Archimedean & Platonic Solids

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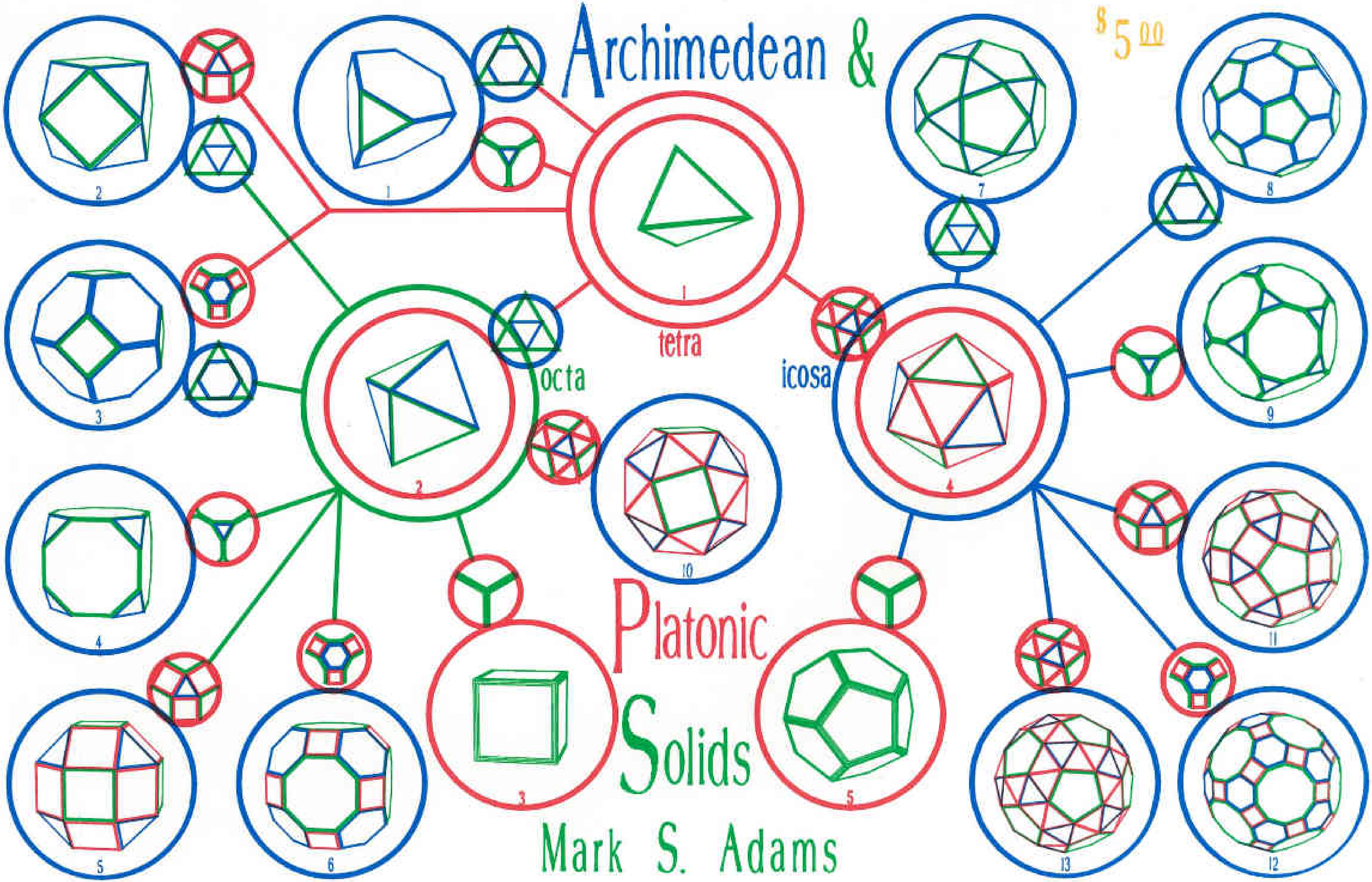

Mark S. Adams

Archimedean and Platonic Solids

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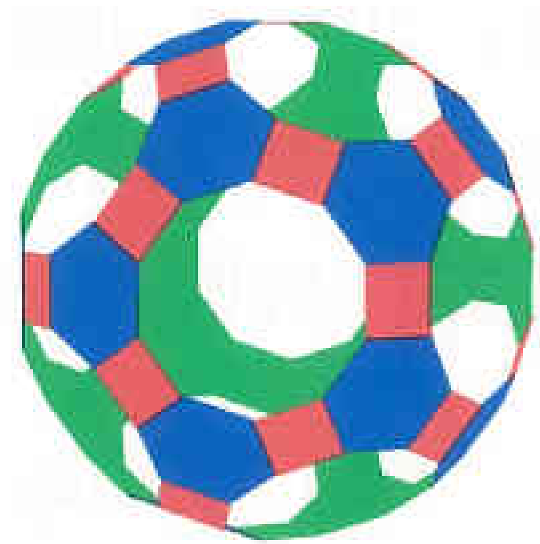
Geodesic Publications
P.O. Box 46156
Baton Rouge, Louisiana. 70895

First Edition.



Introduction

Volumes of the Archimedean and Platonic solids are presented. Proofs to the solids start on page seven, they are ordered by alternate and triacon geodesic breakdown of the tetrahedron, octahedron, and icosahedron base models. The polyhedron is first inscribed on the face planes of the base model. Its edge distance (D) is solved for. Then by dividing the base model radius (R) through by (D), we have the radius for unit edge length. The Pythagorean theorem is used to project the radius to the center of each face. Summing (n) number of volumes of each face pyramid of height (r) yields the complete volume.

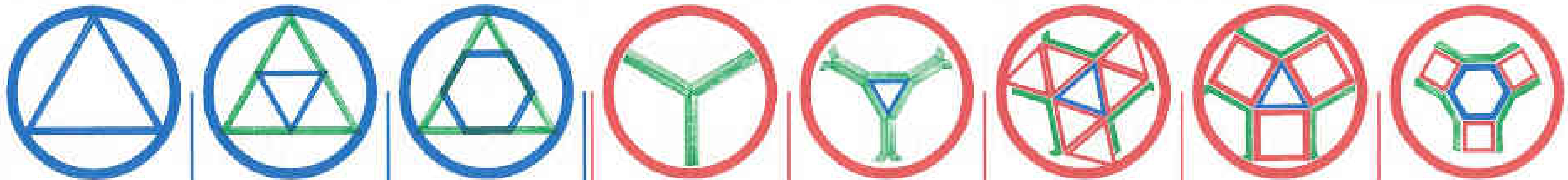
The fifth frequency triacon has the added freedom of curl angle. (α) Positive α is right-handed, negitive α is left-handed. This breakdown creates the four commensurable volume sets within the eighteen volumes.

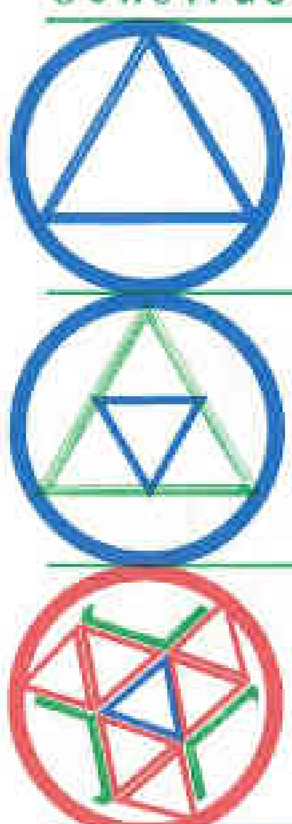
Excluding prisms and anti-prisms there exist twenty one semi-regular finite polyhedra, together possessing thirteen distinct volumes. The 2vA 5vT and $5vT^2$ solids have left and right handed duals. The 7vT, 2vA 7vT, and 5v7vT solids have interweaving edge rings surrounding each green base model vertex site. Spinnability relocation of red and blue faces around one or more sites forms polarized inter-patterning symmetry. A total of nine realizations of the three solids exist, their volumes are unaffected by the spinnability.

Icosahedral based volumes are plotted showing that powers of the Golden Section (τ) divide alternate and triacon regions. The integer part squared minus radical part squared will be equal to one for even powers of τ and minus one for all odd powers. This rule may be extended to the one third power harmonic as plotted.

Spherical Tessellations:

regular {p q}	P Platonic	(o) octa-	(i) icosa-	(r) rhombi
quasi-regular $\left\{ {p \atop q} \right\}$	A Archimedean	(c) cube	(d) dodeca-	(s) snub
	(te) tetra- (hedron)	(co) cubocta-	(id) icosidodeca-	(t) truncated





Wythoff's Construction	1v	2vA	3vA	1vT	3vT	5vT	7vT	9vT
		q	q		p	p	p	p
1vA	{3 3} (te) P1	$\left\{ {3 \atop 3} \right\}$ (o) P2	t{3 3} (t) (te) A1	{3 3} (te) P1	t{3 3} (t) (te) A1	{3 5} (i) P4	$\left\{ {3 \atop 4} \right\}$ (co) A2	t{3 4} (t) (o) A3
2vA	{3 4} (o) P2	$\left\{ {3 \atop 4} \right\}$ (co) A2	t{3 4} (t) (o) A3	{4 3} (c) P3	t{4 3} (t) (c) A4	s$\left\{ {4 \atop 3} \right\}$ (s) (co) A10	r$\left\{ {4 \atop 3} \right\}$ (r)(co) A5	t$\left\{ {4 \atop 3} \right\}$ (t) (co) A6
5vT	{3 5} (i) P4	$\left\{ {3 \atop 5} \right\}$ (id) A7	t{3 5} (t) (i) A8	{5 3} (d) P5	t{5 3} (t) (d) A9	s$\left\{ {5 \atop 3} \right\}$ (s) (id) A13	r$\left\{ {5 \atop 3} \right\}$ (r)(id) A11	t$\left\{ {5 \atop 3} \right\}$ (t) (id) A12

Alternation of Interpatterning Realizations

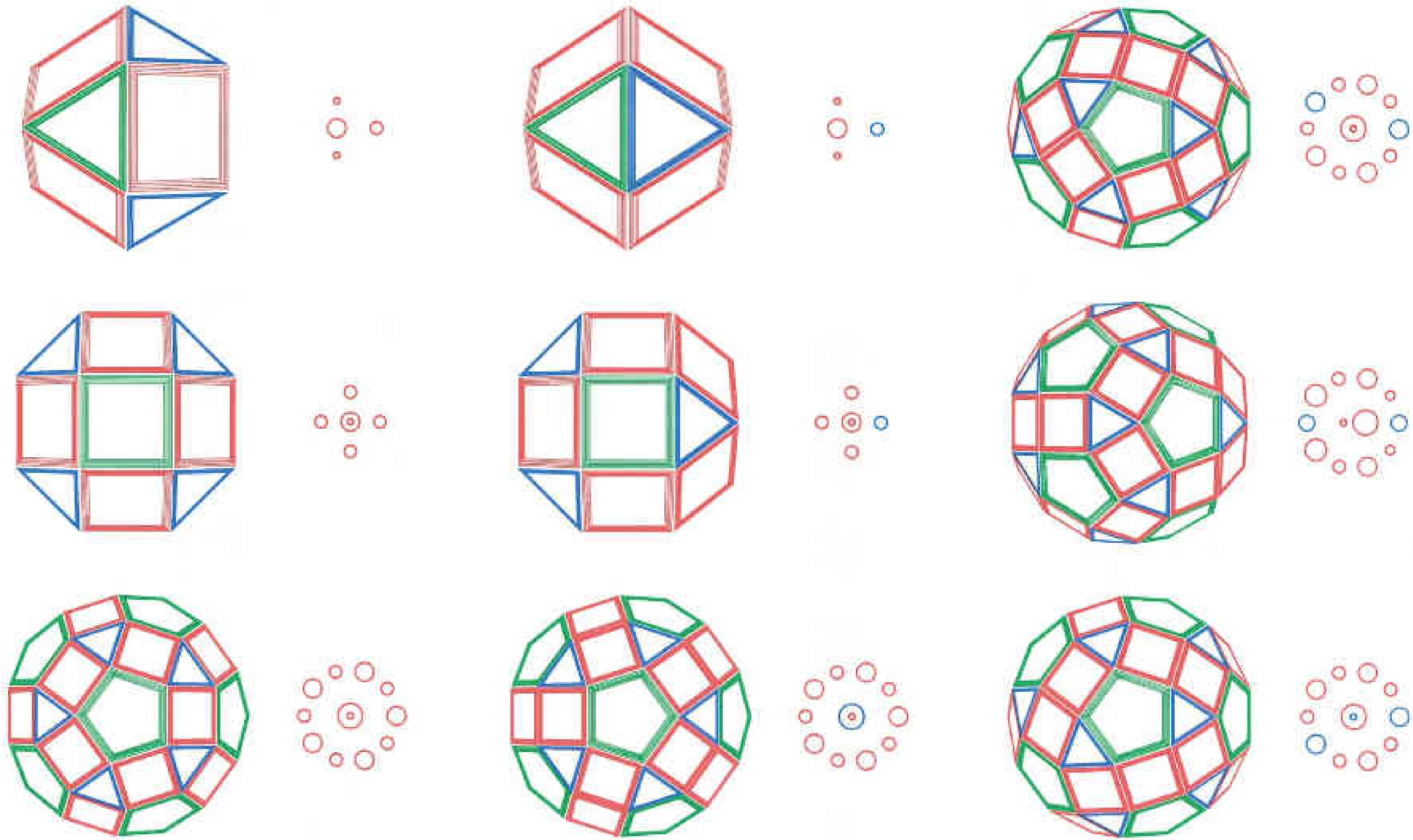
Vector Equilibrium

Vector Polarization

$7_v T$

$2_v A 7_v T$

$5_v 7_v T$



Volumes and Areas

$V_{P1} = 1/6\sqrt{2}$

$V_{P2} = \sqrt{2}/3$

$V_{P3} = 1$

$V_{P4} = 5\tau^2/6$

$V_{P5} = \tau^4\sqrt{5}/2$

$V_{A1} = 23\sqrt{2}/12$

$V_{A2} = 5\sqrt{2}/3$

$V_{A3} = 8\sqrt{2}$

$V_{A4} = (21+14\sqrt{2})/3$

$V_{A5} = \frac{2}{3}(6+5\sqrt{2})$

$V_{A6} = 30+14\sqrt{2}$

$V_{A7} = \frac{1}{6}(45+17\sqrt{5})$

$V_{A8} = \frac{1}{4}(125+43\sqrt{5})$

$V_{A9} = \frac{5}{12}(99+47\sqrt{5})$

$V_{A10} = \frac{4}{3}\sqrt{\frac{3}{2}\upsilon^2+3\upsilon+2} + \sqrt{\upsilon\left(\frac{\upsilon}{2}+1\right)}$

$V_{A11} = \frac{1}{3}(60+29\sqrt{5})$

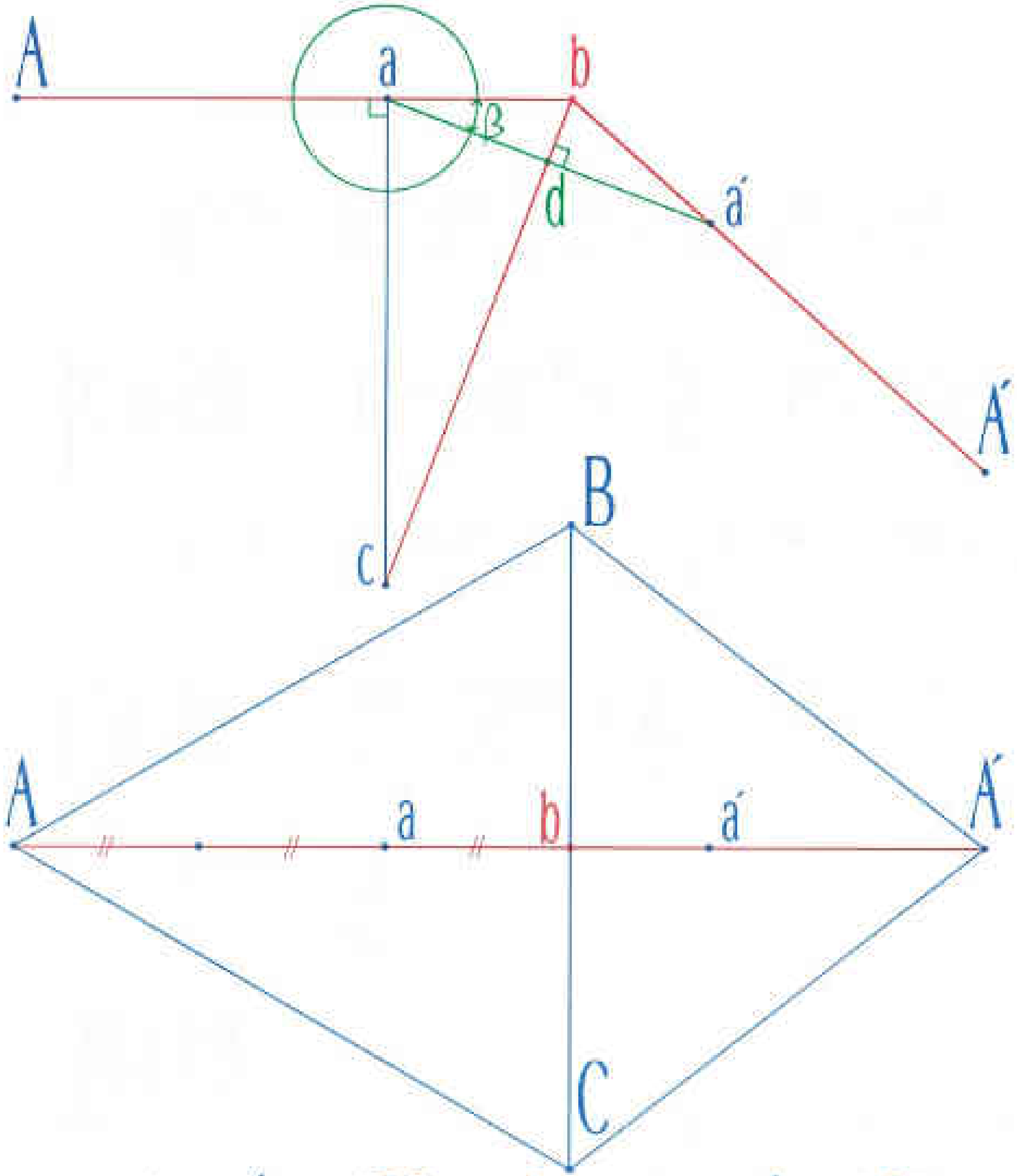
$V_{A12} = 95+50\sqrt{5}$

$V_{A13} = \frac{10\tau}{3}\sqrt{\tau^2+3\phi(\tau+\phi)} + \frac{5\tau^2}{2}\sqrt{\frac{1}{5}+\frac{\tau\phi}{\sqrt{5}}(\tau+\phi)}$

▲	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$a_▲ = \frac{\sqrt{3}}{4}$
■	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$a_■ = 1$
⬟	$36°$	$\sqrt{\frac{\sqrt{5}}{4\tau}}$	$\frac{\tau}{2}$	$\sqrt{\frac{\sqrt{5}}{\tau^3}}$	$a_⬟ = \frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}$
⬢	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{\frac{1}{3}}$	$a_⬢ = \frac{3}{2}\sqrt{3}$
⯃	$22\frac{1}{2}°$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{1}{1+\sqrt{2}}$	$a_⯃ = 2(1+\sqrt{2})$
	$18°$	$\frac{1}{2\tau}$	$\frac{\sqrt{\tau\sqrt{5}}}{2}$	$\sqrt{\frac{1}{\tau^3\sqrt{5}}}$	$a = \frac{5}{2}\sqrt{\tau^3\sqrt{5}}$
	$\frac{180°}{s}$	sin	cos	tan	$\frac{s}{4\tan}$

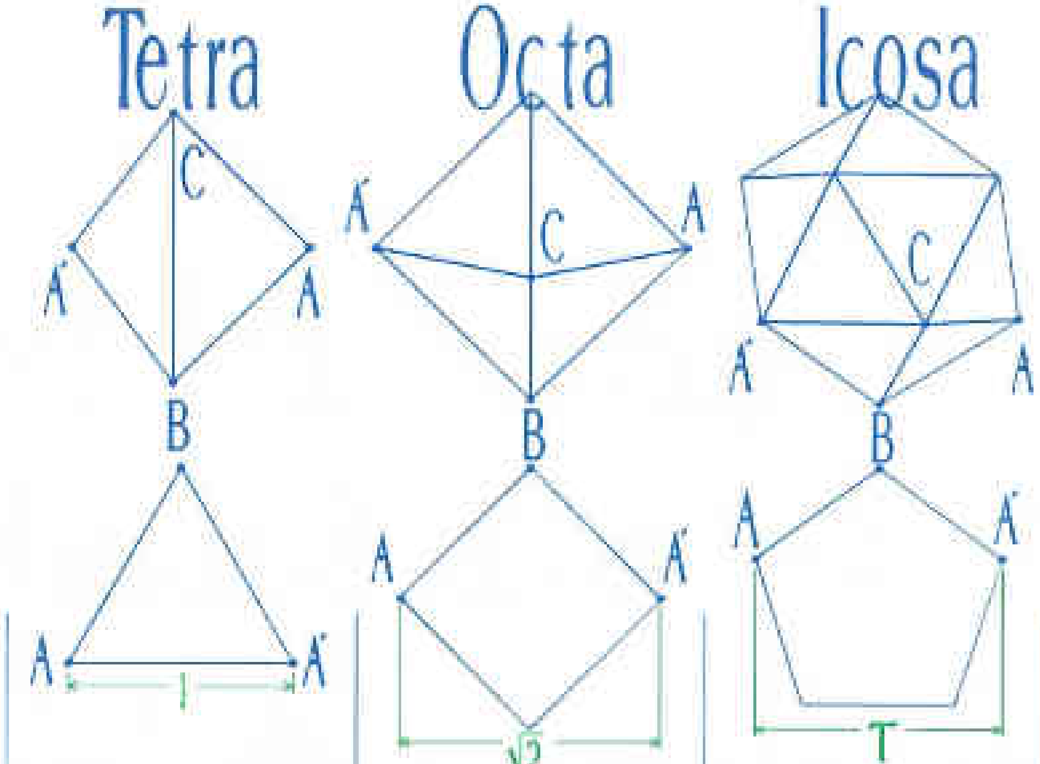
$\tau \equiv \frac{1+\sqrt{5}}{2}$

Define tau upsilon & phi $\upsilon \equiv \sqrt[3]{2+\frac{2}{3}\sqrt{\frac{11}{3}}} + \sqrt[3]{2-\frac{2}{3}\sqrt{\frac{11}{3}}}$ $\phi \equiv \sqrt[3]{\frac{\tau}{2}+\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}} + \sqrt[3]{\frac{\tau}{2}-\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}}$



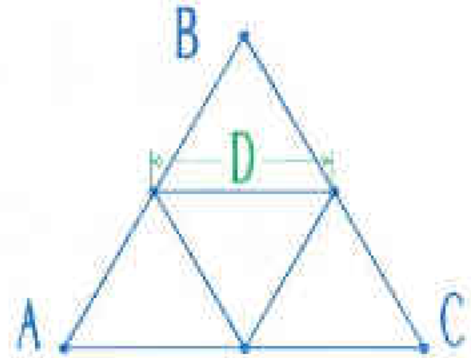
a: center of ▲ABC	a′: center of ▲A′CB
AB=AC=BC=BA′=CA′=1	b: bisecter of $\overline{BC}$
c: center of solid	d: $\overline{aa'}$ intersects $\overline{bc}$

IvA



	Tetra	Octa	Icosa
$\overline{ad} = \overline{AA'}/6$	$\frac{1}{6}$	$\frac{\sqrt{2}}{6}$	$\frac{\tau}{6}$
$\cos\beta = \overline{ad}/\overline{ab}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	$\frac{\tau}{\sqrt{3}}$
$\sin\beta = \sqrt{1-\cos^2\beta}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	$\frac{1}{\tau\sqrt{3}}$
$\overline{bc} = \overline{ab}/\sin\beta$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2}$	$\frac{\tau}{2}$
$\overline{ac} = \overline{bc}\cos\beta$	$R_{T\blacktriangle} = \frac{1}{2\sqrt{6}}$	$R_{O\blacktriangle} = \frac{1}{\sqrt{6}}$	$R_{I\blacktriangle} = \frac{\tau^2}{2\sqrt{3}}$
$n_\blacktriangle$	4	8	20
$n_\blacktriangle\ a_\blacktriangle\ \frac{1}{3}\ R_\blacktriangle$	$V_{1vA} = \frac{1}{6\sqrt{2}}$	$V_{2vA} = \frac{\sqrt{2}}{3}$	$V_{5vT} = \frac{5\tau^2}{6}$

2vA



$D = \frac{1}{2}$

$r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = 2R_{\blacktriangle}$

Tetra:

$V_{2vA} = \frac{\sqrt{2}}{3}$

Octa:

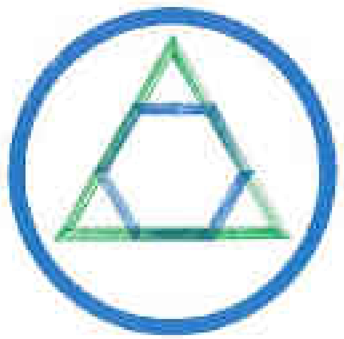
$r_{\blacktriangle} = 2R_{O\blacktriangle} = \sqrt{\frac{2}{3}}$ $r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\frac{8+1-3}{12}} = \frac{1}{\sqrt{2}}$

$V_{2vA^2} = n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3}\, r_{\blacktriangle} + n_{\blacksquare}\, a_{\blacksquare}\, \frac{1}{3}\, r_{\blacksquare} = 8\, \frac{\sqrt{3}}{4}\, \frac{1}{3}\, \sqrt{\frac{2}{3}} + 6\; 1\; \frac{1}{3}\, \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{3}$

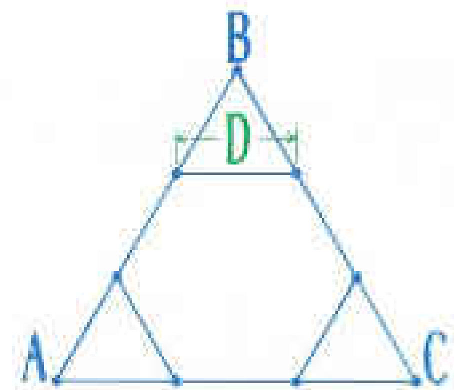
Icosa:

$r_{\blacktriangle} = 2R_{I\blacktriangle} = \frac{\tau^2}{\sqrt{3}}$ $r_{\pentagon} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^{-2} - (2\tan 36^\circ)^{-2}} = \sqrt{\frac{10(7+3\sqrt{5}) + 5 - 3\sqrt{5}(2+\sqrt{5})}{60}} = \sqrt{\frac{\tau^3}{\sqrt{5}}}$

$V_{5vT\,2vA} = n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3}\, r_{\blacktriangle} + n_{\pentagon}\, a_{\pentagon}\, \frac{1}{3}\, r_{\pentagon} = 20\, \frac{\sqrt{3}}{4}\, \frac{1}{3}\, \frac{\tau^2}{\sqrt{3}} + 12\, \frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}\, \frac{1}{3}\, \sqrt{\frac{\tau^3}{\sqrt{5}}} = \frac{45 + 17\sqrt{5}}{6}$



3vA



$$D = \frac{1}{3}$$

$$r_{\bullet} = \frac{R_{\blacktriangle}}{D} = 3R_{\blacktriangle}$$

Tetra: $r_{\bullet} = 3R_{T\blacktriangle} = \sqrt{\frac{3}{8}}$ $\quad r_{\blacktriangle} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 60^\circ)^{-2}} = \sqrt{\frac{9+18-2}{24}} = \frac{5}{2\sqrt{6}}$

$$V_{3vA} = n_{\bullet}\, a_{\bullet}\, \frac{1}{3}\, r_{\bullet} + n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3}\, r_{\blacktriangle} = 4\, \frac{3\sqrt{3}}{2}\, \frac{1}{3}\sqrt{\frac{3}{8}} + 4\, \frac{\sqrt{3}}{4}\, \frac{1}{3}\, \frac{5}{2\sqrt{6}} = \frac{23\sqrt{2}}{12}$$

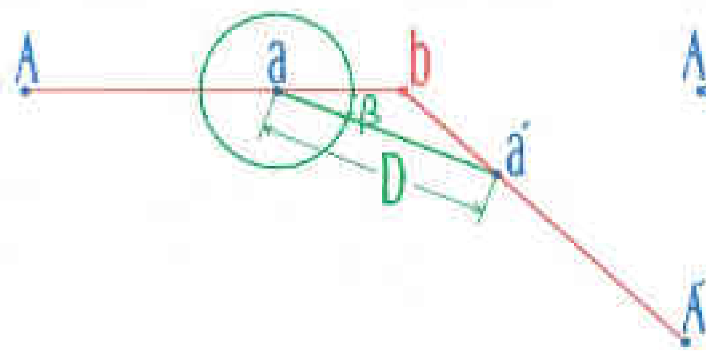
Octa: $r_{\bullet} = 3R_{O\blacktriangle} = \sqrt{\frac{3}{2}}$ $\quad r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\frac{6+3-1}{4}} = \sqrt{2}$

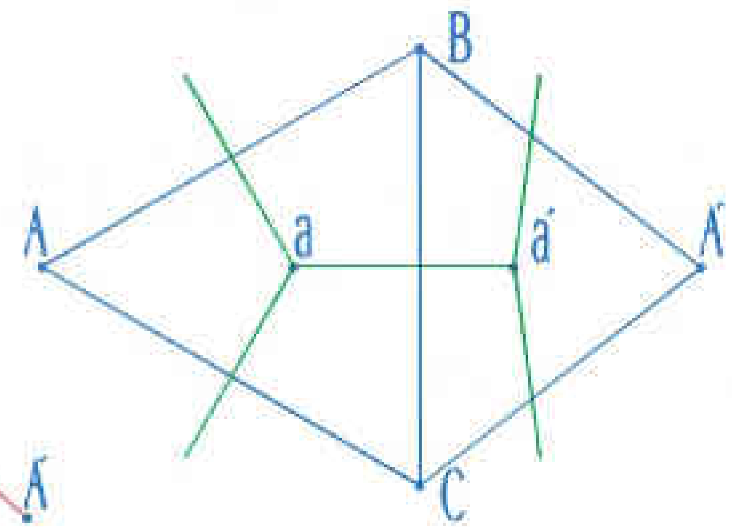
$$V_{2v3vA} = n_{\bullet}\, a_{\bullet}\, \frac{1}{3}\, r_{\bullet} + n_{\blacksquare}\, a_{\blacksquare}\, \frac{1}{3}\, r_{\blacksquare} = 8\, \frac{3\sqrt{3}}{2}\, \frac{1}{3}\sqrt{\frac{3}{2}} + 6\; 1\; \frac{1}{3}\sqrt{2} = 8\sqrt{2}$$

Icosa: $r_{\bullet} = 3R_{I\blacktriangle} = \frac{\tau^2\sqrt{3}}{2}$ $\quad r_{\pentagon} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 36^\circ)^{-2}} = \sqrt{\frac{15(7+3\sqrt{5})+30-10+4\sqrt{5}}{40}} = \sqrt{\frac{41+25\sqrt{5}}{8\sqrt{5}}}$

$$V_{5vT3vA} = n_{\bullet}\, a_{\bullet}\, \frac{1}{3}\, r_{\bullet} + n_{\pentagon}\, a_{\pentagon}\, \frac{1}{3}\, r_{\pentagon} = 20\, \frac{3\sqrt{3}}{2}\, \frac{1}{3}\, \frac{\tau^2\sqrt{3}}{2} + 12\, \frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}\, \frac{1}{3}\sqrt{\frac{41+25\sqrt{5}}{8\sqrt{5}}} = \frac{125+43\sqrt{5}}{4}$$

1vT





$$D = \overline{aa'} = 2\,ab\,\cos\beta = \frac{\cos\beta}{\sqrt{3}}$$

Vertex Radius:

$$r_v = \frac{R_\blacktriangle}{D} = \frac{R_\blacktriangle\sqrt{3}}{\cos\beta}$$

Octa:

$$r_v = \frac{R_{0\blacktriangle}\sqrt{3}}{\cos\beta_0} = \frac{\sqrt{3}}{2}$$

$$r_\blacksquare = \sqrt{r_v^2 - (2\sin 45^\circ)^{-2}} = \sqrt{\frac{3-2}{4}} = \frac{1}{2}$$

$$V_{2vA1vT} = n_\blacksquare\, a_\blacksquare\, \frac{1}{3}\, r_\blacksquare = 6\;1\;\frac{1}{3}\;\frac{1}{2} = 1$$

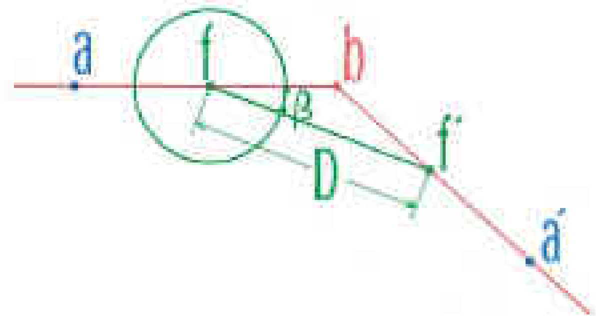
Icosa:

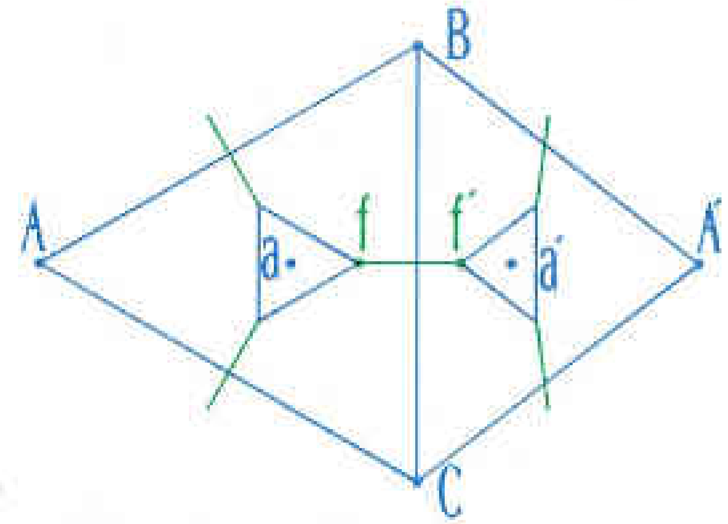
$$r_v = \frac{R_{I\blacktriangle}\sqrt{3}}{\cos\beta_I} = \frac{\tau\sqrt{3}}{2}$$

$$r_{\text{⬟}} = \sqrt{r_v^2 - (2\sin 36^\circ)^{-2}} = \sqrt{\frac{15(3+\sqrt{5}) - 4\sqrt{5}(1+\sqrt{5})}{40}} = \sqrt{\frac{\tau^5}{4\sqrt{5}}}$$

$$V_{5v1vT} = n_{\text{⬟}}\, a_{\text{⬟}}\, \frac{1}{3}\, r_{\text{⬟}} = 12\;\frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}\;\frac{1}{3}\sqrt{\frac{\tau^5}{4\sqrt{5}}} = \frac{\tau^4\sqrt{5}}{2}$$

3vT





$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D\left(\frac{1}{\sqrt{3}} + \frac{1}{2\cos\beta}\right)$$

$$r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = R_{\blacktriangle}\left(2 + \frac{\sqrt{3}}{\cos\beta}\right)$$

Octa:

$$r_{\blacktriangle} = R_{0\blacktriangle}\left(2 + \frac{\sqrt{3}}{\cos\beta_0}\right) = \frac{3+2\sqrt{2}}{2\sqrt{3}}$$

$$r_{\bullet} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^{-2} - (2\tan 22\tfrac{1}{2}^\circ)^{-2}} = \sqrt{\frac{17+12\sqrt{2}+1-3(3+2\sqrt{2})}{12}} = \frac{1+\sqrt{2}}{2}$$

$$V_{2vA3vT} = n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3}\, r_{\blacktriangle} + n_{\bullet}\, a_{\bullet}\, \frac{1}{3}\, r_{\bullet} = 8\, \frac{\sqrt{3}}{4}\, \frac{1}{3}\, \frac{3+2\sqrt{2}}{2\sqrt{3}} + 6\; 2(1+\sqrt{2})\, \frac{1}{3}\, \frac{1+\sqrt{2}}{2} = \frac{7}{3}(3+2\sqrt{2})$$

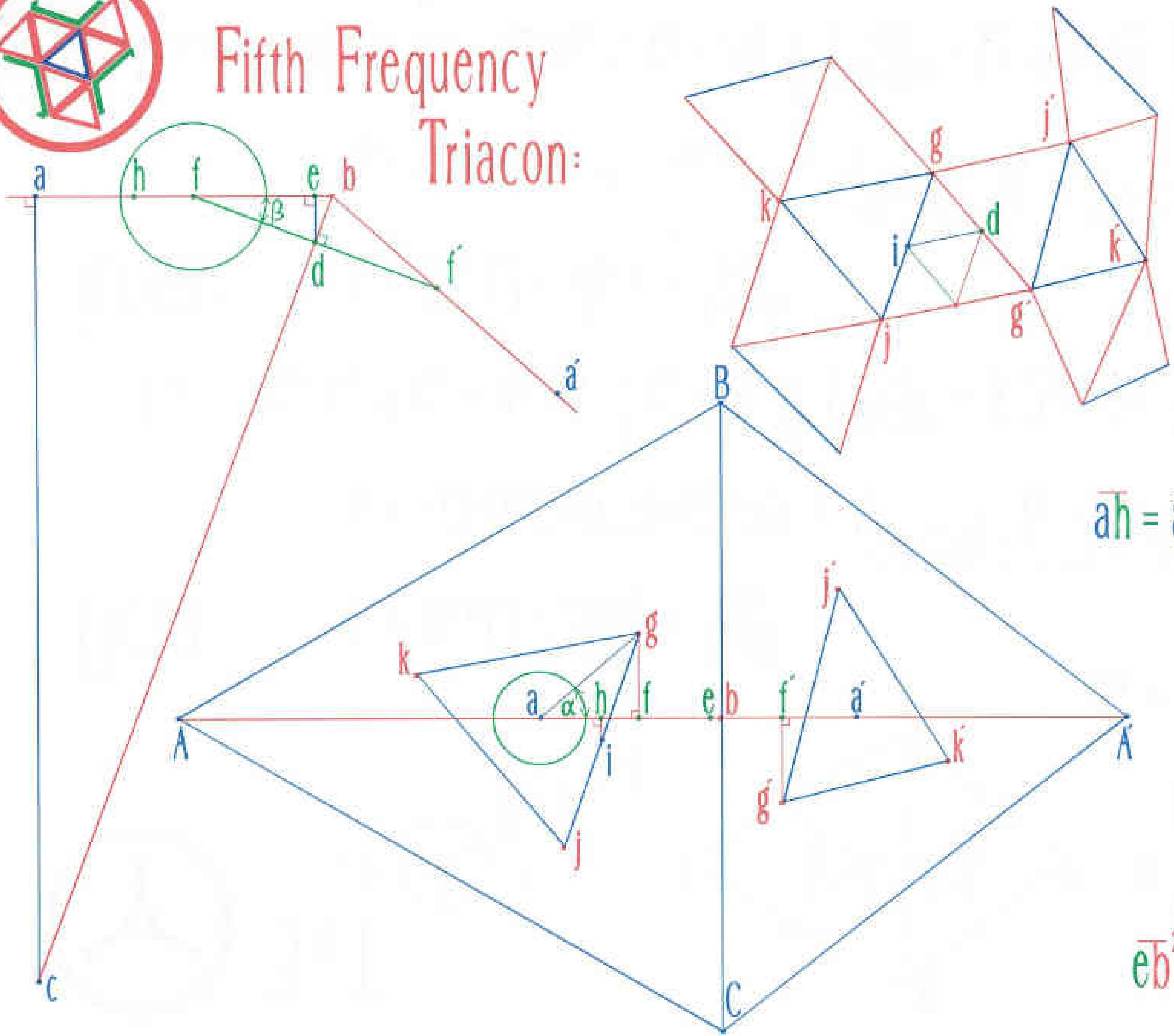
Icosa:

$$r_{\blacktriangle} = R_{I\blacktriangle}\left(2 + \frac{\sqrt{3}}{\cos\beta_I}\right) = \frac{9+5\sqrt{5}}{4\sqrt{3}}$$

$$r_{✱} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^{-2} - (2\tan 18^\circ)^{-2}} = \sqrt{\frac{(9+5\sqrt{5})^2 + 4 - 12\sqrt{5}(2+\sqrt{5})}{48}} = \frac{1}{2}\sqrt{\tau^5\sqrt{5}}$$

$$V_{5v3vT} = n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3}\, r_{\blacktriangle} + n_{✱}\, a_{✱}\, \frac{1}{3}\, r_{✱} = 20\, \frac{\sqrt{3}}{4}\, \frac{1}{3}\, \frac{9+5\sqrt{5}}{4\sqrt{3}} + 12\, \frac{5}{2}\sqrt{\tau^3\sqrt{5}}\, \frac{1}{3}\, \frac{1}{2}\sqrt{\tau^5\sqrt{5}} = \frac{5}{12}(99+47\sqrt{5})$$

Fifth Frequency Triacon:



$$\overline{gj} = \overline{jk} = \overline{kg} = \overline{gg'} = \overline{g'j} = D$$

$$\overline{gi} = \overline{ij} = \overline{id} = \overline{gd} = \overline{dg'} = \frac{D}{2}$$

$$\overline{af}^2 + \overline{fg}^2 = \overline{ag}^2 = \frac{D^2}{3}$$

$$\overline{ah}^2 + \overline{hi}^2 = \overline{ai}^2 = \frac{D^2}{12}$$

$$\overline{ah} = \overline{ai}\cos(60-\alpha) = \frac{D}{4\sqrt{3}}(\cos\alpha + \sqrt{3}\sin\alpha)$$

$$\overline{af} = \overline{ag}\cos\alpha = \frac{D}{\sqrt{3}}\cos\alpha$$

$$\overline{fb} = \overline{ab} - \overline{af} = \frac{1}{2\sqrt{3}}(1 - 2D\cos\alpha)$$

$$\overline{eb} = \overline{db}\sin\beta = \overline{fb}\sin^2\beta$$

$$\overline{eb}^2 + \overline{ed}^2 = \overline{eb}^2(1 + \cot^2\beta) = \overline{eb}\,\overline{fb}$$

$$\overline{gd}^2 - \frac{D^2}{4} = 0 = \overline{gf}^2 + (\overline{ab} - \overline{af} - \overline{eb})^2 + \overline{ed}^2 - \frac{D^2}{4}$$

$$= \overline{ag}^2 + \overline{ab}^2 - 2\,\overline{ab}\,\overline{af} + \overline{eb}\left[2\,\overline{af} - 2\,\overline{ab} + \overline{fb}\right] - \frac{D^2}{4}$$

$$= \frac{D^2}{3} + \frac{1}{12} - \frac{D}{3}\cos\alpha + \sin^2\beta\,\frac{1-2D\cos\alpha}{2\sqrt{3}}\left[\frac{2D\cos\alpha}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1-2D\cos\alpha}{2\sqrt{3}}\right] - \frac{D^2}{4}$$

or, $$D^2 - 4D\cos\alpha + 1 - \sin^2\beta\,(1-2D\cos\alpha)^2 = 0 \qquad \text{Eq. 1}$$

$$\overline{id}^2 - \frac{D^2}{4} = 0 = \overline{hi}^2 + (\overline{ab} - \overline{ah} - \overline{eb})^2 + \overline{ed}^2 - \frac{D^2}{4}$$

$$= \overline{ai}^2 + \overline{ab}^2 - 2\,\overline{ab}\,\overline{ah} + \overline{eb}\left[2\,\overline{ah} - 2\,\overline{ab} + \overline{fb}\right] - \frac{D^2}{4}$$

$$= \frac{D^2}{12} + \frac{1}{12} - \frac{D}{12}(\cos\alpha + \sqrt{3}\sin\alpha) + \sin^2\beta\,\frac{1-2D\cos\alpha}{2\sqrt{3}}\left[\frac{D(\cos\alpha+\sqrt{3}\sin\alpha)}{2\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1-2D\cos\alpha}{2\sqrt{3}}\right] - \frac{D^2}{4}$$

or, $$-2D^2 - D(\cos\alpha + \sqrt{3}\sin\alpha) + 1 - \sin^2\beta\,(1-2D\cos\alpha)\left[1 + D(\cos\alpha - \sqrt{3}\sin\alpha)\right] = 0 \qquad \text{Eq. 2}$$

Define Gamma Operators: $\gamma \equiv \sqrt{3}\tan\alpha \qquad \Gamma \equiv 3\cos\alpha - \sqrt{3}\sin\alpha$

$$\cos^2\alpha + \sin^2\alpha = \cos^2\alpha\left(1 + \frac{\gamma^2}{3}\right) = 1 \quad \text{so,} \quad \cos^2\alpha = \frac{1}{1+\gamma^2/3}$$

$$\Gamma\cos\alpha = (3\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha = (3-\gamma)\cos^2\alpha = \frac{3-\gamma}{1+\gamma^2/3}$$

$$3\left(1+\frac{\gamma^2}{3}\right)\left[\Gamma\cos\alpha - \frac{3-\gamma}{1+\gamma^2/3}\right] = \underbrace{\Gamma\cos\alpha}_{a}\,\gamma^2 + \underbrace{3}_{b}\,\gamma + \underbrace{3(\Gamma\cos\alpha - 3)}_{c} = 0$$

Positive Root of γ:

$$\gamma = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-3 + \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}}{2\Gamma\cos\alpha} \qquad \text{Eq. 3}$$

$$\Gamma^2 = \Gamma\cos\alpha(3-\gamma) = 3\Gamma\cos\alpha - \frac{1}{2}\left[-3 + \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}\right] \qquad \text{Eq. 4}$$

Octa: Eq. 1: $(3-4\cos^2\alpha)D^2 - 8\cos\alpha\, D + 2 \equiv F_{01} = 0$

$\sin^2\beta_0 = \frac{1}{3}$ Eq. 2: $2((\cos\alpha - \sqrt{3}\sin\alpha) - 3)D^2 - 2(\cos\alpha + \sqrt{3}\sin\alpha)D + 2 \equiv F_{02} = 0$

$$\frac{F_{02}-F_{01}}{D} = (2(3\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 9)D + 2(3\cos\alpha - \sqrt{3}\sin\alpha) = (2\Gamma\cos\alpha - 9)D + 2\Gamma = 0 \quad \text{so, } D = \frac{2\Gamma}{9-2\Gamma\cos\alpha} \quad \text{Eq. 5}$$

$$\frac{\Gamma(9-2\Gamma\cos\alpha)}{3D} F_{01} = \frac{\Gamma(9-2\Gamma\cos\alpha)}{3}\left[(3-4\cos^2\alpha)\left(\frac{2\Gamma}{9-2\Gamma\cos\alpha}\right) - 8\cos\alpha + 2\left(\frac{9-2\Gamma\cos\alpha}{2\Gamma}\right)\right] = 0$$

$$= 4(\Gamma\cos\alpha)^2 - 36\Gamma\cos\alpha + 27 + 2\Gamma^2 = 0 \quad \text{from Eq. 4:}$$

$$4(\Gamma\cos\alpha)^2 - 36\Gamma\cos\alpha + 27 + 6\Gamma\cos\alpha + 3 = \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}$$

Square both sides and subtract:

$$16(\Gamma\cos\alpha)^4 - 240(\Gamma\cos\alpha)^3 + 1152(\Gamma\cos\alpha)^2 - 1836\Gamma\cos\alpha + 891 = 0$$

Define: $x \equiv \frac{2}{3}\Gamma\cos\alpha$ and divide by 81: $x^4 - 10x^3 + 32x^2 - 34x + 11 = 0$

First Root of x: $x_{01} = 1$

$$
\begin{array}{r}
\qquad\qquad\quad\; p \qquad\;\; q \qquad\;\; r \\
x^3 - 9x^2 + 23x - 11 \\
(x-1)\,\overline{\big)\, x^4 - 10x^3 + 32x^2 - 34x + 11} \\
\underline{x^4 - x^3} \qquad\qquad\qquad\qquad\quad \\
0 \quad \underline{-9x^3 + 9x^2} \qquad\qquad\quad \\
0 \quad \underline{23x^2 - 23x} \qquad\quad \\
0 \quad \underline{-11x + 11} \\
0 \qquad 0
\end{array}
$$

Second Root: $a = q - \frac{p^2}{3} = 23 - 27 = -4$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -54 + 69 - 11 = 4$$

$$x_{02} = \frac{-p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = 3 - \upsilon$$

From the second root, $\Gamma = \frac{3}{2}(3 - \upsilon)$ Eq. 6

note: $\left(\sqrt[3]{2 + \frac{2}{3}\sqrt{\frac{11}{3}}}\right)\left(\sqrt[3]{2 - \frac{2}{3}\sqrt{\frac{11}{3}}}\right) = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$ therefore, $\upsilon^3 = 4(\upsilon + 1)$ Eq. 7

also note: $(20 + 6\upsilon - 3\upsilon^2)^2 - [(3\upsilon + 2)^2(-3\upsilon^2 + 12\upsilon - 8)] = 0$

$$= 400 + 36\upsilon^2 + 36\upsilon(\upsilon+1) + 240\upsilon - 120\upsilon^2 - 144(\upsilon+1) - [-108\upsilon(\upsilon+1) - 144(\upsilon+1) - 12\upsilon^2 + 432(\upsilon+1) + 144\upsilon^2 + 48\upsilon - 72\upsilon^2 - 96\upsilon - 32] = 0$$

therefore: $\sqrt{-3\upsilon^2 + 12\upsilon - 8} = \frac{20 + 6\upsilon - 3\upsilon^2}{3\upsilon + 2}$ Eq. 8

$$\gamma \underset{\text{Eq.3}}{=} \frac{-3+\sqrt{9-12\Gamma\cos\alpha(\Gamma\cos\alpha-3)}}{2\Gamma\cos\alpha} \underset{\text{Eq.6}}{=} \frac{-1+\sqrt{-3\upsilon^2+12\upsilon-8}}{3-\upsilon} \underset{\text{Eq.8}}{=} \frac{-(3\upsilon+2)+(20+6\upsilon-3\upsilon^2)}{(3-\upsilon)(3\upsilon+2)} = \frac{\upsilon+2}{\upsilon+\frac{2}{3}}$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\frac{\gamma^2}{3}}} = \frac{1}{\sqrt{1+\frac{1}{3}\left(\frac{\upsilon+2}{\upsilon+\frac{2}{3}}\right)^2}} = \frac{3\upsilon+2}{2\sqrt{3\upsilon^2+6\upsilon+4}}$$

$$\Gamma = 3\cos\alpha-\sqrt{3}\sin\alpha = \cos\alpha[3-\gamma] = \frac{(3\upsilon+2)}{2\sqrt{3\upsilon^2+6\upsilon+4}}\left[\frac{3(3\upsilon+2)-3(\upsilon+2)}{3\upsilon+2}\right] = \frac{3\upsilon}{\sqrt{3\upsilon^2+6\upsilon+4}}$$

$$\text{Eq.5}: \quad D = \frac{2\Gamma}{9-2\Gamma\cos\alpha} \underset{\text{Eq.6}}{=} \frac{2\Gamma}{3\upsilon} = \frac{2}{\sqrt{3\upsilon^2+6\upsilon+4}} \qquad r_{\blacktriangle} = \frac{R_{0\blacktriangle}}{D} = \frac{\sqrt{3\upsilon^2+6\upsilon+4}}{2\sqrt{6}}$$

$$r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2+(2\tan 60)^{-2}-(2\tan 45)^{-2}} = \sqrt{\frac{1}{24}(3\upsilon^2+6\upsilon+4+2-6)} = \frac{1}{2}\sqrt{\upsilon\left(\frac{\upsilon}{2}+1\right)}$$

$$V_{2vA5vT} = n_{\blacktriangle}a_{\blacktriangle}\frac{1}{3}r_{\blacktriangle} + n_{\blacksquare}a_{\blacksquare}\frac{1}{3}r_{\blacksquare} = 32\,\frac{\sqrt{3}}{4}\,\frac{1}{3}\,\frac{1}{2\sqrt{3}}\sqrt{3\upsilon^2+6\upsilon+4} + 6\;1\;\frac{1}{3}\;\frac{1}{2}\sqrt{\upsilon\left(\frac{\upsilon}{2}+1\right)}$$

$$= \frac{4}{3}\sqrt{\frac{3}{2}\upsilon^2+3\upsilon+2} + \sqrt{\upsilon\left(\frac{\upsilon}{2}+1\right)}$$

Icosa:

$$3\tau^2(\text{Eq.}1) = (3\tau^2 - 4\cos^2\alpha)D^2 - 4\tau^4\cos\alpha\, D + \tau^4 \equiv F_{11} = 0$$

$\sin^2\beta_1 = \frac{1}{3\tau^2}$

$$3\tau^2(\text{Eq.}2) = 2\left[(\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 3\tau^2\right]D^2 - \tau^4(\cos\alpha + \sqrt{3}\sin\alpha)D + \tau^4 \equiv F_{12} = 0$$

$$F_{13} \equiv F_{12} + yF_{11} = \underbrace{\left[3\tau^2(y-2) + 2((1-2y)\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha\right]}_{i} D^2 - \underbrace{((4y+1)\cos\alpha + \sqrt{3}\sin\alpha)\tau^4}_{j} D + \underbrace{(y+1)\tau^4}_{k} = 0$$

Define eta and lambda: $j^2 - 4ik \equiv (\eta\cos\alpha + \lambda\sqrt{3}\sin\alpha)^2 = \eta^2\cos^2\alpha + (\eta\lambda)2\sqrt{3}\cos\alpha\sin\alpha + \lambda^2 3\sin^2\alpha$

$$= \underbrace{\left[\tau^2(4y+1)^2 - 4\tau^2(y+1)(3\tau^2(y-2) + 2(1-2y))\right]}_{\eta^2}\cos^2\alpha + \underbrace{\left[\tau^8(4y+1) + 4\tau^4(y+1)\right]}_{(\eta\lambda)} 2\sqrt{3}\cos\alpha\sin\alpha + \underbrace{\left[\tau^8 - 4\tau^6(y+1)(y-2)\right]}_{\lambda^2} 3\sin^2\alpha$$

$$(\eta\lambda)^2 - \eta^2\lambda^2 = \left[\tau^8(4y+1) + 4\tau^4(y+1)\right]^2 - \left[\tau^2(4y+1)^2 - 4\tau^2(y+1)(3\tau^2(y-2) + 2(1-2y))\right]\left[\tau^8 - 4\tau^8(y+1)(y-2)\right] = 0$$

=

τ^{16}					16 - 16		8 - 8		1 - 1	
τ^{14}	64		-32		-144		-80	144	-32	-144
τ^{12}	-48	144	96	0	32+128	-288	40-200		8 - 184	144
τ^{10}	64		-32		-192		-32		64	
τ^{8}					16		32		16	
$+144\tau^{12}$		y^4				$-2y^2$		$-\tau^2 y$		$-\tau$

$$\tau^{n-2} + \tau^{n+2} = 3\tau^n$$

First root of y: $y_{11} = -1$

Second root: $p = -1 \quad q = -1 \quad r = -\tau$

$$a = \frac{-p^2}{3} + q = \frac{-1}{3} - 1 = \frac{-4}{3}$$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -\frac{2}{27} - \frac{1}{3} - \tau = \frac{-49 - 27\sqrt{5}}{54}$$

$$y_{12} = \frac{-p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = \phi^2 - \frac{1}{3}$$

$$\begin{array}{r|l} & y^3 - y^2 - y - \tau \\ \hline (y+1) & y^4 \quad -2y^2 - \tau^2 y - \tau \\ & y^4 + y^3 \\ & 0 \quad -y^3 - y^2 \\ & 0 \quad -y^2 - y \\ & 0 \quad -\tau y - \tau \\ & 0 \quad 0 \end{array}$$

From the first root of y: $\frac{F_{13}}{D} = \frac{F_{12} - F_{11}}{D}$

Eq.5

$$= [2(3\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 9\tau^2]D + \tau^4(3\cos\alpha - \sqrt{3}\sin\alpha) = [2\Gamma\cos\alpha - 9\tau^2]D + \tau^4\Gamma = 0 \quad \text{so,} \quad D = \frac{\tau^4\Gamma}{9\tau^2 - 2\Gamma\cos\alpha}$$

$$\frac{\Gamma(9\tau^2 - 2\Gamma\cos\alpha)}{3\tau^2 D} F_{11} = \frac{\Gamma(9\tau^2 - 2\Gamma\cos\alpha)}{3\tau^2}\left[[3\tau^2 - 4\cos^2\alpha]\left(\frac{\tau^4\Gamma}{9\tau^2 - 2\Gamma\cos\alpha}\right) - 4\tau^4\cos\alpha + \tau^4\left(\frac{9\tau^2 - 2\Gamma\cos\alpha}{\tau^4\Gamma}\right)\right] = 0$$

$$= 4(\Gamma\cos\alpha)^2 - 36\tau^2\Gamma\cos\alpha + 27\tau^2 + \tau^4\Gamma^2 = 0$$

from Eq. 4: $4(\Gamma\cos\alpha)^2 + 3\tau^2(\tau^4 - 12)\Gamma\cos\alpha + \frac{3}{2}\tau^2(\tau^2 + 18) = \frac{\tau^4}{2}\sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}$

Square both sides and subtract:

$$16(\Gamma\cos\alpha)^4 + 24\tau^2(\tau^2-12)(\Gamma\cos\alpha)^3 + 36\tau^2(21\tau^2+11)(\Gamma\cos\alpha)^2 + 54\tau^4(\tau^2-36)\Gamma\cos\alpha + 81\tau^4(\tau^2+9) = 0$$

Define x: $x = \frac{2}{3}\Gamma\cos\alpha$ and divide by 81:

$$x^4 + \tau^2(\tau^2-12)x^3 + \tau^2(21\tau^2+11)x^2 + \tau^4(\tau^2-36)x + \tau^4(\tau^2+9) = 0$$

First root of x: $x_{11} = 1$

Second root: $p = -9\tau^2$

$q = \tau^2(21\tau^2+2)$ $\quad r = -\tau^4(\tau^2+9)$

$$\begin{array}{r|lllll}
 & x^3 & -9\tau^2x^2 & +\tau^2(21\tau^2+2)x & -\tau^4(\tau^2+9) & \\
\hline
(x-1) & x^4 + \tau^2(\tau^2-12)x^3 & +\tau^2(21\tau^2+11)x^2 & +\tau^4(\tau^2-36)x & +\tau^4(\tau^2+9) & \\
 & x^4 \quad -x^3 & & & & \\
\cline{2-2}
 & 0 \quad -9\tau^2x^3 & +9\tau^2x^2 & & & \\
\cline{2-3}
 & 0 & \tau^2(21\tau^2+2)x^2 & -\tau^2(21\tau^2+2)x & & \\
\cline{3-4}
 & & 0 & -\tau^4(\tau^2+9)x & +\tau^4(\tau^2+9) & \\
\cline{4-5}
 & & & 0 & 0 &
\end{array}$$

$$a = \frac{-p^2}{3} + q = -27\tau^4 + \tau^2(21\tau^2+2) = -2\tau^6$$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -54\tau^6 + 3\tau^4(21\tau^2+2) - \tau^4(\tau^2+9) = \tau^{10}$$

Eq. 6

$$x_{12} = \frac{-p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = 3\tau^2 - \tau^3\phi \quad \text{so,} \quad \Gamma\cos\alpha = \frac{3}{2}(3\tau^2 - \tau^3\phi)$$

note: $\left(\sqrt[3]{\frac{\tau}{2}+\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}}\right)\left(\sqrt[3]{\frac{\tau}{2}-\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}}\right) = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$ so, $\phi^3 = 2\phi + \tau$ **Eq. 7**

also note: $[9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2]^2 - (\tau + 3\phi)^2(1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2) = 0$

$$= (9\tau^3+\tau)^2 + 36\phi^2 + 9\tau^6\phi(2\phi+\tau) + 2(9\tau^3+\tau)(6\phi - 3\tau^3\phi^2) - 36\tau^3(2\phi+\tau) - \tau^2(1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2))$$

$$- 6\tau(\phi - 3\tau^4(3\sqrt{5}\phi - 2\tau^3\phi^2 + \tau^2(2\phi+\tau))) - 9(\phi^2 - 3\tau^4(3\sqrt{5}\phi^2 - 2\tau^3(2\phi+\tau) + \tau^2\phi(2\phi+\tau)))$$

$$= (81\tau^6 + 18\tau^4 + \tau^2 - 36\tau^4 - \tau^2 + 9\tau^6\sqrt{5} + 18\tau^8 - 54\tau^8)$$

$$+ (9\tau^7 + 12(9\tau^3+\tau) - 72\tau^3 + 6\tau^9 - 6\tau + 54\tau^5\sqrt{5} + 36\tau^7 - 108\tau^8 + 27\tau^7)\phi$$

$$+ (36 + 18\tau^6 - 6\tau^3(9\tau^3+\tau) + 3\tau^8 - 36\tau^8 - 9 + 81\tau^4\sqrt{5} + 54\tau^6)\phi^2 = 0$$

therefore: $\sqrt{1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2)} = \dfrac{9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2}{\tau + 3\phi}$ **Eq. 8**

Eq. 3:

$$\gamma = \frac{-3 + \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}}{2\Gamma\cos\alpha} \underset{\text{Eq.6}}{=} \frac{-1 + \sqrt{1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2)}}{3\tau^2 - \tau^3\phi} \underset{\text{Eq.8}}{=} \frac{-1 + \dfrac{9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2}{\tau + 3\phi}}{3\tau^2 - \tau^3\phi}$$

$$\gamma = \frac{-\tau - 3\phi + 9\tau^3 + \tau + 3\phi + 3(3\tau^2 - \tau^4)\phi - 3\tau^3\phi^2}{(\tau + 3\phi)(3\tau^2 - \tau^3\phi)} = \frac{3\tau + 3\phi}{\tau + 3\phi}$$

$$\cos\alpha = \frac{1}{\sqrt{1 + \tan^2\alpha}} = \frac{1}{\sqrt{1 + \frac{\gamma^2}{3}}} = \frac{1}{\sqrt{1 + \frac{1}{3}\left(\frac{3\tau + 3\phi}{\tau + 3\phi}\right)^2}} = \frac{\tau + 3\phi}{2\sqrt{\tau^2 + 3\phi(\tau + \phi)}}$$

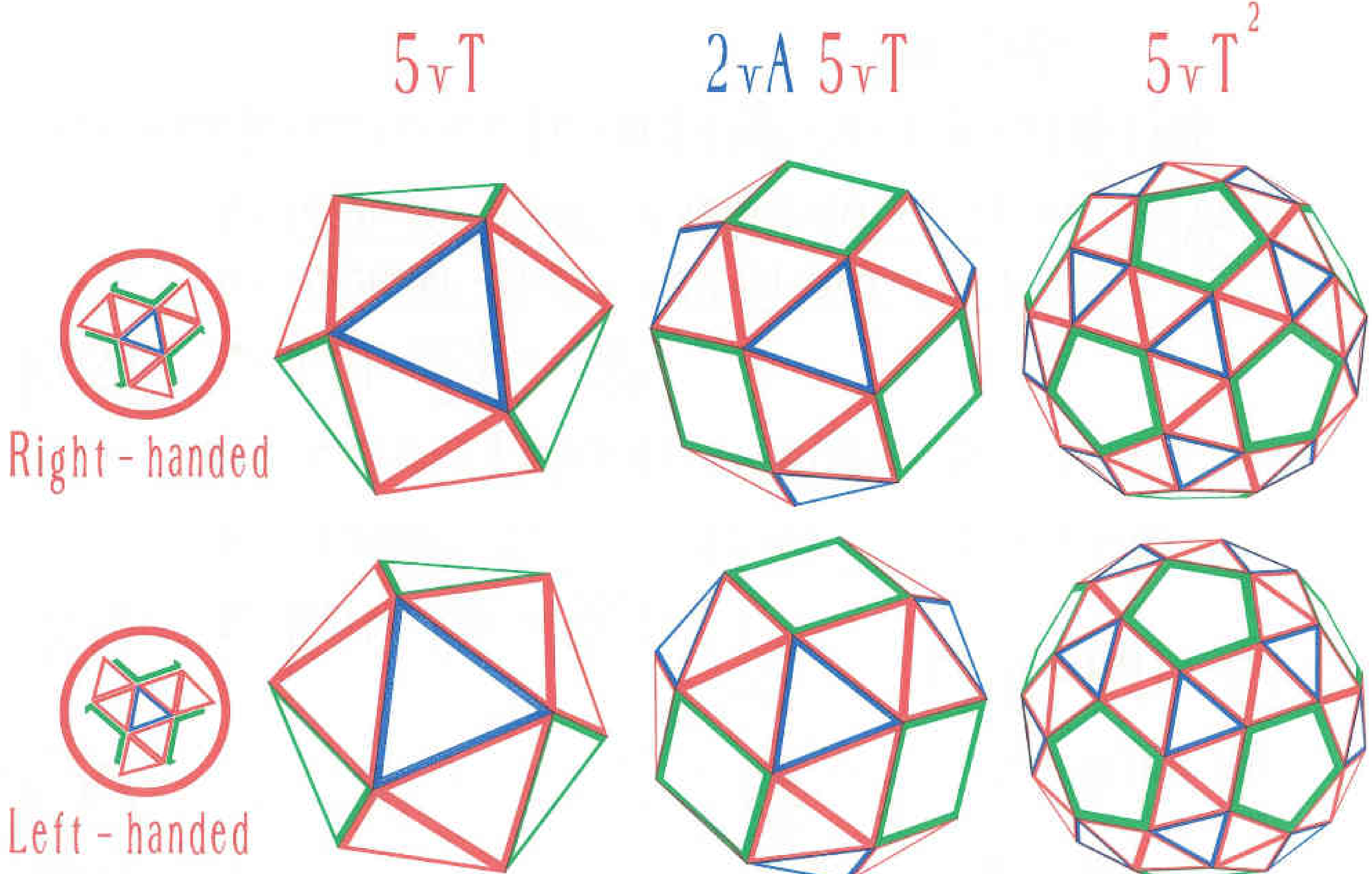
$$\Gamma = \cos\alpha\,[3 - \gamma] = \left(\frac{\tau + 3\phi}{2\sqrt{\tau^2 + 3\phi(\tau + \phi)}}\right)\left[3 - \frac{3\tau + 3\phi}{\tau + 3\phi}\right] = \frac{3\phi}{\sqrt{\tau^2 + 3\phi(\tau + \phi)}}$$

$$\text{Eq. 5:}\quad D = \frac{\tau^4\Gamma}{9\tau^2 - 2\Gamma\cos\alpha} \underset{\text{Eq.6}}{=} \frac{\tau\Gamma}{3\phi} = \frac{\tau}{\sqrt{\tau^2 + 3\phi(\tau + \phi)}} \qquad r_{\blacktriangle} = \frac{R_{I\blacktriangle}}{D} = \frac{\tau}{2\sqrt{3}}\sqrt{\tau^2 + 3\phi(\tau + \phi)}$$

$$r_{\pentagon} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^{-2} - (2\tan 36^\circ)^{-2}} = \sqrt{\frac{5\tau^2(\tau^2 + 3\phi(\tau + \phi)) + 5 - 3\tau^3\sqrt{5}}{60}} = \frac{\tau}{2}\sqrt{\frac{1}{\tau\sqrt{5}} + \phi(\tau + \phi)}$$

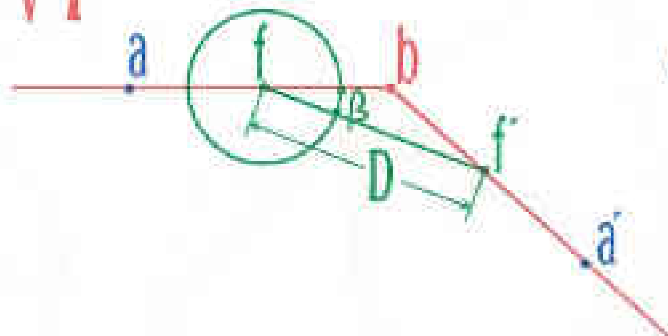
$$V_{5v\tau^2} = n_{\blacktriangle}\, a_{\blacktriangle}\, \frac{1}{3} r_{\blacktriangle} + n_{\pentagon}\, a_{\pentagon}\, \frac{1}{3} r_{\pentagon} = 80\,\frac{\sqrt{3}}{4}\,\frac{1}{3}\,\frac{\tau}{2\sqrt{3}}\sqrt{\tau^2 + 3\phi(\tau + \phi)} + 12\,\frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}\,\frac{1}{3}\,\frac{\tau}{2}\sqrt{\frac{1}{\tau\sqrt{5}} + \phi(\tau + \phi)}$$

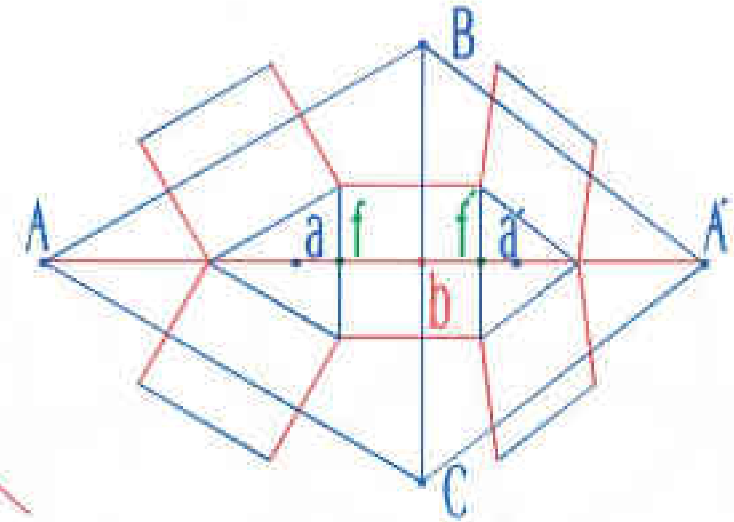
$$= \frac{10\tau}{3}\sqrt{\tau^2 + 3\phi(\tau + \phi)} + \frac{5\tau^2}{2}\sqrt{\frac{1}{5} + \frac{\tau\phi}{\sqrt{5}}(\tau + \phi)}$$



5vT
2vA 5vT
5vT²
Right-handed
Left-handed

7_vT





$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D\left(\frac{1}{2\sqrt{3}} + \frac{1}{2\cos\beta}\right)$$

$$r_\blacktriangle = \frac{R_\blacktriangle}{D} = R_\blacktriangle\left(1 + \frac{\sqrt{3}}{\cos\beta}\right)$$

Octa:

$$r_\blacktriangle = R_{0\blacktriangle}\left(1 + \frac{\sqrt{3}}{\cos\beta_0}\right) = \frac{1}{2\sqrt{3}}\left(3 + \sqrt{2}\right)$$

$$r_\blacksquare = \sqrt{r_\blacktriangle^2 + (2\tan 60^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\frac{1}{12}\left[11 + 6\sqrt{2} + 1 - 3\right]} = \frac{1}{2}\left(1 + \sqrt{2}\right)$$

$$V_{2vA7vT} = n_\blacktriangle a_\blacktriangle \frac{1}{3} r_\blacktriangle + n_\blacksquare a_\blacksquare \frac{1}{3} r_\blacksquare = 8\,\frac{\sqrt{3}}{4}\,\frac{1}{3}\,\frac{1}{2\sqrt{3}}\left(3 + \sqrt{2}\right) + 18\;1\;\frac{1}{3}\,\frac{1}{2}\left(1 + \sqrt{2}\right) = \frac{2}{3}\left(6 + 5\sqrt{2}\right)$$

Icosa:

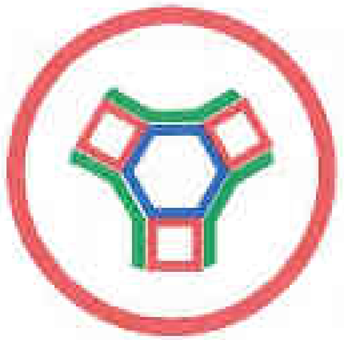
$$r_\blacktriangle = R_{1\blacktriangle}\left(1 + \frac{\sqrt{3}}{\cos\beta_1}\right) = \frac{3 + 2\sqrt{5}}{2\sqrt{3}}$$

$$r_\blacksquare = \sqrt{r_\blacktriangle^2 + (2\tan 60^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\frac{1}{12}\left[\left(29 + 12\sqrt{5}\right) + 1 - 3\right]} = \frac{\tau^3}{2}$$

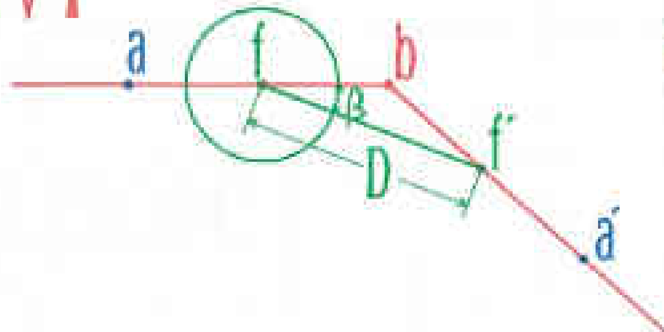
$$r_\bullet = \sqrt{r_\blacktriangle^2 + (2\tan 60^\circ)^{-2} - (2\tan 36^\circ)^{-2}} = \sqrt{\frac{1}{60}\left[5\left(29 + 12\sqrt{5}\right) + 5 - 15 - 6\sqrt{5}\right]} = \frac{3}{2}\sqrt{\frac{\tau^3}{\sqrt{5}}}$$

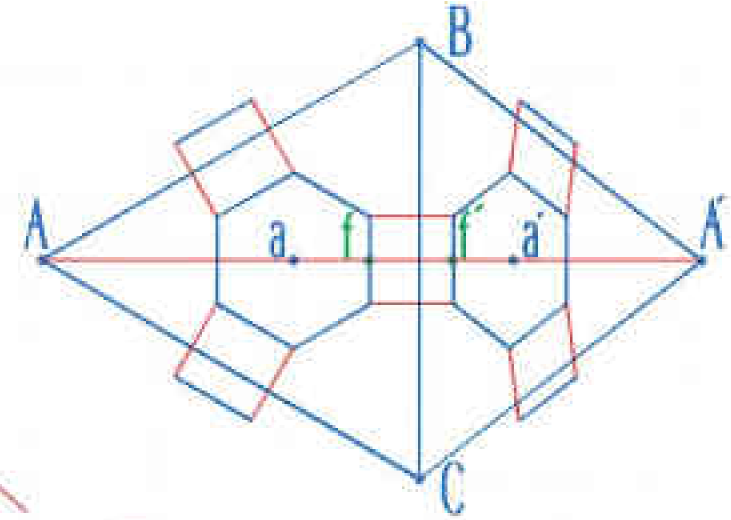
$$V_{5v7vT} = n_\blacktriangle a_\blacktriangle \frac{1}{3} r_\blacktriangle + n_\blacksquare a_\blacksquare \frac{1}{3} r_\blacksquare + n_\bullet a_\bullet \frac{1}{3} r_\bullet = 20\,\frac{\sqrt{3}}{4}\,\frac{1}{3}\,\frac{3 + 2\sqrt{5}}{2\sqrt{3}} + 30\;1\;\frac{1}{3}\,\frac{2 + \sqrt{5}}{2} + 12\,\frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}\,\frac{1}{3}\,\frac{3}{2}\sqrt{\frac{\tau^3}{\sqrt{5}}}$$

$$= \frac{1}{3}\left(60 + 29\sqrt{5}\right)$$



$9_{\nu}T$





$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D\left(\frac{\sqrt{3}}{2} + \frac{1}{2\cos\beta}\right)$$

$$r_{\bullet} = \frac{R_{\blacktriangle}}{D} = R_{\blacktriangle}\left(3 + \frac{\sqrt{3}}{\cos\beta}\right)$$

Octa:

$$r_{\bullet} = R_{0\blacktriangle}\left(3 + \frac{\sqrt{3}}{\cos\beta_0}\right) = \frac{\sqrt{3}}{2}(1+\sqrt{2})$$

$$r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\tfrac{1}{4}[3(3+2\sqrt{2}) + 3 - 1]} = \tfrac{1}{2}(3+\sqrt{2})$$

$$r_{\blacklozenge} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 22\tfrac{1}{2}^\circ)^{-2}} = \sqrt{\tfrac{1}{4}[3(3+2\sqrt{2}) + 3 - 3 - 2\sqrt{2}]} = \tfrac{1}{2}(1+2\sqrt{2})$$

$$V_{2\nu\Lambda 9\nu T} = n_{\bullet}a_{\bullet}\tfrac{1}{3}r_{\bullet} + n_{\blacksquare}a_{\blacksquare}\tfrac{1}{3}r_{\blacksquare} + n_{\blacklozenge}a_{\blacklozenge}\tfrac{1}{3}r_{\blacklozenge} = 8\,\tfrac{3\sqrt{3}}{2}\,\tfrac{1}{3}\,\tfrac{\sqrt{3}}{2}(1+\sqrt{2}) + 12\;1\;\tfrac{1}{3}\,\tfrac{1}{2}(3+\sqrt{2}) + 6\;2(1+\sqrt{2})\,\tfrac{1}{3}\,\tfrac{1}{2}(1+2\sqrt{2})$$

$$= 30 + 14\sqrt{2}$$

Icosa:

$$r_{\bullet} = R_{1\blacktriangle}\left(3 + \frac{\sqrt{3}}{\cos\beta_1}\right) = \frac{\tau^3\sqrt{3}}{2}$$

$$r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 45^\circ)^{-2}} = \sqrt{\tfrac{1}{4}[3(9+4\sqrt{5}) + 3 - 1]} = \tfrac{1}{2}(3+2\sqrt{5})$$

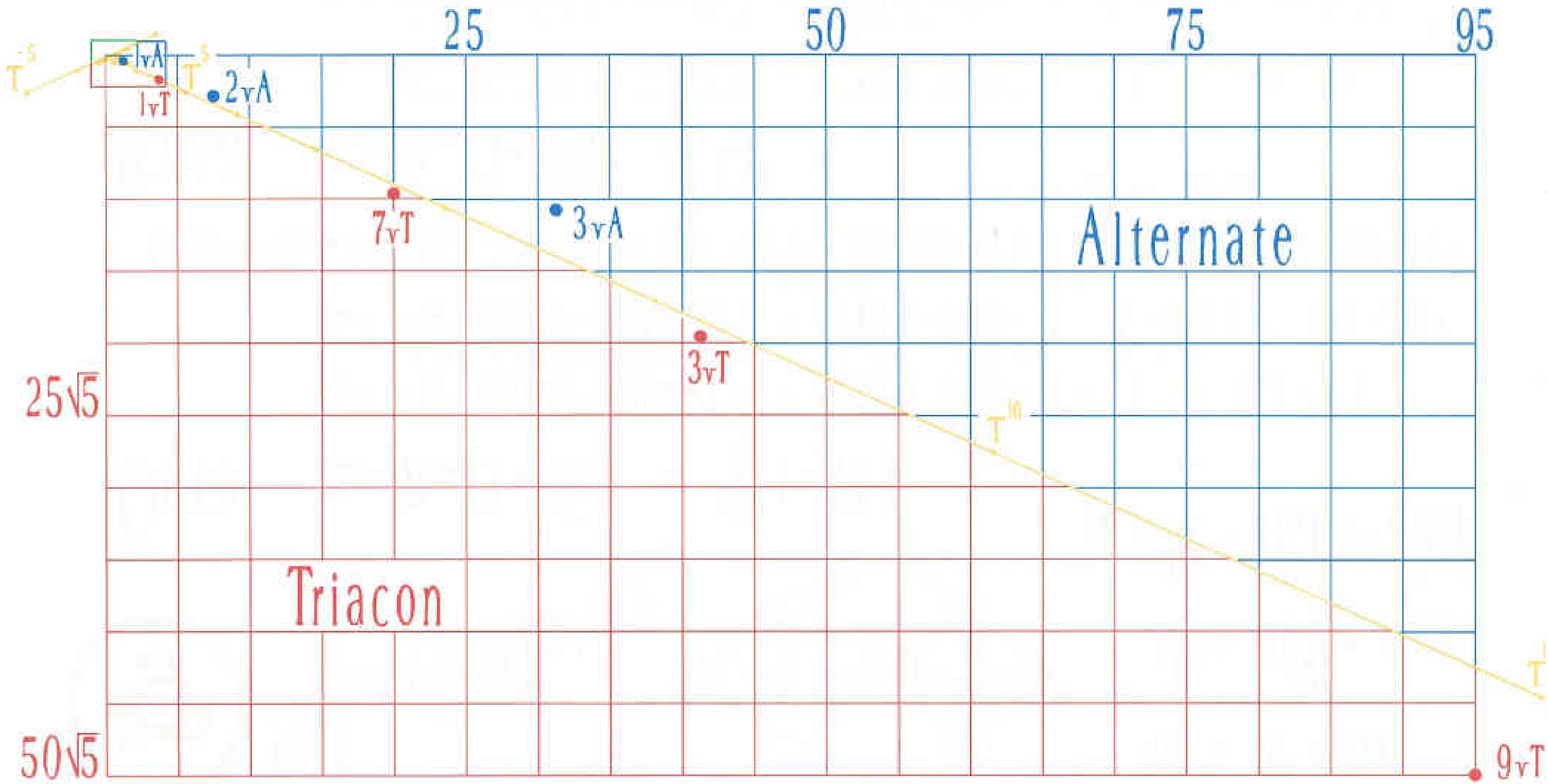
$$r_{\circledast} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^{-2} - (2\tan 18^\circ)^{-2}} = \sqrt{\tfrac{1}{4}[3(9+4\sqrt{5}) + 3 - 5 - 2\sqrt{5}]} = \tfrac{1}{2}\sqrt{5\tau^3\sqrt{5}}$$

$$V_{5\nu 9\nu T} = n_{\bullet}a_{\bullet}\tfrac{1}{3}r_{\bullet} + n_{\blacksquare}a_{\blacksquare}\tfrac{1}{3}r_{\blacksquare} + n_{\circledast}a_{\circledast}\tfrac{1}{3}r_{\circledast} = 20\,\tfrac{3\sqrt{3}}{2}\,\tfrac{1}{3}\,\tfrac{\sqrt{3}}{2}\tau^3 + 30\;1\;\tfrac{1}{3}\,\tfrac{1}{2}(3+2\sqrt{5}) + 12\,\tfrac{5}{2}\sqrt{\tau^3\sqrt{5}}\,\tfrac{1}{3}\,\tfrac{1}{2}\sqrt{5\tau^3\sqrt{5}}$$

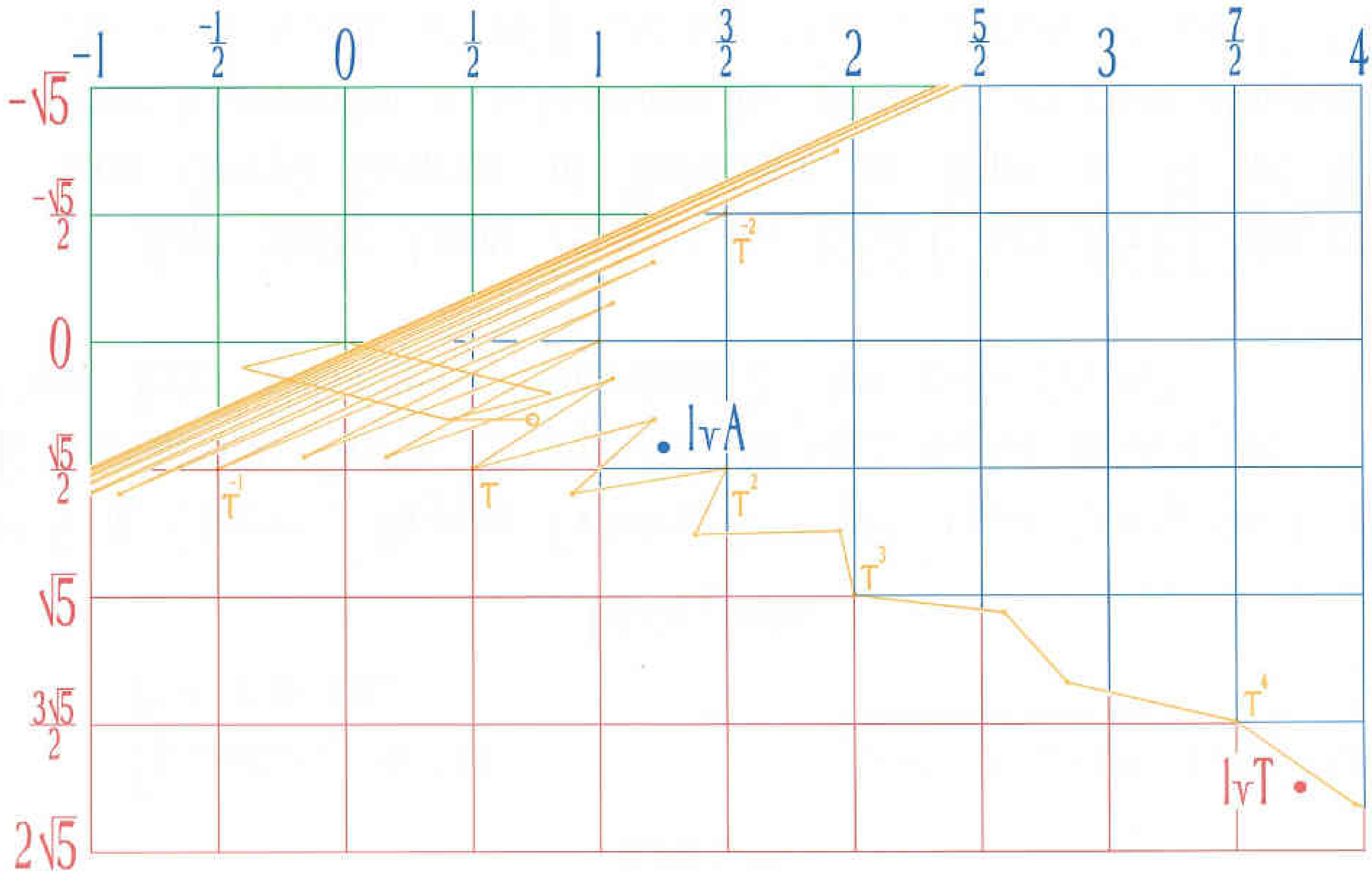
$$= 95 + 50\sqrt{5}$$

Integer powers of the Golden Section

Plotted volumes of icosahedral based solids



One third powers of the Golden Section



Exercise:

Spinnability of the first golden circle leads to a new second root. Plot this root.

Bibliography:

H. S. M. Coxeter, "Regular Complex Polytopes," London: Cambridge Univ. Press, 1974.
R. Buckminster Fuller, "Synergetics," Vols. I and 2. New York: Macmillan, 1982.
Lloya Kahn and others, "Domebook 2," Pacific Domes. Calif. 1971.

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Alternate and Triacon Breakdowns