n) &= c_1L(y_1) + c_2L(y_2) + \cdots + c_nL(y_n) \\ &= c_1f_1 + c_2f_2 + \cdots + c_nf_n. \end{aligned}$$

First, let's look at an nth-order *homogeneous* linear equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

For such an equation, there's a neat algorithm for finding the general solution, a generalization of the procedure we've already seen. First, find the roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$. (*You should see how to form this equation.*)

Focus on the fact that the characteristic equation of an nth-order linear equation is an nth-degree polynomial equation. Realize that once a polynomial has degree greater than or equal to 5; there is no longer a general formula that gives the zeros. (Even the formulas that exist for the zeros of third- and fourth-degree polynomials are very unwieldy.) In general, the only practical way to tackle these equations is to use *approximation* methods. A CAS or a graphing calculator should have various algorithms implemented to solve (or approximate the solutions of) polynomial equations.

Next, group these roots as follows:

1. Distinct real roots
2. Distinct complex conjugate pairs $p \pm qi$
3. Multiple real roots
4. Multiple complex roots

Then the general solution is a sum of n terms of the forms

1. $c_i e^{\lambda_i t}$ for each distinct real root λ_i
2. $e^{pt}(c_1 \cos qt + c_2 \sin qt)$ for each distinct complex pair $p \pm qi$
3. $\left(c_1 + c_2 t + \cdots + c_k t^{k-1}\right) e^{\lambda_i t}$ for each multiple real root λ_i, where k is the multiplicity of that root
4. $e^{pt}(c_1 \cos qt + c_2 \sin qt) + t\,e^{pt}(c_3 \cos qt + c_4 \sin qt) + \cdots + t^{k-1}e^{pt}(c_{2k-1} \cos qt$ $+ c_{2k} \sin qt)$ for each multiple complex pair of roots $p \pm qi$, where k is the multiplicity of the pair $p \pm qi$

Now let's see how to use this procedure to solve some higher-order homogeneous linear equations with constant coefficients.

Example 4.5.1 Solving a Fourth-Order Homogeneous Linear Equation

Let's find the general solution of the fourth-order equation

$$x^{(4)} - 3x'' + 2x' = 0.$$

The characteristic equation is $\lambda^4 - 3\lambda^2 + 2\lambda = \lambda(\lambda^3 - 3\lambda + 2) = 0$, whose roots are 0, 1, 1, and -2. (*Verify this.*) Thus, we have two distinct real roots and another real root of multiplicity 2.

According to the process described previously, the general solution is

$$x = c_1 e^{0 \cdot t} + c_2 e^{-2t} + (c_3 + c_4 t)\, e^{1 \cdot t} = c_1 + c_2 e^{-2t} + (c_3 + c_4 t)\, e^{t}.$$

(*You should check that this is a solution, manually or by using a CAS.*)

Example 4.5.2 Solving an Eighth-Order Homogeneous Linear Equation

The equation $64y^{(8)} + 48y^{(6)} + 12y^{(4)} + y'' = 0$ should be interesting to tackle. The characteristic equation is $64\lambda^8 + 48\lambda^6 + 12\lambda^4 + \lambda^2 = 0$. A CAS gives the roots 0, 0, $i/2$, $-i/2$, $i/2$, $-i/2$, $i/2$, and $-i/2$. Grouping them, we see that 0 is a real root of multiplicity 2, whereas the complex conjugate pair $\pm i/2\,(= 0 \pm i/2)$ has multiplicity 3. Therefore, the form of the general solution of this eighth-order equation is

$$\begin{aligned} y(t) &= (c_1 + c_2 t)\, e^{0\cdot t} + e^{0\cdot t}\left(c_3 \cos\left(\frac{t}{2}\right) + c_4 \sin\left(\frac{t}{2}\right)\right) + t e^{0\cdot t} \\ &\quad \left(c_5 \cos\left(\frac{t}{2}\right) + c_6 \sin\left(\frac{t}{2}\right)\right) + t^2 e^{0\cdot t}\left(c_7 \cos\left(\frac{t}{2}\right) + c_8 \sin\left(\frac{t}{2}\right)\right) \\ &= c_1 + c_2 t + \left(c_3 + c_5 t + c_7 t^2\right) \cos\left(\frac{t}{2}\right) + \left(c_4 + c_6 t + c_8 t^2\right) \sin\left(\frac{t}{2}\right). \end{aligned}$$

For the nonhomogeneous case once again the theory is simple:

The general solution, y_{GNH}, of an nth-order linear nonhomogeneous equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t)$ is obtained by finding a particular solution, y_{PNH}, of the nonhomogeneous equation and adding it to the general solution, y_{GH}, of the associated homogeneous equation: $y_{\mathrm{GNH}} = y_{\mathrm{GH}} + y_{\mathrm{PNH}}$.

As before, the challenge is to find a particular solution of the nonhomogeneous equation. But once again we can use the method of variation of parameters or the method of undetermined coefficients ("educated guessing").

If we look back at Section 4.4 to see the number of calculations required to implement variation of parameters, we realize that the work can be formidable for equations of order 3 and above. But there is no need to do problems of higher order manually because any CAS will use the appropriate method efficiently to give us a general solution or solve an IVP. (One of the best methods for handling single linear equations and systems of linear equations is the *Laplace transform*, which we'll study in Chapter 5.) For now we'll just give an example of solving a higher-order linear equation, with some of the gory details left out.

Example 4.5.3 Solving a Nonhomogeneous Third-Order Equation

Suppose we want to find the general solution of $y''' - y'' - 6y' = 3t^2 + 2$. The first thing to do is to find the general solution of the associated homogeneous equation $y''' - y'' - 6y' = 0$. The characteristic equation is $\lambda^3 - \lambda^2 - 6\lambda = \lambda(\lambda^2 - \lambda - 6) = \lambda(\lambda - 3)(\lambda + 2) = 0$, with roots 0, 3, and -2, so the general solution of the homogeneous equation is $c_1 e^{0\cdot t} + c_2 e^{3t} + c_3 e^{-2t}$, or $c_1 + c_2 e^{3t} + c_3 e^{-2t}$.

Next, we look for a particular solution of the original nonhomogeneous equation. Examining the right-hand side of the equation, we can guess that a particular solution will be a polynomial in t. If the degree of the polynomial we guessed at is n, then the three individual derivative terms making up the differential equation will leave behind polynomials of degrees $n-3$, $n-2$, and $n-1$. In order for the combination $y''' - y'' - 6y'$ to produce the second-degree polynomial $3t^2 + 2$, we must have $n - 1 = 2$—that is, the polynomial we're looking for must be a third-degree polynomial, say $y(t) = At^3 + Bt^2 + Ct + D$, where A, B, C, and D are *undetermined coefficients*. (Think about the reasoning that led to this form for y.)

Substituting this guess into the nonhomogeneous equation, we find that

$$-18At^2 - (12B + 6A)t + (6A - 2B - 6C) = 3t^2 + 2.$$

Equating coefficients of terms of equal degree on both sides, we get the algebraic equations

$$\begin{aligned} -18A &= 3 \quad \text{[Second-degree terms must match.]} \\ -(12B + 6A) &= 0 \quad \text{[First-degree terms must match.]} \\ 6A - 2B - 6C &= 2 \quad \text{[Constant terms must match.]} \end{aligned}$$

Starting from the top, we can solve the equations successively to obtain $A = -\frac{1}{6}$, $B = \frac{1}{12}$, and $C = -\frac{19}{36}$.

Therefore, $y_{\text{PNH}} = -\frac{1}{6}t^3 + \frac{1}{12}t^2 - \frac{19}{36}t$ and the general solution of the nonhomogeneous equation is given by

$$y = y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}} = c_1 + c_2e^{3t} + c_3e^{-2t} - \frac{1}{6}t^3 + \frac{1}{12}t^2 - \frac{19}{36}t.$$

As we might suspect from the discussion at the end of Section 4.4, **variation of parameters can be used for all higher-order linear differential equations, provided that the coefficients are continuous functions of the independent variable**.

Exercises 4.5

A

Find the general solution of each of the higher-order equations in Problems 1–10, using a graphing calculator or CAS only to solve each characteristic equation.

1. $y''' - 2y'' - 3y' = 0$
2. $y''' - 3y'' + 3y' - y = 0$
3. $y''' + 2y'' + y' = 0$
4. $y''' + 4y'' + 13y' = 0$
5. $y''' - 12y'' + 22y' - 20y = 0$
6. $y^{(4)} + 2y'' + y = 0$
7. $y^{(4)} - 13y'' + 36y = 0$
8. $y^{(4)} + 13y'' + 36y = 0$
9. $y^{(4)} - 3y'' + 2y' = 0$
10. $y^{(5)} + 2y''' + y' = 0$

B

11. Find the general solution of the following equation, using technology only to solve a characteristic equation:

$$y^{(7)} - 14y^{(6)} + 80y^{(5)} - 242y^{(4)} + 419y^{(3)} - 416y'' + 220y' - 48y = 0.$$

12. Apply your CAS solver to find the general solution of the equation in the preceding problem.

13. The author of a classic differential equations text[4] once wrote
In preparing problems and examinations … teachers (including the author) must use some restraint. It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as

$$4.317\,\frac{d^4y}{dx^4} + 2.179\,\frac{d^3y}{dx^3} + 1.416\,\frac{d^2y}{dx^2} + 1.295\,\frac{dy}{dx} + 3.169y = 0.$$

Demonstrate that technology has advanced in the past six decades by feeding this equation into your CAS and obtaining the general solution. (You may have to use some "simplify" commands to get a neat answer.)

14. Solve the IVP $3y''' + 5y'' + y' - y = 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.

15. A uniform horizontal beam sags by an amount $y = y(x)$ at a distance x from one end. For a fairly rigid beam with uniform loading, $y(x)$ typically satisfies an equation of the form $d^4y/dx^4 = R$, where R is a constant depending on the load being carried and on the characteristics of the beam itself. If the ends of the beam are supported at $x = 0$ and $x = L$, then $y(0) = y(L) = 0$. The extended beam also behaves as though its profile has an inflection point at each support so that

$$y''(0) = y''(L) = 0.$$

a. Use the multiple root of the associated homogeneous equation's characteristic equation to find the general solution of the homogeneous equation.

b. Show that the sag (vertical deflection) at point x is

$$\frac{1}{24}R\left(x^4 - 2Lx^3 + L^3x\right), \quad 0 \le x \le L.$$

16. Solve the IVP $y^{(5)} = y'$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 1$, $y^{(4)}(0) = 2$.

17. For all positive integers $n \ge 2$, find the general solution of the equation $x^{(n)} = x^{(n-2)}$.

18. Find the general solution of the equation $y''' - 2y'' - y' + 2y = e^x$.

19. Find the general solution of the equation $\dddot{y} + 5\ddot{y} - 6\dot{y} = 9e^{3t}$.

20. Find the general solution of the equation $y''' + 6y'' + 11y' + 6y = 6x - 7$.

21. Find the general solution of $y''' - 5y'' - 2y' + 24y = x^2e^{3x}$.

22. Solve the IVP $y''' + 5y'' - 6y' = 3e^x$; $y(0) = 1$, $y'(0) = 3/7$, $y''(0) = 6/7$.

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[4] Ralph P. Agnew, *Differential Equations*, 2nd ed. (New York: McGraw-Hill, 1960): 176.

C

23. Use the method of undetermined coefficients to solve the equation

$$y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x.$$

24. Consider the IVP $y^{(4)} + 8y'' + 16y = 0$, with $y^{(k)}(0)$ given for $k = 0, 1, 2, 3$. For what initial values $y''(0)$ and $y'''(0)$ will the solutions of this equation be periodic?

25. Consider a third-order differential equation $y''' + p\,y'' + q\,y' + r\,y = g$, where p, q, r, and g are continuous functions. Suppose $y_{GH} = c_1 y_1 + c_2 y_2 + c_3 y_3$ is known.

a. Write down a form for y_{PNH} in terms of the known solutions of the homogeneous problem and unknown coefficient functions C_1, C_2, and C_3.

b. Derive a system of equations that determines C_1, C_2, and C_3. This system should involve C_1', C_2', C_3', y_i, y_i', and y_i'', $i = 1, 2, 3$.

c. Solve $y''' - 2y'' - y' + 2y = e^x$ by using the results of parts (a) and (b).

26. Find a formula involving integrals for a particular solution of the equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = g(x),\ x > 0.$$

[*Hint*: Verify that x, x^2, and x^3 are solutions of the homogeneous equation.]

*4.6 Existence and uniqueness[5]

Once again it's time to ask that important question we first considered in Section 2.8 in the context of first-order equations: How do we know that a given higher-order equation *has* a solution—and do we know that any such solution is *unique*?

We don't want to waste human and computer resources searching for a solution that may not exist or that may merely be one of many solutions. If the differential equation is supposed to model some real-life situation, having no solution or multiple solutions is not satisfactory. For now we'll focus on second-order equations. In Chapter 6 we'll look at generalizations to systems of equations.

The first example shows that when there is one solution of a second-order equation, there may be many.

Example 4.6.1 A Second-Order IVP with Many Solutions

Let's look at the IVP

$$t^2\ddot{x} - 2t\dot{x} + 2x = 0,\ \text{with } x(0) = 0 \text{ and } x'(0) = 0.$$

[5] * Denotes an optional section.

Then $x(t) \equiv 0$ and any function of the form $x(t) = Kt^2$ (where K is any constant) are solutions of the original IVP. (*Verify this.*) What we are saying here is that *our IVP has infinitely many solutions*.

In contrast to the IVP in the preceding example, we can have a second-order differential equation with *no* solution.

Example 4.6.2 A Second-Order IVP with No Solution

Let's look at the IVP $xy'' + y' + y = 0$, with $y(0) = 1$, $y'(0) = 2$. If we try to solve this problem with a CAS, we should get a blank result. This says that there is no solution to this IVP.

What we want in most real-life situations is one and only one solution to an IVP. The next example shows such a case.

Example 4.6.3 A Second-Order IVP with a Unique Solution

The IVP $\frac{d^2x}{dt^2} - x = 0$; $x(0) = 1$, $x'(0) = 0$ has the *unique* solution $x(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$. You may recognize x as the *hyperbolic cosine* (cosh).

4.6.1 An Existence and Uniqueness Theorem

At this point we have seen that the possibilities for second-order IVPs are similar to those we saw in Section 2.8 for first-order IVPs. We can have *no* solution, *infinitely many solutions*, or *exactly one solution*. Once again we would like to determine when there is one and only one solution of an IVP.

The simplest Existence and Uniqueness Theorem (EUT) for second-order differential equations is one that is a natural extension of the result we saw in Section 2.8.

Existence and Uniqueness Theorem (EUT)

Suppose we have a second-order IVP $\frac{d^2y}{dt^2} = f(t, y, \dot{y})$, with $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$. If f, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial \dot{y}}$ are continuous in a closed box B in three-dimensional space (t-y-$\dot{y}$ space) and the point $(t_0, y_0, \dot{y}_0)$ lies inside B, then the IVP has a unique solution $y(t)$ on some t-interval I containing t_0.

4.6.2 Many solutions

We can write the equation in Example 4.6.1 in the form $\ddot{x} = f(t, x, \dot{x}) = \frac{2t\dot{x} - 2x}{t^2}$, so we see that f does not exist in any box in which $t = 0$. Therefore, we should not expect exactly one solution, and, in fact, although there *is* a solution to the IVP with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, any such solution is not unique.

4.6.3 No solution

In Example 4.6.2, we can write the equation in the form $y'' = f(x, y, y') = -\frac{1}{x}y' - \frac{1}{x}y$. Then $\frac{\partial f}{\partial y} = -\frac{1}{x}$ and $\frac{\partial f}{\partial y'} = -\frac{1}{x}$, neither of which is continuous in any region containing the point $x = 0$. This says we are not guaranteed a unique solution—and indeed there is *no* solution.

4.6.4 Exactly one solution

Finally, if we examine the IVP in Example 4.6.3, we should see that in this situation we are guaranteed the existence of one and only one solution of the IVP. (*Check this.*)

The nice thing about these questions is that in most common applied problems, the functions and their derivatives are well behaved (continuous, etc.), so that we *do* have both existence and uniqueness. In particular, under reasonable conditions any IVP involving a second-order *linear* differential equation will have a unique solution.

Existence and Uniqueness Theorem

Consider the initial-value problem

$$y'' + p(t)y' + q(t)y = f(t); \quad y(t_0) = y_0,\ y'(t_0) = y_1.$$

If the functions p, q, and f are continuous on the interval $I : a < t < b$ containing the point $t = t_0$, then there is a unique solution $y = \phi(t)$ of the IVP, and this solution exists throughout the interval I.

Under the given continuity conditions, this theorem guarantees that the given IVP will always have (*existence*) exactly one (*uniqueness*) twice-differentiable solution on any interval containing t_0. In particular, any IVP involving a second-order equation with constant coefficients is certain to have a unique solution valid on $(-\infty, \infty)$. Conversely, neither existence nor uniqueness is guaranteed at a discontinuity of p, q, or f.

As we saw in Section 2.8, it is important to determine the largest interval on which a unique solution is guaranteed to exist.

Example 4.6.4 A Maximal Interval

Let's look at the IVP

$$(t+2)\ddot{x} + t\dot{x} + (\cot x)x = t^2 + 1; \; y(2) = 11, \; y'(2) = -2.$$

Writing this in standard form—so that the coefficient of the second derivative is one, as in the theorem—we have

$$\ddot{x} + \overbrace{\frac{t}{t+2}}^{p(t)}\dot{x} + \overbrace{\frac{\cos t}{(t+2)\sin t}}^{q(t)}x = \overbrace{\frac{t^2+1}{t+2}}^{f(t)},$$

with $t_0 = 2$.

The functions p and f are discontinuous at $t = -2$, whereas the function q is discontinuous at $t = -2$ and $t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Thus, the largest interval containing $t_0 = 2$ but none of the discontinuities is $(0, \pi)$.

Exercises 4.6

A

1. In Example 4.6.3 you saw the IVP $x'' - x = 0$, or $x'' = x$, with $x(0) = 1$ and $x'(0) = 0$. Using the technique in Section 4.1, show that $x(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$ is the solution of the IVP.

2. For each of the following IVPs, find the largest interval in which the solution is certain to exist, assuming that initial conditions are given at $t = 0$. Do not try to find the solution.

a. $\ddot{x} + 2\dot{x} + 6x = e^t \sin t$

b. $x'' + (\tan t)x' + \frac{1}{1-t}x = 0$

c. $(1 - t^2)x'' - 2tx' + 6x = 0$

d. $x'' + (\sec t)x' + \frac{1}{t-3}x = \sec t$

3. For each of the following IVPs, find the largest interval in which the solution is certain to exist. Do not try to find the solution.

a. $(x - 2)(x + 1)y'' + xy' + y = 0;\ y(0) = 1,\ y'(0) = 2$

b. $y'' + 2xy' + \ln(1 - x) = 0;\ y(0) = 1,\ y'(0) = 0$

c. $(x - 2)y'' + y' + (x - 2)(\tan x)y = 0;\ y(3) = 1,\ y'(3) = 2$

d. $(t^2 - 3t)\ddot{y} + t\dot{y} - (t + 3)y = 0;\ y(1) = 2,\ y'(1) = 1$

B

4. Find all solutions of the IVP

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0;\ \ y(t_0) = 0,\ \ \dot{y}(t_0) = 0,$$

where p and q are continuous on an open interval I containing t_0.

5. **a.** Determine the guaranteed interval of existence for the IVP

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0;\ \ y(0) = 1,\ \ \frac{dy}{dx}(0) = 0$$

without solving the problem.

b. Are there initial conditions for which a unique solution is not guaranteed?

c. For each value of $n = 0, 1, 2, 3$, find a polynomial of degree n that solves the IVP $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0;\ y(1) = 1$.

d. Is there a contradiction between theory and the actual result for the solutions in part (c)?

C

6. Think of the natural extension of the EUT for an nth-order differential equations. Consider the equation

$$5x^2y^{(5)} - (6\sin x)y''' + 2xy'' + \pi x^3y' + (3x-5)y = 0.$$

Suppose that $Y(x)$ is a solution of this equation such that $Y(1) = 0$, $Y'(1) = 0$, $Y''(1) = 0$, $Y'''(1) = 0$, $Y^{(4)}(1) = 0$, and $Y^{(5)}(1) = 0$. Why must $Y(x)$ be equal to 0 for *all* values of x?

7. The EUT for second-order linear equations tells us that the IVPs

$$\begin{aligned} &(1) \quad S'' + S = 0;\ S(0) = 0,\ S'(0) = 1 \\ &(2) \quad C'' + C = 0;\ C(0) = 1,\ C'(0) = 0 \end{aligned}$$

define unique functions S and C, respectively, for all real numbers x. Forgetting for the sake of this exercise that you can find the actual solutions of these IVPs, carry out the following tasks:

a. Suppose φ is a solution of $y'' + y = 0$. Show that φ' is also a solution of this equation.

b. Use the EUT to show that S' is actually the function C—that is, if S is the unique solution of IVP (1), then $S' = C$, where C is the unique solution of (2). Also, show that $C' = -S$.

c. Show that if $\varphi(x)$ is any solution of $y'' + y = 0$, then $\varphi(-x)$ is also a solution. Use this to show that C is an *even* function—that is, $C(-x) = C(x)$ for all real values of x.

d. Show that $S^2(x) + C^2(x)$ is a constant, and determine this constant by using initial conditions. [*Hint*: Let $f(x) = S^2(x) + C^2(x)$ and calculate $f'(x)$, using part (b) of this exercise to simplify.]

e. What familiar functions behave the way S and C behave? Verify your guess by solving each of the IVPs using the method given in Section 4.1.

Summary

For second-order homogeneous linear equations with constant coefficients—equations of the form

$$ax'' + bx' + cx = 0,$$

where a, b, and c are constants, $a \neq 0$—we can describe the solutions explicitly in terms of the roots of the associated **characteristic equation** $a\lambda^2 + b\lambda + c = 0$ as follows:

1. If there are two distinct real roots—λ_1, λ_2 with $\lambda_1 \neq \lambda_2$—then the general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

2. If there is a repeated real root λ, then the general solution has the form

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} = (c_1 + c_2 t)\, e^{\lambda t}.$$

3. If the roots form a complex conjugate pair $p \pm qi$, then the (real) general solution has the form $x(t) = e^{pt}\,(c_1 \cos(qt) + c_2 \sin(qt))$. Here, we need Euler's formula to deal with complex exponentials.

The general solution, y_{GNH}, of a linear nonhomogeneous system is obtained by finding a particular solution, y_{PNH}, of the nonhomogeneous system and adding it to the general solution, y_{GH}, of the homogeneous system: $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$. A particular solution can be found using the method of **undetermined coefficients** or **variation of parameters**.

For a linear equation of any order, we have the **Superposition Principle**. If y_j is a solution of $L(y) = f_j$ for $j = 1, 2, \ldots, n$, and $c_1, c_2, \ldots, c_n$ are arbitrary constants, then $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is a solution of $L(y) = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$. That is,

$$\begin{aligned} L\,(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) &= c_1 L\,(y_1) + c_2 L\,(y_2) + \cdots + c_n L\,(y_n) \\ &= c_1 f_1 + c_2 f_2 + \cdots + c_n f_n. \end{aligned}$$

As a consequence of the Superposition Principle, the formula $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$ is valid for a linear equation of any order n. We have an algorithm to find the general solution, y_{GH}, of the associated nth-order homogeneous equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$, where y is a function of the independent variable t and $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants.

First, find the roots of the characteristic equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

Use a CAS to solve the equation if n is greater than or equal to 3. Next, group these roots as follows: (a) distinct real roots; (b) distinct complex conjugate pairs $p \pm qi$; (c) multiple real roots; (d) multiple complex roots. Then the general solution is a sum of n terms of the forms

1. $c_i e^{\lambda_i t}$ for each distinct real root λ_i
2. $e^{pt}\,(c_1 \cos qt + c_2 \sin qt)$ for each distinct complex pair $p \pm qi$
3. $\left(c_1 + c_2 t + \cdots + c_k t^{k-1}\right) e^{\lambda_i t}$ for each multiple real root λ, where k is the multiplicity of that root
4. $e^{pt}\,(c_1 \cos qt + c_2 \sin qt) + t e^{pt}\,(c_3 \cos qt + c_4 \sin qt) + \cdots + t^{k-1} e^{pt}\,(c_{2k-1} \cos qt + c_{2k} \sin qt)$ for each multiple complex pair of roots $p \pm qi$, where k is the multiplicity of the pair $p \pm qi$

To find a particular solution of the nth-order nonhomogeneous equation, we can use the method of **undetermined coefficients** or **variation of parameters** as we did in the second-order case (although more work is involved).

Before getting too immersed in trying to solve higher-order equations, we should determine when solutions *exist* and whether existing solutions are *unique*. A useful result applies to a second-order IVP, $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$ with $y(t_0) = y_0$ and $\frac{dy}{dt}(t_0) = \dot{y}_0$. *If* f, $\frac{\partial f}{\partial y_1}$, *and* $\frac{\partial f}{\partial \dot{y}}$ *are continuous in a closed box B in three-dimensional space* (t-y-$\dot{y}$ *space*) *and the point* $(t_0, y_0, \dot{y}_0)$ *lies inside B, then the IVP has a unique solution* $y(t)$ *on some t-interval I containing* t_0.

For second-order *linear* differential equations, the IVP $y'' + p(t)y' + q(t)y = f(t)$; $y(t_0) = y_0$, $y'(t_0) = y_1$ will have a unique solution if the functions p, q, and f are continuous on the interval $I : a < t < b$ containing the point $t = t_0$, and this solution exists throughout the interval I.

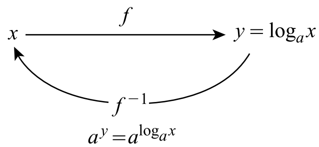
CHAPTER

5

The Laplace transform

Introduction

The idea of a *transform*, or *transformation*, is a very important one in mathematics and problem solving in general. When you are faced with a difficult problem, it is often a good idea to change it in some way into an easier problem, solve that easier problem, and then take your solution and apply it to your original problem. One of the first examples of this process that you have seen involves the idea of a *logarithm*. When John Napier and others developed logarithms in the early 1600s, they served as an aid to calculation. Given a difficult multiplication problem, you could transform it into an addition problem, perform the addition, and then transform the answer back into the answer to the original problem. For example, if you wanted to multiply 8743 by 2591, you could apply the natural logarithm to this product, getting the sum $\ln(8743) + \ln(2591) = 9.07600865918\ldots + 7.85979918056\ldots = 16.9358078397\ldots$. Then you would reverse the process by determining the number whose natural logarithm is $16.9358078397\ldots$. That number, 22,653,113, should be the original product. The process of going from the sum of logs back to the original product is called an *inverse* transformation. Of course, we recognize that the inverse of the logarithmic transformation is the exponential transformation:



In elementary calculus, you evaluated certain integrals by changing variables—transforming an integral in terms of x, say, to what you hope is a simpler integral in another variable, say u. Back in Section 2.2 you encountered a transformation when you introduced an *integrating factor* into a linear equation. By multiplying the equation by the appropriate exponential factor, you transformed the left-hand side into an exact derivative, which could then be integrated to yield the unknown function. You solved the equation by changing it into an equivalent form that was easier to deal with. The entire philosophy of using transformations in solving problems can be stated simply: I. TRANSFORM; II. SOLVE; and III. INVERT.

The important mathematical tool known as the *Laplace transform* is named for the great French mathematician Pierre-Simon de Laplace (1749–1827) who studied

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its properties, but it was probably used earlier by Euler. This transformation will be useful to us because it removes derivatives from differential equations and replaces them with *algebraic* expressions. In this way, differential equations are replaced by algebraic equations. This transformation turns out to be particularly powerful when we are dealing with initial-value problems (IVPs), nonhomogeneous equations with discontinuous forcing terms, and systems of differential equations. The downside is that the use of the Laplace transform is restricted to the solution of *linear* differential equations and *linear* systems of differential equations.

5.1 The Laplace transform of some important functions

We start by assuming that $f(t)$ is a function that is defined for $t \geq 0$. The **Laplace transform** of this function, $\mathcal{L}[f(t)]$, is defined as

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}\,dt, \tag{5.1.1}$$

when this improper integral exists. Note that after we've integrated with respect to t, the result will have the parameter s in it—that is, this integral is a *function* of the parameter s, so we can write $\mathcal{L}[f(t)] = F(s)$. We have transformed our function in t to a function in s.

Before we give some examples, let's just examine the integral in (5.1.1). From basic calculus, we know that the improper integral is defined as $\lim_{b\to\infty}\int_0^b f(t)e^{-st}\,dt$ when this limit exists. There are two important requirements here. First, the ordinary Riemann integral $\int_0^b f(t)e^{-st}\,dt$ must exist for every $b > 0$; and then the limit must exist as $b \to \infty$. Both requirements are taken care of if we stick to *continuous* or *piecewise continuous* functions $f(t)$ for which there are constants a, M, and T such that $|f(t)| < Me^{at}$ for all $t \geq T$. (Here we want M and T to be positive.) This says that the function f doesn't grow faster than an exponential function, so the integrand $f(t)e^{-st}$ in (5.1.1) behaves like the function $Me^{at}\cdot e^{-st} = Me^{-(s-a)t}$ for values of s greater than a and for t large enough. These functions are called **functions of exponential order a**. The improper integral of this kind of function converges. (See Section A.6 for basic definitions and examples.) To summarize,

> ***Let $f(t)$ be piecewise continuous on every finite interval in $[0,\infty)$ and satisfy $|f(t)| < Me^{at}$ for some constants M and a. Then $\mathcal{L}[f(t)]$ exists for $s > a$ and $\lim_{s\to\infty}\mathcal{L}[f(t)] = 0$.***

The limit statement is easily justified.

Suppose f is a piecewise continuous function of exponential order a. Then $F(s) = \lim_{N\to\infty}\int_0^N f(t)e^{-st}\,dt$. Our discussion in the previous paragraphs establishes that the integrals exist for every $N > 0$. Now

$$\left|\int_0^N f(t)\,e^{-st}\,dt\right| \leq \int_0^N |f(t)|\,e^{-st}\,dt \leq \int_0^N Me^{at}e^{-st}\,dt = M\int_0^N e^{-(s-a)t}\,dt.$$

Therefore, for $s > a$ we can take the limit as $N \to \infty$ to see that

$$|F(s)| = |\mathcal{L}[f(t)]| \le \lim_{N\to\infty} \left| \int_0^N f(t)e^{-st}\,dt \right| \le M \int_0^\infty e^{-(s-a)t}\,dt = \frac{M}{s-a}.$$

Thus, $\lim_{s\to\infty} F(s) \le \lim_{s\to\infty} |F(s)| \le M \cdot \lim_{s\to\infty} \left(\frac{1}{s-a}\right) = 0.$

Although being of exponential order is *sufficient* for a function to have a Laplace transform, it is not *necessary*.

Example 5.1.1 A Function Not of Exponential Order Whose Laplace Transform Exists

Consider the function $f(t) = \frac{d}{dt}\sin(e^{t^2}) = 2te^{t^2}\cos(e^{t^2})$. The factor e^{t^2} is the key to seeing that the function is not of exponential order. (*Supply the details.*) Fig. 5.1 shows the graph of $f(t)$.

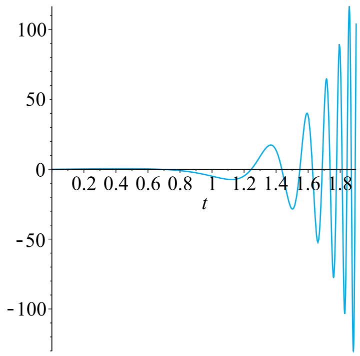


FIGURE 5.1

Graph of $f(t) = \frac{d}{dt}\sin(e^{t^2}) = 2te^{t^2}\cos(e^{t^2})$

Integrating by parts, we see that

$$\int_0^R f(t)e^{-st}\,dt = \int_0^R e^{-st}\frac{d}{dt}\sin(e^{t^2})\,dt = e^{-sR}\sin(e^{R^2}) - \sin 1 + s\int_0^R \sin(e^{t^2})e^{-st}\,dt.$$

The function $\sin(e^{t^2})e^{-st}$ is continuous on the interval $[0, R]$ and is therefore integrable on $[0, R]$ for $s > 0$. Moreover, for $s > 0$ we can write

$$\left| \int_0^R \sin(e^{t^2})e^{-st}\,dt \right| \le \int_0^R |\sin(e^{t^2})|e^{-st}\,dt \le \int_0^R e^{-st}\,dt = \frac{1-e^{-sR}}{R},$$

and $\lim_{R\to\infty} \int_0^R \sin(e^{t^2})e^{-st}\,dt < \lim_{R\to\infty} \frac{1-e^{-sR}}{R} = \frac{1}{s}$.

Therefore for $s > 0$, $\int_0^\infty \sin(e^{t^2})e^{-st}\,dt$ exists as a finite improper Riemann integral—that is, $\int_0^\infty 2te^{t^2}\cos(e^{t^2})e^{-st}\,dt = \mathcal{L}[f(t)]$ also exists and is finite as an improper Riemann integral. Note that $\lim_{s\to\infty} F(s) = 0$.

Now suppose that $f(t) \equiv 1$. Then

$$F(s) = \mathcal{L}[1] \int_0^\infty 1 \cdot e^{-st}\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, dt = \lim_{b\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^b$$
$$= \frac{1}{-s}\left(\lim_{b\to\infty} \left(e^{-sb} - 1 \right) \right) = \frac{1}{s}.$$

From this we can see that the Laplace transform of a constant function $f(t) \equiv C$ is $\frac{C}{s}$ for $s > 0$. (*Yes?*)

Next, we can find $\mathcal{L}[t]$ by using integration by parts and the value of $\mathcal{L}[1]$. For $s > 0$,

$$\mathcal{L}[t] = \int_0^\infty t e^{-st}\, dt = \lim_{b\to\infty} \int_0^b t e^{-st}\, dt$$
$$= \lim_{b\to\infty} \left\{ \left. \frac{te^{-st}}{-s} \right|_0^b - \int_0^b \frac{e^{-st}}{-s}\, dt \right\} = 0 + \frac{1}{s}\mathcal{L}[1] = \frac{1}{s^2}.$$

Similarly, we can show that $\mathcal{L}[t^2] = \frac{2}{s^3}$ and $\mathcal{L}[t^3] = \frac{6}{s^4}$ for $s > 0$. (See Problems 1 and 3 in Exercises 5.1.) In general, for all integers $n \geq 0$, it can be shown that

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \quad (s > 0), \tag{5.1.2}$$

where 0! is defined to be 1.

From the basic properties of integrals, we can see that

$$\mathcal{L}\left[c \cdot f(t)\right] = c\,\mathcal{L}\left[f(t)\right],$$

where c is any real constant, and that

$$\mathcal{L}\left[f(t) + g(t)\right] = \mathcal{L}\left[f(t)\right] + \mathcal{L}\left[g(t)\right],$$

whenever the Laplace transforms of both f and g exist. Any transformation that satisfies the last two properties is called a **linear operator** or a **linear transformation**. (See Section 2.2.) If c_1 and c_2 are constants, then we can combine the two properties to write

$$\mathcal{L}\left[c_1 f(t) + c_2 g(t)\right] = c_1\mathcal{L}\left[f(t)\right] + c_2\mathcal{L}\left[g(t)\right]. \tag{5.1.3}$$

Extending (5.1.3), we can see how to calculate the Laplace transform of any *polynomial function*:

$$\mathcal{L}\left[c_0 + c_1 t + c_2 t^2 + \cdots + c_k t^k + \cdots + c_n t^n\right]$$
$$= \mathcal{L}\left[c_0\right] + \mathcal{L}\left[c_1 t\right] + \mathcal{L}\left[c_2 t^2\right] + \cdots + \mathcal{L}\left[c_k t^k\right] + \cdots + \mathcal{L}\left[c_n t^n\right]$$

$$= c_0\,\mathcal{L}[1] + c_1\mathcal{L}[t] + c_2\mathcal{L}\left[t^2\right] + \cdots + c_k\mathcal{L}\left[t^k\right] + \cdots + c_n\mathcal{L}\left[t^n\right]$$
$$= \frac{c_0}{s} + \frac{c_1}{s^2} + \frac{2c_2}{s^3} + \frac{6c_3}{s^4} + \cdots + \frac{k!c_k}{s^{k+1}} + \cdots + \frac{n!c_n}{s^{n+1}} \quad (s > 0).$$

If a is a real number, let us find the Laplace transform of $f(t) = e^{at}$, an important function for us because of its frequent appearance in differential equations. By definition,

$$\begin{aligned}
\mathcal{L}\left[e^{at}\right] &= \int_0^\infty e^{at}e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{(a-s)t}\,dt \\
&= \lim_{b\to\infty}\left.\frac{e^{(a-s)t}}{(a-s)}\right|_0^b = \frac{1}{a-s}\left(\lim_{b\to\infty}\left(e^{(a-s)b}-1\right)\right) = -\frac{1}{a-s} \\
&= \frac{1}{s-a} \quad \text{for } s > a.
\end{aligned}$$

(*In what step is the assumption about s crucial?*)

To have the tools with which to handle a variety of differential equations, we have to stock our warehouse with different Laplace transforms. Another basic function we need to deal with is sin at, where a is a real number. This transform requires two integrations by parts:

For $s > 0$,

$$\begin{aligned}
\mathcal{L}[\sin at] &= \int_0^\infty \sin at\, e^{-st}\,dt = \lim_{b\to\infty}\sin at\left.\frac{e^{-st}}{-s}\right|_0^b - \int_0^\infty a\,\cos at\frac{e^{-st}}{-s}\,dt \\
&= \frac{a}{s}\int_0^\infty \cos at\, e^{-st}\,dt \\
&= \frac{a}{s}\left(\lim_{b\to\infty}\cos at\left.\frac{e^{-st}}{-s}\right|_0^b - \int_0^\infty -a\,\sin at\frac{e^{-st}}{-s}\,dt\right) \\
&= \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}L\left[\sin at\right]\right),
\end{aligned}$$

so that

$$\left(1+\frac{a^2}{s^2}\right)\mathcal{L}\left[\sin at\right] = \frac{a}{s^2} \text{ and } \mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}.$$

Using one of the steps from this result, we can easily show that $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$. (See Problem 4 in Exercises 5.1.)

To help set the stage for a type of applied differential equation problem that can be handled neatly by using the Laplace transform, let's find $\mathcal{L}[f(t)]$ for the piecewise continuous function defined as follows:

$$f(t) = \begin{cases} t & \text{for } 0 \le t \le 2 \\ 4-t & \text{for } 2 \le t \le 4 \\ 0 & \text{for } t \ge 4. \end{cases}$$

We can sketch the graph of this function. All we have to do is split the integral in definition (5.1.1) into three pieces, one corresponding to the interval [0, 2], another corresponding to [2, 4], and the last matching [4, ∞):

$$\begin{aligned} \mathcal{L}\left[f(t)\right] &= \int_0^2 te^{-st}\,dt + \int_2^4 (4-t)e^{-st}\,dt + \int_4^\infty 0\cdot e^{-st}\,dt \\ &= \frac{1-e^{-2s}-2se^{-2s}}{s^2} + \frac{e^{-4s}-e^{-2s}+2se^{-2s}}{s^2} \\ &= \frac{1+e^{-4s}-2e^{-2s}}{s^2} \end{aligned}$$

for $s > 0$. (Carry out all the integrations yourself!)

Finally, before we can apply Laplace transforms to the solution of differential equations, we have to know the transforms of f', f'', and higher-order derivatives. So suppose that $F(s) = \mathcal{L}[f(t)]$ exists for $s > c$. Then we have, for $s > c$,

$$\begin{aligned} \mathcal{L}\left[f'(t)\right] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \lim_{b\to\infty} f(t)e^{-st}\Big|_0^b + \int_0^\infty s\,f(t)e^{-st}\,dt = -f(0) + s\,\mathcal{L}\left[f(t)\right], \end{aligned}$$

which we can write as

$$\mathcal{L}\left[f'(t)\right] = s\,\mathcal{L}\left[f(t)\right] - f(0). \tag{5.1.4}$$

Note that in this derivation, we are assuming that $f(b)e^{-sb} \to 0$ as $b \to \infty$.

Now if we assume that $f\left(be^{-sb}\right)$ also tends to 0 as $b \to \infty$, we can apply formula (5.1.4) twice, first with f replaced by f', to get

$$\begin{aligned} \mathcal{L}\left[f''(t)\right] &= -f'(0) + s\,\mathcal{L}\left[f'(t)\right] = -f'(0) + s\left[s\,\mathcal{L}\left[f(t)\right] - f(0)\right] \\ &= -f'(0) + s^2\mathcal{L}\left[f(t)\right] - s\,f(0), \end{aligned}$$

so we can write

$$\mathcal{L}\left[f''(t)\right] = s^2\mathcal{L}\left[f(t)\right] - s\,f(0) - f'(0) \;\; (\text{for } s > c)\,. \tag{5.1.5}$$

In general, for any positive integer n, if the nth derivative is continuous (or piecewise continuous), and all the lower-order derivatives are continuous and have the proper growth rate, then

$$\begin{aligned}\mathcal{L}\left[f^{(n)}(t)\right] &= s^n\mathcal{L}\left[f(t)\right] - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0) \\ &= s^n\mathcal{L}\left[f(t)\right] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\, f^{(n-2)}(0) - f^{(n-1)}(0). \qquad (5.1.6)\end{aligned}$$

It is important to note that this last formula implies that a linear differential equation with constant coefficients will be transformed into a purely algebraic equation—that is, an equation without derivatives. (Recognize that $f^{(k)}(0)$ is a number for $k \geq 0$.)

As a hint of what we'll be doing in the next few sections, we'll convert a differential equation into an algebraic expression by using the Laplace transform.

Example 5.1.2 The Laplace Transform of a Differential Equation

Let's look at the IVP

$$x'' + 3x' + 2x = 12e^{2t};\; x(0) = 1,\; x'(0) = -1.$$

We're going to apply the Laplace transform to both sides of the equation and substitute the initial conditions where appropriate,

$$\mathcal{L}\left[x'' + 3x' + 2x\right] = \mathcal{L}\left[12e^{2t}\right],$$

or by using the linearity of the transform (5.1.3),

$$\mathcal{L}[x''] + 3\,\mathcal{L}[x'] + 2\,\mathcal{L}[x] = 12\,\mathcal{L}\left[e^{2t}\right].$$

We have already calculated the Laplace transform of an exponential function. This, together with formulas (5.1.4) and (5.1.5), allows us to write

$$\left\{s^2\mathcal{L}[x(t)] - sx(0) - x'(0)\right\} + 3\left\{s\,\mathcal{L}[x(t)] - x(0)\right\} + 2\,\mathcal{L}[x(t)] = \frac{12}{s-2}.$$

Now we substitute the given initial conditions to get

$$\left\{s^2\mathcal{L}[x(t)] - s + 1\right\} + 3\left\{s\,\mathcal{L}[x(t)] - 1\right\} + 2\,\mathcal{L}[x(t)] = \frac{12}{s-2}.$$

Finally, collecting like terms, we find that

$$\left(s^2 + 3s + 2\right)\mathcal{L}[x(t)] = \frac{12}{s-2} + s + 2 = \frac{s^2+8}{s-2},$$

so we can solve for $\mathcal{L}[x(t)]$:

$$\begin{aligned}\mathcal{L}[x(t)] &= \frac{s^2+8}{s-2}\cdot\frac{1}{s^2+3s+2} = \frac{s^2+8}{(s-2)\left(s^2+3s+2\right)} \\ &= \frac{s^2+8}{(s-2)(s+2)(s+1)}.\end{aligned}$$

Now what? We have an unknown function, the solution of an IVP, whose Laplace transform is known. If we can *reverse* the process and figure out what function has this Laplace transform, we can solve our original IVP. This is what we'll focus on in the next section.

Exercises 5.1

A

Use the definition and properties of the Laplace transform to find the Laplace transform of the functions in Problems 1–16 and specify the values of s for which each transform exists.

1. $f(t) = t^2$
2. $g(t) = t^2 - t$
3. $f(t) = t^3$
4. $h(t) = \cos at$, where a is a real number
5. $F(t) = t\, e^{at}$, where a is a real number
6. $s(t) = 2 \cos 3t$
7. $u(t) = 10 + 100\, e^{2t}$
8. $G(t) = \frac{e^{at} - e^{bt}}{a-b}$, where a and b are real numbers, $a \neq b$
9. $H(t) = 2t^3 - 7t^2 + 5t - 17$
10. $r(t) = 3 \sin 5t - 4 \cos 5t$
11. $U(t) = 2e^t - 3e^{-t} + 4t^2$
12. $S(t) = 3 - 5e^{2t} + 4 \sin t - 7 \cos 3t$
13. $F(t) = \begin{cases} t & \text{for } 0 < t < 4 \\ 0 & \text{for } t > 4 \end{cases}$
14. $f_a(t) = \begin{cases} t/a & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$, where $a \geq 0$
15. $A(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 2 - t & \text{for } 1 \leq t < 2 \\ 0 & \text{for } 2 \leq t \end{cases}$
16. $B(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 2\pi \\ \cos t & \text{for } 2\pi \leq t \leq 7\pi/2 \\ 0 & \text{for } 7\pi/2 \leq t < \infty \end{cases}$

In Problems 17–26, find the Laplace transform of the solution of each IVP, assuming that the Laplace transform exists in each case. (Do not try to solve the IVPs.)

17. $y' - y = 0$; $y(0) = 1$
18. $y' + y = e^{-x}$; $y(0) = 1$
19. $y' = -y + e^{-2t}$; $y(0) = 2$
20. $y' = -y + t^2$; $y(0) = 1$
21. $y'' + y = 0$; $y(0) = 1$, $y'(0) = 0$
22. $y'' + 4y' + 4y = 0$; $y(0) = 1$, $y'(0) = 1$

23. $y'' - y' - 2y = 5\sin x;\ y(0) = 1,\ y'(0) = -1$

24. $2y'' + 3y' - 2y = 1;\ y(0) = 0,\ y'(0) = 1/2$

25. $y''' - 2y'' + y' = 2e^x + 2x;\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$

26. $y^{(4)} - y''' + y = 1 + \sin(2t) - 2e^{3t};\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2,\ y'''(0) = 3$

B

27. Recall that the *hyperbolic sine* and *hyperbolic cosine* are defined as $\sinh(at) = (e^{at} - e^{-at})/2$ and $\cosh(at) = (e^{at} + e^{-at})/2$, respectively. Find the Laplace transform of these two functions and give the values of s for which the transforms exist.

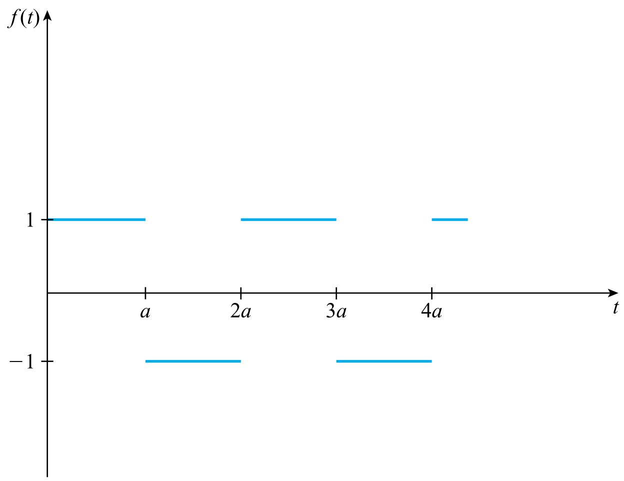
28. Consider the IVP $y'' + 3y = w(t);\ y(0) = 2,\ y'(0) = 0$, where

$$w(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 1 & \text{for } t \ge 1. \end{cases}$$

a. Find $\mathcal{L}[w(t)]$.

b. Find $\mathcal{L}[y(t)]$.

29. Find the Laplace transform of the following periodic function.



30. Determine $\mathcal{L}[\sin a\,t]$ using the fact that $\sin a\,t$ satisfies the differential equation $y'' + a^2 y = 0$. Do the same for $\mathcal{L}[\cos a\,t]$.

31. Apply formula (5.1.4) to the function $f''(t)$ to show that

$$\mathcal{L}[f'''(t)] = s^3\mathcal{L}[f(t)] - s^2 f(0) - s\,f'(0) - f''(0).$$

32. If $\mathcal{L}[f(t)]$ exists for $s = \alpha$, prove that it exists for all $s > \alpha$.

33. If $f(t) = e^{t^2}$ for $t \geq 0$, show that there are no constants M and K such that $|f(t)| < e^{Mt}$ for all $t \geq K$. Thus, show that the Laplace transform of $f(t)$ doesn't exist. [*Hint*: $e^{t^2} < e^{Mt}$ implies that $t^2 < Mt$ for t large enough.]

34. Prove that $\mathcal{L}\left[e^{\sqrt[3]{t}}\right]$ exists, but $\mathcal{L}\left[e^{e^t}\right]$ does not exist.

35. Show that $f(t) = 1$ for $t > 0$ and $g(t) = \begin{cases} 5 & \text{for } t = 3 \\ 1 & \text{for } t \neq 3 \end{cases}$ have the same Laplace transforms, namely $1/s$ for $s > 0$. Can you think of other functions with the same Laplace transform? Explain your answer.

36. Define

$$f(t) = \begin{cases} 1 & \text{for } t = 0 \\ t & \text{for } t > 0. \end{cases}$$

a. Find $\mathcal{L}[f(t)]$ and $\mathcal{L}[f'(t)]$.

b. Is it true for this function that $\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)$? Explain.

C

37. Use definition (5.1.1) and the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ to find the Laplace transform of $f(t) = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}}$. [*Hint*: Make the substitution $t = \frac{u^2}{s}$.]

38. Suppose $f(t)$ is a periodic function with period T—that is, $f(t + T) = f(t)$ for all t—such that $\mathcal{L}[f(t)]$ exists.

a. Show that

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T f(t)\,e^{-sT}\,dt.$$

[*Hint*: Make a substitution and recall geometric series.]

b. Use the result of part (a) to find the Laplace transform of the function given in Problem 29.

39. Suppose that a is any real number and that $F(s) = \mathcal{L}[f(t)]$. Show that

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s - a) \text{ for } s > a.$$

[This is usually called the *First Shift Formula*. See formulas (5.3.1a) and (5.3.1b) for the Second Shift Formula.]

5.2 The inverse transform and the convolution

5.2.1 The inverse Laplace transform

Recall that in Example 5.1.2 we took an IVP, applied the Laplace transform, and then wound up with the Laplace transform of the solution of the IVP. Now we would like to *reverse* the transformation process so that, given $\mathcal{L}[f(t)]$ as a function of the parameter s, we can find $f(t)$. This involves the idea of the **inverse Laplace transform**, $\mathcal{L}^{-1}$.

Now think back to the concept of inverse of a *function*. When you first encountered the inverse in precalculus or calculus, you may have worked with both the formal definition and the graphical interpretation in terms of a "horizontal line test." In any case, the important idea is that to have an inverse function f^{-1} we must guarantee that for any element in the *range* of the original function f, there is one and only one corresponding element in the *domain* of f. Another way of saying this is that *a function has an inverse if and only if it is a one-to-one function*.

For our purposes, the important fact is that *if the Laplace transforms of the continuous functions f and g exist and are equal for $s \geq c$* (c is a constant), *then* $f(t) = g(t)$ *for all $t \geq 0$*. This says that a continuous function can be uniquely recovered from its Laplace transform. (Compare Problem 35 in Exercises 5.1.) Letting $\mathcal{L}[f(t)] = F(s)$, we can express the definition of the inverse Laplace transform as

$$\mathcal{L}^{-1}[F] = f \text{ if and only if } \mathcal{L}[f] = F. \tag{5.2.1}$$

We can easily verify (see Problem 11 in Exercises 5.2) that the inverse Laplace transform is a linear transformation:

$$\mathcal{L}^{-1}[c_1 F(t) + c_2 G(t)] = c_1 \mathcal{L}^{-1}[F(t)] + c_2 \mathcal{L}^{-1}[G(t)]. \tag{5.2.2}$$

Now how do we find the inverse Laplace transform in practice? It turns out that the relationship between calculating a Laplace transform and determining its inverse is similar to that between differentiation and integration.

This means that in calculus, the indefinite integral of a function f answers the question, What is a function whose derivative is f? (Note that in calculus the answer to this question is not unique.) Just as a list of differentiation formulas helps us to construct a list of antidifferentiation formulas (indefinite integrals), so a table of Laplace transforms aids us in finding inverses. In the examples that follow, we will use the information in Table 5.1. Some of these transforms were derived in Section 5.1; others were given as exercises.

Now let's return to Example 5.1.2 and solve the IVP using Laplace transforms and the inverse Laplace transform.

Example 5.2.1 Solving an IVP Using the Inverse Laplace Transform

The IVP was $x'' + 3x' + 2x = 12e^{2t}$, $x(0) = 1$, $x'(0) = -1$, and we found that

$$\mathcal{L}[x(t)] = \frac{s^2 + 8}{(s-2)(s+2)(s+1)}.$$

If we try to work with the given expression for the transform (the single rational expression in s), we would have a tough time figuring out what function $x(t)$ might have this as its Laplace transform. This expression doesn't seem to correspond to any of the forms in the second column of Table 5.1.

Table 5.1 Some Laplace Transforms

	$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	$t^n\,(n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$, $s > 0$
2	e^{at}	$\frac{1}{s-a}$, $s > a$
3	$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
4	$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
5	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
6	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
7	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
8	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
9	$f'(t)$	$sF(s) - f(0)$
10	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
11	$e^{at}f(t)$	$F(s-a)$, $s > a$

However, we can use the partial fractions technique to express the transform as the sum of three simpler terms, each of which matches an entry in the table:

$$\frac{s^2+8}{(s-2)(s+2)(s+1)} = \frac{1}{s-2} + \frac{3}{s+2} - \frac{3}{s+1}.$$

We should be able to see, for example, that the term $\frac{3}{s+2}\left(= \frac{3}{s-(-2)}\right)$ is the Laplace transform of $3e^{-2t}$. Applying the inverse transform to each side of

$$\mathcal{L}[x(t)] = \frac{1}{s-2} + \frac{3}{s+2} - \frac{3}{s+1}$$

and using (5.2.1) and the linearity of $\mathcal{L}^{-1}$, we see that

$$\begin{aligned} x(t) = \mathcal{L}^{-1}[\mathcal{L}[x(t)]] &= \mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{3}{s+2} - \frac{3}{s+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] \\ &= e^{2t} + 3e^{-2t} - 3e^{-t}, \end{aligned}$$

where we have used formula 2 from Table 5.1 three times.

The alternative to using the Laplace transform to solve the IVP in the preceding example is to go back to the method first explained in Section 4.2. First, find the general solution of the homogeneous equation $x'' + 3x' + 2x = 0$; then find a particular solution of the nonhomogeneous equation $x'' + 3x' + 2x = 12e^{2t}$; finally, add these two solutions together to get the general solution of the original *nonhomogeneous* equation. And even then we would not be finished, because we would have to use the initial conditions to determine the two arbitrary constants in the general solution.

Note that the Laplace transform method enables us to handle the nonhomogeneous equation and initial conditions all at once.

Now let's see what the Laplace transform method does in an important applied problem that we first saw as Example 2.2.5.

Example 5.2.2 Solving a Circuit Problem via the Laplace Transform

The current I flowing in a particular electrical circuit can be described by the IVP $L\,\frac{dI}{dt} + RI = v_0 \sin(\omega t)$, $I(0) = 0$. Here, L, R, v_0, and ω are positive constants.

First, we apply the Laplace transform to each side of the differential equation

$$\begin{aligned}
\mathcal{L}\left[L\,\frac{dI}{dt} + R\,I\right] &= \mathcal{L}\left[v_0\, \sin(\omega t)\right] \\
L\,\mathcal{L}\left[\frac{dI}{dt}\right] + R\,\mathcal{L}[I(t)] &= v_0\,\mathcal{L}[\sin(\omega t)] \\
s\,L\,\mathcal{L}[I(t)] - L\,I(0) + R\,\mathcal{L}[I(t)] &= v_0\left(\frac{\omega}{s^2+\omega^2}\right) \\
(L\,s + R)\,\mathcal{L}[I(t)] - L\,I(0) &= v_0\left(\frac{\omega}{s^2+\omega^2}\right) \\
L\left(s + \frac{R}{L}\right)\mathcal{L}[I(t)] &= v_0\left(\frac{\omega}{s^2+\omega^2}\right),
\end{aligned}$$

so that we have $\mathcal{L}[I(t)] = \left(\frac{v_0}{L}\right)\cdot\omega\cdot\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)}$. To find the inverse Laplace transform, we have to use the method of partial fractions on the right-hand side:

$$\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)} = \frac{A}{\left(s+\frac{R}{L}\right)} + \frac{Bs+C}{s^2+\omega^2}.$$

With a little effort, we find that

$$\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)} = \frac{\frac{1}{\left(\frac{R^2}{L^2}+\omega^2\right)}}{s+\frac{R}{L}} + \frac{-\frac{1}{\left(\frac{R^2}{L^2}+\omega^2\right)}\,s + \frac{\left(\frac{R}{L}\right)}{\left(\frac{R^2}{L^2}+\omega^2\right)}}{s^2+\omega^2}$$

and

$$\begin{aligned}
\mathcal{L}[I(t)] &= \left(\frac{v_0}{L}\right)\cdot\omega\cdot\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)} \\
&= \left(\frac{v_0}{L}\right)\cdot\frac{\omega}{\left(\frac{R^2}{L^2}+\omega^2\right)}\left\{\frac{1}{s+\frac{R}{L}} + \frac{-s}{s^2+\omega^2} + \frac{\frac{R}{L}}{s^2+\omega^2}\right\}.
\end{aligned}$$

(*Check the last three equalities.*) The final step is to apply the inverse Laplace transform to both sides of this last equation and then use formulas 2, 3, and 4 from Table 5.1:

$$I(t) = \left(\frac{v_0}{L}\right)\frac{\omega}{\left(\frac{R^2}{L^2}+\omega^2\right)}\left\{\mathcal{L}^{-1}\left[\frac{1}{s+\frac{R}{L}}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] + \frac{R}{L}\,\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right]\right\}$$

$$= \left(\frac{v_0}{L}\right) \frac{\omega}{\left(\frac{R^2}{L^2}+\omega^2\right)} \left\{ e^{-\frac{R}{L}t} - \cos(\omega t) + \frac{R}{L}\frac{1}{\omega}\sin(\omega t) \right\}$$

$$= \left(\frac{v_0}{L}\right) \frac{1}{\left(\frac{R^2}{L^2}+\omega^2\right)} \left\{ \omega e^{-\frac{R}{L}t} - \omega\cos(\omega t) + \frac{R}{L}\sin(\omega t) \right\}.$$

Compare this solution to the one obtained in Example 2.2.5.

Example 5.2.3 Solving an IVP Using the Inverse Laplace Transform

Let's look at the IVP $\ddot{x} - 2\dot{x} = e^t\,(t-3)$; $x(0) = 2 = \dot{x}(0)$. As before, we take Laplace transforms of both sides and use the table. Letting $\mathcal{L}[x(t)] = X(s)$, we get

$$\left\{s^2 X(s) - s x(0) - \dot{x}(0)\right\} - 2\left\{s X(s) - x(0)\right\} = \mathcal{L}\left[e^t\,(t-3)\right]. \qquad (*)$$

To evaluate the right-hand side, first note that

$$\mathcal{L}[t-3] = \frac{1}{s^2} - \frac{3}{s} = F(s),$$

so if we use entry 11 in Table 5.1 (with $a = 1$), we get

$$\mathcal{L}\left[e^t\,(t-3)\right] = F(s-1) = \frac{1}{(s-1)^2} - \frac{3}{s-1} = \frac{4-3s}{(s-1)^2}.$$

If we return to (*) and put in our initial conditions, we get

$$s(s-2)X(s) = 2s - 2 + \frac{4-3s}{(s-1)^2}$$

$$= \frac{2(s-1)^3 + (4-3s)}{(s-1)^2} = \frac{(s-2)\left(2s^2-2s-1\right)}{(s-1)^2}.$$

Therefore, we conclude that

$$X(s) = \frac{2s^2-2s-1}{s(s-1)^2} = \frac{3}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{s}.$$

We go to the table to find the function $x(t) = \mathcal{L}^{-1}[X(t)]$, where we have to use entries 1 and 11 for the second term. The solution of the IVP is $x(t) = 3e^t - te^t - 1$. (*Check that this is the solution.*)

5.2.2 The convolution

In each of the preceding three examples, applying the Laplace transform yielded an expression for $\mathcal{L}[f(t)]$ that seemed to involve the product of two or more transforms. Because we didn't know a way to find the inverse transform of such products, we had to resort to the messiness of a partial fraction decomposition. This, at least, enabled us to use the linearity of the inverse transform.

There is, however, a way to deal with this problem—a method that involves a special product of functions.

The **convolution** of two functions f and g is the integral

$$(f * g)(t) = \int_0^t f(r)g(t-r)\,dr,$$

provided that the integral exists for $t > 0$.

For example the convolution of cos t and t is

$$\begin{aligned}(\cos t) * t &= \int_0^t (\cos r)\,(t-r)\,dr = \int_0^t t\cos r\,dr - \int_0^t r\cos r\,dr \\ &= t\int_0^t \cos r\,dr - \int_0^t r\cos r\,dr = 1 - \cos t\end{aligned}$$

after using integration by parts for the last integral. For this example we should verify that $(\cos t) * (t) = (t) * (\cos t)$.

Convolution has important algebraic properties (see Problem 15 in Exercises 5.2), but the most significant property for us right now is that **the Laplace transform of a convolution of two functions is equal to the product of the Laplace transforms of these two functions**. More precisely, suppose that f and g are two functions whose Laplace transforms exist. Let $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$. Then the **Convolution Theorem** says that

$$\mathcal{L}\big[(f * g)(t)\big] = \mathcal{L}\left[\int_0^t f(r)g(t-r)\,dr\right] = \mathcal{L}[f(t)]\cdot\mathcal{L}[g(t)] = F(s)\cdot G(s).$$

Now let's revisit part of Example 5.2.2 to see how the convolution property helps us.

Example 5.2.4 (Example 5.2.2 Revisited)

How can we find $\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)}\right]$?

The expression inside the brackets is the product of two transforms F and G:

$$F(s)G(s) = \left[\frac{1}{\left(s+\frac{R}{L}\right)(s^2+\omega^2)}\right],$$

where $F(s) = \frac{1}{\left(s+\frac{R}{L}\right)}$ and $G(s) = \frac{1}{(s^2+\omega^2)}$. Entries 2 and 3 of Table 5.1 tell us that $f(t) = \mathcal{L}^{-1}[F(s)] = e^{-\frac{R}{L}t}$ and $g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{\omega}\sin(\omega t)$. Then the Convolution Theorem leads us to conclude that

$$\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(r)g(t-r)\,dr,$$

or

$$\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{R}{L}\right)\left(s^2+\omega^2\right)}\right] = \int_0^t e^{-\frac{R}{L}r}\,\frac{1}{\omega}\sin\omega(t-r)\,dr$$
$$= \frac{1}{\omega}\int_0^t e^{-\frac{R}{L}r}\sin\omega(t-r)\,dr.$$

A computer algebra system evaluates this integral as

$$\frac{L\left(\omega\, L\, e^{\left(-\frac{L R}{t}\right)} - \omega\, L\,\cos\left(\omega t\right) + R\,\sin\left(\omega t\right)\right)}{\left(R^2+\omega^2 L^2\right)\omega}.$$

A bit of algebra will show you that this corresponds to the inverse transform found in Example 5.2.2 via partial fractions. (*Do the work.*)

*5.2.3 Integral equations and integro-differential equations[1]

The Convolution Theorem is certainly useful in solving differential equations, but it can also help us solve **integral equations**, equations involving an integral of the unknown function, and **integro-differential equations**, those involving both a derivative and an integral of the unknown function.

Example 5.2.5 The Convolution Theorem and an Integral Equation

A store manager finds that the proportion of merchandise that remains unsold at time t after she has bought the merchandise is given by $f(t) = e^{-1.5t}$. She wants to find the rate at which she should purchase the merchandise so that the stock in the store remains constant.

Suppose that the store starts off by buying an amount A of the merchandise at time $t = 0$ and buys at a rate $r(t)$ subsequently. Over a short time interval $u \le t \le u + \Delta u$, an amount $r(t)\cdot\Delta u$ is bought by the store, and at time t the portion of this remaining unsold is $e^{-1.5(t-u)}r(u)\Delta u$. Then the amount of previously purchased merchandise remaining unsold at time t is given by

$$Ae^{-1.5t} + \int_0^t e^{-1.5(t-u)} r(u)\,du.$$

Because this is the total stock of the store and the store manager wants it to remain constant at its initial value, we must have

$$A = Ae^{-1.5t} + \int_0^t e^{-1.5(t-u)} r(u)\,du,$$

and the required restocking rate $r(t)$ is the solution of this integral equation.

If we look carefully at the integral on the right-hand side of this last equation, we should recognize something familiar about its form. It looks like a convolution—in fact, it is $e^{-1.5t} * r(t)$. Now we can rewrite the integral equation in the form

$$A = Ae^{-1.5t} + \left(e^{-1.5t} * r(t)\right).$$

[1] * Denotes an optional section.

Taking the Laplace transform of each side and letting $R(s) = \mathcal{L}[r(t)]$, we get

$$\mathcal{L}[A] = A\mathcal{L}\left[e^{-1.5t}\right] + \mathcal{L}\left[e^{-1.5t} * r(t)\right] = \frac{A}{s+1.5} + \frac{1}{s+1.5} \cdot R(s),$$

$$\frac{A}{s} = \frac{A}{s+1.5} + \frac{1}{s+1.5} \cdot R(s),$$

$$(s+1.5)\left(\frac{A}{s} - \frac{A}{s+1.5}\right) = R(s),$$

$$\frac{1.5A}{s} = R(s).$$

Applying the inverse Laplace transform to each side, we find that $r(t) = 1.5A$. That is, the restocking rate should be a constant one and a half times the original amount bought. (*Check that this is a solution of our original integral equation.*)

Example 5.2.6 An Integro-Differential Equation

The following integro-differential equation can also be solved using the properties of the Laplace transform:

$$\frac{dx}{dt} + x(t) - \int_0^t x(r)\sin(t-r)\,dr = -\sin t,\ x(0) = 1.$$

As in the preceding example, we recognize that the integral in our equation represents a convolution, this time $(x * \sin)(t)$. Therefore, taking the Laplace transform of each side of the equation, we get

$$\mathcal{L}[dx/dt] + \mathcal{L}[x(t)] - \mathcal{L}\left[(x * \sin)(t)\right] = \mathcal{L}[-\sin t],$$

or, using formula 10 in Table 5.1 and the Convolution Theorem,

$$[s\mathcal{L}[x(t)] - x(0)] + \mathcal{L}[x(t)] - \mathcal{L}[x(t)] \cdot \mathcal{L}[\sin t] = -\frac{1}{s^2+1},$$

which becomes

$$[s\mathcal{L}[x(t)] - 1] + \mathcal{L}[x(t)] - \mathcal{L}[x(t)] \cdot \frac{1}{s^2+1} = -\frac{1}{s^2+1}.$$

This simplifies to

$$\left(\frac{s^3+s^2+s}{s^2+1}\right)\mathcal{L}[x(t)] = \frac{s^2}{s^2+1},$$

so we wind up with $\mathcal{L}[x(t)] = \frac{s^2}{s^3+s^2+s} = \frac{s}{s^2+s+1}$.

A bit of clever algebra shows us that

$$\frac{s}{s^2+s+1} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{s-\left(-\frac{1}{2}\right)}{\left(s-\left(-\frac{1}{2}\right)\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}} \quad \frac{\frac{\sqrt{3}}{2}}{\left(s-\left(-\frac{1}{2}\right)\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}.$$

Using formulas 5 and 6 from Table 5.1 to invert this transform, we find that

$$x(t) = e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right).$$

(*Checking that this is the solution involves a bit of work, but try it!*)

5.2.4 The Laplace transform and technology

Most computer algebra systems have built-in Laplace transform and inverse transform capabilities. In particular, some systems (for example, *Maple*) have sophisticated differential equation solvers with a "laplace" option for IVPs. If you have such an option at your command, learn to use it. However, realize that the process of how it works is hidden, so you have to develop an understanding of what the system is really doing.

Be aware that some computer algebra systems can find Laplace transforms and their inverses, but have no direct way of solving a linear IVP with these tools. In this case you have to apply the Laplace transform to the differential equation, solve for the transform $\mathcal{L}[x(t)]$ of the solution algebraically (via a *solve* command or by hand), use technology to find the inverse transform $\mathcal{L}^{-1}[\mathcal{L}[x(t)]]$, and finally substitute the initial conditions.

Determine what your options are in using technology to solve IVPs via the Laplace transform. Some of the exercises that follow will help you do this.

Exercises 5.2

A

1. Find the inverse Laplace transform of $\frac{1}{s^2+9}$.
2. Find the inverse Laplace transform of $\frac{s}{s^2-a^2}$.
3. Find the inverse Laplace transform of $\frac{s}{s^2+2}$.
4. Find the inverse Laplace transform of $\frac{a}{s^2(s^2+a^2)}$.
5. Find the inverse Laplace transform of $\frac{1}{s(s^2+2s+2)}$.
6. Find the inverse Laplace transform of $\frac{2s-10}{s^2-4s+20}$.
7. Find the inverse Laplace transform of $\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}$.
8. Find the inverse Laplace transform of $\frac{3s+7}{s^2-2s-3}$.
9. Find the inverse Laplace transform of $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$.
10. Find the inverse Laplace transform of

$$\frac{1}{s-4} + \frac{1}{(s-1)(s-4)}.$$

B

11. If $F(t)$ and $G(t)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, and c_1 and c_2 are constants, show that

$$\mathcal{L}^{-1}[c_1F(t) + c_2G(t)] = c_1\mathcal{L}^{-1}[F(t)] + c_2\mathcal{L}^{-1}[G(t)].$$

12. Suppose $y_1(t)$ is the solution of the IVP $y'' + ay' + by = f_1(t);\ y(0) = y'(0) = 0$, where a and b are constants.

a. Compute an expression for $\mathcal{L}[y_1]$.

b. Suppose $y_2(t)$ is the solution of the IVP

$$y'' + a\,y' + b\,y = f_2(t);\ \ y(0) = y'(0) = 0$$

for a different forcing function $f_2(t)$. Show that

$$\frac{\mathcal{L}[f_2]}{\mathcal{L}[y_2]} = \frac{\mathcal{L}[f_1]}{\mathcal{L}[y_1]}.$$

c. Show that $\mathcal{L}[y_2] = \mathcal{L}[f_2] \cdot \frac{\mathcal{L}[y_1]}{\mathcal{L}[f_1]}$. (This says that we can use the solution with any forcing function and zero initial conditions to compute solutions of other forcing functions.)

13. **a.** Show that the Laplace transform of $t^n f(t)$ is $(-1)^n F^{(n)}(s)$, where $F(s) = \mathcal{L}[f(t)]$.

b. Use the result of part (a) and the derivative of the function $F(s) = \ln(2 + \frac{3}{s})$, $s > 0$, to find its inverse Laplace transform.

14. Find the convolution $f * g$ of each of the following pairs of functions:

a. $f(t) = t^2$, $g(t) = 1$

b. $f(t) = t$, $g(t) = e^{-t}$ for $t \geq 0$

c. $f(t) = t^2$, $g(t) = (t^2 + 1)$ for $t \geq 0$

d. $f(t) = e^{-at}$, $g(t) = e^{-bt}$ (a, b constants)

e. $f(t) = \cos\, t$, $g(t) = \cos\, t$

[*Hint*: For part (e) you need some trigonometric identities.]

15. Prove the following properties of the convolution of functions:

a. $f * g = g * f$ [Commutativity]

b. $(f * g) * h = f * (g * h)$ [Associativity]

c. $f * (g + h) = f * g + f * h$ [Distributivity]

d. $f * 0 = 0$, but $f * 1 \neq f$ and $f * f \neq f^2$ in general. (In particular, $1 * 1 \neq 1$.)

16. **a.** Using property (b) of Problem 15, find $1 * 1 * 1$.

b. Find $1 * t * t^2$.

17. Use the Convolution Theorem to find the Laplace transform of

$$f(t) = \int_0^t \cos(t - r) \sin r \, dr.$$

18. Find the Laplace transform of

$$h(t) = \int_0^t e^{t-v} \sin v \, dv.$$

19. Find the solution of the IVP $y'' + 3y' + 2y = 4t^2$; $y(0) = 0$, $y'(0) = 0$.

20. Solve the IVP $y'' + 4y' + 4y = e^{-2x}$; $y(0) = 0$, $y'(0) = 1$.

21. Solve the IVP $y''' - 2y'' + y' = 2e^x + 2x$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

22. Solve the IVP $y'' + 6y' + 9y = H(x)$; $y(0) = 0$, $y'(0) = 0$, where $H(x)$ is a known function of x. [*Hint*: Use the Convolution Theorem.]

23. An electrical circuit that is initially unforced but is plugged into a particular alternating voltage source at time $t = \pi$ can be modeled by the IVP

$$Q'' + 2Q' + 2Q = \begin{cases} 0 & \text{for } 0 \le t < \pi \\ -\sin t & \text{for } t \ge \pi, \end{cases}$$

with $Q(0) = 0$ and $Q'(0) = 1$. Solve the IVP for $Q(t)$, the charge on the capacitor at time t.

24. The equation $v \frac{dv}{ds} = v \cos(2s) - v^2$ describes the velocity v of a piston moving into an oil-filled cylinder under a variable force. Here, s is the distance moved in time t.

a. Rewrite the given equation as a linear equation with constant coefficients.

b. Assuming that $v(0) = 0$, use the Laplace transform to solve for v as a function of s. Is there a singular solution?

c. Use technology to graph the nontrivial solution found in part (b) for $0 \le s \le 20$.

25. Solve the integral equation for f: $f(t) = 4t + \int_0^t f(t - r) \sin r \, dr$.

26. Solve for g: $g(t) - t = -\int_0^t (t - r) g(r) \, dr$.

C

27. Solve for x : $\dot{x}(t) = 1 - \int_0^t x(t - r) e^{-2r} \, dr$, $x(0) = 1$.

28. Solve the integro-differential equation

$$\dot{y} + y + \int_0^t y(u) \, du = 1, \text{ with } y(0) = 0.$$

29. Solve the equation $\dot{x} - 4x + 4\int_0^t x(u) \, du = t^3 e^{2t}$, with $x(0) = 0$.

30. Solve the equation $f''(x) + \int_0^x e^{2(x-y)} f'(y) \, dy = 1$; $f(0) = 0$, $f'(0) = 0$.

5.3 Transforms of discontinuous functions

Differential equations are often used to model complex systems. In some situations, models have to deal with abrupt changes in these systems. In the circuit problem described in Example 2.2.5 (or Example 5.2.2), we have the equation $L\,\frac{dI}{dt} + RI = v_0\,\sin(\omega t)$, where the right-hand side (the forcing term) represents a continuous alternating current source. Now suppose that the voltage $E(t)$ were applied for only a short period of time and then discontinued. Mathematically, this means that the forcing term would have the form

$$f(t) = \begin{cases} E(t) & \text{for } 0 \le t \le a \\ 0 & \text{for } t > a. \end{cases}$$

Perhaps we have a switch that we can open and close so that the voltage is applied, removed, and then applied again:

$$g(t) = \begin{cases} E(t) & \text{for } 0 \le t \le a \\ 0 & \text{for } a < t < b \\ E(t) & \text{for } t \ge b. \end{cases}$$

Problem 30 in Exercises 2.2, in which advertising expenditure is terminated after a certain period of time, is another illustration of this kind of behavior. The common element here is abrupt change. In mathematical terms, we are dealing with *piecewise continuous functions*.

5.3.1 The Heaviside (unit step) function

In Section 5.1 we saw a simple example of the Laplace transform applied to a piecewise continuous function. We computed the transform directly from the definition, breaking the integral into two parts. This can be tedious if there are several intervals involved in the definition of the function. Now we will see how these kinds of functions can be expressed in such a way that the Laplace transform method doesn't have to consider separate intervals.

We start with the **unit step function** U defined by

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \ge 0. \end{cases}$$

(This is sometimes called the **Heaviside (unit step) function**, named for the English electrical engineer Oliver Heaviside (1850–1925) who developed many of the applications of Laplace transforms that we will see.) We can say that the function U is "off" ($= 0$) for negative values of t and "on" ($= 1$) for values of t greater than or equal to 0. This "switching" aspect makes U an important building block in modeling abrupt changes.

It follows that

$$U(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$$

The function has a jump discontinuity at $t = a$. Fig. 5.2 shows $U(t-3)$ for $t \geq 0$.

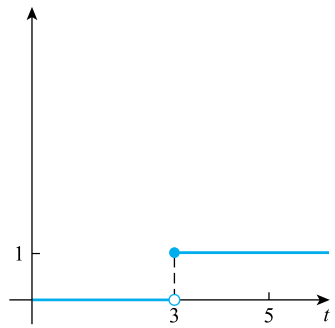


FIGURE 5.2

Graph of $U(t-3)$, $t \geq 0$

The nice thing is that these step functions can be used to express a piecewise continuous function in terms of a single formula. For example, if

$$f(t) = \begin{cases} A(t) & \text{for } t < a \\ B(t) & \text{for } t \geq a, \end{cases}$$

then we can see that $f(t) = A(t) + U(t-a)[B(t) - A(t)]$: If $t < a$, then $U(t-a) = 0$, so $f(t) = A(t)$; whereas if $t \geq a$, then we have $U(t-a) = 1$, so $f(t) = A(t) + [B(t) - A(t)] = B(t)$. (*Is that clear?*)

This technique can be extended to functions such as

$$g(t) = \begin{cases} A(t) & \text{for } a \leq t < b \\ B(t) & \text{for } b \leq t < c \\ C(t) & \text{for } c \leq t < d. \end{cases}$$

We can write $g(t) = U(t-a)A(t) + U(t-b)[B(t) - A(t)] + U(t-c)[C(t) - B(t)]$. (*Make sure that you see how this works.*)

When we are solving differential equations that model abrupt changes, the following result comes in handy:

$$\boxed{\begin{array}{l} \text{If } \mathcal{L}[f(t)] \text{ exists for } s > c \text{ and if } a > 0, \text{ then} \\ \mathcal{L}[f(t-a)U(t-a)] = e^{-as}\mathcal{L}[f(t)] \text{ for } s > c. \end{array}} \tag{5.3.1a}$$

This result is usually called the **Second Shift Formula**—see Problem 39 in Exercises 5.1 for the **First Shift Formula**.

Alternatively, we can write (5.3.1a) as

$$\boxed{f(t-a)U(t-a) = \mathcal{L}^{-1}\left[e^{-as}\mathcal{L}[f(t)]\right].} \tag{5.3.1b}$$

Formula (5.3.1a) follows from a straightforward calculation:

$$\begin{aligned}\mathcal{L}\left[f(t-a)U(t-a)\right] &= \int_0^\infty f(t-a)U(t-a)e^{-st}\,dt = \int_a^\infty f(t-a)e^{-st}\,dt \\ &= \int_0^\infty f(u)e^{-s(u+a)}\,du = e^{-sa}\mathcal{L}[f(t)],\end{aligned}$$

where we have made the substitution $t-a=u$ in the last integral.

The next example shows how to use the Laplace transform of a unit step function to solve an IVP.

Example 5.3.1 An IVP with a Discontinuous Forcing Term

Let's look at the IVP

$$x'(t)+x = \begin{cases} t & \text{for } 0 \le t < 4 \\ 1 & \text{for } 4 \le t \end{cases} \quad ; \quad x(0)=1.$$

Using the unit step function, we can write the differential equation as

$$x'(t)+x = t + (1-t)U(t-4) = t-(t-4)U(t-4)-3U(t-4).$$

[Note that in order to use formula (5.3.1b) later, we have to use algebra to convert the term $(1-t)U(t-4)$ into the form $f(t-a)U(t-a)$.] Now we apply the Laplace transform to both sides of the equation to get

$$\mathcal{L}[x'(t)]+\mathcal{L}[x(t)] = \mathcal{L}[t]-\mathcal{L}\left[(t-4)U(t-4)\right]-3\,\mathcal{L}[U(t-4)],$$

or, using entry 9 in Table 5.1 and then formula (5.3.1a) twice,

$$s\mathcal{L}[x(t)]-1+\mathcal{L}[x(t)] = \frac{1}{s^2}-e^{-4s}\mathcal{L}[t]-3e^{-4s}\mathcal{L}[1],$$

so

$$(s+1)\mathcal{L}[x(t)] = 1+\frac{1}{s^2}-e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right).$$

Therefore,

$$\begin{aligned}\mathcal{L}\left[x(t)\right] &= \frac{1}{s+1}+\frac{1}{s^2(s+1)}-e^{-4s}\left(\frac{3s+1}{s^2(s+1)}\right) \\ &= \left[\text{by partial fractions}\right]\ \frac{2}{s+1}-\frac{1}{s}+\frac{1}{s^2}-e^{-4s}\left(\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s+1}\right).\end{aligned}$$

Finally, applying the inverse transform to both sides and using (5.3.1b), we get

$$\begin{aligned} x(t) &= 2e^{-t} - 1 + t - \mathcal{L}^{-1}\left[e^{-4s}\left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right)\right] \\ &= 2e^{-t} - 1 + t - \left[U(t)\mathcal{L}^{-1}\left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right)\right] \end{aligned}$$

(where t must be replaced by $t-4$ within the brackets before we're finished)

$$\begin{aligned} &= 2e^{-t} - 1 + t - \left[U(t)\left(2 + t - 2e^{-t}\right)\right] \\ &= 2e^{-t} - 1 + t - U(t-4)\left(t - 2 - 2e^{-t+4}\right) \\ &= \begin{cases} 2e^{-t} + t - 1 & \text{for } 0 \le t < 4 \\ 2e^{-t} + 2e^{-t+4} + 1 & \text{for } 4 \le t. \end{cases} \end{aligned}$$

The next example shows the application of the Laplace transform and the unit step function to an important type of applied problem.

Example 5.3.2 A Cantilever Beam Problem

A wooden beam, whose ends are considered to be at $x = 0$ and $x = L$ on a horizontal axis, will "give" (that is, bend) when a vertical load, given by $W(x)$ per unit length, acts on the beam (Fig. 5.3). (Compare Problem 20 in Exercises 1.3.)

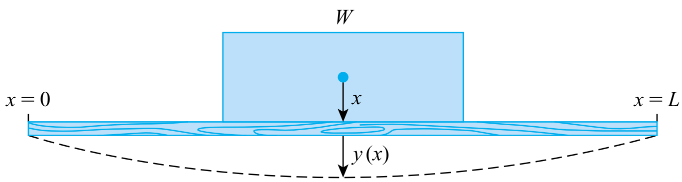


FIGURE 5.3

A loaded beam

It is known that $y(x)$, the amount of bending, or deflection, in the direction of the load force at the point x, satisfies the differential equation $\frac{d^4y}{dx^4} = \frac{W(x)}{EI}$ for $0 < x < L$.

Here, E and I are constants that describe characteristics of the beam. The graph of $y(x)$ is called the *deflection curve* or *elastic curve*.

Now suppose that we have a *cantilever beam* (like a diving board)—clamped at the end $x = 0$ and free at the end $x = L$—and that this beam carries a load per unit length given by

$$W(x) = \begin{cases} W_0 & \text{for } 0 < x < \frac{L}{2} \\ 0 & \text{for } \frac{L}{2} < x < L. \end{cases}$$

Engineering mechanics shows that finding the deflection amounts to solving the boundary-value problem (BVP)

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI} \text{ (for } 0 < x < L); \quad y(0) = 0,\ y'(0) = 0,\ y''(L) = 0,\ y'''(L) = 0.$$

(In physics terms, the quantities $y''(L)$ and $y'''(L)$ are called the *bending moment* and the *shear force*, respectively.)

First of all, note that so far we have applied the technique of Laplace transforms only to IVPs, not to BVPs. Second, to use the Laplace transform, we must assume that $y(x)$ and $W(x)$ are defined on the interval $(0, \infty)$ rather than just on $(0, L)$. This means that we should extend the definition of $W(x)$ as follows:

$$W(x) = \begin{cases} W_0 & \text{for } 0 < x < \frac{L}{2} \\ 0 & \text{for } x > \frac{L}{2}. \end{cases}$$

We can write this function in terms of the unit step function as

$$W(x) = W_0 \left\{ U(x) - U\left(x - \frac{L}{2}\right) \right\}.$$

Now take the Laplace transform of each side of our fourth-order equation, letting $Y = Y(s) = \mathcal{L}[y(x)]$ for convenience. Using (5.1.6), we find that

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{W_0}{EI} \left\{ \frac{1 - e^{-sL/2}}{s} \right\}.$$

Note that in formula (5.1.6) the second and third derivatives of y are evaluated at 0. However, our BVP gives us the values of these derivatives at L. Letting $y''(0) = C_1$ and $y'''(0) = C_2$, we can use all the boundary conditions as given and solve the last equation for $Y : Y = \frac{C_1}{s^3} + \frac{C_2}{s^4} + \frac{W_0}{EIs^5}\left\{1 - e^{2s\,L/2}\right\}$. Using the inverse transform, we find that

$$y(x) = \frac{C_1 x^2}{2!} + \frac{C_2 x^3}{3!} + \frac{W_0}{EI}\frac{x^4}{4!} - \frac{W_0}{EI}\frac{\left(x - \frac{L}{2}\right)^4}{4!} U\left(x - \frac{L}{2}\right),$$

which is equivalent to

$$y(x) = \begin{cases} \frac{C_1 x^2}{2} + \frac{C_2 x^3}{6} + \frac{W_0}{24EI} x^4 & \text{for } 0 \le x < \frac{L}{2} \\ \frac{C_1 x^2}{2} + \frac{C_2 x^3}{6} + \frac{W_0}{24EI} x^4 - \frac{W_0}{24EI}\left(x - \frac{L}{2}\right)^4 & \text{for } x \ge \frac{L}{2}. \end{cases}$$

Now we use the conditions $y''(L) = 0$ and $y'''(L) = 0$ to find that $C_1 = \frac{W_0 L^2}{8EI}$ and $C_2 = -\frac{W_0 L}{2EI}$. (*Be sure to go through the calculations for yourself.*)

Finally, we can write our deflection function as

$$y(x) = \begin{cases} \frac{W_0 L^2}{16EI} x^2 - \frac{W_0 L}{12EI} x^3 + \frac{W_0}{24EI} x^4 & \text{for } 0 \le x < \frac{L}{2} \\ \frac{W_0 L^2}{16EI} x^2 - \frac{W_0 L}{12EI} x^3 + \frac{W_0}{24EI} x^4 - \frac{W_0}{24EI}\left(x - \frac{L}{2}\right)^4 & \text{for } \frac{L}{2} \le x < L. \end{cases}$$

Exercises 5.3

A

In Problems 1–6, (a) sketch the graph of each function $f(t)$ and (b) write each function as a sum of multiples of the unit step function $U(t)$.

1. $f(t)=\begin{cases}1 & \text{for } 1\le t<2\\ 0 & \text{elsewhere}\end{cases}$
2. $f(t)=\begin{cases}t^2 & \text{for } 0<t<2\\ 4t & \text{for } t>2\end{cases}$
3. $f(t)=\begin{cases}1 & \text{for } 1\le t<2\\ -2 & \text{for } 2\le t<3\\ 0 & \text{elsewhere}\end{cases}$
4. $f(t)=\begin{cases}t & \text{for } 0\le t<2\\ 4-t & \text{for } 2\le t<4\\ 0 & \text{elsewhere}\end{cases}$
5. $f(t)=\begin{cases}t & \text{for } 0\le t<2\\ t-2 & \text{for } 2\le t<4\\ 0 & \text{elsewhere}\end{cases}$
6. $f(t)=\begin{cases}\sin t & \text{for } 0<t<\pi\\ \sin 2t & \text{for } \pi<t<2\pi\\ \sin 3t & \text{for } t>2\pi\end{cases}$
7. Show that $\mathcal{L}[t\,U(t-a)]=(1+as)\,s^{-2}e^{-as}$ for $a>0$.
8. Calculate $\mathcal{L}[t^2U(t-1)]$.
9. Show that $\mathcal{L}\left[t^2\,U(t-2)\right]=\frac{2}{s^3}-\frac{2\,e^{-2s}}{s^3}\left(1+2s+2s^2\right)$, $s>0$.
10. Use formula (5.3.1a-b) to compute the Laplace transform of the function in Problem 1.
11. Use formula (5.3.1a-b) to compute the Laplace transform of the function in Problem 2.
12. Use formula (5.3.1a-b) to compute the Laplace transform of the function in Problem 3.
13. Use formula (5.3.1a-b) to compute the Laplace transform of the function in Problem 4.
14. Consider the function

$$F(t)=\begin{cases}e^{-t} & \text{for } 0<t<3\\ 0 & \text{for } t>3.\end{cases}$$

   a. Show that $F(t)$ can be written as $e^{-t}\,[1-U(t-3)]$.
   b. Use formula (5.3.1a) to find $\mathcal{L}[F(t)]$.

B

15. Solve the IVP

$$y''+4y=U(t-\pi)-U(t-3\pi);\;\; y(0)=0,\;\; y'(0)=0.$$

16. Solve the IVP

$$y^{(4)}+5y''+4y=1-U(t-\pi);\;\; y(0)=y'(0)=y''(0)=y'''(0)=0.$$

Solve each of the IVPs in Problems 17–24 by writing each discontinuous forcing function as a linear combination of unit step functions and then using the Laplace transform.

17. $4y' - 5y = \begin{cases} 0 & \text{for } t < 0 \\ -30t & \text{for } 0 \le t < 1 \\ 0 & \text{for } t \ge 1 \end{cases}$; $y(0) = 2$

18. $4y' + 5y = \begin{cases} 0 & \text{for } t < 0 \\ \sin 8t & \text{for } 0 \le t \le 2 \\ 0 & \text{for } t > 2 \end{cases}$; $y(0) = 1$

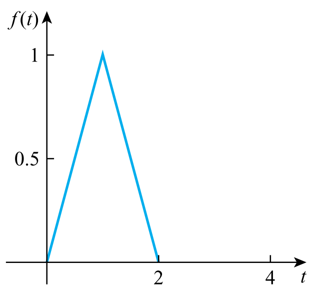
19. $y'' + y = \begin{cases} 1 & \text{for } 0 \le t < \pi/2 \\ 0 & \text{for } t \ge \pi/2 \end{cases}$; $y(0) = 0,\ y'(0) = 1$

20. $y'' + y = \begin{cases} t/2 & \text{for } 0 \le t < 6 \\ 3 & \text{for } t \ge 6 \end{cases}$; $y(0) = 0,\ y'(0) = 1$

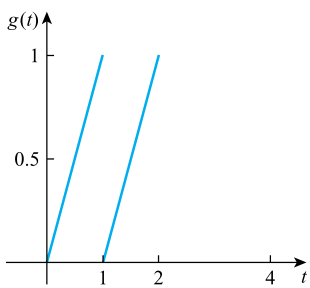
21. $y'' + 5y' + 2y = \begin{cases} 0 & \text{for } t < 0 \\ 8 & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t > 1 \end{cases}$; $y(0) = 0,\ y'(0) = 0$

22. $3y'' + 3y' + 2y = \begin{cases} 0 & \text{for } t < 0 \\ 5 & \text{for } 0 \le t \le 5 \\ 0 & \text{for } t > 5 \end{cases}$; $y(0) = 0,\ y'(0) = 0$

23. $y' - 3y = f(t)$; $y(0) = 0$, where the graph of $f(t)$ is



24. $y' + y = g(t)$; $y(0) = 0$, where the graph of $g(t)$ is



25. Suppose that the population of fish in a large lake is growing too rapidly and the local authorities decide to give out fishing licenses that allow a total of h fish to be caught per day over a 30-day period. A model for such a situation could be

$$P'(t) = k\,P(t) - \begin{Bmatrix} h & \text{for } 0 \le t \le 30 \\ 0 & \text{for } t > 30 \end{Bmatrix},$$

where $P(t)$ denotes the number of fish in the lake at time t (in days) and k is a positive constant describing the natural growth rate of the fish population.

a. Use technology and the Laplace transform to find an expression for $P(t)$ if $P(0) = A$.

b. Find a relation among A, h, and k that guarantees that exactly 330 days after the end of the 30-day fishing season, the fish population will once more be at the level A.

26. Problem 5 of Exercises 2.3 concerns the population of Botswana from 1975 to 1990 under certain basic assumptions. Now consider the situation that occurs if we start with a population of 0.755 million people in 1975 ($t = 0$) and assume that births and deaths, immigration and emigration balance each other until 1977 ($t = 2$). In 1977, an emigration pattern begins in such a way that the population $P(t)$ can be described by the equation

$$P' - k\,P = \begin{cases} 0 & \text{if } 0 < t < 2 \\ -a(t-2) & \text{if } t \ge 2 \end{cases}$$

with $P(0) = 0.755$, $k = 0.0355$, and $a = 1.60625 \times 10^{-3}$.

a. Express the discontinuous function on the right-hand side of the equation in terms of the unit step function.

b. Use technology and the Laplace transform to solve for $P(t)$, expressing the answer as a step function.

c. Graph the solution on the interval $0 \le t \le 35$ and explain what the graph means in terms of the population of Botswana.

C

27. The IVP $y'' + 3y' + 2y = W(t)$; $y(0) = 0$, $y'(0) = 0$ represents a damped spring-mass system subjected to a *square wave* forcing term given by

$$W(t) = U(t-1) - U(t-2).$$

a. Graph $W(t)$.

b. Without using technology, solve the IVP when $W(t) \equiv 0$.

c. Without using technology, solve the given IVP (that is, with $W(t) \not\equiv 0$ as the forcing term).

d. Use technology to graph the solutions to parts (b) and (c) on the same set of axes. What difference does the forcing term make?

28. Solve the following IVP, and find a formula for $y(t)$ that does not involve step functions and represents y on each interval of continuity of f:

$$y'' + 2y' + 2y = f(t); \quad y(0) = 0,\ y'(0) = 0;$$
$$f(t) = (m+1)(\sin t + 2\cos t),\ 2m\pi \le t < 2(m+1)\pi \text{ for } m = 0, 1, 2 \ldots.$$

(You may need the formula $1 + r + r^2 + \cdots + r^m = (1 - r^{m+1})/(1 - r), r \neq 1$.)

5.4 Transforms of impulse functions—the Dirac delta function

In the preceding section we dealt with situations in which some abrupt change occurred. To describe this scenario in general terms, we were dealing with systems acted on by some external force that was applied suddenly. Although the change was sudden, the force was assumed to have been applied for some measurable period of time. Now we want to examine problems in which there is an external force of large magnitude applied suddenly for a very short period of time. For example, think about a baseball being hit by a major-league player. The time of contact of ball with bat is very brief, but enough force can be applied to send the sphere soaring into the stands. More dramatic instances of this phenomenon include an electrical surge caused by a power line that is suddenly struck by lightning and a population that is growing at a certain rate until some sudden disaster strikes the community.

Mathematically, we can start to approach this idea by considering a piecewise continuous function that looks like

$$\delta_b(t) = \begin{cases} \frac{1}{b} & \text{for } 0 \le t \le b \\ 0 & \text{for } t > b. \end{cases}$$

Here, we must assume that $\delta_b(t)$—pronounced "delta sub b of t"—does not exist if $b = 0$. This function can represent a force of magnitude $1/b$ applied for a time period of length b (see Fig. 5.4).

First of all, note that $\int_0^\infty \delta_b(t)\,dt = \int_0^b \frac{1}{b}\,dt = 1$ for all values of $b > 0$. Now look at what happens as we allow the value of b to get smaller and smaller. This situation describes a force whose magnitude $1/b$ is getting larger and larger over a shorter and shorter interval of time $(0, b)$. (*Can you see what's going on*?) More precisely, the unusual nature of this discontinuous function led various physicists, mathematicians, and engineers to consider the limiting behavior of $\delta_b(t)$ as $b \to 0$. In particular, they defined $\delta(t)$ as

$$\delta(t) = \lim_{b \to 0} \delta_b(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0. \end{cases}$$

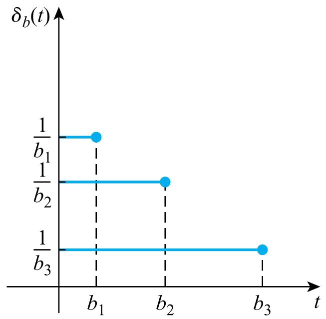


FIGURE 5.4

The graph of $\delta_b(t)$

This "function" δ is called the **unit impulse function** or the **Dirac delta function** (named for the English-Belgian theoretical physicist Paul A.M. Dirac (1902–84), who won the Nobel Prize in 1933 with E. Schrödinger for his work on quantum theory). More generally, we can define

$$\delta(t-a) = \lim_{b\to 0} \delta_b(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a. \end{cases}$$

In the mathematically precise sense of the word, this limit does not exist and so does not define a function. However, such *generalized functions*, or *distributions*, can be put on a firm mathematical foundation and are very useful in modern physics and engineering theory. Before we look at examples of the use of the delta function in solving differential equations, we should try to calculate its Laplace transform. The only reasonable way to do this is to make the formal assumption that

$$\mathcal{L}[\delta(t-a)] = \lim_{b\to 0} \mathcal{L}[\delta_b(t-a)].$$

(Mathematically, this raises an important theoretical question of whether

$$\lim_{b\to 0} \mathcal{L}[\delta_b(t-a)] = \mathcal{L}\left[\lim_{b\to 0} \delta_b(t-a)\right].$$

This question is beyond the scope of this text and will be ignored.)

Now let's write $\delta_b(t-a)$ in terms of the unit step function, as we did for functions in Section 5.3:

$$\delta_b(t-a) = \frac{1}{b}[U(t-a) - U(t-(a+b))].$$

If we use the linearity of the Laplace transform together with formula (5.3.1a)—taking $f(t-a) \equiv 1$—we get

$$\begin{aligned}\mathcal{L}[\delta(t-a)] &= \lim_{b\to 0} \mathcal{L}[\delta_b(t-a)] \\ &= \lim_{b\to 0} \frac{1}{b}\left\{\frac{e^{-sa}}{s} - \frac{e^{-s(a+b)}}{s}\right\} = \lim_{b\to 0} e^{-sa}\left\{\frac{1-e^{-sb}}{bs}\right\} \\ &= e^{-sa} \lim_{b\to 0}\left\{\frac{1-e^{-sb}}{bs}\right\} = e^{-sa},\end{aligned}$$

where we have used L'Hôpital's Rule to evaluate the indeterminate form in this last limit. (Alternatively, we could have used the series expansion of $\left(1-e^{-sb}\right)/bs$ about the point $b = 0$.) Because we have shown that

$$\mathcal{L}[\delta(t-a)] = e^{-sa}, \tag{5.4.1a}$$

it seems reasonable to take $a = 0$ and conclude that

$$\mathcal{L}[\delta(t)] = 1. \tag{5.4.1b}$$

Now let's see how to solve a differential equation involving an impulse function.

Example 5.4.1 Solving an ODE That Involves the Dirac Delta Function

A mass attached to a spring is released from rest one meter below the equilibrium position for the spring-mass system and begins to move up and down. After three seconds, the mass is struck by a hammer in a downward direction. We suppose the undamped system is governed by the IVP

$$\frac{d^2x}{dt^2} + 9x = 3\delta(t-3); \quad x(0) = 1, \; \frac{dx}{dt}(0) = 0,$$

where $x(t)$ denotes the displacement from equilibrium at time t, and we want to determine a formula for $x(t)$. (Note that the impulse force applied at $t = 3$ has magnitude 3.)

Let $X = X(s) = \mathcal{L}[x(t)]$. Then, taking the Laplace transform of both sides of our ordinary differential equation (ODE) and using (5.1.5) and (5.4.1a) with $a = 3$, we get

$$s^2X - s + 9X = 3e^{-3s},$$

so we can solve for X:

$$X = \frac{s}{s^2+9} + e^{-3s}\frac{3}{s^2+9}.$$

Applying the inverse transform yields

$$\begin{aligned}x(t) &= \cos 3t + \sin 3(t-3)\,U(t-3) \\ &= \begin{cases}\cos 3t & \text{for } t < 3 \\ \cos 3t + \sin 3(t-3) & \text{for } 3 \le t.\end{cases}\end{aligned}$$

Fig. 5.5 is the graph of $x(t)$, where the solid curve shows the displacement of the mass if the hammer had not hit it.

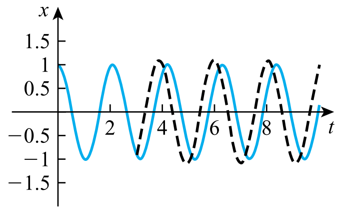


FIGURE 5.5

Graph of $x(t) = \begin{cases} \cos 3t & \text{for } t<3 \\ \cos 3t + \sin 3(t-3) & \text{for } 3 \le t \end{cases}$

Exercises 5.4

A

1. Evaluate the integral $\int_{-\infty}^{\infty} \delta(t - 3\pi/2) \cos 2t \, dt$.

2. Evaluate the integral $\int_0^1 t^3 \, \delta\left(t + \frac{1}{3}\right) dt$.

3. Evaluate $\mathcal{L}\left[\delta(t - \pi) \cos t^3\right]$.

Solve the IVPs in Problems 4–14:

4. $y'' = \delta(t - a)$; $y(0) = 0$, $y'(0) = 0$

5. $y' + 8y = \delta(t - 1) + \delta(t - 2)$; $y(0) = 0$

6. $y'' + y = \delta(t - 2)$; $y(0) = 0$, $y'(0) = 0$

7. $y'' + 2y' - 8y = \delta(t)$; $y(0) = 0$, $y'(0) = 0$

8. $2y'' + y' + 2y = \delta(t - 5)$; $y(0) = 0$, $y'(0) = 0$

9. $y'' + 2y' + y = 2\delta(t - 1)$; $y(0) = 1$, $y'(0) = 1$

10. $y'' + 6y' + 109y = \delta(t - 1) - \delta(t - 7)$; $y(0) = 0$, $y'(0) = 0$

11. $y'' + y = 1 + \delta(t - 2\pi)$; $y(0) = 1$, $y'(0) = 0$

12. $y'' + y = 4\delta\left(t - \frac{3}{2}\pi\right)$; $y(0) = 0$, $y'(0) = 1$

13. $y^{(iv)} - y = \delta(t - 1)$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

14. $y'' + y' = e^t + 3\delta(t - 6)$; $y(0) = -1$, $y'(0) = 4$

B

15. A uniform beam of length L carries a load W concentrated at $x = L/2$. The beam is embedded at its left end and is free at its right end. The deflection $y(x)$ is governed by the equation $EI \frac{d^4 y}{dx^4} = W\delta\left(x - \frac{L}{2}\right)$, where $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, and $y'''(L) = 0$. Use the Laplace transform to determine the deflection $y(x)$.

16. If a, b, and c are constants, show that the solution $x(t)$ of the linear IVP

$$x''(t) + ax'(t) + bx(t) = \delta(t-c);\; x(0) = 0,\; x'(0) = 0$$

is $x(t) = k(t-c)U(t-c)$, where $k(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+as+b}\right]$.

17. Suppose we have the equation $y'' + ay' + by = f(t)$, where a and b are constants and f is a piecewise continuous function whose Laplace transform exists. Show that the effect of replacing $f(t)$ by $f(t) + c\,\delta(t)$, where c is a constant, is the same as increasing the initial value of $y'(0)$ by the constant c.

18. **a.** Show that $L[\delta(t-a)f(t)] = e^{-as} f(a)$.

b. Use the result in part (a) and the result of Problem 20 to solve the IVP

$$y'' + 2y' + y = \delta(t-1)\,t;\; y(0) = 0,\; y'(0) = 0.$$

C

19. If, at time $t = a$, the upper end of an undamped spring-mass system is jerked upward suddenly and then returned to its original position, the equation modeling the situation is $mx'' + kx = kH\,\delta(t-a)$; $x(0) = x_0$, $x'(0) = x_1$, where m is the mass, k is the spring constant, and H is a constant.

a. Solve the IVP manually, with $x(0) = 0 = x'(0)$.

b. Use the solution found in part (a) to explain the significance of the constant H.

c. Choose a value for H so that the mass achieves a prescribed displacement from equilibrium A for $t \geq a$.

20. If the function $g(t)$ is continuous at a, show that $\int_0^\infty \delta(t-a)\,g(t)\,dt = g(a)$. [*Hint*: Use the Mean Value Theorem for integrals (Appendix A.4).]

21. Consider the IVP $y'' + 2y = \sum_{n=1}^{\infty} \delta(t-n)$; $y(0) = 0$, $y'(0) = 0$.

a. Find the Laplace transform of the solution of the IVP.

b. Solve the IVP.

c. What happens to the solution of the IVP as $t \to \infty$?

5.5 Transforms of systems of linear differential equations

We have seen what the Laplace transform does to a single linear equation with constant coefficients. It should be easy to see that when the initial conditions are given, the Laplace transform converts a system of linear differential equations with constant coefficients into a system of simultaneous algebraic equations. Then we can solve the algebraic equations for the *transformed* solution functions. Finally, applying the inverse transform to these functions gives us the solutions of the original system of linear ODEs.

Conceptually, this process is easy. The algebraic details, however, may not be quite so simple. Problems of this kind make us appreciate the availability of technology.

Example 5.5.1 Solving a Linear System via the Laplace Transform

Let's start with the system

$$\frac{dx}{dt} = -3x + y$$
$$\frac{dy}{dt} = x - 3y,$$

where we want the solutions $x(t)$ and $y(t)$ that satisfy $x(0) = 2$ and $y(0) = 3$. (This system was discussed briefly in Example 1.3.5.)

Applying the Laplace transform to each side of each equation gives us the system

$$s\mathcal{L}[x(t)] - x(0) = -3\mathcal{L}[x(t)] + \mathcal{L}[y(t)]$$
$$s\mathcal{L}[y(t)] - y(0) = \mathcal{L}[x(t)] - 3\mathcal{L}[y(t)].$$

Inserting the initial conditions and simplifying the resulting equations, we get the system

$$(s+3)\mathcal{L}[x(t)] - \mathcal{L}[y(t)] = 2$$
$$(s+3)\mathcal{L}[y(t)] - \mathcal{L}[x(t)] = 3.$$

Now we solve the preceding system for $\mathcal{L}[x(t)]$ and $\mathcal{L}[y(t)]$ just as we would solve any algebraic system of two equations in two unknowns. (To simplify things, you could let $\mathcal{L}[x(t)] = X$ and $\mathcal{L}[y(t)] = Y$.) For instance, we can eliminate the variable $\mathcal{L}[y(t)]$ by multiplying the first equation by $(s+3)$ and then adding the result to the second equation. When the dust settles, we get

$$\left\{(s+3)^2 - 1\right\}\mathcal{L}[x(t)] = 2(s+3) + 3,$$

and we find that

$$\mathcal{L}[x(t)] = \frac{2s+9}{(s+3)^2 - 1} = \frac{2s+9}{[(s+3)+1][(s+3)-1]}$$
$$= \frac{2s+9}{(s+4)(s+2)} = \frac{-\frac{1}{2}}{s+4} + \frac{\frac{5}{2}}{s+2} = \frac{-\frac{1}{2}}{s-(-4)} + \frac{\frac{5}{2}}{s-(-2)}$$

and

$$x(t) = -\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-(-4)}\right] + \frac{5}{2}\mathcal{L}^{-1}\left[\frac{1}{s-(-2)}\right]$$
$$= -\frac{1}{2}e^{-4t} + \frac{5}{2}e^{-2t}.$$

Now we could go through this process again to eliminate $\mathcal{L}[x(t)]$ and solve for $y(t)$ this time (see Problem 1 in Exercises 5.5), or we could just substitute our solution for $x(t)$ in the first equation of our original system and solve for y:

$$y(t) = \frac{dx}{dt} + 3x = \frac{d}{dx}\left(-\frac{1}{2}e^{-4t} + \frac{5}{2}e^{-2t}\right) + 3\left(-\frac{1}{2}e^{-4t} + \frac{5}{2}e^{-2t}\right)$$

$$= 2e^{-4t} - 5e^{-2t} - \frac{3}{2}e^{-4t} + \frac{15}{2}e^{-2t} = \frac{1}{2}e^{-4t} + \frac{5}{2}e^{-2t}.$$

The next example shows how we can handle a system of two second-order linear equations. In particular, note that we don't have to write this as a system of four first-order equations. The Laplace transform technique works directly on higher-order derivatives via formula (5.1.6) or, in this case, (5.1.5).

Example 5.5.2 A System of Second-Order Equations

The system IVP we want to solve is

$$\begin{aligned} \frac{d^2x}{dt^2} - 4x + \frac{dy}{dt} &= 0 \\ -4\frac{dx}{dt} + \frac{d^2y}{dt^2} + 2y &= 0, \end{aligned}$$

with $x(0) = 0$, $x'(0) = 1$, $y(0) = -1$, and $y'(0) = 2$.

Applying the Laplace transform to each side of each equation, we get

$$\begin{aligned} \mathcal{L}\left[x''(t)\right] - 4\mathcal{L}\left[x(t)\right] + \mathcal{L}\left[y'(t)\right] &= 0 \\ -4\mathcal{L}\left[x'(t)\right] + \mathcal{L}\left[y''(t)\right] + 2\mathcal{L}\left[y(t)\right] &= 0. \end{aligned}$$

Using (5.1.4) and (5.1.5), we can write the preceding system as

$$\begin{aligned} s^2\mathcal{L}[x(t)] - x'(0) - sx(0) - 4\mathcal{L}[x(t)] + s\mathcal{L}\left[y(t)\right] - y(0) &= 0 \\ -4s\mathcal{L}[x(t)] + 4x(0) + s^2\mathcal{L}[y(t)] - y'(0) - sy(0) + 2\mathcal{L}[y(t)] &= 0. \end{aligned}$$

Now we insert the initial conditions and simplify the resulting equations to get

$$\begin{aligned} \left(s^2 - 4\right)\mathcal{L}[x(t)] + s\mathcal{L}[y(t)] &= 0 \\ -4s\mathcal{L}[x(t)] + \left(s^2 + 2\right)\mathcal{L}[y(t)] &= 2 - s. \end{aligned} \qquad (*)$$

As in the previous example, we can solve these equations by realizing that they constitute a system of ordinary algebraic equations in the unknowns $\mathcal{L}[x(t)]$ and $\mathcal{L}[y(t)]$. If we multiply the first equation of (*) by $4s$, multiply the second by $s^2 - 4$, and then add the resulting equations, we obtain

$$\left(s^4 + 2s^2 - 8\right)\mathcal{L}[y(t)] = -s^3 + 2s^2 + 4s - 8,$$

so

$$\begin{aligned} \mathcal{L}[y(t)] &= \frac{-s^3 + 2s^2 + 4s - 8}{s^4 + 2s^2 - 8} = \frac{-s^3 + 2s^2 + 4s - 8}{\left(s^2 + 4\right)\left(s^2 - 2\right)} \\ &= \frac{-s^3 + 2s^2 + 4s - 8}{\left(s^2 + 4\right)\left(s + \sqrt{2}\right)\left(s - \sqrt{2}\right)} \\ &= \frac{1}{6}\left[\frac{1 + \sqrt{2}}{s + \sqrt{2}} + \frac{1 - \sqrt{2}}{s - \sqrt{2}} - \frac{8(s - 2)}{s^2 + 4}\right] \end{aligned}$$

$$= \frac{1}{6}\left[\frac{1+\sqrt{2}}{s+\sqrt{2}} + \frac{1-\sqrt{2}}{s-\sqrt{2}} - 8\frac{s}{s^2+2^2} + 8\frac{2}{s^2+2^2}\right]. \qquad (**)$$

Using entries 2, 3, and 4 of the table of transforms (Table 5.1), we see that

$$y(t) = \frac{1}{6}\left[\left(1+\sqrt{2}\right)e^{-\sqrt{2}t} + \left(1-\sqrt{2}\right)e^{\sqrt{2}t} - 8\cos 2t + 8\sin 2t\right].$$

To find $\mathcal{L}[x(t)]$, we can go back to system (*) and eliminate $\mathcal{L}[y(t)]$ or we can substitute expression (**) for $\mathcal{L}[y(t)]$ in either equation of (*) and solve for $\mathcal{L}[x(t)]$. Let's try the second method.

Using (**) and the first equation in (*), we find that

$$\left(s^2-4\right)\mathcal{L}[x(t)] + s\left(\frac{-s^3+2s^2+4s-8}{\left(s^2+4\right)\left(s^2-2\right)}\right) = 0.$$

Solving for $\mathcal{L}[x(t)]$, we get

$$\mathcal{L}[x(t)] = -s\left(\frac{-s^3+2s^2+4s-8}{\left(s^2-4\right)\left(s^2+4\right)\left(s^2-2\right)}\right) = \frac{s(s-2)^2(s+2)}{(s-2)(s+2)\left(s^2+4\right)\left(s^2-2\right)}$$

$$= \frac{s(s-2)}{\left(s^2+4\right)\left(s+\sqrt{2}\right)\left(s-\sqrt{2}\right)}$$

$$= -\frac{1}{12}\left[\frac{2+\sqrt{2}}{s+\sqrt{2}} + \frac{2-\sqrt{2}}{s-\sqrt{2}} - 4\left(\frac{s+2}{s^2+4}\right)\right]$$

$$= -\frac{1}{12}\left[\frac{2+\sqrt{2}}{s+\sqrt{2}} + \frac{2-\sqrt{2}}{s-\sqrt{2}} - 4\left(\frac{s}{s^2+2^2} + \frac{2}{s^2+2^2}\right)\right].$$

Entries 2, 3, and 4 from Table 5.1 tell us that

$$x(t) = -\frac{1}{12}\left[\left(2+\sqrt{2}\right)e^{-\sqrt{2}t} + \left(2-\sqrt{2}\right)e^{\sqrt{2}t} - 4\cos 2t - 4\sin 2t\right].$$

You should confirm that these are the solutions to the original IVP.

Exercises 5.5

A

1. Eliminate $\mathcal{L}[x(t)]$ from the algebraic system

$$(s+3)\mathcal{L}[x(t)] - \mathcal{L}[y(t)] = 2$$
$$(s+3)\mathcal{L}[y(t)] - \mathcal{L}[x(t)] = 3$$

and then solve for $y(t)$. (See Example 5.5.1.)

Solve the IVPs in Problems 2–14 by using the Laplace transform.

2. $\{x' = y,\ y' = -x\};\ x(0) = 2,\ y(0) = -1$

3. $\{x' = 2x - 3y,\ y' = y - 2x\};\ x(0) = 8,\ y(0) = 3$

4. $\{x' = 12x + 5y,\ y' = -6x + y\};\ x(0) = 0,\ y(0) = 1$

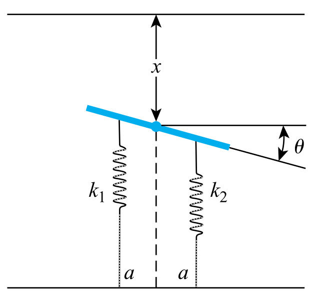
5. $\{x' = -2x + y,\ y' = -9x + 4y\}$; $x(0) = 5$, $y(0) = -3$
6. $\{x' = -6x + 2y,\ y' = -7x + 3y\}$; $x(0) = 1$, $y(0) = 0$
7. $\{x' = x + y,\ y' = -4x + y\}$; $x(0) = 1$, $y(0) = 3$
8. $\left\{x' = y' + 6y,\ y' = \frac{3}{2}x - \frac{1}{2}x'\right\}$; $x(0) = 2$, $y(0) = 3$
9. $\{x' + x - 5y = 0,\ y' + 4x + 5y = 0\}$; $x(0) = -1$, $y(0) = 2$
10. $\{x' + y' = -3x - 2y + e^{-2t},\ 2x' + y' = -2x - y + 1\}$; $x(0) = 0$, $y(0) = 0$
11. $\{x' = x - y - e^{-t},\ y' = 2x + 3y + e^{-t}\}$; $x(0) = 1$, $y(0) = 0$
12. $\{x' + y' = x,\ y' + z' = x, z' + x' = x\}$; $x(0) = 1$, $y(0) = 1$, $z(0) = 1$
13. $\{x' - 3x - 6y = 27t^2,\ x' + y' - 3y = 5e^t\}$; $x(0) = 5$, $y(0) = -1$
14. $\{x' = 2y + e^t,\ y' = 8x - t\}$; $x(0) = 1$, $y(0) = 1$

B

15. Solve the system IVP $\{x'' + y' = 4x,\ 4x' - y'' = 9y\}$; $x(0) = 0$, $x'(0) = 1$, $y(0) = -1$, $y'(0) = 2$ by using the Laplace transform.
16. Solve the system IVP $\{x'' - y' = -t + 1,\ x' - x + 2y' = 4e^t\}$; $x(0) = 0$, $x'(0) = 1$, $y(0) = 0$ by using the Laplace transform.
17. The system

$$mx'' = -k_1(x - a\theta) - k_2(x + a\theta)$$
$$mr^2\theta'' = k_1a(x - a\theta) - k_2\,a\,(x + a\theta)$$

models the motion of a slab of mass m mounted on two springs, as shown in the following figure. Here, x is the vertical displacement of the center of mass and θ is the angle shown. The constant r represents the radius of gyration of the slab about the appropriate axis through the center of mass. Use the Laplace transform and technology to solve the system for x and θ if $m = 1$, $k_1 = 1$, $k_2 = 2$, $a = 1$, $r = 1$, $x(0) = 1$, $x'(0) = 0$, $\theta(0) = 0.1$, and $\theta'(0) = 0$.

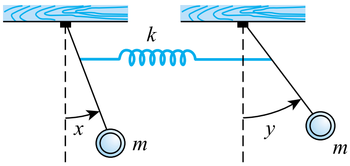


18. The system consisting of two pendulums connected by a spring (see the following figure) has its motion approximated by the system of equations

$$mx'' + m\omega_0^2 x = -k(x - y)$$

$$my'' + m\omega_0^2 y = -k(y - x),$$

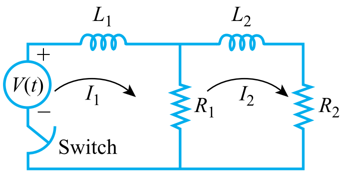
where L is the length of each pendulum, g is the gravitational constant, and $\omega_0^2 = g/L$. Use the Laplace transform and technology to solve this system with $m = 1$, $L = 5$, $g = 32$, $k = 2$, and the initial conditions $x(0) = 0$, $x'(0) = 2$, $y(0) = 0$, and $y'(0) = 2$.



19. The following circuit is described by the system

$$L_1 \dot{I}_1 + R_1 (I_1 - I_2) = v(t)$$
$$L_2 \dot{I}_2 + R_2 I_2 + R_1 (I_2 - I_1) = 0.$$

Determine I_1 and I_2 when the switch is closed if $L_1 = L_2 = 2$ henry, $R_1 = 3$ ohm, $R_2 = 8$ ohm, and $v(t) = 6$ volt. Assume that $I_1(0) = I_2(0) = 0$.



20. In determining the concentration of a chemical in a system consisting of two compartments separated by a membrane, we get the system of equations

$$\dot{x} = ay - bx$$
$$\dot{y} = bx - ay - \beta y,$$

subject to the conditions $x(0) = x^*$ and $y(0) = y^*$, where x^* and y^* are constants. (Here, x and y represent the masses of the chemical in compartments 1 and 2, respectively, at any time t, and the constants a, b, and β are positive constants of proportionality related to the rate of flow of the chemical from one compartment to another.)

a. Solve this system of equations using Laplace transforms.

b. Letting $p = \frac{1}{2}(b + a + \beta)$ and $q = \frac{1}{2}\sqrt{(b + a + \beta)^2 - 4\beta b}$, show that q is a (positive) *real* number and that $p > q$.

c. Using the solution found in part (a) and the results of part (b), show that the chemical masses x and y approach zero steadily.

C

21. Solve the system IVP $\left\{\frac{d^2x}{dt^2} = y + 4e^{-2t}, \frac{d^2y}{dt^2} = x - e^{-2t}\right\}$; $x(0) = y(0) = x'(0) = y'(0) = 0$.

22. Use the Laplace transform to solve the system

$$\frac{d^2x}{dt^2} - 4x + \frac{dy}{dt} = 0$$
$$-4\frac{dx}{dt} + \frac{d^2y}{dt^2} + 2y = 0,$$

for which $x(0) = 0$, $x'(0) = 1$, $y(0) = -1$, and $y'(0) = 2$.

*5.6 Laplace transforms of linear differential equations with variable coefficients[2]

As we have seen, the Laplace transformation is a powerful tool for solving linear differential equations of any order with *constant* coefficients. Although the Laplace transform can be used to solve certain linear differential equations with *variable* coefficients, the calculations may be very complicated and ultimately frustrating. As a general rule, Laplace transform techniques are not useful for most problems of this type.

Example 5.6.1 A Linear Equation with Variable Coefficients

Suppose we're given the IVP $y'' + t^2y = 0$; $y(0) = A$, $y'(0) = B$. First of all, note that this seems to be a fairly simple linear equation with variable coefficients. There is only one variable coefficient, a simple polynomial, and the forcing function is zero.

If we apply the Laplace transform to each side of the equation, we see that

$$\mathcal{L}[y''] + \mathcal{L}[t^2y] = 0. \tag{5.6.1}$$

The first difficulty we encounter is the calculation of $\mathcal{L}[t^2y]$. We have developed no formula for such a product of functions. However, there *is* a formula,

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)], \tag{5.6.2}$$

whose proof (for $n = 1, 2$) is left as an exercise. (Generalize the method indicated for Exercise 2.)

Letting $Y = \mathcal{L}[y]$ and using (5.6.2) with $n = 2$, formula (5.1.5), and the initial conditions, we see that (5.6.1) becomes

$$s^2Y - sA - B + \frac{d^2}{ds^2}Y = 0,$$

[2] * Denotes an optional section.

or

$$Y'' + s^2 Y = As + B. \tag{5.6.3}$$

In this case, the Laplace transform has not simplified our problem to one involving algebraic equations. We see that (5.6.3), the equation in the transform variable Y, has the same level of difficulty as the original.

Example 5.6.2 Another Linear Equation with Variable Coefficients

Consider the nonhomogeneous IVP $x'' + 3tx' - 6x = 2$; $x(0) = 0$, $x'(0) = 0$.

As usual, we apply the Laplace transform to each side of the equation and use linearity to get

$$\mathcal{L}[x''] + 3\mathcal{L}[tx'] - 6\mathcal{L}[x] = \frac{2}{s}. \tag{5.6.4}$$

Using (5.6.2), formulas (5.1.4) and (5.1.5), and the initial condition, we calculate

$$\mathcal{L}[tx'] = -\frac{d}{ds}\mathcal{L}[x'] = -\frac{d}{ds}\left(sX(s) - x(0)\right) = -sX'(s) - X(s),$$

where $X(s) = \mathcal{L}[x(t)]$. Then Eq. (5.6.4) becomes

$$s^2 X(s) - sx(0) - x'(0) + 3\left(-sX'(s) - X(s)\right) - 6X(s) = \frac{2}{s},$$

or

$$X'(s) + \left(\frac{3}{s} - \frac{s}{3}\right) X(s) = -\frac{2}{3s^2}. \tag{5.6.5}$$

(*Whoa!*) Instead of ending up with a Laplace transform to which we could then apply the inverse transform, we get a linear first-order differential equation that we must solve for the function $X(s) = \mathcal{L}[x(t)]$. Fortunately, we know how to solve such an equation. An integrating factor for Eq. (5.6.5) is $\mu(s) = e^{\int\left(\frac{3}{s} - \frac{s}{3}\right) ds} = s^3 e^{-s^2/6}$. Multiplying both sides of (5.6.5) by μ, integrating, and solving for $X(s)$ yields

$$X(s) = \frac{2}{s^3} + C \cdot \frac{e^{s^2/6}}{s^3}. \tag{5.6.6}$$

(*This is getting better and better.*) If we look at Table 5.1, we don't see anything like the right-hand side of (5.6.6). However, *if we assume that the function $x(t)$ is of exponential order*, we can finish the problem. In Section 5.1 we noted that $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \mathcal{L}[f(t)] = 0$. So we know that

$$\lim_{s\to\infty} X(s) = \lim_{s\to\infty}\left(\frac{2}{s^3} + C \cdot \frac{e^{s^2/6}}{s^3}\right) = 0.$$

Since $\lim_{s\to\infty}(2/s^3) = 0$, the only way that the second term in parentheses will vanish as $s \to \infty$ is if $C = 0$. (Use L'Hôpital's Rule or the series expansion of e^x to see that this term is unbounded as s tends to infinity.) Thus Eq. (5.6.6) becomes $X(s) = \frac{2}{s^3}$. Applying the inverse transform gives us $x(t) = t^2$.

In the last example, the initial conditions didn't allow us to determine the arbitrary constant we were left with. We had to resort to a limit argument. Now we'll look at an

IVP where we'll have to work a little harder to determine the values of any constants resulting from applying the Laplace transform.

Example 5.6.3 One More Linear Equation with Variable Coefficients

Let's try to solve $ty'' + y' = t$; $y(0) = 0$, $y'(0) = 0$.

Let $Y = \mathcal{L}[y]$. From formula (5.6.2), we have $\mathcal{L}[ty''] = -\frac{d}{ds}\left(s^2Y - sy(0) - y'(0)\right) = -2sY - s^2Y'$. Therefore, after some simplification, the transformed differential equation is

$$s^2Y' + sY = -\frac{1}{s^2}.$$

In standard form we can write this first-order linear equation as

$$Y' + \left(\frac{1}{s}\right)Y = -\frac{1}{s^4}. \tag{5.6.7}$$

Using the integrating factor $\mu(s) = e^{\int 1/s\, ds} = s$, we solve (5.6.7) to get

$$Y = \frac{1}{2s^3} + \frac{C}{s}. \tag{5.6.8}$$

Note that $\lim_{s\to\infty} Y = 0$, so we can't determine the value of C as we did in Example 5.6.2. However, we can find C by first determining y: $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[1/(2s^3)] + C \cdot \mathcal{L}^{-1}[1/s] = t^2/4 + C$.

Now we use the initial condition $y(0) = 0$ to see that $C = 0$. It is easy to check that $y = t^2/4$ is the solution of the IVP.

Note that in Example 5.6.3, if we replace the initial condition $y'(0) = 0$ by $y'(0) = c \neq 0$, the equation has no solution—the function $y = t^2/4$ wouldn't satisfy both initial conditions. If we write the equation in the standard form $y'' + (1/t)y' = 1$, we see that the function $1/t$ is discontinuous at $t = 0$. However, we *can* find the general solution of the equation $ty'' + y' = t$ by using the familiar formula $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$ and a technique from a previous exercise.

Multiplying the homogeneous equation by t, we have the equation $t^2y'' + ty' = 0$, which is a *Cauchy–Euler equation* (see the explanation after Problem 33 in Exercises 4.3). Using the substitution mentioned there, we find that $y_{\text{GH}} = C_1 + C_2 \ln t$. Since $y = (1/4)t^2$ is a solution of the nonhomogeneous equation, the general solution of the differential equation is $y = C_1 + C_2 \ln t + (1/4)t^2$. Note that if we require $y(0)$ to be finite, we are forced to conclude that $C_2 = 0$.

Formula (5.6.2) suggests that if a second-order equation $a(t)y'' + b(t)y' + c(t)y = f(t)$ has linear coefficients of the form $at + b$, where a and b are constants, then we get a first-order equation for $Y = \mathcal{L}[y]$, which may be simpler than the original equation (sometimes). However, if the differential equation has *quadratic* coefficients $at^2 + bt + c$, we get (letting $n = 2$ in (5.6.2)) a second-order differential equation for $\mathcal{L}[y]$, which hints that the Laplace transform method works well only for special equations with variable coefficients.

As we have tried to imply, the reality is that linear differential equations with variable coefficients are best handled by using alternative methods. The technique of variation of parameters works on equations whose coefficients are continuous functions (see Example 4.4.5). There are various "tricks"—for example, substitutions that reduce some second-order equations to first-order equations. **Power series** methods are effective, and many famous functions in applied mathematics arise as power series solutions of differential equations. Although we will not discuss these series methods further within this textbook, there is a good treatment in Appendix D.

Exercises 5.6

A

1. Consider the first-order linear equation $y' = 1 - xy$.
   a. Try to find the general solution of this equation by using the Laplace transform. Describe the difficulties you encounter.
   b. Solve the equation by using an integrating factor.
2. Let $F(s) = \mathcal{L}[f(t)]$. Prove the formulas $\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$ and $\mathcal{L}[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$ as follows.
   a. Show that $\frac{d}{ds}e^{-st} = -te^{-st}$ and $\frac{d^2}{ds^2}e^{-st} = t^2e^{-st}$.
   b. Using Leibniz's rule for differentiating under an integral sign (Appendix A.7), show that $\frac{dF}{ds} = -\int_0^\infty tf(t)e^{-st}\,dt = -\mathcal{L}[tf(t)]$ and $\frac{d^2F}{ds^2} = \int_0^\infty t^2 f(t)e^{-st}\,dt = \mathcal{L}[t^2 f(t)]$.
3. Use the Laplace transform to solve the IVP $y'' + ty' - y = 0$; $y(0) = 0$, $y'(0) = 1$.

B

4. Consider the equation $ty'' + (1+t)y' + y = t^2$, $t \geq 0$.
   a. Use the Laplace transform to solve the equation.
   b. Explain why your solution contains only one arbitrary constant.
5. Use the Laplace transform to solve the IVP $t^2y'' + 4ty' + 2y = 12t^2$; $y(0) = 0$, $y'(0) = 0$.
6. Use the Laplace transform to solve the IVP $ty'' - ty' + y = 5$; $y(0) = 5$, $y'(0) = 3$.
7. Use the Laplace transform to solve the equation $ty'' - 2(t+1)y' + (t+2)y = 0$.
8. Use the Laplace transform to solve the BVP $ty'' - (t+3)y' + 4y = t - 1$; $y(0) = y(1) = 0$.

C

9. The ***n*th Laguerre polynomial** $l_n(t)$ is defined by the equation

$$l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})$$

for $t > 0$ and $n = 0, 1, 2, 3, \ldots$.

a. Calculate $l_1(t), l_2(t), l_3(t), l_4(t)$.

b. Using entry 2 of Table 5.1 and formula (5.6.2), determine $L_n(s) = \mathcal{L}[l_n(t)]$.

c. Use the Laplace transform to find the general solution of $xy'' + (1-x)y' + ny = 0$, where n is a nonnegative integer.

10. Let $l_n(t)$ denote the nth Laguerre polynomial (see Problem 9).

a. Use the answer to part (b) of the previous exercise (looking at the back of the book if necessary) and the Convolution Theorem to prove that $\int_0^x l_n(t)\,dt = l_n(x) - l_{n+1}(x)$.

b. Prove that $\int_0^x l_n(t)l_m(x-t)\,dt = l_{m+n}(x) - l_{m+n+1}(x)$.

Summary

Transformation methods are important examples of how we can change difficult problems into problems that can be handled more easily. If $f(t)$ is a function that is integrable for $t \geq 0$, then the **Laplace transform** of f is defined by $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$, when this improper integral exists. The integral will exist if we stick to *continuous* or *piecewise continuous* functions $f(t)$ for which there are positive constants M and K such that $|f(t)| < e^{Mt}$ for all $t \geq K$. Note that this integral is a function of the parameter s, so we can write $\mathcal{L}[f(t)] = F(s)$.

Using basic properties of integrals, we can see that $\mathcal{L}[c \cdot f(t)] = c \cdot \mathcal{L}[f(t)]$, where c is any real constant, and that $\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$, whenever the Laplace transforms of both f and g exist. Any transformation that satisfies the last two properties is called a *linear transformation*. If c_1 and c_2 are constants, we can combine the two properties to write $\mathcal{L}[c_1 f(t) + c_2 g(t)] = c_1\mathcal{L}[f(t)] + c_2\mathcal{L}[g(t)]$.

Table 5.1 in Section 5.2 gives the Laplace transform of some important classes of functions, including power functions, exponentials, trigonometric functions, and multiples of these functions. There are also important formulas for the Laplace transforms of f', f'', and higher derivatives. The Laplace transform method lets us handle a linear nonhomogeneous equation with initial conditions all at once.

Once we have calculated the Laplace transform of a function—in particular, once we have transformed a differential equation into an algebraic equation—we have to be able to reverse the process to gain information about the original problem. It is important to remember that *if the Laplace transforms of the continuous functions f and g exist and are equal for $s \geq c$ (c a constant), then $f(t) = g(t)$ for all $t \geq 0$*. This says that a continuous function can be recovered uniquely from its Laplace transform. Letting $\mathcal{L}[f(t)] = F(s)$, we can express the definition of the **inverse Laplace transform** as

$$\mathcal{L}^{-1}[F] = f \quad \text{if and only if} \quad \mathcal{L}[f] = F.$$

It can be shown that the inverse Laplace transform is a linear transformation:

$$\mathcal{L}^{-1}\left[c_1 F(t) + c_2 G(t)\right] = c_1 \mathcal{L}^{-1}[F(t)] + c_2 \mathcal{L}^{-1}[G(t)].$$

In trying to find the inverse transform of an expression that is the product of two or more transforms, we encounter the idea of the convolution of two functions. The **convolution** of two functions f and g is the integral $(f * g)(t) = \int_0^t f(r)g(t-r)\,dr$, provided that the integral exists for $t > 0$. This product has important algebraic properties, and one of the most useful is that the Laplace transform of a convolution of two functions is equal to the product of the Laplace transforms of these two functions. More precisely, suppose that f and g are two functions whose Laplace transforms exist. Let $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$. Then the **Convolution Theorem** says that

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}\left[\int_0^t f(r)g(t-r)\,dr\right] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)] = F(s) \cdot G(s).$$

By using the unit step function (or Heaviside function) U, defined by

$$U(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0, \end{cases}$$

we can model systems in which there are abrupt changes. Mathematically, this means that we can express *piecewise continuous functions* in a simple way, using $U(t)$ as a basic building block.

When we are solving differential equations that model abrupt changes the following result comes in handy. If $\mathcal{L}[f(t)]$ exists for $s > c$ and if $a > 0$, then

$$\mathcal{L}[f(t-a)U(t-a)] = e^{-as}\mathcal{L}[f(t)] \quad \text{for } s > c.$$

Alternatively, we can write the preceding formula as

$$f(t-a)U(t-a) = \mathcal{L}^{-1}\left[e^{-as}\mathcal{L}[f(t)]\right].$$

If we want to consider problems where there is an external force of large magnitude applied suddenly for a very short period of time, we need the idea of the **unit impulse function**, or **Dirac delta function**, defined as

$$\delta(t) = \lim_{b \to 0} \delta_b(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0, \end{cases}$$

where

$$\delta_b(t) = \begin{cases} \frac{1}{b} & \text{for } 0 \leq t \leq b \\ 0 & \text{for } t > b. \end{cases}$$

We can show that $\mathcal{L}[\delta(t-a)] = e^{-sa}$. In particular, $\mathcal{L}[\delta(t)] = 1$.

When initial conditions are given, the Laplace transform converts a system of linear differential equations with constant coefficients to a system of simultaneous algebraic equations. Then we can solve the algebraic equations for the transformed solution functions. Finally, applying the inverse transform to these functions gives us the solutions of the original system of linear ODEs. However neat this sounds conceptually, the algebraic details are often quite messy, and technology comes in handy.

Although the Laplace transform can be used in solving linear equations with *variable* coefficients, the usefulness of this technique is limited, and solving such equations by using *power series* (see Appendix D) is the usual method.

CHAPTER

6 Systems of linear differential equations

Introduction

In Chapters 1–5, we treated single differential equations for the most part, focusing on linear equations of first and higher order. In this chapter we start with the important fact that *any differential equation (linear or nonlinear) of order two or higher can be written as an equivalent system of first-order equations*.

In this chapter, we will explore (mainly) autonomous systems of first-order *linear* differential equations. The theory of these equations is neat and complete, and an important component is the *Superposition Principle,* which we discussed in Chapters 2 and 4 and which we will come to recognize as the distinguishing characteristic of linear systems. This fundamental principle will help us to determine the general solution of linear systems in essentially the same way as we solved single second-order linear equations in Sections 4.1 and 4.2. As we saw in Chapter 5, the Laplace Transform is a useful tool for solving systems of linear differential equations.

To understand the important ideas underlying the theory and application of linear systems, we'll introduce some of the language and concepts from the area of mathematics called *linear algebra* without probing too deeply into the intricacies of this valuable and useful subject. Also, we will develop qualitative (geometric) aspects of second- and third-order systems such as the *phase plane*.

In Chapter 7, we will see how *nonlinear* systems can be analyzed qualitatively in terms of certain related linear systems.

6.1 Higher-order equations and their equivalent systems

To see where we're headed, think back to the first time you had to solve the following kind of word problem:

> Christopher has 21 coins in his pockets, all nickels and dimes. They amount to \$1.75. How many dimes does he have?

The first time you saw this problem, you were probably shown a solution like this one:

A Modern Introduction to Differential Equations. https://doi.org/10.1016/B978-0-12-818217-8.00013-0

> Let x be the number of dimes. Then the total *amount* corresponding to dimes is $10x$ cents. The *number* of nickels must be $21 - x$, so the *amount* corresponding to nickels is $5(21 - x)$ cents. Because the total amount of money in Christopher's pockets is \$1.75, or 175 cents, we have the equation $10x + 5(21 - x) = 175$, equivalent to $5x + 105 = 175$, which has the solution $x = 14$. Thus, Christopher has 14 dimes (and $21 - 14 = 7$ nickels).

A bit later in your algebra course, you might have seen the same problem again, but this time you were probably shown how to turn this problem into a *system* problem:

> Let x be the number of dimes and let y be the number of nickels. Then the words of the problem tell us two things, one fact about the *number* of coins and one fact about the *amount* of money: (1) $x + y = 21$ and (2) $10x + 5y = 175$. In other words, viewed this way, the problem gives us the *system* of equations
>
> $$\begin{aligned} x + y &= 21 \\ 10x + 5y &= 175. \end{aligned}$$
>
> This system can be solved by elimination (multiply the first equation by -5 and then add the result to the second equation) or by substitution (solve the first equation for x, for example, and then substitute for x in the second equation).

The most important consequence of looking at our problem as a system problem is that the system has a very nice geometrical interpretation as a set of two straight lines (Fig. 6.1). The solution of the system (and of our original problem) is given by the coordinates of the point where the lines intersect: $x = 14$, $y = 7$.

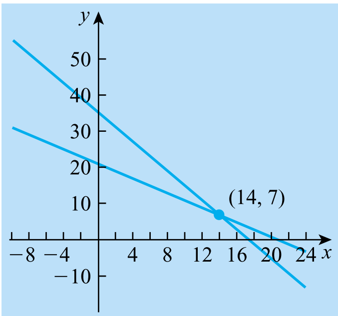


FIGURE 6.1

Graphs of $x + y = 21$ and $10x + 5y = 175$

Similarly, for differential equations, a systems approach has certain advantages, especially the graphical interpretation of a problem and its solution. Also, certain problems may naturally occur in system form. For example, we may want to compute the trajectory of a baseball. In this case, it is natural to consider the components

u and v of the ball's velocity in both its horizontal (x) and vertical (y) directions, respectively. A possible system[1] arising from this problem is

$$mu\,\frac{du}{dx} = -F_L \sin\theta - F_D \cos\theta$$
$$mv\,\frac{dv}{dy} = F_L \cos\theta - F_D \sin\theta - mg.$$

Similarly, in an ecological study, we may want to analyze the interaction of two or more biological species, each of which needs its own equation to represent its growth rate and its relationship to the other species.

6.1.1 Conversion technique I: converting a higher-order equation into a system

Now that our previous discussion has prepared us to see even simple problems as systems, we can tackle some higher-order differential equations. The key here is the following important result:

> Any single nth-order differential equation can be converted into an equivalent system of first-order equations. More precisely, any nth-order differential equation of the form
>
> $$x^{(n)} = F\left(t, x, x', x'', \ldots, x^{(n-1)}\right)$$
>
> can be converted into an equivalent system of n first-order equations by letting
>
> $$x_1 = x, x_2 = x', x_3 = x'', \ldots, x_n = x^{(n-1)}.$$

Here, *equivalent* means that a function $x = u(t)$ is a solution of $x^{(n)} = F(t, x, x', \ldots, x^{(n-1)})$ if and only if the ordered n-tuple of functions $\left(u(t), u'(t), \ldots, u^{(n-1)}(t)\right)$ is a solution of the system $x_1' = x_2$, $x_2' = x_3$, $\ldots$, $x_n' = F\,(t, x_1, x_2, \ldots, x_n)$. In particular, our substitution scheme indicates that *any solution of the single nth-order equation is the first component of the n-tuple that is the solution of the system and vice versa.*

After looking at some examples of how this conversion technique works, we'll introduce the geometric/graphical significance of this method.

Example 6.1.1 Converting a Second-Order Linear Equation

As we will see later in this chapter, a second-order linear equation such as $2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0$ could represent the motion of a weight attached to a spring, the flow of electricity through a circuit, or other important phenomena.

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[1] Robert B. Banks, *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics* (Princeton, NJ: Princeton University Press, 1998).

Using the substitutions described previously, we introduce new variables x_1 and x_2: Let $x_1 = x$ and $x_2 = \frac{dx}{dt}$. Now isolate the highest derivative (the second) in the original equation, and then substitute the new variables in the right-hand side:

$$\text{(1)} \quad \frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + \frac{1}{2}x = 0$$

$$\text{(2)} \quad \frac{d^2x}{dt^2} = -\frac{3}{2}\frac{dx}{dt} - \frac{1}{2}x$$

$$\text{(3)} \quad \frac{d^2x}{dt^2} = -\frac{3}{2}x_2 - \frac{1}{2}x_1$$

In terms of the new variables, we see that $\frac{dx_1}{dt} = \frac{dx}{dt} = x_2$ and $\frac{dx_2}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} =$ [from step (3) above] $-\frac{3}{2}x_2 - \frac{1}{2}x_1$. From this, we see that our original second-order equation leads to the following system of linear first-order equations in two unknown functions x_1 and x_2:

$$\text{(A)} \quad \frac{dx_1}{dt} = x_2$$

$$\text{(B)} \quad \frac{dx_2}{dt} = -\frac{3}{2}x_2 - \frac{1}{2}x_1$$

This system is *equivalent* to the original single differential equation in the sense that any solution $x(t)$ of the original equation yields solutions $x_1(t) = x(t)$ and $x_2(t) = \frac{d}{dt}x(t)$ of the system, and any solution $(x_1(t), x_2(t))$ of the system gives us a solution $x(t) = x_1(t)$ of the original equation.

Let's follow up on the first part of that statement. From our work in Chapter 4, we know that $x(t) = e^{-t/2} + 2e^{-t}$ is a solution of the original second-order equation. Then the pair $x_1(t) = x(t) = e^{-t/2} + 2e^{-t}$ and $x_2(t) = \frac{d}{dt}\,x(t) = -\frac{1}{2}e^{-t/2} - 2e^{-t}$ constitutes a solution of the system. (*Verify this.*)

Let's look at a few more examples of this technique of converting a higher-order equation into a system of first-order equations.

Example 6.1.2 Converting a Second-Order Nonlinear Equation

Suppose we have the second-order nonlinear equation $y'' = y^3 + (y')^3$. Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$, $y'' = x_2'$, $y^3 = x_1^3$, and $(y')^3 = x_2^3$, so we can rewrite $y'' = y^3 + (y')^3$ as $x_2' = x_1^3 + x_2^3$.

Finally, putting these pieces together, we can write the original equation as the following equivalent nonlinear system in x_1 and x_2:

$$x_1' = x_2$$
$$x_2' = x_1^3 + x_2^3.$$

Example 6.1.3 Converting a Third-Order Equation

The nonautonomous third-order linear equation

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - 3x + 6 = 0$$

can be changed into a system of first-order equations as follows: Let $x_1 = x$, $x_2 = \frac{dx}{dt}$, and $x_3 = \frac{d^2x}{dt^2}$. Then $\frac{dx_1}{dt} = \frac{dx}{dt} = x_2$, $\frac{dx_2}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = x_3$, and $\frac{dx_3}{dt} = \frac{d}{dt}\left(\frac{d^2x}{dt^2}\right) = \frac{d^3x}{dt^3}$.

Solving the original equation for $\frac{d^3x}{dt^3}$ and then substituting the new variables x_1, x_2, and x_3, we have

$$\frac{d^3x}{dt^3} = \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 3x - 6 = x_3 - 2tx_2 + 3x_1 - 6.$$

Putting all the information together, we see that the original third-order equation is equivalent to the system of three first-order equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= x_3\\ \frac{dx_3}{dt} &= x_3 - 2tx_2 + 3x_1 - 6.\end{aligned}$$

To be mathematically precise, we can describe this system as a *three-dimensional nonautonomous linear system with independent variable t and dependent variables* x_1, x_2, *and* x_3.

As we'll see later in this chapter, an autonomous system has a nice graphical interpretation that yields a neat qualitative analysis. We lose some of this power when we are dealing with a nonautonomous system. But even when we are confronted with a nonautonomous equation, a simple variation of the conversion technique we've been illustrating will allow us to transform the equation into an autonomous system. To convert a single *nonautonomous* nth-order equation into an equivalent *autonomous* system (one whose equations do not explicitly contain the independent variable t), we need $n+1$ first-order equations: $x_1 = x$, $x_2 = x'$, $x_3 = x''$, $\ldots$, $x_n = x^{(n-1)}$, $x_{n+1} = t$. We'll see this in the next example.

Example 6.1.4 Converting a Nonautonomous Equation into an Autonomous System

The nonautonomous second-order linear equation $2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 50\sin t$ could be handled in the same way as the equation in Example 6.1.3, but instead we'll demonstrate the extension of the conversion technique.

Start by letting $x_1 = x$ and $x_2 = \frac{dx}{dt}$ as before, but also introduce $x_3 = t$. Then

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{dx}{dt} = x_2,\\ \frac{dx_2}{dt} &= \frac{d^2x}{dt^2} = \frac{1}{2}\left(-3\frac{dx}{dt} - x + 50\sin t\right) = \frac{1}{2}\left(-3x_2 - x_1 + 50\sin x_3\right),\\ \frac{dx_3}{dt} &= 1,\end{aligned}$$

so the equivalent system is

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{1}{2}(-x_1 - 3x_2 + 50\sin x_3)$$
$$\frac{dx_3}{dt} = 1.$$

Our second-order nonautonomous equation has been replaced by an equivalent *autonomous three-dimensional system*. If we had not used the third variable x_3 and had written our equation as a system of *two* equations, the second equation would have been nonautonomous. We would have had

$$\frac{dx_1}{dt} = x_2 \quad \text{and} \quad \frac{dx_2}{dt} = \frac{1}{2}(-x_1 - 3x_2 + 50\ \sin\ t)$$

as the system, with the explicit presence of t in the second equation making this equation (and therefore the system) nonautonomous.

Of course, we should be able to convert an initial-value problem (IVP) into a system IVP as well. If we think about this, we would expect that the original initial conditions would have to expand to cover each first-order equation in the system. The next example shows how this works.

Example 6.1.5 Converting a Second-Order Initial-Value Problem

The nonautonomous second-order linear IVP $y'' - xy' - x^2y = 0$; $y(0) = 1$, $y'(0) = 2$ can be transformed into a system IVP as follows. Let $u_1 = y$ and $u_2 = y'$. (We're using a different letter for the new variables to avoid confusion with the original independent variable x.) We see that $u_1' = y' = u_2$ and $u_2' = y'' = xy' + x^2y = xu_2 + x^2u_1$. Then, because $u_1 = y$, $y(0) = 1$ implies that $u_1(0) = 1$, and $y'(0) = 2$ implies that $u_2(0) = 2$ because $u_2 = y'$. Therefore, the original IVP becomes the system IVP

$$u_1' = u_2$$
$$u_2' = xu_2 + x^2u_1; \quad u_1(0) = 1, \quad u_2(0) = 2.$$

Note that because each equation in the system is first-order, we need only one initial condition for each new variable. *What would the equivalent autonomous system look like?*

6.1.2 Conversion technique II: converting a system into a higher-order equation

We showed in Examples 6.1.1–6.1.5 how higher-order equations can always be transformed into equivalent first-order systems. The next example illustrates how a particular linear system can be represented by a single higher-order equation. However, it is *not* true that every system of first-order equations is equivalent to a single higher-order differential equation.[2]

[2] See, e.g., Section 6.4 of *Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems* by J.H. Hubbard and B.H. West (New York: Springer-Verlag, 1995).

Example 6.1.6 Converting a System into a Single Equation

Can we convert the system

$$\begin{aligned} &(1)\quad y' = z \\ &(2)\quad z' = w \\ &(3)\quad w' = x - 3y - 6z - 3w, \end{aligned}$$

where y, z, and w are functions of x, into an equivalent single higher-order equation?

Of course we can. Just look back at what we did in our earlier examples, but start with the *last* equation and work backward. Differentiating Eq. (2) gives us $z'' = w'$, but Eq. (1) says that $z'' = (y')'' = y'''$, so that $w' = y'''$. Now we use this last result to rewrite Eq. (3) as

$$\begin{aligned} y''' &= x - 3y - 6z - 3w \\ &= x - 3y - 6y' - 3z' \quad \text{[from (1) and (2)]} \\ &= x - 3y - 6y' - 3y'' \quad \text{[from (1)]} \end{aligned}$$

or $y''' + 3y'' + 6y' + 3y = x$, a third-order linear nonautonomous differential equation.

Exercises 6.1

A

Write each of the higher-order ordinary differential equations (ODEs) or systems of ODEs in Problems 1–11 as a system of first-order equations. If initial conditions are given, rewrite them in terms of the first-order system.

1. $\frac{d^2x}{dt^2} - x = 1$
2. $\left(x''\right)^2 - (\sin t)x' = x \cos t$
3. $x^2y'' - 3xy' + 4y = 5 \ln x$
4. $\dot{x} + (\dot{x})^2 + x(x - 1) = 0$
5. $x''' - tx'' + x' - 5x + t^2 = 0$
6. $y^{(4)} + y = 0$
7. $w^{(4)} - 2w''' + 5w'' + 3w' - 8w = 6 \sin(4t)$
8. $\ddot{y} + y = t$; $y(0) = 1$, $y'(0) = 0$
9. $x'' + 3x' + 2x = 1$; $x(0) = 1$, $x'(0) = 0$
10. $\frac{d^2x}{dt^2} = -x$, $\frac{d^2y}{dt^2} = y$ (Write each second-order equation as two first-order equations.)
11. $x\frac{d^2y}{dt^2} - y = 4t$, $2\frac{d^2x}{dt^2} + \left(\frac{dy}{dt}\right)^2 = x$ (Convert each second-order equation into two first-order equations.)

Write each of the systems of equations in Problems 12–16 as a single second-order equation, rewriting any initial conditions as necessary.

12. $\frac{dy}{dt} = x$, $\frac{dx}{dt} = -y$; $y(0) = 0$, $x(0) = 1$

13. $\frac{du}{dx} = 2v - 1$, $\frac{dv}{dx} = 1 + 2u$

14. $x' = x + y$, $y' = x - y$

15. $\frac{dx}{dt} = 7y - 4x - 13$, $\frac{dy}{dt} = 2x - 5y + 11$; $x(0) = 2$, $y(0) = 3$

16. $x' = y + \sin x$, $y' = \cos(x + y)$

B

17. The equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$ describes the position $x(t)$ of a particular mass attached to a spring and set in motion by pulling it down 2 ft below its equilibrium position ($x = 0$) and giving it an initial velocity of 2 ft/s in the upward direction. Some air resistance is assumed. Express this equation as a system of first-order equations and describe what each equation of the system represents.

18. In electrical circuit theory, the current I is the derivative of the charge Q. By making this natural substitution $Q' = I$ in the equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

transform the equation into an equivalent system of two first-order equations.

19. An object placed in water, pushed down a certain distance below the water, and then released has its bobbing motion described by the equation

$$\frac{d^2y}{dt^2} + \left(\frac{g}{s_0}\right)y = 0,$$

where y is the vertical displacement from its equilibrium position, g is the acceleration due to gravity, and s_0 is the initial depth. Express this equation as a system of first-order equations.

20. The second-order nonlinear equation $\frac{d^2x}{dt^2} + \frac{g}{L}\sin x = 0$ describes the swinging of a pendulum, where x is the angle the pendulum makes with the vertical, g is the acceleration due to gravity, and L is the length of the pendulum. Convert this equation into a nonlinear system of first-order equations.

21. The equation $y''' + y' - \cos y = 0$ describes a geometrical model of crystal growth. Express this third-order equation as a system of three first-order equations.

22. The equation $y^{(4)} + \lambda(yy''' - y'y'') - y' = 0$, where λ is a positive parameter, arises in a nonlinear "boundary layer" problem in physical oceanography. Write this equation as a system of four first-order equations.

23. Rewrite the system IVP given in Example 6.1.5 as an equivalent *autonomous* system.

24. Consider the equation $y'' + y = 0$.

a. Convert this equation into a system with variables u and v.

b. Use the result of part (a) and the Chain Rule to conclude that $u^2 + v^2$ is a constant.

C

25. Write the following system of equations as a single fourth-order equation, with appropriate initial conditions:

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + 8x = 32t$$
$$\frac{d^2y}{dt^2} + 3\frac{dx}{dt} - 2y = 60e^{-t};\; x(0) = 6,\, x'(0) = 8,\, y(0) = -24,\; \text{and } y'(0) = 0.$$

26. Suppose you are given the linear system of first-order equations

$$t\frac{dx}{dt} = -3x + 4y$$
$$t\frac{dy}{dt} = -2x + 3y.$$

Introduce a new independent variable w by the substitution $w = \ln t$ (or $t = e^w$) and show that this substitution allows you to write the system as a new system with *constant* coefficients.

27. Consider the system

$$x' = f(x, y)$$
$$y' = g(x, y),$$

where x and y are functions of t. Assume you can use the first equation to express y explicitly as a function of x and x', say $y = F(x, x')$ for some function $F(u, v)$ of two variables.

a. Find an expression for $y' = dy/dt$ by differentiating the equation $y = F(x, x')$ via the Chain Rule for functions of two variables (see Section A.7).

b. Substitute the expression for y' found in part (a) into the second equation of the original system and set the right-hand side equal to $g(x, y) = g(x, F(x, x'))$. Observe how the results of parts (a) and (b) yield a second-order equation solely in terms of x and x'.

*6.2 Existence and uniqueness[3]

Now that we've learned how to convert higher-order equations to equivalent systems of first-order equations and we've seen some qualitative analyses of these systems,

[3] * Denotes an optional section.

it's time to ask the important questions we first considered in Section 2.8 in the context of first-order equations: How do we know that a given higher-order equation or equivalent system *has* a solution—and do we know that any such solution is *unique*?

We don't want to waste human and computer resources searching for a solution that may not exist or that may merely be one of many solutions. For now we'll focus on second-order equations and their corresponding systems.

The first example shows that when there is one solution of a system, there may be many.

Example 6.2.1 A System IVP with Many Solutions

Let's look at the IVP

$$t^2x'' - 2tx' + 2x = 0, \text{ with } x(0) = 0 \text{ and } x'(0) = 0.$$

This is equivalent to the system IVP

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= \frac{2}{t}x_2 - \frac{2}{t^2}x_1, \quad \text{with } x_1(0) = 0 = x_2(0). \end{aligned}$$

Then $x(t) \equiv 0$ and any function of the form $x(t) = Kt^2$ (where K is any constant) are solutions of the original IVP. (*Verify this.*) With respect to the equivalent system of equations, $x_1(t) \equiv 0$, $x_2(t) \equiv 0$, is a solution, and any pair of functions $x_1(t) = Kt^2$, $x_2(t) = 2Kt$ is a solution. What we are saying here is that *our IVP has infinitely many solutions*.

In contrast to the IVP in the preceding example, we can have a system of differential equations with *no* solution.

Example 6.2.2 A System IVP with No Solution

Let's look at the IVP

$$x_1' = \frac{1}{x_1^2}, \; x_2' = 2x_1 - x_2, \text{ with } x_1(0) = 0 \text{ and } x_2(0) = 1.$$

When we examine these equations carefully, we see that if $x_1(t)$ is part of a solution pair for this IVP, then x_1' doesn't exist for $t = 0$ because $x_1'(0) = \frac{1}{[x_1(0)]^2}$ and $x_1(0) = 0$. This says that there is no solution to this IVP.

What we want in most real-life situations is one and only one solution to an IVP. The next example shows such a case.

Example 6.2.3 A System IVP with a Unique Solution

The IVP $\left\{\frac{dx}{dt} = y, \; \frac{dy}{dt} = x; \; x(0) = 1, \; y(0) = 0\right\}$ has the *unique* solution $x(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$, $y(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$. You may recognize x and y as the *hyperbolic cosine* (cosh) and *hyperbolic sine* (sinh), respectively.

This system is equivalent to the single equation $\ddot{x} - x = 0$, or $\ddot{x} = x$, with $x(0) = 1$ and $\dot{x}(0) = 0$, and it isn't too difficult to guess what kind of function is equal to its own second derivative. Problem 1 in Exercises 6.2 will ask you to explore this further.

6.2.1 An Existence and Uniqueness Theorem

At this point we have seen that the possibilities for second-order IVPs are similar to those we saw in Section 2.8 for first-order IVPs. We can have *no* solution, *infinitely many solutions*, or *exactly one solution*. Once again we would like to determine when there is one and only one solution of an IVP.

The simplest Existence and Uniqueness Theorem for second-order differential equations or two-dimensional systems of first-order equations is one that is a natural extension of the result we saw in Section 2.8. We'll state two forms of this.

Existence and Uniqueness Theorem I

Suppose we have a second-order IVP $\frac{d^2y}{dt^2} = f(t, y, \dot{y})$, with $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$. If f, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial \dot{y}}$ are continuous in a closed box B in three-dimensional space (t-y-$\dot{y}$ space) and the point $(t_0, y_0, \dot{y}_0)$ lies inside B, then the IVP has a unique solution $y(t)$ on some t-interval I containing t_0.

Equivalently,

Existence and Uniqueness Theorem II

Suppose we have a two-dimensional system of first-order equations

$$\begin{aligned} \frac{dx_1}{dt} &= f(t, x_1, x_2) \\ \frac{dx_2}{dt} &= g(t, x_1, x_2), \end{aligned}$$

where $x_1(t_0) = x_1^0$ and $x_2(t_0) = x_2^0$. If f, g, $\frac{\partial f}{\partial x_1}$, $\frac{\partial g}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, and $\frac{\partial g}{\partial x_2}$ are all continuous in a box B *in* t-$x_1 - x_2$ space containing the point $\left(t_0, x_1^0, x_2^0\right)$, then there is an interval I containing t_0 in which there is a unique solution $x_1 = y_1(t)$, $x_2 = y_2(t)$ of the IVP.

6.2.2 Many solutions

We can write the equation in Example 6.2.1 in the form $x'' = f(t, x, x') = \frac{2tx' - 2x}{t^2}$, so we see that f does not exist in any box in which $t = 0$. Therefore, we should not expect exactly one solution, and, in fact, although there *is* a solution to the IVP with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, any such solution is not unique.

6.2.3 No solution

In Example 6.2.2, we can use the system form of our Existence and Uniqueness Theorem to see that the function $f(t, x_1, x_2) = 1/x_1^2$ does not exist at the point $\left(t, x_1^0, x_2^0\right) = (0, 0, 1)$, so once again we are not guaranteed exactly one solution—and, in fact, there is *no* solution of the IVP.

6.2.4 Exactly one solution

Finally, if we examine the IVP in Example 6.2.3 from either the single-equation or the system point of view, we should see that in this situation we are guaranteed the existence of one and only one solution of the IVP. (*Check this.*)

The nice thing about these questions is that in most common applied problems, the functions and their derivatives are well behaved (continuous, etc.), so that we *do* have both existence and uniqueness.

Exercises 6.2

A

1. In Example 6.2.3, you saw an IVP for a system of equations that was equivalent to the single equation IVP $x'' - x = 0$, or $x'' = x$, with $x(0) = 1$ and $x'(0) = 0$. Using the technique of Section 4.1, show that $x(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$ is the solution of the IVP $x'' - x = 0$ with $x(0) = 1$ and $x'(0) = 0$.
2. Verify that each of the following IVPs has a solution that is guaranteed unique *everywhere* in three-dimensional space.
   a. $x_1' = x_2,\ x_2' = 3x_1 - 5x_2;\ x_1(0) = 1,\ x_2(0) = 0$
   b. $x_1' = x_1^2,\ x_2' = \sin x_1 - x_2^2;\ x_1(0) = 0,\ x_2(0) = 0$
   c. $x_1' = x_2^3,\ x_2' = t x_1 - x_2;\ x_1(0) = 0,\ x_2(0) = 1$
3. For each of the following equations, determine intervals in which solutions are guaranteed to exist.
   a. $y^{(iv)} + 4y''' + 3y = t$
   b. $ty''' + (\sin t)y'' + 3y = \cos t$
   c. $t(t-1)y^{(iv)} + e^t y'' + 4t^2 y = 0$
   d. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
4. Show that the IVP

$$\dot{x} = f(t, x, y) = e^{-0.002t} - 0.08x - xy^2$$
$$\dot{y} = g(t, x, y) = 0.08x - y + xy^2;\ \ x(0) = 0,\ \ y(0) = 0$$

   has a unique solution for all values of t, x, and y.

B

5. Show that the IVP

$$\{yx' = y - 4t,\ (x-3)y' = -4x + \sin t;\ x(0) = 3,\ y(0) = 0\}$$

has no solution. Does this contradict the existence part of the result given in this section? Explain.

6. **a.** Show that $\{x_1(t) = e^{-t}\sin(3t),\ y_1(t) = e^{-t}\cos(3t)\}$ and

$$\left\{x_2(t) = e^{-(t-1)}\sin(3(t-1)),\ y_2(t) = e^{-(t-1)}\cos(3(t-1))\right\}$$

are solutions of the system

$$\frac{dx}{dt} = -x + 3y$$
$$\frac{dy}{dt} = -3x - y.$$

b. Use technology to draw the graphs of each of the solutions in part (a) in the x-y phase plane.

c. Explain why the solutions in part (a) don't contradict the uniqueness part of the result in this section.

C

7. Consider the equation

$$5x^2y^{(5)} - (6\sin x)y''' + 2xy'' + \pi x^3y' + (3x-5)y = 0.$$

Suppose that $Y(x)$ is a solution of this equation such that $Y(1) = 0$, $Y'(1) = 0$, $Y''(1) = 0$, $Y'''(1) = 0$, $Y^{(4)}(1) = 0$, and $Y^{(5)}(1) = 0$. Why must $Y(x)$ be equal to 0 for *all* values of x?

8. Use technology to plot some trajectories of the nonautonomous system

$$\frac{dx}{dt} = (1-t)x - ty$$
$$\frac{dy}{dt} = tx + (1-t)y.$$

Your graph should show some intersecting curves. Does the graph contradict the Existence and Uniqueness Theorem? *Explain.*

9. Consider the system

$$\dot{x} = y$$
$$\dot{y} = -x + (1 - x^2 - y^2)y.$$

a. Let D denote the region in the phase plane defined by $x^2 + y^2 < 4$. Verify that the given system satisfies the hypotheses of the Existence and Uniqueness Theorem throughout D.

b. By substitution, show that $x(t) = \sin t$, $y(t) = \cos t$ is a solution of the system.

c. Now consider a different solution, in this case starting from the initial conditions $x(0) = 1/2$, $y(0) = 0$. Without doing any calculations, explain why this solution *must* satisfy $x(t)^2 + y(t)^2 < 1$ for all real values of t.

6.3 Numerical solutions of systems

Most numerical methods for computing solutions of differential equations are designed for *systems* of first-order equations, and they can be applied to second- and higher-order single differential equations by finding equivalent systems of first-order equations. Any of the numerical approximation methods introduced for first-order equations in Sections 3.1, 3.2, and 3.3 can be extended to systems of first-order equations in a natural way. In fact, calculators that can handle higher-order equations either graphically or numerically require the user to input the equation as a system.

In this section we'll work mainly with two-dimensional linear systems, leaving the obvious generalizations to the last subsection. Even though it is important to be able to solve simple numerical problems by hand, most systems of differential equations are solved using numerical methods implemented on computers. These techniques work for both linear and nonlinear systems.

6.3.1 Euler's method applied to systems

Let's start by recalling *Euler's method* for solving the first-order IVP $y' = f(x, y)$, $y'(x_0) = y_0$. This algorithm was originally given as formula (3.1.3):

$$y_{k+1} = y_k + h \cdot f(x_k, y_k).$$

Here, h is the step size and y_k denotes the approximate value of the solution at the point $x_k = x_0 + kh$.

Now suppose we have a system of two first-order differential equations

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y),$$

with $x(t_0) = x_0$ and $y(t_0) = y_0$. If we let $t_k = t_0 + kh$, $x_k \approx x(t_k)$, and $y_k \approx y(t_k)$, we can apply Euler's algorithm to each equation separately to get the result

$$x_{k+1} = x_k + h \cdot f(t_k, x_k, y_k)$$

$$y_{k+1} = y_k + h \cdot f(t_k, x_k, y_k). \tag{6.3.1}$$

Let's see how this method works on a system we've already seen.

Example 6.3.1 Euler's Method for a System—by Hand

As a simple illustration of Euler's method applied to a system, let's approximate the solution of the IVP of Example 6.2.3 at $t = 0.5$. The system, which we know has a unique solution, is

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = x;\ x(0) = 1,\ y(0) = 0.$$

Equations

When we use a step size $h = 0.1$ the algorithm given by Eq. (6.3.1) looks like

$$\begin{aligned} x_{k+1} &= x_k + (0.1)y_k \\ y_{k+1} &= y_k + (0.1)x_k, \end{aligned}$$

where $x_0 = x(0) = 1$ and $y_0 = y(0) = 0$.

Calculation

We approximate the solution at $t = 0.5$ by taking five steps:

$$\begin{aligned} x_1 &= x_0 + (0.1)y_0 = 1 + (0.1)(0) = 1 \\ y_1 &= y_0 + (0.1)x_0 = 0 + (0.1)(1) = 0.1 \\ x_2 &= x_1 + (0.1)y_1 = 1 + (0.1)(0.1) = 1.01 \\ y_2 &= y_1 + (0.1)x_1 = 0.1 + (0.1)(1) = 0.2 \\ x_3 &= x_2 + (0.1)y_2 = 1.01 + (0.1)(0.2) = 1.03 \\ y_3 &= y_2 + (0.1)x_2 = 0.2 + (0.1)(1.01) = 0.301 \\ x_4 &= x_3 + (0.1)y_3 = 1.03 + (0.1)(0.301) = 1.0601 \\ y_4 &= y_3 + (0.1)x_3 = 0.301 + (0.1)(1.03) = 0.404 \\ x_5 &= x_4 + (0.1)y_4 = 1.0601 + (0.1)(0.404) = 1.1005 \\ y_5 &= y_4 + (0.1)x_4 = 0.404 + (0.1)(1.0601) = 0.51001 \end{aligned}$$

Result

These calculations indicate that $x(0.5) \approx 1.1005$ and $y(0.5) \approx 0.5100$. But to four decimal places, the *exact* solution is $x(0.5) = \cosh(0.5) = \left(\frac{1}{2}\right)(\exp(0.5) + \exp(-0.5)) = 1.1276$ and $y(0.5) = \sinh(0.5) = \left(\frac{1}{2}\right)(\exp(0.5) - \exp(-0.5)) = 0.5211$. Thus, the absolute error is 0.0271 for x and 0.0111 for y.

If we cut our step size in half, letting $h = 0.05$ and using technology, we need 10 steps and find that our approximations are $x(0.5) \approx 1.1138$ and $y(0.5) \approx 0.5151$, to four decimal places. Now the error—0.0130 for x and 0.006 for y—is roughly half of what these errors were when $h = 0.1$. Having computer resources at our command, it's hard to resist another run, this time with $h = 0.01$. Taking 50 steps, we have $x(0.5) \approx 1.1248$ and $y(0.5) \approx 0.5198$, with errors 0.0028 and 0.0013 for x and y, respectively. You should experiment with a few other values of h on your own computer algebra system (CAS) or graphing calculator.

Problem 1 in Exercises 6.3 asks you to write the system form of the *improved Euler method* (*Heun's method*).

6.3.2 The fourth-order Runge–Kutta method for systems

As an additional example, let's look at the system form of the Runge–Kutta algorithm introduced in Section 3.3. (As we mentioned in that discussion, it was Kutta who generalized the basic method to *systems* of ODEs in 1901.)

We start with the same general first-order system we considered before:

$$\begin{aligned}\frac{dx}{dt} &= f(t, x, y)\\ \frac{dy}{dt} &= g(t, x, y),\end{aligned}$$

with $x(t_0) = x_0$ and $y(t_0) = y_0$. Again, we let $t_k = t_0 + kh$, $x_k \approx x(t_k)$, and $y_k \approx y(t_k)$. Then the system version of the classic Runge–Kutta formula (3.3.2) is

$$\begin{aligned}x_{k+1} &= x_k + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)\\ y_{k+1} &= y_k + \frac{1}{6}(M_1 + 2M_2 + 2M_3 + M_4),\end{aligned}$$

where

$$\begin{aligned}m_1 &= hf(t_k, x_k, y_k)\\ m_2 &= hf\left(t_k + \frac{h}{2}, x_k + \frac{m_1}{2}, y_k + \frac{M_1}{2}\right)\\ m_3 &= hf\left(t_k + h, x_k + \frac{m_2}{2}, y_k + \frac{M_2}{2}\right)\\ m_4 &= hf(t_k + h, x_k + m_3, y_k + M_3) = hf(t_{k+1}, x_k + m_3, y_k + M_3),\end{aligned}$$

and

$$\begin{aligned}M_1 &= hg(t_k, x_k, y_k)\\ M_2 &= hg\left(t_k + \frac{h}{2}, x_k + \frac{m_1}{2}, y_k + \frac{M_1}{2}\right)\\ M_3 &= hg\left(t_k + \frac{h}{2}, x_k + \frac{m_2}{2}, y_k + \frac{M_2}{2}\right)\\ M_4 &= hg(t_k + h, x_k + m_3, y_k + M_3) = hg(t_{k+1}, x_k + m_3, y_k + M_3).\end{aligned}$$

Now let's put this algorithm to use—with the aid of technology, of course.

Example 6.3.2 Using Runge–Kutta (RK4) and a CAS

Let's look again at the IVP analyzed in Example 6.2.3. The system IVP is $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$; $x(0) = 1$, $y(0) = 0$, and we want to approximate $x(0.5)$ and $y(0.5)$. Rather than wearing ourselves out trying to implement the fourth-order Runge–Kutta (RK4) method by hand, we can enter the equations and initial conditions into our CAS, specify the method (in whatever way you must describe the RK4 method), and choose a step size $h = 0.1$.

What we get is an approximation for $x(0.5)$ of 1.1276 and an approximation for $y(0.5)$ of 0.5211, both rounded to four decimal places. To four decimal places the absolute error for each approximation is 0.

Our final example shows how the Runge–Kutta–Fehlberg fourth- and fifth-order (rkf45) algorithm works on an interesting nonlinear system application.

Example 6.3.3 Using Runge–Kutta–Fehlberg (rkf45) and a CAS

The Japanese-born British mathematician E.C. Zeeman (1925–2016) developed a simple nonlinear model of the human heartbeat:

$$\varepsilon \frac{dx}{dt} = -(x^3 - Ax + c)$$
$$\frac{dc}{dt} = x,$$

where $x(t)$ is the displacement from equilibrium of the heart's muscle fiber, $c = c(t)$ is the concentration of a chemical control at time t, and ε and A are positive constants. Because the levels of c determine the contraction and expansion (relaxation) of the muscle fibers, we can think of c as a *stimulus* and x as a *response*.

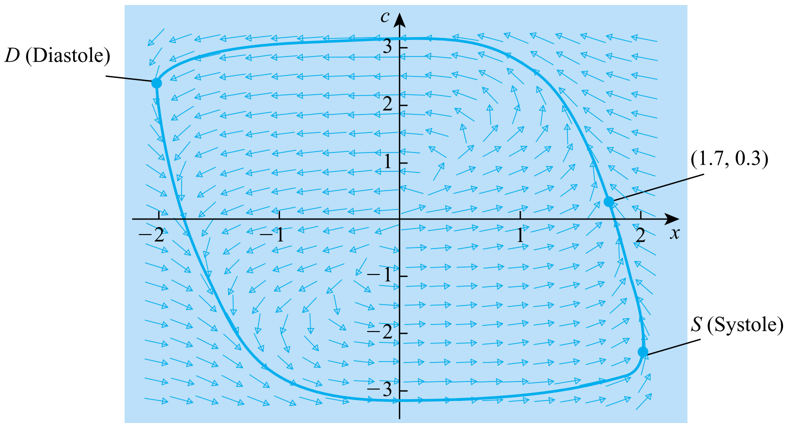


FIGURE 6.2

Slope field and trajectory for the system IVP $\left\{\frac{dx}{dt} = -(x^3 - 3x + c),\ \frac{dc}{dt} = x;\ x(0) = 1.7,\ c(0) = 0.3\right\}$, $0 \le t \le 30$

We want to investigate the nature of the model's solution, and for convenience we'll assume that $\varepsilon \approx 1.0$ and $A \approx 3$. In addition we let $x(0) \approx 1.7$ and $c(0) \approx 0.3$. (The initial conditions were determined after experimenting with various values on a CAS.) The calculations producing the graphs and the values discussed next were carried out using the rkf45 method for the system. Be aware that we have jumped the gun here, anticipating geometrical aspects that will be discussed fully in Section 6.4. For now, just realize that we have eliminated the variable t by dividing the second equation by the first: $\frac{dc}{dx} = \frac{dc}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dc}{dt}}{\frac{dx}{dt}} = x/\left(-(x^3 - 3x + c)\right)$. The slope field in Fig. 6.2 is that of the first-order equation for dc/dx.

Because one important feature of a heartbeat is that it is periodic (lub-*dub*, lub-*dub*, ...), the solution should reveal this in the x-c phase plane—and in fact it does (Fig. 6.2). The *systole*, corresponding to a fully relaxed heart muscle, and the *diastole*, indicating a state of full contraction, are labeled on Fig. 6.2.

We see that the heart muscle starts at $(1.7, 0.3)$ and, under the influence of increasing c, contracts until it is fully contracted at D. Then the muscle begins to relax until it attains systole at S, returns to the initial point, and (we hope) begins the cycle again. Superimposing the trajectory on the slope field makes it easy to see the direction of the trajectory, but the numerical values in Table 6.1 also tell the story.

Table 6.1 Solution Values of $\left\{\frac{dx}{dt} = -(x^3 - 3x + c), \frac{dc}{dt} = x;\ x(0) = 1.7,\ c(0) = 0.3\right\}$

t	$x(t)$	$c(t)$
0	1.7000	0.3000
2	0.7499	2.9990
4	−1.8417	0.4728
6	−1.1132	−2.5911
8	1.9436	−1.2862
10	1.3384	2.0618

If we examine the signs of x and c as t increases, we see that the points (x, c) are moving *counterclockwise* through the quadrants of the x-c plane. Looking carefully at the data in the table, we can see that the trajectory returns to its initial point $(1.7, 0.3)$ at some time between 8 and 10. In fact, a more detailed analysis reveals that the solution of our IVP has period approximately equal to 8.88. (See Problem 12 in Exercises 6.3.)

Solving the system with rkf45 and then plotting x against t (Fig. 6.3a), we see the periodic nature of the heart muscle's expansions and contractions. Fig. 6.3b shows how the electrochemical activity represented by the variable c also varies periodically.

We will investigate interesting nonlinear systems again in Chapter 7.

Just as for a single first-order equation, we can use spreadsheet commands to carry out the calculations needed to approximate the solutions of systems. System versions of the standard numerical techniques may be a bit more difficult to program, may require more intermediate storage, and may take a little more time, but they work well. Graphing calculators also handle systems of differential equations. In fact, as we remarked in the Introduction of this chapter, they usually deal with a single

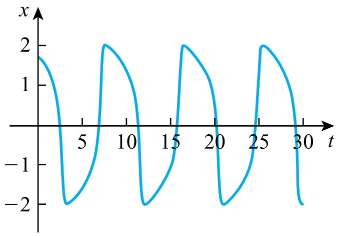


FIGURE 6.3a

Graph of $x(t)$ vs. t for the system IVP $\left\{\frac{dx}{dt} = -(x^3 - 3x + c), \frac{dc}{dt} = x; x(0) = 1.7, c(0) = 0.3\right\}$, $0 \le t \le 30$, $-2.5 \le x \le 2.5$

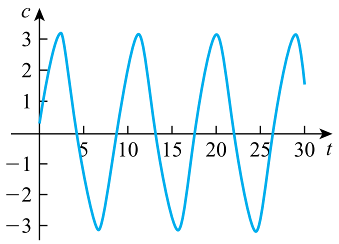


FIGURE 6.3b

Graph of $c(t)$ vs. t for the system IVP $\left\{\frac{dx}{dt} = -\left(x^3 - 3x + c\right), \frac{dc}{dt} = x; x(0) = 1.7, c(0) = 0.3\right\}$, $0 \le t \le 30$, $-4 \le c \le 4$

higher-order equation by requiring the user to write it in terms of a system and then solving the system numerically.

Whatever technology you use, try to understand what methods have been implemented by reading your documentation or checking out your software's "Help" features.

Exercises 6.3

All problems are to be done using technology, unless otherwise indicated.

A

1. **a.** Extend the improved Euler method given by formula (3.2.1) to a system of two first-order equations.

b. *By hand*, re-do Example 6.3.1 using $h = 0.1$ to find approximations to $x(0.5)$ and $y(0.5)$.

c. Calculate the absolute error in part (b).

d. Use technology and the improved Euler method with $h = 0.1$ to check your answers to part (b).

2. Consider the system $x' = x - 4y$, $y' = -x + y$, with $x(0) = 1$ and $y(0) = 0$. The exact solution is $x(t) = \left(e^{-t} + e^{3t}\right)$, $y(t) = \left(e^{-t} - e^{3t}\right)$.

a. Verify that the exact solution of the IVP is the solution given above.

b. Approximate the value of the solution at the point $t = 0.2$ using Euler's method with $h = 0.1$. Compare your result with the values of the exact solution, calculating the absolute error.

c. Approximate the value of the solution at the point $t = 0.2$ using a RK4 method with $h = 0.2$. Calculate the absolute error.

3. Consider the IVP $y'' + y' - 2y = 2x$, with $y(0) = 1$ and $y'(0) = 1$.

a. Convert this problem into a system of two first-order equations. (*Choose your new variables carefully.*)

b. Determine approximate values of the solution at $x = 0.5$ and $x = 1.0$ using Euler's method with $h = 0.1$.

c. Determine approximate values of the solution at $x = 0.5$ and $x = 1.0$ using the RK4 method with $h = 0.1$.

4. Using the method in Section 4.1, we can determine that the solution to the IVP

$$\frac{d^2x}{dt^2} + \frac{1}{4}\frac{dx}{dt} + 2x = 0, \quad \text{with} \quad x(0) = 1,\ \dot{x}(0) = 0$$

is $x(t) = \frac{1}{127}e^{\left(-\frac{1}{8}t\right)}\left(127\cos\left(\frac{1}{8}\sqrt{127}t\right) + \sqrt{127}\sin\left(\frac{1}{8}\sqrt{127}t\right)\right)$.

a. Convert this IVP into a system of first-order equations.

b. Determine the approximate value of the solution at $t = 0.6$ using the *rkf45* method, if available. Otherwise, use the highest-order Runge–Kutta method available to you, with $h = 0.01$. Compare your values with the exact solution above.

B

5. A particle moves in three-dimensional space according to the equations

$$\frac{dx}{dt} = yz, \quad \frac{dy}{dt} = zx, \quad \frac{dz}{dt} = xy.$$

a. Assuming that $x(0) = 0$, $y(0) = 5$, and $z(0) = 0$, use the Runge–Kutta–Fehlberg method, if available, to approximate the solution at $t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 5.0$, and 37. (Otherwise use the highest-order Runge–Kutta method available to you, with $h = 0.01$.) Describe what these values seem to be telling you about the motion of the particle.

b. Now assume that $x(0) = y(0) = 1$ and $z(0) = 0$. Approximate the solution at $t = 0.1, 0.2, 0.3, 1.5, 1.6, 1.7, 1.8$, and 1.9 using the same procedure you used in part (a). What seems to be happening?

6. Consider the system

$$\frac{dx}{dt} = 7y - 4x - 13$$
$$\frac{dy}{dt} = 2x - 5y + 11.$$

a. Suppose that $x(0) = 1$ and $y(0) = 1$. Use technology and the rkf45 method (or a reasonable substitute) to estimate x and y for $t = 1, 2, 3, 4, 5,$ 10, 15, and 20.

b. On the basis of the values found in part (a), guess at $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

7. The nonlinear **Lotka–Volterra system**

$$\dot{x} = 3x - 2xy$$
$$\dot{y} = 0.5xy - y$$

has solutions $(x(t), y(t))$ that are periodic because a given trajectory always returns to its initial point in some finite time t^*: $x(t + t^*) = x(t)$ and $y(t + t^*) = y(t)$. Using technology and the rkf45 method, estimate the smallest value of t^* to two decimal places if $x(0) = 3$ and $y(0) = 2$. (Try different values of $t \neq 0$ until you get $x(t) \approx 3$ and $y(t) \approx 2.0$.)

8. The equations

$$\dot{x} = y$$
$$\dot{y} = -0.25y - 2x;\ x(0) = 1,\ y(0) = 0$$

represent a certain spring-mass system with damping. As usual, assume that the positive direction for $x(t)$ and $y(t)$ is downward and time is measured in seconds.

a. Using technology, approximate $x(t)$ and $y(t)$ for $t = 1, 2, 3, 4$ and interpret the position and velocity in each case.

b. Estimate (to the nearest hundredth of a second) the time when the mass *first* reaches its equilibrium position, $x = 0$.

9. **Emden's equation** $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^n = 0$, where n is a parameter, has been used to model the thermal behavior of a spherical cloud of gas.[4] When the cloud of gas is a star, the first zero of the solution, multiplied by 10^{10} cm, represents the radius of the star. In modeling the bright component of Capella (a star in the constellation Auriga), the astrophysicist A.S. Eddington took $n = 3$ in Emden's equation and used the initial conditions $y(0) = 1,\ y'(0) = 0$.

[4] See H.T. Davis, *Introduction to Nonlinear Differential Equations and Integral Equations* (New York: Dover, 1962): 371ff.

a. Express Eddington's version of Emden's equation as an equivalent system of two first-order equations in the variables u and v.

b. Approximate the radius of Capella by determining (approximating) the first value of x for which $y(x) = 0$.

10. The **Fitzhugh–Nagumo model** arises in the study of the impulses in a nerve fiber:

$$\begin{aligned}\dot{V} &= W \\ \dot{W} &= \left(\frac{1}{3}V^3 - V\right) + R - uW \\ \dot{R} &= \frac{\varepsilon}{u}(bR - V - a).\end{aligned}$$

Assume that $V(0) = 1$, $W(0) = 0.5$, and $R(0) = 1$. Approximate the maximum and minimum values, if they exist, of V, W, and R if $\varepsilon = 0.08$, $a = 0.7$, $b = 0.8$, and $u = 0.6$.

C

11. A famous model for the spread of a disease is the **S-I-R** model. At a given time t, S represents the population of *susceptibles*, those who have never had the disease and can get it; I stands for the *infected*, those who have the disease now and can give it to others; and R denotes the *recovered*, those who have already had the disease and are immune. Suppose these populations are related by the system

$$\begin{aligned}\frac{dS}{dt} &= (-0.00001)SI \\ \frac{dI}{dt} &= (0.00001)SI - \frac{I}{14} \\ \frac{dR}{dt} &= \frac{I}{14},\end{aligned}$$

with $S(0) = 45{,}400$, $I(0) = 2100$, $R(0) = 2500$.

a. Add the three differential equations and interpret the result in terms of a population.

b. Use your CAS to plot S, I, and R as functions of t on separate graphs. [*Warning*: Some mathematical software (such as *Maple*) may reserve the letter I for the imaginary unit $\sqrt{-1}$. If this is your situation, use *IN* to denote the infected population.]

c. Use your CAS to plot phase portraits in the S-I, S-R, and I-R planes.

d. Use a powerful numerical method (with $h = 0.1$ if appropriate) to approximate the values of S, I, and R at $t = 1, 2, 3, 10, 15, 16$, and 17 days. What do you see?

e. Approximate the value of t at which $I = 0$.

12. Use the rkf45 method to show why the period of the trajectory in Fig. 6.2 is approximately 8.88. (Use the method suggested in Problem 7.)

13. Investigate the Zeeman heartbeat model in Example 6.3.3 with $\varepsilon = 0.025$, $A = 0.1575$, and $(x_0, c_0) = (0.45, -0.02025)$.

a. Use the rkf45 method to approximate $x(t)$ and $c(t)$ for $t = 0.01, 0.02, \ldots, 0.10$ seconds. What do your calculations tell you about the direction of the solution curve in the x-c plane?

b. Draw the trajectory corresponding to the initial conditions given above.

c. Approximate the period of the trajectory found in part (b).

d. Estimate the coordinates of the diastole and the systole.

14. A lunar lander is falling freely toward the surface of the moon. If $x(t)$ represents the distance of the lander from the *center* of the moon (in meters, with t in seconds), then $x(t)$ satisfies the IVP

$$\frac{d^2x}{dt^2} = 4 - \frac{4.9044 \times 10^{12}}{x^2},$$

with $x(0) = 1{,}781{,}870$ and $x'(0) = -450$. (The value $x(0)$ represents the fact that the retro rockets are fired at $t = 0$—when the lander is at a height of 41,870 meters from the moon's surface, or 1,781,870 meters from the moon's center.)

a. Determine the value of t when $x(t) = 1{,}740{,}000$—that is, when the craft has landed on the lunar surface.

b. What is the lunar lander's velocity at touchdown?

6.4 The geometry of autonomous systems

The purpose of this section is to introduce some of the basic geometric tools used to analyze the qualitative behavior of systems of first-order equations. Because many systems—especially nonlinear systems—cannot be solved in closed form, the ability to analyze systems geometrically (i.e., graphically) is very important. The first thing we have to realize is that the very useful graphical tool of *slope fields* can't be applied directly to higher-order equations; this technique depends on a knowledge of the first derivative alone. However, as we'll see shortly, there's a clever way of using our knowledge of first-order qualitative methods in the analysis of higher-order differential equations. We already had a quick peek at this technique in our analysis of Example 6.3.3.

For convenience, we'll spend most of our time analyzing autonomous two-dimensional systems (*planar systems*), although we will also tackle some nonautonomous systems and some three-dimensional systems toward the end of this section.

6.4.1 Phase portraits for systems of equations

Suppose we have an autonomous system of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)\\ \frac{dy}{dt} &= g(x, y).\end{aligned} \tag{6.4.1}$$

It is customary to assume that f and g are continuous for all values of x and y and have continuous partial derivatives with respect to x and y. These are realistic assumptions for most applications.

For example, let's take the system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -17x - 2y\end{aligned}$$

and work with it throughout our initial discussions.

First, we can eliminate the variable t by dividing the second equation by the first equation:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x, y)}{f(x, y)}. \tag{6.4.2}$$

(See Section A.2 for a reminder of the Chain Rule used in this process.) For our example $g(x, y) = -17x - 2y$ and $f(x, y) = y$ in (6.4.2), and we get

$$\frac{dy}{dx} = \frac{-17x - 2y}{y}.$$

Now we have a single first-order differential equation $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ in the variables y and x, and we can construct and analyze the **slope field** corresponding to this first-order equation (6.4.2). (Go back to Fig. 6.2 to see one such slope field.) Fig. 6.4a shows the slope field for our current example.

If we could solve Eq. (6.4.2) for y in terms of x, or even implicitly, we would have a solution curve in the x-y plane. The plane of the variables x and y (with x- and y-axes) is called the **phase plane** of the original system of differential equations with t as a parameter. As we saw in Section 1.3, a solution of the system (6.4.1) consists of a pair of functions $x = x(t)$, $y = y(t)$. We can think of a **solution curve** defined parametrically as a set of points $(x(t), y(t))$ as t varies. Each solution curve in the phase plane is called a **trajectory** (or **orbit**) of the system of equations. Although the independent variable t is not present explicitly, the passage of time is represented by the *direction* that a point $(x(t), y(t))$ takes on a particular trajectory. The way the curve is followed as the values of t increase is called the **positive direction** on the trajectory. The collection of plots of the trajectories is called the system's **phase portrait** or **phase-plane diagram**. The concepts of phase plane and phase portrait are natural extensions of the qualitative concepts discussed in Section 2.5.

Usually much less than the complete phase portrait is required (or of interest) for applications. By choosing good representative initial conditions, points through which we want the trajectories to pass, we can draw a fairly useful picture of the system's solutions.

Even if we can't solve the system, we can still look at the slope field of the single Eq. (6.4.2), the outline of the phase portrait of the system.

Let's do this for the system we've been discussing.

Example 6.4.1 Phase Portrait—One Trajectory

Our system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -17x - 2y,$$

which gives us the first-order equation $\frac{dy}{dx} = \frac{-17x-2y}{y}$ when we eliminate the variable t. Using a calculator or CAS to draw a piece of the slope field for this first-order equation can result in an incorrect plot (see Problem 13 in Exercises 6.4). Your technological device may have a problem at points (x, y) with $y = 0$. If you draw the slope field by hand, be sure to place vertical tangent line segments along the x-axis (where $y = 0$), corresponding to an undefined (or infinite) slope when $y = 0$. It is better to use technology that takes the *pair* of equations given previously as input.

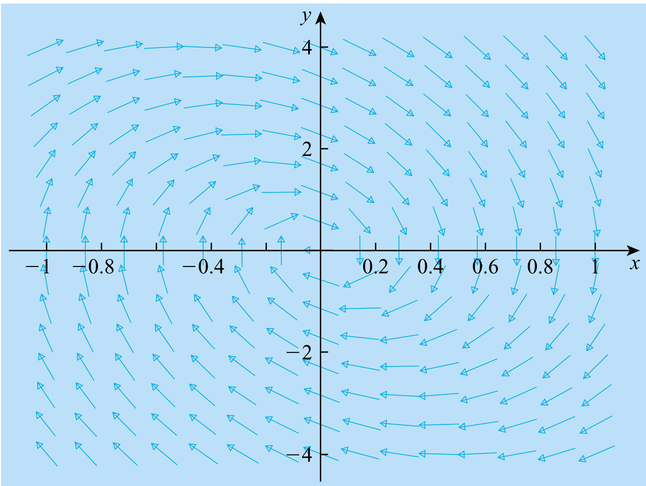


FIGURE 6.4a

Slope field for $\frac{dy}{dx} = \frac{-17x-2y}{y}$, $0 \le t \le 5$; $-1 \le x \le 1$, $-4 \le y \le 4$

Fig. 6.4a shows the slope field, and Fig. 6.4b shows a single trajectory satisfying the initial condition $x(0) = 4$, $y(0) = 0$—that is, a trajectory passing through the point (4, 0) in the x-y (phase) plane—superimposed on the slope field.

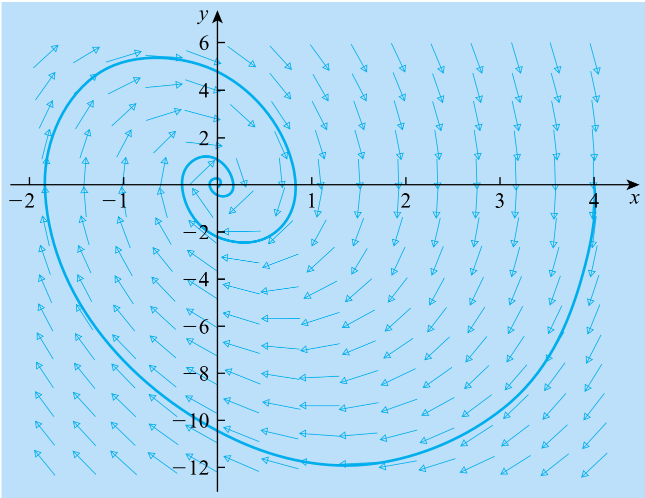


FIGURE 6.4b

Trajectory for $\left\{\frac{dx}{dt} = y,\ \frac{dy}{dt} = -17x - 2y;\ \ x(0) = 4,\ y(0) = 0\right\}$, $0 \le t \le 5$; $-2 \le x \le 4$, $-12 \le y \le 7$

Because the trajectory starts at (4, 0), we can see that the positive direction on the trajectory is clockwise, and the curve seems to spiral into the origin. (Try using technology to draw the trajectory for $0 \le t \le b$, letting b get larger and larger.) To get an accurate phase portrait, you may want to use the slope field to suggest good initial points to use. Each dynamical system has its own appropriate range for t.

Now let's look at a more elaborate phase portrait, one showing several trajectories.

Example 6.4.2 Phase Portrait—Several Trajectories

The system consists of the two equations (1) $\frac{dx}{dt} = x + y$ and (2) $\frac{dy}{dt} = -x + y$. Whatever quantities these equations describe, certain details should be obvious from the nature of the equations. First of all, from Eq. (1), the growth of quantity x depends on itself and on the other quantity y in a positive way. On the other hand, Eq. (2) indicates that quantity y depends on itself positively, but its growth is hampered by the presence of quantity x—a larger value of x leads to a slowdown in the growth of y.

Let's look at the phase portrait corresponding to this problem. For our system, Eq. (6.4.2) looks like

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x + y}{x + y}.$$

This first-order equation is neither separable nor linear, but it is *homogeneous* (see the explanation for Problems 15–18 of Exercises 2.1) and can be solved implicitly. Fig. 6.5a shows several trajectories, obtained by specifying nine initial points $(x(0), y(0))$, superimposed on the slope field for $\frac{dy}{dx} = \frac{-x+y}{x+y}$. Because points on a trajectory are calculated by numerical methods, your CAS may allow you (or require you) to specify a step size and the actual numerical approximation method to be used. Numerical methods for systems of differential equations were discussed in Section 6.3.

Each point on a particular curve in Fig. 6.5a represents a *state* of the system. For each value of t, the point $(x(t), y(t))$ on the curve provides a snapshot of this dynamical system. If the variables x and y are supposed to represent animal or human populations, for example, then the proper place to view the trajectories is the first quadrant. Fig. 6.5b describes the first quadrant of the phase plane for our problem, with four trajectories determined by four initial points.

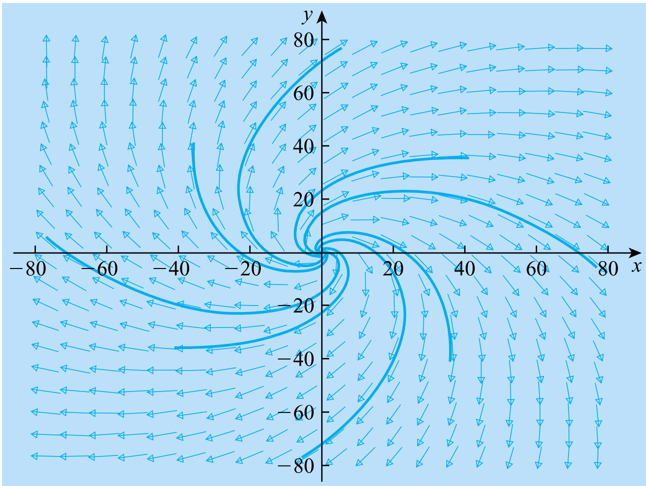


FIGURE 6.5a

Trajectories for $\left\{\frac{dx}{dt} = x + y,\ \frac{dy}{dt} = -x + y\right\}$; $(x(0), y(0)) = (-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0)$, and $(1, 1)$, $0 \le t \le 4$

Using technology again in our preceding example, we can graph $x(t)$ and $y(t)$ in the t-x and t-y planes, respectively. Figs. 6.6a and 6.6b show solution curves with $x(0) = 50$ and $y(0) = 140$, respectively.

These graphs show clearly that the quantity y reaches a maximum of about 164 when $t \approx 0.4$ and that the x quantity hits a peak of about 800 when $t \approx 2$. Note that the horizontal and vertical scales are different for Figs. 6.6a and 6.6b.

Viewed another way, the system in the preceding example had *three* variables: the independent variable t and the dependent variables x and y. To be precise about all this, we state that *a solution of our system is a pair of functions* $x = x(t)$, $y = y(t)$, *and the graphical representation of such a solution is a curve in three-dimensional* t-x-y *space—a set of points of the form* $(t, x(t), y(t))$. Fig. 6.7 shows what the solution with initial point (0, 50, 140) looks like for our problem.

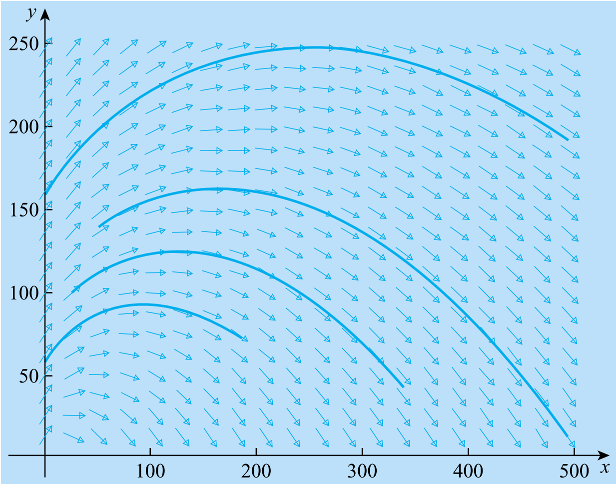


FIGURE 6.5b

Trajectories for $\left\{\frac{dx}{dt} = x + y,\ \frac{dy}{dt} = -x + y\right\}$; $(x(0), y(0)) = (0, 60)$, $(25, 100)$, $(50, 140)$, and $(0, 160)$, $0 \le t \le 1.2$

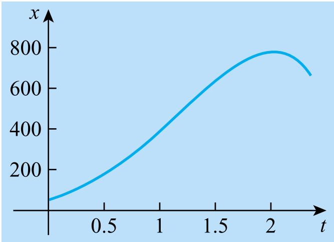


FIGURE 6.6a

$x(t)$; $x(0) = 50$, $0 \le t \le 2.355$

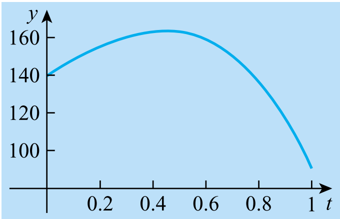


FIGURE 6.6b

$y(t)$; $y(0) = 140$, $0 \le t \le 1$

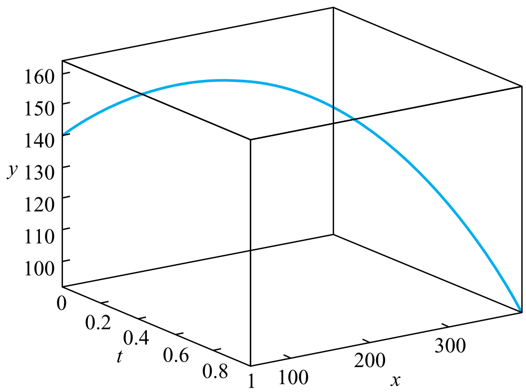


FIGURE 6.7

Solution of $\left\{\frac{dx}{dt} = x + y, \frac{dy}{dt} = -x + y; x(0) = 50,\ y(0) = 140\right\}, 0 \le t \le 1$

Your CAS may allow you to manipulate the axes and get different views of this space curve. Fig. 6.6a represents the *projection* of several such space curves onto the x-y plane, a much less confusing way of viewing the behavior of the system. These projections can be thought of as the shadows that would be cast by the space curves if a very bright light were shining on them from the front (the x-y face) of Fig. 6.7.

Note that because the system we started with in the preceding example is autonomous, the solution curves are *independent of the starting time*. This means that if you pick a starting point (x^*, y^*) at time t^*, then the path of a population starting at this point is the same as the path of a population starting at the same point at any other time t^{**}. Geometrically, this says that there is only one path (or trajectory) through each point of the x-y plane. This is a consequence of an Existence and Uniqueness Theorem for systems that we saw in Section 6.2. (Look back at Section 2.8 for the theorem that applies to first-order ODEs.)

These trajectories tell us that for the initial points chosen, the quantity y increases to a maximum value and then decreases to zero, while the quantity x also increases until it reaches its maximum level after quantity y has disappeared.

6.4.2 Equilibrium points

From the slope field and phase portrait in Fig. 6.5a, it seems clear that all trajectories (solution curves of the single differential equation) are escaping from the origin as t increases. The variable t is behind the scenes in a phase portrait, but you should experiment with different ranges of t in your CAS or graphing calculator to verify the preceding statement.

Recall that in Section 2.6, we found that *equilibrium* (or *constant*) *solutions* were important in the qualitative study of single first-order equations. We will see that the same is true for autonomous systems of equations. A point $(x_0,\ y_0)$ is an **equilibrium**

point (or **critical point**) of the autonomous system (6.4.1)—linear or nonlinear—if $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$. This is equivalent to saying that $x = x_0,\ y = y_0$ is a solution of (6.4.1). For example, it can be easily verified that the origin (0, 0) is an equilibrium point for the system in Example 6.4.2. Both physically and mathematically, *we should be interested primarily in the behavior of trajectories near equilibrium points*. A good deal of information about solutions of an autonomous system can be found without actually solving the system.

The algebra of finding equilibrium points (or equilibrium *solutions*) is trickier now because we must solve a *system* of algebraic equations. For example, if we want to find the equilibrium solutions of the nonlinear system of differential equations $\{\dot{x} = x - y,\ \dot{y} = 1 - xy\}$, we must solve the algebraic system

$$\begin{aligned} &(1) \quad x - y = 0 \\ &(2) \quad 1 - xy = 0. \end{aligned}$$

We can solve Eq. (1) for y, finding that $y = x$, and then substitute for y in the second equation. We get $1 - x^2 = 0$, which implies that $x = \pm 1$. Because $y = x$, the only equilibrium points are $(-1, -1)$ and $(1, 1)$.

Before we move on, let's look at the system $\left\{\dot{x} = 4 - 4x^2 - y^2,\ \dot{y} = 3xy\right\}$. Any equilibrium solution has to satisfy the equations

$$\begin{aligned} &(\text{A}) \quad 4 - 4x^2 - y^2 = 0 \\ &(\text{B}) \quad 3xy = 0. \end{aligned}$$

Eq. (B) tells us that we have two possibilities: (i) $x = 0$ or (ii) $y = 0$. [We can eliminate $x = y = 0$ because (A) wouldn't be satisfied with this choice.] If $x = 0$, substituting in (A) gives us $4 - y^2 = 0$, so $y = \pm 2$. Then we have two equilibrium solutions, (0, 2) and (0, −2). Alternatively, if $y = 0$, substituting in (A) yields $4 - 4x^2 = 0$, so $x = \pm 1$. Now we have the remaining two equilibrium solutions, $(1, 0)$ and $(-1, 0)$.

The next example presents a simple system model of an *arms race*. Models of this general form were proposed by the English scientist Lewis F. Richardson (1881–1953) in the 1930s. As a Quaker, he was greatly interested in the causes and avoidance of war. We'll see how a qualitative analysis helps us to understand the situation being modeled.

Example 6.4.3 An Arms Race Model

Let's look at a nonhomogeneous autonomous linear system:

$$\begin{aligned} \frac{dx}{dt} &= 7y - 4x - 13 \\ \frac{dy}{dt} &= 2x - 5y + 11. \end{aligned}$$

The functions $x(t)$ and $y(t)$ could represent the readiness for war of two nations, X and Y, respectively. This readiness might be measured, for example, in terms of the level of expenditures for weapons for each country at time t. To get the first equation, this model assumes that the rate of increase of x is a linear function of both x and y. In particular, if y increases, then so does the rate at which x increases. (*Clear?*) But the cost of building up and maintaining a supply of weapons also puts the brakes on *too* much expansion. The term $-4x$ in the first equation suggests a sense of restraint proportional to the arms level of nation X. Finally, the constant term -13 can represent some basic, constant relationship of nation X to nation Y—probably some underlying feelings of good will that diminish the threat and therefore decrease dependence on weaponry. The second equation can be interpreted in a similar way, but here the positive constant 11 probably signifies a grievance by Y against X that results in an accumulation of arms. Now what does this model tell us about the situation? We don't know how to solve such a system yet, but we can still learn a lot about the arms race between the two nations.

As in Example 6.4.1, we can start constructing the phase portrait of the system by eliminating the variable t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2x - 5y + 11}{7y - 4x - 13}.$$

Now we can look at the slope field and some trajectories corresponding to this single equation (Fig. 6.8). Several initial points were chosen. (*Try a smaller set of initial points yourself.*) For this to be a realistic model of an arms race, the values of x and y should be positive, and thus our focus is on the first quadrant.

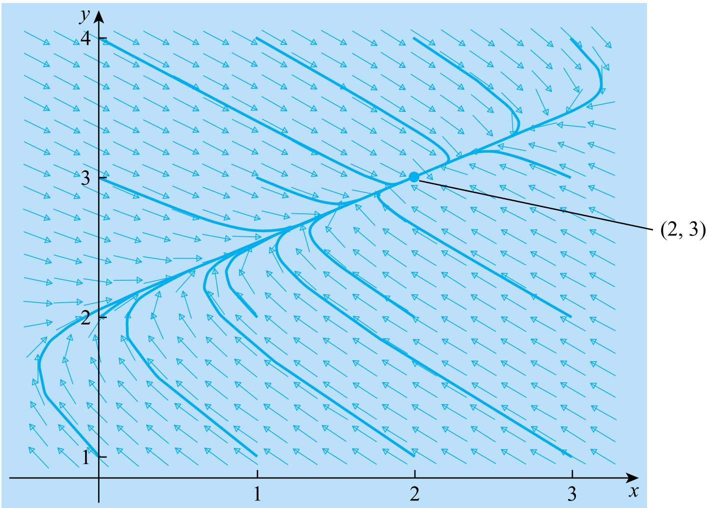


FIGURE 6.8

Trajectories for $\left\{\frac{dx}{dt} = 7y - 4x - 13, \frac{dy}{dt} = 2x - 5y + 11\right\}$. Initial points (i, j), $i = 0, 1, 2, 3$; $j = 1, 2, 3, 4$, $0 \le t \le 5$

First of all, note that one solution of the system is the pair of functions $x(t) \equiv 2$, $y(t) \equiv 3$. In this phase portrait, if we look hard enough, we notice that the points $(x(t), y(t))$ on every trajectory seem to be moving toward the point $(2, 3)$ as t increases. (*To verify this last statement, you should plot the phase portrait for* $0 \le t \le b$ *and let* b *increase.*) The point $(2, 3)$ is an equilibrium point—as we saw previously, a point (x, y) at which both dx/dt and dy/dt equal 0. The behavior of trajectories near this point entitles it to be called a **sink**. In real-life terms, this means that the arms race represented by this system would *stabilize* as time passes, approaching a state in which the level of military expenditures for nation X would be 2 and the level for nation Y would be 3, where the units could be millions or billions.

6.4.3 Three-dimensional systems

We have been focusing on second-order equations and their equivalent systems, but the techniques we have discussed apply to any differential equation of order greater than 1. The main difficulty with equations of order three and higher is that we lose some aspects of the graphical interpretation of the solution. Let's look at the next example, which presents us with a three-dimensional system.

Example 6.4.4 A System of Three First-Order Equations

We want to examine the behavior of the three-dimensional system

$$\dot{x} = -0.1x - y$$
$$\dot{y} = x - 0.1y$$
$$\dot{z} = -0.2z.$$

A Three-Dimensional Trajectory

The complete picture of this linear system is given by the set of points $(t, x(t), y(t), z(t))$, a *four-dimensional* situation. Assuming that x, y, and z are functions of the parameter t and that we have the initial condition $x(0) = 5$, $y(0) = 5$, and $z(0) = 10$, we get the three-dimensional trajectory shown in Fig. 6.9. This is a *projection* of the four-dimensional picture onto three-dimensional space, just as the two-dimensional phase portraits we saw previously are projections of three-dimensional curves onto two-dimensional planes.

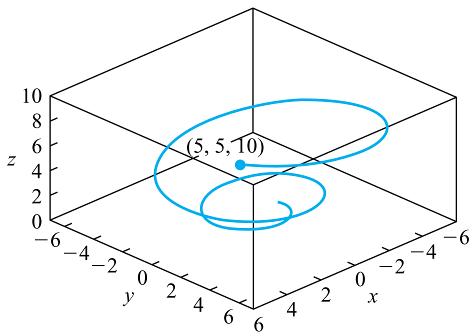


FIGURE 6.9

An x-y-z plane trajectory for the system $\{\dot{x} = -0.1x - y, \dot{y} = x - 0.1y, \dot{z} = -0.2z;\ x(0) = 5, y(0) = 5, z(0) = 10\}$, $0 \le t \le 15$

By plotting this with your CAS and rotating the axes (if possible), you should be able to see that the solution spirals into the origin in the x-y plane, while it moves toward the origin in the variable z as well.

A Two-Dimensional Trajectory

Now we can project the three-dimensional spiral (in x-y-z space) onto the x-y plane (Fig. 6.10).

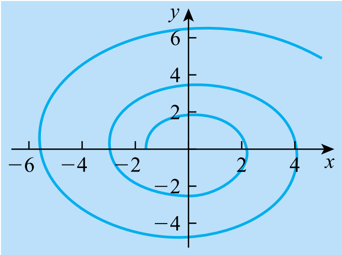


FIGURE 6.10

An x-y plane trajectory for the system $\{\dot{x} = -0.1x - y, \dot{y} = x - 0.1y, \dot{z} = -0.2z;\ x(0) = 5, y(0) = 5, z(0) = 10\}$, $0 \le t \le 15$

If we increase the range of t, we get a tighter spiral and we see that the origin is a *sink* for this system.

Exercises 6.4

A

Assume that each function in Problems 1–8 is a function of time, t. For each of these IVPs, (a) convert to a system, (b) use technology to find the graph of the solution in the phase plane, and (c) show a graph of the two components of the solution relative to the t-axis.

1. $x'' + x' = 0;\ x(0) = 1,\ x'(0) = 2$
2. $\ddot{r} - r = 0;\ r(0) = 0,\ \dot{r}(0) = 1$
3. $\ddot{y} + y = 0;\ y(0) = 2,\ \dot{y}(0) = 0$
4. $y'' = -4;\ y(0) = y'(0) = 0$
5. $\ddot{x} - \dot{x} = 0;\ x(0) = 1,\ \dot{x}(0) = 1$
6. $x'' - 2x' + x = 0;\ x(0) = -1,\ x'(0) = -1$
7. $x'' = x - x^3;\ x(0) = 0,\ x'(0) = 1$
8. $x'' + 0.5x' + 2x = 0;\ x(0) = 0,\ x'(0) = 1$

B

9. Read the explanation before Problem 15 of Exercises 2.1 and solve the equation

$$\frac{dy}{dx} = \frac{-x + y}{x + y}$$

that arises in Example 6.4.2.

10. Consider the following equations:

$$\frac{dx}{dt} = 0.2x - 0.002xy$$
$$\frac{dy}{dt} = -0.1y + 0.001xy.$$

a. Find the *first-order* differential equation that defines the *trajectories* of this system in the phase plane.

b. Solve this separable equation to find the implicit algebraic equation of the trajectories.

11. The equation $\ddot{Q} + 9\dot{Q} + 14\,Q = \frac{1}{2}\sin t$ models an electric circuit with resistance of 180 ohm, capacitance of 1/280 farad, inductance of 20 henry, and an applied voltage given by $E(t) = 10\sin t$. $Q = Q(t)$ denotes the capacitance, the charge on the capacitor at time t, and $\dot{Q}(t)$ denotes the current in the circuit. Assume $Q(0) = 0$ and $\dot{Q}(0) = 0.1$.

a. Express this IVP as a system of two first-order equations using the appropriate initial conditions.

b. Use technology to graph the solution of the system in the phase plane, with $0 \le t \le 8$.

c. Use technology to graph the solution of the original second-order equation relative to the t-axis, considering first the interval $0 \le t \le 2$ and then $0 \le t \le 8$.

d. Describe the behavior of the capacitance as $t \to \infty$.

12. Find all equilibrium solutions of each of the following systems.

a. $\dot{x} = x - 3y,\ \dot{y} = 3x + y$

b. $x' = 2x + 4y,\ y' = 3x + 6y$

c. $\dot{r} = -2rs + 1,\ \dot{s} = 2rs - 3s$

d. $x' = \cos y,\ y' = \sin x - 1$

e. $\dot{x} = x - y^2,\ \dot{y} = x^2 - y$

f. $r' = 1 - s,\ s' = r^3 + s$

13. Find all the equilibrium points of the system

$$x' = x^2y^3$$
$$y' = -x^3y^2$$

and sketch the phase portrait for this system.

14. Consider a population of two species of trout that compete for the same food supply. The system describing the populations for the species x and y is

$$\dot{x} = x(-2x - y + 180)$$

$$\dot{y} = y(-x - 2y + 120).$$

a. Find all equilibrium points of the system.

b. Sketch the phase portrait for this system.

C

15. Convert the equation $x'' + x' - x + x^3 = 0$ to a system and find all equilibrium points.

16. The equation $\frac{d^2\theta}{dt^2} + k^2 \sin\theta = 0$ describes the motion of an *undamped pendulum*, where θ is the angle the pendulum makes with the vertical. Convert this equation to a system and describe all its equilibrium points.

17. Consider the differential equation $x'' + \lambda - e^x = 0$, where λ is a parameter.

a. Sketch the phase-plane diagram for $\lambda > 0$.

b. Sketch the phase-plane diagram for $\lambda < 0$.

c. Describe the significance of the value $\lambda = 0$.

6.5 Systems and matrices

6.5.1 Matrices and vectors

Suppose we look at the homogeneous linear system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned} \tag{6.5.1}$$

where x and y are functions of t, and a, b, c, and d are constants.

There is a useful notation for linear systems that was invented by the English mathematician Arthur Cayley and named by his fellow countryman James Sylvester around 1850. This notation allows us to pick out the coefficients a, b, c, and d in system (6.5.1) and write them in a square array $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ called a **matrix**—in this case, the **matrix of coefficients** of the linear system. (The plural of *matrix* is *matrices*.) In general, a matrix is just a rectangular array of mathematical objects (numbers or functions in this book) and can describe linear systems of all sizes. The size of a matrix is given in terms of the number of its **rows** and **columns**. For example, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is described as a 2×2 matrix because it has two rows, $(a\,b)$ and $(c\,d)$, and two columns, $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$. The matrix $B = \begin{bmatrix} -4 & 0 & 1 & 5 \\ 2 & 6 & 7 & -\pi \\ 0 & \sqrt{5} & 3 & 5/9 \end{bmatrix}$ is a 3×4 matrix because it has three rows and four columns.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then we say the matrices are **equal** and write $A = B$ if $a = e$, $b = f$, $c = g$, and $d = h$. We say that the "corresponding elements must be equal."

In describing the linear system (6.5.1), we can also introduce a **column matrix** or **vector** $X = \begin{bmatrix} x \\ y \end{bmatrix}$. (This is a 2×1 matrix.) If $x(t)$ and $y(t)$ are solutions of the system (6.5.1), we call $X = \begin{bmatrix} x \\ y \end{bmatrix}$ a **solution vector** of the system. We can view X as a point in the x-y plane, or phase plane, whose coordinates are written vertically instead of in the usual horizontal ordered-pair configuration. If a vector is made up of constants, then the *direction* of the vector is taken as the direction of an arrow from the origin to the point (x, y) in the x-y plane. (See Section B.1 for more information.) If $V = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $W = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, we say that $V = W$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

If a vector (or, more generally, a matrix) is made up of objects (**elements** or **entries**) that are *functions*, we can define the *derivative* of such a vector as the vector whose elements are the derivatives of the original elements, provided that all these individual derivatives exist. For example, if $X(t) = \begin{bmatrix} -t^2 \\ \sin t \end{bmatrix}$, then

$$\frac{d}{dt} X(t) = \begin{bmatrix} \frac{d}{dt}(-t^2) \\ \frac{d}{dt}(\sin t) \end{bmatrix} = \begin{bmatrix} -2t \\ \cos t) \end{bmatrix}.$$

In particular, if $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ represents the solution vector of the planar system (6.5.1), then $X'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the **velocity vector** of the point $(x(t), y(t))$ as it moves along a trajectory. We note that velocity vectors have both magnitude and direction, whereas the line segments that make up a slope field only have slope. Another way of saying this is that the system (6.5.1) involves *time*, but the single ODE $dy/dx = (cx + dy)/(ax + by)$, which gives us a slope field, is the result of eliminating t from the original pair of equations.

6.5.2 The matrix representation of a linear system

We can write the system

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

compactly and symbolically as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or $\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The juxtaposition ("product") $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ represents the vector $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. For example, $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ x + 4y \end{bmatrix}$. There is a way to define and interpret this product of a matrix and a vector in the context of linear algebra (see Section B.3 for details), but **we will take this product as a symbolic representation of the system, highlighting the matrix of coefficients and the solution vector**. Soon we will see how a linear system's solutions—its behavior in the phase plane—are determined by the matrix of coefficients. For now, let's look at some examples of the use of matrix notation.

Example 6.5.1 Matrix Form of a Two-Dimensional Linear System

We can write the linear system of ODEs

$$\begin{aligned} \dot{x} &= -3x + 5y \\ \dot{y} &= x - 4y \end{aligned}$$

in matrix terms as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The next example demonstrates how we have to be careful in extracting the correct matrix of coefficients from a linear system problem.

Example 6.5.2 Matrix Form of a Two-Dimensional Linear System

The linear system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$ should be written as

$$\begin{aligned} \frac{dx}{dt} &= 0 \cdot x + 1 \cdot y \\ \frac{dy}{dt} &= -1 \cdot x + 0 \cdot y \end{aligned}$$

first. Then it is clear that the matrix representation of the system is

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

6.5.3 Some matrix algebra

Before we continue, we should discuss some properties of matrix algebra that we'll be using in the rest of this chapter. For example, if we're given the system

$$x' = -4x + 6y = 2(-2x + 3y)$$
$$y' = 2x - 8y = 2(x - 4y)$$

it is natural to write $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 2 & -8 \end{bmatrix} = 2\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$, or $X' = 2AX$, where $A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$. More generally, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and k is a constant (called a **scalar** to distinguish it from a vector or a matrix), then $kA = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$. In particular, for vectors, we have $k\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ku \\ kv \end{bmatrix}$. Put simply, *multiplying a matrix (vector) by a number requires multiplying each element of that matrix (vector) by the number*. For example, if $A = \begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}$ and $k = -2$, then

$$kA = -2A = \begin{bmatrix} -2(2) & -2(-3) \\ -2(5) & -2(0) \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -10 & 0 \end{bmatrix}.$$

Two matrices, A and B, of the same size (that is, having the same number of rows and the same number of columns) can be added element by element. For example, if $A = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} -2+1 & 3+2 \\ 4+3 & (-1)+4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 3 \end{bmatrix},$$
$$A - B = A + (-1)B = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -5 \end{bmatrix}.$$

Similarly, if $V = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $W = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, then $V + W = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$. The vector defined as $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is called the **zero vector**, behaves in the world of vectors the way the *number* 0 acts in arithmetic: $V + \mathbf{0} = V = \mathbf{0} + V$ for any vector V. We can define the **zero matrix**, $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, having the same property for matrix addition. Note that $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is an *equilibrium point* for the system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$—that is, if and only if X is a solution of the matrix equation $AX = \mathbf{0}$.

A particularly useful idea for our future work is a *linear combination* of vectors. Given two vectors $V = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $W = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, any vector of the form $k_1 V + k_2 W$, where k_1 and k_2 are scalars, is called a **linear combination** of V and W. In terms of our given vectors, a linear combination of V and W is any vector of the form $k_1 V + k_2 W = \begin{bmatrix} k_1 x_1 \\ k_1 y_1 \end{bmatrix} + \begin{bmatrix} k_2 x_2 \\ k_2 y_2 \end{bmatrix} = \begin{bmatrix} k_1 x_1 + k_2 x_2 \\ k_1 y_1 + k_2 y_2 \end{bmatrix}$. As an example, for the specific vectors $V = \begin{bmatrix} \sin t \\ 2 \end{bmatrix}$ and $W = \begin{bmatrix} \cos t \\ e^t \end{bmatrix}$, a linear combination has the form $\begin{bmatrix} k_1 \sin t + k_2 \cos t \\ 2k_1 + k_2 e^t \end{bmatrix}$.

It is important to know that the *associative* and *distributive rules* of algebra hold for matrix addition and the product of a matrix and a vector. For example, if A and B are matrices; V and W are vectors; and k, k_1, and k_2 are scalars, then

$$A(kV) = k(AV),$$
$$A(V + W) = AV + AW,$$

and

$$A(k_1 V + k_2 W) = A(k_1 V) + A(k_2 W) = k_1(AV) + k_2(AW).$$

These results are discussed further in Section B.3.

Finally, note that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as an *identity* for multiplication: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for any vector $\begin{bmatrix} x \\ y \end{bmatrix}$. In the context of two-dimensional systems, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix** and is denoted by I.

Shortly we will see how matrix notation gives us insight into the nature of a system's solutions. To understand the solutions more fully, we will introduce some additional concepts from linear algebra.

Exercises 6.5

A

1. Express each of the following systems of linear equations in matrix terms—that is, in the form $AX = B$, where A, X, and B are matrices.

   a. $3x + 4y = -7$
   $-x - 2y = 5$

b. $\pi a - 3b = 4$
$5a + 2b = -3$

c. $x - y + z = 7$
$-x + 2y - 3z = 9$
$2x - 3y + 5z = 11$

[Think about what would make sense in (c).]

2. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $W = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, calculate each of the following:

a. $2A - 3B$

b. AV

c. BW

d. $-2V + 5W$

e. $A(3V - 2W)$

f. $(A - 5I)W$

3. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $V = \begin{bmatrix} x \\ y \end{bmatrix}$, solve the equation $AV = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for V (i.e., find values for x and y).

4. Find the derivative of each of the following vectors:

a. $X(t) = \begin{bmatrix} t^3 - 2t^2 + t \\ e^t \sin t \end{bmatrix}$

b. $V(x) = \begin{bmatrix} 2\cos x \\ -3e^{-2x} \end{bmatrix}$

c. $B(u) = \begin{bmatrix} e^{-u} + e^u \\ 2\cos u - 5\sin u \end{bmatrix}$

d. $Y(t) = \begin{bmatrix} (t^2 + 1)e^{-t} \\ t\sin t \end{bmatrix}$

Convert each system of differential equations in Problems 5–10 to the matrix form $\dot{X} = AX$.

5. $\dot{x} = 2x + y$
$\dot{y} = 3x + 4y$

6. $\dot{x} = x - y$
$\dot{y} = y - 4x$

7. $\dot{x} = 2x + y$
$\dot{y} = 4y - x$

8. $\dot{x} = x$
$\dot{y} = y$

9. $\dot{x} = -2x + y$
$\dot{y} = -2y$

10. $\dot{x} - 8y + x = 0$
$\dot{y} - y - x = 0$

B

11. Using the technique shown in Section 6.1, write each of the following second-order equations as a system of first-order equations and then express the system in matrix form.

a. $y'' - 3y' + 2y = 0$

b. $5y'' + 3y' - y = 0$

c. $y'' + \omega^2 y = 0$, where ω is a constant.

12. Show that the origin is the only equilibrium point of the system

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy,$$

where a, b, c, and d are constants, with $ad - bc \neq 0$.

C

13. If $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$ and $B(t) = \begin{pmatrix} b_{11}(t) \\ b_{21}(t) \end{pmatrix}$ are a matrix and a vector having entries that are differentiable functions of t, show that

$$\frac{d}{dt}[A(t)B(t)] = \frac{dA(t)}{dt}B(t) + A(t)\frac{dB(t)}{dt}.$$

6.6 Two-dimensional systems of first-order linear equations

6.6.1 Eigenvalues and eigenvectors

To begin to understand linear systems of ODEs, including their qualitative behavior and their closed-form solutions, we will focus on linear two-dimensional systems of the form

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy, \end{aligned} \tag{6.6.1}$$

where x and y both depend on the variable t, and a, b, c, and d are constants. Our analysis of such simple (but important) systems will prepare us to understand the treatment of higher-order linear systems in Section 6.12.

First, let's write the system (6.6.1) in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or

$$\dot{X} = AX, \tag{6.6.2}$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Ignoring the fact that the capital letters represent matrices, what does the form of Eq. (6.6.2) remind you of? Have you seen a differential equation of this form before? If we use lowercase letters and write the equation as $\dot{x} = ax$, we get a familiar separable equation representing exponential growth or decay. (See Section 2.1, especially Example 2.1.2.) This observation suggests that the solution of system (6.6.1) or the matrix equation (6.6.2) may have something to do with exponentials.

Let's make a shrewd guess and then examine the consequences of our guess. In particular, let us assume that $x(t) = c_1 e^{\lambda t}$ and $y(t) = c_2 e^{\lambda t}$ for constants λ, c_1, and c_2. (Stating that λ, the coefficient of t, is the same for both x and y is a simplifying assumption.) Substituting $\begin{bmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix}$ for X in (6.6.2), we get

$$\begin{bmatrix} c_1 \lambda e^{\lambda t} \\ c_2 \lambda e^{\lambda t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or

$$\lambda e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

or (dividing out the exponential factor and switching the right and left sides)

$$A\tilde{X} = \lambda \tilde{X}, \tag{6.6.3}$$

where $\tilde{X} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Note that our reasonable guess about x and y has allowed us to replace our original differential equation problem with a pure algebra problem. Eq. (6.6.3) is in matrix terms and has nothing (apparently) to do with differential equations. Given a 2×2 matrix A and a 2×1 column matrix $\tilde{X}$, we can try to solve (6.6.3) for the value or values of λ, each called a **characteristic value** or **eigenvalue** of the matrix A. Eigenvalues will play an important role in solving linear systems and in understanding the qualitative behavior of solutions.

Furthermore, if we have solved Eq. (6.6.3) for its eigenvalues λ, then for each value of λ we can solve (6.6.3) for the corresponding vector or vectors $\tilde{X}$. Each such *nonzero* vector $\tilde{X}$ is called an **eigenvector** (or **characteristic vector**) of the system. We see that if both entries of $\tilde{X}$ are zero, then $\tilde{X}$ satisfies (6.6.3) for any value of λ, but this is the trivial case. **In all the discussion that follows, we will assume that c_1 and c_2 are not both zero**—that is, at least one of the two constants is not zero.

Before getting involved in symbolism, terminology, and the general problem of solving the matrix equation $A\tilde{X} = \lambda\tilde{X}$, let's look at a specific example in detail.

Example 6.6.1 Solving a Linear System with Eigenvalues and Eigenvectors

Suppose we have the system

$$\begin{aligned} \dot{x} &= -2x + y \\ \dot{y} &= -4x + 3y, \end{aligned} \tag{*}$$

which we can write as $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. We want to find the general solution of this system.

Substitution

Assuming, say, that $c_1 \neq 0$, we substitute $x = c_1 e^{\lambda t}$ and $y = c_2 e^{\lambda t}$ into (*) and get $\lambda c_1 e^{\lambda t} = -2c_1 e^{\lambda t} + c_2 e^{\lambda t} = e^{\lambda t}(-2c_1 + c_2)$ and $\lambda c_2 e^{\lambda t} = -4c_1 e^{\lambda t} + 3c_2 e^{\lambda t} = e^{\lambda t}(-4c_1 + 3c_2)$. If we simplify each equation by dividing out the exponential term and moving all terms to the left-hand side, we get

$$\begin{aligned} &(A)\ (\lambda + 2)c_1 - c_2 = 0 \\ &(B)\ 4c_1 + (\lambda - 3)c_2 = 0. \end{aligned} \tag{**}$$

Now we want to solve (**) for λ.

Solving for λ

If we multiply Eq. (A) by $(\lambda - 3)$ and then add the resulting equation to (B), we get $(\lambda - 3)(\lambda + 2)c_1 + 4c_1 = 0$, or $(\lambda^2 - \lambda - 2)c_1 = 0$. Because we have assumed that c_1 is not zero, we must have $\lambda^2 - \lambda - 2 = 0$. This means that the eigenvalues of A are $\lambda = 2$ and $\lambda = -1$. (*Go through all the algebra carefully.*) Note that we didn't have to know c_1 to find λ. We just had to know that c_1 was not zero. It is important that we could have assumed that c_2 was not zero and come to the same conclusion. (*Check this.*)

Solving the System of ODEs

If we take the eigenvalue $\lambda = 2$, we have $x(t) = c_1 e^{2t}$ and $y(t) = c_2 e^{2t}$, so that $X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. But when $\lambda = 2$, the equations in (**) both represent the single equation $4c_1 - c_2 = 0$, so that we have the relation $c_2 = 4c_1$. Then we can write $X_1 = e^{2t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{2t} \begin{bmatrix} c_1 \\ 4c_1 \end{bmatrix} =$

$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{2t}$, which is a one-parameter family of solutions of the system (*). What we mean is that the pair of functions $x(t) = c_1 e^{2t}$ and $y(t) = 4c_1 e^{2t}$ is a nontrivial solution of our system for any nonzero constant c_1.

Similarly, if we take the eigenvalue $\lambda = -1$, then the system (**) reduces to the single equation $c_1 - c_2 = 0$ and we can define $X_2 = e^{-t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{-t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$, which is also a one-parameter family of solutions of the system. In other words, the pair of functions $x(t) = c_1 e^{-t}$ and $y(t) = c_1 e^{-t}$ is also a nontrivial solution of our system for any nonzero constant c_1.

It is easy to see that the Superposition Principle we have been using since Chapter 2 allows us to conclude that

$$X = k_1 X_1 + k_2 X_2 = k_1 \left[c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{2t} \right] + k_2 \left[c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \right] = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

is the general solution of the system $\dot{x} = -2x + y$, $\dot{y} = -4x + 3y$. The constants C_1 and C_2 can be chosen to match arbitrary initial data.

6.6.2 Geometric interpretation of eigenvectors

The vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ that appears in the preceding example is called an **eigenvector** (or **characteristic vector**) corresponding to the eigenvalue (or characteristic value) $\lambda = 2$. This vector is a *nonzero* solution, $\tilde{X}$, of $A\tilde{X} = \lambda\tilde{X}$ when $\lambda = 2$. This means that there are infinitely many eigenvectors corresponding to the eigenvalue $\lambda = 2$—that is, all the vectors $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that $4c_1 - c_2 = 0$, or all the nonzero vectors of the form $\begin{bmatrix} c_1 \\ 4c_1 \end{bmatrix}$ are eigenvectors associated with $\lambda = 2$. Choosing $c_1 = 1$ gives us the simple particular vector $V_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, which can be called the *representative* eigenvector. Graphically, this eigenvector represents a straight line from the origin to the point $(1, 4)$ in the c_1-c_2 plane. Similarly, the vector $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the representative eigenvector corresponding to the eigenvalue $\lambda = -1$ and can be interpreted as a straight line from $(0, 0)$ to $(1, 1)$ in the c_1-c_2 plane. (See the description of vectors in Section B.1.) Fig. 6.11 shows V_1 and V_2 in the c_1-c_2 plane.

6.6.3 The general problem

Now let's consider the equation $A\tilde{X} = \lambda\tilde{X}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\tilde{X} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and at least one of the numbers c_1 and c_2 is nonzero. In the discussion that follows, we'll assume that $c_1 \neq 0$.

Written out as individual equations, $A\tilde{X} = \lambda\tilde{X}$ has the form

$$\begin{aligned} ac_1 + bc_2 &= \lambda c_1 \\ cc_1 + dc_2 &= \lambda c_2 \end{aligned}$$

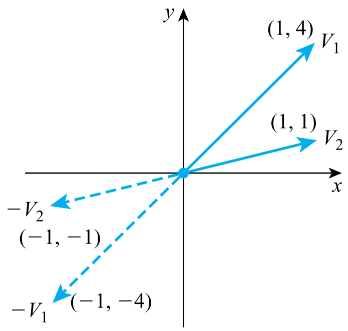


FIGURE 6.11

Representative eigenvectors $V_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

or

$$\begin{aligned} &\text{(A) } (a-\lambda)c_1 + bc_2 = 0 \\ &\text{(B) } cc_1 + (d-\lambda)c_2 = 0, \end{aligned}$$

and we want to determine λ.

We can solve this algebraic system by the method of elimination as follows:

1. Multiply Eq. (A) by $d - \lambda$ to obtain

$$(d-\lambda)(a-\lambda)c_1 + b(d-\lambda)c_2 = 0.$$

2. Multiply Eq. (B) by $-b$ to get

$$-bcc_1 - b(d-\lambda)c_2 = 0.$$

3. Add the equations found in steps 1 and 2 to get

$$(d-\lambda)(a-\lambda)c_1 - bcc_1 = 0,$$

or

$$\left[\lambda^2 - (a+d)\lambda + (ad-bc)\right]c_1 = 0. \text{ (\textit{Check the algebra.})}$$

4. Because we assumed that $c_1 \neq 0$ at the beginning of this discussion, we must have

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0. \quad (6.6.4)$$

This equation is called the **characteristic equation** of the matrix A, and its roots are the **eigenvalues** of A.

Using the quadratic formula, we find that

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

If we had assumed that $c_2 \neq 0$ at the beginning, we would have found the same solution for λ. Then for each distinct value of λ that we find, we can substitute that value into the system

$$\begin{aligned}(a-\lambda)c_1 + bc_2 &= 0\\ cc_1 + (d-\lambda)c_2 &= 0\end{aligned}$$

and solve for $\tilde{X} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, the corresponding eigenvector.

There are two things to notice about the characteristic equation of A,

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

and the resulting formula for λ:

1. The expression $a+d$ is just the sum of the *main diagonal* (upper left, lower right) elements of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In linear algebra, this is called the **trace** of A. For example, if $A = \begin{bmatrix} -7 & 2 \\ 0 & 4 \end{bmatrix}$, then the trace of A is $(-7)+4 = -3$.
2. The expression $ad - bc$ is formed from the matrix of coefficients $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as follows: Multiply the main diagonal elements and then subtract the product of the other diagonal elements (upper right, lower left). The number calculated this way is called the **determinant** of the coefficient matrix. Symbolically, $\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. For example, if $A = \begin{bmatrix} -7 & -3 \\ 2 & 4 \end{bmatrix}$, then $\det(A) = (-7)(4) - (-3)(2) = -28 - (-6) = -28 + 6 = -22$. The determinant of a matrix A is often denoted by the symbol $|A|$, so the rule for calculation in the 2×2 case can be given as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Observations 1 and 2 provide us with an alternative way of viewing the characteristic equation:

$$\lambda^2 - (\text{trace of } A)\lambda + \det(A) = 0. \tag{6.6.5}$$

The roots of the characteristic Eq. (6.6.5)—the eigenvalues—lead to eigenvectors and ultimately to the general solution of a linear system. Let's look at an example using this shortcut.

Example 6.6.2 Solving a Linear System with Eigenvalues and Eigenvectors

The following equations constitute a simple model for detecting diabetes:

$$\frac{dg}{dt} = -2.92g - 4.34h$$
$$\frac{dh}{dt} = 0.208g - 0.780h,$$

where $g(t)$ denotes excess glucose concentration in the bloodstream and $h(t)$ represents excess insulin concentration. “Excess” refers to concentrations above the equilibrium values. We want to determine the solution at any time t.

Eigenvalues

The matrix form of our equations is $\frac{d}{dt}X = \begin{bmatrix} dg/dt \\ dh/dt \end{bmatrix} = \begin{bmatrix} -2.92 & -4.34 \\ 0.208 & -0.780 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}$, so that the matrix of coefficients is $A = \begin{bmatrix} -2.92 & -4.34 \\ 0.208 & -0.780 \end{bmatrix}$. The *trace* of A is $-2.92 + (-0.780) = -3.7$ and the *determinant* of A is $-2.92(-0.780) - (-4.34)(0.208) = 3.18032$. The characteristic equation is

$$\lambda^2 - (\text{trace of } A)\lambda + \det(A) = \lambda^2 + 3.7\lambda + 3.18032 = 0.$$

Solving this by calculator or CAS, we find that the eigenvalues are $\lambda_1 = -2.34212$ and $\lambda_2 = -1.35788$, rounded to five decimal places.

Eigenvectors

Now we substitute each eigenvalue in the equations

$$(a - \lambda)c_1 + bc_2 = 0$$
$$cc_1 + (d - \lambda)c_2 = 0$$

and solve for the corresponding eigenvector $\tilde{X} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. In our problem, we must substitute in the equations

$$(-2.92 - \lambda)c_1 - 4.34c_2 = 0$$
$$0.208c_1 + (-0.780 - \lambda)c_2 = 0.$$

If $\lambda = -2.34212$, then the equations are

$$-0.57788c_1 - 4.34c_2 = 0$$
$$0.208c_1 + 1.56212c_2 = 0.$$

But these two equations are really only one distinct equation, $c_2 = -0.13315c_1$. (*Solve each equation for c_2 and see for yourself.*) Therefore, to ensure that at least one element of the eigenvector is an integer, we can take $c_1 = 1$ and $c_2 = -0.13315$, so that an eigenvector corresponding to the eigenvalue $\lambda = -2.34212$ is

$$\tilde{X}_1 = \begin{bmatrix} 1 \\ -0.13315 \end{bmatrix}.$$

Similarly, if we use the other eigenvalue, $\lambda = -1.35788$, we can take the single equation $(-2.92 - \lambda)c_1 - 4.34c_2 = 0$ and substitute the eigenvalue to get $(-2.92 + 1.35788)c_1 - 4.34c_2 = 0$, so that $c_2 = -0.35994c_1$. If we take $c_1 = 1$, we must have $c_2 = -0.35994$, and an eigenvector corresponding to the eigenvalue $\lambda = -2.34212$ is

$$\tilde{X}_2 = \begin{bmatrix} 1 \\ -0.35994 \end{bmatrix}.$$

The Solution

The Superposition Principle gives the general solution as

$$\tilde{X} = C_1\tilde{X}_1 + C_2\tilde{X}_2 = C_1 \begin{bmatrix} 1 \\ -0.13315 \end{bmatrix} e^{-2.34212t} + C_2 \begin{bmatrix} 1 \\ -0.35994 \end{bmatrix} e^{-1.35788t}.$$

If we were given initial concentrations of glucose and insulin, we could determine the constants C_1 and C_2. (See Problem 12 in Exercises 6.6.)

6.6.4 The geometric behavior of solutions

In the next few examples, we will get a preview of how the behavior of a two-dimensional system of linear differential equations with constant coefficients depends on the eigenvalues and eigenvectors of its matrix of coefficients. We'll illustrate some typical phase portraits. Then in Sections 6.7–6.9 we'll give a systematic description of all the possible behaviors of these linear systems using the nature of their eigenvalues and eigenvectors.

Example 6.6.3 Example 6.6.1 Revisited—A Saddle Point

Let's look again at the system from Example 6.6.1:

$$\dot{x} = -2x + y$$
$$\dot{y} = -4x + 3y.$$

As we saw earlier, the eigenvalues of this system are $\lambda_1 = 2$ and $\lambda_2 = -1$, with corresponding representative eigenvectors $V_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution was given by

$$X = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} C_1e^{2t} + C_2e^{-t} \\ 4C_1e^{2t} + C_2e^{-t} \end{bmatrix}.$$

Fig. 6.12 shows some trajectories for this system of linear equations. These are particular solutions of $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4x+3y}{-2x+y}$. Note in particular that the lines $y = 4x$ and $y = x$ appear as trajectories. These trajectories are actually four *half-lines*: $y = 4x$ for $x > 0$, $y = 4x$ for $x < 0$, $y = x$ for $x > 0$, and $y = x$ for $x < 0$.

A little basic algebra shows us that the origin is the only equilibrium point, and it is called a **saddle point** in this situation. A saddle point is the two-dimensional version of the *node* we discussed in Section 2.6. What characterizes a saddle point is that solutions can approach the equilibrium point along one direction (as though it were a *sink*), yet move away from the point in another direction

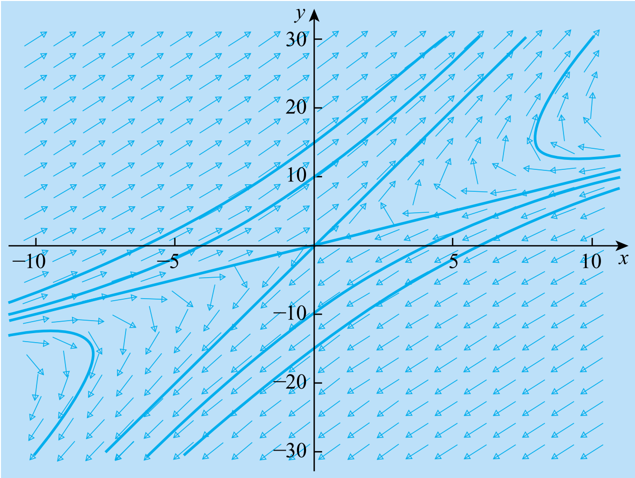


FIGURE 6.12

Phase portrait of the system $\dot{x} = -2x + y$, $\dot{y} = -4x + 3y$

(as though it were a *source*).[5] In particular, it turns out that one trajectory is the half-line $y = 4x$ in the first quadrant, along which the motion is *away* from the origin, and another trajectory is the line $y = x$ also in the first quadrant, along which the movement is *toward* the origin. The straight lines $y = 4x$ and $y = x$ are *asymptotes* for the other trajectories (as $t \to \pm\infty$). You may not be able to see this clearly from the phase portrait that your graphing utility generates unless you play with the range of t and choose initial values carefully, but you can see this and other behavior *analytically* (see Problem 13 in Exercises 6.6).

Example 6.6.4 A Source

Now let's look at the system of differential equations

$$\dot{x} = 2x + y$$
$$\dot{y} = 3x + 4y.$$

First of all, note that the system's only equilibrium point—where $\dot{x} = 0$ and $\dot{y} = 0$—is the origin of the phase plane, $(x, y) = (0, 0)$. (You should verify this using the ordinary algebra of simultaneous equations.)

Using the formula given by Eq. (6.6.4), we see that the characteristic equation of our system is $\lambda^2 - (2+4)\lambda + ((2)(4) - (1)(3)) = 0$, or $\lambda^2 - 6\lambda + 5 = 0$, which has the roots $\lambda_1 = 5$ and $\lambda_2 = 1$. To find the eigenvectors corresponding to these eigenvalues, we must solve the matrix equation

[5] This terminology is usually seen in a multivariable calculus course: If you look at a horse's saddle in the tail-to-head direction, it appears that the center of the saddle is lower than the front or back, so that the center seems to be a *minimum* point on the saddle's surface. However, if you look *across* the saddle from one side of the horse, it appears that the center is at the peak of a stirrup-to-stirrup curve, so the center seems like a *maximum* point. In fact, a *saddle point* is neither a minimum nor a maximum.

$A\tilde{X} = \lambda\tilde{X}$, where $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $\lambda = 5$ or 1, and $\tilde{X} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. This matrix equation is equivalent to the system

$$\begin{aligned} &(1)\ (2-\lambda)c_1 + c_2 = 0 \\ &(2)\ 3c_1 + (4-\lambda)c_2 = 0. \end{aligned} \tag{6.6.6}$$

Substituting the first eigenvalue, $\lambda = 5$, in (6.6.6) gives us

$$\begin{aligned} &(1)\ -3c_1 + c_2 = 0 \\ &(2)\ 3c_1 - c_2 = 0. \end{aligned}$$

There is really only one equation here, and its solution is given by $c_2 = 3c_1$. Thus, the eigenvectors corresponding to the eigenvalue $\lambda = 2$ have the form $\begin{bmatrix} c_1 \\ 3c_1 \end{bmatrix}$, or $c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. If we let $c_1 = 1$, we get the "neat" representative eigenvector $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

When we use the other eigenvalue, $\lambda = 1$, in the system (6.6.6), we find that

$$\begin{aligned} &(1)\ c_1 + c_2 = 0 \\ &(2)\ 3c_1 + 3c_2 = 0, \end{aligned}$$

which has the solution $c_2 = -c_1$. Therefore, the eigenvectors in this case have the form $\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix}$, or $c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, our representative eigenvector can be $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

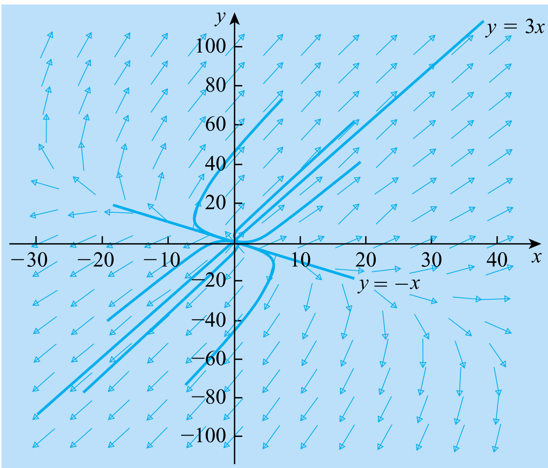


FIGURE 6.13

Phase portrait of the system $\dot{x} = 2x + y,\ \dot{y} = 3x + 4y$

Now let's look at the phase portrait corresponding to the original system, a family of trajectories corresponding to the first-order equation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3x + 4y}{2x + y}.$$

This phase portrait is shown in Fig. 6.13. The curves $y = 3x$ and $y = -x$, which are straight-line trajectories, are labeled so that we can see the significance of the eigenvectors.

If you look carefully (or find your own phase portrait), you will see that the trajectories shown are fleeing the origin (as $t \to \infty$) in such a way that any trajectory is tangent to the line $y = -x$ at the origin—that is, as $t \to -\infty$. In this situation, the origin is called an **unstable node** (specifically, a **source** or **repeller**).

The next example reveals another type of source for a two-dimensional system.

Example 6.6.5 A Spiral Source

Look at the system

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = -4x + y.$$

Note that once again the origin is this system's only equilibrium point. (*Check this for yourself.*) Because the matrix of coefficients has $a = 1$, $b = 1$, $c = -4$, and $d = 1$, we use formula (6.6.4) to

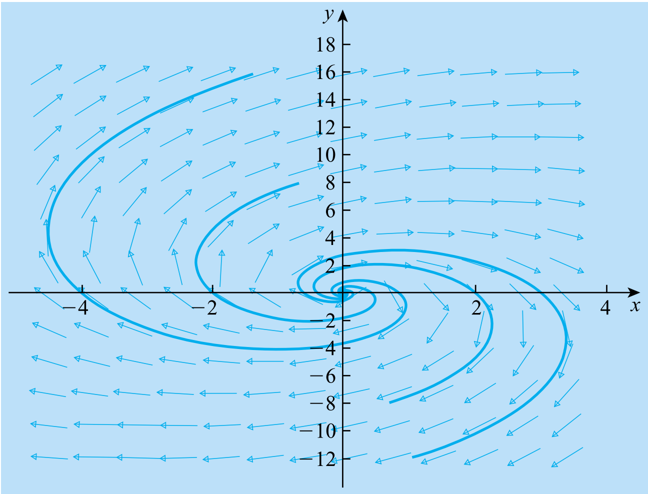


FIGURE 6.14

Phase portrait of the system $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = -4x + y$; $(x(0), y(0)) = (-4, 0)$, $(-2, 0)$, $(2, 0)$, $(3, 0)$; $-5 \le t \le 0.7$

determine that the characteristic equation of this system is $\lambda^2 - 2\lambda + 5 = 0$, so the quadratic formula gives us the eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. When we get complex eigenvalues such as this complex conjugate pair, the eigenvectors will turn out to have complex numbers as entries and to have no useful direct geometric significance.[6] We'll deal with this situation in more detail in Section 6.9. The phase portrait for this system is shown in Fig. 6.14.

We can see that the trajectories are spirals that move outward, away from the equilibrium point, in a clockwise direction. In this case, as in the previous example, the equilibrium point is called a *source* (or a *repeller*). Other systems with complex eigenvalues may correspond to spirals that move in a *counterclockwise* direction or to spirals that move in *toward* the equilibrium point (clockwise or counterclockwise).

These examples should convince you that trajectories can behave quite differently near equilibrium points. In the next section, we will examine how the trajectories of a two-dimensional system can be classified.

Exercises 6.6

A

1. Calculate the determinant of each of the following matrices by hand:

a. $\begin{bmatrix} -3 & 5 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 2 \\ 10 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 6t & -4 \\ \sin t & t^3 \end{bmatrix}$

d. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

2. Find the eigenvalues and eigenvectors of matrices (a) and (b) in Problem 1.

3. Find a 2×2 matrix with eigenvalues 1 and 3 and corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For each system in Problems 4–10, (a) convert to the matrix form $\dot{X} = AX$; (b) find the characteristic equation; (c) find all eigenvalues; (d) describe all eigenvectors corresponding to each eigenvalue found in part (c). Parts (a)–(d) should be done without the aid of a calculator or CAS.

4. $\dot{x} = -x + 4y$
$\dot{y} = 2x - 3y$

5. $\dot{x} = x - y$
$\dot{y} = y - 4x$

[6] See Section 4.4 of *Applied Linear Algebra* by L. Sadun (Upper Saddle River, NJ: Prentice Hall, 2001).

6. $\dot{x} = -4x + 2y$
$\dot{y} = 2x - y$

7. $\dot{x} = x$
$\dot{y} = y$

8. $\dot{x} = -6x + 4y$
$\dot{y} = -3x + y$

9. $\dot{x} = 5x - y$
$\dot{y} = 2x + y$

10. $\dot{x} = 4x - 6y$
$\dot{y} = 3x - 7y$

B

11. Consider the algebraic system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned}$$

where a, b, c, d, e, and f are constants, with $ad - bc \neq 0$.

a. Show that the solution is given by $x = \frac{de-bf}{ad-bc}$, $y = \frac{af-ce}{ad-bc}$.

b. Express your solution in part (a) in terms of the determinants

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix}, \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \text{ and } \begin{vmatrix} e & b \\ f & d \end{vmatrix}.$$

12. In Example 6.6.2, find the solution of the system satisfying the initial conditions $g(0) = g_0$ and $h(0) = 0$. (You may use technology to solve the resulting system of algebraic equations.)

13. In Example 6.6.1, the system $\dot{x} = -2x + y$, $\dot{y} = -4x + 3y$ was shown to have the solution

$$X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1e^{2t} + C_2e^{-t} \\ 4C_1e^{2t} + C_2e^{-t} \end{bmatrix}.$$

a. Substitute for $x(t)$ and $y(t)$ in the right-hand side of the expression

$$\frac{dy}{dx} = \frac{-4x + 3y}{-2x + y}.$$

b. Use the result of part (a) to show that the slope of any trajectory not on either of the lines determined by the eigenvectors approaches 4, the slope of the eigenvector corresponding to the larger of the two distinct eigenvalues. [*Hint*: Factor out e^{2t}, the dominant term for large positive values of t.]

c. Use the result of part (a) to show that the slope of any trajectory not on either of the lines determined by the eigenvectors approaches 1, the slope of the eigenvector corresponding to the smaller of the two eigenvalues. [*Hint*: Factor out e^{-t}, the dominant term for large negative values of t.]

14. Use technology to sketch the phase portrait of the system in Problem 13. Then sketch in the eigenvectors (getting them from the answers in the back of the book if necessary) and comment on the behavior of the trajectories with respect to the origin. Use both positive and negative values of t.

15. Use technology to sketch the phase portrait of the system $\{\dot{x} = -0.1x + 0.075y,\ \dot{y} = 0.1x - 0.2y\}$. Then sketch in the eigenvectors (using your CAS if necessary) and comment on the behavior of the trajectories with respect to the origin. Use both positive and negative values of t.

16. A substance X decays into substance Y at rate $k_1 > 0$, and Y in turn decays into another substance at rate $k_2 > 0$. The system

$$\frac{dx}{dt} = -k_1 x$$
$$\frac{dy}{dt} = k_1 x - k_2 y$$

describes the process, where $x(t)$ and $y(t)$ represent the amount of X and Y, respectively. Assume that $k_1 \neq k_2$.

a. Find the eigenvalues of the system.

b. Find the eigenvectors corresponding to each of the eigenvalues found in part (a).

c. Solve for $x(t)$ and $y(t)$ and then find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$, interpreting your answers in physical terms.

17. The following system models the exchange of nutrients between mother and fetus in the placenta:

$$\frac{dc_1}{dx} = -\alpha_1(c_1 - c_2)$$
$$\frac{dc_2}{dx} = -\alpha_2(c_1 - c_2),$$

where $c_1(x)$ is the concentration of nutrient in the maternal bloodstream at a distance x along the placental membrane and $c_2(x)$ is the concentration of nutrient in the fetal bloodstream at a distance x. Here, α_1 and α_2 are constants, $\alpha_1 \neq \alpha_2$.

a. If $c_1(0) = c_0$ and $c_2(0) = C_0$, use eigenvalues and eigenvectors to solve the system for $c_1(x)$ and $c_2(x)$.

b. Solve for $c_1(x)$ and $c_2(x)$ by converting the system into a single second-order differential equation and using the techniques of Section 4.1.

18. Consider the spring-mass system described by $\ddot{x} + b\dot{x} + kx = 0$.

a. Find all values of b and k for which this system has real, distinct eigenvalues.

b. Find the general solution of the system for the values of b and k found in part (a).

c. Find the solution of the system that satisfies the initial condition $x(0) = 1$.

d. Describe the motion of the mass in the situation described in part (c).

19. The behavior of a *damped pendulum* is described near the lowest point of its trajectory by the linear equation $\ddot{x} = -k\dot{x} - gx/L$, where k is the damping coefficient, g is the acceleration due to gravity, and L is the length of the pendulum.

a. Express the differential equation as a linear system.

b. Find the characteristic equation of the system.

c. Find all eigenvalues of the system.

d. Describe all eigenvectors.

e. If $k^2 > 4g/L$, solve the system and state what happens as $t \to \infty$.

20. Using *Newton's law of cooling* (see Problem 22, Exercises 2.1), the flow of heat in a two-story house just after the furnace fails can be modeled by the IVP

$$\begin{aligned}\dot{x} &= -0.7x + 0.5y\\ \dot{y} &= 0.5x - 0.6y;\ \ x(0) = 70,\ \ y(0) = 60,\end{aligned}$$

where $x(t)$ and $y(t)$ are the temperatures downstairs and upstairs, respectively, at time t (in hours).

a. Find the characteristic equation of the system.

b. Find all eigenvalues of the system.

c. Find all eigenvectors of the system.

d. Determine the values $x(2)$ and $y(2)$.

e. Determine $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

C

21. Consider the system $\dot{x} = Ax$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that the trajectories spiral as in Example 6.6.5, possibly in the opposite direction. The polar form (Section B.1) of a spiral trajectory provides the *polar angle* $\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$; the direction of the spiral will be clockwise if $d\theta/dt < 0$ and counterclockwise if $d\theta/dt > 0$. Show that the direction of a spiral trajectory depends on the sign of c, the lower left-hand entry of the matrix A, as follows.

a. Show that $\det(A) = ad - bc > 0$.

b. Show that (trace of $A)^2 < 4(\det(A))$—that is, $(a+d)^2 < 4(ad-bc)$.

c. Show that $d\theta/dt = (x\dot{y} - y\dot{x})/(x^2 + y^2)$, and that the sign of $d\theta/dt$ equals the sign of $x\dot{y} - y\dot{x}$.

d. Show that

$$x\dot{y} - y\dot{x} = c\left[x + \left(\frac{d-a}{2c}\right)y\right]^2 + \frac{y^2}{4c}\left[4(ad-bc) - (a+d)^2\right]$$

and explain why the sign of $d\theta/dt$ equals the sign of c.

6.7 The stability of homogeneous linear systems: unequal real eigenvalues

First of all, we might be able to guess by now that a linear system $\dot{X} = AX$ of ODEs, where $\det(A) \neq 0$, has exactly one equilibrium point, $(0, 0)$. (See Problem 12 of Exercises 6.5.) If $\det(A) = 0$, however, the system may have many other equilibrium solutions. As promised in the preceding section, the *stability* of a system—the behavior of trajectories with respect to the equilibrium point(s)—will be explained completely in terms of the eigenvalues and eigenvectors of the matrix A.

Because the characteristic equation of a two-dimensional system is a quadratic equation, we know that there are two eigenvalues, λ_1 and λ_2. There are only three possibilities for these eigenvalues:

1. The eigenvalues are both real numbers with $\lambda_1 \neq \lambda_2$.
2. The eigenvalues are real numbers with $\lambda_1 = \lambda_2$.
3. The eigenvalues are complex numbers: $\lambda_1 = p + qi$ and $\lambda_2 = p - qi$, where p and q are real numbers (called the *real part* and the *imaginary part*, respectively) and $i = \sqrt{-1}$.

In situation 3, we say that λ_1 and λ_2 are *complex conjugates* of each other. (You may want to review Appendix C for more information about complex numbers.)

The nature of the eigenvalues will play an important role in the qualitative analysis of systems of linear equations. In this section we will deal with possibility 1, leaving situations 2 and 3 for the next two sections.

6.7.1 Unequal real eigenvalues

First, suppose that the matrix A in the system $\dot{X} = AX$ has two real eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Let V_1 and V_2 be the corresponding representative eigenvectors. Then we'll show that the general solution of the system is given by

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2, \qquad (6.7.1)$$

where c_1 and c_2 are arbitrary constants.

Geometrically, the first term on the right-hand side of (6.7.1) represents a straight-line trajectory parallel[7] to V_1, and the second term describes a line parallel to V_2 (see Fig. 6.15). Note that these trajectories lie in the phase plane (the x-y plane).

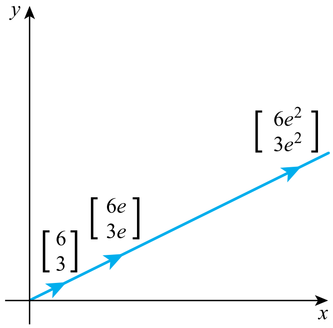


FIGURE 6.15

$V = 3e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $t = 0$, 1, and 2

If both c_1 and c_2 are nonzero, then the solution $X(t)$ is a linear combination of the two basic terms whose relative contributions change with time. In this situation, the trajectories curve in a way that will be described later.

To see why (6.7.1) is the general solution, first note that each term is itself a solution of the system. If, for example, we consider $X_1(t) = c_1 e^{\lambda_1 t} V_1$, then $\dot{X}_1(t) = c_1 \lambda_1 e^{\lambda_1 t} V_1$ and $AX_1 = A(c_1 e^{\lambda_1 t} V_1) = c_1 e^{\lambda_1 t}(AV_1) = c_1 e^{\lambda_1 t}(\lambda_1 V_1) = \lambda_1 c_1 e^{\lambda_1 t} V_1$ because V_1 is an eigenvector corresponding to λ_1. (See Section 6.5 for properties of matrix–vector multiplication.) Therefore, $\dot{X}_1(t) = AX_1$. Now we can see that if X_1 and X_2 are any solutions of the system, then the linear combination $X = k_1 X_1 + k_2 X_2$ is also a solution for any constants k_1 and k_2:

$$\begin{aligned}\dot{X} &= \overbrace{(k_1 X_1 + k_2 X_2)}^{\bullet} = k_1 \dot{X}_1 + k_2 \dot{X}_2 = k_1(AX_1) + k_2(AX_2)\\ &= A(k_1 X_1) + A(k_2 X_2)\\ &= A(k_1 X_1 + k_2 X_2) = AX.\end{aligned}$$

These steps follow from the algebraic properties of matrices, vectors, and of derivatives, and this property of solutions of linear systems is another version of the Superposition Principle that we have encountered several times before.

We can argue (somewhat loosely) that (6.7.1) represents a solution of a two-dimensional system (or its equivalent second-order equation) and has two arbitrary

[7] Two vectors V and W are **parallel** if $W = cV$ for some nonzero constant c. In other words, parallel vectors lie on the same straight line through the origin, pointing in the same direction (if $c > 0$) or in opposite directions (if $c < 0$). See Appendix B.1.

constants and hence is the *general* solution of the system $\dot{X} = AX$. To be rigorous, we can use the fact that any initial condition $X_0 = X(t_0) = \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ for the system can be written as a linear combination of the eigenvectors—$X_0 = k_1 V_1 + k_2 V_2$ for some constants k_1 and k_2—so a solution (6.7.1) can be found to satisfy any initial condition $X(t_0) = X_0$. (You'll be asked to prove these assertions in Problems 24 and 25 in Exercises 6.7.) Finally, Existence and Uniqueness Theorem II of Section 6.2 allows us to say that (6.7.1) is the *only* solution.

6.7.2 The impossibility of dependent eigenvectors

If one of the eigenvectors is a scalar multiple of the other—say V_2 is a multiple of V_1—then the expression in (6.7.1) collapses to a scalar multiple of V_1 and there is only one arbitrary constant. This expression can't represent the general solution of a second-order equation.

Fortunately, this collapse can't happen under our current assumption. It is easy to prove that if a 2×2 matrix A has distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors V_1 and V_2, then neither eigenvector is a scalar multiple of the other. Suppose that $V_2 = cV_1$, where c is a nonzero scalar. Then $V_2 - cV_1 = \mathbf{0}$, the zero vector, and we must have

$$\begin{aligned} \mathbf{0} &= A(V_2 - cV_1) = AV_2 - c(AV_1) = \lambda_2 V_2 - c(\lambda_1 V_1) \\ &= \lambda_2(cV_1) - c(\lambda_1 V_1) = c(\lambda_2 - \lambda_1)V_1. \end{aligned}$$

But then, because $c \neq 0$ and V_1 (as an eigenvector) is nonzero, we must conclude that $(\lambda_2 - \lambda_1) = 0$, which contradicts the assumption that we have distinct eigenvalues.

6.7.3 Unequal positive eigenvalues

In the expression for the general solution, $c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$, suppose that $\lambda_1 > \lambda_2 > 0$. First, note that as t increases, both eigenvector multiples point *away* from the origin so all solutions *grow* with time. (The algebraic signs of the constants c_1 and c_2 influence the quadrants in which the solutions grow.) To understand the *relative* rates at which the individual terms grow, we can factor out the exponential corresponding to the larger eigenvalue and write $X(t) = e^{\lambda_1 t}(c_1 V_1 + c_2 e^{(\lambda_2 - \lambda_1)t} V_2)$.

Note that $e^{(\lambda_2 - \lambda_1)t} \to 0$ as $t \to +\infty$ because $\lambda_2 - \lambda_1 < 0$. Therefore, $X(t) \approx e^{\lambda_1 t} c_1 V_1$ as t gets larger. Noting that $e^{\lambda_1 t} c_1 V_1$ is parallel to V_1, we see that the slope of any trajectory $X(t)$ approaches the slope of the line determined by V_1. This says that trajectories will curve *away* from the origin and their slopes will approach the slope of the line determined by the eigenvector V_1, corresponding to the larger eigenvalue. In this situation, the equilibrium point (0, 0) is called a **source** (**unstable node**, **repeller**). (Recall our discussions in Section 2.6.) In "backward time," as $t \to -\infty$, the trajectories will be asymptotic to the line determined by the eigenvector V_2 because then the first term in the linear combination $c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$ is approaching zero faster than the second term. This says that if we move *backward*, the trajectories *enter* the origin tangent to the line determined by V_2.

We are now ready to re-examine an earlier example in light of the preceding two paragraphs.

Example 6.7.1 Unequal Positive Eigenvalues: A Source

First of all, the system

$$\dot{x} = 2x + y$$
$$\dot{y} = 3x + 4y$$

that we saw in Example 6.6.4 has two positive unequal eigenvalues, $\lambda_1 = 5$ and $\lambda_2 = 1$, with corresponding eigenvectors $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore the general solution is

$$X(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} + c_2 e^{t} \\ 3c_1 e^{5t} - c_2 e^{t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Fig. 6.16 is a more detailed version of Fig. 6.13, the phase portrait of our system. The new graph shows several trajectories and the way in which they curve away from the origin, their slopes approaching the slope of the line determined by the eigenvector $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ corresponding to the larger eigenvalue $\lambda = 5$.

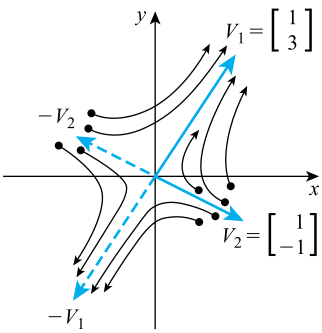


FIGURE 6.16

Trajectories of the system $\dot{x} = 2x + y$, $\dot{y} = 3x + 4y$. Bold points • indicate initial positions ($t = 0$) for trajectories

Analytically, we can examine the equation $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3x+4y}{2x+y}$, whose solutions make up the phase portrait—that is, the equation giving the slopes of trajectories in the x-y plane. Substituting $x(t) = c_1 e^{5t} + c_2 e^{t}$ and $y(t) = 3c_1 e^{5t} - c_2 e^{t}$ from the general solution given previously, we get

$$\frac{dy}{dx} = \frac{15c_1 e^{5t} - c_2 e^{t}}{5c_1 e^{5t} + c_2 e^{t}}.$$

For large values of t, the expression for $\frac{dy}{dx}$ is dominated by the e^{5t} terms, which we can factor out:

$$\frac{dy}{dx} = \frac{e^{5t}(15c_1 - c_2e^{-4t})}{e^{5t}(5c_1 + c_2e^{-4t})} = \frac{15c_1 - c_2e^{-4t}}{5c_1 + c_2e^{-4t}}.$$

The condition $c_1 = 0$ would mean that we are dealing with the straight-line trajectory determined by the eigenvector $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. But if $c_1 \neq 0$, as $t \to \infty$ we see that the slope of any trajectory tends to $\frac{15c_1 - 0}{5c_1 + 0} = 3$, the slope of the line determined by the eigenvector $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

If we consider large *negative* values of t—that is, if we run the trajectories backward in time—then e^t is the dominant term in the expression for $\frac{dy}{dx}$ and we can factor it out:

$$\frac{dy}{dx} = \frac{15c_1e^{5t} - c_2e^t}{5c_1e^{5t} + c_2e^t} = \frac{e^t\left(15c_1e^{4t} - c_2\right)}{e^t\left(5c_1e^{4t} + c_2\right)} = \frac{15c_1e^{4t} - c_2}{5c_1e^{4t} + c_2}.$$

The preceding expression tells us that if $c_2 \neq 0$, then as $t \to -\infty$ the slope of any trajectory tends to $\frac{0 - c_2}{0 + c_2} = -1$, the slope of the line determined by the eigenvector $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If we have $c_2 = 0$, we will be on the straight-line trajectory determined by the eigenvector $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

6.7.4 Unequal negative eigenvalues

If both eigenvalues are *negative* (say $\lambda_1 < \lambda_2 < 0$), then both eigenvector multiples point *toward* the origin, and all solutions *decrease* or *decay* with time. To see this, write (6.7.1) in the form

$$X(t) = \left[\frac{c_1}{e^{-\lambda_1 t}}\right]V_1 + \left[\frac{c_2}{e^{-\lambda_2 t}}\right]V_2 = \left[\frac{c_1}{e^{Kt}}\right]V_1 + \left[\frac{c_2}{e^{Mt}}\right]V_2,$$

where $K = -\lambda_1$ and $M = -\lambda_2$ are *positive* constants. Then clearly, both terms of $X(t)$ approach the origin as $t \to +\infty$. Because $\lambda_1 < \lambda_2$ we have $-\lambda_1 > -\lambda_2$, or $K > M$, so the first term in the expression for $X(t)$ approaches the origin faster than the second term. We will see in the next example that as t increases, trajectories curve *toward* the origin, closer to the eigenvector V_2 (or its negative if $c_2 < 0$), corresponding to the larger eigenvalue. Under these circumstances, we say that $(0, 0)$ is a **stable node**, or **sink**.

Example 6.7.2 Unequal Negative Eigenvalues: A Sink

Here we look at the system

$$\begin{aligned} \dot{x} &= -4x + y \\ \dot{y} &= 3x - 2y. \end{aligned}$$

The characteristic equation is $\lambda^2 + 6\lambda + 5 = 0$ and the eigenvalues are negative and unequal: $\lambda_1 = -5$ and $\lambda_2 = -1$. Using the linear algebra capabilities of a CAS, we find that the corresponding representative eigenvectors are $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. (Don't worry if your CAS produces eigenvectors that are different from these—yours should lie on the same line as the ones given here. Your slopes y/x should be -1 and 3.)

The general solution of our system is

$$X(t) = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -c_1 e^{-5t} + c_2 e^{-t} \\ c_1 e^{-5t} + 3c_2 e^{-t} \end{bmatrix}.$$

It is clear from the negative exponents in the expression for $X(t)$ that $X(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$, so the origin is a *sink*. Fig. 6.17 shows some typical trajectories and seems to indicate that the trajectories are tangent to the line determined by the eigenvector $V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

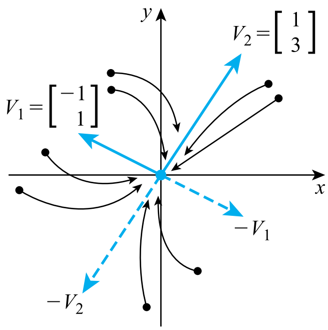


FIGURE 6.17

Trajectories for the system $\dot{x} = -4x + y$, $\dot{y} = 3x - 2y$. Bold points • indicate initial positions $(t = 0)$ for trajectories

Recognizing that e^{-t} is larger than e^{-5t} for large values of t, we look at

$$\frac{dy}{dx} = \frac{3x - 2y}{-4x + y} = \frac{-5c_1 e^{-5t} - 3c_2 e^{-t}}{5c_1 e^{-5t} - c_2 e^{-t}} = \frac{e^{-t}\left(-5c_1 e^{-4t} - 3c_2\right)}{e^{-t}\left(5c_1 e^{-4t} - c_2\right)} = \frac{-5c_1 e^{-4t} - 3c_2}{5c_1 e^{-4t} - c_2}.$$

If $c_2 \neq 0$, then $\frac{dy}{dx}$ approaches $\frac{-3c_2}{-c_2} = 3$, the slope of the eigenvector $V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, as $t \to \infty$. If $c_2 = 0$, then the trajectory is on the straight line determined by the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

6.7.5 Unequal eigenvalues with opposite signs

If the eigenvalues have *opposite* signs (say $\lambda_1 < 0 < \lambda_2$), then look at the general solution $X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$ to see that the term $c_1 e^{\lambda_1 t} V_1$ (corresponding to the negative eigenvalue λ_1) points *toward* the origin, whereas $c_2 e^{\lambda_2 t} V$ points *away* from the origin (Fig. 6.18).

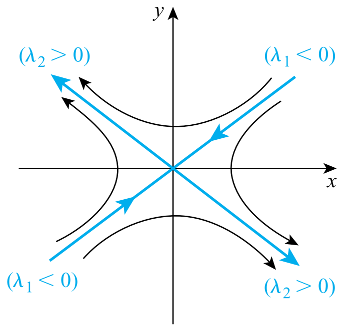


FIGURE 6.18

Typical eigenvectors for the case $\lambda_1 < 0 < \lambda_2$

In this case, trajectories *approach* the origin along one direction and veer *away* from the origin along another. In this situation we describe $(0, 0)$ as a **saddle point.** Look back at Example 6.6.3, especially Fig. 6.12.

Let's consider a new example of what happens when the eigenvalues of a system have opposite signs.

Example 6.7.3 Unequal Eigenvalues with Opposite Signs: A Saddle Point

Let's investigate the system $\frac{dx}{dt} = x + 5y$, $\frac{dy}{dt} = x - 3y$. The characteristic equation is $\lambda^2 + 2\lambda - 8 = 0$. The eigenvalues and their corresponding eigenvectors are $\lambda_1 = -4$, $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 2$, $V_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The general solution is

$$X(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-4t} + 5c_2 e^{2t} \\ -c_1 e^{-4t} + c_2 e^{2t} \end{bmatrix}.$$

We can see that the straight-line trajectory $c_1 e^{-4t} V_1 = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-4t} \\ -c_1 e^{-4t} \end{bmatrix}$ approaches the origin as $t \to \infty$. (There are actually *two* half-line trajectories, one for positive c_1 and one for negative c_1. See Fig. 6.19.) But the half-line trajectories corresponding to $c_2 e^{2t} V_2 = c_2 e^{2t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$ for positive and negative values of c_2 are clearly moving *away* from the origin with increasing t.

Substituting the expressions for $x(t)$ and $y(t)$ in the formula for $\frac{dy}{dx}$ and factoring out the dominant term for large t, we get

$$\frac{dy}{dx} = \frac{x - 3y}{x + 5y} = \frac{4c_1 e^{-4t} + 2c_2 e^{2t}}{-4c_1 e^{-4t} + 10c_2 e^{2t}} = \frac{e^{2t}\left(4c_1 e^{-6t} + 2c_2\right)}{e^{2t}\left(-4c_1 e^{-6t} + 10c_2\right)} = \frac{4c_1 e^{-6t} + 2c_2}{-4c_1 e^{-6t} + 10c_2}.$$

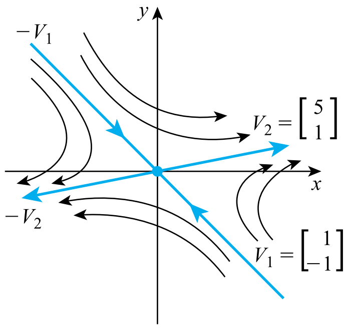


FIGURE 6.19

Trajectories for the system $\frac{dx}{dt} = x + 5y$, $\frac{dy}{dt} = x - 3y$

If $c_2 \neq 0$, we see that as $t \to \infty$, $\frac{dy}{dx}$ tends to $\frac{2c_2}{10c_2} = \frac{1}{5}$, the slope of the eigenvector V_2. This says that the slopes of trajectories not on the straight lines determined by V_1 and V_2 approach the slope of V_2, the eigenvalue associated with the positive eigenvalue. As $t \to -\infty$ the slopes of these trajectories tend to the slope of V_1. Fig. 6.19 shows this partial-source/partial-sink behavior with respect to the origin, which is a *saddle point*.

6.7.6 Unequal eigenvalues, one eigenvalue equal to zero

Finally, we consider the situation in which we have two unequal eigenvalues, but one of them is 0. Suppose that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. This means that the characteristic equation can be written in the form $0 = (\lambda - 0)(\lambda - \lambda_2) = \lambda^2 - \lambda_2\lambda$. We know from Section 6.6 that the constant term of the characteristic equation equals $\det(A)$. Clearly, in this case we have $\det(A) = 0$. Therefore, we should not expect the origin to be the only equilibrium point (see Problem 12 of Exercises 6.5). In fact, *every point* $(x, 0)$ *of the horizontal axis may be an equilibrium point for such a system.* (Problem 21 in Exercises 6.7 asks for a proof of this assertion.) If V_1 is the eigenvector associated with $\lambda_1 = 0$, we know that $A(c_1V_1) = c_1A(V_1) = c_1\lambda_1V_1 = \mathbf{0}$—that is, *each point on the line determined by* V_1 *is an equilibrium point*.

The general solution in this situation has the form

$$X(t) = c_1e^{(0)t}V_1 + c_2e^{\lambda_2 t}V_2 = c_1V_1 + c_2e^{\lambda_2 t}V_2.$$

Note that if $\lambda_2 > 0$ and $t \to \infty$, then $X(t)$ grows without bound. But if $t \to -\infty$, so that we are traveling backward along a trajectory, then the trajectory approaches c_1V_1, the line determined by V_1. Similarly, if $\lambda_2 < 0$ and $t \to \infty$, then $X(t)$ approaches the line determined by V_1, whereas if $t \to -\infty$, then $X(t)$ grows without bound. In any case, each trajectory will be a half-line parallel (in the usual plane geometry sense) to the eigenvector V_2, with one endpoint on the line determined by V_1. (The constant vector c_1V_1 just shifts $c_2e^{\lambda_2 t}V_2$ horizontally and vertically.)

The next example should explain the geometry of the trajectories when we have one eigenvalue equal to 0.

Example 6.7.4 Unequal Eigenvalues, One Eigenvalue Equal to Zero

Fig. 6.20 shows the phase portrait for the system $\dot{x} = y$, $\dot{y} = y$, whose eigenvalues are 0 and 1 and whose corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

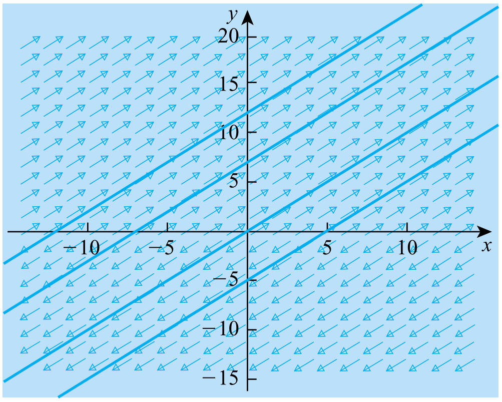


FIGURE 6.20

Phase portrait for the system $\dot{x} = y$, $\dot{y} = y$

The equations of the trajectories are $X(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^t \\ c_2 e^t \end{bmatrix}$. This says (see Exercise 26) that any trajectory not on the line determined by $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has the equation $y(t) = x(t) + k$, so these trajectories form an infinite family of straight lines parallel to $y = x$. Note that the eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to the zero eigenvalue determines two half-line trajectories, the positive x-axis and the negative x-axis. In our example, it is easy to see that every point $(x, 0)$ of the horizontal axis is an equilibrium point: $\dot{x} = y = 0$ and $\dot{y} = y = 0$ imply that $y = 0$ and the x-coordinate is completely arbitrary. The fact that the nonzero eigenvalue is positive makes the points on the x-axis *sources*. (If necessary, review the last full paragraph before this example.)

By looking at Examples 6.6.3–6.6.5 and the examples in this section, we notice that a solution starting in a direction different from those of the eigenvectors is curved, representing [as we know from (6.7.1)] a linear combination, $c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$, of two exponential solutions that have different rates of change (indicated by the eigenvalues). If we look at enough phase portraits, we may also realize that there is a tendency for the "fast" eigenvector (associated with the larger of two unequal eigenvalues) to have the stronger influence on the solutions. Trajectories curve toward the direction of this eigenvector as $t \to \infty$.

In the next section, we'll investigate what happens when there is a repeated real eigenvalue and when there seems to be only one eigenvector corresponding to two real eigenvalues.

Exercises 6.7

A

For each of the Systems 1–12, (a) find the eigenvalues and their corresponding eigenvectors and (b) sketch/plot a few trajectories and show the position(s) of the eigenvector(s). Do part (a) manually, but if the eigenvalues are irrational numbers or decimals, you can use technology to find the corresponding eigenvectors.

1. $\dot{x} = 3x,\ \dot{y} = 2y$
2. $\dot{x} = -x,\ \dot{y} = -2y$
3. $x' = -3x - y,\ y' = 4x + 2y$
4. $\dot{r} = 5r + 4s,\ \dot{s} = -2r - s$
5. $\dot{x} = x + 5y,\ \dot{y} = x - 3y$
6. $\dot{x} = 2x + 3y,\ \dot{y} = x + y$
7. $\dot{x} = -3x + y,\ \dot{y} = 4x - 2y$
8. $x' = -4x + 2y,\ y' = -3x + y$
9. $x' = -2x - y,\ y' = -x + 2y$
10. $\dot{x} = 3y,\ \dot{y} = -3x$
11. $\dot{x} = 3x + 6y,\ \dot{y} = -x - 2y$
12. $x' = 2x + 2y,\ y' = x + 3y$

B

13. Consider the system $\dot{x} = 4x - 3y,\ \dot{y} = 8x - 6y$.
    a. Find the eigenvalues of this system.
    b. Find the eigenvectors corresponding to the eigenvalues in part (a).
    c. Sketch/plot some trajectories and explain what you see.
    d. Write the general solution of the system in the form $X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$, and then re-examine your explanation in part (c).
14. Show that if X is an eigenvector of A corresponding to eigenvalue λ, then any nonzero multiple of X is also an eigenvector of A corresponding to λ.
15. Solve the IVP $X' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} X,\ X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and describe the behavior of the solution as $t \to \infty$. $\left(\text{Here, } X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.\right)$
16. Write a system of first-order linear equations whose trajectories show the following behaviors:

a. $(0, 0)$ is a sink with eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -5$.

b. $(0, 0)$ is a saddle point with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$.

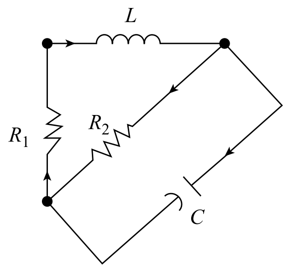
c. $(0, 0)$ is a source with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$.

17. Consider the system $\dot{x} = -x + \alpha y$, $\dot{y} = -2y$, where α is a constant.

a. Show that the origin is a *sink* regardless of the value of α.

b. Assume that $X(t)$ is the solution vector of the system satisfying the initial condition $X(0) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$. Sketch the phase portrait for different values of α and describe how the trajectory $X(t)$, for $t \geq 0$, depends on the value of α.

18. Consider the following circuit.



The current I through the inductor and the voltage V across the capacitor satisfy the system

$$L\frac{dI}{dt} = -R_1 I - V$$
$$C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

a. Find the general solution of the system if $R_1 = 1$ ohm, $R_2 = \frac{3}{5}$ ohm, $L = 2$ henry, and $C = \frac{2}{3}$ farad.

b. Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ regardless of the initial values $I(0)$ and $V(0)$.

19. Consider the system of differential equations in the preceding problem. Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and distinct.

20. Two quantities of a chemical solution are separated by a membrane. If $x(t)$ and $y(t)$ represent the amounts of the chemical at time t on each side of the membrane and V_1 and V_2 represent the (constant) volume of each solution, respectively, then the *diffusion problem* can be modeled by the system

$$\dot{x} = P\left[\frac{y}{V_2} - \frac{x}{V_1}\right]$$

$$\dot{y} = P\left[\frac{x}{V_1} - \frac{y}{V_2}\right],$$

where P is a positive constant called the *permeability* of the membrane. Note that $\frac{x(t)}{V_1}$ and $\frac{y(t)}{V_2}$ represent the *concentrations* of solution on each side.

a. Assuming that $x(0) = x_0$ and $y(0) = y_0$, find the solution of the system IVP without using technology.

b. Calculate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

c. Using part (b), interpret the result $\lim_{t\to\infty}[x(t) + y(t)]$ physically.

d. Notice that if $\frac{y}{V_2} > \frac{x}{V_1}$, then $\dot{x} > 0$. Does this say that the chemical moves across the membrane from the side with a lower concentration to the side with a higher concentration or vice versa? Confirm your answer by considering what happens if $\frac{x}{V_1} > \frac{y}{V_2}$ in the second equation.

21. Consider the system

$$\begin{aligned}\dot{x} &= ax + by\\ \dot{y} &= cx + dy,\end{aligned}$$

where a, b, c, and d are constants. Show that if $ad - bc = 0$, then every point $(x, 0)$ of the horizontal axis is an equilibrium point for the system. [*Hint*: Solve the system $ax + by = 0$, $cx + dy = 0$ for x.]

22. Given the system

$$\begin{aligned}x' &= ax + by\\ y' &= cx + dy,\end{aligned}$$

where a, b, c, and d are constants, express the eigenvalues of this system in terms of its trace and its determinant.

C

23. Consider the system

$$\begin{aligned}\dot{r} &= -r - s\\ \dot{s} &= -\beta r - s,\end{aligned}$$

where β is a parameter.

a. Find the general solution of the system when $\beta = 0.5$. Use the eigenvalues of the coefficient matrix to determine what kind of equilibrium the system has at the origin.

b. Find the general solution of the system when $\beta = 2$. Use the eigenvalues of the coefficient matrix to determine what kind of equilibrium the system has at the origin.

c. The solutions of the system show two rather different types of behavior for the two values of β considered in parts (a) and (b). Find a formula for the eigenvalues in terms of β and determine the value of β between 0.5 and 2 where the transition from one type of behavior to the other occurs. (This critical value of the parameter is called a **bifurcation point**. See Section 2.7.)

24. Suppose that we have the system $\dot{X} = AX$ and that V_1 and V_2 are eigenvectors of A such that neither V_1 nor V_2 is a scalar multiple of the other. Show that any initial condition $X_0 = X(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ can be written as a linear combination of V_1 and V_2. In other words, show that you can always find scalars c_1 and c_2 so that $X_0 = c_1V_1 + c_2V_2$. [*Hint*: Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be the eigenvectors, where you assume that x_1, x_2, y_1, and y_2 are known. Now convert the equation $X_0 = c_1V_1 + c_2V_2$ into a system of algebraic linear equations and go from there.]

25. If the system $\dot{X} = AX$ has two real eigenvalues λ_1 and λ_2, with $\lambda_1 \neq \lambda_2$, and V_1 and V_2 are the corresponding (distinct) eigenvectors, show that $X(t) = c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2, t}V_2$ satisfies the initial condition $X(0) = X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = c_1V_1 + c_2V_2$. (See the preceding problem for the justification of this representation of X_0 for some scalars c_1 and c_2.)

26. As indicated in Example 6.7.4, the system $\dot{x} = y$, $\dot{y} = y$ has the solution $X(t) = \begin{bmatrix} c_1 + c_2e^t \\ c_2e^t \end{bmatrix}$. Show that any trajectory not on the line determined by $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfies the equation $y(t) = x(t) + k$ (in the phase plane) for some constant k. (This says that the trajectories form an infinite family of lines parallel to $y = x$.)

6.8 The stability of homogeneous linear systems: equal real eigenvalues

Now let's see what happens if both eigenvalues are real and equal. In other words, the characteristic equation has a *repeated root*, or *double root*. (See Section 4.1 for the second-order homogeneous linear equation case.) A full understanding of this situation requires more linear algebra than we want to pursue right now. However, the following discussions and examples should give us a clear picture of what's going on.

6.8.1 Equal nonzero eigenvalues, two independent eigenvectors

First, suppose that $\lambda_1 = \lambda_2 \neq 0$. If we can find distinct representative eigenvectors V_1 and V_2 that are not scalar multiples of each other, then we can still write the general solution of the system using (6.7.1): $X(t) = c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2 = c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_1 t}V_2 = e^{\lambda_1 t}(c_1V_1 + c_2V_2)$. If we let $t = 0$, we see that $X(0) = e^{\lambda_1(0)}(c_1V_1 + c_2V_2) = c_1V_1 + c_2V_2$, so we can write $X(t) = e^{\lambda_1 t}X_0$, where $X_0 = X(0)$. (See Problem 18 of Exercises 6.8.) Under these conditions, all trajectories are straight lines through the origin because they are constant multiples of the constant vector $X_0 = c_1V_1 + c_2V_2$. The origin is called a **star node** in this case and will be a *source* if $\lambda_1 > 0$ and a *sink* if $\lambda_1 < 0$. Figs. 6.21a and 6.21b show possible trajectories for various initial vectors X_0.

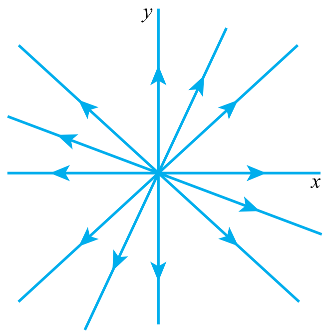


FIGURE 6.21a

Source: $\lambda > 0$

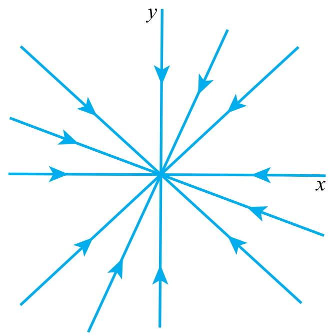


FIGURE 6.21b

Sink: $\lambda < 0$

Let's examine a system for which the origin is a star node.

Example 6.8.1 The Origin as a Star Node (a Source)

Look at the system $\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$. We can write this in matrix form as $\dot{X} = AX$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that A has eigenvalues $\lambda_1 = 1 = \lambda_2$. (*Check this.*) By the way we defined the product of a matrix and a vector earlier in this chapter, we see that our matrix of coefficients A is such that $AV = V = 1 \cdot V = \lambda_1 V$ for *every* vector V. In particular, any *nonzero* vector V is an eigenvector corresponding to the eigenvalue 1. (*Be sure you understand the preceding statement.*) A particularly simple eigenvector to work with is $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is easy to see that the vector $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not a multiple of V_1 because any scalar multiple of V_1 would have the form $\begin{bmatrix} c \\ 0 \end{bmatrix}$, where c is a constant. Therefore, we can write the solution of our system as

$$X(t) = c_1 e^t V_1 + c_2 e^t V_2 = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^t \end{bmatrix}.$$

Of course, because each of our original (separable) differential equations contains only one variable, we could solve each one separately to get the same result in the form $x(t) = c_1 e^t$, $y(t) = c_2 e^t$. As we indicated in the discussion right before this example, the trajectories are straight lines through the origin, and Fig. 6.22 shows that the origin, a star node, is a *source*.

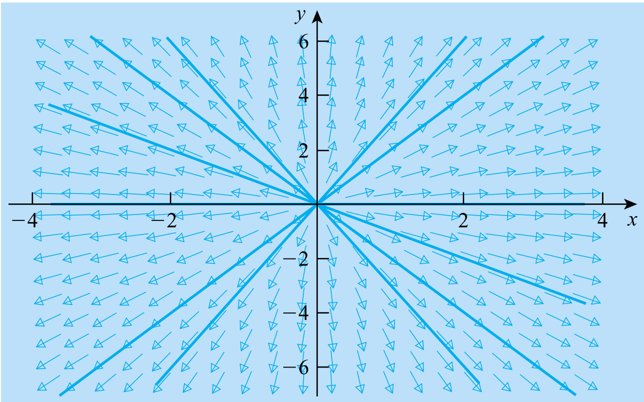


FIGURE 6.22

Phase portrait of the system $\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$

6.8.2 Equal nonzero eigenvalues, only one independent eigenvector

Now suppose that $\lambda_1 = \lambda_2 \neq 0$, but our single eigenvalue has *only one distinct representative eigenvector*. What we mean is that all eigenvectors corresponding to the single distinct eigenvalue are scalar multiples of each other. Geometrically, this says that all eigenvectors lie on the same straight line through the origin. Then if we tried to use the solution form (6.7.1), we would get

$$X(t) = c_1 e^{\lambda_1 t} V + c_2 e^{\lambda_1 t} V = (c_1 + c_2) e^{\lambda_1 t} V = k e^{\lambda_1 t} V.$$

But how can the general solution of a two-dimensional system or second-order equation have only one arbitrary constant?

What we have to do here is find another solution of the system that is *independent* of the one solution we found using the single eigenvalue and its representative eigenvector. This is similar to the technique we used to solve a second-order linear equation whose characteristic equation had a repeated root (see Section 4.1). In our situation, an independent solution is one that is not a scalar multiple of the first solution. If we *do* find another eigenvector corresponding to the single eigenvalue, but one that is independent of the original eigenvector, then the solution can still be written in the form $X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$.

It turns out that we *can* find a substitute for an independent eigenvector. Although we won't go into all the linear algebraic details here, we can at least try to explain the end result. Another (independent) solution of the system must have the form

$$X_2(t) = te^{\lambda t} V + e^{\lambda t} W, \tag{6.8.1}$$

where V is the original eigenvector corresponding to the single eigenvalue λ and W, called a **generalized eigenvector** (**of order 2**), is a vector that satisfies the matrix equation

$$(A - \lambda I)W = V. \tag{6.8.2}$$

(See Problem 13 in Exercises 6.8.)

We can easily see that the vector defined by (6.8.1) is a solution of the system. If $X(t) = te^{\lambda t} V + e^{\lambda t} W$, then $\dot{X}(t) = t(\lambda e^{\lambda t} V) + e^{\lambda t} V + \lambda e^{\lambda t} W = (\lambda t + 1)e^{\lambda t} V + \lambda e^{\lambda t} W$ and, because (6.8.2) implies that $AW = V + \lambda W$,

$$\begin{aligned} AX &= A(te^{\lambda t} V + e^{\lambda t} W) = te^{\lambda t}(AV) + e^{\lambda t}(AW) = te^{\lambda t}(\lambda V) + e^{\lambda t}(V + \lambda W) \\ &= (\lambda t + 1)e^{\lambda t} V + \lambda e^{\lambda t} W. \end{aligned}$$

Thus, $\dot{X} = AX$—that is, (6.8.1) defines a solution of the system.

Next, we must solve Eq. (6.8.2) for W, and then we can write the general solution of the system as

$$X(t) = c_1 e^{\lambda t} V + c_2 \left[te^{\lambda t} V + e^{\lambda t} W \right]. \tag{6.8.3}$$

(The theory of linear algebra shows that we can always solve for W in Eq. (6.8.2) if V is an eigenvector of A corresponding to eigenvalue λ.)

Now let's look at an example in which we have equal nonzero eigenvalues, but only one distinct representative eigenvector.

Example 6.8.2 Equal Nonzero Eigenvalues, Only One Distinct Eigenvector

Consider the system $\dot{x} = -2x + y$, $\dot{y} = -2y$. We can write this in matrix form as $\dot{X} = AX$, where $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$. The characteristic polynomial of A is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$, so $\lambda = -2$ is a repeated root. Then the matrix equation $AV = \lambda V = -2V$ is equivalent to the system

$$-2x + y = -2x$$
$$-2y = -2y,$$

or

$$y = 0$$
$$-2y = -2y.$$

From this we see that any eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ must have the form $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for arbitrary values of x. Therefore we can take $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the only independent eigenvector that corresponds to the eigenvalue -2. Now we must find a vector $W = \begin{bmatrix} r \\ s \end{bmatrix}$ satisfying $(A - \lambda I)W = V$.

In our problem, $(A - \lambda I)W = V$ becomes

$$\left\{ \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\left\{ \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which is equivalent to the algebraic system

$$0 \cdot r + 1 \cdot s = 1$$
$$0 \cdot r + 0 \cdot s = 0.$$

This tells us that $s = 1$ and r is a "free variable"—that is, r is completely arbitrary. For convenience, let $r = 0$ so that our generalized eigenvector is $W = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Finally, we can write the general solution of our system in the form (6.8.3):

$$X(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left[te^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$
$$= \begin{bmatrix} c_1 e^{-2t} + c_2 te^{-2t} \\ c_2 e^{-2t} \end{bmatrix}.$$

Fig. 6.23, generated by a CAS, shows that the trajectories spiral in toward the origin, in such a way that they are tangent to the eigenvector $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or its negative at the origin. (Note that the vector V is part of the positive x-axis.)

Whenever we have a system with equal nonzero eigenvalues but only one distinct eigenvector, the phase portrait will consist of spirals *approaching* the origin when the repeated eigenvalue is *negative*, and the phase portrait will consist of spirals moving *outward* if the eigenvalue is *positive*. A negative eigenvalue makes the origin a **spiral**

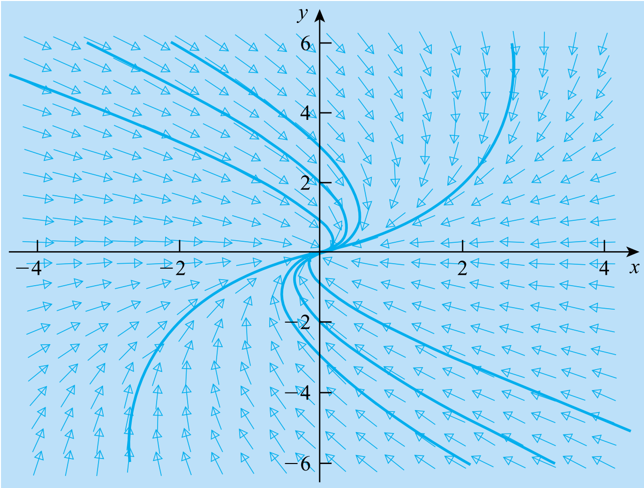


FIGURE 6.23

Trajectories for the system $\dot{x} = -2x + y$, $\dot{y} = -2y$

sink; a positive eigenvalue makes the origin a **spiral source**. Furthermore, if the eigenvalue is negative, the slopes of all trajectories not on the line determined by the one eigenvector approach the slope of this line as $t \to \infty$. A positive eigenvalue indicates that the slopes of all trajectories not on the line determined by the one eigenvector approach the slope of this line as $t \to -\infty$. (Problem 14 in Exercises 6.8 asks for a proof of these two statements.)

6.8.3 Both eigenvalues zero

Finally, let's assume that $\lambda_1 = \lambda_2 = 0$. If there are two linearly independent eigenvectors V_1 and V_2, then the general solution is $X(t) = c_1 e^{0 \cdot t} V_1 + c_2 e^{0 \cdot t} V_2 = c_1 V_1 + c_2 V_2$, a single vector of constants. If there is only one linearly independent eigenvector V corresponding to the eigenvalue 0, then we can find a generalized eigenvector and use formula (6.8.3):

$$X(t) = c_1 e^{\lambda t} V + c_2 \left[t e^{\lambda t} V + e^{\lambda t} W \right].$$

For $\lambda = 0$, we get $X(t) = c_1 V + c_2 [tV + W] = (c_1 + c_2 t) V + c_2 W$. In Exercise 15 you will investigate a system that has both eigenvalues zero.

Exercises 6.8

A

For each of the Systems 1–8, (a) find the eigenvalues and their corresponding linearly independent eigenvectors and (b) sketch/plot a few trajectories and show the position(s) of the eigenvector(s) if they do not have complex entries. Do part (a) manually, but if the eigenvalues are irrational numbers, you may use technology to find the corresponding eigenvectors.

1. $\dot{x} = 3x,\ \dot{y} = 3y$
2. $\dot{x} = -4x,\ \dot{y} = x - 4y$
3. $\dot{x} = 2x + y,\ \dot{y} = 4y - x$
4. $\dot{x} = 3x - y,\ \dot{y} = 4x - y$
5. $\dot{x} = 2y - 3x,\ \dot{y} = y - 2x$
6. $\dot{x} = 5x + 3y,\ \dot{y} = -3x - y$
7. $\dot{x} = -3x - y,\ \dot{y} = x - y$
8. $\dot{x} = \sqrt{2}x + 5y,\ \dot{y} = \sqrt{2}y$

B

9. Given a characteristic polynomial $\lambda^2 + \alpha\lambda + \beta$, what condition on α and β guarantees that there is a repeated eigenvalue?
10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A has only one eigenvalue if and only if $[\text{trace }(A)]^2 - 4\det(A) = 0$.
11. Write a system of first-order linear equations for which $(0, 0)$ is a sink with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -2$.
12. Write a system of first-order linear equations for which $(0, 0)$ is a source with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 3$.
13. Show that if V is an eigenvector of a 2×2 matrix A corresponding to eigenvalue λ and vector W is a solution of $(A - \lambda I)W = V$, then V and W are linearly independent. [See Eqs. (6.8.2)–(6.8.3).] [*Hint*: Suppose that $W = cV$ for some scalar c. Then show that V must be the zero vector.]
14. Suppose that a system $\dot{X} = AX$ has only one eigenvalue λ, and that every eigenvector is a scalar multiple of one fixed eigenvector, V. Then Eq. (6.8.3) tells us that any trajectory has the form $X(t) = c_1 e^{\lambda t} V + c_2 \left[t e^{\lambda t} V + e^{\lambda t} W\right] = t e^{\lambda t} \left[\frac{1}{t}(c_1 V + W) + c_2 V\right]$.
    a. If $\lambda < 0$, show that the slope of $X(t)$ approaches the slope of the line determined by V as $t \to \infty$. [*Hint*: $\frac{e^{-\lambda t}}{t} X(t)$, as a scalar multiple of $X(t)$, is parallel to $X(t)$.]
    b. If $\lambda < 0$, show that the slope of $X(t)$ approaches the slope of the line determined by V as $t \to -\infty$.
15. Consider the system $\dot{x} = 6x + 4y,\ \dot{y} = -9x - 6y$.
    a. Show that the only eigenvalue of the system is 0.
    b. Find the single independent eigenvector V corresponding to $\lambda = 0$.

c. Show that every trajectory of this system is a straight line parallel to V, with trajectories on opposite sides of V moving in opposite directions. [*Hint*: First, for any trajectory not on the line determined by V, look at its slope, dy/dx.]

16. If $\{\dot{x} = ax + by,\ \dot{y} = cx + dy\}$ is a system with a double eigenvalue and $a \neq d$, show that the general solution of the system is

$$c_1 e^{\lambda t} \begin{bmatrix} 2b \\ d - a \end{bmatrix} + c_2 e^{\lambda t} \left(t \begin{bmatrix} 2b \\ d - a \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right),$$

where $\lambda = (a + d)/2$.

C

17. Prove that $c_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}$ is the general solution of $\dot{X} = AX$, where $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

18. Suppose the matrix A has repeated real eigenvalues λ and there is a pair of linearly independent eigenvectors associated with A. Prove that $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

19. A special case of the **Cayley–Hamilton Theorem** states that if $\lambda^2 + \alpha\lambda + \beta = 0$ is the characteristic equation of a matrix A, then $A^2 + \alpha A + \beta I$ is the zero matrix. (We say that a 2×2 matrix always satisfies its own characteristic equation.) Using this result, show that if a 2×2 matrix A has a repeated eigenvalue λ and $V = \begin{bmatrix} x \\ y \end{bmatrix} \neq \mathbf{0}$ (the zero vector), then either V is an eigenvector of A or else $(A - \lambda I)V$ is an eigenvector of A. [See Appendix B.3 if you are not familiar with matrix-matrix multiplication.]

6.9 The stability of homogeneous linear systems: complex eigenvalues

6.9.1 Complex eigenvalues and complex eigenvectors

Now let's examine what occurs when the matrix A in the system $\dot{X} = AX$ has *complex* eigenvalues. As we've already stated, any complex root λ of the quadratic characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ must occur as part of a *complex conjugate pair*: $\lambda = p \pm qi$. As we'll see, the behavior of trajectories in the case of complex eigenvalues depends on the *real part*, p, of the complex eigenvalues. When the eigenvalues of a matrix are complex numbers the eigenvectors will also have complex entries (see Appendix C.1), and therefore the algebra of the situation will be slightly more complicated.

The most important point to realize is that when A has complex eigenvalues the general solution of $\dot{X} = AX$ has the same form as (6.7.1), $X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$. In other words, the Superposition Principle holds, but we have to deal with the fact that this formula will produce vectors whose elements are complex. For example, in the context of the general solution formula, the phrase "multiplying by a scalar" refers to multiplying vectors (whose entries may be complex numbers) by complex numbers.

Fortunately, there are some useful results that aid us in our work with complex eigenvalues and eigenvectors:

1. It is crucial to recall **Euler's formula**, which we saw in Section 4.1:

$$e^{p+qi} = e^p(\cos(q) + i\sin(q)).$$

   This result will be useful in simplifying complex-valued expressions and will show us how to obtain real-valued solutions of $\dot{X} = AX$.

2. Another important fact is that *eigenvectors corresponding to complex conjugate eigenvalues are conjugate to each other*. If the eigenvalue $\lambda_1 = p + qi$ has a corresponding eigenvector $V_1 = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = U + iW$, then $\lambda_2 = \overline{\lambda}_1 = p - qi$ has a corresponding eigenvector $V_2 = \overline{V}_1 = \overline{\begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{bmatrix}} = \begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = U - iW$. The proof of this result follows from the properties of the conjugate: Suppose that $AV_1 = \lambda_1 V_1$. Then $\overline{(AV_1)} = \overline{(\lambda_1 V_1)}$, so $\overline{A}\,\overline{V}_1 = \overline{\lambda}_1 \overline{V}_1$, or (because all elements of A are real) $A\overline{V}_1 = \overline{\lambda}_1 \overline{V}_1 = \lambda_2 \overline{V}_1$. That is, $\overline{V}_1$ is an eigenvector corresponding to $\lambda_2 = \overline{\lambda}_1$.

To see how valuable results 1 and 2 are, let's suppose that $\lambda = p + qi$ is an eigenvalue of the matrix A and that $V = U + iW$ is a corresponding eigenvector. If we define $X(t) = e^{\lambda t} V$, then $AX = A\left(e^{\lambda t} V\right) = e^{\lambda t}(AV) = e^{\lambda t}(\lambda V) = \lambda e^{\lambda t} V = \dot{X}$, so $X(t)$ is a solution of $\dot{X} = AX$. Using Euler's formula and the properties of complex multiplication (see Section C.1), we have

$$\begin{aligned} X(t) &= e^{\lambda t} V = e^{(p+qi)t} V = e^{pt}(\cos qt + i \sin qt)(U + iW) \\ &= e^{pt}\{(\cos qt)U - (\sin qt)W\} + ie^{pt}\{(\cos qt)W + (\sin qt)U\}. \end{aligned}$$

Then the *real part* and the *imaginary part* of $X(t)$ can be considered separately:

$$\begin{aligned} X_1(t) &= \operatorname{Re}\{X(t)\} = e^{pt}\{(\cos qt)U - (\sin qt)W\} \\ X_2(t) &= \operatorname{Im}\{X(t)\} = e^{pt}\{(\cos qt)W + (\sin qt)U\}. \end{aligned}$$

The important observation here is that $X_1(t)$ *and* $X_2(t)$ *are real-valued linearly independent solutions of the system* $\dot{X} = AX$. (Problem 13 in Exercises 6.9 asks for a proof that the same two solutions result from taking the real and imaginary parts of $e^{\overline{\lambda} t}\overline{V}$.)

We will justify this observation for the real part of $X(t)$, leaving the proof for the imaginary part as Problem 14 in Exercises 6.9. First we write $X_R = \text{Re}\{X(t)\} = \frac{X+\overline{X}}{2}$ (see Section C.1 if necessary). Then

$$\begin{aligned} AX_R &= A\left(\frac{X+\overline{X}}{2}\right) = \tfrac{1}{2}A\left(X+\overline{X}\right) = \tfrac{1}{2}\left(AX + A\overline{X}\right) \\ &= \tfrac{1}{2}\left(\dot{X} + \overline{(AX)}\right) = \tfrac{1}{2}\left(\dot{X} + \overline{\dot{X}}\right) = \text{Re}(\dot{X}) = (\dot{X})_R = \overset{\bullet}{\overbrace{(X_R)}}. \end{aligned}$$

Now the Superposition Principle tells us that $c_1X_1(t) + c_2X_2(t)$ is also a solution—in fact, it is the *general solution* of the system. The proofs of this last fact in Section 6.7 are valid here. We can take the scalars c_1 and c_2 to be real numbers.

Let's see how to work with the complexities (pun intended) of a system with complex eigenvalues.

Example 6.9.1 A System with Complex Eigenvalues

As a first example of working with complex eigenvalues and eigenvectors, let's look at the equation $\frac{d^2\theta}{dt^2} + k^2 \sin\theta = 0$, which describes the motion of an *undamped pendulum*. Here, θ is the angle the pendulum makes with the vertical, and $k^2 = \frac{g}{L}$, where g is the acceleration due to gravity and L is the length of the pendulum. This famous equation is nonlinear and will be treated fully in Section 7.5, but for small angles θ, $\sin\theta \approx \theta$, so we can consider the *linearized* equation $\frac{d^2\theta}{dt^2} + k^2\theta = 0$. The system form of the linear pendulum equation has complex eigenvalues.

First, we convert the linearized pendulum equation to a system (see Problem 16 of Exercises 6.4 for the nonlinear case). Letting $x = \theta$ and $y = \frac{d\theta}{dt} = \frac{dx}{dt}$, we convert our linear second-order homogeneous equation into the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -k^2x$. (Be sure that you remember how to carry out this conversion.)

In matrix form, we have the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, with characteristic equation $\lambda^2 + k^2 = 0$ and complex conjugate eigenvalues $\lambda_1 = ki$ and $\lambda_2 = -ki$. (*Verify all the statements in the preceding sentence.*) The equation $AV = \lambda_1 V$ has the form $\begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ki\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kix \\ kiy \end{bmatrix}$, which is equivalent to the algebraic system

$$\begin{aligned} y &= kix \\ -k^2x &= kiy. \end{aligned}$$

Because the second equation is just ki times the first, we see that we can take x as arbitrary and $y = kix$, which gives us the eigenvector $V = \begin{bmatrix} x \\ kix \end{bmatrix} = x\begin{bmatrix} 1 \\ ki \end{bmatrix}$. Letting $x = 1$, we get the representative eigenvector

$$V_1 = \begin{bmatrix} 1 \\ ki \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ k \end{bmatrix}.$$

From the discussion preceding this example, we realize that we don't have to worry about the second (conjugate) eigenvalue and its associated eigenvector. The general solution of our original equation and its system version can be obtained from the information we already have. We start with the solution

$$\begin{aligned}\hat{X}(t) &= e^{kit}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\k\end{bmatrix}\right) = (\cos kt + i\sin kt)\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\k\end{bmatrix}\right)\\ &= \left((\cos kt)\begin{bmatrix}1\\0\end{bmatrix} - (\sin kt)\begin{bmatrix}0\\k\end{bmatrix}\right) + i\left((\cos kt)\begin{bmatrix}0\\k\end{bmatrix} + (\sin kt)\begin{bmatrix}1\\0\end{bmatrix}\right).\end{aligned}$$

Because the real and imaginary parts of the preceding expression are linearly independent solutions of the system, the general solution is given by

$$\begin{aligned}X(t) &= c_1\left((\cos kt)\begin{bmatrix}1\\0\end{bmatrix} - (\sin kt)\begin{bmatrix}0\\k\end{bmatrix}\right) + c_2\left((\cos kt)\begin{bmatrix}0\\k\end{bmatrix} + (\sin kt)\begin{bmatrix}1\\0\end{bmatrix}\right)\\ &= c_1\begin{bmatrix}\cos kt\\-k\sin kt\end{bmatrix} + c_2\begin{bmatrix}\sin kt\\k\cos kt\end{bmatrix} = \begin{bmatrix}c_1\cos kt + c_2\sin kt\\-kc_1\sin kt + kc_2\cos kt\end{bmatrix}.\end{aligned}$$

Fig. 6.24 shows some trajectories for this system when $k = 1$. These curves are circles centered at the origin. We say that the origin is a **center** for the system. You should try to generate your own phase portrait by choosing different values of k and various initial points for each value of k.

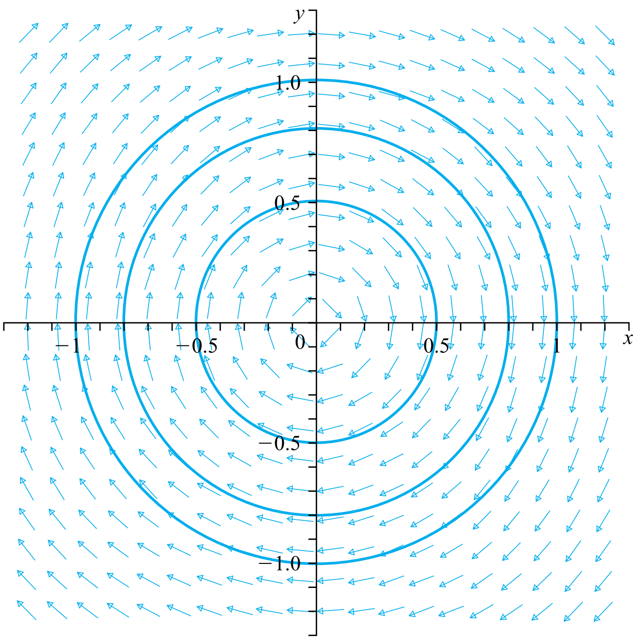


FIGURE 6.24

Trajectories for the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$, $0 \le t \le 7$. Initial points: $(x(0), y(0)) = (1, 0)$, $(0.5, 0)$, $(0, 0.8)$

The next example provides a more challenging problem algebraically.

Example 6.9.2 A System with Complex Eigenvalues

According to *Kirchhoff's Second Law*, an electric circuit with resistance of 2 ohm, capacitance of 0.5 farad, inductance of 1 henry, and no driving electromotive force can be modeled by the second-order linear equation $\ddot{Q} + 2\dot{Q} + 2Q = 0$, where $Q = Q(t)$ is the charge on the capacitor at time t. If $Q(0) = 1$ and $\dot{Q}(0) = 0$, we want to determine the charge on the capacitor at time $t \geq 0$.

We write our second-order equation as a system of first-order equations by introducing new variables: Let $x = Q$ and $y = \dot{x} = \dot{Q}$, so $\dot{y} = \ddot{Q} = -2Q - 2\dot{Q} = -2x - 2y$. Then the original second-order equation is equivalent to the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -2x - 2y,\end{aligned}$$

which can be written in matrix form as $\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix of coefficients has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$, with roots $-1 + i$ and $-1 - i$. Working with the first of these eigenvalues, we see that any eigenvector must satisfy the matrix equation

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (-1 + i)\begin{bmatrix} x \\ y \end{bmatrix},$$

which is equivalent to the equations

$$\begin{aligned} y &= -x + ix\\ -2x - 2y &= -y + iy.\end{aligned}$$

Substituting the first equation in the second equation, we get

$$\begin{aligned} -2x - 2[-x + ix] &= -[-x + ix] + i[-x + ix]\\ -2x + 2x - 2ix &= x - ix - ix - x \text{ (remembering that } \iota^2 = -1)\\ -2ix &= -2ix.\end{aligned}$$

The preceding equation, an identity, says that *any* value of x will be a solution. If we choose $x = 1$ for convenience, then the first equation gives us $y = -1 + i$, so the representative eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ i - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = U + iW.$$

As in the previous example, we work with the solution provided by one of the complex conjugate eigenvalues and its representative eigenvector:

$$\begin{aligned}\hat{X}(t) &= e^{(-1+i)t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{-t}(\cos t + i \sin t)\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\\ &= e^{-t}\left((\cos t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - (\sin t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + ie^{-t}\left((\cos t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (\sin t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).\end{aligned}$$

Extracting the real and imaginary parts of this last complex-valued expression, we express the general solution as

$$X(t) = c_1 e^{-t}\left((\cos t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - (\sin t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2 e^{-t}\left((\cos t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (\sin t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$= c_1 \begin{bmatrix} e^{-t}\cos t \\ -e^{-t}\cos t - e^{-t}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t - e^{-t}\sin t \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ (c_2 - c_1)\cos t - (c_2 + c_1)\sin t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Now, using the initial conditions $x(0) = Q(0) = 1$ and $y(0) = \dot{Q}(0) = 0$ in the general solution just given, we get the condition $\begin{bmatrix} c_1 \\ c_2 - c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which implies that $c_1 = 1$ and $c_2 = 1$. Thus the solution of our original IVP is $Q(t) = x(t) = e^{-t}(\cos t + \sin t)$. (See Fig. 6.25.) Because the current, I, is defined as the rate of change of Q, we get a bonus: $I(t) = \dot{Q}(t) = y(t) = -2e^{-t}\sin t$.

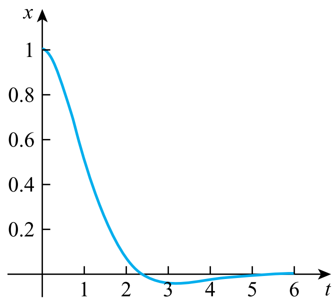


FIGURE 6.25

Graph of $x(t) = e^{-t}(\cos t + \sin t)$, $0 \le t \le 6$

As satisfying as this analytical solution may be, a natural question is what the trajectories for this system look like. Fig. 6.26 shows five trajectories, corresponding to different initial conditions. The trajectory for the IVP we started with is second from the bottom.

Note that in Fig. 6.26 the trajectories are *spirals* moving *toward* the equilibrium solution, the origin. We say that the origin is a **spiral sink**. If we examine the general solution, we can see why the trajectories behave this way. First of all, there is no straight-line direction along which the trajectories approach the origin. In the preceding example, the expressions for both $x(t)$ and $y(t)$ have trigonometric terms that contribute oscillations, movements back and forth across the x-axis. But in addition, each entry of the general solution has a factor of e^{-t}, which *dampens* these oscillations for positive values of t. Thus, as t increases in a positive direction, the amplitudes of these oscillations tend to 0. A look at Euler's formula explains the existence of this decaying exponential: *The real part, p, of the eigenvalue pair is negative*. Fig. 6.25 shows a plot of x against t for the particular solution with $x(0) = 1$ and $y(0) = 0$. The graph of y against t is similar.

As we'll see in some of the exercises following this section, if the eigenvalues are $p \pm qi$ and $p > 0$, then we get spirals that wind *away* from $(0, 0)$ as t increases. Here, we say that the origin is a *spiral source*. This corresponds to oscillatory solutions with increasing amplitudes and describes *resonance*. (See Example 4.3.4, especially Fig. 4.3.)

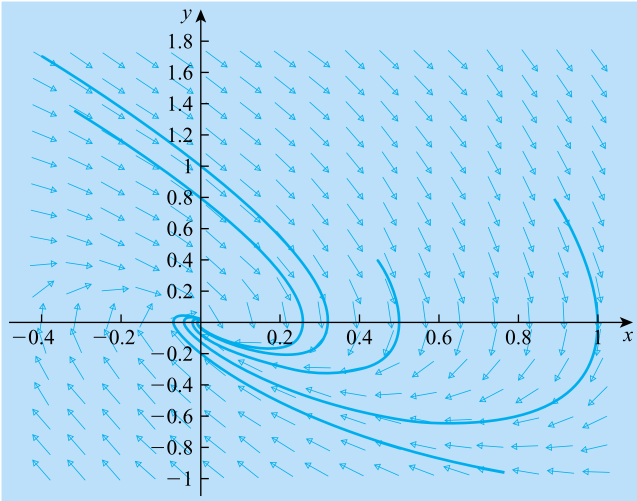


FIGURE 6.26

Trajectories for the system $\dot{x} = y$, $\dot{y} = -2x - 2y$, $-0.3 \le t \le 4$. Initial conditions: $(x(0), y(0)) = (1, 0)$, $(0.5, 0)$, $(0, 0.8)$, $(0, 1)$, $(0.5, -0.8)$

Table 6.2 Summary of Stability Criteria for Two-Dimensional Linear Systems

Eigenvalues	**Stability**	**References**
REAL		
Unequal		
Both > 0	Unstable node (source, repeller)	Examples 6.6.4 and 6.7.1
Both < 0	Stable node (sink, attractor)	Examples 6.6.2 and 6.7.2
Different signs	Saddle point	Examples 6.6.1, 6.6.3, and 6.7.3
One $= 0$, the other $\neq 0$	Whole line of equilibrium points	Example 6.7.4 and Problem 21 of Exercises 6.7
Equal		
Both > 0	Unstable node (source, repeller)	Example 6.8.1
Both < 0	Stable node (sink, attractor)	Example 6.8.2
Both $= 0$	"Algebraically unstable"	Problem 15 of Exercises 6.8
COMPLEX		
Real part > 0	Spiral source (unstable spiral, repeller)	Example 6.6.5
Real part < 0	Spiral sink (stable spiral)	Example 6.9.2
Real part $= 0$	Center (neutral center, stable center)	Example 6.9.1

The case where $p = 0$, so that we have *pure imaginary eigenvalues*, is interesting. Now the trajectories are *closed, nonintersecting curves that encircle the origin*. This corresponds to the situation in which we have *undamped oscillations*. (See Example 6.9.1, especially Fig. 6.24.)

Now let's stand back and summarize all these cases. Table 6.2 categorizes the stability of two-dimensional homogeneous autonomous systems, referring to relevant examples or exercises.

Exercises 6.9

A

For each of the Systems 1–10, (a) find the eigenvalues and their corresponding eigenvectors and (b) sketch/plot a few trajectories and show the position(s) of the eigenvector(s) if they do not have complex entries.

1. $\dot{r} = -r - 2s$, $\dot{s} = 2r - s$
2. $\dot{x} = 3x - 2y$, $\dot{y} = 2x + 3y$
3. $\dot{x} = -0.5x - y$, $\dot{y} = x - 0.5y$
4. $\dot{x} = x + y$, $\dot{y} = -3x - y$
5. $\dot{x} = 2x + y$, $\dot{y} = -3x - y$
6. $\dot{x} = x + 2y$, $\dot{y} = -5x - y$
7. $\dot{x} = y - 7x$, $\dot{y} = -2x - 5y$
8. $\dot{x} = x - 3y$, $\dot{y} = 3x + y$
9. $\dot{x} = 6x - y$, $\dot{y} = 5x + 4y$
10. $\dot{x} + x + 5y = 0$, $\dot{y} - x - y = 0$

B

11. Write systems of first-order linear equations whose trajectories show the following behaviors:
    - **a.** $(0, 0)$ is a spiral source with eigenvalues $\lambda_1 = 2 + 2i$ and $\lambda_2 = 2 - 2i$.
    - **b.** $(0, 0)$ is a stable center with eigenvalues $\lambda_1 = -3i$ and $\lambda_2 = 3i$.
    - **c.** $(0, 0)$ is a spiral sink with eigenvalues $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$.
12. Consider the system

$$\dot{x} = y$$
$$\dot{y} = -x - \beta y,$$

where β is a parameter.

- **a.** Using technology to draw trajectories, examine the stability of the equilibrium solution for $\beta = -1, -0.1, 0, 0.1$, and 1.
- **b.** Does there seem to be a *bifurcation point*—that is, a critical value of β at which the stability changes its nature? (Review Section 2.7.)

c. Find a formula for the eigenvalues of the system, showing their dependence on β.

d. Relate the information found in part (c) to the stability summary in Table 6.2 and answer the question in part (b) with increased confidence.

13. If λ is a complex eigenvalue of matrix A, $V = U + iW$ is a corresponding eigenvector, and $X(t) = e^{\lambda t}V$, then we have seen that

$$X_1(t) = \mathrm{Re}\{X(t)\} = e^{pt}\{(\cos qt)U - (\sin qt)W\}$$
$$X_2(t) = \mathrm{Im}\{X(t)\} = e^{pt}\{(\cos qt)W + (\sin qt)U\}$$

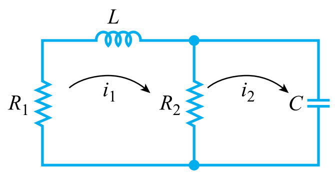
are real-valued linearly independent solutions of the system $\dot{X} = AX$. Show that the same two solutions can be obtained by taking the real and imaginary parts of $e^{\overline{\lambda}t}\overline{V}$. (Thus the second term of the familiar solution formula $c_1e^{\lambda_1 t}V_1 + c_2e^{\lambda_2 t}V_2 = c_1e^{\lambda_1 t}V_1 + c_2e^{\overline{\lambda}_1 t}\overline{V}_1$ is unnecessary.)

14. Show that if $X(t)$ is a complex-valued solution of the system $\dot{X} = AX$, then so is $X_{\mathrm{I}} = \mathrm{Im}(X) = \frac{X-\overline{X}}{2i}$, the imaginary part of $X(t)$.

15. The following two-loop electrical circuit illustration can be modeled by the system

$$\frac{di_1}{dt} = -\left(\frac{R_1 + R_2}{L}\right)i_1 + \frac{R_2}{L}i_2$$
$$\frac{di_2}{dt} = -\left(\frac{R_1 + R_2}{L}\right)i_1 + \left(\frac{R_2}{L} - \frac{1}{R_2C}\right)i_2.$$

Using eigenvalues and eigenvectors, solve the IVP $i_1(0) = 1$, $i_2(0) = 0$, when $R_1 = R_2 = 1$, $L = 1$, and $C = 3$. (Use technology to find the eigenvectors.)



16. The equation $\ddot{P} + \lambda(b - a)P = \lambda(\alpha - \beta)$ is a "cobweb model with stocks (inventory)" used in economic theory to describe the price P of goods under certain circumstances. Here a, b, α, and β are constants with $a < 0$, $b > 0$, $\lambda > 0$.

a. Determine the equilibrium solution $\overline{P}$ of the equation.

b. Letting $p = P - \overline{P}$, find the second-order equation satisfied by p.

c. Solve the equation found in part (b) for p.

d. Express P in terms of $\overline{P}$ and p and describe the behavior of P.

C

17. In Section 6.7.2, we proved the result that the eigenvectors corresponding to distinct eigenvalues are linearly independent. Use this result to show that the real and imaginary parts of complex eigenvectors are linearly independent.

18. The change in the amounts x and y of two substances that enter a certain chemical reaction can be described by the IVP

$$\dot{x} = -3x + \alpha y,\ \dot{y} = \beta x - 2y;\ x(0) = y(0) = 1,$$

where α and β are two parameters that depend on the conditions of reaction (temperature, humidity, etc.). Are there values of α and β for which the solution of the IVP is a periodic function of time?

6.10 Nonhomogeneous systems

6.10.1 The general solution

The linear systems we have been dealing with so far are called **homogeneous** systems. Basically, this means that they can be expressed in the form $\dot{X} = AX$ with no "leftover" terms. If a linear system has to be written as $\dot{X} = AX + B(t)$, where $B(t)$ is a vector of the form $\begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$, then we say that the system is **nonhomogeneous**. For example, in matrix terms, the system $\frac{dx}{dt} = x + \sin t$, $\frac{dy}{dt} = t - y$ must be written as $\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \sin t \\ t \end{bmatrix}$ and so is nonhomogeneous.

Don't confuse the distinction between *autonomous* and *nonautonomous* with that between *homogeneous* and *nonhomogeneous*. For example, if $b_1(t)$ and $b_2(t)$ are constant functions (not both zero), then we have a system that is both autonomous and nonhomogeneous. (See, e.g., Example 6.4.3.)

The techniques that were introduced in Section 4.2 for second-order nonhomogeneous equations generalize to systems, but the calculations are more complicated. To get a handle on solving a nonhomogeneous linear system, we need a fundamental fact about linear systems:

> The general solution, X_{GNH}, of a linear nonhomogeneous system is obtained by finding a *particular* solution, X_{PNH}, of the *nonhomogeneous* system and adding it to the *general* solution, X_{GH}, of the associated *homogeneous* system.

You should see this as an application of the Superposition Principle and as an extension of the result we saw for single linear differential equations (Section 4.2). Symbolically, we can write $X_{\text{GNH}} = X_{\text{GH}} + X_{\text{PNH}}$. Using the definitions of these terms, we can see that this sum of vectors is a solution of the nonhomogeneous system:

$$\dot{X}_{\text{GNH}} = \dot{X}_{\text{GH}} + \dot{X}_{\text{PNH}} = AX_{\text{GH}} + \{AX_{\text{PNH}} + B(t)\}$$
$$= A(X_{\text{GH}} + X_{\text{PNH}}) + B(t) = AX_{\text{GNH}} + B(t).$$

(*Be sure you follow this*.) You should see that X_{GH}, as a general solution, must contain two arbitrary constants, so the expression for X_{GNH} contains two arbitrary constants.

Let's look at a simple example showing the structure of a nonhomogeneous system's solution.

Example 6.10.1 The Solution of a Nonhomogeneous System

The system

$$\dot{x} = x + y + 2e^{-t}$$
$$\dot{y} = 4x + y + 4e^{-t}$$

can be written in the form $\dot{X}(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} X + \begin{bmatrix} 2e^{-t} \\ 4e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} X + 2e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The system has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, with corresponding eigenvectors $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. (*Check this*.) Then the general solution of the associated homogeneous system $\dot{X}(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} X$ is

$$X_{\text{GH}} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We need to verify that a particular solution of the original nonhomogeneous system is given by $X_{\text{PNH}} = e^{-t} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2e^{-t} \end{bmatrix}$. Therefore, the general solution of the nonhomogeneous system is

$$X_{\text{GNH}} = X_{\text{GH}} + X_{\text{PNH}} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2e^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} - 2e^{-t} \end{bmatrix}.$$

(*Check that this is the general solution of the original nonhomogeneous system.*)

6.10.2 The method of undetermined coefficients

The challenge in working with a nonhomogeneous system is to find a particular solution of the nonhomogeneous system. There are various techniques for finding a particular solution. We can use the *variation of parameters* technique seen in Section 4.4, but for systems the calculations involved are very tedious. Therefore, we'll restrict our attention to the method of undetermined coefficients (Section 4.3), which is not as powerful, but is easier to use. As we saw in Section 4.3 and Section 4.4, this method requires intelligent guessing. We have to ask ourselves what terms are contained in $B(t)$ but not in X_{GH}, and then guess at the form of X_{PNH} on the basis of this information.

We should note that this method of undetermined coefficients can be used only when the vector $B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$ contains terms that are constants, exponential functions, sines, cosines, polynomials, or any sum or product of such terms. For other kinds of functions making up $B(t)$, X_{PNH} must be found using some other technique (for example, variation of parameters).

The next example illustrates the method with its resulting algebraic complexities.

Example 6.10.2 Using the Method of Undetermined Coefficients

Let's consider the system $\frac{dx}{dt} = x + \sin t$, $\frac{dy}{dt} = t - y$ that we discussed at the beginning of this section. We have $\dot{X} = AX + B(t)$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B(t) = \begin{bmatrix} \sin t \\ t \end{bmatrix} = \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The eigenvalues of A are 1 and -1, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, the general solution of the homogeneous system can be written as

$$X_{\text{GH}} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(*Verify the statements in this paragraph for yourself.*)

Now we look for a particular solution of the original nonhomogeneous equation. First, we compare the terms of $B(t)$ with the terms of X_{GH} to see whether there is any duplication. In this case, we see that the terms $\sin t$ and t are not terms that can be obtained just from X_{GH}. Because our system is equivalent to a single second-order differential equation, we realize that we have to find a function that can combine with its own first and second derivatives to yield $B(t)$. We take a guess that X_{PNH} must look like $C \sin t + D \cos t + Et + F$, where C, D, E, and F are *vectors of constants*. Our trial solution for X_{PNH} consists of a linear combination of the functions $\sin t$ and t and their derivatives—a linear combination with *undetermined coefficients*.

Let's substitute our guess into the nonhomogeneous system:

$$\overbrace{C \cos t - D \sin t + E}^{\dot{X}_{\text{PNH}}} = A \left(\overbrace{C \sin t + D \cos t + Et + F}^{X_{\text{PNH}}} \right) + \overbrace{\sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B(t)}$$

$$= AC \sin t + AD \cos t + AEt + AF + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

When we collect like terms, matching the coefficients of functions on each side, we get the following system:

(1) $C = AD$ [The coefficients of $\cos t$ must be equal.]

(2) $-D = AC + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ [The coefficients of $\sin t$ must be equal.]

(3) $0 = AE + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ [The coefficients of t must be equal.]

(4) $E = AF$ [The constant terms must be equal.]

Remembering that $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we can solve Eq. (3) for E:

$$AE = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

so $e_1 = 0$ and $e_2 = 1$. (*Check this.*) Now that we know E, we can use Eq. (4) to find F:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

so $f_1 = 0$ and $f_2 = -1$.

If we multiply both sides of (1) by A, we get $AC = A^2D = D$ (because $A^2 = I$, the 2×2 identity matrix), which we can substitute into Eq. (2): $-D = D + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 2D = \begin{bmatrix} 2d_1 \\ 2d_2 \end{bmatrix}$, so $d_1 = -\frac{1}{2}$ and $d_2 = 0$. (*Make sure you follow all this.*) Finally, we solve (1) for C: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$, so $c_1 = -\frac{1}{2}$ and $c_2 = 0$.

We have determined all the coefficients. Putting the pieces together, we have

$$X_{\text{PNH}} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}\cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\sin t + \cos t) \\ t - 1 \end{bmatrix},$$

and we finally obtain

$$\begin{aligned} X_{\text{GNH}} &= X_{\text{GH}} + X_{\text{PNH}} \\ &= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}(\sin t + \cos t) \\ t - 1 \end{bmatrix} = \begin{bmatrix} c_1 e^t - \frac{1}{2}(\sin t + \cos t) \\ t - 1 + c_2 e^{-t} \end{bmatrix} \end{aligned}$$

as the general solution of the original nonhomogeneous equation.

Note that the system in this example is *uncoupled*—that is, each equation contains only one unknown function. Problem 17 in Exercises 6.10 asks you to solve each equation separately to obtain the same answer as the one shown here.

Practice in the technique of undetermined coefficients leads to a more systematic way of guessing a possible solution of the nonhomogeneous system. The second column of Table 6.3 indicates the component of X_{PNH} that corresponds to the matching component $b_i(t)$ of $B(t)$. If $b_i(t)$ is a sum of different functions, then it is a consequence of the Superposition Principle that the matching component of X_{PNH} is a sum of trial solutions.

There is an exception to the neatness of the table. If $b_i(t)$ contains terms that duplicate any corresponding parts of X_{GH}, then each corresponding trial term must be multiplied by t^m, where m is the smallest positive integer that eliminates the duplication.

Table 6.3 Trial Particular Solutions for Nonhomogeneous Systems

$b_i(t)$	Form of Trial Solution
$c \neq 0$, a constant	K, a constant
$P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$	$Q_n(t) = c_n t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$
ce^{at}	Ke^{at}
$a\cos rt + b\sin rt$	$\alpha\cos rt + \beta\sin rt$
$e^{Rt}(a\cos rt + b\sin rt)$	$e^{Rt}(\alpha\cos rt + \beta\sin rt)$
$P_n(t)e^{at}$	$Q_n(t)e^{at}$

In Example 6.10.2, we had $b_1(t) = \sin t$ and $b_2(t) = t$—a trigonometric function $a\cos rt + b\sin rt$, with $a = 0$, $r = 1$, and $b = 1$, and a first-degree polynomial $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, where $n = 1$, $a_1 = 1$, and $a_0 = 0$. There was no duplication between X_{GH} and $B(t)$ because the terms making up X_{GH} are exponential functions. Consequently, our educated guess for X_{PNH} consisted of a linear combination of sine and cosine plus a first-degree polynomial.

Let's use the instant wisdom conferred by Table 6.3 to solve the next problem.

Example 6.10.3 Undetermined Coefficients

Suppose we are trying to solve the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 3y - 2x + 2t^2 + 3e^{2t}$. We can write this system as $\dot{X} = AX + B(t)$, where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and $B(t) = \begin{bmatrix} 0 \\ 2t^2 + 3e^{2t} \end{bmatrix} = \left(2t^2 + 3e^{2t}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The eigenvalues of A are 1 and 2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (*Verify this.*)

We know that the general solution of the homogeneous system is given by

$$X_{\text{GH}} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To find a particular solution of the nonhomogeneous system, we compare the terms of $B(t)$ with the terms of X_{GH} to see whether there is any duplication. In this example, ignoring constants, we see that e^{2t} appears in both X_{GH} and $B(t)$. We also recognize that the term t^2 in $B(t)$ is *not* found in X_{GH}. Using Table 6.3 and the description of how to handle duplicate terms, we guess that X_{PNH} must look like

$$Ct^2 + Dt + E + Fe^{2t} + Gte^{2t},$$

where C, D, E, F, and G are vectors of constants. Note that because there is a second-degree term, our trial particular solution contains a full quadratic polynomial and multiplying e^{2t} by t eliminates the duplication.

If we substitute this guess into the nonhomogeneous system, we get

$$2Ct + D + 2Fe^{2t} + Ge^{2t} + 2Gte^{2t} = A\left(Ct^2 + Dt + E + Fe^{2t} + Gte^{2t}\right) + \left(2t^2 + 3e^{2t}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= ACt^2 + ADt + AE + AFe^{2t} + AGte^{2t} + \left(2t^2 + 3e^{2t}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(AC + \begin{bmatrix}0\\2\end{bmatrix}\right) t^2 + ADt + AE + \left(AF + \begin{bmatrix}0\\3\end{bmatrix}\right) e^{2t} + AGte^{2t}.$$

Matching the coefficients of like terms on each side, we get the system

(1) $0 = AC + \begin{bmatrix}0\\2\end{bmatrix}$ [The coefficients of t^2 must be equal.]

(2) $2C = AD$ [The coefficients of t must be equal.]

(3) $D = AE$ [The constant terms must be equal.]

(4) $2F + G = AF + \begin{bmatrix}0\\3\end{bmatrix}$ [The coefficients of e^{2t} must be equal.]

(5) $2G = AG$ [The coefficients of te^{2t} must be equal.]

Working through these equations (see Problem 19 in Exercises 6.10), we find that

$$C = \begin{bmatrix}1\\0\end{bmatrix}, D = \begin{bmatrix}3\\2\end{bmatrix}, E = \begin{bmatrix}\frac{7}{2}\\3\end{bmatrix}, F = \begin{bmatrix}0\\3\end{bmatrix}, \text{ and } G = \begin{bmatrix}3\\6\end{bmatrix}.$$

Now that we've determined the coefficients C, D, E, F, and G, we can construct the particular solution of the nonhomogeneous equation.

$$X_{\text{PNH}} = \begin{bmatrix}1\\0\end{bmatrix} t^2 + \begin{bmatrix}3\\2\end{bmatrix} t + \begin{bmatrix}\frac{7}{2}\\3\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} e^{2t} + \begin{bmatrix}3\\6\end{bmatrix} te^{2t}.$$

Finally, we get the general solution of the nonhomogeneous equation:

$$\begin{aligned}
X_{\text{GNH}} &= X_{\text{GH}} + X_{\text{PNH}} \\
&= c_1 e^t \begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t} \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} t^2 + \begin{bmatrix}3\\2\end{bmatrix} t + \begin{bmatrix}\frac{7}{2}\\3\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} e^{2t} + \begin{bmatrix}3\\6\end{bmatrix} te^{2t} \\
&= \begin{bmatrix} c_1 e^t + c_2 e^{2t} + t^2 + 3t + \frac{7}{2} + 3te^{2t} \\ c_1 e^t + 2c_2 e^{2t} + 2t + 3 + 3e^{2t} + 6te^{2t} \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^t + (c_2 + 3t)\, e^{2t} + t^2 + 3t + \frac{7}{2} \\ c_1 e^t + (2c_2 + 3 + 6t)\, e^{2t} + 2t + 3 \end{bmatrix}.
\end{aligned}$$

Of course, this means that $x(t) = c_1 e^t + (c_2 + 3t)e^{2t} + t^2 + 3t + 7/2$ and $y(t) = c_1 e^t + (2c_2 + 3 + 6t)e^{2t} + 2t + 3$ are the solutions of our system. You should check to see that these functions satisfy our original system.

When the nonhomogeneous system is also *autonomous*—that is, it has the form $\dot{X} = AX + B(t)$, where the entries of $B(t)$ are *constants*—we can analyze the stability of the system's solutions by finding the equilibrium point(s) (no longer the origin) and considering the eigenvalues and eigenvectors of the matrix A.

Example 6.10.4 Stability of an Autonomous Nonhomogeneous System

We return to the system of Example 6.4.3:

$$\begin{aligned}
\dot{x} &= 7y - 4x - 13 \\
\dot{y} &= 2x - 5y + 11.
\end{aligned}$$

To find the equilibrium point(s), we solve the algebraic system

$$\begin{aligned} -4x + 7y &= 13 \\ 2x - 5y &= -11 \end{aligned}$$

to find that (2, 3) is the only equilibrium point. (The details in this example are left as parts of Problem 21 in Exercises 6.10.)

We can write our system of differential equations in the form

$$\dot{X} = AX + B(t) = \begin{bmatrix} -4 & 7 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -13 \\ 11 \end{bmatrix}.$$

Because the eigenvalues of A are $\lambda_1 = \left(\sqrt{57} - 9\right)/2$ and $\lambda_2 = -\left(\sqrt{57} + 9\right)/2$, both of which are negative real numbers, Table 6.2 at the end of Section 6.9 tells us that the equilibrium point (2, 3) is a *sink*. (Go back to take another look at Fig. 6.8.)

Despite its limitations, the method of undetermined coefficients is very useful. In Chapter 5 we saw another way of solving systems of nonhomogeneous linear equations, by means of the *Laplace transform*. This transform method was particularly useful in solving IVPs.

Exercises 6.10

A

1. Find the particular solution of the system in Example 6.10.2 that satisfies $x(0) = 0$, $y(0) = 1$.
2. Find the particular solution of the system in Example 6.10.3 that satisfies $x(0) = -1$, $y(0) = 2$.

Without using technology, find the general solution of each of the systems in Problems 3–16. You may check your answers using a CAS.

3. $\dot{x} = y + 2e^t$, $\dot{y} = x + t^2$
4. $\dot{x} = y - 5\cos t$, $\dot{y} = 2x + y$
5. $\dot{x} = 3x + 2y + 4e^{5t}$, $\dot{y} = x + 2y$
6. $\dot{x} = 3x - 4y + e^{-2t}$, $\dot{y} = x - 2y - 3e^{-2t}$
7. $\dot{x} = 4x + y - e^{2t}$, $\dot{y} = y - 2x$ [*Hint*: Multiples of both e^{2t} and te^{2t} should appear in your guess for X_{PNH}.]
8. $\dot{x} = 2y - x + 1$, $\dot{y} = 3y - 2x$ [*Hint*: Multiples of both e^t and te^t should appear in your guess forX_{PNH}.]
9. $\dot{x} = 5x - 3y + 2e^{3t}$, $\dot{y} = x + y + 5e^{-t}$
10. $\dot{x} = x + y + 1 + e^t$, $\dot{y} = 3x - y$
11. $\dot{x} = 2x - y$, $\dot{y} = 2y - x - 5e^t \sin t$
12. $\dot{x} = x + 2y$, $\dot{y} = x - 5\sin t$

13. $\dot{x} = y,\ \dot{y} = -2x - 3y + \sin t + e^t$

14. $\dot{x} = -2x + y + 2e^{-t},\ \dot{y} = x - 2y + 3t$

15. $\dot{x} = x + y + e^t,\ \dot{y} = y + e^{-t}$

16. $\dot{x} = x + y + t - 2,\ \dot{y} = 4x + y + 4t - 1$

B

17. Consider each equation in Example 6.10.2 as a first-order linear equation and solve each equation separately, confirming that you get the same answer as in the worked-out example. (You may want to review Section 2.2 and the technique of integration by parts.)

18. **a.** Use technology to draw the phase portrait for the system in Example 6.10.2.

b. Draw a graph of $x(t)$ vs. t.

c. Draw a graph of $y(t)$ vs. t.

19. Assume that you have Eqs. (1)–(5) in Example 6.10.3. Let $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, etc. Solve for the vectors C, D, E, F, and G in the following order:

a. Use Eq. (1) to show that $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b. Use Eq. (2) to show that $D = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

c. Use Eq. (3) to show that $E = \begin{bmatrix} 7/2 \\ 3 \end{bmatrix}$.

d. Assuming that G is not the zero vector, use Eq. (5) to derive a general form for G. (There is an arbitrary constant involved.)

e. Substitute the general form for G found in part (d) into Eq. (4) to determine the concrete form of G. Then use this information to see that a convenient form for F is $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

20. **a.** Use technology to draw the phase portrait for the system in Example 6.10.3.

b. Draw the graph of $x(t)$ vs. t, assuming that $x(0) = 50$.

c. Draw the graph of $y(t)$ vs. t, assuming that $y(0) = 100$.

21. Look at the system in Example 6.10.4.

a. Show that the only equilibrium point is (2, 3).

b. Show that the eigenvalues of the matrix of coefficients A are

$$\lambda_1 = \left(\sqrt{57} - 9\right)/2 \quad \text{and} \quad \lambda_2 = -\left(\sqrt{57} + 9\right)/2.$$

c. Find eigenvectors corresponding to λ_1 and λ_2.

d. Express the general solution of the homogeneous system in terms of the eigenvalues and eigenvectors found in parts (b) and (c).

e. Find a particular solution of the nonhomogeneous system.

f. Put the answers to parts (d) and (e) together to get the general solution of the nonhomogeneous system. Then determine what happens as $t \to \infty$.

22. Newton's laws of motion give the following system as a model for the motion of an object falling under the influence of gravity:

$$\begin{aligned}\frac{dy}{dt} &= v(t)\\ \frac{dv}{dt} &= g - cv(t);\;\; y(0) = 0,\, v(0) = 0\end{aligned}$$

for $0 \le t \le T$, where $y(T) = H$. Here, $y(t)$ denotes the downward distance from the spot where the object was dropped to the place where the falling object is at time t; $v(t)$ is the velocity; g is the gravitational constant; and c is the *drag coefficient*, representing air resistance.

a. Without using technology, solve this nonhomogeneous system for $y(t)$ and $v(t)$.

b. Find $\lim_{t\to\infty} v(t)$ and interpret your answer in physical terms.

23. A cold medication moving through the body can be modeled[8] by the IVP

$$\begin{aligned}\dot{x} &= -k_1 x + I\\ \dot{y} &= k_1 x - k_2 y;\;\; x(0) = 0,\, y(0) = 0,\end{aligned}$$

where $x(t)$ and $y(t)$ are the amounts of medication in the gastrointestinal (GI) tract and the bloodstream, respectively, at time t measured in hours elapsed since the initial dosage. Here, $I > 0$ is the constant dosage rate and k_1, k_2 are positive transfer rates (out of the GI tract and bloodstream, respectively).

a. Without using technology, solve the nonhomogeneous system for $x(t)$ and $y(t)$.

b. Find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

c. Assume that the decongestant part of a continuous acting capsule (such as Contac®) has $k_1 = 1.386/\text{h}$ and $k_2 = 0.1386/\text{h}$ and that the antihistamine portion has $k_1 = 0.6931/\text{h}$ and $k_2 = 0.0231/\text{h}$. Also assume that $I = 1/6$ (i.e., one unit per six hours). Use technology to graph $x(t)$ against t and $y(t)$ against t for the decongestant on the same set of axes.

[8] This model is based on the work of Edward Spitznagel of Washington University and was first communicated to me by Courtney Coleman, Harvey Mudd College.

d. Assuming the data given in part (c), use technology to graph $x(t)$ and $y(t)$ for the antihistamine on the same set of axes.

24. The buying behavior of the public toward a particular product can be modeled by

$$\begin{aligned}\frac{dB}{dt} &= b(M - \beta B)\\ \frac{dM}{dt} &= a(B - \alpha M) + cA,\end{aligned}$$

where $B = B(t)$ is the level of buying, $M = M(t)$ is a measure of the public's motivation or attitude toward the product, and $A = A(t)$ is the advertising policy. The parameters a, b, c, α, and β are all assumed positive.

a. Show that for constant advertising (i.e., $A(t)$ is a constant function) the buying levels tend to a limiting value over time.

b. If $\alpha = \beta = 2$, $a = b = c = 1$, $B(0) = M(0) = 0$, and

$$A(t) = \begin{cases} 100 \text{ units} & \text{for } 0 < t < 10 \\ 0 & \text{for } t > 10 \end{cases}$$

determine the complete forecast for the buying behavior—that is, find $B(t)$ and $M(t)$.

25. A political race between two candidates can be modeled by the system

$$\begin{aligned}\frac{dx}{dt} &= ax - by + e\\ \frac{dy}{dt} &= -cx + dy - e,\end{aligned}$$

where $x(t)$ and $y(t)$ represent the number of supporters of the two candidates and $e > 0$ denotes a gain in supporters for the first candidate based on contact between those committed to the first candidate and those who are uncommitted. The parameters a, b, c, and d are positive.

a. Determine conditions involving a, b, and e guaranteeing that $dx/dt > 0$.

b. Determine conditions involving c, d, and e guaranteeing that $dy/dt > 0$.

c. Determine conditions guaranteeing equilibrium in the system.

26. Consider a closed, two-compartment model in which the initial concentrations of a dye are 2 mg/l in Compartment 1 and 10 mg/l in Compartment 2. The compartments have constant volumes of 10 and 20 liters, respectively, and are separated by a permeable membrane that allows transfer between the compartments at the rate of 0.25 l/h.

a. Determine formulas for the concentrations of dye at any time t in each compartment.

b. Determine what happens to the concentrations in each compartment as $t \to \infty$.

C

27. During World War I, the English scientist F.W. Lanchester (1868–1946) devised several mathematical models for the new art of aerial combat. These models have since been extended and applied to various modern conflicts. One model, describing the interaction of two conventional armies (as opposed to guerrilla forces or a mixture of conventional and guerrilla forces), is given by

$$\frac{dx}{dt} = -ay + f(t) - c$$
$$\frac{dy}{dt} = -bx + g(t) - d; \; x(0) = \alpha, \, y(0) = \beta,$$

where $x(t)$ and $y(t)$ represent the strengths of the opposing forces at time t; a and b denote nonnegative loss rates; c and d are constant noncombat losses per day; and $f(t)$ and $g(t)$ denote reinforcement rates in number of combatants per day.

a. Assuming that $f(t) = k$ and $g(t) = l$ (k and l are constants) during a battle, determine the strengths of each army at time t during the battle.

b. If $\alpha > \frac{l-d}{b} > 0$ and $\beta > \frac{k-c}{a} > 0$, determine the conditions under which the y-force will be wiped out.

c. Assume that $a = 0.006$, $b = 0.008$, $c = d = 1000$, $k = 6000$, $l = 4000$, $\alpha = 90{,}000$, and $\beta = 200{,}000$, where c, d, k, and l are measured in men per day. Use technology to graph $x(t)$ and $y(t)$ for $0 \le t \le 60$. Then use the graphs to determine the time t^* when $x(t^*) = y(t^*)$. Which side is winning after 50 days?

28. A two-compartment model for cholesterol flow yields the following nonhomogeneous system:

$$\frac{dx}{dt} = -(\alpha + \beta)x + \gamma y + K_1 + K_2$$
$$\frac{dy}{dt} = \beta x - \gamma y + K_3,$$

where x and y denote the amounts of cholesterol in the two compartments, β and γ represent rates at which cholesterol moves from one compartment to the other, α represents a rate of excretion, and K_1, K_2, K_3 denote the rates at which cholesterol flows into the compartments.

a. Solve the system using technology for the lengthy algebraic calculations.

b. Describe the behavior of the solutions $x(t)$ and $y(t)$ over a long period of time.

*6.11 Spring-mass problems[9]

6.11.1 Simple harmonic motion

We have seen spring-mass problems before (for example, in the exercises in Section 4.1). The treatment in this section represents a systematic exposition of this important model and demonstrates the usefulness of a qualitative approach.

To start with, suppose we have a spring attached to the ceiling and a weight (mass) hanging from the bottom of the spring, as in Fig. 6.27a.

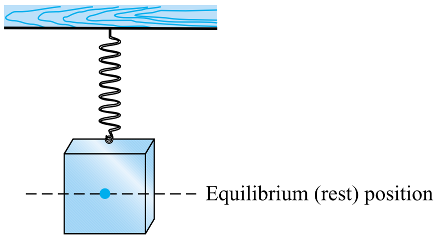


FIGURE 6.27a

Spring-mass system, mass in the equilibrium position

If we set the mass in motion by giving it an upward or downward push, we can use Newtonian mechanics and the qualitative analysis of systems of ODEs to investigate the forces acting on the mass during its motion. We want to describe the state of this system, giving the mass's position and velocity at any time t. First, we'll assume that there's no air resistance, friction, or other impeding force. The resulting situation is called **simple harmonic motion,** or **free undamped motion**.

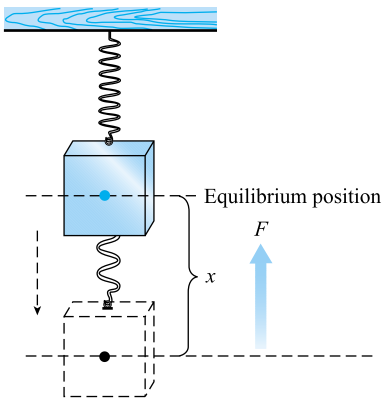


FIGURE 6.27b

Spring-mass system, mass displaced from the equilibrium position

9 * Denotes an optional section.

Fundamental to understanding the mass's movement is **Newton's Second Law of Motion**, which can be stated as $F = m \cdot a$, where F is a force (or sum of forces) acting on a body (such as the weight hanging from the spring), m is the body's mass, and a is the acceleration of the body. If x denotes the displacement (distance) of the mass from its equilibrium (rest) position, where a move downward is considered a *positive* displacement (Fig. 6.27b), we can write this expression for the force as $m \cdot \frac{d^2x}{dt^2}$.

Now note that if you pull *down* on the weight (stretching the spring in the process), you can feel a certain tension—a tendency for the spring to pull the weight back *up*. Similarly, if you push *up* on the weight, thereby compressing the spring, you feel a force that tends to push the weight *down*. This behavior is described by **Hooke's Law**: The force F (called the restoring force) exerted by a spring, tending to restore the weight to the equilibrium position, is proportional to the distance x of the weight from the equilibrium position. Stated simply, *force is proportional to stretch*. Mathematically, we write $F = -kx$, where k is a positive constant called the *spring constant*. Note that if x is *positive*, then the restoring force is *negative*, whereas if x is *negative*, then F is *positive*.

Because we are ignoring any other kind of force acting on the weight, we can equate the two expressions for the force to get

$$m \cdot \frac{d^2x}{dt^2} = -kx,$$

which we can write in the form

$$\frac{d^2x}{dt^2} + \beta x = 0, \text{ where } \beta = \frac{k}{m}. \tag{6.11.1}$$

We saw this kind of homogeneous second-order linear equation in Section 4.1, and from our work in Section 6.1 we know how to convert this equation into an equivalent system of first-order equations. Earlier in this section we learned how to understand what a phase portrait is telling us. Now let's analyze this problem qualitatively.

Example 6.11.1 A Spring-Mass System—Simple Harmonic Motion

Given the equation $\frac{d^2x}{dt^2} + \beta x = 0$, where $\beta = \frac{k}{m}$, we let $x_1 = x$ and $x_2 = \dot{x}$. We see that $\dot{x}_1 = \dot{x} = x_2$ and $\dot{x}_2 = \ddot{x} = -\beta x$ (by solving the second-order equation for the second derivative) $= -\beta x_1$, so we have the two-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_1. \end{aligned} \tag{6.11.2}$$

First of all, note that x_2 represents the *velocity* of the mass, $x_2 = \dot{x}$, the rate of change of the position, or displacement of the mass. Using the language developed in Example 6.4.2, we say that if we could solve the System (6.11.2) for x_1 and x_2, then the ordered pair $(x_1(t), x_2(t))$, consisting of the mass's current position and velocity, would give the *state* of the system at time t.

Now we can look at some trajectories in the phase plane of (6.11.2)—that is, some solution curves in the x_1-x_2 plane. Using initial points $(x_1(0), x_2(0)) = (1, 0)$, $(0, 1)$, and $(2, 0)$, Fig. 6.28

shows what these curves look like when $\beta = \frac{2}{5}$ and we take the interval $0 \le t \le 10$. (*You should use technology to plot your own trajectories with different initial points and smaller ranges for t.*)

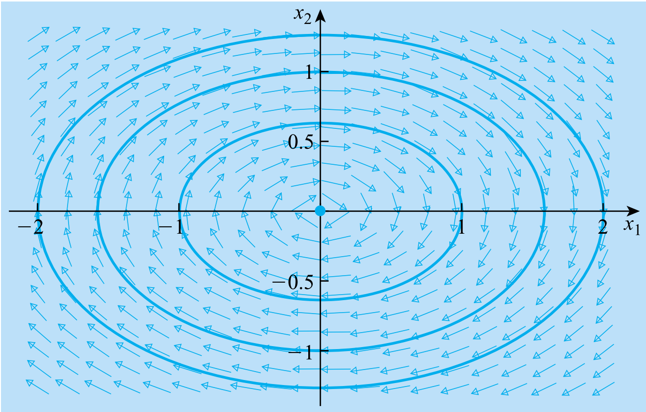


FIGURE 6.28

Trajectories for $\left\{\dot{x}_1 = x_2, \dot{x}_2 = -\frac{2}{5}x_1\right\}$. Initial points $(1, 0)$, $(0, 1)$, $(2, 0)$; $0 \le t \le 10$

6.11.2 Analysis

Is this the behavior expected from a bouncing mass? First of all, note that the origin is a special point, an equilibrium solution, because both equations of our system vanish at $(x_1, x_2) = (0, 0)$. Physically, this means that a mass-spring system that starts at its equilibrium position ($x(0) = 0$) and has no initial push or pull ($\dot{x}(0) = x_2(0) = 0$) will remain at rest forever, which makes sense.

Now we look closely at a typical **closed orbit**, as one of these elliptical trajectories is called. Assume that $x_1 = 0$ and x_2 is positive—that is, the mass is at its equilibrium position and is given an initial tug downward. When the mass is at rest ($x_1 = 0$) and it is pushed or pulled in a downward direction ($dx_1/dt = x_2 > 0$), the flow moves in a clockwise direction (note the direction of the slope field arrows), with x_2 decreasing and x_1 increasing until the trajectory is at the x_1-axis. Physically, this means that the mass moves downward until the spring reaches its maximum extension (x_1 is at its most positive value), depending on how much force was applied initially to pull the mass downward, at which time the mass has lost all its initial velocity ($x_2 = 0$). Then the energy stored in the spring serves to pull the mass back up toward its equilibrium position, so that x_1 is decreasing at the same time that the velocity x_2 is increasing—*but in a negative direction* (*upward*). Graphically, this is taking place in the fourth quadrant of the phase plane. When the flow has reached the state $(0, x_2)$, where x_2 is negative, the mass has reached its original position and has attained its maximum velocity upward.

As the trajectory takes us into the third quadrant, the mass is overshooting its original position but is slowing down: $x_1 < 0$ and $x_2 < 0$. When the trajectory has reached the point $(x_1, 0)$, where x_1 is negative, the spring is most compressed and the mass is (for an instant) not moving.

As the trajectory moves through the second quadrant, the mass is headed back toward its initial position with increasing velocity in a downward (positive) direction: $x_1 < 0$ and $x_2 > 0$. Finally, the mass reaches its initial position with its initial velocity in the positive (downward) direction—$x_1 = 0$, $x_2 > 0$—and the cycle begins all over again.

This analysis seems to say that the mass will never stop, bobbing up and down forever. This apparently nonsensical conclusion is perfectly reasonable when you realize that a real mass-spring system is always subject to some air resistance and some sort of friction that slows the system down and eventually forces the mass to stop moving. Our analysis assumes no such impeding force, so the conclusion is rational, even though the assumption is unrealistic.

6.11.3 Another view—solution curves

As we did in several of the examples in Section 6.4, we can use technology to plot each solution of our system against t. Figs. 6.29a and 6.29b show the solution with $\beta = \frac{2}{5}$, $x_1(0) = 1$, and $x_2(0) = 1$, corresponding to a spring-mass system that starts 1 unit below its equilibrium position and has been given an initial velocity of 1 in a downward direction. We should not be surprised at the appearance of these solution curves. The closed orbits in Fig. 6.28 reflect the periodic nature of the motion of the mass. These motions are called **oscillations**. Using the methods that we saw in Section 4.1, we can determine that when $\beta = \frac{2}{5}$ the general solution of system (6.11.2) is

$$x_1(t) = \frac{C_1}{2}\sqrt{10}\sin\left(\frac{1}{5}\sqrt{10}t\right) + C_2\cos\left(\frac{1}{5}\sqrt{10}t\right)$$
$$x_2(t) = C_1\cos\left(\frac{1}{5}\sqrt{10}t\right) - \frac{C_2}{5}\sqrt{10}\sin\left(\frac{1}{5}\sqrt{10}t\right),$$

and we can see that the explicit source of the oscillations is the trigonometric terms. The particular system solution shown in Figs. 6.29a and 6.29b corresponds to the initial conditions $x_1(0) = 1$, $x_2(0) = 1$, so $C_1 = C_2 = 1$. (*Verify this*.)

Remembering the discussion of the *equivalence* of a second-order equation and a system in Example 6.1.1, we realize that

$$x_1(t) = \frac{C_1}{2}\sqrt{10}\sin\left(\frac{1}{5}\sqrt{10}t\right) + C_2\cos\left(\frac{1}{5}\sqrt{10}t\right)$$

is the general solution of the original single differential equation $\frac{d^2x}{dt^2} + \beta x = 0$, where $\beta = \frac{2}{5}$. (Review Section 6.1. It happens that $x_2(t) = dx_1/dt$ is also a solution, but this is true only because the equation is *homogeneous*.)

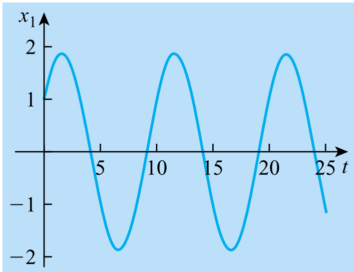


FIGURE 6.29a

$x_1(t)$, displacement; $x_1(0) = 1,\ 0 \leq t \leq 25$

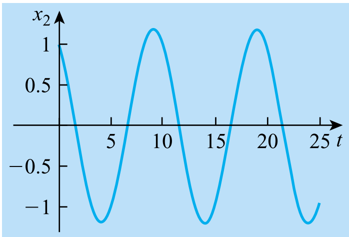


FIGURE 6.29b

$x_2(t)$, velocity; $x_2(0) = 1,\ 0 \leq t \leq 25$

6.11.4 Free damped motion

Now let's look at a more realistic version of a spring-mass system. This time we'll assume the existence of a combination of air resistance and some friction in the spring-mass system, called a **damping force**, to slow the mass down. To dramatize the situation, you may think of the mass as being immersed in a bucket of water, oil, or maple syrup, so that any initial force imparted to the mass is opposed by a force in the opposite direction as the mass meets resistance. The motion that results is called **free damped motion**. For instance, the damping produced by automobile shock absorbers provides a more comfortable ride.

The damping force works *against* the motion of the mass, so when the mass is moving *down* (the positive direction), the damping force acts in an *upward* direction, and when the mass is moving *up* (the negative direction), the damping force acts in a *downward* direction. In algebraic terms, this damping force's sign must be opposite to the sign of the direction of the velocity. For small velocities, experiments have shown that the damping force is proportional to the velocity of the mass. We can express the last two sentences mathematically as $F = -\alpha \frac{dx}{dt}$, where α is a positive constant of proportionality called the **damping constant**. Realizing that both the spring's restoring force and this damping force are opposed to the mass's motion, we can use *Newton's Second Law of Motion* to derive the equation

$$m \cdot \frac{d^2x}{dt^2} = -\alpha \frac{dx}{dt} - kx,$$

which we can write in the form

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cs = 0, \quad \text{where} \quad b = \frac{\alpha}{m} \text{ and } c = \frac{k}{m}. \tag{6.11.3}$$

Now we can convert this second-order differential equation into a system and analyze our problem qualitatively.

Example 6.11.2 A Spring-Mass System—Free Damped Motion

The second-order linear equation $\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ is equivalent to the two-dimensional system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -bx_2 - cx_1. \end{aligned} \tag{6.11.4}$$

Phase Portrait Analysis

To understand the motion of the mass, we'll look first at the trajectory we get when we take $b = \frac{1}{4}$, $c = 2$, $x_1(0) = 1$, and $x_2(0) = 0$ (Fig. 6.30). In particular, we should see that the mass starts off 1 unit *below* its equilibrium position with *no* initial velocity in any direction.

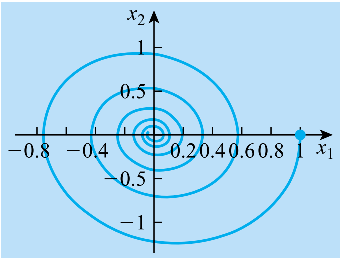


FIGURE 6.30

Trajectory for the system $\left\{\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = -\frac{1}{4}x_2 - 2x_1; x_1(0) = 1, x_2(0) = 0\right\}$, $0 \le t \le 25$

The direction of the trajectory in Fig. 6.30 indicates very dramatically that the state of the system is spiraling into the origin—that is, $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$. Every time the spiral trajectory in Fig. 6.30 crosses the x_2-axis (so that $x_1 = 0$), the mass is at its equilibrium position—on its way up when the velocity x_2 is negative and on its way down when x_2 is positive. (Remember our agreement on which direction is positive and which direction is negative.) This type of spiral clearly indicates why we can say that the origin is a **sink** for the system.

Another View

We can also look at the graphs of $x_1(t)$ against t (Fig. 6.31a) and $x_2(t)$ against t (Fig. 6.31b) for the same system. The oscillations shown in Figs. 6.31a and 6.31b reflect the behavior of the system in a different way.

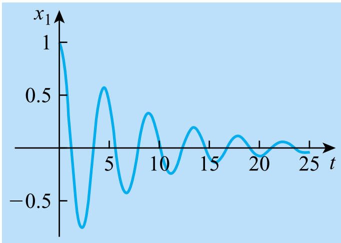


FIGURE 6.31a

$x_1(t)$, displacement; $x_1(0) = 1$, $0 \le t \le 25$

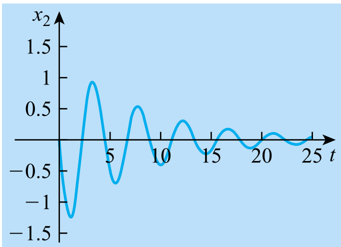


FIGURE 6.31b

$x_2(t)$, velocity; $x_2(0) = 0$, $0 \le t \le 25$

The mass reaches its equilibrium position when the $x_1(t)$ curve crosses the t-axis. If $x_1(t^*) = 0$, then look at Fig. 6.31b to see what the value of $x_2(t^*)$ is. If $x_2(t^*) > 0$, for example, the mass is on its way *down*. Also note how Figs. 6.30, 6.31a, and 6.31b show that the successive rises and falls get progressively smaller.

The figures all reflect the initial conditions and seem to say that the mass eventually comes to rest at its equilibrium position. If you were to hit a brass gong with a special ceremonial hammer, the vibrations would be loud at the beginning, but would gradually fade to nothing. This is roughly what we are seeing here.

A Look at the Actual Solution

The curves in Figs. 6.31a and 6.31b are *not* periodic, despite their resemblance to familiar trigonometric curves that are. In Section 4.1 we saw how to determine that the solution of our IVP $\frac{d^2x}{dt^2} + \frac{1dx}{4dt} + 2x = 0$, with $x(0) = 1$ and $\frac{dx}{dt}(0) = 0$, is given by

$$x(t) = e^{\left(-\frac{1}{8}t\right)}\left(\cos\left(\frac{1}{8}\sqrt{127}t\right) + \frac{\sqrt{127}}{127}\sin\left(\frac{1}{8}\sqrt{127}t\right)\right),$$

which is not a pure trigonometric function because of the exponential factor. (*You should verify that this is a solution.*) In terms of our system (6.11.4), we have $x_1(t) = x(t)$ and $x_2(t) = \frac{dx}{dt}$. (*Do the differentiation to see what $x_2(t)$ looks like.*)

The exponential factor $e^{\left(-\frac{1}{8}t\right)}$, called the **time-varying amplitude**, forces the decay of the oscillations indicated by the trigonometric terms. Fig. 6.32 shows the graph of the solution $x(t)$, together with the graphs of $e^{\left(-\frac{1}{8}t\right)}$ and $-e^{\left(-\frac{1}{8}t;\, 0\le t\le 20\right)}$.

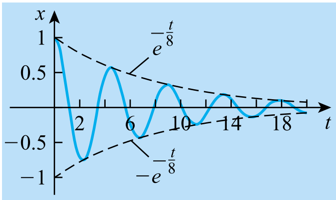


FIGURE 6.32

The graphs of $x(t)$, $e^{\left(-\frac{1}{8}t\right)}$, and $-e^{\left(-\frac{1}{8}t\right)}$

6.11.5 Different kinds of damping

You should be aware that there are different kinds of damped motion. The behavior of a damped system described by the equation $m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = 0$ depends on the relationship among the three constants m, α, and k—the mass, the damping coefficient, and the spring constant, respectively. The example we just analyzed is a case of **underdamped motion**, occurring when the damping coefficient is relatively small compared to the other constants: $\alpha^2 < 4mk$, technically. The other two possibilities, **overdamped motion** ($\alpha^2 > 4mk$) and **critically damped motion** ($\alpha^2 = 4mk$), are explored in Problems 1 and 2 in Exercises 6.11.

6.11.6 Forced motion

Sometimes a physical system is subject to external forces, which must appear in its mathematical representation. For example, the motion of an automobile (whose body-suspension combination can be considered a spring-mass system) is influenced by irregularities in the road surface. Similarly, a tall building may be subjected to strong winds that will cause it to sway in an uncharacteristic way.

We're going to look at an IVP related to Example 2.2.5 and to Problems 27–29 in Exercises 2.2. This discussion will involve an important type of second-order linear equation with a forcing term.

Example 6.11.3 Forced Damped Motion

Suppose we have an electrical circuit with an inductance of 0.5 henry, a resistance of 6 ohm, a capacitance of 0.02 farad, and a generator providing alternating voltage given by 24 sin $(10t)$ for $t \ge 0$.

The alternating voltage is the external force applied to the circuit, and the resistance is a damping coefficient. Then, letting Q denote the instantaneous charge on the capacitor, *Kirchhoff's Law* gives us the equation

$$0.5\frac{d^2Q}{dt^2}+6\frac{dQ}{dt}+50Q=24\sin 10t,$$

or

$$\frac{d^2Q}{dt^2}+12\frac{dQ}{dt}+100Q=48\sin 10t.$$

Let's assume that $Q(0)=0$ and $\frac{dQ}{dt}(0)=0$.

This second-order nonhomogeneous equation is equivalent to the nonautonomous system

$$\frac{dx_1}{dt}=x_2$$
$$\frac{dx_2}{dt}=48\sin 10t-12x_2-100x_1,$$

with initial conditions $x_1(0)=0$ and $x_2(0)=0$. (*You should work this out for yourself.*)

Phase Portrait

The phase portrait (Fig. 6.33a) corresponding to this system, for $0\le t\le 0.94$, is interesting. At first, we suspect that we may get a spiral opening outward. But with an expanded range for t—say, from 0 to 5—the phase portrait resembles a closed orbit around the origin (Fig. 6.33b). We can understand the initial "blip" by using the explicit solution found by the techniques discussed in Section 4.2:

$$Q(t)=\frac{1}{10}e^{-6t}(4\cos 8t+3\sin 8t)-\frac{2}{5}\cos 10t.$$

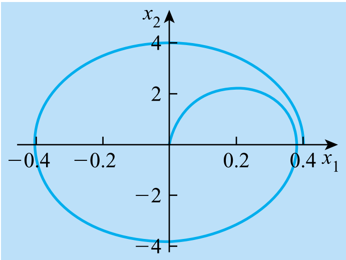


FIGURE 6.33a

Trajectory for the system $\left\{\frac{dx_1}{dt}=x_2,\ \frac{dx_2}{dt}=48\sin 10t-12x_2-100x_1;\ x_1(0)=0=x_2\right\}$, $0\le t\le 0.94$

As in Example 2.2.5, we see that there is a *transient term*, $\frac{1}{10}e^{-6t}(4\cos 8t+3\sin 8t)$, that becomes negligible as t grows large (*Why?*), and a *steady-state term*, $\frac{2}{5}\cos 10t$, that controls the behavior of $Q(t)(=x_1)$ eventually. This steady-state term is periodic with the same period $\left(\frac{2\pi}{10}=\frac{\pi}{5}\right)$ as the forcing term and has the amplitude $\frac{2}{5}$. The *current* in the circuit is given by $I=\frac{dQ}{dt}=x_2$.

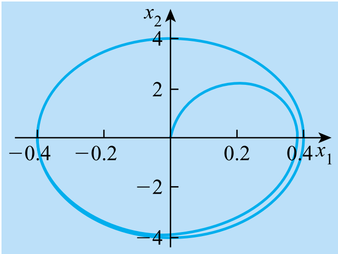


FIGURE 6.33b

Trajectory for the system $\left\{\frac{dx_1}{dt} = x_2,\ \frac{dx_2}{dt} = 48\sin 10t - 12x_2 - 100x_1;\ x_1(0) = 0 = x_2(0)\right\}$, $0 \le t \le 5$

Let's look at one more example of a spring-mass system. First, we suppose that there is no air resistance or friction. Next, we assume that the spring to which the mass is attached is supported by a board. Now we set the mass into motion by moving the supporting board up and down in a periodic manner. This situation is described as **driven undamped motion** or **forced undamped motion**. As in the preceding example, a force external to the spring-mass system itself is being applied to the system, and we want to understand the behavior of the system.

When we apply Newton's Second Law of Motion, an analysis similar to that provided in Example 6.11.1 gives us the equation

$$m \cdot \frac{d^2x}{dt^2} = -kx + f(t),$$

which we can write as

$$\frac{d^2x}{dt^2} + \beta x = F(t) \quad \text{where} \quad \beta = \frac{k}{m} \text{ and } F(t) = \frac{f(t)}{m}. \tag{6.11.5}$$

The **forcing function** $f(t)$ (or $F(t)$) describes the external force that jiggles the supporting board up and down rhythmically. Remember that we are assuming that this force is *periodic*, so $f(t)$ is sometimes positive and sometimes negative—that is, sometimes the board moves downward, and sometimes it moves upward. (Have you ever seen the toy consisting of a paddle with a rubber ball attached to it by an elastic cord?)

The next example gives us the qualitative analysis of this problem.

Example 6.11.4 Forced Undamped Motion

The system equivalent to our problem is

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = F(t) - \beta x_1. \tag{6.11.6}$$

Let's take $\beta = 4$ and assume that the forcing function is $F(t) = \cos(2t)$. Furthermore, let's assume that the mass starts from its equilibrium position, $x_1(0) = x(0) = 0$, and that it has no initial motion before the external force is applied—that is, $x_2(0) = \frac{dx}{dt}(0) = 0$. Fig. 6.32 shows the phase portrait corresponding to this IVP for $0 \le t \le 20$.

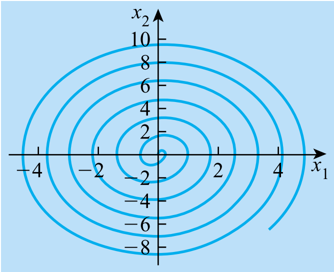


FIGURE 6.34

Trajectory for the system $\left\{\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = \cos 2t - 4x_1; x_1(0) = 0 = x_2(0)\right\}$, $0 \le t \le 20$

Analysis

Note that because the initial point is the origin, it is obvious that the spiral trajectory is moving *outward*—that is, in a clockwise direction. (You should contrast this with Fig. 6.30 in Example 6.11.2.) Fig. 6.34 indicates that both the displacement of the mass and its velocity are growing without bound. The graphs of $x_1(t)$ and $x_2(t)$ against t (Figs. 6.35a and 6.35b) confirm this.

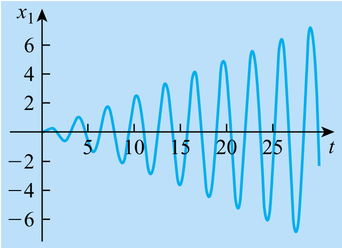


FIGURE 6.35a

$x_1(t)$, displacement, in Example 6.11.4; $0 \le t \le 30$

The Actual Solution

The solution of the system we have chosen as an example is $x_1(t) = x(t) = \frac{t}{4}\sin(2t)$. (*Check that this is a solution of the IVP.*) The sine term contributes an oscillation between -1 and 1, but the factor $\frac{t}{4}$ affects the *amplitude* of the oscillations: $|x(t)| = \left|\frac{t}{4}\right| |\sin(2t)|$, so that $-\frac{t}{4} \le x(t) \le \frac{t}{4}$ for $t \ge 0$, and $x(t)$ gets larger in both the positive and negative directions as t gets larger.

Fig. 6.36 shows how the linear factor $\frac{t}{4}$ magnifies the oscillation caused by the trigonometric factor.

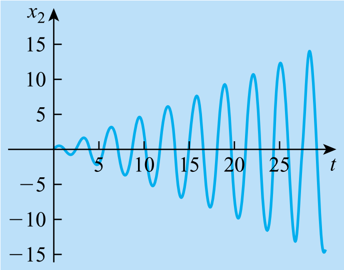


FIGURE 6.35b

$x_2(t)$, velocity; $x_2(t)$ in Example 6.11.4, $0 \le t \le 30$

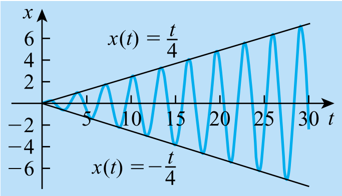


FIGURE 6.36

$x_1(t) = x(t) = \frac{t}{4}\sin(2t)$, $0 \le t \le 30$

6.11.7 Resonance

A situation in which we have unbounded oscillation, as shown in the preceding example, is called **resonance**. This is particularly important because all mechanical systems have **natural** or **characteristic frequencies**—that is, each atom making up the system is vibrating at a particular frequency, and the composite system has its own characteristic frequency. Recall that if a function g is periodic with period T (so that T is the smallest number for which $g(t+T) = g(t)$ for all t), then its **frequency** f is the number of cycles per unit of time: $f = \frac{1}{T}$. Resonance occurs when the frequency of an external force coincides with the natural frequency of the system, thereby amplifying it. You may have noticed that the windows in your home rattle when a heavy vehicle drives by. Going faster than a certain speed in a car may cause a disturbing rattling. In the preceding example, the natural frequency of the system is $\frac{1}{\pi}$ cycles per unit of time, which is equal to the frequency of the forcing function $F(t) = \cos(2t)$.

An unfortunate and frequent physical consequence of such amplified vibration is the destruction of the system. In a spring-mass system, the spring can break. A serious situation can occur when numbers of people march in step over a bridge, and the frequency of the vibrations set in place by the marching feet causes resonance and the collapse of the bridge. (This is why military columns and parade marchers "break step" when crossing a bridge.) As another example, in 1959 and 1960, several models of the same plane crashed, seeming to explode in midair. The Civil Aeronautics Board (CAB) determined that the disintegration of the planes was due to mechanical resonance. A component within the planes, when not fastened securely, generated oscillations that acted as an excessive external force on the wings, breaking them within 30 seconds.[10] Similarly, resonance occurs when the ocean's waves hit a human-made barrier or when wind swirls around a bridge support or tower.

A less disastrous example of resonance is the shattering of a glass by a powerful singer hitting a very high note. The external force here is the sound wave that amplifies the natural frequency of the glass.

It should be pointed out, however, that resonance can also be our friend. The great scientist Galileo (1564–1642) made the following observation about resonance used in the ringing of heavy, free-swinging bells in a tower[11]:

> *Even as a boy, I observed that one man alone by giving these impulses at the right instant was able to ring a bell so large that when four, or even six, men seized the rope and tried to stop it they were lifted from the ground, all of them together being unable to counterbalance the momentum which a single man, by properly timed pulls, had given it.*

A parent pushing a child's swing, timing the pushes to coincide with the swing's motion, is using resonance to increase the amplitude of each swing.[12] A motorist rocking his or her car to get it out of a muddy rut or a snow bank is applying an external force to amplify the car's natural frequency. Tuning a radio depends on resonance.

6.11.8 An analogy

As we've seen, beginning in Chapter 4, linear second-order differential equations with constant coefficients can be applied to both spring-mass systems and to electrical circuit problems. We should reflect on the fact that two apparently different physical situations can be analyzed by means of the same basic differential equation.

Table 6.4 illustrates this mechanical-electrical analogy, one of several that can be used to explain similarities between applied problems. The left-hand side of the table refers to Newton's laws of motion coupled with Hooke's law, while the right-hand side invokes Kirchhoff's second voltage law.

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[10] For examples of resonance, see Alice B. Dickinson, *Differential Equations: Theory and Use in Time and Motion* (Reading, MA: Addison-Wesley, 1972): 100 ff.

[11] Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by H. Crew and A. DeSalvio (New York: Macmillan, 1914), "First Day," 98.

[12] See, e.g., "How to Pump a Swing" by S. Wirkus, R. Rand, and A. Ruina, *College Math. J.* **29** (1998): 266–275.

Table 6.4 Mechanical-Electrical Analogy

Spring-Mass System $m\ddot{x}+\alpha\dot{x}+kx=f(t)$		**Electrical Circuit** $L\ddot{Q}+R\dot{Q}+(1/C)Q=E(t)$	
Displacement	x	Q	Charge
Velocity	$\dot{x}$	$\dot{Q}=I$	Current
Mass	m	L	Inductance
Damping constant	α	R	Resistance
Spring constant	k	$1/C$	1/Capacitance = Elastance
External force	$f(t)$	$E(t)$	Electromotive force (voltage)

Exercises 6.11

A

1. The IVP $\ddot{x}+20\dot{x}+64x=0$, with $x(0)=1/3$ and $\dot{x}(0)=0$, models the motion of a spring-mass system with a damping force. The initial conditions indicate that the mass has been pulled below its equilibrium position and released.

a. Express this IVP as a system of two first-order equations using the appropriate initial conditions.

b. Use technology to graph the solution of the system in the phase plane.

c. Use technology to graph the solution of the original second-order equation relative to the t-axis.

d. Comparing the results of parts (b) and (c) to the appropriate graphs in Examples 6.11.1 and 6.11.2, why do you think that the motion shown in this problem should be called *overdamped*?

2. Consider the spring-mass system modeled by the IVP $\ddot{x}+c\dot{x}+0.25x=0$, with $x(0)=\frac{1}{2}$ and $\dot{x}(0)=\frac{7}{4}$. Here, c is a positive parameter.

a. Express the IVP in terms of a system of first-order equations, including initial conditions.

b. For each of the values $c=0.5$, 1, and 1.5, use technology to graph the solution of the system in the phase plane, $0\le t\le 20$.

c. For each of the values $c=0.5$, 1, and 1.5, use technology to graph the solution of the original equation with respect to t on the interval

$$0\le t\le 20.$$

d. Based on your answers to parts (b) and (c), describe how the nature of the solution changes as the value of c passes through the value 1? (When $c=1$ the system is *critically damped.*)

3. Consider the following model of a spring-mass system: $\ddot{x}+64x=16\cos 8t$, with $x(0)=0$ and $\dot{x}(0)=0$.

a. Express the IVP in terms of a system of first-order equations, including initial conditions.

b. Use technology to graph the solution of the system in the phase plane.

c. Use technology to graph the solution of the original second-order equation relative to the t-axis.

d. What is the relationship of the graph in (c) to the two half-lines $x = t$ and $x = -t$ for $t \geq 0$?

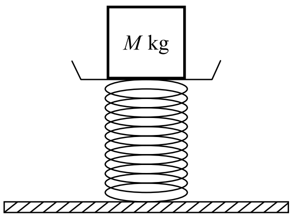
4. A spring-mass system can be modeled by the equation $m\ddot{x} + kx = 0$, with $x(0) = \alpha$ and $\dot{x}(0) = \beta$.

a. Express the IVP in terms of a system of first-order equations (including initial conditions).

b. Determine the maximum velocity of the mass.

B

5. A spring having a spring constant of 250 is used in a simple set of scales to measure the weights of objects placed on the pan. The pan (of mass 0.5 kg) rests on top of the spring (see the following illustration).



A block of mass M kg is placed on the pan, causing the spring to oscillate. There is a damping force of $10v$ Newtons, where v m/s is the speed of the pan.

a. Determine the differential equation of motion for the subsequent damped oscillations.

b. Show that the general solution of the equation found in part (a) has the form $x = A_0 e^{-\frac{10t}{2M+1}} \cos(nt + \varepsilon)$, where A_0 is the initial amplitude. Hence, find an expression for n in terms of M.

c. Find, in terms of M, the time it takes for the system to settle down to oscillations of only 25% of the initial amplitude.

d. What effect would removing the damping force have on the system?

6. A particle is in simple harmonic motion along the y-axis. At $t = 0$, $y = 3$ and $v = dy/dt = 0$. Exactly 1/2 second later, these values repeat themselves. Find $y(t)$ and $v(t)$.

C

7. Convert each of the following systems to a single second-order equation. Then interpret each equation to determine which (if any) *cannot* represent a spring-mass system. Explain your reasoning.

a. $Q' = -6Q + 3R$
$R' = -Q - 2R$

b. $\dot{x} = 3x - y$
$\dot{y} = x + 3y$

6.12 Generalizations: the $n \times n$ case ($n \geq 3$)

6.12.1 Matrix representation

We now extend our previous analysis of systems, first to 3×3 systems and then to nth-order linear systems. We can use matrix notation to represent a homogeneous third-order system with constant coefficients

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3\\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3\\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3\end{aligned}$$

symbolically, in the form $\dot{X} = AX$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix},$$

and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Example 6.12.1 Matrix Representation of a 3×3 System

The system $\dot{x} = -2x + 4y - z$, $\dot{y} = 5x - y + 3z$, $\dot{z} = x + z$ can be written first in the usual vertical way

$$\begin{aligned}\dot{x} &= -2x + 4y - z\\ \dot{y} &= 5x - y + 3z\\ \dot{z} &= x + z\end{aligned} \qquad \text{or} \qquad \begin{aligned}\dot{x} &= -2x + 4y - z\\ \dot{y} &= 5x - y + 3z\\ \dot{z} &= x + 0y + z\end{aligned}$$

and then more compactly as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -2 & 4 & -1 \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

6.12.2 Eigenvalues and eigenvectors

It is important to understand that the concepts of eigenvalue and eigenvector are valid for any system of n equations in n unknowns ($n \geq 2$). Specifically, given a system $\dot{X} = AX$, where X is a nonzero 3×1 column matrix (vector) and A is a 3×3 matrix, then an *eigenvalue* λ is a solution of the equation $AX = \lambda X$. Given an eigenvalue λ, an *eigenvector* associated with λ is a nonzero vector V that satisfies the equation $AV = \lambda V$.

The equation $AX = \lambda X$ can be expressed as $AX - \lambda X = \mathbf{0}$, where $\mathbf{0}$ denotes the 3×1 vector consisting entirely of zeros. This matrix equation is equivalent to the homogeneous algebraic system

$$\begin{aligned}(a_{11} - \lambda)\,x_1 + a_{12}x_2 + a_{13}x_3 &= 0\\ a_{21}x_1 + (a_{22} - \lambda)\,x_2 + a_{23}x_3 &= 0\\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)\,x_3 &= 0,\end{aligned}$$

or

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (6.12.1)$$

Now the matrix of coefficients in (6.12.1) can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A - \lambda I,$$

so the equation $AX - \lambda X = \mathbf{0}$ can be written as $(A - \lambda I)X = \mathbf{0}$, where I is the 3×3 *identity matrix* consisting of ones down the main diagonal and zeros elsewhere. The matrix I is such that $IX = X$ for any 3×1 vector X.

The equation $(A - \lambda I)X = \mathbf{0}$ represents a homogeneous algebraic system of three linear equations in three unknowns, and the theory of linear algebra indicates that there is a number Δ depending on the matrix of coefficients with the following important property:

The system (6.12.1) has only the zero solution $x_1 = x_2 = x_3 = 0$ if $\Delta \neq 0$. However, if $\Delta = 0$, then there is a solution x_1, x_2, x_3 with at least one of the $x_i\,(i = 1, 2, 3)$ different from zero.

This number Δ is the *determinant* of the matrix of coefficients in (6.12.1), denoted by $\det(A - \lambda I)$, and it is the extension to three dimensions of the determinant introduced in Section 6.6. (See Problem 12 of Exercises 6.5 and Problem 11 of Exercises 6.6 for the significance of the 2×2 determinant in the solution of a system of equations.) Therefore, $(A - \lambda I)X = \mathbf{0}$ has a nonzero solution X only if $\det(A - \lambda I) = 0$. An important fact is that $\det(A - \lambda I)$ is a third-degree polynomial in λ, called the **characteristic polynomial** of A, so the *eigenvalues of A are the roots of the characteristic equation* $\det(A - \lambda I) = 0$. There are algorithms for calculating the determinants of 3×3 systems, but they are tedious and any graphing calculator or CAS can evaluate them. In particular, a CAS will provide characteristic polynomials, eigenvalues, and corresponding eigenvectors. There are also formulas for solving cubic equations, but these methods are more complicated than the quadratic formula, and it is advisable to use your calculator or computer to solve these equations.

Let's use technology in the next example to calculate determinants, eigenvalues, and eigenvectors for a three-dimensional system.

Example 6.12.2 Eigenvalues and Eigenvectors via a CAS

Let's look at the matrix of coefficients in Example 6.12.1:

$$A = \begin{bmatrix} -2 & 4 & -1 \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}.$$

A CAS provides the information that $\det(A) = -7$, the characteristic equation is $\lambda^3 + 2\lambda^2 - 20\lambda + 7 = 0$, and the eigenvalues (rounded to four decimal places) are $\lambda_1 = 3.3485$, $\lambda_2 = -5.7143$, and $\lambda_3 = 0.3658$. The corresponding representative eigenvectors are

$$V_1 = \begin{bmatrix} 2.3485 \\ 3.3903 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} -6.7143 \\ 6.4848 \\ 1 \end{bmatrix}, \text{ and } V_3 = \begin{bmatrix} -0.6342 \\ -0.1251 \\ 1 \end{bmatrix}.$$

Don't be concerned if your CAS or calculator gives you eigenvectors that are different from these. You should check to see that each eigenvector you find is a constant multiple of one of the vectors V_1, V_2, and V_3 given here.

6.12.3 Linear independence and linear dependence

At this point you should be asking yourself, What do these eigenvalues and eigenvectors tell me about the system? Just as in the 2×2 case, we can write the general solution of a 3×3 system in terms of the eigenvalues and eigenvectors of the matrix of coefficients. To see what's going on, we'll need to use a few concepts already seen in the 2×2 case. For example, given a number of vectors $v_1, v_2, \ldots, v_k$, a **linear combination** of these vectors is a vector that has the form $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$ for some choice of scalars $a_1, a_2, \ldots, a_k$. The collection of vectors is called **linearly independent** if the only way you can have $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \mathbf{0}$ (the zero vector) is to have $a_1 = a_2 = \cdots = a_k = 0$. If you *could* find scalars a_i, not all zero, so that a linear combination of the vectors v_i was equal to the zero vector, then we say that the collection of vectors is **linearly dependent**. To see what linear dependence means, suppose that $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \mathbf{0}$ and one of the scalars, say a_j, is not zero. Then we can write

$$a_1 v_1 + a_2 v_2 + \cdots + a_j v_j + \cdots + a_k v_k = \mathbf{0},$$

$$a_j v_j = -a_1 v_1 - a_2 v_2 - \cdots - a_{j-1} v_{j-1} - a_{j+1} v_{j+1} - \cdots - a_k v_k,$$

or

$$v_j = \left(-\frac{a_1}{a_j}\right) v_1 + \left(-\frac{a_2}{a_j}\right) v_2 + \cdots + \left(-\frac{a_{j-1}}{a_j}\right) v_{j-1} + \left(-\frac{a_{j+1}}{a_j}\right) v_{j+1} + \cdots + \left(-\frac{a_k}{a_j}\right) v_k.$$

This last line tells us that if a collection of vectors is linearly dependent, then at least one of the vectors is a linear combination of the others.

Let's see some examples of these concepts.

Example 6.12.3 Linearly Independent and Linearly Dependent Vectors

The three vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are linearly *independent* because the equation

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is equivalent to the algebraic system

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_2 + 2a_3 &= 0 \\ 2a_1 + 2a_2 &= 0, \end{aligned}$$

which you can solve to find that $a_1 = a_2 = a_3 = 0$. (*Do the work!*)

On the other hand, the collection of vectors $\begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is linearly *dependent* because the vector equation

$$a_1 \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is equivalent to the algebraic system

$$\begin{aligned} 3a_1 + a_3 &= 0 \\ 4a_1 + a_2 + 2a_3 &= 0 \\ -4a_1 + 2a_2 &= 0, \end{aligned}$$

which has infinitely many solutions of the form $a_1 = K$, $a_2 = 2K$, and $a_3 = -3K$. In particular, we can let $K = 1$, so we have the nonzero solution $a_1 = 1$, $a_2 = 2$, and $a_3 = -3$. Note, for example, that we can write

$$\begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Now suppose that we have the system $\dot{X} = AX$, where X is a 3×1 vector and A is a 3×3 matrix of constants. If A has three distinct real eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then the theory of linear algebra tells us that the corresponding eigenvectors V_1, V_2, V_3 are linearly independent. Furthermore, the vectors $e^{\lambda_1 t}V_1, e^{\lambda_2 t}V_2, e^{\lambda_3 t}V_3$ are linearly independent, and the general solution of $\dot{X} = AX$ is

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + c_3 e^{\lambda_3 t} V_3, \tag{6.12.2}$$

where c_1, c_2, and c_3 are arbitrary constants. Compare this with (6.7.1).

Example 6.12.4 Solving a 3×3 System via Eigenvalues and Eigenvectors

Consider the system

$$\begin{aligned} \dot{x} &= 4x + z \\ \dot{y} &= -2y \\ \dot{z} &= -z. \end{aligned}$$

The matrix of coefficients is $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and a CAS calculates the eigenvalues to be $\lambda_1 = 4$, $\lambda_2 = -1$, and $\lambda_3 = -2$, with corresponding eigenvectors

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \text{ and } V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note that these vectors must be linearly independent because the eigenvalues are distinct real numbers. Thus, via Eq. (6.12.2), the general solution of our system is given by

$$\begin{aligned} X(t) &= c_1 e^{4t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{4t} + c_2 e^{-t} \\ c_3 e^{-2t} \\ -5c_2 e^{-t} \end{bmatrix}. \end{aligned}$$

In this example, we noticed that the second and third differential equations making up our original system were separable. After solving each of them, we could have substituted for z in the first equation, which would then be a simple linear equation in x. (*Do this and compare your answer with the one given previously.*)

A trajectory in x-y-z space (corresponding to the initial conditions $x(0) = 2$, $y(0) = 5$, and $z(0) = -10$) is shown in Fig. 6.37, and the same trajectories in the t-z plane and the y-z plane are shown in Figs. 6.38a and 6.38b, respectively.

Note that the graph of a solution of this system is really four-dimensional, a set of points of the form $(t, x(t), y(t), z(t))$. Therefore, what Fig. 6.37 is showing is a *projection* of a four-dimensional curve onto three-dimensional x-y-z space.

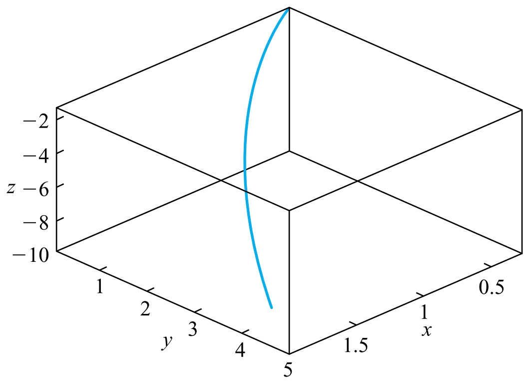


FIGURE 6.37

Solution of $\dot{x} = 4x + z,\ \dot{y} = -2y,\ \dot{z} = -z;\ x(0) = 2,\ y(0) = 5,\ z(0) = -10;\ 0 \leq t \leq 2$

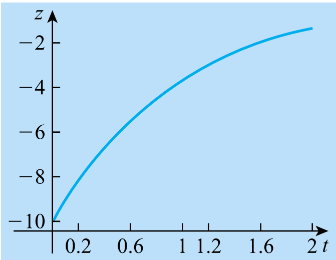


FIGURE 6.38a

Graph of $z(t)$, $0 \leq t \leq 2$

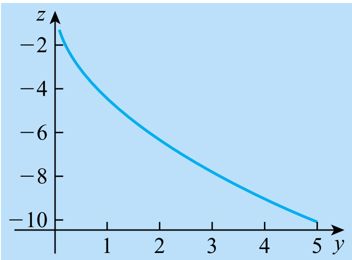


FIGURE 6.38b

Graph of $z(t)$ vs. $y(t)$, $0 \leq t \leq 2$

Accepting the fact that a 3×3 matrix has a cubic characteristic equation, we realize that we can have (1) three distinct real eigenvalues, (2) one distinct real eigenvalue and a different repeated real eigenvalue, (3) one repeated real eigenvalue, or (4) one real eigenvalue and a complex conjugate pair of eigenvalues. Possibilities 1 and 4 are handled easily by formula (6.12.2). However, when we have repeated eigenvalues, we must find linearly independent eigenvectors, sometimes by calculating one or more *generalized eigenvectors*. (Go back to Example 6.8.2 and the discussion preceding it. Also see Problem 18 in Exercises 6.8.)

It should be clear how important the theory of linear algebra is to a full understanding of higher-order differential equations and their equivalent systems, but we will not investigate that theory further in this book.

The next example shows how techniques that we developed for two-dimensional systems with complex eigenvalues (Section 6.9) can be extended to three-dimensional systems.

Example 6.12.5 Solving a 3×3 System—Complex Eigenvalues

Look at the system $\dot{x} = x$, $\dot{y} = 2x + y - 2z$, $\dot{z} = 3x + 2y + z$. The matrix of coefficients is $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$. A CAS provides the characteristic equation $\lambda^3 - 3\lambda^2 + 7\lambda - 5 = (\lambda - 1)(\lambda^2 - 2\lambda + 5) = 0$, which has roots $\lambda_1 = 1$, $\lambda_2 = 1 + 2i$, and $\lambda_3 = 1 - 2i$. A CAS also gives the corresponding eigenvectors $V_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$, and $V_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$. (Remember that your calculator or CAS may give you eigenvectors that look different from these. Just check to see that yours are multiples of the ones used here. Also, note that V_2 and V_3 are conjugates of each other.)

Now we can use (6.12.2) to write the general solution in the form

$$X(t) = c_1 e^t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} + c_3 e^{(1-2i)t} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}.$$

However, we realize that $X_1(t) = e^t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ is a solution of the system by itself.

Furthermore, extending what we saw in Section 6.9, we know that we need work only with the *first* complex eigenvalue-eigenvector pair, because the other eigenvalue and eigenvector are conjugates that produce the same solutions (see Problem 13 of Exercises 6.9). Therefore, we consider only

$$\begin{aligned} \tilde{X}(t) &= e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t (\cos(2t) + i \sin(2t)) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \\ &= e^t \begin{bmatrix} 0 \\ -\sin(2t) + i\cos(2t) \\ \cos(2t) + i \sin(2t) \end{bmatrix} = e^t \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + i e^t \begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix}. \end{aligned}$$

From the preceding expression, we derive two linearly independent real-valued solutions of our system:

$$X_2(t) = \operatorname{Re}\left\{\tilde{X}(t)\right\} = e^t \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} \text{ and } X_3(t) = \operatorname{Im}\left\{\tilde{X}(t)\right\} = e^t \begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix}.$$

Finally, the Superposition Principle tells us that

$$
\begin{aligned}
X(t) &= c_1 X_1 + c_2 X_2 + c_3 X_3 \\
&= c_1 e^t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} 2c_1 e^t \\ -3c_1 e^t - c_2 e^t \sin(2t) + c_3 e^t \cos(2t) \\ 2c_1 e^t + c_2 e^t \cos(2t) + c_3 e^t \sin(2t) \end{bmatrix} \\
&= e^t \begin{bmatrix} 2c_1 \\ -3c_1 - c_2 \sin(2t) + c_3 \cos(2t) \\ 2c_1 + c_2 \cos(2t) + c_3 \sin(2t) \end{bmatrix}
\end{aligned}
$$

is the real-valued general solution of the original system. If you use technology to solve this problem, be aware that your CAS may express the solution functions in a different but equivalent way.

6.12.4 Nonhomogeneous systems

It is important to realize that we can also handle larger *nonhomogeneous* systems in this way using the relationship explored in Section 6.10: $X_{\text{GNH}} = X_{\text{GH}} + X_{\text{PNH}}$. The method of undetermined coefficients becomes algebraically messier as the size of the system increases, and the Laplace transform is a better solution method.

6.12.5 Generalization to $n \times n$ systems

Everything we've done with 2×2 and 3×3 systems of equations can be generalized to $n \times n$ systems. We can express a homogeneous nth-order linear system with constant coefficients

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\
\dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n
\end{aligned}
$$

as $\dot{X} = AX$, where

$$
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.
$$

Example 6.12.6 Matrix Form of a Four-Dimensional System

A compartmental analysis (see Section 2.3) of the processes involved in protein synthesis in animals and humans uses radioactive isotopes as tracers. A particular four-compartment model of this situation could lead to a system such as

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2\\ \dot{x}_2 &= x_1 - 2x_2\\ \dot{x}_3 &= x_1 + x_2 - x_3\\ \dot{x}_4 &= x_3,\end{aligned}$$

where $x_i(t)$ denotes the fraction of the total administered radioactivity attached to the material (albumen) in compartment i $(i = 1, 2, 3, 4)$. The coefficients indicate flow rates of the radioactive material from compartment to compartment.

In matrix terms, we can express this system as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0 & 0\\ 1 & -2 & 0 & 0\\ 1 & 1 & -1 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}.$$

Extending the theory underlying the solution of algebraic systems of three linear equations in three unknowns to systems of n equations in n unknowns, we state that any $n \times n$ matrix has a determinant and that eigenvalues and eigenvectors can be defined for such square matrices. Specifically, given a system $\dot{X} = AX$, where X is an $n \times 1$ column matrix (vector) and A is an $n \times n$ matrix, an eigenvalue λ is a solution of the equation $\det(A - \lambda I) = 0$, where I is the $n \times n$ identity matrix consisting of ones down the main diagonal and zeros elsewhere. Given an eigenvalue λ, an eigenvector associated with λ is a nonzero vector V satisfying the equation $AV = \lambda V$.

The characteristic equation of an $n \times n$ matrix is an nth-degree polynomial. However, *once a polynomial has degree greater than or equal to 5, there is no longer a general formula that gives its zeros*. In general, the only way to tackle such equations is to use *approximation* methods. A CAS—or even a graphing calculator—has various algorithms to do this.

Now suppose that we have the system $\dot{X} = AX$, where X is an $n \times 1$ vector and A is an $n \times n$ matrix of constants. If A has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the theory of linear algebra guarantees that the corresponding eigenvectors $V_1, V_2, \ldots, V_n$ are linearly independent. Furthermore, the vectors $e^{\lambda_1 t}V_1, e^{\lambda_2 t}V_2, \ldots, e^{\lambda_n t}V_n$ are linearly independent, and the general solution of $\dot{X} = AX$ is

$$X(t) = c_1 e^{\lambda_1 t}V_1 + c_2 e^{\lambda_2 t}V_2 + \cdots + c_n e^{\lambda_n t}V_n, \qquad (6.12.3)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. You should expect the usual complications when there are repeated real roots, complex conjugate pairs of roots, and so forth.

If we investigate a mechanical system (Fig. 6.39) consisting of two springs attached to two movable masses, the physics of the situation gives us a pair of second-order linear differential equations. In turn, this system of two equations can be expressed as a system of four first-order linear equations.

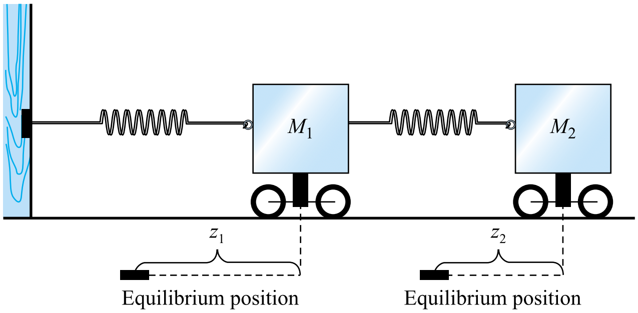


FIGURE 6.39

The spring-mass system for Example 6.12.7

The following example assumes that we start from the equilibrium position by giving one mass an initial velocity. Most of the computational work will be done by a CAS.

Example 6.12.7 A Four-Dimensional System from Mechanics

Let's consider the system

$$\frac{d^2z_1}{dt^2} = -11z_1 + 6z_2$$
$$\frac{d^2z_2}{dt^2} = -6z_2 + 6z_1,$$

where z_1 is the distance of mass 1 from its equilibrium position and z_2 is the distance of mass 2 from equilibrium. We'll assume the initial conditions $z_1(0) = 0$, $z_1'(0) = 0$, $z_2(0) = 0$, and $z_2'(0) = 1$.

Representation as a First-Order System

Introducing the new variables $x_1 = z_1$, $x_2 = \frac{dz_1}{dt}$, $x_3 = z_2$, and $x_4 = \frac{dz_2}{dt}$, we convert our pair of second-order equations into the four-dimensional system of first-order equations

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -11x_1 + 6x_3$$
$$\frac{dx_3}{dt} = x_4$$
$$\frac{dx_4}{dt} = -6x_3 + 6x_1,$$

with $x_1(0) = x_2(0) = x_3(0) = 0$ and $x_3'(0) = x_4(0) = 1$.

Matrix Representation, Eigenvalues, Eigenvectors

We can express the last system as $\frac{d}{dt}X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -11 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = AX$. A CAS gives the characteristic equation of matrix A as $\lambda^4 + 17\lambda^2 + 30 = 0$, which we can factor as $(\lambda^2 + 15)(\lambda^2 + 2) = 0$, so the eigenvalues are $\lambda_1 = \sqrt{15}i$, $\lambda_2 = -\sqrt{15}i$, $\lambda_3 = \sqrt{2}i$, and $\lambda_4 = -\sqrt{2}i$. The corresponding eigenvectors are

$$V_1 = \begin{bmatrix} 3 \\ 3\sqrt{15}i \\ -2 \\ -2\sqrt{15}i \end{bmatrix}, V_2 = \begin{bmatrix} 3 \\ -3\sqrt{15}i \\ -2 \\ 2\sqrt{15}i \end{bmatrix}, V_3 = \begin{bmatrix} 2 \\ 2\sqrt{2}i \\ 3 \\ 3\sqrt{2}i \end{bmatrix}, \text{ and } V_4 = \begin{bmatrix} 2 \\ -2\sqrt{2}i \\ 3 \\ -3\sqrt{2}i \end{bmatrix}.$$

If you check this with a CAS, remember that you may get a different (but equivalent) form for the eigenvectors.

The General Solution

On the basis of our previous experience with complex conjugate pairs of eigenvalues and eigenvectors, we can just work with the pairs λ_1, V_1 and λ_3, V_3. First, we know that

$$\hat{X}(t) = e^{\lambda_1 t} V_1 = e^{\sqrt{15}it} \begin{bmatrix} 3 \\ 3\sqrt{15}i \\ -2 \\ -2\sqrt{15}i \end{bmatrix} = \left(\cos\left(\sqrt{15}t\right) + i\sin\left(\sqrt{15}t\right)\right) \begin{bmatrix} 3 \\ 3\sqrt{15}i \\ -2 \\ -2\sqrt{15}i \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos\left(\sqrt{15}t\right) \\ -3\sqrt{15}\sin\left(\sqrt{15}t\right) \\ -2\cos\left(\sqrt{15}t\right) \\ 2\sqrt{15}\sin\left(\sqrt{15}t\right) \end{bmatrix} + i \begin{bmatrix} 3\sin\left(\sqrt{15}t\right) \\ 3\sqrt{15}\cos\left(\sqrt{15}t\right) \\ -2\sin\left(\sqrt{15}t\right) \\ -2\sqrt{15}\cos\left(\sqrt{15}t\right) \end{bmatrix} = X_1(t) + iX_2(t),$$

where both $X_1(t)$ and $X_2(t)$ are linearly independent real-valued solutions of the system. Then we have

$$\tilde{X}(t) = e^{\lambda_3 t} V_3 = e^{\sqrt{2}it} \begin{bmatrix} 2 \\ 2\sqrt{2}i \\ 3 \\ 3\sqrt{2}i \end{bmatrix} = \left(\cos\left(\sqrt{2}t\right) + i\sin\left(\sqrt{2}t\right)\right) \begin{bmatrix} 2 \\ 2\sqrt{2}i \\ 3 \\ 3\sqrt{2}i \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\left(\sqrt{2}t\right) \\ -2\sqrt{2}\sin\left(\sqrt{2}t\right) \\ 3\cos\left(\sqrt{2}t\right) \\ -3\sqrt{2}\sin\left(\sqrt{2}t\right) \end{bmatrix} + i \begin{bmatrix} 2\sin\left(\sqrt{2}t\right) \\ 2\sqrt{2}\cos\left(\sqrt{2}t\right) \\ 3\sin\left(\sqrt{2}t\right) \\ 3\sqrt{2}\cos\left(\sqrt{2}t\right) \end{bmatrix} = X_3(t) + iX_4(t),$$

where $X_3(t)$ and $X_4(t)$ are linearly independent real-valued solutions of the system. The general solution is

$$X(t) = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4$$

$$= c_1 \begin{bmatrix} 3\cos\left(\sqrt{15}t\right) \\ -3\sqrt{15}\sin\left(\sqrt{15}t\right) \\ -2\cos\left(\sqrt{15}t\right) \\ 2\sqrt{15}\sin\left(\sqrt{15}t\right) \end{bmatrix} + c_2 \begin{bmatrix} 3\sin\left(\sqrt{15}t\right) \\ 3\sqrt{15}\cos\left(\sqrt{15}t\right) \\ -2\sin\left(\sqrt{15}t\right) \\ -2\sqrt{15}\cos\left(\sqrt{15}t\right) \end{bmatrix}$$

$$+ c_3 \begin{bmatrix} 2\cos\left(\sqrt{2}t\right) \\ -2\sqrt{2}\sin\left(\sqrt{2}t\right) \\ 3\cos\left(\sqrt{2}t\right) \\ -3\sqrt{2}\sin\left(\sqrt{2}t\right) \end{bmatrix} + c_4 \begin{bmatrix} 2\sin\left(\sqrt{2}t\right) \\ 2\sqrt{2}\cos\left(\sqrt{2}t\right) \\ 3\sin\left(\sqrt{2}t\right) \\ 3\sqrt{2}\cos\left(\sqrt{2}t\right) \end{bmatrix}.$$

Particular Solutions

The initial conditions $x_1(0) = x_2(0) = x_3(0) = 0$ and $x_3'(0) = x_4(0) = 1$ imply that $c_1 = c_3 = 0$ and $c_2 = -2\sqrt{15}/195$, $c_4 = 3\sqrt{2}/26$. Therefore,

$$z_1(t) = x_1(t) = \frac{3\sqrt{2}}{13}\sin\left(\sqrt{2}t\right) - \frac{2\sqrt{15}}{65}\sin\left(\sqrt{15}t\right)$$

and

$$z_2(t) = x_3(t) = \frac{9\sqrt{2}}{26}\sin\left(\sqrt{2}t\right) + \frac{4\sqrt{15}}{195}\sin\left(\sqrt{15}t\right).$$

If we are interested in the *stability* of an $n \times n$ system rather than its exact solution, we can give a simplified version of the results we have seen for 2×2 systems (see Table 6.2 in Section 6.9): If A is an $n \times n$ matrix of constants, then the equilibrium solution $X = \mathbf{0}$ for the system $\dot{X} = AX$ is asymptotically stable (that is, it is a *sink*) if every eigenvalue of A has a negative real part and is unstable if A has at least one eigenvalue with a positive real part. Furthermore, if *all* eigenvalues have positive real parts, the n-dimensional origin $X = \mathbf{0}$ is a *source*; and if some eigenvalues have positive real parts and others have negative real parts, the equilibrium point is called a *saddle point*.

Exercises 6.12

A

For each of the systems in Problems 1–6, (a) write the system in the form $\dot{X} = AX$; (b) use technology to find eigenvalues and representative eigenvectors; and (c) express the general solution as a single real-valued vector of functions.

1. $\frac{dx}{dt} = x - y + z$
$\frac{dy}{dt} = x + y - z$
$\frac{dz}{dt} = 2x - y$

2. $\frac{dx}{dt} = x - 2y - z$
$\frac{dy}{dt} = -x + y + z$
$\frac{dz}{dt} = x - z$

3. $\frac{dx}{dt} = 3x - y + z$
$\frac{dy}{dx} = x + y + z$
$\frac{dz}{dt} = 4x - y + 4z$

4. $\frac{dx}{dt} = 2x + y$
$\frac{dy}{dt} = x + 3y - z$
$\frac{dz}{dt} = 2y + 3z - x$

5. $\frac{dx}{dt} = 2x - y + z$
$\frac{dy}{dt} = x + 2y - z$
$\frac{dz}{dt} = x - y + 2z$

6. $\frac{dx}{dt} = 2x + 2z - y$
$\frac{dy}{dt} = x + 2z$
$\frac{dz}{dt} = y - 2x - z$

7. For the system in Problem 1, use your CAS to plot the x-y-z space trajectory passing through the point $(0, 1, 0)$ when $t = 0$.

8. For the system in Problem 4, use your CAS to plot the x-y-z space trajectory passing through the point $(1, 1, -1)$ when $t = 0$.

B

9. In Example 6.12.7, use the initial conditions to show that $c_1 = c_3 = 0$ and $c_2 = -2\sqrt{15}/195$, $c_4 = 3\sqrt{2}/26$.

10. **a.** Solve the IVP

$$\frac{dx}{dt} = z + y - x$$
$$\frac{dy}{dt} = z + x - y$$
$$\frac{dz}{dt} = x + y + z,$$
$$x(0) = 1,\ y(0) = -1/3,\ z(0) = 0.$$

b. Use the explicit solution found in part (a) to calculate $x(0.5)$, $y(0.5)$, and $z(0.5)$.

c. Use two or more numerical methods found in your CAS to approximate $x(0.5)$, $y(0.5)$, and $z(0.5)$. Compare the answers to each other and to the exact answers in part (a).

11. Solve the IVP

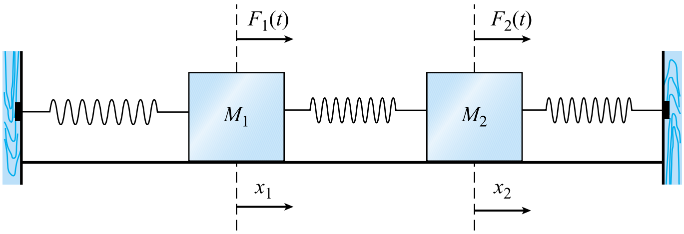
$$\begin{aligned} \frac{dx}{dt} &= y + z \\ \frac{dy}{dt} &= z + x \\ \frac{dz}{dt} &= x + y \end{aligned}$$
$$x(0) = -1,\ y(0) = 1,\ z(0) = 0.$$

12. Consider the system

$$\begin{aligned} \ddot{x} &= 2x + \dot{x} + y \\ \dot{y} &= 4x + 2y. \end{aligned}$$

- **a.** Convert this system to a system of three first-order equations, $\dot{Y} = AY$.
- **b.** Use technology to find the eigenvalues of the matrix A in part (a).
- **c.** Use technology to find two linearly independent eigenvectors corresponding to the eigenvalues found in (b).
- **d.** Take $W = \begin{bmatrix} t \\ 1 \\ -1 - 2t \end{bmatrix}$ as a third eigenvector that is independent of the two found in part (c) and give the general solution of $\dot{Y} = AY$.
- **e.** Find the general solution $x(t)$, $y(t)$ of the original system.

13. Consider the two-mass, three-spring system shown here.



If there are no external forces, and if the masses and spring constants are equal and of unit magnitude, then the equations of motion are

$$x_1'' = -2x_1 + x_2,\ \ x_2'' = x_1 - 2x_2.$$

- **a.** Transform the system of equations into a system of four first-order equations by letting $y_1 = x_1$, $y_2 = x_1'$, $y_3 = x_2$, and $y_4 = x_2'$.
- **b.** Find the eigenvalues of the matrix of coefficients for the system in part (a).

c. Solve the system in part (a) subject to the initial conditions $y_1(0) = 2$, $y_2(0) = 1$, $y_3(0) = 2$, $y_4(0) = 1$. Describe the physical motion of the spring-mass system corresponding to this solution.

d. Solve the system in part (a) subject to the initial conditions $y_1(0) = 2$, $y_2(0) = \sqrt{3}$, $y_3(0) = -2$, $y_4(0) = -\sqrt{3}$. Describe the physical motion of the spring-mass system corresponding to this solution.

e. Observe that the spring-mass system has two natural modes of oscillation in this problem. How are the natural frequencies related to the eigenvalues of the coefficient matrix? Do you think that there might be a third natural mode of oscillation with a different frequency?

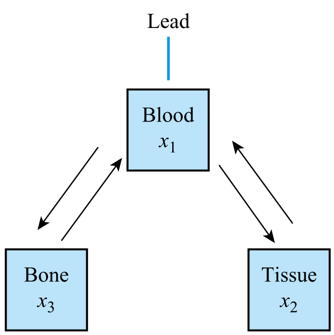
14. In analyzing the flow of lead pollution in a human body among the three compartments bone, blood, and tissue, the following system was developed (see E. Batschelet, L. Brand, and A. Steiner, "On the Kinetics of Lead in the Human Body," *Journal of Mathematical Biology* **8** (1979): 15–23):

$$\dot{x}_1 = -\frac{65}{1800}x_1 + \frac{1088}{87{,}500}x_2 + \frac{7}{200{,}000}x_3 + \frac{6162}{125}$$
$$\dot{x}_2 = \frac{20}{1800}x_1 - \frac{20}{700}x_2$$
$$\dot{x}_3 = \frac{7}{1800}x_1 - \frac{7}{200{,}000}x_3.$$

Here, $x_1(t)$ is the amount of lead in the blood at time t (in years), $x_2(t)$ is the amount of lead in tissue, and $x_3(t)$ is the amount of lead in bone. Assume that $x_1(0) = x_2(0) = x_3(0) = 0$.

a. Use technology to graph the three-dimensional trajectory in x_1-x_2-x_3 space with $0 \le t \le 250$. (Move the axes around to get a good view.)

b. Use technology to graph the solution in the t-x_1 plane, $0 \le t \le 150$. What seems to be the equilibrium level of lead in the blood?



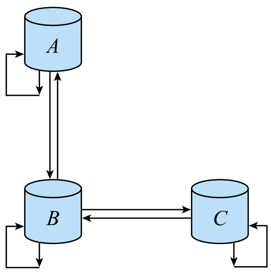
c. Use technology to graph the solution in the t-x_2 plane, $0 \le t \le 250$. What seems to be the equilibrium level of lead in tissue?

d. Use technology to graph the solution in the t-x_3 plane, $0 \leq t \leq 70{,}000$. In your CAS, specify a step size of 50 if you can. (*Warning: It may take a long time for your CAS to produce the graph.*) What seems to be the equilibrium level of lead in bone?

e. What do the graphs in parts (b), (c), and (d) say about the comparative times it takes blood, tissue, and bone to reach their equilibrium levels of lead?

C

15. There are three tanks (see the following figure) that pump fluid back and forth in the following way: Tank A pumps fluid into tank B at a rate of 1% of its volume per hour and also back into itself at a rate of 1% of the volume per hour. Tank B pumps into itself, tank A, and tank C, all at a rate of 2% of its volume per hour. Tank C pumps into tank B at a rate of 2% of its volume per hour and back into itself at the rate of 3% of its volume per hour. Assuming that the initial volumes in tanks A, B, and C are 23,000, 1000, and 1000 liters, respectively, describe the changes in volume of fluid in each tank as functions of time. (Use technology only to find the eigenvalues and corresponding eigenvectors.)



16. Suppose that you have a *double pendulum*—that is, one pendulum suspended from another—as shown in the following figure. The laws of physics, after a simplifying change of variables, give us the following system as a model for small oscillations about the equilibrium position:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \alpha u \\ \dot{u} &= v \\ \dot{v} &= x - u. \end{aligned}$$

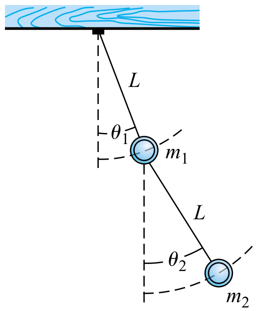
Here, $\alpha = (m_2/m_1)(1 + m_2/m_1)^{-1}$, $x = \theta_1$, $u = \theta_2$, and y and v are the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, respectively. For this problem, let $\alpha = 0.3$.

a. Express the system in matrix form.

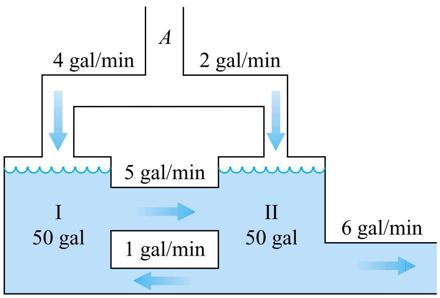
b. Use technology to find the eigenvalues of the system.

c. Use technology to find eigenvectors corresponding to the eigenvalues found in part (b).

d. Find the general real-valued solution of the system.



17. Consider the pair of 50 gallon tanks shown here. Initially, tank I is full of compound B and tank II is full of compound C. Start to introduce compound A into each tank at the rates shown in the figure.



a. Let $x_1(t)$ and $x_2(t)$ denote the amount of compound A in tanks I and II, respectively. Similarly, define $y_1(t)$, $y_2(t)$, $z_1(t)$, and $z_2(t)$ for the amounts of compounds B and C in tanks I and II. Now write a system of six nonhomogeneous differential equations describing the flow of the various substances into and out of tanks I and II, expressing any fractions in decimal form. Be sure to write initial conditions.

b. Use technology to solve the IVP expressed in part (a).

c. Use technology to graph $x_1(t)$, $y_1(t)$, and $z_1(t)$ against t, all on the same set of axes.

d. Use technology to graph $x_2(t)$, $y_2(t)$, and $z_2(t)$ against t, all on the same set of axes.

18. Suppose you have a system $\dot{X} = AX$, where A is a 3×3 matrix that has an eigenvalue λ of multiplicity 3 and corresponding eigenvector V. Then it can be shown that the general solution of the system can be written as $c_1X_1 + c_2X_2 +$

c_3X_3, where $X_1 = e^{\lambda t}V$, $X_2 = e^{\lambda t}(W + tV)$, $X_3 = e^{\lambda t}(U + tW + \frac{t^2}{2}V)$, W satisfies $(A - \lambda I)W = V$, and U satisfies $(A - \lambda I)U = W$.

a. Find the repeated eigenvalue and representative eigenvector for the system

$$x' = x + y + z$$
$$y' = 2x + y - z$$
$$z' = -3x + 2y + 4z.$$

b. Use the method described above and technology to write the general solution of this system.

19. Find the general solution of the nonautonomous, nonhomogeneous system

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = 3x + 2t$$
$$\frac{dz}{dt} = x + 4y + t$$

a. by using ideas from Problem 18, followed by the technique of *undetermined coefficients*. (See Section 6.10.)

b. by solving the first equation and then substituting this solution in the second equation, and so forth.

Summary

The most important thing to remember in this chapter is that **any single nth-order differential equation can be converted into an equivalent system of first-order equations. More precisely, any nth-order differential equation**

$$\boldsymbol{x^{(n)} = F\left(t, x, x', x'', \ldots, x^{(n-1)}\right)}$$

can be converted into an equivalent system of first-order equations by letting $x_1 = x$, $x_2 = x'$, $x_3 = x''$, ..., $x_n = x^{(n-1)}$. However, to convert a single *nonautonomous* nth-order equation into an equivalent *autonomous* system (one whose equations do not explicitly contain the independent variable t), we need $n + 1$ first-order equations: $x_1 = x, x_2 = x', x_3 = x'', \ldots, x_n = x^{(n-1)}, x_{n+1} = t$. The system is linear or nonlinear, autonomous or nonautonomous, according to the nature of the individual equations in the system. Linear systems are easier to calculate with and understand than nonlinear systems. Similarly, autonomous systems are easier to work with than nonautonomous systems.

Before getting too immersed in trying to solve higher-order equations or their equivalent systems, we have to determine when solutions *exist*—and whether existing solutions are *unique*. Suppose we have a two-dimensional system of first-order equations

$$\frac{dx_1}{dt} = f(t, x_1, x_2)$$
$$\frac{dx_2}{dt} = g(t, x_1, x_2),$$

where $x_1(t_0) = x_1^0$ and $x_2(t_0) = x_2^0$. Then if f, g, $\frac{\partial f}{\partial x_1}$, $\frac{\partial g}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, and $\frac{\partial g}{\partial x_2}$ are all continuous in a box B in $t - x_1 - x_2$ space containing the point $\left(t_0, x_1^0, x_2^0\right)$, there is an interval I containing t_0 in which there is a unique solution $x_1 = y_1(t)$, $x_2 = y_2(t)$ of the IVP.

Once we are confident that an IVP involving a higher-order equation or its system equivalent *has* a unique solution, we can apply natural two-dimensional generalizations of the numerical solution methods introduced in Sections 3.1–3.3: Euler's method; the improved Euler method; and higher-order techniques such as the fourth-order Runge–Kutta and Runge–Kutta–Fehlberg methods. Technology is indispensable in the numerical solution of both single equations and systems.

A two-dimensional system has the form

$$x' = F(t, x, y)$$
$$y' = G(t, x, y).$$

A particular **solution** of such a system consists of a *pair* of functions $x(t)$, $y(t)$ that, when substituted into the equations of the system, give true statements. The proper graphical representation of a solution is a curve in three-dimensional t-x-y space, the set of points of the form $(t, x(t), y(t))$, but often it is useful to think of the points $(x(t), y(t))$ as tracing out a path (also called an **orbit** or a **trajectory**) in the x-y plane (called the **phase plane**) as the parameter t varies "offstage." The *positive* direction of the path is the direction that corresponds to increasing values of t. The collection of all trajectories is the **phase portrait** of the system. Technology also enables us to study the graphs of x versus t and y versus t.

For *autonomous* systems $x' = f(x, y)$, $y' = g(x, y)$, we can eliminate any explicit reliance on the parameter t by using the Chain Rule to form the first-order differential equation

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x, y)}{f(x, y)}.$$

This gives the slope of the tangent line at points of the phase plane. The slope field of this first-order equation outlines the phase portrait of the system.

Given any two-dimensional autonomous system $x' = f(x, y)$, $y' = g(x, y)$, an **equilibrium point** is a point (x, y) such that $f(x, y) = 0 = g(x, y)$. This means, for example, that a particle at this point in the phase plane is not moving. The language of **sinks** and **sources** used in Section 2.6 can be extended to apply to equilibrium solutions of systems.

By using **matrices** and their properties, we can write any $n \times n$ autonomous system of linear equations with constant coefficients

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\
\dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n
\end{aligned}$$

in the compact form $\dot{X} = AX$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

For two-dimensional systems in which the matrix of coefficients A has a nonzero determinant, the origin is the only equilibrium point. The qualitative behavior (stability) of such a linear system is completely determined by the **eigenvalues** and **eigenvectors** of A. If the system has a pair of complex conjugate eigenvalues, we have to use **Euler's formula**, $e^{p+qi} = e^p(\cos(q) + i\sin(q))$, to obtain *real-valued* solutions.

Using expressions for the general solution of our system, we can analyze the stability of the system qualitatively in terms of eigenvalues and eigenvectors. The stability results are summarized in Table 6.2 at the end of Section 6.9.

We may have a *nonhomogeneous* system

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n + b_1(t) \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n + b_2(t) \\
\dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n + b_3(t) \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n + b_n(t),
\end{aligned}$$

which can be written as $\dot{X} = AX + B(t)$, with

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad \text{and} \quad B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \\ \vdots \\ b_n(t) \end{bmatrix}.$$

In this situation, we know that the general solution, X_{GNH}, of a linear nonhomogeneous system is obtained by finding a particular solution, X_{PNH}, of the nonhomogeneous system and adding it to the general solution, X_{GH}, of the homogeneous system: $X_{\text{GNH}} = X_{\text{GH}} + X_{\text{PNH}}$. The method of **undetermined coefficients** can be used to make an intelligent guess about the particular solution if the entries of vector $B(t)$ contain terms that are constants, exponential functions, sines, cosines, polynomials, or any sum or product of such terms. For other kinds of functions making up $B(t)$, X_{PNH} must be found using some other technique (for example, **variation of parameters**).

In an optional section, both analytic and qualitative techniques are used to analyze **spring-mass problems**, including one exhibiting the phenomenon of **resonance**.

Although we started with a thorough analysis of the equilibrium points and the stability of the system near these points for *two*-dimensional systems of equations with constant coefficients, we saw eventually that the concepts of eigenvalue and eigenvector were meaningful for systems of n equations. Specifically, given a system $\dot{X} = AX$, where X is an $n \times 1$ column matrix (vector) and A is an $n \times n$ matrix, an eigenvalue λ is a solution of the equation $\det(A - \lambda I) = 0$, where I is the $n \times n$ identity matrix consisting of ones down the main diagonal and zeros elsewhere. We know that $\det(A - \lambda I)$ is an nth-degree polynomial in λ. Given an eigenvalue λ, an eigenvector associated with λ is a nonzero vector V satisfying the equation $AV = \lambda V$.

For values of n greater than 3, we lose the ordinary intuitive geometric interpretation of our results. Also, when n is greater than or equal to 5, there is no general procedure we can follow to solve the characteristic equations. We must use approximation methods, and technology becomes crucial here. The question of the multiplicity of eigenvalues leads to complicated linear-algebra considerations, and the general vector form of the solution of a system is difficult to describe without delving more deeply into linear algebra.

CHAPTER

Systems of nonlinear differential equations

7

Introduction

We have discussed various nonlinear equations throughout previous chapters, especially in Chapters 2 and 3, treating them numerically, graphically, and analytically. In general, we can't expect to find the explicit (closed-form) solution of a nonlinear equation, so we are forced to rely on qualitative and computational methods rather than on purely analytical techniques. This complexity is magnified when we address *systems* of nonlinear equations.

In Chapter 6 we analyzed the *stability* of systems of linear differential equations—that is, the behavior of such systems near equilibrium points—and saw that this stability could be described completely in terms of the eigenvalues and eigenvectors of the system. This kind of analysis can be done for nonlinear systems, but it is not quite as satisfactory and complete. One way of carrying out this study is to examine how closely we can approximate (in some sense) a nonlinear system by a linear system and then apply the linear theory.

The modern qualitative theory of stability discussed in Chapter 6 and in this chapter originated in the late 1800s with the work of the French mathematician Henri Poincaré (1854–1912), who was studying whether the solar system was actually a stable system. The equations involved in Poincaré's study of celestial mechanics could not be solved explicitly, so he and others developed implicit (qualitative) methods to deal with the complicated problems of planetary motion. [An excellent account of this work and its consequences is *Celestial Encounters: The Origins of Chaos and Stability* by F. Diacu and P. Holmes (Princeton: Princeton University Press, 1996).]

7.1 Equilibria of nonlinear systems

Recall that an *equilibrium point* of a differential equation or a system of differential equations is a constant solution. If we look at the two (somewhat similar) equations

$$(1)\ y' = -y$$
$$(2)\ y' = -y(1-y),$$

we see some important differences between linear and nonlinear equations.

A Modern Introduction to Differential Equations. https://doi.org/10.1016/B978-0-12-818217-8.00014-2

Eq. (1) is linear, but more fundamentally it is *separable*, so it is easy to find the general solution, $y = Ce^{-t}$, where C is an arbitrary constant. (We recognize that $C = y(0)$, the initial state of the system being modeled by the equation.)

Now Eq. (2) is nonlinear and separable, and its general solution is $y = \frac{1}{1+Ce^t}$, where $C = 1/y(0) - 1$. (*Verify the solutions to both equations.*)

Let's examine some typical solution curves for Eq. (1). Fig. 7.1 shows that there is only one equilibrium solution, $y \equiv 0$, and this is a *sink*. (Review Section 2.6 if necessary.) If an object described by the equation starts off at zero (that is, if $C = 0$), it remains at zero for all time. If the object's initial state is not zero, then the object will approach the solution $y \equiv 0$ as its asymptotically stable solution (or sink).

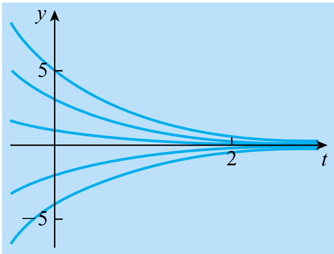


FIGURE 7.1

Solutions of $y' = -y$; $y(0) = 5, 3, 1, -2, -4$

On the other hand, Fig. 7.2 shows the same kind of information for Eq. (2). For such a nonlinear equation there can be more than one equilibrium solution, in this case $y \equiv 0$ and $y \equiv 1$. Also note that some solutions of a nonlinear equation may become unbounded as t approaches some finite value. (*If* $y(0) > 1$, *for what value of* t *does the denominator of the general solution to Eq.* (2) *vanish?*)

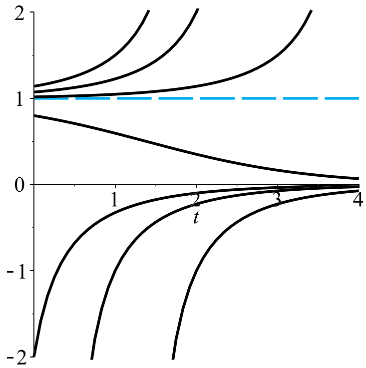


FIGURE 7.2

Solutions of $y' = -y(1 - y)$

In contrast, all solutions of a linear equation or a system of linear equations are defined for all values of the independent variable. Finally, looking closely at the behavior of solutions of Eq. (2) with different initial values, we see that the solutions starting off above 1 behave differently from those solutions with initial values less than 1. The equilibrium solution $y \equiv 0$ is a *sink* if $y(0) < 1$ and $y \equiv 1$ is a *source* if $y(0) > 1$. Furthermore, for solutions with initial values C greater than 1, the line $t = \ln\left(-\frac{1}{C}\right)$ is a vertical asymptote. We say that the solution "blows up in finite time." (Note that when $y(0) > 1$, $C = 1/y(0) - 1 < 0$, so that the blow-up point t is the logarithm of a positive quantity.) The last three types of behavior cannot occur when we are dealing with a linear equation. You should expect that the situation with nonlinear *systems* is appropriately complicated.

Let's look at an example of a nonlinear system and its behavior near its equilibrium points.

Example 7.1.1 Stability of a Nonlinear System

The nonlinear system

$$x' = x - x^2 - xy$$
$$y' = -y - y^2 + 2xy$$

represents two populations interacting in a predator-prey relationship. This is essentially a *Lotka–Volterra system* (see Section 7.4) with "crowding" terms (the squared terms) added for both species.

To calculate the equilibrium points of this system, we solve the system $\{x' = 0, y' = 0\}$, which is the same as the nonlinear algebraic system

$$\text{(A)} \quad x(1 - x - y) = 0$$
$$\text{(B)} \quad y(-1 - y + 2x) = 0.$$

Clearly, the origin, $x = y = 0$, is an equilibrium point. Logically, there are only three other cases to examine: (1) $x = 0$, $y \neq 0$; (2) $x \neq 0$, $y = 0$; and (3) $x \neq 0$, $y \neq 0$. Assuming case 1, we can eliminate Eq. (A) and examine (B), which becomes $y(-1 - y) = 0$. Because $y \neq 0$, we conclude that $-1 - y = 0$, or $y = -1$. Thus, our second equilibrium point is $(0, -1)$. Moving to case 2, we can ignore Eq. (B) and focus on (A), which now looks like $x(1 - x) = 0$. Because we are assuming in case 2 that $x \neq 0$, we can see that $x = 1$, which gives us the third equilibrium point $(1, 0)$. Finally, if $x \neq 0$ and $y \neq 0$, our system of algebraic equations becomes

$$\text{(A2)} \quad x + y = 1$$
$$\text{(B2)} \quad y - 2x = -1.$$

(We have divided out x and y in (A) and (B) and then rearranged the terms of each equation.) Subtracting (B2) from (A2) gives us $3x = 2$, or $x = \frac{2}{3}$. Substituting this value of x in (A2) yields $y = \frac{1}{3}$. Therefore, the last equilibrium point is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

In terms of a population problem, the only interesting equilibrium point is the last one we found. (*Why is this so?*) If we look at a slope field for the original system of nonlinear differential equations near the point $\left(\frac{2}{3}, \frac{1}{3}\right)$, we see an interesting pattern (Fig. 7.3a).

The apparent spiraling of solutions into the equilibrium point can be seen more clearly if we show some numerically generated solution curves (Fig. 7.3b). Fig. 7.3b represents a predator-prey

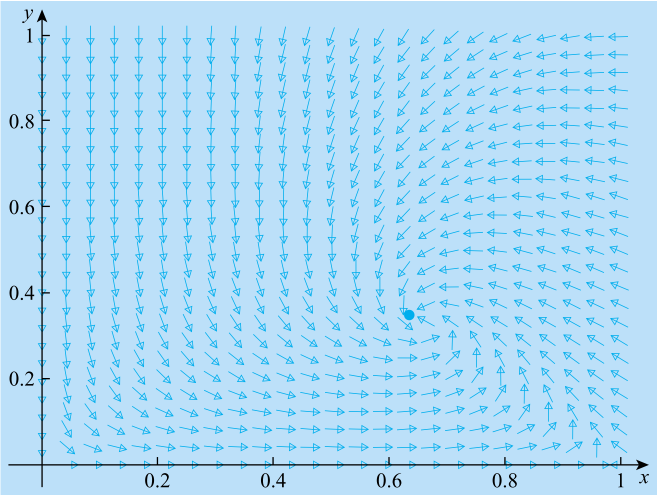


FIGURE 7.3a

Slope field for $x' = x - x^2 - xy,\ y' = -y - y^2 + 2xy$ near $\left(\frac{2}{3}, \frac{1}{3}\right)$

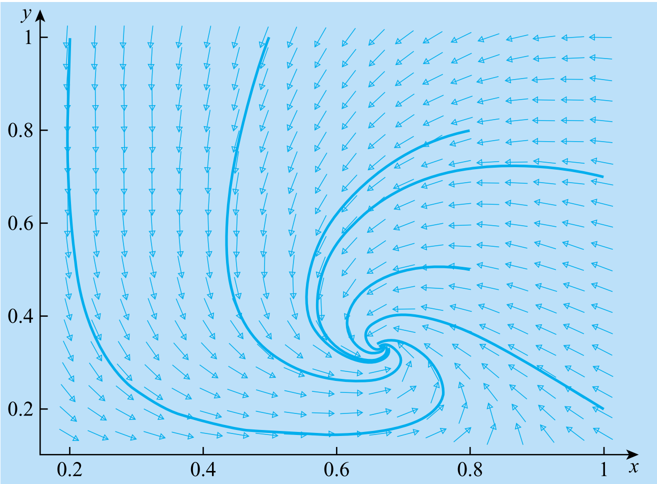


FIGURE 7.3b

Phase portrait for $x' = x - x^2 - xy,\ y' = -y - y^2 + 2xy$ near $\left(\frac{2}{3}, \frac{1}{3}\right)$; $(x(0), y(0)) = (0.2, 1)$, $(0.8, 0.8)$, $(0.8, 0.5)$, $(1, 0.7)$, $(1, 0.2)$, $(0.5, 1)$

population that is stabilizing. If the units are thousands of creatures, then the X population is heading for a steady population of about 667, whereas the Y population has 333 as its stable value.

Mathematically, however, we should look at the entire phase portrait to understand the complex behavior of nonlinear systems. We'll return for a detailed analysis of this system in Example 7.3.1.

Exercises 7.1

A

Find all the equilibrium points for each of the systems in Problems 1–14, using technology if necessary.

1. $x' = -x + xy,\ y' = -y + 2xy$
2. $x' = x - xy,\ y' = y - xy$
3. $x' = x^2 - y^2,\ y' = x - xy$
4. $x' = 1 - y^2,\ y' = 1 - x^2$
5. $x' = x + y + 2xy,\ y' = -2x + y + y^3$
6. $x' = y\left(1 - x^2\right),\ y' = -x\left(1 - y^2\right)$
7. $x' = x - x^2 - xy,\ y' = 3y - xy - 2y^2$
8. $x' = 1 - y,\ y' = x^2 - y^2$
9. $x' = (1 + x)\sin y,\ y' = 1 - x - \cos y$
10. $x' = 3y - e^x,\ y' = 2x - y$ [*Hint*: There are two equilibrium points. Use your CAS to approximate these points.]
11. $x' = y^2 - x^2,\ y' = x - 1$
12. $x' = x^2 - y,\ y' = y^2 - x$
13. $x' = xy(1 - x),\ y' = y\left(1 - \frac{y}{x}\right)$
14. $x' = y,\ y' = -\sin x - 3y$

B

15. Use technology to find all the equilibrium points of the system

$$x' = -y,\ y' = \left(x^4 + 4x^3 - x^2 - 4x + y\right)/8.$$

16. A two-mode laser produces two different kinds of photons, whose numbers are n_1 and n_2. The equations governing the rates of photon production are

$$\dot{n}_1 = G_1 N n_1 - k_1 n_1$$
$$\dot{n}_2 = G_2 N n_2 - k_2 n_2,$$

where $N(t) = N_0 - a_1 n_1 - a_2 n_2$ is the number of excited atoms. The parameters $G_1, G_2, k_1, k_2, a_1, a_2$, and N_0 are all positive. Use a CAS "solve" command to find all the equilibrium points of the system.

17. A *chemostat* is a device for growing and studying bacteria by supplying nutrients and maintaining convenient levels of the bacteria in a culture. One model of a chemostat is the nonlinear system

$$\frac{dN}{dt} = a_1\left(\frac{C}{1+C}\right)N - N$$
$$\frac{dC}{dt} = -\left(\frac{C}{1+C}\right)N - C + a_2,$$

where $N = N(t)$ denotes the bacterial density at time t; $C = C(t)$ denotes the concentration of nutrient; and a_1, a_2 are positive parameters. Use technology to find all the equilibrium solutions (N^*, C^*) of the system.

18. In the absence of damping and any external force, the motion of a pendulum is described by the equation $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$, where θ is the angle between the pendulum and the downward vertical, g is the acceleration due to gravity, and L is the length of the pendulum.

a. Write this equation as a system of two first-order equations.

b. Describe all the equilibrium points of the system.

C

19. Use technology to find all the equilibrium solutions of the system

$$x' = x - y^2 + a + bxy, \quad y' = 0.2y - x + x^3,$$

where $a = 1.28$ and $b = 1.4$. (Round to the nearest thousandth.)

7.2 Linear approximation at equilibrium points

One important aspect of *linear* systems is that the behavior of solutions near an equilibrium point (local behavior) tells you the behavior of solutions in the entire phase plane. However, even though many of the "nice" properties of linear systems are not present when we analyze nonlinear systems, we may be able to understand the local behavior of nonlinear systems by a process of *linearization* or *linear approximation*: We try to replace the original nonlinear system by a linear system that is close to it near an equilibrium point. This is analogous to our discussion of *Euler's method* in Section 3.1, which involved approximating solution curves by tangent lines.

To see how this might work, let's go back to the nonlinear equation $y' = -y(1-y) = -y + y^2$ discussed in Section 7.1. We know that $y = 0$ is an equilibrium point. Now note that for values of y close to zero, y^2 is smaller than y. For example, if $y = 0.00001$, then $y^2 = 0.0000000001$. Then, dropping the squared (nonlinear) terms, we can guess that the linear equation $y' = -y$ is a good approximation for the original equation and that the behavior of this last equation near $y = 0$ should tell us how

$y' = -y(1-y)$ behaves near $y = 0$. A comparison of the solution curves near $y = 0$ in Fig. 7.1 and Fig. 7.2 shows us that this is true. However, it should also be clear that we would be wrong to base our analysis of $y' = -y(1-y)$ on $y' = -y$ for *all* initial values.

If we want to analyze the behavior of $y' = -y(1-y)$ near its other equilibrium point, $y = 1$, we can use a simple change of variable: Let $y = 1 + z$, so that studying the behavior of $y' = -y(1-y)$ near $y = 1$ is the same as analyzing the behavior of the equation $y = 1 + z$ near $z = 0$. (*Make sure you see this.*) With this change of variable we get the new equation $z' = -y(1-y) = (-1-z)(-z) = z + z^2$. Using the same reasoning as before, we can take $z' = z$ as a good linear approximation near $z = 0$. This last equation has the general solution $z = Ce^t$, so solutions of $z' = z + z^2$ move *away from* $z = 0$ as t increases. But because $y = 1 + z$, the solutions of $y' = -y(1-y)$ near $y = 1$ curve away from $y = 1$, behavior we can verify by looking at Fig. 7.2.

As another example, take the second-order nonlinear equation $\frac{d^2x}{dt^2} + \frac{g}{L}\sin x = 0$, which describes the swinging of a pendulum, where x is the angle the pendulum makes with the vertical, g is the acceleration due to gravity, and L is the pendulum's length. This equation is not easy to deal with analytically, so usually the nonlinearity is removed by a substitution. For small values of x—that is, for an oscillation of small amplitude—$\sin x \approx x$, so we can replace our original nonlinear equation by the *linear* equation that approximates it: $\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$. This approximate pendulum model has the same mathematical behavior as the undamped spring-mass system (see Eq. (6.11.1)). Despite our success in approximating a nonlinear equation with one that is linear, this is a limited victory. The next example illustrates how linearization can lead us astray.

Example 7.2.1 Linearization Can Mislead

Let's look at the system

$$\dot{x} = y + ax\left(x^2 + y^2\right)$$
$$\dot{y} = -x + ay\left(x^2 + y^2\right),$$

where a is a real parameter.

Clearly, the origin $(x, y) = (0, 0)$ is an equilibrium point regardless of the value of the parameter a. In our example the obvious linearized system is

$$\dot{x} = y$$
$$\dot{y} = -x.$$

(Look back at the spring-mass system analyzed in Example 6.11.1.) This can be written in the form $\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The characteristic equation is $\lambda^2 + 1 = 0$, so that the eigenvalues of A are purely imaginary: i and $-i$. From Table 6.2 in Section 6.9, we conclude that the origin is a *stable center* of the linearized system. *But this is the wrong conclusion with respect* *to the original nonlinear system*, as the phase portrait of the original nonlinear system near $(0, 0)$ shows. This portrait (Fig. 7.4) corresponds to $a = -1$.

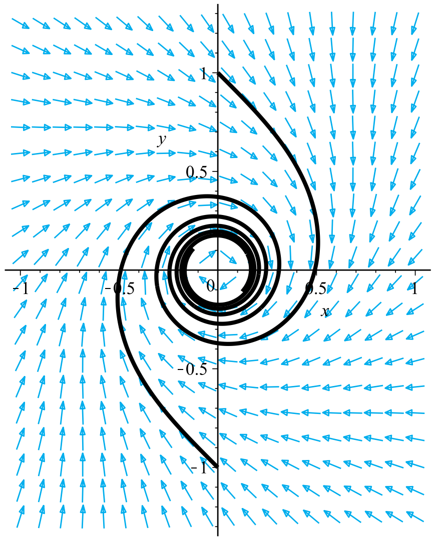


FIGURE 7.4

Trajectories of $\dot{x} = y - x\left(x^2 + y^2\right)$, $\dot{y} = -x - y\left(x^2 + y^2\right)$, $0 \le t \le 18$; $(x(0), y(0)) = (0, 1), (0, -1)$

It seems that the trajectories spiral in toward the equilibrium point, indicating that the origin is actually a *spiral sink* for the nonlinear system. However, appearances can be deceiving, and Problem 21 in Exercises 7.2 suggests a way of proving this claim about the origin.

We begin to suspect that the stability of the original system depends on the value of the parameter a. If $a = 0$, for example, then the nonlinear portion of the system disappears, leaving us with a purely linear system—in fact, the same system analyzed in Example 6.11.1 (with $\beta = 1$). As we've said, the origin is a stable center for this linear system, every trajectory closing perfectly after one cycle. (Problem 21 asks you to explore these ideas further.)

In summary, we can look at this last example as a linear system "perturbed" (disturbed or knocked off kilter) by a nonlinear component. We can write this system as $\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X + \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$, where f and g are nonlinear functions of x and y. As we'll see later, if the nonlinear perturbation is "nice" enough, the behavior of the whole system can be predicted from the behavior of its linear portion near the equilibrium points.

7.2.1 Almost linear systems

If we want to make this discussion of linear approximation mathematically sound, we have to remind ourselves of some basic calculus facts. Back in Section 3.1 we discussed *local linearity*, the idea that if we "zoom" in on a point on a curve $y = f(x)$, the curve looks like a straight line—in fact, like a piece of the tangent line drawn to the curve at that point. More precisely, for values of the independent variable x close to $x = a$, we can write $f(x) \approx f(a) + f'(a)(x - a)$. We should see that this expression consists of the first two terms of an nth-degree ($n \geq 1$) *Taylor polynomial approximation* of f near $x = a$—or, equivalently, the first two terms of the *Taylor series* expansion of f in a neighborhood of $x = a$:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &\quad + \frac{f^{(n)}}{n!}(x-a)^n + \cdots \\ &= f(a) + f'(a)(x-a) + (x-a)^2 \left\{ \frac{f''(a)}{2!} + \frac{f'''(a)}{3!}(x-a) + \cdots \right. \\ &\quad \left. + \frac{f^{(n)}}{n!}(x-a)^{n-2} + \cdots \right\}. \end{aligned}$$

We can write this last result as $f(x) \approx f(a) + f'(a)(x-a) + O\left((x-a)^2\right)$, where the notation $O\left((x-a)^2\right)$ represents the fact that if x is close to a (so that $x - a$ is very small), then the sum of all terms past the second will be bounded by some multiple of $(x-a)^2$. (The series in braces, $\{\cdots\}$, converges to some constant.)

Now assume that we have a general nonlinear autonomous system of the form

$$\begin{aligned} \dot{x} &= F(x, y) \\ \dot{y} &= G(x, y) \end{aligned} \tag{7.2.1}$$

which has the origin as an equilibrium point—that is, $F(0,0) = 0$ and $G(0,0) = 0$. This last assumption is just for convenience as we develop some methodology. If we can write F as $ax + by + f(x,y)$ and G as $cx + dy + g(x,y)$, where f and g are nonlinear functions, then we can express the system in the form

$$\dot{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} X + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

If the nonlinear functions f and g are small enough (in a sense to be explained later) that their effect on the system is negligible, then we can call our system "almost linear." Near the origin, our nonlinear system behaves essentially like the linear system $\dot{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} X$—that is, like the system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy. \end{aligned}$$

Earlier we recalled that the tangent line $y = f(a) + f'(a)(x - a)$ gives the best linear approximation of a single-variable function f near $x = a$. For $F(x, y)$, a function of *two* variables, the best approximation near a point (a, b) is provided by the *tangent plane* given by the approximation formula

$$F(x, y) \approx F(a, b) + \frac{\partial F}{\partial x}(a, b)(x - a) + \frac{\partial F}{\partial y}(a, b)(y - b), \tag{7.2.2}$$

where $\frac{\partial F}{\partial x}(a, b)$ and $\frac{\partial F}{\partial y}(a, b)$ denote the first partial derivatives of F evaluated at the point (a, b). (See Appendix A.8.) For example, if we want to approximate $F(x, y) = x^3 + y^3$ near the point $(1, 1)$, we calculate

$$\begin{aligned} F(1, 1) &= 1^3 + 1^3 = 2 \\ \frac{\partial F}{\partial x} &= 3x^2, \quad \frac{\partial F}{\partial x}(1, 1) = 3(1)^2 = 3 \\ \frac{\partial F}{\partial y} &= 3y^2, \quad \frac{\partial F}{\partial y}(1, 1) = 3(1)^2 = 3, \end{aligned}$$

so the equation of the tangent plane is $z = 2 + 3(x - 1) + 3(y - 1)$.

You should think of the right-hand side of Eq. (7.2.2) as the first-degree Taylor polynomial approximation of F, the linear terms in x and y of the two-variable Taylor series expansion of F. This approximation ignores the rest of the series consisting of the terms in x and y of the second degree and higher, which we can denote by $f(x, y)$. Thus, in our last example, we can write $x^3 + y^3 \approx 2 + 3(x - 1) + 3(y - 1)$ near $(1, 1)$ or $x^3 + y^3 = 2 + 3(x - 1) + 3(y - 1) + f(x, y)$ near $(1, 1)$. (See Appendix A.8 for more information on this topic.)

If we choose the point (a, b) to be the origin, then we can rewrite (7.2.1) as

$$\begin{aligned} \dot{x} &= F(0, 0) + \frac{\partial F}{\partial x}(0, 0)x + \frac{\partial F}{\partial y}(0, 0)y + f(x, y) \\ \dot{y} &= G(0, 0) + \frac{\partial G}{\partial x}(0, 0)x + \frac{\partial G}{\partial y}(0, 0)y + g(x, y), \end{aligned}$$

or (remembering that we have assumed $F(0, 0) = G(0, 0) = 0$) as

$$\begin{aligned} \dot{x} &= ax + by + f(x, y) \\ \dot{y} &= cx + dy + g(x, y), \end{aligned} \tag{7.2.3}$$

where $a = \frac{\partial F}{\partial x}(0, 0)$, $b = \frac{\partial F}{\partial y}(0, 0)$, $c = \frac{\partial G}{\partial x}(0, 0)$, and $d = \frac{\partial G}{\partial y}(0, 0)$.

The technical definition of the "smallness" of f and g near the origin is that

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{g(x, y)}{\sqrt{x^2 + y^2}} = 0. \tag{7.2.4}$$

The limits in (7.2.4) just say that near the origin, f and g are small in comparison to $r = \sqrt{x^2 + y^2}$, which is the radial distance of the point (x, y) from the origin.

We define an **almost linear system** as a nonlinear system (7.2.3) that satisfies (7.2.4). In this situation, the linear part

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \tag{7.2.5}$$

is called the **associated linear system** or **linear approximation** about the equilibrium point (0, 0).

Example 7.2.2 A Linear Approximation

Let's examine the behavior of the following system near the origin:

$$\begin{aligned} \dot{x} &= x + 2y + x\cos y \\ \dot{y} &= -y - \sin y. \end{aligned}$$

First of all, we can see that $(0, 0)$ is an equilibrium point for the system. Now we must find the associated linear system, which is not obvious because $x\cos y$ and $-\sin y$ actually contain linear terms that must be combined with the linear terms already visible in the original system.

Substituting the Taylor (or Maclaurin) expansions for $\cos y$ and $\sin y$ (see Appendix A.3) in the given equations and collecting terms, we have

$$\begin{aligned} \dot{x} &= x + 2y + x\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) = 2x + 2y + x\left(-\frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) \\ \dot{y} &= -y - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) = -2y - \left(-\frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right). \end{aligned}$$

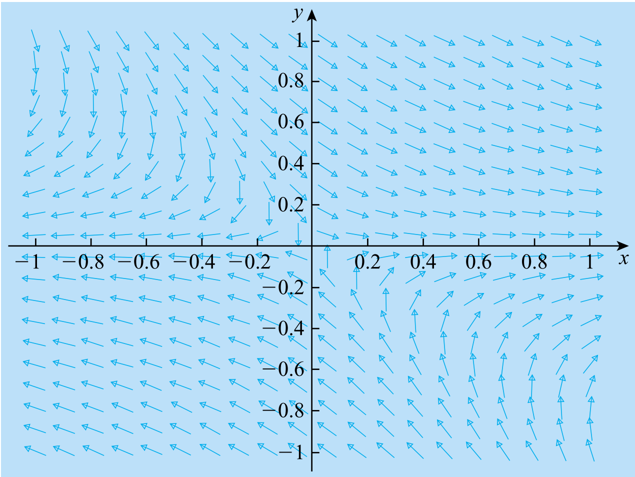


FIGURE 7.5a

Slope field for $\dot{x} = x + 2y + x\cos y$, $\dot{y} = -y - \sin y$

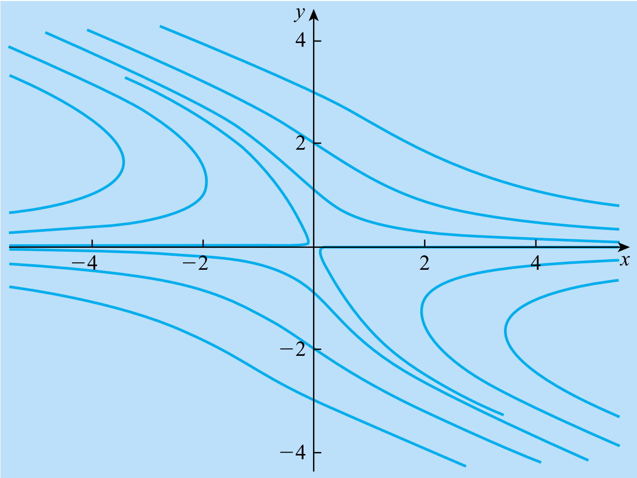


FIGURE 7.5b

Trajectories for $\dot{x} = x + 2y + x\cos y,\ \dot{y} = -y - \sin y$

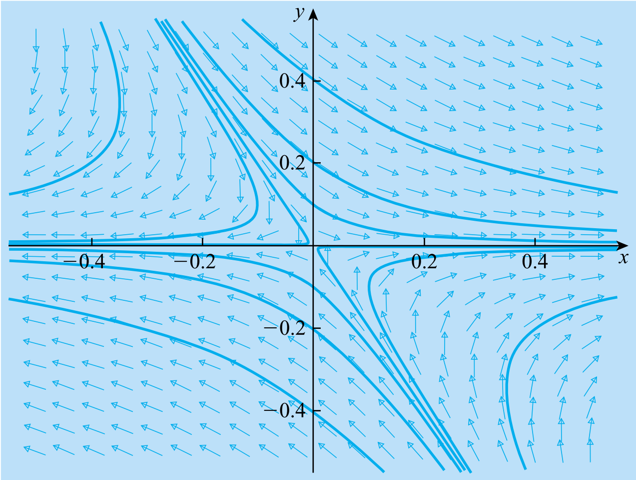


FIGURE 7.5c

Trajectories for $\dot{x} = 2x + 2y,\ \dot{y} = -2y$

Thus, the associated linear system is

$$\begin{aligned}\dot{x} &= 2x + 2y\\ \dot{y} &= -2y,\end{aligned}$$

or

$$\dot{X} = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix} X = AX.$$

The characteristic equation of this linear system is given by $\lambda^2 - 4 = 0$, so the eigenvalues are $\lambda = -2$ and $\lambda = 2$.

Table 6.2 in Section 6.9 tells us that two real eigenvalues opposite in sign indicate that we have a *saddle point*. Fig. 7.5a shows the slope field for the system; Fig. 7.5b shows some trajectories for the nonlinear system around the origin; Fig. 7.5c shows trajectories around the origin for the associated linear system.

We can see from these phase portraits that the linear approximation captures the behavior of the nonlinear system near the origin.

Exercises 7.2

A

In Problems 1–15, (a) verify that $(0, 0)$ is an equilibrium point of each system and (b) show that each system is almost linear.

1. $x' = 3x + y + xy,\ y' = 2x + 2y - 2xy^2$
2. $x' = x - y + x^2,\ y' = x + y$
3. $x' = x - xy - 8x^2,\ y' = -y + xy$
4. $x' = -4x + y - xy^3,\ y' = x - 2y + 3x^2$
5. $x' = 3\sin x + y,\ y' = 4x + \cos y - 1$
6. $x' = x - y,\ y' = 1 - e^x$
7. $x' = -3x - y - xy,\ y' = 5x + y + xy^3$
8. $x' = y\left(1 - x^2\right),\ y' = -x\left(1 - y^2\right)$
9. $x' = -x + x^3,\ y' = -2y$
10. $x' = -2x + 3y + xy,\ y' = -x + y - 2xy^2$
11. $x' = (x - 2y)(y + 4),\ y' = 2x - y$
12. $x' = (x - 2)(y - 3) - 6,\ y' = (x + 2y)(y - 1)$
13. $x' = 5x - 14y + xy,\ y' = 3x - 8y + x^2 + y^2$
14. $x' = 9x + 5y + xy,\ y' = -7x - 3y + x^2$
15. $x' = \frac{1}{2}\left(1 - \frac{1}{2}x - \frac{1}{2}y\right)x,\ y' = \frac{1}{4}\left(1 - \frac{1}{3}x - \frac{2}{3}y\right)y$

B

16. Find the linear approximation of the system $\{\dot{x} = \sin x + 2y,\ \ \dot{y} = xy + 3ye^x + x\}$.
17. Consider the two systems

$$(a) \quad \dot{x} = -y + x(x^2 + y^2), \quad \dot{y} = x + y(x^2 + y^2)$$

$$(b) \quad \dot{x} = -y - x(x^2 + y^2), \quad \dot{y} = x - y(x^2 + y^2).$$

a. Show that both systems are almost linear and determine all the equilibrium solutions.

b. Linearize each system.

c. Convert each system to polar coordinates and use these representations to determine the nature of all the equilibrium solutions for each system.

18. Show that the system

$$x' = -x + 3y + y\cos\sqrt{x^2 + y^2}$$
$$y' = -x - 5y + x\cos\sqrt{x^2 + y^2}$$

is *not* almost linear.

19. Consider the second-order nonlinear equation $\ddot{x} + x - 0.25x^2 = 0$.

a. Convert this equation to a nonlinear system of two first-order equations.

b. Determine if $(0, 0)$ is an equilibrium solution of the system found in part (a).

c. Determine the associated linear system.

20. The interaction between two species is governed by the model $\{\dot{H} = (a_1 - b_1H - c_1P)H,\ \dot{P} = (-a_2 + c_2H)P\}$, where H is the population of the host (prey), P is that of the parasite (or predator), and all constants are positive. Assuming that $a_1c_2 - b_1a_2 > 0$, find the equilibrium solutions for the populations and classify them.

C

21. Let's return to the system in Example 7.2.1:

$$\dot{x} = y + ax\left(x^2 + y^2\right)$$
$$\dot{y} = -x + ay\left(x^2 + y^2\right).$$

a. Introduce polar coordinates. Note that $x^2 + y^2 = r^2$ and use the Chain Rule to show that $x\dot{x} + y\dot{y} = r\dot{r}$.

b. In the expression for $r\dot{r}$ found in part (a), substitute for $\dot{x}$ and $\dot{y}$ using the equations in the system and show that $\dot{r} = ar^3$ for $r > 0$.

c. Show that $\theta = \arctan(y/x)$ and that $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$. Substitute for $\dot{x}$ and $\dot{y}$ in this last formula to see that $\dot{\theta} = 1$.

d. The results of parts (b) and (c) show that our original system is equivalent to the system $\left\{\dot{r} = ar^3, \dot{\theta} = 1\right\}$. The second equation says that all trajectories rotate counterclockwise around the origin with constant angular velocity 1. Recognizing that the first equation describes the radial distance from the origin to a point on the trajectory—for a point $(x(t), y(t))$

on a trajectory, $d(t) = \sqrt{x(t)^2 + y(t)^2}$ is the radial distance—examine what happens to $r(t)$ as $t \to \infty$ in the three cases $a < 0$, $a = 0$, and $a > 0$. What does this say about the nature of the equilibrium point at the origin? Sketch a trajectory (in the x-y plane) for each of the three cases.

7.3 The Hartman–Grobman theorem

In this section, we will expand (pun intended) our view of linear approximation as an aid in understanding the stability of nonlinear systems. More generally, suppose that (a, b) is an equilibrium point for the system

$$\begin{aligned} \dot{x} &= F(x, y) \\ \dot{y} &= G(x, y), \end{aligned}$$

which means that $F(a, b) = 0 = G(a, b)$. Using the tangent plane approximation formula (7.2.2), we can rewrite this system as

$$\begin{aligned} \dot{x} &= F(a, b) + \frac{\partial F}{\partial x}(a, b)(x - a) + \frac{\partial F}{\partial y}(a, b)(y - b) + f(x, y) \\ \dot{y} &= G(a, b) + \frac{\partial G}{\partial x}(a, b)(x - a) + \frac{\partial G}{\partial y}(a, b)(y - b) + g(x, y) \end{aligned}$$

or, because $F(a, b) = 0 = G(a, b)$, as

$$\begin{aligned} \dot{x} &= A(x - a) + B(y - b) + f(x, y) \\ \dot{y} &= C(x - a) + D(y - b) + g(x, y), \end{aligned} \tag{7.3.1}$$

where

$$A = \frac{\partial F}{\partial x}(a, b), \; B = \frac{\partial F}{\partial y}(a, b), \; C = \frac{\partial G}{\partial x}(a, b), \text{ and } D = \frac{\partial G}{\partial y}(a, b).$$

The matrix of coefficients in this situation can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \\ \frac{\partial G}{\partial x}(a, b) & \frac{\partial G}{\partial y}(a, b) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}_{(a,b)}.$$

The matrix of partial derivatives is called the **Jacobian**, denoted by $\boldsymbol{J}(x, y)$, and is named after the mathematician Carl Jacobi (1804–51).

Another way to look at this general situation is to realize that we are translating the equilibrium point (a, b) to the origin by using the change of variables $u = x - a$ and $v = y - b$. Of course, this means that $x = u + a$ and $y = v + b$, so we can rewrite (7.3.1) as

$$\dot{u} = Au + Bv + f(u, v)$$

$$\dot{v} = Cu + Dv + g(u, v),$$

which has $(0, 0)$ as an equilibrium point. **Note that this says that any equilibrium point $(a^*, b^*) \neq (0, 0)$ can be translated to the origin for the purpose of analyzing the stability of the system**. Therefore, we can state an important stability result for nonlinear systems in terms of an equilibrium point at the origin. The theorem is given in terms of the eigenvalues of the associated linear system.

Suppose we have the nonlinear autonomous system

$$\begin{aligned} \dot{x} &= ax + by + f(x, y) \\ \dot{y} &= cx + dy + g(x, y), \end{aligned} \tag{7.3.2}$$

where $ad - bc \neq 0$, $\lim_{(x,y)\to(0,0)} f(x, y)/\sqrt{x^2 + y^2} = \lim_{(x,y)\to(0,0)} g(x, y)/\sqrt{x^2 + y^2} = 0$, and the origin is an equilibrium point. If λ_1 and λ_2 are the eigenvalues of the associated linear system

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= cx + dy, \end{aligned} \tag{7.3.3}$$

then the **Hartman–Grobman theorem** states that the equilibrium points of the two systems (7.3.2) and (7.3.3) are related as follows:

a. If the eigenvalues λ_1 and λ_2 are **not** equal real numbers or are **not** pure imaginary numbers, then the trajectories of the almost linear system (7.3.2) near the equilibrium point $(0, 0)$ behave the same way as the trajectories of the associated linear system (7.3.3) near the origin. That is, we can use the appropriate table entries given in Section 6.9 to determine whether the origin is a node, a saddle point, or a spiral point of both systems.

b. If $\lambda_1 = \lambda_2 > 0$, then the origin is an unstable equilibrium solution for both the nonlinear and linearized systems. If $\lambda_1 = \lambda_2 < 0$, then the origin is asymptotically stable for both systems.

c. If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the origin is unstable for the nonlinear system. If $\lambda_1 = 0$ and $\lambda_2 < 0$, then the origin may be stable or unstable for the nonlinear system, depending on the neglected nonlinear terms.

d. If λ_1 and λ_2 are complex conjugates with a nonzero real part, then the origin is unstable or asymptotically stable, depending on whether the real part of λ_1, λ_2 is positive or negative, respectively.

e. If λ_1 and λ_2 are pure imaginary numbers—that is, $\lambda_1 = i\delta$, $\lambda_2 = -i\delta$, $\delta \neq 0$, then the origin is a *center* for the linearized system, but may not be a center for the nonlinear system. The origin is either a center or a spiral point of the nonlinear system. Also, this spiral point may be asymptotically stable, stable, or unstable, depending on the nonlinear terms neglected during the linearization.

To summarize, there are only two situations in which the behavior of trajectories near an equilibrium solution of a nonlinear system could differ from the behavior of the system's linearization:

1. When the linearized system has zero as an eigenvalue.
2. When the linearized system has pure imaginary eigenvalues, so that the equilibrium solution is a center.

A nonlinear system is said to have a **hyperbolic equilibrium point** if all eigenvalues of the Jacobian matrix $\boldsymbol{J}(x, y)$ are nonzero or have a nonzero real part in the case of complex conjugate eigenvalues. The system has a **nonhyperbolic equilibrium point** if at least one eigenvalue is zero or has a zero real part. In abbreviated form, the Hartman–Grobman result can be stated as follows:

Suppose that (u, v) is a hyperbolic equilibrium point of the nonlinear system (7.3.2). Then there is a neighborhood of this equilibrium point on which the phase portrait of the nonlinear system resembles that of the linearized system (7.3.3). If (u, v) is a nonhyperbolic equilibrium point, no conclusion can be drawn about the trajectories of the nonlinear system near the equilibrium point.

This important result was announced by David Grobman in 1959 and was independently stated and proved by Philip Hartman in 1960. It was based on earlier work on the stability of nonlinear systems by H. Poincaré, A.M. Lyapunov (1857–1918), and others beginning at the end of the 19th century.

The next example shows how to use the Hartman–Grobman theorem.

Example 7.3.1 An Application of the Hartman–Grobman Theorem

Let's return to the system in Example 7.1.1:

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= -y - y^2 + 2xy. \end{aligned}$$

We saw that there were four equilibrium points: $(0, 0)$, $(0, -1)$, $(1, 0)$, and $(\frac{2}{3}, \frac{1}{3})$.

Near the origin, because the terms x^2, y^2, and xy are smaller than the terms x and y, we can replace the nonlinear system by its associated linear system

$$\begin{aligned} x' &= x \\ y' &= -y. \end{aligned}$$

The eigenvalues of this linear system are -1 and 1. According to part (a) of the Hartman–Grobman result, the trajectories of the nonlinear system should behave the same way as the trajectories of this associated linear system. Table 6.2 in Section 6.9 tells us that the origin is a *saddle point* for both systems.

If we want to examine what happens near the equilibrium point $(0, -1)$, we make the change of variables $u = x - 0 = x$ and $v = y - (-1) = y + 1$ so that we can rewrite the original system as

$$\begin{aligned} u' &= x' = u - u^2 - u(v - 1) = 2u - u^2 - uv \\ v' &= y' = -(v - 1) - (v - 1)^2 + 2u(v - 1) = -2u + v - v^2 + 2uv. \end{aligned}$$

Then the associated linear system is

$$u' = 2u$$

$$v' = -2u + v,$$

with eigenvalues 1 and 2. (*Check this.*) Now paragraph (a) of the Hartman–Grobman theorem and the table in Section 6.9 tell us that the equilibrium point $(0, -1)$ is a *source* for the nonlinear system.

The equilibrium point $(1, 0)$ leads us to make the change of variables $u = x - 1$ and $v = y - 0 = y$, so that the nonlinear system is transformed into

$$u' = -u - v - u^2 - uv$$
$$v' = v - v^2 + 2uv,$$

with associated linear system

$$u' = -u - v$$
$$v' = v.$$

The eigenvalues for this last system are -1 and 1, so that $(1, 0)$ is a *saddle point* for both the nonlinear system and its associated linear system.

Finally, we look at the equilibrium point $\left(\frac{2}{3}, \frac{1}{3}\right)$. The transformation $u = x - \frac{2}{3}$, $v = y - \frac{1}{3}$ leads to the system

$$u' = \frac{-2u}{3} - \frac{2v}{3} - u^2 - uv$$
$$v' = \frac{2u}{3} - \frac{v}{3} - v^2 + 2uv.$$

The linear approximation is given by

$$u' = \frac{-2u}{3} - \frac{2v}{3}$$
$$v' = \frac{2u}{3} - \frac{v}{3},$$

which has eigenvalues $-\frac{1}{2} + \frac{\sqrt{15}}{6}i$ and $-\frac{1}{2} - \frac{\sqrt{15}}{6}i$. Therefore, from result (a) and Table 6.2, we know that $\left(\frac{2}{3}, \frac{1}{3}\right)$ is a *spiral sink*. (*Look back at Figs. 7.3a and 7.3b to see this clearly.*) Fig. 7.6a shows some trajectories near the origin, a saddle point. Fig. 7.6b illustrates the behavior of the system near the equilibrium point $(0, -1)$, a source. Finally, Fig. 7.6c makes it clear that $(1, 0)$ is indeed a saddle point.

Now let's examine a system whose stability is not so clear.

Example 7.3.2 Another Application of Hartman–Grobman

The system we'll investigate is

$$\dot{x} = -x^3 - y$$
$$\dot{y} = x - y^3.$$

Set $\dot{x} = 0$ and $\dot{y} = 0$ and then substitute $y = -x^3$ from the first equation into the second equation. We get $x + x^9 = 0$, or $x(1 + x^8) = 0$, so $x = 0$. It follows that $(0, 0)$ is the only equilibrium point of this system.

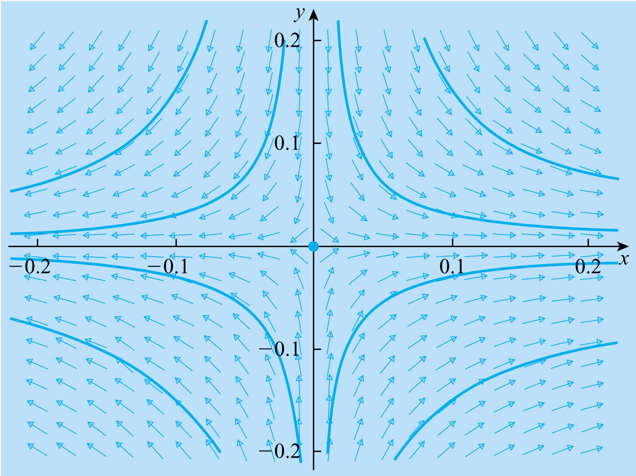


FIGURE 7.6a

Trajectories of $x' = x - x^2 - xy,\ y' = -y - y^2 + 2xy$ near the origin

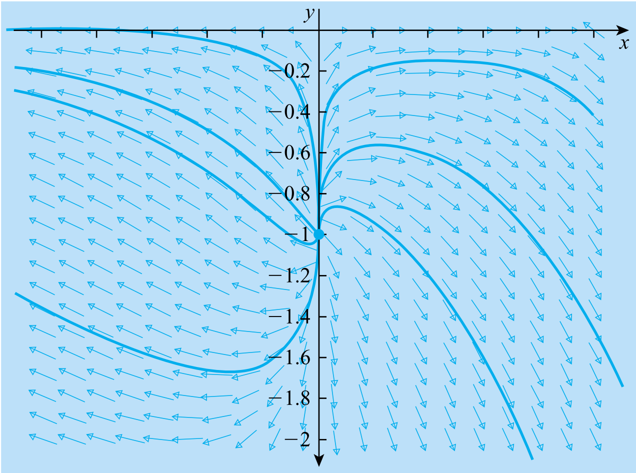


FIGURE 7.6b

Trajectories of $x' = x - x^2 - xy,\ y' = -y - y^2 + 2xy$ near $(0, -1)$

The linearized system is

$$\dot{x} = -y$$
$$\dot{y} = x,$$

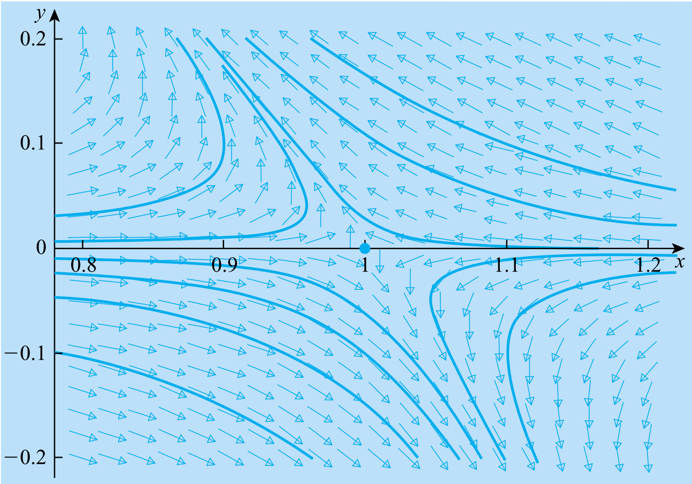


FIGURE 7.6c

Trajectories of $x' = x - x^2 - xy,\ y' = -y - y^2 + 2xy$ near $(1, 0)$

with characteristic equation $\lambda^2 + 1 = 0$ and eigenvalues $-i$ and i. Because the eigenvalues are pure imaginary numbers, case (e) of the Hartman–Grobman result tells us that the origin is either a center or a spiral point of the original nonlinear system. (Note that the origin is a *center* of the associated linear system.) Fig. 7.7 shows a typical trajectory, in this case with initial state $(x(0), y(0)) = (-0.5, 0)$ and t running from -9 to 100.

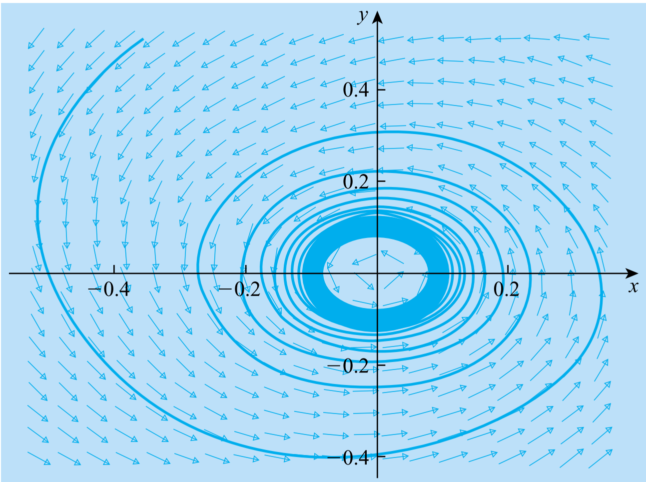


FIGURE 7.7

Trajectories of $\dot{x} = -x^3 - y,\ \dot{y} = x - y^3$ near the origin

From this, we can see that the trajectory appears to spiral in toward the origin—that is, the equilibrium point is *asymptotically stable*. We could have seen this analytically by defining the function

$$d(t) = \sqrt{x^2(t) + y^2(t)},$$

which gives the distance from any point $(x(t), y(t))$ on a trajectory to $(0, 0)$. Differentiating this function and then substituting from our original equations, we get

$$\begin{aligned}\dot{d}(t) &= \frac{1}{2}\left[x^2(t) + y^2(t)\right]^{-1/2}(2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)) \\ &= \frac{x(t)\left(-x^3(t) - y(t)\right) + y(t)\left(x(t) - y^3(t)\right)}{\sqrt{x^2(t) + y^2(t)}} = -\frac{x^4(t) + y^4(t)}{\sqrt{x^2(t) + y^2(t)}} < 0.\end{aligned}$$

This says that the distance between points on the trajectory and the origin is *decreasing* with time—that is, the trajectory is always moving closer and closer to the origin.

The next example shows another type of behavior.

Example 7.3.3 Yet Another Application of Hartman–Grobman

The system

$$\dot{x} = 2x - 6x^2 y$$
$$\dot{y} = 2y + x$$

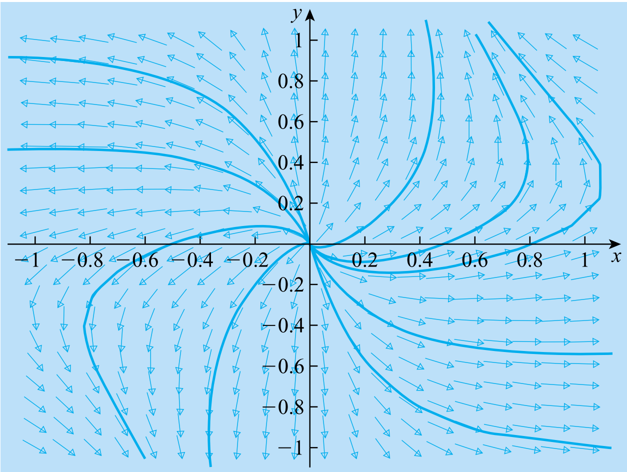


FIGURE 7.8

Trajectories of $\dot{x} = 2x - 6x^2 y$, $\dot{y} = 2y + x$ near the origin has the origin as its only equilibrium point. (*Check this for yourself*.) The linearization of this system is

$$\dot{x} = 2x$$
$$\dot{y} = 2y + x,$$

which has the characteristic equation $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$. Because the eigenvalues are positive and equal, we use part (b) of the Hartman–Grobman theorem to conclude that the origin is an *unstable equilibrium point, a source*. Fig. 7.8 shows this.

It is important to realize that **the classification of equilibrium solutions for nonlinear systems given by the Hartman–Grobman theorem says nothing about the behavior of trajectories of the nonlinear system with initial positions far from an equilibrium point**. The equilibrium solutions of linear systems determine the *global* behavior of solutions, whereas the equilibria of a nonlinear system influence the qualitative picture only *locally*.

Exercises 7.3

A

Each of the almost linear systems in Problems 1–10 has $(0, 0)$ as an equilibrium point. Discuss the type and stability of the origin by examining the associated linear system in each case.

1. $x' = x - y + x^2,\ y' = x + y$
2. $x' = x - xy - 8x^2,\ y' = -y + xy$
3. $x' = -4x + y - xy^3,\ y' = x - 2y + 3x^2$
4. $x' = 3\sin x + y,\ y' = 4x + \cos y - 1$
5. $x' = x - y,\ y' = 1 - e^x$
6. $x' = -3x - y - xy,\ y' = 5x + y + xy^3$
7. $x' = y\left(1 - x^2\right),\ y' = -x\left(1 - y^2\right)$
8. $x' = -x + x^3,\ y' = -2y$
9. $x' = -2x + 3y + xy,\ y' = -x + y - 2xy^2$
10. $x' = 5x - 14y + xy,\ y' = 3x - 8y + x^2 + y^2$

B

11. Consider the nonlinear system

$$\dot{x} = -x + xy, \quad \dot{y} = 2y - xy + 0.5x.$$

   a. Find all the equilibrium points.
   b. Describe the type and stability of each equilibrium point found in part (a) by examining the associated linear system in each case.

12. Consider the nonlinear system

$$\dot{x} = x - y + 5, \quad \dot{y} = x^2 + 6x + 8.$$

a. Find all the equilibrium points.

b. Describe the type and stability of each equilibrium point found in part (a) by examining the associated linear system in each case.

13. Consider the nonlinear system

$$\dot{x} = x(8 - 4x - y), \quad \dot{y} = y(3 - 3x - y),$$

which describes the populations $x(t)$ and $y(t)$ of two species that are competing for the same resources.

a. Find all the equilibrium points of this system.

b. By linearizing about each equilibrium point found in part (a), determine the type and stability of each equilibrium point.

14. Find all the equilibrium points of the system

$$\dot{x} = 2x - 0.2x^2 - 0.4xy$$
$$\dot{y} = 4y - 0.4y^2 - 0.8xy$$

and determine the type and stability of each equilibrium point.

C

15. The **Brusselator** is a simple model of a hypothetical chemical oscillator that first appeared in a 1968 paper by Belgian scientists I. Prigogine (a Nobel laureate) and R. Lefever, and was named for the capital of their home country. One version of the model is

$$\dot{x} = 1 - (a + 1)x + bx^2y$$
$$\dot{y} = ax - bx^2y,$$

where x and y are concentrations of chemicals and a, b are positive parameters.

a. Use technology, if necessary, to find the only equilibrium solution of this system.

b. Linearize the system about the equilibrium point found in part (a).

c. Find the eigenvalues of the associated linear system. (Technology could be useful here.)

d. Using your answers from part (c) and the Hartman–Grobman theorem, discuss the nature of the equilibrium solutions for each of the following cases:

$$(1)\ a = 3, b = 1;$$

(2) $a = 2, b = 7$;

(3) $a = 1, b = 4$.

7.4 Two important nonlinear systems

Now that we know something about the qualitative behavior of nonlinear systems, we can apply this knowledge to the analysis of two important nonlinear systems of differential equations: the **Lotka–Volterra equations** and the system corresponding to an **undamped pendulum**.

7.4.1 A predator-prey model: the Lotka–Volterra equations

An important type of real-life problem that can be modeled by a system of differential equations is a **predator-prey problem**, in which we assume that there are two species of animals, X and Y, in a small geographical region such as an island. One species (the **predator**) thinks of the other species (the **prey**) as food and is very dependent on this food supply for survival.

Let $x(t)$ and $y(t)$ represent the populations of the two species at time t. We can make the following reasonable assumptions:

1. If there are no predators, the prey species will grow at a rate proportional to its own population (assuming an unlimited food supply). [This is a *Malthusian* assumption; see Exercise 20 in Section 2.1.]
2. If there is no prey, the predator species will decline at a rate proportional to the predator population.
3. The presence of both predators and prey is beneficial to the growth of the predator species and is harmful to the growth of the prey species.

The third assumption says that interactions (or close encounters of the hungry kind) between the predator and prey lead to a decrease in the prey population and to a resulting increase in the predator population. As we will see, these contacts are indicated mathematically by a *multiplication* of the variables that represent predator and prey. These assumptions lead to a system of nonlinear first-order differential equations like the following:

$$\dot{x} = 0.2x - 0.002xy, \;\; \dot{y} = -0.1y + 0.001xy. \qquad (7.4.1)$$

For this system, how can we see that $x(t)$ is the size of the *prey* population at any time t and $y(t)$ is the number of *predators* at time t?

First of all, note that if there are *no* predators—that is, if y is always 0—the system reduces to $\dot{x} = 0.2x$, $\dot{y} = 0$. This says that the prey population would increase at a rate that is proportional to the actual prey population at any time. Also, the predator population is constant—at zero. This is realistic and consistent with assumption 1. Furthermore, if there is no *prey*—that is, if $x \equiv 0$—the system becomes $dx/dt = 0$,

$dy/dt = -0.1y$, which means that the number of predators would decrease at a rate proportional to the predator population, where 0.1 is the constant of proportionality, the predator's *intrinsic death rate*. Again, this is realistic because in the absence of a crucial food supply the bottom line would be starvation and a net decline in the predator population.

The intriguing terms in (7.4.1) are the terms involving the product xy. We've already suggested that these terms represent *the number of possible interactions* between the two species. To illustrate this point, suppose there were four foxes and three rabbits on an island. If we label the foxes F_1, F_2, F_3, and F_4 and the rabbits R_1, R_2, and R_3, then we have the following possible one-on-one encounters between foxes and rabbits: (F_1, R_1), (F_1, R_2), (F_1, R_3), (F_2, R_1), (F_2, R_2), (F_2, R_3), (F_3, R_1), (F_3, R_2), (F_3, R_3), (F_4, R_1), (F_4, R_2), and (F_4, R_3). Note that there are $4 \times 3 = 12$, or x times y, possible interactions. Of course, we can have two foxes meeting up with one rabbit or one fox coming upon three rabbits, and so on, but the idea is that *the number of interactions is proportional to the product of the two populations*. The coefficient of xy in the first equation, -0.002, is a measure of the predator's effectiveness in terms of prey capture, whereas the coefficient 0.001 in the second equation is an indicator of the predator's efficiency in terms of prey consumption.

The nonlinear system (7.4.1) is a particular example of a system called the **Lotka–Volterra equations**:

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cy + dxy,$$

where a, b, c, and d are positive constants. Alfred Lotka (1880–1949) was a chemist and demographer, and Vito Volterra (1860–1940) was a mathematical physicist. In the 1920s they derived these equations independently—Lotka from a chemical reaction problem and Volterra from a problem concerning populations of fish in the Adriatic Sea. **In general, there is no explicit solution of the Lotka–Volterra equations in terms of elementary functions**.

However, we can understand this system from the viewpoint of the Hartman–Grobman theorem, by using a *qualitative* analysis.

7.4.2 A qualitative analysis of the Lotka–Volterra equations

The equilibrium points for the system $\{\dot{x} = ax - bxy,\ \dot{y} = -cy + dxy\}$ are solutions of the algebraic system

$$ax - bxy = x(a - by) = 0$$
$$-cy + dxy = y(-c + dx) = 0.$$

Clearly, $x = y = 0$ is a solution—that is, the origin $(0, 0)$ is an equilibrium point. It should also be clear from these last equations that if either x or y is zero, then the other variable must also be zero. Therefore, if there are any other equilibrium points, we must have $x \neq 0$ and $y \neq 0$. In the first algebraic equation, if $x \neq 0$, then we must have $a - by = 0$, so $y = a/b$. From the second equation, we see that if $y \neq 0$, then $-c + dx = 0$, so $x = c/d$. Thus, the only equilibrium points for the Lotka–Volterra system are $(0, 0)$ and $(c/d, a/b)$. (Remember that b and d are assumed to be positive.)

Near the origin, we can replace our original system by the associated linear system

$$\dot{x} = ax$$
$$\dot{y} = -cy,$$

which can be written in matrix form as $\dot{X} = AX$, where $A = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Now the characteristic equation of A is $\lambda^2 + (c - a)\lambda - ac = 0$, so the eigenvalues are a and $-c$. Because the eigenvalues are real and of opposite signs, Table 6.2 in Section 6.9 indicates that the origin is a *saddle point* for the linearized system. The Hartman–Grobman theorem tells us that $(0, 0)$ is also a saddle point for our original nonlinear system.

To study the behavior of the system near the equilibrium point $(c/d, a/b)$, we transform the system by defining $u = x - c/d$ and $v = y - a/b$. Then our original system becomes

$$\dot{u} = a\left(u + \frac{c}{d}\right) - b\left(u + \frac{c}{d}\right)\left(v + \frac{a}{b}\right)$$
$$\dot{v} = -c\left(v + \frac{a}{b}\right) + d\left(u + \frac{c}{d}\right)\left(v + \frac{a}{b}\right),$$

which simplifies to

$$\dot{u} = \left(-\frac{bc}{d}\right)v - buv$$
$$\dot{v} = \left(\frac{ad}{b}\right)u + duv.$$

The associated linear system is given by $\dot{X} = AX$, where $A = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$. The characteristic equation here is $\lambda^2 + ac = 0$, so the eigenvalues are $\lambda_1 = \sqrt{ac}\, i$ and $\lambda_2 = -\sqrt{ac}\, i$. Because we have pure imaginary eigenvalues, part (c) of the Hartman–Grobman theorem tells us that $(c/d, a/b)$ is either a *center* or a *spiral point* for the nonlinear system. (The table in Section 6.9 indicates that $(c/d, a/b)$ is a stable center for the associated linear system, but this doesn't have to be true for our nonlinear system.) Let $a = b = c = d = 1$. Then Fig. 7.9a shows the slope field for the nonlinear system near the equilibrium point $(c/d, a/b) = (1, 1)$, and Fig. 7.9b depicts some trajectories near $(1, 1)$.

These figures suggest (but do not *prove*) that the equilibrium point $(1, 1)$ is a *stable center* for the nonlinear system. (Problem 9 in Exercises 7.4 proposes some investigations in this direction.)

Now let's return to the specific system (7.4.1) and analyze it geometrically.

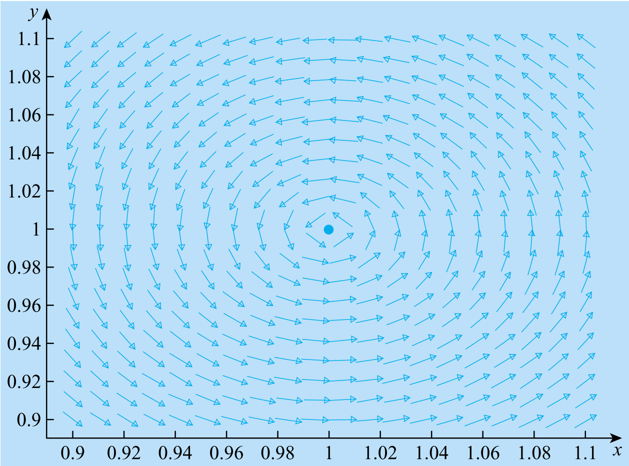


FIGURE 7.9a

Slope field of $\dot{x} = x - xy,\ \dot{y} = -y + xy$ near $(1, 1)$

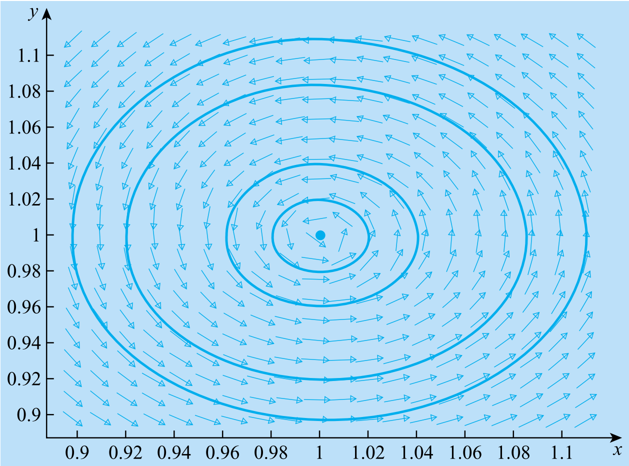


FIGURE 7.9b

Trajectories of $\dot{x} = x - xy,\ \dot{y} = -y + xy$ near $(1, 1)$

Example 7.4.1 Qualitative Analysis of a Predator-Prey Model

Fig. 7.10 shows the trajectory corresponding to our system

$$\frac{dx}{dt} = 0.2x - 0.002xy$$

$$\frac{dy}{dt} = -0.1y + 0.001xy,$$

with $x(0) = 100$, $y(0) = 25$, and $0 \le t \le 52$.

What does this picture tell us? First, we realize that the horizontal axis (x) represents the prey and the vertical axis (y) the predators. Our starting point, corresponding to $t = 0$, is $(100, 25)$, and the direction of the trajectory is counterclockwise. (To see the direction, use technology to look at partial trajectories such as those given by $0 \le t \le 10$, $0 \le t \le 15$, or $0 \le t \le 25$.)

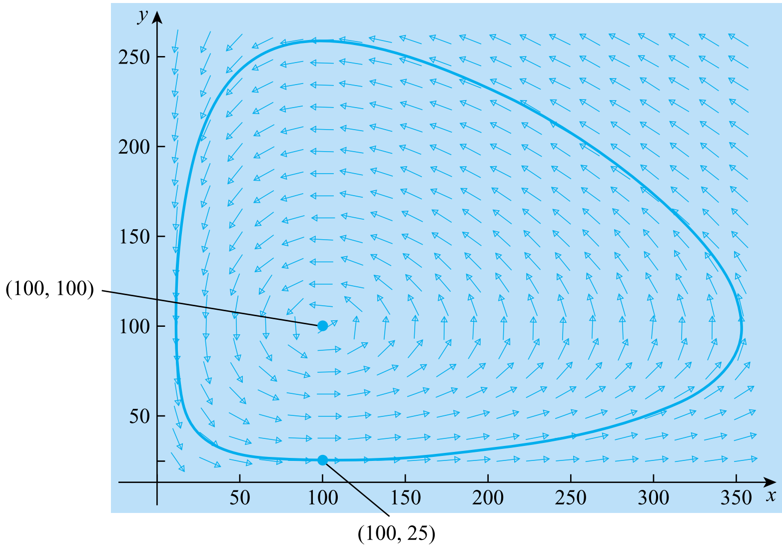


FIGURE 7.10

Trajectory for $\left\{\frac{dx}{dt} = 0.2x - 0.002xy,\ \frac{dy}{dt} = -0.1y + 0.001xy;\ x(0) = 100,\ y(0) = 25\right\}$; $0 \le t \le 52$

Fig. 7.10 illustrates a cyclic behavior that seems a bit too neat to be found in the wild. However, regular population cycles do seem to occur in nature.[1] In our graph, both prey and predator populations increase as the number of prey increases, but when the prey population exceeds about 350, the predators seem to overwhelm their prey to the extent that there are more and more predators, but a declining prey population. The predators continue to increase until their number is about 260, at which time the effect of a dwindling food supply catches up to the predators and their population begins to decline. The predators may starve or start killing each other as competition for diminishing resources grows fierce. Finally, the predator population is low enough for the prey population to recover, and the cycle begins again.

[1] Examination of the records of the Hudson's Bay Company, which trapped fur-bearing animals in Canada for almost 200 years, suggests a periodic pattern in the number of lynx pelts harvested from about 1845 to the 1930s. The lynx, a cat-like predator, has the snowshoe hare as its main prey. For an analysis of the data, see J.D. Murray, *Mathematical Biology I: An Introduction* (*Third Edition*) (New York: Springer-Verlag, 2002): 83–84.

Fig. 7.10 highlights the point (100, 100) because $x = 100$, $y = 100$ is an equilibrium solution of the system, called a **center** in this case. (*Verify the preceding statement.*) If this system were to have initial point (100, 100), neither population would move from this state. The origin is also an equilibrium point.

This discussion highlights the fact that a good model must be simple enough to be analyzed mathematically, but complex enough to represent a system realistically. In modeling, realism is often sacrificed for simplicity, and one of the shortcomings of the Lotka–Volterra model is its reliance on unrealistic assumptions. For example, prey populations are limited by food resources and not just by predation, and no predator can consume infinite quantities of prey. Many other examples of cyclical relationships between predator and prey populations have been demonstrated in the laboratory or observed in nature, but in general they are better fit by models incorporating terms that represent *carrying capacity* (the maximum population size that a given environment can support) for the prey population (see Example 2.5.1), realistic functional responses (how a predator's consumption rate changes as prey densities change) for the predator population, and complexity in the environment.

7.4.3 Other graphical representations

With the aid of technology, we can look at plots of $x(t)$ against t and $y(t)$ against t separately (Figs. 7.11a and 7.11b). We compare these graphs, noting the way in which one population lags behind the other over time. The trajectory (Fig. 7.10) gives the big picture, the state $(x(t), y(t))$ of the ecological system as time marches on, whereas Figs. 7.11a and 7.11b show the individual population fluctuations. Fig. 7.12 exhibits the cyclic nature of the predator fluctuation and that of the prey fluctuation on the same set of axes. Each graph in this example was prepared by a CAS using a numerical approximation to the actual system solution.

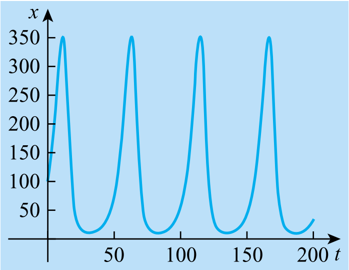


FIGURE 7.11a

$x(t)$, prey population; $x(0) = 100$; $0 \le t \le 200$

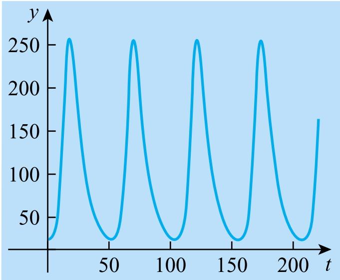


FIGURE 7.11b

$y(t)$, predator population; $y(0) = 25$; $0 \leq t \leq 220$

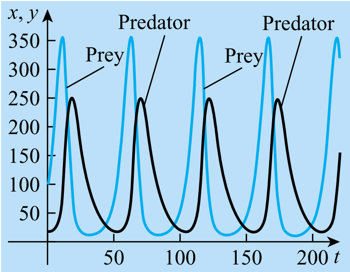


FIGURE 7.12

Predator and prey population vs. t

7.4.4 The undamped pendulum

After our ecological field trip, let's return to the world of physics and look at the motion of a simple pendulum. In Section 7.2, we saw that the second-order nonlinear equation $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ describes the motion of an **undamped pendulum**—that is, a pendulum under the influence of gravity with no friction or air resistance impeding its movement. Here, θ is the angle the pendulum makes with the vertical, g is the acceleration due to gravity, and L is the pendulum's length (Fig. 7.13).

Example 7.4.2 The Undamped Pendulum: A Hartman–Grobman Analysis

Letting $x = \theta$ and $y = \dot{\theta} = \dot{x}$, we can express the single equation $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ as the nonlinear system

$$\dot{x} = y$$
$$\dot{y} = -\frac{g}{L}\sin x.$$

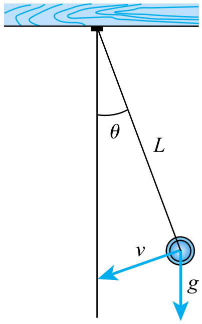


FIGURE 7.13

The undamped pendulum

The first thing we have to do is find the equilibrium points of this system. Clearly, any equilibrium point (x, y) must have $y = 0$. The equation $-\frac{g}{L}\sin x = 0$ has solutions $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Thus, all points of the form $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$ are equilibrium points for the system describing the pendulum's swing. Because the sine function has period 2π—that is, $\sin(x + 2k\pi) = \sin x$ for any integer k—the second equation in the system remains the same for angles differing by integer multiples of 2π. Thus, there is no physical difference in the system for such angles. (*Think about this in physical terms.*) Now all the equilibrium point first coordinates that are *even* multiples of π differ from 0 by multiples of 2π, so we can just study what happens near $(0, 0)$. For example, the point $(-8\pi, 0)$ is the same as $(0 + (-4)\cdot 2\pi, 0)$. Similarly, all the equilibrium point first coordinates that are *odd* multiples of π differ from π by multiples of 2π, so we can just see what happens to the system near $(\pi, 0)$. For example, $(17\pi, 0)$ is the same as $(\pi + (8)\cdot 2\pi, 0)$. Therefore, by analyzing the behavior of the system near the points $(0, 0)$ and $(\pi, 0)$, we can understand the behavior near *any* of the infinite number of equilibrium points.

Near the origin we can replace $\sin x$ by its Taylor series expansion, so our system can be written as

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{g}{L}\sin x = -\frac{g}{L}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\end{aligned}$$

and we see that the linearization of our system is given by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{g}{L}x.\end{aligned}$$

In matrix form this becomes $\dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} X$, with characteristic equation $\lambda^2 + \frac{g}{L} = 0$ and pure imaginary eigenvalues $\lambda = \pm\sqrt{g/L}\,i$. Part (c) of the Hartman–Grobman theorem points to either a *center* or a *spiral point*. Intuitively, we should realize that this is like the situation with the undamped spring-mass system. In the absence of any kind of resistance the object will continue to move periodically about its equilibrium state. In our case we would expect the pendulum to swing back and forth indefinitely. (Compare the pendulum's associated linear system with system (6.11.2) in Example 6.11.1.) Fig. 7.14 shows the phase portrait of the nonlinear system with $g = L$ near the origin.

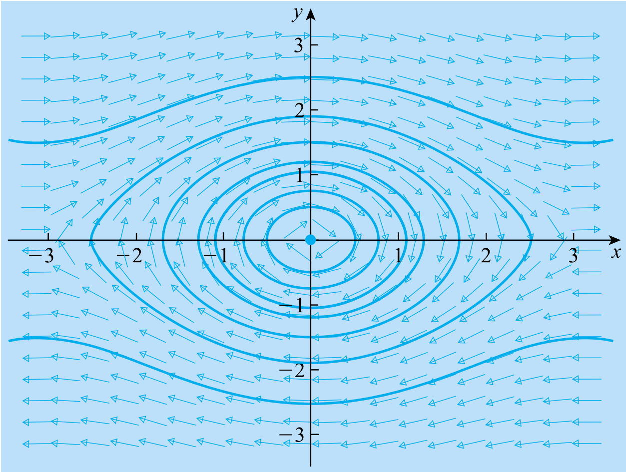


FIGURE 7.14

Trajectories of $\dot{x} = y$, $\dot{y} = -\sin x$ near the origin

Note what the figure tells us. If the pendulum starts with $x_0 = \theta_0$ anywhere between 0 and π and we release the weight at the end (the *bob*), then the pendulum will swing in a clockwise (*negative*) direction toward the vertical position and go past the vertical ($x = \theta = 0$) until it makes the same initial angle on the other side. At this point in time ($x = x_0 = -\theta_0$), the pendulum starts its journey back to the vertical position and then goes past it until $x = \theta_0$ once more. The variable y represents the *angular velocity*, which is zero as we release the pendulum, becomes negative as the velocity increases in a negative (clockwise) direction, attains its maximum as the pendulum swings through the vertical position, and then decreases as the pendulum approaches $x = x_0 = -\theta_0$. At this point, the pendulum begins its swing back toward the center and ultimately back to its initial position, its velocity increasing and decreasing appropriately. (We'll deal with the curves at the top and bottom of Fig. 7.14 shortly.)

Now let's examine the pendulum's behavior near the equilibrium point $(\pi, 0)$. The transformation $u = x - \pi$, $v = y - 0$ results in the nonlinear system

$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= -\frac{g}{L}\sin(u+\pi) = -\frac{g}{L}(-\sin u) = \frac{g}{L}\sin u,\end{aligned}$$

with the associated linear system

$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= \frac{g}{L}u.\end{aligned}$$

(*Make sure you understand how we arrived here.*) This linear system has the characteristic equation $\lambda^2 - \frac{g}{L} = 0$ and eigenvalues $\pm\sqrt{\frac{g}{L}}$. We look to part (a) of the Hartman–Grobman theorem (and

Table 6.2 in Section 6.9) to see that the equilibrium point $(\pi, 0)$ is a *saddle point*. Fig. 7.15 (again with $g = L$) focuses on the system's behavior near this point.

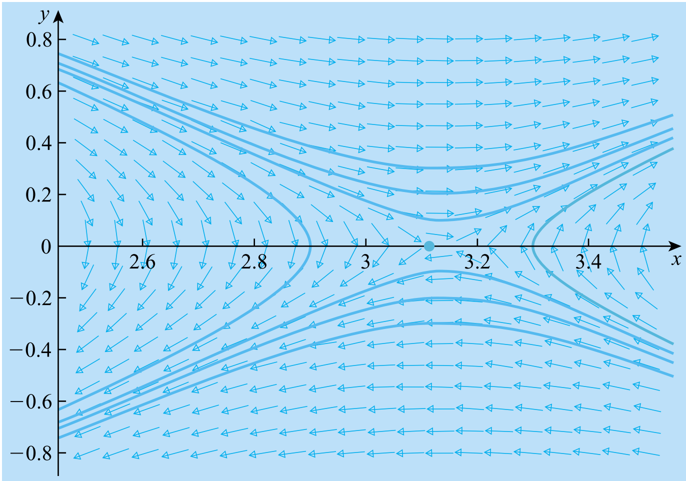


FIGURE 7.15

Trajectories of $\dot{x} = y,\ \dot{y} = -\sin x$ near $(\pi, 0)$

As neat as this analysis seems to be, we've touched on something we haven't explained yet—the strange curves at the top and bottom of Fig. 7.14. If we step back and look at the entire phase portrait (Fig. 7.16), this strangeness becomes more evident.

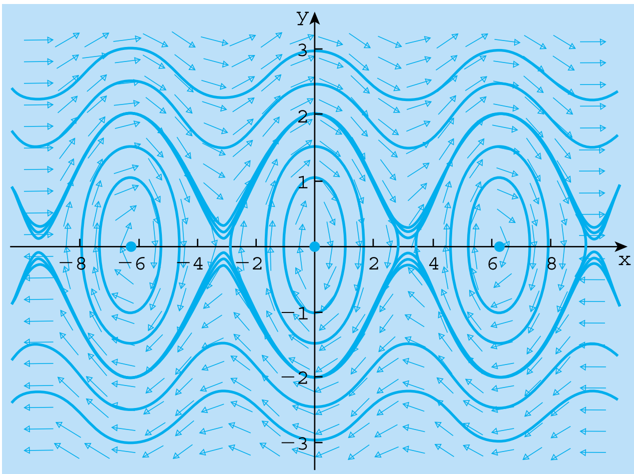


FIGURE 7.16

Phase portrait of $\dot{x} = y,\ \dot{y} = -\sin x$

Clearly, if the initial velocity imparted to the undamped pendulum is low enough, the pendulum swings indefinitely back and forth about its equilibrium point $(0, 0)$. Physically, this equilibrium position corresponds to the pendulum at rest ($y = \dot{\theta} = 0$) and hanging straight down ($x = \theta = 0$). If we give the pendulum a high enough initial velocity, it will whirl up and over the top, over and over again in the absence of any friction or air resistance. Its velocity will vary periodically, attaining its minimum at *odd* multiples of π (when its position is straight up) and its maximum at *even* multiples of π (when it is moving through the straight down position).

The curves joining the saddle points (odd multiples of π on the x-axis) need careful explanation. Fig. 7.17 focuses on the curves connecting the saddle points $(-\pi, 0)$ and $(\pi, 0)$. These are called **separatrices** (the plural of **separatrix**); they separate the regions of "normal" behavior from each other. (More technically, they are called **heteroclinic trajectories** or **saddle connections**.) As we've indicated before, the saddle points represent a pendulum pointed straight up and at rest. Physically, then, these heteroclinic trajectories describe the fact that the pendulum slows down just as it approaches the upside-down position.

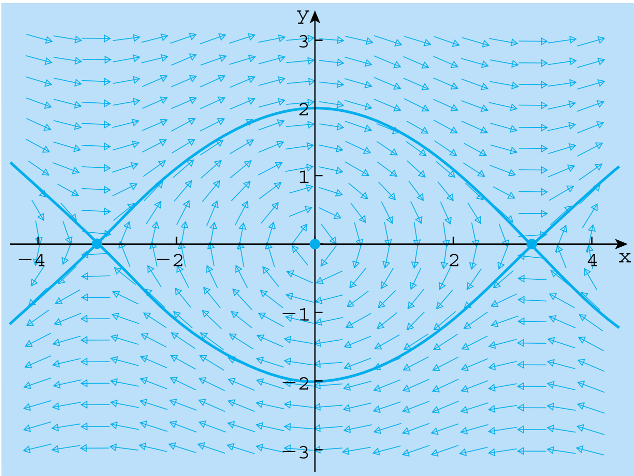


FIGURE 7.17

Separatrices connecting $(-\pi, 0)$ and $(\pi, 0)$

Problems 15 and 16 in Exercises 7.4 suggest a more analytic way of understanding the undamped pendulum's behavior.

Exercises 7.4

A

Find the nontrivial equilibrium point for each of the Lotka–Volterra systems in Problems 1–6. You may need technology.

1. $\dot{x} = 3x - 2xy,\ \dot{y} = -y + 4xy$
2. $\dot{x} = 0.1x - 0.2xy,\ \dot{y} = -0.5y + 0.3xy$
3. $\dot{x} = 0.005x - 0.02xy,\ \dot{y} = -0.3y + 0.4xy$

4. $\dot{x} = x - 2xy$, $\dot{y} = -3y + 4xy$

5. $\dot{x} = 0.2x - 0.2y$, $\dot{y} = -3y + xy$

6. $\dot{x} = 3x - 2xy$, $\dot{y} = -y + \frac{1}{2}xy$

7. Consider the Lotka–Volterra equations $\{\dot{x} = x - xy,\ \dot{y} = -y + xy\}$. To develop some confidence in the power of numerical methods, use whatever Runge–Kutta algorithm and step size your instructor suggests to approximate the solution to the initial value problem with $x(0) = 1$ and $y(0) = 2$ over the interval $[0, 1]$.

8. Find all the equilibrium points for the modified predator-prey system $\{\dot{x} = ax - by - cx^2,\ \dot{y} = -qy + rxy - sy^2\}$. (Here a, b, c, q, r, s are positive constants, and the negative quadratic terms indicate that the growth rates of the two species diminish as either population becomes large due to finite resources for both.)

B

9. Consider the Lotka–Volterra equations for $a = b = c = d = 1$. Fig. 7.9b shows some trajectories corresponding to this situation. Without relying on the graph, we want to show that the trajectories are closed curves—that is, that the equilibrium point $(1, 1)$ is a *stable center*.

- **a.** Show that the slope field for $\frac{dy}{dx}$ is symmetric about the line $y = x$. [*Hint*: Look at what happens if you interchange x and y in the slope equation.]
- **b.** Show that if you start at some point $P = (x, y)$ on the line $y = x$ and travel along the trajectory once around the point $(1, 1)$, you wind up back at the same point P, so that the curve is closed.

10. Consider the system first examined in Example 7.4.1:

$$\dot{x} = 0.2x - 0.002xy$$
$$\dot{y} = -0.1y + 0.001xy.$$

- **a.** Find the equilibrium points for the system.
- **b.** Plot the trajectory corresponding to the initial conditions $x(0) = 100$ and $y(0) = 300$. Interpret these initial values and the shape of the trajectory in terms of the predator and prey populations. (Choose the interval $[0, 55]$ for your independent variable t.)
- **c.** Use the graph of the trajectory found in part (b) to estimate the maximum and minimum values of the populations x and y.
- **d.** Find the slope equation $\frac{dy}{dx}$ and solve it (implicitly) using the initial conditions given in part (b).
- **e.** Use technology to plot the solution found in part (d), using ranges for x and y consistent with your answers to part (c).

11. Recall that the Lotka–Volterra system has the nontrivial equilibrium point $(c/d, a/b)$. To understand the direction of any trajectory for the Lotka–Volterra equations without relying on a graph provided by technology, divide the first quadrant of the x-y plane into four subquadrants via the lines $x = c/d$ and $y = a/b$. (Sketch this situation.)

a. Show that for $x > c/d$ and $y > a/b$, you have $\dot{x} < 0$ and $\dot{y} > 0$.

b. Show that for $x < c/d$ and $y > a/b$, you have $\dot{x} < 0$ and $\dot{y} < 0$.

c. Show that for $x < c/d$ and $y < a/b$, you have $\dot{x} > 0$ and $\dot{y} < 0$.

d. Show that for $x > c/d$ and $y < a/b$, you have $\dot{x} > 0$ and $\dot{y} > 0$.

e. From the results of parts (a)–(d), explain why any point $(x(t), y(t))$ on a trajectory for the Lotka–Volterra equations moves in a *counterclockwise* direction.

12. Recall that in this section the Lotka–Volterra equations $\dot{x} = ax - bxy$, $\dot{y} = -cy + dxy$ were linearized to $\dot{u} = (-bc/d)v$, $\dot{v} = (ad/b)u$ near the equilibrium point $(c/d, a/b)$.

a. Find the slope equation $\frac{du}{dv}$ and show that the linear system has a solution satisfying $ad^2u^2 + b^2cv^2 = K$, where K is a positive constant.

b. Rewrite the solution in part (a) in terms of the original variables x and y and show that you get the equation of an ellipse with center at $(c/d, a/b)$ and with axes parallel to the axes of the x-y plane.

c. Compute the derivative of each equation of the linearized system to get the equations $\ddot{u} = -acu$, $\ddot{v} = -acv$—uncoupled second-order linear equations of the form $\ddot{w} = -Rw$.

d. Show that the solution of the linearized system is a pair of functions $(u(t), v(t))$ with the same period $2\pi/\sqrt{ac}$.

13. Focus on the equation for the *predator* population, $\frac{dy}{dt} = -cy + dxy$.

a. Divide the equation by y and integrate between the initial time t_0 and some arbitrary time t.

b. Assuming that the predator population is periodic (see Fig. 7.10 or 7.11b, for example) with period T, let $t = t_1$ in part (a), so that $t_1 - t_0 = T$ and $y(t_1) = y(t_0)$. Show that the average value of the prey population is c/d, the same as the equilibrium population of the prey. (Recall that the *average value* of a function f on the interval $[a, b]$ is defined as $\frac{1}{b-a}\int_a^b f(r)\,dr$.) [*Hint*: Note that $\dot{y}/y = -c + dx$ and integrate from 0 to T, using the periodicity of $\ln|y(t)|$.]

14. Assuming the result of Problem 13b and that the average value of the predator population $y(t)$ is a/b, the equilibrium population of the predator, and also assuming that both the predator and prey populations have the same period T, show that the average value of $x(t)y(t)$ equals the average value of $x(t)$ times the average value of $y(t)$. [*Hint*: $(\dot{y} + cy)/d = xy$.]

C

15. Consider the simplified pendulum equation used in Fig. 7.14, Example 7.4.2: $\frac{d^2\theta}{dt^2} + \sin\theta = 0$. You should show (analytically) that this equation has periodic solutions—that is, there are closed trajectories in the phase plane corresponding to the system version of the equation.

a. Show that this equation is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\sin x.$$

b. Show that any trajectory in the phase plane is a solution of $\frac{dy}{dx} = -\frac{\sin x}{y}$.

c. Solve the equation in part (b).

d. Show that there are closed trajectories in the x-y plane, and hence that the undamped pendulum problem has periodic solutions. [*Hint*: Find suitable values of the constant of integration you get in part (c).]

16. When you find the general solution of the equation $\frac{dy}{dx} = -\frac{\sin x}{y}$ [as in part (b) of the preceding problem], you have an arbitrary constant C.

a. What values of C give the wavy trajectories at the top and bottom of Fig. 7.16?

b. What values of C give the separatrices, as in Fig. 7.17?

17. For small values of θ, $\sin\theta \approx \theta$, so that the linearized equation of the undamped pendulum is $\ddot{\theta} + \frac{g}{L}\theta = 0$. Work with this equation and the initial conditions $\theta(0) = 0$, $\dot{\theta}(0) = 2$.

a. Find $\theta(t)$ if the length of the pendulum is 8 feet. (Take $g = 32$ ft/s^2.)

b. What is the period of the function found in part (a)?

c. If the pendulum is part of a clock that ticks once for each time the pendulum makes a complete swing, how many ticks does the clock make in one minute?

d. How is the motion of the pendulum affected if the length is changed to $L = 4$?

18. The equation $\ddot{\theta} + k\dot{\theta} + \sin\theta = 0$ describes a particular *damped* pendulum—that is, a pendulum with friction or air resistance. Here, k is a positive constant, the coefficient of friction.

a. Convert this second-order equation to a system of first-order equations.

b. Use technology to produce the phase portrait when $k = 0.1$.

c. Use technology to produce the phase portrait when $k = 0.5$.

d. Compare the phase portraits in parts (b) and (c) and give a physical interpretation of what you see.

7.5 Bifurcations

The concept of a **bifurcation** that we analyzed in Section 2.7 for first-order equations can be extended to two-dimensional (and higher-dimensional) systems.

Since the analysis of a nonlinear system can often be reduced to the study of a related linear system, we'll start by looking at an example of a bifurcation in a linear system.

Example 7.5.1 A Bifurcation in a Linear System

Suppose we're given a spring-mass system described by the second-order linear equation $\ddot{x} + \mu\dot{x} + x = 0$, where μ, the damping constant, is a parameter representing a frictional force. We want to see how the behavior of the system is affected by varying μ.

Writing this equation as a planar system of linear equations, we have

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - \mu y.\end{aligned}$$

Now we write the system in the compact form $\dot{X} = AX$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$, $X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, and $\dot{X} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$.

The only equilibrium point is $(0, 0)$. The eigenvalues of A are $\lambda_1 = -\frac{1}{2}\mu + \frac{1}{2}\sqrt{\mu^2 - 4}$ and $\lambda_2 = -\frac{1}{2}\mu - \frac{1}{2}\sqrt{\mu^2 - 4}$.

If $\mu = 0$, we have $\lambda_1 = i$ and $\lambda_2 = -i$, so the origin is a *stable center*, but not asymptotically stable.

If $\mu = -2$, then $\lambda_1 = \lambda_2$, both real and positive. This means that the origin is an *unstable node*.

If $-2 < \mu < 0$, then λ_1 and λ_2 are complex conjugates with a positive real part. In this case, the origin is an *unstable spiral*.

For $0 < \mu < 2$ we see that λ_1 and λ_2 are complex conjugates with a negative real part, implying that the origin is a *stable spiral* (which is asymptotically stable).

If $\mu < -2$, then λ_1 and λ_2 are positive and unequal, implying that the origin is an *unstable node*.

Finally, if $\mu > 2$ the eigenvalues are negative and unequal, so the origin is a *stable node*.

Looking back over our analysis of the various key values of the parameter μ, we see that $\mu = 0, -2, 2$ are possible bifurcation points. In crossing the value $\mu = 0$ from left to right, we see that an unstable spiral becomes a stable spiral. From the definitions given in Section 2.7, this behavior indicates that we have a **transcritical bifurcation** (Fig. 7.18)

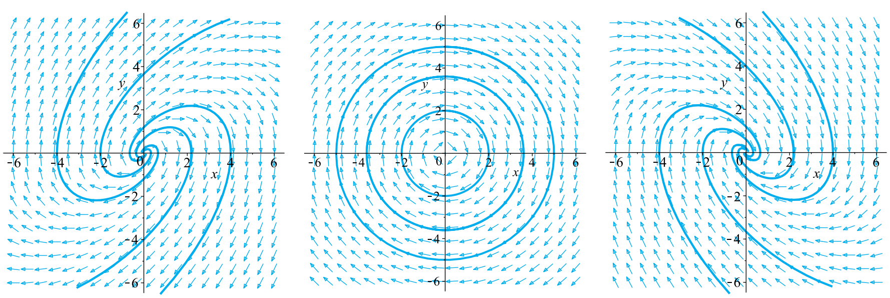


FIGURE 7.18

$\dot{x} = y$, $\dot{y} = -x - \mu$: $\mu = -1, 0, 1$, respectively

When we cross the value $\mu = -2$, there is a change from an unstable node to an unstable spiral. There is no change in the stability of the system, so $\boldsymbol{\mu = -2}$ **is not a bifurcation point**.

Finally, crossing the point $\mu = 2$ produces no change in stability (a stable spiral becomes a stable node), so $\boldsymbol{\mu = 2}$ **is not a bifurcation point**.

In Section 2.7, for a single nonlinear differential equation, we described a *saddle-node bifurcation*, a situation in which two equilibrium solutions collide and annihilate each other. One of the solutions is unstable (the saddle), while the other (the node) is stable. Now we'll see what this looks like in a nonlinear system.

Example 7.5.2 A Saddle-Node Bifurcation

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 - y - \mu. \end{aligned}$$

The equilibrium solutions satisfy

$$\begin{aligned} y &= 0 \\ x^2 - y - \mu &= 0. \end{aligned}$$

From these algebraic equations we get the equilibrium solutions $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$.

When $\mu = 0$, we get a single equilibrium solution $(0, 0)$. For $\mu < 0$, $\sqrt{\mu}$ is complex, and so there are *no* equilibrium solutions. However, for $\mu > 0$, there are *two* equilibrium solutions, $(-\sqrt{\mu}, 0)$ and $(\sqrt{\mu}, 0)$.

To examine the behavior of the system near the equilibrium solution $(\sqrt{\mu}, 0)$, we make the change of variables $u = x - \sqrt{\mu}$ and $v = y - 0 = y$. This gives us the nonlinear system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= (u + \sqrt{\mu})^2 - v - \mu = u^2 + 2\sqrt{\mu}\,u - v \end{aligned}$$

and the associated linear system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= 2\sqrt{\mu}\,u - v. \end{aligned}$$

The matrix of coefficients for the linearized system is $\begin{bmatrix} 0 & 1 \\ 2\sqrt{\mu} & -1 \end{bmatrix}$, with characteristic equation $\lambda^2 + \lambda - 2\sqrt{\mu} = 0$ and eigenvalues $(-1 \pm \sqrt{1 + 8\sqrt{\mu}})/2$.

Since $1 + 8\sqrt{\mu} > 0$, we have two real eigenvalues, one positive and one negative. (Note that $\sqrt{1 + 8\sqrt{\mu}} > 1$, so $-1 + \sqrt{1 + 8\sqrt{\mu}} > 0$ and $-1 - \sqrt{1 + 8\sqrt{\mu}} < 0$.) This indicates that $(\sqrt{\mu}, 0)$ is a **saddle node** (see Table 6.2 in Section 6.9).

On the other hand, as we can easily show (*Do this.*), when we consider the equilibrium solution $(-\sqrt{\mu}, 0)$ the linearized system $\{\dot{u} = v,\ \dot{v} = -2\sqrt{\mu}\,u - v\}$ yields two negative eigenvalues, $-1 \pm \sqrt{1 - 8\sqrt{\mu}}$. Note that for $0 < \sqrt{\mu} \le 1/8$, the radicand $1 - 8\sqrt{\mu}$ is nonnegative, and so we have a stable node (sink). If $\sqrt{\mu} > 1/8$, then $1 - 8\sqrt{\mu} < 0$, and our complex conjugate eigenvalues gives us a stable spiral.

Since the equilibrium points—one unstable, the other stable—merge as μ crosses the origin, we have a **saddle-node bifurcation** (see Fig. 7.19).

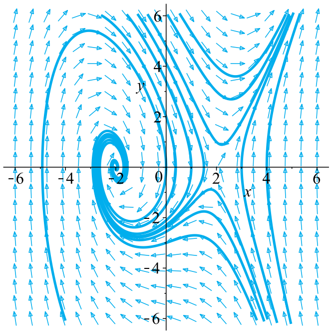


FIGURE 7.19

$\dot{x} = y,\ \dot{y} = -x - \mu$: $\mu = 4$

Example 7.5.3 A Pitchfork Bifurcation

To see another type of bifurcation, we'll analyze the system

$$\dot{x} = y$$
$$\dot{y} = \mu x - x^3 - y,$$

where μ is a real parameter.

The equilibrium points are solutions of $y = 0$, $\mu x - x^3 + y = 0$, or $\mu x - x^3 = x(\mu - x^2) = 0$—that is, $(0, 0)$, $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$ are our equilibrium solutions.

Near the origin, we can approximate the nonlinear system by the linear system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mu & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic equation of the matrix is $\lambda^2 + \lambda - \mu = 0$, yielding eigenvalues $(-1 + \sqrt{1 + 4\mu})/2$ and $(-1 - \sqrt{1 + 4\mu})/2$.

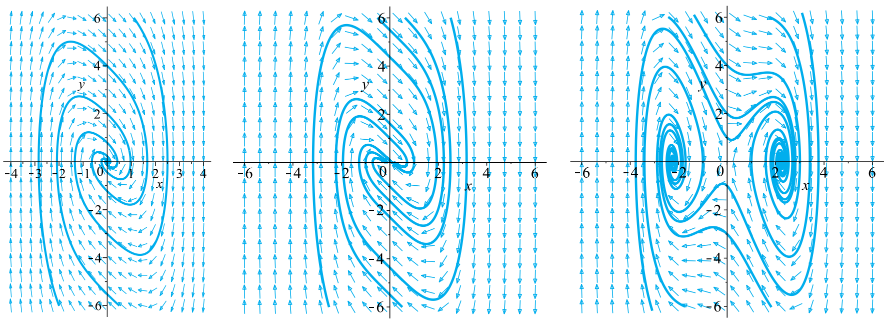


FIGURE 7.20

$\dot{x} = y,\ \dot{y} = \mu x - y$: $\mu = -1, 0, 1$, respectively

According to Table 6.2, if $-1/4 < \mu < 0$, then the origin is a stable node, and if $\mu < -1/4$, then $(0, 0)$ is a stable spiral. For $\mu > 0$, the origin is a saddle point (Fig. 7.20). Note that for $\mu = 0$, the origin is the only equilibrium point, but for $\mu > 0$ there are three: $(-1, 0)$, $(0, 0)$, $(1, 0)$ for $\mu = 1$, for example. Thus, $\mu = 0$ is a bifurcation point, and we have a **pitchfork bifurcation**.

There are many other aspects of bifurcations that we are leaving unexplored. In Section 7.6, we'll encounter the **Hopf bifurcation**, one that can only appear in systems of dimension two and higher.

Exercises 7.5

A

1. In Example 7.5.1, show that the eigenvalues are $\lambda = -\mu/2 \pm (1/2)\sqrt{\mu^2 - 4}$.

2. In Example 7.5.2, verify that the eigenvalues of the linearized system are $\lambda = (-1 \pm \sqrt{1 + 8\sqrt{\mu}})/2$.

3. In Example 7.5.3, show that the eigenvalues of the linearized system are $\lambda = (-1 \pm \sqrt{1 + 4\mu})/2$.

4. Consider the system $\{\dot{x} = \mu x - x^2,\ \dot{y} = -y\}$.

- **a.** Determine all the bifurcation points (if any) of the system.
- **b.** Classify all the bifurcations found in part (a).

5. Consider the system $\{\dot{x} = y - 2x,\ \dot{y} = \mu + x^2 - y\}$.

- **a.** Determine all the bifurcations (if any) of the system.
- **b.** Classify all the bifurcations found in part (a).

B

6. Consider the system $\{\dot{x} = -2x + \frac{1}{4}y,\ \dot{y} = -x + \mu y\}$.

- **a.** Find all the equilibrium solutions.
- **b.** Determine the eigenvalues of the system.
- **c.** Determine all the bifurcation values and classify them.

7. Consider the system $\{\dot{x} = \mu x + y + \sin x,\ \dot{y} = x - y\}$.

- **a.** Find all the equilibrium solutions of the system.
- **b.** Linearize the system about the origin. [*Hint*: Use the power series for $\sin x$ given in Appendix A.3.]
- **c.** Determine the eigenvalues of the linear system found in part (b).
- **d.** Determine all the bifurcation values and classify them.

8. Consider the system $\{\dot{x} = x(\mu - 2x) - xy,\ \dot{y} = y(x - 1)\}$.

- **a.** Find all the equilibrium solutions of the system.
- **b.** Linearize the system about the origin.
- **c.** Determine the eigenvalues of the linear system found in part (b).
- **d.** Determine all the bifurcation values and classify them.

9. Consider the system $\{\dot{x} = \mu x - x^3 + xy^2,\ \dot{y} = -y - y^3 - x^2 y\}$.

a. Find all the equilibrium solutions of the system.
b. Linearize the system about the origin.
c. Determine the eigenvalues of the linear system found in part (b).
d. Determine all the bifurcation values and classify them.

10. Consider the system $\{\dot{x} = \mu x + 10x^2,\ \dot{y} = x - 2y\}$.
a. Find all the equilibrium solutions of the system.
b. Linearize the system about each equilibrium point.
c. Determine the eigenvalues of the linear systems found in part (b).
d. Determine all the bifurcation values and classify them.

C

11. Find and classify all the bifurcations for the system $\{\dot{x} = y - ax,\ \dot{y} = -by + x/(1+x)\}$, where a and b are real parameters. [*Hint*: Consider scenarios depending on the values of ab and $a+b$.]

*7.6 Limit cycles and the Hopf bifurcation[2]

7.6.1 Limit cycles

Recall that a solution $(x(t), y(t))$ of a planar system of differential equations is **periodic** if there is a $T > 0$ such that $x(t+T) = x(t)$ and $y(t+T) = y(t)$ for every t. Periodic solutions are an important aspect of differential equations since many physical phenomena occur roughly periodically. In Section 7.4, for example, we saw how predator-prey interactions could be periodic (Example 7.4.1); and clearly anything connected with daily, monthly, or yearly cycles in nature is approximately periodic. The Zeeman model of the human heartbeat (Example 6.3.3), the Lotka–Volterra equations, and the undamped pendulum (Section 7.4) show that autonomous systems sometimes have periodic solutions whose trajectories are closed curves in the phase plane. As we shall see later in this section, the *van der Pol oscillator*, which can be described as a negatively damped nonlinear oscillator, has solutions whose *limiting* behavior (as $t \to \infty$) is that of a finite periodic solution. Such a nontrivial isolated closed trajectory is called a **limit cycle**. Here "nontrivial" means that the solution curve is not a single point, and "isolated" refers to the fact that no trajectory sufficiently near the limit cycle is also closed.

In general a *linear* system $\dot{X} = AX$ may have closed trajectories, but they won't be *isolated*: If $X(t)$ is a periodic solution, then so is $cX(t)$ for any nonzero constant c. (*Can you show this?*) Therefore, for instance, by choosing $c = (1 - 1/k)$ $(k = 1, 2, 3, \ldots)$, we see that $X(t)$ is being crowded by a one-parameter family of closed trajectories. The closed trajectories of the nonlinear system shown in Fig. 7.9b in Section 7.4 are not isolated and so could not possibly be limit cycles. You can get trajectories as close to each other as you wish.

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[2] * Denotes an optional section.

Example 7.6.1 A Limit Cycle

Consider the system

$$\dot{x} = y + x(1 - x^2 - y^2)$$
$$\dot{y} = -x + y(1 - x^2 - y^2).$$

To make this system easier to analyze, we'll transform it into polar form. Letting $x = r(t)\cos\theta(t)$, $y = r(t)\sin\theta(t)$, we see that $x^2 + y^2 = r^2$. The Chain Rule (Sections A.2, A.7) gives us $r^2\dot{\theta} = x\dot{y} - y\dot{x}$, and we can make the appropriate substitutions to see that

$$\begin{aligned} r\dot{r} &= x\{y + x(1 - r^2)\} + y\{-x + y(1 - r^2)\} \\ &= r^2(1 - r^2), \end{aligned}$$

where $r^2 = x^2 + y^2$. Similarly, $r^2\dot{\theta} = x\{-x + y(1 - r^2)\} - y\{y + x(1 - r^2)\} = -r^2$, and so the polar form of the system is

$$\dot{r} = r(1 + r)(1 - r)$$
$$\dot{\theta} = -1.$$

From this representation, two obvious solutions are $r = 0$ and $r = 1$. (*Why isn't* $r = -1$ *a solution?*) The first solution corresponds to an unstable node at the origin. The second represents the polar form of a circle of radius one centered at the origin. The fact that $\dot{\theta} = -1$ implies the circle is traversed in a *clockwise* direction with a constant angular velocity of one unit. Thus we have a solution of the system that is a closed trajectory (Fig. 7.21, in rectangular coordinates).

We notice that in Fig. 7.21 if $r = \sqrt{x^2 + y^2} < 1$, then $\dot{r} > 0$ and trajectories spiral outwards toward the closed trajectory. If $r > 1$, then $\dot{r} < 0$ and trajectories spiral inwards toward the closed trajectory.

Overall, the trajectories approach an isolated closed trajectory which we call a (**stable**) **limit cycle**.

To repeat, a closed trajectory γ of a nonlinear system is called a **limit cycle** if it is an isolated nonconstant periodic trajectory. Every trajectory that begins sufficiently near a limit cycle approaches it either for $t \to \infty$ or for $t \to -\infty$. Graphically, this means that such a trajectory either winds itself around the limit cycle or unwinds *from* it. A limit cycle is called **stable** if it attracts nearby solutions (from inside and outside) as $t \to \infty$.

As one author has stated,

> *The stable limit cycle is the basic model for all self-sustained oscillators—those which return, or recover, to some fundamental periodic orbit when perturbed from it. The stable oscillations, "beating" of the human heart (which returns to some normal rate after we raise it by sprinting), cycles of predator-prey systems, and various electrical circuits are three among myriad examples. Business cycles and certain periodic outbreaks of social unrest . . . are, quite possibly, others.*[3]

[3] J.M. Epstein, *Nonlinear Dynamics, Mathematical Biology, and Social Science* (Addison-Wesley, 1997): 121.

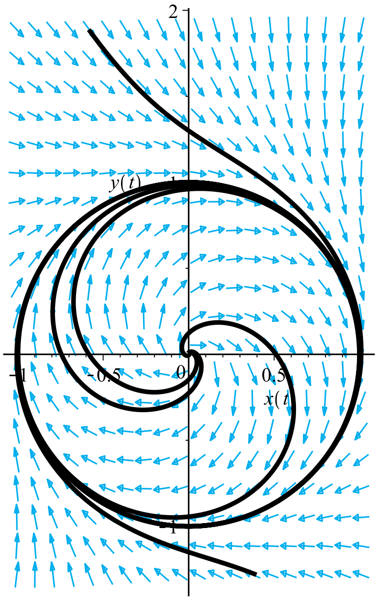


FIGURE 7.21

Phase portrait of $\dot{x} = y + x(1 - x^2 - y^2)$, $\dot{y} = -x + y(1 - x^2 - y^2)$

In this quotation, "self-sustained oscillators" refers to oscillations that are generated and maintained by a source of power that lacks any corresponding periodicity. In other words, there's no external periodic forcing term. In terms of a spring-mass system (Section 6.11), if there is a *negative* damping term, this causes small perturbations to grow exponentially in amplitude.

There are other types of limit cycles that we'll see in the following examples. An **unstable limit cycle** is one from which nearby trajectories spiral away as $t \to \infty$ (Fig. 7.24). (Alternatively, an unstable limit cycle *attracts* nearby solutions as $t \to -\infty$.) Finally, as we expect from our earlier discussions of equilibrium points (Chapter 2), there is a **semistable limit cycle**—trajectories approach one side of it while pulling away from the other side (Fig. 7.28).

Example 7.6.2 An Unstable Limit Cycle

Let's examine the system

$$\dot{x} = -y + x\left(x^2 + y^2 - 1\right)$$
$$\dot{y} = x + y\left(x^2 + y^2 - 1\right).$$

The presence of the algebraic form $x^2 + y^2$, with its suggestion of circularity (rotation), suggests that we may be able to see things more clearly if we switch to polar coordinates. Making the substitutions

$r\cos\theta = r(t)\cos\theta$, $y = r\sin\theta = r(t)\sin\theta(t)$, and $\theta = \arctan(y/x)$, we have $x^2 + y^2 = r^2$. A few algebraic manipulations (see Problem 5 of Exercises 7.6) give us the polar coordinate form of the system we started with:

$$\dot{r} = \left(r^2 - 1\right) r \quad (r \geq 0)$$
$$\dot{\theta} = 1.$$

This system describes the motion of an object in terms of its radial distance $r = r(t)$ from the origin and its (constant) angular velocity $\dot{\theta}$ in a counterclockwise direction. Fig. 7.22 illustrates this in general.

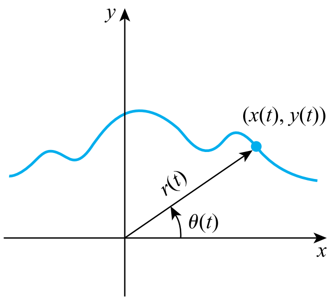


FIGURE 7.22

Motion described in terms of radial distance and angular velocity

Because the equations are independent (or *uncoupled*), each involving only one dependent variable, we can analyze them separately. We can look at the first equation as a first-order nonlinear equation and consider its phase portrait (Fig. 7.23) in the manner of Section 2.5. Recalling that r is nonnegative, we see that the only equilibrium solutions are $r \equiv 0$ and $r \equiv 1$. Note that the first equation tells us that if $r < 1$, then $\dot{r} < 0$, so the trajectory's distance from the origin is decreasing—that is, the trajectory is approaching the origin and moving away from the unit circle ($r \equiv 1, 0 \leq \theta \leq 2\pi$), whereas if $r > 1$, we have $\dot{r} > 0$, so trajectories are also repelled by the unit circle.

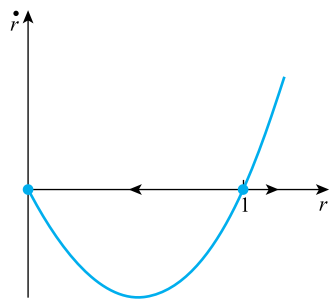


FIGURE 7.23

Phase portrait of $\dot{r} = (r^2 - 1)r$, $r \geq 0$

From this phase portrait we can see that $r \equiv 0$ is a *sink* and $r \equiv 1$ is a *source*. We could have used the *First Derivative Test* of Section 2.6 to see this (see also Problem 6 in Exercises 7.6.). In particular, the origin is a *sink* for the system in its original rectangular coordinate form. Fig. 7.24 shows the phase portrait in x-y space.

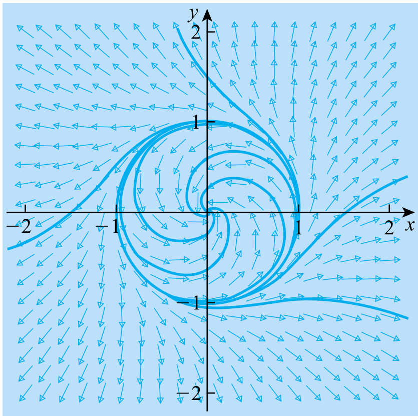


FIGURE 7.24

Phase portrait of $\dot{r} = (r^2 - 1)r$, $r \geq 0$, in the x-y phase plane

From this we see that the unit circle is an *unstable limit cycle*.

Now we're ready for something a bit more complicated, but rewarding.

Example 7.6.3 A System with Two Limit Cycles

Let's look at the system

$$\dot{r} = r(r-1)(r-2),\; r \geq 0$$
$$\dot{\theta} = 1.$$

As in the previous example, the system describes the motion of an object in terms of its radial distance $r = r(t)$ from the origin and its (constant) angular velocity $\dot{\theta}$ in a counterclockwise direction. Let's look at the phase portrait of the first equation (Fig. 7.25), whose equilibrium solutions are $r \equiv 0$, $r \equiv 1$, and $r \equiv 2$.

As we can see, $r \equiv 0$ is a source, $r \equiv 1$ is a sink, and $r \equiv 2$ is a source. This tells us that the system has *two* circular limit cycles: one stable ($r \equiv 1$) and one unstable ($r \equiv 2$). Trajectories starting out inside the unit circle approach the unit circle as $t \to \infty$, as do trajectories with initial points inside the ring formed by the two circles $r \equiv 1$ and $r \equiv 2$. Any trajectory starting outside the circle of radius 2 moves farther away as $t \to \infty$. Fig. 7.26 shows the phase portrait in the usual x-y plane—that is, in rectangular (Cartesian) coordinates.

The next example illustrates a third kind of limit cycle.

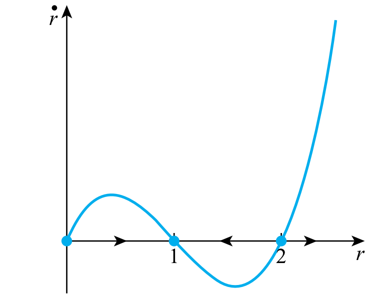


FIGURE 7.25

Phase portrait of $\dot{r} = r(r-1)(r-2)$, $r \geq 0$

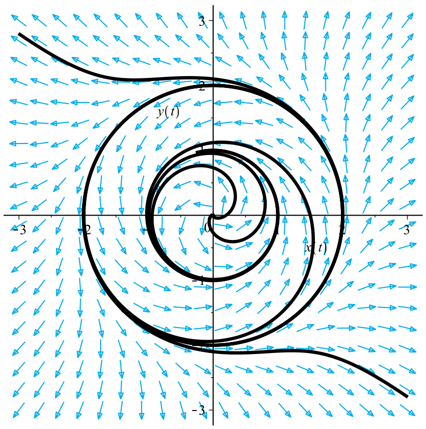


FIGURE 7.26

Phase portrait of $\dot{r} = r(r-1)(r-2)$, $r \geq 0$, in the x-y plane

Example 7.6.4 A Semistable Limit Cycle

What kind of behavior is shown by the following system?

$$\dot{r} = r(r-1)^2, \ r \geq 0$$
$$\dot{\theta} = 1.$$

The phase portrait for the first equation (Fig. 7.27) tells the story.

The equilibrium point $r \equiv 0$ is a source, whereas $r \equiv 1$ is a node because $\dot{r} > 0$ for $0 < r < 1$ and also for $r > 1$. The graphical interpretation of this fact is that the unit circle described by $r \equiv 1$ is a *semistable limit cycle*. Trajectories approach the unit circle from inside it, whereas trajectories that start outside escape the unit circle. Fig. 7.28 shows the phase portrait in the x-y plane.

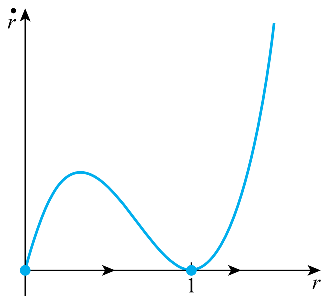


FIGURE 7.27

Phase portrait of $\dot{r} = r(r-1)^2$, $r \geq 0$

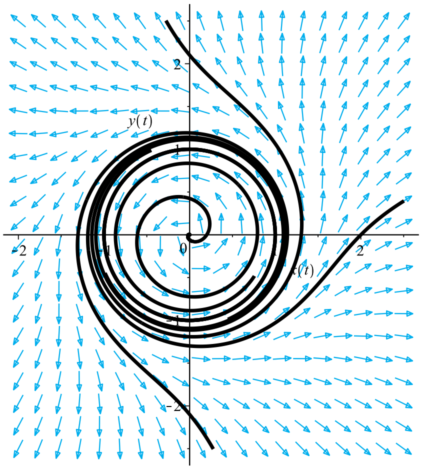


FIGURE 7.28

Phase portrait of $\dot{r} = r(r-1)^2$, $r \geq 0$, in the x-y plane

The next example deals with a famous equation that arose when radios were first developed and that had applications to television as well. The original context was the study of certain electrical circuits containing a vacuum tube (a triode generator), but the work has also had significant biological applications. The pioneering experiments and the first theoretical analysis were conducted by Dutch electrical engineer Balthasar van der Pol (1889–1959) and others in the 1920s.

Example 7.6.5 The Van Der Pol Equation

The **van der Pol equation** (or **van der Pol oscillator**)

$$x'' + \varepsilon\left(x^2 - 1\right)x' + x = 0, \tag{7.6.1}$$

where ε is a positive parameter, can also be interpreted in terms of a spring-mass system with nonlinear resistance (see Problem 1 in Exercises 7.6). Equation (7.6.1) can be written as the equivalent system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + \varepsilon x_2\left(1 - x_1^2\right). \end{aligned} \tag{7.6.2}$$

The first thing we have to do is find the equilibrium points of (7.6.2), the solutions of the system $\{x_1' = 0,\ x_2' = 0\}$. Clearly $x_2 = 0$, and substituting this value into the second equation of (7.6.2) gives us $x_1 = 0$ as well. Thus the origin, $(0, 0)$, is the only equilibrium point.

To get a sense of how this system behaves let's assume that $\varepsilon = 1$. (See Problems 2 and 3 of Exercises 7.6, which ask you to consider other values of ε.) The linearized version of the nonlinear system (7.6.2) is then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + x_2, \end{aligned} \tag{7.6.3}$$

with characteristic equation $\lambda^2 - \lambda + 1 = 0$ and eigenvalues $(1 \pm \sqrt{3}\,i)/2$. This implies that both the nonlinear system (7.6.2) and its linear approximation (7.6.3) have a *spiral source* at the origin. (*Why?*) However, this particular system exhibits some characteristically nonlinear behavior. Fig. 7.29 shows the phase portrait of the nonlinear system (7.6.2) near $(0, 0)$.

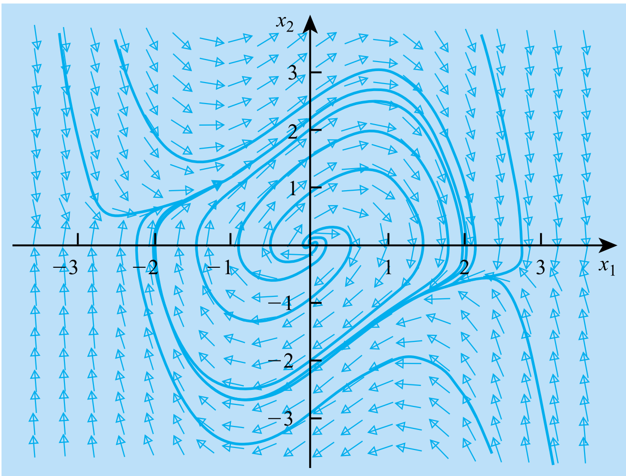


FIGURE 7.29

Phase portrait of $x_1' = x_2$, $x_2' = -x_1 + x_2\left(1 - x_1^2\right)$ near the origin

What is happening here is that several paths starting near the origin spiral *outward* from the origin (as expected) toward a limit cycle, whereas other trajectories starting farther away from $(0, 0)$ also seem to be approaching this limit cycle asymptotically (that is, as $t \to \infty$). Note that the phase portrait of the linearized system (Fig. 7.30) shows no cyclic behavior, only the spiraling away from the origin.

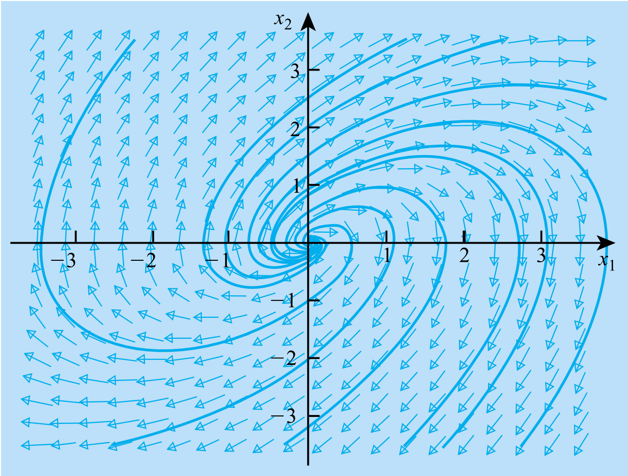


FIGURE 7.30

Phase portrait of $x_1' = x_2$, $x_2' = -x_1 + x_2$ near the origin

Fig. 7.31 shows a plot of x against t with the initial conditions $x(0) = 0.5$, $x'(0) = -0.5$. This graph reflects the eventual periodicity of the solution and the fact that the spirals work their way *outward* (through increasing values of t) to the stable limit cycle. The solution shows *transient* behavior (temporary or short-lived behavior) at the beginning, before settling into its periodic pattern.

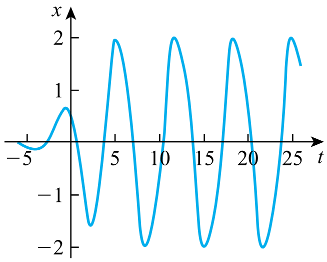


FIGURE 7.31

Plot of $x(t)$ against t, $x(0) = x'(0) = 0.5$, $-6 \le t \le 26$

On the other hand, if we choose an initial point ($x(0) = -3$, $x'(0) = -5$) in a region that appears to be *outside* the limit cycle shown in Fig. 7.29, we see the solution behavior shown by Fig. 7.32. This illustrates how a spiral finds its way *inward* to the stable limit cycle.

Again, we can see that the solution eventually becomes periodic after an initial transient stage.

Of course a nonlinear equation or system may have *no* limit cycles. In general, finding limit cycles is a very difficult problem. Because nonlinear equations and systems are usually too difficult to solve, other qualitative methods have been developed to determine the existence or nonexistence of limit cycles. These methods involve

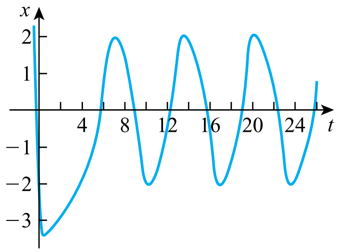


FIGURE 7.32

Plot of $x(t)$ against t, $x(0) = -3$, $x'(0) = -5$, $-0.6 \le t \le 26$

advanced mathematical ideas that we won't discuss in this book, with the exception of a *negative* criterion formulated by the Swedish mathematician Ivar Bendixson (1861–1935). This is explained between exercises 9 and 10, and is illustrated in exercises 10–15.

7.6.2 The Hopf bifurcation

In Section 7.5, we saw transcritical, saddle-node, and pitchfork bifurcations in linear and nonlinear systems. Equilibrium points were created and destroyed and stability properties were changed when parameter values changed. Now we use the following example to introduce some other possibilities.

Example 7.6.6 A Hopf Bifurcation

The system

$$\dot{x} = \mu x + y - x(x^2 + y^2)$$
$$\dot{y} = -x + \mu y - y(x^2 + y^2),$$

where μ is a parameter, exhibits interesting bifurcation behavior.

It is clear that this system has an equilibrium point at the origin, no matter what the value of the parameter μ is. The polar representation of this system is

$$\dot{r} = r(\mu - r^2)$$
$$\dot{\theta} = -1.$$

The equilibrium solution at the origin becomes $r = 0$, and since $\dot{\theta} < 0$ the trajectories move clockwise around the origin. (These polar equations are *uncoupled*; each involves only one variable and so can be analyzed separately.)

If $\mu < 0$, then $\dot{r} < 0$, so the trajectories' distances from the origin is decreasing. This means that the origin is a sink—that is, all trajectories approach the origin as $t \to \infty$ (Fig. 7.33).

If $\mu = 0$, then $\dot{r} = -r^3$. For nonzero r we see that $\dot{r} < 0$, and therefore all trajectories approach the origin as $t \to \infty$. In this case the origin is thus a stable node (a sink) (Fig. 7.34).

However if $\mu > 0$, then $\dot{r} < 0$ for $\sqrt{\mu} < r < \infty$ and $\dot{r} > 0$ for $0 < r < \sqrt{\mu}$. In this situation the origin is an unstable node (a source) surrounded by a stable limit cycle, $r = \sqrt{\mu}$, which grows out

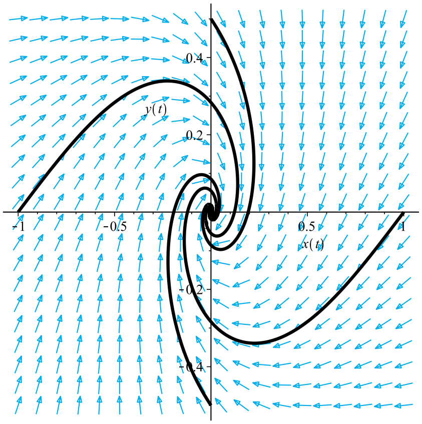


FIGURE 7.33

Phase portrait of $\dot{r} = r(\mu - r^2)$, $r \geq 0$: $\mu < 0$ in the x-y plane

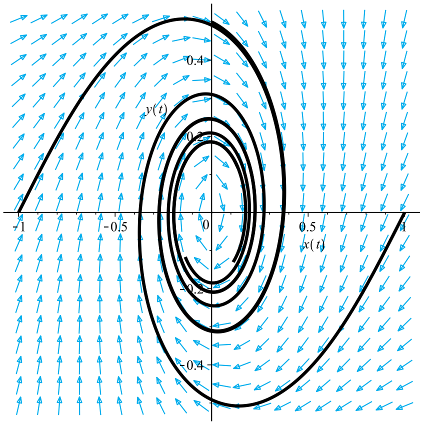


FIGURE 7.34

Phase portrait of $\dot{r} = r(\mu - r^2)$, $r \geq 0$: $\mu = 0$ in the x-y plane

of the origin (Fig. 7.35). *All nonzero trajectories spiral away from the origin and toward the limit cycle as* $t \to \infty$.

What happens in this example is that $\mu = 0$ is a *bifurcation point*, and as μ crosses 0—that is, moves from negative values to positive values—*a stable spiral becomes unstable and a limit cycle is generated*. In this situation, we say that a **(supercritical) Hopf bifurcation** occurs at the parameter value $\mu = 0$.

It is interesting to note that the linearized form of our original system,

$$\dot{x} = \mu x + y$$
$$\dot{y} = -x + \mu y,$$

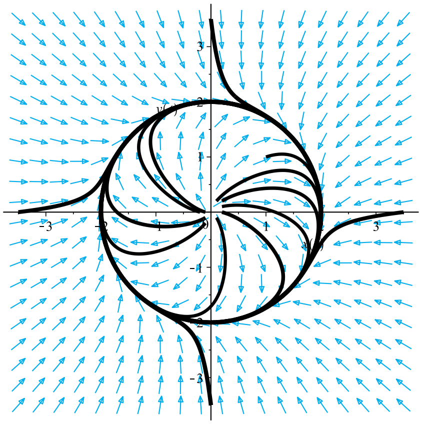


FIGURE 7.35

Phase portrait of $\dot{r} = r(\mu - r^2)$, $r \geq 0$: $\mu > 0$ in the x-y plane

predicts a *center* at the origin, which is not correct (see Exercise 4). In our original system, the origin changes from being asymptotically stable (a sink) to being unstable (a source) without passing through the stage of being a center.

To repeat, a **supercritical Hopf bifurcation** occurs when a stable spiral changes into an unstable spiral surrounded by a limit cycle. In terms of eigenvalues, the way that the stable equilibrium solution became destabilized in Example 7.6.6 was by having the complex conjugate eigenvalues (located in the left half-plane for $\mu < 0$) cross into the right half-plane as μ increased from negative to positive values. The eigenvalues of the Jacobian matrix evaluated at $(0, 0)$ are $\mu \pm i$. [The Hopf bifurcation is also referred to as the *Poincaré–Andronov–Hopf* bifurcation or the *Andronov–Hopf* bifurcation. The phenomenon was discovered by Poincaré in 1892. An important theorem describing the bifurcation was proved by A.A. Andronov for planar systems in 1929, and then proved for n-dimensional systems, $n \geq 2$, by Eberhard Hopf in 1942.]

There is another kind of Hopf bifurcation that we should acknowledge.

Example 7.6.7 A Subcritical Hopf Bifurcation

Let's investigate the behavior of the system

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = x + \mu y + y^3$$

as the parameter μ is varied. We see that this system has a single equilibrium solution at the origin. (*Verify this.*) The linearization is

$$\dot{x} = \mu x - y$$

$$\dot{y} = x + \mu y,$$

and the Jacobian has eigenvalues $\mu \pm i$, which resembles the situation in Example 7.6.6 and suggests the existence of a Hopf bifurcation.

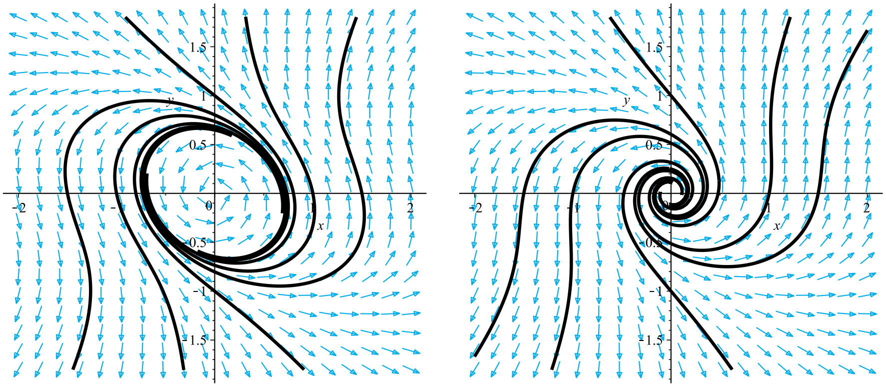


FIGURE 7.36

Phase portrait of $\dot{x} = \mu x - y + xy^2$, $\dot{y} = x + \mu y + y^3$: $\mu = -0.2, 0.2$, respectively

However, if we convert to polar form, we find that the equation for r becomes $\dot{r} = \mu r + ry^2$, which shows that the origin is unstable (*Why?*). Fig. 7.36 shows the phase portraits in rectangular coordinate form for $\mu = -0.2$ and $\mu = 0.2$, respectively. The left phase portrait shows the existence of an unstable limit cycle surrounding the stable focus at the origin. The second graph of Fig. 7.36 shows the unstable focus which occurs when μ is positive. This different type of Hopf bifurcation is referred to as **subcritical**.

To be clear about this, a **subcritical Hopf bifurcation** occurs when an equilibrium solution loses its stability by absorbing an unstable limit cycle.

To summarize:

- A **Hopf bifurcation** occurs when a periodic solution (limit cycle) surrounding an equilibrium point arises or goes away as a parameter μ varies.
- When a stable limit cycle surrounds an unstable equilibrium point, the bifurcation is called **supercritical**. [Examples 7.6.5 and 7.6.6]
- If the limit cycle is unstable and surrounds a stable equilibrium point, then the bifurcation is called **subcritical**. [Example 7.6.7]

Bifurcations in general—and the Hopf bifurcation in particular—may lead to disastrous results in various systems that are essentially nonlinear.

The Hopf bifurcation has been cited to explain the onset of often harmful oscillations in all sorts of natural systems, from human and animal populations and chemical reactions to business cycles and airplane wings and landing gears. Historically, many instances of oscillatory behavior were attributed to *resonance*, but more recent stability analyses have uncovered possible Hopf bifurcations.

For example, the famous *Tacoma Narrows Bridge collapse* in 1940 has long been considered a result of resonance, but now there are mathematicians and engineers who suggest a Hopf bifurcation as the explanation, almost for the same reason as the flutter in airplane wings or the "weave" in bicycle behavior.

Similarly, the *Bhopal disaster*, a 1984 gas leak at a pesticide plant in India in which over 500,000 people were exposed to a toxic gas, has been analyzed as an example of "thermal runaway" that occurred inside a storage tank. (This term describes a process that is accelerated by increased temperature, in turn releasing energy that further increases temperature. For example, lithium-ion batteries are susceptible to this phenomenon.) As one researcher has hypothesized,

> *The stability properties of the model indicate that the thermal runaway may have been due to a large amplitude, hard thermal oscillation initiated at a subcritical Hopf bifurcation.*

[R. Ball, Oscillatory thermal instability and the Bhopal disaster, *Process Safety and Environmental Protection* **89** (No. 5): pp. 317-322.]

Exercises 7.6

A

1. In the discussion of the van der Pol equation (7.6.1), the comment was made that it can be interpreted as a spring-mass system with nonlinear resistance. Specifically, the term $\varepsilon(x^2-1)$ represents a *variable* damping coefficient.

a. Explain why $\varepsilon(x^2-1)<0$ when $-1<x<1$, so that damping is *negative* for the small oscillations corresponding to $-1<x<1$. (This means that small-amplitude oscillations are *amplified* if they become too small.)

b. Explain why $\varepsilon(x^2-1)>0$ when $|x|>1$, so that damping is *positive* for the large oscillations corresponding to $|x|>1$. (This means that large-amplitude oscillations are made to *decay* if they become too large.)

2. Use technology to draw phase portraits of the van der Pol equation for

$$\varepsilon = \frac{1}{4}, \frac{3}{2}, \text{ and } 3.$$

3. Consider the van der Pol equation in the system form (7.6.2), where $x_1(0)=1$ and $x_2(0)=0$.

a. For $\varepsilon=\frac{1}{4}$, graph the trajectory in the x_1-x_2 plane. Then graph $x_1(t)$ against t and $x_2(t)$ against t on different sets of axes. Use technology.

b. For $\varepsilon=4$, graph the trajectory in the x_1-x_2 plane. Then graph $x_1(t)$ against t and $x_2(t)$ against t on different sets of axes. Use technology.

c. Describe the differences between the graphs in part (a) and the graphs in part (b).

4. Use the EUT in Section 6.2.1 to show that the van der Pol equation has a unique solution in any interval containing $t=0$.

B

5. Go back to Example 7.6.2 and look at the trigonometric substitutions suggested there. You will verify the polar coordinate form of the system equations.

a. Use the Chain Rule to show that $r\dot{r} = x\dot{x} + y\dot{y}$.

b. Show that $\dot{\theta} = -\frac{1}{x^2+y^2}\,(y\dot{x} - x\dot{y})$, or $-r^2\dot{\theta} = (y\dot{x} - x\dot{y})$.

c. Show that $x\dot{x} + y\dot{y} = \left(x^2 + y^2\right)\left(x^2 + y^2 - 1\right) = r^2\left(r^2 - 1\right)$. [*Hint*: Multiply the first equation in the system by x and the second equation by y and then add the results.]

d. Use part (a) and part (c) to show that $\dot{r} = r\left(r^2 - 1\right)$.

e. Use part (b) and the general method in part (c) to show that $\dot{\theta} = 1$.

6. Reconsider the uncoupled system (polar coordinate form) in Example 7.6.2.

a. Solve for $r(t)$.

b. Solve for $\theta(t)$.

c. Use your answers to part (a) and part (b) to construct $x(t)$ and $y(t)$.

7. Follow the directions given in the preceding problem for the system in Example 7.6.3.

8. Consider the system $\left\{\dot{r} = r\left(1 - r^2\right), \dot{\theta} = 1\right\}$.

a. Show that this is equivalent to the system

$$\dot{x} = x - y - x\left(x^2 + y^2\right)$$
$$\dot{y} = x + y - y\left(x^2 + y^2\right),$$

where $x = r(t)\cos\theta(t)$ and $y = r(t)\sin\theta(t)$.

b. Use either form of the system to determine its unique limit cycle.

9. Consider the system $\left\{\dot{r} = r\left(4 - r^2\right), \dot{\theta} = 1\right\}$, where $x(t) = r(t)\cos\theta(t)$ and $y = r(t)\sin\theta(t)$. Given the initial conditions $x(0) = 0.1$, $y(0) = 0$, sketch the graph of $x(t)$ without finding an explicit expression for $x(t)$. [*Hint*: Study Example 7.6.2 carefully.]

Suppose we have an autonomous system $\{\dot{x} = f(x, y), \dot{y} = g(x, y)\}$, where f and g have continuous first partial derivatives in some region R of the phase plane that doesn't have any "holes." Then **Bendixson's Theorem** (or **Negative Criterion**) states that if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is always positive or always negative at points of R, then the system has no periodic solutions in R. For example, the system $\left\{\dot{x} = xy^2, \dot{y} = x^2 + 8y\right\}$ has no limit cycles anywhere because $\frac{\partial(xy^2)}{\partial x} + \frac{\partial(x^2+8y)}{\partial y} = y^2 + 8 > 0$ for all values of x and y in the plane.

Use Bendixson's criterion (above) to show that the systems in Problems 10–13 have no limit cycles in the phase plane.

10. $\left\{\dot{x} = x + 2xy + x^3,\ \dot{y} = -y^2 + x^2y\right\}$

11. $\left\{\dot{x} = x^3 + x + 7y,\ \dot{y} = x^2y\right\}$

12. $\{\dot{x} = -2x - x\sin y,\ \dot{y} = -x^2y^3\}$

13. $\{\dot{x} = 2x^3y^4 - 3,\ \dot{y} = 2ye^x + x\}$

14. Show that the system

$$\dot{x} = 12x + 10y + x^2y + y\sin y - x^3$$
$$\dot{y} = x + 14y - xy^2 - y^3$$

has no periodic solution in the disk $x^2 + y^2 \leq 8$.

15. A mechanical system with variable damping can be modeled by the equation

$$\ddot{x} + a(x)\dot{x} + b(x) = 0,$$

where $a(x)$ is a positive function.

a. Write this equation in system form.

b. Use the Bendixson criterion to show that this mechanical system has no nonconstant periodic solution.

16. Consider the system $\{\dot{r} = \mu r - r^3,\ \dot{\theta} = 1\}$.

a. Determine all the equilibrium solutions and analyze their stability.

b. Determine all the bifurcation values of μ and identify the nature of the bifurcations.

17. Consider the system

$$\dot{x} = \mu x - y - \left(x + \frac{3}{2}y\right)(x^2 + y^2)$$
$$\dot{y} = x + \mu y + \left(\frac{3}{2}x - y\right)(x^2 + y^2).$$

a. Use the polar form of the system to find and classify all the bifurcation points.

b. First find all the equilibrium points and then linearize about these points to determine and classify all the bifurcation points.

18. Let $\dot{x} = -\mu x - y + \frac{x}{1+x^2+y^2}$, $\dot{y} = x - \mu y + \frac{y}{1+x^2+y^2}$.

a. Show that this system has a Hopf bifurcation as $\mu > 0$ decreases through $\mu = 1$. [*Hint*: Express the system in polar form.]

b. Find the radius of the limit cycle for $0 < \mu < 1$.

C

19. The system

$$\dot{x} = -y - y^2$$
$$\dot{y} = \frac{1}{2}x - \frac{1}{5}y + xy - \frac{6}{5}y^2$$

was discovered by the Chinese mathematician Tung Chin Chu in the late 1950s in his investigation of a famous unsolved problem on limit cycles.

a. Find the equilibrium point(s) of this system.

b. Use technology to draw a phase portrait for each equilibrium point, focusing on the region around that point. (It's a bit tricky to get a good phase portrait for this problem. Be patient.)

c. Using the phase portrait(s), identify and describe any limit cycle(s) you see with the term *stable* or *unstable*.

20. Find all limit cycles of the system

$$\dot{r} = r(r-1)(r-2)^2(r-3)$$
$$\dot{\theta} = 1$$

and identify them as stable, unstable, or semistable.

21. Show that the system

$$\dot{x} = \mu x - y + x(x^2 + y^2)(2 - x^2 - y^2)$$
$$\dot{y} = x + \mu y + y(x^2 + y^2)(2 - x^2 - y^2)$$

undergoes a subcritical Hopf bifurcation at $\mu = 0$ for a certain range of values of the parameter μ.

Summary

Nonlinear differential equations and systems of nonlinear equations are rarely handled satisfactorily by finding closed-form solutions. In particular, we can't analyze the *stability* of systems of nonlinear equations as easily as we analyzed the stability of linear systems in Chapter 6. The modern study of nonlinear phenomena relies heavily on the qualitative methods pioneered by H. Poincaré and others at the end of the nineteenth century and in the beginning of the twentieth century. Current technology implements the power of these qualitative techniques.

One of the differences between linear and nonlinear equations is that a nonlinear equation may have more than one equilibrium solution. Another difference is that a solution of a nonlinear equation may "blow up in finite time"—that is, they become unbounded as t approaches some finite value. A third difference is that a nonlinear equation or system may be extremely sensitive to initial conditions. A slight change in an initial value may lead to drastic changes in the behavior of the solution or solutions.

A point (a^*, b^*) is an *equilibrium point* of the general nonlinear autonomous system

$$\dot{x} = F(x, y)$$

$$\dot{y} = G(x, y)$$

if $F(a^*, b^*) = 0 = G(a^*, b^*)$. If the origin is an equilibrium point, and the functions F and G are "nice" enough, we may be able to write our system in the form

$$\dot{x} = ax + by + f(x, y)$$
$$\dot{y} = cx + dy + g(x, y),$$

where f and g are nonlinear functions and $a = \frac{\partial F}{\partial x}(0, 0)$, $b = \frac{\partial F}{\partial y}(0, 0)$, $c = \frac{\partial G}{\partial x}(0, 0)$, and $d = \frac{\partial G}{\partial y}(0, 0)$. More generally, if $(a, b) \neq (0, 0)$ is an equilibrium point for the system, we can rewrite the system as

$$\dot{x} = A(x - a) + B(y - b) + f(x, y)$$
$$\dot{y} = C(x - a) + D(y - b) + g(x, y),$$

where f and g are nonlinear and $A = \frac{\partial F}{\partial x}(a, b)$, $B = \frac{\partial F}{\partial y}(a, b)$, $C = \frac{\partial G}{\partial x}(a, b)$, and $D = \frac{\partial G}{\partial y}(a, b)$.

Another way to look at this general situation is to realize that we are translating the equilibrium point (a, b) to the origin by using the change of variables $u = x - a$ and $v = y - b$. Of course, this means that $x = u + a$ and $y = v + b$, so we can rewrite the last system as

$$\dot{u} = Au + Bv + f(u, v)$$
$$\dot{v} = Cu + Dv + g(u, v),$$

which has $(0, 0)$ as an equilibrium point. Note that this says that any equilibrium point $(a, b) \neq (0, 0)$ can be transformed to the origin for the purpose of analyzing the stability of the system.

A nonlinear autonomous system

$$\dot{x} = ax + by + f(x, y)$$
$$\dot{y} = cx + dy + g(x, y),$$

where $ad - bc \neq 0$, $\lim_{(x,y)\to(0,0)} \left(\frac{f(x,y)}{\sqrt{x^2+y^2}} \right) = \lim_{(x,y)\to(0,0)} \left(\frac{g(x,y)}{\sqrt{x^2+y^2}} \right) = 0$ and where the origin is an equilibrium point, is called an **almost linear system**; the reduced system

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

is called the **associated linear system** (or linear approximation) about the origin.

The important **Hartman–Grobman theorem** is a qualitative result describing how the equilibrium points of the nonlinear and linear systems are related according to the nature of eigenvalues of the associated linear system. When the linearized system has zero as an eigenvalue or when the linearized system has pure imaginary eigenvalues (so that the equilibrium solution is a center), further analysis is necessary to determine the nature of the equilibrium points.

The **Lotka–Volterra equations**, the **undamped pendulum**, and the **van der Pol equation** provide important examples of nonlinear systems and their analyses. In particular, the van der Pol oscillator exhibits uniquely nonlinear behavior in having a **stable limit cycle**, an isolated closed trajectory that (in this case) serves as an asymptotic limit for all other trajectories as $t \to \infty$. Some limit cycles, called **unstable limit cycles**, repel nearby trajectories. Finally, if trajectories near a limit cycle approach it from one side while being repelled from the other side, the cycle is called **semistable**.

The concept of a **bifurcation** that was analyzed in Section 2.7 for first-order equations can be extended to two-dimensional systems. Examples of **transcritical**, **pitchfork**, and **saddle-node bifurcations** are provided. A **Hopf bifurcation** occurs when an equilibrium solution loses its stability and a limit cycle appears as a parameter varies. **Supercritical** and **subcritical** Hopf bifurcations are discussed.

APPENDIX

Some calculus concepts and results

A

Appendix A is intended to offer either a brief review of or an introduction to selected key ideas of calculus.

A.1 Local linearity: the tangent line approximation

The concept of **local linearity** says that if the function f is differentiable—that is, if it has a derivative—at $x = a$ and we "zoom in" on the point $(a, f(a))$ on the graph of $y = f(x)$, then the portion of the curve that surrounds the point looks very much like a straight line, at least to the naked eye. Another way of saying this is to say that the tangent line at the point $(a, f(a))$ is a good approximation to the curve for values of x close to a. Fig. A.1 illustrates this.

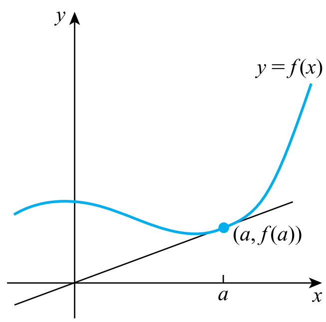


FIGURE A.1

The tangent line approximation

Using the point-slope formula from algebra, we can write the equation of this tangent line as $y = f(a) + f'(a)(x - a)$, and we can express this **tangent line approximation** as $f(x) \approx f(a) + f'(a)(x - a)$ for x close to a.

As x takes on values farther away from a, we expect the **absolute error** $|E(x)| = \left|f(x) - f(a) - f'(a)(x - a)\right|$ to become larger.

For example, the equation of the tangent line drawn to the sine curve at the origin is $y = \sin(0) + \cos(0)(x - 0) = x$. This says that near the origin, $\sin x \approx x$. One consequence of this is that $\frac{\sin x}{x} \approx 1$ for values of x near (but not equal to) zero, so we get the famous result $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

A.2 The chain rule

You should know the rules for finding the derivatives of power functions, polynomials, exponential functions, logarithms, and trigonometric and inverse trigonometric functions. You may also have learned about differentiating certain combinations of exponential functions called *hyperbolic functions*: $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x/\cosh x$, etc. You should know the *Product Rule* and the *Quotient Rule* for differentiation, and how to deal with *implicit functions*.

The Chain Rule applies to *composite functions*. Suppose that a quantity z depends on a quantity y, and that the quantity y depends on the value of quantity x. Using function notation, we can write this as $z = f(y)$, $y = g(x)$, so $z = f(g(x))$. This says that, ultimately, z depends on (is a function of) x. The **Chain Rule** tells us how a change in the value of x affects the value of z. In Leibniz notation,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

This form is useful in many applied problems, and in Chapter 6 where the concepts of *phase plane* and *phase portrait* are introduced.

Example

If $z = y^{57}$ and $y = \sin x$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \left(57y^{56}\right) \cdot \cos x = 57 \sin^{56} x \cos x.$$

You may have learned another way to see the Chain Rule: If $z = f(g(x))$, then $z' = f'(g(x)) \cdot g'(x)$. This alternative point of view uses the idea of an "inside" function and an "outside" function. Try this on the preceding example, where the 57th-power function is outside and the sine function is inside.

A.3 The Taylor polynomial/Taylor series

To extend the idea of the tangent line approximation, we look for a polynomial P_n of degree n that approximates a function f as closely as possible on an interval about a point $x = a$. What this means mathematically is that we want the polynomial to satisfy the following closeness conditions: $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, $P_n''(a) = f''(a)$, $P_n'''(a) = f'''(a)$, …, and $P_n^{(n)}(a) = f^{(n)}(a)$. For a given function f, a point

$x = a$, and degree n, the polynomial that satisfies all these conditions is given by the formula

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the **Taylor polynomial of degree *n*** about $x = a$, and we can write $f(x) \approx P_n(x)$ for x close to a. The closeness of the approximation depends on both the value of x and the value of n. In general, the closer the value of x is to the value a and the higher the degree n, the better the approximation.

If we consider what happens to a Taylor polynomial as we let n get larger, we arrive at the idea of the (infinite) **Taylor series**:

$$\begin{aligned} P(x) &= \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \end{aligned}$$

More precisely, suppose that f is a function with derivatives of all orders in some interval $(a-r, a+r)$. Then the Taylor series given previously represents the function f on the interval $(a-r, a+r)$ if and only if $\lim_{n\to\infty} R_n(x) = 0$, where $R_n(x)$ is the remainder in Taylor's formula:

$$\begin{aligned} R_n(x) &= f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \right. \\ &\quad \left. + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some point } c \text{ in } (a-r, a+r). \end{aligned}$$

As we saw in Section A.1, any approximation process is subject to error. For example, when working with π, which has an infinitely long nonrepeating decimal representation, we lose accuracy by using 3.14159 or even 3.14159265359 as its value. This, in turn, leads to what is called a **propagated error,** the accumulated error resulting from many calculations with rounded values. If each item of data is inaccurate because of rounding of some sort, then the various steps in a calculation process can compound the error. A useful approximation method guarantees that the smaller the round-off error at each stage, the smaller the cumulative round-off error. Of course, it turns out that sometimes round-off errors cancel each other out to a certain extent—approximate values that are too high may be balanced by values that are too low.

A **truncation error** occurs when we stop (or truncate) an approximation process after a certain number of steps. For example, when we approximate the values of $\sin x$ near $x = 0$ by using the first seven nonzero terms of its (infinite) Taylor series,

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!},$$

we are introducing a truncation error. If we write $\sin x = T_{13}(x) + R_{13}(x)$, where $T_{13}(x)$ is the 13th-degree polynomial just given, a formula from calculus gives an upper bound for the absolute truncation error

$$|\sin x - T_{13}(x)| = |R_{13}(x)| = \frac{|\sin c|}{14!}|x|^{14} \leq \frac{|x|^{14}}{14!},$$

where c is a positive number less than x. Even if we use the 1001st-degree Taylor polynomial, we are still only approximating and will therefore have truncation error.

Here are some Taylor series that occur often in applications:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{k+1}\frac{x^k}{k} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .
\end{aligned}$$

Although the first three series are valid (or converge) for any value of x, the logarithmic series is valid only on the interval $(-1, 1]$. The last series, a *geometric series*, converges for $|x| < 1$. In Section C.4, we show how Euler used the exponential series to arrive at a formula for the complex exponential function.

A Taylor series is a special type of *power series*. We can differentiate or integrate a power series term by term for values of x within its interval of convergence. If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to $S(x)$ for x in some interval I, then

$$S'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

and

$$\int_0^x S(t)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

$$= a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1} + \cdots,$$

where both the differentiated and the integrated series converge for x in I.

In Appendix D, we show how to solve certain differential equations using power series methods.

A.4 The fundamental theorem of calculus

A function F is an *antiderivative* of the function f if $F'(x) = f(x)$. A very important connection between derivatives and integrals is expressed by the **Fundamental Theorem of Calculus** (**FTC**). This result comes in two flavors:

A. If $f(x)$ is continuous on the closed interval $[a, b]$ and if $F(x)$ is any antiderivative of $f(x)$ on this interval, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

B. Let $f(x)$ be defined and continuous on a closed interval $[a, b]$ and define the function $G(x)$ on this interval:

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G(x)$ is differentiable there with derivative $f(x)$: $G'(x) = f(x)$—that is, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Version A simplifies the whole business of finding the value of a definite integral: Just find an antiderivative of the integrand. Of course, as we know, this isn't always as simple as it sounds. At least half of your calculus course was probably devoted to techniques of *substitution* and *integration by parts*, trying to recognize integrands as derivatives resulting from the Product Rule, the Chain Rule, and so forth.

A slight twist on version A tells us that if we integrate f', the *rate* function over $[a, b]$, we get the *total change* in f, the *amount* function over the same interval. For example, if $s(t)$, $v(t)$, and $a(t)$ denote the position, velocity, and acceleration, respectively, of a moving object at time t, then we have

$$\int v(t)\,dt = s(t) + C \quad \text{and} \quad \int a(t)\,dt = v(t) + K,$$

where C and K denote arbitrary constants. Consequently, we can write

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(b) - s(a),$$

which says that if we integrate the *velocity* function, we get the *total change in position* of a moving object as t changes from a to b. If we integrate the *speed* function—the absolute value of the velocity—we get the *total distance* traveled by the object. (See Example 1.3.4 in the text.)

As useful as version A is in solving differential equations, version B extends the notion of differentiation (and therefore of integration) to functions defined by integrals. (See, e.g., Problems 9 and 10 in Exercises 1.2.)

Example

Suppose that $Q(x) = \int_{-2}^{x} \cos(u^2)\, du$. Then $Q'(x) = \cos(x^2)$, $Q''(x) = -2x\sin(x^2)$, and so on.

If we are given a curve with equation $y = f(x)$, $a \le x \le b$, then we can calculate the **arclength of** $\boldsymbol{f}$ (the length of the curve $y = f(x)$) by using the formula $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

The **(First) Mean Value Theorem for Integrals** can come in handy: If f and g are continuous functions on $[a, b]$ and $g(x) \ge 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx$ for some $\xi \in [a, b]$.

A.5 Partial fractions

An important and useful result from algebra says that every rational function (quotient of polynomial functions), no matter how complicated, comes from adding simpler fractions. For example, the function

$$\frac{8x+1}{x^2 - x - 6}$$

comes from the following addition of simpler pieces:

$$\frac{3}{x+2} + \frac{5}{x-3} = \frac{3(x-3) + 5(x+2)}{(x+2)(x-3)} = \frac{8x+1}{x^2 - x - 6}.$$

In calculus, when we have an integrand that is a rational function, we can reverse this addition process to find the simpler fractions, fractions that we can integrate easily. Thus, for example,

$$\int \frac{8x+1}{x^2 - x - 6}\, dx = \int \left(\frac{3}{x+2} + \frac{5}{x-3}\right) dx = \int \frac{3}{x+2}\, dx + \int \frac{5}{x-3}\, dx$$
$$= 3\ln|x+2| + 5\ln|x-3| + C.$$

In this example, the algebraic challenge is to find constants A and B such that

$$\frac{8x+1}{x^2 - x - 6} = \frac{A}{x+2} + \frac{B}{x-3}. \qquad (*)$$

The fractions $A/(x+2)$ and $B/(x-3)$ are called **partial fractions** because each contributes a piece of the whole. In particular, the denominators $x+2$ and $x-3$ are parts (factors) of the original denominator x^2-x-6. The numbers A and B are called **undetermined coefficients** (see Section 4.3, Section 6.10.2, and Appendix D). To find A and B, we clear Eq. (*) of fractions by multiplying both sides by x^2-x-6. The result is

$$8x+1 = A(x-3) + B(x+2).$$

This is supposed to be an identity in x. If we let $x=3$, we find that $8(3)+1 = 0+5B$, or $B=5$. Similarly, letting $x=-2$, we get $8(-2)+1 = -5A+0$, so $A=3$.

This technique works for a rational function in lowest terms whose denominator can be factored into distinct linear factors. More complicated denominators can also be handled by this kind of algebraic method, and you can find a more detailed discussion in your calculus text. Most computer algebra systems can produce such "partial-fraction decompositions" and can evaluate integrals with integrands that are rational functions. Partial fractions are particularly useful in Section 2.1, in Chapter 5, and in Chapter 7.

A.6 Improper integrals

In dealing with the definite integral $\int_a^b f(x)\,dx$, we make two basic assumptions: (1) the interval $[a, b]$ is finite and (2) the integrand f is bounded—that is, it does not become infinite—on the closed interval $[a, b]$. If we violate one or both of these assumptions, we encounter a type of **improper integral**.

First, let us assume that we want to consider the interval $[a, \infty)$ or $(-\infty, b]$, where a and b are real numbers. We can define

$$\int_a^{\infty} f(x)\,dx = \lim_{B\to\infty}\int_a^B f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{b} f(x)\,dx = \lim_{A\to\infty}\int_{-A}^{b} f(x)\,dx$$

provided that each limit exists. If the limit exists, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**. Finally,

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{A\to\infty}\int_{-A}^{c} f(x)\,dx + \lim_{B\to\infty}\int_{c}^{B} f(x)\,dx$$

provided that each limit on the right-hand side exists individually. Here, c is an arbitrary real number. It is *not* correct to define $\int_{-\infty}^{\infty} f(x)\,dx$ as $\lim_{C\to\infty}\int_{-C}^{C} f(x)\,dx$.

Example

$$\int_1^{\infty}\frac{dx}{1+x^2} = \lim_{B\to\infty} \arctan x\,|_1^B = \lim_{B\to\infty}(\arctan B - \arctan 1)$$

$$= \lim_{B\to\infty}\left(\arctan B - \frac{\pi}{4}\right) = \lim_{B\to\infty} \arctan B - \frac{\pi}{4}$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Example

Consider $\int_0^\infty \sin x\, dx$. The limit

$$\lim_{B\to\infty}\int_0^B \sin x\, dx = \lim_{B\to\infty}(-\cos(B) + \cos(0)) = -\lim_{B\to\infty}\cos(B) + 1$$

doesn't exist because $\cos(B)$ oscillates from -1 to 1 as B tends to infinity.

When we are dealing with this first type of improper integral, for which the interval is not finite, sometimes a form of **L'Hôpital's Rule** comes in handy: Suppose that as $x \to a$, where a is $\pm\infty$, $f(x) \to \pm\infty$, and $g(x) \to \pm\infty$. If $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$, where L is either a real number or $\pm\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = L$.

Example

Consider Euler's *gamma function*, defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$. Integration by parts tells us that

$$\Gamma(x) = -t^{x-1}e^{-t}\Big]_0^\infty - \int_0^\infty (x-1)t^{x-2}(-e^{-t})\, dt$$
$$= \lim_{c\to\infty}\frac{-t^{c-1}}{e^c} + (x-1)\int_0^\infty t^{x-2}e^{-t}\, dt$$
$$= (x-1)\cdot\Gamma(x-1),$$

where we have used L'Hôpital's Rule several times in evaluating the limit. (Successive differentiations of the numerator and denominator of $\frac{-t^{c-1}}{e^c}$ eventually gives us $-(c-1)!$ in the numerator, whereas the denominator remains e^c, so the limit of the quotient as c tends to infinity is 0.) Note that because $\Gamma(1) = \Gamma(2) = 1$, we can conclude that $\Gamma(x+1) = x \cdot (x-1) \cdot (x-2) \cdot (x-3) \cdots 3 \cdot 2 \cdot 1 = x!$ when x is an integer, so the gamma function provides a generalization of $n!$ to the case in which n is not an integer. (See Section D.3, especially footnote 4.)

Now let's suppose that f is defined and finite on the interval $[a, b]$ except at the endpoint b. Then the integral $\int_a^b f(x)\, dx$

$$\int_a^b f(x)\, dx = \lim_{B\to b^-}\int_a^B f(x)\, dx$$

provided that this *left-hand limit* (or *limit from the left*) exists. Similarly, if f is unbounded at the endpoint a, then we define

$$\int_a^b f(x)\, dx = \lim_{A\to a^+}\int_A^b f(x)\, dx$$

provided that this right-hand limit (or limit from the right) exists.

Example

The function $1/\sqrt{1-x^2}$ is unbounded at $x=1$ (and at $x=-1$). The improper integral of this function on the interval [0, 1] converges:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to 1^-} \int_0^B \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to 1^-} \arcsin(B) - \arcsin(0) = \frac{\pi}{2}.$$

Another possibility is that the function f is defined and finite on $[a, b]$ except at a point ξ *inside* the interval. The improper integral is then defined as

$$\int_a^b f(x)\,dx = \lim_{c\to\xi^-} \int_a^c f(x)\,dx + \lim_{d\to\xi^+} \int_d^b f(x)\,dx$$

provided that both one-sided limits exist.

Example

$$\begin{aligned}\int_0^2 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} + \lim_{d\to 1^+} \int_d^2 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c\to 1^-} 3(x-1)^{1/3}\Big|_0^c + \lim_{d\to 1^+} 3(x-1)^{1/3}\Big|_d^2 \\ &= \lim_{c\to 1^-} \left[3(c-1)^{1/3} - 3\,(-1)^{1/3}\right] + \lim_{d\to 1^+} \left[3\,(1)^{1/3} - 3\,(d-1)^{1/3}\right] \\ &= 3+3=6.\end{aligned}$$

The following **Comparison Test for Improper Integrals** is often useful: Suppose that $0 \le g(x) \le f(x)$. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges. Furthermore, if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

A.7 Functions of several variables/partial derivatives

Sometimes we encounter functions that depend on more than one independent variable. For example, the area of a rectangle depends on both its length and its width. We can express this relationship as $A = f(l, w) = l \cdot w$. In general, if there are two independent variables (x and y) and one dependent variable (z), we can express this situation as $z = f(x, y)$. In other words, the variable z depends on (is a function of) the variables x and y. This means that changes in the value of either x or y (or both) will lead to changes in z. The *instantaneous rate of change of z with respect to x* is given by the **partial derivative of z with respect to x**, which is defined by the formula

$$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, the *instantaneous rate of change of z with respect to y* is given by the **partial derivative of z with respect to y**, which is defined by the formula

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

What this means in terms of practical calculation is that to find $\frac{\partial z}{\partial x}$, you just treat y as a constant and differentiate with respect to x as usual. For $\frac{\partial z}{\partial y}$, you treat x as a constant and regard y as the "live" variable.

The Chain Rule takes different forms in this multivariable environment, among them this version: If $x = f(t)$ and $y = g(t)$, where f and g are differentiable and F is a function of x and y, then $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$.

Example

If $z = f(x, y) = x^2y^2 - 3xy^3 + 5x^4y^2$, then

$$\frac{\partial z}{\partial x} = 2xy^2 - 3y^3 + 20x^3y^2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2x^2y - 9xy^2 + 10x^4y.$$

Example

Suppose $w = e^{2x+3y}\sin(xy)$. Then, using the Product Rule and the Chain Rule, we find that

$$\frac{\partial w}{\partial x} = e^{2x+3y}\cos(xy)y + 2e^{2x+3y}\sin(xy)$$

and

$$\frac{\partial w}{\partial y} = e^{2x+3y}\cos(xy)x + 3e^{2x+3y}\sin(xy).$$

Example

Let $F(x, y) = x^2 + y^2$, with $x = x(t) = \cos t$ and $y = y(t) = \sin t$. Then the multivariable Chain Rule gives us $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = (2x)(-\sin t) + (2y)(\cos t) = -2\cos t \sin t + 2\sin t \cos t = 0$, which is not surprising since $F(x, y) = x^2 + y^2 = \cos^2 t + \sin^2 t \equiv 1$.

In general, if you have a function of n variables, $z = f(x_1, x_2, x_3, \ldots, x_n)$, then you can define the partial derivative of z with respect to x_k, and calculate it by treating x_k as the only true variable and treating the other x_i $(i \neq k)$ as constants. You can define higher derivatives and *mixed* derivatives in the obvious way: $\frac{\partial^2 z}{\partial x_i \partial x_k}$, $\frac{\partial^n z}{\partial x_k^n}$, and so on.

Example

Using $z = f(x, y) = x^2y^2 - 3xy^3 + 5x^4y^2$ and the results of the first example, we have

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^2 - 3y^3 + 20x^3y^2\right) = 2y^2 + 60x^2y^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^2y - 9xy^2 + 10x^4y\right) = 2x^2 - 18xy + 10x^4$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^2y - 9xy^2 + 10x^4y\right) = 4xy - 9y^2 + 40x^3y,$$

and so forth.

If we have an integral of the form $\int_a^b f(x,t)\,dt$, where f is a function of the two variables x and t and a, b are constants, then *this integral is a function of* x: If we integrate with respect to the variable t, the result will still contain the variable x. **Leibniz's rule** for differentiating under the integral sign says that the equation

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

is valid at $x = x_0$, in the sense that both sides exist and are equal, provided that f and its partial derivative $\frac{\partial}{\partial x} f$ are both continuous in some rectangular region $a \le t \le b$, $c \le x \le d$ of the x-t plane. (This was one of the mathematical rules Richard Feynman, the renowned physicist, was fondest of.)

Example

Let $f(x,t) = (2t + x^3)^2$. Then $\int_0^1 f(x,t)\,dt = \int_0^1 (2t + x^3)^2\,dt = 4/3 + 2x^3 + x^6$, so that

$$\frac{d}{dx}\int_0^1 (2t + x^3)^2\,dt = \frac{d}{dx}\left(4/3 + 2x^3 + x^6\right) = 6x^2 + 6x^5.$$

But we can use Leibniz's rule to calculate

$$\frac{d}{dx}\int_0^1 (2t + x^3)^2\,dt = \int_0^1 \frac{\partial}{\partial x}(2t + x^3)^2\,dt = \int_0^1 (6tx^2 + 6x^5)\,dt$$
$$= \int_0^1 \left(12tx^2 + 6x^5\right)\,dt = 6x^2 + 6x^5.$$

A.8 The tangent plane: the Taylor expansion of $F(x, y)$

In Section A.1, we saw that the tangent line $y = f(a) + f'(a)(x - a)$ gives the best linear approximation of a single-variable function f near $x = a$. For $F(x, y)$, a function of two variables, the best approximation near a point (a, b) is provided by the *tangent plane* given by the approximation formula

$$F(x,y) \approx F(a,b) + \frac{\partial F}{\partial x}(a,b)(x - a) + \frac{\partial F}{\partial y}(a,b)(y - b),$$

where $\frac{\partial F}{\partial x}(a,b)$ and $\frac{\partial F}{\partial y}(a,b)$ denote the partial derivatives evaluated at the point (a, b).

Example

Let's calculate the tangent plane approximation of the function $F(x, y) = x^3 - x^2y^2 + y^3$ near the point $(a, b) = (1, 2)$. We have $\frac{\partial F}{\partial x} = 3x^2 - 2xy^2$ and $\frac{\partial F}{\partial y} = -2x^2y + 3y^2$, so $F(1, 2) = 1^3 - 1^2 2^2 + 2^3 = 5$, $\frac{\partial F}{\partial x}(1, 2) = 3(1)^2 - 2(1)(2)^2 = -5$, and $\frac{\partial F}{\partial y}(1, 2) = -2(1)^2(2) + 3(2)^2 = 8$. Putting these results into the tangent plane formula, we get

$$\begin{aligned} F(x, y) &\approx F(1, 2) + \frac{\partial F}{\partial x}(1, 2)(x - 1) + \frac{\partial F}{\partial y}(1, 2)(y - 2) \\ &= 5 - 5(x - 1) + 8(y - 2). \end{aligned}$$

Fig. A.2 shows the three-dimensional picture of the surface and its tangent plane.

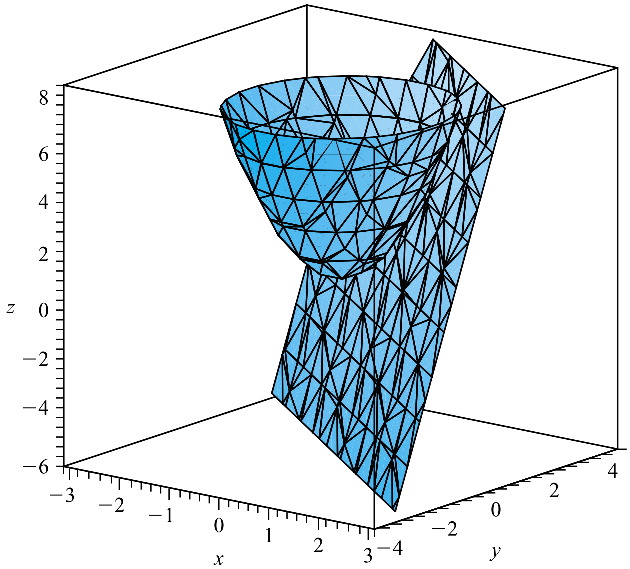


FIGURE A.2

Tangent plane to the surface $z = x^3 - x^2y^2 + y^3$ at $(1, 2)$

For points (x, y) close to $(1, 2)$, the values of z on the tangent plane are close to the values of z on the surface defined by $z = F(x, y)$.

We can define a full Taylor series expansion of a function of several variables, but for this text we only need the idea of the tangent plane (linear) approximation (see Chapter 7).

APPENDIX

B Vectors and matrices

Appendix B provides an expanded view of the vector and matrix algebra needed in this text.

B.1 Vectors and vector algebra; polar coordinates

In more abstract courses, a **vector** is an object in a set whose elements obey certain algebraic rules. For physicists, engineers, and other scientists, a **vector**—more properly, a **geometric vector**—is a quantity that has both magnitude (size) and direction. In two-dimensional physical situations, there are two usual ways to represent a vector: (1) as an ordered pair of real numbers written (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$ and (2) as an *arrow* from the origin (usually) of the x-y plane to a point (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$. The numbers x and y are called the **components** or **coordinates** of the vector. As indicated in Chapter 6, we can also consider vectors with complex-number coordinates and vectors whose components are functions. For the sake of simplicity in this appendix, we'll work with vectors whose components are real numbers.

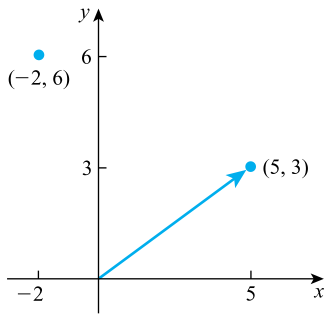


FIGURE B.1

Ways to represent a vector

Both ways of looking at a vector are shown in Fig. B.1. In the "arrow" view, the vector $\mathbf{v} = (x, y)$ is always the hypotenuse of a right triangle, so by the Pythagorean

Theorem its **length**, denoted $|\mathbf{v}|$, is given by the expression $\sqrt{x^2+y^2}$. The direction of a vector is indicated by the direction of the arrow.

Vectors, which often represent forces of various kinds, can be combined to indicate interactions. For example, we can add two vectors as follows: If $\mathbf{v}_1=(x_1,y_1)$ and $\mathbf{v}_2=(x_2,y_2)$, then $\mathbf{v}_1+\mathbf{v}_2=(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$. Subtraction is similar. We can also multiply a vector by a real number (or a complex number or a function), which is called a **scalar** in this situation. To do this, just multiply each component of the vector by the scalar: If $\mathbf{v}=(x,y)$ and r is any real number, then $r\,\mathbf{v}=(rx,ry)$. Because the components of vectors are real numbers, we should expect the usual rules of algebra to apply. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors and r is a scalar, then $\mathbf{v}_1+\mathbf{v}_2=\mathbf{v}_2+\mathbf{v}_1$ [commutative property], $\mathbf{v}_1+(\mathbf{v}_2+\mathbf{v}_3)=(\mathbf{v}_1+\mathbf{v}_2)+\mathbf{v}_3$ [associative property], and $r\,(\mathbf{v}_1+\mathbf{v}_2)=r\,\mathbf{v}_1+r\,\mathbf{v}_2$ [distributive property]. There is a **zero vector**, denoted $\mathbf{0}=(0,0)$, such that $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ for every vector $\mathbf{v}$ [additive identity].

Geometrically, the addition or subtraction of vectors is captured by the **Parallelogram Law** (see Fig. B.2 for the two-dimensional version).

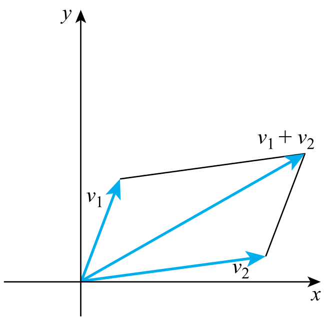


FIGURE B.2

The Parallelogram Law

Another way of representing a vector in two-dimensional space is by using **polar coordinates** (Fig. B.3). If we have a vector corresponding to the point (x,y), then we can describe it using its *length* r (its radial distance from the origin) and the angle θ that the arrow makes with the positive x-axis, measured in a counterclockwise direction. As we saw previously, the length is given by the formula $r=\sqrt{x^2+y^2}$.

As we look at Fig. B.3, simple trigonometry tells us that $x=r\cos\theta$, $y=r\sin\theta$, and $\theta=\arctan\left(\frac{y}{x}\right)$, $x\neq 0$. As indicated in some of the examples in Chapter 7, the polar representation of vectors may be more natural in problems involving expressions that look like x^2, y^2, x^2+y^2, and so on.

There is no reason to restrict our definition of vectors to two dimensions. In three-dimensional space, a vector is an ordered triplet, (x,y,z) or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, of real numbers or an arrow joining the origin $(0,0,0)$ to the point (x,y,z). In general,

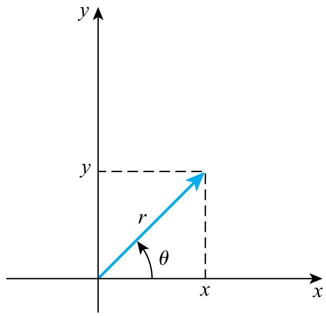


FIGURE B.3

Polar representation of a vector

an ***n*-dimensional vector** is an ordered n-tuple, $(x_1, x_2, x_3, \ldots, x_n)$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, of real numbers. The coordinate-by-coordinate arithmetic or algebra of vectors generalizes to any dimension in the obvious way.

Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, any vector of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$, where $c_1, c_2, \ldots, c_m$ are scalars, is called a **linear combination** of the set of vectors. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called **linearly independent** if the only way we can have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$ (the zero vector) is if $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, the set of vectors is **linearly dependent**. Linear *dependence* implies that at least one vector in the set can be expressed as a linear combination of the others.

Example

We will determine whether the following vectors are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The statement $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ is equivalent to the system of algebraic equations

$$\begin{aligned} c_1 + c_2 \qquad\qquad &= 0 \\ c_1 \qquad\qquad + c_4 &= 0 \\ c_2 + c_3 \qquad &= 0 \\ c_3 + c_4 &= 0. \end{aligned}$$

This system is not difficult to solve by hand using substitution or elimination, but we can also use the capability of a graphing calculator or computer algebra system (CAS) to solve such systems of equations. In any case, we find that $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, and $c_4 = -1$ is a solution. Because the scalars are not all zero, we conclude that the four vectors are *linearly dependent*. Note, for example, that we can write the first vector as a linear combination of the remaining vectors: $\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4$.

B.2 Matrices and basic matrix algebra

A **matrix** (the plural is **matrices**) is simply a rectangular arrangement (array) of numbers or other mathematical objects (such as functions) and is usually denoted by a capital letter. It can be considered a generalization of a vector. For example, we can have the matrix

$$A = \begin{bmatrix} 0 & -4 & 1/2 & 9 \\ \pi & 14/5 & -0.15 & 2 \\ 7 & \sqrt{3} & 0 & -3 \end{bmatrix}.$$

The numbers or objects making up a matrix are called its **elements** or **entries**. Most of the time we'll use real numbers, although complex numbers and even functions can appear as entries of matrices (as they can for components of vectors).

One way to describe a matrix is by indicating how many rows and columns it has. Matrix A in the preceding equation has three rows and four columns and is called a *3 by 4 matrix*, or a *3×4 matrix*. A matrix with m rows and n columns is called an ***m by n matrix*** (***$m \times n$ matrix***). Note that each row or column of a matrix can be considered a vector. An $n \times 1$ matrix is called a **column vector**, whereas a $1 \times n$ matrix is called a **row vector**. Two matrices are called **equal** if they have the same number of rows and columns and their corresponding elements are equal. For example, we can write

$$\begin{bmatrix} 1 & 0 & -5/3 \\ 1/\sqrt{2} & 3 & 0.25 \end{bmatrix} = \begin{bmatrix} 7/7 & 0 & -15/9 \\ \sqrt{2}/2 & 15/5 & 1/4 \end{bmatrix}.$$

We can add and subtract matrices of the same shape by adding or subtracting their corresponding elements, but (for example) we can't add a 3 by 4 matrix and a 4 by 3 matrix or subtract one of these from the other. If we take the matrix A that we have already defined and introduce the matrix B, so that we have

$$A = \begin{bmatrix} 0 & -4 & 1/2 & 9 \\ \pi & 14/5 & -0.15 & 2 \\ 7 & \sqrt{3} & 0 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 5 & -1/2 & 4 \\ 3/4 & 6/5 & 0.65 & 8 \\ -9 & \sqrt{2} & 8 & 3 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 0 + (-3) & -4 + 5 & 1/2 + (-1/2) & 9 + 4 \\ \pi + 3/4 & 14/5 + 6/5 & -0.15 + 0.65 & 2 + 8 \\ 7 + (-9) & \sqrt{3} + \sqrt{2} & 0 + 8 & -3 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & 0 & 13 \\ \pi + 3/4 & 4 & 0.5 & 10 \\ -2 & \sqrt{3}+\sqrt{2} & 8 & 0 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 0-(-3) & -4-5 & 1/2-(-1/2) & 9-4 \\ \pi - 3/4 & 14/5 - 6/5 & -0.15-0.65 & 2-8 \\ 7-(-9) & \sqrt{3}-\sqrt{2} & 0-8 & -3-3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -9 & 1 & 5 \\ \pi - 3/4 & 8/5 & -0.8 & -6 \\ 16 & \sqrt{3}-\sqrt{2} & -8 & -6 \end{bmatrix}.$$

The role of the number zero in matrix algebra is played by the **zero matrix** of the appropriate size—the matrix all of whose entries are zero.

We can also multiply a matrix by a number (or even a function) called a **scalar**, as in the case of vectors. We just multiply every element of the matrix by that scalar:

$$-5 \cdot \begin{bmatrix} 3 & -2 & 0 \\ -7 & 4 & 1/3 \\ 5 & -6 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} -5\,(3) & -5\,(-2) & -5\,(0) \\ -5\,(-7) & -5\,(4) & -5\,(1/3) \\ -5\,(5) & -5\,(-6) & -5\left(\sqrt{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -15 & 10 & 0 \\ 35 & -20 & -5/3 \\ -25 & 30 & -5\sqrt{2} \end{bmatrix}.$$

We've just multiplied a 3 by 3 matrix—one type of **square matrix**—by the scalar (-5).

B.3 Linear transformations and matrix multiplication

The really interesting thing about matrix arithmetic and algebra is how we *multiply* matrices. The natural thing to do—take two matrices with the same shape and multiply their corresponding elements—is not what is meant by matrix multiplication in the theory of linear algebra. Instead, there is a *row-by-column* process that looks strange at first, but becomes more natural when we see its applications.

To motivate the multiplication of matrices, let's return to elementary algebra for a moment and look at a system of two equations in two unknowns:

$$-2x + 3y = 5$$
$$x - 4y = -2.$$

In connection with this system, we can think of a point (x, y) in the plane transformed into another point as follows:

$$T(x, y) = (-2x + 3y, x - 4y).$$

For example,

$$T(1, 0) = (-2(1) + 3(0), 1 - 4(0)) = (-2, 1)$$
$$T(-4, 5) = (-2(-4) + 3(5), -4 - 4(5)) = (23, -24)$$

and

$$T(-2.8, -0.2) = (-2(-2.8) + 3(-0.2), -2.8 - 4(-0.2)) = (5, -2).$$

Note that this last calculation says that the ordered pair $(x, y) = (-2.8, -0.2)$ is a solution of our system of linear equations.

Geometrically, the point $(1, 0)$ has been moved to the location $(-2, 1)$, the point $(-4, 5)$ has been changed to $(23, -24)$, and the point $(-2.8, -0.2)$ has been transformed into $(5, -2)$. If we think of a point (x, y) as defining a vector, then the transformation stretches (or shrinks) the vector and rotates it through some angle θ until it becomes another vector. Fig. B.4 shows this interpretation of the effect of T on the vector $(1, 0)$.

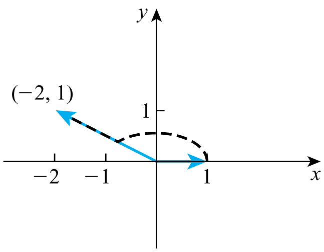


FIGURE B.4

The effect of $T(x, y) = (-2x + 3y, x - 4y)$ on the vector $(1, 0)$

More abstractly, we should be able to see that T is a **linear transformation** of points (x, y) in the plane to other points $(\hat{x}, \hat{y})$ in the plane: If $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$, then $T(c_1\mathbf{u} + c_2\mathbf{v}) = T(c_1\mathbf{u}) + T(c_2\mathbf{v}) = c_1 T(\mathbf{u}) + c_2 T(\mathbf{v})$ for any constants c_1 and c_2.

Matrix notation was invented by the English mathematician Arthur Cayley precisely to describe linear transformations. If $T(x, y) = (\hat{x}, \hat{y})$, where

$$ax + by = \hat{x}$$
$$cx + dy = \hat{y},$$

then we can pick out the coefficients a, b, c, and d and write them in a square array $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ called a **matrix**. If we know what variables $x, y, \hat{x}$, and $\hat{y}$ we're using, then knowing this **matrix of coefficients** enables us to understand what T is doing to points in the plane. We can focus on these variables by introducing the vectors $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\hat{\mathbf{X}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$. Now we can write our system of equations compactly as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix},$$

or $A\mathbf{X} = \hat{\mathbf{X}}$. To make sense, the "product" of A and $\mathbf{X}$ must be the column matrix $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$, which leads to a *row-by-column multiplication*:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax + by \quad \text{and} \quad \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = cx + dy.$$

Furthermore, the multiplication of two matrices of the appropriate sizes can be interpreted as a *composition of transformations*, one transformation followed by another.

Example

Suppose that we have two linear transformations defined by

$$M(x, y) = (x + 2y, 3x + 4y) \quad \text{and} \quad P(x, y) = (-2x, x + 3y).$$

Then

$$\begin{aligned} (M \circ P)(x, y) = M(P(x, y)) &= M(-2x, x + 3y) \\ &= (-2x + 2(x + 3y), 3(-2x) + 4(x + 3y)) \\ &= (6y, -2x + 12y). \end{aligned}$$

In particular, $(M \circ P)(1, 1) = (6, 10)$.

The matrices of coefficients for the transformations M and P look like $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $p = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$, so the composition $M \circ P$ takes the form of a *product of* 2×2 *matrices*:

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2x \\ x + 3y \end{bmatrix} \\ &= \begin{bmatrix} -2x + 2(x + 3y) \\ 3(-2x) + 4(x + 3y) \end{bmatrix} = \begin{bmatrix} 6y \\ -2x + 12y \end{bmatrix} \end{aligned}$$

and when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}.$$

We should check to see that $(M \circ P)(x, y) \neq (P \circ M)(x, y)$ or, equivalently, that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Looking at the preceding example, we see that transforming the vector (x, y) by P and then by M is equivalent to transforming the vector by the single transformation $T(x, y) = (6y, -2x + 12y)$. In matrix terms, we can express the effect of the composition $M \circ P$ as $\begin{bmatrix} 0 & 6 \\ -2 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6y \\ -2x + 12y \end{bmatrix}$. Note what we get when we add the results of multiplying each element of the first *row* of the matrix associated with M, $\begin{bmatrix} 1 & 2 \end{bmatrix}$, by the corresponding element of the first *column* of the matrix associated with P, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$: $(1)(-2) + (2)(1) = 0$, which happens to be the first row, first column element of the matrix corresponding to $M \circ P$. Similarly, for example, combining the second row of the matrix associated with M, $\begin{pmatrix} 3 & 4 \end{pmatrix}$ and the first column of the matrix associated with P, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, we get the element in the second row, first column of $M \circ P : (3)(-2) + (4)(1) = -2$. In this way we can describe the matrix for $M \circ P$ as the **product** of the matrix representing M and the matrix representing P.

In general, if A and B are both 2×2 matrices, we find the element in row i and column j of the product matrix $C = AB$ by adding the products of each element of row i of matrix A and the corresponding element in column j of matrix B. Here's what the matrix product corresponding to $M \circ P$ in the last example looks like in full:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} (1)(-2) + (2)(1) & (1)(0) + (2)(3) \\ (3)(-2) + (4)(1) & (3)(0) + (4)(3) \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -2 & 12 \end{bmatrix}.$$

We should be able to calculate the matrix product corresponding to $P \circ M$. We notice that the *order* of composition/multiplication is important: The matrix corresponding to $M \circ P$ is not necessarily the matrix corresponding to $P \circ M$. In general, **matrix multiplication is not commutative**: *If A and B are two matrices that can be multiplied (see the next paragraph), then $AB \neq BA$ in general.*

This situation of one function or transformation followed by another is the motivation for matrix multiplication. The general multiplication of matrices remains the row-by-column procedure described for 2×2 matrices. In order for us to calculate the matrix product $C = AB$, the number of columns of A must be the same as the number of rows of B. Let $C = AB$, where A is $m \times r$ and B is $r \times n$. Then the product is a matrix with m rows and n columns:

$$A \cdot B = C$$
$$(m \times r) \cdot (r \times n) = m \times n$$

Thus, if A is a 3 by 5 matrix and B is a 5 by 7 matrix, we can find the product AB, which will be a 3 by 7 matrix. However, the product BA does *not* make sense because the number of columns of B (7) does not equal the number of rows of A (3).

If the sizes of A and B are compatible, as described previously, then c_{ij}, the element in row i and column j of the product matrix C, is just the sum of the products of each element of row i of matrix A and the corresponding element in column j of matrix B. Letting a_{ik} denote the entry in row i and column k of matrix A, and letting b_{kj} denote the element in row k and column j of matrix B, we can write the last sentence more concisely as

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}. \tag{B.3.1}$$

Schematically, we can represent this matrix multiplication as follows:

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \dots & \boxed{c_{ij}} & \dots & c_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mj} & \dots & c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \dots \quad a_{ir}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{r1} & b_{r2} & \dots & b_{rj} & \dots & b_{rn} \end{bmatrix}.$$

Here are some more examples of matrix multiplication.

Example

$$\begin{bmatrix} 2 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2(1) - 3(3) + 0(5) & 2(2) - 3(4) + 0(6) \\ 4(1) + 0(3) + 1(5) & 4(2) + 0(4) + 1(6) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -8 \\ 9 & 14 \end{bmatrix},$$

$$\begin{bmatrix} \pi & -2 & 6 \\ 0 & 4 & 1 \\ -3 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 9 & 2 & -6 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \pi(2) - 2(9) + 6(2) & \pi(-3) - 2(2) + 6(1) & \pi(0) - 2(-6) + 6(4) \\ 0(2) + 4(9) + 1(2) & 0(-3) + 4(2) + 1(1) & 0(0) + 4(-6) + 1(4) \\ -3(2) + 5(9) + 7(2) & -3(-3) + 5(2) + 7(1) & -3(0) + 5(-6) + 7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2\pi - 6 & -3\pi + 2 & 36 \\ 38 & 9 & -20 \\ 53 & 26 & -2 \end{bmatrix}.$$

A particularly important and useful 2×2 matrix is the **identity matrix** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We will see that this matrix plays the same role in matrix algebra that the number 1 plays in arithmetic—that is, $I \cdot A = A \cdot I$ for any 2×2 matrix A. If matrix A is $2 \times n$, then $I \cdot A = A$, but $A \cdot I$ is not defined unless $n = 2$. Similarly, if A is an $n \times 2$ matrix, then $A \cdot I = A$, but $I \cdot A$ is not defined unless $n = 2$. In general, for any positive integer n, the $n \times n$ matrix with ones on the main diagonal (upper left corner to lower right corner) and zeros elsewhere serves as the identity matrix I for $n \times n$ matrix multiplication.

Given an $n \times n$ matrix A, the $n \times n$ matrix B is called the (*multiplicative*) *inverse* of A if $AB = I = BA$. If an inverse of A exists, then it is unique and is denoted by A^{-1}.

With the definitions we have seen, matrix addition and multiplication satisfy all the familiar basic rules of algebra, except for commutativity. For example, we have the *associative law for multiplication*: If A is an $m \times r$ matrix, B is an $r \times s$ matrix, and C is an $s \times n$ matrix, then $A(BC) = (AB)C$, an $m \times n$ matrix. We also have the *distributive law*: If A is an $m \times r$ matrix and B and C are $r \times n$ matrices, then $A(B + C) = AB + AC$ (which is an $m \times n$ matrix).

Let's prove the distributive law in the situation in which A is an $m \times r$ matrix and B and C are $r \times 1$ matrices (vectors). We expect the product $A(B + C)$ to be a vector having m rows. Now suppose that a_{ik} denotes the element in row i, column k of A, that b_k and c_k are the elements in row $k (k = 1, 2, \ldots, r)$, and that p_i is the element in row i of the product $A(B + C)$. Then by Eq. (B.3.1) we have (for $i = 1, 2, \ldots, m$)

$$\begin{aligned} p_i &= \sum_{k=1}^{r} a_{ik}\,(b_k + c_k) = \sum_{k=1}^{r} (a_{ik}\,b_k + a_{ik}\,c_k) = \sum_{k=1}^{r} a_{ik}\,b_k + \sum_{k=1}^{r} a_{ik}\,c_k \\ &= (\text{the entry in row } i \text{ of } AB) + (\text{the entry in row } i \text{ of } AC), \end{aligned}$$

so we have shown that $A(B + C) = AB + AC$.

B.4 Eigenvalues and eigenvectors

As we saw in the previous section, if A is an $n \times n$ matrix and $\mathbf{X}$ is a nonzero $n \times 1$ vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we can consider the multiplication of $\mathbf{X}$ by matrix A in the form $A\mathbf{X}$ as somehow transforming or changing the vector $\mathbf{X}$. If there is a scalar λ such that $A\mathbf{X} = \lambda\mathbf{X}$, then λ is called an **eigenvalue** of A, and the vector $\mathbf{X}$ is called an **eigenvector** corresponding to λ. Geometrically, we're saying that *an eigenvector is a nonzero vector that gets changed into a constant multiple of itself*. For example, identifying a vector $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ with the point (x, y) in the familiar Cartesian coordinate system, we can see that an eigenvector is a point (not the origin) such that it and its transformed self lie on the same straight line through the origin. The direction of an eigenvector is either unchanged (if $\lambda > 0$) or reversed (if $\lambda < 0$) when the vector is multiplied by A. The matrix equation $A\mathbf{X} = \lambda\mathbf{X}$ is like the functional equation $f(x) = \lambda x$, which represents a straight line through the origin with slope λ.

If we start with the assumption that $A\mathbf{X} = \lambda\mathbf{X}$, then $A\mathbf{X} - \lambda\mathbf{X} = \mathbf{0}$ (the zero vector), and the distributive property of matrix multiplication allows us to write $(A - \lambda I)\mathbf{X} = \mathbf{0}$. (We must write $A - \lambda I$ instead of $A - \lambda$ because it doesn't make sense to subtract a number from a matrix.) If we can find an inverse for $A - \lambda I$—that is, an $n \times n$ matrix B such that $(A - \lambda I)B = I = B(A - \lambda I)$—then we can cancel the factor $A - \lambda I$ out of the matrix equation $(A - \lambda I)\mathbf{X} = \mathbf{0}$ to get $\mathbf{X} = \mathbf{0}$, the $n \times 1$ vector all of whose elements are 0. Therefore, remembering that an eigenvector was defined as a *nonzero* vector, we see that the only interesting situation occurs when the matrix $A - \lambda I$ does *not* have an inverse. (*Do you follow the logic?*)

The equation $(A - \lambda I)\mathbf{X} = \mathbf{0}$ represents a homogeneous system of n algebraic linear equations in n unknowns, and the theory of linear algebra indicates that there is a number Δ, depending on the matrix $A - \lambda I$, with the following important property: If $\Delta \neq 0$, then the system $(A - \lambda I)\mathbf{X} = \mathbf{0}$ has only the zero solution $x_1 = x_2 = \ldots = x_n = 0$. However, if $\Delta = 0$, then there is a solution $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ with at least one of the x_i different from zero. This number Δ is the **determinant** of the matrix $A - \lambda I$, denoted by $\det(A - \lambda I)$. Therefore $(A - \lambda I)\mathbf{X} = \mathbf{0}$ has a nonzero solution $\mathbf{X}$ only if $\det(A - \lambda I) = 0$. For any $n \times n$ linear system (homogeneous or not), the nature of the solutions depends on (that is, is *determined* by) whether the determinant is zero. The determinant is often calculated by means of successive operations on the rows and/or columns of the matrix. (*Rather than spend time learning tedious algorithms for finding determinants, you should learn how to get these numbers from your CAS. Even a graphing calculator will evaluate a determinant if the matrix is not too large.*) From a more abstract point of view, a determinant is just a special kind of function from a set of square matrices to the real numbers.

For now, let's see how a determinant arises in solving a simple system of algebraic equations.

Example

Suppose we want to solve the following system of two equations in two unknowns:

$$\begin{aligned} 2x - 3y &= 2 \\ x + 4y &= -5. \end{aligned}$$

We can use the method of *elimination* to solve this system. For example, we can subtract twice the second equation from the first equation to eliminate the variable x and get $-11y = 12$, or $y = -\frac{12}{11}$. Then we can substitute this value of y in the second equation and solve for x. We get $x = -\frac{7}{11}$. Note that when we solve this particular system by elimination, the denominator of each component of the solution is 11.

Now write the system in matrix form:

$$\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

What do we get if we take the matrix of coefficients, multiply the *main diagonal* (upper left, lower right) elements 2 and 4, and then subtract the product of the other diagonal elements -3 and 1? We get $(2)(4) - (-3)(1) = 11$. (*Surprise!*) The number calculated this way is the determinant of the coefficient matrix. When solving any system of linear equations in two unknowns, we always wind up dividing by the determinant—if it's not zero. **Cramer's Rule**, which you may have seen in a college algebra course, is a general $n \times n$ linear system solution formula that uses determinants.

For a larger system, a CAS or graphing calculator provides important information about a system easily. Let's use technology in the next example to calculate the determinant, eigenvalues, and eigenvectors for a three-dimensional system.

Example

Suppose we have a system with the matrix of coefficients

$$A = \begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix}.$$

A CAS (*Maple* in this case) tells us that $\det(A) = 24$ and that the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -2 = \lambda_3$. The corresponding (linearly independent) eigenvectors are

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

(*If you try this example using your own CAS, the eigenvectors may not look like those here, but each should be a constant multiple of one of those given previously.*)

APPENDIX

Complex numbers

C

C.1 Complex numbers: the algebraic view

Historically, the need for complex numbers arose when people tried to solve equations such as $x^2 + 1 = 0$ and realized that there was no ordinary number that satisfied this equation. The basic element in the expansion of the number system is the **imaginary unit**, $i = \sqrt{-1}$, a solution of $x^2 + 1 = 0$. There is an interesting pattern to the powers of i: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1, \ldots$. We can use this repetition in groups of four, for example, to calculate a high power of i: $i^{338} = (i^2)^{169} = (-1)^{169} = -1$. A **complex number** is any expression of the form $x + yi$, where x and y are real numbers. If we have a complex number $z = x + yi$, then x is called the **real part**, denoted $\mathrm{Re}(z)$, and y is called the **imaginary part**, denoted $\mathrm{Im}(z)$, of the complex number. (Note that despite its name, y is a real number.) In particular, any real number x is a member of the family of complex numbers because it can be written as $x + 0 \cdot i$. Any complex number of the form yi $(= 0 + yi)$ is called a **pure imaginary number**.

Complex numbers can be added and subtracted in a reasonable way by combining real parts and imaginary parts as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

We can also multiply complex numbers as we would multiply any binomial in algebra, remembering to replace i^2 whenever it occurs by -1:

$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Division of complex numbers is a bit trickier. If $z = x + yi$ is a complex number, then its **complex conjugate**, $\overline{z}$, is defined as $\overline{z} = x - yi$. (Just reverse the sign of the imaginary part.) The complex conjugate is important in division because $z \cdot \overline{z} = x^2 + y^2$, a real number. (*Check this out.*) In division of complex numbers, the conjugate plays much the same role as the conjugate we learned to use in algebra to simplify fractions; for example, if we were asked to simplify the fraction $\frac{3}{\sqrt{5}}$, we would rationalize the denominator as follows:

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Another example from algebra makes the similarity between conjugates more obvious:

$$\frac{2+\sqrt{3}}{3-\sqrt{2}} = \frac{2+\sqrt{3}}{3-\sqrt{2}} \cdot \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{6+2\sqrt{2}+3\sqrt{3}+\sqrt{6}}{9-2}$$
$$= \frac{6+2\sqrt{2}+3\sqrt{3}+\sqrt{6}}{7}.$$

In the preceding example, $3+\sqrt{2}$ is the *conjugate* of $3-\sqrt{2}$; when we multiply these conjugates, the radical sign disappears, leaving us with the integer 7. Now if we have to divide two complex numbers, we use the complex conjugate to get the answer, the quotient, to look like a complex number. For example,

$$\frac{2+3i}{3+5i} = \frac{2+3i}{3+5i} \cdot \frac{3-5i}{3-5i} = \frac{21-i}{9+25} = \frac{21}{34} - \frac{1}{34}i.$$

In general, if $z = a+bi$ and $w = c+di$, then

$$\frac{z}{w} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

If z and w are complex numbers, we should be able to see that $\overline{\overline{z}} = z$, $\overline{(z+w)} = \overline{z}+\overline{w}$, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, and $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ for $w \neq 0$. Also, $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

The important algebraic rules of commutativity, associativity, and distributivity work for complex numbers. Furthermore, all the properties in this section extend to vectors and matrices (Appendix B) with complex-number entries. For example, if $\mathbf{V} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a vector with complex components, then $\overline{\mathbf{V}} = \overline{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}} = \begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \\ \vdots \\ \overline{c}_n \end{bmatrix}$. If $A = [a_{ij}]$ represents a matrix with entry a_{ij} in row i and column j, then $\overline{A} = \overline{[a_{ij}]} = [\overline{a}_{ij}]$.

C.2 Complex numbers: the geometric view

The geometric interpretation of complex numbers occurred at roughly the same time to three people: the Norwegian surveyor and map maker, Caspar Wessel (1745–1818); the French-Swiss mathematician, Jean Robert Argand (1768–1822); and the German mathematician-astronomer-physicist, Karl Friedrich Gauss (1777–1855).

The idea here is to represent a complex number using the familiar Cartesian coordinate system, making the horizontal axis the **real axis** and the vertical axis the **imaginary axis**. Such a system is called the **complex plane**. For example, Fig. C.1 shows how the complex number $3+2i$ would be represented as a point in this way.

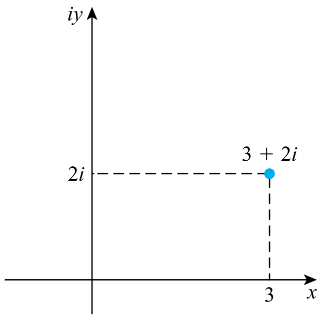


FIGURE C.1

Representation of a complex number

If we join this point to the origin with a straight line, we get a vector. (See Section B.1.) The sum of $z=a+bi$ and $w=c+di$ corresponds to the point (or vector) $(a+c, b+d)$. This implies that the addition/subtraction of complex numbers corresponds to the **Parallelogram Law** of vector algebra (Fig. C.2).

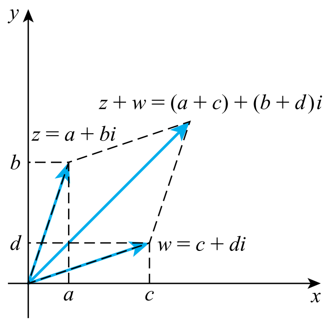


FIGURE C.2

The Parallelogram Law

The **modulus**, or **absolute value**, of the complex number $z=x+yi$, denoted $|z|$, is the nonnegative real number defined by the equation $|z|=\sqrt{x^2+y^2}$. The number $|z|$ represents the distance between the origin and the point (x, y) in the complex plane, the length of the vector representing the complex number $z=x+yi$. Note that $|z|^2=z\cdot\overline{z}$.

C.3 The quadratic formula

Given the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, the solutions are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression inside the radical sign, $b^2 - 4ac$, is called the **discriminant** and enables us to discriminate among the possibilities for solutions. If $b^2 - 4ac > 0$, the quadratic formula yields two real solutions. If $b^2 - 4ac = 0$, we get a single repeated solution, a solution of *multiplicity two*. Finally, if $b^2 - 4ac < 0$, the quadratic formula produces two complex numbers as solutions, a **complex conjugate pair**. To see this last situation, suppose that $b^2 - 4ac = -q$, where q is a positive real number. Then the solution formula looks like

$$x = \frac{-b \pm \sqrt{-q}}{2a} = \frac{-b \pm \sqrt{q(-1)}}{2a} = \frac{-b \pm \sqrt{q}\, i}{2a},$$

so the two solutions are $x_1 = -\frac{b}{2a} + \frac{\sqrt{q}}{2a} i$ and $x_2 = -\frac{b}{2a} - \frac{\sqrt{q}}{2a} i$, which are complex conjugates of each other.

C.4 Euler's formula

Around 1740, while studying differential equations of the form $y'' + y = 0$, Euler discovered his famous formula for complex exponentials:

$$e^{iy} = \cos y + i \sin y.$$

If $z = x + iy$, then we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos\ y + i \sin\ y).$$

Without fully understanding the way infinite series worked, Euler just substituted the complex number iy in the series for e^x (see Section A.3) and then separated the real and imaginary parts:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \cdots \\ &= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}_{\cos y} + i \underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}_{\sin y} = \cos y + i \sin y. \end{aligned}$$

APPENDIX

Series solutions of differential equations

D

Appendix D supplements the treatment of linear equations in Chapters 4, 5, and 6.

D.1 Power series solutions of first-order equations

In Chapters 4, 5, and 6 we discussed solutions for second- and higher-order linear equations with constant coefficients. The methods we discuss in this appendix can be applied to equations—not necessarily linear—with *variable* coefficients (see Section 5.6), equations that in general do not yield closed-form solutions. Among these are equations important in many areas of applied mathematics.

As an illustration of the key idea, we'll solve a simple first-order equation.

Example

Consider the equation $y' = 1 - xy$. We make the fundamental assumption that a solution y can be expanded in a power series (Taylor series, Maclaurin series)

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

that converges in some interval. (See Section A.3 for the basics.) We have chosen an interval around the origin.

Then, because a convergent power series can be differentiated term by term within its interval of convergence (see Section A.3),

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots .$$

Substituting these last two series in the differential equation, we have

$$\begin{aligned} &a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\ &\quad = 1 - x\left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots\right\} \\ &\quad = 1 - a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - \cdots - a_n x^{n+1} - \cdots . \end{aligned}$$

Because these power series are equal, coefficients of equal powers of x on both sides must be equal. (This is really the *method of undetermined coefficients* that we first saw in Section 4.3 of the text.) Therefore, we have

$$a_1 = 1, 2a_2 = -a_0, 3a_3 = -a_1, 4a_4 = -a_2, 5a_5 = -a_3, \ldots, n a_n = -a_{n-2}, \ldots,$$

so

$$a_1 = 1,\ a_2 = -\frac{a_0}{2},\ a_3 = -\frac{a_1}{3} = -\frac{1}{3},\ a_4 = -\frac{a_2}{4} = -\frac{-\frac{a_0}{2}}{4} = \frac{a_0}{2\cdot 4},$$
$$a_5 = -\frac{a_3}{5} = \frac{1}{3\cdot 5}, \ldots,\ a_n = -\frac{a_{n-2}}{n}, \ldots.$$

These formulas, in which we define later coefficients by relating them to earlier coefficients, are called **recurrence** (or **recursion**) **relations**. If we look carefully, we see that for odd indices (subscripts), the pattern is

$$a_1 = 1,\ a_3 = -\frac{1}{3},\ a_5 = \frac{1}{3\cdot 5},\ a_7 = -\frac{1}{3\cdot 5\cdot 7}, \ldots.$$

Similarly, for even indices we find the pattern

$$a_0 = \text{arbitrary},\ a_2 = -\frac{a_0}{2},\ a_4 = \frac{a_0}{2\cdot 4},\ a_6 = -\frac{a_0}{2\cdot 4\cdot 6}, \ldots.$$

In general, the pattern is

$$a_{2k} = \frac{(-1)^k a_0}{2\cdot 4\cdot 6\cdots(2k)} \quad \text{for} \quad k = 1, 2, 3, \ldots;$$
$$a_{2k+1} = \frac{(-1)^k}{1\cdot 3\cdot 5\cdot 7\cdots(2k+1)} \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

Therefore, we can write the power series form of the solution as

$$\begin{aligned} y(x) &= a_0 + x - \frac{a_0}{2}x^2 - \frac{1}{1\cdot 3}x^3 + \left(\frac{a_0}{2\cdot 4}\right)x^4 + \frac{1}{1\cdot 3\cdot 5}x^5 + \cdots \\ &= \left(x - \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} - \cdots\right) + a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4} - \cdots\right), \end{aligned}$$

where $a_0 = y(0)$ is the arbitrary constant that we expect in the general solution of a first-order equation.

To approximate $y(x)$ for a value of x close to zero, we just substitute the value in the series, taking as many terms of this series as are needed to guarantee the accuracy we wish.

If we solve the linear equation in the preceding example using the technique of integrating factors (see Section 2.2.2), we get the answer

$$y = e^{x^2/2}\int e^{-x^2/2}\,dx + Ce^{x^2/2},$$

which can't be expressed in a more elementary way. If we integrate the power series representation of $e^{-x^2/2}$ term by term, multiply by the series form of $e^{x^2/2}$, and then add the series for $Ce^{x^2/2}$, we get the same series solution we found (after collecting terms).

When using this power series method sometimes we can recognize the series in our solution as a representation of an elementary function. (Try using the method on the equation $y' = ay$, where a is a constant, for example. You should recognize the series solution as the Taylor series representation of Ce^{ax} about the origin.)

All computer algebra systems have the ability to work with series expansions, usually truncating the series after a fixed number of terms that the user can control. However, not all systems can give us a power series solution of an ODE directly. For example, *Maple* has a very useful power series package, *powseries*, and the *dsolve* command (in the package *DEtools*) has a *series* option, but other computer algebra systems may require the user to do much more work in finding a series solution.

D.2 Series solutions of second-order linear equations: ordinary points

In this section we'll examine second-order linear equations of the form

$$a(t)y'' + b(t)y' + c(t)y = 0, \tag{D.2.1}$$

where $a(t)$, $b(t)$, and $c(t)$ are polynomial functions. We divide through by $a(t)$ and write Eq. (D.2.1) in the *standard form*

$$y'' + P(t)y' + Q(t)y = 0, \tag{D.2.2}$$

where $P(t) = \frac{b(t)}{a(t)}$ and $Q(t) = \frac{c(t)}{a(t)}$.

A point t_0 is called an **ordinary point** of Eq. (D.2.2) if both P and Q can be expanded in power series centered at t_0 that converge for every t in an open interval containing t_0. Functions that have such power series representations are called **analytic** at the point t_0. If t_0 is not an ordinary point, it is called a **singular point** of the equation.

Example

The point $t = 0$ is an ordinary point of the equation $(t+2)y'' + t^2y' + y = 0$ because each of the functions $P(t) = \frac{t^2}{t+2}$ and $Q(t) = \frac{1}{t+2}$ has its own power series expansion that converges near $t = 0$:

$$Q(t) = \frac{1}{2} - \frac{t}{4} + \frac{t^2}{8} - \frac{t^3}{16} + \cdots \quad \text{and} \quad P(t) = \frac{t^2}{2} - \frac{t^3}{4} + \frac{t^4}{8} - \frac{t^5}{16} + \cdots.$$

(See the *geometric series* in Section A.3.) However, $t = -2$ is a singular point because the denominators of $P(t)$ and $Q(t)$ are zero at $t = -2$.

Let's apply the undetermined coefficient method of the last section to a famous second-order linear equation near an ordinary point. The equation is named for the English mathematician Sir George Bidell Airy (1801–92), who did pioneering work in elasticity and in partial differential equations.

Example

Airy's equation, $y'' + xy = 0$, which occurs in the study of optics and quantum physics, cannot be solved in terms of elementary functions. We can think of the equation as describing a spring-mass system in which the stiffness of the spring is increasing with time. (Perhaps the room containing the system is getting colder.)

Noting that $x = 0$ is an ordinary point of this equation, we assume that we can write a solution as

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots.$$

Then

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots$$

and

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots + n(n-1)a_nx^{n-2} + \cdots.$$

Substituting in the differential equation, we get

$$\begin{aligned}&\left(2a_2 + 6a_3x + 12a_4x^2 + \cdots + n(n-1)a_nx^{n-2} + \cdots\right)\\ &\quad + x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots\right) = 0.\end{aligned}$$

Collecting terms, we can write this last equation as

$$\begin{aligned}&2a_2 + (6a_3 + a_0)\,x + (12a_4 + a_1)\,x^2 + \cdots\\ &\quad + \left(n(n-1)a_n + a_{n-3}\right)x^{n-2} + \cdots = 0.\end{aligned}$$

Equating coefficients of equal powers of x, we see that the preceding equation implies that

$$\begin{aligned}&2a_2 = 0,\ \text{or } a_2 = 0;\ 6a_3 + a_0,\ \text{or } a_3 = -\frac{a_0}{2\cdot 3};\\ &12a_4 + a_1 = 0,\ \text{or } a_4 = -\frac{a_1}{3\cdot 4};\ 20a_5 + a_2 = 0,\ \text{or } a_5 = -\frac{a_2}{4\cdot 5},\end{aligned}$$

and so forth, so we can see the recurrence relation as $a_n = -\frac{a_{n-3}}{(n-1)\cdot n}$ for $n = 3, 4, 5, \ldots$. Note that a_0 and a_1 are arbitrary and that the coefficients are connected by jumps of three in the subscripts. In particular we have $0 = a_2 = a_5 = a_8 = \cdots = a_{2+3k} = \cdots$. We can also see the pattern when the subscript is a multiple of 3:

$$\begin{aligned}&a_3 = -\frac{a_0}{2\cdot 3},\quad a_6 = -\frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},\quad a_9 = -\frac{a_6}{8\cdot 9} = -\frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9},\\ &a_{12} = -\frac{a_9}{11\cdot 12} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdot 11\cdot 12},\end{aligned}$$

and so forth, so the formula is

$$a_{3k} = \frac{(-1)^k a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3k-1)\cdot 3k}.$$

Similarly, we can see that

$$a_4 = -\frac{a_1}{3\cdot 4}, \quad a_7 = -\frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7},$$
$$a_{10} = -\frac{a_7}{9\cdot 10} = -\frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10},$$
$$a_{13} = -\frac{a_{10}}{12\cdot 13} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdot 12\cdot 13},$$

and so forth, so the recurrence relation is

$$a_{3k+1} = \frac{(-1)^k a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots (3k)\cdot (3k+1)}.$$

Putting all the pieces together, we get

$$\begin{aligned} y(x) &= a_0\left[1 - \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} - \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right] \\ &\quad + a_1\left[x - \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} - \frac{x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right] \\ &= y(0)\left[1 - \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} - \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right] \\ &\quad + y'(0)x\left[1 - \frac{x^3}{3\cdot 4} + \frac{x^6}{3\cdot 4\cdot 6\cdot 7} - \frac{x^9}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right] \\ &= y(0)\cdot Ai(x) + y'(0)x\cdot Bi(x), \end{aligned}$$

where the two series (convergent for all values of x) define $Ai(x)$ and $Bi(x)$, the **Airy functions of the first and second kind**, respectively, up to constant multiplicative factors.

With the aid of technology, we can look at the graph of the solution of Airy's equation with initial conditions $y(0) = 0$, $y'(0) = 1$ (Fig. D.1), which is just the graph of $xBi(x)$.

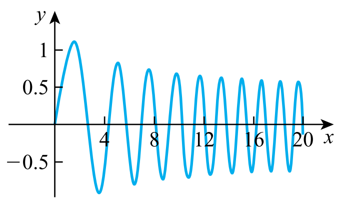


FIGURE D.1

Solution of $y'' + xy = 0$; $y(0) = 0$, $y'(0) = 1$

Both *Maple* and *Mathematica*, for example, have built-in capabilities to deal with Airy functions numerically and graphically. (See the commands $AiryAi(x)$ and $AiryBi(x)$ in *Maple* or $AiryAi[x]$ and $AiryBi[x]$ in *Mathematica*.)

If we want to find a solution near an ordinary point t_0 other than zero, we can use the substitution $u = t - t_0$. This substitution transforms the equation in t to one in the variable u, which we can solve near the ordinary point $u = 0$. When we have solved the equation in u, we can just replace u by $t - t_0$ to return to the original variable.

The method of undetermined coefficients also applies to nonhomogeneous equations and to equations whose coefficients are not polynomials, provided that the function on the right-hand side and the coefficient functions can be expanded in powers of t. When we are trying to solve a nonhomogeneous equation, equating coefficients becomes a little more difficult because some of the coefficients of the solution series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ will include numerical values independent of the two arbitrary constants a_0 and a_1. This part of the general solution y_{GNH} constitutes y_{PNH}. (Check this out for yourself by using series to solve the equation $y'' - y = e^x$. You should recognize your solution as $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$.)

D.3 Regular singular points: the method of Frobenius

Some singular points are such that special series methods have been developed to handle situations in which they occur. The point t_0 is a **regular singular point** of $y'' + P(t)y' + Q(t)y = 0$ if t_0 is a singular point and the functions $(t - t_0)P(t)$ and $(t - t_0)^2 Q(t)$ are both analytic at t_0. If t_0 is a singular point that is not regular, it is called an **irregular singular point**.

For example, $t = 1$ is a singular point of the equation $(t^2 - 1)^2 y'' + (t - 1)y' + y = 0$ because $P(t) = \frac{t-1}{(t^2-1)^2} = \frac{t-1}{(t+1)^2(t-1)^2}$ and $Q(t) = \frac{1}{(t+1)^2(t-1)^2}$ have zero denominators at $t = 1$, so neither $P(t)$ nor $Q(t)$ has a convergent power series expansion in a neighborhood of 1. But if we look at $(t - 1)P(t) = \frac{(t-1)^2}{(t+1)^2(t-1)^2} = \frac{1}{(t+1)^2}$ and $(t - 1)^2 Q(t) = \frac{(t-1)^2}{(t+1)^2(t-1)^2} = \frac{1}{(t+1)^2}$, we see that both $(t - 1)P(t)$ and $(t - 1)^2 Q(t)$ are analytic at $t = 1$, so $t = 1$ is a regular singular point.

Near a regular singular point, say $t = 0$ for convenience, we write Eq. (D.2.2) as

$$t^2 y'' + t p(t) y' + q(t) y = 0, \tag{D.3.1}$$

where $p(t) = tP(t)$ and $q(t) = t^2 Q(t)$. Because $t = 0$ is a regular singular point, p and q are analytic at $t = 0$. The usual power series method will not work, and we use the **method of Frobenius**,[1] which produces at least one solution of the form

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}, \tag{D.3.2}$$

where we assume that $a_0 \neq 0$.

[1] The German mathematician Ferdinand Georg Frobenius (1849–1917) published his method in 1878. It was based on a technique that originated with Euler (*who else?*). Frobenius made many contributions to analysis and especially to algebra.

It is important to note that three of the most popular computer algebra systems (*Maple, Mathematica*, and MATLAB) cannot apply the method of Frobenius directly to get power series solutions near regular singular points. We would have to develop a solution in a step-by-step fashion, using the capabilities of the system to handle power series and recursion relations.

We'll illustrate the method of Frobenius using a famous equation in applied mathematics, one that first arose in an investigation of the motion of a hanging chain and has since appeared in such problems as the analysis of vibrations of a circular membrane and planetary motion.

Example

Bessel's equation of order *p* is $x^2y'' + xy' + \left(x^2 - p^2\right)y = 0$, which is of the form (D.3.1) and has $x = 0$ as a regular singular point.[2] We'll take the parameter p to be an arbitrary nonnegative real number.

Substituting the type of series given in (D.3.2) for y, we find that

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2},$$

so that we have

$$\begin{aligned}
x^2y'' + xy' + \left(x^2 - p^2\right)y &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n p^2 x^{n+r} \\
&= \sum_{n=0}^{\infty} \left\{a_n(n+r)(n+r-1) + a_n(n+r) - a_n p^2\right\} x^{n+r} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\
&= \sum_{n=0}^{\infty} \left\{(n+r)^2 - p^2\right\} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.
\end{aligned}$$

[2] Among other achievements, the German astronomer Friedrich Wilhelm Bessel (1784–1846) was the first to measure accurately the distance to a fixed star.

Transposing series and making the substitution (actually a shift of subscripts) $n+2=n$ on the right-hand side, we get

$$\sum_{n=0}^{\infty}\left\{(n+r)^2-p^2\right\}a_n x^{n+r}=-\sum_{n=0}^{\infty}a_n x^{n+r+2}=-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}.$$

Now we equate coefficients of equal powers. To start, we have

$$n=0:\ \left(r^2-p^2\right)a_0=0$$
$$n=1:\ \left[(1+r)^2-p^2\right]a_1=0.$$

Because we have assumed that $a_0\neq 0$, we must have $r^2-p^2=0$. This last equation is called the **indicial equation**[3] and implies that $r=\pm p$.

Let's assume that $r=p\geq 0$. Then when $n=1$, the equation $\left[(1+r)^2-p^2\right]a_1=0$ reduces to $(2r+1)a_1=0$, so we can conclude $a_1=0$.

For $n\geq 2$, equating coefficients of equal powers of x gives us the recurrence relation

$$\left\{(n+r)^2-p^2\right\}a_n=-a_{n-2}$$

or

$$a_n=-\frac{a_{n-2}}{\left\{(n+r)^2-p^2\right\}}=-\frac{a_{n-2}}{n(n+2r)}$$

because $r^2-p^2=0$. We can look at a few terms to see the pattern:

$$a_2=-\frac{a_0}{2(2+2r)}=-\frac{a_0}{2^2(1+r)},$$
$$a_3=-\frac{a_1}{3(3+2r)}=0,\quad [\text{because } a_1=0]$$
$$a_4=-\frac{a_2}{4(4+2r)}=-\frac{\left(\frac{-a_0}{2^2(1+r)}\right)}{2\cdot 2^2(2+r)}=\frac{a_0}{2^4 2!(1+r)(2+r)},$$
$$a_5=-\frac{a_3}{5(5+2r)}=0,$$
$$a_6=-\frac{a_4}{6(6+2r)}=-\frac{\left(\frac{a_0}{2^4 2!(1+r)(2+r)}\right)}{6(6+2r)}=-\frac{a_0}{2^6 3!(1+r)(2+r)(3+r)}.$$

We can see, for example, that $a_k=0$ for k odd.

Letting $n=2k$ and remembering that we're assuming $r=p$, we can express the even coefficients in the form

$$a_{2k}=\frac{(-1)^k a_0}{2^{2k}k!(r+1)(r+2)\cdots(r+k)}$$
$$=\frac{(-1)^k a_0}{2^{2k}k!(p+1)(p+2)\cdots(p+k)}.$$

[3] In general, for the method of Frobenius, the indicial equation has the form $r(r-1)+r\,p_0+q_0=0$, where p_0 and q_0 are the constant terms of the series expansions of $p(t)$ and $q(t)$ in Eq. (D.3.1).

In working with Bessel's equation, it is common practice to make things neater by taking $a_0 = \frac{1}{2^p p!}$,[4] so that

$$a_{2k} = \frac{(-1)^k}{2^{2k+p}k!(p+k)!}.$$

The final result is the **Bessel function of order *p* of the first kind**, $J_p(x)$:

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p} n!(p+n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!(p+n)!}$$
$$= \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!(p+n)!}.$$

It can be shown that this series converges for all real values of x.

Using technology, we can produce a graph of Bessel functions of order p for $p = 0, 1, 2, 3,$ and 4 (Fig. D.2).

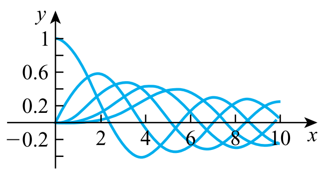


FIGURE D.2

$J_p(x)$ for $p = 0, 1, 2, 3, 4;\ 0 \le x \le 10$

Both *Maple* and *Mathematica*, for example, can deal with Bessel functions of the first kind numerically and graphically via the command *BesselJ* (mu, x) in *Maple* or *BesselJ* $[m, x]$ in *Mathematica*. The parameter mu or m represents the order that we have called p. It is interesting to note that a computer algebra system (CAS) could express the solution of the IVP $y'' + xy = 0$; $y(0) = 0$, $y'(0) = 1$ that we considered in Section D.2 as

$$y(x) = \frac{2}{9} \frac{3^{5/6}\pi}{\Gamma\left(\frac{2}{3}\right)} \sqrt{x}\, \mathit{BesselJ} \left(\frac{1}{3}, \frac{2x^{3/2}}{3}\right).$$

We should make several comments about the preceding example:

1. In our analysis, we have actually assumed that $x > 0$ to avoid the possibility of fractional powers of negative numbers.
2. In the indicial equation $r^2 - p^2 = 0$, we have assumed that $r = p$, a nonnegative number. If r is in fact a nonnegative *integer*, then the Frobenius series is an ordinary power series with first term $a_0 x^n$. For applications, the choices $p = 0$ and $p = 1$ occur most often.

[4] Actually, $a_0 = \frac{1}{2^p \Gamma(p+1)}$, where Γ denotes Euler's *gamma function* (see Section A.6).

3. All our efforts have produced just one solution of Bessel's equation for a fixed value of p. It can be shown that when $2p$ is not a positive integer,

$$J_{-p}(x) = \left(\frac{2}{x}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!(-p+n)!}$$

defines a second, linearly independent solution of Bessel's equation. When p is an integer, it can be shown that $J_p(x) = (-1)^p J_{-p}(x)$, so the two solutions are dependent.
4. If p is not an integer, the general solution of Bessel's equation has the form $y(x) = k_1 J_p(x) + k_2 J_{-p}(x)$ for arbitrary constants k_1 and k_2.
5. The function

$$Y_p(x) = \frac{(\cos p\pi) J_p(x) - J_{-p}(x)}{\sin p\pi}$$

is the standard **Bessel function of the second kind**. Then

$$y(x) = c_1 J_p(x) + c_2 Y_p(x)$$

is the general solution of Bessel's equation in all cases, whether or not p is an integer. Both *Maple* and *Mathematica* have commands, *BesselY* (mu, x) and *BesselY* $[m, x]$, respectively, that enable users to explore Bessel functions of the second kind numerically and graphically.

There are many treatments of the properties and applications of Bessel functions. Accessible sources of information include the books *Differential Equations: Theory, Technique, and Practice* (*Second Edition*) by Steven G. Krantz (New York: Chapman & Hall/CRC, 2014) and *Handbook of Mathematical Formulas and Integrals* (*Fourth Edition*) by Alan Jeffrey and Hui Hui Dai (San Diego: Academic Press, 2008).

D.4 The point at infinity

In some situations we want to determine the behavior of solutions of the equation

$$y'' + P(t)y' + Q(t)y = 0$$

for large values of the independent variable t—the behavior "in the neighborhood of infinity." The way to deal with this problem is to use the substitution $t = \frac{1}{u}$ and investigate the resulting equation near $u = 0$. This substitution converts a problem in large values of t to one in small values of u. Once the "u-problem" is solved near $u = 0$, we make the substitution $t = \frac{1}{u}$ in the u-solution to get the solution near the t point of infinity.

Let $u = \frac{1}{t}$. Then, by the Chain Rule,

$$y' = \frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dt} = \frac{dy}{du}\left(-\frac{1}{t^2}\right) = -u^2.\frac{dy}{du}$$

and

$$y'' = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{du}\left(\frac{dy}{dt}\right)\cdot\frac{du}{dt} = \left(-u^2\frac{d^2y}{du^2} - 2u\frac{dy}{du}\right)\left(-u^2\right).$$

Let's use this transformation method to solve an equation for large values of the independent variable.

Example

Find the general solution of the equation

$$4t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt} + y = 0$$

for large values of t.

First, we write the equation in the standard form

$$\frac{d^2y}{dt^2} + \frac{3}{2t}\frac{dy}{dt} + \frac{1}{4t^3}y = 0.$$

Making the substitution $u = \frac{1}{t}$ and using the calculations for y' and y'' given previously, we transform our equation into

$$\left(-u^2\frac{d^2y}{du^2} - 2u\frac{dy}{du}\right)\left(-u^2\right) + \frac{3u}{2}\left(-u^2\frac{dy}{du}\right) + \frac{u^3}{4}y = 0,$$

or

$$4u\frac{d^2y}{du^2} + 2\frac{dy}{du} + y = 0,$$

which has $u = 0$ as a regular singular point.

If we use the Frobenius method, we find the general solution

$$\Psi(u) = c_1\sum_{n=0}^{\infty}\frac{(-1)^n u^n}{(2n)!} + c_2\sum_{n=0}^{\infty}\frac{(-1)^n u^{n+\frac{1}{2}}}{(2n+1)!}.$$

Substituting $u = \frac{1}{t}$, we get the solution

$$\begin{aligned} y(t) &= c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{t}\right)^n + c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{t}\right)^{n+\frac{1}{2}} \\ &= c_1\cos\left(\frac{1}{\sqrt{t}}\right) + c_2\sin\left(\frac{1}{\sqrt{t}}\right). \end{aligned}$$

(See Section A.3.)

D.5 Some additional special differential equations

Many famous functions, such as the Airy and Bessel functions, arise as power series solutions of second-order differential equations. These functions form a particular class of what are usually called **special functions**.

Some classic references are *Special Functions* by Earl D. Rainville (New York: Chelsea, 1971); *The Functions of Mathematical Physics* by Harry Hochstadt (New York: Dover, 2012); and *Solved Problems in Analysis: As Applied to Gamma, Beta, Legendre and Bessel Functions* by O.J. Farrell and B. Ross (New York: Dover, 2013). More modern books are *Special Functions* by George E. Andrews, Richard Askey, and Ranjan Roy (New York: Cambridge University Press, 1999) and *Special Functions and Orthogonal Polynomials* by Richard Beals and Roderick Wong (New York: Cambridge University Press, 2016).

Among these important second-order equations that have been significant in solving problems in applied mathematics, science, and engineering, which you are invited to investigate using the methods of this appendix, are the following:

Chebyshev's equation: $(1-x^2)y'' - xy' + p^2y = 0$, where p is a constant. (When p is a nonnegative integer, the solution is an nth-degree polynomial.)

Gauss's hypergeometric equation: $x(1-x)y'' + [c - (a+b+1)x]\,y' - aby = 0$, where a, b, and c are constants.

Hermite's equation: $y'' - 2xy' + 2py = 0$, where p is a constant.

Laguerre's equation: $xy'' + (1-x)y' + py = 0$, where p is a constant. [See Problems 9–10 in Exercises 5.6.]

Legendre's equation: $(1-x^2)y'' - 2xy' + \left[k(k+1) - \frac{m^2}{1-x^2}\right]y = 0$, where m and k are constants, $k > 0$.

Answers and hints to odd-numbered exercises

Exercises 1.1

1. **a.** The independent variable is x and the dependent variable is y.
b. First-order.
c. Linear.

3. **a.** The independent variable is not indicated, but the dependent variable is x.
b. Second-order.
c. Nonlinear because of the term $\exp(-x)$; the equation cannot be written in the form (1.1.1), where y is replaced by x and x is replaced by the independent variable.

5. **a.** The independent variable is x and the dependent variable is y.
b. First-order.
c. Nonlinear because you get the terms $x^2(y')^2$ and $x\,y'\,y$ when you remove the parentheses.

7. **a.** The independent variable is x and the dependent variable is y.
b. Fourth-order.
c. Linear.

9. **a.** The independent variable is t and the dependent variable is x.
b. Third-order.
c. Linear.

11. **a.** The independent variable is x, the dependent variable is y; first-order, nonlinear because of the exponent y'.

13. **a.** Nonlinear; the first equation is nonlinear because of the term $4xy = 4x(t)y(t)$.
b. Linear.
c. Nonlinear; the first and second equations are nonlinear because each contains a product of dependent variables.
d. Linear.

15. $a = 1$.

Exercises 1.2

11. **a.** For example, $c\,y' = 1$ is a possible differential equation satisfied by y.
b. For example, $y' - ay = be^{ax}\cos bx$.
c. For example, $y' - y = Be^t$. Other possibilities are the equations $y'' - y = 2Be^t$ and $y'' - y' = Be^t$.
d. One answer is $y' = -3\,e^{-3t} + t\,y$, or $y' - t\,y = -3\,e^{-3t}$.

13. $y' = \left(\frac{y^2+1}{y^2+2}\right)\left(\frac{x^2+2}{x^2+1}\right)$, a first-order nonlinear equation.

15. One solution is $y' = (-2x\,y)/(x^2+4)$.

19. **a.** The given equation is equivalent to $(y')^2 = -1$. Since there is no real-valued function y' whose square is negative, there can be no real-valued function y satisfying the equation.

b. The only way that two absolute values can have a sum equal to zero is if each absolute value is itself zero. This says that y is identically equal to zero, so that the zero function is the only solution. The graph of this solution is the x-axis (if the independent variable is x).

21. If $x > c$ or $x < -c$, then $c^2 - x^2 < 0$ and the functions $y = \pm\sqrt{c^2 - x^2}$ do not exist as real-valued functions. If $x = \pm c$, then each function is the zero function, which is not a solution of the differential equation.

23. One solution is $y(x) = (1/2)(\sin x - \cos x)$.

27. $y'(t) = -\sin t + \int_0^t y(u)du$ or $y'' - y = -\cos t$.

Exercises 1.3

1. $R(t) = -\pi\,(1 + \cos t)$.

3. $r(t) = (a/b)(e^{bt}/b - t - 1/b)$.

5. $A = -1/4$, $B = 1/37$, $C = -6/37$.

9. Recall the Product Rule and the Fundamental Theorem of Calculus (Section A.4).

11. No. An equation of order n requires an n-parameter family of solutions.

13. The length of the runway must be 5/6 mile (five-sixths of a mile).

15. **b.** $x(t) = (1-3t)e^{3t}$, $y(t) = -9te^{3t}$.

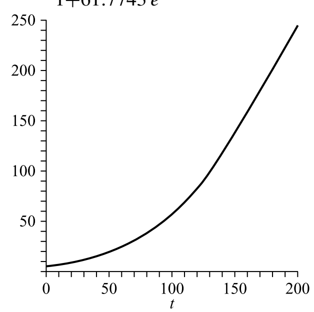
17. **a.** Deriving inspiration from Example 1.2.1, we get $u(t) = u(0)\,e^{kat} = A\,e^{kat}$.

b. We have $w(t) = \frac{(k-1)\,A}{k}(1 - e^{kat})$ for $0 < k \le 1$; $w(t) = a\,At$ for $k = 0$.

19. **b.** As $t \to \infty$, $e^{-kEt} \to 0$, so that $W(t) \to \frac{C}{E}$. Note that C/E is in pounds per day.

c. About 79 days for 20 pounds, 138 days for 30 pounds, and about 177 days for 35 pounds. This says that the weight is a concave up decreasing function of time—that is, the rate of weight loss slows down with time.

21. **b.** Here's the graph of $y(t) = \frac{333.2361}{1+61.7743\,e^{-0.029078\,t}}$:



c.

	Actual Population	Logistic Pop. Value
1800	5,308,483	5,308,480
1980	226,542,199	250,670,896
2000	281,421,906	281,418,481

d. $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{333.2361}{1+61.7743\, e^{-0.029078\, t}} = 333.2361$ million people.

Exercises 2.1

1. $y = \frac{A}{2} + \frac{C}{x^2}$.

3. $y = (t-2)^3 = t^3 - 6t^2 + 12t - 8$. The solution $y \equiv 0$ is a **singular solution** of the basic ODE and satisfies the initial condition.

5. $y = 2 - 3\cos x$. The only possible singular solution is $y \equiv 2$, but this can be obtained by letting $C = 0$.

7. $\frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{x} + C$. The constant function $y \equiv 1$ is a **singular solution**. Note that the implicit solution formula is not defined for $y = 1$.

9. $z = \frac{\ln(C-10^x)}{\ln 10}$. Note that for each particular value of the parameter C, the solution is defined only for $10^x < C$—that is, for $x < \ln C/\ln 10$ (or $x < \log_{10} C$).

11. $y = -\frac{x^2}{2} + C$ or $y = C\,e^{-x}$.

13. $x + 2y - 2\ln|x + 2y + 2| = x + C$; $y = -(x+2)/2$ is a singular solution.

15. $\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) - \ln|x| - C = 0$.

17. We have two one-parameter families of solutions: $y = x\sqrt{2\ln|x| + C}$ and $y = -x\sqrt{2\ln|x| + C}$.

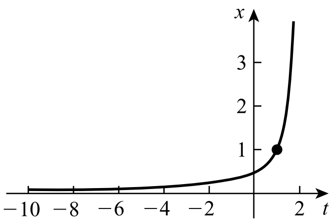
19. \$ 1271.25

21. $m(2) = m(0)e^{-0.0256(2)} \approx 0.95m(0)$. This says that approximately five percent of the original mass has decayed.

25. **a.** $x(t) = \frac{1}{2-t}$.

b. The interval I can be as large as $(-\infty, 2)$ or $(2, \infty)$. Any such interval I cannot include the point $t = 2$, at which $x(t)$ is not defined.

c.



d. The only solution is $x \equiv 0$, a **singular solution** that satisfies the initial condition $x(0) = 0$.

27. $t = 60$.

29. $V(4) = 32\pi$ cubic units.

33. **a.** $10000e^{31.2} \approx 3.5 \times 10^17$

b. 0.533 hours $\approx$ 32 minutes

35. The equation's solution is $N = N(t) = N(0)e^{-0.105t}$ and the half-life is 6.60 hours.

37. $V = 300$ ft/s.

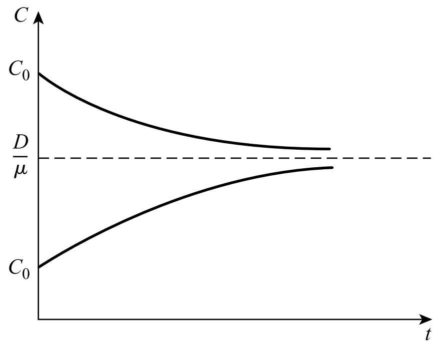
39. **a.** If we let $p(x) = dy/dx$, then the original equation becomes $\frac{dp}{dx} = k\left[1 + p^2\right]^{1/2}$.

b. $y = \frac{C_1}{2k}e^{kx} + \frac{1}{2kC_1}e^{-kx} + C_2 = \frac{1}{2k}\left(C_2e^{kx} + \frac{1}{C_2}e^{-kx}\right) + C_3$.

41. **a.** The equilibrium solution occurs when $C = D/\mu$.

b. The formula for the concentration is $C = \frac{D}{\mu} - \left(\frac{D-\mu C_0}{\mu}\right)e^{-\mu t}$. As $t \to \infty$, $e^{-\mu t} \to 0$, so $C(t) \to \frac{D}{\mu} - 0 = \frac{D}{\mu}$, the equilibrium solution found in part (a).

c.



43. The population is predicted to be approximately 30,400 in 2025.

Exercises 2.2

1. $y = 2x - 1 + Ce^{-2x}$.

3. $x = \frac{t^2}{2} - \frac{1}{2} + Ce^{-t^2}$.

5. $y = \frac{t^3}{6} - \frac{t^2}{5} + \frac{C}{t^3}$.

7. $y = x\sin x + Cx$.

9. $x = e^t\left(\ln|t| + t^2/2 + C\right)$.

11. $y(x) = \frac{e^x + ab - e^a}{x}$.

13. For $m \neq -a$, we have $y = \frac{e^{mx}}{a+m} + C\,e^{-ax}$. If $m = -a$, then

$$e^{ax} y = \int e^{(a-a)x}\,dx = \int 1\,dx = x + C, \text{ so that}$$
$$y = xe^{-ax} + C\,e^{-ax} = (x + C)e^{-ax}.$$

Note: A CAS that can solve ODEs may miss the need for an analysis of two cases.

15. $x(t) = \left(\frac{t}{t+1}\right)(t + \ln|t| - 1)$.

17. $y = -\ln\left(x + C\,x^2\right)$.

19. $y = \frac{t^4}{t^6+C}$; $y \equiv 0$ is a **singular solution**.

21. $y = \frac{\pm 1}{\sqrt{x+\frac{1}{2}+C\,e^{2x}}}$; $y \equiv 0$ is a **singular solution**.

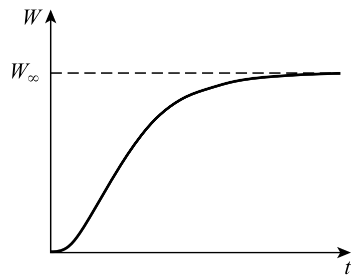
23. $y = 2\,e^{t^2} \Big/ \left(2\,C - e^{t^2}\right)$; $y \equiv 0$ is a **singular solution**.

25. **a.** $W(t) = \left(\frac{\alpha}{\beta} + C\,e^{-\beta t/3}\right)^3$.

b. $W_\infty = \left(\frac{\alpha}{\beta}\right)^3$.

c. $W(t) = W_\infty\left(1 - e^{-\beta t/3}\right)^3$.

d.



27. **a.** $I(t) = \frac{E}{R} - \frac{E}{R}\,e^{-(R/L)t} = \frac{E}{R}\left(1 - e^{-(R/L)t}\right)$.

b. $\lim_{t\to\infty} I(t) = \frac{E}{R}$.

c. $t = \frac{L}{R}\ln 2$.

d. $I(t) \equiv \frac{E}{R}$.

29. $Q(t) = \frac{E_0 C\,[\sin(\omega t) - \omega RC\cos(\omega t)]}{1+(RC\omega)^2} + \frac{\omega E_0 RC^2}{1+(RC\omega)^2}e^{-t/RC}$

$= \frac{E_0 C}{1+(RC\omega)^2}\left\{\sin(\omega t) - \omega RC\cos(\omega t) + \omega RC e^{-t/RC}\right\}$.

31. $p(t) = \frac{v}{\mu+v} + \left(p_0 - \frac{v}{\mu+v}\right)e^{-(\mu+v)t}$

$= \frac{v}{\mu+v}\left[1 - e^{-(\mu+v)t}\right] + p_0 e^{-(\mu+v)t}$

$$
\begin{aligned}
q(t) &= 1 - \left\{ \frac{v}{\mu+v}\left[1 - e^{-(\mu+v)t}\right] + p_0 e^{-(\mu+v)t} \right\} \\
&= \frac{\mu}{\mu+v} + \left(q_0 - \frac{\mu}{\mu+v}\right) e^{-(\mu+v)t} \\
&= \frac{\mu}{\mu+v}\left[1 - e^{-(\mu+v)t}\right] + q_0 e^{-(\mu+v)t}.
\end{aligned}
$$

33. **b.** If $K(t) \equiv 0$, then $V(t) = Ze^{-\int_t^T r(x)\,dx}$, either from the original (separable) differential equation or from the formula for $V(t)$ proved in part (a).

Exercises 2.3

1. $p' = b\,p - d\,p = (b - d)\,p$.

3. $p' = k\,p^2 - d\,p = (k\,p - d)\,p$.

5. **a.** $P = \frac{\alpha t}{k} + \frac{\alpha}{k^2} + \left(1.285 - \frac{\alpha}{k^2}\right) e^{k t}$

$\approx (0.0452\,t + 1.275) + 0.01045\,e^{0.0355\,t}$.

b. $P(35) \approx 2{,}893{,}202$ people.

7. $A(t) = \frac{25}{2} - \frac{25}{2}\,e^{-t/50} = \frac{25}{2}\left(1 - e^{-t/50}\right)$.

9. **a.** 100 gallons.

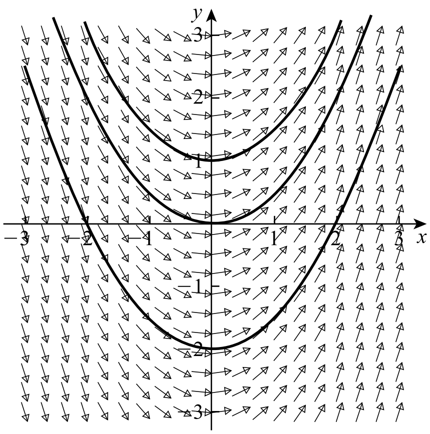
b. 175 pounds.

11. 165.12 minutes.

15. The rate of ventilation should be greater than 50 cubic feet per minute.

Exercises 2.4

1.



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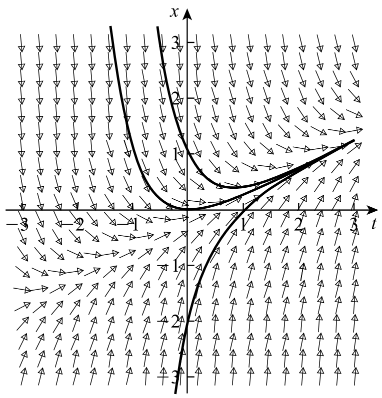

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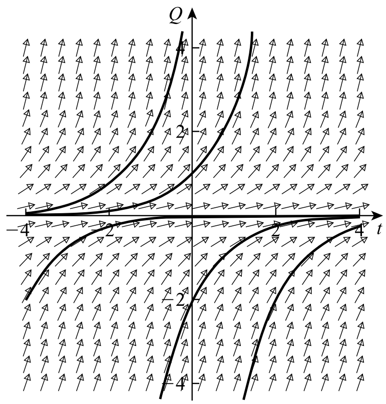

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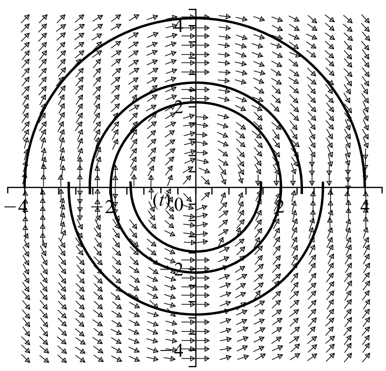

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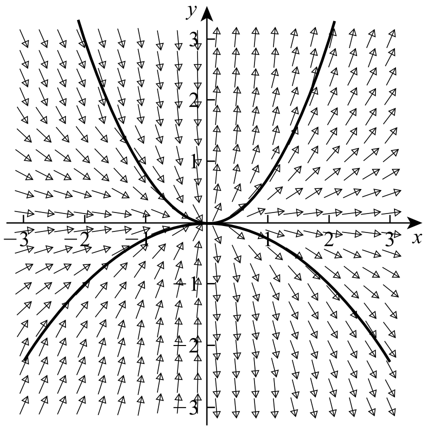

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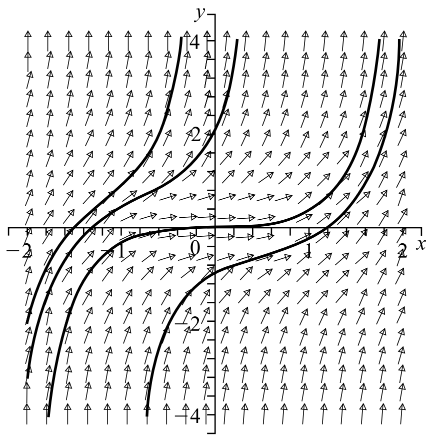
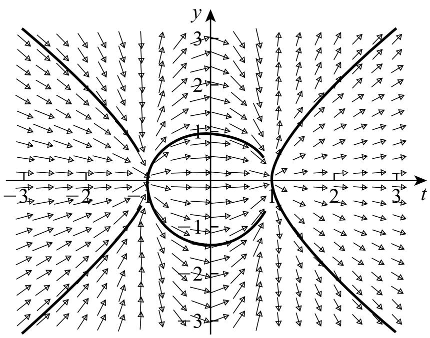
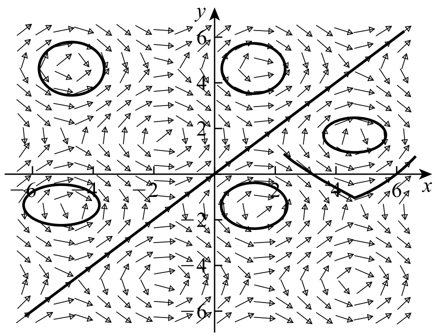


13.

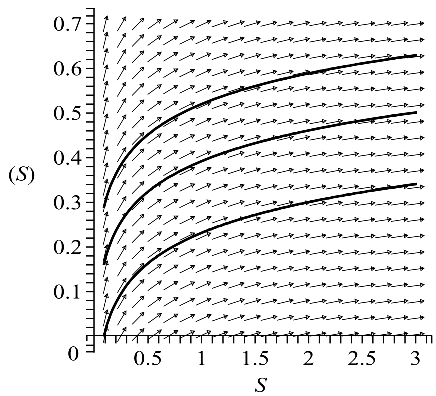


15.



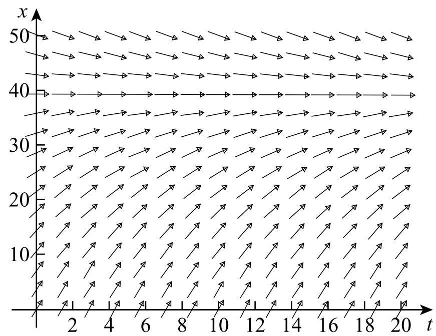
The nullclines are all points (x, y) such that $x = \pm(2k+1)\pi/2$ (an odd multiple of $\pi/2$) and y is *not* an odd multiple of $\pi/2$.

17.



19. The equations in Problems 4, 5, 8, and 14 are autonomous.

21. **a.**



b. The amount of substance X approaches 40.

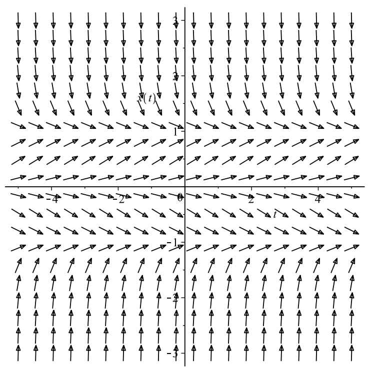
23. Circles centered at the origin with radii $\sqrt{1/C^2 - 1}$, where $C \in [-1, 0) \cup (0, 1]$ is the slope.

25. The horizontal lines $x = \alpha$ and $x = \beta$.

27. **a.** Field 3.
b. Field 1.
c. Field 2.

29. **a.** If the initial point is above the t-axis (i.e., $y(t_0) = y_0 < 0$), then $y \to \infty$ as $t \to \infty$. If the initial point is *on* the t-axis, the solution curve *is* the t-axis—that is, $y(t) = 0$ for *all* values of t. Finally, if the initial point is below the t-axis, then $y \to -\infty$ as $t \to \infty$.
b. If the initial value of P, P_0, is above 1, then $P \to 1$ as $t \to \infty$. For $0 < P_0 < 1$, we have $P \to 1$ as $t \to \infty$. If $P_0 < 0$, then $P \to -\infty$ as $t \to \infty$.
c. A careful examination of the slope field reveals that when $y(0) < 1/2$ we seem to have $y(t) \to -\infty$ as $t \to \infty$, and when $y(0) > 1/2$ we have $y(t) \to \infty$ as $t \to \infty$. When $y(0) = 1/2$ the solutions tend to 0.
d. Some solutions seem to be unbounded (positively or negatively) as t tends to infinity, while others seem to be periodic. The initial condition is essential in determining which of these behaviors to expect.

31. **a.**

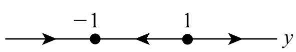


b. $x(t) \to 1$ as $t \to \infty$
c. $x(t) \to 1$ as $t \to \infty$
d. $x(t) \to -1$ as $t \to \infty$
e. $x(t) \to -1$ as $t \to \infty$

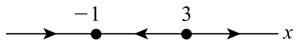
33. **c.** No.

Exercises 2.5

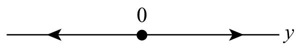
1.



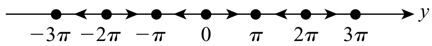
3.



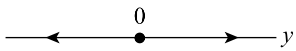
5.



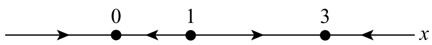
7.



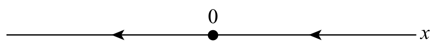
9.



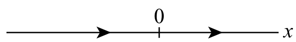
11.



13.

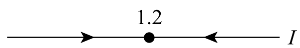


15. a.



b. There are no critical points. Any solution must be an increasing function.

17. a.

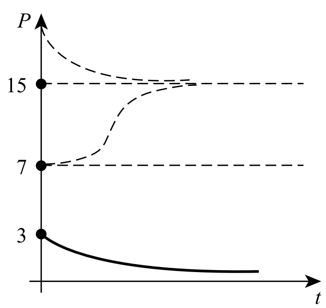


b. If the initial current, $I(0)$, is 3 amp, it is to the right of the critical point, so that the current tends to *decrease* toward 1.2 amp as t gets larger.

19. a. $y(t) \to 0$ as $t \to \infty$.

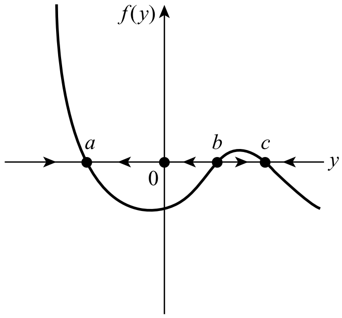
b. $y(t) \to \infty$ as $t \to \infty$.

21. a.



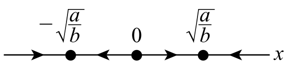
b. $P(t) \to 0$ as $t \to \infty$

23.



25. If $\alpha < 1/2$, the equilibrium solution $(1-2\alpha)^{-1}$ is a sink. If $\alpha > 1/2$, the equilibrium solution $(1-2\alpha)^{-1}$ is a source.

27. **a.**



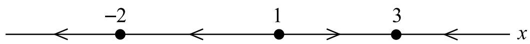
b. $x(t) \to \sqrt{a/b}$.
c. $x(t)$ stays at zero.
d. $x(t) \to \sqrt{a/b}$.

Exercises 2.6

1. The equilibrium points are $y = 0$ and $y = 1$. Both are nodes.
3. The only equilibrium point is $y = 0$, a source.
5. The equilibrium points are $x = -a/b$ and 0. We find that $x = -a/b$ is a sink and $x = 0$ is a source.
7. The equilibrium points are $x = 0$ and 1. The solution $x = 1$ is a sink, but $x = 0$ is a node.
9. The equilibrium points are $x = 0$, 2, and 4. We see that $x = 0$ is a source, $x = 2$ is a sink, and $x = 4$ is a source.
11. The only equilibrium point is $x = 0$. A careful examination reveals that $x = 0$ is a source.
13. There is only one equilibrium point, $x \approx 0.74$, which is a source.
15. We see that $x = -1$ is a sink, $x = 0$ is a source, and $x = 0.5$ is a node.
17. For example, $\dot{x} = e^x$ or $\dot{x} = \sin x + 2$. Neither of these functions $f(x)$ can ever be equal to zero.
19. **a.** $u = \sqrt[3]{\frac{8P}{bS}} = 2\sqrt[3]{\frac{P}{bS}}$.
b. The equilibrium speed is a sink.
c. A rower may start from rest with maximum acceleration but then tire a bit so that his or her speed would level off at the equilibrium speed. If the rower's speed is *greater* than the equilibrium speed, we can reasonably believe that

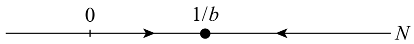
he or she may tire or that the "drag force" bSu^2 may exceed the "tractive force" $\frac{8P}{u}$ and so slow the boat down.

21. a.

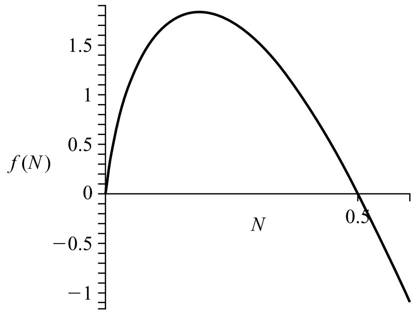


b. The equilibrium solutions are $x = -2, 1, 3$; they are a node, a source, and a sink, respectively.

23. a.



b. With $a = 10$ and $b = 2$, the graph is

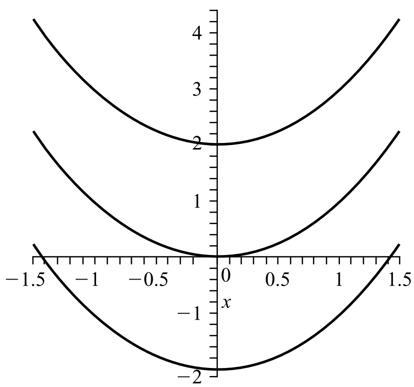


c. The only equilibrium point is $N = 1/b$. The phase portrait given in (a) indicates that this is a sink.

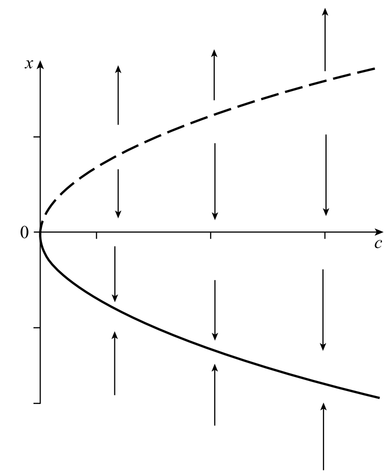
d. The graph of $N(t)$ is concave up on the interval $[0, 1/be)$ and concave down on the interval $[1/be, 1]$.

Exercises 2.7

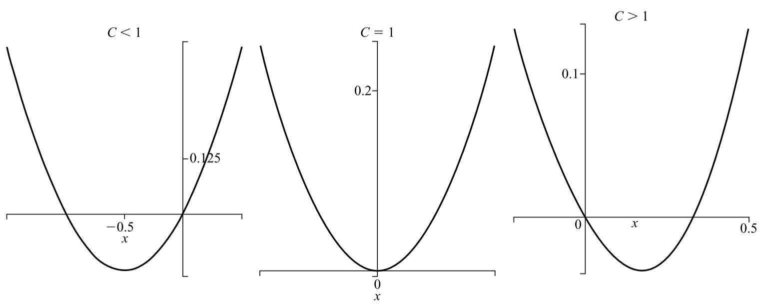
1. (1).



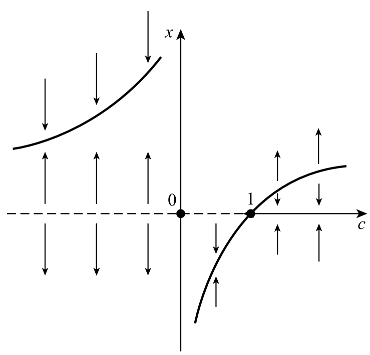
(2). The only bifurcation point is $c = 0$.
(3). The bifurcation diagram is



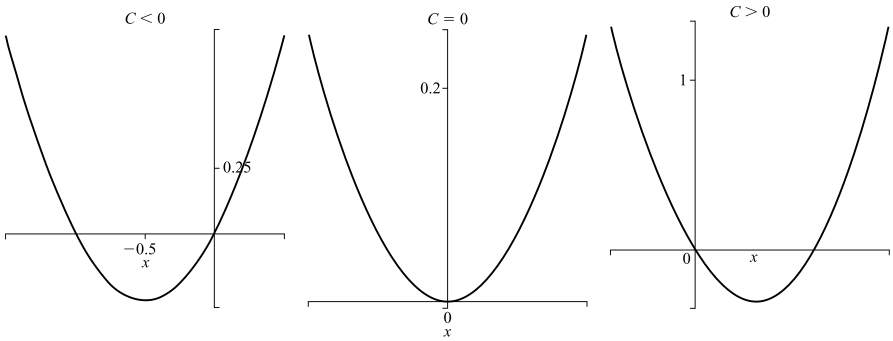
3. (1).



(2). The only bifurcation point is $c = 1$.
(3).

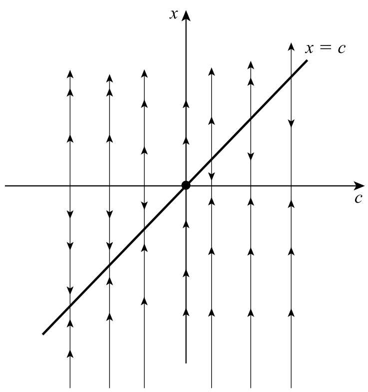


5. (1).



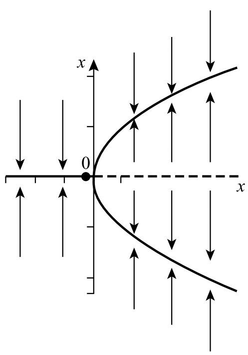
(2). There is one bifurcation point, $c = 0$.

(3).

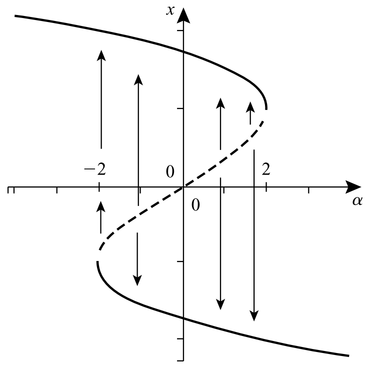


7. We find that $h^* = 25/4$ is the maximum harvest rate beyond which any population will become extinct.

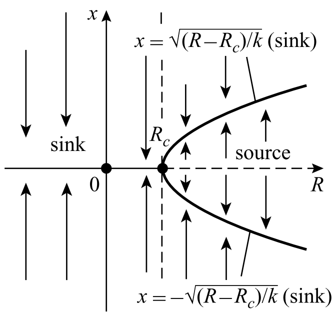
9. The value $c = 0$ is a *pitchfork bifurcation*.



11. The values $\alpha = -2$ and $\alpha = 2$ are the only bifurcation points:



13. c. The bifurcation point $R = R_c$ is a *sink*.



Exercises 2.8

1. For example, take any rectangle centered at $(0, 3)$ that avoids the t-axis $(x = 0)$.
3. There is no rectangle R containing the origin that does not also include points of the x-axis, where $t = 0$.
5. There is no rectangle R satisfying the requirements of the Existence and Uniqueness Theorem.
7. For example, take any rectangle in the t-y plane that does not include part of the line $t = -1$.
9. The solution's domain is $I = (-\pi/2, \pi/2)$, an interval of length π.
11. $x(t) = \left(\frac{t}{3} + \sqrt[3]{x_0}\right)^3$; the initial condition of Example 2.8.2 is $x(0) = 0$, so we don't expect uniqueness in that case. In the current exercise, both f and $\frac{\partial f}{\partial x}$ are continuous at $(0, x_0)$ if $x_0 < 0$, so we are guaranteed existence and uniqueness on some t-interval I.
13. **a.** $\frac{\partial f}{\partial Q}$ is not defined at $Q = 1$.

b. The constant function $Q \equiv 1$ is a solution because $Q' = 0 = |Q - 1|$ and $Q(0) = 1$. This solution is in fact unique.

c. Consider the possible cases $Q < 1$ and $Q > 1$ separately.

15. The conditions of the Existence and Uniqueness Theorem are not satisfied, so uniqueness is not guaranteed.

17. The conditions of the Existence and Uniqueness Theorem are satisfied, and so we expect to find an interval $I = (2 - h, 2 + h)$ centered at $x = 2$ such that the IVP has a unique solution on I.

19. No. If a solution near $P \equiv b$ were to equal the equilibrium solution—that is, if another solution curve intersects the horizontal line $P \equiv b$ at the point (t^*, b)—then we would have *two* solutions of the IVP $\frac{dP}{dt} = kP(b - P)$, $P(t^*) = b$.

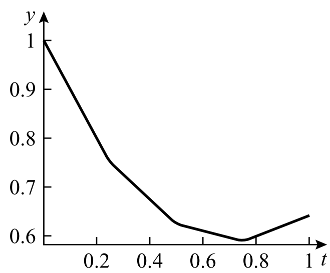
21. **a.** Show that $[y(t)\, y(-t)]' = 0$.

b. What does part (a) imply about the signs of $y(t)$ and $y(-t)$?

Exercises 3.1

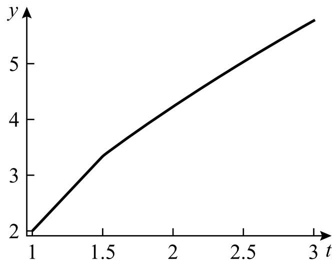
1.

t_k	y_k
0	1.000000
0.25	0.750000
0.50	0.625000
0.75	0.589844
1.00	0.643490



3.

t_k	y_k
1.0	2.000000
1.5	3.359141
2.0	4.266010
2.5	5.065065
3.0	5.807155



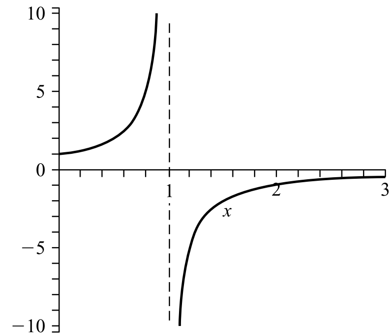
5. $y(\pi/2) \approx 1.14884140143$. The absolute error of the approximation is $|1 - 1.14884140143| = 0.14884140143$.
7. $y(1) \approx 1.38556107091$.
9. $y(1) \approx 0.80998149723$; since the solution of the equation is $y(x) = \arctan x$, we have $y(1) = \arctan 1 = \pi/4$. Multiplying our approximation for $y(1)$ by 4, we get an approximation for π.
11. **a.**

$$
\begin{aligned}
y_1 &= y_0 + 0.2{y_0}^2 = 1 + 0.2(1)^2 = 1.2\\
y_2 &= y_1 + 0.2{y_1}^2 = 1.2 + 0.2(1.2)^2 = 1.488\\
y_3 &= y_2 + 0.2{y_2}^2 = 1.488 + 0.2(1.488)^2 = 1.9308288\\
y_4 &= y_3 + 0.2{y_3}^2 = 1.9308288 + 0.2(1.9308288)^2 = 2.67644877098\\
y_5 &= y_4 + 0.2{y_4}^2 = 2.67644877098 + 0.2(2.67644877098)^2\\
&= 4.10912437572\\
y_6 &= y_5 + 0.2{y_5}^2 = 4.10912437572 + 0.2(4.10912437572)^2\\
&= 7.48610500275.
\end{aligned}
$$

b. The equation is separable.
c. The following table compares approximate and actual values:

t_k	y_k	Actual $y(t_k)$
0	1	1
0.2	1.2	1.25
0.4	1.488	1.6667
0.6	1.9308	2.5000
0.8	2.6764	5.0000
1.0	4.1091	UNDEFINED
1.2	7.4861	−5.0000

The solution graph indicates the difficulty:



13. **a.** $P(1) \approx 1.330624$ million people $= 1{,}330{,}624$ people.

b. $P(0) = 1.284999\ldots$ people $\approx 1{,}285{,}000$ people.

15. $V(0) = 166.390541\ldots \approx 166.39$ meters per second.

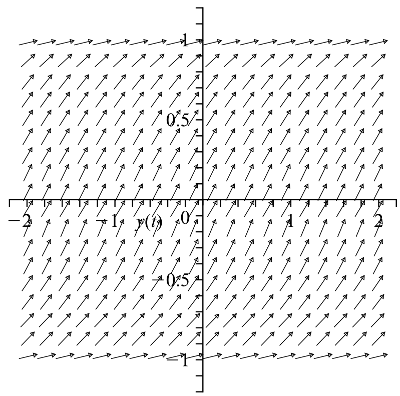
17. The exact solution is $x(t) = \sqrt{t^3 + 1}$. With $h = 0.5$, we get $x(2) \approx 2.746746$, with an absolute error of about 0.253254. With $h = 0.25$, we get $x(2) \approx 2.870814$, with an absolute error of about 0.129186.

19. **a.** $x'' = 3x^5$.

b. If $x(0) = 1$, then $x'' = 3x^5 > 0$ for all $x > 0$, implying that the solution curve is concave up.

c. Euler's method *underestimates* the true value of the solution at $t = 0.1$.

21. **a.** Note that the direction field is not meaningful for $y < -1$ or $y > 1$.



c. We see that $y(1) \approx 0.8950$ and $y(1.2) \approx 1.0235$. However $y(1.3)$ and $y(1.3)$, for example, can't be calculated because the values would involve the square roots of negative numbers. With a relatively large step size of 0.4, once you get a little past $t = 1.2$, Euler's method produces values of y that are greater than 1.

23. The only way a solution curve can coincide with its tangent line segments is if the solution curve is a straight line—that is, if $y(x) = Cx + D$, so that the differential equation is $\frac{dy}{dx} = C$.

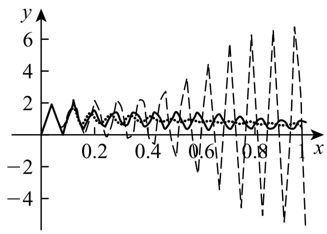
25. **a.** $\frac{2500}{2501}\cos x + \frac{50}{2501}\sin x - \frac{2500}{2501}e^{-50x}$; $y(0.2) = 0.9836011240\ldots$.

b. $y(0.2) \approx 1.7466146068$. The absolute error is 0.7630134828.

c. $y(0.2) \approx 1.1761983279$. The absolute error is 0.1925972039.

d. $y(0.2) \approx 1.8623800769$. The absolute error is 0.8787789529.

e. No.



Exercises 3.2

1. The table of approximations

	TRUE VALUE	Euler's Method	Absolute Error	Improved Euler Method	Absolute Error
$h = 0.1$	5.93977	5.69513	0.24464	5.93266	0.00711
$h = 0.05$	5.93977	5.81260	0.12717	5.93791	0.00186
$h = 0.025$	5.93977	5.87490	0.06487	5.93930	0.00047

3. **a.** $x(t) = -t - 1 + 2e^t$.

b. $x(1) \approx 3.42816$.

c. The following table shows the absolute error at each step of part (b):

t_k	x_k	TRUE VALUE	Absolute Error
0	1	1	0
0.1	1.11000	1.11034	0.00034
0.2	1.24205	1.24281	0.00076
0.3	1.39847	1.39972	0.00125
0.4	1.58180	1.58365	0.00185
0.5	1.79489	1.79744	0.00255
0.6	2.04086	2.04424	0.00338
0.7	2.32315	2.32751	0.00436
0.8	2.64558	2.65108	0.00550
0.9	3.01236	3.01921	0.00685
1.0	3.42816	3.43656	0.00840

5. **a.** $P(1) \approx 1.330624$ million people $= 1{,}330{,}624$ people.

b. $P(0) \approx 1.285363$ million people $= 1{,}285{,}363$ people.

7. a. $y(t) = [(1-\alpha)\,t]^{1/(1-\alpha)}$.

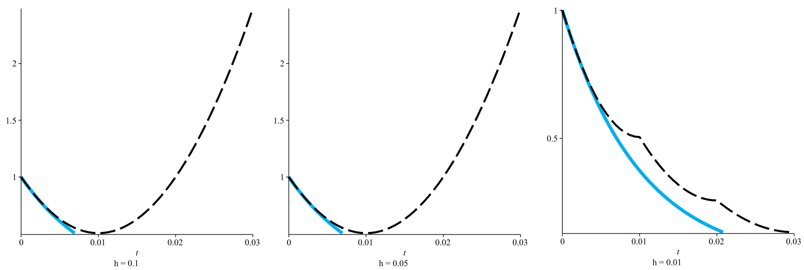
b. Take $f(t, y) = y^{\alpha}$, $t_0 = 0$, and $y_0 = 0$ in the improved Euler method. Then we have, for any h, $y_1 = y_0 + \frac{h}{2}\{f(t_0, y_0) + f(t_1, y_0 + h\,f(t_0, y_0))\} = \frac{h}{2}\,f(t_1, 0) = 0 = y_2 = \cdots = y_k = y_{k+1}$ for all positive integer values of k.

c. If $y_0 = y(0) = 0.01$, then for any h, the improved Euler's method produces a nonzero sequence of approximate values.

9. a. We find that $y(t) = 1/100 + (99/100)e^{-100\,t}$ and $y(1) = 0.01$.

b. With $h = 0.1$, we get $y(1) \approx 1.3288 \times 10^{16}$; with $h = 0.05$, we get $y(1) \approx 3.8372 \times 10^{18}$; and with $h = 0.01$, we get $y(1) = 0.01$.

c.



Exercises 3.3

1.

	TRUE VALUE	Euler's Method	Improved Euler Method	RK4 Method
$h = 0.1$	2.7182818	2.5937425	2.7140808	2.7182797
$h = 0.05$	2.7182818	2.6532977	2.7171911	2.7182817
$h = 0.025$	2.7182818	2.6850638	2.7180039	2.7182818

3. $y(1) = e \approx 2.718281811394140 93$.

5. a. $x = \frac{2}{t^2 + C}$.

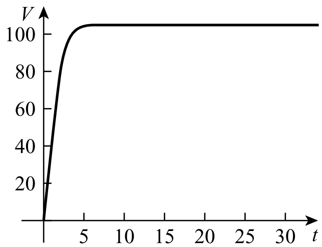
b. The *rkf45* method yields $x(1) \approx 0.99999999727228860$.

7. a. The following table provides the required data:

t	$V(t)$
5	100.163
10	104.984
15	105.045
16	105.046
17	105.046
18	105.046
19	105.046
20	105.046

We guess that the terminal velocity is 105.046 ft/s.

b.



11. a.

t	(0.5, 1)	(0.5, 2)	(1.5, 1)	(1.5, 2)	(2, 2)
1	1.845	1.707	2.320	1.918	2.051
2	2.663	2.069	4.207	2.215	2.233
3	3.349	2.190	4.893	2.235	2.236
4	3.872	2.224	4.988	2.236	2.236
5	4.246	2.233	4.999	2.236	2.236
6	4.504	2.235	5.000	2.236	2.236
7	4.677	2.236	5.000	2.236	2.236
8	4.791	2.236	5.000	2.236	2.236
9	4.865	2.236	5.000	2.236	2.236
10	4.913	2.236	5.000	2.236	2.236

b. $(r, q) \approx (0.5, 1.4)$.

Exercises 4.1

1. $y = (c_1 + c_2 t)e^{2t}$.

3. $x = e^t(c_1 \cos t + c_2 \sin t)$.

5. $x = c_1 + c_2 e^{-2t}$.

7. $y = c_1 \cos 2t + c_2 \sin 2t$.

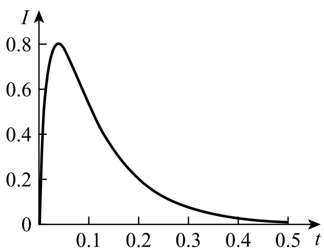
9. $r(t) = e^{2t}(c_1 \cos 4t + c_2 \sin 4t)$.

11. $x(t) = -e^{2t} + 2e^t$.

13. $y(t) = \frac{1}{4} e^{(2t-\pi)} \sin 4t$.

17. a. $I(t) = \frac{3}{2}\left(e^{-10t} - e^{-50t}\right)$.

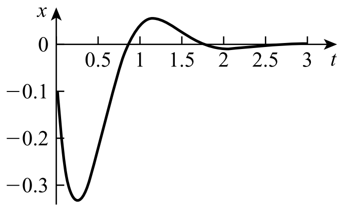
b.

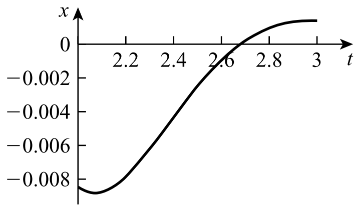


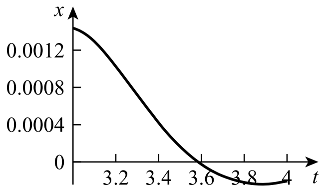
c. The maximum value of I is approximately 0.8.
d. The maximum value of I is achieved when $t \approx 0.04$ seconds.

19. $x(t) = 3\cos 12t + \frac{5}{6}\sin 12t$.

21. **a.** $x(t) = -\frac{1}{30}e^{-2t}\left(11\sqrt{3}\sin(2\sqrt{3}t) + 3\cos(2\sqrt{3}t)\right)$.
b. The graphs are







c. The greatest distance is approximately 33 cm.

25. **c.** $u(t) = C_1 t + C_2$.

Exercises 4.2

7. $y_p = 3x^2$.

9. $y(t) = \frac{3}{4}x - \frac{1}{16} + e^{-3t/2}\left(C_1\cos\left(\frac{1}{2}\sqrt{7}x\right) + C_2\sin\left(\frac{1}{2}\sqrt{7}x\right)\right)$.

11. $y(x) = c_1 e^{-x} + c_2 e^{2x/3} - \frac{5}{13}\cos x + \frac{1}{13}\sin x$.

13. $y(x) = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$.

15. $x(t) = c_1 e^{-t} + c_2 + \frac{1}{2}e^t - te^{-t}$.

17. $x(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < 1 \\ -\cos \pi t & \text{if} \quad t \ge 1. \end{cases}$

Exercises 4.3

1. $y_{GH} = c_1 + c_2 e^{-3t}$; $y_{PNH} = Kt$.

3. $y_{GH} = c_1 + c_2 e^{-7x}$; $y_{PNH} = K x e^{-7x}$.

5. $y_{GH} = c_1 \cos 5x + c_2 \sin 5x$; $y_{PNH} = Ax(\cos 5x + \sin 5x)$.

7. $y_{GH} = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$; $y_{PNH} = xe^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$.

9. $y_{GH} = c_1 \cos kt + c_2 \sin kt$; $y_{PNH} = C$.

11. $y_{GNH} = C_1 e^{3t} + C_2 e^{-t} + \frac{1}{5} e^{4t}$.

13. $x_{GNH} = C_1 e^t \cos t + C_2 e^t \sin t + \left(-\frac{2}{5}t - \frac{14}{25}\right) \sin t + \left(\frac{1}{5}t + \frac{2}{25}\right) \cos t + e^t$.

15. $x_{GNH} = C_1 + C_2 e^{-t} - 2\cos t - 2\sin t$.

17. $y_{GNH} = (C_1 + C_2 x)e^{-5x} + 2x^2 e^{-5x}$.

19. $x_{GNH} = C_1 e^{-t} + C_2 e^{-2t} + \left(-\frac{3}{10}t + \frac{17}{50}\right) \cos t + \left(\frac{1}{10}t + \frac{3}{25}\right) \sin t$.

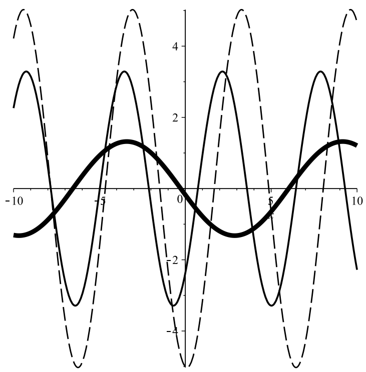
21. $y(x) = \frac{3}{5} x e^{4x} - \frac{3}{25} e^{4x} + \frac{3}{25} e^{-x}$.

23. $y(t) = e^{-t/2}\left(7\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{11\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$
$+ \left(-t^2 e^{-t} - 6e^{-t} - 4te^{-t}\right)\sin t + \left(-2te^{-t} - 6e^{-t}\right)\cos t$.

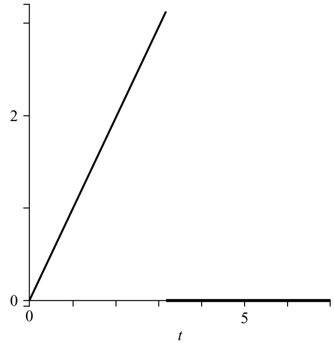
25. Assuming that a particular solution of the equation has the form $y_p = A\cos(\omega x) + B\sin(\omega x)$, we find that

$$y_p = -\frac{25(1-\omega^2)\sin(\omega x)}{25 - 49\omega^2 + 25\omega^4} - \frac{5\cos(\omega x)\omega}{25 - 49\omega^2 + 25\omega^4}.$$

The amplitude of a particular solution is a maximum when $\omega \approx 0.99$. This situation is very close to resonance, but the oscillations do not build up without limit.



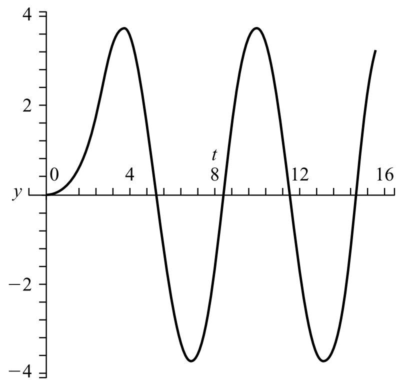
27. a.



b. $y(t) = t - \sin t$.

c. $y(t) = -\pi \cos t - 2 \sin t$.

d. $y(t) = \begin{cases} t - \sin t, & 0 \leq t \leq \pi \\ -\pi \cos t - 2 \sin t, & t \geq \pi. \end{cases}$



29. $y(x) = c_1 e^x + c_2 e^{2x} + \frac{3}{130}(9 \cos 3x - 7 \sin 3x) - \frac{1}{6970}(27 \cos 9x - 79 \sin 9x)$.

31. $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{12} - \frac{1}{24} \cos^2 2x - \frac{1}{32} \cos^3 2x$.

33. b. No, any solution is unbounded unless the constant is zero.

c. If $c = 0$, the solution is unbounded unless the constant is zero. If $b = 0$ also, the solution is a polynomial and thus unbounded as $t \to \infty$.

35. $y(x) = c_1 x^{(\sqrt{10}-2)} + c_2 x^{(-\sqrt{10}-2)} + (10x^2 - 2)/5x^2$.

37. $y(x) = x(c_1 \cos(\ln x) + c_2 \sin(\ln x)) + x \ln x$.

Exercises 4.4

1. $x_{\text{GNH}} = C_1 t\, e^t + C_2 e^t + t \ln t\, e^t - t\, e^t = K_1 t\, e^t + K_2 e^t + t\, e^t \ln t$.

3. $r_{\mathrm{GNH}} = C_1 t\, e^t + C_2 e^t + t \ln t\, e^t - t\, e^t = K_1 t\, e^t + K_2 e^t + t\, e^t \ln t.$
5. $y_{\mathrm{GNH}} = c_1 x\, e^{-2x} + c_2 e^{-2x} + \frac{1}{2} x^3 e^{-2x}.$
7. $y_{\mathrm{GNH}} = c_1 x\, e^{-x} + c_2 e^{-x} + \frac{1}{4} x^2 e^{-x} (2 \ln x - 3).$
9. $y_{\mathrm{GNH}} = e^x \left(c_1 e^x + c_2 - e^x \cos(e^{-x}) \right).$
11. $y_{\mathrm{GNH}} = c_1 x \ln x + c_2 x + \frac{1}{2} x \ln^2 x.$
13. $y_{\mathrm{GNH}} = c_1 x^2 + c_2 x + \frac{1}{4} x^3 (2 \ln x - 3).$
15. $y_{\mathrm{GNH}} = c_1 \tan x + c_2 + \frac{1}{2} x \tan x.$
17. **b.** $y_{\mathrm{GNH}} = c_1 x^3 + c_2 x^2 + \frac{1}{2} x^4.$

Exercises 4.5

1. $y(x) = C_1 e^{3x} + C_2 e^{-x} + C_3.$
3. $y(x) = C_1 x e^{-x} + C_2 e^{-x} + C_3.$
5. $y(t) = C_1 e^{10t} + e^t \,(C_2 \cos t + C_3 \sin t).$
7. $y(t) = C_1 e^{-2t} + C_2 e^{2t} + C_3 e^{-3t} + C_4 e^{3t}.$
9. $y(t) = C_1 + (C_2 t + C_3)\, e^t + C_4 e^{-2t}.$
11. $y(t) = (C_1 t^2 + C_2 t + C_3) e^t + (C_4 t + C_5) e^{2t} + C_6 e^{3t} + C_7 e^{4t}.$
13. $y(x) = e^{-0.7289x} (C_1 \cos(0.6186\,t) + C_2 \sin(0.6186\,t))$

$$+ e^{0.4765x} (C_3 \cos(0.7591t) + C_2 \sin(0.7591t)).$$

15. **a.** $y_{\mathrm{GH}} = C_1 x^3 + C^2 x^2 + C_3 x + C_4.$
b. An intelligent guess would be $y_{\mathrm{PNH}} = (R/24) x^4.$
17. $y(t) = (C_1 t^{n-3} + C_2 t^{n-4} + \cdots + C_{n-3} t + C_{n-2}) + C_{n-1} e^{-t} + C_n e^t.$
19. $y(t) = \frac{1}{6} e^{3t} + C_1 + C_2 e^{-6t} + C_3 e^t.$
21. $y(x) = C_1 e^{4x} + C_2 e^{3x} + C_3 e^{-2x} - \frac{x e^{3x}}{375} (25x^2 + 60x + 126).$
23. $y(x) = C_1 \cos x + C_2 \sin x + C_3 x + C_4 + x \sin x + 2x \cos x + \frac{1}{4} x^4 - 3x^2.$
25. **a.** $y_{\mathrm{PNH}} = C_1 y_1 + C_2 y_2 + C_3 y_3.$
b. $\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \\ C_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}.$
c. $y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} - \frac{1}{2} x e^x.$

Exercises 4.6

3. **a.** $(-1, 2)$
b. $(-\infty, 1)$
c. $\left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$
d. $(0, 3).$

5. **a.** $(-1, 1)$
b. Any initial conditions given at $x = -1$ or $x = 1$ would not guarantee a unique solution.
c. $n = 0$: $y = 1$; $n = 1$: $y = x$; $n = 2$: $y = (3x^2 - 1)/2$; $n = 3$: $y = (5x^3 - 3x)/2$.
d. There is no contradiction.

7. **d.** The constant is 1.
e. The functions S and C behave the way the **sine** and **cosine**, respectively, behave.

Exercises 5.1

1. $2/s^3$, $s > 0$.
3. $3!/s^4$, $s > 0$.
5. $1/(s-a)^2$, $s > a$.
7. $10/s + 100/(s-2)$, $s > 2$.
9. $\left(12 - 14s + 5s^2 - 17s^3\right)/s^4$, $s > 0$.
11. $2/(s-1) - 3/(s+1) + 8/s^3$, $s > 1$.
13. $\left(1 - e^{-4s} - 4se^{-4s}\right)/s^2$, $s > 0$.
15. $\left(s + e^{-2s} - e^{-s}\right)/s^2$, $s > 0$.
17. $\mathcal{L}[y(t)] = 1/(s-1)$.
19. $\mathcal{L}[y(t)] = (2s+5)/\left((s+1)(s+2)\right)$.
21. $\mathcal{L}[y(t)] = s/\left(s^2+1\right)$.
23. $\mathcal{L}[y(x)] = \left(s^3 - 2s^2 + s + 3\right)/\left((s^2+1)(s^2-s-2)\right)$.
25. $\mathcal{L}[y(x)] = 2\left(s^2 + s - 1\right)/\left(s^3(s-1)^3\right)$.
27. $\mathcal{L}[\sin h(at)] = 1/[2(s-a)] - 1/[2(s+a)] = a/\left(s^2 - a^2\right)$; $\mathcal{L}[\cos h(at)] = s/\left(s^2 - a^2\right)$. The transforms exist for $s > a$.
29. $\frac{1}{s}\left(\frac{e^{as}-1}{e^{as}+1}\right) = \frac{1}{s}\tan h\left(\frac{as}{2}\right)$.
35. For example, $h(t) = \pi$ for $t = 1$, $h(t) = 1$ for $t \neq 1$. Functions that differ at only a finite number of points have equal integrals.
37. $F(s) = \sqrt{\frac{\pi}{s}}$, $s > 0$.

Exercises 5.2

1. $\frac{1}{3}\sin 3t$.
3. $\cos\sqrt{2}t$.
5. $\frac{1}{2}\left\{1 - e^{-t}(\cos t + \sin t)\right\}$.
7. $4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$.
9. $-\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$.

13. b. $\left(1 - e^{-3t/2}\right)/t$.

17. $s/\left(s^2+1\right)^2$.

19. $y(t) = 2t^2 - 6t + 7 + e^{-2t} - 8e^{-t}$.

21. $y(x) = x^2 + 4x + 4 + x^2e^x - 4e^x$.

23. $Q(t) = \begin{cases} e^{-t}\sin t & \text{for} \quad 0 \le t < \pi \\ \frac{2}{5}\cos t - \frac{1}{5}\sin t & \\ \quad -\frac{1}{5}e^{(-t+\pi)}\{2\cos t + \sin t\} + e^{-t}\sin t & \text{for} \quad t \ge \pi. \end{cases}$

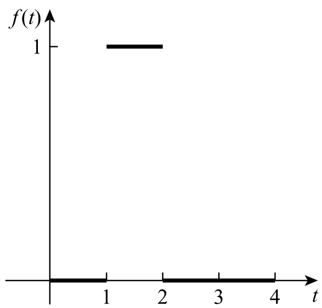
25. $f(t) = 4t + \frac{2}{3}t^3$.

27. $x(t) = 2 - e^{-t}$.

29. $x(t) = \frac{1}{10}t^5e^{2t} + \frac{1}{4}t^4e^{2t} = \frac{1}{20}t^4e^{2t}(2t+5)$.

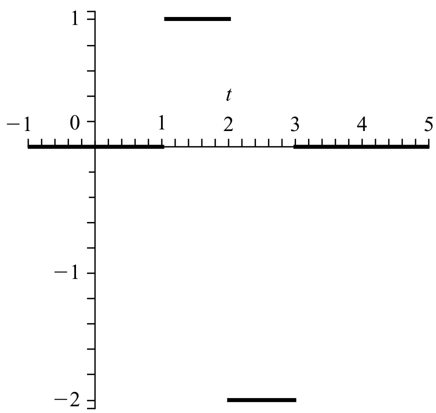
Exercises 5.3

1. a.



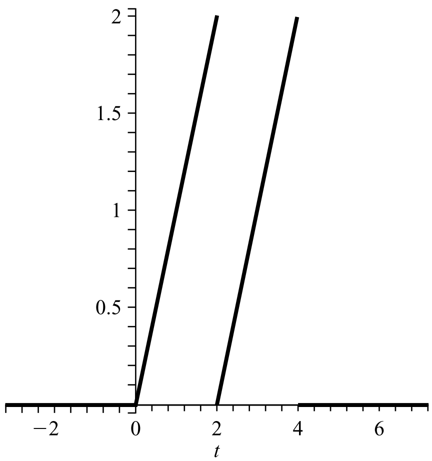
b. $f(t) = 1 \cdot U(t-1) + U(t-2)[0-1] = U(t-1) - U(t-2)$.

3. a.



b. $f(t) = U(t-1) - 3U(t-2) + 2U(t-3)$.

5. a.



b. $f(t) = t \cdot U(t) + U(t-2)[(t-2) - t] + U(t-4)[0 - (t-2)]$
$= tU(t) - 2U(t-2) + (2-t)U(t-4).$

11. $\mathcal{L}[f(t)] = \left(2 - 2e^{-2s} + 4s^2 e^{-2s}\right)/s^3.$

13. $\mathcal{L}[f(t)] = \left(1 - 2e^{-2s} + e^{-4s}\right)/s^2.$

15. $\left[\frac{1}{2}U(t-\pi) - \frac{1}{2}U(t-3\pi)\right]\sin^2 t = \frac{1}{2}\sin^2 t$ for $\pi < t < 3\pi$ and 0 elsewhere.

17. $$y(t) = \begin{cases} -\frac{14}{5}e^{5t/4} + 6t + \frac{24}{5} & \text{for} \quad 0 \le t < 1 \\ -\frac{14}{5}e^{5t/4} + \frac{54}{5}e^{\frac{5}{4}(t-1)} & \text{for} \quad t \ge 1. \end{cases}$$

19. $y(t) = 1 - \cos t + \sin t - U\left(t - \frac{\pi}{2}\right)(1 - \sin t)$
$$= \begin{cases} 1 - \cos t + \sin t & \text{for} \quad t < \pi/2 \\ -\cos t + 2\sin t & \text{for} \quad t \ge \pi/2. \end{cases}$$

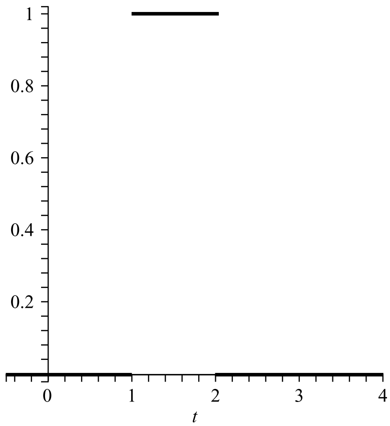
21. $$y(t) = \begin{cases} 4 + \left(\frac{10\sqrt{17}}{17} - 2\right)e^{-(\sqrt{17}+5)t/2} - \left(2 + \frac{10\sqrt{17}}{17}\right)e^{(\sqrt{17}-5)t/2} & \text{for } t \le 1 \\ 4 - e^{t-1} + \left(\frac{1}{2} - \frac{7\sqrt{17}}{34} - 2 + \frac{10\sqrt{17}}{17}\right)e^{-\left((\sqrt{17}+5)(t-1)/2\right)} & \\ + \left(\frac{1}{2} + \frac{7\sqrt{17}}{34}\right)e^{(\sqrt{17}-5)(t-1)/2} + \left(\frac{10\sqrt{17}}{17} - 2\right)e^{-\left((\sqrt{17}+5)t/2\right)} & \\ - \left(\frac{10\sqrt{17}}{17} + 2\right)e^{-(\sqrt{17}+5)t/2} & \text{for } t > 1. \end{cases}$$

23. $$y(t) = \begin{cases} \frac{1}{9}e^{3t} - \frac{1}{3}t - \frac{1}{9} & \text{for} \quad 0 \le t \le 1 \\ \frac{1}{9}e^{3t} - \frac{2}{9}e^{3(t-1)} + \frac{1}{3}t - \frac{5}{9} & \text{for} \quad 1 \le t \le 2 \\ \frac{1}{9}e^{3t} - \frac{2}{9}e^{3(t-1)} + \frac{1}{9}e^{3(t-2)} & \text{for} \quad t > 2. \end{cases}$$

25. **a.** $P(t) = \begin{cases} Ae^{kt} + \frac{h}{k}\left(1 - e^{kt}\right) & \text{for} \quad 0 \le t \le 30 \\ Ae^{kt} + \frac{h}{k}\left(e^{-k(30-t)} - e^{kt}\right) & \text{for} \quad t > 30. \end{cases}$

b. $A = \frac{h}{k}\left(e^{330k} - e^{360k}\right) \Big/ \left(1 - e^{360k}\right).$

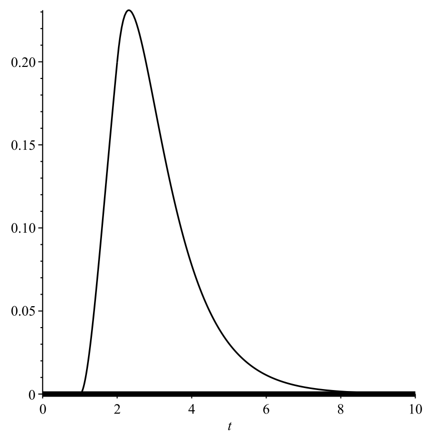
27. **a.** The graph of $W(t)$ is



b. $y(t) \equiv 0.$

c. $y(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < 1 \\ \frac{1}{2}e^{2(1-t)} + \frac{1}{2} - e^{1-t} & \text{for} \quad 1 \le t \le 2 \\ \frac{1}{2}e^{2(1-t)} - e^{1-t} + e^{2-t} - \frac{1}{2}e^{2(2-t)} & \text{for} \quad t > 2. \end{cases}$

d. The forcing term creates a temporary "blip" in the zero function. However, the solution begins decaying exponentially shortly after 2 units of time:



Exercises 5.4

1. -1.
3. $e^{-\pi s}\cos\pi^3$.
5. $y(t)=\begin{cases}0 & \text{for } t<1\\ e^{-8(t-1)} & \text{for } 1\le t<2\\ e^{-8(t-1)}+e^{-8(t-2)} & \text{for } t\ge 2.\end{cases}$
7. $y(t)=\frac{1}{6}\left(e^{2t}-e^{-4t}\right)$.
9. $y(t)=e^{-t}+2te^{-t}+\left[2(t-1)e^{-(t-1)}\right]U(t-1)$.
11. $y(t)=\begin{cases}1 & \text{for } t<2\pi\\ 1+\sin t & \text{for } t\ge 2\pi.\end{cases}$
13. $y(t)=-\frac{1}{4}\left(2\sin(t-1)-e^{t-1}+e^{-(t-1)}\right)U(t-1)$.
15. $y(x)=$ [after expanding and simplifying] $\begin{cases}\frac{W}{6EI}x^2\left(\frac{3}{2}L-x\right) & \text{for } 0\le x<\frac{L}{2}\\ \frac{WL^2}{24EI}\left(3x-\frac{L}{2}\right) & \text{for } x\ge\frac{L}{2}.\end{cases}$
19. **a.** $x(t)=\begin{cases}0 & \text{for } t<a\\ \frac{H\sqrt{mk}}{m}\sin\left(\sqrt{\frac{k}{m}}(t-a)\right) & \text{for } t\ge a.\end{cases}$

    b. Because m and k are constants directly related to the spring-mass system, varying the value of H affects the *amplitude* of the oscillations, the maximum distance of the mass from its equilibrium position. Viewed another way, in the original equation, the quantity kH represents the *magnitude* of the jerk upward.

    c. If we want A to be the maximum displacement of the mass from equilibrium, we must have $\frac{H\sqrt{mk}}{m}=A$, or $H=\frac{A\sqrt{mk}}{k}$.
21. **a.** $\mathcal{L}[y(t)]=\left(1/\left(s^2+2\right)\right)\sum_{n=1}^{\infty}e^{-ns}$.

    b. $y(t)=\left(1/\sqrt{2}\right)\sum_{n=1}^{\infty}U(t-n)\sin\sqrt{2}(t-n)$.

    c. The solution oscillates indefinitely.

Exercises 5.5

1. $y(t)=\frac{5}{2}e^{-2t}+\frac{1}{2}e^{-4t}$.
3. $x(t)=3e^{4t}+5e^{-t}$, $y(t)=-2e^{4t}+5e^{-t}$.
5. $x(t)=5e^t-18te^t$, $y(t)=-3e^t-54te^t$.
7. $x(t)=e^t\cos 2t+\frac{1}{2}e^t\sin 2t$, $y(t)=x'-x=e^t\cos 2t-2e^t\sin 2t$.
9. $x(t)=-e^{-3t}(-2\sin 4t+\cos 4t)$, $y(t)=2e^{-3t}\cos 4t$.
11. $x(t)=\frac{3}{10}e^{-t}+\frac{7}{10}e^{2t}\cos t-\frac{11}{10}e^{2t}\sin t$, $y(t)=-\frac{2}{5}e^{-t}+\frac{2}{5}e^{t}\cos t+\frac{9}{5}e^{2t}\sin t$.
13. $x(t)=3e^t-9t^2+6t+2$, $y(t)=-e^t-6t$.

15. $x(t) = \frac{4}{15}\cos^2\sqrt{3}t - \frac{2}{15} + \frac{4}{15}\sqrt{3}\sin\sqrt{3}t\cos\sqrt{3}t - \frac{2}{15}\cos h\sqrt{3}t$
$+ \frac{1}{15}\sqrt{3}\sin h\sqrt{3}t$
$y(t) = -\frac{32}{15}\cos^2\sqrt{3}t + \frac{16}{15} + \frac{32}{45}\sqrt{3}\sin\sqrt{3}t\cos\sqrt{3}t + \frac{1}{15}\cos h\sqrt{3}t$
$- \frac{2}{45}\sqrt{3}\sin h\sqrt{3}t.$

17. $x(t) = \frac{11}{20}\cos 2t + \frac{9}{20}\cos\left(\sqrt{2}t\right), \theta(t) = \frac{11}{20}\cos 2t - \frac{9}{20}\cos\left(\sqrt{2}t\right).$

19. $I_1(t) = \frac{11}{4} - \frac{1}{20}e^{-6t} - \frac{27}{10}e^{-t}, I_2(t) = \frac{3}{4} + \frac{3}{20}e^{-6t} - \frac{9}{10}e^{-t}.$

21. $x(t) = \frac{1}{4}e^t - \frac{3}{4}e^{-t} - \frac{1}{2}\cos t + \sin t + e^{-2t}$
$y(t) = \frac{1}{2}\cos t - \sin t + \frac{1}{4}e^t - \frac{3}{4}e^{-t}.$

Exercises 5.6

1. **b.** $y(x) = e^{-x^2/2}\int e^{x^2/2}\,dx + Ce^{-x^2/2}.$

3. $y(t) = t.$

5. $y(t) = t^2.$

7. $y(t) = Ct^3e^t.$

9. **a.** $l_1(t) = -t + 1;\ l_2(t) = \frac{1}{2}(t^2 - 4t + 2);\ l_3(t) = \frac{1}{6}(-t^3 + 9t^2 - 18t + 6);$
$l_4(t) = \frac{1}{24}(t^4 - 16t^3 + 72t^2 - 96t + 24).$

b. $L_n(s) = \mathcal{L}[l_n(t)] = (s-1)^n/s^{n+1}.$

c. $y(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) = l_n(x).$

Exercises 6.1

1. $\left\{\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = 1 + x_1\right\}.$

3. The nonautonomous system is $\{y_1' = y_2, y_2' = (5\ln x + 3xy_2 - 4y_1)/x^2\}$. Replacing x by y_3 and adding the equation $y_3' = 1$ yields an autonomous system.

5. $\{x_1' = x_2, x_2' = x_3, x_3' = t\,x_3 - x_2 + 5x_1 - t^2\}.$

7. The nonautonomous system is $\{w_1' = w_2, w_2' = w_3, w_3' = w_4, w_4' = 6\sin(4t) + 2w_4 - 5w_3 - 3w_2 + 8w_1\}$. To get an *autonomous* system, replace t by w_5 and add the equation $w_5' = 1$.

9. $\{x_1' = x_2, x_2' = 1 - 3x_2 - 2x_1;\ x_1(0) = 1, x_2(0) = 0\}.$

11. $\frac{dx_1}{dt} = x_2,\ \frac{dx_2}{dt} = \frac{1}{2}\{x_1 - y_2^2\},\ \frac{dy_1}{dt} = y_2$, and $\frac{dy_2}{dt} = \frac{4t + y_1}{x_1}.$

13. $\frac{d^2u}{dx^2} - 4u - 2 = 0$ or $\frac{d^2v}{dx^2} - 4v + 2 = 0.$

15. $\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 6x - 12 = 0$ or $\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 6y - 18 = 0.$

17. $\left\{\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = -4x_1 - 4x_2;\ x_1(0) = 2, x_2(0) = -2\right\}$; the first equation represents the *velocity* of the mass at time t, whereas the second equation represents the *acceleration* of the mass.

19. $\left\{\frac{dy_1}{dt} = y_2, \frac{dy_2}{dt} = -\left(\frac{g}{s_0}\right) y_1\right\}$.

21. $\{w_1' = w_2, w_2' = w_3, w_3' = w_4, w_4' = 6\sin(4t) + 2w_4 - 5w_3 - 3w_2 + 8w_1\}$; to get an *autonomous* system, replace t by w_5 and add the equation $w_5' = 1$.

23. $\{u_1' = u_2, u_2' = u_3u_2 + u_3^2 u_1, u_3' = 1;\ u_1(0) = 1, u_2(0) = 2, u_3(0) = 0\}$.

25. $\frac{d^4x}{dt^4} - 16x = 120e^{-t} - 64t$; $x(0) = 6$, $x'(0) = 8$, $x''(0) = -48$, and $x'''(0) = -8$ or $\frac{d^4y}{dt^4} - 16y = 540e^{-t} - 96$, $y(0) = -24$, $y'(0) = 0$, $y''(0) = -12$, and $y'''(0) = 84$.

Exercises 6.2

3. **a.** $(-\infty, \infty)$.
b. $(-\infty, 0)$ or $(0, \infty)$.
c. $(-\infty, 0)$ or $(0, 1)$ or $(1, \infty)$.
d. $(0, \infty)$.

5. There is no contradiction.

7. Extend the Existence and Uniqueness Theorem to six dimensions.

9. **c.** If the solution with initial condition $x(0) = 1/2$, $y(0) = 0$ satisfies $x^2(t) + y^2(t) \geq 1$ for any finite value of t, this means that the solution intersects the solution given in part (b) at some point $(x(t^*), y(t^*))$ on the unit circle. Thus, two solutions of the system pass through the same point in the open disk $x^2 + y^2 < 4$, contradicting the uniqueness established in part (a).

Exercises 6.3

1. **a.**
$$x_{k+1} = x_k + \frac{h}{2}\{f(t_k, x_k, y_k) + f(t_{k+1}, x_k + hf(t_k, x_k, y_k)), y_k + hg(t_k, x_k, y_k)\}$$
$$y_{k+1} = y_k + \frac{h}{2}\{g(t_k, x_k, y_k) + g(t_{k+1}, x_k + hf(t_k, x_k, y_k)), y_k + hg(t_k, x_k, y_k)\}.$$
b. $x(0.5) \approx 1.1273$, $y(0.5) \approx 0.5202$.
c. For $x(0.5)$ the absolute error is approximately 0.0003, while for $y(0.5)$ the absolute error is approximately 0.0009.

3. **a.** $\{u_1' = u_2,\ u_2' = 2x + 2u_1 - u_2;\ u_1(0) = 1, u_2(0) = 1\}$.
b. Using Euler's method with $h = 0.1$, we find that $u_1(0.5) \approx 1.8774$ and $u_2(0.5) \approx 4.1711$; $u_1(1.0) \approx 5.5515$ and $u_2(1.0) \approx 13.3031$.
c. Using a fourth-order Runge–Kutta method with $h = 0.1$, we get $u_1(0.5) \approx 2.1784$ and $u_2(0.5) \approx 4.7536$; $u_1(1.0) \approx 6.7731$ and $u_2(1.0) \approx 14.7205$.

5. **a.** $(x(t), y(t), z(t)) = (0, 5, 0)$ for all the values of t specified. The particle doesn't seem to be moving.

b. The values of x, y, and z seem to be increasing without bound as t grows larger, with the values of x, y, and z approaching each other.

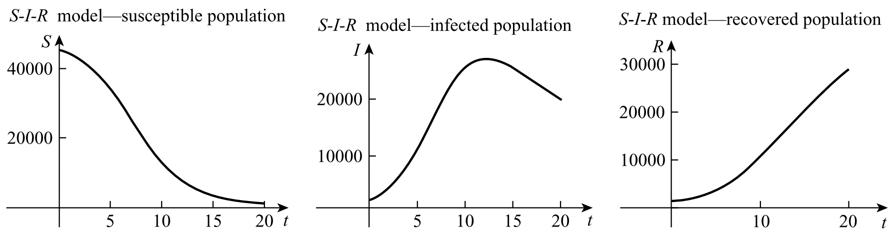
7. $t^* = 3.72$, to two decimal places.

9. a. $\left\{\frac{du}{dx} = v,\ \frac{dv}{dx} = -\frac{2}{x}v - u^3;\ u(0) = 1, v(0) = 0\right\}$.

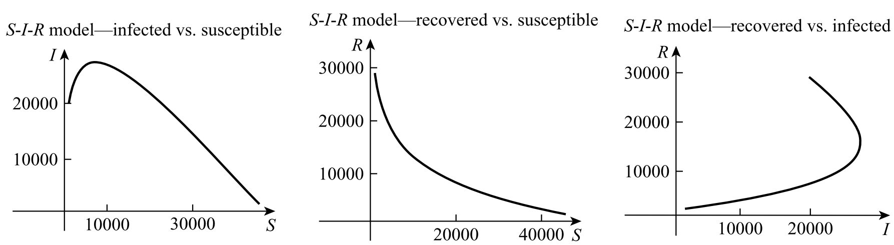
b. $x \approx 6.9$.

11. a. $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{du} = \frac{d}{dt}(S + I + R) = 0$. This means that the total population does not change.

b.



c.



d. We used the *rkf45* method and all values were rounded to the nearest whole number. The values show the steady increase in the number of people who recovered, the decreasing number of susceptible people, and that the number of infected people probably peaks between days 10 and 15.

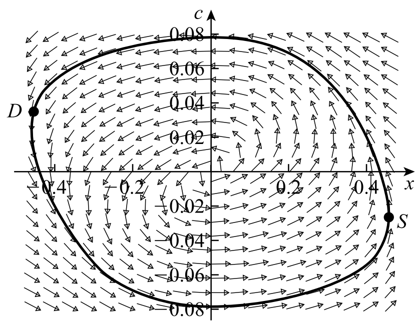
t	*S*	*I*	*R*
1	44,255	3062	2682
2	42,649	4405	2947
3	40,460	6217	3323
10	13,044	25,547	11,408
15	3447	25,638	20,915
16	2681	24,609	22,710
17	2108	23,464	24,428

e. We conclude that $t \approx 161$ if we round down, but $t \approx 171$ if we round I to the nearest integer.

13. a.

t	$x(t)$	$y(t)$
0.01	0.4492	−0.0158
0.02	0.4468	−0.0113
0.03	0.4432	−0.0068
0.04	0.4385	−0.0024
0.05	0.4330	0.0019
0.06	0.4266	0.0062
0.07	0.4196	0.0105
0.08	0.4120	0.0146
0.09	0.4039	0.0187
0.10	0.3952	0.0227

b. The direction of the solution curve is *counterclockwise*.



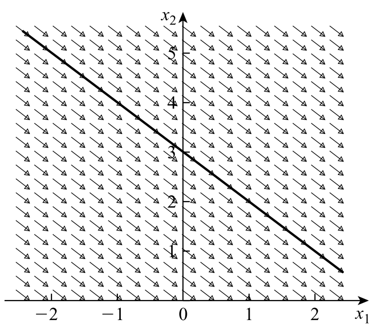
c. $t \approx 1.1$.

d. Diastole: $(x, c) \approx (-0.46, 0.02)$ when $t \approx 0.52$; systole: $(x, c) \approx (0.46, -0.02)$ when $t \approx 1.05$.

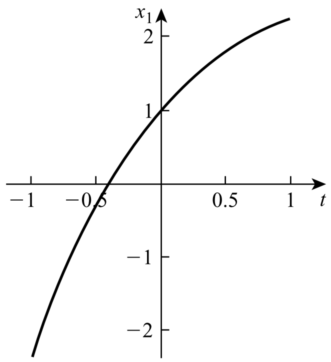
Exercises 6.4

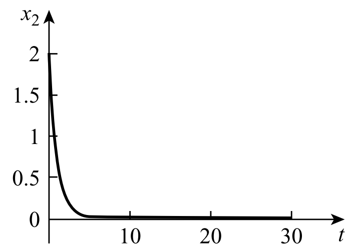
1. a. $\{x_1' = x_2, x_2' = -x_2;\ x_1(0) = 1, x_2(0) = 2\}$.

b.



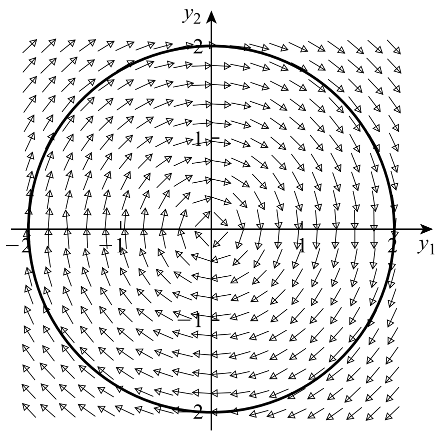
c.



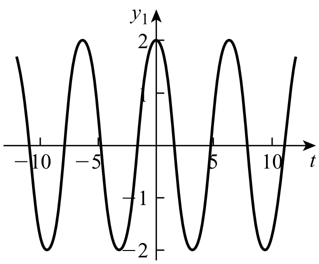


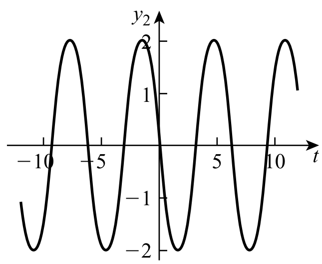
3. a. $\{\dot{y}_1 = y_2, \dot{y}_2 = -y_1;\ y_1(0) = 2, y_2(0) = 0\}$.

b.



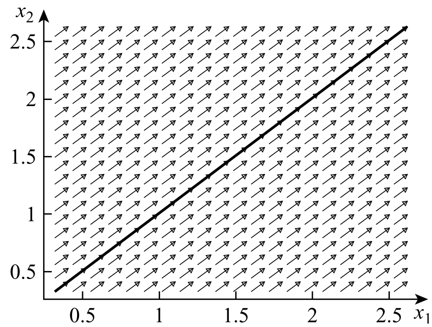
c.



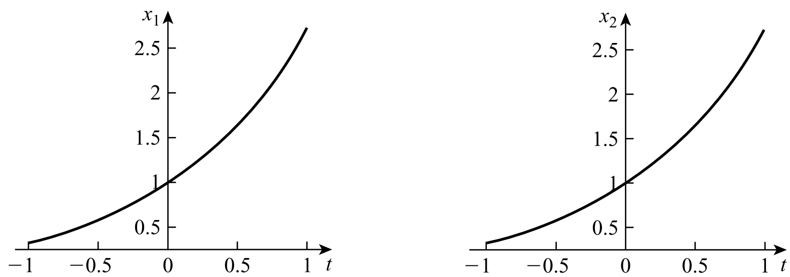


5. a. $\{\dot{x}_1 = x_2, \dot{x}_2 = x_2;\ x_1(0) = 1 = x_2(0)\}$.

b.

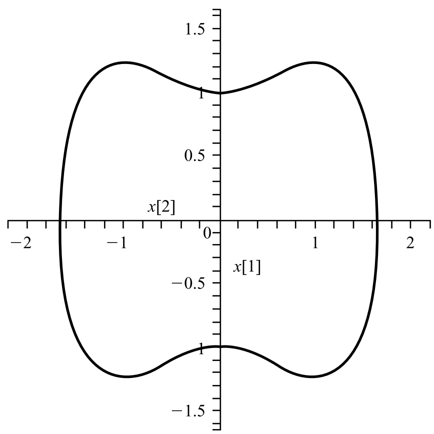


c.

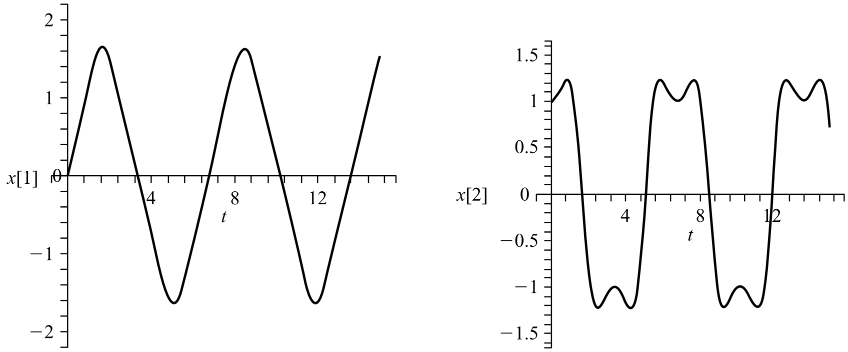


7. **a.** $\{x_1' = x_2, x_2' = x_1 - x_1^3;\ x_1(0) = 0, x_2(0) = 1\}$.

b.



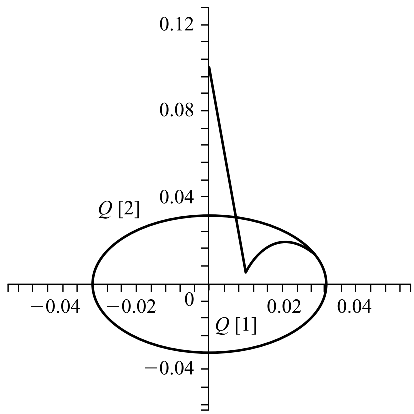
c.



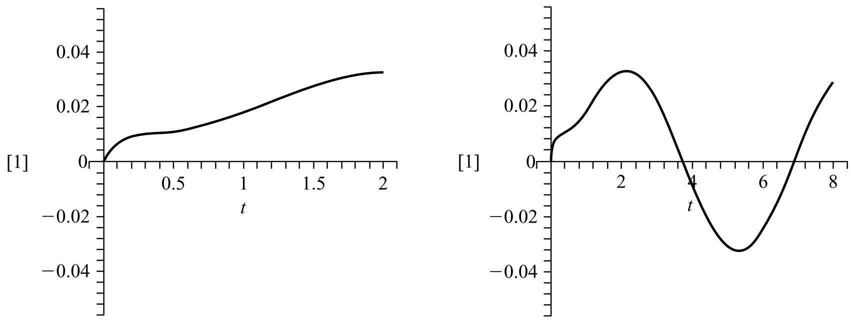
9. $2\arctan\left(\frac{y}{x}\right)+\ln\left(x^2+y^2\right)-C=0.$

11. **a.** $\{\dot{Q}_1=Q_2,\,\dot{Q}_2=\frac{1}{2}\sin t-14Q_1-9Q_2;\;\;Q_1(0)=0,\,Q_2(0)=0.1\}.$

b.

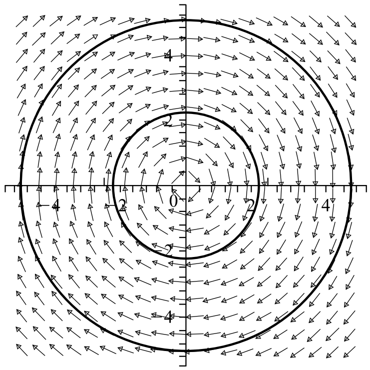


c.



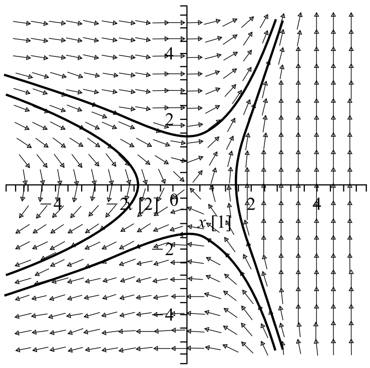
d. The capacitance appears to be periodic.

13. Every point on the x- and y-axes is an equilibrium point. The phase portrait:

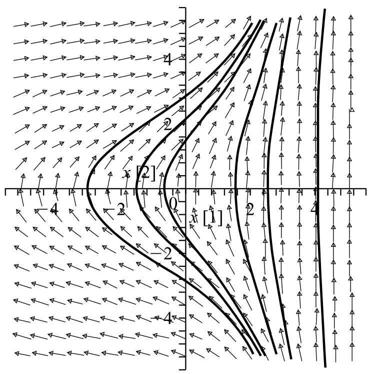


15. The system form of the equation is $\{x_1' = x_2,\ x_2' = -x_2 + x_1 - x_1^3\}$. The equilibrium solutions are $(0, 0)$, $(-1, 0)$, and $(1, 0)$.

17. **a.** With $\lambda = 1$ the phase portrait is



b. With $\lambda = -1$ the phase portrait is



c. The value $\lambda = 0$ is a *bifurcation point*, a point at which the phase portrait changes drastically.

Exercises 6.5

1. a. $\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$;

b. $\begin{bmatrix} \pi & -3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

c. $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ 2 & -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}$.

3. $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

5. $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

7. $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

9. $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

11. a. The system is $\{y_1' = y_2,\ y_2' = 3y_2 - 2y_1\}$, which can be written as $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

b. The system is $\left\{y_1' = y_2,\ y_2' = \frac{1}{5}y_1 - \frac{3}{5}y_2\right\}$, which can be written as $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

c. The system is $\{y_1' = y_2,\ y_2' = -\omega^2 y_1\}$, which can be written as $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Exercises 6.6

1. a. 17

b. 0

c. $6t^4 + 4\sin t$

d. $\cos^2\theta + \sin^2\theta = 1$.

3. $\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}$, for example.

5. a. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

b. The characteristic equation is $\lambda^2 - (1+1)\lambda + (1(1) - (-1)(-4)) = \lambda^2 - 2\lambda - 3 = 0$.

c. $\lambda_1 = 3$ and $\lambda_2 = -1$.

d. Any nonzero vector of the form $\begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 3$. Any nonzero vector of the form $\begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$.

7. a. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

b. The characteristic equation is $\lambda^2 - (1+1)\lambda + (1-0) = \lambda^2 - 2\lambda + 1 = 0$.

c. $\lambda_1 = 1 = \lambda_2$.

d. Clearly, any nonzero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$. As we will see in Section 6.8, a 2×2 system that has a repeated eigenvalue (an eigenvalue of "multiplicity two") must be handled carefully. In this problem, we can find at least two eigenvectors that do not lie on the same straight line—for example, $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

9. a. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

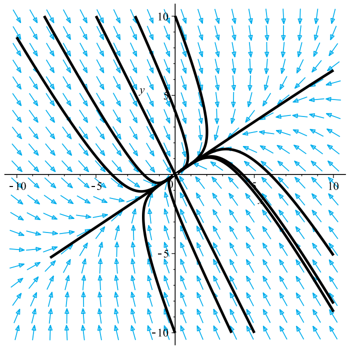
b. $\lambda^2 - 6\lambda + 7 = 0$.

c. $\lambda_1 = 3 + \sqrt{2}, \lambda_2 = 3 - \sqrt{2}$.

d. $V_1 = \begin{bmatrix} 1 \\ 2 - \sqrt{2} \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 2 + \sqrt{2} \end{bmatrix}$.

11. b. $x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $y = \begin{vmatrix} a & e \\ c & f \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

15. The phase portrait corresponding to the system is



The trajectories are moving toward the origin as t increases. Algebraically, this is a consequence of the fact that both eigenvalues are negative. Furthermore, the trajectories approach the line determined by the eigenvector associated with the larger (less negative) eigenvalue as $t \to \infty$. The trajectories approach the line determined by the eigenvector associated with the smaller (more negative) eigenvalue as $t \to -\infty$.

17. **a.** $c_1(x) = \alpha_1 \left(\frac{C_0 - c_0}{\alpha_2 - \alpha_1}\right) e^{(\alpha_2 - \alpha_1)x} + \frac{\alpha_2 c_0 - \alpha_1 C_0}{\alpha_2 - \alpha_1}$ and

$c_2(x) = \alpha_2 \left(\frac{C_0 - c_0}{\alpha_2 - \alpha_1}\right) e^{(\alpha_2 - \alpha_1)x} + \frac{\alpha_2 c_0 - \alpha_1 C_0}{\alpha_2 - \alpha_1}$.

19. **a.** The system is $\left\{\dot{x}_1 = x_2,\ \dot{x}_2 = -kx_2 - \frac{g}{L}x_1\right\}$.

b. $\lambda^2 + k\lambda + \frac{g}{L} = 0$.

c. $\lambda_1 = \left(-kL + \sqrt{k^2L^2 - 4gL}\right)\Big/2L$, $\lambda_2 = \left(-kL - \sqrt{k^2L^2 - 4gL}\right)\Big/2L$.

d. $$V_1 = \begin{bmatrix} 1 \\ \left(-kL + \sqrt{k^2L^2 - 4gL}\right)\Big/2L \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 1 \\ \left(-kL - \sqrt{k^2L^2 - 4gL}\right)\Big/2L \end{bmatrix}.$$

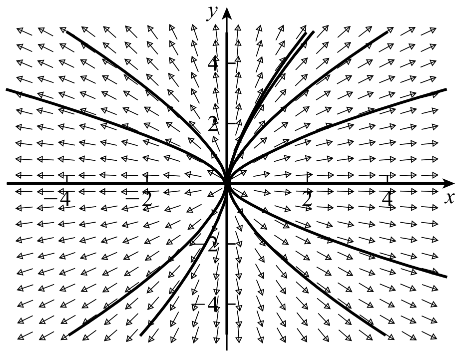
e. As $t \to \infty$ the pendulum tends to its equilibrium position $(0, 0)$.

21. **d.** Part (c) shows that the sign of $d\theta/dt$ equals the sign of $x\dot{y} - y\dot{x}$, which has just been shown to equal $c\left[x + \left(\frac{d-a}{2c}\right)y\right]^2 + \frac{y^2}{4c}\left[4(ad - bc) - (a+d)^2\right]$. The first term of this last expression is c times a perfect square, while in the second term the bracketed expression is positive according to part (b). Given the presence of another perfect square, $y^2/4$, it is clear that $x\dot{y} - y\dot{x}$ (and so $d\theta/dt$) must have the same sign as c.

Exercises 6.7

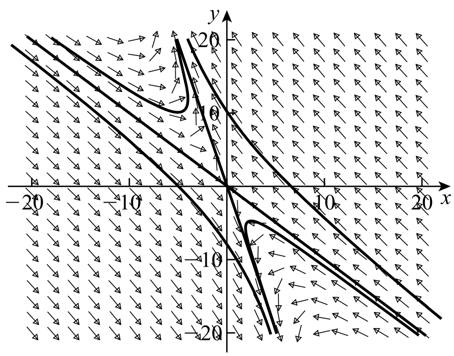
1. **a.** $\lambda_1 = 3$ and λ_2: Any nonzero vector of the form $\begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 3$. Any nonzero vector of the form $\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

b. Here is a plot of several trajectories with the eigenvectors (essentially, the x- and y-axes) shown:



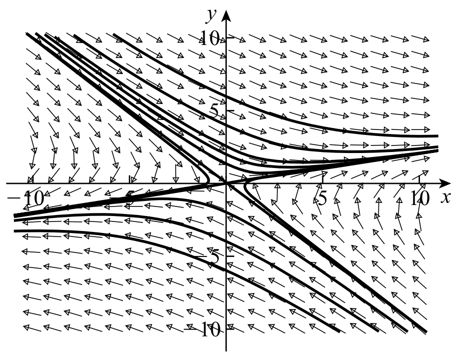
3. a. $\lambda_1 = 1$ and $\lambda_2 = -2$: Any nonzero vector of the form $\begin{bmatrix} x \\ -4x \end{bmatrix} = x \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. Any nonzero vector of the form $\begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$.

b. Here's the plot of several trajectories and the eigenvectors:



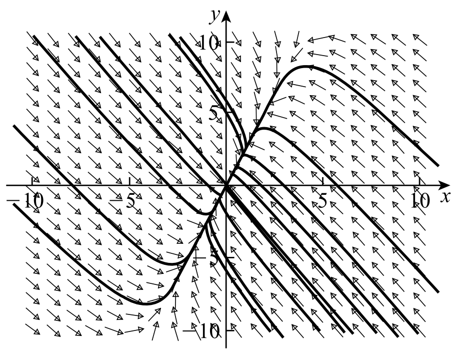
5. a. $\lambda_1 = 2$ and $\lambda_2 = -4$: Any nonzero vector of the form $\begin{bmatrix} x \\ \frac{1}{5}x \end{bmatrix} = x \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} = x \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$. Any nonzero vector of the form $\begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -4$.

b. Trajectories and eigenvectors are shown here:



7. a. $\lambda_1 = \frac{-5+\sqrt{17}}{2}$ and $\lambda_2 = \frac{-5-\sqrt{17}}{2}$, both irrational numbers; using a CAS we find corresponding representative eigenvectors $V_1 = \begin{bmatrix} \frac{-1+\sqrt{17}}{8} \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} \frac{-1-\sqrt{17}}{8} \\ 1 \end{bmatrix}$.

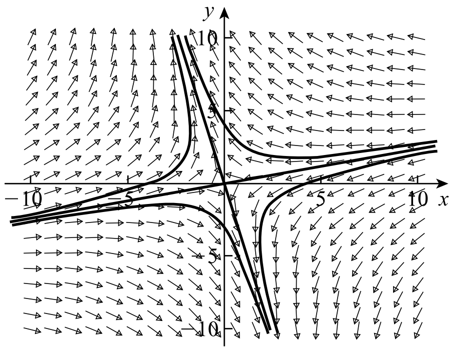
b. Here are some trajectories and the eigenvectors (which are difficult to pick out):



9. a. $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$, irrational numbers: Any nonzero vector of the form $\begin{bmatrix} x \\ -\left(2+\sqrt{5}\right) \end{bmatrix} = x \begin{bmatrix} 1 \\ -\left(2+\sqrt{5}\right) \end{bmatrix} = x \begin{bmatrix} 2-\sqrt{5} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = \sqrt{5}$. Any nonzero vector of the form

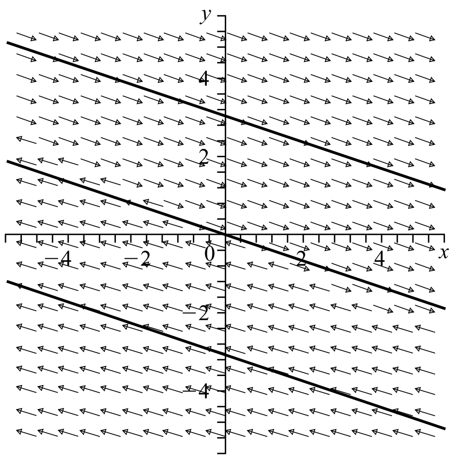
$$\begin{bmatrix} x \\ -\left(2-\sqrt{5}\right)x \end{bmatrix} = x\begin{bmatrix} 1 \\ -\left(2-\sqrt{5}\right) \end{bmatrix} = x\begin{bmatrix} 2+\sqrt{5} \\ 1 \end{bmatrix}$$ is an eigenvector corresponding to the eigenvalue $\lambda = -\sqrt{5}$.

b. Here are some trajectories and the eigenvectors:



11. a. $\lambda_1 = 0,\ V_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix};\ \lambda_2 = 1,\ V_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

b. Here are the eigenvectors and some trajectories:

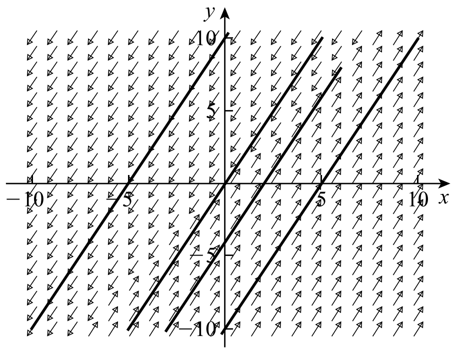


The two eigenvectors are very close together and may be difficult to see as separate vectors.

13. a. $\lambda = 0$ and $\lambda = -2$.

b. Representative eigenvectors corresponding to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$ are $V_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively.

c. Here's a plot of some trajectories:

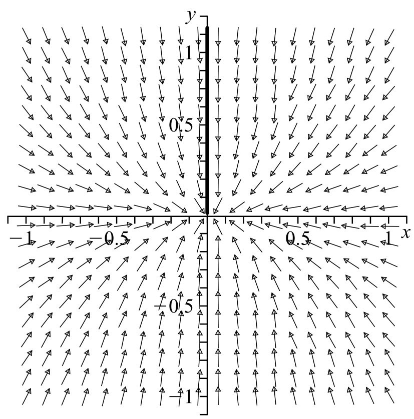


Every point of the line $y = \frac{4}{3}x$ is an equilibrium point. The origin is a *sink*, while every other point on the line is a *node*. All the other trajectories (straight lines) seem to be parallel to the trajectory determined by the eigenvector V_1.

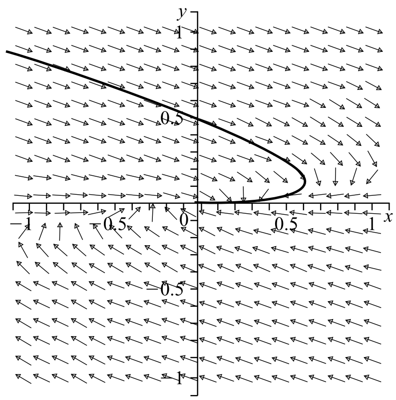
d. $X(t) = c_1 e^{0 \cdot t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

15. $X(t) = \frac{1}{2} e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; the solution approaches $\frac{1}{2} e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (the line $y = 5x$) asymptotically.

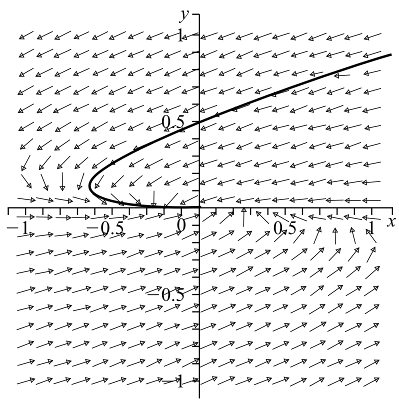
17. b. Here are some trajectories for different values of α:



$\alpha = 0$



$\alpha = 5$



$\alpha = -5$

When $\alpha = 0$ the trajectory is a straight line from $(0, 0.5)$ to the origin. For $\alpha > 0$ the trajectory swirls down from $(0, 0.5)$ toward the origin in a clockwise direction, flattening as α increases. For $\alpha < 0$ the trajectory swirls from $(0, 0.5)$ toward the origin in a counterclockwise direction, flattening as α increases in the negative direction.

19. $\left(\frac{1}{CR_2} - \frac{R_1}{L}\right)^2 - \frac{4}{CL} > 0$ [Also: $L + CR_1R_2 > 2R_2\sqrt{LC}$, or $L > R_2\left(2\sqrt{LC} - CR_1\right)$.

23. a. $$X(t) = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix} = c_1 e^{\left(\frac{-2+\sqrt{2}}{2}\right)t} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{\left(\frac{-2-\sqrt{2}}{2}\right)t} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}c_1 e^{\left(\frac{-2+\sqrt{2}}{2}\right)t} + \sqrt{2}c_2 e^{\left(\frac{-2-\sqrt{2}}{2}\right)t} \\ c_1 e^{\left(\frac{-2+\sqrt{2}}{2}\right)t} + c_2 e^{\left(\frac{-2-\sqrt{2}}{2}\right)t} \end{bmatrix}$$: The origin is a *sink*.

b. $X(t) = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix} = c_1 e^{\left(\sqrt{2}-1\right)t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + c_2 e^{\left(-1-\sqrt{2}\right)t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$

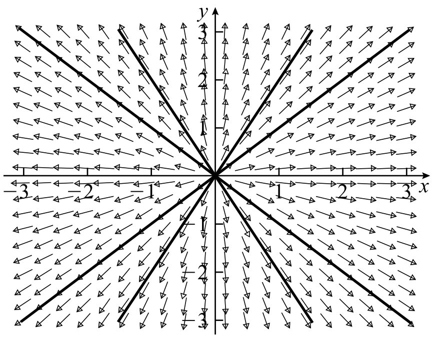
$= \begin{bmatrix} c_1 e^{\left(\sqrt{2}-1\right)t} + c_2 e^{\left(-1-\sqrt{2}\right)t} \\ -\sqrt{2} c_1 e^{\left(\sqrt{2}-1\right)t} + \sqrt{2} c_2 e^{\left(-1-\sqrt{2}\right)t} \end{bmatrix}$: The origin is a *saddle point*.

c. $\lambda_1 = -1 + \sqrt{\beta}$ and $\lambda_2 = -1 - \sqrt{\beta}$; the bifurcation point occurs at $\beta = 1$.

Exercises 6.8

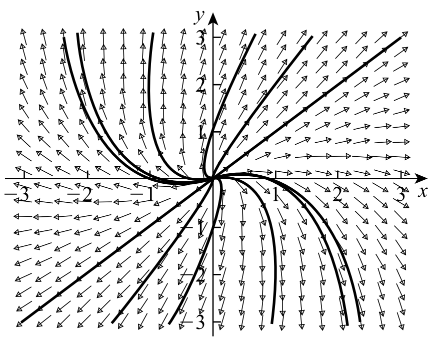
1. a. $\lambda_1 = 3 = \lambda_2$; $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b. Here's a plot of the eigenvectors and some trajectories:



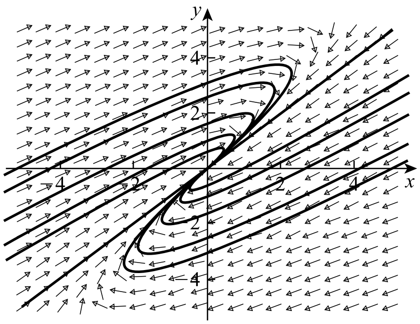
3. a. $\lambda_1 = 3 = \lambda_2$: Any nonzero vector of the form $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the repeated eigenvalue $\lambda = 3$. All eigenvectors lie on the straight line determined by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and there is only one linearly independent eigenvector.

b. Here's a plot of the eigenvector and some trajectories:



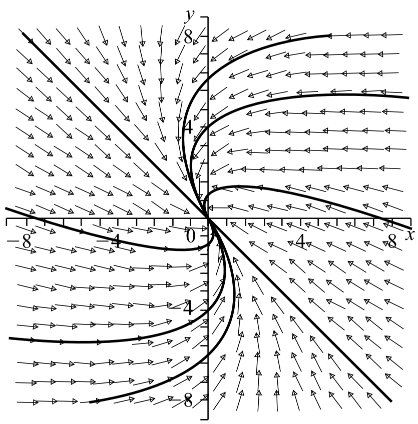
5. a. $\lambda_1 = -1 = \lambda_2$: Any nonzero vector of the form $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the repeated eigenvalue $\lambda = -1$. All eigenvectors lie on the straight line determined by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and there is only one linearly independent eigenvector.

b. Here's a plot of the eigenvector and some trajectories:



7. a. $\lambda_1 = -2 = \lambda_2$: Any nonzero vector of the form $\begin{bmatrix} -x \\ x \end{bmatrix} = x \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the repeated eigenvalue $\lambda = -2$. All eigenvectors lie on the straight line determined by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and there is only one linearly independent eigenvector.

b. Here's a plot of the eigenvector and some trajectories:



9. $\alpha^2 = 4\beta$.

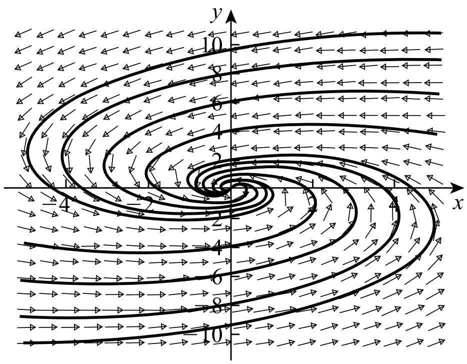
11. Two such systems are $\{\dot{x} = -2x,\ \dot{y} = -2y\}$ and $\{\dot{x} = x + 3y,\ \dot{y} = -3x - 5y\}$.

15. b. The sole linearly independent eigenvector is $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Exercises 6.9

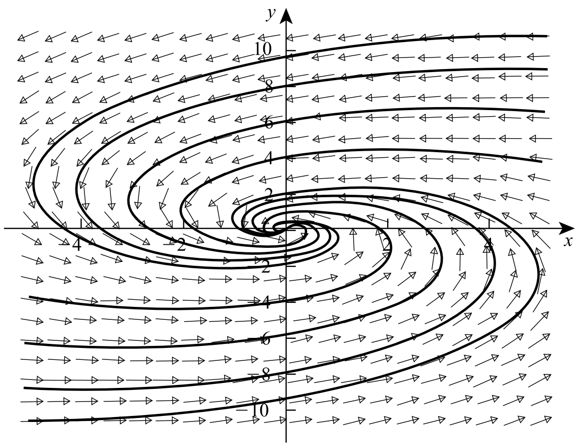
1. **a.** $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$; $V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $V_2 = \overline{V}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

b. Here's a plot of some trajectories, spirals swirling into the origin (a sink):



3. **a.** $\lambda_1 = -0.5 + i$ and $\lambda_2 = -0.5 - i$; $V_1 = \begin{bmatrix} -1 \\ i \end{bmatrix}$ and $V_2 = \begin{bmatrix} -1 \\ -i \end{bmatrix}$.

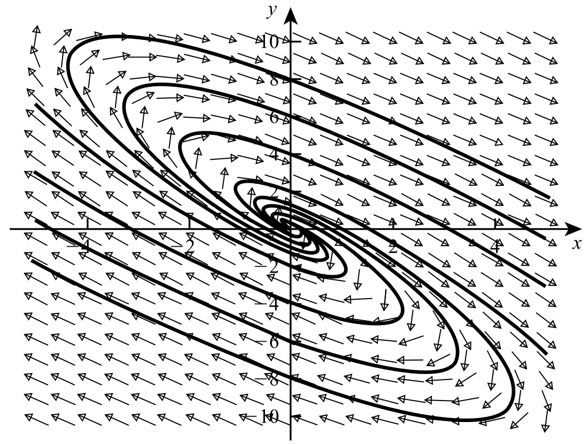
b. Some trajectories, spirals swirling toward the origin (a sink):



5. **a.** $\lambda_1 = \left(1 + \sqrt{3}i\right)/2$ and $\lambda_2 = \left(1 - \sqrt{3}i\right)/2$; $V_1 = \begin{bmatrix} 1 \\ \left(-3 + \sqrt{3}i\right)/2 \end{bmatrix}$ and

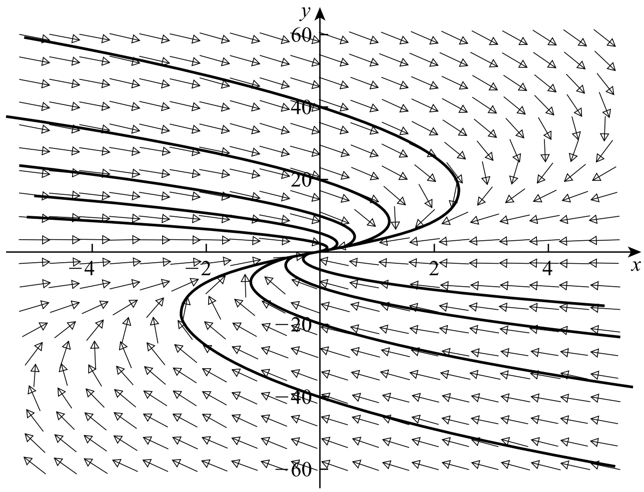
$V_2 = \begin{bmatrix} 1 \\ \left(-3 - \sqrt{3}\,i\right)/2 \end{bmatrix}$.

b. Here are some trajectories, spirals swirling away from the origin (a source):



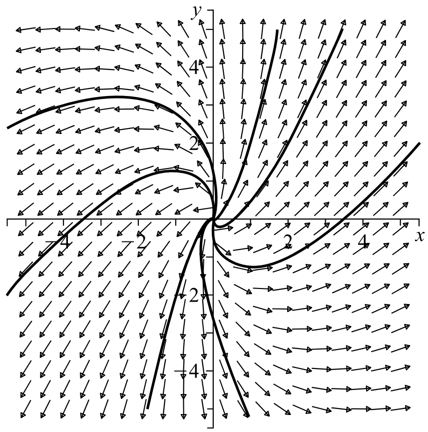
7. a. $\lambda_1 = -6 + i$ and $\lambda_2 = -6 - i$; $V_1 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$.

b. Some trajectories, spirals swirling toward the origin (a sink):



9. a. $\lambda_1 = 5 + 2i$ and $\lambda_2 = 5 - 2i$; $V_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$.

b. Some trajectories, spirals swirling away from the origin (a source):



11. **a.** One such system has the matrix of coefficients $\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$, which yields the system $\{\dot{x} = 2x - 4y,\ \dot{y} = x + 2y\}$.

b. One such system has the matrix of coefficients $\begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}$, which yields the system $\{\dot{x} = x - 2y,\ \dot{y} = 5x - y\}$.

c. One such system has the matrix of coefficients $\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$, which yields the system $\{\dot{x} = -x + 2y, \dot{y} = -2x - y\}$.

15. $X(t) = e^{-\frac{2}{3}t} \begin{bmatrix} \cos\left(\frac{\sqrt{2}}{3}t\right) - 2\sqrt{2}\sin\left(\frac{\sqrt{2}}{3}t\right) \\ -3\sqrt{2}\sin\left(\frac{\sqrt{2}}{3}t\right) \end{bmatrix}$.

Exercises 6.10

1. $\left\{x(t) = \frac{1}{2}e^t - \frac{1}{2}(\sin t + \cos t),\ y(t) = t - 1 + 2e^{-t}\right\}$.

3. $$X_{\text{GNH}} = \begin{bmatrix} c_1e^t - c_2e^{-t} + te^t - \frac{1}{2}e^t - t^2 - 2 \\ c_1e^t + c_2e^{-t} + te^t - \frac{3}{2}e^t - 2t \end{bmatrix}$$
$$= \begin{bmatrix} \left(c_1 - \frac{1}{2}\right)e^t - c_2e^{-t} + te^t - t^2 - 2 \\ \left(c_1 - \frac{3}{2}\right)e^t + c_2e^{-t} + te^t - 2t \end{bmatrix}.$$

5. $$X_{\text{GNH}} = \begin{bmatrix} 2c_1e^{4t} - c_2e^t + 3e^{5t} \\ c_1e^{4t} + c_2e^t + e^{5t} \end{bmatrix}.$$

7. $X_{\mathrm{GNH}} = \begin{bmatrix} -c_1 e^{3t} + c_2 e^{2t} + t e^{2t} \\ c_1 e^{3t} - 2c_2 e^{2t} + 2e^{2t} - 2t e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{3t} + (t + c_2)\, e^{2t} \\ c_1 e^{3t} - (2t - 2 + 2c_2)\, e^{2t} \end{bmatrix}.$

9. $X_{\mathrm{GNH}} = \begin{bmatrix} 3c_1 e^{4t} + c_2 e^{2t} - 4e^{3t} - e^{-t} \\ c_1 e^{4t} + c_2 e^{2t} - 2e^{3t} - 2e^{-t} \end{bmatrix}.$

11. $X_{\mathrm{GNH}} = \begin{bmatrix} -c_1 e^{3t} + c_2 e^{t} + 2e^{t} \cos t - e^{t} \sin t \\ c_1 e^{3t} + c_2 e^{t} + 3e^{t} \cos t + e^{t} \sin t \end{bmatrix}.$

13. $X_{\mathrm{GNH}} = \begin{bmatrix} -\frac{1}{2} c_1 e^{-2t} - c_2 e^{-t} + \frac{1}{10} \sin t - \frac{3}{10} \cos t + \frac{1}{6} e^{t} \\ -c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10} \cos t + \frac{3}{10} \sin t + \frac{1}{6} e^{t} \end{bmatrix}.$

15. $X_{\mathrm{GNH}} = \begin{bmatrix} (c_1 + (c_2 + 1)\, t)\, e^{t} + \frac{1}{4} e^{-t} \\ c_2 e^{t} - \frac{1}{2} e^{-t} \end{bmatrix}.$

19. d. $G = g_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where $g_1 \neq 0$ is arbitrary.

e. $G = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Then, letting $f_1 = 0$ for convenience, the equations imply that $f_2 = 3$, so that $F = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

21. c. The eigenvectors corresponding to λ_1 and λ_2 are $V_1 = \begin{bmatrix} (1 + \sqrt{57})/4 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} (1 - \sqrt{57})/4 \\ 1 \end{bmatrix}$, respectively.

d. $X_{\mathrm{GNH}} = c_1 e^{(-9+\sqrt{57})\, t/2} \begin{bmatrix} (1 + \sqrt{57})/4 \\ 1 \end{bmatrix} + c_2 e^{(-9-\sqrt{57})\, t/2} \begin{bmatrix} (1 - \sqrt{57})/4 \\ 1 \end{bmatrix}$

e. Any particular solution has to be a vector of constants; for example, $X_{\mathrm{PNH}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ works.

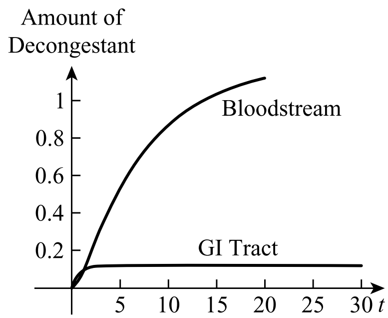
f. $X_{\mathrm{GH}}(t) = X_{\mathrm{GNH}}(t) + X_{\mathrm{PNH}} \to \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as $t \to \infty$ because all exponential terms have negative exponents.

23. a. $X_{\mathrm{GNH}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{I}{k_1} e^{-k_1 t} + I/k_1 \\ \frac{I}{k_1 - k_2} e^{-k_1 t} - \frac{k_1 I}{k_2 (k_1 - k_2)} e^{-k_2 t} + I/k_2 \end{bmatrix}$

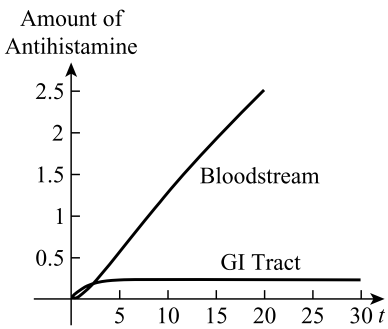
$$= \begin{bmatrix} \frac{I}{k_1} \left(1 - e^{-k_1 t}\right) \\ \frac{I}{k_2} \left(1 + \frac{k_2}{k_1 - k_2} e^{-k_1 t} - \frac{k_1}{k_1 - k_2} e^{-k_2 t}\right) \end{bmatrix}.$$

b. $\lim_{t \to \infty} x(t) = \frac{I}{k_1}$ and $\lim_{t \to \infty} y(t) = \frac{I}{k_2}$.

c. The graphs of $x(t)$ and $y(t)$ for the decongestant:



d. The graphs of $x(t)$ and $y(t)$ for the antihistamine:



25. a. $ax - by + e > 0.$

b. $-cx + dy - e > 0.$

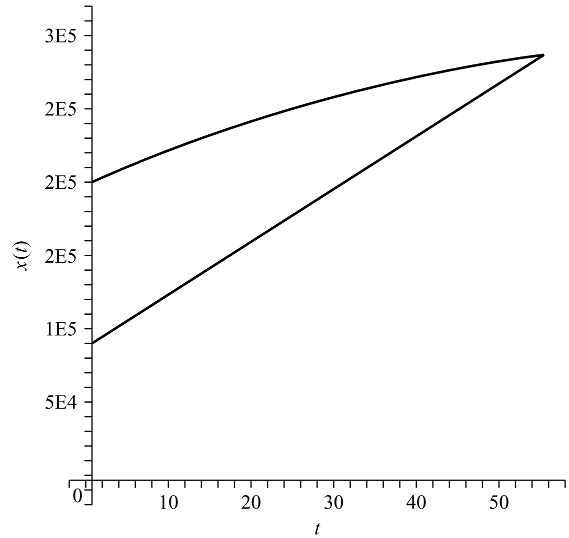
c. We have the equilibrium point $(x^*, y^*) = \left(\frac{e(b-d)}{ad-bc}, \frac{e(a-c)}{ad-bc}\right)$, provided that $b \geq d, a \geq c$, and $ad - bc \neq 0$. (We can't have a negative number of supporters.)

27. a. $X_{\text{GNH}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$= \begin{bmatrix} \left[\frac{1}{2}\left(\alpha - \frac{l-d}{b}\right) - \frac{\sqrt{ab}}{2b}\left(\beta - \frac{k-c}{a}\right)\right] e^{\sqrt{ab}t} \\ \quad + \left[\frac{1}{2}\left(\alpha - \frac{l-d}{b}\right) + \frac{\sqrt{ab}}{2b}\left(\beta - \frac{k-c}{a}\right)\right] e^{-\sqrt{ab}t} + \frac{l-d}{b} \\ \left[-\frac{\sqrt{ab}}{2a}\left(\alpha - \frac{l-d}{b}\right) + \frac{1}{2}\left(\beta - \frac{k-c}{a}\right)\right] e^{\sqrt{ab}t} \\ \quad + \left[\frac{\sqrt{ab}}{2a}\left(\alpha - \frac{l-d}{b}\right) + \frac{1}{2}\left(\beta - \frac{k-c}{a}\right)\right] e^{-\sqrt{ab}t} + \frac{k-c}{a} \end{bmatrix}.$$

b. $\beta - \frac{k-c}{a} < \frac{\sqrt{ab}}{a}\left(\alpha - \frac{l-d}{b}\right).$

c. Here are the graphs of $x(t)$ and $y(t)$ for $0 \le t \le 50$:

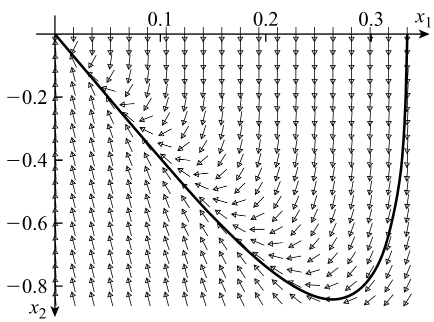


Extending the time axis a bit and using the "trace" or "zoom" capabilities of a CAS or graphing calculator, we find that $x(t^*) = y(t^*)$ when $t^* \approx$ 55 days. From the graph we see that side "y" is winning after 50 days.

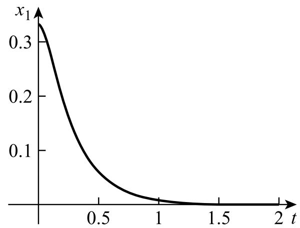
Exercises 6.11

1. **a.** $\{\dot{x}_1 = x_2,\ \dot{x}_2 = -64x_1 - 20x_2;\ x_1(0) = 1/3,\ x_2(0) = 0\}$.

b.



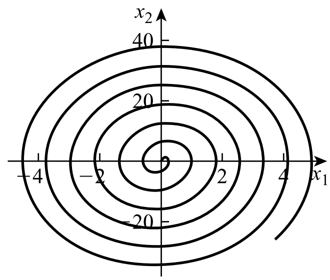
c.



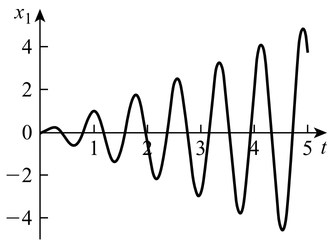
d. The mass approaches its equilibrium position but doesn't quite reach it because of the large damping force. In particular, the mass doesn't overshoot its equilibrium position.

3. a. $\{\dot{x}_1 = x_2,\ \dot{x}_2 = 16\cos 8t - 64x_1;\ x_1(0) = 0,\ x_2(0) = 0\}$.

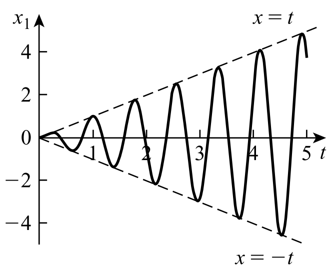
b.



c.



d. The half-lines $x = t$ and $x = -t$ are asymptotes for the graph in (c) for $t \geq 0$:



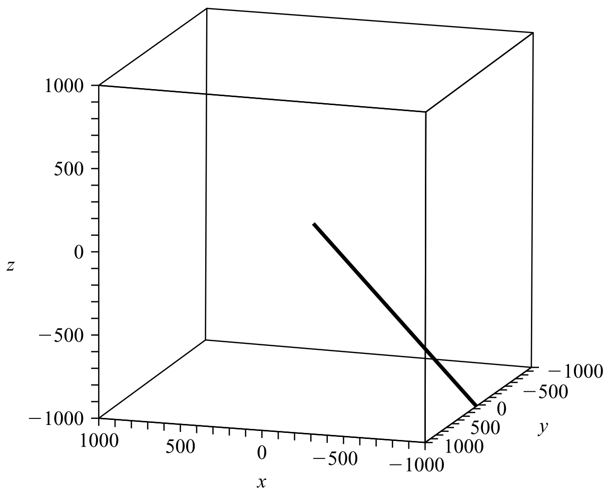
5. a. $(M+0.5)\ddot{x}+10\dot{x}+kx=0$.
   b. $n=\sqrt{1000\,M+400}/(2\,M+1)$.
   c. $t=\frac{2M+1}{10}\ln 4$.
   d. Removing the damping force means that the scales would continue to oscillate and would not settle down to allow a reading to be taken.
7. a. $Q''+8Q'+15Q=0$: This equation *can* represent a spring-mass system with spring constant 15 and damping constant 8. (We could also have derived a single second-order equation in R.)
   b. $\ddot{x}-6\dot{x}+10x=0$: This second-order equation *cannot* represent a spring-mass system because the equation implies that any damping force works in the same direction as the mass's motion. (We could also have derived a single second-order equation in y.)

Exercises 6.12

1. a. $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 2 & -1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.
   b. $\lambda_1=-1,\ V_1=\begin{bmatrix}1\\ -3\\ -5\end{bmatrix};\ \lambda_2=1,\ V_2=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix};\ \lambda_3=2,\ V_3=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$.
   c. $X(t)=\begin{bmatrix}c_1e^{-t}+c_2e^{t}+c_3e^{2t}\\ -3c_1e^{-t}+c_2e^{t}\\ -5c_1e^{-t}+c_2e^{t}+c_3e^{2t}\end{bmatrix}$.
3. a. $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}3 & -1 & 1\\ 1 & 1 & 1\\ 4 & -1 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.
   b. $\lambda_1=1,\ V_1=\begin{bmatrix}1\\ 1\\ -1\end{bmatrix};\ \lambda_2=2,\ V_2=\begin{bmatrix}1\\ -2\\ -3\end{bmatrix};\ \lambda_3=5,\ V_3=\begin{bmatrix}1\\ 1\\ 3\end{bmatrix}$.
   c. $X(t)=\begin{bmatrix}c_1e^{t}+c_2e^{2t}+c_3e^{5t}\\ c_1e^{t}-2c_2e^{2t}+c_3e^{5t}\\ -c_1e^{t}-3c_2e^{2t}+3c_3e^{5t}\end{bmatrix}$.
5. a. $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.
   b. $\lambda_1=2,\ V_1=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix};\ \lambda_2=3,\ V_2=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix};\ \lambda_3=1,\ V_3=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$.

c. $X(t) = \begin{bmatrix} c_1 e^{3t} + c_3 e^{2t} \\ c_2 e^{t} + c_3 e^{2t} \\ c_1 e^{3t} + c_2 e^{t} + c_3 e^{2t} \end{bmatrix}.$

7. The space trajectory through (0, 1, 0) when $t = 0$ is



11. $X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}.$

13. a. $\{y_1' = y_2,\ y_2' = -2y_1 + y_3,\ y_3' = y_4,\ y_4' = y_1 - 2y_3\}.$

b. $\lambda_1 = i,\ \lambda_2 = -i,\ \lambda_3 = \sqrt{3}i,\ \lambda_4 = -\sqrt{3}i.$

c. $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_1'(t) \\ x_2(t) \\ x_2'(t) \end{bmatrix} \begin{bmatrix} 2\cos t + \sin t \\ -2\sin t + \cos t \\ 2\cos t + \sin t \\ -2\sin t + \cos t \end{bmatrix}.$

The masses oscillate in sync because $x_1(t) = x_2(t)$ and $x_1'(t) = x_2'(t)$ for all positive values of t.

d. $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_1'(t) \\ x_2(t) \\ x_2'(t) \end{bmatrix} \begin{bmatrix} 2\cos\left(\sqrt{3}t\right) + \sin\left(\sqrt{3}t\right) \\ -2\sqrt{3}\sin\left(\sqrt{3}t\right) + \sqrt{3}\cos\left(\sqrt{3}t\right) \\ -2\cos\left(\sqrt{3}t\right) - \sin\left(\sqrt{3}t\right) \\ 2\sqrt{3}\sin\left(\sqrt{3}t\right) - \sqrt{3}\cos\left(\sqrt{3}t\right) \end{bmatrix}.$

The masses are now out of sync because at any time t the masses are located at opposite sides of their respective equilibrium positions and are moving either toward each other or away from each other.

e. There are two natural frequencies, $1/2\pi$ for the system in part (c), and $\sqrt{3}/2\pi$. The "1" and the "$\sqrt{3}$" in the numerators of the frequencies are the imaginary parts of the eigenvalues. A third mode of oscillation is possible that combines the two natural frequencies already found.

15. $$X(t) = \begin{bmatrix} A(t) \\ B(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} 11{,}000e^{-0.02t} + (11{,}000/3)e^{-0.06t} + 25{,}000/3 \\ -(22{,}000/3)e^{-0.06t} + 25{,}000/3 \\ -11{,}000e^{-0.02t} + (11{,}000/3)\,e^{-0.06t} + 25{,}000/3 \end{bmatrix}.$$

17. a. $$\dot{X}(t) = \begin{bmatrix} -0.1 & 0.02 & 0 & 0 & 0 & 0 \\ 0.1 & -0.14 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 0.02 & 0 & 0 \\ 0 & 0 & 0.1 & -0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 & 0.02 \\ 0 & 0 & 0 & 0 & 0.1 & -0.14 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

b. $x_1(t) = 50 - 4.59e^{-0.169t} - 45.41e^{-0.071t}$,

$x_2(t) = 50 + 15.82e^{-0.169t} - 65.82e^{-0.071t}$,

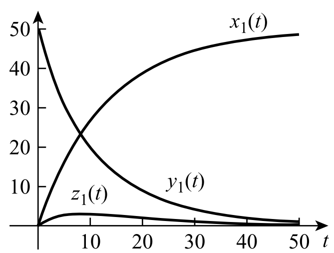
$y_1(t) = 14.79e^{-0.169t} + 35.21e^{-0.071t}$,

$y_2(t) = -51.03e^{-0.169t} + 51.03e^{-0.071t}$,

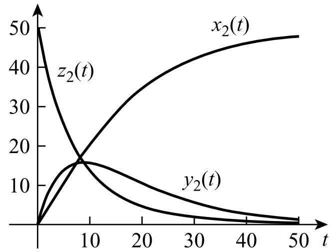
$z_1(t) = -10.21e^{-0.169t} + 10.21e^{-0.071t}$,

$z_2(t) = 35.21e^{-0.169t} + 14.79e^{-0.071t}$.

c. Here are $x_1(t)$, $y_1(t)$, and $z_1(t)$ on the same set of axes:



d. Here are $x_2(t)$, $y_2(t)$, and $z_2(t)$ on the same set of axes:



19. $X_{\mathrm{GNH}} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t^2 + \frac{c_3}{12} \\ t^3 + t^2 + \frac{c_3}{4}t + \frac{c_2}{4} - \frac{c_3}{48} \\ t^4 + \frac{5}{3}t^3 + \frac{1}{2}(c_3+1)t^2 + c_2 t + c_1 \end{bmatrix}.$

Exercises 7.1

1. $(0,0)$ and $\left(\frac{1}{2}, 1\right)$.

3. $(0,0)$, $(1,1)$, and $(-1,1)$.

5. $(0,0)$ and $(-1,-1)$.

7. $(0,0)$, $\left(0, \frac{3}{2}\right)$, $(1,0)$, and $(-1,2)$.

9. $(0, 2n\pi)$ and $(2, (2n+1)\pi)$, $n = 0, \pm 1, \pm 2, \ldots$.

11. $(1,-1)$, and $(1,1)$.

13. The entire x-axis except the origin, plus the point $(1,1)$.

15. $(0,0)$, $(1,0)$, $(-1,0)$, and $(-4,0)$.

17. $(0, a_2)$ and $\left(\frac{a_1(-1+a_1a_2-a_2)}{a_1-1}, \frac{1}{a_1-1}\right)$, provided that $a_1 \neq 1$.

19. $(-1.016, 0.166)$, $(-0.798, -1.450)$, $(-0.259, -1.208)$, $(0.355, 1.551)$, $(0.634, 1.900)$, and $(1.085, -0.956)$.

Exercises 7.2

17. **a.** The origin, $(0,0)$, is the only equilibrium solution for each system.

b. (a) $\{\dot{x} = -y,\ \dot{y} = x\}$; (b) $\{\dot{x} = -y,\ \dot{y} = x\}$.

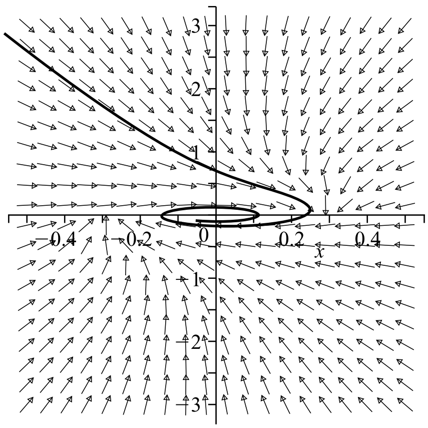
c. For system (a), the origin is a source (unstable node), whereas for system (b) the origin is a sink (stable node).

19. **a.** $\left\{\dot{x} = y, \dot{y} = 0.25x^2 - x\right\}$.

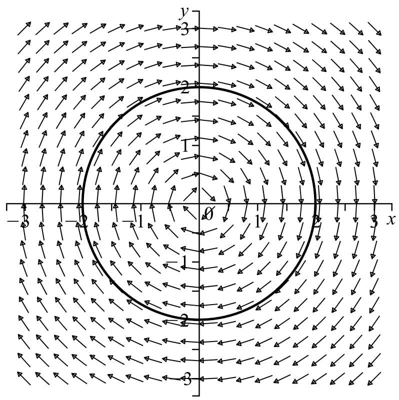
b. Yes, $(0,0)$ is an equilibrium solution.

c. $\{\dot{x} = y, \dot{y} = -x\}$.

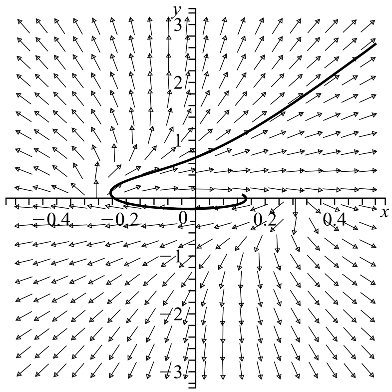
21. **d.** If $a < 0$, then $\dot{r} < 0$, implying that a trajectory will spiral into $(0,0)$, so that the origin is a *stable spiral point* (a sink). If $a = 0$, then $\dot{r} = 0$, so that r is a constant and the origin is a *stable center*. If $a > 0$, the origin is an *unstable spiral point* (a source). In the language of Section 2.7, the parameter value $a = 0$ is a *bifurcation point*.



$a < 0$



$a = 0$



$a > 0$

Exercises 7.3

1. The origin is a *spiral source*.

3. The origin is a *stable node*, a *sink*.

5. The origin is a *saddle point*.

7. The origin is a *center*.

9. The origin is a *spiral sink*.

11. **a.** $(0, 0)$ and $(4, 1)$.

b. $(0, 0)$ is a *saddle point*, whereas $(4, 1)$ is a *sink*.

13. **a.** $(0, 0)$, $(2, 0)$, and $(0, 3)$.

b. $(0, 0)$ is a *source*, $(2, 0)$ is a *sink*, and $(0, 3)$ is a *saddle point*.

15. **a.** The only equilibrium point is $\left(1, \frac{a}{b}\right)$.

b. $\dot{x} = (a-1)(x-1) + b\left(y - \frac{a}{b}\right)$, $\dot{y} = -a(x-1) - b\left(y - \frac{a}{b}\right)$. Letting $u = x - 1$ and $v = y - \frac{a}{b}$, this becomes the system $\{\dot{u} = (a-1)u + bv,\ \dot{v} = -au - bv\}$.

c. $\lambda = -\frac{1}{2} - \frac{b}{2} + \frac{a}{2} \pm \frac{1}{2}\sqrt{(a-b)^2 - 2(a+b) + 1}$.

d. (1) $(1, 3)$ is a *spiral source*; (2) $(1, 2/7)$ is a *sink*; (3) $(1, 1/4)$ is a *sink*.

Exercises 7.4

1. The only nontrivial equilibrium point is $\left(\frac{1}{4}, \frac{3}{2}\right)$.

3. The only nontrivial equilibrium point is $\left(\frac{3}{4}, \frac{1}{4}\right)$.

5. The only nontrivial equilibrium point is $(3, 3)$.

7. $[t = .1, x(t) = .905130981942487424, y(t) = 1.99036351756532892]$

$[t = .2, x(t) = .820792722038854339, y(t) = 1.96310057595484810]$

$[t = .3, x(t) = .746918149815353428, y(t) = 1.92096012363624435]$

$[t = .4, x(t) = .682988624646712062, y(t) = 1.8668278660690446]$

$[t = .5, x(t) = .628223724693027674, y(t) = 1.80349863147592626]$

$[t = .6, x(t) = .581725902364229608, y(t) = 1.73353191257205697]$

$[t = .8, x(t) = .509916830817235156, y(t) = 1.58232127656877064]$

$[t = .9, x(t) = .482944734671241827, y(t) = 1.504546664756024171]$

$[t = 1, x(t) = .460967197688796904, y(t) = 1.42710548153511119]$.

15. c. $y = \pm\sqrt{2\cos x + C}$.

17. a. $\theta(t) = \sin 2t$.

b. The period of $\sin 2t$ is $2\pi/2 = \pi$.

c. 19 ticks.

d. Halving the length of the pendulum reduces the amplitude and the period by a factor of $\frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}}$. In other words, shortening the pendulum makes it run faster, yielding more ticks per minute. (In this case, the clock will tick 27 times per minute.)

Exercises 7.5

5. a. The only bifurcation value is $\mu = 0$.

b. The bifurcation is transcritical.

7. a. The only equilibrium solution is $(0, 0)$.

b. $\{\dot{x} = (\mu + 1)x + y,\ \dot{y} = x - y\}$.

c. The eigenvalues are $\frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 + 4\mu + 8}$.

d. We have $\mu = -2$ as a bifurcation value, giving us a *pitchfork bifurcation*.

9. a. The origin, $(0, 0)$, is the only equilibrium solution.

b. $\{\dot{x} = \mu x,\ \dot{y} = -y\}$

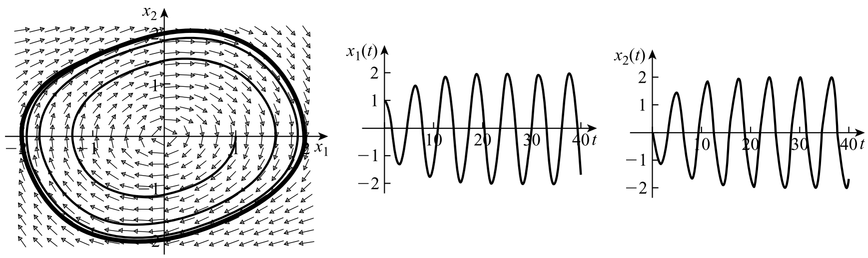
c. The eigenvalues are -1 and μ.

d. There is a pitchfork bifurcation at the origin.

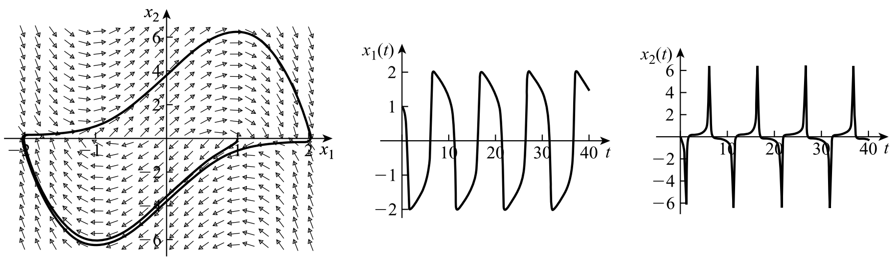
11. There are transcritical bifurcations along the curve $a = \frac{1}{b}$.

Exercises 7.6

3. a.



b.



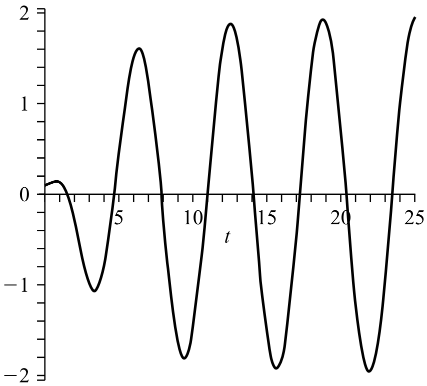
c. Each trajectory indicates the existence of a stable limit cycle. However, the shapes of the trajectories and the limit cycles change as ε changes. Similarly, $x_1(t)$ and $x_2(t)$ are periodic but not trigonometric; when ε changes from 1/4 to 4, $x_1(t)$ changes to a flatter shape, while $x_2(t)$ develops spikes.

7. a. $r(t) = 1 + \sqrt{\frac{1}{Ce^{2t} - 1}}$.

b. $\theta(t) = t + C$.

c. $(x(t), y(t)) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$.

9.

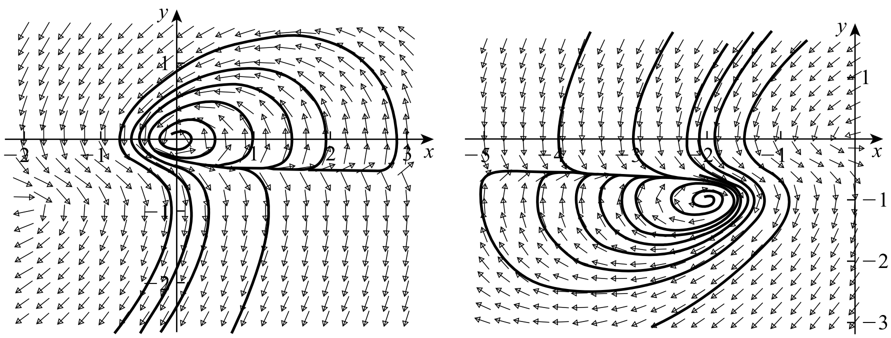


15. **a.** $\{\dot{x}_1 = x_2, \dot{x}_2 = -a(x_1)\,x_2 - b(x_1)\}$.

17. **a–b.** The only bifurcation value is $\mu = 0$, and the bifurcation is a supercritical Hopf bifurcation.

19. **a.** The only equilibrium points are $(0, 0)$ and $(-2, -1)$.

b. The phase portrait near the origin and near the point $(-2, -1)$:



c. There seems to be an unstable limit cycle around $(0, 0)$ (a source) and a stable limit cycle (a sink) around $(-2, -1)$.

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