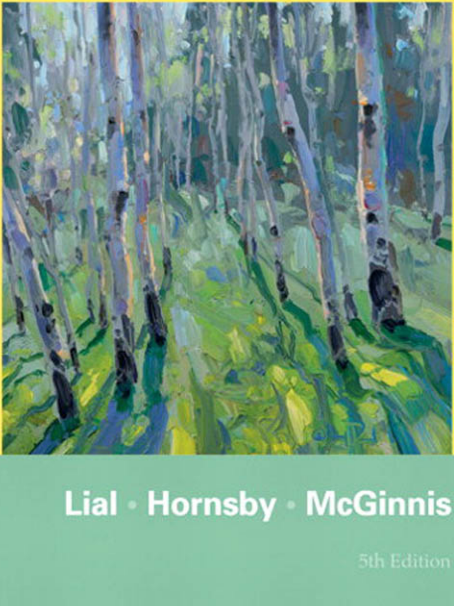
Beginning & Intermediate

# Algebra



**Lial · Hornsby · McGinnis**

5th Edition

5TH EDITION

# BEGINNING AND INTERMEDIATE ALGEBRA

5TH EDITION

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**Library of Congress Cataloging-in-Publication Data**
Lial, Margaret L.
Beginning and intermediate algebra / Margaret L. Lial, John Hornsby,
Terry McGinnis. — 5th ed.
p. cm.
ISBN-13: 978-0-321-71542-5
(student edition)
ISBN-10: 0-321-71542-X
(student edition)
1. Algebra—Textbooks. I. Hornsby, John. II. McGinnis, Terry. III. Title.
QA152.3.L52 2012
512—dc22
2010002285

1 2 3 4 5 6 7 8 9 10—CRK—14 13 12 11 10

Addison-Wesley
is an imprint of

PEARSON

**www.pearsonhighered.com**

ISBN 13: 978-0-321-71542-5
ISBN 10: 0-321-71542-X

*To Callie, Kurt, Clayton, and Grady—*
*Welcome to our family.*

Marge, John, and Terry

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# Preface

It is with pleasure that we offer the fifth edition of *Beginning and Intermediate Algebra*. With each new edition, the text has been shaped and adapted to meet the changing needs of both students and educators, and this edition faithfully continues that process. As always, we have taken special care to respond to the specific suggestions of users and reviewers through enhanced discussions, new and updated examples and exercises, helpful features, updated figures and graphs, and an extensive package of supplements and study aids. We believe the result is an easy-to-use, comprehensive text that is the best edition yet.

Students who have never studied algebra—as well as those who require further review of basic algebraic concepts before taking additional courses in mathematics, business, science, nursing, or other fields—will benefit from the text's student-oriented approach. Of particular interest to students and instructors will be the **NEW** Study Skills activities and Now Try Exercises.

This text is part of a series that also includes the following books:

- *Beginning Algebra,* Eleventh Edition, by Lial, Hornsby, and McGinnis
- *Intermediate Algebra,* Eleventh Edition, by Lial, Hornsby, and McGinnis
- *Algebra for College Students,* Seventh Edition, by Lial, Hornsby, and McGinnis

## NEW IN THIS EDITION

We are pleased to offer the following new student-oriented features and study aids:

**Lial Video Library** This collection of video resources helps students navigate the road to success. It is available in MyMathLab and on Video Resources on DVD.

**MyWorkBook** This helpful guide provides extra practice exercises for every chapter of the text and includes the following resources for every section:

- Key vocabulary terms and vocabulary practice problems
- Guided Examples with step-by-step solutions and similar Practice Exercises, keyed to the text by Learning Objective
- References to textbook Examples and Section Lecture Videos for additional help
- Additional Exercises with ample space for students to show their work, keyed to the text by Learning Objective

**Study Skills** Poor study skills are a major reason why students do not succeed in mathematics. In these short activities, we provide helpful information, tips, and strategies on a variety of essential study skills, including *Reading Your Math Textbook, Tackling Your Homework, Taking Math Tests,* and *Managing Your Time.* While most of the activities are concentrated in the early chapters of the text, each has been designed independently to allow flexible use with individuals or small groups of students, or as a source of material for in-class discussions. (See pages 48 and 223.)

**Now Try Exercises** To actively engage students in the learning process, we now include a parallel margin exercise juxtaposed with each numbered example. These all-new exercises enable students to immediately apply and reinforce the concepts and skills presented in the corresponding examples. Answers are conveniently located on the same page so students can quickly check their results. (See pages 3 and 87.)

**Revised Exposition** As each section of the text was being revised, we paid special attention to the exposition, which has been tightened and polished. (See Section 1.4 Real Numbers and the Number Line, for example.) We believe this has improved discussions and presentations of topics.

**Specific Content Changes** These include the following:

- We gave the exercise sets special attention. There are over 1000 new and updated exercises, including problems that check conceptual understanding, focus on skill development, and provide review. We also worked to improve the even-odd pairing of exercises.
- Real-world data in over 150 applications in the examples and exercises have been updated.
- There is an increased emphasis on the difference between expressions and equations, including a new Caution at the beginning of Section 2.1. Throughout the text, we have reformatted many example solutions to use a “drop down” layout in order to further emphasize for students the difference between simplifying expressions and solving equations.
- We increased the emphasis on checking solutions and answers, as indicated by the new *CHECK* tag and ✓ in the exposition and examples.
- The presentation on solving linear equations in Sections 2.1–2.3 now includes five new examples and corresponding exercises.
- Section 2.6 includes entirely new discussion and examples on percent, percent equations, and percent applications, plus corresponding exercises.
- Section 3.4 on writing and graphing equations of lines provides increased development and coverage of the slope-intercept form, including two new examples.
- Section 6.5 includes new coverage of simplifying rational expressions with negative exponents.
- Section 7.3 Introduction to Functions from the previous edition has been expanded and split into two sections.
- Presentations of the following topics have also been enhanced and expanded:

  Dividing real numbers involving zero (Section 1.6)
  Solving applications involving consecutive integers and finding angle measures (Section 2.4)
  Solving formulas for specified variables (Sections 2.5 and 6.7)
  Using interval notation (Section 2.8)
  Graphing linear equations in two variables (Section 3.2)
  Dividing polynomials (Section 4.7)
  Factoring trinomials (Section 5.2)
  Solving quadratic equations by factoring (Sections 5.6 and 11.1)
  Solving systems of linear equations with decimal coefficients (Section 8.2)

Solving systems of linear equations in three variables (Section 8.4)
Graphing linear inequalities in two variables (Section 9.3)
Solving quadratic equations by substitution (Section 11.4)
Evaluating expressions involving the greatest integer (Section 13.1)
Graphing hyperbolas (Section 13.3)
Evaluating factorials and binomial coefficients (Section 14.4)

## HALLMARK FEATURES

We have included the following helpful features, each of which is designed to increase ease-of-use by students and/or instructors.

**Annotated Instructor's Edition** For convenient reference, we include answers to the exercises "on page" in the *Annotated Instructor's Edition,* using an enhanced, easy-to-read format. In addition, we have added approximately 30 new Teaching Tips and over 40 new and updated Classroom Examples.

**Relevant Chapter Openers** In the new and updated chapter openers, we feature real-world applications of mathematics that are relevant to students and tied to specific material within the chapters. Examples of topics include Americans' personal savings rate, the Olympics, and student credit card debt. Each opener also includes a section outline. (See pages 1, 85, and 175.)

**Helpful Learning Objectives** We begin each section with clearly stated, numbered objectives, and the included material is directly keyed to these objectives so that students and instructors know exactly what is covered in each section. (See pages 2 and 130.)

**Popular Cautions and Notes** One of the most popular features of previous editions, we include information marked ⚠ **CAUTION** and **NOTE** to warn students about common errors and emphasize important ideas throughout the exposition. The updated text design makes them easy to spot. (See pages 2 and 56.)

**Comprehensive Examples** The new edition of this text features a multitude of step-by-step, worked-out examples that include pedagogical color, helpful side comments, and special pointers. We give increased attention to checking example solutions—more checks, designated using a special *CHECK* tag, are included than in past editions. (See pages 87 and 333.)

**More Pointers** Well received by both students and instructors in the previous edition, we incorporate more pointers in examples and discussions throughout this edition of the text. They provide students with important on-the-spot reminders and warnings about common pitfalls. (See pages 192 and 281.)

**Updated Figures, Photos, and Hand-Drawn Graphs** Today's students are more visually oriented than ever. As a result, we have made a concerted effort to include appealing mathematical figures, diagrams, tables, and graphs, including a "hand-drawn" style of graphs, whenever possible. (See pages 188 and 261.) Many of the graphs also use a style similar to that seen by students in today's print and electronic media. We have incorporated new photos to accompany applications in examples and exercises. (See pages 109 and 176.)

**Relevant Real-Life Applications** We include many new or updated applications from fields such as business, pop culture, sports, technology, and the life sciences that show the relevance of algebra to daily life. (See pages 116 and 541.)

**Emphasis on Problem-Solving** We introduce our six-step problem-solving method in Chapter 2 and integrate it throughout the text. The six steps, *Read, Assign a Variable, Write an Equation, Solve, State the Answer,* and *Check,* are emphasized in boldface type and repeated in examples and exercises to reinforce the problem-solving process for students. (See pages 108 and 337.) We also provide students with **PROBLEM-SOLVING HINT** boxes that feature helpful problem-solving tips and strategies. (See pages 139 and 338.)

**Connections** We include these to give students another avenue for making connections to the real world, graphing technology, or other mathematical concepts, as well as to provide historical background and thought-provoking questions for writing, class discussion, or group work. (See pages 195 and 251.)

**Ample and Varied Exercise Sets** One of the most commonly mentioned strengths of this text is its exercise sets. We include a wealth of exercises to provide students with opportunities to practice, apply, connect, review, and extend the algebraic concepts and skills they are learning. We also incorporate numerous illustrations, tables, graphs, and photos to help students visualize the problems they are solving. Problem types include writing ✎, graphing calculator 🖩, multiple-choice, true/false, matching, and fill-in-the-blank problems, as well as the following:

- ***Concept Check*** exercises facilitate students' mathematical thinking and conceptual understanding. (See pages 96 and 196.)
- ***WHAT WENT WRONG?*** exercises ask students to identify typical errors in solutions and work the problems correctly. (See pages 208 and 335.)
- ***Brain Busters*** exercises challenge students to go beyond the section examples. (See pages 119 and 246.)
- ***RELATING CONCEPTS*** exercises help students tie together topics and develop problem-solving skills as they compare and contrast ideas, identify and describe patterns, and extend concepts to new situations. These exercises make great collaborative activities for pairs or small groups of students. (See pages 209 and 264.)
- ***TECHNOLOGY INSIGHTS*** exercises provide an opportunity for students to interpret typical results seen on graphing calculator screens. Actual screens from the TI-83/84 Plus graphing calculator are featured. (See pages 210 and 336.)
- ***PREVIEW EXERCISES*** allow students to *review* previously-studied concepts and *preview* skills needed for the upcoming section. These make good oral warm-up exercises to open class discussions. (See pages 92 and 199.)

**Special Summary Exercises** We include a set of these popular in-chapter exercises in selected chapters. They provide students with the all-important ***mixed* review problems** they need to master topics and often include summaries of solution methods and/or additional examples. (See pages 247 and 404.)

**Extensive Review Opportunities** We conclude each chapter with the following review components:

- A **Chapter Summary** that features a helpful list of **Key Terms,** organized by section, **New Symbols, Test Your Word Power** vocabulary quiz (with answers immediately following), and a **Quick Review** of each section's contents, complete with additional examples (See pages 224–226.)

- A comprehensive set of **Chapter Review Exercises,** keyed to individual sections for easy student reference, as well as a set of **Mixed Review Exercises** that helps students further synthesize concepts (See pages 227–228.)
- A **Chapter Test** that students can take under test conditions to see how well they have mastered the chapter material (See page 229.)
- A set of **Cumulative Review Exercises** (beginning in Chapter 2) that covers material going back to Chapter 1 (See page 230.)

**Glossary** For easy reference at the back of the book, we include a comprehensive glossary featuring key terms and definitions from throughout the text. (See pages G-1 to G-8.)

## SUPPLEMENTS

For a comprehensive list of the supplements and study aids that accompany *Beginning and Intermediate Algebra,* Fifth Edition, see pages xix–xxi.

## ACKNOWLEDGMENTS

The comments, criticisms, and suggestions of users, nonusers, instructors, and students have positively shaped this textbook over the years, and we are most grateful for the many responses we have received. Thanks to the following people for their review work, feedback, assistance at various meetings, and additional media contributions:

Barbara Aaker, *Community College of Denver*
Viola Lee Bean, *Boise State University*
Kim Bennekin, *Georgia Perimeter College*
Dixie Blackinton, *Weber State University*
Tim Caldwell, *Meridian Community College*
Sally Casey, *Shawnee Community College*
Callie Daniels, *St. Charles Community College*
Cheryl Davids, *Central Carolina Technical College*
Robert Diaz, *Fullerton College*
Chris Diorietes, *Fayetteville Technical Community College*
Sylvia Dreyfus, *Meridian Community College*
Lucy Edwards, *Las Positas College*
Sabine Eggleston, *Edison College*
LaTonya Ellis, *Bishop State Community College*
Jacqui Fields, *Wake Technical Community College*
Beverly Hall, *Fayetteville Technical Community College*
Sandee House, *Georgia Perimeter College*
Lynette King, *Gadsden State Community College*
Linda Kodama, *Windward Community College*
Ted Koukounas, *Suffolk Community College*
Karen McKarnin, *Allen County Community College*
James Metz, *Kapi´olani Community College*
Barbara Meyers, *Cameron University*

Jean Millen, *Georgia Perimeter College*
Molly Misko, *Gadsden State Community College*
Jane Roads, *Moberly Area Community College*
Cindy Scofield, *Polk State College*
Lisa Scott, *Texas Wesleyan University*
Melanie Smith, *Bishop State Community College*
Linda Smoke, *Central Michigan University*
Erik Stubsten, *Chattanooga State Technical Community College*
Tong Wagner, *Greenville Technical College*
Sessia Wyche, *University of Texas at Brownsville*

Special thanks are due the many instructors at Broward College who provided insightful comments.

Over the years, we have come to rely on an extensive team of experienced professionals. Our sincere thanks go to these dedicated individuals at Addison-Wesley, who worked long and hard to make this revision a success: Chris Hoag, Maureen O'Connor, Michelle Renda, Adam Goldstein, Kari Heen, Courtney Slade, Kathy Manley, Stephanie Green, Lin Mahoney, and Mary St. Thomas.

We are especially grateful to Callie Daniels for her excellent work on the new Now Try Exercises. Abby Tanenbaum did a terrific job helping us revise real-data applications. Kathy Diamond provided expert guidance through all phases of production and rescued us from one snafu or another on multiple occasions. Marilyn Dwyer and Nesbitt Graphics, Inc., provided some of the highest quality production work we have experienced on the challenging format of these books.

Special thanks are due Jeff Cole, who continues to supply accurate, helpful solutions manuals; David Atwood, who wrote the comprehensive *Instructor's Resource Manual with Tests;* Beverly Fusfield, who provided the new MyWorkBook; Beth Anderson, who provided wonderful photo research; and Lucie Haskins, for yet another accurate, useful index. De Cook, Shannon d'Hemecourt, Paul Lorczak, and Sarah Sponholz did a thorough, timely job accuracy checking manuscript and page proofs. It has indeed been a pleasure to work with such an outstanding group of professionals.

As an author team, we are committed to providing the best possible text and supplements package to help instructors teach and students succeed. As we continue to work toward this goal, we would welcome any comments or suggestions you might have via e-mail to math@pearson.com.

**Margaret L. Lial**
**John Hornsby**
**Terry McGinnis**

## STUDENT SUPPLEMENTS

### Student's Solutions Manual

- By Jeffery A. Cole, *Anoka-Ramsey Community College*
- Provides detailed solutions to the odd-numbered, section-level exercises and to all Now Try Exercises, Relating Concepts, Summary, Chapter Review, Chapter Test, and Cumulative Review Exercises

ISBNs: 0-321-71565-9, 978-0-321-71565-4

### NEW Video Resources on DVD featuring the Lial Video Library

- Provides a wealth of video resources to help students navigate the road to success
- Available in MyMathLab (with optional subtitles in English)
- Includes the following resources:

  **Section Lecture Videos** that offer a new navigation menu for easy focus on key examples and exercises needed for review in each section (with optional subtitles in Spanish and English)

  **Solutions Clips** that feature an instructor working through selected exercises marked in the text with a DVD icon 💿

  **Quick Review Lectures** that provide a short summary lecture of each key concept from Quick Reviews at the end of every chapter in the text

  **Chapter Test Prep Videos** that include step-by-step solutions to all Chapter Test exercises and give guidance and support when needed most—the night before an exam. Also available on YouTube (searchable using author name and book title)

ISBNs: 0-321-71572-1, 978-0-321-71572-2

### NEW MyWorkBook

- Provides Guided Examples and corresponding Now Try Exercises for each text objective
- Refers students to correlated Examples, Lecture Videos, and Exercise Solution Clips
- Includes extra practice exercises for every section of the text with ample space for students to show their work
- Lists the learning objectives and key vocabulary terms for every text section, along with vocabulary practice problems

ISBNs: 0-321-71573-X, 978-0-321-71573-9

## INSTRUCTOR SUPPLEMENTS

### Annotated Instructor's Edition

- Provides "on-page" answers to all text exercises in an easy-to-read margin format, along with Teaching Tips and extensive Classroom Examples
- Includes icons to identify writing ✎ and calculator 🖩 exercises. These are in the Student Edition also.

ISBNs: 0-321-71569-1, 978-0-321-71569-2

### Instructor's Solutions Manual

- By Jeffery A. Cole, *Anoka-Ramsey Community College*
- Provides complete answers to all text exercises, including all Classroom Examples and Now Try Exercises

ISBNs: 0-321-71566-7, 978-0-321-71566-1

### Instructor's Resource Manual with Tests

- By David Atwood, *Rochester Community and Technical College*
- Contains two diagnostic pretests, four free-response and two multiple-choice test forms per chapter, and two final exams
- Includes a mini-lecture for each section of the text with objectives, key examples, and teaching tips
- Provides a correlation guide from the fourth to the fifth edition

ISBNs: 0-321-71567-5, 978-0-321-71567-8

### PowerPoint® Lecture Slides

- Present key concepts and definitions from the text
- Available for download at www.pearsonhighered.com/irc

ISBNs: 0-321-71571-3, 978-0-321-71571-5

### TestGen® (www.pearsonhighered.com/testgen)

- Enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all text objectives
- Allows instructors to create multiple but equivalent versions of the same question or test with the click of a button
- Allows instructors to modify test bank questions or add new questions
- Available for download from Pearson Education's online catalog

ISBNs: 0-321-71568-3, 978-0-321-71568-5

## STUDENT SUPPLEMENTS

### InterAct Math Tutorial Website http://www.interactmath.com

- Provides practice and tutorial help online
- Provides algorithmically generated practice exercises that correlate directly to the exercises in the textbook
- Allows students to retry an exercise with new values each time for unlimited practice and mastery
- Includes an interactive guided solution for each exercise that gives helpful feedback when an incorrect answer is entered
- Enables students to view the steps of a worked-out sample problem similar to the one being worked on

## INSTRUCTOR SUPPLEMENTS

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(http://www.pearsontutorservices.com/math-adjunct.html)

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Assistance is provided for faculty in the following areas:

- Suggested syllabus consultation
- Tips on using materials packed with your book
- Book-specific content assistance
- Teaching suggestions, including advice on classroom strategies

## Available for Students and Instructors

**MyMathLab® Online Course (Access code required.)**

MyMathLab® is a text-specific, easily customizable online course that integrates interactive multimedia instruction with textbook content. MyMathLab gives instructors the tools they need to deliver all or a portion of their course online, whether their students are in a lab setting or working from home.

- **Interactive homework exercises,** correlated to the textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and tutorial learning aids for extra help.
- **Personalized homework** assignments can be designed to meet the needs of the class. MyMathLab tailors the assignment for each student based on their test or quiz scores so that each student's homework assignment contains only the problems they still need to master.
- **Personalized Study Plan,** generated when students complete a test or quiz or homework, indicates which topics have been mastered and links to tutorial exercises for topics students have not mastered. Instructors can customize the Study Plan so that the topics available match their course content.
- **Multimedia learning aids,** such as video lectures and podcasts, animations, and a complete multimedia textbook, help students independently improve their understanding and performance. Instructors can assign these multimedia learning aids as homework to help their students grasp the concepts.
- **Homework and Test Manager** lets instructors assign homework, quizzes, and tests that are automatically graded. They can select just the right mix of questions from the MyMathLab exercise bank, instructor-created custom exercises, and/or TestGen® test items.
- **Gradebook,** designed specifically for mathematics and statistics, automatically tracks students' results, lets instructors stay on top of student performance, and gives them control over how to calculate final grades. They can also add offline (paper-and-pencil) grades to the gradebook.

- **MathXL Exercise Builder** allows instructors to create static and algorithmic exercises for their online assignments. They can use the library of sample exercises as an easy starting point, or they can edit any course-related exercise.
- **Pearson Tutor Center** (www.pearsontutorservices.com) access is automatically included with MyMathLab. The Tutor Center is staffed by qualified math instructors who provide textbook-specific tutoring for students via toll-free phone, fax, email, and interactive Web sessions.

Students do their assignments in the Flash®-based MathXL Player, which is compatible with almost any browser (Firefox®, Safari™, or Internet Explorer®) on almost any platform (Macintosh® or Windows®). MyMathLab is powered by CourseCompass™, Pearson Education's online teaching and learning environment, and by MathXL®, our online homework, tutorial, and assessment system. MyMathLab is available to qualified adopters. For more information, visit our website at www.mymathlab.com or contact your Pearson representative.

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- **Interactive homework exercises,** correlated to the textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and learning aids for extra help.
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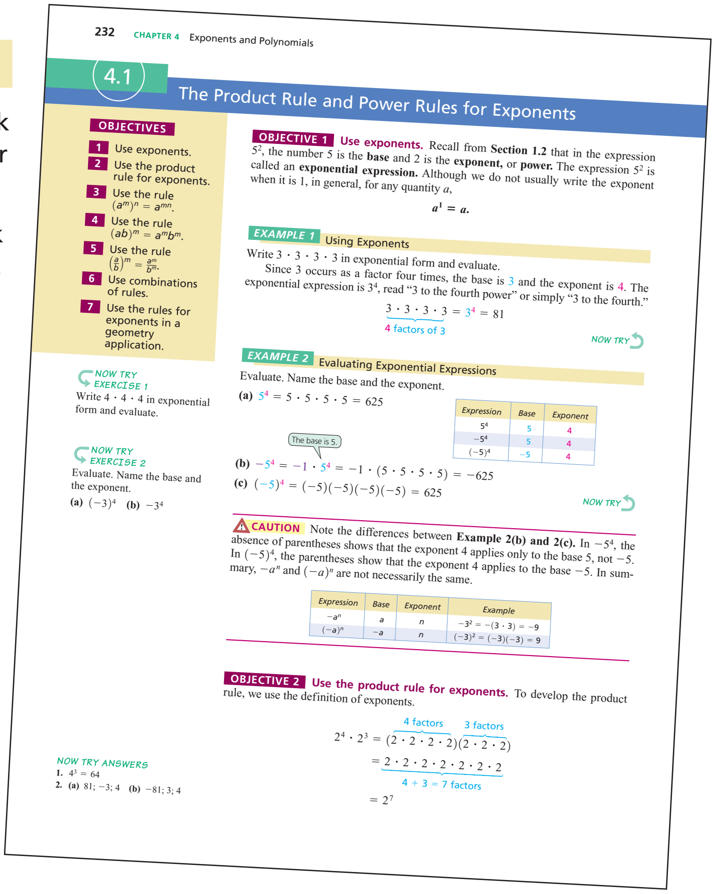
- **MathXL Exercise Builder** allows instructors to create static and algorithmic exercises for their online assignments. They can use the library of sample exercises as an easy starting point or the Exercise Builder to edit any of the course-related exercises.
- **Homework and Test Manager** lets instructors create online homework, quizzes, and tests that are automatically graded. They can select just the right mix of questions from the MathXL exercise bank, instructor-created custom exercises, and/or TestGen test items.

The new, Flash®-based MathXL Player is compatible with almost any browser (Firefox®, Safari™, or Internet Explorer®) on almost any platform (Macintosh® or Windows®). MathXL is available to qualified adopters. For more information, visit our website at www.mathxl.com, or contact your Pearson representative.

STUDY SKILLS

# Using Your Math Textbook

***Your textbook is a valuable resource.*** You will learn more if you fully make use of the features it offers.



232 CHAPTER 4 Exponents and Polynomials

4.1 The Product Rule and Power Rules for Exponents

**OBJECTIVES**

1. Use exponents.
2. Use the product rule for exponents.
3. Use the rule $(a^m)^n = a^{mn}$.
4. Use the rule $(ab)^m = a^m b^m$.
5. Use the rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.
6. Use combinations of rules.
7. Use the rules for exponents in a geometry application.

**OBJECTIVE 1** **Use exponents.** Recall from **Section 1.2** that in the expression $5^2$, the number 5 is the **base** and 2 is the **exponent,** or **power.** The expression $5^2$ is called an **exponential expression.** Although we do not usually write the exponent when it is 1, in general, for any quantity $a$,

$$a^1 = a.$$

**EXAMPLE 1** Using Exponents

Write $3 \cdot 3 \cdot 3 \cdot 3$ in exponential form and evaluate.

Since 3 occurs as a factor four times, the base is 3 and the exponent is 4. The exponential expression is $3^4$, read "3 to the fourth power" or simply "3 to the fourth."

$$\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{4 \text{ factors of } 3} = 3^4 = 81$$

NOW TRY

NOW TRY EXERCISE 1
Write $4 \cdot 4 \cdot 4$ in exponential form and evaluate.

**EXAMPLE 2** Evaluating Exponential Expressions

Evaluate. Name the base and the exponent.

**(a)** $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$

| Expression | Base | Exponent |
|---|---|---|
| $5^4$ | 5 | 4 |
| $-5^4$ | 5 | 4 |
| $(-5)^4$ | $-5$ | 4 |

The base is 5.

**(b)** $-5^4 = -1 \cdot 5^4 = -1 \cdot (5 \cdot 5 \cdot 5 \cdot 5) = -625$

**(c)** $(-5)^4 = (-5)(-5)(-5)(-5) = 625$

NOW TRY

NOW TRY EXERCISE 2
Evaluate. Name the base and the exponent.
**(a)** $(-3)^4$ **(b)** $-3^4$

CAUTION Note the differences between **Example 2(b) and 2(c).** In $-5^4$, the absence of parentheses shows that the exponent 4 applies only to the base 5, not $-5$. In $(-5)^4$, the parentheses show that the exponent 4 applies to the base $-5$. In summary, $-a^n$ and $(-a)^n$ are not necessarily the same.

| Expression | Base | Exponent | Example |
|---|---|---|---|
| $-a^n$ | $a$ | $n$ | $-3^2 = -(3 \cdot 3) = -9$ |
| $(-a)^n$ | $-a$ | $n$ | $(-3)^2 = (-3)(-3) = 9$ |

**OBJECTIVE 2** **Use the product rule for exponents.** To develop the product rule, we use the definition of exponents.

$$\begin{aligned} 2^4 \cdot 2^3 &= \overbrace{(2 \cdot 2 \cdot 2 \cdot 2)}^{4 \text{ factors}}\overbrace{(2 \cdot 2 \cdot 2)}^{3 \text{ factors}} \\ &= \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{4 + 3 = 7 \text{ factors}} \\ &= 2^7 \end{aligned}$$

NOW TRY ANSWERS
1. $4^3 = 64$
2. (a) $81; -3; 4$ (b) $-81; 3; 4$

## General Features

- **Table of Contents** Find this at the front of the text. Mark the chapters and sections you will cover, as noted on your course syllabus.
- **Answer Section** Tab this section at the back of the book so you can refer to it frequently when doing homework. Answers to odd-numbered section exercises are provided. Answers to ALL summary, chapter review, test, and cumulative review exercises are given.
- **Glossary** Find this feature after the answer section at the back of the text. It provides an alphabetical list of the key terms found in the text, with definitions and section references.
- **List of Formulas** Inside the back cover of the text is a helpful list of geometric formulas, along with review information on triangles and angles. Use these for reference throughout the course.

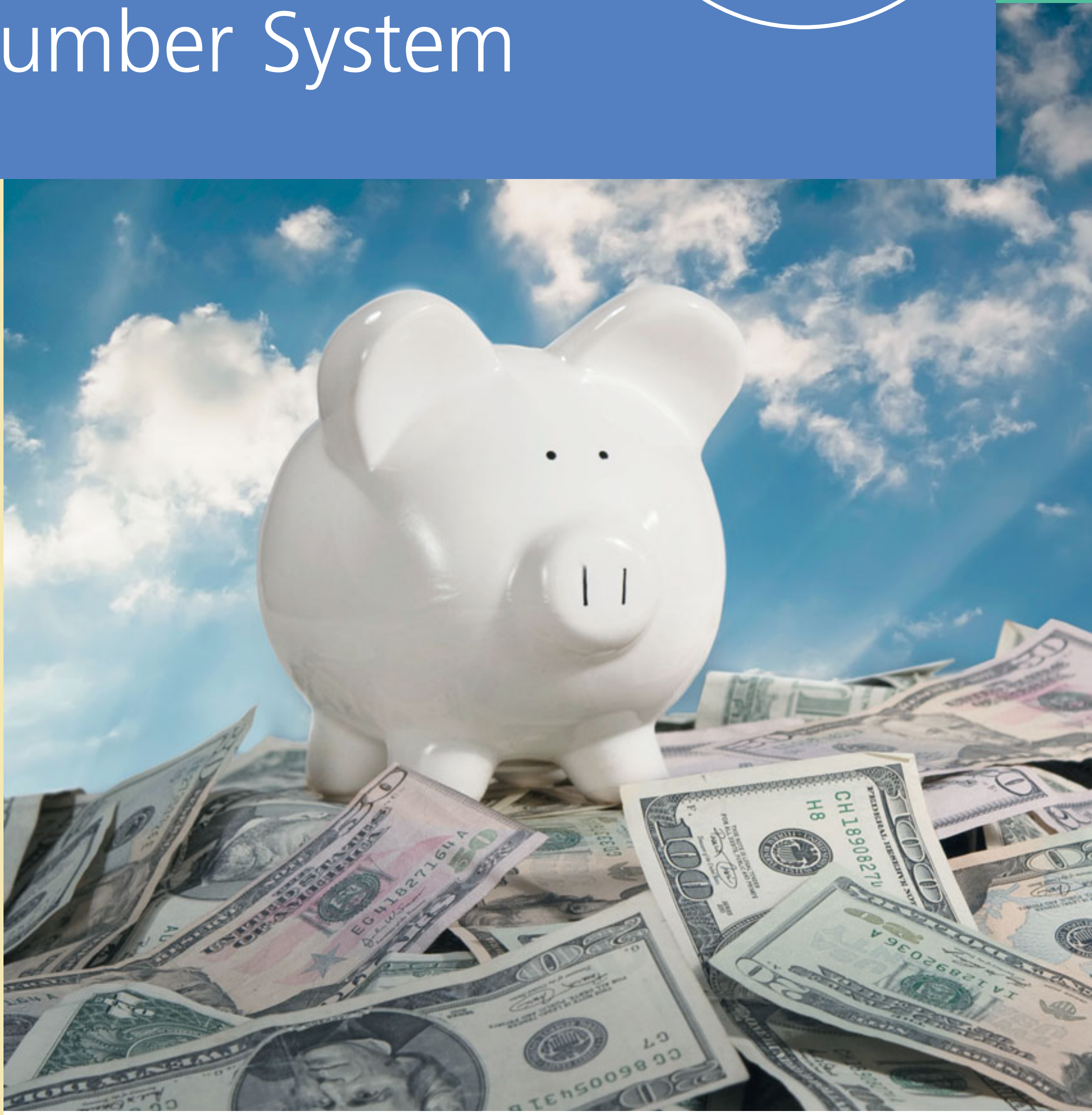
## Specific Features

- **Objectives** The objectives are listed at the beginning of each section and again within the section as the corresponding material is presented. Once you finish a section, ask yourself if you have accomplished them.
- **Now Try Exercises** These margin exercises allow you to immediately practice the material covered in the examples and prepare you for the exercises. Check your results using the answers at the bottom of the page.
- **Pointers** These small shaded balloons provide on-the-spot warnings and reminders, point out key steps, and give other helpful tips.
- **Cautions** These provide warnings about common errors that students often make or trouble spots to avoid.
- **Notes** These provide additional explanations or emphasize important ideas.
- **Problem-Solving Hints** These green boxes give helpful tips or strategies to use when you work applications.

***Find an example of each of these features in your textbook.***

CHAPTER 1

# The Real Number System



The personal savings rate of Americans has fluctuated over time. It stood at a hefty 10.8% of after-tax income in 1984, but dropped to $-0.5\%$ by 2005 when Americans actually spent more than they earned. This was the first negative savings rate since the Great Depression of the 1930s. In recent years, Americans have spent less and saved more, and personal savings rates have returned to positive territory, reaching 6.9% in May 2009. (*Source:* U.S. Bureau of Economic Analysis.)

In this chapter, we examine *signed numbers* and apply them to situations such as the personal savings rate of Americans in **Exercise 115** of **Section 1.5.**

# 1.1 Fractions

**OBJECTIVES**

1. Learn the definition of *factor.*
2. Write fractions in lowest terms.
3. Multiply and divide fractions.
4. Add and subtract fractions.
5. Solve applied problems that involve fractions.
6. Interpret data in a circle graph.

In everyday life, the numbers seen most often are the **natural numbers,**

$$1, 2, 3, 4, \ldots,$$

the **whole numbers,**

$$0, 1, 2, 3, 4, \ldots,$$

and **fractions,** such as

$$\frac{1}{2}, \quad \frac{2}{3}, \quad \text{and} \quad \frac{15}{7}.$$

The parts of a fraction are named as shown.

$$\text{Fraction bar} \rightarrow \frac{4}{7} \begin{array}{l} \leftarrow \text{Numerator} \\ \leftarrow \text{Denominator} \end{array}$$

***The fraction bar represents division*** $\left(\frac{a}{b} = a \div b\right)$.

A fraction is classified as being either a **proper fraction** or an **improper fraction.**

| | | |
|---|---|---|
| Proper fractions | $\frac{1}{5}, \frac{2}{7}, \frac{9}{10}, \frac{23}{25}$ | Numerator is **less than** denominator. Value is less than 1. |
| Improper fractions | $\frac{3}{2}, \frac{5}{5}, \frac{11}{7}, \frac{28}{4}$ | Numerator is **greater than or equal to** denominator. Value is greater than or equal to 1. |

A **mixed number** is a single number that represents the sum of a natural number and a proper fraction.

$$\text{Mixed number} \rightarrow 5\frac{3}{4} = 5 + \frac{3}{4}$$

**OBJECTIVE 1 Learn the definition of *factor*.** In the statement $3 \times 6 = 18$, the numbers 3 and 6 are called **factors** of 18. Other factors of 18 include 1, 2, 9, and 18. The result of the multiplication, 18, is called the **product.** We can represent the product of two numbers, such as 3 and 6, in several ways.

$$3 \times 6, \quad 3 \cdot 6, \quad (3)(6), \quad (3)6, \quad 3(6) \qquad \text{Products}$$

We *factor* a number by writing it as the product of two or more numbers. Factoring is the reverse of multiplying two numbers to get the product.

Multiplication

$$\underbrace{\underset{\uparrow}{3} \cdot \underset{\uparrow}{6}}_{\text{Factors}} = \underset{\text{Product}}{\underset{\uparrow}{18}}$$

Factoring

$$\underset{\text{Product}}{\underset{\uparrow}{18}} = \underbrace{\underset{\uparrow}{3} \cdot \underset{\uparrow}{6}}_{\text{Factors}}$$

---

**NOTE** In algebra, a raised dot $\cdot$ is often used instead of the $\times$ symbol to indicate multiplication because $\times$ may be confused with the letter $x$.

---

A natural number greater than 1 is **prime** if it has only itself and 1 as factors. "Factors" are understood here to mean natural number factors.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 \qquad \text{First dozen prime numbers}$$

A natural number greater than 1 that is not prime is called a **composite number.**

4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21 First dozen composite numbers

***By agreement, the number 1 is neither prime nor composite.***

Sometimes we must find all **prime factors** of a number—those factors which are prime numbers.

NOW TRY EXERCISE 1

Write 60 as the product of prime factors.

### EXAMPLE 1 Factoring Numbers

Write each number as the product of prime factors.

**(a)** 35

Write 35 as the product of the prime factors 5 and 7, or as

$$35 = 5 \cdot 7.$$

**(b)** 24

We show a factor tree on the right. The prime factors are circled.

Divide by the least prime factor of 24, which is 2.

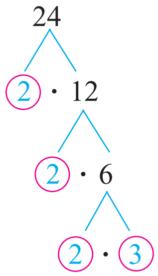
$$24 = 2 \cdot 12$$

Divide 12 by 2 to find two factors of 12.

$$24 = 2 \cdot 2 \cdot 6$$

Now factor 6 as 2 · 3.

$$24 = \underbrace{2 \cdot 2 \cdot 2 \cdot 3}_{\text{All factors are prime.}}$$



NOW TRY

**NOTE** When factoring, we need not start with the least prime factor. No matter which prime factor we start with, we will *always* obtain the same prime factorization. Verify this in **Example 1(b)** by starting with 3 instead of 2.

**OBJECTIVE 2** **Write fractions in lowest terms.** Recall the following **basic principle of fractions,** which is used to write a fraction in *lowest terms*.

### Basic Principle of Fractions

If the numerator and denominator of a fraction are multiplied or divided by the same nonzero number, the value of the fraction is not changed.

A fraction is in **lowest terms** when the numerator and denominator have no factors in common (other than 1).

### Writing a Fraction in Lowest Terms

***Step 1*** Write the numerator and the denominator as the product of prime factors.

***Step 2*** Divide the numerator and the denominator by the **greatest common factor,** the product of all factors common to both.

NOW TRY ANSWER

**1.** $2 \cdot 2 \cdot 3 \cdot 5$

*NOW TRY EXERCISE 2*

Write $\frac{30}{42}$ in lowest terms.

### EXAMPLE 2 Writing Fractions in Lowest Terms

Write each fraction in lowest terms.

**(a)** $$\frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{2 \cdot 1}{3 \cdot 1} = \frac{2}{3}$$

The factored form shows that 5 is the greatest common factor of 10 and 15. Dividing both numerator and denominator by 5 gives $\frac{10}{15}$ in lowest terms as $\frac{2}{3}$.

**(b)** $$\frac{15}{45}$$

By inspection, the greatest common factor of 15 and 45 is 15.

$$\frac{15}{45} = \frac{15}{3 \cdot 15} = \frac{1}{3 \cdot 1} = \frac{1}{3}$$

Remember to write 1 in the numerator.

If the greatest common factor is not obvious, factor the numerator and denominator into prime factors.

$$\frac{15}{45} = \frac{3 \cdot 5}{3 \cdot 3 \cdot 5} = \frac{1 \cdot 1}{3 \cdot 1 \cdot 1} = \frac{1}{3}$$

The same answer results.

*NOW TRY*

---

**CAUTION** When writing fractions like $\frac{15}{45}$ from **Example 2(b)** in lowest terms, be sure to include the factor 1 in the numerator.

---

## OBJECTIVE 3 Multiply and divide fractions.

### Multiplying Fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then $$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

That is, to multiply two fractions, multiply their numerators and then multiply their denominators.

### EXAMPLE 3 Multiplying Fractions

Find each product, and write it in lowest terms.

**(a)** $$\frac{3}{8} \cdot \frac{4}{9} = \frac{3 \cdot 4}{8 \cdot 9}$$

Multiply numerators.
Multiply denominators.

$$= \frac{3 \cdot 4}{2 \cdot 4 \cdot 3 \cdot 3}$$

Factor the denominator.

Remember to write 1 in the numerator.

$$= \frac{1}{2 \cdot 3}$$

Divide numerator and denominator by 3 · 4, or 12.

$$= \frac{1}{6}$$

Lowest terms

*NOW TRY ANSWER*

**2.** $\frac{5}{7}$

**NOW TRY EXERCISE 3**

Find each product, and write it in lowest terms.

**(a)** $\frac{4}{7}\cdot\frac{5}{8}$ **(b)** $3\frac{2}{5}\cdot 6\frac{2}{3}$

Think: 4 · 5 = 20, and 20 + 1 = 21, so $5\frac{1}{4}=\frac{21}{4}$.

**(b)** $$2\frac{1}{3}\cdot 5\frac{1}{4}=\frac{7}{3}\cdot\frac{21}{4}$$ Write each mixed number as an improper fraction.

Think: 3 · 2 = 6, and 6 + 1 = 7, so $2\frac{1}{3}=\frac{7}{3}$.

$$=\frac{7\cdot 21}{3\cdot 4}$$ Multiply numerators. Multiply denominators.

$$=\frac{7\cdot 3\cdot 7}{3\cdot 4}$$ Factor the numerator.

$$=\frac{49}{4},\quad\text{or}\quad 12\frac{1}{4}$$ Write in lowest terms and as a mixed number.

Think: $\frac{49}{4}$ means 49 ÷ 4.

$$\begin{array}{r} 12\\ 4\overline{)49}\\ \underline{4\phantom{9}}\\ 9\\ \underline{8}\\ 1\end{array}\quad\text{gives}\quad 12\frac{1}{4}.$$

NOW TRY

---

**NOTE** Some students prefer to factor and divide out any common factors *before* multiplying.

$$\frac{3}{8}\cdot\frac{4}{9}=\frac{3}{2\cdot 4}\cdot\frac{4}{3\cdot 3}$$ Example 3(a)

$$=\frac{1}{2\cdot 3}$$ Divide out common factors. Multiply.

$$=\frac{1}{6}$$ The same answer results.

---

Two fractions are **reciprocals** of each other if their product is 1. See the table in the margin. Because division is the opposite (or inverse) of multiplication, we use reciprocals to divide fractions.

| *Number* | *Reciprocal* |
|---|---|
| $\frac{3}{4}$ | $\frac{4}{3}$ |
| $\frac{11}{7}$ | $\frac{7}{11}$ |
| $\frac{1}{5}$ | 5, or $\frac{5}{1}$ |
| 9, or $\frac{9}{1}$ | $\frac{1}{9}$ |

***A number and its reciprocal have a product of* 1.** For example,

$$\frac{3}{4}\cdot\frac{4}{3}=\frac{12}{12}=1.$$

**Dividing Fractions**

If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}.$$

That is, to divide by a fraction, multiply by its reciprocal.

As an example of why this method works, we know that 20 ÷ 10 = 2 and also that $20\cdot\frac{1}{10}=2$. The answer to a division problem is called a **quotient.** For example, the quotient of 20 and 10 is 2.

## EXAMPLE 4 Dividing Fractions

Find each quotient, and write it in lowest terms.

**(a)** $$\frac{3}{4}\div\frac{8}{5}=\frac{3}{4}\cdot\frac{5}{8}=\frac{3\cdot 5}{4\cdot 8}=\frac{15}{32}$$ Make sure the answer is in lowest terms.

Multiply by the reciprocal of the second fraction.

**(b)** $$\frac{3}{4}\div\frac{5}{8}=\frac{3}{4}\cdot\frac{8}{5}=\frac{3\cdot 8}{4\cdot 5}=\frac{3\cdot 4\cdot 2}{4\cdot 5}=\frac{6}{5},\quad\text{or}\quad 1\frac{1}{5}$$

**NOW TRY ANSWERS**

**3. (a)** $\frac{5}{14}$ **(b)** $\frac{68}{3}$, or $22\frac{2}{3}$

**NOW TRY EXERCISE 4**

Find each quotient, and write it in lowest terms.

**(a)** $\frac{2}{7} \div \frac{8}{9}$ **(b)** $3\frac{3}{4} \div 4\frac{2}{7}$

**(c)** $$\frac{5}{8} \div 10 = \frac{5}{8} \div \underset{\substack{\uparrow \\ \text{Write 10 as } \frac{10}{1}.}}{\frac{10}{1}} = \frac{5}{8} \cdot \frac{1}{10} = \frac{5 \cdot 1}{8 \cdot 10} = \frac{5 \cdot 1}{8 \cdot 5 \cdot 2} = \frac{1}{16}$$

Remember to write 1 in the numerator.

**(d)** $$1\frac{2}{3} \div 4\frac{1}{2} = \frac{5}{3} \div \frac{9}{2}$$ Write each mixed number as an improper fraction.

$$= \frac{5}{3} \cdot \frac{2}{9}$$ Multiply by the reciprocal of the second fraction.

$$= \frac{10}{27}$$ Multiply numerators. Multiply denominators.

NOW TRY

**OBJECTIVE 4** **Add and subtract fractions.** The result of adding two numbers is called the **sum** of the numbers. For example, $2 + 3 = 5$, so 5 is the sum of 2 and 3.

### Adding Fractions

If $\frac{a}{b}$ and $\frac{c}{b}$ are fractions, then $$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

That is, to find the sum of two fractions having the *same* denominator, add the numerators and ***keep the same denominator.***

**NOW TRY EXERCISE 5**

Find the sum, and write it in lowest terms.

$$\frac{1}{8} + \frac{3}{8}$$

### EXAMPLE 5 Adding Fractions with the Same Denominator

Find each sum, and write it in lowest terms.

**(a)** $$\frac{3}{7} + \frac{2}{7} = \frac{3 + 2}{7} = \frac{5}{7}$$ Add numerators. Keep the same denominator.

**(b)** $$\frac{2}{10} + \frac{3}{10} = \frac{2 + 3}{10} = \frac{5}{10} = \frac{1}{2}$$ Write in lowest terms.

NOW TRY

If the fractions to be added do *not* have the same denominators, we must first rewrite them with a common denominator. For example, to rewrite $\frac{3}{4}$ as an equivalent fraction with denominator 32, think,

$$\frac{3}{4} = \frac{?}{32}.$$

We must find the number that can be multiplied by 4 to give 32. Since $4 \cdot 8 = 32$, we multiply numerator and denominator by 8.

$$\frac{3}{4} = \frac{3 \cdot 8}{4 \cdot 8} = \frac{24}{32}$$

$\frac{3}{4}$ and $\frac{24}{32}$ are equivalent fractions.

### Finding the Least Common Denominator

To add or subtract fractions with different denominators, find the **least common denominator (LCD)** as follows.

***Step 1*** Factor each denominator.

***Step 2*** For the LCD, use every factor that appears in any factored form. If a factor is repeated, use the largest number of repeats in the LCD.

**NOW TRY ANSWERS**

**4.** **(a)** $\frac{9}{28}$ **(b)** $\frac{7}{8}$

**5.** $\frac{1}{2}$

**NOW TRY EXERCISE 6**

Find each sum, and write it in lowest terms.

**(a)** $\frac{5}{12} + \frac{3}{8}$ **(b)** $3\frac{1}{4} + 5\frac{5}{8}$

## EXAMPLE 6 Adding Fractions with Different Denominators

Find each sum, and write it in lowest terms.

**(a)** $\frac{4}{15} + \frac{5}{9}$

To find the least common denominator, first factor both denominators.

$$15 = 5 \cdot 3 \quad \text{and} \quad 9 = 3 \cdot 3$$

Since 5 and 3 appear as factors, and 3 is a factor of 9 twice, the LCD is

$$\overset{15\quad 9}{5 \cdot 3 \cdot 3}, \quad \text{or} \quad 45.$$

Write each fraction with 45 as denominator.

$$\frac{4}{15} = \frac{4 \cdot 3}{15 \cdot 3} = \frac{12}{45} \quad \text{and} \quad \frac{5}{9} = \frac{5 \cdot 5}{9 \cdot 5} = \frac{25}{45}$$

At this stage, the fractions are *not* in lowest terms.

$$\frac{4}{15} + \frac{5}{9} = \frac{12}{45} + \frac{25}{45} = \frac{37}{45}$$

Add the two equivalent fractions.

**(b)** $3\frac{1}{2} + 2\frac{3}{4}$

***Method 1***

$$3\frac{1}{2} + 2\frac{3}{4} = \frac{7}{2} + \frac{11}{4}$$

Write each mixed number as an improper fraction.

$$= \frac{14}{4} + \frac{11}{4}$$

Think: $\frac{7 \cdot 2}{2 \cdot 2} = \frac{14}{4}$

Find a common denominator. The LCD is 4.

$$= \frac{25}{4}, \quad \text{or} \quad 6\frac{1}{4}$$

Add. Write as a mixed number.

***Method 2***

$$\begin{array}{rcl} 3\frac{1}{2} & = & 3\frac{2}{4} \\ +\ 2\frac{3}{4} & = & 2\frac{3}{4} \\ \hline \end{array}$$

Write $3\frac{1}{2}$ as $3\frac{2}{4}$. Then add vertically. Add the whole numbers and the fractions separately.

$$5\frac{5}{4} = 5 + 1\frac{1}{4} = 6\frac{1}{4}, \quad \text{or} \quad \frac{25}{4}$$

NOW TRY

The **difference** between two numbers is found by subtracting the numbers. For example, $9 - 5 = 4$, so the difference between 9 and 5 is 4.

### Subtracting Fractions

If $\frac{a}{b}$ and $\frac{c}{b}$ are fractions, then $\boldsymbol{\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}}$.

That is, to find the difference between two fractions having the *same* denominator, subtract the numerators and ***keep the same denominator.***

**NOW TRY ANSWERS**

**6. (a)** $\frac{19}{24}$ **(b)** $\frac{71}{8}$, or $8\frac{7}{8}$

**NOW TRY EXERCISE 7**

Find each difference, and write it in lowest terms.

**(a)** $\frac{5}{11} - \frac{2}{9}$ **(b)** $4\frac{1}{3} - 2\frac{5}{6}$

## EXAMPLE 7 Subtracting Fractions

Find each difference, and write it in lowest terms.

**(a)** $\frac{15}{8} - \frac{3}{8} = \frac{15 - 3}{8}$ Subtract numerators. Keep the same denominator.

$= \frac{12}{8}$

$= \frac{3}{2}$, or $1\frac{1}{2}$ Write in lowest terms and as a mixed number.

**(b)** $\frac{7}{18} - \frac{4}{15} = \frac{7 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 5} - \frac{4 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3 \cdot 5}$ $18 = 2 \cdot 3 \cdot 3$ and $15 = 3 \cdot 5$, so the LCD is $2 \cdot 3 \cdot 3 \cdot 5 = 90$.

$= \frac{35}{90} - \frac{24}{90}$ Write the equivalent fractions.

$= \frac{11}{90}$ Subtract. The answer is in lowest terms.

**(c)** $\frac{15}{32} - \frac{11}{45}$

Since $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ and $45 = 3 \cdot 3 \cdot 5$, there are no common factors. The LCD is $32 \cdot 45 = 1440$.

$\frac{15}{32} - \frac{11}{45} = \frac{15 \cdot 45}{32 \cdot 45} - \frac{11 \cdot 32}{45 \cdot 32}$ Find a common denominator.

$= \frac{675}{1440} - \frac{352}{1440}$ Write the equivalent fractions.

$= \frac{323}{1440}$ Subtract numerators. Keep the common denominator.

**(d)** $4\frac{1}{2} - 1\frac{3}{4}$

***Method 1*** $4\frac{1}{2} - 1\frac{3}{4} = \frac{9}{2} - \frac{7}{4}$ Write each mixed number as an improper fraction.

$= \frac{18}{4} - \frac{7}{4}$ Find a common denominator. The LCD is 4.

Think: $\frac{9 \cdot 2}{2 \cdot 2} = \frac{18}{4}$

$= \frac{11}{4}$, or $2\frac{3}{4}$ Subtract. Write as a mixed number.

***Method 2***

$$\begin{array}{r} 4\frac{1}{2} = 4\frac{2}{4} = 3\frac{6}{4} \\ -1\frac{3}{4} = 1\frac{3}{4} = 1\frac{3}{4} \\ \hline 2\frac{3}{4}, \text{ or } \frac{11}{4} \end{array}$$

$4\frac{2}{4} = 3 + 1 + \frac{2}{4} = 3 + \frac{4}{4} + \frac{2}{4} = 3\frac{6}{4}$

NOW TRY

**NOW TRY ANSWERS**

7. **(a)** $\frac{23}{99}$ **(b)** $\frac{3}{2}$, or $1\frac{1}{2}$

**NOW TRY EXERCISE 8**

A board is $10\frac{1}{2}$ ft long. If it must be divided into four pieces of equal length for shelves, how long must each piece be?

## OBJECTIVE 5 Solve applied problems that involve fractions.

### EXAMPLE 8 Adding Fractions to Solve an Applied Problem

The diagram in **FIGURE 1** appears in the book *Woodworker's 39 Sure-Fire Projects.* Find the height of the bookcase/desk to the top of the writing surface.

We must add these measures. (" means inches.)

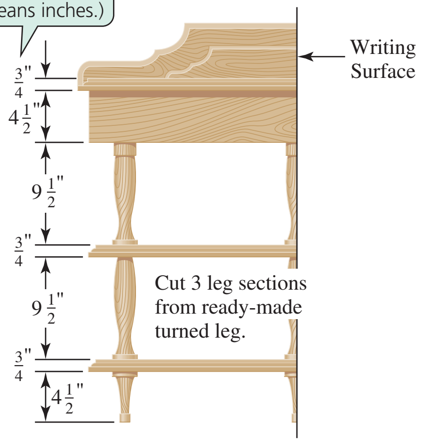


**FIGURE 1**

$$\begin{array}{rcl} \frac{3}{4} & \rightarrow & \frac{3}{4} \\ 4\frac{1}{2} & = & 4\frac{2}{4} \\ 9\frac{1}{2} & = & 9\frac{2}{4} \\ \frac{3}{4} & \rightarrow & \frac{3}{4} \\ 9\frac{1}{2} & = & 9\frac{2}{4} \\ \frac{3}{4} & \rightarrow & \frac{3}{4} \\ +\,4\frac{1}{2} & = & 4\frac{2}{4} \\ \hline & & 26\frac{17}{4} \end{array}$$

Use Method 2 from **Example 6(b).** The common denominator is 4.

Because $\frac{17}{4}$ is an improper fraction, this is *not* the final answer.

Think: $\frac{17}{4}$ means $17 \div 4$.

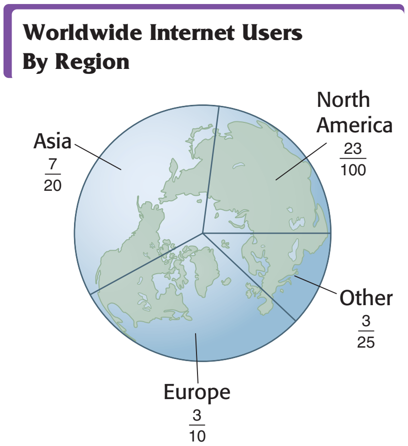
Since $\frac{17}{4} = 4\frac{1}{4}$, $26\frac{17}{4} = 26 + 4\frac{1}{4} = 30\frac{1}{4}$. The height is $30\frac{1}{4}$ in.

NOW TRY

**OBJECTIVE 6** **Interpret data in a circle graph.** In a **circle graph,** or **pie chart,** a circle is used to indicate the total of all the data categories represented. The circle is divided into *sectors*, or wedges, whose sizes show the relative magnitudes of the categories. The sum of all the fractional parts must be 1 (for 1 whole circle).

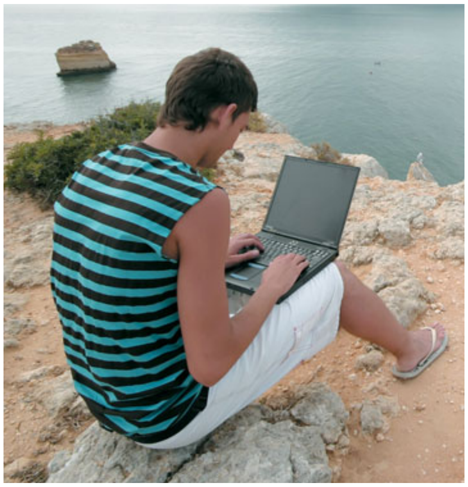
### EXAMPLE 9 Using a Circle Graph to Interpret Information

Recently there were about 970 million Internet users worldwide. The circle graph in **FIGURE 2** shows the fractions of these users living in various regions of the world.



***Source:*** www.internetworldstats.com

**FIGURE 2**



**NOW TRY ANSWER**

**8.** $2\frac{5}{8}$ ft

**NOW TRY EXERCISE 9**

Refer to the circle graph in **FIGURE 2** on the preceding page.

**(a)** Which region had the least number of Internet users?

**(b)** Estimate the number of Internet users in Asia.

**(c)** How many actual Internet users were there in Asia?

**(a)** Which region had the largest share of Internet users? What was that share?

The sector for Asia is the largest, so Asia had the largest share of Internet users, $\frac{7}{20}$.

**(b)** Estimate the number of Internet users in North America.

A share of $\frac{23}{100}$ can be rounded to $\frac{25}{100}$, or $\frac{1}{4}$, and the total number of Internet users, 970 million, can be rounded to 1000 million (1 billion). We multiply $\frac{1}{4}$ by 1000. The number of Internet users in North America would be about

$$\frac{1}{4}(1000) = 250 \text{ million.}$$

**(c)** How many actual Internet users were there in North America?

We multiply the actual fraction from the graph for North America, $\frac{23}{100}$, by the number of users, 970 million.

$$\frac{23}{100}(970) = \frac{23}{100} \cdot \frac{970}{1} = \frac{22{,}310}{100} = 223\frac{1}{10}$$

This is reasonable, given our estimate in part (b).

Thus, $223\frac{1}{10}$ million, or 223,100,000 (since $\frac{1}{10}$ million $= \frac{1}{10} \cdot 1{,}000{,}000 = 100{,}000$), people in North America used the Internet. NOW TRY

**NOW TRY ANSWERS**

9. **(a)** other
 **(b)** 333 million ($\frac{7}{20}$ is about $\frac{1}{3}$.)
 **(c)** $339\frac{1}{2}$ million, or 339,500,000

## 1.1 EXERCISES

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***Concept Check*** *Decide whether each statement is* true *or* false. *If it is false, say why.*

**1.** In the fraction $\frac{5}{8}$, 5 is the numerator and 8 is the denominator.

**2.** The mixed number equivalent of $\frac{31}{5}$ is $6\frac{1}{5}$.

**3.** The fraction $\frac{7}{7}$ is proper.

**4.** The number 1 is prime.

**5.** The fraction $\frac{13}{39}$ is in lowest terms.

**6.** The reciprocal of $\frac{6}{2}$ is $\frac{3}{1}$.

**7.** The product of 10 and 2 is 12.

**8.** The difference between 10 and 2 is 5.

*Identify each number as* prime, composite, *or* neither. *If the number is composite, write it as the product of prime factors.* ***See Example 1.***

**9.** 19 **10.** 31 **11.** 30 **12.** 50

◉ **13.** 64 **14.** 81 **15.** 1 **16.** 0

**17.** 57 **18.** 51 **19.** 79 **20.** 83 **21.** 124

**22.** 138 **23.** 500 **24.** 700 **25.** 3458 **26.** 1025

*Write each fraction in lowest terms.* ***See Example 2.***

**27.** $\frac{8}{16}$ **28.** $\frac{4}{12}$ ◉ **29.** $\frac{15}{18}$ **30.** $\frac{16}{20}$ **31.** $\frac{64}{100}$

**32.** $\frac{55}{200}$ **33.** $\frac{18}{90}$ **34.** $\frac{16}{64}$ **35.** $\frac{144}{120}$ **36.** $\frac{132}{77}$

**37.** ***Concept Check*** Which choice shows the correct way to write $\frac{16}{24}$ in lowest terms?

**A.** $\frac{16}{24} = \frac{8+8}{8+16} = \frac{8}{16} = \frac{1}{2}$

**B.** $\frac{16}{24} = \frac{4 \cdot 4}{4 \cdot 6} = \frac{4}{6}$

**C.** $\frac{16}{24} = \frac{8 \cdot 2}{8 \cdot 3} = \frac{2}{3}$

**D.** $\frac{16}{24} = \frac{14+2}{21+3} = \frac{2}{3}$

**38.** *Concept Check* Which fraction is *not* equal to $\frac{5}{9}$?

**A.** $\frac{15}{27}$ **B.** $\frac{30}{54}$ **C.** $\frac{40}{74}$ **D.** $\frac{55}{99}$

*Find each product or quotient, and write it in lowest terms.* ***See Examples 3 and 4.***

**39.** $\frac{4}{5} \cdot \frac{6}{7}$ **40.** $\frac{5}{9} \cdot \frac{2}{7}$ **41.** $\frac{2}{3} \cdot \frac{15}{16}$ **42.** $\frac{3}{5} \cdot \frac{20}{21}$

**43.** $\frac{1}{10} \cdot \frac{12}{5}$ **44.** $\frac{1}{8} \cdot \frac{10}{7}$ **45.** $\frac{15}{4} \cdot \frac{8}{25}$ **46.** $\frac{21}{8} \cdot \frac{4}{7}$

**47.** $21 \cdot \frac{3}{7}$ **48.** $36 \cdot \frac{4}{9}$ **49.** $3\frac{1}{4} \cdot 1\frac{2}{3}$ **50.** $2\frac{2}{3} \cdot 1\frac{3}{5}$

**51.** $2\frac{3}{8} \cdot 3\frac{1}{5}$ **52.** $3\frac{3}{5} \cdot 7\frac{1}{6}$ **53.** $\frac{5}{4} \div \frac{3}{8}$ **54.** $\frac{7}{5} \div \frac{3}{10}$

**55.** $\frac{32}{5} \div \frac{8}{15}$ **56.** $\frac{24}{7} \div \frac{6}{21}$ **57.** $\frac{3}{4} \div 12$ **58.** $\frac{2}{5} \div 30$

**59.** $6 \div \frac{3}{5}$ **60.** $8 \div \frac{4}{9}$ **61.** $6\frac{3}{4} \div \frac{3}{8}$ **62.** $5\frac{3}{5} \div \frac{7}{10}$

**63.** $2\frac{1}{2} \div 1\frac{5}{7}$ **64.** $2\frac{2}{9} \div 1\frac{2}{5}$ **65.** $2\frac{5}{8} \div 1\frac{15}{32}$ **66.** $2\frac{3}{10} \div 1\frac{4}{5}$

**67.** *Concept Check* For the fractions $\frac{p}{q}$ and $\frac{r}{s}$, which one of the following can serve as a common denominator?

**A.** $q \cdot s$ **B.** $q + s$ **C.** $p \cdot r$ **D.** $p + r$

**68.** *Concept Check* Write a fraction with denominator 24 that is equivalent to $\frac{5}{8}$.

*Find each sum or difference, and write it in lowest terms.* ***See Examples 5–7.***

**69.** $\frac{7}{15} + \frac{4}{15}$ **70.** $\frac{2}{9} + \frac{5}{9}$ **71.** $\frac{7}{12} + \frac{1}{12}$ **72.** $\frac{3}{16} + \frac{5}{16}$

**73.** $\frac{5}{9} + \frac{1}{3}$ **74.** $\frac{4}{15} + \frac{1}{5}$ **75.** $\frac{3}{8} + \frac{5}{6}$ **76.** $\frac{5}{6} + \frac{2}{9}$

**77.** $3\frac{1}{8} + 2\frac{1}{4}$ **78.** $4\frac{2}{3} + 2\frac{1}{6}$ **79.** $3\frac{1}{4} + 1\frac{4}{5}$ **80.** $5\frac{3}{4} + 1\frac{1}{3}$

**81.** $\frac{7}{9} - \frac{2}{9}$ **82.** $\frac{8}{11} - \frac{3}{11}$ **83.** $\frac{13}{15} - \frac{3}{15}$ **84.** $\frac{11}{12} - \frac{3}{12}$

**85.** $\frac{7}{12} - \frac{1}{3}$ **86.** $\frac{5}{6} - \frac{1}{2}$ **87.** $\frac{7}{12} - \frac{1}{9}$ **88.** $\frac{11}{16} - \frac{1}{12}$

**89.** $4\frac{3}{4} - 1\frac{2}{5}$ **90.** $3\frac{4}{5} - 1\frac{4}{9}$ **91.** $6\frac{1}{4} - 5\frac{1}{3}$ **92.** $5\frac{1}{3} - 4\frac{1}{2}$

*Use the table to answer Exercises 93 and 94.*

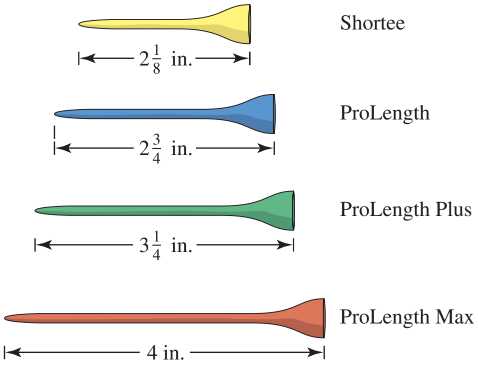
**93.** How many cups of water would be needed for eight microwave servings?

**94.** How many teaspoons of salt would be needed for five stove-top servings? (*Hint:* 5 is halfway between 4 and 6.)

| *Microwave* | | *Stove Top* | | |
|---|---|---|---|---|
| *Servings* | 1 | 1 | 4 | 6 |
| Water | $\frac{3}{4}$ cup | 1 cup | 3 cups | 4 cups |
| Grits | 3 Tbsp | 3 Tbsp | $\frac{3}{4}$ cup | 1 cup |
| Salt (optional) | Dash | Dash | $\frac{1}{4}$ tsp | $\frac{1}{2}$ tsp |

*Source:* Package of Quaker Quick Grits.

*The Pride Golf Tee Company, the only U.S. manufacturer of wooden golf tees, has created the Professional Tee System, shown in the figure. Use the information given to work Exercises 95 and 96. (Source: The Gazette.)*



**95.** Find the difference in length between the ProLength Plus and the once-standard Shortee.

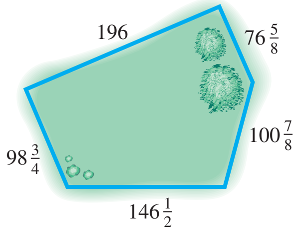
**96.** The ProLength Max tee is the longest tee allowed by the U.S. Golf Association's *Rules of Golf.* How much longer is the ProLength Max than the Shortee?

*Solve each problem.* ***See Example 8.***

**97.** A hardware store sells a 40-piece socket wrench set. The measure of the largest socket is $\frac{3}{4}$ in. The measure of the smallest is $\frac{3}{16}$ in. What is the difference between these measures?

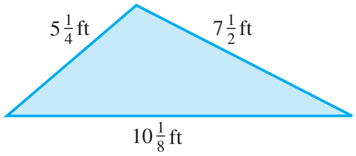
**98.** Two sockets in a socket wrench set have measures of $\frac{9}{16}$ in. and $\frac{3}{8}$ in. What is the difference between these two measures?

**99.** A piece of property has an irregular shape, with five sides, as shown in the figure. Find the total distance around the piece of property. (This distance is called the **perimeter** of the figure.)

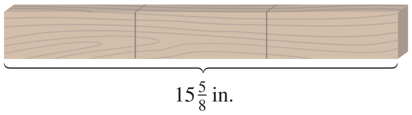


Measurements in feet

**100.** Find the perimeter of the triangle in the figure.



**101.** A board is $15\frac{5}{8}$ in. long. If it must be divided into three pieces of equal length, how long must each piece be?



**102.** Paul Beaulieu's favorite recipe for barbecue sauce calls for $2\frac{1}{3}$ cups of tomato sauce. The recipe makes enough barbecue sauce to serve seven people. How much tomato sauce is needed for one serving?

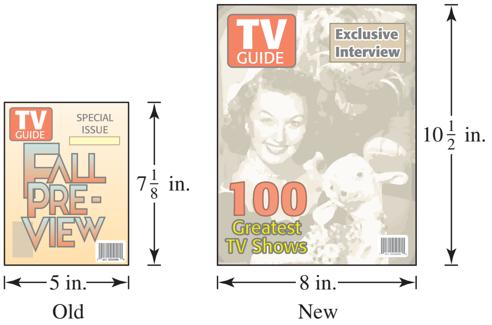
**103.** A cake recipe calls for $1\frac{3}{4}$ cups of sugar. A caterer has $15\frac{1}{2}$ cups of sugar on hand. How many cakes can he make?

**104.** Kyla Williams needs $2\frac{1}{4}$ yd of fabric to cover a chair. How many chairs can she cover with $23\frac{2}{3}$ yd of fabric?

**105.** It takes $2\frac{3}{8}$ yd of fabric to make a costume for a school play. How much fabric would be needed for seven costumes?

**106.** A cookie recipe calls for $2\frac{2}{3}$ cups of sugar. How much sugar would be needed to make four batches of cookies?

**107.** First published in 1953, the digest-sized *TV Guide* has changed to a full-sized magazine. The full-sized magazine is 3 in. wider than the old guide. What is the difference in their heights? (*Source: TV Guide.*)



**108.** Under existing standards, most of the holes in Swiss cheese must have diameters between $\frac{11}{16}$ and $\frac{13}{16}$ in. To accommodate new high-speed slicing machines, the U.S. Department of Agriculture wants to reduce the minimum size to $\frac{3}{8}$ in. How much smaller is $\frac{3}{8}$ in. than $\frac{11}{16}$ in.? (*Source:* U.S. Department of Agriculture.)

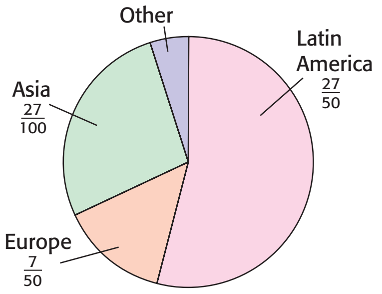
*Approximately 38 million people living in the United States in 2006 were born in other countries. The circle graph gives the fractional number from each region of birth for these people. Use the graph to answer each question.* ***See Example 9.***

**109.** What fractional part of the foreign-born population was from other regions?

**110.** What fractional part of the foreign-born population was from Latin America or Asia?

**111.** How many people (in millions) were born in Europe?

**U.S. Foreign-Born Population By Region of Birth**



***Source:*** U.S. Census Bureau.

**112.** At the conclusion of the Pearson Education softball league season, batting statistics for five players were as follows:

| *Player* | *At-Bats* | *Hits* | *Home Runs* |
|---|---|---|---|
| Courtney Slade | 36 | 12 | 3 |
| Kari Heen | 40 | 9 | 2 |
| Adam Goldstein | 11 | 5 | 1 |
| Nathaniel Koven | 16 | 8 | 0 |
| Jonathan Wooding | 20 | 10 | 2 |

Use the table to answer each question. Estimate as necessary.

**(a)** Which player got a hit in exactly $\frac{1}{3}$ of his or her at-bats?

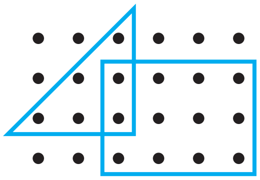
**(b)** Which player got a hit in just less than $\frac{1}{2}$ of his or her at-bats?

**(c)** Which player got a home run in just less than $\frac{1}{10}$ of his or her at-bats?

**(d)** Which player got a hit in just less than $\frac{1}{4}$ of his or her at-bats?

**(e)** Which two players got hits in exactly the same fractional parts of their at-bats? What was the fractional part, expressed in lowest terms?

**113.** For each description, write a fraction in lowest terms that represents the region described.



**(a)** The dots in the rectangle as a part of the dots in the entire figure

**(b)** The dots in the triangle as a part of the dots in the entire figure

**(c)** The dots in the overlapping region of the triangle and the rectangle as a part of the dots in the triangle alone

**(d)** The dots in the overlapping region of the triangle and the rectangle as a part of the dots in the rectangle alone

**114.** *Concept Check* Estimate the best approximation for the sum.

$$\frac{14}{26} + \frac{98}{99} + \frac{100}{51} + \frac{90}{31} + \frac{13}{27}$$

**A.** 6 **B.** 7 **C.** 5 **D.** 8

STUDY SKILLS

# Reading Your Math Textbook

***Take time to read each section and its examples before doing your homework.*** You will learn more and be better prepared to work the exercises your instructor assigns.

## Approaches to Reading Your Math Textbook

**Student A** learns best by listening to her teacher explain things. She "gets it" when she sees the instructor work problems. She previews the section before the lecture, so she knows generally what to expect. **Student A carefully reads the section in her text *AFTER* she hears the classroom lecture on the topic.**

**Student B** learns best by reading on his own. He reads the section and works through the examples before coming to class. That way, he knows what the teacher is going to talk about and what questions he wants to ask. **Student B carefully reads the section in his text *BEFORE* he hears the classroom lecture on the topic.**

***Which reading approach works best for you—that of Student A or Student B?***

## Tips for Reading Your Math Textbook

- **Turn off your cell phone.** You will be able to concentrate more fully on what you are reading.
- **Read slowly.** Read only one section—or even part of a section—at a sitting, with paper and pencil in hand.
- **Pay special attention to important information given in colored boxes or set in boldface type.**
- **Study the examples carefully.** Pay particular attention to the blue side comments and pointers.
- **Do the Now Try exercises in the margin on separate paper as you go.** These mirror the examples and prepare you for the exercise set. The answers are given at the bottom of the page.
- **Make study cards as you read.** (See **page 48.**) Make cards for new vocabulary, rules, procedures, formulas, and sample problems.
- **Mark anything you don't understand.** ***ASK QUESTIONS*** in class—everyone will benefit. Follow up with your instructor, as needed.

***Select several reading tips to try this week.***

# 1.2 Exponents, Order of Operations, and Inequality

**OBJECTIVES**

1. Use exponents.
2. Use the rules for order of operations.
3. Use more than one grouping symbol.
4. Know the meanings of ≠, <, >, ≤, and ≥.
5. Translate word statements to symbols.
6. Write statements that change the direction of inequality symbols.

**OBJECTIVE 1** **Use exponents.** Consider the prime factored form of 81.

$$81 = 3 \cdot 3 \cdot 3 \cdot 3 \qquad \text{The factor 3 appears four times.}$$

In algebra, repeated factors are written with an *exponent*, so the product $3 \cdot 3 \cdot 3 \cdot 3$ is written as $3^4$ and read as "3 to the fourth power."

Exponent

$$\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{\text{4 factors of 3}} = 3^4$$

Base

The number 4 is the **exponent,** or **power,** and 3 is the **base** in the **exponential expression** $3^4$. A natural number exponent, then, tells how many times the base is used as a factor. ***A number raised to the first power is simply that number.*** For example,

$$5^1 = 5 \quad \text{and} \quad \left(\frac{1}{2}\right)^1 = \frac{1}{2}.$$

## EXAMPLE 1 Evaluating Exponential Expressions

Find the value of each exponential expression.

**(a)** $5^2 = \underbrace{5 \cdot 5}_{\text{5 is used as a factor 2 times.}} = 25$

Read $5^2$ as "5 to the second power" or, more commonly, "5 squared."

**(b)** $6^3 = \underbrace{6 \cdot 6 \cdot 6}_{\text{6 is used as a factor 3 times.}} = 216$

Read $6^3$ as "6 to the third power" or, more commonly, "6 cubed."

**(c)** $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ 2 is used as a factor 5 times.

Read $2^5$ as "2 to the fifth power."

**(d)** $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$ $\frac{2}{3}$ is used as a factor 3 times.

**(e)** $(0.3)^2 = 0.3(0.3) = 0.09$ 0.3 is used as a factor 2 times.

*NOW TRY*

*NOW TRY EXERCISE 1*

Find the value of each exponential expression.

**(a)** $6^2$ **(b)** $\left(\frac{4}{5}\right)^3$

---

**CAUTION** ***Squaring, or raising a number to the second power, is NOT the same as doubling the number.*** For example,

$$\mathbf{3^2 \quad \text{means} \quad 3 \cdot 3,} \quad \textit{not} \quad 2 \cdot 3.$$

Thus $3^2 = 9$, *not* 6. Similarly, cubing, or raising a number to the third power, does *not* mean tripling the number.

---

**OBJECTIVE 2** **Use the rules for order of operations.** When a problem involves more than one operation, we often use **grouping symbols,** such as parentheses ( ), to indicate the order in which the operations should be performed.

Consider the expression $5 + 2 \cdot 3$. To show that the multiplication should be performed before the addition, we use parentheses to group $2 \cdot 3$.

$$5 + (2 \cdot 3) \quad \text{equals} \quad 5 + 6, \quad \text{or} \quad 11.$$

*NOW TRY ANSWERS*

1. **(a)** 36 **(b)** $\frac{64}{125}$

If addition is to be performed first, the parentheses should group $5 + 2$.

$$(5 + 2) \cdot 3 \quad \text{equals} \quad 7 \cdot 3, \quad \text{or} \quad 21.$$

Other grouping symbols are brackets [ ], braces { }, and fraction bars. (For example, in $\frac{8-2}{3}$, the expression $8 - 2$ is "grouped" in the numerator.)

To work problems with more than one operation, we use the following **order of operations.** This order is used by most calculators and computers.

## Order of Operations

***If grouping symbols are present,*** simplify within them, innermost first (and above and below fraction bars separately), in the following order.

***Step 1*** Apply all **exponents.**

***Step 2*** Do any **multiplications** or **divisions** in the order in which they occur, working from left to right.

***Step 3*** Do any **additions** or **subtractions** in the order in which they occur, working from left to right.

***If no grouping symbols are present,*** start with Step 1.

**NOTE** In expressions such as $3(7)$ or $(-5)(-4)$, multiplication is understood.

## EXAMPLE 2 Using the Rules for Order of Operations

Find the value of each expression.

**(a)** $4 + 5 \cdot 6$ Be careful! Multiply first.

$= 4 + 30$ Multiply.

$= 34$ Add.

**(b)** $9(6 + 11)$

$= 9(17)$ Work inside parentheses.

$= 153$ Multiply.

**(c)** $6 \cdot 8 + 5 \cdot 2$

$= 48 + 10$ Multiply, working from left to right.

$= 58$ Add.

**(d)** $2(5 + 6) + 7 \cdot 3$

$= 2(11) + 7 \cdot 3$ Work inside parentheses.

$= 22 + 21$ Multiply.

$= 43$ Add.

**(e)** $9 - 2^3 + 5$ $2^3 = 2 \cdot 2 \cdot 2$, not $2 \cdot 3$.

$= 9 - 2 \cdot 2 \cdot 2 + 5$ Apply the exponent.

$= 9 - 8 + 5$ Multiply.

$= 1 + 5$ Subtract.

$= 6$ Add.

**NOW TRY EXERCISE 2**

Find the value of each expression.

**(a)** $15 - 2 \cdot 6$

**(b)** $6(2 + 4) - 7 \cdot 5$

**(c)** $8 \cdot 10 \div 4 - 2^3 + 3 \cdot 4^2$

**(f)** $72 \div 2 \cdot 3 + 4 \cdot 2^3 - 3^3$ Think: $3^3 = 3 \cdot 3 \cdot 3$

$$
\begin{aligned}
&= \underbrace{72 \div 2}_{} \cdot 3 + 4 \cdot 8 - 27 && \text{Apply the exponents.}\\
&= \underbrace{36 \cdot 3}_{} + \underbrace{4 \cdot 8}_{} - 27 && \text{Divide.}\\
&= 108 + 32 - 27 && \text{Multiply.}\\
&= 140 - 27 && \text{Add.}\\
&= 113 && \text{Subtract.}
\end{aligned}
$$

***Multiplications and divisions are done from left to right as they appear. Then additions and subtractions are done from left to right as they appear.*** NOW TRY

**OBJECTIVE 3** **Use more than one grouping symbol.** In an expression such as $2(8 + 3(6 + 5))$, we often use brackets, [ ], in place of one pair of parentheses.

**NOW TRY EXERCISE 3**

Simplify each expression.

**(a)** $7[(3^2 - 1) + 4]$

**(b)** $\dfrac{9(14 - 4) - 2}{4 + 3 \cdot 6}$

### EXAMPLE 3 Using Brackets and Fraction Bars as Grouping Symbols

Simplify each expression.

**(a)** $2[8 + 3(6 + 5)]$

$$
\begin{aligned}
&= 2[8 + 3(11)] && \text{Add inside parentheses.}\\
&= 2[8 + 33] && \text{Multiply inside brackets.}\\
&= 2[41] && \text{Add inside brackets.}\\
&= 82 && \text{Multiply.}
\end{aligned}
$$

**(b)** $\dfrac{4(5 + 3) + 3}{2(3) - 1}$ Simplify the numerator and denominator separately.

$$
\begin{aligned}
&= \frac{4(8) + 3}{2(3) - 1} && \text{Work inside parentheses.}\\
&= \frac{32 + 3}{6 - 1} && \text{Multiply.}\\
&= \frac{35}{5}, \quad \text{or} \quad 7 && \text{Add and subtract. Then divide.}
\end{aligned}
$$

NOW TRY

**NOTE** The expression $\frac{4(5 + 3) + 3}{2(3) - 1}$ in **Example 3(b)** can be written as the quotient

$$[4(5 + 3) + 3] \div [2(3) - 1],$$

which shows that the fraction bar "groups" the numerator and denominator separately.

**OBJECTIVE 4** **Know the meanings of $\neq$, $<$, $>$, $\leq$, and $\geq$.** So far, we have used the equality symbol $=$. The symbols $\neq$, $<$, $>$, $\leq$, and $\geq$ are used to express an **inequality,** a statement that two expressions may not be equal. The equality symbol with a slash through it, $\neq$, means "is not equal to."

$7 \neq 8$ 7 is not equal to 8.

If two numbers are not equal, then one of the numbers must be less than the other. The symbol $<$ represents "is less than."

$7 < 8$ 7 is less than 8.

**NOW TRY ANSWERS**

**2.** **(a)** 3 **(b)** 1 **(c)** 60

**3.** **(a)** 84 **(b)** 4

The symbol $>$ means "is greater than."

$8 > 2$ 8 is greater than 2.

***To keep the meanings of the symbols $<$ and $>$ clear, remember that the symbol always points to the lesser number.***

Lesser number → $8 < 15$

$15 > 8$ ← Lesser number

The symbol $\leq$ means "is less than or equal to."

$5 \leq 9$ 5 is less than or equal to 9.

***If either the $<$ part or the $=$ part is true, then the inequality $\leq$ is true.*** The statement $5 \leq 9$ is true, since $5 < 9$ is true.

The symbol $\geq$ means "is greater than or equal to."

$9 \geq 5$ 9 is greater than or equal to 5.

**NOW TRY EXERCISE 4**

Determine whether each statement is *true* or *false*.

**(a)** $12 \neq 10 - 2$

**(b)** $5 > 4 \cdot 2$

**(c)** $7 \leq 7$

**(d)** $\frac{5}{9} > \frac{7}{11}$

### EXAMPLE 4 Using Inequality Symbols

Determine whether each statement is *true* or *false.*

**(a)** $6 \neq 5 + 1$ This statement is false because $6 = 5 + 1$.

**(b)** $5 + 3 < 19$ The statement $5 + 3 < 19$ is true, since $8 < 19$.

**(c)** $15 \leq 20 \cdot 2$ The statement $15 \leq 20 \cdot 2$ is true, since $15 < 40$.

**(d)** $25 \geq 30$ Both $25 > 30$ and $25 = 30$ are false, so $25 \geq 30$ is false.

**(e)** $12 \geq 12$ Since $12 = 12$, this statement is true.

**(f)** $9 < 9$ Since $9 = 9$, this statement is false.

**(g)** $\frac{6}{15} \geq \frac{2}{3}$

$\frac{6}{15} \geq \frac{10}{15}$ Get a common denominator.

Both statements $\frac{6}{15} > \frac{10}{15}$ and $\frac{6}{15} = \frac{10}{15}$ are false. Therefore, $\frac{6}{15} \geq \frac{2}{3}$ is false.

NOW TRY

## OBJECTIVE 5 Translate word statements to symbols.

**NOW TRY EXERCISE 5**

Write each word statement in symbols.

**(a)** Ten is not equal to eight minus two.

**(b)** Fifty is greater than fifteen.

**(c)** Eleven is less than or equal to twenty.

### EXAMPLE 5 Translating from Words to Symbols

Write each word statement in symbols.

**(a)** Twelve equals ten plus two.

$12 = 10 + 2$

**(b)** Nine is less than ten.

$9 < 10$

**(c)** Fifteen is not equal to eighteen.

$15 \neq 18$

**(d)** Seven is greater than four.

$7 > 4$

**(e)** Thirteen is less than or equal to forty.

$13 \leq 40$

**(f)** Eleven is greater than or equal to eleven.

$11 \geq 11$

NOW TRY

**NOW TRY ANSWERS**

4. **(a)** true **(b)** false **(c)** true **(d)** false
5. **(a)** $10 \neq 8 - 2$ **(b)** $50 > 15$ **(c)** $11 \leq 20$

**OBJECTIVE 6** **Write statements that change the direction of inequality symbols.** Any statement with $<$ can be converted to one with $>$, and any statement with $>$ can be converted to one with $<$. ***We do this by reversing the order of the numbers and the direction of the symbol.*** For example,

Interchange numbers.

$$6 < 10 \qquad \text{becomes} \qquad 10 > 6.$$

Reverse symbol.

**NOW TRY EXERCISE 6**

Write the statement as another true statement with the inequality symbol reversed.

$$8 < 9$$

### EXAMPLE 6 Converting between Inequality Symbols

Parts (a)–(c) each show a statement written in two equally correct ways. In each inequality, the inequality symbol points toward the lesser number.

**(a)** $5 > 2, \quad 2 < 5$ **(b)** $3 \leq 8, \quad 8 \geq 3$ **(c)** $12 \geq 5, \quad 5 \leq 12$

NOW TRY

Here is a summary of the symbols discussed in this section.

| *Symbol* | *Meaning* | *Example* |
|---|---|---|
| $=$ | Is equal to | $0.5 = \frac{1}{2}$ means 0.5 is equal to $\frac{1}{2}$. |
| $\neq$ | Is not equal to | $3 \neq 7$ means 3 is not equal to 7. |
| $<$ | Is less than | $6 < 10$ means 6 is less than 10. |
| $>$ | Is greater than | $15 > 14$ means 15 is greater than 14. |
| $\leq$ | Is less than or equal to | $4 \leq 8$ means 4 is less than or equal to 8. |
| $\geq$ | Is greater than or equal to | $1 \geq 0$ means 1 is greater than or equal to 0. |

**CAUTION** Equality and inequality symbols are used to write mathematical ***sentences,*** while operation symbols ($+, -, \cdot,$ and $\div$) are used to write mathematical ***expressions*** that represent a number. Compare the following.

***Sentence:*** $4 < 10 \leftarrow$ Gives the relationship between 4 and 10

***Expression:*** $4 + 10 \leftarrow$ Tells how to operate on 4 and 10 to get 14

**NOW TRY ANSWER**

**6.** $9 > 8$

## 1.2 EXERCISES

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*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Decide whether each statement is* true *or* false. *If it is false, explain why.*

**1.** The expression $6^2$ means that 2 is used as a factor 6 times.

**2.** $3^2 = 6$

**3.** $1^3 = 3$

**4.** $3^1 = 1$

**5.** When evaluated, $4 + 3(8 - 2)$ is equal to 42.

**6.** When evaluated, $12 \div 2 \cdot 3$ is equal to 2.

*Find the value of each exponential expression.* ***See Example 1.***

**7.** $3^2$
**8.** $8^2$
**9.** $7^2$
**10.** $4^2$
**11.** $12^2$
**12.** $14^2$
**13.** $4^3$
**14.** $5^3$
**15.** $10^3$
**16.** $11^3$
**17.** $3^4$
**18.** $6^4$
**19.** $4^5$
**20.** $3^5$
**21.** $\left(\frac{1}{6}\right)^2$
**22.** $\left(\frac{1}{3}\right)^2$
**23.** $\left(\frac{2}{3}\right)^4$
**24.** $\left(\frac{3}{4}\right)^3$
**25.** $(0.4)^3$
**26.** $(0.5)^4$

*Find the value of each expression.* ***See Examples 2 and 3.***

**27.** $64 \div 4 \cdot 2$
**28.** $250 \div 5 \cdot 2$
**29.** $13 + 9 \cdot 5$
**30.** $11 + 7 \cdot 6$
**31.** $25.2 - 12.6 \div 4.2$
**32.** $12.4 - 9.3 \div 3.1$
**33.** $\frac{1}{4} \cdot \frac{2}{3} + \frac{2}{5} \cdot \frac{11}{3}$
**34.** $\frac{9}{4} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{5}{3}$
**35.** $9 \cdot 4 - 8 \cdot 3$
**36.** $11 \cdot 4 + 10 \cdot 3$
**37.** $20 - 4 \cdot 3 + 5$
**38.** $18 - 7 \cdot 2 + 6$
**39.** $10 + 40 \div 5 \cdot 2$
**40.** $12 + 64 \div 8 - 4$
**41.** $18 - 2(3 + 4)$
**42.** $30 - 3(4 + 2)$
**43.** $3(4 + 2) + 8 \cdot 3$
**44.** $9(1 + 7) + 2 \cdot 5$
**45.** $18 - 4^2 + 3$
**46.** $22 - 2^3 + 9$
**47.** $2 + 3[5 + 4(2)]$
**48.** $5 + 4[1 + 7(3)]$
**49.** $5[3 + 4(2^2)]$
**50.** $6[2 + 8(3^3)]$
**51.** $3^2[(11 + 3) - 4]$
**52.** $4^2[(13 + 4) - 8]$
**53.** $\frac{6(3^2 - 1) + 8}{8 - 2^2}$
**54.** $\frac{2(8^2 - 4) + 8}{29 - 3^3}$
**55.** $\frac{4(6 + 2) + 8(8 - 3)}{6(4 - 2) - 2^2}$
**56.** $\frac{6(5 + 1) - 9(1 + 1)}{5(8 - 6) - 2^3}$

*First simplify both sides of each inequality. Then tell whether the given statement is* true *or* false. ***See Examples 2–4.***

**57.** $9 \cdot 3 - 11 \leq 16$
**58.** $6 \cdot 5 - 12 \leq 18$
**59.** $5 \cdot 11 + 2 \cdot 3 \leq 60$
**60.** $9 \cdot 3 + 4 \cdot 5 \geq 48$
**61.** $0 \geq 12 \cdot 3 - 6 \cdot 6$
**62.** $10 \leq 13 \cdot 2 - 15 \cdot 1$
**63.** $45 \geq 2[2 + 3(2 + 5)]$
**64.** $55 \geq 3[4 + 3(4 + 1)]$
**65.** $[3 \cdot 4 + 5(2)] \cdot 3 > 72$
**66.** $2 \cdot [7 \cdot 5 - 3(2)] \leq 58$
**67.** $\frac{3 + 5(4 - 1)}{2 \cdot 4 + 1} \geq 3$
**68.** $\frac{7(3 + 1) - 2}{3 + 5 \cdot 2} \leq 2$
**69.** $3 \geq \frac{2(5 + 1) - 3(1 + 1)}{5(8 - 6) - 4 \cdot 2}$
**70.** $7 \leq \frac{3(8 - 3) + 2(4 - 1)}{9(6 - 2) - 11(5 - 2)}$

***Concept Check*** *Insert one pair of parentheses so that the left side of each equation is equal to the right side.*

**71.** $3 \cdot 6 + 4 \cdot 2 = 60$
**72.** $2 \cdot 8 - 1 \cdot 3 = 42$
**73.** $10 - 7 - 3 = 6$
**74.** $15 - 10 - 2 = 7$
**75.** $8 + 2^2 = 100$
**76.** $4 + 2^2 = 36$

*Write each statement in words and decide whether it is* true *or* false. ***See Examples 4 and 5.***

**77.** $5 < 17$
**78.** $8 < 12$
**79.** $5 \neq 8$
**80.** $6 \neq 9$
**81.** $7 \geq 14$
**82.** $6 \geq 12$
**83.** $15 \leq 15$
**84.** $21 \leq 21$

*Write each word statement in symbols.* ***See Example 5.***

**85.** Fifteen is equal to five plus ten.

**86.** Twelve is equal to twenty minus eight.

**87.** Nine is greater than five minus four.

**88.** Ten is greater than six plus one.

**89.** Sixteen is not equal to nineteen.

**90.** Three is not equal to four.

**91.** One-half is less than or equal to two-fourths.

**92.** One-third is less than or equal to three-ninths.

*Write each statement with the inequality symbol reversed while keeping the same meaning.* ***See Example 6.***

**93.** $5 < 20$

**94.** $30 > 9$

**95.** $2.5 \geq 1.3$

**96.** $4.1 \leq 5.3$

*One way to measure a person's cardiofitness is to calculate how many METs, or metabolic units, he or she can reach at peak exertion. One MET is the amount of energy used when sitting quietly. To calculate ideal METs, we can use the following expressions.*

$14.7 - \text{age} \cdot 0.13$ For women

$14.7 - \text{age} \cdot 0.11$ For men

(*Source: New England Journal of Medicine.*)

**97.** A 40-yr-old woman wishes to calculate her ideal MET.

**(a)** Write the expression, using her age.

**(b)** Calculate her ideal MET. (*Hint:* Use the rules for order of operations.)

**(c)** Researchers recommend that a person reach approximately 85% of his or her MET when exercising. Calculate 85% of the ideal MET from part (b). Then refer to the following table. What activity can the woman do that is approximately this value?

| *Activity* | *METs* | *Activity* | *METs* |
|---|---|---|---|
| Golf (with cart) | 2.5 | Skiing (water or downhill) | 6.8 |
| Walking (3 mph) | 3.3 | Swimming | 7.0 |
| Mowing lawn (power mower) | 4.5 | Walking (5 mph) | 8.0 |
| Ballroom or square dancing | 5.5 | Jogging | 10.2 |
| Cycling | 5.7 | Skipping rope | 12.0 |

*Source:* Harvard School of Public Health.

**98.** Repeat parts (a)–(c) of **Exercise 97** for a 55-yr-old man.

**99.** Repeat parts (a)–(c) of **Exercise 97** using your age.

**100.** The table shows the number of pupils per teacher in U.S. public schools in selected states.

**(a)** Which states had a figure greater than 13.9?

**(b)** Which states had a figure that was at most 14.7?

**(c)** Which states had a figure not less than 13.9?

| *State* | *Pupils per Teacher* |
|---|---|
| Alaska | 16.7 |
| Texas | 14.7 |
| California | 20.5 |
| Wyoming | 12.5 |
| Maine | 12.3 |
| Idaho | 17.8 |
| Missouri | 13.9 |

*Source:* National Center for Education Statistics.

STUDY SKILLS

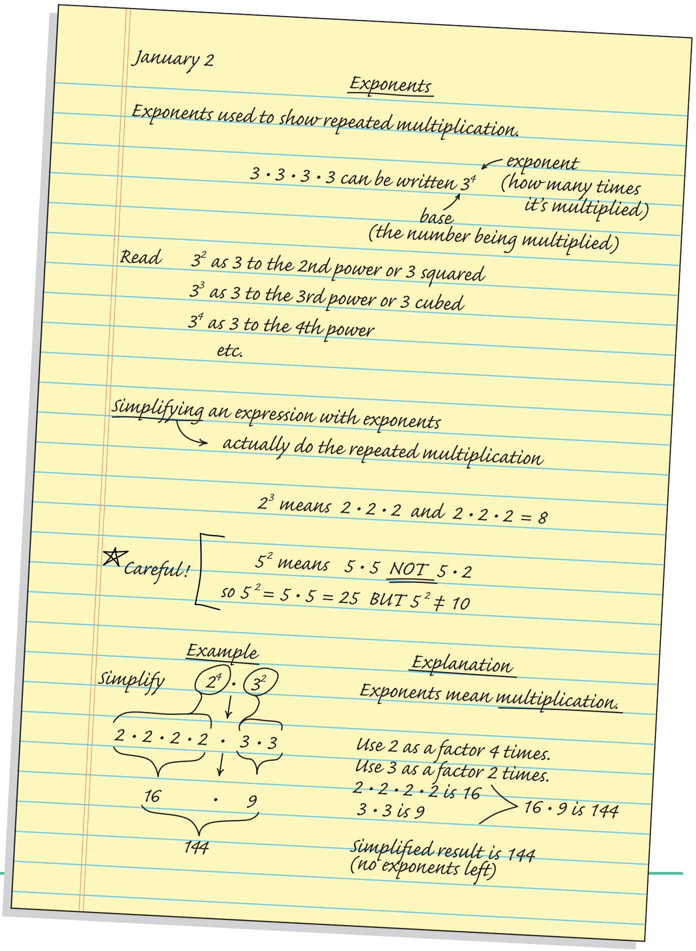
## Taking Lecture Notes

Study the set of sample math notes given here.

- **Include the date and title** of the day's lecture topic.
- **Include definitions,** written here in parentheses—don't trust your memory.
- **Skip lines and write neatly** to make reading easier.
- **Emphasize direction words** (like *simplify*) with their explanations.
- **Mark important concepts with stars, underlining, etc.**
- **Use two columns,** which allows an example and its explanation to be close together.
- **Use brackets and arrows** to clearly show steps, related material, etc.

***With a partner or in a small group, compare lecture notes.***

1. What are you doing to show main points in your notes (such as boxing, using stars, etc.)?
2. In what ways do you set off explanations from worked problems and subpoints (such as indenting, using arrows, circling, etc.)?
3. What new ideas did you learn by examining your classmates' notes?
4. What new techniques will you try in your notes?



January 2

Exponents

Exponents used to show repeated multiplication.

$3 \cdot 3 \cdot 3 \cdot 3$ can be written $3^4$

exponent (how many times it's multiplied)

base (the number being multiplied)

Read $3^2$ as 3 to the 2nd power or 3 squared

$3^3$ as 3 to the 3rd power or 3 cubed

$3^4$ as 3 to the 4th power

etc.

Simplifying an expression with exponents

→ actually do the repeated multiplication

$2^3$ means $2 \cdot 2 \cdot 2$ and $2 \cdot 2 \cdot 2 = 8$

☆ Careful! $5^2$ means $5 \cdot 5$ NOT $5 \cdot 2$

so $5^2 = 5 \cdot 5 = 25$ BUT $5^2 \neq 10$

| Example | Explanation |
|---|---|
| Simplify $2^4 \cdot 3^2$ | Exponents mean multiplication. |
| $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ | Use 2 as a factor 4 times.<br>Use 3 as a factor 2 times. |
| $16 \cdot 9$ | $2 \cdot 2 \cdot 2 \cdot 2$ is 16<br>$3 \cdot 3$ is 9<br>$16 \cdot 9$ is 144 |
| 144 | Simplified result is 144<br>(no exponents left) |

## 1.3 Variables, Expressions, and Equations

**OBJECTIVES**

1. Evaluate algebraic expressions, given values for the variables.
2. Translate word phrases to algebraic expressions.
3. Identify solutions of equations.
4. Identify solutions of equations from a set of numbers.
5. Distinguish between *expressions* and *equations*.

A **variable** is a symbol, usually a letter such as

$$x, \quad y, \quad \text{or} \quad z, \qquad \text{Variables}$$

used to represent any unknown number. An **algebraic expression** is a sequence of numbers, variables, operation symbols, and/or grouping symbols formed according to the rules of algebra.

$$x + 5, \quad 2m - 9, \quad 8p^2 + 6(p - 2) \qquad \text{Algebraic expressions}$$

$2m$ means $2 \cdot m$, the product of 2 and $m$.

$6(p - 2)$ means the product of 6 and $p - 2$.

**OBJECTIVE 1 Evaluate algebraic expressions, given values for the variables.** An algebraic expression has different numerical values for different values of the variables.

**NOW TRY EXERCISE 1**

Find the value of each algebraic expression for $k = 6$.

**(a)** $9k$ **(b)** $4k^2$

## EXAMPLE 1 Evaluating Expressions

Find the value of each algebraic expression for $x = 5$.

**(a)** $8x$

$$\begin{aligned} &= 8 \cdot x \\ &= 8 \cdot 5 && \text{Let } x = 5. \\ &= 40 && \text{Multiply.} \end{aligned}$$

**(b)** $3x^2$

$$\begin{aligned} &= 3 \cdot x^2 \\ &= 3 \cdot 5^2 && \text{Let } x = 5. \\ &= 3 \cdot 25 && \text{Square 5.} \\ &= 75 && \text{Multiply.} \end{aligned}$$

$5^2 = 5 \cdot 5$

NOW TRY

**CAUTION** In **Example 1(b),** $3x^2$ means $3 \cdot x^2$, ***not*** $3x \cdot 3x$. ***Unless parentheses are used, the exponent refers only to the variable or number just before it.*** Use parentheses to write $3x \cdot 3x$ with exponents as $(3x)^2$.

**NOW TRY EXERCISE 2**

Find the value of each expression for $x = 4$ and $y = 7$.

**(a)** $3x + 4y$ **(b)** $\dfrac{6x - 2y}{2y - 9}$

**(c)** $4x^2 - y^2$

## EXAMPLE 2 Evaluating Expressions

Find the value of each expression for $x = 5$ and $y = 3$.

**(a)** $2x + 7y$

We could use parentheses and write $2(5) + 7(3)$.

$$\begin{aligned} &= 2 \cdot 5 + 7 \cdot 3 && \text{Let } x = 5 \text{ and } y = 3. \\ &= 10 + 21 && \text{Multiply.} \\ &= 31 && \text{Add.} \end{aligned}$$

Follow the rules for order of operations.

**(b)** $\dfrac{9x - 8y}{2x - y}$

$$\begin{aligned} &= \frac{9 \cdot 5 - 8 \cdot 3}{2 \cdot 5 - 3} && \text{Let } x = 5 \text{ and } y = 3. \\ &= \frac{45 - 24}{10 - 3} && \text{Multiply.} \\ &= \frac{21}{7}, \quad \text{or} \quad 3 && \text{Subtract, and then divide.} \end{aligned}$$

**(c)** $x^2 - 2y^2$

$3^2 = 3 \cdot 3$

$$\begin{aligned} &= 5^2 - 2 \cdot 3^2 && \text{Let } x = 5 \text{ and } y = 3. \\ &= 25 - 2 \cdot 9 && \text{Apply the exponents.} \\ &= 25 - 18 && \text{Multiply.} \\ &= 7 && \text{Subtract.} \end{aligned}$$

$5^2 = 5 \cdot 5$

NOW TRY

## OBJECTIVE 2 Translate word phrases to algebraic expressions.

## EXAMPLE 3 Using Variables to Write Word Phrases as Algebraic Expressions

Write each word phrase as an algebraic expression, using $x$ as the variable.

**(a)** The sum of a number and 9

$x + 9$, or $9 + x$ "Sum" is the answer to an addition problem.

**(b)** 7 minus a number

$7 - x$ "Minus" indicates subtraction.

***$x - 7$ is incorrect. We cannot subtract in either order and get the same result.***

**NOW TRY ANSWERS**

1. **(a)** 54 **(b)** 144
2. **(a)** 40 **(b)** 2 **(c)** 15

**NOW TRY EXERCISE 3**

Write each word phrase as an algebraic expression, using $x$ as the variable.

**(a)** The sum of a number and 10

**(b)** A number divided by 7

**(c)** The product of 3 and the difference between 9 and a number

**(c)** A number subtracted **from 12**

$$12 - x$$

Be careful with order.

Compare this result with "12 subtracted from a number," which is $x - 12$.

**(d)** The product of 11 and a number

$$11 \cdot x, \quad \text{or} \quad 11x$$

**(e)** 5 divided by a number

$$5 \div x, \quad \text{or} \quad \frac{5}{x}$$

$\frac{x}{5}$ is *not* correct here.

**(f)** The product of 2 and the difference between a number and 8

We are multiplying 2 times "something." This "something" is the difference between a number and 8, written $x - 8$. We use parentheses around this difference.

$$2 \cdot (x - 8), \quad \text{or} \quad 2(x - 8)$$

$8 - x$, which means the difference between 8 and a number, is not correct.

NOW TRY

**OBJECTIVE 3** **Identify solutions of equations.** An **equation** is a statement that two algebraic expressions are equal. ***An equation always includes the equality symbol, =.***

$$\left.\begin{array}{lll} x + 4 = 11, & 2y = 16, & 4p + 1 = 25 - p, \\ \frac{3}{4}x + \frac{1}{2} = 0, & z^2 = 4, & 4(m - 0.5) = 2m \end{array}\right\} \text{Equations}$$

To **solve** an equation means to find the values of the variable that make the equation true. Such values of the variable are called the **solutions** of the equation.

**NOW TRY EXERCISE 4**

Decide whether the given number is a solution of the equation.

$$8k + 5 = 61; \quad 7$$

## EXAMPLE 4 Deciding Whether a Number Is a Solution of an Equation

Decide whether the given number is a solution of the equation.

**(a)** $5p + 1 = 36; \quad 7$

$$\begin{aligned} 5p + 1 &= 36 \\ 5 \cdot 7 + 1 &\stackrel{?}{=} 36 && \text{Let } p = 7. \\ 35 + 1 &\stackrel{?}{=} 36 && \text{Multiply.} \\ 36 &= 36 \quad ✓ && \text{True—the left side of the equation equals the right side.} \end{aligned}$$

Be careful! Multiply first.

The number 7 is a solution of the equation.

**(b)** $9m - 6 = 32; \quad 4$

$$\begin{aligned} 9m - 6 &= 32 \\ 9 \cdot 4 - 6 &\stackrel{?}{=} 32 && \text{Let } m = 4. \\ 36 - 6 &\stackrel{?}{=} 32 && \text{Multiply.} \\ 30 &= 32 && \text{False—the left side does } \textit{not} \text{ equal the right side.} \end{aligned}$$

The number 4 is not a solution of the equation.

NOW TRY

**NOW TRY ANSWERS**

**3.** **(a)** $x + 10$, or $10 + x$ **(b)** $\frac{x}{7}$
**(c)** $3(9 - x)$

**4.** yes

**OBJECTIVE 4** **Identify solutions of equations from a set of numbers.** A **set** is a collection of objects. In mathematics, these objects are most often numbers. The objects that belong to the set, called **elements** of the set, are written between **braces.**

$$\{1, 2, 3, 4, 5\} \leftarrow \text{The set containing the numbers 1, 2, 3, 4, and 5}$$

**NOW TRY EXERCISE 5**

Write the word statement as an equation. Then find all solutions of the equation from the set $\{0, 2, 4, 6, 8, 10\}$.

The sum of a number and nine is equal to the difference between 25 and the number.

### EXAMPLE 5 Finding a Solution from a Given Set

Write each word statement as an equation. Use $x$ as the variable. Then find all solutions of the equation from the set

$$\{0, 2, 4, 6, 8, 10\}.$$

**(a)** The sum of a number and four is six.

$$\underset{\text{a number and four}}{\overset{\text{The sum of}}{\phantom{x}}} \quad \text{is} \quad \text{six.}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ x + 4 & = & 6 \end{array}$$

The word *is* translates as =.

Use $x$ for the unknown number.

One by one, mentally substitute each number from the given set $\{0, 2, 4, 6, 8, 10\}$ in $x + 4 = 6$. Since $2 + 4 = 6$ is true, 2 is the only solution.

**(b)** Nine more than five times a number is 49.

Start with $5x$, and then add 9 to it. The word *is* translates as =.

$$\begin{array}{ccc} \downarrow & \downarrow & \\ 5x + 9 & = & 49 \end{array} \qquad 5 \cdot x = 5x$$

Substitute each of the given numbers. The solution is 8, since $5 \cdot 8 + 9 = 49$ is true.

**(c)** The sum of a number and 12 is equal to four times the number.

$$\begin{array}{ccc} \text{The sum of a number and 12} & \text{is equal to} & \text{four times the number.} \\ \downarrow & \downarrow & \downarrow \\ x + 12 & = & 4x \end{array}$$

Substituting each of the given numbers in the equation leads to a true statement only for $x = 4$, since $4 + 12 = 4(4)$ is true. NOW TRY

**OBJECTIVE 5** **Distinguish between *expressions* and *equations*.** Students often have trouble distinguishing between equations and expressions. ***An equation is a sentence—it has something on the left side, an = symbol, and something on the right side. An expression is a phrase that represents a number.***

$$\underbrace{4x + 5}_{\text{Left side}} \overset{\uparrow}{=} \underbrace{9}_{\text{Right side}} \qquad\qquad 4x + 5$$

$$\begin{array}{cc} \text{Equation} & \text{Expression} \\ \text{(to solve)} & \text{(to simplify or evaluate)} \end{array}$$

**NOW TRY EXERCISE 6**

Decide whether each of the following is an *expression* or an *equation*.

**(a)** $2x + 5 = 6$

**(b)** $2x + 5 - 6$

### EXAMPLE 6 Distinguishing between Equations and Expressions

Decide whether each of the following is an *equation* or an *expression*.

**(a)** $2x - 5y$ There is no equals symbol. This is an expression.

**(b)** $2x = 5y$ There is an equals symbol with something on either side of it. This is an equation. NOW TRY

**NOW TRY ANSWERS**

**5.** $x + 9 = 25 - x$; 8

**6.** **(a)** equation **(b)** expression

## 1.3 EXERCISES

MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

◉ *Complete solution available on the Video Resources on DVD*

***Concept Check*** *Choose the letter(s) of the correct response.*

**1.** The expression $8x^2$ means ______.

**A.** $8 \cdot x \cdot 2$ **B.** $8 \cdot x \cdot x$ **C.** $8 + x^2$ **D.** $8x^2 \cdot 8x^2$

**2.** If $x = 2$ and $y = 1$, then the value of $xy$ is ______.

**A.** $\frac{1}{2}$ **B.** 1 **C.** 2 **D.** 3

**3.** The sum of 15 and a number $x$ is represented by ______.

**A.** $15 + x$ **B.** $15 - x$ **C.** $x - 15$ **D.** $15x$

**4.** Which of the following are expressions?

**A.** $6x = 7$ **B.** $6x + 7$ **C.** $6x - 7$ **D.** $6x - 7 = 0$

✎ *In Exercises 5–8, give a short explanation.*

**5.** Explain why $2x^3$ is not the same as $2x \cdot 2x \cdot 2x$.

**6.** Why are "7 less than a number" and "7 is less than a number" translated differently?

**7.** When evaluating the expression $5x^2$ for $x = 4$, explain why 4 must be squared *before* multiplying by 5.

**8.** There are many pairs of values of $x$ and $y$ for which $2x + y$ will equal 6. Name two such pairs and describe how you determined them.

*Find the value for **(a)** $x = 4$ and **(b)** $x = 6$.* ***See Example 1.***

**9.** $x + 7$ **10.** $x - 3$ **11.** $4x$ **12.** $6x$ ◉ **13.** $4x^2$

**14.** $5x^2$ **15.** $\frac{x + 1}{3}$ **16.** $\frac{x - 2}{5}$ **17.** $\frac{3x - 5}{2x}$ **18.** $\frac{4x - 1}{3x}$

**19.** $3x^2 + x$ **20.** $2x + x^2$ **21.** $6.459x$ **22.** $3.275x$

*Find the value for **(a)** $x = 2$ and $y = 1$ and **(b)** $x = 1$ and $y = 5$.* ***See Example 2.***

◉ **23.** $8x + 3y + 5$ **24.** $4x + 2y + 7$ **25.** $3(x + 2y)$ **26.** $2(2x + y)$

**27.** $x + \frac{4}{y}$ **28.** $y + \frac{8}{x}$ **29.** $\frac{x}{2} + \frac{y}{3}$ **30.** $\frac{x}{5} + \frac{y}{4}$

**31.** $\frac{2x + 4y - 6}{5y + 2}$ **32.** $\frac{4x + 3y - 1}{x}$ **33.** $2y^2 + 5x$ **34.** $6x^2 + 4y$

**35.** $\frac{3x + y^2}{2x + 3y}$ **36.** $\frac{x^2 + 1}{4x + 5y}$ **37.** $0.841x^2 + 0.32y^2$ **38.** $0.941x^2 + 0.25y^2$

*Write each word phrase as an algebraic expression, using $x$ as the variable.* ***See Example 3.***

◉ **39.** Twelve times a number

**40.** Fifteen times a number

**41.** Nine added to a number

**42.** Six added to a number

**43.** Four subtracted from a number

**44.** Seven subtracted from a number

**45.** A number subtracted from seven

**46.** A number subtracted from four

**47.** The difference between a number and 8

**48.** The difference between 8 and a number

**49.** 18 divided by a number

**50.** A number divided by 18

**51.** The product of 6 and four less than a number

**52.** The product of 9 and five more than a number

**53.** Suppose that the directions on a test read "*Solve the following expressions*." How would you politely correct the person who wrote these directions?

**54.** Suppose that, for the equation $3x - y = 9$, the value of $x$ is given as 4. What would be the corresponding value of $y$? How do you know this?

*Decide whether the given number is a solution of the equation.* ***See Example 4.***

**55.** $4m + 2 = 6; \quad 1$

**56.** $2r + 6 = 8; \quad 1$

**57.** $2y + 3(y - 2) = 14; \quad 3$

**58.** $6x + 2(x + 3) = 14; \quad 2$

**59.** $6p + 4p + 9 = 11; \quad \frac{1}{5}$

**60.** $2x + 3x + 8 = 20; \quad \frac{12}{5}$

**61.** $3r^2 - 2 = 46; \quad 4$

**62.** $2x^2 + 1 = 19; \quad 3$

**63.** $\frac{3}{8}x + \frac{1}{4} = 1; \quad 2$

**64.** $\frac{7}{10}x + \frac{1}{2} = 4; \quad 5$

**65.** $0.5(x - 4) = 80; \quad 20$

**66.** $0.2(x - 5) = 70; \quad 40$

*Write each word statement as an equation. Use x as the variable. Find all solutions from the set* $\{2, 4, 6, 8, 10\}$. ***See Example 5.***

**67.** The sum of a number and 8 is 18.

**68.** A number minus three equals 1.

**69.** Sixteen minus three-fourths of a number is 13.

**70.** The sum of six-fifths of a number and 2 is 14.

**71.** One more than twice a number is 5.

**72.** The product of a number and 3 is 6.

**73.** Three times a number is equal to 8 more than twice the number.

**74.** Twelve divided by a number equals $\frac{1}{3}$ times that number.

*Identify each as an* expression *or an* equation. ***See Example 6.***

**75.** $3x + 2(x - 4)$

**76.** $8y - (3y + 5)$

**77.** $7t + 2(t + 1) = 4$

**78.** $9r + 3(r - 4) = 2$

**79.** $x + y = 9$

**80.** $x + y - 9$

*A* ***mathematical model*** *is an equation that describes the relationship between two quantities. For example, the life expectancy at birth of Americans can be approximated by the equation*

$$y = 0.212x - 347,$$

*where x is a year between 1943 and 2005 and y is age in years. (Source: Centers for Disease Control and Prevention.)*

*Use this model to approximate life expectancy (to the nearest tenth of a year) in each of the following years.*

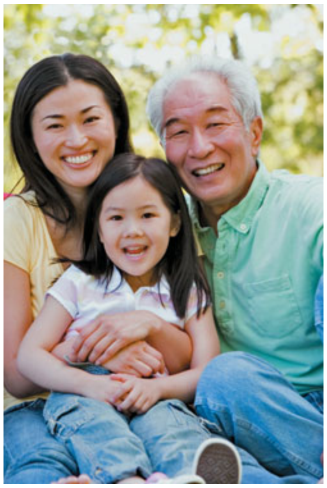
**81.** 1943

**82.** 1960

**83.** 1985

**84.** 2005

**85.** How has the life expectancy at birth of Americans changed in the years from 1943 to 2005?



# 1.4 Real Numbers and the Number Line

**OBJECTIVES**

1. Classify numbers and graph them on number lines.
2. Tell which of two real numbers is less than the other.
3. Find the additive inverse of a real number.
4. Find the absolute value of a real number.
5. Interpret the meanings of real numbers from a table of data.

**OBJECTIVE 1** **Classify numbers and graph them on number lines.** In **Section 1.1,** we introduced the set of *natural numbers* and the set of *whole numbers*.

**Natural Numbers**

$\{1, 2, 3, 4, \ldots\}$ is the set of **natural numbers** (or **counting numbers**).

**Whole Numbers**

$\{0, 1, 2, 3, 4, \ldots\}$ is the set of **whole numbers.**

**NOTE** The three dots ( . . . ) show that the list of numbers continues in the same way indefinitely.

We can represent numbers on a **number line** like the one in **FIGURE 3**.

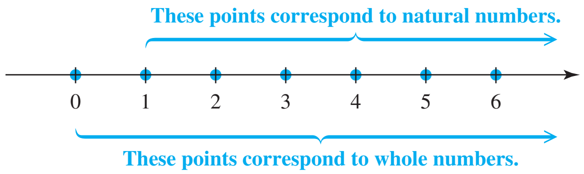


**FIGURE 3**

To draw a number line, choose any point on the line and label it 0. Then choose any point to the right of 0 and label it 1. Use the distance between 0 and 1 as the scale to locate, and then label, other points.

The natural numbers are located to the right of 0 on the number line. For each natural number, we can place a corresponding number to the left of 0, labeling the points $-1$, $-2$, $-3$, and so on, as shown in **FIGURE 4**. Each is the **opposite,** or **negative,** of a natural number. The natural numbers, their opposites, and 0 form the set of *integers*.

**Integers**

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of **integers.**

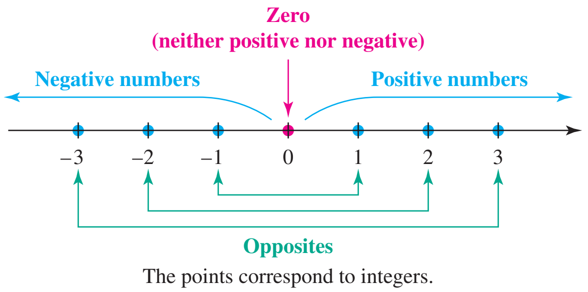


**FIGURE 4**

*Positive numbers* and *negative numbers* are called **signed numbers.**

**NOW TRY EXERCISE 1**

Use an integer to express the number in boldface italics in the following statement.

At its deepest point, the floor of West Okoboji Lake sits ***136*** ft below the water's surface. (*Source:* www.watersafetycouncil.org)

### EXAMPLE 1 Using Negative Numbers in Applications

Use an integer to express the number in boldface italics in each application.

**(a)** The lowest Fahrenheit temperature ever recorded was ***129***° below zero at Vostok, Antarctica, on July 21, 1983. (*Source: World Almanac and Book of Facts.*)
Use −129 because "below zero" indicates a negative number.

**(b)** General Motors had a loss of about $***31*** billion in 2008. (*Source: The Wall Street Journal.*)
Here, a loss indicates a negative "profit," −31.

NOW TRY

Fractions, introduced in **Section 1.1,** are examples of *rational numbers.*

### Rational Numbers

$\{x \mid x$ is a quotient of two integers, with denominator not $0\}$ is the set of **rational numbers.**

(Read the part in the braces as "the set of all numbers $x$ such that $x$ is a quotient of two integers, with denominator not 0.")

**NOTE** The set symbolism used in the definition of rational numbers,

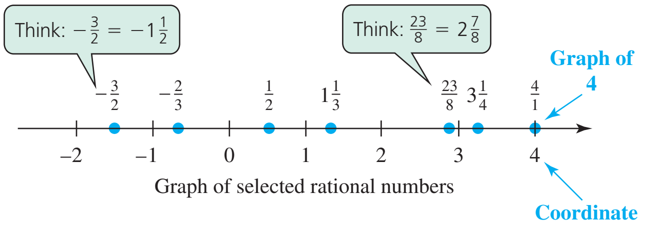
$$\{x \mid x \textbf{ has a certain property}\},$$

is called **set-builder notation.** We use this notation when it is not possible to list all the elements of a set.

Since any number that can be written as the quotient of two integers (that is, as a fraction) is a rational number, ***all integers, mixed numbers, terminating (or ending) decimals, and repeating decimals are rational.*** The table gives examples.

| *Rational Number* | *Equivalent Quotient of Two Integers* |
|---|---|
| −5 | $\frac{-5}{1}$ (means −5 ÷ 1) |
| $1\frac{3}{4}$ | $\frac{7}{4}$ (means 7 ÷ 4) |
| 0.23 (terminating decimal) | $\frac{23}{100}$ (means 23 ÷ 100) |
| 0.3333 . . . , or $0.\overline{3}$ (repeating decimal) | $\frac{1}{3}$ (means 1 ÷ 3) |
| 4.7 | $\frac{47}{10}$ (means 47 ÷ 10) |

To **graph** a number, we place a dot on the number line at the point that corresponds to the number. The number is called the **coordinate** of the point. See **FIGURE 5.**



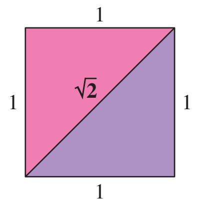
Graph of selected rational numbers

**FIGURE 5**

Think of the graph of a set of numbers as a picture of the set.

**NOW TRY ANSWER**

**1.** −136



This square has diagonal of length $\sqrt{2}$. The number $\sqrt{2}$ is an irrational number.

**FIGURE 6**

Not all numbers are rational. For example, the square root of 2, written $\sqrt{2}$, cannot be written as a quotient of two integers. Because of this, $\sqrt{2}$ is an *irrational number*. (See **FIGURE 6**.)

### Irrational Numbers

$\{x \mid x$ is a nonrational number represented by a point on the number line$\}$ is the set of **irrational numbers.**

The decimal form of an irrational number neither terminates nor repeats.

Both rational and irrational numbers can be represented by points on the number line and together form the set of *real numbers.*

### Real Numbers

$\{x \mid x$ is a rational or an irrational number$\}$ is the set of **real numbers.***

The relationships among the various sets of numbers are shown in **FIGURE 7**.

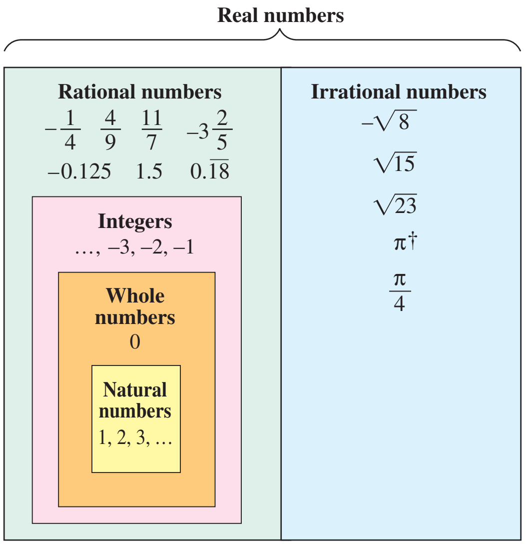


**FIGURE 7**

## EXAMPLE 2 Determining Whether a Number Belongs to a Set

List the numbers in the following set that belong to each set of numbers.

$$\left\{-5, -\frac{2}{3}, 0, 0.\overline{6}, \sqrt{2}, 3\frac{1}{4}, 5, 5.8\right\}$$

**(a)** Natural numbers: 5

**(b)** Whole numbers: 0 and 5

The whole numbers consist of the natural (counting) numbers and 0.

---

*An example of a number that is not a real number is the square root of a negative number, such as $\sqrt{-5}$.

†The value of $\pi$ (pi) is approximately 3.141592654. The decimal digits continue forever with no repeated pattern.

**NOW TRY EXERCISE 2**

List the numbers in the following set that belong to each set of numbers.

$$\left\{-7, -\frac{4}{5}, 0, \sqrt{3}, 2.7, \pi, 13\right\}$$

**(a)** Whole numbers
**(b)** Integers
**(c)** Rational numbers
**(d)** Irrational numbers

**(c)** Integers: $-5$, 0, and 5

**(d)** Rational numbers: $-5, -\frac{2}{3}, 0, 0.\overline{6}\left(\text{or } \frac{2}{3}\right), 3\frac{1}{4}\left(\text{or } \frac{13}{4}\right)$, 5, and $5.8\left(\text{or } \frac{58}{10}\right)$

Each of these numbers can be written as the quotient of two integers.

**(e)** Irrational numbers: $\sqrt{2}$

**(f)** Real numbers: All the numbers in the set are real numbers.

NOW TRY

**OBJECTIVE 2** **Tell which of two real numbers is less than the other.** Given any two positive integers, you probably can tell which number is less than the other. Positive numbers decrease as the corresponding points on the number line go to the left. For example, $8 < 12$ because 8 is to the left of 12 on the number line. This ordering is extended to all real numbers by definition.

*a* lies to the left of *b*,
or $a < b$.

**FIGURE 8**

### Ordering of Real Numbers

For any two real numbers *a* and *b*, ***a* is less than *b*** if *a* lies to the left of *b* on the number line. See **FIGURE 8**.

This means that any negative number is less than 0, and any negative number is less than any positive number. Also, 0 is less than any positive number.

**NOW TRY EXERCISE 3**

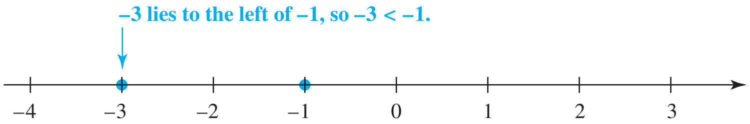
Determine whether the statement is *true* or *false*.

$$-8 \le -9$$

### EXAMPLE 3 Determining the Order of Real Numbers

Is the statement $-3 < -1$ *true* or *false*?

Locate $-3$ and $-1$ on a number line, as shown in **FIGURE 9**. Since $-3$ lies to the left of $-1$ on the number line, $-3$ is less than $-1$. The statement $-3 < -1$ is true.



−3 lies to the left of −1, so −3 < −1.

**FIGURE 9**

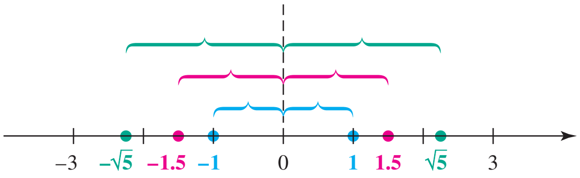
NOW TRY

We can also say that, for any two real numbers *a* and *b*, ***a* is greater than *b*** if *a* lies to the right of *b* on the number line. See **FIGURE 10**.

*a* lies to the right of *b*,
or $a > b$.

**FIGURE 10**

**OBJECTIVE 3** **Find the additive inverse of a real number.** By a property of the real numbers, for any real number *x* (except 0), there is exactly one number on the number line the same distance from 0 as *x*, but on the *opposite* side of 0. See **FIGURE 11**. Such pairs of numbers are called *additive inverses,* or *opposites,* of each other.



**Pairs of additive inverses, or opposites**

**FIGURE 11**

**NOW TRY ANSWERS**

**2.** **(a)** 0, 13 **(b)** $-7$, 0, 13
**(c)** $-7, -\frac{4}{5}, 0, 2.7, 13$
**(d)** $\sqrt{3}, \pi$
**3.** false

### Additive Inverse

The **additive inverse** of a number $x$ is the number that is the same distance from 0 on the number line as $x$, but on the *opposite* side of 0.

We indicate the additive inverse of a number by writing the symbol $-$ in front of the number. For example, the additive inverse of 7 is written $-7$. We could write the additive inverse of $-3$ as $-(-3)$, but we know that 3 is the additive inverse of $-3$. Since a number can have only one additive inverse, 3 and $-(-3)$ must represent the same number, so

$$-(-3) = 3.$$

This idea can be generalized.

| *Number* | *Additive Inverse* |
|---|---|
| 7 | $-7$ |
| $-3$ | $-(-3)$, or 3 |
| 0 | 0 |
| 19 | $-19$ |
| $-\frac{2}{3}$ | $\frac{2}{3}$ |
| 0.52 | $-0.52$ |

***The additive inverse of a nonzero number is found by changing the sign of the number.***

### Double Negative Rule

For any real number $x$, $\quad \mathbf{-(-x) = x.}$

The table in the margin shows several numbers and their additive inverses.

**OBJECTIVE 4** **Find the absolute value of a real number.** Because additive inverses are the same distance from 0 on a number line, a number and its additive inverse have the same *absolute value*. The **absolute value** of a real number $x$, written $|x|$ and read **"the absolute value of $x$,"** can be defined as the distance between 0 and the number on a number line. For example,

$|2| = 2,$ The distance between 2 and 0 on a number line is 2 units.

$|-2| = 2.$ The distance between $-2$ and 0 on a number line is also 2 units.

Distance is a physical measurement, which is never negative. ***Therefore, the absolute value of a number is never negative.***

In symbols, the absolute value of $x$ is defined as follows.

### Absolute Value

For any real number $x$,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

***By this definition, if x is a positive number or 0, then its absolute value is x itself.*** For example, since 8 is a positive number,

$$|8| = 8.$$

***If x is a negative number, then its absolute value is the additive inverse of x.***

$|-8| = -(-8) = 8$ The additive inverse of $-8$ is 8.

**CAUTION** The "$-x$" in the second part of the definition of absolute value does ***NOT*** represent a negative number. Since $x$ is negative in the second part, $-x$ represents the opposite of a negative number—that is, a positive number. ***The absolute value of a number is never negative.***

NOW TRY EXERCISE 4

Simplify by finding the absolute value.

**(a)** $|4|$ **(b)** $|-4|$ **(c)** $-|-4|$

### EXAMPLE 4 Finding the Absolute Value

Simplify by finding the absolute value.

**(a)** $|0| = 0$

**(b)** $|5| = 5$

**(c)** $|-5| = -(-5) = 5$

**(d)** $-|5| = -(5) = -5$

**(e)** $-|-5| = -(5) = -5$

**(f)** $|8 - 2| = |6| = 6$

**(g)** $-|8 - 2| = -|6| = -6$

Parts (f) and (g) show that absolute value bars are grouping symbols. We perform any operations inside absolute value symbols *before* finding the absolute value.

NOW TRY

## OBJECTIVE 5 Interpret the meanings of real numbers from a table of data.

NOW TRY EXERCISE 5

In the table for **Example 5,** which category represents a decrease for both years?

### EXAMPLE 5 Interpreting Data

The Consumer Price Index (CPI) measures the average change in prices of goods and services purchased by urban consumers in the United States.

The table shows the percent change in the Consumer Price Index for selected categories of goods and services from 2005 to 2006 and from 2006 to 2007. Use the table to answer each question.

| *Category* | *Change from 2005 to 2006* | *Change from 2006 to 2007* |
|---|---|---|
| Education | 6.2 | 5.7 |
| Food | 3.2 | 2.8 |
| Gasoline | 12.9 | 8.2 |
| Medical care | 4.0 | 4.4 |
| New cars | −0.2 | −1.0 |

*Source:* U.S. Bureau of Labor Statistics.

**(a)** What category in which year represents the greatest percent decrease?

We must find the negative number with the greatest absolute value. The number that satisfies this condition is $-1.0$, so the greatest percent decrease was shown by new cars from 2006 to 2007.

**(b)** Which category in which year represents the least change?

We must find the number (either positive, negative, or zero) with the least absolute value. From 2005 to 2006, new cars showed the least change, a decrease of 0.2%.

NOW TRY

NOW TRY ANSWERS

4. **(a)** 4 **(b)** 4 **(c)** $-4$
5. new cars

## 1.4 EXERCISES

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*In Exercises 1–4, use an integer to express each number in boldface italics representing a change. In Exercises 5–8, use a rational number.* ***See Example 1.***

**1.** Between July 1, 2006, and July 1, 2007, the population of the United States increased by approximately ***2,866,000.*** (*Source:* U.S. Census Bureau.)

**2.** Between 2006 and 2007, the number of movie screens in the United States increased by ***409.*** (*Source:* Motion Picture Association of America.)

**3.** From 2006 to 2007, attendance at the World Series went from 225,000 to 173,000, a decrease of ***52,000.*** (*Source:* Major League Baseball.)

**4.** In 1935, there were 15,295 banks in the United States. By 2008, the number was 8441, representing a decrease of ***6854*** banks. (*Source:* Federal Deposit Insurance Corporation.)

**5.** The number of bachelor's degrees in computer and information sciences in the United States declined ***11.2***% from the 2005–2006 academic year to the 2006–2007 year, while the number of bachelor's degrees in biological and biomedical sciences rose ***8.6***%. (*Source:* National Center for Education Statistics.)

**6.** Between 2006 and 2007, print advertising revenue in the United States declined ***9.4***%, while online advertising rose ***18.8***%. (*Source:* Newspaper Association of America.)

**7.** On Tuesday, August 18, 2009, the Dow Jones Industrial Average (DJIA) closed at 9217.94. On the previous day it had closed at 9135.34. Thus, on Tuesday, it closed up ***82.60*** points. (*Source: The Washington Post.*)

**8.** On Monday, August 17, 2009, the NASDAQ closed at 1930.84. On the previous Friday, it had closed at 1985.52. Thus, on Monday, it closed down ***54.68*** points. (*Source: The Washington Post.*)

*Concept Check* *In Exercises 9–14, give a number that satisfies the given condition.*

**9.** An integer between 3.6 and 4.6

**10.** A rational number between 2.8 and 2.9

**11.** A whole number that is not positive and is less than 1

**12.** A whole number greater than 3.5

**13.** An irrational number that is between $\sqrt{12}$ and $\sqrt{14}$

**14.** A real number that is neither negative nor positive

*Concept Check* *In Exercises 15–20, decide whether each statement is* true *or* false.

**15.** Every natural number is positive.

**16.** Every whole number is positive.

**17.** Every integer is a rational number.

**18.** Every rational number is a real number.

**19.** Some numbers are both rational and irrational.

**20.** Every terminating decimal is a rational number.

*Concept Check* *Give three numbers between* $-6$ *and* 6 *that satisfy each given condition.*

**21.** Positive real numbers but not integers

**22.** Real numbers but not positive numbers

**23.** Real numbers but not whole numbers

**24.** Rational numbers but not integers

**25.** Real numbers but not rational numbers

**26.** Rational numbers but not negative numbers

*For Exercises 27 and 28,* ***see Example 2.*** *List all numbers from each set that are*

***(a)*** *natural numbers* ***(b)*** *whole numbers* ***(c)*** *integers*
***(d)*** *rational numbers* ***(e)*** *irrational numbers* ***(f)*** *real numbers.*

**27.** $\left\{-9, -\sqrt{7}, -1\frac{1}{4}, -\frac{3}{5}, 0, 0.\overline{1}, \sqrt{5}, 3, 5.9, 7\right\}$

**28.** $\left\{-5.3, -5, -\sqrt{3}, -1, -\frac{1}{9}, 0, 0.\overline{27}, 1.2, 1.8, 3, \sqrt{11}\right\}$

*Graph each group of numbers on a number line. See* **FIGURE 4** *and* **FIGURE 5**.

**29.** $0, 3, -5, -6$ **30.** $2, 6, -2, -1$ **31.** $-2, -6, -4, 3, 4$

**32.** $-5, -3, -2, 0, 4$ **33.** $\frac{1}{4}, 2\frac{1}{2}, -3\frac{4}{5}, -4, -1\frac{5}{8}$ **34.** $5\frac{1}{4}, 4\frac{5}{9}, -2\frac{1}{3}, 0, -3\frac{2}{5}$

**35.** ***Concept Check*** Match each expression in Column I with its value in Column II. Choices in Column II may be used once, more than once, or not at all.

| I | II |
|---|---|
| **(a)** $\lvert -9 \rvert$ | **A.** 9 |
| **(b)** $-(-9)$ | **B.** $-9$ |
| **(c)** $-\lvert -9 \rvert$ | **C.** Neither A nor B |
| **(d)** $-\lvert -(-9) \rvert$ | **D.** Both A and B |

**36.** ***Concept Check*** Fill in the blanks with the correct values: The opposite of $-5$ is ______, while the absolute value of $-5$ is ______. The additive inverse of $-5$ is ______, while the additive inverse of the absolute value of $-5$ is ______.

*Find* ***(a)*** *the opposite (or additive inverse) of each number and* ***(b)*** *the absolute value of each number.* ***See Objective 3 and Example 4.***

**37.** $-7$ **38.** $-4$ **39.** $8$ **40.** $10$ **41.** $-\frac{3}{4}$ **42.** $-\frac{2}{5}$

*Simplify by finding the absolute value.* ***See Example 4.***

**43.** $\lvert -6 \rvert$ **44.** $\lvert -14 \rvert$ **45.** $-\lvert 12 \rvert$ **46.** $-\lvert 19 \rvert$

**47.** $-\left\lvert -\frac{2}{3} \right\rvert$ **48.** $-\left\lvert -\frac{4}{5} \right\rvert$ **49.** $\lvert 6 - 3 \rvert$ **50.** $-\lvert 6 - 3 \rvert$

**51.** Students often say "Absolute value is always positive." Is this true? Explain.

**52.** ***Concept Check*** *True* or *false:* If $a$ is negative, then $\lvert a \rvert = -a$.

*Select the lesser of the two given numbers.* ***See Examples 3 and 4.***

**53.** $-11, -3$ **54.** $-8, -13$ **55.** $-7, -6$

**56.** $-16, -17$ **57.** $4, \lvert -5 \rvert$ **58.** $4, \lvert -3 \rvert$

**59.** $\lvert -3.5 \rvert, \lvert -4.5 \rvert$ **60.** $\lvert -8.9 \rvert, \lvert -9.8 \rvert$ **61.** $-\lvert -6 \rvert, -\lvert -4 \rvert$

**62.** $-\lvert -2 \rvert, -\lvert -3 \rvert$ **63.** $\lvert 5 - 3 \rvert, \lvert 6 - 2 \rvert$ **64.** $\lvert 7 - 2 \rvert, \lvert 8 - 1 \rvert$

*Decide whether each statement is* true *or* false. ***See Examples 3 and 4.***

**65.** $-5 < -2$ **66.** $-8 > -2$ **67.** $-4 \le -(-5)$

**68.** $-6 \le -(-3)$ **69.** $\lvert -6 \rvert < \lvert -9 \rvert$ **70.** $\lvert -12 \rvert < \lvert -20 \rvert$

**71.** $-\lvert 8 \rvert > \lvert -9 \rvert$ **72.** $-\lvert 12 \rvert > \lvert -15 \rvert$ **73.** $-\lvert -5 \rvert \ge -\lvert -9 \rvert$

**74.** $-\lvert -12 \rvert \le -\lvert -15 \rvert$ **75.** $\lvert 6 - 5 \rvert \ge \lvert 6 - 2 \rvert$ **76.** $\lvert 13 - 8 \rvert \le \lvert 7 - 4 \rvert$

*The table shows the percent change in the Consumer Price Index (CPI) for selected categories of goods and services from 2004 to 2005 and from 2006 to 2007. Use the table to answer Exercises 77–80.* ***See Example 5.***

**77.** Which category in which year represents the greatest percentage increase?

**78.** Which category in which year represents the greatest percentage decrease?

**79.** Which category in which year represents the least change?

**80.** Which categories represent a decrease for both years?

| *Category* | *Change from 2004 to 2005* | *Change from 2006 to 2007* |
|---|---|---|
| Shelter | 2.6 | 3.7 |
| Apparel and upkeep | −0.7 | −0.4 |
| Fuel and other utilities | 10.6 | 3.0 |
| Medical care | 4.0 | 4.4 |
| Public transportation | 3.9 | 1.5 |

*Source:* U.S. Bureau of Labor Statistics.

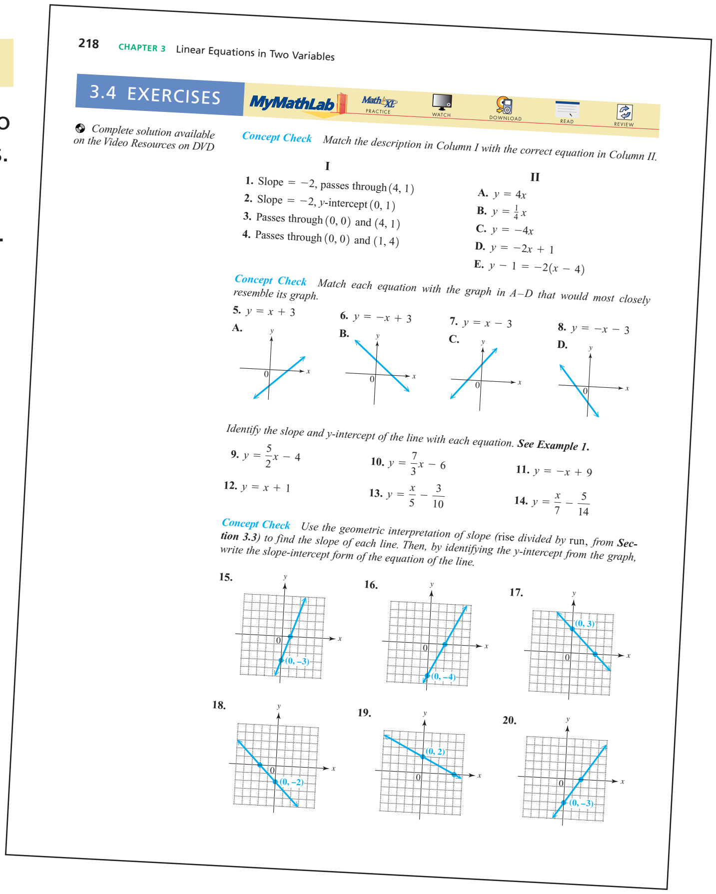
STUDY SKILLS

# Tackling Your Homework

You are ready to do your homework ***AFTER*** you have read the corresponding textbook section and worked through the examples and Now Try exercises.

## Homework Tips

- **Work problems neatly.** Use pencil and write legibly, so others can read your work. Skip lines between steps. Clearly separate problems from each other.
- **Show all your work.** It is tempting to take shortcuts. Include ALL steps.
- **Check your work frequently to make sure you are on the right track.** It is hard to unlearn a mistake. For all odd-numbered problems, answers are given in the back of the book.
- **If you have trouble with a problem, refer to the corresponding worked example in the section.** The exercise directions will often reference specific examples to review. Pay attention to every line of the worked example to see how to get from step to step.
- **If you are having trouble with an even-numbered problem, work the corresponding odd-numbered problem.** Check your answer in the back of the book, and apply the same steps to work the even-numbered problem.
- **Mark any problems you don't understand.** Ask your instructor about them.



218 CHAPTER 3 Linear Equations in Two Variables

3.4 EXERCISES

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*Concept Check* *Match the description in Column I with the correct equation in Column II.*

| I | II |
|---|---|
| **1.** Slope = $-2$, passes through $(4, 1)$ | **A.** $y = 4x$ |
| **2.** Slope = $-2$, $y$-intercept $(0, 1)$ | **B.** $y = \frac{1}{4}x$ |
| **3.** Passes through $(0, 0)$ and $(4, 1)$ | **C.** $y = -4x$ |
| **4.** Passes through $(0, 0)$ and $(1, 4)$ | **D.** $y = -2x + 1$ |
| | **E.** $y - 1 = -2(x - 4)$ |

*Concept Check* *Match each equation with the graph in A–D that would most closely resemble its graph.*

**5.** $y = x + 3$ **6.** $y = -x + 3$ **7.** $y = x - 3$ **8.** $y = -x - 3$

**A.** **B.** **C.** **D.**

*Identify the slope and y-intercept of the line with each equation.* ***See Example 1.***

**9.** $y = \frac{5}{2}x - 4$ **10.** $y = \frac{7}{3}x - 6$ **11.** $y = -x + 9$

**12.** $y = x + 1$ **13.** $y = \frac{x}{5} - \frac{3}{10}$ **14.** $y = \frac{x}{7} - \frac{5}{14}$

*Concept Check* *Use the geometric interpretation of slope* (rise *divided by* run, *from* ***Section 3.3****) to find the slope of each line. Then, by identifying the y-intercept from the graph, write the slope-intercept form of the equation of the line.*

**15.** **16.** **17.**

**18.** **19.** **20.**

***Select several homework tips to try this week.***

# 1.5 Adding and Subtracting Real Numbers

**OBJECTIVES**

1. Add two numbers with the same sign.
2. Add two numbers with different signs.
3. Use the definition of subtraction.
4. Use the rules for order of operations with real numbers.
5. Translate words and phrases involving addition and subtraction.
6. Use signed numbers to interpret data.

**OBJECTIVE 1 Add two numbers with the same sign.** Recall that the answer to an addition problem is called a **sum.** A number line can be used to add real numbers.

**EXAMPLE 1** Adding Numbers on a Number Line

Use a number line to find each sum.

**(a)** $2 + 3$

***Step 1*** Start at 0 and draw an arrow 2 units to the *right*. See **FIGURE 12**.

***Step 2*** From the right end of that arrow, draw another arrow 3 units to the right.

The number below the end of this second arrow is 5, so $2 + 3 = 5$.

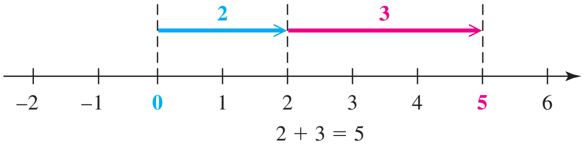


**FIGURE 12**

*NOW TRY EXERCISE 1*

Use a number line to find each sum.

**(a)** $3 + 5$ **(b)** $-1 + (-3)$

**(b)** $-2 + (-4)$

(We put parentheses around $-4$ due to the $+$ and $-$ next to each other.)

***Step 1*** Start at 0 and draw an arrow 2 units to the *left*. See **FIGURE 13**.

***Step 2*** From the left end of the first arrow, draw a second arrow 4 units to the *left* to represent the addition of a *negative* number.

The number below the end of this second arrow is $-6$, so $-2 + (-4) = -6$.

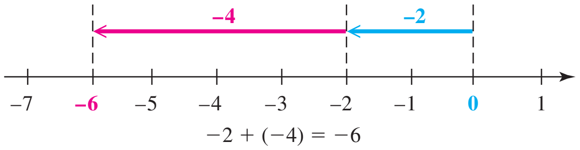


**FIGURE 13**

*NOW TRY*

In **Example 1(b),** the sum of the two negative numbers $-2$ and $-4$ is a negative number whose distance from 0 is the sum of the distance of $-2$ from 0 and the distance of $-4$ from 0. ***That is, the sum of two negative numbers is the negative of the sum of their absolute values.***

$$-2 + (-4) = -(|-2| + |-4|) = -(2 + 4) = -6$$

*NOW TRY ANSWERS*

**1. (a)** 8 **(b)** $-4$

### Adding Numbers with the Same Sign

To add two numbers with the *same* sign, add the absolute values of the numbers. The sum has the same sign as the numbers being added.

*Example:* $-4 + (-3) = -7$

**NOW TRY EXERCISE 2**

Find the sum.

$$-6 + (-11)$$

### EXAMPLE 2 Adding Two Negative Numbers

Find each sum.

**(a)** $-2 + (-9) = -(|-2| + |-9|) = -(2 + 9) = -11$

**(b)** $-8 + (-12) = -20$ **(c)** $-15 + (-3) = -18$

NOW TRY

## OBJECTIVE 2 Add two numbers with different signs.

**NOW TRY EXERCISE 3**

Use a number line to find the sum.

$$4 + (-8)$$

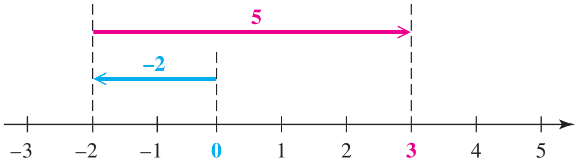
### EXAMPLE 3 Adding Numbers with Different Signs

Use a number line to find the sum $-2 + 5$.

***Step 1*** Start at 0 and draw an arrow 2 units to the left. See **FIGURE 14**.

***Step 2*** From the left end of this arrow, draw a second arrow 5 units to the right.

The number below the end of the second arrow is 3, so $-2 + 5 = 3$.



$-2 + 5 = 3$

**FIGURE 14**

NOW TRY

### Adding Numbers with Different Signs

To add two numbers with *different* signs, find the absolute values of the numbers and subtract the lesser absolute value from the greater. Give the answer the same sign as the number having the greater absolute value.

*Example:* $-12 + 6 = -6$

**NOW TRY EXERCISE 4**

Find the sum.

$$8 + (-17)$$

### EXAMPLE 4 Adding Numbers with Different Signs

Find the sum $-12 + 5$.

Find the absolute value of each number.

$$|-12| = 12 \quad \text{and} \quad |5| = 5$$

Then find the difference between these absolute values: $12 - 5 = 7$. The sum will be negative, since $|-12| > |5|$.

$$-12 + 5 = -7$$

NOW TRY

**NOW TRY ANSWERS**

**2.** $-17$ **3.** $-4$ **4.** $-9$

**NOW TRY EXERCISE 5**

Check each answer.

**(a)** $\frac{2}{3}+\left(-2\frac{1}{9}\right)=-1\frac{4}{9}$

**(b)** $3.7+(-5.7)=-2$

### EXAMPLE 5 Adding Mentally

Check each answer by adding mentally. If necessary, use a number line.

**(a)** $7+(-4)=3$ **(b)** $-8+12=4$

**(c)** $-\frac{1}{2}+\frac{1}{8}=-\frac{4}{8}+\frac{1}{8}=-\frac{3}{8}$

Find a common denominator.

**(d)** $\frac{5}{6}+\left(-1\frac{1}{3}\right)=\frac{5}{6}+\left(-\frac{4}{3}\right)=\frac{5}{6}+\left(-\frac{8}{6}\right)=-\frac{3}{6}=-\frac{1}{2}$

**(e)** $-4.6+8.1=3.5$ **(f)** $-16+16=0$ **(g)** $42+(-42)=0$

Notice in parts (f) and (g) that ***when additive inverses are added, the sum is 0.***

NOW TRY

The rules for adding signed numbers are summarized as follows.

**Adding Signed Numbers**

***Same sign*** Add the absolute values of the numbers. The sum has the same sign as the given numbers being added.

***Different signs*** Find the absolute values of the numbers and subtract the lesser absolute value from the greater. Give the answer the same sign as the number having the greater absolute value.

**OBJECTIVE 3** **Use the definition of subtraction.** Recall that the answer to a subtraction problem is called a **difference.** In the subtraction $x-y$, $x$ is called the **minuend** and $y$ is called the **subtrahend.**

To illustrate subtracting 4 from 7, written $7-4$, with a number line, we begin at 0 and draw an arrow 7 units to the right. See **FIGURE 15**. From the right end of this arrow, we draw an arrow 4 units to the *left*. The number at the end of the second arrow shows that $7-4=3$.

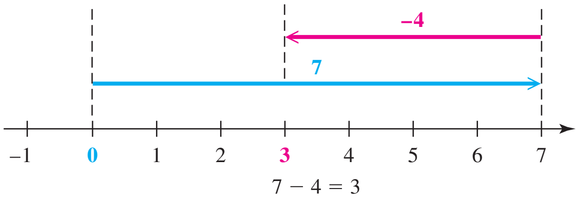


**FIGURE 15**

The procedure used to find the difference $7-4$ is exactly the same procedure that would be used to find the sum $7+(-4)$, so

$$7-4=7+(-4).$$

This equation suggests that *subtracting* a positive number from a greater positive number is the same as *adding* the additive inverse of the lesser number to the greater. This result leads to the definition of subtraction for all real numbers.

**NOW TRY ANSWER**

**5.** Both are correct.

### Definition of Subtraction

For any real numbers $x$ and $y$,

$$x - y = x + (-y).$$

***To subtract y from x, add the additive inverse (or opposite) of y to x.*** That is, change the subtrahend to its opposite and add.

*Example:* $4 - 9 = 4 + (-9) = -5$

NOW TRY EXERCISE 6

Subtract.

**(a)** $-5 - (-11)$

**(b)** $4 - 15$

**(c)** $-\frac{5}{7} - \frac{1}{3}$

### EXAMPLE 6 Using the Definition of Subtraction

Subtract.

Change − to +.

No change

Additive inverse of 3

**(a)** $12 - 3 = 12 + (-3) = 9$

**(b)** $5 - 7 = 5 + (-7) = -2$ **(c)** $-6 - 9 = -6 + (-9) = -15$

Change − to +.

No change

Additive inverse of −5

**(d)** $-3 - (-5) = -3 + (5) = 2$

**(e)** $\frac{4}{3} - \left(-\frac{1}{2}\right) = \frac{4}{3} + \frac{1}{2} = \frac{8}{6} + \frac{3}{6} = \frac{11}{6}$, or $1\frac{5}{6}$

NOW TRY

### Uses of the Symbol −

We use the symbol − for three purposes:

1. ***to represent subtraction,*** as in $9 - 5 = 4$;
2. ***to represent negative numbers,*** such as $-10$, $-2$, and $-3$;
3. ***to represent the opposite (or negative) of a number,*** as in "the opposite (or negative) of 8 is $-8$."

We may see more than one use of − in the same expression, such as $-6 - (-9)$, where $-9$ is subtracted from $-6$. The meaning of the − symbol depends on its position in the expression.

## OBJECTIVE 4 Use the rules for order of operations with real numbers.

### EXAMPLE 7 Adding and Subtracting with Grouping Symbols

Perform each indicated operation.

**(a)**

| | |
|---|---|
| $-6 - [2 - (8 + 3)]$ | Start within the innermost parentheses. |
| $= -6 - [2 - 11]$ | Add. |
| $= -6 - [2 + (-11)]$ | Definition of subtraction |
| $= -6 - [-9]$ | Add. |
| $= -6 + (9)$ | Definition of subtraction |
| $= 3$ | Add. |

NOW TRY ANSWERS

6. **(a)** 6 **(b)** $-11$
**(c)** $-\frac{22}{21}$, or $-1\frac{1}{21}$

NOW TRY EXERCISE 7

Perform each indicated operation.

**(a)** $8 - [(-3 + 7) - (3 - 9)]$

**(b)** $3|6 - 9| - |4 - 12|$

**(b)**
$$\begin{aligned} &5 + [(-3 - 2) - (4 - 1)] \\ &= 5 + [(-3 + (-2)) - 3] \quad \text{Work within each set of parentheses inside the brackets.} \\ &= 5 + [(-5) - 3] \\ &= 5 + [(-5) + (-3)] \quad \text{Show all steps to avoid sign errors.} \\ &= 5 + [-8] \\ &= -3 \end{aligned}$$

**(c)**
$$\begin{aligned} &\frac{2}{3} - \left[\frac{1}{12} - \left(-\frac{1}{4}\right)\right] \\ &= \frac{8}{12} - \left[\frac{1}{12} - \left(-\frac{3}{12}\right)\right] \quad \text{Find a common denominator.} \\ &= \frac{8}{12} - \left[\frac{1}{12} + \frac{3}{12}\right] \quad \text{Definition of subtraction} \\ &= \frac{8}{12} - \frac{4}{12} \quad \text{Add.} \\ &= \frac{4}{12}, \text{ or } \frac{1}{3} \quad \text{Subtract. Write in lowest terms.} \end{aligned}$$

**(d)**
$$\begin{aligned} &|4 - 7| + 2|6 - 3| \\ &= |-3| + 2|3| \quad \text{Work within absolute value bars.} \\ &= 3 + 2 \cdot 3 \quad \text{Evaluate absolute values.} \\ &= 3 + 6 \quad \text{Multiply.} \\ &= 9 \quad \text{Add.} \end{aligned}$$

Be careful! Multiply first.

NOW TRY

**OBJECTIVE 5 Translate words and phrases involving addition and subtraction.** The table lists words and phrases that indicate addition.

| *Word or Phrase* | *Example* | *Numerical Expression and Simplification* |
|---|---|---|
| Sum of | The *sum of* −3 and 4 | $-3 + 4$, or 1 |
| Added to | 5 *added to* −8 | $-8 + 5$, or −3 |
| More than | 12 *more than* −5 | $-5 + 12$, or 7 |
| Increased by | −6 *increased by* 13 | $-6 + 13$, or 7 |
| Plus | 3 *plus* 14 | $3 + 14$, or 17 |

NOW TRY EXERCISE 8

Write a numerical expression for the phrase, and simplify the expression.

The sum of −3 and 7,
increased by 10

### EXAMPLE 8 Translating Words and Phrases (Addition)

Write a numerical expression for each phrase, and simplify the expression.

**(a)** The sum of −8 and 4 and 6

$$-8 + 4 + 6 \quad \text{simplifies to} \quad -4 + 6, \quad \text{or} \quad 2.$$

Add in order from left to right.

**(b)** 3 more than −5, increased by 12

$$(-5 + 3) + 12 \quad \text{simplifies to} \quad -2 + 12, \quad \text{or} \quad 10.$$

NOW TRY

NOW TRY ANSWERS

7. **(a)** −2 **(b)** 1
8. $(-3 + 7) + 10$; 14

The table lists words and phrases that indicate subtraction in problem solving.

| *Word, Phrase, or Sentence* | *Example* | *Numerical Expression and Simplification* |
|---|---|---|
| Difference between | The *difference between* $-3$ and $-8$ | $-3 - (-8)$ simplifies to $-3 + 8$, or 5 |
| Subtracted from* | 12 *subtracted from* 18 | $18 - 12$, or 6 |
| From . . . , subtract . . . . | From 12, subtract 8. | $12 - 8$ simplifies to $12 + (-8)$, or 4 |
| Less | 6 *less* 5 | $6 - 5$, or 1 |
| Less than* | 6 *less than* 5 | $5 - 6$ simplifies to $5 + (-6)$, or $-1$ |
| Decreased by | 9 *decreased by* $-4$ | $9 - (-4)$ simplifies to $9 + 4$, or 13 |
| Minus | 8 *minus* 5 | $8 - 5$, or 3 |

*Be careful with order when translating.

**CAUTION** When subtracting two numbers, be careful to write them in the correct order, because, in general,

$$x - y \neq y - x.$$

For example, $5 - 3 \neq 3 - 5$. ***Think carefully before interpreting an expression involving subtraction.***

## EXAMPLE 9 Translating Words and Phrases (Subtraction)

Write a numerical expression for each phrase, and simplify the expression.

**(a)** The difference between $-8$ and 5

When "difference between" is used, write the numbers in the order given.*

$$-8 - 5 \quad \text{simplifies to} \quad -8 + (-5), \quad \text{or} \quad -13.$$

**(b)** 4 subtracted from the sum of 8 and $-3$

First, add 8 and $-3$. Next, subtract 4 from this sum.

$$[8 + (-3)] - 4 \quad \text{simplifies to} \quad 5 - 4, \quad \text{or} \quad 1.$$

**(c)** 4 less than $-6$

Here, 4 must be taken *from* $-6$, so write $-6$ first.

Be careful with order.

$$-6 - 4 \quad \text{simplifies to} \quad -6 + (-4), \quad \text{or} \quad -10.$$

Notice that "4 less than $-6$" differs from "4 *is less than* $-6$." The second of these is symbolized $4 < -6$ (which is a false statement).

**(d)** 8, decreased by 5 less than 12

First, write "5 less than 12" as $12 - 5$. Next, subtract $12 - 5$ from 8.

$$8 - (12 - 5) \quad \text{simplifies to} \quad 8 - 7, \quad \text{or} \quad 1.$$

NOW TRY

**NOW TRY EXERCISE 9**

Write a numerical expression for each phrase, and simplify the expression.

**(a)** The difference between 5 and $-8$, decreased by 4

**(b)** 7 less than $-2$

**NOW TRY ANSWERS**

**9. (a)** $[5 - (-8)] - 4$; 9

**(b)** $-2 - 7$; $-9$

---

*In some cases, people interpret "the difference between" (at least for two positive numbers) to represent the larger minus the smaller. However, we will not do so in this book.

**NOW TRY EXERCISE 10**

Find the difference between a gain of 226 yd on the football field by the Chesterfield Bears and a loss of 7 yd by the New London Wildcats.

### EXAMPLE 10 Solving a Problem Involving Subtraction

The record-high temperature in the United States is 134°F, recorded at Death Valley, California, in 1913. The record low is −80°F, at Prospect Creek, Alaska, in 1971. See **FIGURE 16**. What is the difference between these highest and lowest temperatures? (*Source: National Climatic Data Center*.)

We must subtract the lowest temperature from the highest temperature.

Order of numbers matters in subtraction.

$$\begin{aligned} &134 - (-80) & \\ &= 134 + 80 & \text{Definition of subtraction} \\ &= 214 & \text{Add.} \end{aligned}$$

The difference between the two temperatures is 214°F.

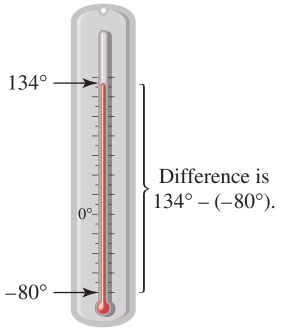


**FIGURE 16**

NOW TRY

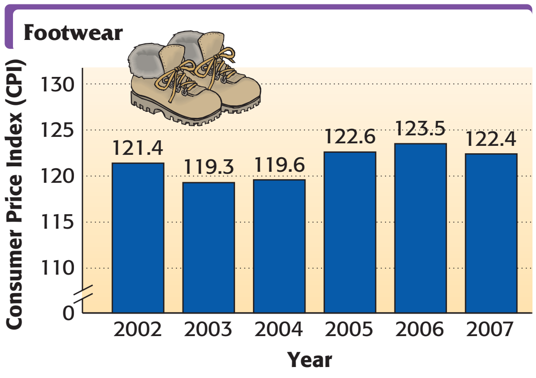
## OBJECTIVE 6 Use signed numbers to interpret data.

**NOW TRY EXERCISE 11**

Refer to **FIGURE 17** and use a signed number to represent the change in the CPI from 2003 to 2004.

### EXAMPLE 11 Using a Signed Number to Interpret Data

The bar graph in **FIGURE 17** gives the Consumer Price Index (CPI) for footwear between 2002 and 2007.



***Source:*** U.S. Bureau of Labor Statistics.

**FIGURE 17**

**(a)** Use a signed number to represent the change in the CPI from 2005 to 2006.

Start with the index number for 2006. Subtract from it the index number for 2005.

$$\underbrace{123.5}_{\text{2006 index}} - \underbrace{122.6}_{\text{2005 index}} = \underbrace{+0.9}_{\text{A positive number indicates an increase.}}$$

**(b)** Use a signed number to represent the change in the CPI from 2006 to 2007.

$$\underbrace{122.4}_{\text{2007 index}} - \underbrace{123.5}_{\text{2006 index}} = 122.4 + (-123.5) = \underbrace{-1.1}_{\text{A negative number indicates a decrease.}}$$

NOW TRY

**NOW TRY ANSWERS**

**10.** 233 yd

**11.** 0.3

## 1.5 EXERCISES

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⊙ *Complete solution available on the Video Resources on DVD*

***Concept Check*** *Fill in each blank with the correct response.*

⊙ **1.** The sum of two negative numbers will always be a ________________ number.
(positive/negative)
Give a number-line illustration using the sum $-2 + (-3)$.

**2.** The sum of a number and its opposite will always be ________.

⊙ **3.** When adding a positive number and a negative number, where the negative number has the greater absolute value, the sum will be a ________________ number. Give
(positive/negative)
a number-line illustration using the sum $-4 + 2$.

**4.** To simplify the expression $8 + [-2 + (-3 + 5)]$, one should begin by adding ________ and ________, according to the rule for order of operations.

**5.** By the definition of subtraction, in order to perform the subtraction $-6 - (-8)$, we must add the opposite of ________ to ________ to get ________.

**6.** “The difference between 7 and 12” translates as ________, while “the difference between 12 and 7” translates as ________.

***Concept Check*** *In Exercises 7–10, suppose that $x$ represents a positive number and $y$ represents a negative number. Determine whether the given expression must represent a positive number or a negative number.*

**7.** $x - y$
**8.** $y - x$
**9.** $y - |x|$
**10.** $x + |y|$

*Find each sum.* ***See Examples 1–7.***

⊙ **11.** $-6 + (-2)$
**12.** $-9 + (-2)$
**13.** $-5 + (-7)$
**14.** $-11 + (-5)$
**15.** $6 + (-4)$
**16.** $11 + (-8)$
**17.** $4 + (-6)$
**18.** $3 + (-7)$
**19.** $-3.5 + 12.4$
**20.** $-12.5 + 21.3$
**21.** $4 + [13 + (-5)]$
**22.** $6 + [2 + (-13)]$
**23.** $8 + [-2 + (-1)]$
**24.** $12 + [-3 + (-4)]$
**25.** $-2 + [5 + (-1)]$
**26.** $-8 + [9 + (-2)]$
**27.** $-6 + [6 + (-9)]$
**28.** $-3 + [3 + (-8)]$
**29.** $[(-9) + (-3)] + 12$
**30.** $[(-8) + (-6)] + 14$
⊙ **31.** $-\frac{1}{6} + \frac{2}{3}$
**32.** $-\frac{6}{25} + \frac{19}{20}$
**33.** $\frac{5}{8} + \left(-\frac{17}{12}\right)$
**34.** $\frac{9}{10} + \left(-\frac{3}{5}\right)$
**35.** $2\frac{1}{2} + \left(-3\frac{1}{4}\right)$
**36.** $-4\frac{3}{8} + 6\frac{1}{2}$
**37.** $-6.1 + [3.2 + (-4.8)]$
**38.** $-9.4 + [-5.8 + (-1.4)]$
**39.** $[-3 + (-4)] + [5 + (-6)]$
**40.** $[-8 + (-3)] + [-7 + (-6)]$
**41.** $[-4 + (-3)] + [8 + (-1)]$
**42.** $[-5 + (-9)] + [16 + (-21)]$
**43.** $[-4 + (-6)] + [(-3) + (-8)] + [12 + (-11)]$
**44.** $[-2 + (-11)] + [12 + (-2)] + [18 + (-6)]$

*Find each difference.* ***See Examples 1–7.***

**45.** $4 - 7$
**46.** $8 - 13$
⊙ **47.** $5 - 9$
**48.** $6 - 11$
**49.** $-7 - 1$
**50.** $-9 - 4$
**51.** $-8 - 6$
**52.** $-9 - 5$
**53.** $7 - (-2)$
**54.** $9 - (-2)$
**55.** $-6 - (-2)$
**56.** $-7 - (-5)$

**57.** $2 - (3 - 5)$
**58.** $-3 - (4 - 11)$
**59.** $\frac{1}{2} - \left(-\frac{1}{4}\right)$
**60.** $\frac{1}{3} - \left(-\frac{4}{3}\right)$
**61.** $-\frac{3}{4} - \frac{5}{8}$
**62.** $-\frac{5}{6} - \frac{1}{2}$
**63.** $\frac{5}{8} - \left(-\frac{1}{2} - \frac{3}{4}\right)$
**64.** $\frac{9}{10} - \left(\frac{1}{8} - \frac{3}{10}\right)$
**65.** $3.4 - (-8.2)$
**66.** $5.7 - (-11.6)$
**67.** $-6.4 - 3.5$
**68.** $-4.4 - 8.6$

*Perform each indicated operation.* ***See Examples 1–7.***

**69.** $(4 - 6) + 12$
**70.** $(3 - 7) + 4$
**71.** $(8 - 1) - 12$
**72.** $(9 - 3) - 15$
**73.** $6 - (-8 + 3)$
**74.** $8 - (-9 + 5)$
**75.** $2 + (-4 - 8)$
**76.** $6 + (-9 - 2)$
**77.** $|-5 - 6| + |9 + 2|$
**78.** $|-4 + 8| + |6 - 1|$
**79.** $|-8 - 2| - |-9 - 3|$
**80.** $|-4 - 2| - |-8 - 1|$
**81.** $\left(-\frac{3}{4} - \frac{5}{2}\right) - \left(-\frac{1}{8} - 1\right)$
**82.** $\left(-\frac{3}{8} - \frac{2}{3}\right) - \left(-\frac{9}{8} - 3\right)$
**83.** $\left(-\frac{1}{2} + 0.25\right) - \left(-\frac{3}{4} + 0.75\right)$
**84.** $\left(-\frac{3}{2} - 0.75\right) - \left(0.5 - \frac{1}{2}\right)$
**85.** $-9 + [(3 - 2) - (-4 + 2)]$
**86.** $-8 - [(-4 - 1) + (9 - 2)]$
**87.** $-3 + [(-5 - 8) - (-6 + 2)]$
**88.** $-4 + [(-12 + 1) - (-1 - 9)]$
**89.** $-9.1237 + [(-4.8099 - 3.2516) + 11.27903]$
**90.** $-7.6247 - [(-3.9928 + 1.42773) - (-2.80981)]$

*Write a numerical expression for each phrase and simplify.* ***See Examples 8 and 9.***

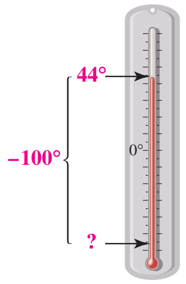
**91.** The sum of $-5$ and 12 and 6
**92.** The sum of $-3$ and 5 and $-12$
**93.** 14 added to the sum of $-19$ and $-4$
**94.** $-2$ added to the sum of $-18$ and 11
**95.** The sum of $-4$ and $-10$, increased by 12
**96.** The sum of $-7$ and $-13$, increased by 14
**97.** $\frac{2}{7}$ more than the sum of $\frac{5}{7}$ and $-\frac{9}{7}$
**98.** 1.85 more than the sum of $-1.25$ and $-4.75$
**99.** The difference between 4 and $-8$
**100.** The difference between 7 and $-14$
**101.** 8 less than $-2$
**102.** 9 less than $-13$
**103.** The sum of 9 and $-4$, decreased by 7
**104.** The sum of 12 and $-7$, decreased by 14
**105.** 12 less than the difference between 8 and $-5$
**106.** 19 less than the difference between 9 and $-2$

*Solve each problem.* ***See Example 10.***

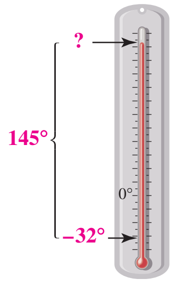
**107.** Based on 2020 population projections, New York will lose 5 seats in the U.S. House of Representatives, Pennsylvania will lose 4 seats, and Ohio will lose 3. Write a signed number that represents the total number of seats these three states are projected to lose. (*Source:* Population Reference Bureau.)

**108.** Michigan is projected to lose 3 seats in the U.S. House of Representatives and Illinois 2 in 2020. The states projected to gain the most seats are California with 9, Texas with 5, Florida with 3, Georgia with 2, and Arizona with 2. Write a signed number that represents the algebraic sum of these changes. (*Source:* Population Reference Bureau.)

**109.** The largest change in temperature ever recorded within a 24-hr period occurred in Montana, on January 23–24, 1916. The temperature fell 100°F from a starting temperature of 44°F. What was the low temperature during this period? (*Source: Guinness World Records.*)



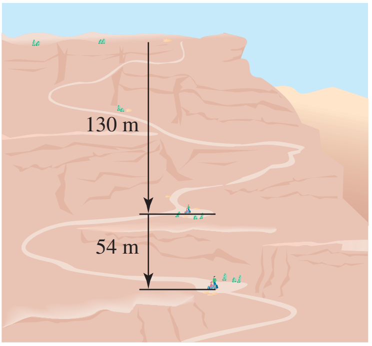
**110.** The lowest temperature ever recorded in Tennessee was −32°F. The highest temperature ever recorded there was 145°F more than the lowest. What was this highest temperature? (*Source:* National Climatic Data Center.)



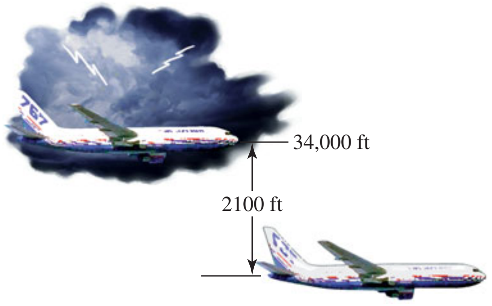
**111.** The lowest temperature ever recorded in Illinois was −36°F on January 5, 1999. The lowest temperature ever recorded in Utah was on February 1, 1985, and was 33°F lower than Illinois's record low. What is the record low temperature for Utah? (*Source:* National Climatic Data Center.)

**112.** The top of Mt. Whitney, visible from Death Valley, has an altitude of 14,494 ft above sea level. The bottom of Death Valley is 282 ft below sea level. Using 0 as sea level, find the difference between these two elevations. (*Source: World Almanac and Book of Facts.*)

**113.** The surface, or rim, of a canyon is at altitude 0. On a hike down into the canyon, a party of hikers stops for a rest at 130 m below the surface. The hikers then descend another 54 m. Write the new altitude as a signed number.



**114.** A pilot announces to the passengers that the current altitude of their plane is 34,000 ft. Because of turbulence, the pilot is forced to descend 2100 ft. Write the new altitude as a signed number.



**115.** In 2005, Americans saved −0.5% of their after-tax incomes. In May 2009, they saved 6.9%. (*Source:* U.S. Bureau of Economic Analysis.)

**(a)** Find the difference between the 2009 and the 2005 amounts.

**(b)** How could Americans have a negative personal savings rate in 2005?

**116.** In 2000, the U.S. federal budget had a surplus of $236 billion. In 2008, the federal budget had a deficit of $455 billion. Find the difference between the 2008 and the 2000 amounts. (*Source:* U.S. Treasury Department.)

**117.** In 1998, undergraduate college students had an average (mean) credit card balance of \$1879. The average balance increased \$869 by 2000, then dropped \$579 by 2004, and then increased \$1004 by 2008. What was the average credit card balance of undergraduate college students in 2008? (*Source:* Sallie Mae.)

**118.** Among entertainment expenditures, the average annual spending per U.S. household on fees and admissions was \$526 in 2001. This amount decreased \$32 by 2003 and then increased \$112 by 2006. What was the average household expenditure for fees and admissions in 2006? (*Source:* U.S. Bureau of Labor Statistics.)

**119.** Nadine Blackwood enjoys playing Triominoes every Wednesday night. Last Wednesday, on four successive turns, her scores were $-19$, 28, $-5$, and 13. What was her final score for the four turns?

**120.** Bruce Buit also enjoys playing Triominoes. On five successive turns, his scores were $-13$, 15, $-12$, 24, and 14. What was his total score for the five turns?

**121.** In August, Susan Goodman began with a checking account balance of \$904.89. Her checks and deposits for August are as follows:

| *Checks* | *Deposits* |
|---|---|
| \$35.84 | \$85.00 |
| \$26.14 | \$120.76 |
| \$3.12 | |

Assuming no other transactions, what was her account balance at the end of August?

**122.** In September, Jeffery Cooper began with a checking account balance of \$904.89. His checks and deposits for September are as follows:

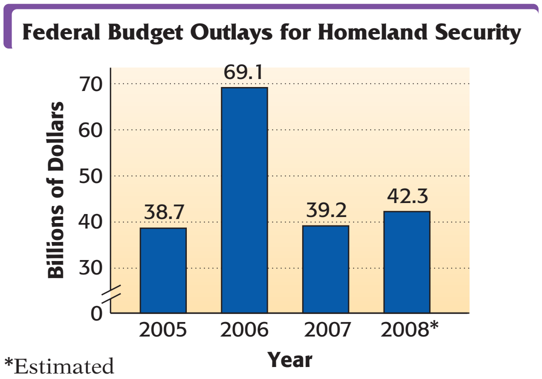
| *Checks* | *Deposits* |
|---|---|
| \$41.29 | \$80.59 |
| \$13.66 | \$276.13 |
| \$84.40 | |

Assuming no other transactions, what was his account balance at the end of September?

**123.** Linda Des Jardines owes \$870.00 on her MasterCard account. She returns two items costing \$35.90 and \$150.00 and receives credit for these on the account. Next, she makes a purchase of \$82.50 and then two more purchases of \$10.00 each. She makes a payment of \$500.00. She then incurs a finance charge of \$37.23. How much does she still owe?

**124.** Marcial Echenique owes \$679.00 on his Visa account. He returns three items costing \$36.89, \$29.40, and \$113.55 and receives credit for these on the account. Next, he makes purchases of \$135.78 and \$412.88 and two purchases of \$20.00 each. He makes a payment of \$400. He then incurs a finance charge of \$24.57. How much does he still owe?

*The bar graph shows federal budget outlays for the U.S. Department of Homeland Security for the years 2005 through 2008. In Exercises 125–128, use a signed number to represent the change in outlay for each period.* ***See Example 11.***



*Source:* U.S. Department of Management and Budget.

**125.** 2005 to 2006

**126.** 2006 to 2007

**127.** 2007 to 2008

**128.** 2005 to 2008

*The two tables show the heights of some selected mountains and the depths of some selected trenches. Use the information given to answer Exercises 129–134.*

| *Mountain* | *Height (in feet)* |
|---|---|
| Foraker | 17,400 |
| Wilson | 14,246 |
| Pikes Peak | 14,110 |

| *Trench* | *Depth (in feet, as a negative number)* |
|---|---|
| Philippine | −32,995 |
| Cayman | −24,721 |
| Java | −23,376 |

*Source: World Almanac and Book of Facts.*

**129.** What is the difference between the height of Mt. Foraker and the depth of the Philippine Trench?

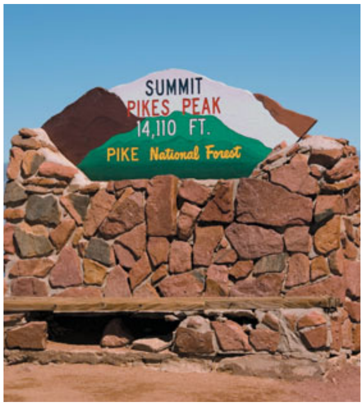
**130.** What is the difference between the height of Pikes Peak and the depth of the Java Trench?

**131.** How much deeper is the Cayman Trench than the Java Trench?

**132.** How much deeper is the Philippine Trench than the Cayman Trench?

**133.** How much higher is Mt. Wilson than Pikes Peak?

**134.** If Mt. Wilson and Pikes Peak were stacked one on top of the other, how much higher would they be than Mt. Foraker?



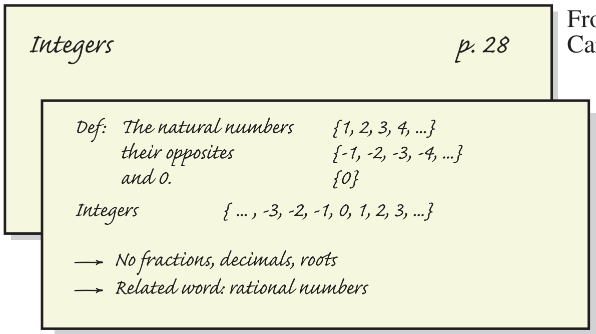
## STUDY SKILLS

## Using Study Cards

You may have used "flash cards" in other classes. In math, "study cards" can help you remember terms and definitions, procedures, and concepts. Use study cards to

- Quickly review when you have a few minutes;
- Review before a quiz or test.

***One of the advantages of study cards is that you learn while you are making them.***



Front of Card

*Integers* *p. 28*

Back of Card

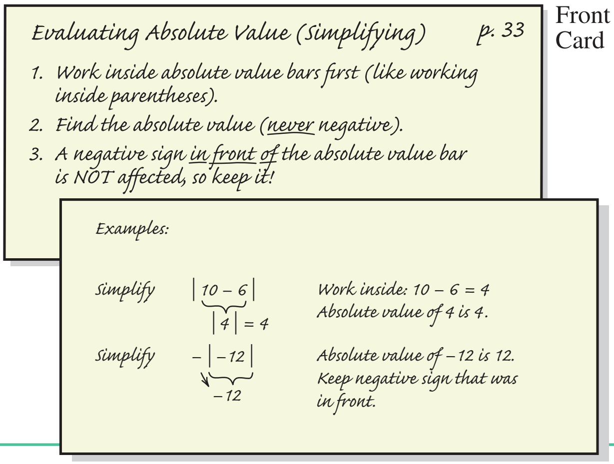
*Def: The natural numbers {1, 2, 3, 4, ...}*
*their opposites {-1, -2, -3, -4, ...}*
*and 0. {0}*

*Integers { ... , -3, -2, -1, 0, 1, 2, 3, ...}*

*→ No fractions, decimals, roots*
*→ Related word: rational numbers*

### Vocabulary Cards

Put the word and a page reference on the front of the card. On the back, write the definition, an example, any related words, and a sample problem (if appropriate).



Front of Card

*Evaluating Absolute Value (Simplifying)* *p. 33*

1. *Work inside absolute value bars first (like working inside parentheses).*
2. *Find the absolute value (never negative).*
3. *A negative sign in front of the absolute value bar is NOT affected, so keep it!*

Back of Card

*Examples:*

*Simplify* $\underbrace{|10-6|}_{|4| = 4}$ *Work inside: 10 − 6 = 4*
*Absolute value of 4 is 4.*

*Simplify* $\underbrace{-|-12|}_{-12}$ *Absolute value of −12 is 12.*
*Keep negative sign that was in front.*

### Procedure ("Steps") Cards

Write the name of the procedure on the front of the card. Then write each step in words. On the back of the card, put an example showing each step.

***Make a vocabulary card and a procedure card for material you are learning now.***

# 1.6 Multiplying and Dividing Real Numbers

**OBJECTIVES**

1. Find the product of a positive number and a negative number.
2. Find the product of two negative numbers.
3. Identify factors of integers.
4. Use the reciprocal of a number to apply the definition of division.
5. Use the rules for order of operations when multiplying and dividing signed numbers.
6. Evaluate expressions involving variables.
7. Translate words and phrases involving multiplication and division.
8. Translate simple sentences into equations.

The result of multiplication is called the **product.** We know that the product of two positive numbers is positive. We also know that the product of 0 and any positive number is 0, so we extend that property to all real numbers.

**Multiplication by Zero**

For any real number $x$, $\quad \boldsymbol{x \cdot 0 = 0.}$

**OBJECTIVE 1 Find the product of a positive number and a negative number.** Look at the following pattern.

$$\begin{aligned} 3 \cdot 5 &= 15 \\ 3 \cdot 4 &= 12 \\ 3 \cdot 3 &= 9 \\ 3 \cdot 2 &= 6 \\ 3 \cdot 1 &= 3 \\ 3 \cdot 0 &= 0 \\ 3 \cdot (-1) &= ? \end{aligned}$$

The products decrease by 3.

What should $3(-1)$ equal? The product $3(-1)$ represents the sum

$$-1 + (-1) + (-1) = -3,$$

so the product should be $-3$. Also,

$$3(-2) = -2 + (-2) + (-2) = -6$$

and

$$3(-3) = -3 + (-3) + (-3) = -9.$$

These results maintain the pattern in the list, which suggests the following rule.

**Multiplying Numbers with Different Signs**

For any positive real numbers $x$ and $y$,

$$\boldsymbol{x(-y) = -(xy)} \quad \text{and} \quad \boldsymbol{(-x)y = -(xy).}$$

That is, the product of two numbers with opposite signs is negative.

*Examples:* $6(-3) = -18$ and $(-6)3 = -18$

**EXAMPLE 1** Multiplying a Positive Number and a Negative Number

Find each product, using the multiplication rule given in the box.

**(a)** $8(-5) = -(8 \cdot 5) = -40$

**(b)** $(-5)4 = -(5 \cdot 4) = -20$

**(c)** $-9\left(\frac{1}{3}\right) = -\left(9 \cdot \frac{1}{3}\right) = -3$

**(d)** $6.2(-4.1) = -(6.2 \cdot 4.1) = -25.42$

NOW TRY

*NOW TRY EXERCISE 1*

Find each product.

**(a)** $-11(9)$ **(b)** $3.1(-2.5)$

*NOW TRY ANSWERS*

**1. (a)** $-99$ **(b)** $-7.75$

**OBJECTIVE 2** **Find the product of two negative numbers.** Look at another pattern.

$$\begin{aligned} -5(4) &= -20 \\ -5(3) &= -15 \\ -5(2) &= -10 \\ -5(1) &= -5 \\ -5(0) &= 0 \\ -5(-1) &= ? \end{aligned}$$

The products increase by 5.

The numbers in color on the left of the equals symbol decrease by 1 for each step down the list. The products on the right increase by 5 for each step down the list. To maintain this pattern, $-5(-1)$ should be 5 more than $-5(0)$, or 5 more than 0, so

$$-5(-1) = 5.$$

The pattern continues with

$$\begin{aligned} -5(-2) &= 10 \\ -5(-3) &= 15 \\ -5(-4) &= 20 \\ -5(-5) &= 25, \end{aligned}$$

and so on, which suggests the next rule.

### Multiplying Two Negative Numbers

For any positive real numbers $x$ and $y$,

$$-x(-y) = xy.$$

That is, the product of two negative numbers is positive.

*Example:* $-5(-4) = 20$

**NOW TRY EXERCISE 2**
Find the product.

$$-\frac{1}{7}\left(-\frac{5}{2}\right)$$

### EXAMPLE 2 Multiplying Two Negative Numbers

Find each product, using the multiplication rule given in the box.

**(a)** $-9(-2) = 9 \cdot 2 = 18$

**(b)** $-6(-12) = 6 \cdot 12 = 72$

**(c)** $-8(-1) = 8 \cdot 1 = 8$

**(d)** $-\frac{2}{3}\left(-\frac{3}{2}\right) = \frac{2}{3} \cdot \frac{3}{2} = 1$

NOW TRY

The following box summarizes multiplying signed numbers.

### Multiplying Signed Numbers

The product of two numbers having the *same* sign is *positive.*

The product of two numbers having *different* signs is *negative.*

**OBJECTIVE 3** **Identify factors of integers.** The definition of **factor** from **Section 1.1** can be extended to integers. If the product of two integers is a third integer, then each of the two integers is a *factor* of the third. The table on the next page shows examples.

NOW TRY ANSWER

2. $\frac{5}{14}$

| *Integer* | 18 | 20 | 15 | 7 | 1 |
|---|---|---|---|---|---|
| *Pairs of factors* | 1, 18<br>2, 9<br>3, 6<br>−1, −18<br>−2, −9<br>−3, −6 | 1, 20<br>2, 10<br>4, 5<br>−1, −20<br>−2, −10<br>−4, −5 | 1, 15<br>3, 5<br>−1, −15<br>−3, −5 | 1, 7<br>−1, −7 | 1, 1<br>−1, −1 |

**OBJECTIVE 4** **Use the reciprocal of a number to apply the definition of division.** Recall that the result of division is called the **quotient.** The quotient of two numbers is found by multiplying by the *reciprocal,* or *multiplicative inverse,* of the second number.

| *Number* | *Multiplicative Inverse (Reciprocal)* |
|---|---|
| 4 | $\frac{1}{4}$ |
| 0.3, or $\frac{3}{10}$ | $\frac{10}{3}$ |
| −5 | $\frac{1}{-5}$, or $-\frac{1}{5}$ |
| $-\frac{5}{8}$ | $-\frac{8}{5}$ |

***A number and its multiplicative inverse have a product of* 1.** For example,

$$4 \cdot \frac{1}{4} = \frac{4}{4} = 1.$$

### Reciprocal or Multiplicative Inverse

Pairs of numbers whose product is 1 are called **reciprocals,** or **multiplicative inverses,** of each other.

The table in the margin shows several numbers and their multiplicative inverses.

### Definition of Division

For any real numbers $x$ and $y$, with $y \neq 0$, $\quad \boldsymbol{\frac{x}{y} = x \cdot \frac{1}{y}}.$

That is, to divide two numbers, multiply the first by the reciprocal, or multiplicative inverse, of the second.

*Example:* $\quad \frac{-8}{4} = -8 \cdot \frac{1}{4} = 2$

**NOTE** Recall that an equivalent form of $\frac{x}{y}$ is $x \div y$, where $x$ is called the **dividend** and $y$ is called the **divisor.** For example, $\frac{-8}{4} = -8 \div 4$.

Since division is defined in terms of multiplication, all the rules for multiplying signed numbers also apply to dividing them.

NOW TRY EXERCISE 3

Find each quotient, using the definition of division.

**(a)** $\frac{15}{-3}$ **(b)** $\frac{9.81}{-0.9}$

**(c)** $-\frac{5}{6} \div \frac{17}{9}$

## EXAMPLE 3 Using the Definition of Division

Find each quotient, using the definition of division.

**(a)** $\frac{12}{3} = 12 \cdot \frac{1}{3} = 4 \quad \frac{x}{y} = x \cdot \frac{1}{y}$

**(b)** $\frac{-10}{2} = -10 \cdot \frac{1}{2} = -5$

**(c)** $\frac{-1.47}{-7} = -1.47\left(-\frac{1}{7}\right) = 0.21$

**(d)** $-\frac{2}{3} \div \left(-\frac{4}{5}\right) = -\frac{2}{3} \cdot \left(-\frac{5}{4}\right) = \frac{5}{6}$

Remember to write in lowest terms.

NOW TRY

NOW TRY ANSWERS

3. **(a)** −5 **(b)** −10.9 **(c)** $-\frac{15}{34}$

We can use multiplication to check a division problem. Consider **Example 3(a).**

$$\frac{12}{3} = 4, \quad \text{since} \quad 4 \cdot 3 = 12.$$

Multiply to check a division problem.

This relationship between multiplication and division allows us to investigate division by 0. Consider the quotient $\frac{0}{3}$.

$$\frac{0}{3} = 0, \quad \text{since} \quad 0 \cdot 3 = 0.$$

Now consider $\frac{3}{0}$.

$$\frac{3}{0} = ?$$

We need to find a number that when multiplied by 0 will equal 3, that is, $? \cdot 0 = 3$. *No* real number satisfies this equation, since the product of any real number and 0 must be 0. Thus,

> **$\frac{x}{0}$ is not a number, and *division by 0 is undefined*. If a division problem involves division by 0, write "undefined."**

### Division Involving 0

For any real number $x$, with $x \neq 0$,

$$\frac{0}{x} = 0 \quad \text{and} \quad \frac{x}{0} \text{ is undefined.}$$

*Examples:* $\frac{0}{-10} = 0$ and $\frac{-10}{0}$ is undefined.

When dividing fractions, multiplying by the reciprocal works well. However, using the definition of division directly with integers may be awkward. It is easier to divide in the usual way and then determine the sign of the answer.

### Dividing Signed Numbers

The quotient of two numbers having the *same* sign is *positive.*

The quotient of two numbers having *different* signs is *negative.*

*Examples:* $\frac{15}{5} = 3$, $\frac{-15}{-5} = 3$, $\frac{15}{-5} = -3$, and $\frac{-15}{5} = -3$

**NOW TRY EXERCISE 4**

Find each quotient.

**(a)** $\frac{-10}{5}$ **(b)** $\frac{-1.44}{-0.12}$

**(c)** $-\frac{3}{8} \div \frac{7}{10}$

### EXAMPLE 4 Dividing Signed Numbers

Find each quotient.

**(a)** $\frac{8}{-2} = -4$ **(b)** $\frac{-100}{5} = -20$ **(c)** $\frac{-4.5}{-0.09} = 50$

**(d)** $-\frac{1}{8} \div \left(-\frac{3}{4}\right) = -\frac{1}{8} \cdot \left(-\frac{4}{3}\right) = \frac{1}{6}$

Remember to write in lowest terms.

NOW TRY

**NOW TRY ANSWERS**

4. **(a)** $-2$ **(b)** $12$ **(c)** $-\frac{15}{28}$

From the definitions of multiplication and division of real numbers,

$$\frac{-40}{8} = -40 \cdot \frac{1}{8} = -5 \quad \text{and} \quad \frac{40}{-8} = 40\left(\frac{1}{-8}\right) = -5, \quad \text{so} \quad \frac{-40}{8} = \frac{40}{-8}.$$

Based on this example, the quotient of a positive and a negative number can be expressed in any of the following three forms.

**Equivalent Forms**

For any positive real numbers $x$ and $y$, $\quad \dfrac{-x}{y} = \dfrac{x}{-y} = -\dfrac{x}{y}.$

Similarly, the quotient of two negative numbers can be expressed as a quotient of two positive numbers.

**Equivalent Forms**

For any positive real numbers $x$ and $y$, $\quad \dfrac{-x}{-y} = \dfrac{x}{y}.$

## OBJECTIVE 5 Use the rules for order of operations when multiplying and dividing signed numbers.

*NOW TRY EXERCISE 5*

Perform each indicated operation.

**(a)** $-4(6) - (-5)(5)$

**(b)** $\dfrac{12(-4) - 6(-3)}{-4(7 - 16)}$

### EXAMPLE 5 Using the Rules for Order of Operations

Perform each indicated operation.

**(a)** $-9(2) - (-3)(2)$

$= -18 - (-6)$ Multiply.

$= -18 + 6$ Definition of subtraction

$= -12$ Add.

**(b)** $-5(-2 - 3)$

$= -5(-5)$ Work inside the parentheses.

$= 25$ Multiply.

**(c)** $-6 + 2(3 - 5)$ Begin inside the parentheses.

Do *not* add first.

$= -6 + 2(-2)$ Subtract inside the parentheses.

$= -6 + (-4)$ Multiply.

$= -10$ Add.

**(d)** $\dfrac{5(-2) - 3(4)}{2(1 - 6)}$

$= \dfrac{-10 - 12}{2(-5)}$ Simplify the numerator and denominator separately.

$= \dfrac{-22}{-10}$, or $\dfrac{11}{5}$ Subtract in the numerator. Multiply in the denominator. Write in lowest terms.

*NOW TRY*

*NOW TRY ANSWERS*

5. **(a)** 1 **(b)** $-\frac{5}{6}$

## OBJECTIVE 6 Evaluate expressions involving variables.

**NOW TRY EXERCISE 6**

Evaluate $\frac{3x^2 - 12}{y}$ for $x = -4$ and $y = -3$.

### EXAMPLE 6 Evaluating Expressions for Numerical Values

Evaluate each expression for $x = -1$, $y = -2$, and $m = -3$.

**(a)** $(3x + 4y)(-2m)$

Use parentheses around substituted negative values to avoid errors.

$$\begin{aligned} &= [3(-1) + 4(-2)][-2(-3)] && \text{Substitute the given values for the variables.} \\ &= [-3 + (-8)][6] && \text{Multiply.} \\ &= [-11]6 && \text{Add inside the brackets.} \\ &= -66 && \text{Multiply.} \end{aligned}$$

**(b)** $2x^2 - 3y^2$

Think: $(-2)^2 = -2(-2) = 4$

Think: $(-1)^2 = -1(-1) = 1$

$$\begin{aligned} &= 2(-1)^2 - 3(-2)^2 && \text{Substitute.} \\ &= 2(1) - 3(4) && \text{Apply the exponents.} \\ &= 2 - 12 && \text{Multiply.} \\ &= -10 && \text{Subtract.} \end{aligned}$$

**(c)** $\frac{4y^2 + x}{m}$

$$\begin{aligned} &= \frac{4(-2)^2 + (-1)}{-3} && \text{Substitute.} \\ &= \frac{4(4) + (-1)}{-3} && \text{Apply the exponent.} \\ &= \frac{16 + (-1)}{-3} && \text{Multiply.} \\ &= \frac{15}{-3}, \quad \text{or} \quad -5 && \text{Add, and then divide.} \end{aligned}$$

NOW TRY

## OBJECTIVE 7 Translate words and phrases involving multiplication and division.

The table gives words and phrases that indicate multiplication.

| *Word or Phrase* | *Example* | *Numerical Expression and Simplification* |
|---|---|---|
| Product of | The *product of* $-5$ and $-2$ | $-5(-2)$, or $10$ |
| Times | 13 *times* $-4$ | $13(-4)$, or $-52$ |
| Twice (meaning "2 times") | *Twice* 6 | $2(6)$, or $12$ |
| Of (used with fractions) | $\frac{1}{2}$ *of* 10 | $\frac{1}{2}(10)$, or $5$ |
| Percent of | 12*% of* $-16$ | $0.12(-16)$, or $-1.92$ |
| As much as | $\frac{2}{3}$ *as much as* 30 | $\frac{2}{3}(30)$, or $20$ |

### EXAMPLE 7 Translating Words and Phrases (Multiplication)

Write a numerical expression for each phrase, and simplify the expression.

**(a)** The product of 12 and the sum of 3 and $-6$

$$12[3 + (-6)] \quad \text{simplifies to} \quad 12[-3], \quad \text{or} \quad -36.$$

**(b)** Twice the difference between 8 and $-4$

$$2[8 - (-4)] \quad \text{simplifies to} \quad 2[12], \quad \text{or} \quad 24.$$

**NOW TRY ANSWER**

**6.** $-12$

**NOW TRY EXERCISE 7**

Write a numerical expression for each phrase, and simplify the expression.

**(a)** Twice the sum of $-10$ and 7

**(b)** 40% of the difference between 45 and 15

**(c)** Two-thirds of the sum of $-5$ and $-3$

$$\frac{2}{3}[-5+(-3)] \quad \text{simplifies to} \quad \frac{2}{3}[-8], \quad \text{or} \quad -\frac{16}{3}.$$

**(d)** 15% of the difference between 14 and $-2$

Remember that 15% = 0.15.

$$0.15[14-(-2)] \quad \text{simplifies to} \quad 0.15[16], \quad \text{or} \quad 2.4.$$

**(e)** Double the product of 3 and 4

$$2\cdot(3\cdot 4) \quad \text{simplifies to} \quad 2(12), \quad \text{or} \quad 24.$$

NOW TRY

In algebra, quotients are usually represented with a fraction bar. The symbol ÷ is seldom used. The table gives some phrases associated with division.

| *Phrase* | *Example* | *Numerical Expression and Simplification* |
|---|---|---|
| Quotient of | The *quotient of* $-24$ and 3 | $\frac{-24}{3}$, or $-8$ |
| Divided by | $-16$ *divided by* $-4$ | $\frac{-16}{-4}$, or 4 |
| Ratio of | The *ratio of* 2 to 3 | $\frac{2}{3}$ |

***When translating a phrase involving division, we write the first number named as the numerator and the second as the denominator.***

**NOW TRY EXERCISE 8**

Write a numerical expression for the phrase, and simplify the expression.

The quotient of 21 and the sum of 10 and $-7$

### EXAMPLE 8 Interpreting Words and Phrases Involving Division

Write a numerical expression for each phrase, and simplify the expression.

**(a)** The quotient of 14 and the sum of $-9$ and 2

"Quotient" indicates division.

$$\frac{14}{-9+2} \quad \text{simplifies to} \quad \frac{14}{-7}, \quad \text{or} \quad -2.$$

**(b)** The product of 5 and $-6$, divided by the difference between $-7$ and 8

$$\frac{5(-6)}{-7-8} \quad \text{simplifies to} \quad \frac{-30}{-15}, \quad \text{or} \quad 2.$$

NOW TRY

## OBJECTIVE 8 Translate simple sentences into equations.

### EXAMPLE 9 Translating Sentences into Equations

Write each sentence as an equation, using $x$ as the variable. Then find the solution from the list of integers between $-12$ and 12, inclusive.

**(a)** Three times a number is $-18$.

The word *times* indicates multiplication.

The word *is* translates as =.

$$\underset{\uparrow}{3\cdot x} = -18, \quad \text{or} \quad 3x=-18 \qquad 3\cdot x = 3x$$

The integer between $-12$ and 12, inclusive, that makes this statement true is $-6$, since $3(-6) = -18$. The solution of the equation is $-6$.

**NOW TRY ANSWERS**

**7.** **(a)** $2(-10+7)$; $-6$
**(b)** $0.40(45-15)$; 12

**8.** $\frac{21}{10+(-7)}$; 7

**NOW TRY EXERCISE 9**

Write each sentence as an equation, using $x$ as the variable. Then find the solution from the list of integers between $-12$ and 12, inclusive.

**(a)** The sum of a number and $-4$ is 7.

**(b)** The difference between $-8$ and a number is $-11$.

**(b)** The sum of a number and 9 is 12.

$$x + 9 = 12$$

Since $3 + 9 = 12$, the solution of this equation is 3.

**(c)** The difference between a number and 5 is 0.

$$x - 5 = 0$$

Since $5 - 5 = 0$, the solution of this equation is 5.

**(d)** The quotient of 24 and a number is $-2$.

$$\frac{24}{x} = -2$$

Here, $x$ must be a negative number, since the numerator is positive and the quotient is negative. Since $\frac{24}{-12} = -2$, the solution is $-12$. NOW TRY

**CAUTION** In **Examples 7 and 8,** the *phrases* translate as *expressions,* while in **Example 9,** the *sentences* translate as *equations.* ***An expression is a phrase. An equation is a sentence with something on the left side, an = symbol, and something on the right side.***

$$\frac{5(-6)}{-7 \underset{\substack{\uparrow \\ \text{Expression}}}{-} 8} \qquad 3x \underset{\substack{\uparrow \\ \text{Equation}}}{=} -18$$

**NOW TRY ANSWERS**

**9.** **(a)** $x + (-4) = 7; 11$
**(b)** $-8 - x = -11; 3$

## 1.6 EXERCISES

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◉ *Complete solution available on the Video Resources on DVD*

*Concept Check* *Fill in each blank with one of the following:* greater than 0, less than 0, equal to 0.

**1.** The product or the quotient of two numbers with the same sign is ______________.

**2.** The product or the quotient of two numbers with different signs is ______________.

**3.** If three negative numbers are multiplied, the product is ______________.

**4.** If two negative numbers are multiplied and then their product is divided by a negative number, the result is ______________.

**5.** If a negative number is squared and the result is added to a positive number, the result is ______________.

**6.** The reciprocal of a negative number is ______________.

**7.** If three positive numbers, five negative numbers, and zero are multiplied, the product is ______________.

**8.** The cube of a negative number is ______________.

◉ **9.** *Concept Check* Complete this statement: The quotient formed by any nonzero number divided by 0 is ______________, and the quotient formed by 0 divided by any nonzero number is ______________. Give an example of each quotient.

**10.** *Concept Check* Which expression is undefined?

**A.** $\frac{4+4}{4+4}$ **B.** $\frac{4-4}{4+4}$ **C.** $\frac{4-4}{4-4}$ **D.** $\frac{4-4}{4}$

*Find each product.* ***See Examples 1 and 2.***

**11.** $5(-6)$
**12.** $-3(4)$
**13.** $-5(-6)$
**14.** $-3(-4)$
**15.** $-10(-12)$
**16.** $-9(-5)$
**17.** $3(-11)$
**18.** $3(-15)$
**19.** $-0.5(0)$
**20.** $-0.3(0)$
**21.** $-6.8(0.35)$
**22.** $-4.6(0.24)$
**23.** $-\frac{3}{8} \cdot \left(-\frac{10}{9}\right)$
**24.** $-\frac{5}{4} \cdot \left(-\frac{5}{8}\right)$
**25.** $\frac{2}{15}\left(-1\frac{1}{4}\right)$
**26.** $\frac{3}{7}\left(-1\frac{5}{9}\right)$
**27.** $-8\left(-\frac{3}{4}\right)$
**28.** $-6\left(-\frac{5}{3}\right)$

*Find all integer factors of each number.* ***See Objective 3.***

**29.** 32
**30.** 36
**31.** 40
**32.** 50
**33.** 31
**34.** 17

*Find each quotient.* ***See Examples 3 and 4.***

**35.** $\frac{15}{5}$
**36.** $\frac{35}{5}$
**37.** $\frac{-42}{6}$
**38.** $\frac{-28}{7}$
**39.** $\frac{-32}{-4}$
**40.** $\frac{-35}{-5}$
**41.** $\frac{96}{-16}$
**42.** $\frac{38}{-19}$
**43.** $-\frac{4}{3} \div \left(-\frac{1}{8}\right)$
**44.** $-\frac{6}{5} \div \left(-\frac{1}{3}\right)$
**45.** $\frac{-8.8}{2.2}$
**46.** $\frac{-4.6}{0.23}$
**47.** $\frac{0}{-5}$
**48.** $\frac{0}{-9}$
**49.** $\frac{11.5}{0}$
**50.** $\frac{15.2}{0}$

*Perform each indicated operation.* ***See Example 5.***

**51.** $7 - 3 \cdot 6$
**52.** $8 - 2 \cdot 5$
**53.** $-10 - (-4)(2)$
**54.** $-11 - (-3)(6)$
**55.** $-7(3 - 8)$
**56.** $-5(4 - 7)$
**57.** $7 + 2(4 - 1)$
**58.** $5 + 3(6 - 4)$
**59.** $-4 + 3(2 - 8)$
**60.** $-8 + 4(5 - 7)$
**61.** $(12 - 14)(1 - 4)$
**62.** $(8 - 9)(4 - 12)$
**63.** $(7 - 10)(10 - 4)$
**64.** $(5 - 12)(19 - 4)$
**65.** $(-2 - 8)(-6) + 7$
**66.** $(-9 - 4)(-2) + 10$
**67.** $3(-5) + |3 - 10|$
**68.** $4(-8) + |4 - 15|$
**69.** $\frac{-5(-6)}{9 - (-1)}$
**70.** $\frac{-12(-5)}{7 - (-5)}$
**71.** $\frac{-21(3)}{-3 - 6}$
**72.** $\frac{-40(3)}{-2 - 3}$
**73.** $\frac{-10(2) + 6(2)}{-3 - (-1)}$
**74.** $\frac{-12(4) + 5(3)}{-14 - (-3)}$
**75.** $\frac{3^2 - 4^2}{7(-8 + 9)}$
**76.** $\frac{5^2 - 7^2}{2(3 + 3)}$
**77.** $\frac{8(-1) - |(-4)(-3)|}{-6 - (-1)}$
**78.** $\frac{-27(-2) - |6 \cdot 4|}{-2(3) - 2(2)}$
**79.** $\frac{-13(-4) - (-8)(-2)}{(-10)(2) - 4(-2)}$
**80.** $\frac{-5(2) + [3(-2) - 4]}{-3 - (-1)}$

*Evaluate each expression for $x = 6$, $y = -4$, and $a = 3$.* ***See Example 6.***

**81.** $5x - 2y + 3a$
**82.** $6x - 5y + 4a$
**83.** $(2x + y)(3a)$
**84.** $(5x - 2y)(-2a)$
**85.** $\left(\frac{1}{3}x - \frac{4}{5}y\right)\left(-\frac{1}{5}a\right)$
**86.** $\left(\frac{5}{6}x + \frac{3}{2}y\right)\left(-\frac{1}{3}a\right)$
**87.** $(-5 + x)(-3 + y)(3 - a)$
**88.** $(6 - x)(5 + y)(3 + a)$
**89.** $-2y^2 + 3a$
**90.** $5x - 4a^2$
**91.** $\frac{2y^2 - x}{a + 10}$
**92.** $\frac{xy + 8a}{x - y}$

*Write a numerical expression for each phrase and simplify.* ***See Examples 7 and 8.***

**93.** The product of $-9$ and 2, added to 9

**94.** The product of 4 and $-7$, added to $-12$

**95.** Twice the product of $-1$ and 6, subtracted from $-4$

**96.** Twice the product of $-8$ and 2, subtracted from $-1$

**97.** Nine subtracted from the product of 1.5 and $-3.2$

**98.** Three subtracted from the product of 4.2 and $-8.5$

**99.** The product of 12 and the difference between 9 and $-8$

**100.** The product of $-3$ and the difference between 3 and $-7$

**101.** The quotient of $-12$ and the sum of $-5$ and $-1$

**102.** The quotient of $-20$ and the sum of $-8$ and $-2$

**103.** The sum of 15 and $-3$, divided by the product of 4 and $-3$

**104.** The sum of $-18$ and $-6$, divided by the product of 2 and $-4$

**105.** Two-thirds of the difference between 8 and $-1$

**106.** Three-fourths of the sum of $-8$ and 12

**107.** 20% of the product of $-5$ and 6

**108.** 30% of the product of $-8$ and 5

**109.** The sum of $\frac{1}{2}$ and $\frac{5}{8}$, times the difference between $\frac{3}{5}$ and $\frac{1}{3}$

**110.** The sum of $\frac{3}{4}$ and $\frac{1}{2}$, times the difference between $\frac{2}{3}$ and $\frac{1}{6}$

**111.** The product of $-\frac{1}{2}$ and $\frac{3}{4}$, divided by $-\frac{2}{3}$

**112.** The product of $-\frac{2}{3}$ and $-\frac{1}{5}$, divided by $\frac{1}{7}$

*Write each sentence as an equation, using x as the variable. Then find the solution from the set of integers between* $-12$ *and* 12, *inclusive.* ***See Example 9.***

**113.** The quotient of a number and 3 is $-3$.

**114.** The quotient of a number and 4 is $-1$.

**115.** 6 less than a number is 4.

**116.** 7 less than a number is 2.

**117.** When 5 is added to a number, the result is $-5$.

**118.** When 6 is added to a number, the result is $-3$.

*To find the* ***average (mean)*** *of a group of numbers, we add the numbers and then divide the sum by the number of terms added. For example, to find the average of* 14, 8, 3, 9, *and* 1, *we add them and then divide by* 5.

$$\frac{14 + 8 + 3 + 9 + 1}{5} = \frac{35}{5} = 7 \leftarrow \text{Average}$$

*Find the average of each group of numbers.*

**119.** 23, 18, 13, $-4$, and $-8$

**120.** 18, 12, 0, $-4$, and $-10$

**121.** $-15$, 29, 8, $-6$

**122.** $-17$, 34, 9, $-2$

**123.** All integers between $-10$ and 14, inclusive

**124.** All even integers between $-18$ and 4, inclusive

*The operation of division is used in* ***divisibility tests.*** *A divisibility test allows us to determine whether a given number is divisible (without remainder) by another number.*

**125.** An integer is divisible by 2 if its last digit is divisible by 2, and not otherwise. Show that

**(a)** 3,473,986 is divisible by 2 and **(b)** 4,336,879 is not divisible by 2.

**126.** An integer is divisible by 3 if the sum of its digits is divisible by 3, and not otherwise. Show that

**(a)** 4,799,232 is divisible by 3 and **(b)** 2,443,871 is not divisible by 3.

**127.** An integer is divisible by 4 if its last two digits form a number divisible by 4, and not otherwise. Show that

**(a)** 6,221,464 is divisible by 4 and **(b)** 2,876,335 is not divisible by 4.

**128.** An integer is divisible by 5 if its last digit is divisible by 5, and not otherwise. Show that

**(a)** 3,774,595 is divisible by 5 and **(b)** 9,332,123 is not divisible by 5.

**129.** An integer is divisible by 6 if it is divisible by both 2 and 3, and not otherwise. Show that

**(a)** 1,524,822 is divisible by 6 and **(b)** 2,873,590 is not divisible by 6.

**130.** An integer is divisible by 8 if its last three digits form a number divisible by 8, and not otherwise. Show that

**(a)** 2,923,296 is divisible by 8 and **(b)** 7,291,623 is not divisible by 8.

**131.** An integer is divisible by 9 if the sum of its digits is divisible by 9, and not otherwise. Show that

**(a)** 4,114,107 is divisible by 9 and **(b)** 2,287,321 is not divisible by 9.

**132.** An integer is divisible by 12 if it is divisible by both 3 and 4, and not otherwise. Show that

**(a)** 4,253,520 is divisible by 12 and **(b)** 4,249,474 is not divisible by 12.

## *SUMMARY EXERCISES* on Operations with Real Numbers

### Operations with Signed Numbers

**Addition**

***Same sign*** Add the absolute values of the numbers. The sum has the same sign as the numbers being added.

***Different signs*** Find the absolute values of the numbers, and subtract the lesser absolute value from the greater. Give the answer the same sign as the number having the greater absolute value.

**Subtraction**

Add the additive inverse (or opposite) of the subtrahend to the minuend.

**Multiplication and Division**

***Same sign*** The product or quotient of two numbers with the same sign is positive.

***Different signs*** The product or quotient of two numbers with different signs is negative.

***Division by 0 is undefined.***

*Perform each indicated operation.*

**1.** $14 - 3 \cdot 10$

**2.** $-3(8) - 4(-7)$

**3.** $(3 - 8)(-2) - 10$

**4.** $-6(7 - 3)$

**5.** $7 + 3(2 - 10)$

**6.** $-4[(-2)(6) - 7]$

**7.** $(-4)(7) - (-5)(2)$

**8.** $-5[-4 - (-2)(-7)]$

**9.** $40 - (-2)[8 - 9]$

(continued)

**10.** $\frac{5(-4)}{-7-(-2)}$

**11.** $\frac{-3-(-9+1)}{-7-(-6)}$

**12.** $\frac{5(-8+3)}{13(-2)+(-7)(-3)}$

**13.** $\frac{6^2-8}{-2(2)+4(-1)}$

**14.** $\frac{16(-8+5)}{15(-3)+(-7-4)(-3)}$

**15.** $\frac{9(-6)-3(8)}{4(-7)+(-2)(-11)}$

**16.** $\frac{2^2+4^2}{5^2-3^2}$

**17.** $\frac{(2+4)^2}{(5-3)^2}$

**18.** $\frac{4^3-3^3}{-5(-4+2)}$

**19.** $\frac{-9(-6)+(-2)(27)}{3(8-9)}$

**20.** $|-4(9)|-|-11|$

**21.** $\frac{6(-10+3)}{15(-2)-3(-9)}$

**22.** $\frac{3^2-5^2}{(-9)^2-9^2}$

**23.** $\frac{(-10)^2+10^2}{-10(5)}$

**24.** $-\frac{3}{4}\div\left(-\frac{5}{8}\right)$

**25.** $\frac{1}{2}\div\left(-\frac{1}{2}\right)$

**26.** $\frac{8^2-12}{(-5)^2+2(6)}$

**27.** $\left[\frac{5}{8}-\left(-\frac{1}{16}\right)\right]+\frac{3}{8}$

**28.** $\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{5}{6}$

**29.** $-0.9(-3.7)$

**30.** $-5.1(-0.2)$

**31.** $-3^2-2^2$

**32.** $|-2(3)+4|-|-2|$

**33.** $40+2(-5-3)$

*Evaluate each expression for* $x=-2$, $y=3$, *and* $a=4$.

**34.** $-x+y-3a$

**35.** $(x+6)^3-y^3$

**36.** $(x-y)-(a-2y)$

**37.** $\left(\frac{1}{2}x+\frac{2}{3}y\right)\left(-\frac{1}{4}a\right)$

**38.** $\frac{2x+3y}{a-xy}$

**39.** $\frac{x^2-y^2}{x^2+y^2}$

**40.** $-x^2+3y$

**41.** $\left(\frac{x}{y}\right)^3$

**42.** $\left(\frac{a}{x}\right)^2$

## 1.7 Properties of Real Numbers

**OBJECTIVES**

1. Use the commutative properties.
2. Use the associative properties.
3. Use the identity properties.
4. Use the inverse properties.
5. Use the distributive property.

In the basic properties covered in this section, $a$, $b$, and $c$ represent real numbers.

**OBJECTIVE 1 Use the commutative properties.** The word *commute* means to go back and forth. Many people commute to work or to school. If you travel from home to work and follow the same route from work to home, you travel the same distance each time.

The **commutative properties** say that if two numbers are added or multiplied in any order, the result is the same.

**Commutative Properties**

$a+b=b+a$ Addition

$ab=ba$ Multiplication

**NOW TRY EXERCISE 1**

Use a commutative property to complete each statement.

**(a)** $7 + (-3) = -3 + \underline{\qquad}$

**(b)** $(-5)4 = 4 \cdot \underline{\qquad}$

### EXAMPLE 1 Using the Commutative Properties

Use a commutative property to complete each statement.

**(a)** $-8 + 5 = 5 + \underline{\ ?\ }$

Notice that the “order” changed.

$-8 + 5 = 5 + (-8)$ Commutative property of addition

**(b)** $(-2)7 = \underline{\ ?\ }(-2)$

$-2(7) = 7(-2)$ Commutative property of multiplication

NOW TRY

**OBJECTIVE 2** **Use the associative properties.** When we *associate* one object with another, we think of those objects as being grouped together.

The **associative properties** say that when we add or multiply three numbers, we can group the first two together or the last two together and get the same answer.

### Associative Properties

$(a + b) + c = a + (b + c)$ Addition

$(ab)c = a(bc)$ Multiplication

**NOW TRY EXERCISE 2**

Use an associative property to complete each statement.

**(a)** $-9 + (3 + 7) = \underline{\qquad}$

**(b)** $5[(-4) \cdot 9] = \underline{\qquad}$

### EXAMPLE 2 Using the Associative Properties

Use an associative property to complete each statement.

**(a)** $-8 + (1 + 4) = (-8 + \underline{\ ?\ }) + 4$

The “order” is the same.
The “grouping” changed.

$-8 + (1 + 4) = (-8 + 1) + 4$ Associative property of addition

**(b)** $[2 \cdot (-7)] \cdot 6 = 2 \cdot \underline{\ ?\ }$

$[2 \cdot (-7)] \cdot 6 = 2 \cdot [(-7) \cdot 6]$ Associative property of multiplication

NOW TRY

By the associative property, the sum (or product) of three numbers will be the same no matter how the numbers are “associated” in groups. Parentheses can be left out if a problem contains only addition (or multiplication). For example,

$$(-1 + 2) + 3 \quad \text{and} \quad -1 + (2 + 3) \quad \text{can be written as} \quad -1 + 2 + 3.$$

### EXAMPLE 3 Distinguishing Between Properties

Is each statement an example of the associative or the commutative property?

**(a)** $(2 + 4) + 5 = 2 + (4 + 5)$

The order of the three numbers is the same on both sides of the equals symbol. The only change is in the *grouping,* or association, of the numbers. This is an example of the associative property.

**(b)** $6 \cdot (3 \cdot 10) = 6 \cdot (10 \cdot 3)$

The same numbers, 3 and 10, are grouped on each side. On the left, the 3 appears first, but on the right, the 10 appears first. Since the only change involves the *order* of the numbers, this is an example of the commutative property.

**NOW TRY ANSWERS**

**1.** **(a)** 7 **(b)** $-5$

**2.** **(a)** $(-9 + 3) + 7$
**(b)** $[5 \cdot (-4)] \cdot 9$

NOW TRY EXERCISE 3

Is $5 + (7 + 6) = 5 + (6 + 7)$ an example of the associative property or the commutative property?

**(c)** $(8 + 1) + 7 = 8 + (7 + 1)$

Both the order and the grouping are changed. On the left, the order of the three numbers is 8, 1, and 7. On the right, it is 8, 7, and 1. On the left, the 8 and 1 are grouped. On the right, the 7 and 1 are grouped. Therefore, *both* properties are used.

NOW TRY

NOW TRY EXERCISE 4

Find each sum or product.

**(a)** $8 + 54 + 7 + 6 + 32$

**(b)** $5(37)(20)$

## EXAMPLE 4 Using the Commutative and Associative Properties

Find each sum or product.

**(a)**
$$\begin{aligned} &23 + 41 + 2 + 9 + 25 \\ &= (41 + 9) + (23 + 2) + 25 \\ &= 50 + 25 + 25 \\ &= 100 \end{aligned}$$

**(b)**
$$\begin{aligned} &25(69)(4) \\ &= 25(4)(69) \\ &= 100(69) \\ &= 6900 \end{aligned}$$

Use the commutative and associative properties.

NOW TRY

**OBJECTIVE 3** **Use the identity properties.** If a child wears a costume on Halloween, the child's appearance is changed, but his or her *identity* is unchanged. The identity of a real number is left unchanged when identity properties are applied.

The **identity properties** say that the sum of 0 and any number equals that number, and the product of 1 and any number equals that number.

### Identity Properties

$\boldsymbol{a + 0 = a}$ and $\boldsymbol{0 + a = a}$ Addition

$\boldsymbol{a \cdot 1 = a}$ and $\boldsymbol{1 \cdot a = a}$ Multiplication

The number 0 leaves the identity, or value, of any real number unchanged by addition, so 0 is called the **identity element for addition,** or the **additive identity.** Since multiplication by 1 leaves any real number unchanged, 1 is the **identity element for multiplication,** or the **multiplicative identity.**

NOW TRY EXERCISE 5

Use an identity property to complete each statement.

**(a)** $\frac{2}{5} \cdot \underline{\qquad} = \frac{2}{5}$

**(b)** $8 + \underline{\qquad} = 8$

## EXAMPLE 5 Using the Identity Properties

Use an identity property to complete each statement.

**(a)** $-3 + \underline{\ ?\ } = -3$

$-3 + 0 = -3$

Identity property of addition

**(b)** $\underline{\ ?\ } \cdot \frac{1}{2} = \frac{1}{2}$

$1 \cdot \frac{1}{2} = \frac{1}{2}$

Identity property of multiplication

NOW TRY

NOW TRY ANSWERS

3. commutative
4. **(a)** 107 **(b)** 3700
5. **(a)** 1 **(b)** 0

**NOW TRY EXERCISE 6**

Simplify.

**(a)** $\frac{16}{20}$ **(b)** $\frac{2}{5} + \frac{3}{20}$

## EXAMPLE 6 Using the Identity Property to Simplify Expressions

Simplify.

**(a)** $\frac{49}{35}$

$$= \frac{7 \cdot 7}{5 \cdot 7} \quad \text{Factor.}$$

$$= \frac{7}{5} \cdot \frac{7}{7} \quad \text{Write as a product.}$$

$$= \frac{7}{5} \cdot 1 \quad \text{Divide.}$$

$$= \frac{7}{5} \quad \text{Identity property}$$

**(b)** $\frac{3}{4} + \frac{5}{24}$

$$= \frac{3}{4} \cdot 1 + \frac{5}{24} \quad \text{Identity property}$$

$$= \frac{3}{4} \cdot \frac{6}{6} + \frac{5}{24} \quad \text{Use } 1 = \tfrac{6}{6} \text{ to get a common denominator.}$$

$$= \frac{18}{24} + \frac{5}{24} \quad \text{Multiply.}$$

$$= \frac{23}{24} \quad \text{Add.}$$

NOW TRY

**OBJECTIVE 4** **Use the inverse properties.** Each day before you go to work or school, you probably put on your shoes. Before you go to sleep at night, you probably take them off, and this leads to the same situation that existed before you put them on. These operations from everyday life are examples of *inverse* operations.

The **inverse properties** of addition and multiplication lead to the additive and multiplicative identities, respectively. Recall that $-a$ is the **additive inverse,** or **opposite,** of $a$ and $\frac{1}{a}$ is the **multiplicative inverse,** or **reciprocal,** of the nonzero number $a$. The sum of the numbers $a$ and $-a$ is 0, and the product of the nonzero numbers $a$ and $\frac{1}{a}$ is 1.

### Inverse Properties

$$a + (-a) = 0 \quad \text{and} \quad -a + a = 0 \qquad \text{Addition}$$

$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1 \quad (a \neq 0) \qquad \text{Multiplication}$$

**NOW TRY EXERCISE 7**

Use an inverse property to complete each statement.

**(a)** $10 + \underline{\qquad} = 0$

**(b)** $-9 \cdot \underline{\qquad} = 1$

## EXAMPLE 7 Using the Inverse Properties

Use an inverse property to complete each statement.

**(a)** $\underline{\ ?\ } + \frac{1}{2} = 0$

$-\frac{1}{2} + \frac{1}{2} = 0$

**(b)** $4 + \underline{\ ?\ } = 0$

$4 + (-4) = 0$

**(c)** $-0.75 + \frac{3}{4} = \underline{\ ?\ }$

$-0.75 + \frac{3}{4} = 0$

The inverse property of addition is used in parts (a)–(c).

**(d)** $\underline{\ ?\ } \cdot \frac{5}{2} = 1$

$\frac{2}{5} \cdot \frac{5}{2} = 1$

**(e)** $-5(\underline{\ ?\ }) = 1$

$-5\left(-\frac{1}{5}\right) = 1$

**(f)** $4(0.25) = \underline{\ ?\ }$

$4(0.25) = 1$

The inverse property of multiplication is used in parts (d)–(f).

NOW TRY

**NOW TRY ANSWERS**

**6.** **(a)** $\frac{4}{5}$ **(b)** $\frac{11}{20}$

**7.** **(a)** $-10$ **(b)** $-\frac{1}{9}$

NOW TRY EXERCISE 8

Simplify.

$$-\frac{1}{3}x + 7 + \frac{1}{3}x$$

## EXAMPLE 8 Using Properties to Simplify an Expression

Simplify.

$$
\begin{aligned}
&-2x + 10 + 2x \\
&= (-2x + 10) + 2x && \text{Order of operations} \\
&= [10 + (-2x)] + 2x && \text{Commutative property} \\
&= 10 + [(-2x) + 2x] && \text{Associative property} \\
&= 10 + 0 && \text{Inverse property} \\
&= 10 && \text{Identity property}
\end{aligned}
$$

For *any* value of $x$, $-2x$ and $2x$ are additive inverses.

NOW TRY

**NOTE** The steps of **Example 8** may be skipped when we actually do the simplification.

**OBJECTIVE 5** **Use the distributive property.** The word *distribute* means "to give out from one to several." Look at the value of the following expressions:

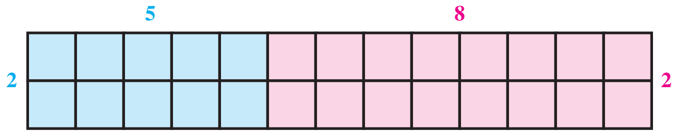
$$2(5 + 8), \quad \text{which equals} \quad 2(13), \quad \text{or} \quad 26$$

$$2(5) + 2(8), \quad \text{which equals} \quad 10 + 16, \quad \text{or} \quad 26.$$

Since both expressions equal 26,

$$2(5 + 8) = 2(5) + 2(8).$$

This result is an example of the *distributive property of multiplication with respect to addition,* the only property involving *both* addition and multiplication. With this property, a product can be changed to a sum or difference. This idea is illustrated in **FIGURE 18**.



**The area of the left part is 2(5) = 10.**
**The area of the right part is 2(8) = 16.**
The total area is 2(5 + 8) = 2(13) = 26,
or the total area is 2(5) + 2(8) = 10 + 16 = 26.
Thus, 2(5 + 8) = 2(5) + 2(8).

FIGURE 18

The **distributive property** says that multiplying a number $a$ by a sum of numbers $b + c$ gives the same result as multiplying $a$ by $b$ and $a$ by $c$ and then adding the two products.

### Distributive Property

$$\boldsymbol{a(b + c) = ab + ac} \quad \text{and} \quad \boldsymbol{(b + c)a = ba + ca}$$

As the arrows show, the $a$ outside the parentheses is "distributed" over the $b$ and $c$ inside. The distributive property is also valid for multiplication over subtraction.

$$\boldsymbol{a(b - c) = ab - ac} \quad \text{and} \quad \boldsymbol{(b - c)a = ba - ca}$$

The distributive property can be extended to more than two numbers.

$$\boldsymbol{a(b + c + d) = ab + ac + ad}$$

The distributive property can also be used "in reverse."

$$\boldsymbol{ac + bc = (a + b)c}$$

NOW TRY ANSWER

**8.** 7

**NOW TRY EXERCISE 9**

Use the distributive property to rewrite each expression.

**(a)** $-5(4x + 1)$

**(b)** $6(2r + t - 5z)$

**(c)** $5x - 5y$

### EXAMPLE 9 Using the Distributive Property

Use the distributive property to rewrite each expression.

**(a)**

$$\begin{aligned} &5(9 + 6) \\ &= 5 \cdot 9 + 5 \cdot 6 && \text{Distributive property} \\ &= 45 + 30 && \text{Multiply.} \\ &= 75 && \text{Add.} \end{aligned}$$

Multiply first.

**(b)**

$$\begin{aligned} &4(x + 5 + y) \\ &= 4x + 4 \cdot 5 + 4y && \text{Distributive property} \\ &= 4x + 20 + 4y && \text{Multiply.} \end{aligned}$$

**(c)**

$$\begin{aligned} &-\frac{1}{2}(4x + 3) \\ &= -\frac{1}{2}(4x) + \left(-\frac{1}{2}\right)(3) && \text{Distributive property} \\ &= -2x - \frac{3}{2} && \text{Multiply.} \end{aligned}$$

Think: $-\frac{1}{2}(4x) = \left(-\frac{1}{2} \cdot 4\right)x = \left(-\frac{1}{2} \cdot \frac{4}{1}\right)x$

**(d)**

$$\begin{aligned} &3(k - 9) \\ &= 3[k + (-9)] && \text{Definition of subtraction} \\ &= 3k + 3(-9) && \text{Distributive property} \\ &= 3k - 27 && \text{Multiply.} \end{aligned}$$

**(e)**

$$\begin{aligned} &8(3r + 11t + 5z) \\ &= 8(3r) + 8(11t) + 8(5z) && \text{Distributive property} \\ &= (8 \cdot 3)r + (8 \cdot 11)t + (8 \cdot 5)z && \text{Associative property} \\ &= 24r + 88t + 40z && \text{Multiply.} \end{aligned}$$

**(f)**

$$\begin{aligned} &6 \cdot 8 + 6 \cdot 2 \\ &= 6(8 + 2) && \text{Distributive property in reverse} \\ &= 6(10) && \text{Add.} \\ &= 60 && \text{Multiply.} \end{aligned}$$

**(g)**

$$\begin{aligned} &4x - 4m \\ &= 4(x - m) && \text{Distributive property in reverse} \end{aligned}$$

**(h)**

$$\begin{aligned} &6x - 12 \\ &= 6 \cdot x - 6 \cdot 2 \\ &= 6(x - 2) && \text{Distributive property in reverse} \end{aligned}$$

NOW TRY

---

**CAUTION** In practice, we often omit the first step in **Example 9(d),** where we rewrote the subtraction as addition of the additive inverse.

$$\begin{aligned} &3(k - 9) \\ &= 3k - 3(9) && \text{Be careful not to make a sign error.} \\ &= 3k - 27 && \text{Multiply.} \end{aligned}$$

---

**NOW TRY ANSWERS**

**9. (a)** $-20x - 5$

**(b)** $12r + 6t - 30z$

**(c)** $5(x - y)$

The symbol $-a$ may be interpreted as $-1 \cdot a$. Using this result and the distributive property, we can remove (or clear) parentheses from some expressions.

NOW TRY EXERCISE 10

Write each expression without parentheses.

**(a)** $-(2 - r)$

**(b)** $-(2x - 5y - 7)$

### EXAMPLE 10 Using the Distributive Property to Remove (Clear) Parentheses

Write each expression without parentheses.

**(a)**

$$\begin{aligned} &-(2y + 3) \\ &= -1 \cdot (2y + 3) && -a = -1 \cdot a \\ &= -1 \cdot 2y + (-1) \cdot 3 && \text{Distributive property} \\ &= -2y - 3 && \text{Multiply.} \end{aligned}$$

The $-$ symbol indicates a factor of $-1$.

**(b)**

$$\begin{aligned} &-(-9w - 2) \\ &= -1(-9w - 2) \\ &= -1(-9w) - 1(-2) \\ &= 9w + 2 \end{aligned}$$

We can also interpret the negative sign in front of the parentheses to mean the *opposite* of each of the terms within the parentheses.

**(c)**

$$\begin{aligned} &-(-x - 3y + 6z) \\ &= -1(-1x - 3y + 6z) \\ &= -1(-1x) - 1(-3y) - 1(6z) \\ &= x + 3y - 6z && -1(-1x) = 1x = x \end{aligned}$$

Be careful with signs.

NOW TRY

Here is a summary of the properties of real numbers discussed in this section.

### Properties of Addition and Multiplication

For any real numbers $a$, $b$, and $c$, the following properties hold.

***Commutative Properties*** $\quad a + b = b + a \qquad ab = ba$

***Associative Properties*** $\quad (a + b) + c = a + (b + c)$

$$(ab)c = a(bc)$$

***Identity Properties*** There is a real number 0 such that

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

There is a real number 1 such that

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a.$$

***Inverse Properties*** For each real number $a$, there is a single real number $-a$ such that

$$a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0.$$

For each nonzero real number $a$, there is a single real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1.$$

***Distributive Properties*** $\quad a(b + c) = ab + ac \qquad (b + c)a = ba + ca$

NOW TRY ANSWERS

10. (a) $-2 + r$
    (b) $-2x + 5y + 7$

## 1.7 EXERCISES

◉ *Complete solution available on the Video Resources on DVD*

**1.** *Concept Check* Match each item in Column I with the correct choice(s) from Column II. Choices may be used once, more than once, or not at all.

**I**

- **(a)** Identity element for addition
- **(b)** Identity element for multiplication
- **(c)** Additive inverse of $a$
- **(d)** Multiplicative inverse, or reciprocal, of the nonzero number $a$
- **(e)** The number that is its own additive inverse
- **(f)** The two numbers that are their own multiplicative inverses
- **(g)** The only number that has no multiplicative inverse
- **(h)** An example of the associative property
- **(i)** An example of the commutative property
- **(j)** An example of the distributive property

**II**

- **A.** $(5 \cdot 4) \cdot 3 = 5 \cdot (4 \cdot 3)$
- **B.** $0$
- **C.** $-a$
- **D.** $-1$
- **E.** $5 \cdot 4 \cdot 3 = 60$
- **F.** $1$
- **G.** $(5 \cdot 4) \cdot 3 = 3 \cdot (5 \cdot 4)$
- **H.** $5(4 + 3) = 5 \cdot 4 + 5 \cdot 3$
- **I.** $\frac{1}{a}$

**2.** *Concept Check* Fill in the blanks: The commutative property allows us to change the ________ of the terms in a sum or the factors in a product. The associative property allows us to change the ________ of the terms in a sum or the factors in a product.

*Concept Check* *Tell whether or not the following everyday activities are commutative.*

**3.** Washing your face and brushing your teeth

**4.** Putting on your left sock and putting on your right sock

**5.** Preparing a meal and eating a meal

**6.** Starting a car and driving away in a car

**7.** Putting on your socks and putting on your shoes

**8.** Getting undressed and taking a shower

**9.** *Concept Check* Use parentheses to show how the associative property can be used to give two different meanings to the phrase "foreign sales clerk."

**10.** *Concept Check* Use parentheses to show how the associative property can be used to give two different meanings to the phrase "defective merchandise counter."

*Use the commutative or the associative property to complete each statement. State which property is used.* ***See Examples 1 and 2.***

**11.** $-15 + 9 = 9 + \underline{\qquad}$

**12.** $6 + (-2) = -2 + \underline{\qquad}$

**13.** $-8 \cdot 3 = \underline{\qquad} \cdot (-8)$

**14.** $-12 \cdot 4 = 4 \cdot \underline{\qquad}$

**15.** $(3 + 6) + 7 = 3 + (\underline{\qquad} + 7)$

**16.** $(-2 + 3) + 6 = -2 + (\underline{\qquad} + 6)$

**17.** $7 \cdot (2 \cdot 5) = (\underline{\qquad} \cdot 2) \cdot 5$

**18.** $8 \cdot (6 \cdot 4) = (8 \cdot \underline{\qquad}) \cdot 4$

**19.** *Concept Check* Evaluate $25 - (6 - 2)$ and evaluate $(25 - 6) - 2$. Do you think subtraction is associative?

**20.** *Concept Check* Evaluate $180 \div (15 \div 3)$ and evaluate $(180 \div 15) \div 3$. Do you think division is associative?

**21.** *Concept Check* Complete the table and the statement beside it.

| *Number* | *Additive Inverse* | *Multiplicative Inverse* |
|---|---|---|
| 5 | | |
| $-10$ | | |
| $-\frac{1}{2}$ | | |
| $\frac{3}{8}$ | | |
| $x$ | | $(x \neq 0)$ |
| $-y$ | | $(y \neq 0)$ |

In general, a number and its additive inverse have ______ signs.
(the same/opposite)

A number and its multiplicative inverse have ______ signs.
(the same/opposite)

**22.** *Concept Check* The following conversation actually took place between one of the authors of this book and his son, Jack, when Jack was 4 years old:

DADDY: "Jack, what is 3 + 0?"
JACK: "3."
DADDY: "Jack, what is 4 + 0?"
JACK: "4. And Daddy, *string* plus zero equals *string*!"

What property of addition did Jack recognize?

*Decide whether each statement is an example of the* commutative, associative, identity, inverse, *or* distributive property. ***See Examples 1, 2, 3, 5, 6, 7, and 9.***

**23.** $4 + 15 = 15 + 4$

**24.** $3 + 12 = 12 + 3$

**25.** $5 \cdot (13 \cdot 7) = (5 \cdot 13) \cdot 7$

**26.** $-4 \cdot (2 \cdot 6) = (-4 \cdot 2) \cdot 6$

**27.** $-6 + (12 + 7) = (-6 + 12) + 7$

**28.** $(-8 + 13) + 2 = -8 + (13 + 2)$

**29.** $-9 + 9 = 0$

**30.** $1 + (-1) = 0$

**31.** $\frac{2}{3}\left(\frac{3}{2}\right) = 1$

**32.** $\frac{5}{8}\left(\frac{8}{5}\right) = 1$

**33.** $1.75 + 0 = 1.75$

**34.** $-8.45 + 0 = -8.45$

**35.** $(4 + 17) + 3 = 3 + (4 + 17)$

**36.** $(-8 + 4) + 12 = 12 + (-8 + 4)$

**37.** $2(x + y) = 2x + 2y$

**38.** $9(t + s) = 9t + 9s$

**39.** $-\frac{5}{9} = -\frac{5}{9} \cdot \frac{3}{3} = -\frac{15}{27}$

**40.** $-\frac{7}{12} = -\frac{7}{12} \cdot \frac{7}{7} = -\frac{49}{84}$

**41.** $4(2x) + 4(3y) = 4(2x + 3y)$

**42.** $6(5t) - 6(7r) = 6(5t - 7r)$

*Find each sum or product.* ***See Example 4.***

**43.** $97 + 13 + 3 + 37$

**44.** $49 + 199 + 1 + 1$

**45.** $1999 + 2 + 1 + 8$

**46.** $2998 + 3 + 2 + 17$

**47.** $159 + 12 + 141 + 88$

**48.** $106 + 8 + (-6) + (-8)$

**49.** $843 + 627 + (-43) + (-27)$

**50.** $1846 + 1293 + (-46) + (-93)$

**51.** $5(47)(2)$

**52.** $2(79)5$

**53.** $-4 \cdot 5 \cdot 93 \cdot 5$

**54.** $2 \cdot 25 \cdot 67 \cdot (-2)$

*Simplify each expression.* ***See Examples 7 and 8.***

**55.** $6t + 8 - 6t + 3$

**56.** $9r + 12 - 9r + 1$

**57.** $\frac{2}{3}x - 11 + 11 - \frac{2}{3}x$

**58.** $\frac{1}{5}y + 4 - 4 - \frac{1}{5}y$

**59.** $\left(\frac{9}{7}\right)(-0.38)\left(\frac{7}{9}\right)$

**60.** $\left(\frac{4}{5}\right)(-0.73)\left(\frac{5}{4}\right)$

**61.** $t + (-t) + \frac{1}{2}(2)$

**62.** $w + (-w) + \frac{1}{4}(4)$

**63.** *Concept Check* Suppose that a student simplifies the expression $-3(4-6)$ as shown.

$$\begin{aligned} &-3(4-6)\\ &\quad=-3(4)-3(6)\\ &\quad=-12-18\\ &\quad=-30 \end{aligned}$$

*WHAT WENT WRONG?* Work the problem correctly.

**64.** Explain how the procedure of changing $\frac{3}{4}$ to $\frac{9}{12}$ requires the use of the multiplicative identity element, 1.

*Use the distributive property to rewrite each expression. Simplify if possible.* ***See Example 9.***

**65.** $5(9+8)$
**66.** $6(11+8)$
**67.** $4(t+3)$
**68.** $5(w+4)$
**69.** $7(z-8)$
**70.** $8(x-6)$
**71.** $-8(r+3)$
**72.** $-11(x+4)$
**73.** $-\frac{1}{4}(8x+3)$
**74.** $-\frac{1}{3}(9x+5)$
**75.** $-5(y-4)$
**76.** $-9(g-4)$
**77.** $-\frac{4}{3}(12y+15z)$
**78.** $-\frac{2}{5}(10b+20a)$
**79.** $8z+8w$
**80.** $4s+4r$
**81.** $7(2v)+7(5r)$
**82.** $13(5w)+13(4p)$
**83.** $8(3r+4s-5y)$
**84.** $2(5u-3v+7w)$
**85.** $-3(8x+3y+4z)$
**86.** $-5(2x-5y+6z)$
**87.** $5x+15$
**88.** $9p+18$

*Write each expression without parentheses.* ***See Example 10.***

**89.** $-(4t+3m)$
**90.** $-(9x+12y)$
**91.** $-(-5c-4d)$
**92.** $-(-13x-15y)$
**93.** $-(-q+5r-8s)$
**94.** $-(-z+5w-9y)$

# 1.8 Simplifying Expressions

**OBJECTIVES**

1. Simplify expressions.
2. Identify terms and numerical coefficients.
3. Identify like terms.
4. Combine like terms.
5. Simplify expressions from word phrases.

**OBJECTIVE 1 Simplify expressions.** We use the properties of **Section 1.7** to do this.

**EXAMPLE 1** Simplifying Expressions

Simplify each expression.

**(a)** $4x+8+9$ simplifies to $4x+17$.

**(b)** $4(3m-2n)$

To simplify, we clear the parentheses.

$$\begin{aligned} &=4(3m)-4(2n) && \text{Distributive property}\\ &=(4\cdot 3)m-(4\cdot 2)n && \text{Associative property}\\ &=12m-8n && \text{Multiply.} \end{aligned}$$

NOW TRY EXERCISE 1

Simplify each expression.

**(a)** $3(2x - 4y)$

**(b)** $-4 - (-3y + 5)$

**(c)**

$$\begin{aligned} & 6 + 3(4k + 5) & \\ &= 6 + 3(4k) + 3(5) & \text{Distributive property} \\ &= 6 + (3 \cdot 4)k + 3(5) & \text{Associative property} \\ &= 6 + 12k + 15 & \text{Multiply.} \\ &= 6 + 15 + 12k & \text{Commutative property} \\ &= 21 + 12k & \text{Add.} \end{aligned}$$

Don't start by adding!

**(d)**

$$\begin{aligned} & 5 - (2y - 8) & \\ &= 5 - 1(2y - 8) & -a = -1 \cdot a \\ &= 5 - 1(2y) - 1(-8) & \text{Distributive property} \\ &= 5 - 2y + 8 & \text{Multiply.} \\ &= 5 + 8 - 2y & \text{Commutative property} \\ &= 13 - 2y & \text{Add.} \end{aligned}$$

Be careful with signs.

NOW TRY

**NOTE** The steps using the commutative and associative properties will not be shown in the rest of the examples. However, be aware that they are usually involved.

**OBJECTIVE 2** **Identify terms and numerical coefficients.** A **term** is a number, a variable, or a product or quotient of numbers and variables raised to powers, such as

$$9x, \quad 15y^2, \quad -3, \quad -8m^2n, \quad \frac{2}{p}, \quad \text{and} \quad k. \qquad \text{Terms}$$

In the term $9x$, the **numerical coefficient,** or simply **coefficient,** of the variable $x$ is $9$. Additional examples are shown in the table in the margin.

| *Term* | *Numerical Coefficient* |
|---|---|
| $8$ | $8$ |
| $-7y$ | $-7$ |
| $34r^3$ | $34$ |
| $-26x^5yz^4$ | $-26$ |
| $-k$, or $-1k$ | $-1$ |
| $r$, or $1r$ | $1$ |
| $\frac{3x}{8} = \frac{3}{8}x$ | $\frac{3}{8}$ |
| $\frac{x}{3} = \frac{1x}{3} = \frac{1}{3}x$ | $\frac{1}{3}$ |

**CAUTION** It is important to be able to distinguish between ***terms*** and ***factors.*** Consider the following expressions.

$8x^3 + 12x^2$ This expression has **two terms,** $8x^3$ and $12x^2$. Terms are separated by a + or − symbol.

$(8x^3)(12x^2)$ This is a **one-term** expression. The **factors** $8x^3$ and $12x^2$ are multiplied.

**OBJECTIVE 3** **Identify like terms.** Terms with exactly the same variables that have the same exponents are **like terms.** Here are some examples.

| **Like Terms** | **Unlike Terms** | |
|---|---|---|
| $9t$ and $4t$ | $4y$ and $7t$ | Different variables |
| $6x^2$ and $-5x^2$ | $17x$ and $-8x^2$ | Different exponents |
| $-2pq$ and $11pq$ | $4xy^2$ and $4xy$ | Different exponents |
| $3x^2y$ and $5x^2y$ | $-7wz^3$ and $2xz^3$ | Different variables |

NOW TRY ANSWERS

1. **(a)** $6x - 12y$ **(b)** $3y - 9$

**OBJECTIVE 4** **Combine like terms.** Recall that the distributive property

$$a(b+c) = ab + ac \quad \text{can be written “in reverse” as} \quad ab + ac = a(b+c).$$

This last form, which may be used to find the sum or difference of like terms, provides justification for **combining like terms.**

## EXAMPLE 2 Combining Like Terms

Combine like terms in each expression.

**(a)** $-9m + 5m$

$$\begin{aligned} &= (-9+5)m \\ &= -4m \end{aligned}$$

**(b)** $6r + 3r + 2r$

$$\begin{aligned} &= (6+3+2)r \\ &= 11r \end{aligned}$$

**(c)** $4x + x$

$$\begin{aligned} &= 4x + 1x && x = 1x \\ &= (4+1)x \\ &= 5x \end{aligned}$$

**(d)** $16y^2 - 9y^2$

$$\begin{aligned} &= (16-9)y^2 \\ &= 7y^2 \end{aligned}$$

**(e)** $32y + 10y^2$ These unlike terms cannot be combined. NOW TRY

**NOW TRY EXERCISE 2**

Combine like terms in each expression.

**(a)** $4x + 6x - 7x$ **(b)** $z + z$

**(c)** $4p^2 - 3p^2$

**CAUTION** ***Remember that only like terms may be combined.***

## EXAMPLE 3 Simplifying Expressions Involving Like Terms

Simplify each expression.

**(a)** $14y + 2(6+3y)$

$$\begin{aligned} &= 14y + 2(6) + 2(3y) && \text{Distributive property} \\ &= 14y + 12 + 6y && \text{Multiply.} \\ &= 20y + 12 && \text{Combine like terms.} \end{aligned}$$

**(b)** $9k - 6 - 3(2-5k)$

Be careful with signs.

$$\begin{aligned} &= 9k - 6 - 3(2) - 3(-5k) && \text{Distributive property} \\ &= 9k - 6 - 6 + 15k && \text{Multiply.} \\ &= 24k - 12 && \text{Combine like terms.} \end{aligned}$$

**(c)** $-(2-r) + 10r$

$$\begin{aligned} &= -1(2-r) + 10r && -a = -1 \cdot a \\ &= -1(2) - 1(-r) + 10r && \text{Distributive property} \\ &= -2 + 1r + 10r && \text{Multiply.} \\ &= -2 + 11r && \text{Combine like terms.} \end{aligned}$$

Be careful with signs.

**(d)** $100[0.03(x+4)]$

$$\begin{aligned} &= [(100)(0.03)](x+4) && \text{Associative property} \\ &= 3(x+4) && \text{Multiply.} \\ &= 3x + 12 && \text{Distributive property} \end{aligned}$$

NOW TRY ANSWERS

**2.** **(a)** $3x$ **(b)** $2z$ **(c)** $p^2$

**NOW TRY EXERCISE 3**

Simplify each expression.

**(a)** $5k - 6 - (3 - 4k)$

**(b)** $\frac{1}{4}x - \frac{2}{3}(x - 9)$

**(e)** $5(2a - 6) - 3(4a - 9)$

$= 10a - 30 - 12a + 27$ Distributive property

$= -2a - 3$ Combine like terms.

**(f)** $-\frac{2}{3}(x - 6) - \frac{1}{6}x$

$= -\frac{2}{3}x - \frac{2}{3}(-6) - \frac{1}{6}x$ Distributive property

$= -\frac{2}{3}x + 4 - \frac{1}{6}x$ Multiply.

$= -\frac{4}{6}x + 4 - \frac{1}{6}x$ Get a common denominator.

$= -\frac{5}{6}x + 4$ Combine like terms. NOW TRY

**NOTE** **Examples 2 and 3** suggest that like terms may be combined by adding or subtracting the coefficients of the terms and keeping the same variable factors.

## OBJECTIVE 5 Simplify expressions from word phrases.

**NOW TRY EXERCISE 4**

Translate the phrase into a mathematical expression and simplify.

Twice a number, subtracted
from the sum of the number
and 5

### EXAMPLE 4 Translating Words into a Mathematical Expression

Translate the phrase into a mathematical expression and simplify.

The sum of 9, five times a number,
four times the number, and
six times the number

The word "sum" indicates that the terms should be added. Use $x$ for the number.

$9 + 5x + 4x + 6x$ simplifies to $9 + 15x$. Combine like terms.

This is an expression to be simplified, *not* an equation to be solved.

NOW TRY

**NOW TRY ANSWERS**

3. **(a)** $9k - 9$ **(b)** $-\frac{5}{12}x + 6$
4. $(x + 5) - 2x$; $-x + 5$

## 1.8 EXERCISES

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***Concept Check*** *In Exercises 1–4, choose the letter of the correct response.*

1. Which expression is a simplified form of $-(6x - 3)$?

   **A.** $-6x - 3$ **B.** $-6x + 3$ **C.** $6x - 3$ **D.** $6x + 3$

2. Which is an example of a term with numerical coefficient 5?

   **A.** $5x^3y^7$ **B.** $x^5$ **C.** $\frac{x}{5}$ **D.** $5^2xy^3$

**3.** Which is an example of a pair of like terms?

**A.** $6t, 6w$ **B.** $-8x^2y, 9xy^2$ **C.** $5ry, 6yr$ **D.** $-5x^2, 2x^3$

**4.** Which is a correct translation for "six times a number, subtracted from the product of eleven and the number" (if $x$ represents the number)?

**A.** $6x - 11x$ **B.** $11x - 6x$ **C.** $(11 + x) - 6x$ **D.** $6x - (11 + x)$

*Simplify each expression.* ***See Example 1.***

**5.** $4r + 19 - 8$ **6.** $7t + 18 - 4$

**7.** $5 + 2(x - 3y)$ **8.** $8 + 3(s - 6t)$

**9.** $-2 - (5 - 3p)$ **10.** $-10 - (7 - 14r)$

**11.** $6 + (4 - 3x) - 8$ **12.** $-12 + (7 - 8x) + 6$

*In each term, give the numerical coefficient of the variable(s).* ***See Objective 2.***

**13.** $-12k$ **14.** $-11y$ **15.** $3m^2$ **16.** $9n^6$

**17.** $xw$ **18.** $pq$ **19.** $-x$ **20.** $-t$

**21.** $\frac{x}{2}$ **22.** $\frac{x}{6}$ **23.** $\frac{2x}{5}$

**24.** $\frac{8x}{9}$ **25.** $10$ **26.** $15$

*Identify each group of terms as* like *or* unlike. ***See Objective 3.***

**27.** $8r, -13r$ **28.** $-7x, 12x$ **29.** $5z^4, 9z^3$ **30.** $8x^5, -10x^3$

**31.** $4, 9, -24$ **32.** $7, 17, -83$ **33.** $x, y$ **34.** $t, s$

**35.** ***Concept Check*** A student simplified the expression $7x - 2(3 - 2x)$ as shown.

$$\begin{aligned} &7x - 2(3 - 2x)\\ &\quad = 7x - 2(3) - 2(2x)\\ &\quad = 7x - 6 - 4x\\ &\quad = 3x - 6 \end{aligned}$$

***WHAT WENT WRONG?*** Find the correct simplified answer.

**36.** ***Concept Check*** A student simplified the expression $3 + 2(4x - 5)$ as shown.

$$\begin{aligned} &3 + 2(4x - 5)\\ &\quad = 5(4x - 5)\\ &\quad = 20x - 25 \end{aligned}$$

***WHAT WENT WRONG?*** Find the correct simplified answer.

*Simplify each expression.* ***See Examples 1–3.***

**37.** $7y + 6y$ **38.** $5m + 2m$ **39.** $-6x - 3x$

**40.** $-4z - 8z$ **41.** $12b + b$ **42.** $19x + x$

**43.** $3k + 8 + 4k + 7$ **44.** $1 + 15z + 2 + 4z$

**45.** $-5y + 3 - 1 + 5 + y - 7$ **46.** $2k - 7 - 5k + 7k - 3 - k$

**47.** $-2x + 3 + 4x - 17 + 20$ **48.** $r - 6 - 12r - 4 + 6r$

**49.** $16 - 5m - 4m - 2 + 2m$ **50.** $6 - 3z - 2z - 5 + z - 3z$

**51.** $-10 + x + 4x - 7 - 4x$ **52.** $-p + 10p - 3p - 4 - 5p$

**53.** $1 + 7x + 11x - 1 + 5x$ **54.** $-r + 2 - 5r + 3 + 4r$

**55.** $-\frac{4}{3} + 2t + \frac{1}{3}t - 8 - \frac{8}{3}t$ **56.** $-\frac{5}{6} + 8x + \frac{1}{6}x - 7 - \frac{7}{6}$

**57.** $6y^2 + 11y^2 - 8y^2$ **58.** $-9m^3 + 3m^3 - 7m^3$

**59.** $2p^2 + 3p^2 - 8p^3 - 6p^3$

**60.** $5y^3 + 6y^3 - 3y^2 - 4y^2$

**61.** $2(4x + 6) + 3$

**62.** $4(6y - 9) + 7$

**63.** $100[0.05(x + 3)]$

**64.** $100[0.06(x + 5)]$

**65.** $-6 - 4(y - 7)$

**66.** $-4 - 5(t - 13)$

**67.** $-\frac{4}{3}(y - 12) - \frac{1}{6}y$

**68.** $-\frac{7}{5}(t - 15) - \frac{1}{2}t$

**69.** $-5(5y - 9) + 3(3y + 6)$

**70.** $-3(2t + 4) + 8(2t - 4)$

**71.** $-3(2r - 3) + 2(5r + 3)$

**72.** $-4(5y - 7) + 3(2y - 5)$

**73.** $8(2k - 1) - (4k - 3)$

**74.** $6(3p - 2) - (5p + 1)$

**75.** $-2(-3k + 2) - (5k - 6) - 3k - 5$

**76.** $-2(3r - 4) - (6 - r) + 2r - 5$

**77.** $-4(-3x + 3) - (6x - 4) - 2x + 1$

**78.** $-5(8x + 2) - (5x - 3) - 3x + 17$

**79.** $-7.5(2y + 4) - 2.9(3y - 6)$

**80.** $8.4(6t - 6) + 2.4(9 - 3t)$

*Translate each phrase into a mathematical expression. Use x as the variable. Combine like terms when possible.* ***See Example 4.***

**81.** Five times a number, added to the sum of the number and three

**82.** Six times a number, added to the sum of the number and six

**83.** A number multiplied by $-7$, subtracted from the sum of 13 and six times the number

**84.** A number multiplied by 5, subtracted from the sum of 14 and eight times the number

**85.** Six times a number added to $-4$, subtracted from twice the sum of three times the number and 4 (*Hint: Twice* means two times.)

**86.** Nine times a number added to 6, subtracted from triple the sum of 12 and 8 times the number (*Hint: Triple* means three times.)

## RELATING CONCEPTS EXERCISES 87–90

### FOR INDIVIDUAL OR GROUP WORK

*A manufacturer has fixed costs of* \$1000 *to produce widgets. Each widget costs* \$5 *to make. The fixed cost to produce gadgets is* \$750, *and each gadget costs* \$3 *to make.* ***Work Exercises 87–90 in order.***

**87.** Write an expression for the cost to make $x$ widgets. (*Hint:* The cost will be the sum of the fixed cost and the cost per item times the number of items.)

**88.** Write an expression for the cost to make $y$ gadgets.

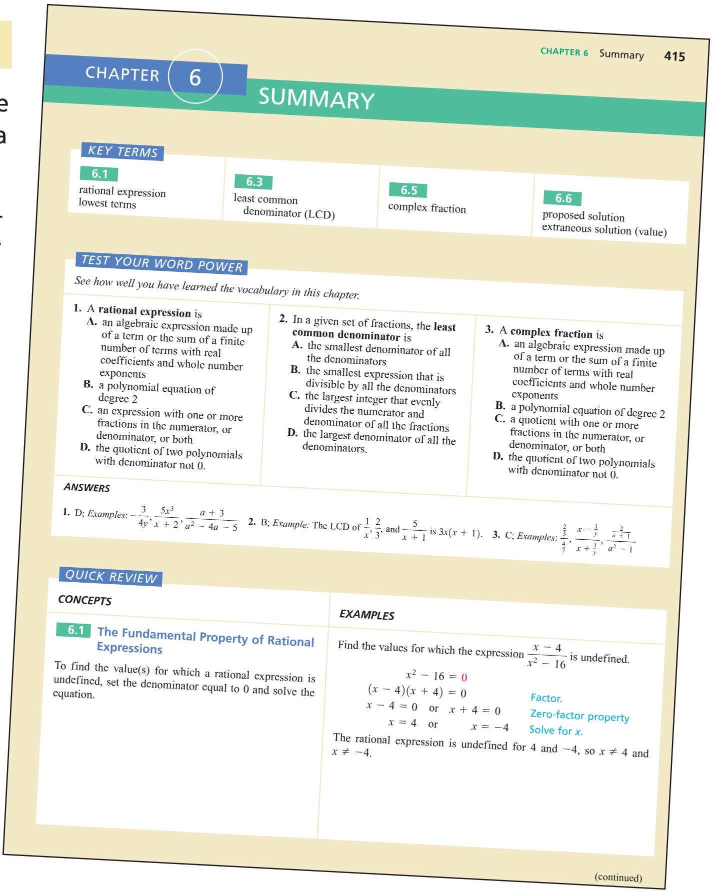
**89.** Use your answers from **Exercises 87 and 88** to write an expression for the total cost to make $x$ widgets and $y$ gadgets.

**90.** Simplify the expression you wrote in **Exercise 89.**

STUDY SKILLS

# Reviewing a Chapter

Your textbook provides material to help you prepare for quizzes or tests in this course. Refer to a **Chapter Summary** as you read through the following techniques.



CHAPTER 6 Summary 415

CHAPTER 6 SUMMARY

**KEY TERMS**

| 6.1 | 6.3 | 6.5 | 6.6 |
|---|---|---|---|
| rational expression<br>lowest terms | least common denominator (LCD) | complex fraction | proposed solution<br>extraneous solution (value) |

**TEST YOUR WORD POWER**

*See how well you have learned the vocabulary in this chapter.*

1. A **rational expression** is
   - **A.** an algebraic expression made up of a term or the sum of a finite number of terms with real coefficients and whole number exponents
   - **B.** a polynomial equation of degree 2
   - **C.** an expression with one or more fractions in the numerator, or denominator, or both
   - **D.** the quotient of two polynomials with denominator not 0.
2. In a given set of fractions, the **least common denominator** is
   - **A.** the smallest denominator of all the denominators
   - **B.** the smallest expression that is divisible by all the denominators
   - **C.** the largest integer that evenly divides the numerator and denominator of all the fractions
   - **D.** the largest denominator of all the denominators.
3. A **complex fraction** is
   - **A.** an algebraic expression made up of a term or the sum of a finite number of terms with real coefficients and whole number exponents
   - **B.** a polynomial equation of degree 2
   - **C.** a quotient with one or more fractions in the numerator, or denominator, or both
   - **D.** the quotient of two polynomials with denominator not 0.

**ANSWERS**

**1.** D; *Examples:* $-\frac{3}{4y}, \frac{5x^3}{x+2}, \frac{a+3}{a^2-4a-5}$ **2.** B; *Example:* The LCD of $\frac{1}{x}, \frac{2}{3},$ and $\frac{5}{x+1}$ is $3x(x+1)$. **3.** C; *Examples:* $\frac{\frac{2}{3}}{\frac{4}{7}}, \frac{x-\frac{1}{y}}{x+\frac{1}{y}}, \frac{\frac{2}{a+1}}{a^2-1}$

**QUICK REVIEW**

| CONCEPTS | EXAMPLES |
|---|---|
| **6.1 The Fundamental Property of Rational Expressions**<br>To find the value(s) for which a rational expression is undefined, set the denominator equal to 0 and solve the equation. | Find the values for which the expression $\frac{x-4}{x^2-16}$ is undefined.<br>$x^2 - 16 = 0$<br>$(x-4)(x+4) = 0$ Factor.<br>$x - 4 = 0$ or $x + 4 = 0$ Zero-factor property<br>$x = 4$ or $x = -4$ Solve for $x$.<br>The rational expression is undefined for 4 and $-4$, so $x \neq 4$ and $x \neq -4$. |

(continued)

## Chapter Reviewing Techniques

- **Review the Key Terms.** Make a study card for each. Include a definition, an example, a sketch (if appropriate), and a section or page reference.
- **Take the Test Your Word Power quiz** to check your understanding of new vocabulary. The answers immediately follow.
- **Read the Quick Review.** Pay special attention to the headings. Study the explanations and examples given for each concept. Try to think about the whole chapter.
- **Reread your lecture notes.** Focus on what your instructor has emphasized in class, and review that material in your text.
- **Work the Review Exercises.** They are grouped by section.
  - ✓ Pay attention to direction words, such as *simplify*, *solve*, and *estimate.*
  - ✓ After you've done each section of exercises, check your answers in the answer section.
  - ✓ Are your answers exact and complete? Did you include the correct labels, such as \$, cm$^2$, ft, etc.?
  - ✓ Make study cards for difficult problems.
- **Work the Mixed Review Exercises.** They are in mixed-up order. Check your answers in the answer section.
- **Take the Chapter Test under test conditions.**
  - ✓ Time yourself.
  - ✓ Use a calculator or notes (if your instructor permits them on tests).
  - ✓ Take the test in one sitting.
  - ✓ Show all your work.
  - ✓ Check your answers in the back of the book. Section references are provided.

***Reviewing a chapter will take some time.*** Avoid rushing through your review in one night. Use the suggestions over a few days or evenings to better understand the material and remember it longer.

***Follow these reviewing techniques for your next test. Evaluate how they worked for you.***

# CHAPTER 1 SUMMARY

## KEY TERMS

**1.1**

natural (counting) numbers
whole numbers
fractions
numerator
denominator
proper fraction
improper fraction
mixed number
factor
product
prime number
composite number
prime factors
basic principle of fractions
lowest terms
greatest common factor
reciprocal
quotient
sum
least common denominator (LCD)
difference

**1.2**

exponent (power)
base
exponential expression
grouping symbols
order of operations
inequality

**1.3**

variable
algebraic expression
equation
solution
set
element

**1.4**

number line
integers
signed numbers
rational numbers
set-builder notation
graph
coordinate
irrational numbers
real numbers
additive inverse (opposite)
absolute value

**1.5**

minuend
subtrahend

**1.6**

multiplicative inverse (reciprocal)
dividend
divisor

**1.7**

commutative property
associative property
identity property
identity element for addition (additive identity)
identity element for multiplication (multiplicative identity)
inverse property
distributive property

**1.8**

term
numerical coefficient
like terms
unlike terms
combining like terms

## NEW SYMBOLS

| | |
|---|---|
| $a^n$ | $n$ factors of $a$ |
| [ ] | brackets |
| $=$ | is equal to |
| $\neq$ | is not equal to |
| $<$ | is less than |
| $>$ | is greater than |
| $\leq$ | is less than or equal to |
| $\geq$ | is greater than or equal to |
| { } | set braces |
| **$\{x \mid x$ has a certain property$\}$** | set-builder notation |
| $-x$ | the additive inverse, or opposite, of $x$ |
| $\lvert x \rvert$ | absolute value of $x$ |
| $\frac{1}{x}$ | the multiplicative inverse, or reciprocal, of the nonzero number $x$ |
| **$a(b)$, $(a)b$, $(a)(b)$, $a \cdot b$, or $ab$** | $a$ times $b$ |
| **$a \div b$, $\frac{a}{b}$, $a/b$, or $b\overline{)a}$** | $a$ divided by $b$ |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **factor** is
**A.** the answer in an addition problem
**B.** the answer in a multiplication problem
**C.** one of two or more numbers that are added to get another number
**D.** one of two or more numbers that are multiplied to get another number.

**2.** A number is **prime** if
**A.** it cannot be factored
**B.** it has just one factor
**C.** it has only itself and 1 as factors
**D.** it has at least two different factors.

**3.** An **exponent** is
**A.** a symbol that tells how many numbers are being multiplied
**B.** a number raised to a power
**C.** a number that tells how many times a factor is repeated
**D.** one of two or more numbers that are multiplied.

**4.** A **variable** is
**A.** a symbol used to represent an unknown number
**B.** a value that makes an equation true
**C.** a solution of an equation
**D.** the answer in a division problem.

(continued)

**5.** An **integer** is
- **A.** a positive or negative number
- **B.** a natural number, its opposite, or zero
- **C.** any number that can be graphed on a number line
- **D.** the quotient of two numbers.

**6.** The **absolute value** of a number is
- **A.** the graph of the number
- **B.** the reciprocal of the number
- **C.** the opposite of the number
- **D.** the distance between 0 and the number on a number line.

**7.** A **term** is
- **A.** a numerical factor
- **B.** a number, a variable, or a product or quotient of numbers and variables raised to powers
- **C.** one of several variables with the same exponents
- **D.** a sum of numbers and variables raised to powers.

**8.** A **numerical coefficient** is
- **A.** the numerical factor of the variable(s) in a term
- **B.** the number of terms in an expression
- **C.** a variable raised to a power
- **D.** the variable factor in a term.

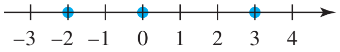
***ANSWERS***

**1.** D; *Example:* Since $2 \times 5 = 10$, the numbers 2 and 5 are factors of 10. Other factors of 10 are $-10$, $-5$, $-2$, $-1$, 1, and 10. **2.** C; *Examples:* 2, 3, 11, 41, 53 **3.** C; *Example:* In $2^3$, the number 3 is the exponent (or power), so 2 is a factor three times, and $2^3 = 2 \cdot 2 \cdot 2 = 8$. **4.** A; *Examples:* $a, b, c$ **5.** B; *Examples:* $-9, 0, 6$ **6.** D; *Examples:* $|2| = 2$ and $|-2| = 2$ **7.** B; *Examples:* $6, \frac{x}{2}, -4ab^2$ **8.** A; *Examples:* The term 3 has numerical coefficient 3, $8z$ has numerical coefficient 8, and $-10x^4y$ has numerical coefficient $-10$.

## QUICK REVIEW

| CONCEPTS | EXAMPLES |
|---|---|
| **1.1 Fractions**<br>**Operations with Fractions**<br>***Addition/Subtraction***<br>**1.** ***Same denominator:*** Add/subtract the numerators and keep the same denominator.<br>**2.** ***Different denominators:*** Find the LCD, and write each fraction with this LCD. Then follow the procedure above.<br>***Multiplication:*** Multiply numerators and multiply denominators.<br>***Division:*** Multiply the first fraction by the reciprocal of the second fraction. | Perform each operation.<br>$\frac{2}{5}+\frac{7}{5}=\frac{2+7}{5}=\frac{9}{5}$, or $1\frac{4}{5}$<br>$\frac{2}{3}-\frac{1}{2}=\frac{4}{6}-\frac{3}{6}$ 6 is the LCD.<br>$=\frac{4-3}{6}=\frac{1}{6}$<br>$\frac{4}{3}\cdot\frac{5}{6}=\frac{20}{18}=\frac{10}{9}$, or $1\frac{1}{9}$<br>$\frac{6}{5}\div\frac{1}{4}=\frac{6}{5}\cdot\frac{4}{1}=\frac{24}{5}$, or $4\frac{4}{5}$ |
| **1.2 Exponents, Order of Operations, and Inequality**<br>**Order of Operations**<br>Simplify within any parentheses or brackets and above and below fraction bars first. Always follow this order.<br>***Step 1*** Apply all exponents.<br>***Step 2*** Do any multiplications or divisions from left to right.<br>***Step 3*** Do any additions or subtractions from left to right. | Simplify $36 - 4(2^2 + 3)$.<br>$36 - 4(2^2 + 3)$<br>$= 36 - 4(4 + 3)$ Apply the exponent.<br>$= 36 - 4(7)$ Add inside the parentheses.<br>$= 36 - 28$ Multiply.<br>$= 8$ Subtract. |
| **1.3 Variables, Expressions, and Equations**<br>Evaluate an expression with a variable by substituting a given number for the variable. | Evaluate $2x + y^2$ for $x = 3$ and $y = -4$.<br>$2x + y^2$<br>$= 2(3) + (-4)^2$ Substitute.<br>$= 6 + 16$ Multiply. Apply the exponent.<br>$= 22$ Add. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| Values of a variable that make an equation true are solutions of the equation. | Is 2 a solution of $5x + 3 = 18$?<br>$5(2) + 3 \overset{?}{=} 18$ Let $x = 2$.<br>$13 = 18$ False<br>2 is not a solution. |
| **1.4 Real Numbers and the Number Line**<br>**Ordering Real Numbers**<br>$a$ is less than $b$ if $a$ is to the left of $b$ on the number line.<br>The additive inverse of $x$ is $-x$.<br>The absolute value of $x$, written $\lvert x \rvert$, is the distance between $x$ and 0 on the number line. | Graph −2, 0, and 3.<br><br> <br>$-2 < 3$ $3 > 0$ $0 < 3$<br>$-(5) = -5$ $-(-7) = 7$ $-0 = 0$<br>$\lvert 13 \rvert = 13$ $\lvert 0 \rvert = 0$ $\lvert -5 \rvert = 5$ |
| **1.5 Adding and Subtracting Real Numbers**<br>**Adding Two Signed Numbers**<br>***Same sign*** Add their absolute values. The sum has that same sign.<br>***Different signs*** Subtract their absolute values. The sum has the sign of the number with greater absolute value.<br>**Definition of Subtraction**<br>$\mathbf{x - y = x + (-y)}$ | Add. $9 + 4 = 13$<br>$-8 + (-5) = -13$<br>$7 + (-12) = -5$<br>$-5 + 13 = 8$<br>Subtract. $-3 - 4 = -3 + (-4) = -7$<br>$-2 - (-6) = -2 + 6 = 4$<br>$13 - (-8) = 13 + 8 = 21$ |
| **1.6 Multiplying and Dividing Real Numbers**<br>**Multiplying and Dividing Two Signed Numbers**<br>***Same sign*** The product (or quotient) is *positive.*<br>***Different signs*** The product (or quotient) is *negative.*<br>**Definition of Division**<br>$\frac{x}{y} = x \cdot \frac{1}{y}, \quad y \neq 0$<br>***0 divided by a nonzero number equals 0.***<br>***Division by 0 is undefined.*** | Multiply or divide.<br>$6 \cdot 5 = 30$ $-7(-8) = 56$ $\frac{20}{4} = 5$<br>$\frac{-24}{-6} = 4$ $-6(5) = -30$ $6(-5) = -30$<br>$\frac{-18}{9} = -2$ $\frac{49}{-7} = -7$<br>$\frac{10}{2} = 10 \cdot \frac{1}{2} = 5$<br>$\frac{0}{5} = 0$ $\frac{5}{0}$ is undefined. |
| **1.7 Properties of Real Numbers**<br>**Commutative Properties**<br>$a + b = b + a$<br>$ab = ba$<br>**Associative Properties**<br>$(a + b) + c = a + (b + c)$<br>$(ab)c = a(bc)$<br>**Identity Properties**<br>$a + 0 = a$ $0 + a = a$<br>$a \cdot 1 = a$ $1 \cdot a = a$ | $7 + (-1) = -1 + 7$<br>$5(-3) = (-3)5$<br>$(3 + 4) + 8 = 3 + (4 + 8)$<br>$[-2(6)]4 = -2[(6)4]$<br>$-7 + 0 = -7$ $0 + (-7) = -7$<br>$9 \cdot 1 = 9$ $1 \cdot 9 = 9$ |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Inverse Properties**<br>$a + (-a) = 0 \qquad -a + a = 0$<br>$a \cdot \frac{1}{a} = 1 \qquad \frac{1}{a} \cdot a = 1 \quad (a \neq 0)$ | $7 + (-7) = 0 \qquad -7 + 7 = 0$<br>$-2\left(-\frac{1}{2}\right) = 1 \qquad -\frac{1}{2}(-2) = 1$ |
| **Distributive Properties**<br>$a(b + c) = ab + ac$<br>$(b + c)a = ba + ca$<br>$a(b - c) = ab - ac$ | $5(4 + 2) = 5(4) + 5(2)$<br>$(4 + 2)5 = 4(5) + 2(5)$<br>$9(5 - 4) = 9(5) - 9(4)$ |
| **1.8 Simplifying Expressions**<br>***Only like terms may be combined.*** We use the distributive property to combine like terms. | $-3y^2 + 6y^2 + 14y^2$<br>$= (-3 + 6 + 14)y^2$<br>$= 17y^2$<br><br>$4(3 + 2x) - 6(5 - x)$<br>$= 4(3) + 4(2x) - 6(5) - 6(-x)$<br>$= 12 + 8x - 30 + 6x$<br>$= 14x - 18$ |

# CHAPTER 1 REVIEW EXERCISES

**1.1** *Perform each indicated operation.*

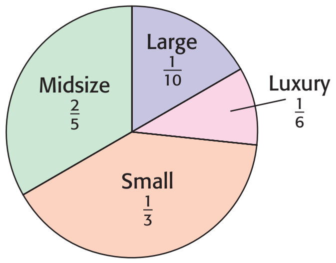
**1.** $\frac{8}{5} \div \frac{32}{15}$ **2.** $2\frac{4}{5} \cdot 1\frac{1}{4}$ **3.** $\frac{5}{8} - \frac{1}{6}$ **4.** $\frac{3}{8} + 3\frac{1}{2} - \frac{3}{16}$

*The circle graph indicates the fraction of cars in different size categories sold in the United States in 2007. There were approximately 7618 thousand cars sold that year.*

**5.** About how many luxury cars, to the nearest thousand, were sold?

**6.** To the nearest thousand, how many of the cars sold were *not* small cars?

**U.S. Car Sales by Size, 2007**



***Source:*** *World Almanac and Book of Facts.*

**1.2** *Find the value of each exponential expression.*

**7.** $5^4$ **8.** $\left(\frac{3}{5}\right)^3$ **9.** $(0.02)^2$ **10.** $(0.1)^3$

*Find the value of each expression.*

**11.** $8 \cdot 5 - 13$ **12.** $16 + 12 \div 4 - 2$ **13.** $20 - 2(5 + 3)$

**14.** $7[3 + 6(3^2)]$ **15.** $\frac{9(4^2 - 3)}{4 \cdot 5 - 17}$ **16.** $\frac{6(5 - 4) + 2(4 - 2)}{3^2 - (4 + 3)}$

*Tell whether each statement is* true *or* false.

**17.** $12 \cdot 3 - 6 \cdot 6 \leq 0$ **18.** $3[5(2) - 3] > 20$ **19.** $9 \leq 4^2 - 8$

*Write each word statement in symbols.*

**20.** Thirteen is less than seventeen. **21.** Five plus two is not equal to ten.

**22.** Two-thirds is greater than or equal to four-sixths.

**1.3** *Evaluate each expression for* $x = 6$ *and* $y = 3$.

**23.** $2x + 6y$ **24.** $4(3x - y)$ **25.** $\frac{x}{3} + 4y$ **26.** $\frac{x^2 + 3}{3y - x}$

*Write each word phrase as an algebraic expression, using x as the variable.*

**27.** Six added to a number **28.** A number subtracted from eight

**29.** Nine subtracted from six times a number **30.** Three-fifths of a number added to 12

*Decide whether the given number is a solution of the given equation.*

**31.** $5x + 3(x + 2) = 22; \quad 2$ **32.** $\frac{t + 5}{3t} = 1; \quad 6$

*Write each word statement as an equation. Use x as the variable. Then find the solution from the set* $\{0, 2, 4, 6, 8, 10\}$.

**33.** Six less than twice a number is 10. **34.** The product of a number and 4 is 8.

**1.4** *Graph each group of numbers on a number line.*

**35.** $-4, -\frac{1}{2}, 0, 2.5, 5$ **36.** $-2, |-3|, -3, |-1|$

*Classify each number, using the sets* natural numbers, whole numbers, integers, rational numbers, irrational numbers, *and* real numbers.

**37.** $\frac{4}{3}$ **38.** $0.\overline{63}$ **39.** $19$ **40.** $\sqrt{6}$

*Select the lesser number in each pair.*

**41.** $-10, 5$ **42.** $-8, -9$ **43.** $-\frac{2}{3}, -\frac{3}{4}$ **44.** $0, -|23|$

*Decide whether each statement is* true *or* false.

**45.** $12 > -13$ **46.** $0 > -5$ **47.** $-9 < -7$ **48.** $-13 \geq -13$

*For each number,* **(a)** *find the opposite of the number and* **(b)** *find the absolute value of the number.*

**49.** $-9$ **50.** $0$ **51.** $6$ **52.** $-\frac{5}{7}$

*Simplify.*

**53.** $|-12|$ **54.** $-|3|$ **55.** $-|-19|$ **56.** $-|9 - 2|$

**1.5** *Perform each indicated operation.*

**57.** $-10 + 4$ **58.** $14 + (-18)$ **59.** $-8 + (-9)$

**60.** $\frac{4}{9} + \left(-\frac{5}{4}\right)$ **61.** $-13.5 + (-8.3)$ **62.** $(-10 + 7) + (-11)$

**63.** $[-6 + (-8) + 8] + [9 + (-13)]$ **64.** $(-4 + 7) + (-11 + 3) + (-15 + 1)$

**65.** $-7 - 4$

**66.** $-12 - (-11)$

**67.** $5 - (-2)$

**68.** $-\frac{3}{7} - \frac{4}{5}$

**69.** $2.56 - (-7.75)$

**70.** $(-10 - 4) - (-2)$

**71.** $(-3 + 4) - (-1)$

**72.** $-(-5 + 6) - 2$

*Write a numerical expression for each phrase, and simplify the expression.*

**73.** 19 added to the sum of −31 and 12

**74.** 13 more than the sum of −4 and −8

**75.** The difference between −4 and −6

**76.** Five less than the sum of 4 and −8

*Find the solution of each equation from the set* $\{-3, -2, -1, 0, 1, 2, 3\}$.

**77.** $x + (-2) = -4$

**78.** $12 + x = 11$

*Solve each problem.*

**79.** George Fagley found that his checkbook balance was −\$23.75, so he deposited \$50.00. What is his new balance?

**80.** The low temperature in Yellowknife, in the Canadian Northwest Territories, one January day was −26°F. It rose 16° that day. What was the high temperature?

**81.** Reginald Fulwood owed a friend \$28. He repaid \$13, but then borrowed another \$14. What positive or negative amount represents his present financial status?

**82.** If the temperature drops 7° below its previous level of −3°, what is the new temperature?

**83.** Mark Sanchez of the New York Jets passed for a gain of 8 yd, was sacked for a loss of 12 yd, and then threw a 42 yd touchdown pass. What positive or negative number represents the total net yardage for the plays?

**84.** On Monday, August 31, 2009, the Dow Jones Industrial Average closed at 9496.28, down 47.92 from the previous Friday. What was the closing value the previous Friday? (*Source: The Washington Post.*)

**1.6** *Perform each indicated operation.*

**85.** $(-12)(-3)$

**86.** $15(-7)$

**87.** $-\frac{4}{3}\left(-\frac{3}{8}\right)$

**88.** $(-4.8)(-2.1)$

**89.** $5(8 - 12)$

**90.** $(5 - 7)(8 - 3)$

**91.** $2(-6) - (-4)(-3)$

**92.** $3(-10) - 5$

**93.** $\frac{-36}{-9}$

**94.** $\frac{220}{-11}$

**95.** $-\frac{1}{2} \div \frac{2}{3}$

**96.** $-33.9 \div (-3)$

**97.** $\frac{-5(3) - 1}{8 - 4(-2)}$

**98.** $\frac{5(-2) - 3(4)}{-2[3 - (-2)] - 1}$

**99.** $\frac{10^2 - 5^2}{8^2 + 3^2 - (-2)}$

**100.** $\frac{(0.6)^2 + (0.8)^2}{(-1.2)^2 - (-0.56)}$

*Evaluate each expression if* $x = -5$, $y = 4$, *and* $z = -3$.

**101.** $6x - 4z$

**102.** $5x + y - z$

**103.** $5x^2$

**104.** $z^2(3x - 8y)$

*Write a numerical expression for each phrase, and simplify the expression.*

**105.** Nine less than the product of −4 and 5

**106.** Five-sixths of the sum of 12 and −6

**107.** The quotient of 12 and the sum of 8 and −4

**108.** The product of −20 and 12, divided by the difference between 15 and −15

*Write each sentence in symbols, using x as the variable, and find the solution from the list of integers between* $-12$ *and* 12.

**109.** 8 times a number is $-24$.

**110.** The quotient of a number and 3 is $-2$.

*Find the average of each group of numbers.*

**111.** 26, 38, 40, 20, 4, 14, 96, 18

**112.** $-12, 28, -36, 0, 12, -10$

**1.7** *Decide whether each statement is an example of the* commutative, associative, identity, inverse, *or* distributive *property.*

**113.** $6 + 0 = 6$

**114.** $5 \cdot 1 = 5$

**115.** $-\frac{2}{3}\left(-\frac{3}{2}\right) = 1$

**116.** $17 + (-17) = 0$

**117.** $5 + (-9 + 2) = [5 + (-9)] + 2$

**118.** $w(xy) = (wx)y$

**119.** $3x + 3y = 3(x + y)$

**120.** $(1 + 2) + 3 = 3 + (1 + 2)$

*Use the distributive property to rewrite each expression. Simplify if possible.*

**121.** $7y + 14$

**122.** $-12(4 - t)$

**123.** $3(2s) + 3(5y)$

**124.** $-(-4r + 5s)$

**1.8** *Combine like terms whenever possible.*

**125.** $2m + 9m$

**126.** $15p^2 - 7p^2 + 8p^2$

**127.** $5p^2 - 4p + 6p + 11p^2$

**128.** $-2(3k - 5) + 2(k + 1)$

**129.** $7(2m + 3) - 2(8m - 4)$

**130.** $-(2k + 8) - (3k - 7)$

*Translate each phrase into a mathematical expression. Use x to represent the number, and combine like terms when possible.*

**131.** Seven times a number, subtracted from the product of $-2$ and three times the number

**132.** A number multiplied by 8, added to the sum of 5 and four times the number

## MIXED REVIEW EXERCISES*

*Perform each indicated operation.*

**133.** $\frac{6(-4) + 2(-12)}{5(-3) + (-3)}$

**134.** $\frac{3}{8} - \frac{5}{12}$

**135.** $\frac{8^2 + 6^2}{7^2 + 1^2}$

**136.** $-\frac{12}{5} \div \frac{9}{7}$

**137.** $2\frac{5}{6} - 4\frac{1}{3}$

**138.** $\left(-\frac{5}{6}\right)^2$

**139.** $[(-2) + 7 - (-5)] + [-4 - (-10)]$

**140.** $-16(-3.5) - 7.2(-3)$

**141.** $-8 + [(-4 + 17) - (-3 - 3)]$

**142.** $-4(2t + 1) - 8(-3t + 4)$

**143.** $5x^2 - 12y^2 + 3x^2 - 9y^2$

**144.** $(-8 - 3) - 5(2 - 9)$

**145.** Write a sentence or two explaining the special considerations involving 0 in division.

**146.** The highest temperature ever recorded in Iowa was 118°F at Keokuk on July 20, 1934. The lowest temperature ever recorded in the state was at Elkader on February 3, 1996, and was 165° lower than the highest temperature. What is the record low temperature for Iowa? (*Source:* National Climatic Data Center.)

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*The order of exercises in this final group does not correspond to the order in which topics occur in the chapter. This random ordering should help you prepare for the chapter test in yet another way.

(continued)

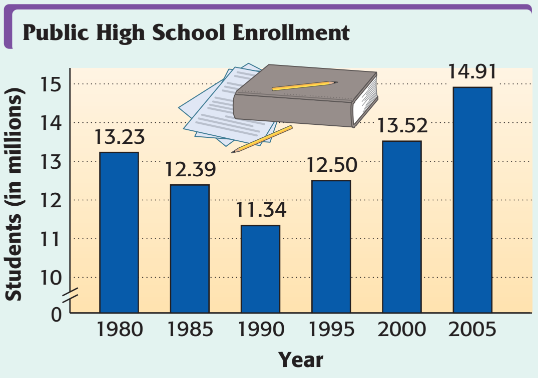
*The bar graph shows public high school (grades 9–12) enrollment in millions for selected years from 1980 to 2005 in the United States. Use a signed number to represent the change in enrollment for each period.*

**147.** 1980 to 1985

**148.** 1985 to 1990

**149.** 1995 to 2000

**150.** 2000 to 2005



***Source:*** U.S. National Center for Education Statistics.

# CHAPTER 1 TEST

Chapter Test Prep Videos

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

**1.** Write $\frac{63}{99}$ in lowest terms. **2.** Add: $\frac{5}{8} + \frac{11}{12} + \frac{7}{15}$. **3.** Divide: $\frac{19}{15} \div \frac{6}{5}$.

**4.** *True* or *false*? $4[-20 + 7(-2)] \leq 135$

**5.** Graph the group of numbers $-1, -3, |-4|, |-1|$ on a number line.

**6.** To which of the following sets does $-\frac{2}{3}$ belong: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers?

**7.** Explain how a number line can be used to show that $-8$ is less than $-1$.

**8.** Write in symbols: The quotient of $-6$ and the sum of 2 and $-8$. Simplify the expression.

*Perform each indicated operation.*

**9.** $-2 - (5 - 17) + (-6)$ **10.** $-5\frac{1}{2} + 2\frac{2}{3}$

**11.** $-6 - [-7 + (2 - 3)]$ **12.** $4^2 + (-8) - (2^3 - 6)$

**13.** $(-5)(-12) + 4(-4) + (-8)^2$ **14.** $\frac{30(-1 - 2)}{-9[3 - (-2)] - 12(-2)}$

*Find the solution of each equation from the set* $\{-6, -4, -2, 0, 2, 4, 6\}$.

**15.** $-x + 3 = -3$ **16.** $-3x = -12$

*Evaluate each expression, given* $x = -2$ *and* $y = 4$.

**17.** $3x - 4y^2$ **18.** $\frac{5x + 7y}{3(x + y)}$

*Solve each problem.*

**19.** The highest elevation in Argentina is Mt. Aconcagua, which is 6960 m above sea level. The lowest point in Argentina is the Valdés Peninsula, 40 m below sea level. Find the difference between the highest and lowest elevations.

**20.** For a certain system of rating relief pitchers, 3 points are awarded for a save, 3 points are awarded for a win, 2 points are subtracted for a loss, and 2 points are subtracted for a blown save. If Brad Lidge of the Philadelphia Phillies has 4 saves, 3 wins, 2 losses, and 1 blown save, how many points does he have?

**21.** For 2009, the U.S. federal government collected \$2.10 trillion in revenues, but spent \$3.52 trillion. Write the federal budget deficit as a signed number. (*Source: The Gazette*.)

*Match each property in Column I with the example of it in Column II.*

| I | II |
|---|---|
| **22.** Commutative property | **A.** $3x + 0 = 3x$ |
| **23.** Associative property | **B.** $(5 + 2) + 8 = 8 + (5 + 2)$ |
| **24.** Inverse property | **C.** $-3(x + y) = -3x + (-3y)$ |
| **25.** Identity property | **D.** $-5 + (3 + 2) = (-5 + 3) + 2$ |
| **26.** Distributive property | **E.** $-\frac{5}{3}\left(-\frac{3}{5}\right) = 1$ |

**27.** What property is used to clear parentheses and write $3(x + 1)$ as $3x + 3$?

**28.** Consider the expression $-6[5 + (-2)]$.

**(a)** Evaluate it by first working within the brackets.

**(b)** Evaluate it by using the distributive property.

**(c)** Why must the answers in parts (a) and (b) be the same?

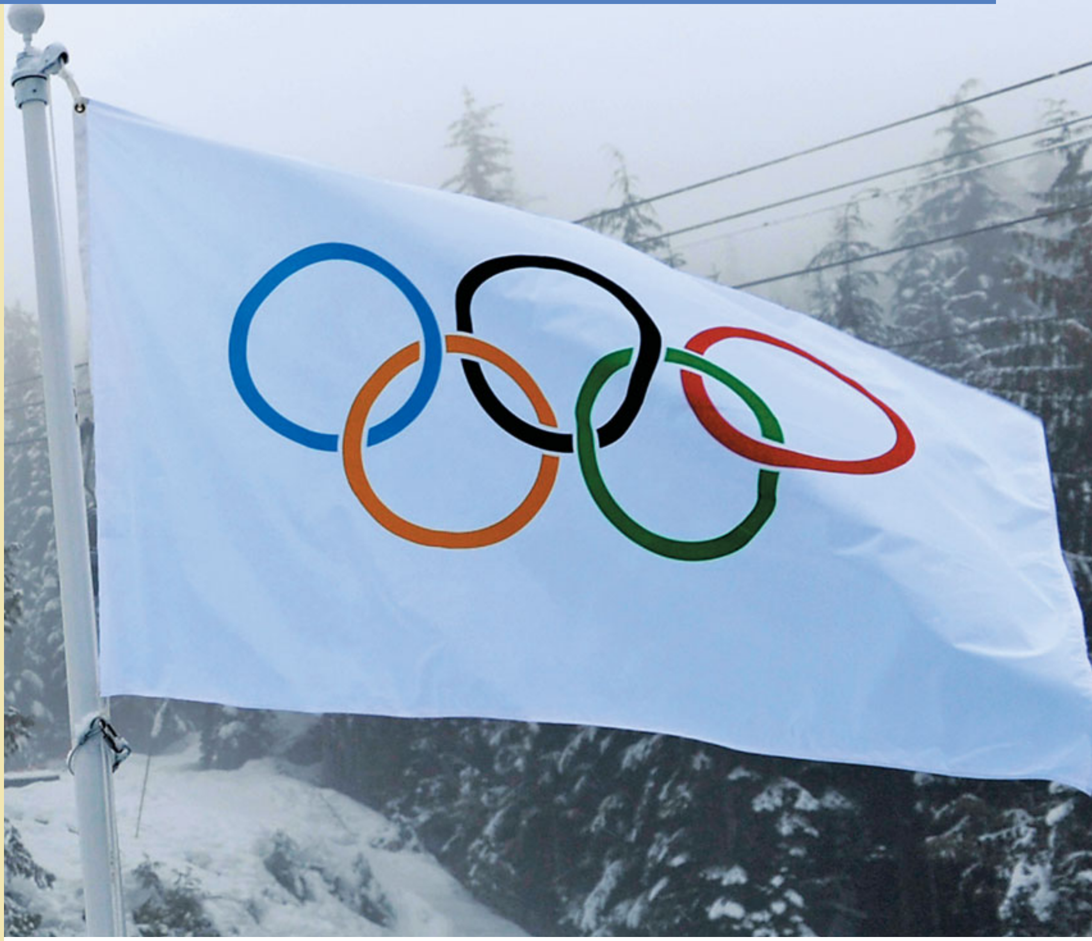
*Simplify by combining like terms.*

**29.** $8x + 4x - 6x + x + 14x$

**30.** $5(2x - 1) - (x - 12) + 2(3x - 5)$

CHAPTER 2

# Linear Equations and Inequalities in One Variable



In 1924, 258 competitors gathered in Chamonix, France, for the 16 events of the first Olympic Winter Games. This small, mainly European, sports competition has become the world's largest global sporting event. The XXI Olympic Winter Games, hosted in 2010 by Vancouver, British Columbia, attracted 2500 athletes, who competed in 86 events. First introduced at the 1920 Games in Antwerp, Belgium, the five interlocking rings on the Olympic flag symbolize unity among the nations of Africa, the Americas, Asia, Australia, and Europe. (*Source:* www.olympic.org)

Throughout this chapter we use *linear equations* to solve applications about the Olympics.

## 2.1 The Addition Property of Equality

**OBJECTIVES**

1. Identify linear equations.
2. Use the addition property of equality.
3. Simplify, and then use the addition property of equality.

An **equation** is a statement asserting that two algebraic expressions are equal.

**CAUTION** ***Remember that an equation includes an equals symbol.***

$$\underbrace{x - 5}_{\text{Left side}} \overset{\substack{\text{Equation (to solve)}\\ \downarrow}}{=} \underbrace{2}_{\text{Right side}} \qquad\qquad \overset{\substack{\text{Expression (to simplify or evaluate)}\\ \downarrow}}{x - 5}$$

**OBJECTIVE 1** **Identify linear equations.** The simplest type of equation is a *linear equation.*

**Linear Equation in One Variable**

A **linear equation in one variable** can be written in the form

$$Ax + B = C,$$

where $A$, $B$, and $C$ are real numbers, and $A \neq 0$.

$$4x + 9 = 0, \quad 2x - 3 = 5, \quad \text{and} \quad x = 7 \qquad \text{Linear equations}$$

$$x^2 + 2x = 5, \quad \frac{1}{x} = 6, \quad \text{and} \quad |2x + 6| = 0 \qquad \textit{Non}\text{linear equations}$$

A **solution** of an equation is a number that makes the equation true when it replaces the variable. An equation is solved by finding its **solution set,** the set of all solutions. Equations with exactly the same solution sets are **equivalent equations.**

A linear equation in $x$ is solved by using a series of steps to produce a simpler equivalent equation of the form

$$x = \textbf{a number} \qquad \text{or} \qquad \textbf{a number} = x.$$

**OBJECTIVE 2** **Use the addition property of equality.** In the linear equation $x - 5 = 2$, both $x - 5$ and 2 represent the same number because that is the meaning of the equals symbol. To solve the equation, we change the left side from $x - 5$ to just $x$, as follows.

| | |
|---|---|
| $x - 5 = 2$ | Given equation |
| $x - 5 + 5 = 2 + 5$ | Add 5 to *each* side to keep them equal. |
| $x + 0 = 7$ | Additive inverse property |
| $x = 7$ | Additive identity property |

Add 5. It is the opposite (additive inverse) of $-5$, and $-5 + 5 = 0$.

The solution is 7. We check by replacing $x$ with 7 in the original equation.

*CHECK*

| | |
|---|---|
| $x - 5 = 2$ | Original equation |
| $7 - 5 \overset{?}{=} 2$ | Let $x = 7$. |
| $2 = 2$ ✓ | True |

The left side equals the right side.

Since the final equation is true, 7 checks as the solution and $\{7\}$ is the solution set.

To solve the equation $x - 5 = 2$, we used the **addition property of equality.**

**Addition Property of Equality**

If $A$, $B$, and $C$ represent real numbers, then the equations

$$A = B \qquad \text{and} \qquad A + C = B + C$$

are equivalent equations.

That is, we can add the same number to each side of an equation without changing the solution.

In this property, $C$ represents a real number. Any quantity that represents a real number can be added to each side of an equation to obtain an equivalent equation.

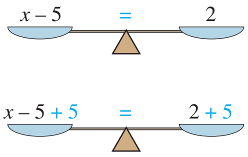


**FIGURE 1**

**NOTE** Equations can be thought of in terms of a balance. Thus, adding the same quantity to each side does not affect the balance. See **FIGURE 1**.

*NOW TRY EXERCISE 1*

Solve $x - 13 = 4$.

### EXAMPLE 1 Applying the Addition Property of Equality

Solve $x - 16 = 7$.

Our goal is to get an equivalent equation of the form $x =$ a number.

$$x - 16 = 7$$

$$x - 16 + 16 = 7 + 16 \qquad \text{Add 16 to each side.}$$

$$x = 23 \qquad \text{Combine like terms.}$$

*CHECK* Substitute 23 for $x$ in the *original* equation.

$$x - 16 = 7 \qquad \text{Original equation}$$

$$23 - 16 \stackrel{?}{=} 7 \qquad \text{Let } x = 23.$$

7 is *not* the solution.

$$7 = 7 \ \checkmark \qquad \text{True}$$

Since a true statement results, 23 is the solution and $\{23\}$ is the solution set.

*NOW TRY*

**CAUTION** ***The final line of the check does*** **not** ***give the solution to the problem,*** only a confirmation that the solution found is correct.

*NOW TRY EXERCISE 2*

Solve $t - 5.7 = -7.2$.

### EXAMPLE 2 Applying the Addition Property of Equality

Solve $x - 2.9 = -6.4$.

Our goal is to isolate $x$.

$$x - 2.9 = -6.4$$

$$x - 2.9 + 2.9 = -6.4 + 2.9 \qquad \text{Add 2.9 to each side.}$$

$$x = -3.5$$

*CHECK*

$$x - 2.9 = -6.4 \qquad \text{Original equation}$$

$$-3.5 - 2.9 \stackrel{?}{=} -6.4 \qquad \text{Let } x = -3.5.$$

$$-6.4 = -6.4 \ \checkmark \qquad \text{True}$$

Since a true statement results, the solution set is $\{-3.5\}$.

*NOW TRY*

*NOW TRY ANSWERS*

**1.** $\{17\}$ **2.** $\{-1.5\}$

The addition property of equality says that the same number may be *added* to each side of an equation. In **Section 1.5,** subtraction was defined as addition of the opposite. Thus, we can also use the following rule when solving an equation.

**The same number may be *subtracted* from each side of an equation without changing the solution.**

*NOW TRY EXERCISE 3*

Solve $-15 = x + 12$.

## EXAMPLE 3 Applying the Addition Property of Equality

Solve $-7 = x + 22$.

Here, the variable $x$ is on the right side of the equation.

$$-7 = x + 22$$

The variable can be isolated on *either* side.

$$-7 - 22 = x + 22 - 22$$ Subtract 22 from each side.

$$-29 = x, \quad \text{or} \quad x = -29$$ Rewrite; a number $= x$, or $x =$ a number.

*CHECK*

$$-7 = x + 22$$ Original equation

$$-7 \overset{?}{=} -29 + 22$$ Let $x = -29$.

$$-7 = -7 \ \checkmark$$ True

The check confirms that the solution set is $\{-29\}$. *NOW TRY*

---

**NOTE** In **Example 3,** what happens if we subtract $-7 - 22$ incorrectly, obtaining $x = -15$, instead of $x = -29$, as the last line of the solution? A check should indicate an error.

*CHECK*

$$-7 = x + 22$$ Original equation from **Example 3**

$$-7 \overset{?}{=} -15 + 22$$ Let $x = -15$.

$$-7 = 7$$ False

The left side does *not* equal the right side.

The false statement indicates that $-15$ is *not* a solution of the equation. If this happens, rework the problem.

---

*NOW TRY EXERCISE 4*

Solve $\frac{2}{3}x - 4 = \frac{5}{3}x$.

## EXAMPLE 4 Subtracting a Variable Expression

Solve $\frac{3}{5}x + 17 = \frac{8}{5}x$.

$$\frac{3}{5}x + 17 = \frac{8}{5}x$$ Original equation

$$\frac{3}{5}x + 17 - \frac{3}{5}x = \frac{8}{5}x - \frac{3}{5}x$$ Subtract $\frac{3}{5}x$ from each side.

$$17 = 1x$$ $\frac{3}{5}x - \frac{3}{5}x = 0$; $\frac{8}{5}x - \frac{3}{5}x = \frac{5}{5}x = 1x$

From now on we will skip this step.

$$17 = x$$ Multiplicative identity property

Check by replacing $x$ with 17 in the original equation. The solution set is $\{17\}$. *NOW TRY*

*NOW TRY ANSWERS*

**3.** $\{-27\}$ **4.** $\{-4\}$

What happens in **Example 4** if we start by subtracting $\frac{8}{5}x$ from each side?

$$\frac{3}{5}x + 17 = \frac{8}{5}x \qquad \text{Original equation from **Example 4**}$$

$$\frac{3}{5}x + 17 - \frac{8}{5}x = \frac{8}{5}x - \frac{8}{5}x \qquad \text{Subtract } \tfrac{8}{5}x \text{ from each side.}$$

$$17 - x = 0 \qquad \tfrac{3}{5}x - \tfrac{8}{5}x = -\tfrac{5}{5}x = -1x = -x;\ \tfrac{8}{5}x - \tfrac{8}{5}x = 0$$

$$17 - x - 17 = 0 - 17 \qquad \text{Subtract 17 from each side.}$$

$$-x = -17 \qquad \text{Combine like terms; additive inverse}$$

This result gives the value of $-x$, but not of $x$ itself. However, it does say that the additive inverse of $x$ is $-17$, which means that $x$ must be 17.

$$x = 17 \qquad \text{Same result as in **Example 4**}$$

We can make the following generalization:

**If $a$ is a number and $-x = a$, then $x = -a$.**

*NOW TRY EXERCISE 5*
Solve $6x - 8 = 12 + 5x$.

### EXAMPLE 5 Applying the Addition Property of Equality Twice

Solve $8 - 6p = -7p + 5$.

$$8 - 6p = -7p + 5$$

$$8 - 6p + 7p = -7p + 5 + 7p \qquad \text{Add } 7p \text{ to each side.}$$

$$8 + p = 5 \qquad \text{Combine like terms.}$$

$$8 + p - 8 = 5 - 8 \qquad \text{Subtract 8 from each side.}$$

$$p = -3 \qquad \text{Combine like terms.}$$

*CHECK*

$$8 - 6p = -7p + 5 \qquad \text{Original equation}$$

$$8 - 6(-3) \stackrel{?}{=} -7(-3) + 5 \qquad \text{Let } p = -3.$$

$$8 + 18 \stackrel{?}{=} 21 + 5 \qquad \text{Multiply.}$$

$$26 = 26 \ ✓ \qquad \text{True}$$

Use parentheses when substituting to avoid errors.

The check results in a true statement, so the solution set is $\{-3\}$. *NOW TRY*

**NOTE** ***There are often several correct ways to solve an equation.*** In **Example 5,** we could begin by adding $6p$ to each side. Combining like terms and subtracting 5 from each side gives $3 = -p$. (Try this.) If $3 = -p$, then $-3 = p$, and the variable has been isolated on the right side of equation. The same solution results.

## OBJECTIVE 3 Simplify, and then use the addition property of equality.

### EXAMPLE 6 Combining Like Terms When Solving

Solve $3t - 12 + t + 2 = 5 + 3t + 2$.

$$3t - 12 + t + 2 = 5 + 3t + 2$$

$$4t - 10 = 7 + 3t \qquad \text{Combine like terms.}$$

$$4t - 10 - 3t = 7 + 3t - 3t \qquad \text{Subtract } 3t \text{ from each side.}$$

$$t - 10 = 7 \qquad \text{Combine like terms.}$$

$$t - 10 + 10 = 7 + 10 \qquad \text{Add 10 to each side.}$$

$$t = 17 \qquad \text{Combine like terms.}$$

*NOW TRY ANSWER*
**5.** $\{20\}$

**NOW TRY EXERCISE 6**

Solve.

$$5x - 10 - 12x = 4 - 8x - 9$$

$$\begin{aligned}
&\textbf{CHECK} & 3t - 12 + t + 2 &= 5 + 3t + 2 && \text{Original equation}\\
&& 3(17) - 12 + 17 + 2 &\overset{?}{=} 5 + 3(17) + 2 && \text{Let } t = 17.\\
&& 51 - 12 + 17 + 2 &\overset{?}{=} 5 + 51 + 2 && \text{Multiply.}\\
&& 58 &= 58 \checkmark && \text{True}
\end{aligned}$$

The check results in a true statement, so the solution set is $\{17\}$. NOW TRY

**NOW TRY EXERCISE 7**

Solve.

$$4(3x - 2) - (11x - 4) = 3$$

### EXAMPLE 7 Using the Distributive Property When Solving

Solve $3(2 + 5x) - (1 + 14x) = 6$.

Be sure to distribute to *all* terms within the parentheses.

Be careful here!

$$\begin{aligned}
3(2 + 5x) - (1 + 14x) &= 6 &&\\
3(2 + 5x) - 1(1 + 14x) &= 6 && -(1 + 14x) = -1(1 + 14x)\\
3(2) + 3(5x) - 1(1) - 1(14x) &= 6 && \text{Distributive property}\\
6 + 15x - 1 - 14x &= 6 && \text{Multiply.}\\
x + 5 &= 6 && \text{Combine like terms.}\\
x + 5 - 5 &= 6 - 5 && \text{Subtract 5 from each side.}\\
x &= 1 && \text{Combine like terms.}
\end{aligned}$$

Check by substituting 1 for $x$ in the original equation. The solution set is $\{1\}$.

NOW TRY

**NOW TRY ANSWERS**

**6.** $\{5\}$ **7.** $\{7\}$

---

⚠ **CAUTION** ***Be careful to apply the distributive property correctly*** in a problem like that in **Example 7,** or a sign error may result.

---

## 2.1 EXERCISES

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**1.** *Concept Check* Decide whether each of the following is an *expression* or an *equation*. If it is an expression, simplify it. If it is an equation, solve it.

**(a)** $5x + 8 - 4x + 7$ **(b)** $-6y + 12 + 7y - 5$

**(c)** $5x + 8 - 4x = 7$ **(d)** $-6y + 12 + 7y = -5$

**2.** *Concept Check* Which pairs of equations are equivalent equations?

**A.** $x + 2 = 6$ and $x = 4$ **B.** $10 - x = 5$ and $x = -5$

**C.** $x + 3 = 9$ and $x = 6$ **D.** $4 + x = 8$ and $x = -4$

**3.** *Concept Check* Which of the following are *not* linear equations in one variable?

**A.** $x^2 - 5x + 6 = 0$ **B.** $x^3 = x$ **C.** $3x - 4 = 0$ **D.** $7x - 6x = 3 + 9x$

✎ **4.** Explain how to check a solution of an equation.

*Solve each equation, and check your solution.* ***See Examples 1–5.***

**5.** $x - 3 = 9$ **6.** $x - 9 = 8$ ◉ **7.** $x - 12 = 19$

**8.** $x - 18 = 22$ **9.** $x - 6 = -9$ **10.** $x - 5 = -7$

**11.** $r + 8 = 12$

**12.** $x + 7 = 11$

**13.** $x + 28 = 19$

**14.** $x + 47 = 26$

**15.** $x + \frac{1}{4} = -\frac{1}{2}$

**16.** $x + \frac{2}{3} = -\frac{1}{6}$

**17.** $7 + r = -3$

**18.** $8 + k = -4$

**19.** $2 = p + 15$

**20.** $5 = z + 19$

**21.** $-4 = x - 14$

**22.** $-7 = x - 22$

**23.** $-\frac{1}{3} = x - \frac{3}{5}$

**24.** $-\frac{1}{4} = x - \frac{2}{3}$

**25.** $x - 8.4 = -2.1$

**26.** $x - 15.5 = -5.1$

**27.** $t + 12.3 = -4.6$

**28.** $x + 21.5 = -13.4$

**29.** $3x = 2x + 7$

**30.** $5x = 4x + 9$

**31.** $10x + 4 = 9x$

**32.** $8t + 5 = 7t$

**33.** $3x + 7 = 2x + 4$

**34.** $9x + 1 = 8x + 4$

**35.** $8t + 6 = 7t + 6$

**36.** $13t + 9 = 12t + 9$

**37.** $-4x + 7 = -5x + 9$

**38.** $-6x + 3 = -7x + 10$

**39.** $\frac{2}{5}w - 6 = \frac{7}{5}w$

**40.** $\frac{2}{7}z - 2 = \frac{9}{7}z$

**41.** $5.6x + 2 = 4.6x$

**42.** $9.1x + 5 = 8.1x$

**43.** $1.4x - 3 = 0.4x$

**44.** $1.9t - 6 = 0.9t$

**45.** $5p = 4p$

**46.** $8z = 7z$

**47.** $1.2y - 4 = 0.2y - 4$

**48.** $7.7r - 6 = 6.7r - 6$

**49.** $\frac{1}{2}x + 5 = -\frac{1}{2}x$

**50.** $\frac{1}{5}x + 7 = -\frac{4}{5}x$

**51.** $3x + 7 - 2x = 0$

**52.** $5x + 4 - 4x = 0$

*Solve each equation, and check your solution.* ***See Examples 6 and 7.***

**53.** $5t + 3 + 2t - 6t = 4 + 12$

**54.** $4x + 3x - 6 - 6x = 10 + 3$

**55.** $6x + 5 + 7x + 3 = 12x + 4$

**56.** $4x - 3 - 8x + 1 = -5x + 9$

**57.** $5.2q - 4.6 - 7.1q = -0.9q - 4.6$

**58.** $-4.0x + 2.7 - 1.6x = -4.6x + 2.7$

**59.** $\frac{5}{7}x + \frac{1}{3} = \frac{2}{5} - \frac{2}{7}x + \frac{2}{5}$

**60.** $\frac{6}{7}s - \frac{3}{4} = \frac{4}{5} - \frac{1}{7}s + \frac{1}{6}$

**61.** $(5y + 6) - (3 + 4y) = 10$

**62.** $(8r - 3) - (7r + 1) = -6$

**63.** $2(p + 5) - (9 + p) = -3$

**64.** $4(k - 6) - (3k + 2) = -5$

**65.** $-6(2b + 1) + (13b - 7) = 0$

**66.** $-5(3w - 3) + (1 + 16w) = 0$

**67.** $10(-2x + 1) = -19(x + 1)$

**68.** $2(2 - 3r) = -5(r - 3)$

***Brain Busters*** *Solve each equation, and check your solution.* ***See Examples 6 and 7.***

**69.** $-2(8p + 2) - 3(2 - 7p) - 2(4 + 2p) = 0$

**70.** $-5(1 - 2z) + 4(3 - z) - 7(3 + z) = 0$

**71.** $4(7x - 1) + 3(2 - 5x) - 4(3x + 5) = -6$

**72.** $9(2m - 3) - 4(5 + 3m) - 5(4 + m) = -3$

**73.** ***Concept Check*** Write an equation that requires the use of the addition property of equality, in which 6 must be added to each side to solve the equation and the solution is a negative number.

**74.** ***Concept Check*** Write an equation that requires the use of the addition property of equality, in which $\frac{1}{2}$ must be subtracted from each side and the solution is a positive number.

*Write an equation using the information given in the problem. Use x as the variable. Then solve the equation.*

**75.** Three times a number is 17 more than twice the number. Find the number.

**76.** One added to three times a number is three less than four times the number. Find the number.

**77.** If six times a number is subtracted from seven times the number, the result is $-9$. Find the number.

**78.** If five times a number is added to three times the number, the result is the sum of seven times the number and 9. Find the number.

**"Preview Exercises" are designed to *review* ideas introduced earlier, as well as *preview* ideas needed for the next section.**

## PREVIEW EXERCISES

*Simplify each expression.* ***See Section 1.8.***

**79.** $\frac{2}{3}\left(\frac{3}{2}\right)$

**80.** $\frac{5}{6}\left(\frac{6}{5}\right)$

**81.** $-\frac{5}{4}\left(-\frac{4}{5}x\right)$

**82.** $-\frac{9}{7}\left(-\frac{7}{9}x\right)$

**83.** $9\left(\frac{r}{9}\right)$

**84.** $6\left(\frac{t}{6}\right)$

# 2.2

# The Multiplication Property of Equality

**OBJECTIVES**

1. Use the multiplication property of equality.
2. Simplify, and then use the multiplication property of equality.

**OBJECTIVE 1** **Use the multiplication property of equality.** The addition property of equality from **Section 2.1** is not enough to solve some equations, such as $3x + 2 = 17$.

$$3x + 2 = 17$$

$$3x + 2 - 2 = 17 - 2 \quad \text{Subtract 2 from each side.}$$

$$3x = 15 \quad \text{Combine like terms.}$$

The coefficient of $x$ is 3, not 1 as desired. Another property, the **multiplication property of equality,** is needed to change $3x = 15$ to an equation of the form

$$\boldsymbol{x} = \textbf{a number}.$$

Since $3x = 15$, both $3x$ and 15 must represent the same number. Multiplying both $3x$ and 15 by the same number will also result in an equality.

### Multiplication Property of Equality

If $A$, $B$, and $C$ $(C \neq 0)$ represent real numbers, then the equations

$$\boldsymbol{A = B} \qquad \text{and} \qquad \boldsymbol{AC = BC}$$

are equivalent equations.

That is, we can multiply each side of an equation by the same nonzero number without changing the solution.

In $3x = 15$, we must change $3x$ to $1x$, or $x$. To do this, we multiply each side of the equation by $\frac{1}{3}$, the reciprocal of 3, because $\frac{1}{3} \cdot 3 = \frac{3}{3} = 1$.

$$\begin{aligned} 3x &= 15 & & \\ \frac{1}{3}(3x) &= \frac{1}{3} \cdot 15 & & \text{Multiply each side by } \tfrac{1}{3}. \\ \underbrace{\left(\frac{1}{3} \cdot 3\right)}x &= \frac{1}{3} \cdot 15 & & \text{Associative property} \\ 1x &= 5 & & \text{Multiplicative inverse property} \\ x &= 5 & & \text{Multiplicative identity property} \end{aligned}$$

The product of a number and its reciprocal is 1.

The solution is 5. We can check this result in the original equation.

Just as the addition property of equality permits *subtracting* the same number from each side of an equation, the multiplication property of equality permits *dividing* each side of an equation by the same nonzero number.

$$\begin{aligned} 3x &= 15 & & \\ \frac{3x}{3} &= \frac{15}{3} & & \text{Divide each side by 3.} \\ x &= 5 & & \text{Same result as above} \end{aligned}$$

We can divide each side of an equation by the same nonzero number without changing the solution. ***Do not, however, divide each side by a variable, since the variable might be equal to* 0.**

---

**NOTE** In practice, it is usually easier to multiply on each side if the coefficient of the variable is a fraction, and divide on each side if the coefficient is an integer. For example, to solve

$\frac{3}{4}x = 12$, it is easier to multiply by $\frac{4}{3}$ than to divide by $\frac{3}{4}$.

On the other hand, to solve

$5x = 20$, it is easier to divide by 5 than to multiply by $\frac{1}{5}$.

---

**NOW TRY EXERCISE 1**

Solve $8x = 80$.

## EXAMPLE 1 Applying the Multiplication Property of Equality

Solve $5x = 60$.

$$\begin{aligned} 5x &= 60 & & \\ \frac{5x}{5} &= \frac{60}{5} & & \text{Divide each side by 5, the coefficient of } x. \\ x &= 12 & & \frac{5x}{5} = \frac{5}{5}x = 1x = x \end{aligned}$$

Our goal is to isolate $x$.

Dividing by 5 is the same as multiplying by $\frac{1}{5}$.

*CHECK* Substitute 12 for $x$ in the original equation.

$$\begin{aligned} 5x &= 60 & & \text{Original equation} \\ 5(12) &\overset{?}{=} 60 & & \text{Let } x = 12. \\ 60 &= 60 \ ✓ & & \text{True} \end{aligned}$$

Since a true statement results, the solution set is $\{12\}$. NOW TRY

**NOW TRY ANSWER**

**1.** $\{10\}$

*NOW TRY EXERCISE 2*

Solve $10x = -24$.

## EXAMPLE 2 Applying the Multiplication Property of Equality

Solve $25x = -30$.

$$25x = -30$$

$$\frac{25x}{25} = \frac{-30}{25} \qquad \text{Divide each side by 25, the coefficient of } x.$$

$$x = \frac{-30}{25} = -\frac{6}{5} \qquad \frac{-a}{b} = -\frac{a}{b}\text{; Write in lowest terms.}$$

*CHECK*

$$25x = -30 \qquad \text{Original equation}$$

$$\frac{25}{1}\left(-\frac{6}{5}\right) \stackrel{?}{=} -30 \qquad \text{Let } x = -\frac{6}{5}.$$

$$-30 = -30 \checkmark \qquad \text{True}$$

The check confirms that the solution set is $\left\{-\frac{6}{5}\right\}$. *NOW TRY*

*NOW TRY EXERCISE 3*

Solve $-1.3x = 7.02$.

## EXAMPLE 3 Solving an Equation with Decimals

Solve $-2.1x = 6.09$.

$$-2.1x = 6.09$$

$$\frac{-2.1x}{-2.1} = \frac{6.09}{-2.1} \qquad \text{Divide each side by } -2.1.$$

$$x = -2.9 \qquad \text{Divide.}$$

Check by replacing $x$ with $-2.9$ in the original equation. The solution set is $\{-2.9\}$. *NOW TRY*

*NOW TRY EXERCISE 4*

Solve $\frac{x}{5} = -7$.

## EXAMPLE 4 Applying the Multiplication Property of Equality

Solve $\frac{x}{4} = 3$.

$$\frac{x}{4} = 3$$

$$\frac{1}{4}x = 3 \qquad \frac{x}{4} = \frac{1x}{4} = \frac{1}{4}x$$

$$4 \cdot \frac{1}{4}x = 4 \cdot 3 \qquad \text{Multiply each side by 4, the reciprocal of } \frac{1}{4}.$$

$4 \cdot \frac{1}{4}x = 1x = x$

$$x = 12 \qquad \text{Multiplicative inverse property; multiplicative identity property}$$

*CHECK*

$$\frac{x}{4} = 3 \qquad \text{Original equation}$$

$$\frac{12}{4} \stackrel{?}{=} 3 \qquad \text{Let } x = 12.$$

$$3 = 3 \checkmark \qquad \text{True}$$

Since a true statement results, the solution set is $\{12\}$. *NOW TRY*

*NOW TRY ANSWERS*

**2.** $\left\{-\frac{12}{5}\right\}$ **3.** $\{-5.4\}$

**4.** $\{-35\}$

**NOW TRY EXERCISE 5**

Solve $\frac{4}{7}z = -16$.

## EXAMPLE 5 Applying the Multiplication Property of Equality

Solve $\frac{3}{4}w = 6$.

$$\begin{aligned} \frac{3}{4}w &= 6 \\ \frac{4}{3}\cdot\frac{3}{4}w &= \frac{4}{3}\cdot 6 && \text{Multiply each side by } \tfrac{4}{3}\text{, the reciprocal of } \tfrac{3}{4}. \\ 1\cdot w &= \frac{4}{3}\cdot\frac{6}{1} && \text{Multiplicative inverse property} \\ w &= 8 && \text{Multiplicative identity property; multiply fractions.} \end{aligned}$$

Check to confirm that the solution set is $\{8\}$. NOW TRY

In **Section 2.1,** we obtained $-x = -17$ in our alternative solution to **Example 4.** We reasoned that since the additive inverse (or opposite) of $x$ is $-17$, then $x$ must equal 17. We can use the multiplication property of equality to obtain the same result.

**NOW TRY EXERCISE 6**

Solve $-x = 9$.

## EXAMPLE 6 Applying the Multiplication Property of Equality

Solve $-x = -17$.

$$\begin{aligned} -x &= -17 \\ -1x &= -17 && -x = -1x \\ -1(-1x) &= -1(-17) && \text{Multiply each side by } -1. \\ [-1(-1)]x &= 17 && \text{Associative property; multiply.} \\ 1x &= 17 && \text{Multiplicative inverse property} \\ x &= 17 && \text{Multiplicative identity property} \end{aligned}$$

These steps are usually omitted.

*CHECK*

$$\begin{aligned} -x &= -17 && \text{Original equation} \\ -(17) &\stackrel{?}{=} -17 && \text{Let } x = 17. \\ -17 &= -17 \ \checkmark && \text{True} \end{aligned}$$

The solution, 17, checks, so $\{17\}$ is the solution set. NOW TRY

## OBJECTIVE 2 Simplify, and then use the multiplication property of equality.

**NOW TRY EXERCISE 7**

Solve $9n - 6n = 21$.

## EXAMPLE 7 Combining Like Terms When Solving

Solve $5m + 6m = 33$.

$$\begin{aligned} 5m + 6m &= 33 \\ 11m &= 33 && \text{Combine like terms.} \\ \frac{11m}{11} &= \frac{33}{11} && \text{Divide by 11.} \\ m &= 3 && \text{Multiplicative identity property; divide.} \end{aligned}$$

*CHECK*

$$\begin{aligned} 5m + 6m &= 33 && \text{Original equation} \\ 5(3) + 6(3) &\stackrel{?}{=} 33 && \text{Let } m = 3. \\ 15 + 18 &\stackrel{?}{=} 33 && \text{Multiply.} \\ 33 &= 33 \ \checkmark && \text{True} \end{aligned}$$

Since a true statement results, the solution set is $\{3\}$. NOW TRY

**NOW TRY ANSWERS**

**5.** $\{-28\}$ **6.** $\{-9\}$ **7.** $\{7\}$

## 2.2 EXERCISES

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💿 *Complete solution available on the Video Resources on DVD*

**1.** ***Concept Check*** Tell whether you would use the addition or multiplication property of equality to solve each equation. *Do not actually solve.*

**(a)** $3x = 12$ **(b)** $3 + x = 12$ **(c)** $-x = 4$ **(d)** $-12 = 6 + x$

**2.** ***Concept Check*** Which equation does *not* require the use of the multiplication property of equality?

**A.** $3x - 5x = 6$ **B.** $-\frac{1}{4}x = 12$ **C.** $5x - 4x = 7$ **D.** $\frac{x}{3} = -2$

✎ **3.** How would you find the solution of a linear equation with next-to-last step "$-x = 5$?"

✎ **4.** In the statement of the multiplication property of equality in this section, there is a restriction that $C \neq 0$. What would happen if you multiplied each side of an equation by 0?

***Concept Check*** *By what number is it necessary to multiply both sides of each equation to isolate x on the left side? Do not actually solve.*

**5.** $\frac{4}{5}x = 8$ **6.** $\frac{2}{3}x = 6$ **7.** $\frac{x}{10} = 5$ **8.** $\frac{x}{100} = 10$

**9.** $-\frac{9}{2}x = -4$ **10.** $-\frac{8}{3}x = -11$ **11.** $-x = 0.75$ **12.** $-x = 0.48$

***Concept Check*** *By what number is it necessary to divide both sides of each equation to isolate x on the left side? Do not actually solve.*

**13.** $6x = 5$ **14.** $7x = 10$ **15.** $-4x = 16$ **16.** $-13x = 26$

**17.** $0.12x = 48$ **18.** $0.21x = 63$ **19.** $-x = 25$ **20.** $-x = 50$

*Solve each equation, and check your solution.* ***See Examples 1–6.***

**21.** $6x = 36$ **22.** $8x = 64$ **23.** $2m = 15$ **24.** $3m = 10$

**25.** $4x = -20$ **26.** $5x = -60$ **27.** $-7x = 28$ **28.** $-9x = 36$

💿 **29.** $10t = -36$ **30.** $10s = -54$ **31.** $-6x = -72$ **32.** $-4x = -64$

**33.** $4r = 0$ **34.** $7x = 0$ 💿 **35.** $-x = 12$ **36.** $-t = 14$

**37.** $-x = -\frac{3}{4}$ **38.** $-x = -\frac{1}{2}$ **39.** $0.2t = 8$ **40.** $0.9x = 18$

💿 **41.** $-2.1m = 25.62$ **42.** $-3.9x = 32.76$ **43.** $\frac{1}{4}x = -12$ **44.** $\frac{1}{5}p = -3$

💿 **45.** $\frac{z}{6} = 12$ **46.** $\frac{x}{5} = 15$ **47.** $\frac{x}{7} = -5$ **48.** $\frac{r}{8} = -3$

💿 **49.** $\frac{2}{7}p = 4$ **50.** $\frac{3}{8}x = 9$ **51.** $-\frac{5}{6}t = -15$ **52.** $-\frac{3}{4}z = -21$

**53.** $-\frac{7}{9}x = \frac{3}{5}$ **54.** $-\frac{5}{6}x = \frac{4}{9}$ **55.** $-0.3x = 9$ **56.** $-0.5x = 20$

*Solve each equation, and check your solution.* ***See Example 7.***

💿 **57.** $4x + 3x = 21$ **58.** $8x + 3x = 121$ **59.** $6r - 8r = 10$

**60.** $3p - 7p = 24$ **61.** $\frac{2}{5}x - \frac{3}{10}x = 2$ **62.** $\frac{2}{3}x - \frac{5}{9}x = 4$

**63.** $7m + 6m - 4m = 63$ **64.** $9r + 2r - 7r = 68$ **65.** $-6x + 4x - 7x = 0$

**66.** $-5x + 4x - 8x = 0$ **67.** $8w - 4w + w = -3$ **68.** $9x - 3x + x = -4$

**69.** $\frac{1}{3}x - \frac{1}{4}x + \frac{1}{12}x = 3$ **70.** $\frac{2}{5}x + \frac{1}{10}x - \frac{1}{20}x = 18$

**71.** *Concept Check* Write an equation that requires the use of the multiplication property of equality, where each side must be multiplied by $\frac{2}{3}$ and the solution is a negative number.

**72.** *Concept Check* Write an equation that requires the use of the multiplication property of equality, where each side must be divided by 100 and the solution is not an integer.

*Write an equation using the information given in the problem. Use x as the variable. Then solve the equation.*

**73.** When a number is multiplied by 4, the result is 6. Find the number.

**74.** When a number is multiplied by $-4$, the result is 10. Find the number.

**75.** When a number is divided by $-5$, the result is 2. Find the number.

**76.** If twice a number is divided by 5, the result is 4. Find the number.

## PREVIEW EXERCISES

*Simplify each expression.* ***See Section 1.8.***

**77.** $-(3m + 5)$ **78.** $-4(-1 + 6x)$

**79.** $4(-5 + 2p) - 3(p - 4)$ **80.** $2(4k - 7) - 4(-k + 3)$

*Solve each equation.* ***See Section 2.1.***

**81.** $4x + 5 + 2x = 7x$ **82.** $2x + 5x - 3x + 4 = 3x + 2$

# 2.3

# More on Solving Linear Equations

**OBJECTIVES**

1. Learn and use the four steps for solving a linear equation.
2. Solve equations with fractions or decimals as coefficients.
3. Solve equations with no solution or infinitely many solutions.
4. Write expressions for two related unknown quantities.

### OBJECTIVE 1 Learn and use the four steps for solving a linear equation.

We now apply *both* properties of equality to solve linear equations.

**Solving a Linear Equation**

*Step 1* **Simplify each side separately.** Clear (eliminate) parentheses, fractions, and decimals, using the distributive property as needed, and combine like terms.

*Step 2* **Isolate the variable term on one side.** Use the addition property if necessary so that the variable term is on one side of the equation and a number is on the other.

*Step 3* **Isolate the variable.** Use the multiplication property if necessary to get the equation in the form $x$ = a number, or a number = $x$. (Other letters may be used for variables.)

*Step 4* **Check.** Substitute the proposed solution into the *original* equation to see if a true statement results. If not, rework the problem.

***Remember that when we solve an equation, our primary goal is to isolate the variable on one side of the equation.***

NOW TRY EXERCISE 1

Solve $7 + 2m = -3$.

## EXAMPLE 1 Applying Both Properties of Equality to Solve an Equation

Solve $-6x + 5 = 17$.

***Step 1*** There are no parentheses, fractions, or decimals in this equation, so this step is not necessary.

Our goal is to isolate $x$.

$$-6x + 5 = 17$$

***Step 2***

$$\begin{aligned} -6x + 5 - 5 &= 17 - 5 && \text{Subtract 5 from each side.} \\ -6x &= 12 && \text{Combine like terms.} \end{aligned}$$

***Step 3***

$$\begin{aligned} \frac{-6x}{-6} &= \frac{12}{-6} && \text{Divide each side by } -6. \\ x &= -2 \end{aligned}$$

***Step 4*** Check by substituting $-2$ for $x$ in the original equation.

CHECK

$$\begin{aligned} -6x + 5 &= 17 && \text{Original equation} \\ -6(-2) + 5 &\overset{?}{=} 17 && \text{Let } x = -2. \\ 12 + 5 &\overset{?}{=} 17 && \text{Multiply.} \\ 17 &= 17 \ \checkmark && \text{True} \end{aligned}$$

The solution, $-2$, checks, so the solution set is $\{-2\}$. NOW TRY

NOW TRY EXERCISE 2

Solve $2q + 3 = 4q - 9$.

## EXAMPLE 2 Applying Both Properties of Equality to Solve an Equation

Solve $3x + 2 = 5x - 8$.

***Step 1*** There are no parentheses, fractions, or decimals in the equation.

Our goal is to isolate $x$.

$$3x + 2 = 5x - 8$$

***Step 2***

$$\begin{aligned} 3x + 2 - 5x &= 5x - 8 - 5x && \text{Subtract } 5x \text{ from each side.} \\ -2x + 2 &= -8 && \text{Combine like terms.} \\ -2x + 2 - 2 &= -8 - 2 && \text{Subtract 2 from each side.} \\ -2x &= -10 && \text{Combine like terms.} \end{aligned}$$

***Step 3***

$$\begin{aligned} \frac{-2x}{-2} &= \frac{-10}{-2} && \text{Divide each side by } -2. \\ x &= 5 \end{aligned}$$

***Step 4*** Check by substituting 5 for $x$ in the original equation.

CHECK

$$\begin{aligned} 3x + 2 &= 5x - 8 && \text{Original equation} \\ 3(5) + 2 &\overset{?}{=} 5(5) - 8 && \text{Let } x = 5. \\ 15 + 2 &\overset{?}{=} 25 - 8 && \text{Multiply.} \\ 17 &= 17 \ \checkmark && \text{True} \end{aligned}$$

The solution, 5, checks, so the solution set is $\{5\}$. NOW TRY

NOW TRY ANSWERS

**1.** $\{-5\}$ **2.** $\{6\}$

---

**NOTE** ***Remember that the variable can be isolated on either side of the equation.*** In **Example 2,** $x$ will be isolated on the right if we begin by subtracting $3x$.

$$\begin{aligned}
3x + 2 &= 5x - 8 && \text{Equation from \textbf{Example 2}}\\
3x + 2 - 3x &= 5x - 8 - 3x && \text{Subtract } 3x \text{ from each side.}\\
2 &= 2x - 8 && \text{Combine like terms.}\\
2 + 8 &= 2x - 8 + 8 && \text{Add 8 to each side.}\\
10 &= 2x && \text{Combine like terms.}\\
\frac{10}{2} &= \frac{2x}{2} && \text{Divide each side by 2.}\\
5 &= x && \text{The same solution results.}
\end{aligned}$$

***There are often several equally correct ways to solve an equation.***

---

**NOW TRY EXERCISE 3**

Solve.

$$3(z - 6) - 5z = -7z + 7$$

### EXAMPLE 3 Using the Four Steps to Solve an Equation

Solve $4(k - 3) - k = k - 6$.

***Step 1*** Clear parentheses using the distributive property.

$$\begin{aligned}
4(k - 3) - k &= k - 6 &&\\
4(k) + 4(-3) - k &= k - 6 && \text{Distributive property}\\
4k - 12 - k &= k - 6 && \text{Multiply.}\\
3k - 12 &= k - 6 && \text{Combine like terms.}
\end{aligned}$$

***Step 2***

$$\begin{aligned}
3k - 12 - k &= k - 6 - k && \text{Subtract } k.\\
2k - 12 &= -6 && \text{Combine like terms.}\\
2k - 12 + 12 &= -6 + 12 && \text{Add 12.}\\
2k &= 6 && \text{Combine like terms.}
\end{aligned}$$

***Step 3***

$$\begin{aligned}
\frac{2k}{2} &= \frac{6}{2} && \text{Divide by 2.}\\
k &= 3 &&
\end{aligned}$$

***Step 4*** CHECK

$$\begin{aligned}
4(k - 3) - k &= k - 6 && \text{Original equation}\\
4(3 - 3) - 3 &\overset{?}{=} 3 - 6 && \text{Let } k = 3.\\
4(0) - 3 &\overset{?}{=} 3 - 6 && \text{Work inside the parentheses.}\\
-3 &= -3 \ \checkmark && \text{True}
\end{aligned}$$

The solution set of the equation is $\{3\}$.

NOW TRY

### EXAMPLE 4 Using the Four Steps to Solve an Equation

Solve $8z - (3 + 2z) = 3z + 1$.

***Step 1***

$$\begin{aligned}
8z - (3 + 2z) &= 3z + 1 &&\\
8z - 1(3 + 2z) &= 3z + 1 && \text{Multiplicative identity property}\\
8z - 3 - 2z &= 3z + 1 && \text{Distributive property}\\
6z - 3 &= 3z + 1 && \text{Combine like terms.}
\end{aligned}$$

Be careful with signs.

**NOW TRY ANSWER**

**3.** $\{5\}$

NOW TRY EXERCISE 4

Solve.

$$5x - (x + 9) = x - 4$$

$$\begin{aligned}
&\textit{Step 2} & 6z - 3 - 3z &= 3z + 1 - 3z && \text{Subtract } 3z.\\
& & 3z - 3 &= 1 && \text{Combine like terms.}\\
& & 3z - 3 + 3 &= 1 + 3 && \text{Add 3.}\\
& & 3z &= 4 && \text{Combine like terms.}\\
&\textit{Step 3} & \frac{3z}{3} &= \frac{4}{3} && \text{Divide by 3.}\\
& & z &= \frac{4}{3}
\end{aligned}$$

***Step 4*** Check that $\left\{\frac{4}{3}\right\}$ is the solution set. NOW TRY

---

**CAUTION** In an expression such as $8z - (3 + 2z)$ in **Example 4,** the $-$ sign acts like a factor of $-1$ and affects the sign of *every* term within the parentheses.

$$\begin{aligned}
&8z - (3 + 2z)\\
&= 8z - 1(3 + 2z)\\
&= 8z + (-1)(3 + 2z)\\
&= 8z \underset{\uparrow}{-} 3 \underset{\uparrow}{-} 2z
\end{aligned}$$

Change to $-$ in *both* terms.

---

NOW TRY EXERCISE 5

Solve.

$$24 - 4(7 - 2t) = 4(t - 1)$$

## EXAMPLE 5 Using the Four Steps to Solve an Equation

Solve $4(4 - 3x) = 32 - 8(x + 2)$.

$$\begin{aligned}
&\textit{Step 1} & 4(4 - 3x) &= 32 - 8(x + 2) && \text{Be careful with signs.}\\
& & 16 - 12x &= 32 - 8x - 16 && \text{Distributive property}\\
& & 16 - 12x &= 16 - 8x && \text{Combine like terms.}\\
&\textit{Step 2} & 16 - 12x + 8x &= 16 - 8x + 8x && \text{Add } 8x.\\
& & 16 - 4x &= 16 && \text{Combine like terms.}\\
& & 16 - 4x - 16 &= 16 - 16 && \text{Subtract 16.}\\
& & -4x &= 0 && \text{Combine like terms.}\\
&\textit{Step 3} & \frac{-4x}{-4} &= \frac{0}{-4} && \text{Divide by } -4.\\
& & x &= 0
\end{aligned}$$

$$\begin{aligned}
&\textit{Step 4}\ \ \textit{CHECK} & 4(4 - 3x) &= 32 - 8(x + 2) && \text{Original equation}\\
& & 4[4 - 3(0)] &\stackrel{?}{=} 32 - 8(0 + 2) && \text{Let } x = 0.\\
& & 4(4 - 0) &\stackrel{?}{=} 32 - 8(2) && \text{Multiply and add.}\\
& & 4(4) &\stackrel{?}{=} 32 - 16 && \text{Subtract and multiply.}\\
& & 16 &= 16\ \checkmark && \text{True}
\end{aligned}$$

Since the solution 0 checks, the solution set is $\{0\}$. NOW TRY

## OBJECTIVE 2 Solve equations with fractions or decimals as coefficients.

To avoid messy computations, we clear an equation of fractions by multiplying each side by the least common denominator (LCD) of all the fractions in the equation.

NOW TRY ANSWERS

**4.** $\left\{\frac{5}{3}\right\}$ **5.** $\{0\}$

**CAUTION** ***When clearing an equation of fractions, be sure to multiply every term on each side of the equation by the LCD.***

**NOW TRY EXERCISE 6**

Solve.

$$\frac{1}{2}x + \frac{5}{8}x = \frac{3}{4}x - 6$$

### EXAMPLE 6 Solving an Equation with Fractions as Coefficients

Solve $\frac{2}{3}x - \frac{1}{2}x = -\frac{1}{6}x - 2$.

***Step 1*** The LCD of all the fractions in the equation is 6.

$$\frac{2}{3}x - \frac{1}{2}x = -\frac{1}{6}x - 2$$

Pay particular attention here.

$$6\left(\frac{2}{3}x - \frac{1}{2}x\right) = 6\left(-\frac{1}{6}x - 2\right)$$

Multiply each side by 6, the LCD.

$$6\left(\frac{2}{3}x\right) + 6\left(-\frac{1}{2}x\right) = 6\left(-\frac{1}{6}x\right) + 6(-2)$$

Distributive property; multiply *each* term inside the parentheses by 6.

The fractions have been cleared.

$$4x - 3x = -x - 12 \qquad \text{Multiply.}$$

$$x = -x - 12 \qquad \text{Combine like terms.}$$

***Step 2***

$$x + x = -x - 12 + x \qquad \text{Add } x.$$

$$2x = -12 \qquad \text{Combine like terms.}$$

***Step 3***

$$\frac{2x}{2} = \frac{-12}{2} \qquad \text{Divide by 2.}$$

$$x = -6$$

***Step 4*** *CHECK*

$$\frac{2}{3}x - \frac{1}{2}x = -\frac{1}{6}x - 2 \qquad \text{Original equation}$$

$$\frac{2}{3}(-6) - \frac{1}{2}(-6) \stackrel{?}{=} -\frac{1}{6}(-6) - 2 \qquad \text{Let } x = -6.$$

$$-4 + 3 \stackrel{?}{=} 1 - 2 \qquad \text{Multiply.}$$

$$-1 = -1 \ ✓ \qquad \text{True}$$

The solution, −6, checks, so the solution set is $\{-6\}$.

**NOW TRY**

### EXAMPLE 7 Solving an Equation with Fractions as Coefficients

Solve $\frac{1}{3}(x + 5) - \frac{3}{5}(x + 2) = 1$.

***Step 1***

$$\frac{1}{3}(x + 5) - \frac{3}{5}(x + 2) = 1$$

$$15\left[\frac{1}{3}(x + 5) - \frac{3}{5}(x + 2)\right] = 15(1) \qquad \text{Clear the fractions. Multiply by 15, the LCD.}$$

$$15\left[\frac{1}{3}(x + 5)\right] + 15\left[-\frac{3}{5}(x + 2)\right] = 15(1) \qquad \text{Distributive property}$$

$$5(x + 5) - 9(x + 2) = 15 \qquad \text{Multiply.}$$

$$15\left[\frac{1}{3}(x + 5)\right] = 15 \cdot \frac{1}{3} \cdot (x + 5) = 5(x + 5)$$

$$5x + 25 - 9x - 18 = 15 \qquad \text{Distributive property}$$

$$-4x + 7 = 15 \qquad \text{Combine like terms.}$$

**NOW TRY ANSWER**

**6.** $\{-16\}$

NOW TRY EXERCISE 7

Solve.

$$\frac{2}{3}(x + 2) - \frac{1}{2}(3x + 4) = -4$$

**Step 2**
$$-4x + 7 - 7 = 15 - 7 \qquad \text{Subtract 7.}$$
$$-4x = 8 \qquad \text{Combine like terms.}$$

**Step 3**
$$\frac{-4x}{-4} = \frac{8}{-4} \qquad \text{Divide by } -4.$$
$$x = -2$$

**Step 4** Check to confirm that $\{-2\}$ is the solution set. NOW TRY

---

**CAUTION** Be sure you understand how to multiply by the LCD to clear an equation of fractions. ***Study Step 1 in Examples 6 and 7 carefully.***

---

NOW TRY EXERCISE 8

Solve.

$$0.05(13 - t) - 0.2t = 0.08(30)$$

### EXAMPLE 8 Solving an Equation with Decimals as Coefficients

Solve $0.1t + 0.05(20 - t) = 0.09(20)$.

**Step 1** The decimals here are expressed as tenths $(0.1)$ and hundredths $(0.05$ and $0.09)$. We choose the least exponent on 10 needed to eliminate the decimals. Here, we use $10^2 = 100$.

$$0.1t + 0.05(20 - t) = 0.09(20)$$
$$0.10t + 0.05(20 - t) = 0.09(20) \qquad 0.1 = 0.10$$
$$100[0.10t + 0.05(20 - t)] = 100[0.09(20)] \qquad \text{Multiply by 100.}$$
$$100(0.10t) + 100[0.05(20 - t)] = 100[0.09(20)] \qquad \text{Distributive property}$$
$$10t + 5(20 - t) = 9(20) \qquad \text{Multiply.}$$
$$10t + 5(20) + 5(-t) = 180 \qquad \text{Distributive property}$$
$$10t + 100 - 5t = 180 \qquad \text{Multiply.}$$
$$5t + 100 = 180 \qquad \text{Combine like terms.}$$

**Step 2**
$$5t + 100 - 100 = 180 - 100 \qquad \text{Subtract 100.}$$
$$5t = 80 \qquad \text{Combine like terms.}$$

**Step 3**
$$\frac{5t}{5} = \frac{80}{5} \qquad \text{Divide by 5.}$$
$$t = 16$$

**Step 4** Check to confirm that $\{16\}$ is the solution set. NOW TRY

---

**NOTE** In **Example 8,** multiplying by 100 is the same as moving the decimal point two places to the right.

$$0\underset{\llcorner\!\uparrow}{.10}t + 0\underset{\llcorner\!\uparrow}{.05}(20 - t) = 0\underset{\llcorner\!\uparrow}{.09}(20)$$
$$10t + 5(20 - t) = 9(20) \qquad \text{Multiply by 100.}$$

---

**OBJECTIVE 3** **Solve equations with no solution or infinitely many solutions.** Each equation so far has had exactly one solution. An equation with exactly one solution is a **conditional equation** because it is only true under certain conditions. Some equations may have no solution or infinitely many solutions.

NOW TRY ANSWERS

**7.** $\{4\}$ **8.** $\{-7\}$

NOW TRY EXERCISE 9

Solve.

$$-3(x-7) = 2x - 5x + 21$$

### EXAMPLE 9 Solving an Equation That Has Infinitely Many Solutions

Solve $5x - 15 = 5(x - 3)$.

$$\begin{aligned}
5x - 15 &= 5(x-3) \\
5x - 15 &= 5x - 15 && \text{Distributive property} \\
5x - 15 - 5x &= 5x - 15 - 5x && \text{Subtract } 5x. \\
-15 &= -15 && \text{Combine like terms.} \\
-15 + 15 &= -15 + 15 && \text{Add 15.} \\
0 &= 0 && \text{True}
\end{aligned}$$

Notice that the variable "disappeared."

**Solution set: {all real numbers}**

Since the last statement $(0 = 0)$ is true, *any* real number is a solution. We could have predicted this from the second line in the solution,

$$5x - 15 = 5x - 15. \leftarrow \text{This is true for } \textit{any} \text{ value of } x.$$

Try several values for $x$ in the original equation to see that they all satisfy it.

An equation with both sides exactly the same, like $0 = 0$, is called an **identity.** An identity is true for all replacements of the variables. As shown above, we write the solution set as **{all real numbers}.** NOW TRY

**CAUTION** In **Example 9,** do not write $\{0\}$ as the solution set. While 0 is a solution, there are infinitely many other solutions. ***For* {0} *to be the solution set, the last line must include a variable, such as x, and read x = 0, not 0 = 0.***

NOW TRY EXERCISE 10

Solve.

$$-4x + 12 = 3 - 4(x - 3)$$

### EXAMPLE 10 Solving an Equation That Has No Solution

Solve $2x + 3(x + 1) = 5x + 4$.

$$\begin{aligned}
2x + 3(x+1) &= 5x + 4 \\
2x + 3x + 3 &= 5x + 4 && \text{Distributive property} \\
5x + 3 &= 5x + 4 && \text{Combine like terms.} \\
5x + 3 - 5x &= 5x + 4 - 5x && \text{Subtract } 5x. \\
3 &= 4 && \text{False}
\end{aligned}$$

Again, the variable "disappeared."

**There is no solution. Solution set: ∅**

A false statement $(3 = 4)$ results. The original equation, called a **contradiction,** has no solution. Its solution set is the **empty set,** or **null set,** symbolized **∅.** NOW TRY

**CAUTION** **DO NOT** write $\{\emptyset\}$ to represent the empty set.

The table summarizes the solution sets of the equations in this section.

| *Type of Equation* | *Final Equation in Solution* | *Number of Solutions* | *Solution Set* |
|---|---|---|---|
| Conditional<br>**(See Examples 1–8.)** | $x =$ a number | One | {a number} |
| Identity<br>**(See Example 9.)** | A true statement with no variable, such as $0 = 0$ | Infinite | {all real numbers} |
| Contradiction<br>**(See Example 10.)** | A false statement with no variable, such as $3 = 4$ | None | ∅ |

NOW TRY ANSWERS

**9.** {all real numbers} **10.** ∅

## OBJECTIVE 4 Write expressions for two related unknown quantities.

**NOW TRY EXERCISE 11**
Two numbers have a sum of 18. If one of the numbers is represented by *m*, find an expression for the other number.

### EXAMPLE 11 Translating a Phrase into an Algebraic Expression

Perform each translation.

**(a)** Two numbers have a sum of 23. If one of the numbers is represented by *x*, find an expression for the other number.

First, suppose that the sum of two numbers is 23, and one of the numbers is 10. How would you find the other number? You would subtract 10 from 23.

$$23 - 10 \leftarrow \text{This gives 13 as the other number.}$$

Instead of using 10 as one of the numbers, use $x$. The other number would be obtained in the same way—by subtracting $x$ from 23.

$$23 - x. \quad \text{x − 23 is } \textit{not} \text{ correct.}$$

To check, find the sum of the two numbers:

$$x + (23 - x) = 23, \qquad \text{as required.}$$

**(b)** Two numbers have a product of 24. If one of the numbers is represented by *x*, find an expression for the other number.

Suppose that one of the numbers is 4. To find the other number, we would divide 24 by 4.

$$\frac{24}{4} \leftarrow \begin{array}{l}\text{This gives 6 as the other number.}\\ \text{The product } 6 \cdot 4 \text{ is 24.}\end{array}$$

In the same way, if $x$ is one of the numbers, then we divide 24 by $x$ to find the other number.

$$\frac{24}{x} \leftarrow \text{The other number}$$

NOW TRY

**NOW TRY ANSWER**
**11.** $18 - m$

## 2.3 EXERCISES

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

*Complete solution available on the Video Resources on DVD*

*Using the methods of this section, what should we do first when solving each equation? Do not actually solve.*

**1.** $7x + 8 = 1$
**2.** $7x - 5x + 15 = 8 + x$
**3.** $3(2t - 4) = 20 - 2t$
**4.** $\frac{3}{4}z = -15$
**5.** $\frac{2}{3}x - \frac{1}{6} = \frac{3}{2}x + 1$
**6.** $0.9x + 0.3(x + 12) = 6$

**7.** ***Concept Check*** Which equation does *not* have {all real numbers} as its solution set?

**A.** $5x = 4x + x$ **B.** $2(x + 6) = 2x + 12$ **C.** $\frac{1}{2}x = 0.5x$ **D.** $3x = 2x$

**8.** ***Concept Check*** The expression $100[0.03(x - 10)]$ is equivalent to which of the following?

**A.** $0.03x - 0.3$ **B.** $3x - 3$ **C.** $3x - 10$ **D.** $3x - 30$

*Solve each equation, and check your solution.* ***See Examples 1–5, 9, and 10.***

**9.** $3x + 2 = 14$
**10.** $4x + 3 = 27$
**11.** $-5z - 4 = 21$
**12.** $-7w - 4 = 10$
**13.** $4p - 5 = 2p$
**14.** $6q - 2 = 3q$

**15.** $2x + 9 = 4x + 11$

**16.** $7p + 8 = 9p - 2$

**17.** $5m + 8 = 7 + 3m$

**18.** $4r + 2 = r - 6$

**19.** $-12x - 5 = 10 - 7x$

**20.** $-16w - 3 = 13 - 8w$

**21.** $12h - 5 = 11h + 5 - h$

**22.** $-4x - 1 = -5x + 1 + 3x$

**23.** $7r - 5r + 2 = 5r + 2 - r$

**24.** $9p - 4p + 6 = 7p + 6 - 3p$

**25.** $3(4x + 2) + 5x = 30 - x$

**26.** $5(2m + 3) - 4m = 2m + 25$

**27.** $-2p + 7 = 3 - (5p + 1)$

**28.** $4x + 9 = 3 - (x - 2)$

**29.** $6(3w + 5) = 2(10w + 10)$

**30.** $4(2x - 1) = -6(x + 3)$

**31.** $-(4x + 2) - (-3x - 5) = 3$

**32.** $-(6k - 5) - (-5k + 8) = -3$

**33.** $6(4x - 1) = 12(2x + 3)$

**34.** $6(2x + 8) = 4(3x - 6)$

**35.** $3(2x - 4) = 6(x - 2)$

**36.** $3(6 - 4x) = 2(-6x + 9)$

**37.** $11x - 5(x + 2) = 6x + 5$

**38.** $6x - 4(x + 1) = 2x + 4$

*Solve each equation, and check your solution.* ***See Examples 6–8.***

**39.** $\frac{3}{5}t - \frac{1}{10}t = t - \frac{5}{2}$

**40.** $-\frac{2}{7}r + 2r = \frac{1}{2}r + \frac{17}{2}$

**41.** $\frac{3}{4}x - \frac{1}{3}x + 5 = \frac{5}{6}x$

**42.** $\frac{1}{5}x - \frac{2}{3}x - 2 = -\frac{2}{5}x$

**43.** $\frac{1}{7}(3x + 2) - \frac{1}{5}(x + 4) = 2$

**44.** $\frac{1}{4}(3x - 1) + \frac{1}{6}(x + 3) = 3$

**45.** $-\frac{1}{4}(x - 12) + \frac{1}{2}(x + 2) = x + 4$

**46.** $\frac{1}{9}(p + 18) + \frac{1}{3}(2p + 3) = p + 3$

**47.** $\frac{2}{3}k - \left(k - \frac{1}{2}\right) = \frac{1}{6}(k - 51)$

**48.** $-\frac{5}{6}q - (q - 1) = \frac{1}{4}(-q + 80)$

**49.** $0.2(60) + 0.05x = 0.1(60 + x)$

**50.** $0.3(30) + 0.15x = 0.2(30 + x)$

**51.** $1.00x + 0.05(12 - x) = 0.10(63)$

**52.** $0.92x + 0.98(12 - x) = 0.96(12)$

**53.** $0.6(10{,}000) + 0.8x = 0.72(10{,}000 + x)$

**54.** $0.2(5000) + 0.3x = 0.25(5000 + x)$

*Solve each equation, and check your solution.* ***See Examples 1–10.***

**55.** $10(2x - 1) = 8(2x + 1) + 14$

**56.** $9(3k - 5) = 12(3k - 1) - 51$

**57.** $\frac{1}{2}(x + 2) + \frac{3}{4}(x + 4) = x + 5$

**58.** $\frac{1}{3}(x + 3) + \frac{1}{6}(x - 6) = x + 3$

**59.** $0.1(x + 80) + 0.2x = 14$

**60.** $0.3(x + 15) + 0.4(x + 25) = 25$

**61.** $4(x + 8) = 2(2x + 6) + 20$

**62.** $4(x + 3) = 2(2x + 8) - 4$

**63.** $9(v + 1) - 3v = 2(3v + 1) - 8$

**64.** $8(t - 3) + 4t = 6(2t + 1) - 10$

*Write the answer to each problem in terms of the variable.* ***See Example 11.***

**65.** Two numbers have a sum of 11. One of the numbers is $q$. What expression represents the other number?

**66.** Two numbers have a sum of 34. One of the numbers is $r$. What expression represents the other number?

**67.** The product of two numbers is 9. One of the numbers is $x$. What expression represents the other number?

**68.** The product of two numbers is $-6$. One of the numbers is $m$. What expression represents the other number?

**69.** A football player gained $x$ yards rushing. On the next down, he gained 9 yd. What expression represents the number of yards he gained altogether?

**70.** A football player gained $y$ yards on a punt return. On the next return, he gained 6 yd. What expression represents the number of yards he gained altogether?

**71.** A baseball player got 65 hits one season. He got $h$ of the hits in one game. What expression represents the number of hits he got in the rest of the games?

**72.** A hockey player scored 42 goals in one season. He scored $n$ goals in one game. What expression represents the number of goals he scored in the rest of the games?

**73.** Monica is $x$ years old. What expression represents her age 15 yr from now? 5 yr ago?

**74.** Chandler is $y$ years old. What expression represents his age 4 yr ago? 11 yr from now?

**75.** Cliff has $r$ quarters. Express the value of the quarters in cents.

**76.** Claire has $y$ dimes. Express the value of the dimes in cents.

**77.** A bank teller has $t$ dollars, all in \$5 bills. What expression represents the number of \$5 bills the teller has?

**78.** A clerk has $v$ dollars, all in \$10 bills. What expression represents the number of \$10 bills the clerk has?

**79.** A plane ticket costs $x$ dollars for an adult and $y$ dollars for a child. Find an expression that represents the total cost for 3 adults and 2 children.

**80.** A concert ticket costs $p$ dollars for an adult and $q$ dollars for a child. Find an expression that represents the total cost for 4 adults and 6 children.

## PREVIEW EXERCISES

*Write each phrase as a mathematical expression using x as the variable.* ***See Sections 1.3, 1.5, 1.6, and 1.8.***

**81.** A number added to $-6$

**82.** A number decreased by 9

**83.** The difference between $-5$ and a number

**84.** The quotient of $-6$ and a nonzero number

**85.** The product of 12 and the difference between a number and 9

**86.** The quotient of 9 more than a number and 6 less than the number

## SUMMARY EXERCISES on Solving Linear Equations

*This section provides practice in solving all the types of linear equations introduced in* ***Sections 2.1–2.3.***

*Solve each equation, and check your solution.*

**1.** $x + 2 = -3$

**2.** $2m + 8 = 16$

**3.** $12.5x = -63.75$

**4.** $-x = -12$

**5.** $\frac{4}{5}x = -20$

**6.** $7m - 5m = -12$

**7.** $5x - 9 = 3(x - 3)$

**8.** $\frac{x}{-2} = 8$

**9.** $-x = 6$

**10.** $\frac{2}{3}x + 8 = \frac{1}{4}x$

**11.** $4x + 2(3 - 2x) = 6$

**12.** $-6z = -14$

**13.** $-3(m - 4) + 2(5 + 2m) = 29$

**14.** $-0.3x + 2.1(x - 4) = -6.6$

**15.** $0.08x + 0.06(x + 9) = 1.24$

**16.** $x - 16.2 = 7.5$

**17.** $7m - (2m - 9) = 39$

**18.** $7(p - 2) + p = 2(p + 2)$

**19.** $-2t + 5t - 9 = 3(t - 4) - 5$

**20.** $3(m + 5) - 1 + 2m = 5(m + 2)$

**21.** $0.2(50) + 0.8r = 0.4(50 + r)$

**22.** $2.3x + 13.7 = 1.3x + 2.9$

**23.** $2(3 + 7x) - (1 + 15x) = 2$

**24.** $6q - 9 = 12 + 3q$

**25.** $2(4 + 3r) = 3(r + 1) + 11$

**26.** $r + 9 + 7r = 4(3 + 2r) - 3$

**27.** $\frac{1}{4}x - 4 = \frac{3}{2}x + \frac{3}{4}x$

**28.** $0.6(100 - x) + 0.4x = 0.5(92)$

**29.** $\frac{3}{4}(z - 2) - \frac{1}{3}(5 - 2z) = -2$

**30.** $2 - (m + 4) = 3m - 2$

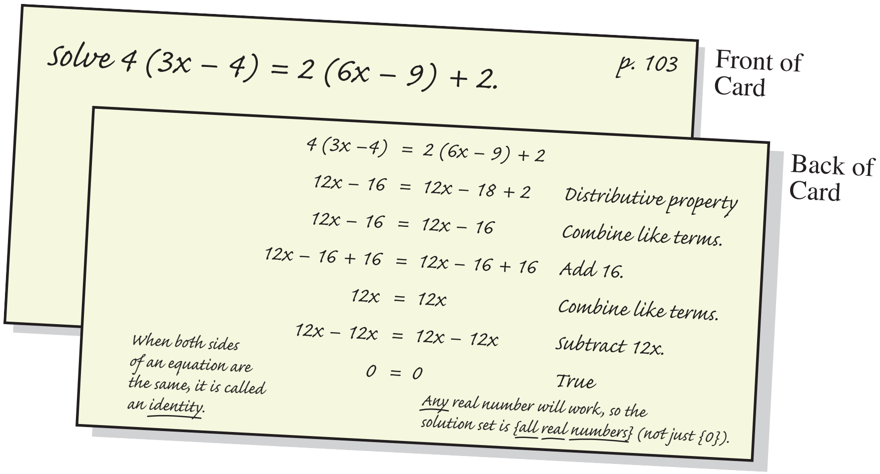
## STUDY SKILLS

# Using Study Cards Revisited

We introduced study cards on **page 48.** Another type of study card follows.

## Practice Quiz Cards

Write a problem with direction words (like *solve, simplify*) on the front of the card, and work the problem on the back. Make one for each type of problem you learn.



Solve $4(3x - 4) = 2(6x - 9) + 2.$ p. 103

$4(3x - 4) = 2(6x - 9) + 2$

$12x - 16 = 12x - 18 + 2$ Distributive property

$12x - 16 = 12x - 16$ Combine like terms.

$12x - 16 + 16 = 12x - 16 + 16$ Add 16.

$12x = 12x$ Combine like terms.

$12x - 12x = 12x - 12x$ Subtract 12x.

$0 = 0$ True

When both sides of an equation are the same, it is called an identity.

Any real number will work, so the solution set is {all real numbers} (not just {0}).

***Make a practice quiz card for material you are learning now.***

# 2.4 An Introduction to Applications of Linear Equations

**OBJECTIVES**

1. Learn the six steps for solving applied problems.
2. Solve problems involving unknown numbers.
3. Solve problems involving sums of quantities.
4. Solve problems involving consecutive integers.
5. Solve problems involving supplementary and complementary angles.

**OBJECTIVE 1** **Learn the six steps for solving applied problems.** To solve applied problems, the following six-step method is often applicable.

**Solving an Applied Problem**

***Step 1*** **Read** the problem carefully. What information is given? What are you asked to find?

***Step 2*** **Assign a variable** to represent the unknown value. Use a sketch, diagram, or table, as needed. If necessary, express any other unknown values in terms of the variable.

***Step 3*** **Write an equation** using the variable expression(s).

***Step 4*** **Solve** the equation.

***Step 5*** **State the answer.** Label it appropriately. Does it seem reasonable?

***Step 6*** **Check** the answer in the words of the *original* problem.

**OBJECTIVE 2** **Solve problems involving unknown numbers.**

*NOW TRY EXERCISE 1*

If 5 is added to a number, the result is 7 less than 3 times the number. Find the number.

### *EXAMPLE 1* Finding the Value of an Unknown Number

If 4 is multiplied by a number decreased by 7, the product is 100. Find the number.

***Step 1*** **Read** the problem carefully. We are asked to find a number.

***Step 2*** **Assign a variable** to represent the unknown quantity.

Let $x =$ the number.

***Step 3*** **Write an equation.**

Writing a "word equation" is often helpful.

$$\begin{array}{cccccc} \text{If 4 is multiplied by} & \text{a number} & \text{decreased by} & 7, & \text{the product is} & 100. \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4\cdot & (x & - & 7) & = & 100 \end{array}$$

Note the careful use of parentheses.

***Step 4*** **Solve** the equation.

$$\begin{aligned} 4(x-7) &= 100 && \text{Equation from Step 3} \\ 4x-28 &= 100 && \text{Distributive property} \\ 4x-28+28 &= 100+28 && \text{Add 28.} \\ 4x &= 128 && \text{Combine like terms.} \\ \frac{4x}{4} &= \frac{128}{4} && \text{Divide by 4.} \\ x &= 32 \end{aligned}$$

***Step 5*** **State the answer.** The number is 32.

***Step 6*** **Check.** When 32 is decreased by 7, we get $32 - 7 = 25$. If 4 is multiplied by 25, we get 100, as required. The answer, 32, is correct. *NOW TRY*

*NOW TRY ANSWER*

**1.** 6

## OBJECTIVE 3 Solve problems involving sums of quantities.

**PROBLEM-SOLVING HINT**

To solve problems involving sums of quantities, choose a variable to represent one of the unknowns. ***Then represent the other quantity in terms of the same variable.*** (See **Example 11** in **Section 2.3.**)

NOW TRY EXERCISE 2

In the 2006 Winter Olympics in Torino, Italy, Russia won 7 fewer medals than Germany. The two countries won a total of 51 medals. How many medals did each country win? (*Source:* U.S. Olympic Committee.)

### EXAMPLE 2 Finding Numbers of Olympic Medals

In the 2006 Winter Olympics in Torino, Italy, the United States won 11 more medals than Sweden. The two countries won a total of 39 medals. How many medals did each country win? (*Source:* U.S. Olympic Committee.)

***Step 1*** **Read** the problem carefully. We are given information about the total number of medals and asked to find the number each country won.

***Step 2*** **Assign a variable.**

Let $x =$ the number of medals Sweden won.

Then $x + 11 =$ the number of medals the United States won.

***Step 3*** **Write an equation.**

| The total | is | the number of medals Sweden won | plus | the number of medals the United States won. |
|---|---|---|---|---|
| ↓ | ↓ | ↓ | ↓ | ↓ |
| $39$ | $=$ | $x$ | $+$ | $(x + 11)$ |

***Step 4*** **Solve the equation.**

$$39 = 2x + 11 \qquad \text{Combine like terms.}$$

$$39 - 11 = 2x + 11 - 11 \qquad \text{Subtract 11.}$$

$$28 = 2x \qquad \text{Combine like terms.}$$

$$\frac{28}{2} = \frac{2x}{2} \qquad \text{Divide by 2.}$$

$$14 = x, \quad \text{or} \quad x = 14$$

***Step 5*** **State the answer.** The variable $x$ represents the number of medals Sweden won, so Sweden won 14 medals. The number of medals the United States won is

$$x + 11 = 14 + 11 = 25.$$

***Step 6*** **Check.** Since the United States won 25 medals and Sweden won 14, the total number of medals was $25 + 14 = 39$. Because $25 - 14 = 11$, the United States won 11 more medals than Sweden. This information agrees with what is given in the problem, so the answer checks. NOW TRY

NOW TRY ANSWER

**2.** Germany: 29 medals; Russia: 22 medals

**NOTE** The problem in **Example 2** could also be solved by letting $x$ represent the number of medals the United States won. Then $x - 11$ would represent the number of medals Sweden won. The equation would be different.

$$39 = x + (x - 11)$$

The solution of this equation is 25, which is the number of U.S. medals. The number of Swedish medals would be $25 - 11 = 14$. ***The answers are the same,*** whichever approach is used, even though the equation and its solution are different.

**NOW TRY EXERCISE 3**

In one week, the owner of Carly's Coffeehouse found that the number of orders for bagels was $\frac{2}{3}$ the number of orders for chocolate scones. If the total number of orders for the two items was 525, how many orders were placed for bagels?

### EXAMPLE 3 Finding the Number of Orders for Tea

The owner of Terry's Coffeehouse found that on one day the number of orders for tea was $\frac{1}{3}$ the number of orders for coffee. If the total number of orders for the two drinks was 76, how many orders were placed for tea?

***Step 1*** **Read** the problem. It asks for the number of orders for tea.

***Step 2*** **Assign a variable.** Because of the way the problem is stated, let the variable represent the number of orders for coffee.

Let $x$ = the number of orders for coffee.

Then $\frac{1}{3}x$ = the number of orders for tea.

***Step 3*** **Write an equation.** Use the fact that the total number of orders was 76.

$$\begin{array}{ccccc} \text{The total} & \text{is} & \text{orders for coffee} & \text{plus} & \text{orders for tea.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 76 & = & x & + & \frac{1}{3}x \end{array}$$

***Step 4*** **Solve.**

$$76 = \frac{4}{3}x \qquad x = 1x = \tfrac{3}{3}x; \text{ Combine like terms.}$$

$$\frac{3}{4}(76) = \frac{3}{4}\left(\frac{4}{3}x\right) \qquad \text{Multiply by } \tfrac{3}{4}.$$

$$57 = x$$

Be careful! This is *not* the answer.

***Step 5*** **State the answer.** In this problem, ***x does not represent the quantity that we are asked to find.*** The number of orders for tea was $\frac{1}{3}x$. So $\frac{1}{3}(57) = 19$ is the number of orders for tea.

***Step 6*** **Check.** The number of orders for tea, 19, is one-third the number of orders for coffee, 57, and $19 + 57 = 76$. Since this agrees with the information given in the problem, the answer is correct. NOW TRY

**PROBLEM-SOLVING HINT**

In **Example 3,** it was easier to let the variable represent the quantity that was *not* specified. This required extra work in Step 5 to find the number of orders for tea. In some cases, this approach is easier than letting the variable represent the quantity that we are asked to find.

**NOW TRY ANSWER**

**3.** 210 bagel orders

**NOW TRY EXERCISE 4**

At the Sherwood Estates pool party, each resident brought four guests. If a total of 175 people visited the pool that day, how many were residents and how many were guests?

## EXAMPLE 4 Analyzing a Gasoline-Oil Mixture

A lawn trimmer uses a mixture of gasoline and oil. The mixture contains 16 oz of gasoline for each 1 ounce of oil. If the tank holds 68 oz of the mixture, how many ounces of oil and how many ounces of gasoline does it require when it is full?

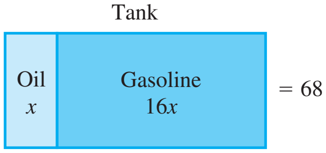
***Step 1*** **Read** the problem. We must find how many ounces of oil and gasoline are needed to fill the tank.

***Step 2*** **Assign a variable.**

Let $x =$ the number of ounces of oil required.

Then $16x =$ the number of ounces of gasoline required.

A diagram like the following is sometimes helpful.



***Step 3*** **Write an equation.**

$$\begin{array}{ccccc} \text{Amount of gasoline} & \text{plus} & \text{amount of oil} & \text{is} & \text{total amount in tank.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 16x & + & x & = & 68 \end{array}$$

***Step 4*** **Solve.**

$$17x = 68 \quad \text{Combine like terms.}$$

$$\frac{17x}{17} = \frac{68}{17} \quad \text{Divide by 17.}$$

$$x = 4$$

***Step 5*** **State the answer.** The lawn trimmer requires 4 oz of oil, and $16(4) = 64$ oz of gasoline when full.

***Step 6*** **Check.** Since $4 + 64 = 68$, and 64 is 16 times 4, the answer checks.

NOW TRY

## PROBLEM-SOLVING HINT

Sometimes we must find three unknown quantities. When the three unknowns are compared in *pairs,* ***let the variable represent the unknown found in both pairs.***

## EXAMPLE 5 Dividing a Board into Pieces

A project calls for three pieces of wood. The longest piece must be twice the length of the middle-sized piece. The shortest piece must be 10 in. shorter than the middle-sized piece. If a board 70 in. long is to be used, how long can each piece be?

***Step 1*** **Read** the problem. There will be three answers.

***Step 2*** **Assign a variable.** Since the middle-sized piece appears in both pairs of comparisons, let $x$ represent the length, in inches, of the middle-sized piece.

Let $x =$ the length of the middle-sized piece.

Then $2x =$ the length of the longest piece,

and $x - 10 =$ the length of the shortest piece.

**NOW TRY ANSWER**

**4.** 35 residents; 140 guests

**NOW TRY EXERCISE 5**

A basketball player spent 6 hr watching game films, practicing free throws, and lifting weights. He spent twice as much time lifting weights as practicing free throws and 2 hr longer watching game films than practicing free throws. How many hours did he spend on each task?

A sketch is helpful here. See **FIGURE 2**.

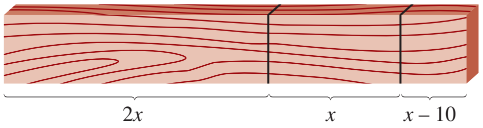


**FIGURE 2**

***Step 3*** **Write an equation.**

| Longest | plus | middle-sized | plus | shortest | is | total length. |
|---|---|---|---|---|---|---|
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| $2x$ | $+$ | $x$ | $+$ | $(x - 10)$ | $=$ | $70$ |

***Step 4*** **Solve.**

$$\begin{aligned} 4x - 10 &= 70 && \text{Combine like terms.} \\ 4x - 10 + 10 &= 70 + 10 && \text{Add 10.} \\ 4x &= 80 && \text{Combine like terms.} \\ \frac{4x}{4} &= \frac{80}{4} && \text{Divide by 4.} \\ x &= 20 \end{aligned}$$

***Step 5*** **State the answer.** The middle-sized piece is 20 in. long, the longest piece is $2(20) = 40$ in. long, and the shortest piece is $20 - 10 = 10$ in. long.

***Step 6*** **Check.** The lengths sum to 70 in. All problem conditions are satisfied.

NOW TRY

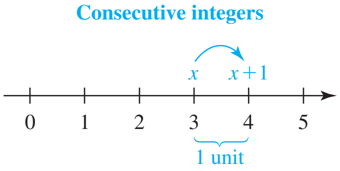


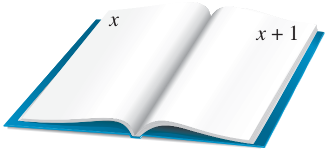
**FIGURE 3**

**OBJECTIVE 4** **Solve problems involving consecutive integers.** Two integers that differ by 1 are called **consecutive integers.** For example, 3 and 4, 6 and 7, and −2 and −1 are pairs of consecutive integers. See **FIGURE 3**.

***In general, if x represents an integer, $x + 1$ represents the next greater consecutive integer.***

### EXAMPLE 6 Finding Consecutive Integers

Two pages that face each other in this book have 225 as the sum of their page numbers. What are the page numbers?



***Step 1*** **Read** the problem. Because the two pages face each other, they must have page numbers that are consecutive integers.

***Step 2*** **Assign a variable.**

Let $x =$ the lesser page number.

Then $x + 1 =$ the greater page number.

***Step 3*** **Write an equation.** The sum of the page numbers is 225.

$$x + (x + 1) = 225$$

***Step 4*** **Solve.**

$$\begin{aligned} 2x + 1 &= 225 && \text{Combine like terms.} \\ 2x &= 224 && \text{Subtract 1.} \\ x &= 112 && \text{Divide by 2.} \end{aligned}$$

**NOW TRY ANSWER**

**5.** practicing free throws: 1 hr; lifting weights: 2 hr; watching game films: 3 hr

**NOW TRY EXERCISE 6**

Two pages that face each other have 593 as the sum of their page numbers. What are the page numbers?

***Step 5*** **State the answer.** The lesser page number is 112, and the greater page number is $112 + 1 = 113$. (Your book is opened to these two pages.)

***Step 6*** **Check.** The sum of 112 and 113 is 225. The answer is correct.

NOW TRY

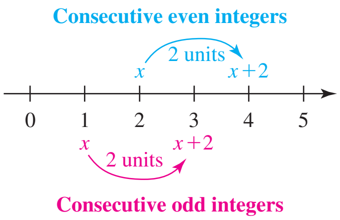


**FIGURE 4**

**Consecutive *even* integers,** such as 8 and 10, differ by 2. Similarly, **consecutive *odd* integers,** such as 9 and 11, also differ by 2. See **FIGURE 4**.

***In general, if x represents an even or odd integer, x + 2 represents the next greater consecutive even or odd integer, respectively.***

In this book, we list consecutive integers in increasing order.

## PROBLEM-SOLVING HINT

If $x =$ the lesser integer, then, for any

| | |
|---|---|
| two consecutive integers, use | $\mathbf{x, \quad x + 1;}$ |
| two consecutive *even* integers, use | $\mathbf{x, \quad x + 2;}$ |
| two consecutive *odd* integers, use | $\mathbf{x, \quad x + 2.}$ |

**NOW TRY EXERCISE 7**

Find two consecutive odd integers such that the sum of twice the lesser and three times the greater is 191.

## *EXAMPLE 7* Finding Consecutive Odd Integers

If the lesser of two consecutive odd integers is doubled, the result is 7 more than the greater of the two integers. Find the two integers.

Let $x$ be the lesser integer. Since the two numbers are consecutive *odd* integers, then $x + 2$ is the greater. Now we write an equation.

$$\underset{\downarrow}{\text{If the lesser is doubled,}} \quad \underset{\downarrow}{\text{the result is}} \quad \underset{\downarrow}{7} \quad \underset{\downarrow}{\text{more than}} \quad \underset{\downarrow}{\text{the greater.}}$$

$$2x \qquad = \qquad 7 \qquad + \qquad (x + 2)$$

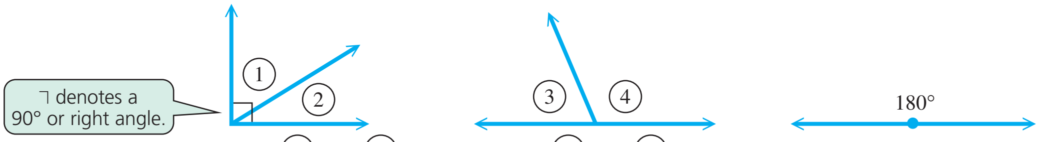
$$2x = 9 + x \qquad \text{Combine like terms.}$$

$$x = 9 \qquad \text{Subtract } x.$$

The lesser integer is 9 and the greater is $9 + 2 = 11$. As a check, when 9 is doubled, we get 18, which is 7 more than the greater odd integer, 11. The answers are correct.

NOW TRY

**OBJECTIVE 5** **Solve problems involving supplementary and complementary angles.** An angle can be measured by a unit called the **degree** (°), which is $\frac{1}{360}$ of a complete rotation. Two angles whose sum is 90° are said to be **complementary,** or *complements* of each other. An angle that measures 90° is a **right angle.** Two angles whose sum is 180° are said to be **supplementary,** or *supplements* of each other. One angle *supplements* the other to form a **straight angle** of 180°. See **FIGURE 5**.



Angles ① and ② are complementary. They form a right angle.

Angles ③ and ④ are supplementary. They form a straight angle.

Straight angle

**FIGURE 5**

**NOW TRY ANSWERS**

**6.** 296, 297 **7.** 37, 39

**PROBLEM-SOLVING HINT**

If $x$ represents the degree measure of an angle, then

$\mathbf{90 - x}$ represents the degree measure of its complement.

$\mathbf{180 - x}$ represents the degree measure of its supplement.

*NOW TRY EXERCISE 8*

Find the measure of an angle whose complement is twice its measure.

## EXAMPLE 8 Finding the Measure of an Angle

Find the measure of an angle whose complement is five times its measure.

***Step 1*** **Read** the problem. We must find the measure of an angle, given information about the measure of its complement.

***Step 2*** **Assign a variable.**

Let $x =$ the degree measure of the angle.

Then $90 - x =$ the degree measure of its complement.

***Step 3*** **Write an equation.**

$$\begin{array}{ccc} \text{Measure of the complement} & \text{is} & \text{5 times the measure of the angle.} \\ \downarrow & \downarrow & \downarrow \\ 90 - x & = & 5x \end{array}$$

***Step 4*** **Solve.**

$$\begin{aligned} 90 - x + x &= 5x + x && \text{Add } x. \\ 90 &= 6x && \text{Combine like terms.} \\ \frac{90}{6} &= \frac{6x}{6} && \text{Divide by 6.} \\ 15 &= x, \quad \text{or} \quad x = 15 \end{aligned}$$

***Step 5*** **State the answer.** The measure of the angle is 15°.

***Step 6*** **Check.** If the angle measures 15°, then its complement measures $90° - 15° = 75°$, which is equal to five times 15°, as required.

*NOW TRY*

## EXAMPLE 9 Finding the Measure of an Angle

Find the measure of an angle whose supplement is 10° more than twice its complement.

***Step 1*** **Read** the problem. We are to find the measure of an angle, given information about its complement and its supplement.

***Step 2*** **Assign a variable.**

Let $x =$ the degree measure of the angle.

Then $90 - x =$ the degree measure of its complement,

and $180 - x =$ the degree measure of its supplement.

We can visualize this information using a sketch. See **FIGURE 6**.

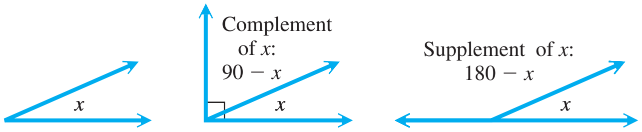


**FIGURE 6**

*NOW TRY ANSWER*

**8.** 30°

NOW TRY EXERCISE 9

Find the measure of an angle whose supplement is 46° less than three times its complement.

***Step 3*** **Write an equation.**

$$\begin{array}{cccccc} \text{Supplement} & \text{is} & 10 & \text{more than} & \text{twice} & \text{its complement.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 180 - x & = & 10 & + & 2 \cdot & (90 - x) \end{array}$$

Be sure to use parentheses here.

***Step 4*** **Solve.**

$$\begin{aligned} 180 - x &= 10 + 180 - 2x && \text{Distributive property} \\ 180 - x &= 190 - 2x && \text{Combine like terms.} \\ 180 - x + 2x &= 190 - 2x + 2x && \text{Add } 2x. \\ 180 + x &= 190 && \text{Combine like terms.} \\ 180 + x - 180 &= 190 - 180 && \text{Subtract 180.} \\ x &= 10 \end{aligned}$$

***Step 5*** **State the answer.** The measure of the angle is 10°.

***Step 6*** **Check.** The complement of 10° is 80° and the supplement of 10° is 170°. 170° is equal to 10° more than twice 80° (that is, $170 = 10 + 2(80)$ is true). Therefore, the answer is correct. NOW TRY

NOW TRY ANSWER

**9.** 22°

## 2.4 EXERCISES

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**1.** ***Concept Check*** A problem requires finding the number of cars on a dealer's lot. Which would *not* be a reasonable answer? Justify your response.

**A.** 0 **B.** 45 **C.** 1 **D.** $6\frac{1}{2}$

**2.** ***Concept Check*** A problem requires finding the number of hours a lightbulb is on during a day. Which would *not* be a reasonable answer? Justify your response.

**A.** 0 **B.** 4.5 **C.** 13 **D.** 25

**3.** ***Concept Check*** A problem requires finding the distance traveled in miles. Which would *not* be a reasonable answer? Justify your response.

**A.** $-10$ **B.** 1.8 **C.** $10\frac{1}{2}$ **D.** 50

**4.** ***Concept Check*** A problem requires finding the time in minutes. Which would *not* be a reasonable answer? Justify your response.

**A.** 0 **B.** 10.5 **C.** $-5$ **D.** 90

*Solve each problem.* ***See Example 1.***

**5.** The product of 8, and a number increased by 6, is 104. What is the number?

**6.** The product of 5, and 3 more than twice a number, is 85. What is the number?

**7.** If 2 is added to five times a number, the result is equal to 5 more than four times the number. Find the number.

**8.** If four times a number is added to 8, the result is three times the number, added to 5. Find the number.

**9.** If 2 is subtracted from a number and this difference is tripled, the result is 6 more than the number. Find the number.

**10.** If 3 is added to a number and this sum is doubled, the result is 2 more than the number. Find the number.

**11.** The sum of three times a number and 7 more than the number is the same as the difference between $-11$ and twice the number. What is the number?

**12.** If 4 is added to twice a number and this sum is multiplied by 2, the result is the same as if the number is multiplied by 3 and 4 is added to the product. What is the number?

*Solve each problem.* ***See Example 2.***

**13.** Pennsylvania and Ohio were the states with the most remaining drive-in movie screens in the United States in 2007. Pennsylvania had 2 more screens than Ohio, and there were 68 screens total in the two states. How many drive-in movie screens remained in each state? (*Source:* www.drive-ins.com)

**14.** As of 2008, the two most highly watched episodes in the history of television were the final episode of *M*A*S*H* and the final episode of *Cheers*. The number of viewers for these original broadcasts in 1983 was about 92 million, with 8 million more people watching the *M*A*S*H* episode than the *Cheers* episode. How many people watched each show? (*Source:* Nielsen Media Research.)

**15.** In August 2009, the U.S. Senate had a total of 98 Democrats and Republicans. There were 18 more Democrats than Republicans. How many members of each party were there? (*Source:* www.thegreenpapers.com)

**16.** In August 2009, the total number of Democrats and Republicans in the U.S. House of Representatives was 434. There were 78 more Democrats than Republicans. How many members of each party were there? (*Source:* www.thegreenpapers.com)

**17.** Bon Jovi and Bruce Springsteen had the two top-grossing North American concert tours for 2008, together generating \$415.3 million in ticket sales. If Bruce Springsteen took in \$6.1 million less than Bon Jovi, how much did each tour generate? (*Source:* www.billboard.com)

**18.** The Toyota Camry was the top-selling passenger car in the United States in 2007, followed by the Honda Accord. Accord sales were 81 thousand less than Camry sales, and 865 thousand of the two types of cars were sold. How many of each make of car were sold? (*Source: World Almanac and Book of Facts*.)

**19.** In the 2008–2009 NBA regular season, the Boston Celtics won two more than three times as many games as they lost. The Celtics played 82 games. How many wins and losses did the team have? (*Source:* www.NBA.com)

**20.** In the 2008 regular baseball season, the Tampa Bay Rays won 33 fewer than twice as many games as they lost. They played 162 regular-season games. How many wins and losses did the team have? (*Source:* www.MLB.com)

**21.** A one-cup serving of orange juice contains 3 mg less than four times the amount of vitamin C as a one-cup serving of pineapple juice. Servings of the two juices contain a total of 122 mg of vitamin C. How many milligrams of vitamin C are in a serving of each type of juice? (*Source:* U.S. Agriculture Department.)

**22.** A one-cup serving of pineapple juice has 9 more than three times as many calories as a one-cup serving of tomato juice. Servings of the two juices contain a total of 173 calories. How many calories are in a serving of each type of juice? (*Source:* U.S. Agriculture Department.)

*Solve each problem.* ***See Examples 3 and 4.***

**23.** In one day, a store sold $\frac{8}{5}$ as many DVDs as CDs. The total number of DVDs and CDs sold that day was 273. How many DVDs were sold?

**24.** A workout that combines weight training and aerobics burns a total of 374 calories. If doing aerobics burns $\frac{12}{5}$ as many calories as weight training, how many calories does each activity burn?

**25.** The world's largest taco contained approximately 1 kg of onion for every 6.6 kg of grilled steak. The total weight of these two ingredients was 617.6 kg. To the nearest tenth of a kilogram, how many kilograms of each ingredient were used to make the taco? (*Source: Guinness World Records.*)

**26.** As of 2005, the combined population of China and India was estimated at 2.4 billion. If there were about 0.8 as many people living in India as China, what was the population of each country, to the nearest tenth of a billion? (*Source:* U.S. Census Bureau.)

**27.** The value of a "Mint State-63" (uncirculated) 1950 Jefferson nickel minted at Denver is twice the value of a 1945 nickel in similar condition minted at Philadelphia. Together, the total value of the two coins is $24.00. What is the value of each coin? (*Source:* Yeoman, R., *A Guide Book of United States Coins,* 62nd edition, 2009.)

**28.** U.S. five-cent coins are made from a combination of two metals: nickel and copper. For every 1 pound of nickel, 3 lb of copper are used. How many pounds of copper would be needed to make 560 lb of five-cent coins? (*Source:* The United States Mint.)

**29.** A recipe for whole-grain bread calls for 1 oz of rye flour for every 4 oz of whole-wheat flour. How many ounces of each kind of flour should be used to make a loaf of bread weighing 32 oz?

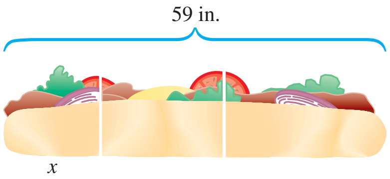
**30.** A medication contains 9 mg of active ingredients for every 1 mg of inert ingredients. How much of each kind of ingredient would be contained in a single 250-mg caplet?

*Solve each problem.* ***See Example 5.***

**31.** An office manager booked 55 airline tickets, divided among three airlines. He booked 7 more tickets on American Airlines than United Airlines. On Southwest Airlines, he booked 4 more than twice as many tickets as on United. How many tickets did he book on each airline?

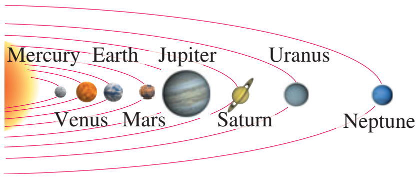
**32.** A mathematics textbook editor spent 7.5 hr making telephone calls, writing e-mails, and attending meetings. She spent twice as much time attending meetings as making telephone calls and 0.5 hr longer writing e-mails than making telephone calls. How many hours did she spend on each task?

**33.** A party-length submarine sandwich that is 59 in. long is cut into three pieces. The middle piece is 5 in. longer than the shortest piece, and the shortest piece is 9 in. shorter than the longest piece. How long is each piece?



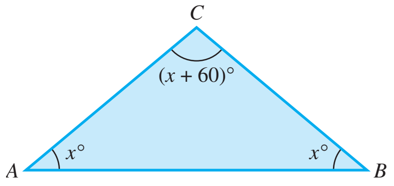
**34.** China earned a total of 100 medals at the 2008 Beijing Summer Olympics. The number of gold medals earned was 23 more than the number of bronze medals. The number of bronze medals earned was 7 more than the number of silver medals. How many of each kind of medal did China earn? (*Source: World Almanac and Book of Facts*.)

**35.** Venus is 31.2 million mi farther from the sun than Mercury, while Earth is 57 million mi farther from the sun than Mercury. If the total of the distances from these three planets to the sun is 196.2 million mi, how far away from the sun is Mercury? (All distances given here are *mean* (*average*) distances.) (*Source: The New York Times Almanac.*)

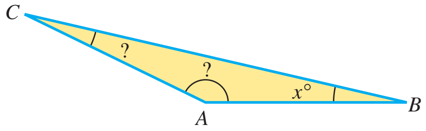


**36.** Together, Saturn, Jupiter, and Uranus have a total of 137 known satellites (moons). Jupiter has 16 more satellites than Saturn, and Uranus has 20 fewer satellites than Saturn. How many known satellites does Uranus have? (*Source: The New York Times Almanac.*)

**37.** The sum of the measures of the angles of any triangle is 180°. In triangle $ABC$, angles $A$ and $B$ have the same measure, while the measure of angle $C$ is 60° greater than each of $A$ and $B$. What are the measures of the three angles?



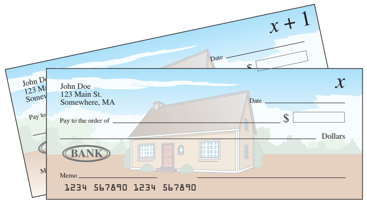
**38.** In triangle $ABC$, the measure of angle $A$ is 141° more than the measure of angle $B$. The measure of angle $B$ is the same as the measure of angle $C$. Find the measure of each angle. (*Hint:* **See Exercise 37.**)



*Solve each problem.* ***See Examples 6 and 7.***

**39.** The numbers on two consecutively numbered gym lockers have a sum of 137. What are the locker numbers?

**40.** The numbers on two consecutive checkbook checks have a sum of 357. What are the numbers?



**41.** Two pages that are back-to-back in this book have 203 as the sum of their page numbers. What are the page numbers?

**42.** Two apartments have numbers that are consecutive integers. The sum of the numbers is 59. What are the two apartment numbers?

**43.** Find two consecutive even integers such that the lesser added to three times the greater gives a sum of 46.

**44.** Find two consecutive odd integers such that twice the greater is 17 more than the lesser.

**45.** When the lesser of two consecutive integers is added to three times the greater, the result is 43. Find the integers.

**46.** If five times the lesser of two consecutive integers is added to three times the greater, the result is 59. Find the integers.

***Brain Busters*** *Solve each problem.*

**47.** If the sum of three consecutive even integers is 60, what is the first of the three even integers? (*Hint:* If $x$ and $x + 2$ represent the first two consecutive even integers, how would you represent the third consecutive even integer?)

**48.** If the sum of three consecutive odd integers is 69, what is the third of the three odd integers?

**49.** If 6 is subtracted from the third of three consecutive odd integers and the result is multiplied by 2, the answer is 23 less than the sum of the first and twice the second of the integers. Find the integers.

**50.** If the first and third of three consecutive even integers are added, the result is 22 less than three times the second integer. Find the integers.

*Solve each problem.* ***See Examples 8 and 9.***

**51.** Find the measure of an angle whose complement is four times its measure.

**52.** Find the measure of an angle whose complement is five times its measure.

**53.** Find the measure of an angle whose supplement is eight times its measure.

**54.** Find the measure of an angle whose supplement is three times its measure.

**55.** Find the measure of an angle whose supplement measures 39° more than twice its complement.

**56.** Find the measure of an angle whose supplement measures 38° less than three times its complement.

**57.** Find the measure of an angle such that the difference between the measures of its supplement and three times its complement is 10°.

**58.** Find the measure of an angle such that the sum of the measures of its complement and its supplement is 160°.

## PREVIEW EXERCISES

*Use the given values to evaluate each expression.* ***See Section 1.3.***

**59.** $LW$; $L = 6, W = 4$

**60.** $rt$; $r = 25, t = 4.5$

**61.** $2L + 2W$; $L = 8, W = 2$

**62.** $\frac{1}{2}h(b + B)$; $h = 10, b = 4, B = 12$

# 2.5

# Formulas and Additional Applications from Geometry

**OBJECTIVES**

1. Solve a formula for one variable, given values of the other variables.
2. Use a formula to solve an applied problem.
3. Solve problems involving vertical angles and straight angles.
4. Solve a formula for a specified variable.

*NOW TRY EXERCISE 1*

Find the value of the remaining variable.

$$P = 2a + 2b; \quad P = 78,\ a = 12$$

A **formula** is an equation in which variables are used to describe a relationship. For example, formulas exist for finding perimeters and areas of geometric figures, calculating money earned on bank savings, and converting among measurements.

$$P = 4s, \quad \mathcal{A} = \pi r^2, \quad I = prt, \quad F = \frac{9}{5}C + 32 \qquad \text{Formulas}$$

***Many of the formulas used in this book are given on the inside covers.***

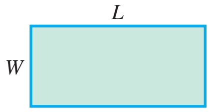
**OBJECTIVE 1** **Solve a formula for one variable, given values of the other variables.** In **Example 1,** we use the idea of *area*. The **area** of a plane (two-dimensional) geometric figure is a measure of the surface covered by the figure.

### EXAMPLE 1 Using Formulas to Evaluate Variables

Find the value of the remaining variable in each formula.

**(a)** $\mathcal{A} = LW; \quad \mathcal{A} = 64,\ L = 10$

As shown in **FIGURE 7**, this formula gives the area $\mathcal{A}$ of a rectangle with length $L$ and width $W$. Substitute the given values into the formula.



**Rectangle**
$\mathcal{A} = LW$

**FIGURE 7**

In this book, $\mathcal{A}$ denotes area.

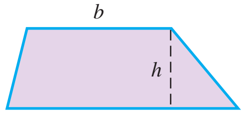
Solve for $W$.

$$\begin{aligned} \mathcal{A} &= LW \\ 64 &= 10W && \text{Let } \mathcal{A} = 64 \text{ and } L = 10. \\ \frac{64}{10} &= \frac{10W}{10} && \text{Divide by 10.} \\ 6.4 &= W \end{aligned}$$

The width is 6.4. Since $10(6.4) = 64$, the given area, the answer checks.

**(b)** $\mathcal{A} = \frac{1}{2}h(b + B); \quad \mathcal{A} = 210,\ B = 27,\ h = 10$

This formula gives the area of a trapezoid. See **FIGURE 8**.



**Trapezoid**
$\mathcal{A} = \frac{1}{2}h(b + B)$

**FIGURE 8**

Solve for $b$.

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}h(b + B) \\ 210 &= \frac{1}{2}(10)(b + 27) && \text{Let } \mathcal{A} = 210,\ h = 10,\ B = 27. \\ 210 &= 5(b + 27) && \text{Multiply.} \\ 210 &= 5b + 135 && \text{Distributive property} \\ 210 - 135 &= 5b + 135 - 135 && \text{Subtract 135.} \\ 75 &= 5b && \text{Combine like terms.} \\ \frac{75}{5} &= \frac{5b}{5} && \text{Divide by 5.} \\ 15 &= b \end{aligned}$$

The length of the shorter parallel side, $b$, is 15. This answer checks, since

$$\frac{1}{2}(10)(15 + 27) = 210, \quad \text{as required.}$$

*NOW TRY*

*NOW TRY ANSWER*

1. $b = 27$

**OBJECTIVE 2** **Use a formula to solve an applied problem.** ***When solving an applied problem that involves a geometric figure, it is a good idea to draw a sketch.*** **Examples 2 and 3** use the idea of *perimeter*. The **perimeter** of a plane (two-dimensional) geometric figure is the distance around the figure. For a polygon (e.g., a rectangle, square, or triangle), it is the sum of the lengths of its sides.

*NOW TRY EXERCISE 2*

Kurt's garden is in the shape of a rectangle. The length is 10 ft less than twice the width, and the perimeter is 160 ft. Find the dimensions of the garden.

## EXAMPLE 2 Finding the Dimensions of a Rectangular Yard

Cathleen Horne's backyard is in the shape of a rectangle. The length is 5 m less than twice the width, and the perimeter is 80 m. Find the dimensions of the yard.

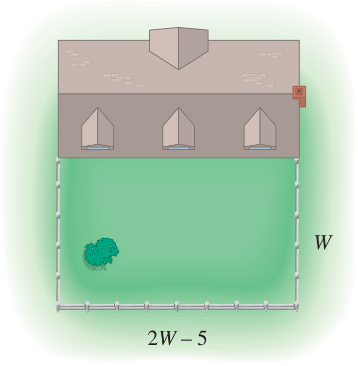


FIGURE 9

***Step 1*** **Read** the problem. We must find the dimensions of the yard.

***Step 2*** **Assign a variable.** Let $W =$ the width of the lot, in meters. Since the length is 5 meters less than twice the width, the length is $L = 2W - 5$. See FIGURE 9.

***Step 3*** **Write an equation.** Use the formula for the perimeter of a rectangle.

$$P = 2L + 2W \qquad \text{Perimeter of a rectangle}$$

$$\underset{\displaystyle 80}{\overset{\text{Perimeter}}{\downarrow}} \;\; \underset{\displaystyle = 2(2W-5)}{\overset{= 2 \cdot \text{Length}}{\downarrow}} \;\; + \;\; \underset{\displaystyle 2W}{\overset{+\, 2 \cdot \text{Width}}{\downarrow}} \qquad \text{Substitute } 2W - 5 \text{ for length } L.$$

***Step 4*** **Solve.**

$$\begin{aligned} 80 &= 4W - 10 + 2W && \text{Distributive property} \\ 80 &= 6W - 10 && \text{Combine like terms.} \\ 80 + 10 &= 6W - 10 + 10 && \text{Add 10.} \\ 90 &= 6W && \text{Combine like terms.} \\ \frac{90}{6} &= \frac{6W}{6} && \text{Divide by 6.} \\ 15 &= W \end{aligned}$$

***Step 5*** **State the answer.** The width is 15 m and the length is $2(15) - 5 = 25$ m.

***Step 6*** **Check.** If the width is 15 m and the length is 25 m, the perimeter is

$$2(25) + 2(15) = 50 + 30 = 80 \text{ m}, \quad \text{as required.}$$

*NOW TRY*

## EXAMPLE 3 Finding the Dimensions of a Triangle

The longest side of a triangle is 3 ft longer than the shortest side. The medium side is 1 ft longer than the shortest side. If the perimeter of the triangle is 16 ft, what are the lengths of the three sides?

***Step 1*** **Read** the problem. We must find the lengths of the sides of a triangle.

***Step 2*** **Assign a variable.**

Let $s =$ the length of the shortest side, in feet,

$s + 1 =$ the length of the medium side, in feet, and,

$s + 3 =$ the length of the longest side in feet.

See FIGURE 10.

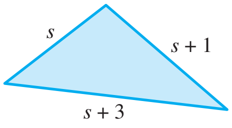


FIGURE 10

*NOW TRY ANSWER*

2. width: 30 ft; length: 50 ft

NOW TRY EXERCISE 3

The perimeter of a triangle is 30 ft. The longest side is 1 ft longer than the medium side, and the shortest side is 7 ft shorter than the medium side. What are the lengths of the three sides?

***Step 3*** **Write an equation.** Use the formula for the perimeter of a triangle.

$P = a + b + c$ Perimeter of a triangle

$16 = s + (s + 1) + (s + 3)$ Substitute.

***Step 4*** **Solve.** $16 = 3s + 4$ Combine like terms.

$12 = 3s$ Subtract 4.

$4 = s$ Divide by 3.

***Step 5*** **State the answer.** The shortest side, $s$, has length 4 ft. Then

$s + 1 = 4 + 1 = 5$ ft, Length of medium side

and $s + 3 = 4 + 3 = 7$ ft. Length of longest side

***Step 6*** **Check.** The medium side, 5 ft, is 1 ft longer than the shortest side, and the longest side, 7 ft, is 3 ft longer than the shortest side. Futhermore, the perimeter is $4 + 5 + 7 = 16$ ft, as required. NOW TRY

NOW TRY EXERCISE 4

The area of a triangle is 77 $\text{cm}^2$. The base is 14 cm. Find the height of the triangle.

## EXAMPLE 4 Finding the Height of a Triangular Sail

The area of a triangular sail of a sailboat is 126 $\text{ft}^2$. (Recall that "$\text{ft}^2$" means "square feet.") The base of the sail is 12 ft. Find the height of the sail.

***Step 1*** **Read** the problem. We must find the height of the triangular sail.

***Step 2*** **Assign a variable.** Let $h =$ the height of the sail, in feet. See **FIGURE 11**.

***Step 3*** **Write an equation.** The formula for the area of a triangle is $\mathcal{A} = \frac{1}{2}bh$, where $\mathcal{A}$ is the area, $b$ is the base, and $h$ is the height.

**FIGURE 11**

$\mathcal{A} = \frac{1}{2}bh$ Area of a triangle

$126 = \frac{1}{2}(12)h$ $\mathcal{A} = 126$, $b = 12$

***Step 4*** **Solve.** $126 = 6h$ Multiply.

$21 = h$ Divide by 6.

***Step 5*** **State the answer.** The height of the sail is 21 ft.

***Step 6*** **Check** to see that the values $\mathcal{A} = 126$, $b = 12$, and $h = 21$ satisfy the formula for the area of a triangle. NOW TRY

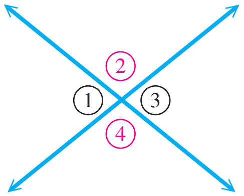


**FIGURE 12**

## OBJECTIVE 3 Solve problems involving vertical angles and straight angles.

**FIGURE 12** shows two intersecting lines forming angles that are numbered ①, ②, ③, and ④. Angles ① and ③ lie "opposite" each other. They are called **vertical angles.** Another pair of vertical angles is ② and ④. ***Vertical angles have equal measures.***

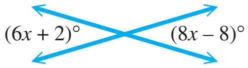
Now look at angles ① and ②. When their measures are added, we get 180°, the measure of a straight angle. There are three other such pairs of angles: ② and ③, ③ and ④, and ① and ④.

NOW TRY ANSWERS

**3.** 5 ft, 12 ft, 13 ft **4.** 11 cm

**NOW TRY EXERCISE 5**

Find the measure of each marked angle in the figure.



## EXAMPLE 5 Finding Angle Measures

Refer to the appropriate figure in each part.

**(a)** Find the measure of each marked angle in **FIGURE 13**.

Since the marked angles are vertical angles, they have equal measures.

$$\begin{aligned} 4x + 19 &= 6x - 5 && \text{Set } 4x + 19 \text{ equal to } 6x - 5. \\ 19 &= 2x - 5 && \text{Subtract } 4x. \\ 24 &= 2x && \text{Add 5.} \\ 12 &= x && \text{Divide by 2.} \end{aligned}$$

This is *not* the answer.

Replace $x$ with 12 in the expression for the measure of each angle.

$$\begin{aligned} 4x + 19 &= 4(12) + 19 = 48 + 19 = 67 \\ 6x - 5 &= 6(12) - 5 = 72 - 5 = 67 \end{aligned}$$

The angles have equal measures, as required.

Each angle measures 67°.

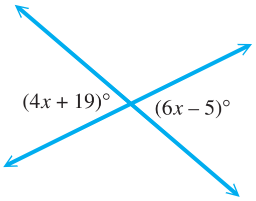


**FIGURE 13**

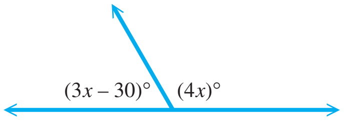


**FIGURE 14**

**(b)** Find the measure of each marked angle in **FIGURE 14**.

The measures of the marked angles must add to 180° because together they form a straight angle. (They are also *supplements* of each other.)

$$\begin{aligned} (3x - 30) + 4x &= 180 \\ 7x - 30 &= 180 && \text{Combine like terms.} \\ 7x &= 210 && \text{Add 30.} \\ x &= 30 && \text{Divide by 7.} \end{aligned}$$

Don't stop here!

Replace $x$ with 30 in the expression for the measure of each angle.

$$\begin{aligned} 3x - 30 &= 3(30) - 30 = 90 - 30 = 60 \\ 4x &= 4(30) = 120 \end{aligned}$$

The measures of the angles add to 180°, as required.

The two angle measures are 60° and 120°.

NOW TRY

**CAUTION** In **Example 5,** the answer is *not* the value of $x$. ***Remember to substitute the value of the variable into the expression given for each angle.***

**OBJECTIVE 4** **Solve a formula for a specified variable.** Sometimes we want to rewrite a formula in terms of a *different* variable in the formula. For example, consider $\mathcal{A} = LW$, the formula for the area of a rectangle.

How can we rewrite $\mathcal{A} = LW$ in terms of $W$?

The process whereby we do this is called **solving for a specified variable,** or **solving a literal equation.**

**NOW TRY ANSWER**

**5.** 32°, 32°

To solve a formula for a specified variable, we use the *same* steps that we used to solve an equation with just one variable. For example, solve the following for $x$.

| | | | |
|---|---|---|---|
| $3x + 4 = 13$ | | $ax + b = c$ | |
| $3x + 4 - 4 = 13 - 4$ | Subtract 4. | $ax + b - b = c - b$ | Subtract $b$. |
| $3x = 9$ | | $ax = c - b$ | |
| $\frac{3x}{3} = \frac{9}{3}$ | Divide by 3. | $\frac{ax}{a} = \frac{c-b}{a}$ | Divide by $a$. |
| $x = 3$ | Equation solved for $x$ | $x = \frac{c-b}{a}$ | Formula solved for $x$ |

***When solving a formula for a specified variable, we treat the specified variable as if it were the ONLY variable, and treat the other variables as if they were numbers.***

NOW TRY EXERCISE 6
Solve $W = Fd$ for $F$.

## EXAMPLE 6 Solving for a Specified Variable

Solve $\mathcal{A} = LW$ for $W$.

$W$ is multiplied by $L$, so undo the multiplication by dividing each side by $L$.

$$\mathcal{A} = LW$$

Our goal is to isolate $W$.

$$\frac{\mathcal{A}}{L} = \frac{LW}{L} \qquad \text{Divide by } L.$$

$$\frac{\mathcal{A}}{L} = W, \quad \text{or} \quad W = \frac{\mathcal{A}}{L} \qquad \frac{LW}{L} = \frac{L}{L} \cdot W = 1 \cdot W = W$$

NOW TRY

NOW TRY EXERCISE 7
Solve $Ax + By = C$ for $A$.

## EXAMPLE 7 Solving for a Specified Variable

Solve $P = 2L + 2W$ for $L$.

$$P = 2L + 2W$$

Our goal is to isolate $L$.

$$P - 2W = 2L + 2W - 2W \qquad \text{Subtract } 2W.$$

$$P - 2W = 2L \qquad \text{Combine like terms.}$$

$$\frac{P - 2W}{2} = \frac{2L}{2} \qquad \text{Divide by 2.}$$

$$\frac{P - 2W}{2} = L, \quad \text{or} \quad L = \frac{P - 2W}{2} \qquad \frac{2L}{2} = \frac{2}{2} \cdot L = 1 \cdot L = L$$

NOW TRY

NOW TRY EXERCISE 8
Solve $x = u + zs$ for $z$.

## EXAMPLE 8 Solving for a Specified Variable

Solve $F = \frac{9}{5}C + 32$ for $C$.

Our goal is to isolate $C$.

$$F = \frac{9}{5}C + 32 \qquad \text{This is the formula for converting temperatures from Celsius to Fahrenheit.}$$

$$F - 32 = \frac{9}{5}C + 32 - 32 \qquad \text{Subtract 32.}$$

$$F - 32 = \frac{9}{5}C$$

Be sure to use parentheses.

$$\frac{5}{9}(F - 32) = \frac{5}{9} \cdot \frac{9}{5}C \qquad \text{Multiply by } \tfrac{5}{9}.$$

$$\frac{5}{9}(F - 32) = C, \quad \text{or} \quad C = \frac{5}{9}(F - 32) \qquad \text{This is the formula for converting temperatures from Fahrenheit to Celsius.}$$

NOW TRY

NOW TRY ANSWERS

6. $F = \frac{W}{d}$ 7. $A = \frac{C - By}{x}$

8. $z = \frac{x - u}{s}$

**NOW TRY EXERCISE 9**

Solve $S = \frac{1}{2}(a + b + c)$ for $a$.

## EXAMPLE 9 Solving for a Specified Variable

Solve $\mathscr{A} = \frac{1}{2}h(b + B)$ for $B$.

Our goal is to isolate $B$.

$$\mathscr{A} = \frac{1}{2}h(b + B)$$

Multiplying 2 times $\frac{1}{2}$ here is *not* an application of the distributive property.

$$2\mathscr{A} = 2 \cdot \frac{1}{2}h(b + B) \qquad \text{Multiply by 2 to clear the fraction.}$$

$$2\mathscr{A} = h(b + B) \qquad 2 \cdot \tfrac{1}{2} = \tfrac{2}{2} = 1$$

$$2\mathscr{A} = hb + hB \qquad \text{Distributive property}$$

$$2\mathscr{A} - hb = hb + hB - hb \qquad \text{Subtract } hb.$$

$$2\mathscr{A} - hb = hB \qquad \text{Combine like terms.}$$

$$\frac{2\mathscr{A} - hb}{h} = \frac{hB}{h} \qquad \text{Divide by } h.$$

$$\frac{2\mathscr{A} - hb}{h} = B, \quad \text{or} \quad B = \frac{2\mathscr{A} - hb}{h}$$

NOW TRY

---

**NOTE** The result in **Example 9** can be written in a different form as follows:

$$B = \frac{2\mathscr{A} - hb}{h} = \frac{2\mathscr{A}}{h} - \frac{hb}{h} = \frac{2\mathscr{A}}{h} - b. \qquad \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$$

---

**NOW TRY ANSWER**

9. $a = 2S - b - c$

## 2.5 EXERCISES

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1. In your own words, explain what is meant by each term.
   - **(a)** Perimeter of a plane geometric figure
   - **(b)** Area of a plane geometric figure

2. ***Concept Check*** In parts (a)–(c), choose one of the following words to make the statement true: *linear, square*, or *cubic*.
   - **(a)** If the dimensions of a plane geometric figure are given in feet, then the **area** is given in __________ feet.
   - **(b)** If the dimensions of a rectangle are given in yards, then the **perimeter** is given in __________ yards.
   - **(c)** If the dimensions of a pyramid are given in meters, then the **volume** is given in __________ meters.

3. ***Concept Check*** The measure of a straight angle is __________. Vertical angles have __________ measures.
(the same/different)

4. ***Concept Check*** If a formula has exactly five variables, how many values would you need to be given in order to find the value of any one variable?

***Concept Check*** *Decide whether perimeter or area would be used to solve a problem concerning the measure of the quantity.*

**5.** Carpeting for a bedroom

**6.** Sod for a lawn

**7.** Fencing for a yard

**8.** Baseboards for a living room

**9.** Tile for a bathroom

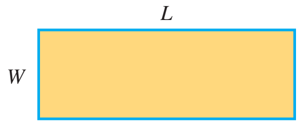
**10.** Fertilizer for a garden

**11.** Determining the cost of replacing a linoleum floor with a wood floor

**12.** Determining the cost of planting rye grass in a lawn for the winter

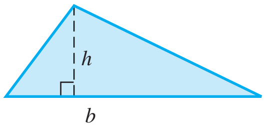
*A formula is given along with the values of all but one of the variables. Find the value of the variable that is not given. Use 3.14 as an approximation for $\pi$ (pi).* ***See Example 1.***

**13.** $P = 2L + 2W$ (perimeter of a rectangle); $L = 8$, $W = 5$



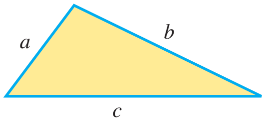
**14.** $P = 2L + 2W$; $L = 6$, $W = 4$

**15.** $\mathcal{A} = \frac{1}{2}bh$ (area of a triangle); $b = 8$, $h = 16$



**16.** $\mathcal{A} = \frac{1}{2}bh$; $b = 10$, $h = 14$

**17.** $P = a + b + c$ (perimeter of a triangle); $P = 12$, $a = 3$, $c = 5$



**18.** $P = a + b + c$; $P = 15$, $a = 3$, $b = 7$

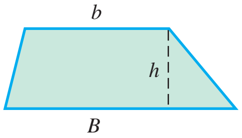
**19.** $d = rt$ (distance formula); $d = 252$, $r = 45$

**20.** $d = rt$; $d = 100$, $t = 2.5$

**21.** $I = prt$ (simple interest); $p = 7500$, $r = 0.035$, $t = 6$

**22.** $I = prt$; $p = 5000$, $r = 0.025$, $t = 7$

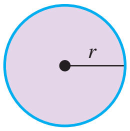
**23.** $\mathcal{A} = \frac{1}{2}h(b + B)$ (area of a trapezoid); $\mathcal{A} = 91$, $h = 7$, $b = 12$



**24.** $\mathcal{A} = \frac{1}{2}h(b + B)$; $\mathcal{A} = 75$, $b = 19$, $B = 31$

**25.** $C = 2\pi r$ (circumference of a circle); $C = 16.328$

**26.** $C = 2\pi r$; $C = 8.164$



**27.** $C = 2\pi r$; $C = 20\pi$

**28.** $C = 2\pi r$; $C = 100\pi$

**29.** $\mathcal{A} = \pi r^2$ (area of a circle); $r = 4$

**30.** $\mathcal{A} = \pi r^2$; $r = 12$

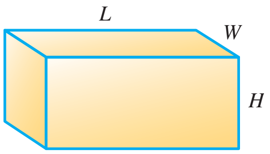
**31.** $S = 2\pi rh$; $S = 120\pi$, $h = 10$

**32.** $S = 2\pi rh$; $S = 720\pi$, $h = 30$

*The* ***volume*** *of a three-dimensional object is a measure of the space occupied by the object. For example, we would need to know the volume of a gasoline tank in order to find how many gallons of gasoline it would take to completely fill the tank.*

*In the following exercises, a formula for the volume (V) of a three-dimensional object is given, along with values for the other variables. Evaluate V. (Use 3.14 as an approximation for $\pi$.)* ***See Example 1.***

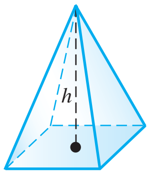
**33.** $V = LWH$ (volume of a rectangular box); $L = 10$, $W = 5$, $H = 3$



**34.** $V = LWH$; $L = 12$, $W = 8$, $H = 4$

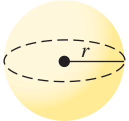
**35.** $V = \frac{1}{3}Bh$ (volume of a pyramid); $B = 12, h = 13$

**36.** $V = \frac{1}{3}Bh$; $B = 36, h = 4$



**37.** $V = \frac{4}{3}\pi r^3$ (volume of a sphere); $r = 12$

**38.** $V = \frac{4}{3}\pi r^3$; $r = 6$

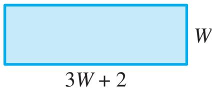


*Solve each problem.* ***See Examples 2 and 3.***

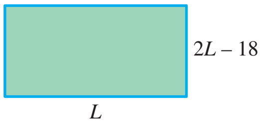
**39.** The length of a rectangle is 9 in. more than the width. The perimeter is 54 in. Find the length and the width of the rectangle.

**40.** The width of a rectangle is 3 ft less than the length. The perimeter is 62 ft. Find the length and the width of the rectangle.

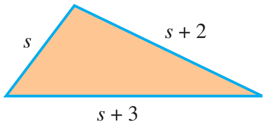
**41.** The perimeter of a rectangle is 36 m. The length is 2 m more than three times the width. Find the length and the width of the rectangle.



**42.** The perimeter of a rectangle is 36 yd. The width is 18 yd less than twice the length. Find the length and the width of the rectangle.



**43.** The longest side of a triangle is 3 in. longer than the shortest side. The medium side is 2 in. longer than the shortest side. If the perimeter of the triangle is 20 in., what are the lengths of the three sides?



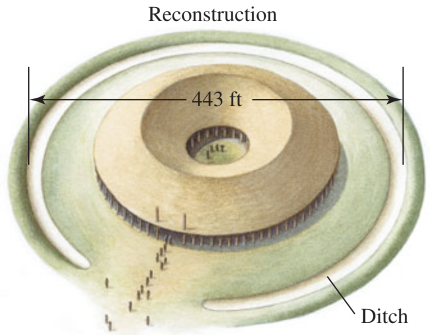
**44.** The perimeter of a triangle is 28 ft. The medium side is 4 ft longer than the shortest side, while the longest side is twice as long as the shortest side. What are the lengths of the three sides?

**45.** Two sides of a triangle have the same length. The third side measures 4 m less than twice that length. The perimeter of the triangle is 24 m. Find the lengths of the three sides.

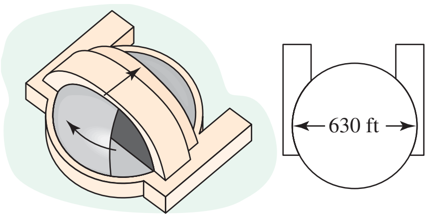
**46.** A triangle is such that its medium side is twice as long as its shortest side and its longest side is 7 yd less than three times its shortest side. The perimeter of the triangle is 47 yd. What are the lengths of the three sides?

*Use a formula to solve each problem. (Use* 3.14 *as an approximation for* $\pi$*.)* ***Formulas are found on the inside covers of this book. See Examples 2–4.***

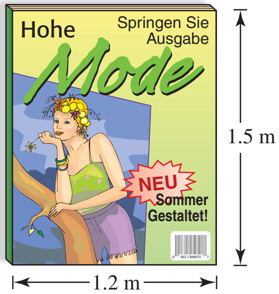
**47.** A prehistoric ceremonial site dating to about 3000 B.C. was discovered in southwestern England. The site is a nearly perfect circle, consisting of nine concentric rings that probably held upright wooden posts. Around this timber temple is a wide, encircling ditch enclosing an area with a diameter of 443 ft. Find this enclosed area to the nearest thousand square feet. (*Hint:* Find the radius. Then use $\mathcal{A} = \pi r^2$.) (*Source: Archaeology,* vol. 51, no. 1, Jan./Feb. 1998.)



**48.** The Rogers Centre in Toronto, Canada, is the first stadium with a hard-shell, retractable roof. The steel dome is 630 ft in diameter. To the nearest foot, what is the circumference of this dome? (*Source:* www.ballparks.com)



**49.** The largest fashion catalogue in the world was published in Hamburg, Germany. Each of the 212 pages in the catalogue measured 1.2 m by 1.5 m. What was the perimeter of a page? What was the area? (*Source: Guinness World Records.*)



**50.** The world's largest sand painting was created by Buddhist monks in the Singapore Expo Hall in May 2004. The painting measured 12.24 m by 12.24 m. What was the perimeter of the sand painting? To the nearest hundredth of a square meter, what was the area? (*Source: Guinness World Records.*)

**51.** The area of a triangular road sign is 70 $\text{ft}^2$. If the base of the sign measures 14 ft, what is the height of the sign?

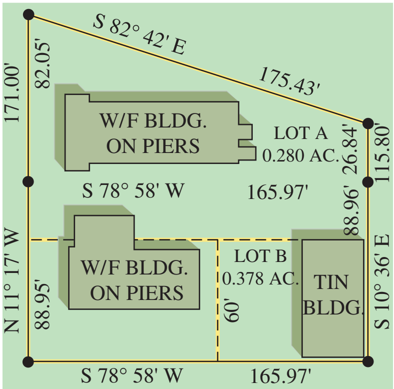
**52.** The area of a triangular advertising banner is 96 $\text{ft}^2$. If the height of the banner measures 12 ft, what is the measure of the base?

**53.** The largest drum ever constructed was made from Japanese cedar and cowhide, with diameter 15.74 ft. What was the area of the circular face of the drum? What was the circumference of the drum? Round your answers to the nearest hundredth. (*Source: Guinness World Records.*)

**54.** A drum played at the Royal Festival Hall in London had diameter 13 ft. What was the area of the circular face of the drum? What was the circumference of the drum? (*Source: Guinness World Records.*)

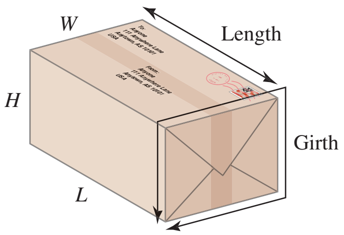
**55.** The survey plat depicted here shows two lots that form a trapezoid. The measures of the parallel sides are 115.80 ft and 171.00 ft. The height of the trapezoid is 165.97 ft. Find the combined area of the two lots. Round your answer to the nearest hundredth of a square foot.

**56.** Lot A in the survey plat is in the shape of a trapezoid. The parallel sides measure 26.84 ft and 82.05 ft. The height of the trapezoid is 165.97 ft. Find the area of Lot A. Round your answer to the nearest hundredth of a square foot.

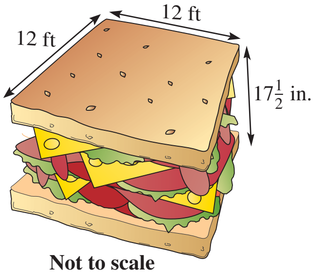


*Source:* Property survey in New Roads, Louisiana.

**57.** The U.S. Postal Service requires that any box sent by Priority Mail® have length plus girth (distance around) totaling no more than 108 in. The maximum volume that meets this condition is contained by a box with a square end 18 in. on each side. What is the length of the box? What is the maximum volume? (*Source:* United States Postal Service.)



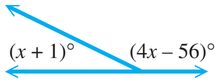
**58.** The world's largest sandwich, made by Wild Woody's Chill and Grill in Roseville, Michigan, was 12 ft long, 12 ft wide, and $17\frac{1}{2}$ in. $\left(1\frac{11}{24}\text{ ft}\right)$ thick. What was the volume of the sandwich? (*Source: Guinness World Records.*)



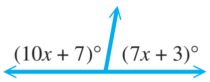
**Not to scale**

*Find the measure of each marked angle.* ***See Example 5.***

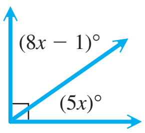
**59.**



**60.**

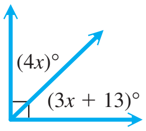


**61.**

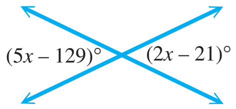


(*Hint:* These angles are *complements* of each other.)

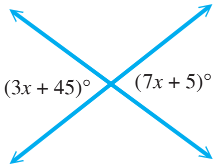
**62.**



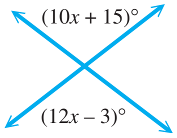
**63.**



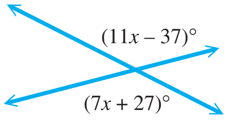
**64.**



**65.**



**66.**



*Solve each formula for the specified variable.* ***See Examples 6–9.***

**67.** $d = rt$ for $t$

**68.** $d = rt$ for $r$

**69.** $\mathcal{A} = bh$ for $b$

**70.** $\mathcal{A} = LW$ for $L$

**71.** $C = \pi d$ for $d$

**72.** $P = 4s$ for $s$

**73.** $V = LWH$ for $H$

**74.** $V = LWH$ for $W$

**75.** $I = prt$ for $r$

**76.** $I = prt$ for $p$

**77.** $\mathcal{A} = \frac{1}{2}bh$ for $h$

**78.** $\mathcal{A} = \frac{1}{2}bh$ for $b$

**79.** $V = \frac{1}{3}\pi r^2 h$ for $h$

**80.** $V = \pi r^2 h$ for $h$

**81.** $P = a + b + c$ for $b$

**82.** $P = a + b + c$ for $a$

**83.** $P = 2L + 2W$ for $W$

**84.** $A = p + prt$ for $r$

**85.** $y = mx + b$ for $m$

**86.** $y = mx + b$ for $x$

**87.** $Ax + By = C$ for $y$

**88.** $Ax + By = C$ for $x$

**89.** $M = C(1 + r)$ for $r$

**90.** $C = \frac{5}{9}(F - 32)$ for $F$

**91.** $P = 2(a + b)$ for $a$

**92.** $P = 2(a + b)$ for $b$

## *PREVIEW EXERCISES*

*Solve each equation.* ***See Section 2.2.***

**93.** $0.06x = 300$

**94.** $0.4x = 80$

**95.** $\frac{3}{4}x = 21$

**96.** $-\frac{5}{6}x = 30$

**97.** $-3x = \frac{1}{4}$

**98.** $4x = \frac{1}{3}$

# 2.6 Ratio, Proportion, and Percent

**OBJECTIVES**

1. Write ratios.
2. Solve proportions.
3. Solve applied problems by using proportions.
4. Find percents and percentages.

**OBJECTIVE 1 Write ratios.** A **ratio** is a comparison of two quantities using a quotient.

**Ratio**

The ratio of the number $a$ to the number $b$ (where $b \neq 0$) is written

$$\boldsymbol{a} \textbf{ to } \boldsymbol{b}, \qquad \boldsymbol{a}:\boldsymbol{b}, \qquad \text{or} \qquad \frac{\boldsymbol{a}}{\boldsymbol{b}}.$$

Writing a ratio as a quotient $\frac{a}{b}$ is most common in algebra.

NOW TRY EXERCISE 1

Write a ratio for each word phrase.

**(a)** 7 in. to 4 in.

**(b)** 45 sec to 2 min

## EXAMPLE 1 Writing Word Phrases as Ratios

Write a ratio for each word phrase.

**(a)** 5 hr to 3 hr $\quad \frac{5 \text{ hr}}{3 \text{ hr}} = \frac{5}{3}$

**(b)** 6 hr to 3 days

First convert 3 days to hours.

$$3 \text{ days} = 3 \cdot 24 = 72 \text{ hr} \qquad 1 \text{ day} = 24 \text{ hr}$$

Now write the ratio using the common unit of measure, hours.

$$\frac{6 \text{ hr}}{3 \text{ days}} = \frac{6 \text{ hr}}{72 \text{ hr}} = \frac{6}{72} = \frac{1}{12} \qquad \text{Write in lowest terms.}$$

NOW TRY

An application of ratios is in *unit pricing,* to see which size of an item offered in different sizes produces the best price per unit.

## EXAMPLE 2 Finding Price per Unit

A Cub Foods supermarket charges the following prices for a jar of extra crunchy peanut butter.

PEANUT BUTTER

| *Size* | *Price* |
|---|---|
| 18 oz | $1.78 |
| 28 oz | $2.97 |
| 40 oz | $3.98 |

Which size is the best buy? That is, which size has the lowest unit price?

NOW TRY ANSWERS

1. **(a)** $\frac{7}{4}$ **(b)** $\frac{3}{8}$

**NOW TRY EXERCISE 2**

A supermarket charges the following prices for a certain brand of liquid detergent.

| Size | Price |
|---|---|
| 150 oz | \$19.97 |
| 100 oz | \$13.97 |
| 75 oz | \$ 8.94 |

Which size is the best buy? What is the unit cost for that size?

To find the best buy, write ratios comparing the price for each size of jar to the number of units (ounces) per jar. Then divide to obtain the price per unit (ounce).

| Size | Unit Cost (dollars per ounce) |
|---|---|
| 18 oz | $\frac{\$1.78}{18} = \$0.099$ |
| 28 oz | $\frac{\$2.97}{28} = \$0.106$ |
| 40 oz | $\frac{\$3.98}{40} = \$0.100$ |

The best buy

(Results are rounded to the nearest thousandth.)

Because the 18-oz size produces the lowest unit cost, it is the best buy. This example shows that buying the largest size does not always provide the best buy.

NOW TRY

**OBJECTIVE 2** **Solve proportions.** A ratio is used to compare two numbers or amounts. A **proportion** says that two ratios are equal. For example, the proportion

$$\frac{3}{4} = \frac{15}{20}$$

A proportion is a special type of equation.

says that the ratios $\frac{3}{4}$ and $\frac{15}{20}$ are equal. In the proportion

$$\frac{a}{b} = \frac{c}{d} \quad (\text{where } b, d \neq 0),$$

$a$, $b$, $c$, and $d$ are the **terms** of the proportion. The terms $a$ and $d$ are called the **extremes,** and the terms $b$ and $c$ are called the **means.** We read the proportion $\frac{a}{b} = \frac{c}{d}$ as **"*a* is to *b* as *c* is to *d*."** Multiplying each side of this proportion by the common denominator, $bd$, gives the following.

$$bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}$$ Multiply each side by *bd*.

$$\frac{b}{b}(d \cdot a) = \frac{d}{d}(b \cdot c)$$ Associative and commutative properties

$$ad = bc$$ Commutative and identity properties

We can also find the products $ad$ and $bc$ by multiplying diagonally.

$$ad = bc$$

$$\frac{a}{b} = \frac{c}{d}$$

For this reason, $ad$ and $bc$ are called **cross products.**

### Cross Products

If $\frac{a}{b} = \frac{c}{d}$, then the cross products $ad$ and $bc$ are equal—that is, ***the product of the extremes equals the product of the means.***

Also, if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ (where $b, d \neq 0$).

**NOW TRY ANSWER**

**2.** 75 oz; \$0.119 per oz

**NOTE** If $\frac{a}{c} = \frac{b}{d}$, then $ad = cb$, or $ad = bc$. This means that the two proportions are equivalent, and the proportion

$$\frac{a}{b} = \frac{c}{d} \quad \text{can also be written as} \quad \frac{a}{c} = \frac{b}{d} \quad (\text{where } c, d \neq 0).$$

Sometimes one form is more convenient to work with than the other.

**NOW TRY EXERCISE 3**

Decide whether each proportion is *true* or *false*.

**(a)** $\frac{1}{3} = \frac{33}{100}$ **(b)** $\frac{4}{13} = \frac{16}{52}$

### EXAMPLE 3 Deciding Whether Proportions Are True

Decide whether each proportion is *true* or *false*.

**(a)** $\frac{3}{4} = \frac{15}{20}$

Check to see whether the cross products are equal.

$$3 \cdot 20 = 60 \qquad 4 \cdot 15 = 60$$

$$\frac{3}{4} = \frac{15}{20}$$

The cross products are equal, so the proportion is true.

**(b)** $\frac{6}{7} = \frac{30}{32}$

The cross products, $6 \cdot 32 = 192$ and $7 \cdot 30 = 210$, are not equal, so the proportion is false.

NOW TRY

Four numbers are used in a proportion. If any three of these numbers are known, the fourth can be found.

**NOW TRY EXERCISE 4**

Solve the proportion.

$$\frac{9}{7} = \frac{x}{56}$$

### EXAMPLE 4 Finding an Unknown in a Proportion

Solve the proportion $\frac{5}{9} = \frac{x}{63}$.

$$\frac{5}{9} = \frac{x}{63} \qquad \text{Solve for } x.$$

$$\begin{aligned} 5 \cdot 63 &= 9 \cdot x && \text{Cross products must be equal.} \\ 315 &= 9x && \text{Multiply.} \\ \frac{315}{9} &= \frac{9x}{9} && \text{Divide by 9.} \\ 35 &= x \end{aligned}$$

Check by substituting 35 for $x$ in the proportion. The solution set is $\{35\}$.

NOW TRY

**CAUTION** ***The cross-product method cannot be used directly if there is more than one term on either side of the equals symbol.***

**NOW TRY ANSWERS**

3. **(a)** false **(b)** true
4. $\{72\}$

NOW TRY EXERCISE 5

Solve the equation.

$$\frac{k-3}{6}=\frac{3k+2}{4}$$

## EXAMPLE 5 Solving an Equation by Using Cross Products

Solve the equation $\frac{m-2}{5}=\frac{m+1}{3}$.

$$\begin{aligned}
\frac{m-2}{5} &= \frac{m+1}{3} && \\
3(m-2) &= 5(m+1) && \text{Cross products} \\
3m-6 &= 5m+5 && \text{Distributive property} \\
3m &= 5m+11 && \text{Add 6.} \\
-2m &= 11 && \text{Subtract } 5m. \\
m &= -\frac{11}{2} && \text{Divide by } -2.
\end{aligned}$$

Be sure to use parentheses.

The solution set is $\left\{-\frac{11}{2}\right\}$.

NOW TRY

---

**NOTE** When you set cross products equal to each other, you are really multiplying each ratio in the proportion by a common denominator.

---

## OBJECTIVE 3 Solve applied problems by using proportions.

NOW TRY EXERCISE 6

Twenty gallons of gasoline costs \$49.80. How much would 27 gal of the same gasoline cost?

## EXAMPLE 6 Applying Proportions

After Lee Ann Spahr had pumped 5.0 gal of gasoline, the display showing the price read \$16.60. When she finished pumping the gasoline, the price display read \$48.14. How many gallons did she pump?

To solve this problem, set up a proportion, with prices in the numerators and gallons in the denominators. Let $x =$ the number of gallons she pumped.

$$\begin{aligned}
\text{Price} \rightarrow \frac{\$16.60}{5.0} &= \frac{\$48.14}{x} \leftarrow \text{Price} \\
\text{Gallons} \rightarrow \qquad & \qquad \leftarrow \text{Gallons} \\
16.60x &= 5.0(48.14) && \text{Cross products} \\
16.60x &= 240.70 && \text{Multiply.} \\
x &= 14.5 && \text{Divide by 16.60.}
\end{aligned}$$

Be sure that numerators represent the *same* quantities and denominators represent the *same* quantities.

She pumped 14.5 gal. Check this answer. (Using a calculator reduces the possibility of error.) Notice that the way the proportion was set up uses the fact that the unit price is the same, no matter how many gallons are purchased.

NOW TRY

## OBJECTIVE 4 Find percents and percentages.

***A percent is a ratio where the second number is always 100.*** For example,

50% represents the ratio of 50 to 100, that is, $\frac{50}{100}$, or, as a decimal, 0.50.

27% represents the ratio of 27 to 100, that is, $\frac{27}{100}$, or, as a decimal, 0.27.

Since the word **percent** means **"per 100,"** one percent means "one per one hundred."

$$\mathbf{1\% = 0.01,} \quad \text{or} \quad \mathbf{1\% = \frac{1}{100}}$$

NOW TRY ANSWERS

**5.** $\left\{-\frac{12}{7}\right\}$ **6.** \$67.23

**NOW TRY EXERCISE 7**

Convert.

**(a)** 16% to a decimal

**(b)** 1.5 to a percent

### EXAMPLE 7 Converting Between Decimals and Percents

**(a)** Write 75% as a decimal.

$$75\% = 75 \cdot 1\% = 75 \cdot 0.01 = 0.75$$

The fraction form $1\% = \frac{1}{100}$ can also be used to convert 75% to a decimal.

$$75\% = 75 \cdot 1\% = 75 \cdot \frac{1}{100} = \frac{75}{100} = 0.75$$

**(b)** Write 3% as a decimal.

$$3\% = 3 \cdot 1\% = 3 \cdot 0.01 = 0.03$$

**(c)** Write 0.375 as a percent.

$$0.375 = 37.5 \cdot 0.01 = 37.5 \cdot 1\% = 37.5\%$$

**(d)** Write 2.63 as a percent.

$$2.63 = 263 \cdot 0.01 = 263 \cdot 1\% = 263\%$$

NOW TRY

We can solve a percent problem involving $x\%$ by writing it as the proportion

$$\frac{\textbf{amount}}{\textbf{base}} = \frac{x}{100}.$$

The amount, or **percentage,** is compared to the **base** (the whole amount). Another way to write this proportion is

$$\frac{\text{amount}}{\text{base}} = \text{percent (as a decimal)}$$

$$\textbf{amount} = \textbf{percent (as a decimal)} \cdot \textbf{base.} \qquad \text{Basic percent equation}$$

### EXAMPLE 8 Solving Percent Equations

Solve each problem.

**(a)** What is 15% of 600?
Let $n$ = the number. The word *of* indicates multiplication.

$$\begin{array}{ccccc} \text{What} & \text{is} & 15\% & \text{of} & 600? \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ n & = & 0.15 & \cdot & 600 \end{array} \qquad \text{Write the percent equation.}$$

Translate each word or phrase to write the equation.

$$n = 90 \qquad \text{Multiply.}$$

Write 15% as a decimal.

Thus, 90 is 15% of 600.

**(b)** 32% of what number is 64?

$$\begin{array}{ccccc} 32\% & \text{of} & \text{what number} & \text{is} & 64? \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0.32 & \cdot & n & = & 64 \end{array} \qquad \text{Write the percent equation.}$$

Write 32% as a decimal.

$$n = \frac{64}{0.32} \qquad \text{Divide by 0.32.}$$

$$n = 200 \qquad \text{Simplify. Use a calculator.}$$

32% of 200 is 64.

**NOW TRY ANSWERS**

7. **(a)** 0.16 **(b)** 150%

**NOW TRY EXERCISE 8**

Solve each problem.

**(a)** What is 20% of 70?

**(b)** 40% of what number is 130?

**(c)** 121 is what percent of 484?

**(c)** 90 is what percent of 360?

$$\begin{array}{ccccc} 90 & \text{is} & \text{what percent} & \text{of} & 360? \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 90 & = & p & \cdot & 360 \end{array}$$

Write the percent equation.

$$\frac{90}{360} = p$$

Divide by 360.

$$0.25 = p, \quad \text{or} \quad 25\% = p$$

Simplify. Write 0.25 as a percent.

Thus, 90 is 25% of 360.

NOW TRY

**NOW TRY EXERCISE 9**

A winter coat is on a clearance sale for $48. The regular price is $120. What percent of the regular price is the savings?

### EXAMPLE 9 Solving Applied Percent Problems

Solve each problem.

**(a)** A DVD with a regular price of $18 is on sale this week at 22% off. Find the amount of the discount and the sale price of the disc.

The discount is 22% of 18, so we must find the number that is 22% of 18.

$$\begin{array}{ccccc} \text{What number} & \text{is} & 22\% & \text{of} & 18? \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ n & = & 0.22 & \cdot & 18 \end{array}$$

Write the percent equation.

$$n = 3.96$$

Multiply.

The discount is $3.96, so the sale price is found by subtracting.

$$\$18.00 - \$3.96 = \$14.04$$

Original price − discount = sales price

**(b)** A newspaper ad offered a set of tires at a sales price of $258. The regular price was $300. What percent of the regular price was the savings?

The savings amounted to $300 − $258 = $42. We can now restate the problem: What percent of 300 is 42?

$$\begin{array}{ccccc} \text{What percent} & \text{of} & 300 & \text{is} & 42? \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ p & \cdot & 300 & = & 42 \end{array}$$

Write the percent equation.

$$p = \frac{42}{300}$$

Divide by 300.

$$p = 0.14, \quad \text{or} \quad 14\%$$

Simplify. Write 0.14 as a percent.

The sale price represents a 14% savings.

NOW TRY

**NOW TRY ANSWERS**

**8. (a)** 14 **(b)** 325 **(c)** 25%

**9.** 60%

## 2.6 EXERCISES

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*Complete solution available on the Video Resources on DVD*

**1.** ***Concept Check*** Match each ratio in Column I with the ratio equivalent to it in Column II.

| I | II |
|---|---|
| **(a)** 75 to 100 | **A.** 80 to 100 |
| **(b)** 5 to 4 | **B.** 50 to 100 |
| **(c)** $\frac{1}{2}$ | **C.** 3 to 4 |
| **(d)** 4 to 5 | **D.** 15 to 12 |

**2.** Which one of the following represents a ratio of 3 days to 2 weeks?

**A.** $\frac{3}{2}$ **B.** $\frac{3}{7}$

**C.** $\frac{1.5}{1}$ **D.** $\frac{3}{14}$

*Write a ratio for each word phrase. Write fractions in lowest terms.* ***See Example 1.***

**3.** 40 mi to 30 mi

**4.** 60 ft to 70 ft

**5.** 120 people to 90 people

**6.** 72 dollars to 220 dollars

**7.** 20 yd to 8 ft

**8.** 30 in. to 8 ft

**9.** 24 min to 2 hr

**10.** 16 min to 1 hr

**11.** 60 in. to 2 yd

**12.** 5 days to 40 hr

*Find the best buy for each item. Give the unit price to the nearest thousandth.* ***See Example 2.*** (*Source:* Cub Foods.)

**13.** GRANULATED SUGAR

| *Size* | *Price* |
|---|---|
| 4 lb | $1.78 |
| 10 lb | $4.29 |

**14.** GROUND COFFEE

| *Size* | *Price* |
|---|---|
| 15 oz | $3.43 |
| 34.5 oz | $6.98 |

**15.** SALAD DRESSING

| *Size* | *Price* |
|---|---|
| 16 oz | $2.44 |
| 32 oz | $2.98 |
| 48 oz | $4.95 |

**16.** BLACK PEPPER

| *Size* | *Price* |
|---|---|
| 2 oz | $2.23 |
| 4 oz | $2.49 |
| 8 oz | $6.59 |

**17.** VEGETABLE OIL

| *Size* | *Price* |
|---|---|
| 16 oz | $1.66 |
| 32 oz | $2.59 |
| 64 oz | $4.29 |
| 128 oz | $6.49 |

**18.** MOUTHWASH

| *Size* | *Price* |
|---|---|
| 8.5 oz | $0.99 |
| 16.9 oz | $1.87 |
| 33.8 oz | $2.49 |
| 50.7 oz | $2.99 |

**19.** TOMATO KETCHUP

| *Size* | *Price* |
|---|---|
| 14 oz | $1.39 |
| 24 oz | $1.55 |
| 36 oz | $1.78 |
| 64 oz | $3.99 |

**20.** GRAPE JELLY

| *Size* | *Price* |
|---|---|
| 12 oz | $1.05 |
| 18 oz | $1.73 |
| 32 oz | $1.84 |
| 48 oz | $2.88 |

**21.** LAUNDRY DETERGENT

| *Size* | *Price* |
|---|---|
| 87 oz | $7.88 |
| 131 oz | $10.98 |
| 263 oz | $19.96 |

**22.** SPAGHETTI SAUCE

| *Size* | *Price* |
|---|---|
| 15.5 oz | $1.19 |
| 32 oz | $1.69 |
| 48 oz | $2.69 |

*Decide whether each proportion is* true *or* false. ***See Example 3.***

**23.** $\frac{5}{35} = \frac{8}{56}$

**24.** $\frac{4}{12} = \frac{7}{21}$

**25.** $\frac{120}{82} = \frac{7}{10}$

**26.** $\frac{27}{160} = \frac{18}{110}$

**27.** $\frac{\frac{1}{2}}{5} = \frac{1}{10}$

**28.** $\frac{\frac{1}{3}}{6} = \frac{1}{18}$

*Solve each equation.* ***See Examples 4 and 5.***

**29.** $\frac{k}{4} = \frac{175}{20}$

**30.** $\frac{x}{6} = \frac{18}{4}$

**31.** $\frac{49}{56} = \frac{z}{8}$

**32.** $\frac{20}{100} = \frac{z}{80}$

**33.** $\frac{x}{24} = \frac{15}{16}$

**34.** $\frac{x}{4} = \frac{12}{30}$

**35.** $\frac{z}{2} = \frac{z+1}{3}$

**36.** $\frac{m}{5} = \frac{m-2}{2}$

**37.** $\frac{3y-2}{5} = \frac{6y-5}{11}$

**38.** $\frac{2r+8}{4} = \frac{3r-9}{3}$

**39.** $\frac{5k+1}{6} = \frac{3k-2}{3}$

**40.** $\frac{x+4}{6} = \frac{x+10}{8}$

**41.** $\frac{2p+7}{3} = \frac{p-1}{4}$

**42.** $\frac{3m-2}{5} = \frac{4-m}{3}$

*Solve each problem. (Assume that all items are equally priced.) (In Exercises 53–56, round answers to the nearest tenth.)* ***See Example 6.***

**43.** If 16 candy bars cost $20.00, how much do 24 candy bars cost?

**44.** If 12 ring tones cost $30.00, how much do 8 ring tones cost?

**45.** Eight quarts of oil cost \$14.00. How much do 5 qt of oil cost?

**46.** Four tires cost \$398.00. How much do 7 tires cost?

**47.** If 9 pairs of jeans cost \$121.50, find the cost of 5 pairs.

**48.** If 7 shirts cost \$87.50, find the cost of 11 shirts.

**49.** If 6 gal of premium unleaded gasoline costs \$19.56, how much would it cost to completely fill a 15-gal tank?

**50.** If sales tax on a \$16.00 DVD is \$1.32, find the sales tax on a \$120.00 DVD player.

**51.** The distance between Kansas City, Missouri, and Denver is 600 mi. On a certain wall map, this is represented by a length of 2.4 ft. On the map, how many feet would there be between Memphis and Philadelphia, two cities that are actually 1000 mi apart?

**52.** The distance between Singapore and Tokyo is 3300 mi. On a certain wall map, this distance is represented by 11 in. The actual distance between Mexico City and Cairo is 7700 mi. How far apart are they on the same map?

**53.** A wall map of the United States has a distance of 8.5 in. between Memphis and Denver, two cities that are actually 1040 mi apart. The actual distance between St. Louis and Des Moines is 333 mi. How far apart are St. Louis and Des Moines on the map?

**54.** A wall map of the United States has a distance of 8.0 in. between New Orleans and Chicago, two cities that are actually 912 mi apart. The actual distance between Milwaukee and Seattle is 1940 mi. How far apart are Milwaukee and Seattle on the map?

**55.** On a world globe, the distance between Capetown and Bangkok, two cities that are actually 10,080 km apart, is 12.4 in. The actual distance between Moscow and Berlin is 1610 km. How far apart are Moscow and Berlin on this globe?

**56.** On a world globe, the distance between Rio de Janeiro and Hong Kong, two cities that are actually 17,615 km apart, is 21.5 in. The actual distance between Paris and Stockholm is 1605 km. How far apart are Paris and Stockholm on this globe?

**57.** According to the directions on a bottle of Armstrong® Concentrated Floor Cleaner, for routine cleaning, $\frac{1}{4}$ cup of cleaner should be mixed with 1 gal of warm water. How much cleaner should be mixed with $10\frac{1}{2}$ gal of water?

**58.** The directions on the bottle mentioned in **Exercise 57** also specify that, for extra-strength cleaning, $\frac{1}{2}$ cup of cleaner should be used for each gallon of water. For extra-strength cleaning, how much cleaner should be mixed with $15\frac{1}{2}$ gal of water?

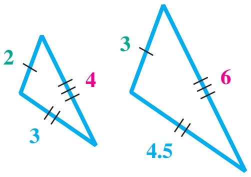
**59.** The euro is the common currency used by most European countries, including Italy. On August 15, 2009, the exchange rate between euros and U.S. dollars was 1 euro to \$1.4294. Ashley went to Rome and exchanged her U.S. currency for euros, receiving 300 euros. How much in U.S. dollars did she exchange? (*Source:* www.xe.com/ucc)

**60.** If 8 U.S. dollars can be exchanged for 103.0 Mexican pesos, how many pesos can be obtained for \$65? (Round to the nearest tenth.)

**61.** Biologists tagged 500 fish in North Bay on August 20. At a later date, they found 7 tagged fish in a sample of 700. Estimate the total number of fish in North Bay to the nearest hundred.

**62.** On June 13, researchers at West Okoboji Lake tagged 840 fish. A few weeks later, a sample of 1000 fish contained 18 that were tagged. Approximate the fish population to the nearest hundred.

*Two triangles are* ***similar*** *if they have the same shape (but not necessarily the same size). Similar triangles have sides that are proportional. The figure shows two similar triangles. Notice that the ratios of the corresponding sides all equal* $\frac{3}{2}$:

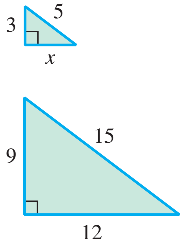


$$\frac{3}{2} = \frac{3}{2}, \qquad \frac{4.5}{3} = \frac{3}{2}, \qquad \frac{6}{4} = \frac{3}{2}.$$

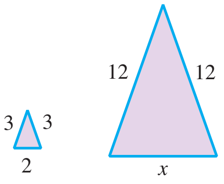
*If we know that two triangles are similar, we can set up a proportion to solve for the length of an unknown side.*

*Use a proportion to find the lengths x and y in each pair of similar triangles.*

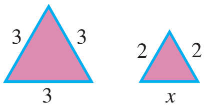
**63.**



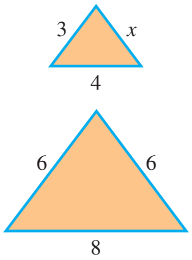
**64.**



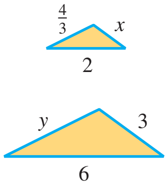
**65.**



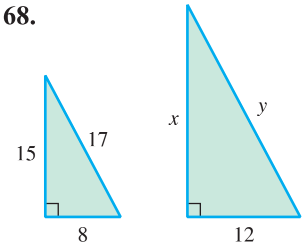
**66.**



**67.**



**68.**



*For Exercises 69 and 70,* ***(a)*** *draw a sketch consisting of two right triangles depicting the situation described, and* ***(b)*** *solve the problem. (Source: Guinness World Records.)*

**69.** An enlarged version of the chair used by George Washington at the Constitutional Convention casts a shadow 18 ft long at the same time a vertical pole 12 ft high casts a shadow 4 ft long. How tall is the chair?

**70.** One of the tallest candles ever constructed was exhibited at the 1897 Stockholm Exhibition. If it cast a shadow 5 ft long at the same time a vertical pole 32 ft high cast a shadow 2 ft long, how tall was the candle?

*The Consumer Price Index (CPI) provides a means of determining the purchasing power of the U.S. dollar from one year to the next. Using the period from 1982 to 1984 as a measure of* 100.0, *the CPI for selected years from 1995 through 2007 is shown in the table. To use the CPI to predict a price in a particular year, we set up a proportion and compare it with a known price in another year*:

$$\frac{\text{price in year } A}{\text{index in year } A} = \frac{\text{price in year } B}{\text{index in year } B}.$$

| *Year* | *Consumer Price Index* |
|---|---|
| 1995 | 152.4 |
| 1997 | 160.5 |
| 1999 | 166.6 |
| 2001 | 177.1 |
| 2003 | 184.0 |
| 2005 | 195.3 |
| 2007 | 207.3 |

*Source:* Bureau of Labor Statistics.

*Use the CPI figures in the table to find the amount that would be charged for using the same amount of electricity that cost* $225 *in 1995. Give your answer to the nearest dollar.*

**71.** in 1997
**72.** in 1999
**73.** in 2003
**74.** in 2007

*Convert each percent to a decimal.* ***See Examples 7(a) and 7(b).***

**75.** 53%
**76.** 38%
**77.** 96%
**78.** 11%
**79.** 9%
**80.** 7%
**81.** 129%
**82.** 174%

*Convert each decimal to a percent.* ***See Examples 7(c) and 7(d).***

**83.** 0.80
**84.** 0.75
**85.** 0.02
**86.** 0.06
**87.** 0.125
**88.** 0.983
**89.** 2.2
**90.** 1.4

*Solve each problem.* ***See Examples 8 and 9.***

**91.** What is 14% of 780?
**92.** What is 26% of 480?
**93.** 42% of what number is 294?
**94.** 18% of what number is 108?
**95.** 120% of what number is 510?
**96.** 140% of what number is 315?
**97.** 4 is what percent of 50?
**98.** 8 is what percent of 64?
**99.** What percent of 30 is 36?
**100.** What percent of 48 is 96?

**101.** Find the discount on a leather recliner with a regular price of $795 if the recliner is 15% off. What is the sale price of the recliner?

**102.** A laptop computer with a regular price of $597 is on sale at 20% off. Find the amount of the discount and the sale price of the computer.

**103.** Clayton earned 48 points on a 60-point geometry project. What percent of the total points did he earn?

**104.** On a 75-point algebra test, Grady scored 63 points. What percent of the total points did he score?

**105.** Anna saved $1950, which was 65% of the amount she needed for a used car. What was the total amount she needed for the car?

**106.** Bryn had $525, which was 70% of the total amount she needed for a deposit on an apartment. What was the total deposit she needed?

## PREVIEW EXERCISES

*Solve each equation.* ***See Section 2.3.***

**107.** $0.15x + 0.30(3) = 0.20(3 + x)$
**108.** $0.20(60) + 0.05x = 0.10(60 + x)$
**109.** $0.92x + 0.98(12 - x) = 0.96(12)$
**110.** $0.10(7) + 1.00x = 0.30(7 + x)$

# 2.7 Further Applications of Linear Equations

### OBJECTIVES

1. Use percent in solving problems involving rates.
2. Solve problems involving mixtures.
3. Solve problems involving simple interest.
4. Solve problems involving denominations of money.
5. Solve problems involving distance, rate, and time.

**OBJECTIVE 1 Use percent in solving problems involving rates.** Recall from **Section 2.6** that the word "percent" means "per 100."

$$1\% = 0.01, \qquad \text{or} \qquad 1\% = \frac{1}{100}$$

### PROBLEM-SOLVING HINT

Mixing different concentrations of a substance or different interest rates involves percents. To get the amount of pure substance or the interest, we multiply.

| Mixture Problems | Interest Problems (annual) |
|---|---|
| base × rate (%) = percentage | principal × rate (%) = interest |
| $b \times r = p$ | $p \times r = I$ |

***In an equation, percent is always written as a decimal or a fraction.***

*NOW TRY EXERCISE 1*

**(a)** How much pure alcohol is in 70 L of a 20% alcohol solution?

**(b)** Find the annual simple interest if $3200 is invested at 2%.

### EXAMPLE 1 Using Percents to Find Percentages

**(a)** If a chemist has 40 L of a 35% acid solution, then the amount of pure acid in the solution is

Write 35% as a decimal.

$$\underset{\text{Amount of solution}}{40 \text{ L}} \times \underset{\text{Rate of concentration}}{0.35} = \underset{\text{Amount of pure acid}}{14 \text{ L}}.$$

**(b)** If $1300 is invested for one year at 7% simple interest, the amount of interest earned in the year is

$$\underset{\text{Principal}}{\$1300} \times \underset{\text{Interest rate}}{0.07} = \underset{\text{Interest earned}}{\$91}.$$

*NOW TRY*

## OBJECTIVE 2 Solve problems involving mixtures.

### PROBLEM-SOLVING HINT

Using a table helps organize the information in a problem and more easily set up an equation, which is usually the most difficult step.

### EXAMPLE 2 Solving a Mixture Problem

A chemist needs to mix 20 L of a 40% acid solution with some 70% acid solution to obtain a mixture that is 50% acid. How many liters of the 70% acid solution should be used?

***Step 1*** **Read** the problem. Note the percent of each solution and of the mixture.

***Step 2*** **Assign a variable.**

Let $x =$ the number of liters of 70% acid solution needed.

Recall from **Example 1(a)** that the amount of pure acid in this solution is the product of the percent of strength and the number of liters of solution, or

$$0.70x. \quad \text{Liters of pure acid in } x \text{ liters of 70\% solution}$$

The amount of pure acid in the 20 L of 40% solution is

$$0.40(20) = 8. \quad \text{Liters of pure acid in the 40\% solution}$$

The new solution will contain $(x + 20)$ liters of 50% solution. The amount of pure acid in this solution is

$$0.50(x + 20). \quad \text{Liters of pure acid in the 50\% solution}$$

**FIGURE 15** illustrates this information, which is summarized in the table.

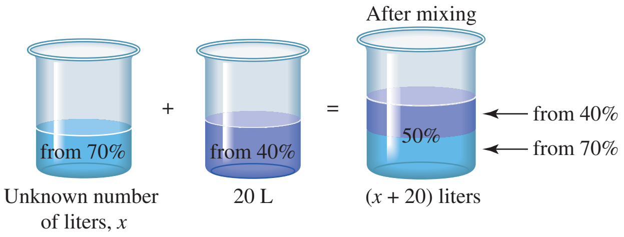


**FIGURE 15**

| *Liters of Solution* | *Rate (as a decimal)* | *Liters of Pure Acid* |
|---|---|---|
| $x$ | 0.70 | $0.70x$ |
| 20 | 0.40 | $0.40(20) = 8$ |
| $x + 20$ | 0.50 | $0.50(x + 20)$ |

*NOW TRY ANSWERS*

**1. (a)** 14 L **(b)** $64

NOW TRY EXERCISE 2

A certain seasoning is 70% salt. How many ounces of this seasoning must be mixed with 30 oz of dried herbs containing 10% salt to obtain a seasoning that is 50% salt?

***Step 3*** **Write an equation.** The number of liters of pure acid in the 70% solution added to the number of liters of pure acid in the 40% solution will equal the number of liters of pure acid in the final mixture.

$$\overset{\substack{\text{Pure acid}\\ \text{in 70\%}\\ \downarrow}}{0.70x} \quad \overset{\substack{\text{plus}\\ \downarrow}}{+} \quad \overset{\substack{\text{pure acid}\\ \text{in 40\%}\\ \downarrow}}{0.40(20)} \quad \overset{\substack{\text{is}\\ \downarrow}}{=} \quad \overset{\substack{\text{pure acid}\\ \text{in 50\%.}\\ \downarrow}}{0.50(x + 20)}$$

***Step 4*** **Solve** the equation.

$$\begin{aligned} 0.70x + 0.40(20) &= 0.50x + 0.50(20) && \text{Distributive property} \\ 70x + 40(20) &= 50x + 50(20) && \text{Multiply by 100.} \\ 70x + 800 &= 50x + 1000 && \text{Multiply.} \\ 20x + 800 &= 1000 && \text{Subtract } 50x. \\ 20x &= 200 && \text{Subtract 800.} \\ x &= 10 && \text{Divide by 20.} \end{aligned}$$

***Step 5*** **State the answer.** The chemist needs to use 10 L of 70% solution.

***Step 6*** **Check.** The answer checks, since

$$0.70(10) + 0.40(20) = 7 + 8 = 15 \qquad \text{Sum of two solutions}$$

and

$$0.50(10 + 20) = 0.50(30) = 15. \qquad \text{Mixture}$$

NOW TRY

---

**NOTE** In a mixture problem, the concentration of the final mixture must be *between* the concentrations of the two solutions making up the mixture.

---

**OBJECTIVE 3** **Solve problems involving simple interest.** The formula for simple interest, $I = prt$, becomes $I = pr$ when time $t = 1$ (for annual interest), as shown in the Problem-Solving Hint at the beginning of this section. Multiplying the total amount (principal) by the rate (rate of interest) gives the percentage (amount of interest).

### EXAMPLE 3 Solving a Simple Interest Problem

Susan Grody plans to invest some money at 6% and \$2000 more than this amount at 7%. To earn \$790 per year in interest, how much should she invest at each rate?

***Step 1*** **Read** the problem again. There will be two answers.

***Step 2*** **Assign a variable.**

Let $x =$ the amount invested at 6% (in dollars).

Then $x + 2000 =$ the amount invested at 7% (in dollars).

| *Amount Invested in Dollars* | *Rate of Interest* | *Interest for One Year* |
|---|---|---|
| $x$ | 0.06 | $0.06x$ |
| $x + 2000$ | 0.07 | $0.07(x + 2000)$ |

Use a table to arrange the given information.

NOW TRY ANSWER

**2.** 60 oz

NOW TRY EXERCISE 3

A financial advisor invests some money in a municipal bond paying 3% annual interest and \$5000 more than that amount in a certificate of deposit paying 4% annual interest. To earn \$410 per year in interest, how much should he invest at each rate?

***Step 3*** **Write an equation.** Multiply amount by rate to get the interest earned.

$$\begin{array}{ccccc} \text{Interest at 6\%} & \text{plus} & \text{interest at 7\%} & \text{is} & \text{total interest.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0.06x & + & 0.07(x+2000) & = & 790 \end{array}$$

***Step 4*** **Solve.**

$0.06x + 0.07x + 0.07(2000) = 790$ Distributive property

$6x + 7x + 7(2000) = 79{,}000$ Multiply by 100.

$6x + 7x + 14{,}000 = 79{,}000$ Multiply.

$13x + 14{,}000 = 79{,}000$ Combine like terms.

$13x = 65{,}000$ Subtract 14,000.

$x = 5000$ Divide by 13.

***Step 5*** **State the answer.** She should invest \$5000 at 6% and \$5000 + \$2000 = \$7000 at 7%.

***Step 6*** **Check.** Investing \$5000 at 6% and \$7000 at 7% gives total interest of

$$0.06(\$5000) + 0.07(\$7000) = \$300 + \$490 = \$790, \quad \text{as required.}$$

NOW TRY

## OBJECTIVE 4 Solve problems involving denominations of money.

### PROBLEM-SOLVING HINT

To get the total value in problems that involve different denominations of money or items with different monetary values, we multiply.

**Money Denominations Problems**

**number × value of one item = total value**

For example, 30 dimes have a monetary value of $30(\$0.10) = \$3$. Fifteen \$5 bills have a value of $15(\$5) = \$75$. A table is also helpful for these problems.

## EXAMPLE 4 Solving a Money Denominations Problem

A bank teller has 25 more \$5 bills than \$10 bills. The total value of the money is \$200. How many of each denomination of bill does she have?

***Step 1*** **Read** the problem. We must find the number of each denomination of bill.

***Step 2*** **Assign a variable.**

Let $x =$ the number of \$10 bills.

Then $x + 25 =$ the number of \$5 bills.

| *Number of Bills* | *Denomination* | *Total Value* |
|---|---|---|
| $x$ | 10 | $10x$ |
| $x + 25$ | 5 | $5(x + 25)$ |

Organize the given information in a table.

NOW TRY ANSWER

**3.** \$3000 at 3%; \$8000 at 4%

**NOW TRY EXERCISE 4**

Clayton has saved \$5.65 in dimes and quarters. He has 10 more quarters than dimes. How many of each denomination of coin does he have?

***Step 3*** **Write an equation.** Multiplying the number of bills by the denomination gives the monetary value. The value of the tens added to the value of the fives must be \$200.

$$\overset{\substack{\text{Value of}\\ \text{tens}\\ \downarrow}}{10x} \quad \overset{\substack{\text{plus}\\ \downarrow}}{+} \quad \overset{\substack{\text{value of}\\ \text{fives}\\ \downarrow}}{5(x+25)} \quad \overset{\substack{\text{is}\\ \downarrow}}{=} \quad \overset{\substack{\$200.\\ \downarrow}}{200}$$

***Step 4*** **Solve.**

$$\begin{aligned} 10x + 5x + 125 &= 200 && \text{Distributive property} \\ 15x + 125 &= 200 && \text{Combine like terms.} \\ 15x &= 75 && \text{Subtract 125.} \\ x &= 5 && \text{Divide by 15.} \end{aligned}$$

***Step 5*** **State the answer.** The teller has 5 tens and $5 + 25 = 30$ fives.

***Step 6*** **Check.** The teller has $30 - 5 = 25$ more fives than tens. The value of the money is

$$5(\$10) + 30(\$5) = \$200, \quad \text{as required.}$$

NOW TRY

**OBJECTIVE 5** **Solve problems involving distance, rate, and time.** If your car travels at an average rate of 50 mph for 2 hr, then it travels $50 \times 2 = 100$ mi. This is an example of the basic relationship between distance, rate, and time,

$$\textbf{distance} = \textbf{rate} \times \textbf{time},$$

given by the formula $d = rt$. By solving, in turn, for $r$ and $t$ in the formula, we obtain two other equivalent forms of the formula. The three forms are given here.

**Distance, Rate, and Time Relationship**

$$d = rt \qquad r = \frac{d}{t} \qquad t = \frac{d}{r}$$

### EXAMPLE 5 Finding Distance, Rate, or Time

Solve each problem using a form of the distance formula.

**(a)** The speed of sound is 1088 ft per sec at sea level at 32°F. Find the distance sound travels in 5 sec under these conditions.
We must find distance, given rate and time, using $d = rt$ (or $rt = d$).

$$\underset{\substack{\uparrow\\ \text{Rate}}}{1088} \quad \underset{\substack{\\ \times}}{\times} \quad \underset{\substack{\uparrow\\ \text{Time}}}{5} \quad \underset{\substack{\\ =}}{=} \quad \underset{\substack{\uparrow\\ \text{Distance}}}{5440 \text{ ft}}$$

**(b)** The winner of the first Indianapolis 500 race (in 1911) was Ray Harroun, driving a Marmon Wasp at an average rate (speed) of 74.59 mph. (*Source: Universal Almanac.*) How long did it take him to complete the 500 mi?

We must find time, given rate and distance, using $t = \frac{d}{r}$ $\left(\text{or } \frac{d}{r} = t\right)$.

$$\frac{\text{Distance} \rightarrow 500}{\text{Rate} \rightarrow 74.59} = 6.70 \text{ hr} \quad \text{(rounded)} \leftarrow \text{Time}$$

To convert 0.70 hr to minutes, we multiply by 60 to get $0.70(60) = 42$. It took Harroun about 6 hr, 42 min, to complete the race.

**NOW TRY ANSWER**

**4.** dimes: 9; quarters: 19

NOW TRY EXERCISE 5

It took a driver 6 hr to travel from St. Louis to Fort Smith, a distance of 400 mi. What was the driver's rate, to the nearest hundredth?

**(c)** At the 2008 Olympic Games, Australian swimmer Leisel Jones set an Olympic record of 65.17 sec in the women's 100-m breaststroke swimming event. (*Source: World Almanac and Book of Facts.*) Find her rate.

We must find rate, given distance and time, using $r = \frac{d}{t}$ $\left(\text{or } \frac{d}{t} = r\right)$.

$$\begin{array}{l} \text{Distance} \rightarrow \\ \text{Time} \rightarrow \end{array} \frac{100}{65.17} = 1.53 \text{ m per sec (rounded)} \leftarrow \text{Rate}$$

NOW TRY

NOW TRY EXERCISE 6

From a point on a straight road, two bicyclists ride in the same direction. One travels at a rate of 18 mph, the other at a rate of 20 mph. In how many hours will they be 5 mi apart?

## EXAMPLE 6 Solving a Motion Problem

Two cars leave Iowa City, Iowa, at the same time and travel east on Interstate 80. One travels at a constant rate of 55 mph. The other travels at a constant rate of 63 mph. In how many hours will the distance between them be 24 mi?

***Step 1*** **Read** the problem. We must find the time it will take for the distance between the cars to be 24 mi.

***Step 2*** **Assign a variable.** We are looking for time.

Let $t =$ the number of hours until the distance between them is 24 mi.

The sketch in **FIGURE 16** shows what is happening in the problem.

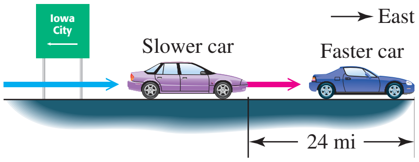


**FIGURE 16**

To construct a table, we fill in the information given in the problem, using $t$ for the time traveled by each car. We multiply rate by time to get the expressions for distances traveled.

| | *Rate* | *Time* | *Distance* |
|---|---|---|---|
| *Faster Car* | 63 | $t$ | $63t$ |
| *Slower Car* | 55 | $t$ | $55t$ |

The quantities $63t$ and $55t$ represent the two distances. The *difference* between the larger distance and the smaller distance is 24 mi.

***Step 3*** **Write an equation.**

$$63t - 55t = 24$$

***Step 4*** **Solve.**

$$8t = 24 \quad \text{Combine like terms.}$$

$$t = 3 \quad \text{Divide by 8.}$$

***Step 5*** **State the answer.** It will take the cars 3 hr to be 24 mi apart.

***Step 6*** **Check.** After 3 hr, the faster car will have traveled $63 \times 3 = 189$ mi and the slower car will have traveled $55 \times 3 = 165$ mi. The difference is

$$189 - 165 = 24, \quad \text{as required.}$$

NOW TRY

### PROBLEM-SOLVING HINT

In motion problems, once we have filled in two pieces of information in each row of the table, we can automatically fill in the third piece of information, using the appropriate form of the distance formula. Then we set up the equation based on our sketch and the information in the table.

NOW TRY ANSWERS

**5.** 66.67 mph **6.** 2.5 hr

### EXAMPLE 7 Solving a Motion Problem

Two planes leave Memphis at the same time. One heads south to New Orleans. The other heads north to Chicago. The Chicago plane flies 50 mph faster than the New Orleans plane. In $\frac{1}{2}$ hr, the planes are 275 mi apart. What are their rates?

***Step 1*** **Read** the problem carefully.

***Step 2*** **Assign a variable.**

Let $r =$ the rate of the slower plane.

Then $r + 50 =$ the rate of the faster plane.

| | *Rate* | *Time* | *Distance* |
|---|---|---|---|
| *Slower plane* | $r$ | $\frac{1}{2}$ | $\frac{1}{2}r$ |
| *Faster plane* | $r + 50$ | $\frac{1}{2}$ | $\frac{1}{2}(r + 50)$ |

Sum is 275 mi.

***Step 3*** **Write an equation.** As **FIGURE 17** shows, the planes are headed in *opposite* directions. The *sum* of their distances equals 275 mi.

$$\frac{1}{2}r + \frac{1}{2}(r + 50) = 275$$

***Step 4*** **Solve.**

$$r + (r + 50) = 550 \qquad \text{Multiply by 2.}$$

$$2r + 50 = 550 \qquad \text{Combine like terms.}$$

$$2r = 500 \qquad \text{Subtract 50.}$$

$$r = 250 \qquad \text{Divide by 2.}$$

***Step 5*** **State the answer.** The slower plane (headed south) has a rate of 250 mph. The rate of the faster plane is $250 + 50 = 300$ mph.

***Step 6*** **Check.** Verify that $\frac{1}{2}(250) + \frac{1}{2}(300) = 275$ mi.

NOW TRY

NOW TRY EXERCISE 7

Two cars leave a parking lot at the same time, one traveling east and the other traveling west. The westbound car travels 6 mph faster than the eastbound car. In $\frac{1}{4}$ hr, they are 35 mi apart. What are their rates?

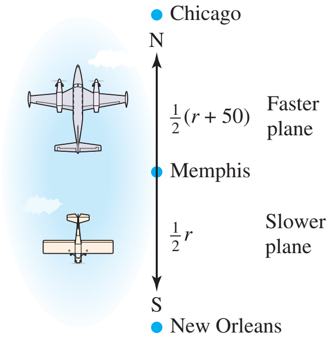


FIGURE 17

NOW TRY ANSWER

7. 67 mph; 73 mph

## 2.7 EXERCISES

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*Answer each question.* ***See Example 1 and the Problem-Solving Hint preceding Example 4.***

**1.** How much pure alcohol is in 150 L of a 30% alcohol solution?

**2.** How much pure acid is in 250 mL of a 14% acid solution?

**3.** If \$25,000 is invested for 1 yr at 3% simple interest, how much interest is earned?

**4.** If \$10,000 is invested for 1 yr at 3.5% simple interest, how much interest is earned?

**5.** What is the monetary value of 35 half-dollars?

**6.** What is the monetary value of 283 nickels?

*Concept Check* *Solve each percent problem. Remember that base* × *rate* = *percentage.*

**7.** The population of New Mexico in 2007 was about 1,917,000, with 44.4% Hispanic. What is the best estimate of the Hispanic population in New Mexico? (*Source:* U.S. Census Bureau.)

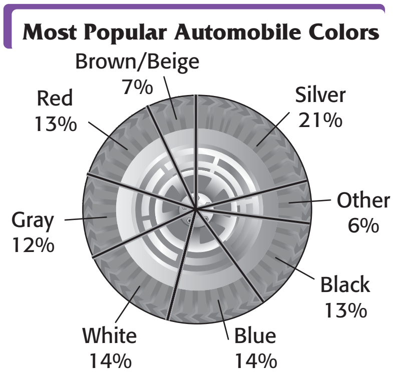
**A.** 850,000 **B.** 85,000 **C.** 650,000 **D.** 44,000

**8.** The population of Alabama in 2007 was about 4,628,000, with 26.5% represented by African-Americans. What is the best estimate of the African-American population in Alabama? (*Source:* U.S. Census Bureau.)

**A.** 600,000 **B.** 750,000 **C.** 1,200,000 **D.** 1,500,000

**9.** The graph shows the breakdown, by approximate percents, of the colors chosen for new 2007 model-year full-size and intermediate cars sold in the United States. If approximately 3.8 million of these cars were sold, about how many were each color? (*Source:* Ward's Communications.)

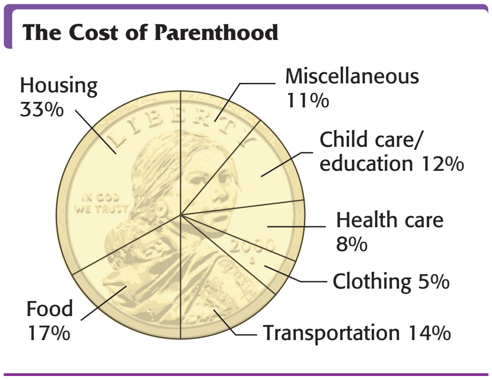
**(a)** White **(b)** Silver **(c)** Red



***Source:*** DuPont Automotive Products.

**10.** An average middle-income family will spend $221,190 to raise a child born in 2008 from birth to age 18. The graph shows the breakdown, by approximate percents, for various expense categories. To the nearest dollar, about how much will be spent to provide the following?

**(a)** Housing
**(b)** Food
**(c)** Health care



***Source:*** U.S. Department of Agriculture.

***Concept Check*** *Answer each question.*

**11.** Suppose that a chemist is mixing two acid solutions, one of 20% concentration and the other of 30% concentration. Which concentration could *not* be obtained?

**A.** 22% **B.** 24% **C.** 28% **D.** 32%

**12.** Suppose that pure alcohol is added to a 24% alcohol mixture. Which concentration could *not* be obtained?

**A.** 22% **B.** 26% **C.** 28% **D.** 30%

*Work each mixture problem.* ***See Example 2.***

**13.** How many liters of 25% acid solution must a chemist add to 80 L of 40% acid solution to obtain a solution that is 30% acid?

| *Liters of Solution* | *Rate* | *Liters of Acid* |
|---|---|---|
| $x$ | 0.25 | $0.25x$ |
| 80 | 0.40 | $0.40(80)$ |
| $x + 80$ | 0.30 | $0.30(x + 80)$ |

**14.** How many gallons of 50% antifreeze must be mixed with 80 gal of 20% antifreeze to obtain a mixture that is 40% antifreeze?

| *Gallons of Mixture* | *Rate* | *Gallons of Antifreeze* |
|---|---|---|
| $x$ | 0.50 | $0.50x$ |
| 80 | 0.20 | $0.20(80)$ |
| $x + 80$ | 0.40 | $0.40(x + 80)$ |

**15.** A pharmacist has 20 L of a 10% drug solution. How many liters of 5% solution must be added to get a mixture that is 8%?

| *Liters of Solution* | *Rate* | *Liters of Pure Drug* |
|---|---|---|
| 20 | | 20(0.10) |
| | 0.05 | |
| | 0.08 | |

**16.** A certain metal is 20% tin. How many kilograms of this metal must be mixed with 80 kg of a metal that is 70% tin to get a metal that is 50% tin?

| *Kilograms of Metal* | *Rate* | *Kilograms of Pure Tin* |
|---|---|---|
| $x$ | 0.20 | |
| | 0.70 | |
| | 0.50 | |

**17.** In a chemistry class, 12 L of a 12% alcohol solution must be mixed with a 20% solution to get a 14% solution. How many liters of the 20% solution are needed?

**18.** How many liters of a 10% alcohol solution must be mixed with 40 L of a 50% solution to get a 40% solution?

**19.** Minoxidil is a drug that has recently proven to be effective in treating male pattern baldness. Water must be added to 20 mL of a 4% minoxidil solution to dilute it to a 2% solution. How many milliliters of water should be used? (*Hint:* Water is 0% minoxidil.)

**20.** A pharmacist wishes to mix a solution that is 2% minoxidil. She has on hand 50 mL of a 1% solution, and she wishes to add some 4% solution to it to obtain the desired 2% solution. How much 4% solution should she add?

**21.** How many liters of a 60% acid solution must be mixed with a 75% acid solution to get 20 L of a 72% solution?

**22.** How many gallons of a 12% indicator solution must be mixed with a 20% indicator solution to get 10 gal of a 14% solution?

*Work each investment problem using simple interest.* ***See Example 3.***

**23.** Arlene Frank is saving money for her college education. She deposited some money in a savings account paying 5% and $1200 less than that amount in a second account paying 4%. The two accounts produced a total of $141 interest in 1 yr. How much did she invest at each rate?

**24.** Margaret Fennell won a prize for her work. She invested part of the money in a certificate of deposit at 4% and $3000 more than that amount in a bond paying 6%. Her annual interest income was $780. How much did Margaret invest at each rate?

**25.** An artist invests in a tax-free bond paying 6%, and $6000 more than three times as much in mutual funds paying 5%. Her total annual interest income from the investments is $825. How much does she invest at each rate?

**26.** With income earned by selling the rights to his life story, an actor invests some of the money at 3% and $30,000 more than twice as much at 4%. The total annual interest earned from the investments is $5600. How much is invested at each rate?

*Work each problem involving monetary values.* ***See Example 4.***

**27.** A coin collector has \$1.70 in dimes and nickels. She has two more dimes than nickels. How many nickels does she have?

| Number of Coins | Denomination | Total Value |
|---|---|---|
| $x$ | 0.05 | $0.05x$ |
|  | 0.10 |  |

**28.** A bank teller has \$725 in \$5 bills and \$20 bills. The teller has five more twenties than fives. How many \$5 bills does the teller have?

| Number of Bills | Denomination | Total Value |
|---|---|---|
| $x$ | 5 |  |
| $x + 5$ | 20 |  |

**29.** In May 2009, U.S. first-class mail rates increased to 44 cents for the first ounce, plus 17 cents for each additional ounce. If Sabrina spent \$14.40 for a total of 45 stamps of these two denominations, how many stamps of each denomination did she buy? (*Source:* U.S. Postal Service.)

**30.** A movie theater has two ticket prices: \$8 for adults and \$5 for children. If the box office took in \$4116 from the sale of 600 tickets, how many tickets of each kind were sold?

**31.** Harriet Amato operates a coffee shop. One of her customers wants to buy two kinds of beans: Arabian Mocha and Colombian Decaf. If she wants twice as much Mocha as Colombian Decaf, how much of each can she buy for a total of \$87.50? (Prices are listed on the sign.)

| | |
|---|---|
| Arabian Mocha | \$8.50/lb |
| Chocolate Mint | \$10.50/lb |
| Colombian Decaf | \$8.00/lb |
| French Roast | \$7.50/lb |
| Guatemalan Spice | \$9.50/lb |
| Hazelnut Decaf | \$10.00/lb |
| Italian Espresso | \$9.00/lb |
| Kona Deluxe | \$11.50/lb |

**32.** Harriet's Special Blend contains a combination of French Roast and Kona Deluxe beans. How many pounds of Kona Deluxe should she mix with 12 lb of French Roast to get a blend to be sold for \$10 a pound?

*Solve each problem involving distance, rate, and time.* ***See Example 5.***

**33.** ***Concept Check*** Which choice is the best estimate for the average rate of a bus trip of 405 mi that lasted 8.2 hr?

**A.** 50 mph **B.** 30 mph **C.** 60 mph **D.** 40 mph

**34.** Suppose that an automobile averages 45 mph and travels for 30 min. Is the distance traveled $45 \times 30 = 1350$ mi? If not, explain why not, and give the correct distance.

**35.** A driver averaged 53 mph and took 10 hr to travel from Memphis to Chicago. What is the distance between Memphis and Chicago?

**36.** A small plane traveled from Warsaw to Rome, averaging 164 mph. The trip took 2 hr. What is the distance from Warsaw to Rome?

**37.** The winner of the 2008 Indianapolis 500 (mile) race was Scott Dixon, who drove his Dellara-Honda to victory at a rate of 143.567 mph. What was his time (to the nearest thousandth of an hour)? (*Source: World Almanac and Book of Facts*.)

**38.** In 2008, Jimmie Johnson drove his Chevrolet to victory in the Brickyard 400 (mile) race at a rate of 115.117 mph. What was his time (to the nearest thousandth of an hour)? (*Source: World Almanac and Book of Facts*.)

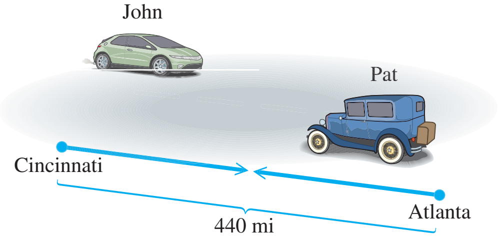
*In Exercises 39–42, find the rate on the basis of the information provided. Use a calculator and round your answers to the nearest hundredth. All events were at the 2008 Olympics. (Source: World Almanac and Book of Facts.)*

| | *Event* | *Participant* | *Distance* | *Time* |
|---|---|---|---|---|
| **39.** | 100-m hurdles, women | Dawn Harper, USA | 100 m | 12.54 sec |
| **40.** | 400-m hurdles, women | Melanie Walker, Jamaica | 400 m | 52.64 sec |
| **41.** | 400-m hurdles, men | Angelo Taylor, USA | 400 m | 47.25 sec |
| **42.** | 400-m run, men | LaShawn Merritt, USA | 400 m | 43.75 sec |

*Solve each motion problem.* ***See Examples 6 and 7.***

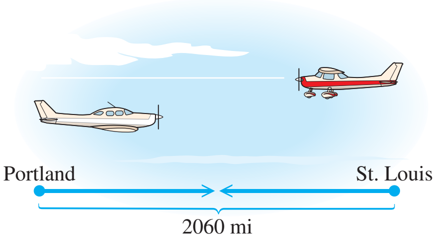
**43.** Atlanta and Cincinnati are 440 mi apart. John leaves Cincinnati, driving toward Atlanta at an average rate of 60 mph. Pat leaves Atlanta at the same time, driving toward Cincinnati in her antique auto, averaging 28 mph. How long will it take them to meet?

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| *John* | 60 | $t$ | $60t$ |
| *Pat* | 28 | $t$ | $28t$ |



**44.** St. Louis and Portland are 2060 mi apart. A small plane leaves Portland, traveling toward St. Louis at an average rate of 90 mph. Another plane leaves St. Louis at the same time, traveling toward Portland and averaging 116 mph. How long will it take them to meet?

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| *Plane Leaving Portland* | 90 | $t$ | $90t$ |
| *Plane Leaving St. Louis* | 116 | $t$ | $116t$ |



**45.** A train leaves Kansas City, Kansas, and travels north at 85 km per hr. Another train leaves at the same time and travels south at 95 km per hour. How long will it take before they are 315 km apart?

**46.** Two steamers leave a port on a river at the same time, traveling in opposite directions. Each is traveling at 22 mph. How long will it take for them to be 110 mi apart?

**47.** From a point on a straight road, Marco and Celeste ride bicycles in the same direction. Marco rides at 10 mph and Celeste rides at 12 mph. In how many hours will they be 15 mi apart?

**48.** At a given hour, two steamboats leave a city in the same direction on a straight canal. One travels at 18 mph and the other travels at 24 mph. In how many hours will the boats be 9 mi apart?

**49.** Two planes leave an airport at the same time, one flying east, the other flying west. The eastbound plane travels 150 mph slower. They are 2250 mi apart after 3 hr. Find the rate of each plane.

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| *Eastbound* | $x - 150$ | 3 | |
| *Westbound* | $x$ | 3 | |

**50.** Two trains leave a city at the same time. One travels north, and the other travels south 20 mph faster. In 2 hr, the trains are 280 mi apart. Find their rates.

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| *Northbound* | $x$ | 2 | |
| *Southbound* | $x + 20$ | 2 | |

**51.** Two cars start from towns 400 mi apart and travel toward each other. They meet after 4 hr. Find the rate of each car if one travels 20 mph faster than the other.

**52.** Two cars leave towns 230 km apart at the same time, traveling directly toward one another. One car travels 15 km per hr slower than the other. They pass one another 2 hr later. What are their rates?

***Brains Busters*** *Solve each problem.*

**53.** Kevin is three times as old as Bob. Three years ago the sum of their ages was 22 yr. How old is each now? (*Hint:* Write an expression first for the age of each now and then for the age of each three years ago.)

**54.** A store has 39 qt of milk, some in pint cartons and some in quart cartons. There are six times as many quart cartons as pint cartons. How many quart cartons are there? (*Hint:* 1 qt = 2 pt)

**55.** A table is three times as long as it is wide. If it were 3 ft shorter and 3 ft wider, it would be square (with all sides equal). How long and how wide is the table?

**56.** Elena works for \$6 an hour. A total of 25% of her salary is deducted for taxes and insurance. How many hours must she work to take home \$450?

**57.** Paula received a paycheck for \$585 for her weekly wages less 10% deductions. How much was she paid before the deductions were made?

**58.** At the end of a day, the owner of a gift shop had \$2394 in the cash register. This amount included sales tax of 5% on all sales. Find the amount of the sales.

## PREVIEW EXERCISES

*Decide whether each statement is* true *or* false. ***See Section 1.4.***

**59.** $6 > 6$ **60.** $10 \leq 10$ **61.** $-4 \leq -3$ **62.** $-11 > -9$ **63.** $0 > -\frac{1}{2}$

**64.** Graph the numbers $-3, -\frac{2}{3}, 0, 2, \frac{7}{2}$ on a number line. **See Section 1.4.**

# 2.8 Solving Linear Inequalities

**OBJECTIVES**

1. Graph intervals on a number line.
2. Use the addition property of inequality.
3. Use the multiplication property of inequality.
4. Solve linear inequalities by using both properties of inequality.
5. Solve applied problems by using inequalities.
6. Solve linear inequalities with three parts.

An **inequality** is an algebraic expression related by

| | | | | |
|---|---|---|---|---|
| $<$ | "is less than," | | $\leq$ | "is less than or equal to," |
| $>$ | "is greater than," | or | $\geq$ | "is greater than or equal to." |

**Linear Inequality in One Variable**

A **linear inequality in one variable** can be written in the form

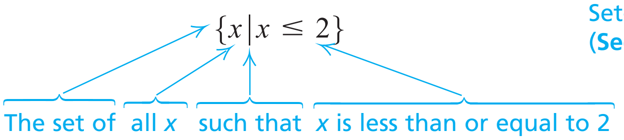
$$\boldsymbol{Ax + B < C,} \quad \boldsymbol{Ax + B \leq C,} \quad \boldsymbol{Ax + B > C,} \quad \text{or} \quad \boldsymbol{Ax + B \geq C,}$$

where $A$, $B$, and $C$ represent real numbers, and $A \neq 0$.

Some examples of linear inequalities in one variable follow.

$$x + 5 < 2, \quad z - \frac{3}{4} \geq 5, \quad \text{and} \quad 2k + 5 \leq 10 \qquad \text{Linear inequalities}$$

We solve a linear inequality by finding all real number solutions of it. For example, the solution set



Set-builder notation **(Section 1.4)**

includes *all real numbers* that are less than or equal to 2, not just the *integers* less than or equal to 2.

**OBJECTIVE 1** **Graph intervals on a number line.** Graphing is a good way to show the solution set of an inequality. To graph all real numbers belonging to the set $\{x|x \leq 2\}$, we place a square bracket at 2 on a number line and draw an arrow extending from the bracket to the left (since all numbers *less than* 2 are also part of the graph). See **FIGURE 18**.

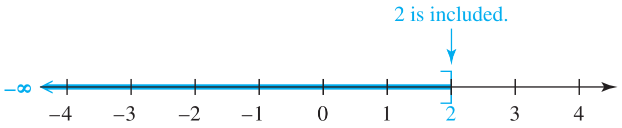


**FIGURE 18** Graph of the interval $(-\infty, 2]$

The set of numbers less than or equal to 2 is an example of an **interval** on the number line. We can write this interval using **interval notation.**

$$(-\infty, 2] \qquad \text{Interval notation}$$

The **negative infinity** symbol $\boldsymbol{-\infty}$ does not indicate a number, but shows that the interval includes *all* real numbers less than 2. Again, the square bracket indicates that 2 is part of the solution.

**NOW TRY EXERCISE 1**

Write each inequality in interval notation, and graph the interval.

**(a)** $x < -1$ **(b)** $-2 \leq x$

## EXAMPLE 1 Graphing Intervals on a Number Line

Write each inequality in interval notation, and graph the interval.

**(a)** $x > -5$

The statement $x > -5$ says that $x$ can represent any number greater than $-5$ but cannot equal $-5$. The interval is written $(-5, \infty)$. We graph this interval by placing a parenthesis at $-5$ and drawing an arrow to the right, as in **FIGURE 19**. The parenthesis at $-5$ indicates that $-5$ is *not* part of the graph.

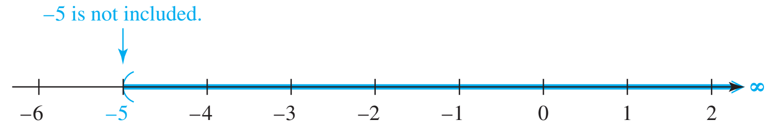


**FIGURE 19** Graph of the interval $(-5, \infty)$

**(b)** $3 > x$

The statement $3 > x$ means the same as $x < 3$. ***The inequality symbol continues to point toward the lesser number.*** The graph of $x < 3$, written in interval notation as $(-\infty, 3)$, is shown in **FIGURE 20**.

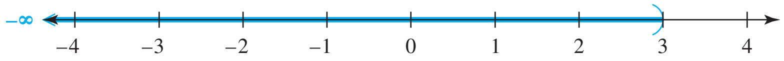


**FIGURE 20** Graph of the interval $(-\infty, 3)$

NOW TRY

***Keep the following important concepts regarding interval notation in mind:***

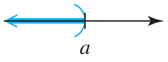
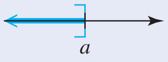
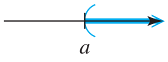
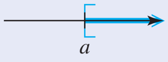
1. A parenthesis indicates that an endpoint is *not included* in a solution set.
2. A bracket indicates that an endpoint is *included* in a solution set.
3. A parenthesis is *always* used next to an infinity symbol, $-\infty$ or $\infty$.
4. The set of all real numbers is written in interval notation as $\boldsymbol{(-\infty, \infty)}$.

---

**NOTE** Some texts use a solid circle ● rather than a square bracket to indicate that an endpoint is included in a number line graph. An open circle ○ is used to indicate noninclusion, rather than a parenthesis.

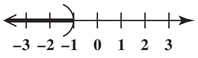
---

The table summarizes methods of expressing solution sets of linear inequalities.

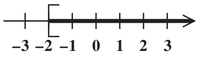
| *Set-Builder Notation* | *Interval Notation* | *Graph* |
|---|---|---|
| $\{x \mid x < a\}$ | $(-\infty, a)$ |  a |
| $\{x \mid x \leq a\}$ | $(-\infty, a]$ |  a |
| $\{x \mid x > a\}$ | $(a, \infty)$ |  a |
| $\{x \mid x \geq a\}$ | $[a, \infty)$ |  a |
| $\{x \mid x$ is a real number$\}$ | $(-\infty, \infty)$ |  |

**NOW TRY ANSWERS**

1. **(a)** $(-\infty, -1)$



**(b)** $[-2, \infty)$



**OBJECTIVE 2** **Use the addition property of inequality.** Consider the true inequality $2 < 5$. If 4 is added to each side, the result is

$$2 + 4 < 5 + 4 \qquad \text{Add 4.}$$

$$6 < 9, \qquad \text{True}$$

also a true sentence. This example suggests the **addition property of inequality.**

### Addition Property of Inequality

If $A$, $B$, and $C$ represent real numbers, then the inequalities

$$\boldsymbol{A < B} \quad \text{and} \quad \boldsymbol{A + C < B + C}$$

have exactly the same solutions.

That is, the same number may be added to each side of an inequality without changing the solutions.

***As with the addition property of equality, the same number may be* subtracted *from each side of an inequality.***

**NOW TRY EXERCISE 2**

Solve the inequality, and graph the solution set.

$$5 + 5x \geq 4x + 3$$

### EXAMPLE 2 Using the Addition Property of Inequality

Solve $7 + 3x \geq 2x - 5$, and graph the solution set.

$$7 + 3x \geq 2x - 5$$

$$7 + 3x - 2x \geq 2x - 5 - 2x \qquad \text{Subtract } 2x.$$

$$7 + x \geq -5 \qquad \text{Combine like terms.}$$

$$7 + x - 7 \geq -5 - 7 \qquad \text{Subtract 7.}$$

$$x \geq -12 \qquad \text{Combine like terms.}$$

The solution set is $[-12, \infty)$. Its graph is shown in **FIGURE 21**.

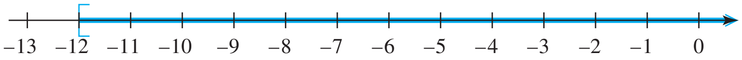


**FIGURE 21**

NOW TRY

**NOTE** Because an inequality has many solutions, we cannot check all of them by substitution as we did with the single solution of an equation. To check the solutions in **Example 2,** we first substitute $-12$ for $x$ in the related *equation*.

*CHECK*

$$7 + 3x = 2x - 5 \qquad \text{Related equation}$$

$$7 + 3(-12) \overset{?}{=} 2(-12) - 5 \qquad \text{Let } x = -12.$$

$$7 - 36 \overset{?}{=} -24 - 5 \qquad \text{Multiply.}$$

$$-29 = -29 \ ✓ \qquad \text{True}$$

A true statement results, so $-12$ is indeed the "boundary" point. Next we test a number other than $-12$ from the interval $[-12, \infty)$. We choose 0.

*CHECK*

$$7 + 3x \geq 2x - 5 \qquad \text{Original inequality}$$

$$7 + 3(0) \overset{?}{\geq} 2(0) - 5 \qquad \text{Let } x = 0.$$

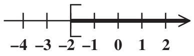
0 is easy to substitute.

$$7 \geq -5 \ ✓ \qquad \text{True}$$

The checks confirm that solutions to the inequality are in the interval $[-12, \infty)$. Any number "outside" the interval $[-12, \infty)$, that is, any number in $(-\infty, -12)$, will give a false statement when tested. (Try this.)

**NOW TRY ANSWER**

2. $[-2, \infty)$



**OBJECTIVE 3** **Use the multiplication property of inequality.** Consider the true inequality $3 < 7$. Multiply each side by the positive number 2.

$$\begin{aligned} 3 &< 7 \\ 2(3) &< 2(7) && \text{Multiply each side by 2.} \\ 6 &< 14 && \text{True} \end{aligned}$$

Now multiply each side of $3 < 7$ by the negative number $-5$.

$$\begin{aligned} 3 &< 7 \\ -5(3) &< -5(7) && \text{Multiply each side by } -5. \\ -15 &< -35 && \text{False} \end{aligned}$$

To get a true statement when multiplying each side by $-5$, ***we must reverse the direction of the inequality symbol.***

$$\begin{aligned} 3 &< 7 \\ -5(3) &> -5(7) && \text{Multiply by } -5. \text{ Reverse the symbol.} \\ -15 &> -35 && \text{True} \end{aligned}$$

---

**NOTE** The above illustrations began with the inequality $3 < 7$, a true statement involving two positive numbers. Similar results occur when one or both of the numbers is negative. Verify this with

$$-3 < 7, \quad 3 > -7, \quad \text{and} \quad -7 < -3$$

by multiplying each inequality first by 2 and then by $-5$.

---

These observations suggest the **multiplication property of inequality.**

**Multiplication Property of Inequality**

If $A$, $B$, and $C$ represent real numbers, with $C \neq 0$, and

1. if $C$ is *positive,* then the inequalities

$$A < B \quad \text{and} \quad AC < BC$$

have exactly the same solutions;

2. if $C$ is *negative,* then the inequalities

$$A < B \quad \text{and} \quad AC > BC$$

have exactly the same solutions.

That is, each side of an inequality may be multiplied by the same positive number without changing the solutions. ***If the multiplier is negative, we must reverse the direction of the inequality symbol.***

***As with the multiplication property of equality, the same nonzero number may be divided into each side of an inequality.***

Note the following differences for positive and negative numbers.

1. When each side of an inequality is multiplied or divided by a ***positive number,*** the direction of the inequality symbol *does not change.*
2. ***Reverse the direction of the inequality symbol ONLY when multiplying or dividing each side of an inequality by a NEGATIVE NUMBER.***

**NOW TRY EXERCISE 3**

Solve the inequality, and graph the solution set.

$$-5k \geq 15$$

## EXAMPLE 3 Using the Multiplication Property of Inequality

Solve each inequality, and graph the solution set.

**(a)** $3x < -18$

We divide each side by 3, a positive number, so the direction of the inequality symbol *does not* change. ***(It does not matter that the number on the right side of the inequality is negative.)***

$$3x < -18$$

3 is *positive*. Do NOT reverse the direction of the symbol.

$$\frac{3x}{3} < \frac{-18}{3} \qquad \text{Divide by 3.}$$

$$x < -6$$

The solution set is $(-\infty, -6)$. The graph is shown in **FIGURE 22**.

**FIGURE 22**

**(b)** $-4x \geq 8$

Here, each side of the inequality must be divided by $-4$, a negative number, which *does* require changing the direction of the inequality symbol.

$$-4x \geq 8$$

$-4$ is *negative*. Change $\geq$ to $\leq$.

$$\frac{-4x}{-4} \leq \frac{8}{-4} \qquad \text{Divide by } -4. \text{ Reverse the symbol.}$$

$$x \leq -2$$

The solution set $(-\infty, -2]$ is graphed in **FIGURE 23**.

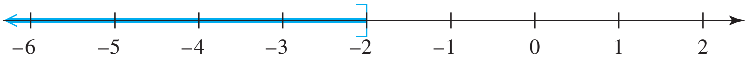


**FIGURE 23**

NOW TRY

## OBJECTIVE 4 Solve linear inequalities by using both properties of inequality.

### Solving a Linear Inequality

*Step 1* **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms on each side as needed.

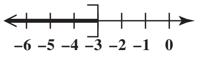
*Step 2* **Isolate the variable terms on one side.** Use the addition property of inequality to get all terms with variables on one side of the inequality and all numbers on the other side.

*Step 3* **Isolate the variable.** Use the multiplication property of inequality to change the inequality to the form "variable $< k$" or "variable $> k$," where $k$ is a number.

***Remember:*** ***Reverse the direction of the inequality symbol only when multiplying or dividing each side of an inequality by a negative number.***

**NOW TRY ANSWER**

**3.** $(-\infty, -3]$



**NOW TRY EXERCISE 4**

Solve the inequality, and graph the solution set.

$$6 - 2t + 5t \leq 8t - 4$$

## EXAMPLE 4 Solving a Linear Inequality

Solve $3x + 2 - 5 > -x + 7 + 2x$, and graph the solution set.

***Step 1*** Combine like terms and simplify.

$$3x + 2 - 5 > -x + 7 + 2x$$

$$3x - 3 > x + 7$$

***Step 2*** Use the addition property of inequality.

$$3x - 3 + 3 > x + 7 + 3 \qquad \text{Add 3.}$$

$$3x > x + 10$$

$$3x - x > x + 10 - x \qquad \text{Subtract } x.$$

$$2x > 10$$

***Step 3*** Use the multiplication property of inequality.

Because 2 is positive, keep the symbol >.

$$\frac{2x}{2} > \frac{10}{2} \qquad \text{Divide by 2.}$$

$$x > 5$$

The solution set is $(5, \infty)$. Its graph is shown in **FIGURE 24**.

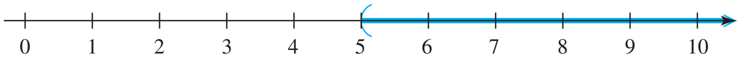


**FIGURE 24**

NOW TRY

**NOW TRY EXERCISE 5**

Solve the inequality, and graph the solution set.

$$2x - 3(x - 6) < 4(x + 7)$$

## EXAMPLE 5 Solving a Linear Inequality

Solve $5(x - 3) - 7x \geq 4(x - 3) + 9$, and graph the solution set.

***Step 1***

$$5(x - 3) - 7x \geq 4(x - 3) + 9$$

$$5x - 15 - 7x \geq 4x - 12 + 9 \qquad \text{Distributive property}$$

$$-2x - 15 \geq 4x - 3 \qquad \text{Combine like terms.}$$

***Step 2***

$$-2x - 15 - 4x \geq 4x - 3 - 4x \qquad \text{Subtract } 4x.$$

$$-6x - 15 \geq -3 \qquad \text{Combine like terms.}$$

$$-6x - 15 + 15 \geq -3 + 15 \qquad \text{Add 15.}$$

$$-6x \geq 12 \qquad \text{Combine like terms.}$$

***Step 3***

Because −6 is negative, change ≥ to ≤.

$$\frac{-6x}{-6} \leq \frac{12}{-6} \qquad \text{Divide by } -6. \text{ Reverse the symbol.}$$

$$x \leq -2$$

The solution set is $(-\infty, -2]$. Its graph is shown in **FIGURE 25**.

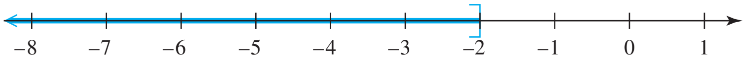


**FIGURE 25**

NOW TRY

## EXAMPLE 6 Solving a Linear Inequality with Fractions

Solve $-\frac{2}{3}(x - 3) - \frac{1}{2} < \frac{1}{2}(5 - x)$, and graph the solution set.

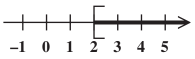
To clear fractions, multiply each side by the least common denominator, 6.

$$-\frac{2}{3}(x - 3) - \frac{1}{2} < \frac{1}{2}(5 - x)$$

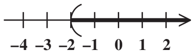
$$6\left[-\frac{2}{3}(x - 3) - \frac{1}{2}\right] < 6\left[\frac{1}{2}(5 - x)\right] \qquad \text{Multiply by 6, the LCD.}$$

**NOW TRY ANSWERS**

**4.** $[2, \infty)$



**5.** $(-2, \infty)$



**NOW TRY EXERCISE 6**

Solve and graph the solution set.

$$\frac{3}{4}(x-2)+\frac{1}{2}>\frac{1}{5}(x-8)$$

Be careful here.

$$6\left[-\frac{2}{3}(x-3)\right]-6\left(\frac{1}{2}\right)<6\left[\frac{1}{2}(5-x)\right]$$ Distributive property

$$-4(x-3)-3<3(5-x)$$ Multiply.

**Step 1** $$-4x+12-3<15-3x$$ Distributive property

$$-4x+9<15-3x$$

**Step 2** $$-4x+9+3x<15-3x+3x$$ Add $3x$.

$$-x+9<15$$

$$-x+9-9<15-9$$ Subtract 9.

$$-x<6$$

**Step 3** $$-1(-x)>-1(6)$$ Multiply by $-1$.
Change $<$ to $>$.

Reverse the inequality symbol when multiplying by a *negative* number.

$$x>-6$$

Check that the solution set is $(-6, \infty)$. See the graph in **FIGURE 26**.

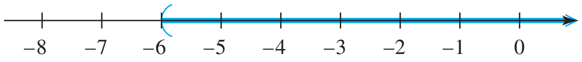


**FIGURE 26**

NOW TRY

## OBJECTIVE 5 Solve applied problems by using inequalities.

The table gives some common phrases that suggest inequality.

| *Phrase/Word* | *Example* | *Inequality* |
|---|---|---|
| Is more than | A number *is more than* 4 | $x > 4$ |
| Is less than | A number *is less than* −12 | $x < -12$ |
| Exceeds | A number *exceeds* 3.5 | $x > 3.5$ |
| Is at least | A number *is at least* 6 | $x \geq 6$ |
| Is at most | A number *is at most* 8 | $x \leq 8$ |

**NOW TRY EXERCISE 7**

A local health club charges a $40 one-time enrollment fee, plus $35 per month for a membership. Sara can spend no more than $355 on this exercise expense. What is the *maximum* number of months that Sara can belong to this health club?

### EXAMPLE 7 Using a Linear Inequality to Solve a Rental Problem

A rental company charges $15 to rent a chain saw, plus $2 per hr. Tom Ruhberg can spend no more than $35 to clear some logs from his yard. What is the *maximum* amount of time he can use the rented saw?

***Step 1*** **Read** the problem again.

***Step 2*** **Assign a variable.** Let $x =$ the number of hours he can rent the saw.

***Step 3*** **Write an inequality.** He must pay $15, plus $2x, to rent the saw for $x$ hours, and this amount must be *no more than* $35.

$$\overbrace{15+2x}^{\text{Cost of renting}} \quad \overbrace{\leq}^{\text{is no more than}} \quad \overbrace{35}^{\text{35 dollars.}}$$

Since "is more than" translates as $>$, "is *no* more than" translates as $\leq$.

***Step 4*** **Solve.** $$2x \leq 20$$ Subtract 15.

$$x \leq 10$$ Divide by 2.

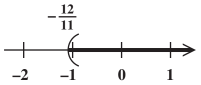
***Step 5*** **State the answer.** He can use the saw for a maximum of 10 hr. (Of course, he may use it for less time, as indicated by the inequality $x \leq 10$.)

***Step 6*** **Check.** If Tom uses the saw for 10 hr, he will spend $15 + 2(10) = 35$ dollars, the maximum amount.

NOW TRY

**NOW TRY ANSWERS**

**6.** $\left(-\frac{12}{11}, \infty\right)$



**7.** 9 months

**NOTE** In **Example 7,** we used the six problem-solving steps from **Section 2.4,** changing Step 3 from

"Write an equation" to "Write an inequality."

The next example uses the idea of finding the average of a number of scores. ***In general, to find the average of n numbers, add the numbers and divide by n.***

**NOW TRY EXERCISE 8**

Kristine has grades of 98 and 85 on her first two tests in algebra. If she wants an average of at least 90 after her third test, what score must she make on that test?

### EXAMPLE 8 Finding an Average Test Score

John Baker has grades of 86, 88, and 78 on his first three tests in geometry. If he wants an average of at least 80 after his fourth test, what are the possible scores he can make on that test?

***Step 1*** **Read** the problem again.

***Step 2*** **Assign a variable.** Let $x =$ John's score on his fourth test.

***Step 3*** **Write an inequality.**

$$\underbrace{\frac{86+88+78+x}{4}}_{\text{Average}} \; \underbrace{\geq 80}_{\text{is at least 80.}}$$

To find his average after four tests, add the test scores and divide by 4.

***Step 4*** **Solve.**

$$\frac{252+x}{4} \geq 80 \qquad \text{Add in the numerator.}$$

$$4\left(\frac{252+x}{4}\right) \geq 4(80) \qquad \text{Multiply by 4.}$$

$$252+x \geq 320$$

$$252+x-252 \geq 320-252 \qquad \text{Subtract 252.}$$

$$x \geq 68 \qquad \text{Combine like terms.}$$

***Step 5*** **State the answer.** He must score 68 or more on the fourth test to have an average of *at least* 80.

***Step 6*** **Check.**

$$\frac{86+88+78+68}{4} = \frac{320}{4} = 80$$

(Also show that a score greater than 68 gives an average greater than 80.)

NOW TRY

**CAUTION** In applied problems, remember that

| | | | |
|---|---|---|---|
| | at least | translates as | is greater than or equal to |
| and | at most | translates as | is less than or equal to. |

**OBJECTIVE 6** **Solve linear inequalities with three parts.** Inequalities that say that one number is *between* two other numbers are **three-part inequalities.** For example,

$$-3 < 5 < 7 \qquad \text{says that 5 is } between\ -3 \text{ and } 7.$$

**NOW TRY ANSWER**

**8.** 87 or more

**NOW TRY EXERCISE 9**

Write the inequality in interval notation, and graph the interval.

$$0 \le x < 2$$

## EXAMPLE 9 Graphing a Three-Part Inequality

Write the inequality in interval notation, and graph the interval.

$$-1 \le x < 3$$

The statement is read "$-1$ is less than or equal to $x$ *and* $x$ is less than 3." We want the set of numbers *between* $-1$ and 3, with $-1$ included and 3 excluded. In interval notation, we write $[-1, 3)$, using a square bracket at $-1$ because $-1$ is part of the graph and a parenthesis at 3 because 3 is not part of the graph. See **FIGURE 27**.

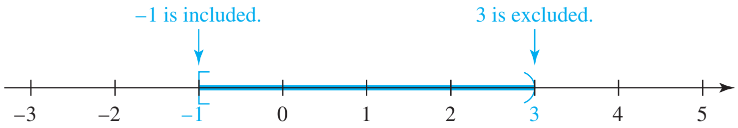


**FIGURE 27** Graph of the interval $[-1, 3)$

NOW TRY

The three-part inequality

$$3 < x + 2 < 8 \qquad \text{says that } x + 2 \text{ is } \textit{between} \text{ 3 and 8.}$$

We solve this inequality as follows.

$$3 - 2 < x + 2 - 2 < 8 - 2 \qquad \text{Subtract 2 from each part.}$$
$$1 < x < 6$$

The idea is to get the inequality in the form

$$\text{a number} < x < \text{another number.}$$

---

**CAUTION** ***Three-part inequalities are written so that the symbols point in the same direction and both point toward the lesser number.*** It would be *wrong* to write an inequality as $8 < x + 2 < 3$, since this would imply that $8 < 3$, a false statement.

---

## EXAMPLE 10 Solving Three-Part Inequalities

Solve each inequality, and graph the solution set.

**(a)**

$$4 < 3x - 5 \le 10$$
$$4 + 5 < 3x - 5 + 5 \le 10 + 5 \qquad \text{Add 5 to each part.}$$
$$9 < 3x \le 15$$
$$\frac{9}{3} < \frac{3x}{3} \le \frac{15}{3} \qquad \text{Divide each part by 3.}$$
$$3 < x \le 5$$

Remember to divide all *three* parts by 3.

The solution set is $(3, 5]$. Its graph is shown in **FIGURE 28**.

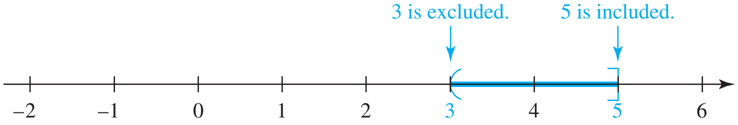
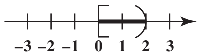


**FIGURE 28** Graph of the interval $(3, 5]$

**NOW TRY ANSWER**

**9.** $[0, 2)$



**NOW TRY EXERCISE 10**

Solve the inequality, and graph the solution set.

$$-4 \le \frac{3}{2}x - 1 \le 0$$

**(b)**

$$\begin{aligned}
-4 \le{}& \tfrac{2}{3}m - 1 < 8 \\
3(-4) \le{}& 3\left(\tfrac{2}{3}m - 1\right) < 3(8) && \text{Multiply each part by 3 to clear the fraction.} \\
-12 \le{}& 2m - 3 < 24 && \text{Distributive property} \\
-12 + 3 \le{}& 2m - 3 + 3 < 24 + 3 && \text{Add 3 to each part.} \\
-9 \le{}& 2m < 27 \\
\frac{-9}{2} \le{}& \frac{2m}{2} < \frac{27}{2} && \text{Divide each part by 2.} \\
-\frac{9}{2} \le{}& m < \frac{27}{2}
\end{aligned}$$

The solution set is $\left[-\frac{9}{2}, \frac{27}{2}\right)$. Its graph is shown in **FIGURE 29**.

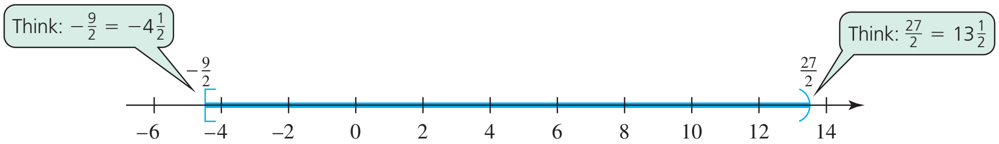


**FIGURE 29**

NOW TRY

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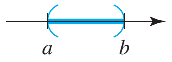
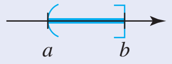
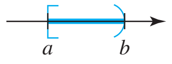
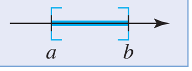
**NOTE** The inequality in **Example 10(b),** $-4 \le \frac{2}{3}m - 1 < 8$, can also be solved by first adding 1 to each part and then multiplying each part by $\frac{3}{2}$. Try this.

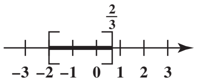
---

The table summarizes methods of expressing solution sets of three-part inequalities.

| *Set-Builder Notation* | *Interval Notation* | *Graph* |
|---|---|---|
| $\{x \mid a < x < b\}$ | $(a, b)$ |  |
| $\{x \mid a < x \le b\}$ | $(a, b]$ |  |
| $\{x \mid a \le x < b\}$ | $[a, b)$ |  |
| $\{x \mid a \le x \le b\}$ | $[a, b]$ |  |

**NOW TRY ANSWER**

**10.** $\left[-2, \frac{2}{3}\right]$



## 2.8 EXERCISES

*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Work each problem.*

**1.** When graphing an inequality, use a parenthesis if the inequality symbol is ________ or ________. Use a square bracket if the inequality symbol is ________ or ________.

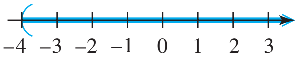
**2.** *True* or *false*? In interval notation, a square bracket is sometimes used next to an infinity symbol.

**3.** In interval notation, the set $\{x \mid x > 0\}$ is written ________.

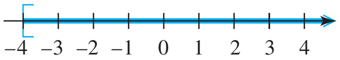
**4.** How does the graph of $x \ge -7$ differ from the graph of $x > -7$?

***Concept Check*** *Write an inequality involving the variable x that describes each set of numbers graphed.*

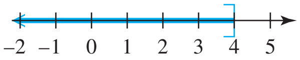
**5.**



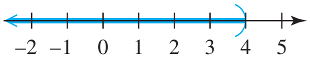
**6.**



**7.**



**8.**



*Write each inequality in interval notation, and graph the interval.* ***See Example 1.***

**9.** $k \leq 4$
**10.** $x \leq 3$
**11.** $x < -3$
**12.** $r < -11$
**13.** $t > 4$
**14.** $m > 5$
**15.** $0 \geq x$
**16.** $1 \geq x$
**17.** $-\frac{1}{2} \leq x$
**18.** $-\frac{3}{4} \leq x$

*Solve each inequality. Write the solution set in interval notation, and graph it.* ***See Example 2.***

**19.** $z - 8 \geq -7$
**20.** $p - 3 \geq -11$
**21.** $2k + 3 \geq k + 8$
**22.** $3x + 7 \geq 2x + 11$
**23.** $3n + 5 < 2n - 6$
**24.** $5x - 2 < 4x - 5$

*Solve each inequality. Write the solution set in interval notation, and graph it.* ***See Example 3.***

**25.** $3x < 18$
**26.** $5x < 35$
**27.** $2y \geq -20$
**28.** $6m \geq -24$
**29.** $-8t > 24$
**30.** $-7x > 49$
**31.** $-x \geq 0$
**32.** $-k < 0$
**33.** $-\frac{3}{4}r < -15$
**34.** $-\frac{7}{8}t < -14$
**35.** $-0.02x \leq 0.06$
**36.** $-0.03v \geq -0.12$

*Solve each inequality. Write the solution set in interval notation, and graph it.* ***See Examples 4–6.***

**37.** $8x + 9 \leq -15$
**38.** $6x + 7 \leq -17$
**39.** $-4x - 3 < 1$
**40.** $-5x - 4 < 6$
**41.** $5r + 1 \geq 3r - 9$
**42.** $6t + 3 < 3t + 12$
**43.** $6x + 3 + x < 2 + 4x + 4$
**44.** $-4w + 12 + 9w \geq w + 9 + w$
**45.** $-x + 4 + 7x \leq -2 + 3x + 6$
**46.** $14y - 6 + 7y > 4 + 10y - 10$
**47.** $5(t - 1) > 3(t - 2)$
**48.** $7(m - 2) < 4(m - 4)$
**49.** $5(x + 3) - 6x \leq 3(2x + 1) - 4x$
**50.** $2(x - 5) + 3x < 4(x - 6) + 1$
**51.** $4x - (6x + 1) \leq 8x + 2(x - 3)$
**52.** $2y - (4y + 3) > 6y + 3(y + 4)$
**53.** $\frac{2}{3}(p + 3) > \frac{5}{6}(p - 4)$
**54.** $\frac{7}{9}(y - 4) \leq \frac{4}{3}(y + 5)$
**55.** $\frac{2}{3}(3x - 1) \geq \frac{3}{2}(2x - 3)$
**56.** $\frac{7}{5}(10x - 1) < \frac{2}{3}(6x + 5)$
**57.** $-\frac{1}{4}(p + 6) + \frac{3}{2}(2p - 5) < 10$
**58.** $\frac{3}{5}(t - 2) - \frac{1}{4}(2t - 7) \leq 3$

## RELATING CONCEPTS EXERCISES 59–62

### FOR INDIVIDUAL OR GROUP WORK

***Work Exercises 59–62 in order,*** *to see how the solutions of an inequality are closely connected to the solution of the corresponding equation.*

**59.** Solve the equation $3x + 2 = 14$, and graph the solution set on a number line.

**60.** Solve the inequality $3x + 2 > 14$, and graph the solution set on a number line.

**61.** Solve the inequality $3x + 2 < 14$, and graph the solution set on a number line.

**62.** If we were to graph all the solution sets from **Exercises 59–61** on the same number line, describe the graph. (This is called the **union** of all the solution sets.)

*Concept Check* *Translate each statement into an inequality. Use x as the variable.*

**63.** You must be at least 18 yr old to vote.

**64.** Less than 1 in. of rain fell.

**65.** Chicago received more than 5 in. of snow.

**66.** A full-time student must take at least 12 credits.

**67.** Tracy could spend at most \$20 on a gift.

**68.** The car's speed exceeded 60 mph.

*Solve each problem.* ***See Examples 7 and 8.***

**69.** Christy Heinrich has scores of 76 and 81 on her first two algebra tests. If she wants an average of at least 80 after her third test, what possible scores can she make on that test?

**70.** Joseph Despagne has scores of 96 and 86 on his first two geometry tests. What possible scores can he make on his third test so that his average is at least 90?

**71.** When 2 is added to the difference between six times a number and 5, the result is greater than 13 added to five times the number. Find all such numbers.

**72.** When 8 is subtracted from the sum of three times a number and 6, the result is less than 4 more than the number. Find all such numbers.

**73.** The formula for converting Fahrenheit temperature to Celsius is

$$C = \frac{5}{9}(F - 32).$$

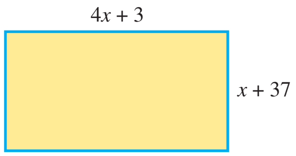
If the Celsius temperature on a certain winter day in Minneapolis is never less than $-25°$, how would you describe the corresponding Fahrenheit temperatures? (*Source:* National Climatic Data Center.)

**74.** The formula for converting Celsius temperature to Fahrenheit is

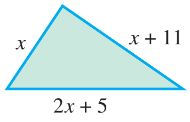
$$F = \frac{9}{5}C + 32.$$

The Fahrenheit temperature of Phoenix has never exceeded 122°. How would you describe this using Celsius temperature? (*Source:* National Climatic Data Center.)

**75.** For what values of $x$ would the rectangle have a perimeter of at least 400?



**76.** For what values of $x$ would the triangle have a perimeter of at least 72?



**77.** For a certain provider, an international phone call costs \$2.00 for the first 3 min, plus \$0.30 per minute for each minute or fractional part of a minute after the first 3 min. If $x$ represents the number of minutes of the length of the call after the first 3 min, then $2 + 0.30x$ represents the cost of the call. If Alan Lebovitz has \$5.60 to spend on a call, what is the maximum total time he can use the phone?

**78.** At the Speedy Gas'n Go, a car wash costs \$4.50 and gasoline is selling for \$3.20 per gallon. Carla Arriola has \$38.10 to spend, and her car is so dirty that she must have it washed. What is the maximum number of gallons of gasoline that she can purchase?

*A company that produces DVDs has found that revenue from the sales of the DVDs is* \$5 *per DVD, less sales costs of* \$100. *Production costs are* \$125, *plus* \$4 *per DVD. Profit (P) is given by revenue (R) less cost (C), so the company must find the production level x that makes*

$$P > 0, \quad \text{that is,} \quad R - C > 0. \qquad P = R - C$$

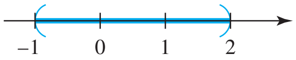
**79.** Write an expression for revenue $R$, letting $x$ represent the production level (number of DVDs to be produced).

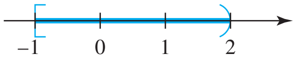
**80.** Write an expression for production costs $C$ in terms of $x$.

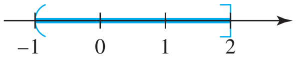
**81.** Write an expression for profit $P$, and then solve the inequality $P > 0$.

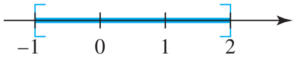
**82.** Describe the solution in terms of the problem.

***Concept Check*** *Write a three-part inequality involving the variable x that describes each set of numbers graphed.*

**83.**


**84.**


**85.**


**86.**


*Write each inequality in interval notation, and graph the interval.* ***See Example 9.***

**87.** $8 \le x \le 10$

**88.** $3 \le x \le 5$

**89.** $0 < y \le 10$

**90.** $-3 \le x < 0$

**91.** $4 > x > -3$

**92.** $6 \ge x \ge -4$

*Solve each inequality. Write the solution set in interval notation, and graph it.* ***See Example 10.***

**93.** $-5 \le 2x - 3 \le 9$

**94.** $-7 \le 3x - 4 \le 8$

**95.** $5 < 1 - 6m < 12$

**96.** $-1 \le 1 - 5q \le 16$

**97.** $10 < 7p + 3 < 24$

**98.** $-8 \le 3r - 1 \le -1$

**99.** $-12 \le \frac{1}{2}z + 1 \le 4$

**100.** $-6 \le 3 + \frac{1}{3}x \le 5$

**101.** $1 \le 3 + \frac{2}{3}p \le 7$

**102.** $2 < 6 + \frac{3}{4}x < 12$

**103.** $-7 \le \frac{5}{4}r - 1 \le -1$

**104.** $-12 \le \frac{3}{7}x + 2 \le -4$

## PREVIEW EXERCISES

*Find the value of y when* ***(a)*** $x = -2$ *and* ***(b)*** $x = 4$. ***See Sections 1.3 and 2.3.***

**105.** $y = 5x + 3$

**106.** $y = 4 - 3x$

**107.** $6x - 2 = y$

**108.** $4x + 7y = 11$

**109.** $2x - 5y = 10$

**110.** $y + 3x = 8$

STUDY SKILLS

# Taking Math Tests

| Techniques To Improve Your Test Score | Comments |
|---|---|
| ***Come prepared*** with a pencil, eraser, paper, and calculator, if allowed. | Working in pencil lets you erase, keeping your work neat and readable. |
| ***Scan the entire test, note the point values of different problems, and plan your time accordingly.*** | To do 20 problems in 50 minutes, allow $50 \div 20 = 2.5$ minutes per problem. Spend less time on the easier problems. |
| ***Do a "knowledge dump" when you get the test.*** Write important notes to yourself in a corner of the test, such as formulas. | Writing down tips and things that you've memorized at the beginning allows you to relax later. |
| ***Read directions carefully, and circle any significant words.*** When you finish a problem, read the directions again to make sure you did what was asked. | Pay attention to announcements written on the board or made by your instructor. Ask if you don't understand. |
| ***Show all your work.*** Many teachers give partial credit if some steps are correct, even if the final answer is wrong. ***Write neatly.*** | If your teacher can't read your writing, you won't get credit for it. If you need more space to work, ask to use extra paper. |
| ***Write down anything that might help solve a problem: a formula, a diagram, etc.*** If you can't get it, circle the problem and come back to it later. Do *not* erase anything you wrote down. | If you know even a little bit about the problem, write it down. The answer may come to you as you work on it, or you may get partial credit. Don't spend too long on any one problem. |
| ***If you can't solve a problem, make a guess.*** Do not change it unless you find an obvious mistake. | Have a good reason for changing an answer. Your first guess is usually your best bet. |
| ***Check that the answer to an application problem is reasonable and makes sense.*** Read the problem again to make sure you've answered the question. | Use common sense. Can the father really be seven years old? Would a month's rent be $32,140? Label your answer: $, years, inches, etc. |
| ***Check for careless errors. Rework the problem without looking at your previous work.*** Compare the two answers. | Reworking the problem from the beginning forces you to rethink it. If possible, use a different method to solve the problem. |

***Select several tips to try when you take your next math test.***

# CHAPTER 2

# SUMMARY

## KEY TERMS

### 2.1

equation
linear equation in one variable
solution
solution set
equivalent equations

### 2.3

conditional equation
identity
contradiction
empty (null) set

### 2.4

consecutive integers
degree
complementary angles
right angle
supplementary angles
straight angle

### 2.5

formula
area
perimeter
vertical angles
volume

### 2.6

ratio
proportion
terms of a proportion
extremes
means
cross products

### 2.8

inequality
linear inequality in one variable
interval on a number line
interval notation
three-part inequality

## NEW SYMBOLS

| | |
|---|---|
| $\emptyset$ | empty set |
| $1°$ | one degree |
| $a$ to $b$, $a:b$, or $\frac{a}{b}$ | the ratio of $a$ to $b$ |
| $\infty$ | infinity |
| $-\infty$ | negative infinity |
| $(-\infty, \infty)$ | set of all real numbers |
| $(a, b)$ | interval notation for $a < x < b$ |
| $[a, b]$ | interval notation for $a \leq x \leq b$ |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

1. A **solution set** is the set of numbers that
   **A.** make an expression undefined
   **B.** make an equation false
   **C.** make an equation true
   **D.** make an expression equal to 0.

2. **Complementary angles** are angles
   **A.** formed by two parallel lines
   **B.** whose sum is 90°
   **C.** whose sum is 180°
   **D.** formed by perpendicular lines.

3. **Supplementary angles** are angles
   **A.** formed by two parallel lines
   **B.** whose sum is 90°
   **C.** whose sum is 180°
   **D.** formed by perpendicular lines.

4. A **ratio**
   **A.** compares two quantities using a quotient
   **B.** says that two quotients are equal
   **C.** is a product of two quantities
   **D.** is a difference between two quantities.

5. A **proportion**
   **A.** compares two quantities using a quotient
   **B.** says that two quotients are equal
   **C.** is a product of two quantities
   **D.** is a difference between two quantities.

6. An **inequality** is
   **A.** a statement that two algebraic expressions are equal
   **B.** a point on a number line
   **C.** an equation with no solutions
   **D.** a statement with algebraic expressions related by $<, \leq, >$, or $\geq$.

7. **Interval notation** is
   **A.** a portion of a number line
   **B.** a special notation for describing a point on a number line
   **C.** a way to use symbols to describe an interval on a number line
   **D.** a notation to describe unequal quantities.

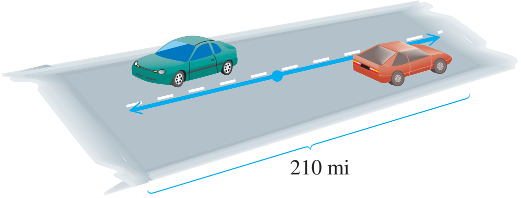
***ANSWERS***

**1.** C; *Example:* $\{8\}$ is the solution set of $2x + 5 = 21$. **2.** B; *Example:* Angles with measures 35° and 55° are complementary angles.
**3.** C; *Example:* Angles with measures 112° and 68° are supplementary angles. **4.** A; *Example:* $\frac{7 \text{ in.}}{12 \text{ in.}}$, or $\frac{7}{12}$ **5.** B; *Example:* $\frac{2}{3} = \frac{8}{12}$
**6.** D; *Examples:* $x < 5$, $7 + 2y \geq 11$, $-5 < 2z - 1 \leq 3$ **7.** C; *Examples:* $(-\infty, 5]$, $(1, \infty)$, $[-3, 13)$, $(-\infty, \infty)$

## QUICK REVIEW

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **2.1 The Addition Property of Equality**<br>The same number may be added to (or subtracted from) each side of an equation without changing the solution. | Solve. $x - 6 = 12$<br>$x - 6 + 6 = 12 + 6$ Add 6.<br>$x = 18$ Combine like terms.<br>Solution set: $\{18\}$ |
| **2.2 The Multiplication Property of Equality**<br>Each side of an equation may be multiplied (or divided) by the same nonzero number without changing the solution. | Solve. $\frac{3}{4}x = -9$<br>$\frac{4}{3} \cdot \left(\frac{3}{4}x\right) = \frac{4}{3}(-9)$ Multiply by $\frac{4}{3}$.<br>$x = -12$<br>Solution set: $\{-12\}$ |
| **2.3 More on Solving Linear Equations** | |
| ***Step 1*** Simplify each side separately. | Solve.<br>$2x + 2(x + 1) = 14 + x$<br>$2x + 2x + 2 = 14 + x$ Distributive property<br>$4x + 2 = 14 + x$ Combine like terms. |
| ***Step 2*** Isolate the variable term on one side. | $4x + 2 - x - 2 = 14 + x - x - 2$ Subtract *x*. Subtract 2.<br>$3x = 12$ Combine like terms. |
| ***Step 3*** Isolate the variable. | $\frac{3x}{3} = \frac{12}{3}$ Divide by 3.<br>$x = 4$ |
| ***Step 4*** Check. | *CHECK* $2(4) + 2(4 + 1) \overset{?}{=} 14 + 4$ Let $x = 4$.<br>$18 = 18$ ✓ True<br>Solution set: $\{4\}$ |
| **2.4 An Introduction to Applications of Linear Equations** | |
| ***Step 1*** Read. | One number is five more than another. Their sum is 21. What are the numbers? |
| ***Step 2*** Assign a variable. | Let $x =$ the lesser number.<br>Then $x + 5 =$ the greater number. |
| ***Step 3*** Write an equation. | $x + (x + 5) = 21$ |
| ***Step 4*** Solve the equation. | $2x + 5 = 21$ Combine like terms.<br>$2x = 16$ Subtract 5.<br>$x = 8$ Divide by 2. |
| ***Step 5*** State the answer. | The numbers are 8 and 13. |
| ***Step 6*** Check. | 13 is five more than 8, and $8 + 13 = 21$. It checks. |

(continued)

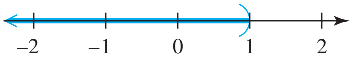
| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **2.5 Formulas and Additional Applications from Geometry** | |
| To find the value of one of the variables in a formula, given values for the others, substitute the known values into the formula. | Find $L$ if $\mathcal{A} = LW$, given that $\mathcal{A} = 24$ and $W = 3$.<br>$24 = L \cdot 3$ $\quad$ $\mathcal{A} = 24, W = 3$<br>$\frac{24}{3} = \frac{L \cdot 3}{3}$ $\quad$ Divide by 3.<br>$8 = L$ |
| To solve a formula for one of the variables, isolate that variable by treating the other variables as numbers and using the steps for solving equations. | Solve $P = 2a + 2b$ for $b$.<br>$P - 2a = 2a + 2b - 2a$ $\quad$ Subtract $2a$.<br>$P - 2a = 2b$ $\quad$ Combine like terms.<br>$\frac{P - 2a}{2} = \frac{2b}{2}$ $\quad$ Divide by 2.<br>$\frac{P - 2a}{2} = b$, or $b = \frac{P - 2a}{2}$ |
| **2.6 Ratio, Proportion, and Percent** | |
| To write a ratio, express quantities in the same units. | 4 ft to 8 in. = 48 in. to 8 in. $= \frac{48}{8} = \frac{6}{1}$ |
| To solve a proportion, use the method of cross products. | Solve. $\frac{x}{12} = \frac{35}{60}$<br>$60x = 12 \cdot 35$ $\quad$ Cross products<br>$60x = 420$ $\quad$ Multiply.<br>$x = 7$ $\quad$ Divide by 60.<br>Solution set: $\{7\}$ |
| To solve a percent problem, use the percent equation.<br>**amount = percent (as a decimal) · base** | 65 is what percent of 325?<br>$65 = p \cdot 325$<br>$\frac{65}{325} = p$<br>$0.2 = p$, or $20\% = p$<br>65 is 20% of 325. |
| **2.7 Further Applications of Linear Equations** | |
| ***Step 1*** Read. | Two cars leave from the same point, traveling in opposite directions. One travels at 45 mph and the other at 60 mph. How long will it take them to be 210 mi apart? |
| ***Step 2*** Assign a variable. Make a table and/or draw a sketch to help solve the problem.<br>The three forms of the formula relating distance, rate, and time are<br>$d = rt, \quad r = \frac{d}{t}, \quad \text{and} \quad t = \frac{d}{r}.$ | Let $t$ = time it takes for them to be 210 mi apart.<br><br> |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|

| | Rate | Time | Distance |
|---|---|---|---|
| One Car | 45 | $t$ | $45t$ |
| Other Car | 60 | $t$ | $60t$ |

The sum of the distances is 210 mi.

| | |
|---|---|
| ***Step 3*** Write an equation. | $45t + 60t = 210$ |
| ***Step 4*** Solve the equation. | $105t = 210$ Combine like terms.<br>$t = 2$ Divide by 105. |
| ***Steps 5 and 6*** State the answer and check the solution. | It will take them 2 hr to be 210 mi apart. |
| **2.8 Solving Linear Inequalities** | |
| ***Step 1*** Simplify each side separately. | Solve the inequality, and graph the solution set.<br>$3(1 - x) + 5 - 2x > 9 - 6$<br>$3 - 3x + 5 - 2x > 9 - 6$ Clear parentheses.<br>$8 - 5x > 3$ Combine like terms. |
| ***Step 2*** Isolate the variable term on one side. | $8 - 5x - 8 > 3 - 8$ Subtract 8.<br>$-5x > -5$ Combine like terms. |
| ***Step 3*** Isolate the variable.<br>***Be sure to reverse the direction of the inequality symbol when multiplying or dividing by a negative number.*** | $\frac{-5x}{-5} < \frac{-5}{-5}$ Divide by $-5$. Change $>$ to $<$.<br>$x < 1$<br>Solution set: $(-\infty, 1)$  |
| To solve a three-part inequality such as $4 < 2x + 6 < 8$, work with all three expressions at the same time. | Solve. $4 < 2x + 6 < 8$<br>$4 - 6 < 2x + 6 - 6 < 8 - 6$ Subtract 6.<br>$-2 < 2x < 2$<br>$\frac{-2}{2} < \frac{2x}{2} < \frac{2}{2}$ Divide by 2.<br>$-1 < x < 1$<br>Solution set: $(-1, 1)$ |

# CHAPTER 2 REVIEW EXERCISES

**2.1–2.3** *Solve each equation.*

**1.** $x - 5 = 1$

**2.** $x + 8 = -4$

**3.** $3t + 1 = 2t + 8$

**4.** $5z = 4z + \frac{2}{3}$

**5.** $(4r - 2) - (3r + 1) = 8$

**6.** $3(2x - 5) = 2 + 5x$

**7.** $7x = 35$

**8.** $12r = -48$

**9.** $2p - 7p + 8p = 15$

**10.** $\frac{x}{12} = -1$

**11.** $\frac{5}{8}q = 8$

**12.** $12m + 11 = 59$

**13.** $3(2x + 6) - 5(x + 8) = x - 22$

**14.** $5x + 9 - (2x - 3) = 2x - 7$

**15.** $\frac{1}{2}r - \frac{r}{3} = \frac{r}{6}$

**16.** $0.1(x + 80) + 0.2x = 14$

**17.** $3x - (-2x + 6) = 4(x - 4) + x$

**18.** $\frac{1}{2}(x + 3) - \frac{2}{3}(x - 2) = 3$

**2.4** *Solve each problem.*

**19.** If 7 is added to five times a number, the result is equal to three times the number. Find the number.

**20.** In 2009, Illinois had 118 members in its House of Representatives, consisting of only Democrats and Republicans. There were 22 more Democrats than Republicans. How many representatives from each party were there? (*Source:* www.ilga.gov)

**21.** The land area of Hawaii is 5213 $\text{mi}^2$ greater than the area of Rhode Island. Together, the areas total 7637 $\text{mi}^2$. What is the area of each of the two states?

**22.** The height of Seven Falls in Colorado is $\frac{5}{2}$ the height of Twin Falls in Idaho. The sum of the heights is 420 ft. Find the height of each. (*Source: World Almanac and Book of Facts.*)

**23.** The supplement of an angle measures 10 times the measure of its complement. What is the measure of the angle?

**24.** Find two consecutive odd integers such that when the lesser is added to twice the greater, the result is 24 more than the greater integer.

**2.5** *A formula is given along with the values for all but one of the variables. Find the value of the variable that is not given. Use 3.14 as an approximation for $\pi$.*

**25.** $\mathcal{A} = \frac{1}{2}bh; \quad \mathcal{A} = 44, b = 8$

**26.** $\mathcal{A} = \frac{1}{2}h(b + B); \quad h = 8, b = 3, B = 4$

**27.** $C = 2\pi r; \quad C = 29.83$

**28.** $V = \frac{4}{3}\pi r^3; \quad r = 6$

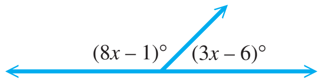
*Solve each formula for the specified variable.*

**29.** $\mathcal{A} = bh$ for $h$

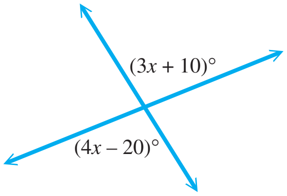
**30.** $\mathcal{A} = \frac{1}{2}h(b + B)$ for $h$

*Find the measure of each marked angle.*

**31.**

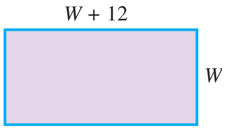


**32.**



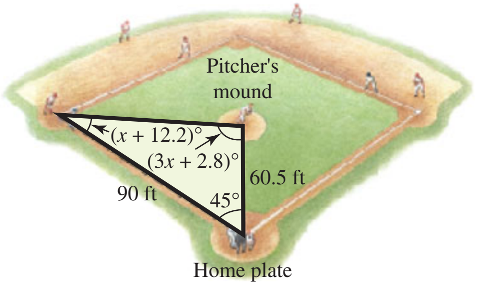
*Solve each problem.*

**33.** The perimeter of a certain rectangle is 16 times the width. The length is 12 cm more than the width. Find the width of the rectangle.



**34.** The Ziegfield Room in Reno, Nevada, has a circular turntable on which its showgirls dance. The circumference of the table is 62.5 ft. What is the diameter? What is the radius? What is the area? (Use $\pi = 3.14$.) (*Source: Guinness World Records.*)

**35.** A baseball diamond is a square with a side of 90 ft. The pitcher's mound is located 60.5 ft from home plate, as shown in the figure. Find the measures of the angles marked in the figure. (*Hint:* Recall that the sum of the measures of the angles of any triangle is 180°.)



**2.6** *Give a ratio for each word phrase, writing fractions in lowest terms.*

**36.** 60 cm to 40 cm

**37.** 5 days to 2 weeks

**38.** 90 in. to 10 ft

*Solve each equation.*

**39.** $\frac{p}{21} = \frac{5}{30}$

**40.** $\frac{5 + x}{3} = \frac{2 - x}{6}$

*Solve each problem.*

**41.** The tax on a \$24.00 item is \$2.04. How much tax would be paid on a \$36.00 item?

**42.** The distance between two cities on a road map is 32 cm. The two cities are actually 150 km apart. The distance on the map between two other cities is 80 cm. How far apart are these cities?

**43.** In the 2008 Olympics in Beijing, China, Japanese athletes earned 25 medals. Two of every 5 medals were bronze. How many bronze medals did Japan earn? (*Source: World Almanac and Book of Facts.*)

**44.** Find the best buy. Give the unit price to the nearest thousandth for that size. (*Source:* Cub Foods.)

CEREAL

| *Size* | *Price* |
|---|---|
| 15 oz | \$2.69 |
| 20 oz | \$3.29 |
| 25.5 oz | \$3.49 |

**45.** What is 8% of 75?

**46.** What percent of 12 is 21?

**47.** 36% of what number is 900?

**2.7** *Solve each problem.*

**48.** A nurse must mix 15 L of a 10% solution of a drug with some 60% solution to obtain a 20% mixture. How many liters of the 60% solution will be needed?

**49.** Robert Kay invested \$10,000, from which he earns an annual income of \$550 per year. He invested part of the \$10,000 at 5% annual interest and the remainder in bonds paying 6% interest. How much did he invest at each rate?

**50.** In 1846, the vessel *Yorkshire* traveled from Liverpool to New York, a distance of 3150 mi, in 384 hr. What was the *Yorkshire's* average rate? Round your answer to the nearest tenth.

**51.** Janet Hartnett drove from Louisville to Dallas, a distance of 819 mi, averaging 63 mph. What was her driving time?

**52.** Two planes leave St. Louis at the same time. One flies north at 350 mph and the other flies south at 420 mph. In how many hours will they be 1925 mi apart?

**2.8** *Write each inequality in interval notation, and graph it.*

**53.** $x \geq -4$

**54.** $x < 7$

**55.** $-5 \leq x < 6$

**56.** ***Concept Check*** Which inequality requires reversing the inequality symbol when it is solved?

**A.** $4x \geq -36$ **B.** $-4x \leq 36$ **C.** $4x < 36$ **D.** $4x > 36$

*Solve each inequality. Write the solution set in interval notation, and graph it.*

**57.** $x + 6 \geq 3$

**58.** $5x < 4x + 2$

**59.** $-6x \leq -18$

**60.** $8(x - 5) - (2 + 7x) \geq 4$

**61.** $4x - 3x > 10 - 4x + 7x$

**62.** $3(2x + 5) + 4(8 + 3x) < 5(3x + 7)$

**63.** $-3 \leq 2x + 1 \leq 4$

**64.** $9 < 3x + 5 \leq 20$

*Solve each problem.*

**65.** Awilda Delgado has grades of 94 and 88 on her first two calculus tests. What possible scores on a third test will give her an average of at least 90?

**66.** If nine times a number is added to 6, the result is at most 3. Find all such numbers.

## MIXED REVIEW EXERCISES

*Solve.*

**67.** $\frac{x}{7} = \frac{x - 5}{2}$

**68.** $I = prt$ for $r$

**69.** $-2x > -4$

**70.** $2k - 5 = 4k + 13$

**71.** $0.05x + 0.02x = 4.9$

**72.** $2 - 3(x - 5) = 4 + x$

**73.** $9x - (7x + 2) = 3x + (2 - x)$

**74.** $\frac{1}{3}s + \frac{1}{2}s + 7 = \frac{5}{6}s + 5 + 2$

**75.** A family of four with a monthly income of \$3800 plans to spend 8% of this amount on entertainment. How much will be spent on entertainment?

**76.** Athletes in vigorous training programs can eat 50 calories per day for every 2.2 lb of body weight. To the nearest hundred, how many calories can a 175-lb athlete consume per day? (*Source: The Gazette.*)

**77.** The Golden Gate Bridge in San Francisco is 2604 ft longer than the Brooklyn Bridge. Together, their spans total 5796 ft. How long is each bridge? (*Source: World Almanac and Book of Facts.*)

**78.** Find the best buy. Give the unit price to the nearest thousandth for that size. (*Source:* Cub Foods.)

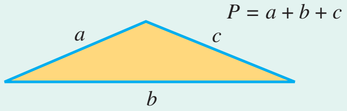
LAUNDRY DETERGENT

| *Size* | *Price* |
|---|---|
| 50 oz | \$ 4.69 |
| 100 oz | \$ 5.98 |
| 200 oz | \$13.68 |

**79.** If 1 qt of oil must be mixed with 24 qt of gasoline, how much oil would be needed for 192 qt of gasoline?

**80.** Two trains are 390 mi apart. They start at the same time and travel toward one another, meeting 3 hr later. If the rate of one train is 30 mph more than the rate of the other train, find the rate of each train.

**81.** The perimeter of a triangle is 96 m. One side is twice as long as another, and the third side is 30 m long. What is the length of the longest side?



**82.** The perimeter of a certain square cannot be greater than 200 m. Find the possible values for the length of a side.

## CHAPTER 2 TEST

Chapter Test Prep Videos

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*Solve each equation.*

**1.** $5x + 9 = 7x + 21$

**2.** $-\frac{4}{7}x = -12$

**3.** $7 - (x - 4) = -3x + 2(x + 1)$

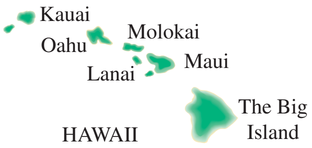
**4.** $0.6(x + 20) + 0.8(x - 10) = 46$

**5.** $-8(2x + 4) = -4(4x + 8)$

*Solve each problem.*

**6.** In the 2008 baseball season, the Los Angeles Angels of Anaheim won the most games of any major league team. The Angels won 24 less than twice as many games as they lost. They played 162 regular-season games. How many wins and losses did the Angels have? (*Source:* www.MLB.com)

**7.** Three islands in the Hawaiian island chain are Hawaii (the Big Island), Maui, and Kauai. Together, their areas total 5300 mi$^2$. The island of Hawaii is 3293 mi$^2$ larger than the island of Maui, and Maui is 177 mi$^2$ larger than Kauai. What is the area of each island?



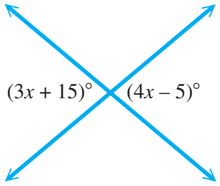
**8.** Find the measure of an angle if its supplement measures 10° more than three times its complement.

**9.** The formula for the perimeter of a rectangle is $P = 2L + 2W$.

**(a)** Solve for $W$.

**(b)** If $P = 116$ and $L = 40$, find the value of $W$.

**10.** Find the measure of each marked angle.



*Solve each equation.*

**11.** $\dfrac{z}{8} = \dfrac{12}{16}$

**12.** $\dfrac{x + 5}{3} = \dfrac{x - 3}{4}$

*Solve each problem.*

**13.** Find the best buy. Give the unit price to the nearest thousandth for that size.

PROCESSED
CHEESE SLICES

| *Size* | *Price* |
|---|---|
| 8 oz | \$2.79 |
| 16 oz | \$4.99 |
| 32 oz | \$7.99 |

**14.** The distance between Milwaukee and Boston is 1050 mi. On a certain map, this distance is represented by 42 in. On the same map, Seattle and Cincinnati are 92 in. apart. What is the actual distance between Seattle and Cincinnati?

**15.** Carlos Periu invested some money at 3% simple interest and \$6000 more than that amount at 4.5% simple interest. After 1 yr, his total interest from the two accounts was \$870. How much did he invest at each rate?

**16.** Two cars leave from the same point, traveling in opposite directions. One travels at a constant rate of 50 mph, while the other travels at a constant rate of 65 mph. How long will it take for them to be 460 mi apart?

*Solve each inequality. Write the solution set in interval notation, and graph it.*

**17.** $-4x + 2(x - 3) \geq 4x - (3 + 5x) - 7$

**18.** $-10 < 3x - 4 \leq 14$

**19.** Susan Jacobson has grades of 76 and 81 on her first two algebra tests. If she wants an average of at least 80 after her third test, what score must she make on that test?

✎ **20.** Write a short explanation of the additional (extra) rule that must be remembered when solving an inequality (as opposed to solving an equation).

# CHAPTERS 1–2 CUMULATIVE REVIEW EXERCISES

*Perform each indicated operation.*

**1.** $\dfrac{5}{6} + \dfrac{1}{4} - \dfrac{7}{15}$

**2.** $\dfrac{9}{8} \cdot \dfrac{16}{3} \div \dfrac{5}{8}$

*Translate from words to symbols. Use x as the variable.*

**3.** The difference between half a number and 18

**4.** The quotient of 6 and 12 more than a number is 2.

**5.** *True* or *false?* $\dfrac{8(7) - 5(6 + 2)}{3 \cdot 5 + 1} \geq 1$

*Perform each indicated operation.*

**6.** $\dfrac{-4(9)(-2)}{-3^2}$

**7.** $(-7 - 1)(-4) + (-4)$

**8.** Find the value of $\dfrac{3x^2 - y^3}{-4z}$ when $x = -2$, $y = -4$, and $z = 3$.

*Name each property illustrated.*

**9.** $7(p + q) = 7p + 7q$

**10.** $3 + (5 + 2) = 3 + (2 + 5)$

*Solve each equation, and check the solution.*

**11.** $2r - 6 = 8r$

**12.** $4 - 5(s + 2) = 3(s + 1) - 1$

**13.** $\frac{2}{3}x + \frac{3}{4}x = -17$

**14.** $\frac{2x + 3}{5} = \frac{x - 4}{2}$

**15.** Solve $3x + 4y = 24$ for $y$.

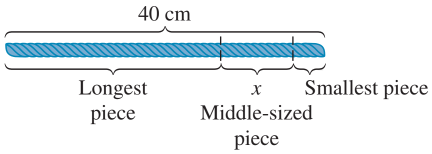
*Solve each inequality. Write the solution set in interval notation, and graph it.*

**16.** $6(r - 1) + 2(3r - 5) \leq -4$

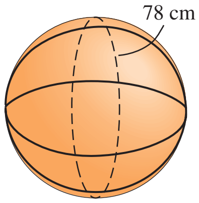
**17.** $-18 \leq -9z < 9$

*Solve each problem.*

**18.** A 40-cm piece of yarn must be cut into three pieces. The longest piece is to be three times as long as the middle-sized piece, and the shortest piece is to be 5 cm shorter than the middle-sized piece. Find the length of each piece.



**19.** A fully inflated professional basketball has a circumference of 78 cm. What is the radius of a circular cross section through the center of the ball? (Use 3.14 as the approximation for $\pi$.) Round your answer to the nearest hundredth.



**20.** Two cars are 400 mi apart. Both start at the same time and travel toward one another. They meet 4 hr later. If the rate of one car is 20 mph faster than the other, what is the rate of each car?

CHAPTER 3

# Linear Equations in Two Variables

In recent years, college students, like U.S. consumers as a whole, have increased their dependency on credit cards. In 2008, 84% of undergraduates had at least one credit card, up from 76% in 2004. The average (mean) outstanding balance for undergraduates grew from $946 in 2004 to a record-high $3173 in 2008, with 92% of these students using credit cards to pay direct education expenses. (*Source:* Sallie Mae.)

In **Example 7** of **Section 3.2,** we examine a *linear equation in two variables* that models credit card debt in the United States.

# 3.1

# Linear Equations in Two Variables; The Rectangular Coordinate System

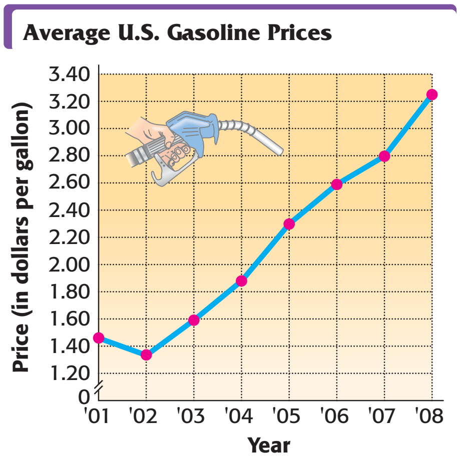
**OBJECTIVES**

1. Interpret graphs.
2. Write a solution as an ordered pair.
3. Decide whether a given ordered pair is a solution of a given equation.
4. Complete ordered pairs for a given equation.
5. Complete a table of values.
6. Plot ordered pairs.

**OBJECTIVE 1** **Interpret graphs.** A line graph is used to show changes or trends in data over time. To form a **line graph,** we connect a series of points representing data with line segments.

### EXAMPLE 1 Interpreting a Line Graph

The line graph in **FIGURE 1** shows average prices of a gallon of regular unleaded gasoline in the United States for the years 2001 through 2008.



*Source:* U.S. Department of Energy.

**FIGURE 1**

*NOW TRY EXERCISE 1*

Refer to the line graph in **FIGURE 1**.

**(a)** Estimate the average price of a gallon of gasoline in 2006.

**(b)** About how much did the average price of a gallon of gasoline increase from 2006 to 2008?

**(a)** Between which years did the average price of a gallon of gasoline decrease?

The line between 2001 and 2002 falls, so the average price of a gallon of gasoline decreased from 2001 to 2002.

**(b)** What was the general trend in the average price of a gallon of gasoline from 2002 through 2008?

The line graph rises from 2002 to 2008, so the average price of a gallon of gasoline increased over those years.

**(c)** Estimate the average price of a gallon of gasoline in 2002 and 2008. About how much did the price increase between 2002 and 2008?

Move up from 2002 on the horizontal scale to the point plotted for 2002. Looking across at the vertical scale, this point is about three-fourths of the way between the lines on the vertical scale for \$1.20 and \$1.40. Halfway between the lines for \$1.20 and \$1.40 would be \$1.30. So, it cost about \$1.35 for a gallon of gasoline in 2002.

Similarly, move up from 2008 on the horizontal scale to the point plotted for 2008. Then move across to the vertical scale. The price for a gallon of gasoline in 2008 was about \$3.25.

Between 2002 and 2008, the average price of a gallon of gasoline increased by about

$$\$3.25 - \$1.35 = \$1.90.$$

*NOW TRY*

*NOW TRY ANSWERS*

1. **(a)** about \$2.60
   **(b)** about \$0.65

| Year | Average Price (in dollars per gallon) |
|---|---|
| 2001 | 1.46 |
| 2002 | 1.36 |
| 2003 | 1.59 |
| 2004 | 1.88 |
| 2005 | 2.30 |
| 2006 | 2.59 |
| 2007 | 2.80 |
| 2008 | 3.25 |

*Actual Data Source:* U.S. Department of Energy.

The line graph in **FIGURE 1** relates years to average prices for a gallon of gasoline. We can also represent these two related quantities using a table of data, as shown in the margin. In table form, we can see more precise data rather than estimating it. Trends in the data are easier to see from the graph, which gives a "picture" of the data.

We can extend these ideas to the subject of this chapter, *linear equations in* ***two*** *variables*. A linear equation in two variables, one for each of the quantities being related, can be used to represent the data in the table or graph. ***The graph of a linear equation in two variables is a line.***

### Linear Equation in Two Variables

A **linear equation in two variables** is an equation that can be written in the form

$$Ax + By = C,$$

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0.

Some examples of linear equations in two variables in this form, called *standard form,* are

$$3x + 4y = 9, \quad x - y = 0, \quad \text{and} \quad x + 2y = -8.$$

Linear equations in two variables

**NOTE** Other linear equations in two variables, such as

$$y = 4x + 5 \quad \text{and} \quad 3x = 7 - 2y,$$

are not written in standard form, but could be algebraically rewritten in this form. We discuss the forms of linear equations in more detail in **Section 3.4.**

**OBJECTIVE 2** **Write a solution as an ordered pair.** Recall from **Section 1.3** that a *solution* of an equation is a number that makes the equation true when it replaces the variable. For example, the linear equation in *one* variable

$$x - 2 = 5$$

has solution 7, since replacing $x$ with 7 gives a true statement.

***A solution of a linear equation in*** **two** ***variables requires*** **two** ***numbers, one for each variable.*** For example, a true statement results when we replace $x$ with 2 and $y$ with 13 in the equation $y = 4x + 5$, since

$$13 = 4(2) + 5. \qquad \text{Let } x = 2 \text{ and } y = 13.$$

The pair of numbers $x = 2$ and $y = 13$ gives a solution of the equation $y = 4x + 5$. The phrase "$x = 2$ and $y = 13$" is abbreviated

$$\underbrace{(\overset{x\text{-value}}{2}, \overset{y\text{-value}}{13})}_{\text{Ordered pair}}$$

with the $x$-value, 2, and the $y$-value, 13, given as a pair of numbers written inside parentheses. ***The x-value is always given first.*** A pair of numbers such as $(2, 13)$ is called an **ordered pair.**

---

**CAUTION** The ordered pairs $(2, 13)$ and $(13, 2)$ are *not* the same. In the first pair, $x = 2$ and $y = 13$. In the second pair, $x = 13$ and $y = 2$. ***The order in which the numbers are written in an ordered pair is important.***

---

**OBJECTIVE 3** **Decide whether a given ordered pair is a solution of a given equation.** We substitute the $x$- and $y$-values of an ordered pair into a linear equation in two variables to see whether the ordered pair is a solution. An ordered pair that is a solution of an equation is said to *satisfy* the equation.

**NOW TRY EXERCISE 2**

Decide whether each ordered pair is a solution of the equation.

$$3x - 7y = 19$$

**(a)** $(3, 4)$ **(b)** $(-3, -4)$

### EXAMPLE 2 Deciding Whether Ordered Pairs Are Solutions of an Equation

Decide whether each ordered pair is a solution of the equation $2x + 3y = 12$.

**(a)** $(3, 2)$

Substitute 3 for $x$ and 2 for $y$ in the equation.

$$
\begin{aligned}
2x + 3y &= 12 \\
2(3) + 3(2) &\overset{?}{=} 12 && \text{Let } x = 3 \text{ and } y = 2. \\
6 + 6 &\overset{?}{=} 12 && \text{Multiply.} \\
12 &= 12 \ \checkmark && \text{True}
\end{aligned}
$$

This result is true, so $(3, 2)$ is a solution of $2x + 3y = 12$.

**(b)** $(-2, -7)$

$$
\begin{aligned}
2x + 3y &= 12 \\
2(-2) + 3(-7) &\overset{?}{=} 12 && \text{Let } x = -2 \text{ and } y = -7. \\
-4 + (-21) &\overset{?}{=} 12 && \text{Multiply.} \\
-25 &= 12 && \text{False}
\end{aligned}
$$

Use parentheses to avoid errors.

This result is false, so $(-2, -7)$ is *not* a solution of $2x + 3y = 12$. NOW TRY

**OBJECTIVE 4** **Complete ordered pairs for a given equation.** Substituting a number for one variable in a linear equation makes it possible to find the value of the other variable.

### EXAMPLE 3 Completing Ordered Pairs

Complete each ordered pair for the equation $y = 4x + 5$.

**(a)** $(7, \_\_)$

The $x$-value always comes first.

In this ordered pair, $x = 7$. To find the corresponding value of $y$, replace $x$ with 7 in the equation.

$$
\begin{aligned}
y &= 4x + 5 \\
y &= 4(7) + 5 && \text{Let } x = 7. \\
y &= 28 + 5 && \text{Multiply.} \\
y &= 33 && \text{Add.}
\end{aligned}
$$

The ordered pair is $(7, 33)$.

**NOW TRY ANSWERS**

2. **(a)** no **(b)** yes

**NOW TRY EXERCISE 3**

Complete each ordered pair for the equation.

$$y = 3x - 12$$

**(a)** $(4, \_\_)$ **(b)** $(\_\_, 3)$

**(b)** $(\_\_, -3)$

In this ordered pair, $y = -3$. Find the corresponding value of $x$ by replacing $y$ with $-3$ in the equation.

$$\begin{aligned} y &= 4x + 5 & & \\ -3 &= 4x + 5 & & \text{Let } y = -3. \\ -8 &= 4x & & \text{Subtract 5 from each side.} \\ -2 &= x & & \text{Divide each side by 4.} \end{aligned}$$

The ordered pair is $(-2, -3)$. NOW TRY

**OBJECTIVE 5** **Complete a table of values.** Ordered pairs are often displayed in a **table of values.** Although we usually write tables of values vertically, they may be written horizontally.

### EXAMPLE 4 Completing Tables of Values

Complete the table of values for each equation. Write the results as ordered pairs.

**(a)** $x - 2y = 8$

| $x$ | $y$ |
|---|---|
| 2 | |
| 10 | |
| | 0 |
| | −2 |

To complete the first two ordered pairs, let $x = 2$ and $x = 10$, respectively, in the equation.

$$\begin{array}{rr|rr} \text{If} & x = 2, & \text{If} & x = 10, \\ \text{then} & x - 2y = 8 & \text{then} & x - 2y = 8 \\ \text{becomes} & 2 - 2y = 8 & \text{becomes} & 10 - 2y = 8 \\ & -2y = 6 & & -2y = -2 \\ & y = -3. & & y = 1. \end{array}$$

The first two ordered pairs are $(2, -3)$ and $(10, 1)$. Complete the last two ordered pairs by letting $y = 0$ and $y = -2$, respectively.

$$\begin{array}{rr|rr} \text{If} & y = 0, & \text{If} & y = -2, \\ \text{then} & x - 2y = 8 & \text{then} & x - 2y = 8 \\ \text{becomes} & x - 2(0) = 8 & \text{becomes} & x - 2(-2) = 8 \\ & x - 0 = 8 & & x + 4 = 8 \\ & x = 8. & & x = 4. \end{array}$$

The last two ordered pairs are $(8, 0)$ and $(4, -2)$. The completed table of values and corresponding ordered pairs follow.

| $x$ | $y$ | | Ordered Pairs |
|---|---|---|---|
| 2 | −3 | ⟶ | (2, −3) |
| 10 | 1 | ⟶ | (10, 1) |
| 8 | 0 | ⟶ | (8, 0) |
| 4 | −2 | ⟶ | (4, −2) |

Each ordered pair is a solution of the given equation $x - 2y = 8$.

**NOW TRY ANSWERS**

**3.** **(a)** $(4, 0)$ **(b)** $(5, 3)$

**NOW TRY EXERCISE 4**

Complete the table of values for the equation. Write the results as ordered pairs.

$$5x - 4y = 20$$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| 2 | |

**(b)** $x = 5$

| $x$ | $y$ |
|---|---|
| | $-2$ |
| | 6 |
| | 3 |

The given equation is $x = 5$. No matter which value of $y$ is chosen, the value of $x$ is always 5.

| $x$ | $y$ | | Ordered Pairs |
|---|---|---|---|
| 5 | $-2$ | $\longrightarrow$ | $(5, -2)$ |
| 5 | 6 | $\longrightarrow$ | $(5, 6)$ |
| 5 | 3 | $\longrightarrow$ | $(5, 3)$ |

NOW TRY

**NOTE** We can think of $x = 5$ in **Example 4(b)** as an equation in two variables by rewriting $x = 5$ as $x + 0y = 5$. This form of the equation shows that, for any value of $y$, the value of $x$ is 5. Similarly, $y = 4$ can be written $0x + y = 4$.

**OBJECTIVE 6** **Plot ordered pairs.** In **Section 2.3,** we saw that linear equations in *one* variable had either one, zero, or an infinite number of real number solutions. These solutions could be graphed on *one* number line. For example, the linear equation in one variable $x - 2 = 5$ has solution 7, which is graphed on the number line in **FIGURE 2**.

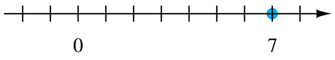


**FIGURE 2**

Every linear equation in *two* variables has an infinite number of ordered pairs $(x, y)$ as solutions. To graph these solutions, we need *two* number lines, one for each variable, drawn at right angles as in **FIGURE 3**. The horizontal number line is called the ***x*-axis,** and the vertical line is called the ***y*-axis.** The point at which the $x$-axis and $y$-axis intersect is called the **origin.** Together, the $x$-axis and $y$-axis form a **rectangular coordinate system.**

**René Descartes (1596–1650)**

The rectangular coordinate system is also called the **Cartesian coordinate system,** in honor of René Descartes, the French mathematician credited with its invention.

The rectangular coordinate system is divided into four regions, called **quadrants.** These quadrants are numbered counterclockwise, as shown in **FIGURE 3**.

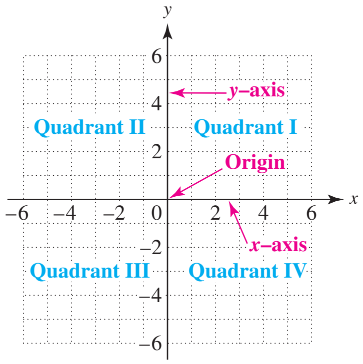


**FIGURE 3** Rectangular Coordinate System

The $x$-axis and $y$-axis determine a **plane**—a flat surface illustrated by a sheet of paper. By referring to the two axes, we can associate every point in the plane with an ordered pair. The numbers in the ordered pair are called the **coordinates** of the point.

**NOW TRY ANSWER**

4.

| $x$ | $y$ |
|---|---|
| 0 | $-5$ |
| 4 | 0 |
| 2 | $-\frac{5}{2}$ |

$(0, -5)$, $(4, 0)$, $\left(2, -\frac{5}{2}\right)$

**NOTE** In a plane, *both* numbers in the ordered pair are needed to locate a point. The ordered pair is a name for the point.

**NOW TRY EXERCISE 5**

Plot the given points in a coordinate system.

$(-3, 1), (2, -4), (0, -1),$
$\left(\frac{5}{2}, 3\right), (-4, -3), (-4, 0)$

## EXAMPLE 5 Plotting Ordered Pairs

Plot the given points in a coordinate system.

**(a)** $(2, 3)$ **(b)** $(-1, -4)$ **(c)** $(-2, 3)$ **(d)** $(3, -2)$ **(e)** $\left(\frac{3}{2}, 2\right)$

**(f)** $(4, -3.75)$ **(g)** $(5, 0)$ **(h)** $(0, -3)$ **(i)** $(0, 0)$

The point $(2, 3)$ from part (a) is **plotted** (graphed) in **FIGURE 4**. The other points are plotted in **FIGURE 5**.

In each case, begin at the origin. Move right or left the number of units that corresponds to the $x$-coordinate in the ordered pair—***right if the x-coordinate is positive or left if it is negative.*** Then turn and move up or down the number of units that corresponds to the $y$-coordinate—***up if the y-coordinate is positive or down if it is negative.***

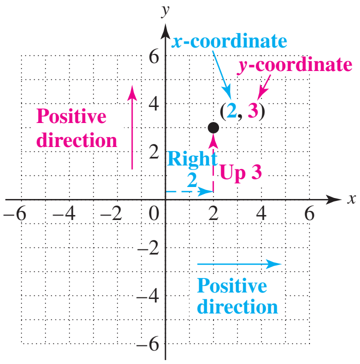


FIGURE 4

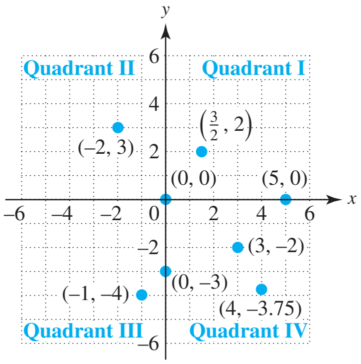


FIGURE 5

Notice the difference in the locations of the points $(-2, 3)$ and $(3, -2)$ in parts (c) and (d). The point $(-2, 3)$ is in quadrant II, whereas the point $(3, -2)$ is in quadrant IV. ***The order of the coordinates is important. The x-coordinate is always given first in an ordered pair.***

To plot the point $\left(\frac{3}{2}, 2\right)$ in part (e), think of the improper fraction $\frac{3}{2}$ as the mixed number $1\frac{1}{2}$ and move $\frac{3}{2}$ $\left(\text{or } 1\frac{1}{2}\right)$ units to the right along the $x$-axis. Then turn and go 2 units up, parallel to the $y$-axis. The point $(4, -3.75)$ in part (f) is plotted similarly, by approximating the location of the decimal $y$-coordinate.

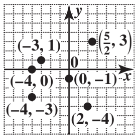
In part (g), the point $(5, 0)$ lies on the $x$-axis since the $y$-coordinate is 0. In part (h), the point $(0, -3)$ lies on the $y$-axis since the $x$-coordinate is 0. In part (i), the point $(0, 0)$ is at the origin. ***Points on the axes themselves are not in any quadrant.***

NOW TRY

Sometimes we can use a linear equation to mathematically describe, or *model,* a real-life situation, as shown in the next example.

**NOW TRY ANSWER**

5.



**NOW TRY EXERCISE 6**

Use the linear equation in **Example 6** to estimate the number of twin births in 2004. Interpret the results.

## EXAMPLE 6 Completing Ordered Pairs to Estimate the Number of Twin Births

The annual number of twin births in the United States from 2001 through 2006 can be closely approximated by the linear equation

$$\overset{\text{Number of twin births}}{y} = 3.049\overset{\text{Year}}{x} - 5979.0,$$

which relates $x$, the year, and $y$, the number of twin births in thousands. (*Source:* Department of Health and Human Services.)

**(a)** Complete the table of values for the given linear equation.

| *x (Year)* | *y (Number of Twin Births, in thousands)* |
|---|---|
| 2001 | |
| 2003 | |
| 2006 | |

To find $y$ when $x = 2001$, we substitute into the equation.

$$y = 3.049(2001) - 5979.0 \qquad \text{Let } x = 2001.$$

$$y \approx 122 \qquad \text{Use a calculator.}$$

≈ means "is approximately equal to."

This means that in 2001, there were about 122 thousand (or 122,000) twin births.

We substitute the years 2003 and 2006 in the same way to complete the table.

| *x (Year)* | *y (Number of Twin Births, in thousands)* | Ordered Pairs $(x, y)$ |
|---|---|---|
| 2001 | 122 | → $(2001, 122)$ |
| 2003 | 128 | → $(2003, 128)$ |
| 2006 | 137 | → $(2006, 137)$ |

Here each year $x$ is paired with the number of twin births $y$ (in thousands).

**(b)** Graph the ordered pairs found in part (a).

The ordered pairs are graphed in **FIGURE 6**. This graph of ordered pairs of data is called a **scatter diagram.**

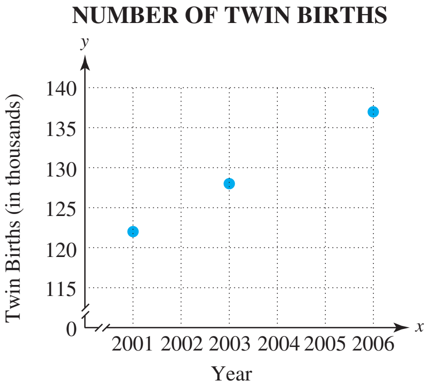


**FIGURE 6**

Notice the axes labels and scales. Each square represents 1 unit in the horizontal direction and 5 units in the vertical direction. Because the numbers in the first ordered pair are large, we show a break in the axes near the origin.

A scatter diagram enables us to tell whether two quantities are related to each other. In **FIGURE 6**, the plotted points could be connected to approximate a straight ***line,*** so the variables $x$ (year) and $y$ (number of twin births) have a ***lin***ear relationship. The increase in the number of twin births is also reflected.

NOW TRY

**NOW TRY ANSWER**

6. $y \approx 131$; There were approximately 131 thousand (or 131,000) twin births in the U.S. in 2004.

**CAUTION** The equation in **Example 6** is valid only for the years 2001 through 2006, because it was based on data for those years. ***Do not assume that this equation would provide reliable data for other years, since the data for those years may not follow the same pattern.***

## 3.1 EXERCISES

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*The line graph shows the overall unemployment rate in the U.S. civilian labor force in August of the years 2003 through 2009. Use the graph to work Exercises 1–4.* ***See Example 1.***

**1.** Between which pairs of consecutive years did the unemployment rate decrease?

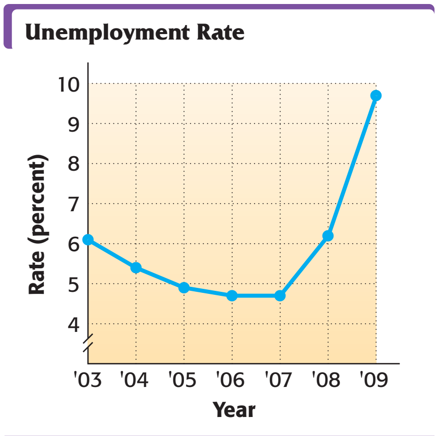
**2.** What was the general trend in the unemployment rate between 2007 and 2009?

**3.** Estimate the overall unemployment rate in 2003 and 2004. About how much did the unemployment rate decline between 2003 and 2004?

**4.** During which year(s)

**(a)** was the unemployment rate greater than 6%, but less than 7%?

**(b)** did the unemployment rate stay the same?



***Source:*** U.S. Bureau of Labor Statistics.

*Concept Check* *Fill in each blank with the correct response.*

**5.** The symbol $(x, y)$ __________ (does/does not) represent an ordered pair, while the symbols $[x, y]$ and $\{x, y\}$ __________ (do/do not) represent ordered pairs.

**6.** The ordered pair $(3, 2)$ is a solution of the equation $2x - 5y =$ _____.

**7.** The point whose graph has coordinates $(-4, 2)$ is in quadrant _____.

**8.** The point whose graph has coordinates $(0, 5)$ lies on the _____-axis.

**9.** The ordered pair $(4, \_\_)$ is a solution of the equation $y = 3$.

**10.** The ordered pair $(\_\_, -2)$ is a solution of the equation $x = 6$.

*Decide whether the given ordered pair is a solution of the given equation.* ***See Example 2.***

**11.** $x + y = 8$; $(0, 8)$

**12.** $x + y = 9$; $(0, 9)$

**13.** $2x + y = 5$; $(3, -1)$

**14.** $2x - y = 6$; $(4, 2)$

**15.** $5x - 3y = 15$; $(5, 2)$

**16.** $4x - 3y = 6$; $(2, 1)$

**17.** $x = -4y$; $(-8, 2)$

**18.** $y = 3x$; $(2, 6)$

**19.** $y = 2$; $(4, 2)$

**20.** $x = -6$; $(-6, 5)$

**21.** $x - 6 = 0$; $(4, 2)$

**22.** $x + 4 = 0$; $(-6, 2)$

*Complete each ordered pair for the equation* $y = 2x + 7$. ***See Example 3.***

**23.** $(5, \_\_)$

**24.** $(2, \_\_)$

**25.** $(\_\_, -3)$

**26.** $(\_\_, 0)$

*Complete each ordered pair for the equation* $y = -4x - 4$. ***See Example 3.***

**27.** $(\_\_, 0)$

**28.** $(0, \_\_)$

**29.** $(\_\_, 24)$

**30.** $(\_\_, 16)$

*Complete each table of values. Write the results as ordered pairs.* ***See Example 4.***

**31.** $4x + 3y = 24$

| x | y |
|---|---|
| 0 | |
| | 0 |
| | 4 |

**32.** $2x + 3y = 12$

| x | y |
|---|---|
| 0 | |
| | 0 |
| | 8 |

**33.** $4x - 9y = -36$

| x | y |
|---|---|
| | 0 |
| 0 | |
| | 8 |

**34.** $3x - 5y = -15$

| x | y |
|---|---|
| 0 | |
| | 0 |
| | −6 |

**35.** $x = 12$

| x | y |
|---|---|
| | 3 |
| | 8 |
| | 0 |

**36.** $x = -9$

| x | y |
|---|---|
| | 6 |
| | 2 |
| | −3 |

**37.** $y = -10$

| x | y |
|---|---|
| 4 | |
| 0 | |
| −4 | |

**38.** $y = -6$

| x | y |
|---|---|
| 8 | |
| 4 | |
| −2 | |

**39.** $y + 2 = 0$

| x | y |
|---|---|
| 9 | |
| 2 | |
| 0 | |

**40.** $y + 6 = 0$

| x | y |
|---|---|
| 6 | |
| 3 | |
| 0 | |

**41.** $x - 4 = 0$

| x | y |
|---|---|
| | 4 |
| | 0 |
| | −4 |

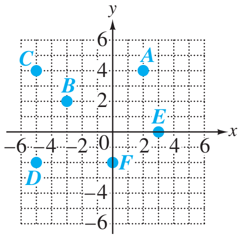
**42.** $x - 8 = 0$

| x | y |
|---|---|
| | 8 |
| | 3 |
| | 0 |

**43.** Do $(3, 4)$ and $(4, 3)$ correspond to the same point in the plane? Explain.

**44.** Do $(4, -1)$ and $(-1, 4)$ represent the same ordered pair? Explain.

**45.** Give the ordered pairs for the points labeled *A–F* in the figure. (All coordinates are integers.) Tell the quadrant in which each point is located. **See Example 5.**



**46.** ***Concept Check*** The origin is represented by the ordered pair __________.

*Plot and label each point in a rectangular coordinate system.* ***See Example 5.***

**47.** $(6, 2)$

**48.** $(5, 3)$

**49.** $(-4, 2)$

**50.** $(-3, 5)$

**51.** $\left(-\frac{4}{5}, -1\right)$

**52.** $\left(-\frac{3}{2}, -4\right)$

**53.** $(3, -1.75)$

**54.** $(5, -4.25)$

**55.** $(0, 4)$

**56.** $(0, -3)$

**57.** $(4, 0)$

**58.** $(-3, 0)$

***Concept Check*** *Fill in each blank with the word* positive *or the word* negative.

The point with coordinates $(x, y)$ is in

**59.** quadrant III if $x$ is __________ and $y$ is __________.

**60.** quadrant II if $x$ is __________ and $y$ is __________.

**61.** quadrant IV if $x$ is __________ and $y$ is __________.

**62.** quadrant I if $x$ is __________ and $y$ is __________.

**63.** A point $(x, y)$ has the property that $xy < 0$. In which quadrant(s) must the point lie? Explain.

**64.** A point $(x, y)$ has the property that $xy > 0$. In which quadrant(s) must the point lie? Explain.

*Complete each table of values. Then plot and label the ordered pairs.* ***See Examples 4 and 5.***

**65.** $x - 2y = 6$

| x | y |
|---|---|
| 0 | |
| | 0 |
| 2 | |
| | −1 |

**66.** $2x - y = 4$

| x | y |
|---|---|
| 0 | |
| | 0 |
| 1 | |
| | −6 |

**67.** $3x - 4y = 12$

| x | y |
|---|---|
| 0 | |
| | 0 |
| −4 | |
| | −4 |

**68.** $2x - 5y = 10$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| −5 | |
| | −3 |

**69.** $y + 4 = 0$

| $x$ | $y$ |
|---|---|
| 0 | |
| 5 | |
| −2 | |
| −3 | |

**70.** $x - 5 = 0$

| $x$ | $y$ |
|---|---|
| | 1 |
| | 0 |
| | 6 |
| | −4 |

**71.** Look at your graphs of the ordered pairs in **Exercises 65–70.** Describe the pattern indicated by the plotted points.

**72. (a)** A line through the plotted points in **Exercise 69** would be horizontal. What do you notice about the $y$-coordinates of the ordered pairs?

**(b)** A line through the plotted points in **Exercise 70** would be vertical. What do you notice about the $x$-coordinates of the ordered pairs?

*Solve each problem.* ***See Example 6.***

**73.** It costs a flat fee of \$20 plus \$5 per day to rent a pressure washer. Therefore, the cost to rent the pressure washer for $x$ days is given by

$$y = 5x + 20,$$

where $y$ is in dollars. Express each of the following as an ordered pair.

**(a)** When the washer is rented for 5 days, the cost is \$45.

**(b)** I paid \$50 when I returned the washer, so I must have rented it for 6 days.

**74.** Suppose that it costs \$5000 to start up a business selling snow cones. Furthermore, it costs \$0.50 per cone in labor, ice, syrup, and overhead. Then the cost to make $x$ snow cones is given by $y$ dollars, where

$$y = 0.50x + 5000.$$

Express each of the following as an ordered pair.

**(a)** When 100 snow cones are made, the cost is \$5050.

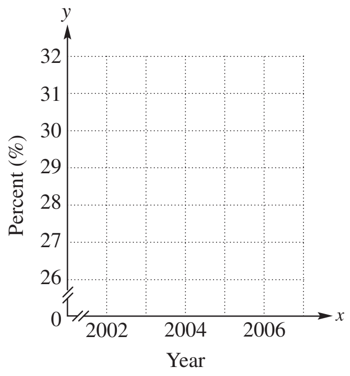
**(b)** When the cost is \$6000, the number of snow cones made is 2000.

**75.** The table shows the rate (in percent) at which 2-year college students (public) completed a degree within 3 years.

| *Year* | *Percent* |
|---|---|
| 2002 | 31.6 |
| 2003 | 30.1 |
| 2004 | 29.0 |
| 2005 | 27.5 |
| 2006 | 26.6 |
| 2007 | 26.9 |

*Source:* ACT.

**2-YEAR COLLEGE STUDENTS COMPLETING A DEGREE WITHIN 3 YEARS**



**(a)** Write the data from the table as ordered pairs $(x, y)$, where $x$ represents the year and $y$ represents the percent.

**(b)** What does the ordered pair $(2007, 26.9)$ mean in the context of this problem?

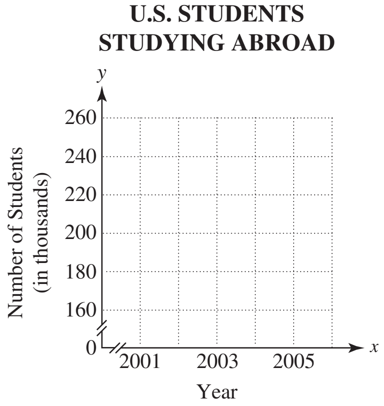
**(c)** Make a scatter diagram of the data, using the ordered pairs from part (a) and the given grid.

**(d)** Describe the pattern indicated by the points on the scatter diagram. What is happening to rates at which 2-year college students complete a degree within 3 years?

**76.** The table shows the number of U.S. students who studied abroad (in thousands) for several academic years.

| *Academic Year* | *Number of Students (in thousands)* |
|---|---|
| 2001 | 161 |
| 2002 | 175 |
| 2003 | 191 |
| 2004 | 206 |
| 2005 | 224 |
| 2006 | 242 |

*Source:* Institute of International Education.



**(a)** Write the data from the table as ordered pairs $(x, y)$, where $x$ represents the year and $y$ represents the number of U.S. students (in thousands) studying abroad.

**(b)** What does the ordered pair $(2006, 242)$ mean in the context of this problem?

**(c)** Make a scatter diagram of the data, using the ordered pairs from part (a) and the given grid.

**(d)** Describe the pattern indicated by the points on the scatter diagram. What was the trend in the number of U.S. students studying abroad during these years?

**77.** The maximum benefit for the heart from exercising occurs if the heart rate is in the target heart rate zone. The lower limit of this target zone can be approximated by the linear equation

$$y = -0.65x + 143,$$

where $x$ represents age and $y$ represents heartbeats per minute. (*Source: The Gazette.*)

| *Age* | *Heartbeats (per minute)* |
|---|---|
| 20 | |
| 40 | |
| 60 | |
| 80 | |

**(a)** Complete the table of values for this linear equation.

**(b)** Write the data from the table of values as ordered pairs.

**(c)** Make a scatter diagram of the data. Do the points lie in an approximately linear pattern?

**78.** (See **Exercise 77.**) The upper limit of the target heart rate zone can be approximated by the linear equation

$$y = -0.85x + 187,$$

where $x$ represents age and $y$ represents heartbeats per minute. (*Source: The Gazette.*)

| *Age* | *Heartbeats (per minute)* |
|---|---|
| 20 | |
| 40 | |
| 60 | |
| 80 | |

**(a)** Complete the table of values for this linear equation.

**(b)** Write the data from the table of values as ordered pairs.

**(c)** Make a scatter diagram of the data. Describe the pattern indicated by the data.

**79.** See **Exercises 77 and 78.** What is the target heart rate zone for age 20? Age 40?

**80.** See **Exercises 77 and 78.** What is the target heart rate zone for age 60? Age 80?

## PREVIEW EXERCISES

*Solve each equation.* ***See Section 2.3.***

**81.** $3x + 6 = 0$ **82.** $4 + 2x = 10$ **83.** $9 - x = -4$ **84.** $-5 + t = 3$

## STUDY SKILLS

## Managing Your Time

Many college students juggle a difficult schedule and multiple responsibilities, including school, work, and family demands.

### Time Management Tips

- **Read the syllabus for each class.** Understand class policies, such as attendance, late homework, and make-up tests. Find out how you are graded.
- **Make a semester or quarter calendar**. Put test dates and major due dates for *all* your classes on the *same* calendar. Try using a different color pen for each class.
- **Make a weekly schedule**. After you fill in your classes and other regular responsibilities, block off some study periods. Aim for 2 hours of study for each 1 hour in class.
- **Choose a regular study time and place** (such as the campus library). Routine helps.
- **Make "to-do" lists.** Number tasks in order of importance. Cross off tasks as you complete them.
- **Break big assignments into smaller chunks**. Make deadlines for each smaller chunk so that you stay on schedule.
- **Take breaks when studying.** Do not try to study for hours at a time. Take a 10-minute break each hour or so.
- **Ask for help when you need it.** Talk with your instructor during office hours. Make use of the learning center, tutoring center, counseling office, or other resources available at your school.

*Select several tips to help manage your time this week.*

## 3.2

# Graphing Linear Equations in Two Variables

**OBJECTIVES**

1. Graph linear equations by plotting ordered pairs.
2. Find intercepts.
3. Graph linear equations of the form $Ax + By = 0$.
4. Graph linear equations of the form $y = b$ or $x = a$.
5. Use a linear equation to model data.

**OBJECTIVE 1** **Graph linear equations by plotting ordered pairs.** We know that infinitely many ordered pairs satisfy a linear equation in two variables. We find these ordered-pair solutions by choosing as many values of $x$ (or $y$) as we wish and then completing each ordered pair. For example, consider the equation

$$x + 2y = 7.$$

If we choose $x = 1$, then $y = 3$, so the ordered pair $(1, 3)$ is a solution.

$$1 + 2(3) = 7 \qquad (1, 3) \text{ is a solution.}$$

This ordered pair and other solutions of $x + 2y = 7$ are graphed in **FIGURE 7**.

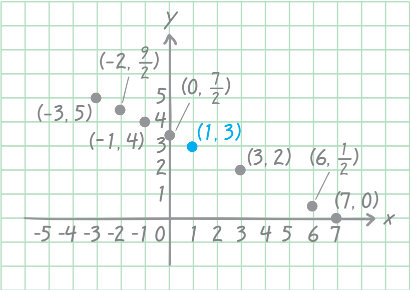


**FIGURE 7**

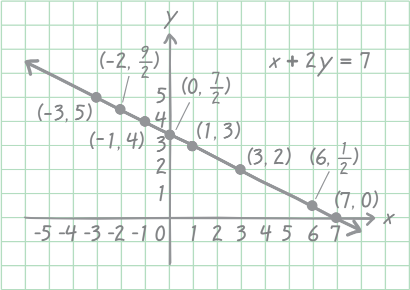


**FIGURE 8**

Notice that the points plotted in **FIGURE 7** all appear to lie on a straight line, as shown in **FIGURE 8**. In fact, the following is true.

> ***Every point on the line represents a solution of the equation $x + 2y = 7$, and every solution of the equation corresponds to a point on the line.***

The line gives a "picture" of all the solutions of the equation $x + 2y = 7$. The line extends indefinitely in both directions, as suggested by the arrowhead on each end. The line is called the **graph** of the equation, and the process of plotting the ordered pairs and drawing the line through the corresponding points is called **graphing.**

### Graph of a Linear Equation

The graph of any linear equation in two variables is a straight line.

Notice that the word ***line*** appears in the name "***line***ar equation."

### EXAMPLE 1 Graphing a Linear Equation

Graph $4x - 5y = 20$.

At least two different points are needed to draw the graph. First let $x = 0$ and then let $y = 0$ in the equation to complete two ordered pairs.

$$\begin{aligned} 4x - 5y &= 20 & & \\ 4(0) - 5y &= 20 & & \text{Let } x = 0. \\ 0 - 5y &= 20 & & \text{Multiply.} \\ -5y &= 20 & & \text{Subtract.} \\ y &= -4 & & \text{Divide by } -5. \end{aligned}$$

$$\begin{aligned} 4x - 5y &= 20 & & \\ 4x - 5(0) &= 20 & & \text{Let } y = 0. \\ 4x - 0 &= 20 & & \text{Multiply.} \\ 4x &= 20 & & \text{Subtract.} \\ x &= 5 & & \text{Divide by 4.} \end{aligned}$$

**NOW TRY EXERCISE 1**

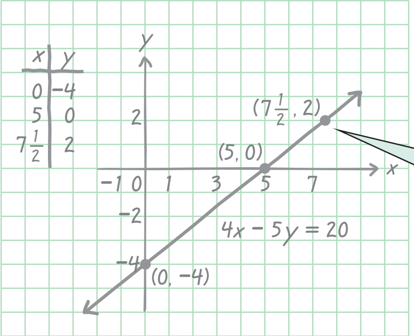
Graph $2x - 4y = 8$.

Write each $x$-value first.

The ordered pairs are $(0, -4)$ and $(5, 0)$. We find a third ordered pair (as a check) by choosing some other number for $x$ or $y$. We choose $y = 2$.

$$
\begin{aligned}
4x - 5y &= 20 && \\
4x - 5(2) &= 20 && \text{Let } y = 2. \\
4x - 10 &= 20 && \text{Multiply.} \\
4x &= 30 && \text{Add 10.} \\
x &= \frac{30}{4}, \text{ or } \frac{15}{2} && \text{Divide by 4. Write in lowest terms.}
\end{aligned}
$$

This gives the ordered pair $\left(\frac{15}{2}, 2\right)$, or $\left(7\frac{1}{2}, 2\right)$. We plot the three ordered pairs $(0, -4)$, $(5, 0)$, and $\left(7\frac{1}{2}, 2\right)$, and draw a line through them, as shown in **FIGURE 9**.



All three points should lie on the same straight line. If they don't, double-check the ordered pairs.

**FIGURE 9**

NOW TRY

## EXAMPLE 2 Graphing a Linear Equation

Graph $y = -\frac{3}{2}x + 3$.

Although this linear equation is not in standard form $(Ax + By = C)$, it *could* be written in that form. To find two different points on the graph, we first let $x = 0$ and then let $y = 0$.

$$
\begin{aligned}
y &= -\frac{3}{2}x + 3 && \\
y &= -\frac{3}{2}(0) + 3 && \text{Let } x = 0. \\
y &= 0 + 3 && \text{Multiply.} \\
y &= 3 && \text{Add.}
\end{aligned}
$$

$$
\begin{aligned}
y &= -\frac{3}{2}x + 3 && \\
0 &= -\frac{3}{2}x + 3 && \text{Let } y = 0. \\
\frac{3}{2}x &= 3 && \text{Add } \tfrac{3}{2}x. \\
x &= 2 && \text{Multiply by } \tfrac{2}{3}.
\end{aligned}
$$

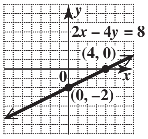
This gives the ordered pairs $(0, 3)$ and $(2, 0)$. To find a third point, we let $x = -2$.

$$
\begin{aligned}
y &= -\frac{3}{2}x + 3 && \\
y &= -\frac{3}{2}(-2) + 3 && \text{Let } x = -2. \\
y &= 3 + 3 && \text{Multiply.} \\
y &= 6 && \text{Add.}
\end{aligned}
$$

Choosing a multiple of 2 makes multiplying by $-\frac{3}{2}$ easier.

**NOW TRY ANSWER**

1.



**NOW TRY EXERCISE 2**

Graph $y = \frac{1}{3}x + 1$.

This gives the ordered pair $(-2, 6)$. We plot the three ordered pairs and draw a line through them, as shown in **FIGURE 10**.

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 2 | 0 |
| −2 | 6 |

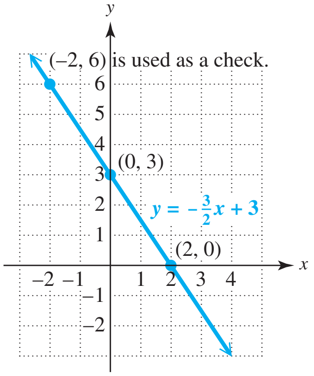


**FIGURE 10**

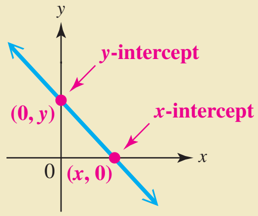
NOW TRY

**OBJECTIVE 2** **Find intercepts.** In **FIGURE 10**, the graph intersects (crosses) the $y$-axis at $(0, 3)$ and the $x$-axis at $(2, 0)$. For this reason, $(0, 3)$ is called the ***y*-intercept** and $(2, 0)$ is called the ***x*-intercept** of the graph. The intercepts are particularly useful for graphing linear equations.

### Finding Intercepts

To find the $x$-intercept, let $y = 0$ in the given equation and solve for $x$. Then $(x, 0)$ is the $x$-intercept.

To find the $y$-intercept, let $x = 0$ in the given equation and solve for $y$. Then $(0, y)$ is the $y$-intercept.



### EXAMPLE 3 Finding Intercepts

Find the intercepts for the graph of $2x + y = 4$. Then draw the graph.

To find the $y$-intercept, let $x = 0$.

$$\begin{aligned} 2x + y &= 4 \\ 2(0) + y &= 4 \quad \text{Let } x = 0. \\ 0 + y &= 4 \\ y &= 4 \quad y\text{-intercept is } (0, 4). \end{aligned}$$

To find the $x$-intercept, let $y = 0$.

$$\begin{aligned} 2x + y &= 4 \\ 2x + 0 &= 4 \quad \text{Let } y = 0. \\ 2x &= 4 \\ x &= 2 \quad x\text{-intercept is } (2, 0). \end{aligned}$$

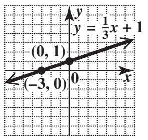
The intercepts are $(0, 4)$ and $(2, 0)$. To find a third point, we let $x = 4$.

$$\begin{aligned} 2x + y &= 4 \\ 2(4) + y &= 4 \quad \text{Let } x = 4. \\ 8 + y &= 4 \quad \text{Multiply.} \\ y &= -4 \quad \text{Subtract 8.} \end{aligned}$$

This gives the ordered pair $(4, -4)$. The graph, with the two intercepts in red, is shown in **FIGURE 11** on the next page.

**NOW TRY ANSWER**

2.



**NOW TRY EXERCISE 3**
Find the intercepts for the graph of $x + 2y = 2$. Then draw the graph.

|  | $x$ | $y$ |
|---|---|---|
| $y$-intercept → | 0 | 4 |
| $x$-intercept → | 2 | 0 |
|  | 4 | −4 |

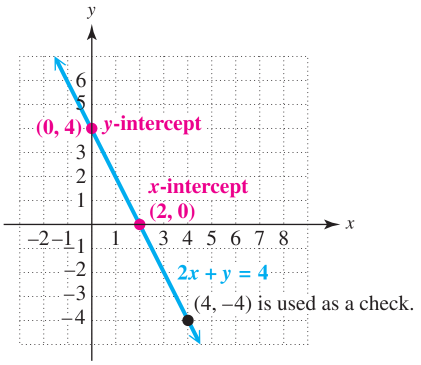


**FIGURE 11**

NOW TRY

**CAUTION** ***When choosing x- or y-values to find ordered pairs to plot, be careful to choose so that the resulting points are not too close together.*** For example, using $(-1, -1)$, $(0, 0)$, and $(1, 1)$ to graph $x - y = 0$ may result in an inaccurate line. It is better to choose points whose $x$-values differ by at least 2.

## OBJECTIVE 3 Graph linear equations of the form $Ax + By = 0$.

**NOW TRY EXERCISE 4**
Graph $2x + y = 0$.

### EXAMPLE 4 Graphing an Equation with $x$- and $y$-Intercepts $(0, 0)$

Graph $x - 3y = 0$.

To find the $y$-intercept, let $x = 0$.

$$x - 3y = 0$$
$$0 - 3y = 0 \quad \text{Let } x = 0.$$
$$-3y = 0$$
$$y = 0 \quad y\text{-intercept is } (0, 0).$$

To find the $x$-intercept, let $y = 0$.

$$x - 3y = 0$$
$$x - 3(0) = 0 \quad \text{Let } y = 0.$$
$$x - 0 = 0$$
$$x = 0 \quad x\text{-intercept is } (0, 0).$$

The $x$- and $y$-intercepts are the *same* point, $(0, 0)$. We must select *two other values* for $x$ or $y$ to find two other points on the graph. We choose $x = 6$ and $x = -6$.

$$x - 3y = 0$$
$$6 - 3y = 0 \quad \text{Let } x = 6.$$
$$-3y = -6$$
$$y = 2 \quad \text{Gives } (6, 2)$$

$$x - 3y = 0$$
$$-6 - 3y = 0 \quad \text{Let } x = -6.$$
$$-3y = 6$$
$$y = -2 \quad \text{Gives } (-6, -2)$$

We use the ordered pairs $(-6, -2)$, $(0, 0)$, and $(6, 2)$ to draw the graph in **FIGURE 12**.

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 6 | 2 |
| −6 | −2 |

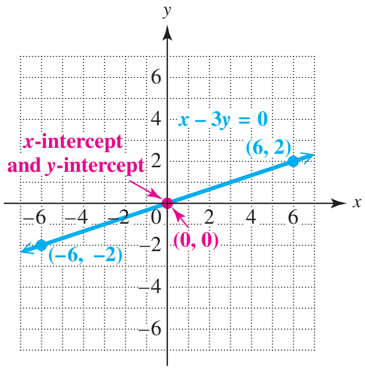
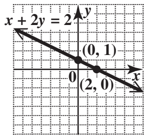


**FIGURE 12**

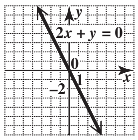
NOW TRY

**NOW TRY ANSWERS**

**3.** $x$-intercept: $(2, 0)$;
$y$-intercept: $(0, 1)$



**4.**



### Line through the Origin

If $A$ and $B$ are nonzero real numbers, the graph of a linear equation of the form

$$Ax + By = 0$$

passes through the origin $(0, 0)$.

**OBJECTIVE 4** **Graph linear equations of the form $y = b$ or $x = a$.** Consider the following linear equations:

$$y = -4, \quad \text{which can be written} \quad 0x + y = -4;$$
$$x = 3, \quad \text{which can be written} \quad x + 0y = 3.$$

When the coefficient of $x$ or $y$ is 0, the graph of the linear equation is a horizontal or vertical line.

NOW TRY EXERCISE 5

Graph $y = 2$.

### EXAMPLE 5 Graphing an Equation of the Form $y = b$ (Horizontal Line)

Graph $y = -4$.

For any value of $x$, $y$ is always equal to $-4$. Three ordered pairs that satisfy the equation are shown in the table of values. Drawing a line through these points gives the **horizontal line** shown in **FIGURE 13**. The $y$-intercept is $(0, -4)$. There is no $x$-intercept.

$x$ can be any real number.

| $x$ | $y$ |
|---|---|
| $-2$ | $-4$ |
| 0 | $-4$ |
| 3 | $-4$ |

$y$ must be $-4$.

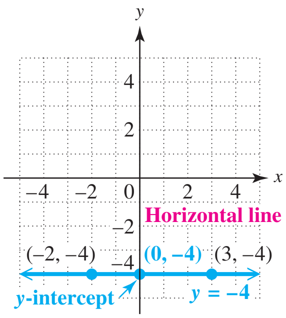


FIGURE 13

NOW TRY

### Horizontal Line

The graph of the linear equation $y = b$, where $b$ is a real number, is the horizontal line with $y$-intercept $(0, b)$. There is no $x$-intercept (unless the horizontal line is the $x$-axis itself).

NOW TRY EXERCISE 6

Graph $x + 4 = 0$.

### EXAMPLE 6 Graphing an Equation of the Form $x = a$ (Vertical Line)

Graph $x - 3 = 0$.

First we add 3 to each side of the equation $x - 3 = 0$ to get $x = 3$. All ordered-pair solutions of this equation have $x$-coordinate 3. Any number can be used for $y$. We show three ordered pairs that satisfy the equation in the table of values. The graph is the **vertical line** shown in **FIGURE 14**. The $x$-intercept is $(3, 0)$. There is no $y$-intercept.

$y$ can be any real number.

| $x$ | $y$ |
|---|---|
| 3 | 3 |
| 3 | 0 |
| 3 | $-2$ |

$x$ must be 3.

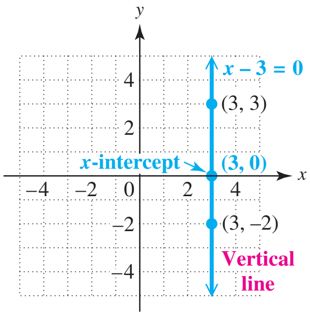
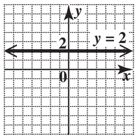


FIGURE 14

NOW TRY

NOW TRY ANSWERS

5.



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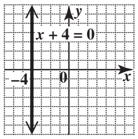


### Vertical Line

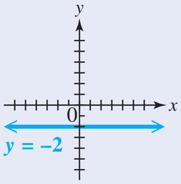
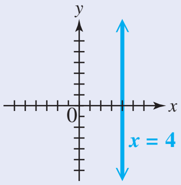
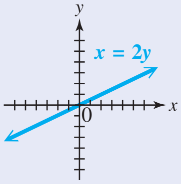
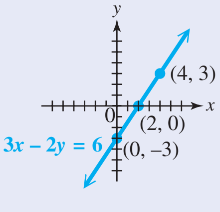
The graph of the linear equation $\boldsymbol{x = a}$**,** where $a$ is a real number, is the vertical line with $x$-intercept $(a, 0)$. There is no $y$-intercept (unless the vertical line is the $y$-axis itself).

***The equation of the x-axis is the horizontal line*** $\boldsymbol{y = 0}$***, and the equation of the y-axis is the vertical line*** $\boldsymbol{x = 0}$***.***

**CAUTION** The equations of horizontal and vertical lines are often confused with each other. Remember that the graph of $y = b$ is parallel to the $x$-axis and the graph of $x = a$ is parallel to the $y$-axis (for $a \neq 0$ and $b \neq 0$).

A summary of the forms of linear equations from this section follows.

### Graphing a Linear Equation

| ***Equation*** | ***To Graph*** | ***Example*** |
|---|---|---|
| $\boldsymbol{y = b}$ | Draw a horizontal line, through $(0, b)$. |  |
| $\boldsymbol{x = a}$ | Draw a vertical line, through $(a, 0)$. |  |
| $\boldsymbol{Ax + By = 0}$ | The graph passes through $(0, 0)$. To find additional points that lie on the graph, choose any value for $x$ or $y$, except 0. |  |
| $\boldsymbol{Ax + By = C}$<br>**(but not of the types above)** | Find any two points on the line. A good choice is to find the intercepts. Let $x = 0$, and find the corresponding value of $y$. Then let $y = 0$, and find $x$.<br>As a check, get a third point by choosing a value for $x$ or $y$ that has not yet been used. |  |

## OBJECTIVE 5 Use a linear equation to model data.

**NOW TRY EXERCISE 7**

Use **(a)** the graph and **(b)** the equation in **Example 7** to approximate credit card debt in 2006.

### EXAMPLE 7 Using a Linear Equation to Model Credit Card Debt

Credit card debt in the United States increased steadily from 2000 through 2008. The amount of debt $y$ in billions of dollars can be modeled by the linear equation

$$y = 32.0x + 684,$$

where $x = 0$ represents 2000, $x = 1$ represents 2001, and so on. (*Source: The Nilson Report.*)

**(a)** Use the equation to approximate credit card debt in the years 2000, 2004, and 2008.

For 2000: $y = 32.0(0) + 684$ Replace $x$ with 0.
$y = 684$ billion dollars

For 2004: $y = 32.0(4) + 684$ $2004 - 2000 = 4$
$y = 812$ billion dollars Replace $x$ with 4.

For 2008: $y = 32.0(8) + 684$ $2008 - 2000 = 8$
$y = 940$ billion dollars Replace $x$ with 8.

**(b)** Write the information from part (a) as three ordered pairs, and use them to graph the given linear equation.

Since $x$ represents the year and $y$ represents the debt, the ordered pairs are

$$(0, 684), \quad (4, 812), \quad \text{and} \quad (8, 940).$$

See **FIGURE 15**. (Arrowheads are not included with the graphed line, since the data are for the years 2000 to 2008 only—that is, from $x = 0$ to $x = 8$.)

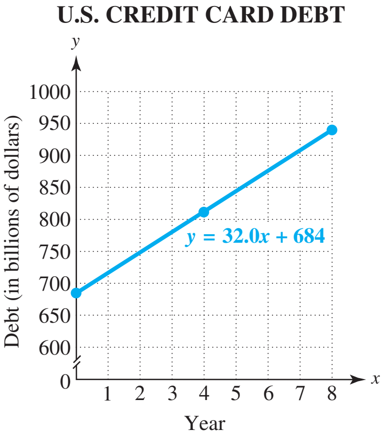


**FIGURE 15**

**(c)** Use the graph and then the equation to approximate credit card debt in 2002.

For 2002, $x = 2$. On the graph, find 2 on the horizontal axis, move up to the graphed line and then across to the vertical axis. It appears that credit card debt in 2002 was about 750 billion dollars. To use the equation, substitute 2 for $x$.

$y = 32.0x + 684$ Given linear equation

$y = 32.0(2) + 684$ Let $x = 2$.

$y = 748$ billion dollars Multiply, and then add.

This result for 2002 is close to our estimate of 750 billion dollars from the graph.

NOW TRY

**NOW TRY ANSWERS**

**7.** **(a)** about 875 billion dollars
**(b)** 876 billion dollars

## CONNECTIONS

Among the basic features of graphing calculators is their ability to graph equations. We must solve the equation for $y$ in order to enter it into the calculator. Also, we must select a "window" for the graph, determined by the minimum and maximum values of $x$ and $y$. The *standard window* is from $x = -10$ to $x = 10$ and from $y = -10$ to $y = 10$, written $[-10, 10]$, $[-10, 10]$, with the $x$-interval first.

To graph $2x + y = 4$, discussed in **Example 3,** we first solve for $y$.

$$y = -2x + 4 \qquad \text{Subtract } 2x.$$

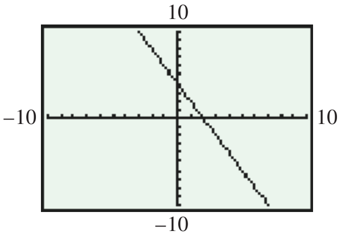


**FIGURE 16**

We enter this equation into the calculator and choose the standard window to get the graph in **FIGURE 16**. The line intersects the $x$-axis at $(2, 0)$, indicating that 2 is the solution of the equation

$$-2x + 4 = 0.$$

### For Discussion or Writing

Rewrite each equation with the left side equal to 0, the form required for a graphing calculator. (It is not necessary to clear parentheses or combine like terms.)

**1.** $3x + 4 - 2x - 7 = 4x + 3$

**2.** $5x - 15 = 3(x - 2)$

## 3.2 EXERCISES

MyMathLab  Math XL PRACTICE  WATCH  DOWNLOAD  READ  REVIEW

⊙ *Complete solution available on the Video Resources on DVD*

*Use the given equation to complete the given ordered pairs. Then graph each equation by plotting the points and drawing a line through them.* ***See Examples 1 and 2.***

⊙ **1.** $x + y = 5$
$(0, \_\_), (\_\_, 0), (2, \_\_)$

**2.** $x - y = 2$
$(0, \_\_), (\_\_, 0), (5, \_\_)$

⊙ **3.** $y = \frac{2}{3}x + 1$
$(0, \_\_), (3, \_\_), (-3, \_\_)$

**4.** $y = -\frac{3}{4}x + 2$
$(0, \_\_), (4, \_\_), (-4, \_\_)$

**5.** $3x = -y - 6$
$(0, \_\_), (\_\_, 0), \left(-\frac{1}{3}, \_\_\right)$

**6.** $x = 2y + 3$
$(\_\_, 0), (0, \_\_), \left(\_\_, \frac{1}{2}\right)$

**7.** ***Concept Check*** Match the information about each graph in Column I with the correct linear equation in Column II.

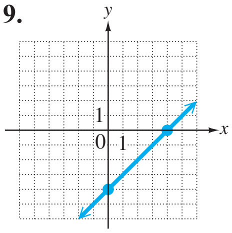
| I | II |
|---|---|
| **(a)** The graph of the equation has $y$-intercept $(0, -4)$. | **A.** $3x + y = -4$ |
| **(b)** The graph of the equation has $(0, 0)$ as $x$-intercept and $y$-intercept. | **B.** $x - 4 = 0$ |
| **(c)** The graph of the equation does not have an $x$-intercept. | **C.** $y = 4x$ |
| **(d)** The graph of the equation has $x$-intercept $(4, 0)$. | **D.** $y = 4$ |

**8.** ***Concept Check*** Which of these equations have a graph with only one intercept?

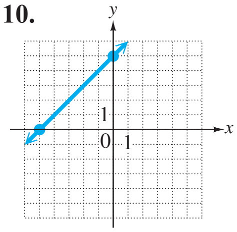
**A.** $x + 8 = 0$ **B.** $x - y = 3$ **C.** $x + y = 0$ **D.** $y = 4$

***Concept Check*** *Find the intercepts of each graph. (All coordinates are integers.)*

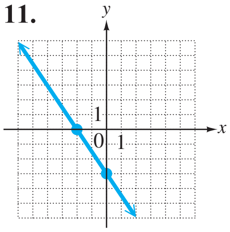
**9.**



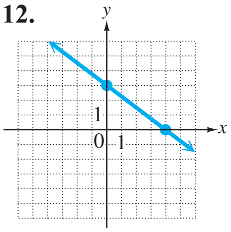
**10.**



**11.**



**12.**



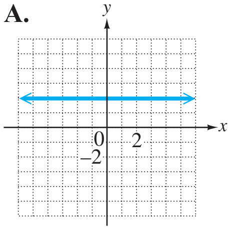
*Find the x-intercept and the y-intercept for the graph of each equation.* ***See Examples 1–6.***

**13.** $x - y = 8$
**14.** $x - y = 7$
**15.** $5x - 2y = 20$
**16.** $-3x + 2y = 12$
**17.** $x + 6y = 0$
**18.** $3x + y = 0$
**19.** $y = -2x + 4$
**20.** $y = 3x + 6$
**21.** $y = \frac{1}{3}x - 2$
**22.** $y = \frac{1}{4}x - 1$
**23.** $2x - 3y = 0$
**24.** $4x - 5y = 0$
**25.** $x - 4 = 0$
**26.** $x - 5 = 0$
**27.** $y = 2.5$
**28.** $y = -1.5$

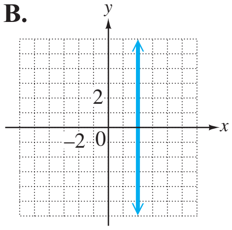
**29.** ***Concept Check*** Match each equation in (a)–(d) with its graph in A–D.

**(a)** $x = -2$
**(b)** $y = -2$
**(c)** $x = 2$
**(d)** $y = 2$

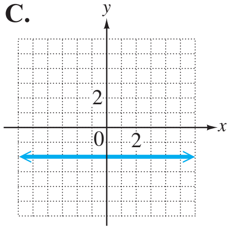
**A.**



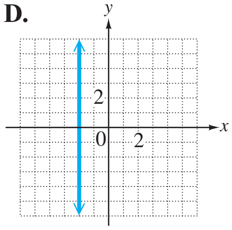
**B.**



**C.**



**D.**



**30.** ***Concept Check*** What is the equation of the $x$-axis? What is the equation of the $y$-axis?

*Graph each linear equation.* ***See Examples 1–6.***

**31.** $x = y + 2$
**32.** $x = -y + 6$
**33.** $x - y = 4$
**34.** $x - y = 5$
**35.** $2x + y = 6$
**36.** $-3x + y = -6$
**37.** $y = 2x - 5$
**38.** $y = 4x + 3$
**39.** $3x + 7y = 14$
**40.** $6x - 5y = 18$
**41.** $y = -\frac{3}{4}x + 3$
**42.** $y = -\frac{2}{3}x - 2$
**43.** $y - 2x = 0$
**44.** $y + 3x = 0$
**45.** $y = -6x$
**46.** $y = 4x$
**47.** $y = -1$
**48.** $y = 3$
**49.** $x + 2 = 0$
**50.** $x - 4 = 0$
**51.** $-3y = 15$
**52.** $-2y = 12$
**53.** $x + 2 = 8$
**54.** $x - 1 = -4$

***Concept Check*** *In Exercises 55–62, describe what the graph of each linear equation will look like in the coordinate plane.* (Hint: *Rewrite the equation if necessary so that it is in a more recognizable form.*)

**55.** $3x = y - 9$
**56.** $2x = y - 4$
**57.** $x - 10 = 1$
**58.** $x + 4 = 3$
**59.** $3y = -6$
**60.** $5y = -15$
**61.** $2x = 4y$
**62.** $3x = 9y$

***Concept Check*** *Plot each set of points, and draw a line through them. Then give the equation of the line.*

**63.** $(3, 5)$, $(3, 0)$, and $(3, -3)$
**64.** $(1, 3)$, $(1, 0)$, and $(1, -1)$
**65.** $(-3, -3)$, $(0, -3)$, and $(4, -3)$
**66.** $(-5, 5)$, $(0, 5)$, and $(3, 5)$

*Solve each problem.* ***See Example 7.***

**67.** The weight $y$ (in pounds) of a man taller than 60 in. can be approximated by the linear equation

$$y = 5.5x - 220,$$

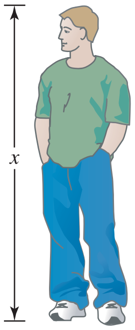
where $x$ is the height of the man in inches.

**(a)** Use the equation to approximate the weights of men whose heights are 62 in., 66 in., and 72 in.

**(b)** Write the information from part (a) as three ordered pairs.

**(c)** Graph the equation, using the data from part (b).

**(d)** Use the graph to estimate the height of a man who weighs 155 lb. Then use the equation to find the height of this man to the nearest inch.



**68.** The height $y$ (in centimeters) of a woman is related to the length of her radius bone $x$ (from the wrist to the elbow) and is approximated by the linear equation

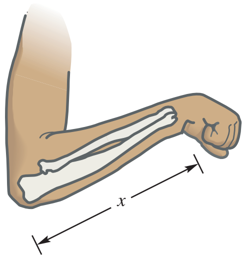
$$y = 3.9x + 73.5.$$

**(a)** Use the equation to approximate the heights of women with radius bone of lengths 20 cm, 26 cm, and 22 cm.

**(b)** Write the information from part (a) as three ordered pairs.

**(c)** Graph the equation, using the data from part (b).

**(d)** Use the graph to estimate the length of the radius bone in a woman who is 167 cm tall. Then use the equation to find the length of the radius bone to the nearest centimeter.



**69.** As a fundraiser, a club is selling posters. The printer charges a \$25 set-up fee, plus \$0.75 for each poster. The cost $y$ in dollars to print $x$ posters is given by

$$y = 0.75x + 25.$$

**(a)** What is the cost $y$ in dollars to print 50 posters? To print 100 posters?

**(b)** Find the number of posters $x$ if the printer billed the club for costs of \$175.

**(c)** Write the information from parts (a) and (b) as three ordered pairs.

**(d)** Use the data from part (c) to graph the equation.

**70.** A gas station is selling gasoline for \$3.50 per gallon and charges \$7 for a car wash. The cost $y$ in dollars for $x$ gallons of gasoline and a car wash is given by

$$y = 3.50x + 7.$$

**(a)** What is the cost $y$ in dollars for 9 gallons of gasoline and a car wash? For 4 gallons of gasoline and a car wash?

**(b)** Find the number of gallons of gasoline $x$ if the cost for gasoline and a car wash is \$35.

**(c)** Write the information from parts (a) and (b) as three ordered pairs.

**(d)** Use the data from part (c) to graph the equation.

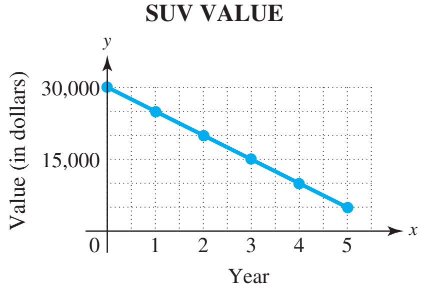
**71.** The graph shows the value of a sport-utility vehicle (SUV) over the first 5 yr of ownership. Use the graph to do the following.

**(a)** Determine the initial value of the SUV.

**(b)** Find the **depreciation** (loss in value) from the original value after the first 3 yr.

**(c)** What is the annual or yearly depreciation in each of the first 5 yr?

**(d)** What does the ordered pair $(5, 5000)$ mean in the context of this problem?


**SUV VALUE**

**72.** Demand for an item is often closely related to its price. As price increases, demand decreases, and as price decreases, demand increases. Suppose demand for a video game is 2000 units when the price is $40 and is 2500 units when the price is $30.

**(a)** Let $x$ be the price and $y$ be the demand for the game. Graph the two given pairs of prices and demands.

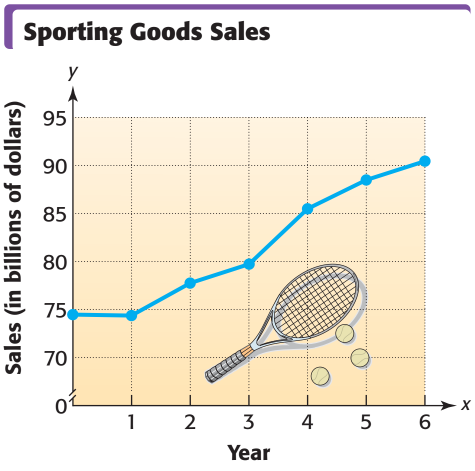
**(b)** Assume that the relationship is linear. Draw a line through the two points from part (a). From your graph, estimate the demand if the price drops to $20.

**(c)** Use the graph to estimate the price if the demand is 3500 units.

**73.** In the United States, sporting goods sales $y$ (in billions of dollars) from 2000 through 2006 are shown in the graph and modeled by the linear equation

$$y = 3.018x + 72.52,$$

where $x = 0$ corresponds to 2000, $x = 1$ corresponds to 2001, and so on.



***Source:*** National Sporting Goods Association.

**(a)** Use the equation to approximate sporting goods sales in 2000, 2004, and 2006. Round your answers to the nearest billion dollars.

**(b)** Use the graph to estimate sales for the same years.

**(c)** How do the approximations using the equation compare with the estimates from the graph?

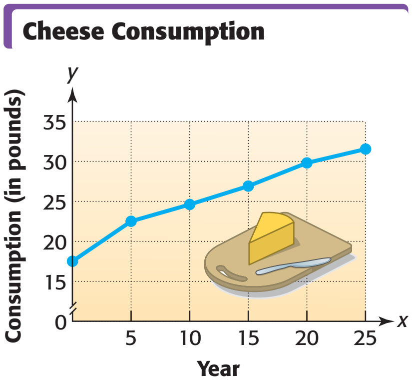
**74.** U.S. per capita consumption of cheese increased for the years 1980 through 2005 as shown in the graph. If $x = 0$ represents 1980, $x = 5$ represents 1985, and so on, per capita consumption $y$ in pounds can be modeled by the linear equation

$$y = 0.5383x + 18.74.$$

**(a)** Use the equation to approximate cheese consumption (to the nearest tenth) in 1990, 2000, and 2005.

**(b)** Use the graph to estimate consumption for the same years.

**(c)** How do the approximations using the equation compare with the estimates from the graph?



***Source:*** U.S. Department of Agriculture.

## PREVIEW EXERCISES

*Find each quotient.* ***See Section 1.8.***

**75.** $\frac{4-2}{8-5}$ **76.** $\frac{-3-5}{2-7}$ **77.** $\frac{-2-(-4)}{3-(-1)}$ **78.** $\frac{5-(-7)}{-4-(-1)}$

# 3.3

# The Slope of a Line

**OBJECTIVES**

1. Find the slope of a line, given two points.
2. Find the slope from the equation of a line.
3. Use slopes to determine whether two lines are parallel, perpendicular, or neither.

An important characteristic of the lines we graphed in **Section 3.2** is their slant, or “steepness.” See **FIGURE 17**.

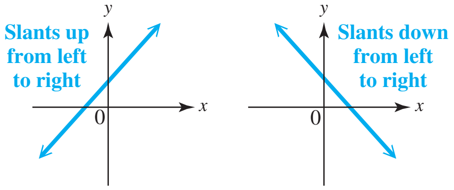


**FIGURE 17**

One way to measure the steepness of a line is to compare the vertical change in the line with the horizontal change while moving along the line from one fixed point to another. This measure of steepness is called the *slope* of the line.

**OBJECTIVE 1 Find the slope of a line, given two points.** To find the steepness, or slope, of the line in **FIGURE 18**, we begin at point $Q$ and move to point $P$. The vertical change, or **rise,** is the change in the $y$-values, which is the difference

$$6 - 1 = 5 \text{ units.}$$

The horizontal change, or **run,** is the change in the $x$-values, which is the difference

$$5 - 2 = 3 \text{ units.}$$

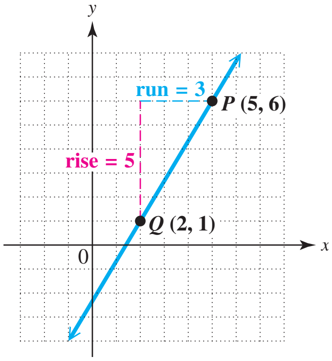


**FIGURE 18**

Remember from **Section 2.6** that one way to compare two numbers is by using a ratio. **Slope** is the ratio of the vertical change in $y$ to the horizontal change in $x$. The line in **FIGURE 18** has

$$\text{slope} = \frac{\text{vertical change in } y \text{ (rise)}}{\text{horizontal change in } x \text{ (run)}} = \frac{5}{3}.$$

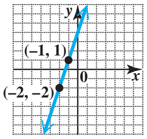
To confirm this ratio, we can count grid squares. We start at point $Q$ in **FIGURE 18** and count *up* 5 grid squares to find the vertical change (rise). To find the horizontal change (run) and arrive at point $P$, we count to the *right* 3 grid squares. The slope is $\frac{5}{3}$, as found analytically.

We can summarize this discussion as follows.

***Slope is a single number that allows us to determine the direction in which a line is slanting from left to right, as well as how much slant there is to the line.***

NOW TRY EXERCISE 1

Find the slope of the line.



## EXAMPLE 1 Finding the Slope of a Line

Find the slope of the line in **FIGURE 19**.

We use the two points shown on the line. The vertical change is the difference in the $y$-values, or $-1 - 3 = -4$, and the horizontal change is the difference in the $x$-values, or $6 - 2 = 4$. Thus, the line has

$$\text{slope} = \frac{\text{change in } y \text{ (rise)}}{\text{change in } x \text{ (run)}} = \frac{-4}{4}, \quad \text{or} \quad -1.$$

Counting grid squares, we begin at point $P$ and count *down* 4 grid squares. Then we count to the *right* 4 grid squares to reach point $Q$. Because we counted down, we write the vertical change as a negative number, $-4$ here. The slope is $\frac{-4}{4}$, or $-1$.

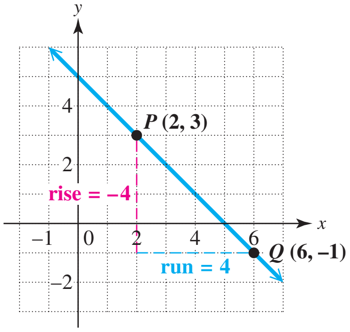


**FIGURE 19**

NOW TRY

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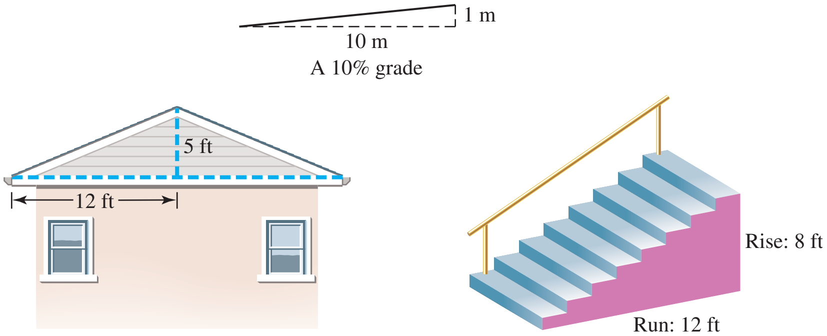
**NOTE** ***The slope of a line is the same for any two points on the line.*** To see this, refer to **FIGURE 19**. Find the points $(3, 2)$ and $(5, 0)$ on the line. If we start at $(3, 2)$ and count *down* 2 units and then to the *right* 2 units, we arrive at $(5, 0)$. The slope is $\frac{-2}{2}$, or $-1$, the same slope we found in **Example 1.**

---

The idea of slope is used in many everyday situations. See **FIGURE 20**. A highway with a 10%, or $\frac{1}{10}$, grade (or slope) rises 1 m for every 10 m horizontally. Architects specify the pitch of a roof by using slope. A $\frac{5}{12}$ roof pitch means that the roof rises 5 ft for every 12 ft that it runs horizontally. The slope of a stairwell indicates the ratio of the vertical rise to the horizontal run. The slope of the stairwell is $\frac{8}{12}$, or $\frac{2}{3}$.



A 10% grade

$\frac{5}{12}$ roof pitch

Slope of a stairwell

**FIGURE 20**

We can generalize the preceding discussion and find the slope of a line through two nonspecific points $(x_1, y_1)$ and $(x_2, y_2)$. (This notation is called **subscript notation.**) Read $x_1$ as **"x-sub-one"** and $x_2$ as **"x-sub-two."**) See **FIGURE 21** on the next page.

NOW TRY ANSWER

**1.** 3

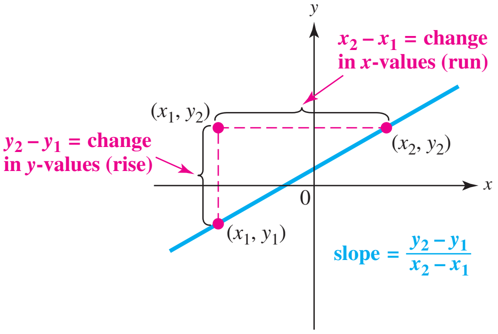


FIGURE 21

Moving along the line from the point $(x_1, y_1)$ to the point $(x_2, y_2)$, we see that $y$ changes by $y_2 - y_1$ units. This is the vertical change (rise). Similarly, $x$ changes by $x_2 - x_1$ units, which is the horizontal change (run). The slope of the line is the ratio of $y_2 - y_1$ to $x_2 - x_1$.

---

**NOTE** Subscript notation is used to identify a point. It does *not* indicate any operation. Note the difference between $x_2$, a nonspecific value, and $x^2$, which means $x \cdot x$. Read $x_2$ as "$x$-sub-two," *not* "$x$ squared."

---

### Slope Formula

The **slope** $m$ of the line through the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (\text{where } x_1 \neq x_2).$$

***The slope gives the change in y for each unit of change in x.***

### EXAMPLE 2 Finding Slopes of Lines

Find the slope of each line.

**(a)** The line through $(-4, 7)$ and $(1, -2)$

Use the slope formula. Let $(-4, 7) = (x_1, y_1)$ and $(1, -2) = (x_2, y_2)$.

$$\text{slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 7}{1 - (-4)} = \frac{-9}{5} = -\frac{9}{5}$$

Substitute carefully here.

Begin at $(-4, 7)$ and count grid squares in **FIGURE 22** to confirm your calculation.

What happens if we let $(1, -2) = (x_1, y_1)$ and $(-4, 7) = (x_2, y_2)$?

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-2)}{-4 - 1} = \frac{9}{-5} = -\frac{9}{5}$$

The same slope is obtained.

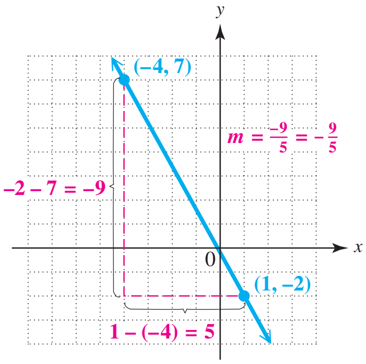


FIGURE 22

NOW TRY EXERCISE 2

Find the slope of the line through $(4, -5)$ and $(-2, -4)$.

**(b)** The line through $(-9, -2)$ and $(12, 5)$

*y*-value

$$m = \frac{\overset{\downarrow}{5} - (-2)}{\underset{\uparrow}{12} - (-9)} = \frac{7}{21} = \frac{1}{3}$$

Corresponding *x*-value

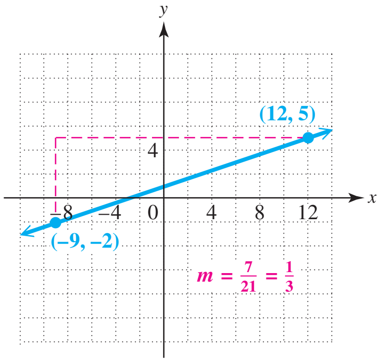


**FIGURE 23**

See **FIGURE 23**. Again, note that the same slope is obtained by subtracting in reverse order.

*y*-value

$$m = \frac{\overset{\downarrow}{-2} - 5}{\underset{\uparrow}{-9} - 12} = \frac{-7}{-21} = \frac{1}{3}$$

Corresponding *x*-value

NOW TRY

---

**CAUTION** ***It makes no difference which point is $(x_1, y_1)$ or $(x_2, y_2)$. Be consistent, however.*** Start with the $x$- and $y$-values of one point (either one), and subtract the corresponding values of the other point.

---

The slopes we found for the lines in **FIGURES 22** and **23** suggest the following.

**Orientation of Lines with Positive and Negative Slopes**

A line with positive slope rises (slants up) from left to right.

A line with negative slope falls (slants down) from left to right.

NOW TRY EXERCISE 3

Find the slope of the line through $(1, -3)$ and $(4, -3)$.

### EXAMPLE 3 Finding the Slope of a Horizontal Line

Find the slope of the line through $(-5, 4)$ and $(2, 4)$.

$$m = \frac{4 - 4}{-5 - 2} = \frac{0}{-7} = 0 \qquad \text{Slope } 0$$

As shown in **FIGURE 24**, the line through these two points is horizontal, with equation $y = 4$. ***All horizontal lines have slope 0,*** since the difference in $y$-values is 0.

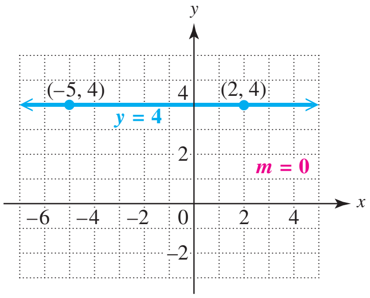


**FIGURE 24**

NOW TRY

NOW TRY ANSWERS

**2.** $-\frac{1}{6}$

**3.** 0

**NOW TRY EXERCISE 4**

Find the slope of the line through $(-2, 1)$ and $(-2, -4)$.

## EXAMPLE 4 Finding the Slope of a Vertical Line

Find the slope of the line through $(6, 2)$ and $(6, -4)$.

$$m = \frac{2-(-4)}{6-6} = \frac{6}{0} \qquad \text{Undefined slope}$$

Since division by 0 is undefined, the slope is undefined. The graph in **FIGURE 25** shows that the line through the given two points is vertical with equation $x = 6$. All points on a vertical line have the same $x$-value, so ***the slope of any vertical line is undefined.***

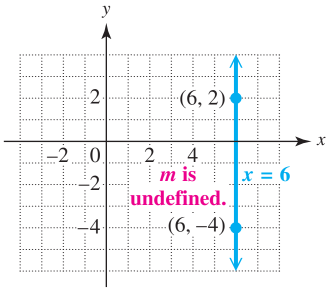


**FIGURE 25**

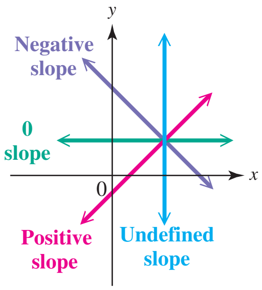
NOW TRY

### Slopes of Horizontal and Vertical Lines

**Horizontal lines,** with equations of the form $y = b$, have **slope 0.**

**Vertical lines,** with equations of the form $x = a$, have **undefined slope.**

**FIGURE 26** summarizes the four cases for slopes of lines.



Slopes of lines

**FIGURE 26**

**OBJECTIVE 2** **Find the slope from the equation of a line.** Consider this linear equation.

$$y = -3x + 5$$

We can find the slope of the line using any two points on the line. We get these two points by first choosing two different values of $x$ and then finding the corresponding values of $y$. We choose $x = -2$ and $x = 4$.

$$\begin{aligned} y &= -3x + 5 & \\ y &= -3(-2) + 5 & &\text{Let } x = -2. \\ y &= 6 + 5 & &\text{Multiply.} \\ y &= 11 & &\text{Add.} \end{aligned}$$

$$\begin{aligned} y &= -3x + 5 & \\ y &= -3(4) + 5 & &\text{Let } x = 4. \\ y &= -12 + 5 & &\text{Multiply.} \\ y &= -7 & &\text{Add.} \end{aligned}$$

The ordered pairs are $(-2, 11)$ and $(4, -7)$. Now we use the slope formula.

$$m = \frac{11-(-7)}{-2-4} = \frac{18}{-6} = -3$$

***The slope, $-3$, is the same number as the coefficient of x in the given equation*** $\boldsymbol{y = -3x + 5}$**.** It can be shown that this always happens, *as long as the equation is solved for y.* This fact is used to find the slope of a line from its equation.

**NOW TRY ANSWER**

**4.** undefined slope

### Finding the Slope of a Line from Its Equation

***Step 1*** Solve the equation for $y$.

***Step 2*** The slope is given by the coefficient of $x$.

**NOW TRY EXERCISE 5**

Find the slope of the line.

$$3x + 5y = -1$$

### EXAMPLE 5 Finding Slopes from Equations

Find the slope of each line.

**(a)** $2x - 5y = 4$

***Step 1*** Solve the equation for $y$.

$$2x - 5y = 4$$

Isolate $y$ on one side.

$$-5y = -2x + 4 \qquad \text{Subtract } 2x \text{ from each side.}$$

$$y = \underset{\substack{\uparrow\\ \text{Slope}}}{\frac{2}{5}}x - \frac{4}{5} \qquad \text{Divide by } -5.$$

***Step 2*** The slope is given by the coefficient of $x$, so the slope is $\frac{2}{5}$.

**(b)**

$$8x + 4y = 1$$

Solve for $y$.

$$4y = -8x + 1 \qquad \text{Subtract } 8x.$$

$$y = -2x + \frac{1}{4} \qquad \text{Divide by 4.}$$

The slope of this line is given by the coefficient of $x$, which is $-2$. NOW TRY

---

**NOTE** We can solve the equation $Ax + By = C$ (with $B \neq 0$) for $y$ to show that, in general, the slope of the line is $m = -\frac{A}{B}$.

---

**OBJECTIVE 3 Use slopes to determine whether two lines are parallel, perpendicular, or neither.** Two lines in a plane that never intersect are **parallel**. We use slopes to tell whether two lines are parallel.

**FIGURE 27** on the next page shows the graphs of $x + 2y = 4$ and $x + 2y = -6$. These lines appear to be parallel. We solve each equation for $y$ to find the slope.

$$x + 2y = 4$$

$$2y = -x + 4 \qquad \text{Subtract } x.$$

$$y = \frac{-x}{2} + 2 \qquad \text{Divide by 2.}$$

$$y = \underset{\substack{\uparrow\\ \text{Slope}}}{-\frac{1}{2}}x + 2 \qquad \frac{-x}{2} = \frac{-1x}{2} = -\frac{1}{2}x$$

The slope is $-\frac{1}{2}$, not $-\frac{x}{2}$.

$$x + 2y = -6$$

$$2y = -x - 6 \qquad \text{Subtract } x.$$

$$y = \frac{-x}{2} - 3 \qquad \text{Divide by 2.}$$

$$y = \underset{\substack{\uparrow\\ \text{Slope}}}{-\frac{1}{2}}x - 3 \qquad \frac{-x}{2} = \frac{-1x}{2} = -\frac{1}{2}x$$

Both lines have slope $-\frac{1}{2}$. ***Nonvertical parallel lines always have equal slopes.***

**NOW TRY ANSWER**

5. $-\frac{3}{5}$

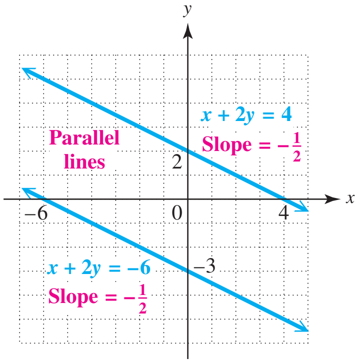


FIGURE 27

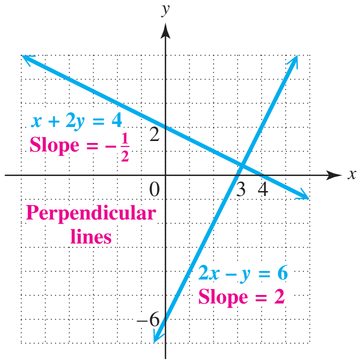


FIGURE 28

**FIGURE 28** shows the graphs of $x + 2y = 4$ and $2x - y = 6$. These lines appear to be **perpendicular** (that is, they intersect at a 90° angle). As shown earlier, solving $x + 2y = 4$ for $y$ gives $y = -\frac{1}{2}x + 2$, with slope $-\frac{1}{2}$. We must solve $2x - y = 6$ for $y$.

$$\begin{aligned} 2x - y &= 6 \\ -y &= -2x + 6 && \text{Subtract } 2x. \\ y &= \underset{\substack{\uparrow \\ \text{Slope}}}{2}x - 6 && \text{Multiply by } -1. \end{aligned}$$

The product of the two slopes $-\frac{1}{2}$ and 2 is

$$-\frac{1}{2}(2) = -1.$$

***The product of the slopes of two perpendicular lines, neither of which is vertical, is always −1.*** This means that the slopes of perpendicular lines are negative (or opposite) reciprocals—if one slope is the nonzero number $a$, the other is $-\frac{1}{a}$. The table in the margin shows several examples.

| *Number* | *Negative Reciprocal* |
|---|---|
| $\frac{3}{4}$ | $-\frac{4}{3}$ |
| $\frac{1}{2}$ | $-\frac{2}{1}$, or $-2$ |
| $-6$, or $-\frac{6}{1}$ | $\frac{1}{6}$ |
| $-0.4$, or $-\frac{4}{10}$ | $\frac{10}{4}$, or 2.5 |

***The product of each number and its negative reciprocal is −1.***

### Slopes of Parallel and Perpendicular Lines

Two lines with the same slope are parallel.

Two lines whose slopes have a product of −1 are perpendicular.

### EXAMPLE 6 Deciding Whether Two Lines Are Parallel or Perpendicular

Decide whether each pair of lines is *parallel, perpendicular,* or *neither.*

**(a)**
$$\begin{aligned} x + 3y &= 7 \\ -3x + y &= 3 \end{aligned}$$

Find the slope of each line by first solving each equation for $y$.

$$\begin{aligned} x + 3y &= 7 \\ 3y &= -x + 7 && \text{Subtract } x. \\ y &= -\frac{1}{3}x + \frac{7}{3} && \text{Divide by 3.} \end{aligned}$$

Slope is $-\frac{1}{3}$.

$$\begin{aligned} -3x + y &= 3 \\ y &= 3x + 3 && \text{Add } 3x. \end{aligned}$$

Slope is 3.

**NOW TRY EXERCISE 6**

Decide whether the pair of lines is *parallel, perpendicular,* or *neither*.

$$2x - 3y = 1$$
$$4x + 6y = 5$$

Since the slopes $-\frac{1}{3}$ and 3 are not equal, the lines are not parallel. Check the product of the slopes.

$$-\frac{1}{3}(3) = -1 \qquad \text{The slopes are negative reciprocals.}$$

The two lines are perpendicular because the product of their slopes is $-1$.

**(b)**
$$4x - y = 4 \xrightarrow{\text{Solve for } y.} y = 4x - 4$$
$$8x - 2y = -12 \longrightarrow y = 4x + 6$$

Both lines have slope 4, so the lines are parallel.

**(c)**
$$4x + 3y = 6 \xrightarrow{\text{Solve for } y.} y = -\frac{4}{3}x + 2$$
$$2x - y = 5 \longrightarrow y = 2x - 5$$

Here the slopes are $-\frac{4}{3}$ and 2. These lines are neither parallel nor perpendicular, because $-\frac{4}{3} \neq 2$ and $-\frac{4}{3} \cdot 2 \neq -1$.

**(d)**
$$6x - y = 1 \xrightarrow{\text{Solve for } y.} y = 6x - 1$$
$$x - 6y = -12 \longrightarrow y = \frac{1}{6}x + 2$$

The slopes are 6 and $\frac{1}{6}$. The lines are not parallel, nor are they perpendicular. $\left(\textbf{\textit{Be careful!}}\ \mathbf{6\left(\frac{1}{6}\right) = 1,}\ \textbf{\textit{not}}\ \mathbf{-1.}\right)$ NOW TRY

**NOW TRY ANSWER**

**6.** neither

## 3.3 EXERCISES

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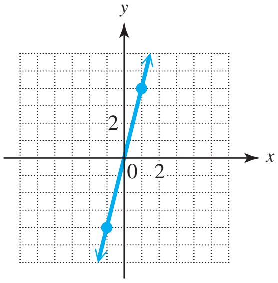
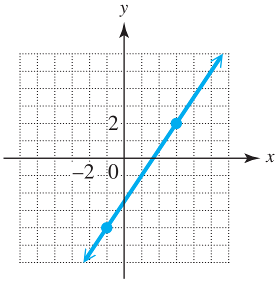
*Use the indicated points to find the slope of each line.* ***See Example 1.***

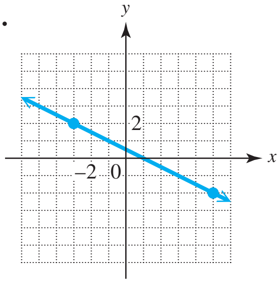
**1.**



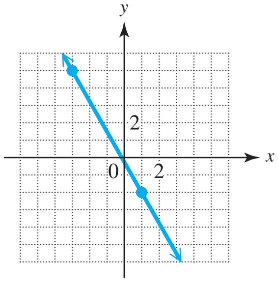
**2.**



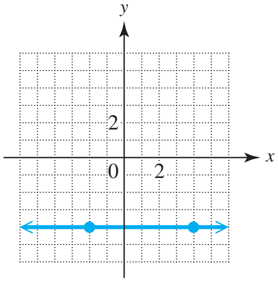
**3.**



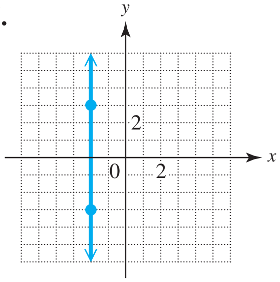
**4.**



**5.**



**6.**

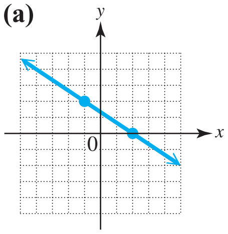


**7.** In the context of the graph of a straight line, what is meant by "rise"? What is meant by "run"?

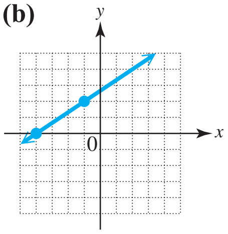
**8.** Explain in your own words what is meant by *slope* of a line.

**9.** ***Concept Check*** Match the graph of each line in (a)–(d) with its slope in A–D.

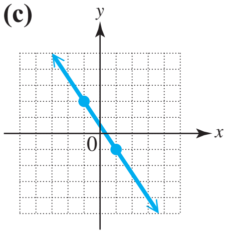
**(a)**



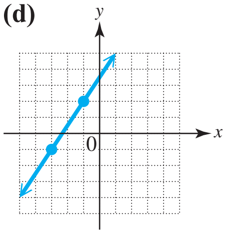
**(b)**



**(c)**



**(d)**

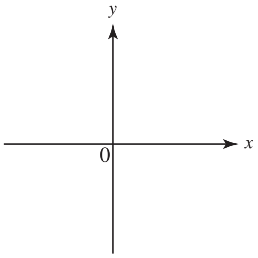


**A.** $\frac{2}{3}$ **B.** $\frac{3}{2}$ **C.** $-\frac{2}{3}$ **D.** $-\frac{3}{2}$

**10.** ***Concept Check*** Decide whether the line with the given slope rises from left to right, falls from left to right, is horizontal, or is vertical.

**(a)** $m = -4$ **(b)** $m = 0$ **(c)** $m$ is undefined. **(d)** $m = \frac{3}{7}$

***Concept Check*** *On a pair of axes similar to the one shown, sketch the graph of a straight line having the indicated slope.*



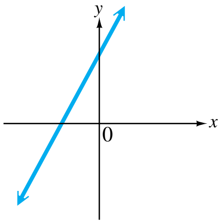
**11.** Negative

**12.** Positive

**13.** Undefined

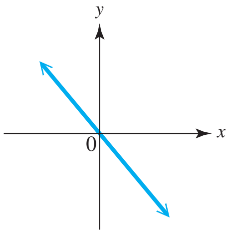
**14.** Zero

***Concept Check*** *The figure at the right shows a line that has a positive slope (because it rises from left to right) and a positive y-value for the y-intercept (because it intersects the y-axis above the origin).*

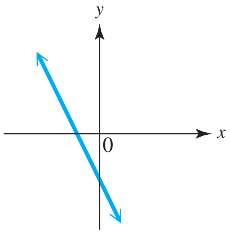


*For each line in Exercises 15–20, decide whether **(a)** the slope is positive, negative, or zero and **(b)** the y-value of the y-intercept is positive, negative, or zero.*

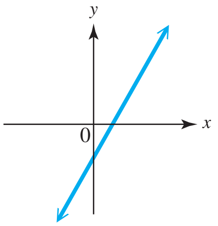
**15.**



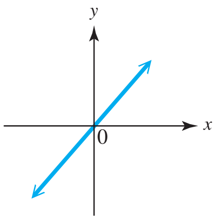
**16.**



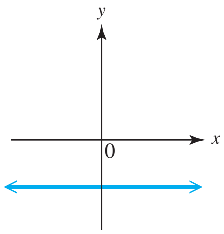
**17.**



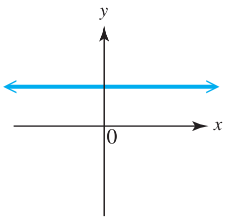
**18.**



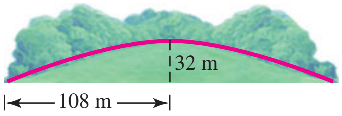
**19.**



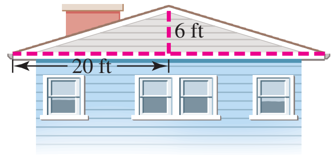
**20.**



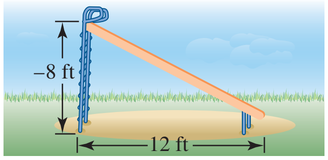
**21.** ***Concept Check*** What is the slope (or grade) of this hill?



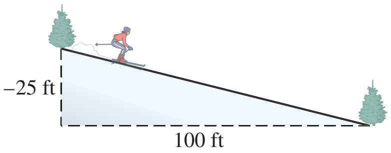
**22.** ***Concept Check*** What is the slope (or pitch) of this roof?



**23.** ***Concept Check*** What is the slope of the slide? (*Hint:* The slide *drops* 8 ft vertically as it extends 12 ft horizontally.)



**24.** ***Concept Check*** What is the slope (or grade) of this ski slope? (*Hint:* The ski slope drops 25 ft vertically for every 100 horizontal feet.)



**25.** ***Concept Check*** A student was asked to find the slope of the line through the points $(2, 5)$ and $(-1, 3)$. His answer, $-\frac{2}{3}$, was incorrect. He showed his work as

$$\frac{3-5}{2-(-1)} = \frac{-2}{3} = -\frac{2}{3}.$$

***WHAT WENT WRONG?*** Give the correct slope.

**26.** ***Concept Check*** A student was asked to find the slope of the line through the points $(-2, 4)$ and $(6, -1)$. Her answer, $-\frac{8}{5}$, was incorrect. She showed her work as

$$\frac{6-(-2)}{-1-4} = \frac{8}{-5} = -\frac{8}{5}.$$

***WHAT WENT WRONG?*** Give the correct slope.

*Find the slope of the line through each pair of points.* ***See Examples 2–4.***

**27.** $(1, -2)$ and $(-3, -7)$
**28.** $(4, -1)$ and $(-2, -8)$
**29.** $(0, 3)$ and $(-2, 0)$
**30.** $(8, 0)$ and $(0, -5)$
**31.** $(4, 3)$ and $(-6, 3)$
**32.** $(6, 5)$ and $(-12, 5)$
**33.** $(-2, 4)$ and $(-3, 7)$
**34.** $(-4, 5)$ and $(-5, 8)$
**35.** $(-12, 3)$ and $(-12, -7)$
**36.** $(-8, 6)$ and $(-8, -1)$
**37.** $(4.8, 2.5)$ and $(3.6, 2.2)$
**38.** $(3.1, 2.6)$ and $(1.6, 2.1)$
**39.** $\left(-\frac{7}{5}, \frac{3}{10}\right)$ and $\left(\frac{1}{5}, -\frac{1}{2}\right)$
**40.** $\left(-\frac{4}{3}, \frac{1}{2}\right)$ and $\left(\frac{1}{3}, -\frac{5}{6}\right)$

*Find the slope of each line.* ***See Example 5.***

**41.** $y = 5x + 12$
**42.** $y = 2x + 3$
**43.** $4y = x + 1$
**44.** $2y = x + 4$
**45.** $3x - 2y = 3$
**46.** $6x - 4y = 4$
**47.** $-3x + 2y = 5$
**48.** $-2x + 4y = 5$
**49.** $y = -5$
**50.** $y = 4$
**51.** $x = 6$
**52.** $x = -2$

**53.** ***Concept Check*** What is the slope of a line whose graph is parallel to the graph of $3x + y = 7$? Perpendicular to the graph of $3x + y = 7$?

**54.** ***Concept Check*** What is the slope of a line whose graph is parallel to the graph of $-5x + y = -3$? Perpendicular to the graph of $-5x + y = -3$?

**55.** ***Concept Check*** If two lines are both vertical or both horizontal, which of the following are they?

**A.** Parallel **B.** Perpendicular **C.** Neither parallel nor perpendicular

**56.** ***Concept Check*** If a line is vertical, what is true of any line that is perpendicular to it?

*For each pair of equations, give the slopes of the lines and then determine whether the two lines are* parallel, perpendicular, *or* neither. ***See Example 6.***

**57.** $2x + 5y = 4$
$4x + 10y = 1$

**58.** $-4x + 3y = 4$
$-8x + 6y = 0$

**59.** $8x - 9y = 6$
$8x + 6y = -5$

**60.** $5x - 3y = -2$
$3x - 5y = -8$

**61.** $3x - 2y = 6$
$2x + 3y = 3$

**62.** $3x - 5y = -1$
$5x + 3y = 2$

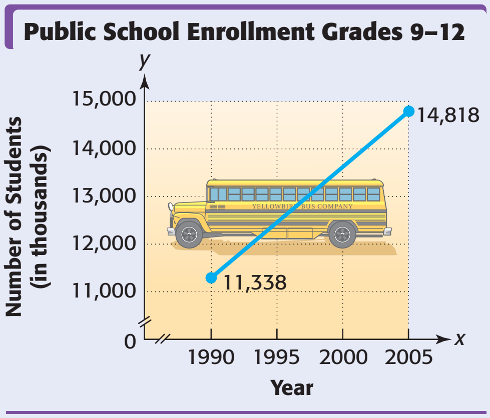
**63.** $5x - y = 1$
$x - 5y = -10$

**64.** $3x - 4y = 12$
$4x + 3y = 12$

## *RELATING CONCEPTS* EXERCISES 65–70

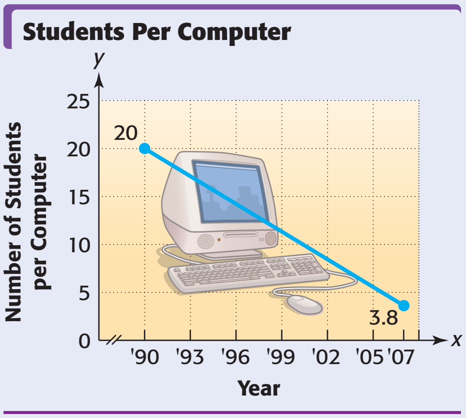
### FOR INDIVIDUAL OR GROUP WORK

**FIGURE A** *gives public school enrollment (in thousands) in grades 9–12 in the United States.* **FIGURE B** *gives the (average) number of public school students per computer.*



***Source:*** U.S. Department of Education.

**FIGURE A**



***Source:*** Quality Education Data, Inc.

**FIGURE B**

***Work Exercises 65–70 in order.***

**65.** Use the ordered pairs $(1990, 11{,}338)$ and $(2005, 14{,}818)$ to find the slope of the line in **FIGURE A**.

**66.** The slope of the line in **FIGURE A** is ____________________. This means that
(positive/negative)
during the period represented, enrollment ____________________.
(increased/decreased)

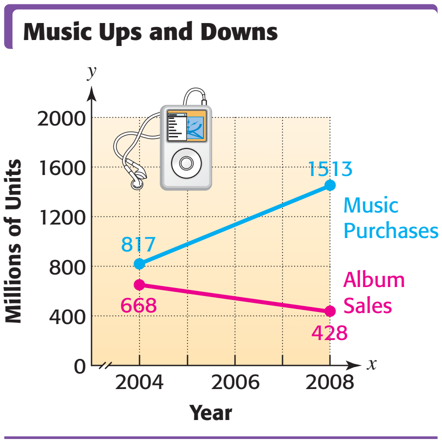
**67.** The slope of a line represents the *rate of change of the line.* On the basis of **FIGURE A**, what was the increase in students *per year* during the period shown?

**68.** Use the given information to find the slope, to the nearest hundredth, of the line in **FIGURE B**.

**69.** The slope of the line in **FIGURE B** is ____________________. This means that
(positive/negative)
the number of students per computer ____________________ during the period represented.
(increased/decreased)

**70.** On the basis of **FIGURE B**, what was the decrease in students per computer *per year* during the period shown?

*The graph shows album sales (which include CD, vinyl, cassette, and digital albums) and music purchases (which include digital tracks, albums, singles, and music videos) in millions of units from 2004 through 2008. Use the graph to work Exercises 71 and 72.*


**Music Ups and Downs**

***Source:*** Nielsen SoundScan.

**71.** Locate the line on the graph that represents music purchases.

**(a)** Write two ordered pairs $(x, y)$, where $x$ is the year and $y$ is purchases in millions of units, to represent the data for the years 2004 and 2008.

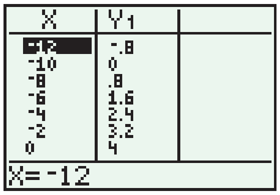
**(b)** Use the ordered pairs from part (a) to find the slope of the line.

**(c)** Interpret the meaning of the slope in the context of this problem.

**72.** Locate the line on the graph that represents album sales. Repeat parts (a)–(c) of **Exercise 71.** For part (a), $x$ is the year and $y$ is sales in millions of units.

## TECHNOLOGY INSIGHTS EXERCISES 73–76

*Some graphing calculators have the capability of displaying a table of points for a graph. The table shown here gives several points that lie on a line designated* $Y_1$.



| X | Y1 |
|---|---|
| -12 | -.8 |
| -10 | 0 |
| -8 | .8 |
| -6 | 1.6 |
| -4 | 2.4 |
| -2 | 3.2 |
| 0 | 4 |

X=-12

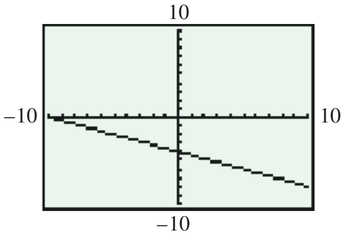
**73.** Use any two of the ordered pairs displayed to find the slope of the line.

**74.** What is the $x$-intercept of the line?

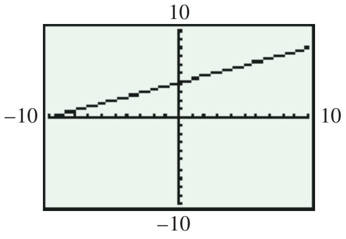
**75.** What is the $y$-intercept of the line?

**76.** Which one of the two lines shown is the graph of $Y_1$?

**A.**



**B.**



## PREVIEW EXERCISES

*Solve each equation for y.* ***See Section 2.5.***

**77.** $2x + 5y = 15$

**78.** $-4x + 3y = 8$

**79.** $10x = 30 + 3y$

**80.** $8x = 8 - 2y$

**81.** $y - (-8) = 2(x - 4)$

**82.** $y - 3 = 4[x - (-6)]$

# 3.4 Writing and Graphing Equations of Lines

**OBJECTIVES**

1. Use the slope-intercept form of the equation of a line.
2. Graph a line by using its slope and a point on the line.
3. Write an equation of a line by using its slope and any point on the line.
4. Write an equation of a line by using two points on the line.
5. Write an equation of a line that fits a data set.

**OBJECTIVE 1 Use the slope-intercept form of the equation of a line.** In **Section 3.3,** we found the slope (steepness) of a line by solving the equation of the line for $y$. In that form, the slope is the coefficient of $x$. For example, the slope of the line with equation $y = 2x + 3$ is $2$. What does the number $3$ represent?

To find out, suppose a line has slope $m$ and $y$-intercept $(0, b)$. We can find an equation of this line by choosing another point $(x, y)$ on the line, as shown in **FIGURE 29**. Then we use the slope formula.

$$m = \frac{y - b \leftarrow \text{Change in } y\text{-values}}{x - 0 \leftarrow \text{Change in } x\text{-values}}$$

$$m = \frac{y - b}{x} \qquad \text{Subtract in the denominator.}$$

$$mx = y - b \qquad \text{Multiply by } x.$$

$$mx + b = y \qquad \text{Add } b.$$

$$y = mx + b \qquad \text{Rewrite.}$$

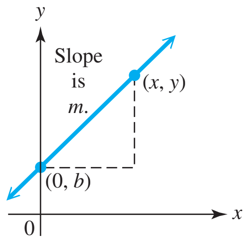


**FIGURE 29**

This result is the *slope-intercept form* of the equation of a line, because both the slope and the $y$-intercept of the line can be read directly from the equation. For the line with equation $y = 2x + 3$, the number 3 gives the $y$-intercept $(0, 3)$.

**Slope-Intercept Form**

The **slope-intercept form** of the equation of a line with slope $m$ and $y$-intercept $(0, b)$ is

$$y = \underset{\text{Slope}}{\underset{\uparrow}{m}}x + \underset{(0,\,b) \text{ is the } y\text{-intercept.}}{\underset{\uparrow}{b}}.$$

***Remember: The intercept given by slope-intercept form is the y-intercept.***

**NOW TRY EXERCISE 1**

Identify the slope and $y$-intercept of the line with each equation.

**(a)** $y = -\frac{3}{5}x - 9$

**(b)** $y = -\frac{x}{3} + \frac{7}{3}$

**EXAMPLE 1** Identifying Slopes and $y$-Intercepts

Identify the slope and $y$-intercept of the line with each equation.

**(a)** $y = \underset{\text{Slope}}{\underset{\uparrow}{-4}}x + \underset{y\text{-intercept } (0,\,1)}{\underset{\uparrow}{1}}$

**(b)** $y = x - 8$ can be written as $y = \underset{\text{Slope}}{\underset{\uparrow}{1}}x + \underset{y\text{-intercept } (0,\,-8)}{\underset{\uparrow}{(-8)}}.$

**(c)** $y = 6x$ can be written as $y = \underset{\text{Slope}}{\underset{\uparrow}{6}}x + \underset{y\text{-intercept } (0,\,0)}{\underset{\uparrow}{0}}.$

**(d)** $y = \frac{x}{4} - \frac{3}{4}$ can be written as $y = \underset{\text{Slope}}{\underset{\uparrow}{\frac{1}{4}}}x + \underset{y\text{-intercept } \left(0,\,-\frac{3}{4}\right)}{\underset{\uparrow}{\left(-\frac{3}{4}\right)}}.$

NOW TRY

**NOW TRY ANSWERS**

1. **(a)** slope: $-\frac{3}{5}$; $y$-intercept: $(0, -9)$
   **(b)** slope: $-\frac{1}{3}$; $y$-intercept: $\left(0, \frac{7}{3}\right)$

Given the slope and $y$-intercept of a line, we can write an equation of the line.

**NOW TRY EXERCISE 2**

Write an equation of the line with slope $-4$ and $y$-intercept $(0, 2)$.

## EXAMPLE 2 Writing an Equation of a Line

Write an equation of the line with slope $\frac{2}{3}$ and $y$-intercept $(0, -1)$.

Here, $m = \frac{2}{3}$ and $b = -1$, so the equation is

$$y = \overset{\substack{\text{Slope}\\ \downarrow}}{m}x + \overset{\substack{y\text{-intercept is }(0,\, b).\\ \downarrow}}{b} \qquad \text{Slope-intercept form}$$

$$y = \frac{2}{3}x + (-1), \quad \text{or} \quad y = \frac{2}{3}x - 1.$$

NOW TRY

**OBJECTIVE 2** **Graph a line by using its slope and a point on the line.** We can use the slope and $y$-intercept to graph a line.

### Graphing a Line by Using the Slope and $y$-Intercept

***Step 1*** Write the equation in slope-intercept form, if necessary, by solving for $y$.

***Step 2*** Identify the $y$-intercept. Graph the point $(0, b)$.

***Step 3*** Identify slope $m$ of the line. Use the geometric interpretation of slope ("rise over run") to find another point on the graph by counting from the $y$-intercept.

***Step 4*** Join the two points with a line to obtain the graph. (If desired, obtain a third point, such as the $x$-intercept, as a check.)

## EXAMPLE 3 Graphing Lines by Using Slopes and $y$-intercepts

Graph the equation of each line by using the slope and $y$-intercept.

**(a)** $y = \frac{2}{3}x - 1$

***Step 1*** The equation is in slope-intercept form.

$$y = \underset{\substack{\uparrow\\ \text{Slope}}}{\frac{2}{3}}x - \underset{\substack{\uparrow\\ \text{Value of } b \text{ in } y\text{-intercept } (0,\, b)}}{1}$$

***Step 2*** The $y$-intercept is $(0, -1)$. Graph this point. See **FIGURE 30**.

***Step 3*** The slope is $\frac{2}{3}$. By the definition of slope,

$$m = \frac{\text{change in } y \text{ (rise)}}{\text{change in } x \text{ (run)}} = \frac{2}{3}.$$

From the $y$-intercept, count up 2 units and to the right 3 units to obtain the point $(3, 1)$.

***Step 4*** Draw the line through the points $(0, -1)$ and $(3, 1)$ to obtain the graph in **FIGURE 30**.

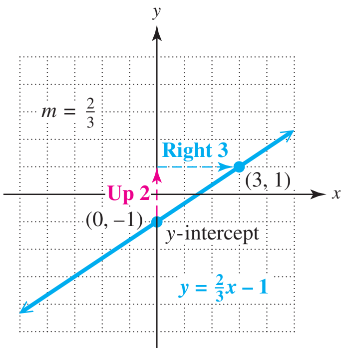


**FIGURE 30**

**NOW TRY ANSWER**

**2.** $y = -4x + 2$

**NOW TRY EXERCISE 3**

Graph $3x + 2y = 8$ by using the slope and $y$-intercept.

**(b)** $3x + 4y = 8$

***Step 1*** Solve for $y$ to write the equation in slope-intercept form.

Isolate $y$ on one side.

$$\begin{aligned} 3x + 4y &= 8 \\ 4y &= -3x + 8 && \text{Subtract } 3x. \\ \text{Slope-intercept form} \longrightarrow y &= -\frac{3}{4}x + 2 && \text{Divide by 4.} \end{aligned}$$

***Step 2*** The $y$-intercept is $(0, 2)$. Graph this point. See **FIGURE 31**.

***Step 3*** The slope is $-\frac{3}{4}$, which can be written as either $\frac{-3}{4}$ or $\frac{3}{-4}$. We use $\frac{-3}{4}$ here.

$$m = \frac{\text{change in } y \text{ (rise)}}{\text{change in } x \text{ (run)}} = \frac{-3}{4}$$

From the $y$-intercept, count *down* 3 units (because of the negative sign) and to the right 4 units, to obtain the point $(4, -1)$.

***Step 4*** Draw the line through the two points $(0, 2)$ and $(4, -1)$ to obtain the graph in **FIGURE 31**.

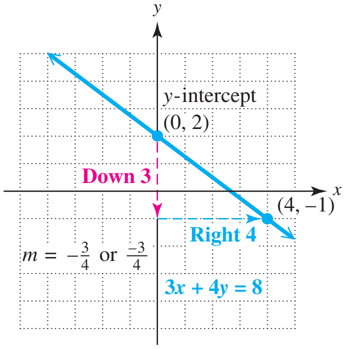


**FIGURE 31**

NOW TRY

**NOTE** In Step 3 of **Example 3(b),** we could use $\frac{3}{-4}$ for the slope. From the $y$-intercept, count up 3 units and to the *left* 4 units (because of the negative sign) to obtain the point $(-4, 5)$. Confirm that this produces the same line.

**NOW TRY EXERCISE 4**

Graph the line through $(-3, -4)$ with slope $\frac{5}{2}$.

### EXAMPLE 4 Graphing a Line by Using the Slope and a Point

Graph the line through $(-2, 3)$ with slope $-4$.

First, locate the point $(-2, 3)$. Write the slope as

$$m = \frac{\text{change in } y \text{ (rise)}}{\text{change in } x \text{ (run)}} = -4 = \frac{-4}{1}.$$

Locate another point on the line by counting *down* 4 units and then to the right 1 unit. Finally, draw the line through this new point $P$ and the given point $(-2, 3)$. See **FIGURE 32**.

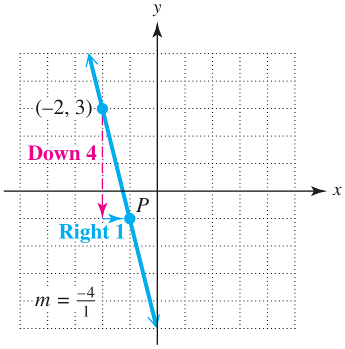


**FIGURE 32**

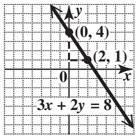
NOW TRY

**NOTE** In **Example 4,** we could have written the slope as $\frac{4}{-1}$ instead. Verify that this produces the same line.

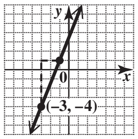
**OBJECTIVE 3 Write an equation of a line by using its slope and any point on the line.** We can use the slope-intercept form to write the equation of a line if we know the slope and any point on the line.

**NOW TRY ANSWERS**

3.



4.



NOW TRY EXERCISE 5

Write an equation, in slope-intercept form, of the line having slope 3 and passing through the point $(-2, 1)$.

## EXAMPLE 5 Using the Slope-Intercept Form to Write an Equation

Write an equation, in slope-intercept form, of the line having slope 4 passing through the point $(2, 5)$.

Since the line passes through the point $(2, 5)$, we can substitute $x = 2$, $y = 5$, and the given slope $m = 4$ into $y = mx + b$ and solve for $b$.

$$\begin{aligned} y &= mx + b && \text{Slope-intercept form} \\ 5 &= 4(2) + b && \text{Let } x = 2,\ y = 5, \text{ and } m = 4. \\ 5 &= 8 + b && \text{Multiply.} \\ -3 &= b && \text{Subtract 8.} \end{aligned}$$

(0, $b$) is the $y$-intercept. Don't stop here.

Now substitute the values of $m$ and $b$ into slope-intercept form.

$$\begin{aligned} y &= mx + b && \text{Slope-intercept form} \\ y &= 4x - 3 && m = 4 \text{ and } b = -3 \end{aligned}$$

NOW TRY

There is another form that can be used to write the equation of a line. To develop this form, let $m$ represent the slope of a line and let $(x_1, y_1)$ represent a given point on the line. Let $(x, y)$ represent any other point on the line. See **FIGURE 33**. Then,

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} && \text{Definition of slope} \\ m(x - x_1) &= y - y_1 && \text{Multiply each side by } x - x_1. \\ y - y_1 &= m(x - x_1). && \text{Rewrite.} \end{aligned}$$

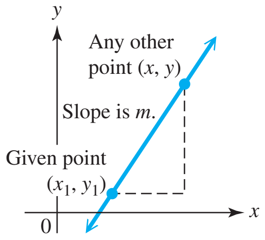


**FIGURE 33**

This result is the *point-slope form* of the equation of a line.

### Point-Slope Form

The **point-slope form** of the equation of a line with slope $m$ passing through the point $(x_1, y_1)$ is

Slope

$$\boldsymbol{y - y_1 = m(x - x_1).}$$

Given point

## EXAMPLE 6 Using the Point-Slope Form to Write Equations

Write an equation of each line. Give the final answer in slope-intercept form.

**(a)** Through $(-2, 4)$, with slope $-3$

The given point is $(-2, 4)$ so $x_1 = -2$ and $y_1 = 4$. Also, $m = -3$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 4 &= -3[x - (-2)] && \text{Let } y_1 = 4,\ m = -3,\ x_1 = -2. \\ y - 4 &= -3(x + 2) && \text{Definition of subtraction} \\ y - 4 &= -3x - 6 && \text{Distributive property} \\ y &= -3x - 2 && \text{Add 4.} \end{aligned}$$

Only $y_1$, $m$, and $x_1$ are replaced with numbers.

NOW TRY ANSWER

**5.** $y = 3x + 7$

NOW TRY EXERCISE 6

Write an equation of the line through $(3, -1)$, with slope $-\frac{2}{5}$. Give the final answer in slope-intercept form.

**(b)** Through $(4, 2)$, with slope $\frac{3}{5}$

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point-slope form} \\
y - 2 &= \frac{3}{5}(x - 4) && \text{Let } y_1 = 2,\ m = \tfrac{3}{5},\ x_1 = 4. \\
y - 2 &= \frac{3}{5}x - \frac{12}{5} && \text{Distributive property} \\
y &= \frac{3}{5}x - \frac{12}{5} + \frac{10}{5} && \text{Add } 2 = \tfrac{10}{5} \text{ to each side.} \\
y &= \frac{3}{5}x - \frac{2}{5} && \text{Combine like terms.}
\end{aligned}$$

NOW TRY

## OBJECTIVE 4 Write an equation of a line by using two points on the line.

Many of the linear equations in **Sections 3.1–3.3** were given in the form

$$\boldsymbol{Ax + By = C,} \qquad \text{Standard form}$$

called **standard form**, where $A$, $B$, and C are real numbers and $A$ and $B$ are not both 0. In most cases, $A$, $B$, and $C$ are rational numbers. For consistency in this book, we give answers so that $A$, $B$, and $C$ are integers with greatest common factor 1 and $A \geq 0$.

---

**NOTE** The definition of standard form is not the same in all texts. A linear equation can be written in many different, equally correct, ways. For example,

$$3x + 4y = 12, \quad 6x + 8y = 24, \quad \text{and} \quad -9x - 12y = -36$$

all represent the same set of ordered pairs. When giving answers in standard form, let us agree that $3x + 4y = 12$ is preferable to the other forms because the greatest common factor of 3, 4, and 12 is 1 and $A \geq 0$.

---

NOW TRY EXERCISE 7

Write an equation of the line through the points $(4, 1)$ and $(6, -2)$. Give the final answer in

**(a)** slope-intercept form and

**(b)** standard form.

### EXAMPLE 7 Writing the Equation of a Line by Using Two Points

Write an equation of the line through the points $(-2, 5)$ and $(3, 4)$. Give the final answer in slope-intercept form and then in standard form.

First, find the slope of the line, using the slope formula.

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 4}{-2 - 3} = \frac{1}{-5} = -\frac{1}{5}$$

Now use either $(-2, 5)$ or $(3, 4)$ and either slope-intercept or point-slope form.

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{We choose } (3, 4) \text{ and point-slope form.} \\
y - 4 &= -\frac{1}{5}(x - 3) && \text{Let } y_1 = 4,\ m = -\tfrac{1}{5},\ x_1 = 3. \\
y - 4 &= -\frac{1}{5}x + \frac{3}{5} && \text{Distributive property} \\
y &= -\frac{1}{5}x + \frac{3}{5} + \frac{20}{5} && \text{Add } 4 = \tfrac{20}{5} \text{ to each side.} \\
\text{Slope-intercept form} \longrightarrow y &= -\frac{1}{5}x + \frac{23}{5} && \text{Combine like terms.} \\
5y &= -x + 23 && \text{Multiply by 5 to clear fractions.} \\
\text{Standard form} \longrightarrow x + 5y &= 23 && \text{Add } x.
\end{aligned}$$

NOW TRY

NOW TRY ANSWERS

6. $y = -\frac{2}{5}x + \frac{1}{5}$

7. (a) $y = -\frac{3}{2}x + 7$

(b) $3x + 2y = 14$

**NOTE** In **Example 7,** the same result would be found by using $(-2, 5)$ for $(x_1, y_1)$. We could also substitute the slope and either given point in slope-intercept form $y = mx + b$ and then solve for $b$, as in **Example 5.**

A summary of the forms of linear equations follows.

**Forms of Linear Equations**

| *Equation* | *Description* | *Example* |
|---|---|---|
| $x = a$ | **Vertical line**<br>Slope is undefined.<br>$x$-intercept is $(a, 0)$. | $x = 3$ |
| $y = b$ | **Horizontal line**<br>Slope is 0.<br>$y$-intercept is $(0, b)$. | $y = 3$ |
| $y = mx + b$ | **Slope-intercept form**<br>Slope is $m$.<br>$y$-intercept is $(0, b)$. | $y = \frac{3}{2}x - 6$ |
| $y - y_1 = m(x - x_1)$ | **Point-slope form**<br>Slope is $m$.<br>Line passes through $(x_1, y_1)$. | $y + 3 = \frac{3}{2}(x - 2)$ |
| $Ax + By = C$ | **Standard form**<br>Slope is $-\frac{A}{B}$.<br>$x$-intercept is $\left(\frac{C}{A}, 0\right)$.<br>$y$-intercept is $\left(0, \frac{C}{B}\right)$. | $3x - 2y = 12$ |

**NOTE** Slope-intercept form is an especially useful form for a linear equation because of the information we can determine from it. It is also the form used by graphing calculators and the one that describes *a linear function.*

**OBJECTIVE 5** **Write an equation of a line that fits a data set.** If a given set of data fits a linear pattern—that is, if its graph consists of points lying close to a straight line—we can write a linear equation that models the data.

### EXAMPLE 8 Writing an Equation of a Line That Describes Data

The table lists the average annual cost (in dollars) of tuition and fees for in-state students at public 4-year colleges and universities for selected years. Year 1 represents 2001, year 3 represents 2003, and so on. Plot the data and write an equation that approximates it.

| *Year* | *Cost (in dollars)* |
|---|---|
| 1 | 3766 |
| 3 | 4645 |
| 5 | 5491 |
| 7 | 6185 |

*Source:* The College Board.

Letting $y$ represent the cost in year $x$, we plot the data as shown in **FIGURE 34** on the next page.

**NOW TRY EXERCISE 8**

Use the points $(3, 4645)$ and $(5, 5491)$ to write an equation in slope-intercept form that approximates the data of **Example 8.** How well does this equation approximate the cost in 2007?

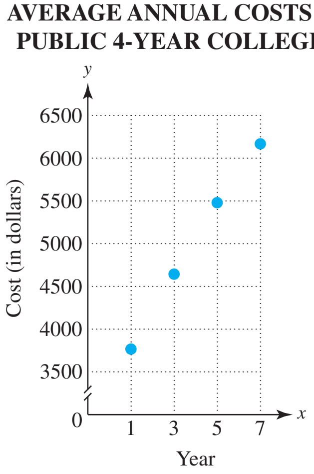


**FIGURE 34**

The points appear to lie approximately in a straight line. We choose the ordered pairs $(5, 5491)$ and $(7, 6185)$ from the table and find the slope of the line through these points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6185 - 5491}{7 - 5} = 347 \qquad \text{Let } (7, 6185) = (x_2, y_2) \text{ and } (5, 5491) = (x_1, y_1).$$

The slope, 347, is positive, indicating that tuition and fees *increased* \$347 each year. Now use this slope and the point $(5, 5491)$ in the slope-intercept form to find an equation of the line.

$$\begin{aligned} y &= mx + b && \text{Slope-intercept form} \\ 5491 &= 347(5) + b && \text{Substitute for } x, y, \text{ and } m. \\ 5491 &= 1735 + b && \text{Multiply.} \\ 3756 &= b && \text{Subtract 1735.} \end{aligned}$$

Thus, $m = 347$ and $b = 3756$, so we can write an equation of the line.

$$y = 347x + 3756$$

To see how well this equation approximates the ordered pairs in the data table, let $x = 3$ (for 2003) and find $y$.

$$\begin{aligned} y &= 347x + 3756 && \text{Equation of the line} \\ y &= 347(3) + 3756 && \text{Substitute 3 for } x. \\ y &= 4797 && \text{Multiply and then add.} \end{aligned}$$

The corresponding value in the table for $x = 3$ is 4645, so the equation approximates the data reasonably well. With caution, the equation could be used to predict values for years that are not included in the table. NOW TRY

**NOTE** In **Example 8,** if we had chosen two different data points, we would have found a slightly different equation.

**NOW TRY ANSWER**

**8.** $y = 423x + 3376$; The equation gives $y = 6337$ when $x = 7$, which approximates the data reasonably well.

## 3.4 EXERCISES

MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Match the description in Column I with the correct equation in Column II.*

| I | II |
|---|---|
| **1.** Slope $= -2$, passes through $(4, 1)$ | **A.** $y = 4x$ |
| **2.** Slope $= -2$, $y$-intercept $(0, 1)$ | **B.** $y = \frac{1}{4}x$ |
| **3.** Passes through $(0, 0)$ and $(4, 1)$ | **C.** $y = -4x$ |
| **4.** Passes through $(0, 0)$ and $(1, 4)$ | **D.** $y = -2x + 1$ |
| | **E.** $y - 1 = -2(x - 4)$ |

***Concept Check*** *Match each equation with the graph in A–D that would most closely resemble its graph.*

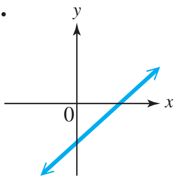
**5.** $y = x + 3$

**6.** $y = -x + 3$

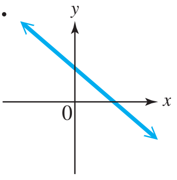
**7.** $y = x - 3$

**8.** $y = -x - 3$

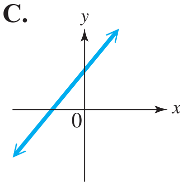
**A.**



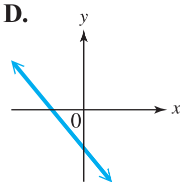
**B.**



**C.**



**D.**



*Identify the slope and y-intercept of the line with each equation.* ***See Example 1.***

**9.** $y = \frac{5}{2}x - 4$

**10.** $y = \frac{7}{3}x - 6$

**11.** $y = -x + 9$

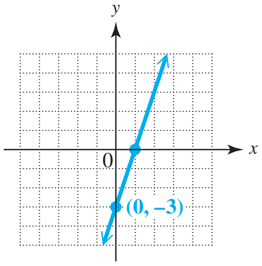
**12.** $y = x + 1$

**13.** $y = \frac{x}{5} - \frac{3}{10}$

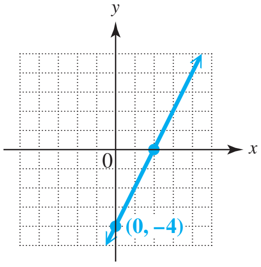
**14.** $y = \frac{x}{7} - \frac{5}{14}$

***Concept Check*** *Use the geometric interpretation of slope (*rise *divided by* run, *from* ***Section 3.3****) to find the slope of each line. Then, by identifying the y-intercept from the graph, write the slope-intercept form of the equation of the line.*

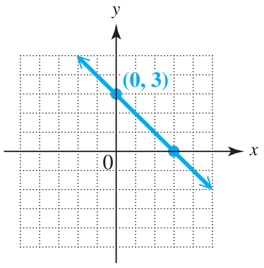
**15.**



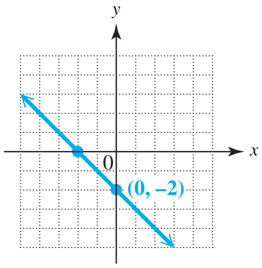
**16.**



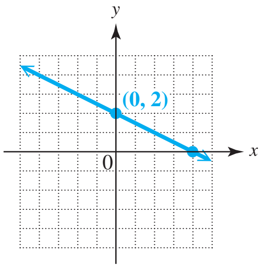
**17.**



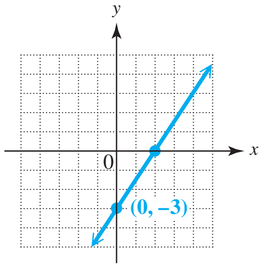
**18.**



**19.**



**20.**



*Write the equation of each line with the given slope and y-intercept.* ***See Example 2.***

**21.** $m = 4, (0, -3)$
**22.** $m = -5, (0, 6)$
**23.** $m = -1, (0, -7)$
**24.** $m = 1, (0, -9)$
**25.** $m = 0, (0, 3)$
**26.** $m = 0, (0, -4)$
**27.** Undefined slope, $(0, -2)$
**28.** Undefined slope, $(0, 5)$

*Graph each equation by using the slope and y-intercept.* ***See Example 3.***

**29.** $y = 3x + 2$
**30.** $y = 4x - 4$
**31.** $y = -\frac{1}{3}x + 4$
**32.** $y = -\frac{1}{2}x + 2$
**33.** $2x + y = -5$
**34.** $3x + y = -2$
**35.** $4x - 5y = 20$
**36.** $6x - 5y = 30$

*Graph each line passing through the given point and having the given slope. (In Exercises 45–48, recall the types of lines having slope 0 and undefined slope.)* ***See Example 4.***

**37.** $(0, 1), m = 4$
**38.** $(0, -5), m = -2$
**39.** $(1, -5), m = -\frac{2}{5}$
**40.** $(2, -1), m = -\frac{1}{3}$
**41.** $(-1, 4), m = \frac{2}{5}$
**42.** $(-2, 2), m = \frac{3}{2}$
**43.** $(0, 0), m = -2$
**44.** $(0, 0), m = -3$
**45.** $(-2, 3), m = 0$
**46.** $(3, 2), m = 0$
**47.** $(2, 4)$, undefined slope
**48.** $(3, -2)$, undefined slope

**49.** ***Concept Check*** What is the common name given to a vertical line whose $x$-intercept is the origin?

**50.** ***Concept Check*** What is the common name given to a line with slope 0 whose $y$-intercept is the origin?

*Write an equation for each line passing through the given point and having the given slope. Give the final answer in slope-intercept form.* ***See Examples 5 and 6.***

**51.** $(4, 1), m = 2$
**52.** $(2, 7), m = 3$
**53.** $(-1, 3), m = -4$
**54.** $(-3, 1), m = -2$
**55.** $(9, 3), m = 1$
**56.** $(8, 4), m = 1$
**57.** $(-4, 1), m = \frac{3}{4}$
**58.** $(2, 1), m = \frac{5}{2}$
**59.** $(-2, 5), m = \frac{2}{3}$
**60.** $(4, 2), m = -\frac{1}{3}$
**61.** $(6, -3), m = -\frac{4}{5}$
**62.** $(7, -2), m = -\frac{7}{2}$

**63.** ***Concept Check*** Which equations are equivalent to $2x - 3y = 6$?

**A.** $y = \frac{2}{3}x - 2$
**B.** $-2x + 3y = -6$
**C.** $y = -\frac{3}{2}x + 3$
**D.** $y - 2 = \frac{2}{3}(x - 6)$

**64.** ***Concept Check*** In the summary box on **page 216,** we give the equations

$$y = \frac{3}{2}x - 6 \quad \text{and} \quad y + 3 = \frac{3}{2}(x - 2)$$

as examples of equations in slope-intercept form and point-slope form, respectively. Write each of these equations in standard form. What do you notice?

*Write an equation for each line passing through the given pair of points. Give the final answer in* **(a)** *slope-intercept form and* **(b)** *standard form.* ***See Example 7.***

**65.** $(4, 10)$ and $(6, 12)$

**66.** $(8, 5)$ and $(9, 6)$

**67.** $(-4, 0)$ and $(0, 2)$

**68.** $(0, -2)$ and $(-3, 0)$

**69.** $(-2, -1)$ and $(3, -4)$

**70.** $(-1, -7)$ and $(-8, -2)$

**71.** $\left(-\frac{2}{3}, \frac{8}{3}\right)$ and $\left(\frac{1}{3}, \frac{7}{3}\right)$

**72.** $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(-\frac{1}{4}, \frac{5}{4}\right)$

*Write an equation of the line satisfying the given conditions. Give the final answer in slope-intercept form.* (Hint: *Recall the relationships among slopes of parallel and perpendicular lines in* ***Section 3.3.***)

**73.** Perpendicular to $x - 2y = 7$; $y$-intercept $(0, -3)$

**74.** Parallel to $5x - y = 10$; $y$-intercept $(0, -2)$

**75.** Through $(2, 3)$; parallel to $4x - y = -2$

**76.** Through $(4, 2)$; perpendicular to $x - 3y = 7$

**77.** Through $(2, -3)$; parallel to $3x = 4y + 5$

**78.** Through $(-1, 4)$; perpendicular to $2x = -3y + 8$

*The cost y of producing x items is, in some cases, expressed as* $y = mx + b$*. The number b gives the* ***fixed cost*** *(the cost that is the same no matter how many items are produced), and the number m is the* ***variable cost*** *(the cost of producing an additional item). Use this information to work Exercises 79 and 80.*

**79.** It costs \$400 to start up a business selling snow cones. Each snow cone costs \$0.25 to produce.

**(a)** What is the fixed cost?

**(b)** What is the variable cost?

**(c)** Write the cost equation.

**(d)** What will be the cost of producing 100 snow cones, based on the cost equation?

**(e)** How many snow cones will be produced if the total cost is \$775?

**80.** It costs \$2000 to purchase a copier, and each copy costs \$0.02 to make.

**(a)** What is the fixed cost?

**(b)** What is the variable cost?

**(c)** Write the cost equation.

**(d)** What will be the cost of producing 10,000 copies, based on the cost equation?

**(e)** How many copies will be produced if the total cost is \$2600?

*Solve each problem.* ***See Example 8.***

**81.** The table lists the average annual cost (in dollars) of tuition and fees at 2-year colleges for selected years, where year 1 represents 2004, year 2 represents 2005, and so on.

| *Year* | *Cost (in dollars)* |
|---|---|
| 1 | 2079 |
| 2 | 2182 |
| 3 | 2272 |
| 4 | 2361 |
| 5 | 2402 |

*Source:* The College Board.

**(a)** Write five ordered pairs from the data.

**(b)** Plot the ordered pairs. Do the points lie approximately in a straight line?

**(c)** Use the ordered pairs $(1, 2079)$ and $(4, 2361)$ to write an equation of a line that approximates the data. Give the final equation in slope-intercept form.

**(d)** Use the equation from part (c) to estimate the average annual cost at 2-year colleges in 2009 to the nearest dollar. (*Hint:* What is the value of $x$ for 2009?)

**82.** The table gives heavy-metal nuclear waste (in thousands of metric tons) from spent reactor fuel stored temporarily at reactor sites, awaiting permanent storage. Let $x = 0$ represent 1995, $x = 5$ represent 2000 (since $2000 - 1995 = 5$), and so on. (*Source:* "Burial of Radioactive Nuclear Waste Under the Seabed," *Scientific American*.)

| *Year x* | *Waste y* |
|---|---|
| 1995 | 32 |
| 2000 | 42 |
| 2010* | 61 |
| 2020* | 76 |

*Estimated by the U.S. Department of Energy.

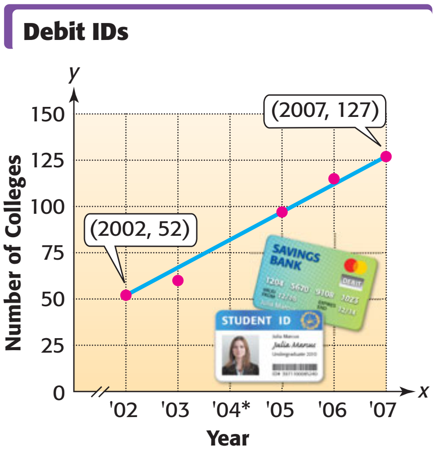
**(a)** For 1995, the ordered pair is $(0, 32)$. Write ordered pairs for the data for the other years given in the table.

**(b)** Plot the ordered pairs $(x, y)$. Do the points lie approximately in a straight line?

**(c)** Use the ordered pairs $(0, 32)$ and $(25, 76)$ to write the equation of a line that approximates the other ordered pairs. Give the equation in slope-intercept form.

**(d)** Use the equation from part (c) to estimate the amount of nuclear waste in 2015. (*Hint:* What is the value of $x$ for 2015?)

*The points on the graph show the number of colleges that teamed up with banks to issue student ID cards which doubled as debit cards from 2002 through 2007. The graph of a linear equation that models the data is also shown.*



***Source:*** CR80News.
* Data for 2004 unavailable.

**83.** Use the ordered pairs shown on the graph to write an equation of the line that models the data. Give the equation in slope-intercept form.

**84.** Use the equation from **Exercise 83** to estimate the number of colleges that teamed up with banks to offer debit IDs in 2004, the year with unavailable data.

## PREVIEW EXERCISES

*Evaluate each expression.* ***See Section 1.2.***

**85.** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

**86.** $3 \cdot 3 \cdot 3$

**87.** $5 \cdot 5 \cdot 5 \cdot 5$

**88.** $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

**89.** $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$

**90.** $\frac{5}{8} \cdot \frac{5}{8}$

## *SUMMARY EXERCISES* on Linear Equations and Graphs

*Graph each line, using the given information or equation.*

**1.** $x - 2y = -4$

**2.** $2x + 3y = 12$

**3.** $m = 1$, $y$-intercept $(0, -2)$

**4.** $y = -2x + 6$

**5.** $m = -\frac{2}{3}$, passes through $(3, -4)$

**6.** Undefined slope, passes through $(-3.5, 0)$

**7.** $x - 4y = 0$

**8.** $y - 4 = -9$

**9.** $8x = 6y + 24$

**10.** $m = 1$, $y$-intercept $(0, -4)$

**11.** $5x + 2y = 10$

**12.** $m = -\frac{3}{4}$, passes through $(4, -4)$

**13.** $m = 0$, passes through $\left(0, \frac{3}{2}\right)$

**14.** $x + 5y = 0$

**15.** $y = -x + 6$

**16.** $4x = 3y - 24$

**17.** $x + 4 = 0$

**18.** $x - 3y = 6$

**19.** ***Concept Check*** Match the description in Column I with the correct equation in Column II.

| I | II |
|---|---|
| **(a)** Slope $-0.5$, $b = -2$ | **A.** $y = -\frac{1}{2}x$ |
| **(b)** $x$-intercept $(4, 0)$, $y$-intercept $(0, 2)$ | **B.** $y = -\frac{1}{2}x - 2$ |
| **(c)** Passes through $(4, -2)$ and $(0, 0)$ | **C.** $x - 2y = 2$ |
| **(d)** $m = \frac{1}{2}$, passes through $(-2, -2)$ | **D.** $x + 2y = 4$ |
| | **E.** $x = 2y$ |

**20.** ***Concept Check*** Which equations are equivalent to $2x + 5y = 20$?

**A.** $y = -\dfrac{2}{5}x + 4$

**B.** $y - 2 = -\dfrac{2}{5}(x - 5)$

**C.** $y = \dfrac{5}{2}x - 4$

**D.** $2x = 5y - 20$

*Write an equation for each line. Give the final answer in slope-intercept form if possible.*

**21.** $m = -3$, $b = -6$

**22.** $m = \frac{3}{2}$, through $(-4, 6)$

**23.** Through $(1, -7)$ and $(-2, 5)$

**24.** Through $(0, 0)$ and $(5, 3)$

**25.** Through $(0, 0)$, undefined slope

**26.** Through $(3, 0)$ and $(0, -3)$

**27.** Through $(0, 0)$ and $(3, 2)$

**28.** $m = -2$, $b = -4$

**29.** Through $(5, 0)$ and $(0, -5)$

**30.** Through $(0, 0)$, $m = 0$

**31.** $m = \frac{5}{3}$, through $(-3, 0)$

**32.** Through $(1, -13)$ and $(-2, 2)$

STUDY SKILLS

# Analyzing Your Test Results

***An exam is a learning opportunity—learn from your mistakes.*** After a test is returned, do the following:

- **Note what you got wrong and why you had points deducted.**
- **Figure out how to solve the problems you missed.** Check your textbook or notes, or ask your instructor. Rework the problems correctly.
- **Keep all quizzes and tests that are returned to you.** Use them to study for future tests and the final exam.

## Typical Reasons for Errors on Math Tests

**These are test taking errors.** They are easy to correct if you read carefully, show all your work, proofread, and double-check units and labels.

1. You read the directions wrong.
2. You read the question wrong or skipped over something.
3. You made a computation error.
4. You made a careless error. (For example, you incorrectly copied a correct answer onto a separate answer sheet.)
5. Your answer is not complete.
6. You labeled your answer wrong. (For example, you labeled an answer “ft” instead of “ft$^2$.”)
7. You didn’t show your work.

**These are test preparation errors.** You must practice the kinds of problems that you will see on tests.

8. You didn’t understand a concept.
9. You were unable to set up the problem (in an application).
10. You were unable to apply a procedure.

***Below are sample charts for tracking your test taking progress. Use them to find out if you tend to make certain kinds of errors on tests. Check the appropriate box when you’ve made an error in a particular category.***

**Test Taking Errors**

| Test | Read directions wrong | Read question wrong | Computation error | Not exact or accurate | Not complete | Labeled wrong | Didn’t show work |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | | | | | | | |

**Test Preparation Errors**

| Test | Didn’t understand concept | Didn’t set up problem correctly | Couldn’t apply concept to new situation |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |

***What will you do to avoid these kinds of errors on your next test?***

# CHAPTER 3 SUMMARY

## KEY TERMS

**3.1**

line graph
linear equation in two variables
ordered pair
table of values
$x$-axis
$y$-axis
origin
rectangular (Cartesian) coordinate system
quadrant
plane
coordinates
plot
scatter diagram

**3.2**

graph, graphing
$y$-intercept
$x$-intercept
horizontal line
vertical line

**3.3**

rise
run
slope
subscript notation
parallel lines
perpendicular lines

## NEW SYMBOLS

| | |
|---|---|
| $(a, b)$ | an ordered pair |
| $m$ | slope |
| $(x_1, y_1)$ | $x$-sub-one, $y$-sub-one |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **linear equation in two variables** is an equation that can be written in the form
- **A.** $Ax + By < C$
- **B.** $ax = b$
- **C.** $y = x^2$
- **D.** $Ax + By = C$.

**2.** An **ordered pair** is a pair of numbers written
- **A.** in numerical order between brackets
- **B.** between parentheses or brackets
- **C.** between parentheses in which order is important
- **D.** between parentheses in which order does not matter.

**3.** An **intercept** is
- **A.** the point where the $x$-axis and $y$-axis intersect
- **B.** a pair of numbers written in parentheses in which order matters
- **C.** one of the four regions determined by a rectangular coordinate system
- **D.** the point where a graph intersects the $x$-axis or the $y$-axis.

**4.** The **slope** of a line is
- **A.** the measure of the run over the rise of the line
- **B.** the distance between two points on the line
- **C.** the ratio of the change in $y$ to the change in $x$ along the line
- **D.** the horizontal change compared with the vertical change of two points on the line.

**5.** Two lines in a plane are **parallel** if
- **A.** they represent the same line
- **B.** they never intersect
- **C.** they intersect at a 90° angle
- **D.** one has a positive slope and one has a negative slope.

**6.** Two lines in a plane are **perpendicular** if
- **A.** they represent the same line
- **B.** they never intersect
- **C.** they intersect at a 90° angle
- **D.** one has a positive slope and one has a negative slope.

***ANSWERS***

**1.** D; *Examples:* $3x + 2y = 6$, $x = y - 7$, $y = 4x$ **2.** C; *Examples:* $(0, 3)$, $(-3, 8)$, $(4, 0)$ **3.** D; *Example:* The graph of the equation $4x - 3y = 12$ has $x$-intercept $(3, 0)$ and $y$-intercept $(0, -4)$. **4.** C; *Example:* The line through $(3, 6)$ and $(5, 4)$ has slope $\frac{4-6}{5-3} = \frac{-2}{2} = -1$. **5.** B; *Example:* See **FIGURE 27** in **Section 3.3.** **6.** C; *Example:* See **FIGURE 28** in **Section 3.3.**

## QUICK REVIEW

| *CONCEPTS* | *EXAMPLES* |
|---|---|

## 3.1 Linear Equations in Two Variables; The Rectangular Coordinate System

An ordered pair is a solution of an equation if it satisfies the equation.

Is $(2, -5)$ or $(0, -6)$ a solution of $4x - 3y = 18$?

$4(2) - 3(-5) = 23 \neq 18$

$(2, -5)$ is not a solution.

$4(0) - 3(-6) = 18$

$(0, -6)$ is a solution.

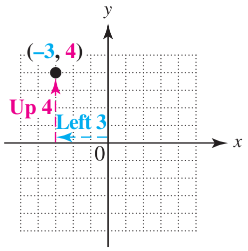
If a value of either variable in an equation is given, then the other variable can be found by substitution.

Complete the ordered pair $(0, \_\_)$ for $3x = y + 4$.

$$
\begin{aligned}
3(0) &= y + 4 && \text{Let } x = 0. \\
0 &= y + 4 && \text{Multiply.} \\
-4 &= y && \text{Subtract 4.}
\end{aligned}
$$

The ordered pair is $(0, -4)$.

Plot the ordered pair $(-3, 4)$ by starting at the origin, moving 3 units to the left, and then moving 4 units up.



## 3.2 Graphing Linear Equations in Two Variables

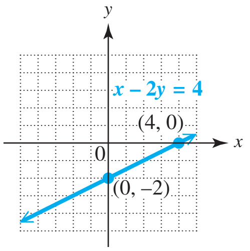
To graph a linear equation, follow these steps.

***Step 1*** Find at least two ordered pairs that satisfy the equation.

***Step 2*** Plot the corresponding points.

***Step 3*** Draw a straight line through the points.

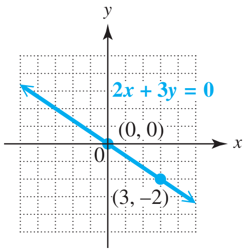
| x | y |
|---|---|
| 0 | −2 |
| 4 | 0 |

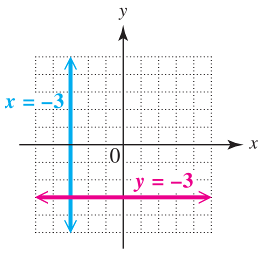


The graph of $Ax + By = 0$ passes through the origin. Find and plot another point that satisfies the equation. Then draw the line through the two points.

The graph of $y = b$ is a horizontal line through $(0, b)$.

The graph of $x = a$ is a vertical line through $(a, 0)$.





(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **3.3 The Slope of a Line** | |
| The slope of the line through $(x_1, y_1)$ and $(x_2, y_2)$ is<br>$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$ (where $x_1 \neq x_2$). | The line through $(-2, 3)$ and $(4, -5)$ has slope as follows.<br>$m = \frac{-5 - 3}{4 - (-2)} = \frac{-8}{6} = -\frac{4}{3}$ |
| **Horizontal lines have slope 0.** | The line $y = -2$ has slope $0$. |
| **Vertical lines have undefined slope.** | The line $x = 4$ has undefined slope. |
| To find the slope of a line from its equation, solve for $y$. The slope is the coefficient of $x$. | Find the slope of $3x - 4y = 12$.<br>$-4y = -3x + 12$ Add $-3x$.<br>$y = \frac{3}{4}x - 3$ Divide by $-4$.<br>Slope |
| Parallel lines have the same slope. | The lines $y = 3x - 1$ and $y = 3x + 4$ are parallel because both have slope $3$. |
| The slopes of perpendicular lines are negative reciprocals (that is, their product is $-1$). | The lines $y = -3x - 1$ and $y = \frac{1}{3}x + 4$ are perpendicular because their slopes are $-3$ and $\frac{1}{3}$, and $-3\left(\frac{1}{3}\right) = -1$. |
| **3.4 Writing and Graphing Equations of Lines** | |
| **Slope-Intercept Form**<br>$y = mx + b$<br>$m$ is the slope.<br>$(0, b)$ is the $y$-intercept. | Write an equation of the line with slope $2$ and $y$-intercept $(0, -5)$.<br>$y = 2x - 5$ |
| **Point-Slope Form**<br>$y - y_1 = m(x - x_1)$<br>$m$ is the slope.<br>$(x_1, y_1)$ is a point on the line. | Write an equation of the line with slope $-\frac{1}{2}$ through $(-4, 5)$.<br>$y - 5 = -\frac{1}{2}[x - (-4)]$ Substitute.<br>$y - 5 = -\frac{1}{2}(x + 4)$ Definition of subtraction<br>$y - 5 = -\frac{1}{2}x - 2$ Distributive property<br>$y = -\frac{1}{2}x + 3$ Add 5. |
| **Standard Form**<br>$Ax + By = C$<br>$A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0. | The equation $y = -\frac{1}{2}x + 3$ is written in standard form as<br>$x + 2y = 6$,<br>with $A = 1$, $B = 2$, and $C = 6$. |

# CHAPTER 3

# REVIEW EXERCISES

## 3.1

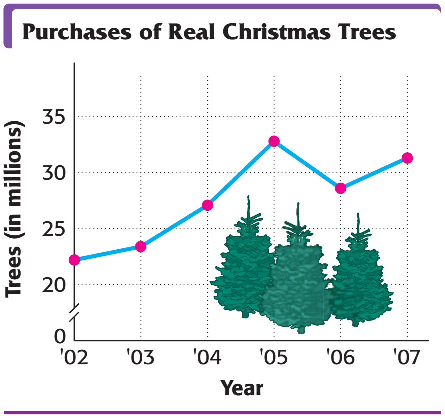
**1.** The line graph shows the number, in millions, of real Christmas trees purchased for the years 2002 through 2007.

**(a)** Between which years did the number of real trees purchased increase?

**(b)** Between which years did the number of real trees purchased decrease?

**(c)** Estimate the number of real trees purchased in 2005 and 2006.

**(d)** By about how much did the number of real trees purchased between 2005 and 2006 decrease?



***Source:*** National Christmas Tree Association.

*Complete the given ordered pairs for each equation.*

**2.** $y = 3x + 2$; $(-1, \_\_), (0, \_\_), (\_\_, 5)$

**3.** $4x + 3y = 6$; $(0, \_\_), (\_\_, 0), (-2, \_\_)$

**4.** $x = 3y$; $(0, \_\_), (8, \_\_), (\_\_, -3)$

**5.** $x - 7 = 0$; $(\_\_, -3), (\_\_, 0), (\_\_, 5)$

*Determine whether the given ordered pair is a solution of the given equation.*

**6.** $x + y = 7$; $(2, 5)$ **7.** $2x + y = 5$; $(-1, 3)$ **8.** $3x - y = 4$; $\left(\frac{1}{3}, -3\right)$

*Name the quadrant in which each ordered pair lies. Then plot each pair in a rectangular coordinate system.*

**9.** $(2, 3)$ **10.** $(-4, 2)$ **11.** $(3, 0)$ **12.** $(0, -6)$

**13.** ***Concept Check*** If $xy > 0$, in what quadrant or quadrants must $(x, y)$ lie?

## 3.2

*Find the x- and y-intercepts for the line that is the graph of each equation, and graph the line.*

**14.** $y = 2x + 5$ **15.** $3x + 2y = 8$ **16.** $x + 2y = -4$

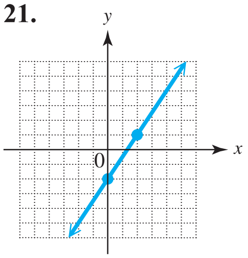
## 3.3

*Find the slope of each line.*

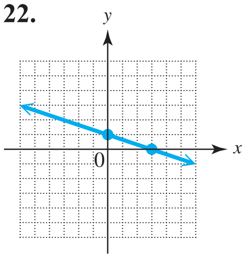
**17.** Through $(2, 3)$ and $(-4, 6)$ **18.** Through $(2, 5)$ and $(2, 8)$

**19.** $y = 3x - 4$ **20.** $y = 5$

**21.**



**22.**



**23.** The line passing through these points

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 2 | 4 |
| 6 | 10 |

**24.** **(a)** A line parallel to the graph of $y = 2x + 3$

**(b)** A line perpendicular to the graph of $y = -3x + 3$

*Decide whether each pair of lines is* parallel, perpendicular, *or* neither.

**25.** $3x + 2y = 6$
$6x + 4y = 8$

**26.** $x - 3y = 1$
$3x + y = 4$

**27.** $x - 2y = 8$
$x + 2y = 8$

**3.4** *Write an equation for each line. Give the final answer in slope-intercept form if possible.*

**28.** $m = -1, b = \frac{2}{3}$

**29.** Through $(2, 3)$ and $(-4, 6)$

**30.** Through $(4, -3), m = 1$

**31.** Through $(-1, 4), m = \frac{2}{3}$

**32.** Through $(1, -1), m = -\frac{3}{4}$

**33.** $m = -\frac{1}{4}, b = \frac{3}{2}$

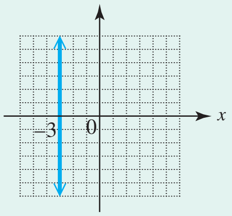
**34.** Slope 0, through $(-4, 1)$

**35.** Through $\left(\frac{1}{3}, -\frac{5}{4}\right)$, undefined slope

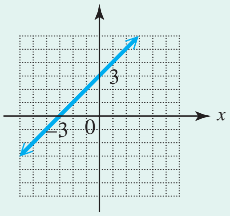
## MIXED REVIEW EXERCISES

***Concept Check*** *In Exercises 36–41, match each statement to the appropriate graph or graphs in A–D. Graphs may be used more than once.*

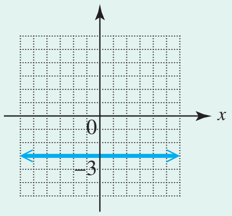
**A.**



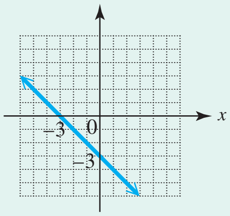
**B.**



**C.**



**D.**



**36.** The line shown in the graph has undefined slope.

**37.** The graph of the equation has $y$-intercept $(0, -3)$.

**38.** The graph of the equation has $x$-intercept $(-3, 0)$.

**39.** The line shown in the graph has negative slope.

**40.** The graph is that of the equation $y = -3$.

**41.** The line shown in the graph has slope 1.

*Find the intercepts and the slope of each line. Then graph the line.*

**42.** $y = -2x - 5$

**43.** $x + 3y = 0$

**44.** $y - 5 = 0$

*Write an equation for each line. Give the final answer in slope-intercept form.*

**45.** $m = -\frac{1}{4}, b = -\frac{5}{4}$

**46.** Through $(8, 6), m = -3$

**47.** Through $(3, -5)$ and $(-4, -1)$

**48.** Slope 0, through $(5, -5)$

# CHAPTER 3 TEST

Chapter Test Prep Videos

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

**1.** Complete the ordered pairs $(0, \_\_)$, $(\_\_, 0)$, $(\_\_, -3)$ for the equation $3x + 5y = -30$.

**2.** Is $(4, -1)$ a solution of $4x - 7y = 9$?

**3.** How do you find the $x$-intercept of the graph of a linear equation in two variables? How do you find the $y$-intercept?

*Graph each linear equation. Give the x- and y-intercepts.*

**4.** $3x + y = 6$

**5.** $y - 2x = 0$

**6.** $x + 3 = 0$

**7.** $y = 1$

**8.** $x - y = 4$

*Find the slope of each line.*

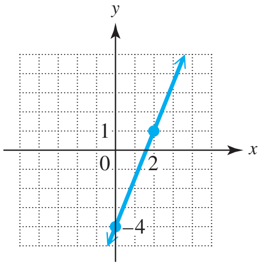
**9.** Through $(-4, 6)$ and $(-1, -2)$

**10.** $2x + y = 10$

**11.** $x + 12 = 0$

**12.** A line parallel to the graph of $y - 4 = 6$

**13.**



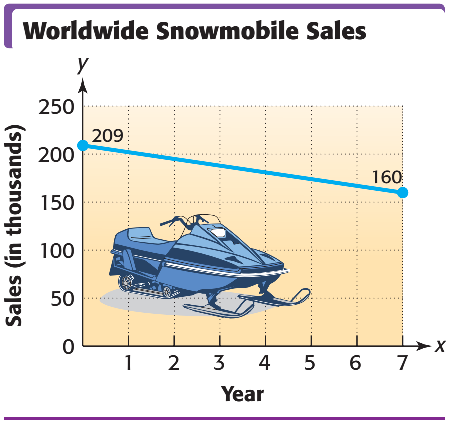
*Write an equation for each line. Give the final answer in slope-intercept form.*

**14.** Through $(-1, 4)$, $m = 2$

**15.** The line in **Exercise 13**

**16.** Through $(2, -6)$ and $(1, 3)$

*The graph shows worldwide snowmobile sales from 2000 through 2007, where 2000 corresponds to x = 0. Use the graph to work Exercises 17–20.*



**Source:** www.snowmobile.org

**17.** Is the slope of the line in the graph positive or negative? Explain.

**18.** Write two ordered pairs for the data points shown in the graph. Use them to write an equation of a line that models the data. Give the equation in slope-intercept form.

**19.** Use the equation from **Exercise 18** to approximate worldwide snowmobile sales for 2005. How does your answer compare to the actual sales of 173.7 thousand?

**20.** What does the ordered pair $(7, 160)$ mean in the context of this problem?

# CHAPTERS 1–3 CUMULATIVE REVIEW EXERCISES

*Perform each indicated operation.*

**1.** $10\frac{5}{8} - 3\frac{1}{10}$

**2.** $\frac{3}{4} \div \frac{1}{8}$

**3.** $5 - (-4) + (-2)$

**4.** $\frac{(-3)^2 - (-4)(2^4)}{5(2) - (-2)^3}$

**5.** *True* or *false*? $\frac{4(3 - 9)}{2 - 6} \geq 6$

**6.** Find the value of $xz^3 - 5y^2$ when $x = -2$, $y = -3$, and $z = -1$.

**7.** What property does $3(-2 + x) = -6 + 3x$ illustrate?

**8.** Simplify $-4p - 6 + 3p + 8$ by combining like terms.

*Solve.*

**9.** $V = \frac{1}{3}\pi r^2 h$ for $h$

**10.** $6 - 3(1 + x) = 2(x + 5) - 2$

**11.** $-(m - 3) = 5 - 2m$

**12.** $\frac{x - 2}{3} = \frac{2x + 1}{5}$

*Solve each inequality, and graph the solution set.*

**13.** $-2.5x < 6.5$

**14.** $4(x + 3) - 5x < 12$

**15.** $\frac{2}{3}x - \frac{1}{6}x \leq -2$

*Solve each problem.*

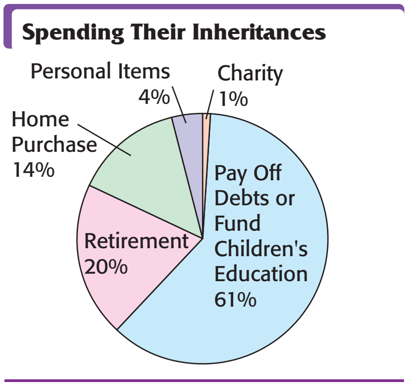
**16.** The gap in average annual earnings by level of education has increased over time. In 2008, the average full-time worker age 25 or over with a bachelor's degree earned \$20,124 more than a full-time worker with only a high school diploma. Together, these two individuals earned a total of \$81,588. How much did the average worker at each level of education earn? (*Source:* U.S. Bureau of Labor Statistics.)

**17.** Baby boomers are expected to inherit \$10.4 trillion from their parents over the next 45 yr, an average of \$50,000 each. The circle graph shows how they plan to spend their inheritances.

**(a)** How much of the \$50,000 is expected to go toward a home purchase?

**(b)** How much is expected to go toward retirement?

**18.** Use the answer from **Exercise 17(b)** to estimate the amount expected to go toward paying off debts or funding children's education.



***Source:*** First Interstate Bank Trust and Private Banking Group.

*Consider the linear equation* $-3x + 4y = 12$. *Find the following.*

**19.** The $x$- and $y$-intercepts

**20.** The slope

**21.** The graph

**22.** Are the lines with equations $x + 5y = -6$ and $y = 5x - 8$ *parallel, perpendicular,* or *neither*?

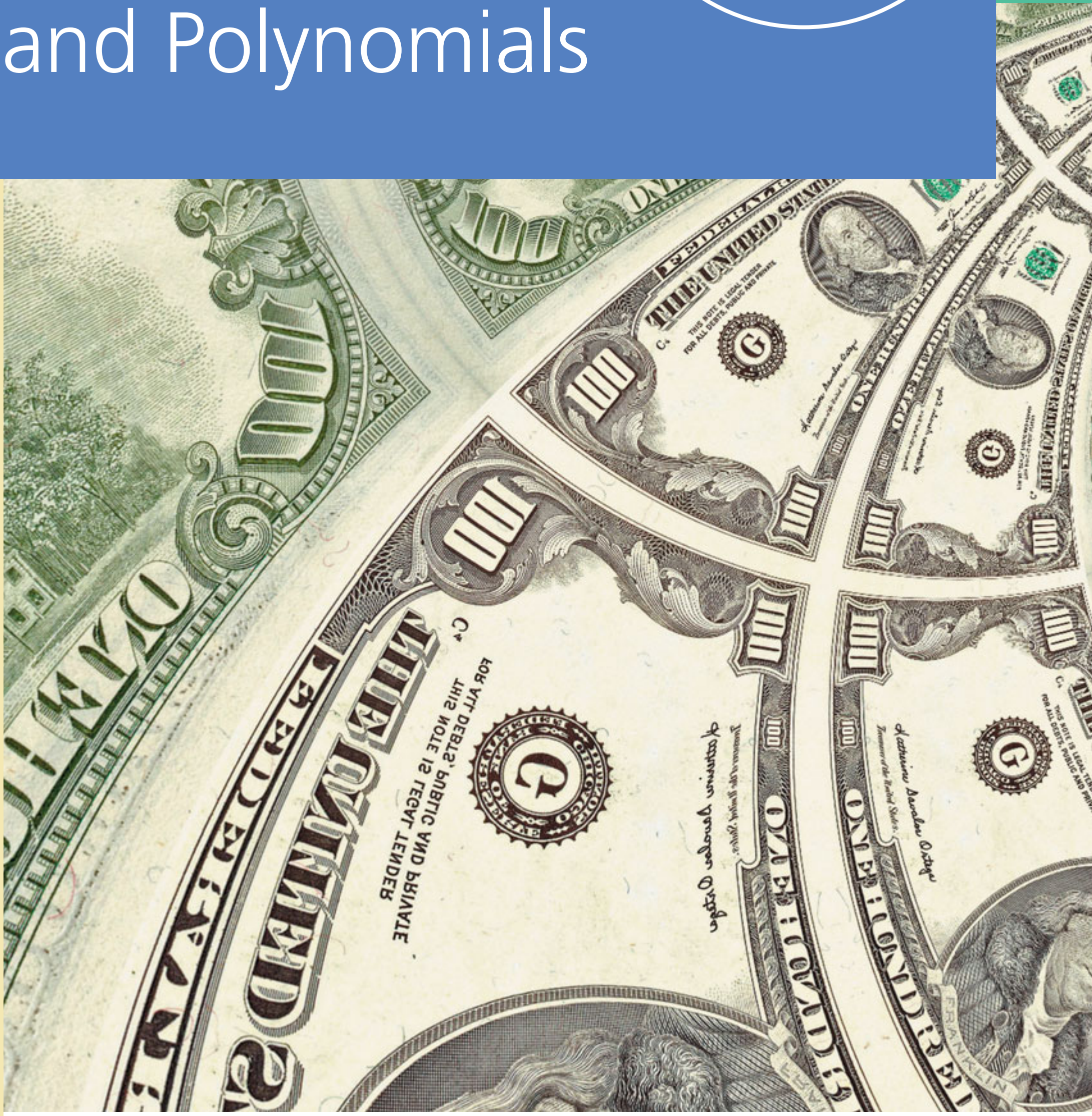
*Write an equation for each line. Give the final answer in slope-intercept form if possible.*

**23.** Through $(2, -5)$, slope 3

**24.** Through $(0, 4)$ and $(2, 4)$

CHAPTER 4

# Exponents and Polynomials



Just how much is a *trillion*? A trillion, written 1,000,000,000,000, is a million million, or a thousand billion. A trillion seconds would last more than 31,000 years—that is, 310 centuries. The U.S. government projects that by 2017, consumers and taxpayers will spend more than $4 trillion on health care, accounting for $1 of every $5 spent. (*Source:* Centers for Medicare and Medicaid Services.)

In **Section 4.3,** we use *exponents* and *scientific notation* to write and calculate with large numbers, such as the national debt, tax revenue, and the distances from Earth to celestial objects.

# 4.1

# The Product Rule and Power Rules for Exponents

## OBJECTIVES

1. Use exponents.
2. Use the product rule for exponents.
3. Use the rule $(a^m)^n = a^{mn}$.
4. Use the rule $(ab)^m = a^m b^m$.
5. Use the rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.
6. Use combinations of rules.
7. Use the rules for exponents in a geometry application.

**OBJECTIVE 1** **Use exponents.** Recall from **Section 1.2** that in the expression $5^2$, the number 5 is the **base** and 2 is the **exponent,** or **power.** The expression $5^2$ is called an **exponential expression.** Although we do not usually write the exponent when it is 1, in general, for any quantity $a$,

$$a^1 = a.$$

### EXAMPLE 1 Using Exponents

Write $3 \cdot 3 \cdot 3 \cdot 3$ in exponential form and evaluate.

Since 3 occurs as a factor four times, the base is 3 and the exponent is 4. The exponential expression is $3^4$, read "3 to the fourth power" or simply "3 to the fourth."

$$\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{\text{4 factors of 3}} = 3^4 = 81$$

NOW TRY

**NOW TRY EXERCISE 1**

Write $4 \cdot 4 \cdot 4$ in exponential form and evaluate.

### EXAMPLE 2 Evaluating Exponential Expressions

Evaluate. Name the base and the exponent.

**(a)** $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$

| Expression | Base | Exponent |
|---|---|---|
| $5^4$ | 5 | 4 |
| $-5^4$ | 5 | 4 |
| $(-5)^4$ | $-5$ | 4 |

The base is 5.

**(b)** $-5^4 = -1 \cdot 5^4 = -1 \cdot (5 \cdot 5 \cdot 5 \cdot 5) = -625$

**(c)** $(-5)^4 = (-5)(-5)(-5)(-5) = 625$

NOW TRY

**NOW TRY EXERCISE 2**

Evaluate. Name the base and the exponent.

**(a)** $(-3)^4$ **(b)** $-3^4$

---

**CAUTION** Note the differences between **Example 2(b) and 2(c).** In $-5^4$, the absence of parentheses shows that the exponent 4 applies only to the base 5, not $-5$. In $(-5)^4$, the parentheses show that the exponent 4 applies to the base $-5$. In summary, $-a^n$ and $(-a)^n$ are not necessarily the same.

| Expression | Base | Exponent | Example |
|---|---|---|---|
| $-a^n$ | $a$ | $n$ | $-3^2 = -(3 \cdot 3) = -9$ |
| $(-a)^n$ | $-a$ | $n$ | $(-3)^2 = (-3)(-3) = 9$ |

---

**OBJECTIVE 2** **Use the product rule for exponents.** To develop the product rule, we use the definition of exponents.

$$2^4 \cdot 2^3 = \overbrace{(2 \cdot 2 \cdot 2 \cdot 2)}^{\text{4 factors}}\overbrace{(2 \cdot 2 \cdot 2)}^{\text{3 factors}}$$

$$= \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{4 + 3 = 7 \text{ factors}}$$

$$= 2^7$$

**NOW TRY ANSWERS**

1. $4^3 = 64$
2. **(a)** 81; $-3$; 4 **(b)** $-81$; 3; 4

Also,
$$\begin{aligned} 6^2 \cdot 6^3 &= (6 \cdot 6)(6 \cdot 6 \cdot 6) \\ &= 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \\ &= 6^5. \end{aligned}$$

Generalizing from these examples, we have

$$2^4 \cdot 2^3 = 2^{4+3} = 2^7 \quad \text{and} \quad 6^2 \cdot 6^3 = 6^{2+3} = 6^5.$$

This suggests the **product rule for exponents.**

### Product Rule for Exponents

For any positive integers $m$ and $n$, $\boldsymbol{a^m \cdot a^n = a^{m+n}}$.
(Keep the same base and add the exponents.)

*Example:* $6^2 \cdot 6^5 = 6^{2+5} = 6^7$

**CAUTION** Do not multiply the bases when using the product rule. ***Keep the same base and add the exponents.*** For example,

$$6^2 \cdot 6^5 = 6^7, \qquad \textit{\textbf{not}} \qquad 36^7.$$

**NOW TRY EXERCISE 3**

Use the product rule for exponents to find each product if possible.

**(a)** $(-5)^2(-5)^4$

**(b)** $y^2 \cdot y \cdot y^5$

**(c)** $(2x^3)(4x^7)$

**(d)** $2^4 \cdot 5^3$

**(e)** $3^2 + 3^3$

### EXAMPLE 3 Using the Product Rule

Use the product rule for exponents to find each product if possible.

**(a)** $6^3 \cdot 6^5 = 6^{3+5} = 6^8$

Keep the same base.

**(b)** $(-4)^7(-4)^2 = (-4)^{7+2} = (-4)^9$

**(c)** $x^2 \cdot x = x^2 \cdot x^1 = x^{2+1} = x^3$

**(d)** $m^4m^3m^5 = m^{4+3+5} = m^{12}$

**(e)** $2^3 \cdot 3^2$

The product rule does not apply, since the bases are different.

$2^3 \cdot 3^2 = 8 \cdot 9 = 72$ Evaluate $2^3$ and $3^2$. Then multiply.

Think: $2^3 = 2 \cdot 2 \cdot 2$ Think: $3^2 = 3 \cdot 3$

**(f)** $2^3 + 2^4$

The product rule does not apply, since this is a *sum,* not a *product*.

$2^3 + 2^4 = 8 + 16 = 24$ Evaluate $2^3$ and $2^4$. Then add.

**(g)** $(2x^3)(3x^7)$ $2x^3$ means $2 \cdot x^3$ and $3x^7$ means $3 \cdot x^7$.

$= (2 \cdot 3) \cdot (x^3 \cdot x^7)$ Commutative and associative properties

$= 6x^{3+7}$ Multiply; product rule

$= 6x^{10}$ Add.

NOW TRY

**CAUTION** Be sure that you understand the difference between *adding* and *multiplying* exponential expressions. For example, consider the following.

$$8x^3 + 5x^3 = (8 + 5)x^3 = 13x^3$$
$$(8x^3)(5x^3) = (8 \cdot 5)x^{3+3} = 40x^6$$

**NOW TRY ANSWERS**

**3.** **(a)** $(-5)^6$ **(b)** $y^8$ **(c)** $8x^{10}$ **(d)** The product rule does not apply; 2000 **(e)** The product rule does not apply; 36

## OBJECTIVE 3 Use the rule $(a^m)^n = a^{mn}$.

Consider the following.

$$(8^3)^2 = (8^3)(8^3) = 8^{3+3} = 8^6 \qquad \text{Product rule for exponents}$$

The product of the exponents in $(8^3)^2$, $3 \cdot 2$, gives the exponent in $8^6$. Also

$$\begin{aligned} (5^2)^4 &= 5^2 \cdot 5^2 \cdot 5^2 \cdot 5^2 && \text{Definition of exponent} \\ &= 5^{2+2+2+2} && \text{Product rule} \\ &= 5^8, && \text{Add the exponents.} \end{aligned}$$

and $2 \cdot 4 = 8$. These examples suggest **power rule (a) for exponents.**

### Power Rule (a) for Exponents

For any positive integers $m$ and $n$, $\quad \boldsymbol{(a^m)^n = a^{mn}}$.
(Raise a power to a power by multiplying exponents.)

*Example:* $(3^2)^4 = 3^{2 \cdot 4} = 3^8$

NOW TRY EXERCISE 4

Simplify.

**(a)** $(4^7)^5$ **(b)** $(y^4)^7$

### EXAMPLE 4 Using Power Rule (a)

Use power rule (a) for exponents to simplify.

**(a)** $(2^5)^3 = 2^{5 \cdot 3} = 2^{15}$ **(b)** $(5^7)^2 = 5^{7(2)} = 5^{14}$ **(c)** $(x^2)^5 = x^{2(5)} = x^{10}$

NOW TRY

## OBJECTIVE 4 Use the rule $(ab)^m = a^m b^m$.

Consider the following.

$$\begin{aligned} (4x)^3 &= (4x)(4x)(4x) && \text{Definition of exponent} \\ &= (4 \cdot 4 \cdot 4)(x \cdot x \cdot x) && \text{Commutative and associative properties} \\ &= 4^3 \cdot x^3 && \text{Definition of exponent} \end{aligned}$$

This example suggests **power rule (b) for exponents.**

### Power Rule (b) for Exponents

For any positive integer $m$, $\quad \boldsymbol{(ab)^m = a^m b^m}$.
(Raise a product to a power by raising each factor to the power.)

*Example:* $(2p)^5 = 2^5 p^5$

### EXAMPLE 5 Using Power Rule (b)

Use power rule (b) for exponents to simplify.

**(a)** $(3xy)^2$

$$\begin{aligned} &= 3^2x^2y^2 && \text{Power rule (b)} \\ &= 9x^2y^2 && 3^2 = 3 \cdot 3 = 9 \end{aligned}$$

**(b)** $5(pq)^2$

$$\begin{aligned} &= 5(p^2q^2) && \text{Power rule (b)} \\ &= 5p^2q^2 && \text{Multiply.} \end{aligned}$$

**(c)** $3(2m^2p^3)^4$

$$\begin{aligned} &= 3[2^4(m^2)^4(p^3)^4] && \text{Power rule (b)} \\ &= 3 \cdot 2^4m^8p^{12} && \text{Power rule (a)} \\ &= 48m^8p^{12} && 3 \cdot 2^4 = 3 \cdot 16 = 48 \end{aligned}$$

NOW TRY ANSWERS

4. **(a)** $4^{35}$ **(b)** $y^{28}$

**NOW TRY EXERCISE 5**

Simplify.

**(a)** $(-5ab)^3$ **(b)** $3(4t^3p^5)^2$

**(d)** $(-5^6)^3$

$= (-1 \cdot 5^6)^3$ $\quad -a = -1 \cdot a$

$= (-1)^3 \cdot (5^6)^3$ $\quad$ Power rule (b)

$= -1 \cdot 5^{18}$ $\quad$ Power rule (a)

$= -5^{18}$

Raise −1 to the designated power.

NOW TRY

**CAUTION** ***Power rule (b) does not apply to a sum.*** For example,

$$(4x)^2 = 4^2x^2, \quad \text{but} \quad (4 + x)^2 \neq 4^2 + x^2.$$

**OBJECTIVE 5** **Use the rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.** Since the quotient $\frac{a}{b}$ can be written as $a\left(\frac{1}{b}\right)$, we use this fact and power rule (b) to get **power rule (c) for exponents.**

### Power Rule (c) for Exponents

For any positive integer $m$, $\quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0).$

(Raise a quotient to a power by raising both numerator and denominator to the power.)

*Example:* $\left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2}$

**NOW TRY EXERCISE 6**

Simplify.

**(a)** $\left(\frac{p}{q}\right)^5$ $(q \neq 0)$ **(b)** $\left(\frac{1}{4}\right)^3$

### EXAMPLE 6 Using Power Rule (c)

Use power rule (c) for exponents to simplify.

**(a)** $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5} = \frac{32}{243}$ **(b)** $\left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3} \quad (n \neq 0)$

**(c)** $\left(\frac{1}{5}\right)^4 = \frac{1^4}{5^4} = \frac{1}{5^4} = \frac{1}{625}$ $\quad 1^4 = 1 \cdot 1 \cdot 1 \cdot 1 = 1$

NOW TRY

**NOTE** In **Example 6(c),** we used the fact that $1^4 = 1$.

***In general, $1^n = 1$, for any integer n.***

### Rules for Exponents

For positive integers $m$ and $n$, the following are true.

| | | **Examples** |
|---|---|---|
| **Product rule** | $a^m \cdot a^n = a^{m+n}$ | $6^2 \cdot 6^5 = 6^{2+5} = 6^7$ |
| **Power rules (a)** | $(a^m)^n = a^{mn}$ | $(3^2)^4 = 3^{2 \cdot 4} = 3^8$ |
| **(b)** | $(ab)^m = a^mb^m$ | $(2p)^5 = 2^5p^5$ |
| **(c)** | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$ | $\left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2}$ |

**NOW TRY ANSWERS**

5. **(a)** $-125a^3b^3$ **(b)** $48t^6p^{10}$

6. **(a)** $\frac{p^5}{q^5}$ **(b)** $\frac{1}{64}$

## OBJECTIVE 6 Use combinations of rules.

*NOW TRY EXERCISE 7*

Simplify.

**(a)** $\left(\frac{3}{5}\right)^3 \cdot 3^2$ **(b)** $(8k)^5(8k)^4$

**(c)** $(x^4y)^5(-2x^2y^5)^3$

### EXAMPLE 7 Using Combinations of Rules

Simplify.

**(a)** $\left(\frac{2}{3}\right)^2 \cdot 2^3$

$$
\begin{aligned}
&= \frac{2^2}{3^2} \cdot \frac{2^3}{1} && \text{Power rule (c)} \\
&= \frac{2^2 \cdot 2^3}{3^2 \cdot 1} && \text{Multiply fractions.} \\
&= \frac{2^{2+3}}{3^2} && \text{Product rule} \\
&= \frac{2^5}{3^2}, \quad \text{or} \quad \frac{32}{9}
\end{aligned}
$$

**(b)** $(5x)^3(5x)^4$

$$
\begin{aligned}
&= (5x)^7 && \text{Product rule} \\
&= 5^7x^7 && \text{Power rule (b)}
\end{aligned}
$$

**(c)** $(2x^2y^3)^4(3xy^2)^3$

$$
\begin{aligned}
&= 2^4(x^2)^4(y^3)^4 \cdot 3^3x^3(y^2)^3 && \text{Power rule (b)} \\
&= 2^4x^8y^{12} \cdot 3^3x^3y^6 && \text{Power rule (a)} \\
&= 2^4 \cdot 3^3x^8x^3y^{12}y^6 && \text{Commutative and associative properties} \\
&= 16 \cdot 27x^{11}y^{18}, \quad \text{or} \quad 432x^{11}y^{18} && \text{Product rule; multiply.}
\end{aligned}
$$

Notice that $(2x^2y^3)^4$ means $2^4x^{2\cdot 4}y^{3\cdot 4}$, ***not*** $(2 \cdot 4)x^{2\cdot 4}y^{3\cdot 4}$.

**(d)** $(-x^3y)^2(-x^5y^4)^3$

Don't forget each factor of $-1$.

$$
\begin{aligned}
&= (-1 \cdot x^3y)^2(-1 \cdot x^5y^4)^3 && -a = -1 \cdot a \\
&= (-1)^2(x^3)^2y^2 \cdot (-1)^3(x^5)^3(y^4)^3 && \text{Power rule (b)} \\
&= (-1)^2(x^6)(y^2)(-1)^3(x^{15})(y^{12}) && \text{Power rule (a)} \\
&= (-1)^5(x^{21})(y^{14}) && \text{Product rule} \\
&= -x^{21}y^{14} && \text{Simplify.}
\end{aligned}
$$

*NOW TRY*

---

**CAUTION** Be aware of the distinction between $(2y)^3$ and $2y^3$.

$$(2y)^3 = 2y \cdot 2y \cdot 2y = 8y^3, \quad \text{while} \quad 2y^3 = 2 \cdot y \cdot y \cdot y.$$

---

## OBJECTIVE 7 Use the rules for exponents in a geometry application.

### EXAMPLE 8 Using Area Formulas

Find an expression that represents the area in **(a) FIGURE 1** and **(b) FIGURE 2**.

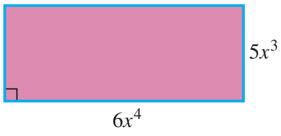


**FIGURE 1**

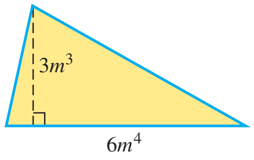


**FIGURE 2**

Assume $x > 0$, $m > 0$.

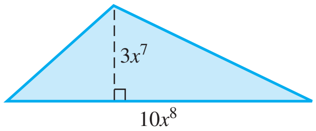
*NOW TRY ANSWERS*

7. **(a)** $\frac{243}{125}$ **(b)** $8^9k^9$

**(c)** $-8x^{26}y^{20}$

**NOW TRY EXERCISE 8**

Write an expression that represents the area of the figure. Assume $x > 0$.



**(a)** For **FIGURE 1**, use the formula for the area of a rectangle, $\mathcal{A} = LW$.

$$\begin{aligned} \mathcal{A} &= (6x^4)(5x^3) && \text{Area formula} \\ \mathcal{A} &= 6 \cdot 5 \cdot x^{4+3} && \text{Commutative property; product rule} \\ \mathcal{A} &= 30x^7 && \text{Multiply. Add the exponents.} \end{aligned}$$

**(b)** **FIGURE 2** is a triangle with base $6m^4$ and height $3m^3$.

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}bh && \text{Area formula} \\ \mathcal{A} &= \frac{1}{2}(6m^4)(3m^3) && \text{Substitute.} \\ \mathcal{A} &= \frac{1}{2}(18m^7), \quad \text{or} \quad 9m^7 && \text{Product rule; multiply.} \end{aligned}$$

NOW TRY

**NOW TRY ANSWER**

**8.** $15x^{15}$

## 4.1 EXERCISES

*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Decide whether each statement is* true *or* false. *If false, tell why.*

**1.** $3^3 = 9$ **2.** $(-3)^4 = 3^4$ **3.** $(x^2)^3 = x^5$ **4.** $\left(\frac{1}{5}\right)^2 = \frac{1}{5^2}$

*Write each expression by using exponents.* ***See Example 1.***

**5.** $w \cdot w \cdot w \cdot w \cdot w \cdot w$ **6.** $t \cdot t \cdot t \cdot t \cdot t \cdot t \cdot t$

**7.** $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ **8.** $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

**9.** $(-4)(-4)(-4)(-4)$ **10.** $(-3)(-3)(-3)(-3)(-3)(-3)$

**11.** $(-7y)(-7y)(-7y)(-7y)$ **12.** $(-8p)(-8p)(-8p)(-8p)(-8p)$

**13.** Explain how the expressions $(-3)^4$ and $-3^4$ are different.

**14.** Explain how the expressions $(5x)^3$ and $5x^3$ are different.

*Identify the base and the exponent for each exponential expression. In Exercises 15–18, also evaluate each expression.* ***See Example 2.***

**15.** $3^5$ **16.** $2^7$ **17.** $(-3)^5$ **18.** $(-2)^7$

**19.** $(-6x)^4$ **20.** $(-8x)^4$ **21.** $-6x^4$ **22.** $-8x^4$

**23.** Explain why the product rule does not apply to the expression $5^2 + 5^3$. Then evaluate the expression by finding the individual powers and adding the results.

**24.** Repeat **Exercise 23** for the expression $(-4)^3 + (-4)^4$.

*Use the product rule, if possible, to simplify each expression. Write each answer in exponential form.* ***See Example 3.***

**25.** $5^2 \cdot 5^6$ **26.** $3^6 \cdot 3^7$ **27.** $4^2 \cdot 4^7 \cdot 4^3$

**28.** $5^3 \cdot 5^8 \cdot 5^2$ **29.** $(-7)^3(-7)^6$ **30.** $(-9)^8(-9)^5$

**31.** $t^3 \cdot t^8 \cdot t^{13}$ **32.** $n^5 \cdot n^6 \cdot n^9$ **33.** $(-8r^4)(7r^3)$

**34.** $(10a^7)(-4a^3)$ **35.** $(-6p^5)(-7p^5)$ **36.** $(-5w^8)(-9w^8)$

**37.** $(5x^2)(-2x^3)(3x^4)$ **38.** $(12y^3)(4y)(-3y^5)$ **39.** $3^8 + 3^9$

**40.** $4^{12} + 4^5$ **41.** $5^8 \cdot 3^9$ **42.** $6^3 \cdot 8^9$

*Use the power rules for exponents to simplify each expression. Write each answer in exponential form.* ***See Examples 4–6.***

**43.** $(4^3)^2$ **44.** $(8^3)^6$ **45.** $(t^4)^5$ **46.** $(y^6)^5$

**47.** $(7r)^3$ **48.** $(11x)^4$ **49.** $(5xy)^5$ **50.** $(9pq)^6$

**51.** $(-5^2)^6$ **52.** $(-9^4)^8$ **53.** $(-8^3)^5$ **54.** $(-7^5)^7$

**55.** $8(qr)^3$ **56.** $4(vw)^5$ **57.** $\left(\frac{9}{5}\right)^8$

**58.** $\left(\frac{12}{7}\right)^3$ **59.** $\left(\frac{1}{2}\right)^3$ **60.** $\left(\frac{1}{3}\right)^5$

**61.** $\left(\frac{a}{b}\right)^3 \quad (b \neq 0)$ **62.** $\left(\frac{r}{t}\right)^4 \quad (t \neq 0)$ **63.** $\left(\frac{x}{2}\right)^3$

**64.** ***Concept Check*** Will $(-a)^n$ ever equal $a^n$? If so, when?

*Simplify each expression.* ***See Example 7.***

**65.** $\left(\frac{5}{2}\right)^3 \cdot \left(\frac{5}{2}\right)^2$ **66.** $\left(\frac{3}{4}\right)^5 \cdot \left(\frac{3}{4}\right)^6$ **67.** $\left(\frac{9}{8}\right)^3 \cdot 9^2$

**68.** $\left(\frac{8}{5}\right)^4 \cdot 8^3$ **69.** $(2x)^9(2x)^3$ **70.** $(6y)^5(6y)^8$

**71.** $(-6p)^4(-6p)$ **72.** $(-13q)^3(-13q)$ **73.** $(6x^2y^3)^5$

**74.** $(5r^5t^6)^7$ **75.** $(x^2)^3(x^3)^5$ **76.** $(y^4)^5(y^3)^5$

**77.** $(2w^2x^3y)^2(x^4y)^5$ **78.** $(3x^4y^2z)^3(yz^4)^5$ **79.** $(-r^4s)^2(-r^2s^3)^5$

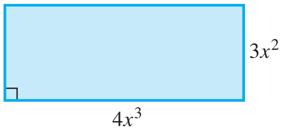
**80.** $(-ts^6)^4(-t^3s^5)^3$ **81.** $\left(\frac{5a^2b^5}{c^6}\right)^3 \quad (c \neq 0)$ **82.** $\left(\frac{6x^3y^9}{z^5}\right)^4 \quad (z \neq 0)$

**83.** ***Concept Check*** A student simplified $(10^2)^3$ as $1000^6$. **WHAT WENT WRONG?**

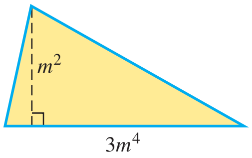
**84.** Explain why $(3x^2y^3)^4$ is *not* equivalent to $(3 \cdot 4)x^8y^{12}$.

*Find an expression that represents the area of each figure.* ***See Example 8.*** *(If necessary, refer to the formulas on the inside covers. The ⌉ in the figures indicate 90° right angles.)*

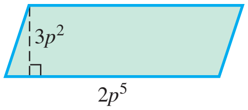
**85.**



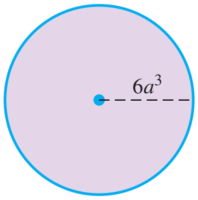
**86.**



**87.**

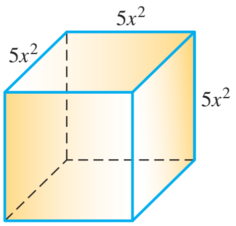


**88.**

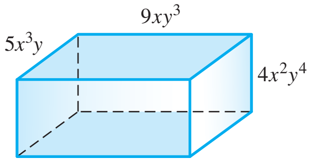


*Find an expression that represents the volume of each figure. (If necessary, refer to the formulas on the inside covers.)*

**89.**



**90.**



**91.** Assume that $a$ is a number greater than 1. Arrange the following terms in order from least to greatest: $-(-a)^3$, $-a^3$, $(-a)^4$, $-a^4$. Explain how you decided on the order.

**92.** Devise a rule that tells whether an exponential expression with a negative base is positive or negative.

**Compound interest** is interest paid on the principal and the interest earned earlier. The formula for compound interest, which involves an exponential expression, is

$$A = P(1 + r)^n,$$

where $A$ is the amount accumulated from a principal of $P$ dollars left untouched for $n$ years with an annual interest rate $r$ (expressed as a decimal).

*In Exercises 93–96, use the preceding formula and a calculator to find A to the nearest cent.*

**93.** $P = \$250$, $r = 0.04$, $n = 5$

**94.** $P = \$400$, $r = 0.04$, $n = 3$

**95.** $P = \$1500$, $r = 0.035$, $n = 6$

**96.** $P = \$2000$, $r = 0.025$, $n = 4$

## PREVIEW EXERCISES

*Give the reciprocal of each number.* ***See Section 1.1.***

**97.** 9

**98.** $-3$

**99.** $-\frac{1}{8}$

**100.** 0.5

*Perform each subtraction.* ***See Section 1.5.***

**101.** $8 - (-4)$

**102.** $-4 - 8$

**103.** Subtract $-6$ from $-3$.

**104.** Subtract $-3$ from $-6$.

# 4.2

# Integer Exponents and the Quotient Rule

**OBJECTIVES**

1. Use 0 as an exponent.
2. Use negative numbers as exponents.
3. Use the quotient rule for exponents.
4. Use combinations of rules.

Consider the following list.

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$

Each time we reduce the exponent by 1, the value is divided by 2 (the base). Using this pattern, we can continue the list to lesser and lesser integer exponents.

$$2^1 = 2$$
$$2^0 = 1$$
$$2^{-1} = \frac{1}{2}$$
$$2^{-2} = \frac{1}{4}$$
$$2^{-3} = \frac{1}{8}$$

From the preceding list, it appears that we should define $2^0$ as 1 and bases raised to negative exponents as reciprocals of those bases.

**OBJECTIVE 1** **Use 0 as an exponent.** The definitions of 0 and negative exponents must satisfy the rules for exponents from **Section 4.1.** For example, if $6^0 = 1$, then

$$6^0 \cdot 6^2 = 1 \cdot 6^2 = 6^2 \quad \text{and} \quad 6^0 \cdot 6^2 = 6^{0+2} = 6^2,$$

so that the product rule is satisfied. Check that the power rules are also valid for a 0 exponent. Thus, we define a 0 exponent as follows.

### Zero Exponent

For any nonzero real number $a$, $\mathbf{a^0 = 1.}$

*Example:* $17^0 = 1$

**NOW TRY EXERCISE 1**

Evaluate.

**(a)** $6^0$

**(b)** $-12^0$

**(c)** $(-12x)^0 \quad (x \neq 0)$

**(d)** $14^0 - 12^0$

### EXAMPLE 1 Using Zero Exponents

Evaluate.

**(a)** $60^0 = 1$

**(b)** $(-60)^0 = 1$

**(c)** $-60^0 = -(1) = -1$

**(d)** $y^0 = 1 \quad (y \neq 0)$

**(e)** $6y^0 = 6(1) = 6 \quad (y \neq 0)$

**(f)** $(6y)^0 = 1 \quad (y \neq 0)$

**(g)** $8^0 + 11^0 = 1 + 1 = 2$

**(h)** $-8^0 - 11^0 = -1 - 1 = -2$

NOW TRY

---

**CAUTION** Look again at **Examples 1(b) and 1(c).** In $(-60)^0$, the base is $-60$, and since any nonzero base raised to the 0 exponent is 1, $(-60)^0 = 1$. In $-60^0$, which can be written $-(60)^0$, the base is 60, so $-60^0 = -1$.

---

**OBJECTIVE 2** **Use negative numbers as exponents.** From the lists at the beginning of this section, since $2^{-2} = \frac{1}{4}$ and $2^{-3} = \frac{1}{8}$, we can deduce that $2^{-n}$ should equal $\frac{1}{2^n}$. Is the product rule valid in such cases? For example,

$$6^{-2} \cdot 6^2 = 6^{-2+2} = 6^0 = 1.$$

The expression $6^{-2}$ behaves as if it were the reciprocal of $6^2$, since their product is 1. The reciprocal of $6^2$ is also $\frac{1}{6^2}$, leading us to define $6^{-2}$ as $\frac{1}{6^2}$.

### Negative Exponents

For any nonzero real number $a$ and any integer $n$, $\boldsymbol{a^{-n} = \frac{1}{a^n}}.$

*Example:* $3^{-2} = \frac{1}{3^2}$

By definition, $a^{-n}$ and $a^n$ are reciprocals, since

$$a^n \cdot a^{-n} = a^n \cdot \frac{1}{a^n} = 1.$$

Because $1^n = 1$, the definition of $a^{-n}$ can also be written

$$a^{-n} = \frac{1}{a^n} = \frac{1^n}{a^n} = \left(\frac{1}{a}\right)^n.$$

For example, $\quad 6^{-3} = \left(\frac{1}{6}\right)^3 \quad$ and $\quad \left(\frac{1}{3}\right)^{-2} = 3^2.$

**NOW TRY ANSWERS**

1. (a) 1 (b) −1 (c) 1 (d) 0

**NOW TRY EXERCISE 2**

Simplify.

**(a)** $2^{-3}$ **(b)** $\left(\frac{1}{7}\right)^{-2}$

**(c)** $\left(\frac{3}{2}\right)^{-4}$ **(d)** $3^{-2} + 4^{-2}$

**(e)** $p^{-4}$ $(p \neq 0)$

## EXAMPLE 2 Using Negative Exponents

Simplify by writing with positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ **(b)** $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

**(c)** $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$ $\frac{1}{2}$ and 2 are reciprocals.

***Notice that we can change the base to its reciprocal if we also change the sign of the exponent.***

**(d)** $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}$ **(e)** $\left(\frac{4}{3}\right)^{-5} = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$

$\frac{2}{5}$ and $\frac{5}{2}$ are reciprocals.

**(f)** $4^{-1} - 2^{-1} = \frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4}$ Apply the exponents first, and then subtract.

Remember to find a common denominator.

**(g)** $p^{-2} = \frac{1}{p^2}$

**(h)** $\frac{1}{x^{-4}} = \frac{1^{-4}}{x^{-4}}$

It is convenient to write 1 as $1^{-4}$ here, because $-4$ is the exponent in the denominator.

$= \left(\frac{1}{x}\right)^{-4}$ Power rule (c)

$= x^4$ $\frac{1}{x}$ and $x$ are reciprocals.

**(i)** $x^3y^{-4} = \frac{x^3}{y^4}$

NOW TRY

Consider the following.

$$\frac{2^{-3}}{3^{-4}} = \frac{\frac{1}{2^3}}{\frac{1}{3^4}} = \frac{1}{2^3} \div \frac{1}{3^4} = \frac{1}{2^3} \cdot \frac{3^4}{1} = \frac{3^4}{2^3}$$

To divide by a fraction, multiply by its reciprocal.

Therefore, $\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}$.

### Changing from Negative to Positive Exponents

For any nonzero numbers $a$ and $b$ and any integers $m$ and $n$, the following are true.

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \quad \text{and} \quad \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$$

*Examples:* $\frac{3^{-5}}{2^{-4}} = \frac{2^4}{3^5}$ and $\left(\frac{4}{5}\right)^{-3} = \left(\frac{5}{4}\right)^3$

**NOW TRY ANSWERS**

**2.** **(a)** $\frac{1}{8}$ **(b)** 49 **(c)** $\frac{16}{81}$

**(d)** $\frac{25}{144}$ **(e)** $\frac{1}{p^4}$

**NOW TRY EXERCISE 3**

Simplify by writing with positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $\frac{5^{-3}}{6^{-2}}$ **(b)** $m^2n^{-4}$

**(c)** $\frac{x^2y^{-3}}{5z^{-4}}$

### EXAMPLE 3 Changing from Negative to Positive Exponents

Simplify by writing with positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $\frac{4^{-2}}{5^{-3}} = \frac{5^3}{4^2} = \frac{125}{16}$ **(b)** $\frac{m^{-5}}{p^{-1}} = \frac{p^1}{m^5} = \frac{p}{m^5}$

**(c)** $\frac{a^{-2}b}{3d^{-3}} = \frac{bd^3}{3a^2}$ Notice that *b* in the numerator and 3 in the denominator are not affected.

**(d)** $\left(\frac{x}{2y}\right)^{-4} = \left(\frac{2y}{x}\right)^4 = \frac{2^4y^4}{x^4} = \frac{16y^4}{x^4}$

NOW TRY

---

**CAUTION** Be careful. We cannot use this rule to change negative exponents to positive exponents if the exponents occur in a *sum or difference* of terms. For example,

$$\frac{5^{-2}+3^{-1}}{7-2^{-3}} \quad \text{would be written with positive exponents as} \quad \frac{\frac{1}{5^2}+\frac{1}{3}}{7-\frac{1}{2^3}}.$$

---

**OBJECTIVE 3** **Use the quotient rule for exponents.** Consider the following.

$$\frac{6^5}{6^3} = \frac{6\cdot6\cdot6\cdot6\cdot6}{6\cdot6\cdot6} = 6^2$$

The difference between the exponents, $5 - 3 = 2$, is the exponent in the quotient.

Also, $$\frac{6^2}{6^4} = \frac{6\cdot6}{6\cdot6\cdot6\cdot6} = \frac{1}{6^2} = 6^{-2}.$$

Here, $2 - 4 = -2$. These examples suggest the **quotient rule for exponents.**

### Quotient Rule for Exponents

For any nonzero real number *a* and any integers *m* and *n*,

$$\frac{a^m}{a^n} = a^{m-n}.$$

(Keep the same base and subtract the exponents.)

*Example:* $\frac{5^8}{5^4} = 5^{8-4} = 5^4$

---

**CAUTION** A common **error** is to write $\frac{5^8}{5^4} = 1^{8-4} = 1^4$. By the quotient rule, the quotient must have the *same base,* 5, just as in the product rule.

$$\frac{5^8}{5^4} = 5^{8-4} = 5^4$$

If you are not sure, use the definition of an exponent to write out the factors.

$$\frac{5^8}{5^4} = \frac{5\cdot5\cdot5\cdot5\cdot5\cdot5\cdot5\cdot5}{5\cdot5\cdot5\cdot5} = 5^4$$

---

**NOW TRY ANSWERS**

3. **(a)** $\frac{6^2}{5^3}$, or $\frac{36}{125}$ **(b)** $\frac{m^2}{n^4}$

**(c)** $\frac{x^2z^4}{5y^3}$

NOW TRY EXERCISE 4

Simplify by writing with positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $\frac{6^3}{6^4}$ **(b)** $\frac{t^4}{t^{-5}}$

**(c)** $\frac{(p+q)^{-3}}{(p+q)^{-7}} \quad (p \neq -q)$

**(d)** $\frac{5^2xy^{-3}}{3^{-1}x^{-2}y^2}$

## EXAMPLE 4 Using the Quotient Rule

Simplify by writing with positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $\frac{5^8}{5^6} = 5^{8-6} = 5^2 = 25$

Keep the same base.

**(b)** $\frac{4^2}{4^9} = 4^{2-9} = 4^{-7} = \frac{1}{4^7}$

**(c)** $\frac{5^{-3}}{5^{-7}} = 5^{-3-(-7)} = 5^4 = 625$

Be careful with signs.

**(d)** $\frac{q^5}{q^{-3}} = q^{5-(-3)} = q^8$

**(e)** $\frac{3^2x^5}{3^4x^3}$

$$= \frac{3^2}{3^4} \cdot \frac{x^5}{x^3}$$

$$= 3^{2-4} \cdot x^{5-3} \qquad \text{Quotient rule}$$

$$= 3^{-2}x^2 \qquad \text{Subtract.}$$

$$= \frac{x^2}{3^2}, \quad \text{or} \quad \frac{x^2}{9}$$

**(f)** $\frac{(m+n)^{-2}}{(m+n)^{-4}}$

$$= (m+n)^{-2-(-4)}$$

$$= (m+n)^{-2+4}$$

$$= (m+n)^2, \quad m \neq -n$$

**(g)** $\frac{7x^{-3}y^2}{2^{-1}x^2y^{-5}}$

$$= \frac{7 \cdot 2^1y^2y^5}{x^2x^3} \qquad \text{Negative-to-positive rule}$$

$$= \frac{14y^7}{x^5} \qquad \text{Product rule}$$

NOW TRY

The definitions and rules for exponents are summarized here.

### Definitions and Rules for Exponents

For any integers $m$ and $n$, the following are true.

| | | Examples |
|---|---|---|
| **Product rule** | $a^m \cdot a^n = a^{m+n}$ | $7^4 \cdot 7^5 = 7^{4+5} = 7^9$ |
| **Zero exponent** | $a^0 = 1 \quad (a \neq 0)$ | $(-3)^0 = 1$ |
| **Negative exponent** | $a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$ | $5^{-3} = \frac{1}{5^3}$ |
| **Quotient rule** | $\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$ | $\frac{2^2}{2^5} = 2^{2-5} = 2^{-3} = \frac{1}{2^3}$ |
| **Power rule (a)** | $(a^m)^n = a^{mn}$ | $(4^2)^3 = 4^{2 \cdot 3} = 4^6$ |
| **Power rule (b)** | $(ab)^m = a^mb^m$ | $(3k)^4 = 3^4k^4$ |
| **Power rule (c)** | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$ | $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$ |
| **Negative-to-positive rules** | $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \quad (a \neq 0, b \neq 0)$<br>$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ | $\frac{2^{-4}}{5^{-3}} = \frac{5^3}{2^4}$<br>$\left(\frac{4}{7}\right)^{-2} = \left(\frac{7}{4}\right)^2$ |

NOW TRY ANSWERS

4. **(a)** $\frac{1}{6}$ **(b)** $t^9$

**(c)** $(p+q)^4$ **(d)** $\frac{75x^3}{y^5}$

**NOW TRY EXERCISE 5**

Simplify. Assume that all variables represent nonzero real numbers.

**(a)** $\dfrac{3^{15}}{(3^3)^4}$ **(b)** $(4t)^5(4t)^{-3}$

**(c)** $\left(\dfrac{7y^4}{10}\right)^{-3}$ **(d)** $\dfrac{(a^2b^{-2}c)^{-3}}{(2ab^3c^{-4})^5}$

## OBJECTIVE 4 Use combinations of rules.

### EXAMPLE 5 Using Combinations of Rules

Simplify. Assume that all variables represent nonzero real numbers.

**(a)** $\dfrac{(4^2)^3}{4^5}$

$$= \frac{4^6}{4^5} \qquad \text{Power rule (a)}$$

$$= 4^{6-5} \qquad \text{Quotient rule}$$

$$= 4^1$$

$$= 4$$

**(b)** $(2x)^3(2x)^2$

$$= (2x)^5 \qquad \text{Product rule}$$

$$= 2^5x^5 \qquad \text{Power rule (b)}$$

$$= 32x^5$$

**(c)** $\left(\dfrac{2x^3}{5}\right)^{-4}$

$$= \left(\frac{5}{2x^3}\right)^4 \qquad \text{Negative-to-positive rule}$$

$$= \frac{5^4}{2^4x^{12}} \qquad \text{Power rules (a)–(c)}$$

$$= \frac{625}{16x^{12}}$$

**(d)** $\left(\dfrac{3x^{-2}}{4^{-1}y^3}\right)^{-3}$

$$= \frac{3^{-3}x^6}{4^3y^{-9}} \qquad \text{Power rules (a)–(c)}$$

$$= \frac{x^6y^9}{4^3 \cdot 3^3} \qquad \text{Negative-to-positive rule}$$

$$= \frac{x^6y^9}{1728}$$

**(e)** $\dfrac{(4m)^{-3}}{(3m)^{-4}}$

$$= \frac{4^{-3}m^{-3}}{3^{-4}m^{-4}} \qquad \text{Power rule (b)}$$

$$= \frac{3^4m^4}{4^3m^3} \qquad \text{Negative-to-positive rule}$$

$$= \frac{3^4m^{4-3}}{4^3} \qquad \text{Quotient rule}$$

$$= \frac{3^4m}{4^3}, \quad \text{or} \quad \frac{81m}{64}$$

NOW TRY

**NOW TRY ANSWERS**

**5.** **(a)** $3^3$, or 27 **(b)** $16t^2$

**(c)** $\dfrac{1000}{343y^{12}}$ **(d)** $\dfrac{c^{17}}{32a^{11}b^9}$

## 4.2 EXERCISES

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◉ *Complete solution available on the Video Resources on DVD*

*Decide whether each expression is equal to 0, 1, or −1.* ***See Example 1.***

◉ **1.** $9^0$ **2.** $3^0$ **3.** $(-2)^0$ **4.** $(-12)^0$

**5.** $-8^0$ **6.** $-6^0$ **7.** $-(-6)^0$ **8.** $-(-13)^0$

**9.** $(-4)^0 - 4^0$ **10.** $(-11)^0 - 11^0$ **11.** $\dfrac{0^{10}}{12^0}$ **12.** $\dfrac{0^5}{2^0}$

**13.** $8^0 - 12^0$ **14.** $6^0 - 13^0$ **15.** $\frac{0^2}{2^0 + 0^2}$ **16.** $\frac{2^0}{0^2 + 2^0}$

***Concept Check*** *In Exercises 17 and 18, match each expression in Column I with the equivalent expression in Column II. Choices in Column II may be used once, more than once, or not at all. (In Exercise 17, $x \neq 0$.)*

**17.**

| I | II |
|---|---|
| **(a)** $x^0$ | **A.** $0$ |
| **(b)** $-x^0$ | **B.** $1$ |
| **(c)** $7x^0$ | **C.** $-1$ |
| **(d)** $(7x)^0$ | **D.** $7$ |
| **(e)** $-7x^0$ | **E.** $-7$ |
| **(f)** $(-7x)^0$ | **F.** $\frac{1}{7}$ |

**18.**

| I | II |
|---|---|
| **(a)** $-2^{-4}$ | **A.** $8$ |
| **(b)** $(-2)^{-4}$ | **B.** $16$ |
| **(c)** $2^{-4}$ | **C.** $-\frac{1}{16}$ |
| **(d)** $\frac{1}{2^{-4}}$ | **D.** $-8$ |
| **(e)** $\frac{1}{-2^{-4}}$ | **E.** $-16$ |
| **(f)** $\frac{1}{(-2)^{-4}}$ | **F.** $\frac{1}{16}$ |

*Evaluate each expression.* ***See Examples 1 and 2.***

**19.** $6^0 + 8^0$ **20.** $4^0 + 2^0$ **21.** $4^{-3}$

**22.** $5^{-4}$ **23.** $\left(\frac{1}{2}\right)^{-4}$ **24.** $\left(\frac{1}{3}\right)^{-3}$

**25.** $\left(\frac{6}{7}\right)^{-2}$ **26.** $\left(\frac{2}{3}\right)^{-3}$ **27.** $(-3)^{-4}$

**28.** $(-4)^{-3}$ **29.** $5^{-1} + 3^{-1}$ **30.** $6^{-1} + 2^{-1}$

**31.** $3^{-2} - 2^{-1}$ **32.** $6^{-2} - 3^{-1}$

**33.** $\left(\frac{1}{2}\right)^{-1} + \left(\frac{2}{3}\right)^{-1}$ **34.** $\left(\frac{1}{3}\right)^{-1} + \left(\frac{4}{3}\right)^{-1}$

*Simplify by writing each expression with positive exponents. Assume that all variables represent nonzero real numbers.* ***See Examples 2–4.***

**35.** $\frac{5^8}{5^5}$ **36.** $\frac{11^6}{11^3}$ **37.** $\frac{3^{-2}}{5^{-3}}$ **38.** $\frac{4^{-3}}{3^{-2}}$

**39.** $\frac{5}{5^{-1}}$ **40.** $\frac{6}{6^{-2}}$ **41.** $\frac{x^{12}}{x^{-3}}$ **42.** $\frac{y^4}{y^{-6}}$

**43.** $\frac{1}{6^{-3}}$ **44.** $\frac{1}{5^{-2}}$ **45.** $\frac{2}{r^{-4}}$ **46.** $\frac{3}{s^{-8}}$

**47.** $\frac{4^{-3}}{5^{-2}}$ **48.** $\frac{6^{-2}}{5^{-4}}$ **49.** $p^5q^{-8}$ **50.** $x^{-8}y^4$

**51.** $\frac{r^5}{r^{-4}}$ **52.** $\frac{a^6}{a^{-4}}$ **53.** $\frac{x^{-3}y}{4z^{-2}}$ **54.** $\frac{p^{-5}q^{-4}}{9r^{-3}}$

**55.** $\frac{(a+b)^{-3}}{(a+b)^{-4}}$ **56.** $\frac{(x+y)^{-8}}{(x+y)^{-9}}$ **57.** $\frac{(x+2y)^{-3}}{(x+2y)^{-5}}$ **58.** $\frac{(p-3q)^{-2}}{(p-3q)^{-4}}$

## RELATING CONCEPTS EXERCISES 59–62

### FOR INDIVIDUAL OR GROUP WORK

*In **Objective 1,** we showed how* $6^0$ *acts as* 1 *when it is applied to the product rule, thus motivating the definition of* 0 *as an exponent. We can also use the quotient rule to motivate this definition.* ***Work Exercises 59–62 in order.***

**59.** Consider the expression $\frac{25}{25}$. What is its simplest form?

**60.** Because $25 = 5^2$, the expression $\frac{25}{25}$ can be written as the quotient of powers of 5. Write the expression in this way.

**61.** Apply the quotient rule for exponents to the expression you wrote in **Exercise 60.** Give the answer as a power of 5.

**62.** Your answers in **Exercises 59 and 61** must be equal because they both represent $\frac{25}{25}$. Write this equality. What definition does this result support?

*Simplify by writing each expression with positive exponents. Assume that all variables represent nonzero real numbers.* ***See Example 5.***

**63.** $\frac{(7^4)^3}{7^9}$

**64.** $\frac{(5^3)^2}{5^2}$

**65.** $x^{-3} \cdot x^5 \cdot x^{-4}$

**66.** $y^{-8} \cdot y^5 \cdot y^{-2}$

**67.** $\frac{(3x)^{-2}}{(4x)^{-3}}$

**68.** $\frac{(2y)^{-3}}{(5y)^{-4}}$

**69.** $\left(\frac{x^{-1}y}{z^2}\right)^{-2}$

**70.** $\left(\frac{p^{-4}q}{r^{-3}}\right)^{-3}$

**71.** $(6x)^4(6x)^{-3}$

**72.** $(10y)^9(10y)^{-8}$

**73.** $\frac{(m^7n)^{-2}}{m^{-4}n^3}$

**74.** $\frac{(m^8n^{-4})^2}{m^{-2}n^5}$

**75.** $\frac{(x^{-1}y^2z)^{-2}}{(x^{-3}y^3z)^{-1}}$

**76.** $\frac{(a^{-2}b^{-3}c^{-4})^{-5}}{(a^2b^3c^4)^{-4}}$

**77.** $\left(\frac{xy^{-2}}{x^2y}\right)^{-3}$

**78.** $\left(\frac{wz^{-5}}{w^{-3}z}\right)^{-2}$

***Brain Busters*** *Simplify by writing each expression wth positive exponents. Assume that all variables represent nonzero real numbers.*

**79.** $\frac{(4a^2b^3)^{-2}(2ab^{-1})^3}{(a^3b)^{-4}}$

**80.** $\frac{(m^6n)^{-2}(m^2n^{-2})^3}{m^{-1}n^{-2}}$

**81.** $\frac{(2y^{-1}z^2)^2(3y^{-2}z^{-3})^3}{(y^3z^2)^{-1}}$

**82.** $\frac{(3p^{-2}q^3)^2(5p^{-1}q^{-4})^{-1}}{(p^2q^{-2})^{-3}}$

**83.** $\frac{(9^{-1}z^{-2}x)^{-1}(4z^2x^4)^{-2}}{(5z^{-2}x^{-3})^2}$

**84.** $\frac{(4^{-1}a^{-1}b^{-2})^{-2}(5a^{-3}b^4)^{-2}}{(3a^{-3}b^{-5})^2}$

**85.** ***Concept Check*** A student simplified $\frac{16^3}{2^2}$ as shown.

$$\frac{16^3}{2^2} = \left(\frac{16}{2}\right)^{3-2} = 8^1 = 8$$

***WHAT WENT WRONG?*** Give the correct answer.

**86.** ***Concept Check*** A student simplified $-5^4$ as shown.

$$-5^4 = (-5)^4 = 625$$

***WHAT WENT WRONG?*** Give the correct answer.

## PREVIEW EXERCISES

*Evaluate.*

**87.** 10(6428)

**88.** 100(72.79)

**89.** 1000(1.53)

**90.** 10,000(36.94)

**91.** $38 \div 10$

**92.** $6504 \div 100$

**93.** $277 \div 1000$

**94.** $49 \div 10{,}000$

## *SUMMARY EXERCISES* on the Rules for Exponents

*Simplify each expression. Use only positive exponents in your answers. Assume that all variables represent nonzero real numbers.*

**1.** $(10x^2y^4)^2(10xy^2)^3$

**2.** $(-2ab^3c)^4(-2a^2b)^3$

**3.** $\left(\dfrac{9wx^3}{y^4}\right)^3$

**4.** $(4x^{-2}y^{-3})^{-2}$

**5.** $\dfrac{c^{11}(c^2)^4}{(c^3)^3(c^2)^{-6}}$

**6.** $\left(\dfrac{k^4t^2}{k^2t^{-4}}\right)^{-2}$

**7.** $5^{-1} + 6^{-1}$

**8.** $\dfrac{(3y^{-1}z^3)^{-1}(3y^2)}{(y^3z^2)^{-3}}$

**9.** $\dfrac{(2xy^{-1})^3}{2^3x^{-3}y^2}$

**10.** $-4^0 + (-4)^0$

**11.** $(z^4)^{-3}(z^{-2})^{-5}$

**12.** $\left(\dfrac{r^2st^5}{3r}\right)^{-2}$

**13.** $\dfrac{(3^{-1}x^{-3}y)^{-1}(2x^2y^{-3})^2}{(5x^{-2}y^2)^{-2}}$

**14.** $\left(\dfrac{5x^2}{3x^{-4}}\right)^{-1}$

**15.** $\left(\dfrac{-9x^{-2}}{9x^2}\right)^{-2}$

**16.** $\dfrac{(x^{-4}y^2)^3(x^2y)^{-1}}{(xy^2)^{-3}}$

**17.** $\dfrac{(a^{-2}b^3)^{-4}}{(a^{-3}b^2)^{-2}(ab)^{-4}}$

**18.** $(2a^{-30}b^{-29})(3a^{31}b^{30})$

**19.** $5^{-2} + 6^{-2}$

**20.** $\left[\dfrac{(x^{43}y^{23})^2}{x^{-26}y^{-42}}\right]^0$

**21.** $\left(\dfrac{7a^2b^3}{2}\right)^3$

**22.** $-(-19^0)$

**23.** $-(-13)^0$

**24.** $\dfrac{0^{13}}{13^0}$

**25.** $\dfrac{(2xy^{-3})^{-2}}{(3x^{-2}y^4)^{-3}}$

**26.** $\left(\dfrac{a^2b^3c^4}{a^{-2}b^{-3}c^{-4}}\right)^{-2}$

**27.** $(6x^{-5}z^3)^{-3}$

**28.** $(2p^{-2}qr^{-3})(2p)^{-4}$

**29.** $\dfrac{(xy)^{-3}(xy)^5}{(xy)^{-4}}$

**30.** $52^0 - (-8)^0$

**31.** $\dfrac{(7^{-1}x^{-3})^{-2}(x^4)^{-6}}{7^{-1}x^{-3}}$

**32.** $\left(\dfrac{3^{-4}x^{-3}}{3^{-3}x^{-6}}\right)^{-2}$

**33.** $(5p^{-2}q)^{-3}(5pq^3)^4$

**34.** $8^{-1} + 6^{-1}$

**35.** $\left[\dfrac{4r^{-6}s^{-2}t}{2r^8s^{-4}t^2}\right]^{-1}$

**36.** $(13x^{-6}y)(13x^{-6}y)^{-1}$

**37.** $\dfrac{(8pq^{-2})^4}{(8p^{-2}q^{-3})^3}$

**38.** $\left(\dfrac{mn^{-2}p}{m^2np^4}\right)^{-2}\left(\dfrac{mn^{-2}p}{m^2np^4}\right)^3$

**39.** $-(-8^0)^0$

**40.** ***Concept Check*** *Match each expression (a)–(j) in Column I with the equivalent expression A–J in Column II. Choices in Column II may be used once, more than once, or not at all.*

**I**

**(a)** $2^0 + 2^0$

**(b)** $2^1 \cdot 2^0$

**(c)** $2^0 - 2^{-1}$

**(d)** $2^1 - 2^0$

**(e)** $2^0 \cdot 2^{-2}$

**(f)** $2^1 \cdot 2^1$

**(g)** $2^{-2} - 2^{-1}$

**(h)** $2^0 \cdot 2^0$

**(i)** $2^{-2} \div 2^{-1}$

**(j)** $2^0 \div 2^{-2}$

**II**

**A.** $0$

**B.** $1$

**C.** $-1$

**D.** $2$

**E.** $\dfrac{1}{2}$

**F.** $4$

**G.** $-2$

**H.** $-4$

**I.** $-\dfrac{1}{4}$

**J.** $\dfrac{1}{4}$

# 4.3 An Application of Exponents: Scientific Notation

**OBJECTIVES**

1. Express numbers in scientific notation.
2. Convert numbers in scientific notation to numbers without exponents.
3. Use scientific notation in calculations.

**OBJECTIVE 1 Express numbers in scientific notation.** Numbers occurring in science are often extremely large (such as the distance from Earth to the sun, 93,000,000 mi) or extremely small (the wavelength of yellow-green light, approximately 0.0000006 m). Because of the difficulty of working with many zeros, scientists often express such numbers with exponents, using a form called *scientific notation.*

### Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$\boldsymbol{a \times 10^n}, \qquad \text{where } 1 \le |a| < 10 \text{ and } n \text{ is an integer.}$$

In scientific notation, there is always one nonzero digit before the decimal point.

| | |
|---|---|
| $3.19 \times 10^1 = 3.19 \times 10 = 31.9$ | Decimal point moves 1 place to the right. |
| $3.19 \times 10^2 = 3.19 \times 100 = 319.$ | Decimal point moves 2 places to the right. |
| $3.19 \times 10^3 = 3.19 \times 1000 = 3190.$ | Decimal point moves 3 places to the right. |
| $3.19 \times 10^{-1} = 3.19 \times 0.1 = 0.319$ | Decimal point moves 1 place to the left. |
| $3.19 \times 10^{-2} = 3.19 \times 0.01 = 0.0319$ | Decimal point moves 2 places to the left. |
| $3.19 \times 10^{-3} = 3.19 \times 0.001 = 0.00319$ | Decimal point moves 3 places to the left. |

**NOTE** In work with scientific notation, the times symbol, ×, is commonly used.

***A number in scientific notation is always written with the decimal point after the first nonzero digit and then multiplied by the appropriate power of 10.*** For example, 56,200 is written $5.62 \times 10^4$, since

$$56{,}200 = 5.62 \times 10{,}000 = 5.62 \times 10^4.$$

Other examples include

| | | | |
|---|---|---|---|
| | 42,000,000 | written | $4.2 \times 10^7$, |
| | 0.000586 | written | $5.86 \times 10^{-4}$, |
| and | 2,000,000,000 | written | $2 \times 10^9$. |

It is not necessary to write 2.0.

To write a number in scientific notation, follow these steps. (For a negative number, follow these steps using the *absolute value* of the number. Then make the result negative.)

### Writing a Number in Scientific Notation

***Step 1*** Move the decimal point to the right of the first nonzero digit.

***Step 2*** Count the number of places you moved the decimal point.

***Step 3*** The number of places in Step 2 is the absolute value of the exponent on 10.

***Step 4*** The exponent on 10 is positive if the original number is greater than the number in Step 1. The exponent is negative if the original number is less than the number in Step 1. If the decimal point is not moved, the exponent is 0.

**NOW TRY EXERCISE 1**

Write each number in scientific notation.

**(a)** 12,600,000

**(b)** 0.00027

**(c)** $-0.0000341$

### EXAMPLE 1 Using Scientific Notation

Write each number in scientific notation.

**(a)** 93,000,000

Move the decimal point to follow the first nonzero digit (the 9). Count the number of places the decimal point was moved.

$$9\underbrace{3{,}000{,}000}_{\text{7 places}}. \longleftarrow \text{Decimal point}$$

The number will be written in scientific notation as $9.3 \times 10^n$. To find the value of $n$, first compare the original number, 93,000,000, with 9.3. Since 93,000,000 is *greater* than 9.3, we must multiply by a *positive* power of 10 so that the product $9.3 \times 10^n$ will equal the larger number.

Since the decimal point was moved seven places, and since $n$ is positive,

$$93{,}000{,}000 = 9.3 \times 10^7.$$

**(b)** $63{,}200{,}000{,}000 = 6.\underbrace{3200000000}_{\text{10 places}} = 6.32 \times 10^{10}$

**(c)** 0.00462

Move the decimal point to the right of the first nonzero digit, and count the number of places the decimal point was moved.

$$0.\underbrace{004}_{}62 \qquad \text{3 places}$$

Since 0.00462 is *less* than 4.62, the exponent must be *negative*.

$$0.\underbrace{004}_{}62 = 4.62 \times 10^{-3}$$

**(d)** $-0.\underbrace{00007}_{\text{5 places}}62 = -7.62 \times 10^{-5}$

Remember the negative sign.

NOW TRY

**NOTE** To choose the exponent when you write a positive number in scientific notation, think as follows.

1. If the original number is "large," like 93,000,000, use a *positive* exponent on 10, since positive is greater than negative.
2. If the original number is "small," like 0.00462, use a *negative* exponent on 10, since negative is less than positive.

**OBJECTIVE 2 Convert numbers in scientific notation to numbers without exponents.** To do this, we work in reverse. ***Multiplying a number by a positive power of 10 will make the number greater. Multiplying by a negative power of 10 will make the number less.***

### EXAMPLE 2 Writing Numbers without Exponents

Write each number without exponents.

**(a)** $6.2 \times 10^3$

Since the exponent is positive, we make 6.2 greater by moving the decimal point three places to the right. We attach two zeros.

$$6.2 \times 10^3 = 6.\underbrace{200}_{} = 6200$$

**NOW TRY ANSWERS**

1. **(a)** $1.26 \times 10^7$
   **(b)** $2.7 \times 10^{-4}$
   **(c)** $-3.41 \times 10^{-5}$

**NOW TRY EXERCISE 2**

Write each number without exponents.

**(a)** $5.71 \times 10^4$

**(b)** $2.72 \times 10^{-5}$

**(b)** $4.283 \times 10^6 = 4.\underbrace{283000} = 4{,}283{,}000$ Move 6 places to the right. Attach zeros as necessary.

**(c)** $7.04 \times 10^{-3} = 0.\underbrace{007}04$ Move 3 places to the left.

***The exponent tells the number of places and the direction that the decimal point is moved.*** NOW TRY

## OBJECTIVE 3 Use scientific notation in calculations.

**NOW TRY EXERCISE 3**

Perform each calculation. Write answers in scientific notation and also without exponents.

**(a)** $(6 \times 10^7)(7 \times 10^{-4})$

**(b)** $\dfrac{18 \times 10^{-3}}{6 \times 10^4}$

### EXAMPLE 3 Multiplying and Dividing with Scientific Notation

Perform each calculation.

**(a)**

$$\begin{aligned}
&(7 \times 10^3)(5 \times 10^4) \\
&= (7 \times 5)(10^3 \times 10^4) && \text{Commutative and associative properties} \\
&= 35 \times 10^7 && \text{Multiply; product rule} \\
&= (3.5 \times 10^1) \times 10^7 && \text{Write 35 in scientific notation.} \\
&= 3.5 \times (10^1 \times 10^7) && \text{Associative property} \\
&= 3.5 \times 10^8 && \text{Product rule} \\
&= 350{,}000{,}000 && \text{Write without exponents.}
\end{aligned}$$

Don't stop! This number is *not* in scientific notation, since 35 is not between 1 and 10.

**(b)** $\dfrac{4 \times 10^{-5}}{2 \times 10^3} = \dfrac{4}{2} \times \dfrac{10^{-5}}{10^3} = 2 \times 10^{-8} = 0.00000002$ NOW TRY

---

**NOTE** Multiplying or dividing numbers written in scientific notation may produce an answer in the form $a \times 10^0$. Since $10^0 = 1$, $a \times 10^0 = a$. For example,

$$(8 \times 10^{-4})(5 \times 10^4) = 40 \times 10^0 = 40. \qquad 10^0 = 1$$

Also, if $a = 1$, then $a \times 10^n = 10^n$. For example, we could write 1,000,000 as $10^6$ instead of $1 \times 10^6$.

---

**NOW TRY EXERCISE 4**

See **Example 4.** About how much would 8,000,000 nanometers measure in inches?

### EXAMPLE 4 Using Scientific Notation to Solve an Application

A *nanometer* is a very small unit of measure that is equivalent to about 0.00000003937 in. About how much would 700,000 nanometers measure in inches? (*Source: World Almanac and Book of Facts*.)

Write each number in scientific notation, and then multiply.

$$\begin{aligned}
&700{,}000(0.00000003937) \\
&= (7 \times 10^5)(3.937 \times 10^{-8}) && \text{Write in scientific notation.} \\
&= (7 \times 3.937)(10^5 \times 10^{-8}) && \text{Properties of real numbers} \\
&= 27.559 \times 10^{-3} && \text{Multiply; product rule} \\
&= (2.7559 \times 10^1) \times 10^{-3} && \text{Write 27.559 in scientific notation.} \\
&= 2.7559 \times 10^{-2} && \text{Product rule} \\
&= 0.027559 && \text{Write without exponents.}
\end{aligned}$$

Don't stop here.

Thus, 700,000 nanometers would measure

$$2.7559 \times 10^{-2} \text{ in.,} \quad \text{or} \quad 0.027559 \text{ in.}$$

NOW TRY

**NOW TRY ANSWERS**

2\. **(a)** 57,100 **(b)** 0.0000272

3\. **(a)** $4.2 \times 10^4$, or 42,000
**(b)** $3 \times 10^{-7}$, or 0.0000003

4\. $3.1496 \times 10^{-1}$ in., or 0.31496 in.

NOW TRY EXERCISE 5

The land area of California is approximately $1.6 \times 10^5$ mi$^2$, and the 2008 estimated population of California was approximately $4 \times 10^7$ people. Use this information to estimate the number of square miles per California resident in 2008. (*Source:* U.S. Census Bureau.)

### EXAMPLE 5 Using Scientific Notation to Solve an Application

In 2008, the national debt was \$$1.0025 \times 10^{13}$ (which is more than \$10 trillion). The population of the United States was approximately 304 million that year. About how much would each person have had to contribute in order to pay off the national debt? (*Source*: Bureau of Public Land; U.S. Census Bureau.)

Write the population in scientific notation. Then divide to obtain the per person contribution.

$$\frac{1.0025 \times 10^{13}}{304{,}000{,}000} = \frac{1.0025 \times 10^{13}}{3.04 \times 10^8} \qquad \text{Write 304 million in scientific notation.}$$

$$= \frac{1.0025}{3.04} \times 10^5 \qquad \text{Quotient rule}$$

$$= 0.32977 \times 10^5 \qquad \text{Divide. Round to 5 decimal places.}$$

$$= 32{,}977 \qquad \text{Write without exponents.}$$

Each person would have to pay about \$32,977. NOW TRY

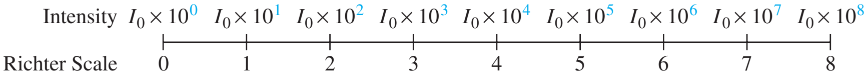
## CONNECTIONS

In 1935, Charles F. Richter devised a scale to compare the intensities of earthquakes. The *intensity* of an earthquake is measured relative to the intensity of a standard *zero-level* earthquake of intensity $I_0$. The relationship is equivalent to $I = I_0 \times 10^R$, where $R$ is the **Richter scale** measure.

For example, if an earthquake has magnitude 5.0 on the Richter scale, then its intensity is calculated as

$$I = I_0 \times 10^{5.0} = I_0 \times 100{,}000,$$

which is 100,000 times as intense as a zero-level earthquake.




To compare two earthquakes, a ratio of the intensities is calculated. For example, to compare an earthquake that measures 8.0 on the Richter scale with one that measures 5.0, find the ratio of the intensities.

$$\frac{\text{intensity } 8.0}{\text{intensity } 5.0} = \frac{I_0 \times 10^{8.0}}{I_0 \times 10^{5.0}} = \frac{10^8}{10^5} = 10^{8-5} = 10^3 = 1000$$

Therefore, an earthquake that measures 8.0 on the Richter scale is 1000 times as intense as one that measures 5.0.

### For Discussion or Writing

| *Year* | *Earthquake Location* | *Richter Scale Measurement* |
|---|---|---|
| 1964 | Prince William Sound, Alaska | 9.2 |
| 2004 | Sumatra, Indonesia | 9.0 |
| 2007 | Central Peru | 8.0 |
| 2008 | E. Sichuan Province, China | 7.9 |
| 2002 | Hindu Kush, Afghanistan | 5.9 |

*Source*: U.S. Geological Survey.

NOW TRY ANSWER

**5.** $4 \times 10^{-3}$ mi$^2$, or 0.004 mi$^2$

**1.** Compare the intensity of the 2004 Indonesia earthquake with the 2007 Peru earthquake.

**2.** Compare the intensity of the 2002 Afghanistan earthquake with the 2008 China earthquake.

**3.** Compare the intensity of the 1964 Alaska earthquake with the 2008 China earthquake. (*Hint*: Use a calculator.)

**4.** Suppose an earthquake measures 7.2 on the Richter scale. How would the intensity of a second earthquake compare if its Richter scale measure differed by $+3.0$? By $-1.0$?

## 4.3 EXERCISES

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

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***Concept Check*** *Match each number written in scientific notation in Column I with the correct choice from Column II. Not all choices in Column II will be used.*

| I | II | I | II |
|---|---|---|---|
| **1. (a)** $4.6 \times 10^{-4}$ | **A.** 46,000 | **2. (a)** $1 \times 10^{9}$ | **A.** 1 billion |
| **(b)** $4.6 \times 10^{4}$ | **B.** 460,000 | **(b)** $1 \times 10^{6}$ | **B.** 100 million |
| **(c)** $4.6 \times 10^{5}$ | **C.** 0.00046 | **(c)** $1 \times 10^{8}$ | **C.** 1 million |
| **(d)** $4.6 \times 10^{-5}$ | **D.** 0.000046 | **(d)** $1 \times 10^{10}$ | **D.** 10 billion |
| | **E.** 4600 | | **E.** 100 billion |

***Concept Check*** *Determine whether or not each number is written in scientific notation as defined in* ***Objective 1.*** *If it is not, write it as such.*

**3.** $4.56 \times 10^{4}$ **4.** $7.34 \times 10^{6}$ **5.** 5,600,000 **6.** 34,000
**7.** $0.8 \times 10^{2}$ **8.** $0.9 \times 10^{3}$ **9.** 0.004 **10.** 0.0007

✎ **11.** Explain what it means for a number to be written in scientific notation. Give examples.

✎ **12.** Explain how to multiply a number by a positive power of 10. Then explain how to multiply a number by a negative power of 10.

*Write each number in scientific notation.* ***See Example 1.***

⊛ **13.** 5,876,000,000 **14.** 9,994,000,000 **15.** 82,350 **16.** 78,330
**17.** 0.000007 **18.** 0.0000004 **19.** 0.00203 **20.** 0.0000578
**21.** $-13{,}000{,}000$ **22.** $-25{,}000{,}000{,}000$ **23.** $-0.006$ **24.** $-0.01234$

*Write each number without exponents.* ***See Example 2.***

⊛ **25.** $7.5 \times 10^{5}$ **26.** $8.8 \times 10^{6}$ **27.** $5.677 \times 10^{12}$ **28.** $8.766 \times 10^{9}$
**29.** $1 \times 10^{12}$ **30.** $1 \times 10^{7}$ **31.** $6.21 \times 10^{0}$ **32.** $8.56 \times 10^{0}$
**33.** $7.8 \times 10^{-4}$ **34.** $8.9 \times 10^{-5}$ **35.** $5.134 \times 10^{-9}$ **36.** $7.123 \times 10^{-10}$
**37.** $-4 \times 10^{-3}$ **38.** $-6 \times 10^{-4}$ **39.** $-8.1 \times 10^{5}$ **40.** $-9.6 \times 10^{6}$

*Perform the indicated operations. Write each answer* ***(a)*** *in scientific notation and* ***(b)*** *without exponents.* ***See Example 3.***

**41.** $(2 \times 10^{8})(3 \times 10^{3})$ **42.** $(4 \times 10^{7})(3 \times 10^{3})$
⊛ **43.** $(5 \times 10^{4})(3 \times 10^{2})$ **44.** $(8 \times 10^{5})(2 \times 10^{3})$

**45.** $(3 \times 10^{-4})(-2 \times 10^{8})$

**46.** $(4 \times 10^{-3})(-2 \times 10^{7})$

**47.** $(6 \times 10^{3})(4 \times 10^{-2})$

**48.** $(7 \times 10^{5})(3 \times 10^{-4})$

**49.** $(9 \times 10^{4})(7 \times 10^{-7})$

**50.** $(6 \times 10^{4})(8 \times 10^{-8})$

**51.** $\dfrac{9 \times 10^{-5}}{3 \times 10^{-1}}$

**52.** $\dfrac{12 \times 10^{-4}}{4 \times 10^{-3}}$

**53.** $\dfrac{8 \times 10^{3}}{-2 \times 10^{2}}$

**54.** $\dfrac{15 \times 10^{4}}{-3 \times 10^{3}}$

**55.** $\dfrac{2.6 \times 10^{-3}}{2 \times 10^{2}}$

**56.** $\dfrac{9.5 \times 10^{-1}}{5 \times 10^{3}}$

**57.** $\dfrac{4 \times 10^{5}}{8 \times 10^{2}}$

**58.** $\dfrac{3 \times 10^{9}}{6 \times 10^{5}}$

**59.** $\dfrac{-4.5 \times 10^{4}}{1.5 \times 10^{-2}}$

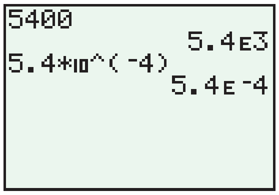
**60.** $\dfrac{-7.2 \times 10^{3}}{6.0 \times 10^{-1}}$

**61.** $\dfrac{-8 \times 10^{-4}}{-4 \times 10^{3}}$

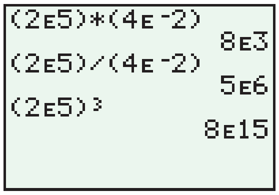
**62.** $\dfrac{-5 \times 10^{-6}}{-2 \times 10^{2}}$

## TECHNOLOGY INSIGHTS EXERCISES 63–68

*Graphing calculators such as the TI-83/84 Plus can display numbers in scientific notation (when in scientific mode), using the format shown in the screen on the left. For* 5400, *the calculator displays* 5.4E3 *to represent* $5.4 \times 10^3$. *The display* 5.4E-4 *means* $5.4 \times 10^{-4}$. *The calculator will also perform operations with numbers entered in scientific notation, as shown in the screen on the right. Notice how the rules for exponents are applied.*



```
5400
          5.4E3
5.4*10^(-4)
         5.4E-4
```



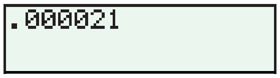
```
(2E5)*(4E-2)
            8E3
(2E5)/(4E-2)
            5E6
(2E5)³
           8E15
```

*Predict the display the calculator would give for the expression shown in each screen.*

**63.**

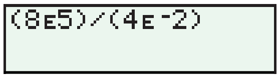
```
.00000047
```

**64.**



```
.000021
```

**65.**

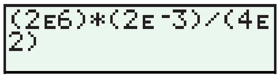


```
(8E5)/(4E-2)
```

**66.**

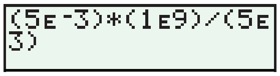
```
(9E-4)/(3E3)
```

**67.**



```
(2E6)*(2E-3)/(4E
2)
```

**68.**



```
(5E-3)*(1E9)/(5E
3)
```

***Brain Busters*** *Use scientific notation to calculate the answer to each problem. Write answers in scientific notation.*

**69.** $\dfrac{650{,}000{,}000(0.0000032)}{0.00002}$

**70.** $\dfrac{3{,}400{,}000{,}000(0.000075)}{0.00025}$

**71.** $\dfrac{0.00000072(0.00023)}{0.000000018}$

**72.** $\dfrac{0.000000081(0.000036)}{0.00000048}$

**73.** $\dfrac{0.0000016(240{,}000{,}000)}{0.00002(0.0032)}$

**74.** $\dfrac{0.000015(42{,}000{,}000)}{0.000009(0.000005)}$

*Each statement comes from* Astronomy! A Brief Edition *by James B. Kaler (Addison-Wesley). If the number in boldface italics is in scientific notation, write it without exponents. If the number is written without exponents, write it in scientific notation.*

**75.** Multiplying this view over the whole sky yields a galaxy count of more than ***10 billion.*** (page 496)

**76.** The circumference of the solar orbit is . . . about ***4.7 million*** km . . . . (in reference to the orbit of Jupiter, page 395)

**77.** The solar luminosity requires that $\boldsymbol{2 \times 10^{9}}$ kg of mass be converted into energy every second. (page 327)

**78.** At maximum, a cosmic ray particle—a mere atomic nucleus of only $\boldsymbol{10^{-13}}$ cm across—can carry the energy of a professionally pitched baseball. (page 445)

*Each statement contains a number in boldface italics. Write the number in scientific notation.*

**79.** At the end of 2007, the total number of cellular telephone subscriptions in the world reached about ***3,305,000,000.*** (*Source*: International Telecommunications Union.)

**80.** In 2007, the leading U.S. advertiser was the Procter and Gamble Company, which spent approximately ***$5,230,000,000.*** (*Source*: Crain Communications, Inc.)

**81.** During 2008, worldwide motion picture box office receipts (in U.S. dollars) totaled ***$28,100,000,000.*** (*Source*: Motion Picture Association of America.)

**82.** In 2007, assets of the insured commercial banks in the United States totaled about ***$13,039,000,000,000.*** (*Source*: U.S. Federal Deposit Insurance Corporation.)

*Use scientific notation to calculate the answer to each problem.* ***See Examples 3–5.***

**83.** The body of a 150-lb person contains about $2.3 \times 10^{-4}$ 1b of copper. How much copper is contained in the bodies of 1200 such people?

**84.** In 2007, the state of Minnesota had about $7.9 \times 10^{4}$ farms with an average of $3.5 \times 10^{2}$ acres per farm. What was the total number of acres devoted to farmland in Minnesota that year? (*Source*: U.S. Department of Agriculture.)

**85.** Venus is $6.68 \times 10^{7}$ mi from the sun. If light travels at a speed of $1.86 \times 10^{5}$ mi per sec, how long does it take light to travel from the sun to Venus? (*Source*: *World Almanac and Book of Facts.*)

**86.** **(a)** The distance to Earth from Pluto is $4.58 \times 10^{9}$ km. In April 1983, *Pioneer 10* transmitted radio signals from Pluto to Earth at the speed of light, $3.00 \times 10^{5}$ km per sec. How long (in seconds) did it take for the signals to reach Earth?

**(b)** How many hours did it take for the signals to reach Earth?

**87.** During the 2007–2008 season, Broadway shows grossed a total of $9.38 \times 10^{8}$ dollars. Total attendance for the season was $1.23 \times 10^{7}$. What was the average ticket price for a Broadway show? (*Source*: The Broadway League.)

**88.** In 2007, $9.63 \times 10^{9}$ dollars were spent to attend motion pictures in the United States. Domestic admissions (the total number of tickets sold) for that year totaled 1.4 billion. What was the average ticket price? (*Source*: Motion Picture Association of America.)

**89.** On February 17, 2009, Congress raised the U.S. government's debt limit to $\$1.2 \times 10^{13}$. When this national debt limit is reached, about how much will it be for every man, women, and child in the country? Use 300 million as the population of the United States. (*Source:* The Concord Coalition.)

**90.** In theory there are $1 \times 10^9$ possible Social Security numbers. The population of the United States is about $3 \times 10^8$. How many Social Security numbers are available for each person? (*Source*: U.S. Census Bureau.)

**91.** Astronomers using the Spitzer Space Telescope discovered a twisted double-helix nebula, a conglomeration of dust and gas stretching across the center of the Milky Way galaxy. This nebula is 25,000 light-years from Earth. If one light-year is about 6,000,000,000,000 (that is, 6 trillion) miles, about how many miles is the twisted double-helix nebula from Earth? (*Source*: http://articles.news.aol.com)

**92.** A computer can perform 466,000,000 calculations per second. How many calculations can it perform per minute? Per hour?

**93.** In 2008, the U.S. government collected about $4013 per person in personal income taxes. If the population was 304,000,000, how much did the government collect in taxes for 2008? (*Source*: U.S. Office of Management and Budget.)

**94.** Pollux, one of the brightest stars in the night sky, is 33.7 light-years from Earth. If one light-year is about 6,000,000,000,000 mi, about how many miles is Pollux from Earth? (*Source: World Almanac and Book of Facts.*)

**95.** In September of 2009, the population of the United States was about 307.5 million. To the nearest dollar, calculate how much each person in the United States would have had to contribute in order to make one lucky person a trillionaire (that is, to give that person $1,000,000,000,000). (*Source*: U.S. Census Bureau.)

**96.** In 2006, national expenditures for health care reached $2,106,000,000,000. Using 300 million as the population of the United States, about how much, to the nearest dollar, was spent on health care per person in 2006? (*Source*: U.S. Centers for Medicare and Medicaid Services.)

## PREVIEW EXERCISES

*Simplify.* ***See Section 1.8.***

**97.** $-3(2x + 4) + 4(2x - 6)$

**98.** $-8(-3x + 7) - 4(2x + 3)$

*Evaluate each expression for* $x = 3$. ***See Sections 1.3 and 1.6.***

**99.** $2x^2 - 3x + 10$

**100.** $3x^2 - 3x + 4$

**101.** $4x^3 - 5x^2 + 2x - 5$

**102.** $-4x^3 + 2x^2 - 9x - 2$

# 4.4

# Adding and Subtracting Polynomials; Graphing Simple Polynomials

**OBJECTIVES**

1. Identify terms and coefficients.
2. Add like terms.
3. Know the vocabulary for polynomials.
4. Evaluate polynomials.
5. Add and subtract polynomials.
6. Graph equations defined by polynomials of degree 2.

**OBJECTIVE 1** **Identify terms and coefficients.** In an expression such as

$$4x^3 + 6x^2 + 5x + 8,$$

the quantities $4x^3$, $6x^2$, $5x$, and 8 are called **terms.** (See **Section 1.8.**) In the first (or *leading*) term $4x^3$, the number 4 is called the **numerical coefficient,** or simply the **coefficient,** of $x^3$. In the same way, 6 is the coefficient of $x^2$ in the term $6x^2$, and 5 is the coefficient of $x$ in the term $5x$. The constant term 8 can be thought of as $8 \cdot 1 = 8x^0$, since $x^0 = 1$, so 8 is the coefficient in the term 8.

## EXAMPLE 1 Identifying Coefficients

Name the coefficient of each term in these expressions.

**(a)** $x - 6x^4$ can be written as $1x + (-6x^4)$.

The coefficients are 1 and −6.

**(b)** $5 - v^3$ can be written as $5v^0 + (-1v^3)$.

The coefficients are 5 and −1.

NOW TRY

**NOW TRY EXERCISE 1**
Name the coefficient of each term in the expression.

$$t - 10t^2$$

**OBJECTIVE 2** **Add like terms.** Recall from **Section 1.8** that **like terms** have exactly the same combination of variables, with the same exponents on the variables. ***Only the coefficients may differ.***

$$\left.\begin{array}{l} 19m^5 \quad \text{and} \quad 14m^5 \\ 6y^9, \quad -37y^9, \quad \text{and} \quad y^9 \\ 3pq \quad \text{and} \quad -2pq \\ 2xy^2 \quad \text{and} \quad -xy^2 \end{array}\right\} \text{Examples of like terms} \qquad \left.\begin{array}{l} 7x \quad \text{and} \quad 7y \\ z^4 \quad \text{and} \quad z \\ 2pq \quad \text{and} \quad 2p \\ -4xy^2 \quad \text{and} \quad 5x^2y \end{array}\right\} \text{Examples of unlike terms}$$

Using the distributive property, we combine, or add, like terms by adding their coefficients.

## EXAMPLE 2 Adding Like Terms

Simplify by adding like terms.

**(a)**
$$\begin{aligned} &-4x^3 + 6x^3 \\ &= (-4 + 6)x^3 && \text{Distributive property} \\ &= 2x^3 && \text{Add.} \end{aligned}$$

**(b)**
$$\begin{aligned} &9x^6 - 14x^6 + x^6 \\ &= (9 - 14 + 1)x^6 && x^6 = 1x^6 \\ &= -4x^6 \end{aligned}$$

**(c)**
$$\begin{aligned} &12m^2 + 5m + 4m^2 \\ &= (12 + 4)m^2 + 5m \\ &= 16m^2 + 5m \end{aligned}$$

**(d)**
$$\begin{aligned} &3x^2y + 4x^2y - x^2y \\ &= (3 + 4 - 1)x^2y \\ &= 6x^2y \end{aligned}$$

NOW TRY

**NOW TRY EXERCISE 2**
Simplify by adding like terms.

$$3x^2 - x^2 + 2x$$

**NOW TRY ANSWERS**

**1.** 1; −10
**2.** $2x^2 + 2x$

**CAUTION** In **Example 2(c),** we cannot combine $16m^2$ and $5m$, because the exponents on the variables are different. ***Unlike terms have different variables or different exponents on the same variables.***

**OBJECTIVE 3** **Know the vocabulary for polynomials.** A **polynomial in $x$** is a term or the sum of a finite number of terms of the form $ax^n$, for any real number $a$ and any whole number $n$. For example,

$$16x^8 - 7x^6 + 5x^4 - 3x^2 + 4 \qquad \text{Polynomial in } x \; (\text{The 4 can be written as } 4x^0.)$$

is a polynomial in $x$. This polynomial is written in **descending powers** of the variable, since the exponents on $x$ decrease from left to right. By contrast,

$$2x^3 - x^2 + \frac{4}{x}, \quad \text{or} \quad 2x^3 - x^2 + 4x^{-1}, \qquad \text{Not a polynomial}$$

is not a polynomial in $x$. A variable appears in a denominator or to a negative power.

**NOTE** We can define *polynomial* using any variable and not just $x$, as in **Example 2(c).** Polynomials may have terms with more than one variable, as in **Example 2(d).**

The **degree of a term** is the sum of the exponents on the variables. The **degree of a polynomial** is the greatest degree of any nonzero term of the polynomial. The table gives several examples.

| *Term* | *Degree* | *Polynomial* | *Degree* |
|---|---|---|---|
| $3x^4$ | 4 | $3x^4 - 5x^2 + 6$ | 4 |
| $5x$, or $5x^1$ | 1 | $5x + 7$ | 1 |
| $-7$, or $-7x^0$ | 0 | $x^2y + xy - 5y^2$ | 3 |
| $2x^2y$, or $2x^2y^1$ | $2 + 1 = 3$ | $x^5 + 3x^6$ | 6 |

Three types of polynomials are common and are given special names. A polynomial with only one term is called a **monomial.** (*Mono* means "one," as in *mono*rail.)

$$9m, \quad -6y^5, \quad a^2, \quad \text{and} \quad 6 \qquad \text{Monomials}$$

A polynomial with exactly two terms is called a **binomial.** (*Bi-* means "two," as in *bi*cycle.)

$$-9x^4 + 9x^3, \quad 8m^2 + 6m, \quad \text{and} \quad 3m^5 - 9m^2 \qquad \text{Binomials}$$

A polynomial with exactly three terms is called a **trinomial.** (*Tri-* means "three," as in *tri*angle.)

$$9m^3 - 4m^2 + 6, \quad \frac{19}{3}y^2 + \frac{8}{3}y + 5, \quad \text{and} \quad -3m^5 - 9m^2 + 2 \qquad \text{Trinomials}$$

## EXAMPLE 3 Classifying Polynomials

For each polynomial, first simplify, if possible. Then give the degree and tell whether the polynomial is a *monomial,* a *binomial,* a *trinomial,* or *none of these.*

**(a)** $2x^3 + 5$ The polynomial cannot be simplified. It is a binomial of degree 3.

**(b)** $4xy - 5xy + 2xy$

Add like terms: $4xy - 5xy + 2xy = xy$, which is a monomial of degree 2.

NOW TRY

NOW TRY EXERCISE 3

Simplify, give the degree, and tell whether the simplified polynomial is a *monomial,* a *binomial,* a *trinomial,* or *none of these.*

$$x^2 + 4x - 2x - 8$$

NOW TRY ANSWER

**3.** $x^2 + 2x - 8$; degree 2; trinomial

**OBJECTIVE 4** **Evaluate polynomials.** A polynomial usually represents different numbers for different values of the variable.

**NOW TRY EXERCISE 4**

Find the value for $t = -3$.

$$4t^3 - t^2 - t$$

**EXAMPLE 4** Evaluating a Polynomial

Find the value of $3x^4 + 5x^3 - 4x - 4$ for **(a)** $x = -2$ and **(b)** $x = 3$.

**(a)** First, substitute $-2$ for $x$.

$$\begin{aligned} &3x^4 + 5x^3 - 4x - 4 \\ &= 3(-2)^4 + 5(-2)^3 - 4(-2) - 4 && \text{Let } x = -2. \\ &= 3(16) + 5(-8) - 4(-2) - 4 && \text{Apply the exponents.} \\ &= 48 - 40 + 8 - 4 && \text{Multiply.} \\ &= 12 && \text{Add and subtract.} \end{aligned}$$

Use parentheses to avoid errors.

**(b)**

$$\begin{aligned} &3x^4 + 5x^3 - 4x - 4 \\ &= 3(3)^4 + 5(3)^3 - 4(3) - 4 && \text{Let } x = 3. \\ &= 3(81) + 5(27) - 12 - 4 && \text{Apply the exponents.} \\ &= 243 + 135 - 12 - 4 && \text{Multiply.} \\ &= 362 && \text{Add and subtract.} \end{aligned}$$

Replace $x$ with 3.

NOW TRY

---

**CAUTION** Use parentheses around the numbers that are substituted for the variable, as in **Example 4.** ***Be particularly careful when substituting a negative number for a variable that is raised to a power, or a sign error may result.***

---

**OBJECTIVE 5** **Add and subtract polynomials.**

**Adding Polynomials**

To add two polynomials, add like terms.

**NOW TRY EXERCISE 5**

Add $4y^3 - 2y^2 + y - 1$ and $y^3 - y - 7$ vertically.

**EXAMPLE 5** Adding Polynomials Vertically

**(a)** Add $6x^3 - 4x^2 + 3$ and $-2x^3 + 7x^2 - 5$.

$$\begin{array}{r} 6x^3 - 4x^2 + 3 \\ \underline{-2x^3 + 7x^2 - 5} \end{array} \qquad \text{Write like terms in columns.}$$

Now add, column by column.

$$\begin{array}{r} 6x^3 \\ \underline{-2x^3} \\ 4x^3 \end{array} \qquad \begin{array}{r} -4x^2 \\ \underline{7x^2} \\ 3x^2 \end{array} \qquad \begin{array}{r} 3 \\ \underline{-5} \\ -2 \end{array}$$

Combine the coefficients only. Do *not* add the exponents.

Add the three sums together.

$$4x^3 + 3x^2 + (-2) = 4x^3 + 3x^2 - 2 \longleftarrow \text{Final sum}$$

**(b)** Add $2x^2 - 4x + 3$ and $x^3 + 5x$.

Write like terms in columns and add column by column.

$$\begin{array}{rrrr} & 2x^2 & -\ 4x & +\ 3 \\ x^3 & & +\ 5x & \\ \hline x^3 & +\ 2x^2 & +\ \ x & +\ 3 \end{array}$$

Leave spaces for missing terms.

NOW TRY

**NOW TRY ANSWERS**

**4.** $-114$

**5.** $5y^3 - 2y^2 - 8$

The polynomials in **Example 5** also can be added horizontally.

**NOW TRY EXERCISE 6**

Add $10x^4 - 3x^2 - x$ and $x^4 - 3x^2 + 5x$ horizontally.

## EXAMPLE 6 Adding Polynomials Horizontally

**(a)** Add $6x^3 - 4x^2 + 3$ and $-2x^3 + 7x^2 - 5$.
Combine like terms.

$$(6x^3 - 4x^2 + 3) + (-2x^3 + 7x^2 - 5) = 4x^3 + 3x^2 - 2$$

Same answer as found in **Example 5(a)**

**(b)** Add $2x^2 - 4x + 3$ and $x^3 + 5x$.

$$(2x^2 - 4x + 3) + (x^3 + 5x)$$

$$= x^3 + 2x^2 - 4x + 5x + 3 \qquad \text{Commutative property}$$

$$= x^3 + 2x^2 + x + 3 \qquad \text{Combine like terms.}$$

NOW TRY

In **Section 1.5,** we defined the difference $x - y$ as $x + (-y)$. (We find the difference $x - y$ by adding $x$ and the opposite of $y$.) For example,

$$7 - 2 = 7 + (-2) = 5 \quad \text{and} \quad -8 - (-2) = -8 + 2 = -6.$$

A similar method is used to subtract polynomials.

### Subtracting Polynomials

To subtract two polynomials, change all the signs in the second polynomial and add the result to the first polynomial.

**NOW TRY EXERCISE 7**

Subtract $5t^4 - 3t^2 + 1$ from $4t^4 - t^2 + 7$.

## EXAMPLE 7 Subtracting Polynomials Horizontally

**(a)** Perform the subtraction $(5x - 2) - (3x - 8)$.

$$(5x - 2) - (3x - 8)$$

$$= (5x - 2) + [-(3x - 8)] \qquad \text{Definition of subtraction}$$

$$= (5x - 2) + [-1(3x - 8)] \qquad -a = -1a$$

$$= (5x - 2) + (-3x + 8) \qquad \text{Distributive property}$$

$$= 2x + 6 \qquad \text{Combine like terms.}$$

**(b)** Subtract $6x^3 - 4x^2 + 2$ from $11x^3 + 2x^2 - 8$.

$$(11x^3 + 2x^2 - 8) - (6x^3 - 4x^2 + 2)$$

Be careful to write the problem in the correct order.

$$= (11x^3 + 2x^2 - 8) + (-6x^3 + 4x^2 - 2)$$

$$= 5x^3 + 6x^2 - 10 \qquad \text{Answer}$$

*CHECK* To check a subtraction problem, use the fact that

$$\text{if} \quad a - b = c, \quad \text{then} \quad a = b + c.$$

Here, add $6x^3 - 4x^2 + 2$ and $5x^3 + 6x^2 - 10$.

$$(6x^3 - 4x^2 + 2) + (5x^3 + 6x^2 - 10)$$

$$= 11x^3 + 2x^2 - 8 \quad ✓$$

NOW TRY

**NOW TRY ANSWERS**

**6.** $11x^4 - 6x^2 + 4x$

**7.** $-t^4 + 2t^2 + 6$

We use vertical subtraction in **Section 4.7** when we divide polynomials.

**NOW TRY EXERCISE 8**

Subtract by columns.

$$(12x^2 - 9x + 4) \\ - (-10x^2 - 3x + 7)$$

### EXAMPLE 8 Subtracting Polynomials Vertically

Subtract by columns to find

$$(14y^3 - 6y^2 + 2y - 5) - (2y^3 - 7y^2 - 4y + 6).$$

$$\begin{array}{r} 14y^3 - 6y^2 + 2y - 5 \\ 2y^3 - 7y^2 - 4y + 6 \\ \hline \end{array}$$

Arrange like terms in columns.

Change all signs in the second row, and then add.

$$\begin{array}{rrrr} 14y^3 & - \;6y^2 & + \;2y & - \;\;5 \\ -2y^3 & + \;7y^2 & + \;4y & - \;\;6 \\ \hline 12y^3 & + \;\;\;y^2 & + \;6y & - \;11 \end{array}$$

Change all signs.

Add.

NOW TRY

**NOW TRY EXERCISE 9**

Subtract.

$$(4x^2 - 2xy + y^2) \\ - (6x^2 - 7xy + 2y^2)$$

### EXAMPLE 9 Adding and Subtracting Polynomials with More Than One Variable

Add or subtract as indicated.

**(a)** $(4a + 2ab - b) + (3a - ab + b)$

$= 4a + 2ab - b + 3a - ab + b$

$= 7a + ab$ Combine like terms.

**(b)** $(2x^2y + 3xy + y^2) - (3x^2y - xy - 2y^2)$

$= 2x^2y + 3xy + y^2 - 3x^2y + xy + 2y^2$

$= -x^2y + 4xy + 3y^2$

Be careful with signs.
The coefficient of *xy* is 1.

NOW TRY

**OBJECTIVE 6** **Graph equations defined by polynomials of degree 2.** In **Chapter 3,** we introduced graphs of linear equations (which are actually polynomial equations of degree 1). By plotting points selectively, we can graph polynomial equations of degree 2.

### EXAMPLE 10 Graphing Equations Defined by Polynomials of Degree 2

Graph each equation.

**(a)** $y = x^2$

Select values for $x$. Then find the corresponding $y$-values. Selecting $x = 2$ gives

$$y = x^2 = 2^2 = 4,$$

so the point $(2, 4)$ is on the graph of $y = x^2$. (Recall that in an ordered pair such as $(2, 4)$, ***the x-value comes first and the y-value second.***) We show some ordered pairs that satisfy $y = x^2$ in the table with **FIGURE 3** on the next page. If we plot the ordered pairs from the table on a coordinate system and draw a smooth curve through them, we obtain the graph shown in **FIGURE 3**.

**NOW TRY ANSWERS**

**8.** $22x^2 - 6x - 3$

**9.** $-2x^2 + 5xy - y^2$

NOW TRY EXERCISE 10

Graph $y = -x^2 - 1$.

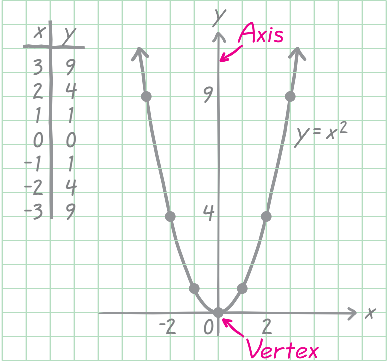


FIGURE 3

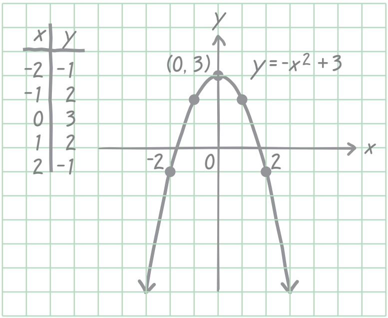


FIGURE 4

The graph of $y = x^2$ in **FIGURE 3** is called a **parabola.** The point $(0, 0)$, the *lowest* point on this graph, is called the **vertex** of the parabola. The vertical line through the vertex (the $y$-axis here) is called the **axis** of the parabola. The axis of a parabola is a **line of symmetry** for the graph. If the graph is folded on this line, the two halves will match.

**(b)** $y = -x^2 + 3$

Once again, plot points to obtain the graph. For example, if $x = -2$, then

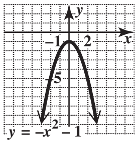
$$y = -(-2)^2 + 3 = -4 + 3 = -1.$$

The point $(-2, -1)$ and several others are shown in the table that accompanies the graph in **FIGURE 4**. The vertex of this parabola is $(0, 3)$. Now the vertex is the *highest* point on the graph. The graph opens downward because $x^2$ has a negative coefficient.

NOW TRY

NOW TRY ANSWER

**10.**



**NOTE** ***All polynomials of degree 2 have parabolas as their graphs.*** When graphing, find points until the vertex and points on either side of it are located. (In this section, all parabolas have their vertices on the $x$-axis or the $y$-axis.)

## 4.4 EXERCISES

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*Complete solution available on the Video Resources on DVD*

*Concept Check* *Fill in each blank with the correct response.*

**1.** In the term $4x^6$, the coefficient is ______ and the exponent is ______.

**2.** The expression $4x^3 - 5x^2$ has ____________ term(s). (how many?)

**3.** The degree of the term $-3x^9$ is ______.

**4.** The polynomial $4x^2 + y^2$ ____________ an example of a trinomial. (is/is not)

**5.** When $x^2 + 10$ is evaluated for $x = 3$, the result is ______.

**6.** $5x^{\_\_\_} + 3x^3 - 7x$ is a trinomial of degree 6.

**7.** $-3xy - 2xy + 5xy =$ ______

**8.** ______ is an example of a monomial with coefficient 8, in the variable $x$, having degree 5.

*For each polynomial, determine the number of terms and name the coefficients of the terms.* ***See Example 1.***

**9.** $6x^4$ **10.** $-9y^5$ **11.** $t^4$ **12.** $s^7$

**13.** $-19r^2 - r$ **14.** $2y^3 - y$ **15.** $x + 8x^2 + 5x^3$ **16.** $v - 2v^3 - v^7$

*In each polynomial, add like terms whenever possible. Write the result in descending powers of the variable.* ***See Example 2.***

**17.** $-3m^5 + 5m^5$ **18.** $-4y^3 + 3y^3$ **19.** $2r^5 + (-3r^5)$

**20.** $9y^2 + (-19y^2)$ **21.** $0.2m^5 - 0.5m^2$ **22.** $-0.9y + 0.9y^2$

**23.** $-3x^5 + 3x^5 - 5x^5$ **24.** $6x^3 - 9x^3 + 10x^3$

**25.** $-4p^7 + 8p^7 + 5p^9$ **26.** $-3a^8 + 4a^8 - 3a^2$

**27.** $-4xy^2 + 3xy^2 - 2xy^2 + xy^2$ **28.** $3pr^5 - 8pr^5 + pr^5 + 2pr^5$

*For each polynomial, first simplify, if possible, and write it in descending powers of the variable. Then give the degree of the resulting polynomial and tell whether it is a* monomial, *a* binomial, *a* trinomial, *or* none of these. ***See Example 3.***

**29.** $6x^4 - 9x$ **30.** $7t^3 - 3t$

**31.** $5m^4 - 3m^2 + 6m^4 - 7m^3$ **32.** $6p^5 + 4p^3 - 8p^5 + 10p^2$

**33.** $\frac{5}{3}x^4 - \frac{2}{3}x^4$ **34.** $\frac{4}{5}r^6 + \frac{1}{5}r^6$

**35.** $0.8x^4 - 0.3x^4 - 0.5x^4 + 7$ **36.** $1.2t^3 - 0.9t^3 - 0.3t^3 + 9$

*Find the value of each polynomial for* ***(a)*** $x = 2$ *and* ***(b)*** $x = -1$. ***See Example 4.***

**37.** $2x^2 - 3x - 5$ **38.** $x^2 + 5x - 10$

**39.** $-3x^2 + 14x - 2$ **40.** $-2x^2 + 5x - 1$

**41.** $2x^5 - 4x^4 + 5x^3 - x^2$ **42.** $x^4 - 6x^3 + x^2 - x$

*Add.* ***See Example 5.***

**43.** $\begin{array}{r} 2x^2 - 4x \\ 3x^2 + 2x \\ \hline \end{array}$ **44.** $\begin{array}{r} -5y^3 + 3y \\ 8y^3 - 4y \\ \hline \end{array}$ **45.** $\begin{array}{r} 3m^2 + 5m + 6 \\ 2m^2 - 2m - 4 \\ \hline \end{array}$

**46.** $\begin{array}{r} 4a^3 - 4a^2 - 4 \\ 6a^3 + 5a^2 - 8 \\ \hline \end{array}$ **47.** $\begin{array}{r} \frac{2}{3}x^2 + \frac{1}{5}x + \frac{1}{6} \\ \frac{1}{2}x^2 - \frac{1}{3}x + \frac{2}{3} \\ \hline \end{array}$ **48.** $\begin{array}{r} \frac{4}{7}y^2 - \frac{1}{5}y + \frac{7}{9} \\ \frac{1}{3}y^2 - \frac{1}{3}y + \frac{2}{5} \\ \hline \end{array}$

**49.** $9m^3 - 5m^2 + 4m - 8$ and $-3m^3 + 6m^2 - 6$

**50.** $12r^5 + 11r^4 - 7r^3 - 2r^2$ and $-8r^5 + 3r^3 + 2r^2$

*Subtract.* ***See Example 8.***

**51.** $\begin{array}{r} 5y^3 - 3y^2 \\ 2y^3 + 8y^2 \\ \hline \end{array}$ **52.** $\begin{array}{r} -6t^3 + 4t^2 \\ 8t^3 - 6t^2 \\ \hline \end{array}$

**53.** $\begin{array}{r} 12x^4 - \phantom{3}x^2 + \phantom{3}x \\ 8x^4 + 3x^2 - 3x \\ \hline \end{array}$ **54.** $\begin{array}{r} 13y^5 - \phantom{5}y^3 - 8y^2 \\ 7y^5 + 5y^3 + \phantom{8}y^2 \\ \hline \end{array}$

**55.** $\begin{array}{r} 12m^3 - 8m^2 + 6m + 7 \\ -3m^3 + 5m^2 - 2m - 4 \\ \hline \end{array}$ **56.** $\begin{array}{r} 5a^4 - 3a^3 + 2a^2 - a + 6 \\ -6a^4 + \phantom{3}a^3 - \phantom{2}a^2 + a - 1 \\ \hline \end{array}$

**57.** After reading **Examples 5–8,** do you have a preference regarding horizontal or vertical addition and subtraction of polynomials? Explain your answer.

**58.** Write a paragraph explaining how to add and subtract polynomials. Give an example using addition.

*Perform each indicated operation.* ***See Examples 6 and 7.***

**59.** $(8m^2 - 7m) - (3m^2 + 7m - 6)$

**60.** $(x^2 + x) - (3x^2 + 2x - 1)$

**61.** $(16x^3 - x^2 + 3x) + (-12x^3 + 3x^2 + 2x)$

**62.** $(-2b^6 + 3b^4 - b^2) + (b^6 + 2b^4 + 2b^2)$

**63.** Subtract $18y^4 - 5y^2 + y$ from $7y^4 + 3y^2 + 2y$.

**64.** Subtract $19t^5 - 6t^3 + t$ from $8t^5 + 3t^3 + 5t$.

**65.** $(9a^4 - 3a^2 + 2) + (4a^4 - 4a^2 + 2) + (-12a^4 + 6a^2 - 3)$

**66.** $(4m^2 - 3m + 2) + (5m^2 + 13m - 4) + (-16m^2 - 4m + 3)$

**67.** $[(8m^2 + 4m - 7) - (2m^2 - 5m + 2)] - (m^2 + m + 1)$

**68.** $[(9b^3 - 4b^2 + 3b + 2) - (-2b^3 - 3b^2 + b)] - (8b^3 + 6b + 4)$

**69.** $[(3x^2 - 2x + 7) - (4x^2 + 2x - 3)] - [(9x^2 + 4x - 6) + (-4x^2 + 4x + 4)]$

**70.** $[(6t^2 - 3t + 1) - (12t^2 + 2t - 6)] - [(4t^2 - 3t - 8) + (-6t^2 + 10t - 12)]$

**71.** ***Concept Check*** Without actually performing the operations, determine mentally the coefficient of the $x^2$-term in the simplified form of

$$(-4x^2 + 2x - 3) - (-2x^2 + x - 1) + (-8x^2 + 3x - 4).$$

**72.** ***Concept Check*** Without actually performing the operations, determine mentally the coefficient of the $x$-term in the simplified form of

$$(-8x^2 - 3x + 2) - (4x^2 - 3x + 8) - (-2x^2 - x + 7).$$

*Add or subtract as indicated.* ***See Example 9.***

**73.** $(6b + 3c) + (-2b - 8c)$

**74.** $(-5t + 13s) + (8t - 3s)$

**75.** $(4x + 2xy - 3) - (-2x + 3xy + 4)$

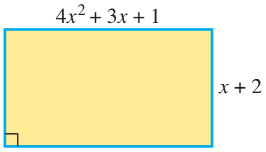
**76.** $(8ab + 2a - 3b) - (6ab - 2a + 3b)$

**77.** $(5x^2y - 2xy + 9xy^2) - (8x^2y + 13xy + 12xy^2)$

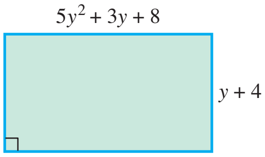
**78.** $(16t^3s^2 + 8t^2s^3 + 9ts^4) - (-24t^3s^2 + 3t^2s^3 - 18ts^4)$

*Find a polynomial that represents the perimeter of each rectangle, square, or triangle.*

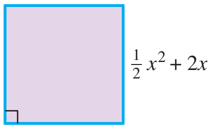
**79.**



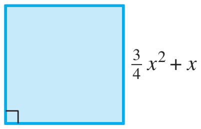
**80.**



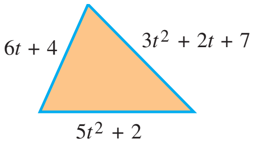
**81.**



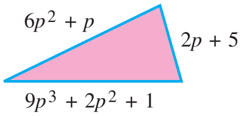
**82.**



**83.**

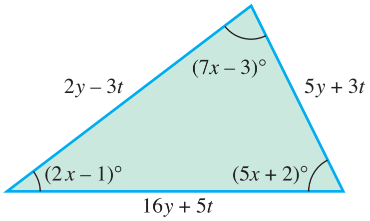


**84.**

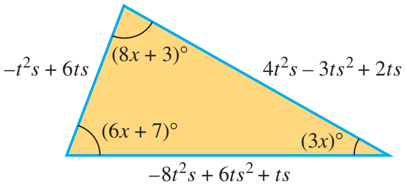


*Find* ***(a)*** *a polynomial that represents the perimeter of each triangle and* ***(b)*** *the degree measures of the angles of the triangle.*

**85.**



**86.**



*Perform each indicated operation.*

**87.** Find the difference between the sum of $5x^2 + 2x - 3$ and $x^2 - 8x + 2$ and the sum of $7x^2 - 3x + 6$ and $-x^2 + 4x - 6$.

**88.** Subtract the sum of $9t^3 - 3t + 8$ and $t^2 - 8t + 4$ from the sum of $12t + 8$ and $t^2 - 10t + 3$.

*Graph each equation by completing the table of values.* ***See Example 10.***

**89.** $y = x^2 - 4$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**90.** $y = x^2 - 6$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**91.** $y = 2x^2 - 1$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**92.** $y = 2x^2 + 2$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**93.** $y = -x^2 + 4$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**94.** $y = -x^2 + 2$

| x | y |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**95.** $y = (x + 3)^2$

| x | −5 | −4 | −3 | −2 | −1 |
|---|---|---|---|---|---|
| y | | | | | |

**96.** $y = (x - 4)^2$

| x | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| y | | | | | |

## *RELATING CONCEPTS* EXERCISES 97–100

### FOR INDIVIDUAL OR GROUP WORK

*The polynomial equation*

$$y = -0.0545x^2 + 5.047x + 11.78$$

*gives a good approximation of the age of a dog in human years y, where x represents age in dog years. Each time we evaluate this polynomial for a value of x, we get one and only one output value y. For example, if a dog is 4 in dog years, let x = 4 to find that y ≈ 31.1. (Verify this.) This means that the dog is about 31 yr old in human years. This illustrates the concept of a* ***function,*** *one of the most important topics in mathematics.*

***Exercises 97–100*** *further illustrate the function concept with polynomials.* ***Work these exercises in order.***

**97.** It used to be thought that each dog year was about 7 human years, so that $y = 7x$ gave the number of human years for $x$ dog years. Evaluate $y$ for $x = 9$, and interpret the result.

**98.** Use the polynomial equation given in the directions above to find the number of human years equivalent to 3 dog years.

**99.** If an object is projected upward under certain conditions, its height in feet is given by the trinomial

$$-16x^2 + 60x + 80,$$

where $x$ is in seconds. Evaluate this polynomial for $x = 2.5$. Use the result to fill in the blanks: If ______ seconds have elapsed, the height of the object is ______ feet.

**100.** If it costs \$15 to rent a chain saw, plus \$2 per day, the binomial $2x + 15$ gives the cost to rent the chain saw for $x$ days. Evaluate this polynomial for $x = 6$. Use the result to fill in the blanks: If the saw is rented for ______ days, the cost is ______.

### PREVIEW EXERCISES

*Multiply. See* ***Section 1.8.***

**101.** $5(x+4)$ **102.** $-3(x^2+7)$ **103.** $4(2a+6b)$ **104.** $\frac{1}{2}(4m-8n)$

*Multiply. See* ***Section 4.1.***

**105.** $(2a)(-5ab)$ **106.** $(3xz)(4x)$ **107.** $(-m^2)(m^5)$ **108.** $(2c)(3c^2)$

## 4.5 Multiplying Polynomials

**OBJECTIVES**

- **1** Multiply a monomial and a polynomial.
- **2** Multiply two polynomials.
- **3** Multiply binomials by the FOIL method.

**OBJECTIVE 1 Multiply a monomial and a polynomial.** As shown in **Section 4.1,** we find the product of two monomials by using the rules for exponents and the commutative and associative properties. Consider this example.

$$\begin{aligned}&-8m^6(-9n^6)\\&\quad=-8(-9)(m^6)(n^6)\\&\quad=72m^6n^6\end{aligned}$$

**CAUTION** ***Do not confuse addition of terms with multiplication of terms.*** For instance,

$$7q^5+2q^5=9q^5, \qquad \text{but} \qquad (7q^5)(2q^5)=7\cdot 2q^{5+5}=14q^{10}.$$

**NOW TRY EXERCISE 1**

Find the product.

$$-3x^5(2x^3-5x^2+10)$$

**EXAMPLE 1 Multiplying Monomials and Polynomials**

Find each product.

**(a)** $4x^2(3x+5)$

$$\begin{aligned}4x^2(3x+5)&=4x^2(3x)+4x^2(5) &&\text{Distributive property}\\&=12x^3+20x^2 &&\text{Multiply monomials.}\end{aligned}$$

**(b)** $-8m^3(4m^3+3m^2+2m-1)$

$$\begin{aligned}&=-8m^3(4m^3)+(-8m^3)(3m^2)\\&\quad+(-8m^3)(2m)+(-8m^3)(-1) &&\text{Distributive property}\\&=-32m^6-24m^5-16m^4+8m^3 &&\text{Multiply monomials.}\end{aligned}$$

NOW TRY

**OBJECTIVE 2 Multiply two polynomials.** To find the product of the polynomials $x^2+3x+5$ and $x-4$, we can think of $x-4$ as a single quantity and use the distributive property as follows.

$$\begin{aligned}&(x^2+3x+5)(x-4)\\&\quad=x^2(x-4)+3x(x-4)+5(x-4) &&\text{Distributive property}\\&\quad=x^2(x)+x^2(-4)+3x(x)+3x(-4)+5(x)+5(-4)\\& &&\text{Distributive property again}\\&\quad=x^3-4x^2+3x^2-12x+5x-20 &&\text{Multiply monomials.}\\&\quad=x^3-x^2-7x-20 &&\text{Combine like terms.}\end{aligned}$$

**NOW TRY ANSWER**

**1.** $-6x^8+15x^7-30x^5$

### Multiplying Polynomials

To multiply two polynomials, multiply each term of the second polynomial by each term of the first polynomial and add the products.

NOW TRY EXERCISE 2

Multiply.

$$(x^2 - 4)(2x^2 - 5x + 3)$$

### EXAMPLE 2 Multiplying Two Polynomials

Multiply $(m^2 + 5)(4m^3 - 2m^2 + 4m)$.

$$\begin{aligned} &(m^2 + 5)(4m^3 - 2m^2 + 4m) \\ &= m^2(4m^3) + m^2(-2m^2) + m^2(4m) + 5(4m^3) + 5(-2m^2) + 5(4m) \\ &= 4m^5 - 2m^4 + 4m^3 + 20m^3 - 10m^2 + 20m \\ &= 4m^5 - 2m^4 + 24m^3 - 10m^2 + 20m \end{aligned}$$

Multiply each term of the second polynomial by each term of the first.

Combine like terms. NOW TRY

NOW TRY EXERCISE 3

Multiply.

$$\begin{array}{r} 5t^2 - 7t + 4 \\ 2t - 6 \\ \hline \end{array}$$

### EXAMPLE 3 Multiplying Polynomials Vertically

Multiply $(x^3 + 2x^2 + 4x + 1)(3x + 5)$ vertically.

$$\begin{array}{r} x^3 + 2x^2 + 4x + 1 \\ 3x + 5 \\ \hline \end{array}$$

Write the polynomials vertically.

Begin by multiplying each of the terms in the top row by 5.

$$\begin{array}{rrrr} x^3 & +\,2x^2 & +\,4x & +\,1 \\ & & 3x & +\,5 \\ \hline 5x^3 & +\,10x^2 & +\,20x & +\,5 \end{array}$$

$5(x^3 + 2x^2 + 4x + 1)$

Now multiply each term in the top row by $3x$. Then add like terms.

$$\begin{array}{rrrrr} & x^3 & +\,2x^2 & +\,4x & +\,1 \\ & & & 3x & +\,5 \\ \hline & 5x^3 & +\,10x^2 & +\,20x & +\,5 \\ 3x^4 & +\,6x^3 & +\,12x^2 & +\,3x & \\ \hline 3x^4 & +\,11x^3 & +\,22x^2 & +\,23x & +\,5 \end{array} \leftarrow \text{Product}$$

This process is similar to multiplication of whole numbers.

Place like terms in columns so they can be added.

$3x(x^3 + 2x^2 + 4x + 1)$

NOW TRY

NOW TRY EXERCISE 4

Find the product of

$9x^3 - 12x^2 + 3$ and $\frac{1}{3}x^2 - \frac{2}{3}$.

### EXAMPLE 4 Multiplying Polynomials with Fractional Coefficients Vertically

Find the product of $4m^3 - 2m^2 + 4m$ and $\frac{1}{2}m^2 + \frac{5}{2}$.

$$\begin{array}{rrrrr} & & 4m^3 & -\,2m^2 & +\,4m \\ & & & \frac{1}{2}m^2 & +\,\frac{5}{2} \\ \hline & & 10m^3 & -\,5m^2 & +\,10m \\ 2m^5 & -\,m^4 & +\,2m^3 & & \\ \hline 2m^5 & -\,m^4 & +\,12m^3 & -\,5m^2 & +\,10m \end{array}$$

Terms of top row are multiplied by $\frac{5}{2}$.

Terms of top row are multiplied by $\frac{1}{2}m^2$.

Add. NOW TRY

We can use a rectangle to model polynomial multiplication. For example, to find

$$(2x + 1)(3x + 2),$$

label a rectangle with each term as shown next on the left. Then put the product of each pair of monomials in the appropriate box, as shown on the right.

| | $3x$ | $2$ |
|---|---|---|
| $2x$ | | |
| $1$ | | |

| | $3x$ | $2$ |
|---|---|---|
| $2x$ | $6x^2$ | $4x$ |
| $1$ | $3x$ | $2$ |

NOW TRY ANSWERS

**2.** $2x^4 - 5x^3 - 5x^2 + 20x - 12$

**3.** $10t^3 - 44t^2 + 50t - 24$

**4.** $3x^5 - 4x^4 - 6x^3 + 9x^2 - 2$

The product of the binomials is the sum of the four monomial products.

$$\begin{aligned}&(2x + 1)(3x + 2)\\&= 6x^2 + 4x + 3x + 2\\&= 6x^2 + 7x + 2\end{aligned}$$

This approach can be extended to polynomials with any number of terms.

**OBJECTIVE 3** **Multiply binomials by the FOIL method.** When multiplying binomials, the **FOIL method** reduces the rectangle method to a systematic approach without the rectangle. Consider this example.

$$\begin{aligned}&(x + 3)(x + 5)\\&= (x + 3)x + (x + 3)5 && \text{Distributive property}\\&= x(x) + 3(x) + x(5) + 3(5) && \text{Distributive property again}\\&= x^2 + 3x + 5x + 15 && \text{Multiply.}\\&= x^2 + 8x + 15 && \text{Combine like terms.}\end{aligned}$$

The letters of the word FOIL originate as shown.

$(x + 3)(x + 5)$ Multiply the **First terms:** $x(x)$. **F**

$(x + 3)(x + 5)$ Multiply the **Outer terms:** $x(5)$. **O**
This is the **outer product.**

$(x + 3)(x + 5)$ Multiply the **Inner terms:** $3(x)$. **I**
This is the **inner product.**

$(x + 3)(x + 5)$ Multiply the **Last terms:** $3(5)$. **L**

The outer product, $5x$, and the inner product, $3x$, should be added mentally to get $8x$ so that the three terms of the answer can be written without extra steps.

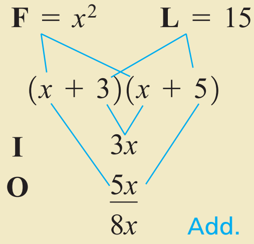
$$\begin{aligned}&(x + 3)(x + 5)\\&= x^2 + 8x + 15\end{aligned}$$

**Multiplying Binomials by the FOIL Method**

***Step 1*** Multiply the two **F**irst terms of the binomials to get the first term of the answer.

***Step 2*** Find the **O**uter product and the **I**nner product and add them (when possible) to get the middle term of the answer.

***Step 3*** Multiply the two **L**ast terms of the binomials to get the last term of the answer.



**NOW TRY EXERCISE 5**

Use the FOIL method to find the product.

$(t-6)(t+5)$

## EXAMPLE 5 Using the FOIL Method

Use the FOIL method to find the product $(x+8)(x-6)$.

***Step 1*** **F** Multiply the First terms: $x(x)=x^2$.

***Step 2*** **O** Find the Outer product: $x(-6)=-6x$.

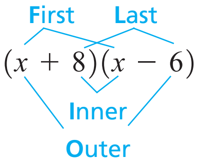
**I** Find the Inner product: $8(x)=8x$.

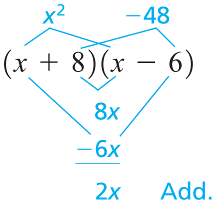
Add the outer and inner products mentally: $-6x+8x=2x$.

***Step 3*** **L** Multiply the Last terms: $8(-6)=-48$.

$(x+8)(x-6)=x^2+2x-48$ Add the terms found in Steps 1–3.

Shortcut:





NOW TRY

**NOW TRY EXERCISE 6**

Multiply.

$(7y-3)(2x+5)$

## EXAMPLE 6 Using the FOIL Method

Multiply $(9x-2)(3y+1)$.

| | | |
|---|---|---|
| *First* | $(9x-2)(3y+1)$ | $27xy$ |
| *Outer* | $(9x-2)(3y+1)$ | $9x$ |
| *Inner* | $(9x-2)(3y+1)$ | $-6y$ |
| *Last* | $(9x-2)(3y+1)$ | $-2$ |

These unlike terms cannot be combined.

The product $(9x-2)(3y+1)$ is $\overset{F}{27xy}+\overset{O}{9x}-\overset{I}{6y}-\overset{L}{2}$.

NOW TRY

**NOW TRY EXERCISE 7**

Find each product.

**(a)** $(3p-5q)(4p-q)$

**(b)** $5x^2(3x+1)(x-5)$

## EXAMPLE 7 Using the FOIL Method

Find each product.

**(a)** $(2k+5y)(k+3y)$

$$=\overset{F}{2k(k)}+\overset{O}{2k(3y)}+\overset{I}{5y(k)}+\overset{L}{5y(3y)}$$

$$=2k^2+6ky+5ky+15y^2 \quad \text{Multiply.}$$

$$=2k^2+11ky+15y^2 \quad \text{Combine like terms.}$$

**(b)** $(7p+2q)(3p-q)$

$$=21p^2-pq-2q^2 \quad \text{FOIL}$$

**(c)** $2x^2(x-3)(3x+4)$

$$=2x^2(3x^2-5x-12) \quad \text{FOIL}$$

$$=6x^4-10x^3-24x^2 \quad \text{Distributive property}$$

NOW TRY

**NOTE** Alternatively, **Example 7(c)** can be solved as follows.

$$2x^2(x-3)(3x+4) \quad \text{Multiply } 2x^2 \text{ and } x-3 \text{ first.}$$

$$=(2x^3-6x^2)(3x+4) \quad \text{Multiply that product and } 3x+4.$$

$$=6x^4-10x^3-24x^2 \quad \text{Same answer}$$

**NOW TRY ANSWERS**

**5.** $t^2-t-30$

**6.** $14yx+35y-6x-15$

**7. (a)** $12p^2-23pq+5q^2$

**(b)** $15x^4-70x^3-25x^2$

## 4.5 EXERCISES

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***Concept Check*** *In Exercises 1 and 2, match each product in Column I with the correct polynomial in Column II.*

| I | II | I | II |
|---|---|---|---|
| **1. (a)** $5x^3(6x^7)$ | **A.** $125x^{21}$ | **2. (a)** $(x-5)(x+4)$ | **A.** $x^2+9x+20$ |
| **(b)** $-5x^7(6x^3)$ | **B.** $30x^{10}$ | **(b)** $(x+5)(x+4)$ | **B.** $x^2-9x+20$ |
| **(c)** $(5x^7)^3$ | **C.** $-216x^9$ | **(c)** $(x-5)(x-4)$ | **C.** $x^2-x-20$ |
| **(d)** $(-6x^3)^3$ | **D.** $-30x^{10}$ | **(d)** $(x+5)(x-4)$ | **D.** $x^2+x-20$ |

*Find each product.* ***See Objective 1.***

**3.** $5y^4(3y^7)$
**4.** $10p^2(5p^3)$
**5.** $-15a^4(-2a^5)$
**6.** $-3m^6(-5m^4)$
**7.** $5p(3q^2)$
**8.** $4a^3(3b^2)$
**9.** $-6m^3(3n^2)$
**10.** $9r^3(-2s^2)$
**11.** $y^5 \cdot 9y \cdot y^4$
**12.** $x^2 \cdot 3x^3 \cdot 2x$
**13.** $(4x^3)(2x^2)(-x^5)$
**14.** $(7t^5)(3t^4)(-t^8)$

*Find each product.* ***See Example 1.***

◉ **15.** $2m(3m+2)$
**16.** $4x(5x+3)$
**17.** $3p(-2p^3+4p^2)$
**18.** $4x(3+2x+5x^3)$
**19.** $-8z(2z+3z^2+3z^3)$
**20.** $-7y(3+5y^2-2y^3)$
**21.** $2y^3(3+2y+5y^4)$
**22.** $2m^4(6+5m+3m^2)$
**23.** $-4r^3(-7r^2+8r-9)$
**24.** $-9a^5(-3a^6-2a^4+8a^2)$
**25.** $3a^2(2a^2-4ab+5b^2)$
**26.** $4z^3(8z^2+5zy-3y^2)$
**27.** $7m^3n^2(3m^2+2mn-n^3)$
**28.** $2p^2q(3p^2q^2-5p+2q^2)$

*Find each product.* ***See Examples 2–4.***

◉ **29.** $(6x+1)(2x^2+4x+1)$
**30.** $(9a+2)(9a^2+a+1)$
**31.** $(9y-2)(8y^2-6y+1)$
**32.** $(2r-1)(3r^2+4r-4)$
◉ **33.** $(4m+3)(5m^3-4m^2+m-5)$
**34.** $(2y+8)(3y^4-2y^2+1)$
**35.** $(2x-1)(3x^5-2x^3+x^2-2x+3)$
**36.** $(2a+3)(a^4-a^3+a^2-a+1)$
**37.** $(5x^2+2x+1)(x^2-3x+5)$
**38.** $(2m^2+m-3)(m^2-4m+5)$
◉ **39.** $(6x^4-4x^2+8x)\left(\frac{1}{2}x+3\right)$
**40.** $(8y^6+4y^4-12y^2)\left(\frac{3}{4}y^2+2\right)$

*Find each product. Use the FOIL method.* ***See Examples 5–7.***

◉ **41.** $(m+7)(m+5)$
**42.** $(n+9)(n+3)$
**43.** $(n-1)(n+4)$
**44.** $(t-3)(t+8)$
**45.** $(x+5)(x-5)$
**46.** $(y+8)(y-8)$
**47.** $(2x+3)(6x-4)$
**48.** $(3y+5)(8y-6)$
**49.** $(9+t)(9-t)$
**50.** $(10+r)(10-r)$
**51.** $(3x-2)(3x-2)$
**52.** $(4m+3)(4m+3)$
**53.** $(5a+1)(2a+7)$
**54.** $(b+8)(6b-2)$
**55.** $(6-5m)(2+3m)$
**56.** $(8-3a)(2+a)$
**57.** $(5-3x)(4+x)$
**58.** $(6-5x)(2+x)$
**59.** $(3t-4s)(t+3s)$
**60.** $(2m-3n)(m+5n)$
◉ **61.** $(4x+3)(2y-1)$
**62.** $(5x+7)(3y-8)$
◉ **63.** $(3x+2y)(5x-3y)$
**64.** $(5a+3b)(5a-4b)$

**65.** $3y^3(2y + 3)(y - 5)$

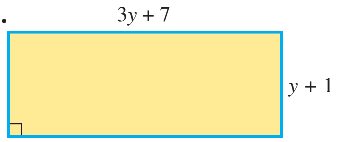
**66.** $2x^2(2x - 5)(x + 3)$

**67.** $-8r^3(5r^2 + 2)(5r^2 - 2)$

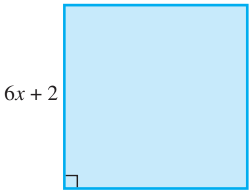
**68.** $-5t^4(2t^4 + 1)(2t^4 - 1)$

*Find polynomials that represent* ***(a)*** *the area and* ***(b)*** *the perimeter of each square or rectangle. (If necessary, refer to the formulas on the inside covers.)*

**69.**



**70.**



*Find each product. In Exercises 81–84, 89, and 90, apply the meaning of exponents.*

**71.** $\left(3p + \frac{5}{4}q\right)\left(2p - \frac{5}{3}q\right)$

**72.** $\left(2x + \frac{2}{3}y\right)\left(3x - \frac{3}{4}y\right)$

**73.** $(x + 7)^2$

**74.** $(m + 6)^2$

**75.** $(a - 4)(a + 4)$

**76.** $(b - 10)(b + 10)$

**77.** $(2p - 5)^2$

**78.** $(3m - 1)^2$

**79.** $(5k + 3q)^2$

**80.** $(8m + 3n)^2$

**81.** $(m - 5)^3$

**82.** $(p - 3)^3$

**83.** $(2a + 1)^3$

**84.** $(3m + 1)^3$

**85.** $-3a(3a + 1)(a - 4)$

**86.** $-4r(3r + 2)(2r - 5)$

**87.** $7(4m - 3)(2m + 1)$

**88.** $5(3k - 7)(5k + 2)$

**89.** $(3r - 2s)^4$

**90.** $(2z - 5y)^4$

**91.** $3p^3(2p^2 + 5p)(p^3 + 2p + 1)$

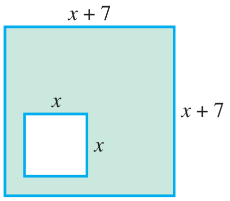
**92.** $5k^2(k^3 - 3)(k^2 - k + 4)$

**93.** $-2x^5(3x^2 + 2x - 5)(4x + 2)$

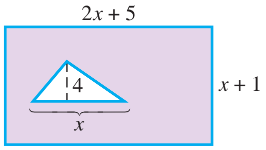
**94.** $-4x^3(3x^4 + 2x^2 - x)(-2x + 1)$

*The figures in Exercises 95–98 are composed of triangles, squares, rectangles, and circles. Find a polynomial that represents the area of each shaded region. In Exercises 97 and 98, leave π in your answers. (If necessary, refer to the formulas on the inside covers.)*

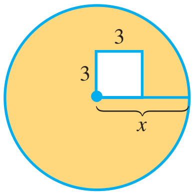
**95.**



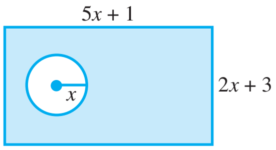
**96.**



**97.**



**98.**



## PREVIEW EXERCISES

*Apply a power rule for exponents.* ***See Section 4.1.***

**99.** $(3m)^2$

**100.** $(5p)^2$

**101.** $(-2r)^2$

**102.** $(-5a)^2$

**103.** $(4x^2)^2$

**104.** $(8y^3)^2$

# 4.6 Special Products

**OBJECTIVES**

1. Square binomials.
2. Find the product of the sum and difference of two terms.
3. Find greater powers of binomials.

**NOW TRY EXERCISE 1**

Find $(x + 5)^2$.

**OBJECTIVE 1 Square binomials.** The square of a binomial can be found quickly by using the method suggested by **Example 1.**

**EXAMPLE 1 Squaring a Binomial**

Find $(m + 3)^2$.

$$(m + 3)(m + 3)$$

$(m + 3)^2$ means $(m + 3)(m + 3)$.

$$= m^2 + 3m + 3m + 9 \quad \text{FOIL}$$

$$= m^2 + 6m + 9$$

Combine like terms. This is the answer.

This result has the squares of the first and the last terms of the binomial.

$$m^2 = m^2 \quad \text{and} \quad 3^2 = 9$$

***The middle term, 6m, is twice the product of the two terms of the binomial,*** since the outer and inner products are $m(3)$ and $3(m)$. Then we find their sum.

$$m(3) + 3(m) = 2(m)(3) = 6m$$

NOW TRY

**Square of a Binomial**

The square of a binomial is a trinomial consisting of

the square of the first term + twice the product of the two terms + the square of the last term.

For $x$ and $y$, the following are true.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

**EXAMPLE 2 Squaring Binomials**

Find each binomial square and simplify.

$$\begin{array}{rcl} (x - y)^2 & = & x^2 - 2 \cdot x \cdot y + y^2 \\ \downarrow \quad \downarrow & & \downarrow \qquad\; \downarrow \;\; \downarrow \;\; \downarrow \quad\;\; \downarrow \\ (5z - 1)^2 & = & (5z)^2 - 2(5z)(1) + (1)^2 \end{array}$$

**(a)** $(5z - 1)^2 = (5z)^2 - 2(5z)(1) + (1)^2$

$= 25z^2 - 10z + 1$ $\quad (5z)^2 = 5^2z^2 = 25z^2$

**(b)** $(3b + 5r)^2$

$= (3b)^2 + 2(3b)(5r) + (5r)^2$

$= 9b^2 + 30br + 25r^2$

**(c)** $(2a - 9x)^2$

$= (2a)^2 - 2(2a)(9x) + (9x)^2$

$= 4a^2 - 36ax + 81x^2$

**(d)** $\left(4m + \frac{1}{2}\right)^2$

$= (4m)^2 + 2(4m)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$

$= 16m^2 + 4m + \frac{1}{4}$

**NOW TRY ANSWER**

**1.** $x^2 + 10x + 25$

**NOW TRY EXERCISE 2**

Find each binomial square and simplify.

**(a)** $(3x - 1)^2$

**(b)** $(4p - 5q)^2$

**(c)** $\left(6t - \frac{1}{3}\right)^2$

**(d)** $m(2m + 3)^2$

**(e)** $x(4x - 3)^2$ Remember the middle term.

$$\begin{aligned} &= x(16x^2 - 24x + 9) && \text{Square the binomial.} \\ &= 16x^3 - 24x^2 + 9x && \text{Distributive property} \end{aligned}$$

NOW TRY

***In the square of a sum, all of the terms are positive,*** **as in Examples 2(b) and (d).** ***In the square of a difference, the middle term is negative,*** **as in Examples 2(a), (c), and (e).**

**CAUTION** A common error when squaring a binomial is to forget the middle term of the product. In general,

$$(x + y)^2 = x^2 + 2xy + y^2, \quad \textit{not} \quad x^2 + y^2,$$

and

$$(x - y)^2 = x^2 - 2xy + y^2, \quad \textit{not} \quad x^2 - y^2.$$

## OBJECTIVE 2 Find the product of the sum and difference of two terms.

In binomial products of the form $(x + y)(x - y)$, one binomial is the sum of two terms and the other is the difference of the *same* two terms. Consider $(x + 2)(x - 2)$.

$$\begin{aligned} &(x + 2)(x - 2) \\ &= x^2 - 2x + 2x - 4 && \text{FOIL} \\ &= x^2 - 4 && \text{Combine like terms.} \end{aligned}$$

Thus, the product of $x + y$ and $x - y$ is the difference of two squares.

**Product of the Sum and Difference of Two Terms**

$$(x + y)(x - y) = x^2 - y^2$$

**NOW TRY EXERCISE 3**

Find the product.

$(t + 10)(t - 10)$

### EXAMPLE 3 Finding the Product of the Sum and Difference of Two Terms

Find each product.

**(a)** $(x + 4)(x - 4)$

Use the rule for the product of the sum and difference of two terms.

$$\begin{aligned} &(x + 4)(x - 4) \\ &= x^2 - 4^2 \\ &= x^2 - 16 \end{aligned}$$

**(b)** $\left(\frac{2}{3} - w\right)\left(\frac{2}{3} + w\right)$

$$\begin{aligned} &= \left(\frac{2}{3} + w\right)\left(\frac{2}{3} - w\right) && \text{Commutative property} \\ &= \left(\frac{2}{3}\right)^2 - w^2 && (x + y)(x - y) = x^2 - y^2 \\ &= \frac{4}{9} - w^2 && \text{Square } \tfrac{2}{3}. \end{aligned}$$

NOW TRY

**NOW TRY ANSWERS**

**2.** **(a)** $9x^2 - 6x + 1$
**(b)** $16p^2 - 40pq + 25q^2$
**(c)** $36t^2 - 4t + \frac{1}{9}$
**(d)** $4m^3 + 12m^2 + 9m$

**3.** $t^2 - 100$

NOW TRY EXERCISE 4

Find each product.

**(a)** $(4x - 6)(4x + 6)$

**(b)** $\left(5r - \frac{4}{5}\right)\left(5r + \frac{4}{5}\right)$

**(c)** $y(3y + 1)(3y - 1)$

### EXAMPLE 4 Finding the Product of the Sum and Difference of Two Terms

Find each product.

**(a)** $(\overset{\overset{x}{\downarrow}}{5m} + \overset{\overset{y}{\downarrow}}{3})(\overset{\overset{x}{\downarrow}}{5m} - \overset{\overset{y}{\downarrow}}{3})$

Use the rule for the product of the sum and difference of two terms.

$$\begin{aligned} &(5m + 3)(5m - 3) \\ &= (5m)^2 - 3^2 && (x + y)(x - y) = x^2 - y^2 \\ &= 25m^2 - 9 && \text{Apply the exponents.} \end{aligned}$$

**(b)** $(4x + y)(4x - y)$

$$\begin{aligned} &= (4x)^2 - y^2 \\ &= 16x^2 - y^2 \end{aligned}$$

**(c)** $\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)$

$$= z^2 - \frac{1}{16}$$

**(d)** $p(2p + 1)(2p - 1)$

$$\begin{aligned} &= p(4p^2 - 1) \\ &= 4p^3 - p && \text{Distributive property} \end{aligned}$$

NOW TRY

**OBJECTIVE 3** **Find greater powers of binomials.** The methods used in the previous section and this section can be combined to find greater powers of binomials.

NOW TRY EXERCISE 5

Find the product.

$(2m - 1)^3$

### EXAMPLE 5 Finding Greater Powers of Binomials

Find each product.

**(a)** $(x + 5)^3$

$$\begin{aligned} &= (x + 5)^2(x + 5) && a^3 = a^2 \cdot a \\ &= (x^2 + 10x + 25)(x + 5) && \text{Square the binomial.} \\ &= x^3 + 10x^2 + 25x + 5x^2 + 50x + 125 && \text{Multiply polynomials.} \\ &= x^3 + 15x^2 + 75x + 125 && \text{Combine like terms.} \end{aligned}$$

**(b)** $(2y - 3)^4$

$$\begin{aligned} &= (2y - 3)^2(2y - 3)^2 && a^4 = a^2 \cdot a^2 \\ &= (4y^2 - 12y + 9)(4y^2 - 12y + 9) && \text{Square each binomial.} \\ &= 16y^4 - 48y^3 + 36y^2 - 48y^3 + 144y^2 && \text{Multiply polynomials.} \\ &\quad - 108y + 36y^2 - 108y + 81 \\ &= 16y^4 - 96y^3 + 216y^2 - 216y + 81 && \text{Combine like terms.} \end{aligned}$$

**(c)** $-2r(r + 2)^3$

$$\begin{aligned} &= -2r(r + 2)(r + 2)^2 && a^3 = a \cdot a^2 \\ &= -2r(r + 2)(r^2 + 4r + 4) && \text{Square the binomial.} \\ &= -2r(r^3 + 4r^2 + 4r + 2r^2 + 8r + 8) && \text{Multiply polynomials.} \\ &= -2r(r^3 + 6r^2 + 12r + 8) && \text{Combine like terms.} \\ &= -2r^4 - 12r^3 - 24r^2 - 16r && \text{Multiply.} \end{aligned}$$

NOW TRY

NOW TRY ANSWERS

**4.** **(a)** $16x^2 - 36$
  **(b)** $25r^2 - \frac{16}{25}$
  **(c)** $9y^3 - y$

**5.** $8m^3 - 12m^2 + 6m - 1$

# 4.6 EXERCISES

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**1.** ***Concept Check*** Consider the square $(4x + 3)^2$.

**(a)** What is the simplest form of the square of the first term, $(4x)^2$?

**(b)** What is the simplest form of twice the product of the two terms, $2(4x)(3)$?

**(c)** What is the simplest form of the square of the last term, $3^2$?

**(d)** Write the final product, which is a trinomial, using your results in parts (a)–(c).

✎ **2.** Explain in your own words how to square a binomial. Give an example.

*Find each product.* ***See Examples 1 and 2.***

**3.** $(m + 2)^2$ **4.** $(x + 8)^2$ **5.** $(r - 3)^2$

**6.** $(z - 5)^2$ ⊛ **7.** $(x + 2y)^2$ **8.** $(p - 3m)^2$

**9.** $(5p + 2q)^2$ **10.** $(8a + 3b)^2$ **11.** $(4a + 5b)^2$

**12.** $(9y + 4z)^2$ ⊛ **13.** $\left(6m - \frac{4}{5}n\right)^2$ **14.** $\left(5x + \frac{2}{5}y\right)^2$

**15.** $t(3t - 1)^2$ **16.** $x(2x + 5)^2$ **17.** $3t(4t + 1)^2$

**18.** $2x(7x - 2)^2$ **19.** $-(4r - 2)^2$ **20.** $-(3y - 8)^2$

**21.** ***Concept Check*** Consider the product $(7x + 3y)(7x - 3y)$.

**(a)** What is the simplest form of the product of the first terms, $7x(7x)$?

**(b)** Multiply the outer terms, $7x(-3y)$. Then multiply the inner terms, $3y(7x)$. Add the results. What is this sum?

**(c)** What is the simplest form of the product of the last terms, $3y(-3y)$?

**(d)** Write the final product, using your results in parts (a) and (c). Why is the sum found in part (b) omitted here?

✎ **22.** Explain in your own words how to find the product of the sum and the difference of two terms. Give an example.

*Find each product.* ***See Examples 3 and 4.***

⊛ **23.** $(k + 5)(k - 5)$ **24.** $(a + 8)(a - 8)$ **25.** $(4 - 3t)(4 + 3t)$

**26.** $(7 - 2x)(7 + 2x)$ **27.** $(5x + 2)(5x - 2)$ **28.** $(2m + 5)(2m - 5)$

**29.** $(5y + 3x)(5y - 3x)$ **30.** $(3x + 4y)(3x - 4y)$ ⊛ **31.** $(10x + 3y)(10x - 3y)$

**32.** $(13r + 2z)(13r - 2z)$ **33.** $(2x^2 - 5)(2x^2 + 5)$ **34.** $(9y^2 - 2)(9y^2 + 2)$

⊛ **35.** $\left(\frac{3}{4} - x\right)\left(\frac{3}{4} + x\right)$ **36.** $\left(\frac{2}{3} + r\right)\left(\frac{2}{3} - r\right)$ **37.** $\left(9y + \frac{2}{3}\right)\left(9y - \frac{2}{3}\right)$

**38.** $\left(7x + \frac{3}{7}\right)\left(7x - \frac{3}{7}\right)$ ⊛ **39.** $q(5q - 1)(5q + 1)$ **40.** $p(3p + 7)(3p - 7)$

✎ **41.** Does $(a + b)^2$ equal $a^2 + b^2$ in general? Explain.

✎ **42.** Does $(a + b)^3$ equal $a^3 + b^3$ in general? Explain.

*Find each product.* ***See Example 5.***

⊛ **43.** $(x + 1)^3$ **44.** $(y + 2)^3$ **45.** $(t - 3)^3$ **46.** $(m - 5)^3$

**47.** $(r + 5)^3$ **48.** $(p + 3)^3$ **49.** $(2a + 1)^3$ **50.** $(3m + 1)^3$

⊛ **51.** $(4x - 1)^4$ **52.** $(2x - 1)^4$ **53.** $(3r - 2t)^4$ **54.** $(2z + 5y)^4$

**55.** $2x(x + 1)^3$ **56.** $3y(y + 2)^3$ **57.** $-4t(t + 3)^3$

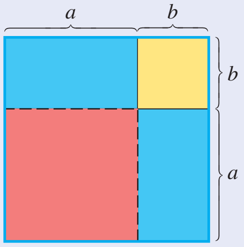
**58.** $-5r(r + 1)^3$ **59.** $(x + y)^2(x - y)^2$ **60.** $(s + 2)^2(s - 2)^2$

## RELATING CONCEPTS EXERCISES 61–70

### FOR INDIVIDUAL OR GROUP WORK

*Special products can be illustrated by using areas of rectangles. Use the figure, and* ***work Exercises 61–66 in order*** *to justify the special product*

$$(a + b)^2 = a^2 + 2ab + b^2.$$



**61.** Express the area of the large square as the square of a binomial.

**62.** Give the monomial that represents the area of the red square.

**63.** Give the monomial that represents the sum of the areas of the blue rectangles.

**64.** Give the monomial that represents the area of the yellow square.

**65.** What is the sum of the monomials you obtained in **Exercises 62–64?**

**66.** Explain why the binomial square you found in **Exercise 61** must equal the polynomial you found in **Exercise 65.**

*To understand how the special product $(a + b)^2 = a^2 + 2ab + b^2$ can be applied to a purely numerical problem,* ***work Exercises 67–70 in order.***

**67.** Evaluate $35^2$, using either traditional paper-and-pencil methods or a calculator.

**68.** The number 35 can be written as $30 + 5$. Therefore, $35^2 = (30 + 5)^2$. Use the special product for squaring a binomial with $a = 30$ and $b = 5$ to write an expression for $(30 + 5)^2$. Do not simplify at this time.

**69.** Use the order of operations to simplify the expression you found in **Exercise 68.**

**70.** How do the answers in **Exercises 67 and 69** compare?

*The special product*

$$(x + y)(x - y) = x^2 - y^2$$

*can be used to perform some multiplication problems. Here are two examples.*

$$\begin{aligned} 51 \times 49 &= (50 + 1)(50 - 1) \\ &= 50^2 - 1^2 \\ &= 2500 - 1 \\ &= 2499 \end{aligned}$$

$$\begin{aligned} 102 \times 98 &= (100 + 2)(100 - 2) \\ &= 100^2 - 2^2 \\ &= 10{,}000 - 4 \\ &= 9996 \end{aligned}$$

*Once these patterns are recognized, multiplications of this type can be done mentally. Use this method to calculate each product mentally.*

**71.** $101 \times 99$

**72.** $103 \times 97$

**73.** $201 \times 199$

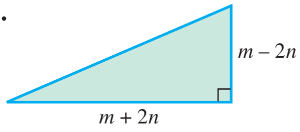
**74.** $301 \times 299$

**75.** $20\frac{1}{2} \times 19\frac{1}{2}$

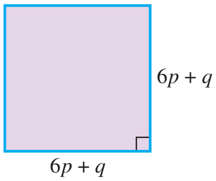
**76.** $30\frac{1}{3} \times 29\frac{2}{3}$

*Determine a polynomial that represents the area of each figure. (If necessary, refer to the formulas on the inside covers.)*

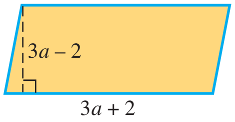
**77.**



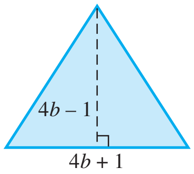
**78.**



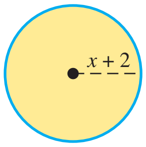
**79.**



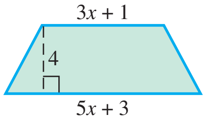
**80.**



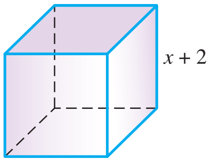
**81.**



**82.**



*In Exercises 83 and 84, refer to the figure shown here.*



**83.** Find a polynomial that represents the volume of the cube (in cubic units).

**84.** If the value of $x$ is 6, what is the volume of the cube (in cubic units)?

## PREVIEW EXERCISES

*Write each product as a sum of terms. Write answers with positive exponents only. Simplify each term.* ***See Section 1.8.***

**85.** $\frac{1}{2p}(4p^2 + 2p + 8)$

**86.** $\frac{1}{5x}(5x^2 - 10x + 45)$

**87.** $\frac{1}{3m}(m^3 + 9m^2 - 6m)$

**88.** $\frac{1}{4y}(y^4 + 6y^2 + 8)$

*Find each product.* ***See Section 4.5.***

**89.** $-3k(8k^2 - 12k + 2)$

**90.** $(3r + 5)(2r + 1)$

**91.** $(-2k + 1)(8k^2 + 9k + 3)$

**92.** $(x^2 - 2)(3x^2 + x + 4)$

*Subtract.* ***See Section 4.4.***

**93.** $\begin{array}{r} 5t^2 + 2t - 6 \\ \underline{5t^2 - 3t - 9} \end{array}$

**94.** $\begin{array}{r} x^5 + x^3 - 2x^2 + 3 \\ \underline{-4x^5 \qquad\quad + 3x^2 - 8} \end{array}$

# 4.7 Dividing Polynomials

**OBJECTIVES**

1. Divide a polynomial by a monomial.
2. Divide a polynomial by a polynomial.

**OBJECTIVE 1** **Divide a polynomial by a monomial.** We add two fractions with a common denominator as follows.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

In reverse, this statement gives a rule for dividing a polynomial by a monomial.

**Dividing a Polynomial by a Monomial**

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

$$\frac{\boldsymbol{a+b}}{\boldsymbol{c}} = \frac{\boldsymbol{a}}{\boldsymbol{c}} + \frac{\boldsymbol{b}}{\boldsymbol{c}} \quad (c \neq 0)$$

*Examples:* $\frac{2+5}{3} = \frac{2}{3} + \frac{5}{3}$ and $\frac{x+3z}{2y} = \frac{x}{2y} + \frac{3z}{2y}$

The parts of a division problem are named here.

$$\begin{matrix}\text{Dividend} \rightarrow \\ \text{Divisor} \rightarrow\end{matrix} \frac{12x^2 + 6x}{6x} = 2x + 1 \leftarrow \text{Quotient}$$

**NOW TRY EXERCISE 1**

Divide $16a^6 - 12a^4$ by $4a^2$.

## EXAMPLE 1 Dividing a Polynomial by a Monomial

Divide $5m^5 - 10m^3$ by $5m^2$.

$$\frac{5m^5 - 10m^3}{5m^2}$$

$$= \frac{5m^5}{5m^2} - \frac{10m^3}{5m^2} \qquad \text{Use the preceding rule, with + replaced by } -.$$

$$= m^3 - 2m \qquad \text{Quotient rule}$$

CHECK Multiply: $\underset{\substack{\uparrow \\ \text{Divisor}}}{5m^2} \cdot \underset{\substack{\uparrow \\ \text{Quotient}}}{(m^3 - 2m)} = 5m^5 - 10m^3. \leftarrow$ Original polynomial (Dividend)

Because division by 0 is undefined, the quotient $\frac{5m^5 - 10m^3}{5m^2}$ is undefined if $5m^2 = 0$, or $m = 0$. From now on, we assume that no denominators are 0. NOW TRY

**NOW TRY EXERCISE 2**

Divide.

$$\frac{36x^5 + 24x^4 - 12x^3}{6x^4}$$

## EXAMPLE 2 Dividing a Polynomial by a Monomial

Divide.

$$\frac{16a^5 - 12a^4 + 8a^2}{4a^3}$$

$$= \frac{16a^5}{4a^3} - \frac{12a^4}{4a^3} + \frac{8a^2}{4a^3} \qquad \text{Divide each term by } 4a^3.$$

This becomes $\frac{2}{a}$, **not** $2a$.

$$= 4a^2 - 3a + \frac{2}{a} \qquad \text{Quotient rule}$$

The quotient $4a^2 - 3a + \frac{2}{a}$ is *not* a polynomial because of the presence of the expression $\frac{2}{a}$, which has a variable in the denominator. While the sum, difference, and product of two polynomials are always polynomials, the quotient of two polynomials may not be a polynomial.

CHECK

$$4a^3\left(4a^2 - 3a + \frac{2}{a}\right) \qquad \text{Divisor} \times \text{Quotient should equal Dividend.}$$

$$= 4a^3(4a^2) + 4a^3(-3a) + 4a^3\left(\frac{2}{a}\right) \qquad \text{Distributive property}$$

$$= 16a^5 - 12a^4 + 8a^2 \ \checkmark \qquad \text{Dividend}$$

NOW TRY

⚠ **CAUTION** The most frequent error in a problem like that in **Example 2** is with the last term of the quotient.

$$\frac{8a^2}{4a^3} = \frac{8}{4}a^{2-3} = 2a^{-1} = 2\left(\frac{1}{a}\right) = \frac{2}{a}$$

**NOW TRY ANSWERS**

1. $4a^4 - 3a^2$
2. $6x + 4 - \frac{2}{x}$

**NOW TRY EXERCISE 3**

Divide $7y^4 - 40y^5 + 100y^2$ by $-5y^2$.

### EXAMPLE 3 Dividing a Polynomial by a Monomial with a Negative Coefficient

Divide $-7x^3 + 12x^4 - 4x$ by $-4x$.

Write the polynomial in descending powers as $12x^4 - 7x^3 - 4x$ before dividing.

Write in descending powers.

$$\frac{12x^4 - 7x^3 - 4x}{-4x}$$

$$= \frac{12x^4}{-4x} - \frac{7x^3}{-4x} - \frac{4x}{-4x} \qquad \text{Divide each term by } -4x.$$

$$= -3x^3 - \frac{7x^2}{-4} - (-1) \qquad \text{Quotient rule}$$

$$= -3x^3 + \frac{7x^2}{4} + 1$$

Be sure to include 1 in the answer.

*Check* by multiplying. NOW TRY

**NOW TRY EXERCISE 4**

Divide $35m^5n^4 - 49m^2n^3 + 12mn$ by $7m^2n$.

### EXAMPLE 4 Dividing a Polynomial by a Monomial

Divide $180x^4y^{10} - 150x^3y^8 + 120x^2y^6 - 90xy^4 + 100y$ by $30xy^2$.

$$\frac{180x^4y^{10} - 150x^3y^8 + 120x^2y^6 - 90xy^4 + 100y}{30xy^2}$$

$$= \frac{180x^4y^{10}}{30xy^2} - \frac{150x^3y^8}{30xy^2} + \frac{120x^2y^6}{30xy^2} - \frac{90xy^4}{30xy^2} + \frac{100y}{30xy^2}$$

$$= 6x^3y^8 - 5x^2y^6 + 4xy^4 - 3y^2 + \frac{10}{3xy}$$

NOW TRY

**OBJECTIVE 2** **Divide a polynomial by a polynomial.** As shown in the box, we use a method of "long division" to divide a polynomial by a polynomial (other than a monomial). ***Both polynomials must first be written in descending powers.***

| Dividing Whole Numbers | Dividing Polynomials |
|---|---|
| ***Step 1***<br>Divide 6696 by 27.<br>$27\overline{)6696}$ | Divide $8x^3 - 4x^2 - 14x + 15$ by $2x + 3$.<br>$2x + 3\overline{)8x^3 - 4x^2 - 14x + 15}$ |
| ***Step 2***<br>66 divided by $27 = 2$.<br>$2 \cdot 27 = 54$<br>$\begin{array}{r} 2\phantom{96} \\ 27\overline{)6696} \\ \underline{54}\phantom{96} \end{array}$ | $8x^3$ divided by $2x = 4x^2$.<br>$4x^2(2x + 3) = 8x^3 + 12x^2$<br>$\begin{array}{r} 4x^2\phantom{- 14x + 15} \\ 2x + 3\overline{)8x^3 - 4x^2 - 14x + 15} \\ \underline{8x^3 + 12x^2}\phantom{- 14x + 15} \end{array}$<br>(continued) |

**NOW TRY ANSWERS**

3. $8y^3 - \frac{7y^2}{5} - 20$

4. $5m^3n^3 - 7n^2 + \frac{12}{7m}$

***Step 3***

Subtract. Then bring down the next digit.

$$\begin{array}{rcccc}
 & & 2 & & \\ \cline{2-5}
27\big) & 6 & 6 & 9 & 6 \\
 & 5 & 4 & \downarrow & \\ \cline{2-3}
 & 1 & 2 & 9 &
\end{array}$$

Subtract. Then bring down the next term.

$$\begin{array}{rrrrr}
 & & 4x^2 & & \\ \cline{2-5}
2x+3\big) & 8x^3 & -\;4x^2 & -\;14x & +\;15 \\
 & 8x^3 & +\;12x^2 & \downarrow & \\ \cline{2-3}
 & & -16x^2 & -\;14x &
\end{array}$$

***(To subtract two polynomials, change the signs of the second and then add.)***

***Step 4***

129 divided by 27 $=$ 4.
4 · 27 $=$ 108

$$\begin{array}{rcccc}
 & & 2 & 4 & \\ \cline{2-5}
27\big) & 6 & 6 & 9 & 6 \\
 & 5 & 4 & & \\ \cline{2-3}
 & 1 & 2 & 9 & \\
 & 1 & 0 & 8 & \\ \cline{2-4}
\end{array}$$

$-16x^2$ divided by $2x = -8x$.
$-8x(2x+3) = -16x^2 - 24x$

$$\begin{array}{rrrrr}
 & & 4x^2 & -\;8x & \\ \cline{2-5}
2x+3\big) & 8x^3 & -\;4x^2 & -\;14x & +\;15 \\
 & 8x^3 & +\;12x^2 & & \\ \cline{2-3}
 & & -16x^2 & -\;14x & \\
 & & -16x^2 & -\;24x & \\ \cline{3-4}
\end{array}$$

***Step 5***

Subtract. Then bring down the next digit.

$$\begin{array}{rcccc}
 & & 2 & 4 & \\ \cline{2-5}
27\big) & 6 & 6 & 9 & 6 \\
 & 5 & 4 & & \downarrow \\ \cline{2-3}
 & 1 & 2 & 9 & \\
 & 1 & 0 & 8 & \\ \cline{2-4}
 & & 2 & 1 & 6
\end{array}$$

Subtract. Then bring down the next term.

$$\begin{array}{rrrrr}
 & & 4x^2 & -\;8x & \\ \cline{2-5}
2x+3\big) & 8x^3 & -\;4x^2 & -\;14x & +\;15 \\
 & 8x^3 & +\;12x^2 & & \downarrow \\ \cline{2-3}
 & & -16x^2 & -\;14x & \\
 & & -16x^2 & -\;24x & \\ \cline{3-4}
 & & & 10x & +\;15
\end{array}$$

***Step 6***

216 divided by 27 $=$ 8.
8 · 27 $=$ 216

$$\begin{array}{rcccc}
 & & 2 & 4 & 8 \\ \cline{2-5}
27\big) & 6 & 6 & 9 & 6 \\
 & 5 & 4 & & \\ \cline{2-3}
 & 1 & 2 & 9 & \\
 & 1 & 0 & 8 & \\ \cline{2-4}
 & & 2 & 1 & 6 \\
 & & 2 & 1 & 6 \\ \cline{3-5}
 & & \text{Remainder} & \rightarrow & 0
\end{array}$$

$10x$ divided by $2x = 5$.
$5(2x+3) = 10x + 15$

$$\begin{array}{rrrrr}
 & & 4x^2 & -\;8x & +\;5 \\ \cline{2-5}
2x+3\big) & 8x^3 & -\;4x^2 & -\;14x & +\;15 \\
 & 8x^3 & +\;12x^2 & & \\ \cline{2-3}
 & & -16x^2 & -\;14x & \\
 & & -16x^2 & -\;24x & \\ \cline{3-4}
 & & & 10x & +\;15 \\
 & & & 10x & +\;15 \\ \cline{4-5}
 & & \text{Remainder} & \rightarrow & 0
\end{array}$$

6696 divided by 27 is 248. The remainder is 0.

$8x^3 - 4x^2 - 14x + 15$ divided by $2x + 3$ is $4x^2 - 8x + 5$. The remainder is 0.

***Step 7***

CHECK Multiply.

$$27 \cdot 248 = 6696 \quad ✓$$

CHECK Multiply.

$$\begin{aligned}
&(2x+3)(4x^2 - 8x + 5) \\
&= 8x^3 - 4x^2 - 14x + 15 \quad ✓
\end{aligned}$$

**NOW TRY EXERCISE 5**

Divide.

$$\frac{4x^2 + x - 18}{x - 2}$$

## EXAMPLE 5 Dividing a Polynomial by a Polynomial

Divide. $\dfrac{3x^2 - 5x - 28}{x - 4}$

$$\begin{array}{r}
3x + 7 \longleftarrow \text{Quotient} \\
\overset{\overset{\text{Divisor}}{\downarrow}}{x - 4}\,\big)\overline{3x^2 - 5x - 28} \longleftarrow \text{Dividend} \\
\underline{3x^2 - 12x} \phantom{{}-28} \\
7x - 28 \\
\underline{7x - 28} \\
0
\end{array}$$

***Step 1*** $3x^2$ divided by $x$ is $3x$.
$3x(x - 4) = 3x^2 - 12x$

***Step 2*** Subtract $3x^2 - 12x$ from $3x^2 - 5x$. Bring down $-28$.

***Step 3*** $7x$ divided by $x$ is 7.
$7(x - 4) = 7x - 28$

***Step 4*** Subtract $7x - 28$ from $7x - 28$. The remainder is 0.

**CHECK** Multiply the divisor, $x - 4$, by the quotient, $3x + 7$. The product must be the original dividend, $3x^2 - 5x - 28$.

$$\begin{aligned}
(\underset{\underset{\text{Divisor}}{\uparrow}}{x - 4})(\underset{\underset{\text{Quotient}}{\uparrow}}{3x + 7}) &= 3x^2 + 7x - 12x - 28 \\
&= \underset{\underset{\text{Dividend}}{\uparrow}}{3x^2 - 5x - 28} \quad \checkmark
\end{aligned}$$

**NOW TRY**

## EXAMPLE 6 Dividing a Polynomial by a Polynomial

Divide. $\dfrac{5x + 4x^3 - 8 - 4x^2}{2x - 1}$

The first polynomial must be written in descending powers as $4x^3 - 4x^2 + 5x - 8$. Then divide by $2x - 1$.

$$\begin{array}{r}
2x^2 - x + 2 \\
2x - 1\,\big)\overline{4x^3 - 4x^2 + 5x - 8} \\
\underline{4x^3 - 2x^2} \phantom{{}+5x-8} \\
-2x^2 + 5x \phantom{{}-8} \\
\underline{-2x^2 + x} \phantom{{}-8} \\
4x - 8 \\
\underline{4x - 2} \\
-6 \longleftarrow \text{Remainder}
\end{array}$$

Write in descending powers.

Each time you subtract, add the opposite.

***Step 1*** $4x^3$ divided by $2x$ is $2x^2$. $2x^2(2x - 1) = 4x^3 - 2x^2$

***Step 2*** Subtract. Bring down the next term.

***Step 3*** $-2x^2$ divided by $2x$ is $-x$. $-x(2x - 1) = -2x^2 + x$

***Step 4*** Subtract. Bring down the next term.

***Step 5*** $4x$ divided by $2x$ is 2. $2(2x - 1) = 4x - 2$

***Step 6*** Subtract. The remainder is $-6$. Write the remainder as the numerator of a fraction that has $2x - 1$ as its denominator. Because of the nonzero remainder, the answer is not a polynomial.

Remember to add $\frac{\text{remainder}}{\text{divisor}}$. Don't forget the + sign.

$$\begin{array}{r} \text{Dividend} \rightarrow \\ \text{Divisor} \rightarrow \end{array} \frac{4x^3 - 4x^2 + 5x - 8}{2x - 1} = \underbrace{2x^2 - x + 2}_{\substack{\text{Quotient} \\ \text{polynomial}}} + \underbrace{\frac{-6}{2x - 1}}_{\substack{\text{Fractional part} \\ \text{of quotient}}} \begin{array}{l} \leftarrow \text{Remainder} \\ \leftarrow \text{Divisor} \end{array}$$

**NOW TRY ANSWER**

**5.** $4x + 9$

NOW TRY EXERCISE 6

Divide.

$$\frac{6k^3 - 20k - k^2 + 1}{2k - 3}$$

*Step 7* CHECK

$$(2x - 1)\left(2x^2 - x + 2 + \frac{-6}{2x - 1}\right)$$

Multiply Divisor × (Quotient including the Remainder).

$$= (2x - 1)(2x^2) + (2x - 1)(-x) + (2x - 1)(2) + (2x - 1)\left(\frac{-6}{2x - 1}\right)$$

$$= 4x^3 - 2x^2 - 2x^2 + x + 4x - 2 - 6$$

$$= 4x^3 - 4x^2 + 5x - 8 \quad ✓$$

NOW TRY

---

**CAUTION** Remember to include "$+ \frac{\text{remainder}}{\text{divisor}}$" as part of the answer.

---

NOW TRY EXERCISE 7

Divide $m^3 - 1000$ by $m - 10$.

## EXAMPLE 7 Dividing into a Polynomial with Missing Terms

Divide $x^3 - 1$ by $x - 1$.

Here, the dividend, $x^3 - 1$, is missing the $x^2$-term and the $x$-term. We use $0$ as the coefficient for each missing term. Thus, $x^3 - 1 = x^3 + 0x^2 + 0x - 1$.

$$\begin{array}{rr}
 & x^2 + x + 1 \\
x - 1 & \overline{)\,x^3 + 0x^2 + 0x - 1} \\
 & \underline{x^3 - x^2 \qquad\qquad} \\
 & x^2 + 0x \qquad \\
 & \underline{x^2 - x \qquad} \\
 & x - 1 \\
 & \underline{x - 1} \\
 & 0
\end{array}$$

Insert placeholders for the missing terms.

The remainder is 0. The quotient is $x^2 + x + 1$.

CHECK $(x - 1)(x^2 + x + 1) = x^3 - 1$ ✓ Divisor × Quotient = Dividend

NOW TRY

NOW TRY EXERCISE 8

Divide.

$$y^4 - 5y^3 + 6y^2 + y - 4$$

by $y^2 + 2$.

## EXAMPLE 8 Dividing by a Polynomial with Missing Terms

Divide $x^4 + 2x^3 + 2x^2 - x - 1$ by $x^2 + 1$.

Since the divisor, $x^2 + 1$, has a missing $x$-term, write it as $x^2 + 0x + 1$.

$$\begin{array}{rr}
 & x^2 + 2x + 1 \\
x^2 + 0x + 1 & \overline{)\,x^4 + 2x^3 + 2x^2 - x - 1} \\
 & \underline{x^4 + 0x^3 + x^2 \qquad\qquad} \\
 & 2x^3 + x^2 - x \qquad \\
 & \underline{2x^3 + 0x^2 + 2x \qquad} \\
 & x^2 - 3x - 1 \\
 & \underline{x^2 + 0x + 1} \\
 & -3x - 2 \longleftarrow \text{Remainder}
\end{array}$$

Insert a placeholder for the missing term.

When the result of subtracting ($-3x - 2$ here) is a constant or a polynomial of degree less than the divisor $(x^2 + 0x + 1)$, that constant or polynomial is the remainder. The answer is

$$x^2 + 2x + 1 + \frac{-3x - 2}{x^2 + 1}.$$

Remember to write "$+ \frac{\text{remainder}}{\text{divisor}}$."

*Multiply* to check that this is correct.

NOW TRY

NOW TRY ANSWERS

6. $3k^2 + 4k - 4 + \frac{-11}{2k - 3}$
7. $m^2 + 10m + 100$
8. $y^2 - 5y + 4 + \frac{11y - 12}{y^2 + 2}$

**NOW TRY EXERCISE 9**

Divide $10x^3 + 21x^2 + 5x - 8$ by $2x + 4$.

## EXAMPLE 9 Dividing a Polynomial When the Quotient Has Fractional Coefficients

Divide $4x^3 + 2x^2 + 3x + 2$ by $4x - 4$.

$\frac{6x^2}{4x} = \frac{3}{2}x$

$$\begin{array}{r} x^2 + \frac{3}{2}x + \frac{9}{4} \\ 4x - 4\overline{)4x^3 + 2x^2 + 3x + 2} \\ \underline{4x^3 - 4x^2} \qquad\qquad\qquad \\ 6x^2 + 3x \qquad\quad \\ \underline{6x^2 - 6x} \qquad\quad \\ 9x + 2 \\ \underline{9x - 9} \\ 11 \end{array}$$

$\leftarrow \frac{9x}{4x} = \frac{9}{4}$

The answer is $x^2 + \frac{3}{2}x + \frac{9}{4} + \frac{11}{4x - 4}$.

NOW TRY

**NOW TRY ANSWER**

**9.** $5x^2 + \frac{1}{2}x + \frac{3}{2} + \frac{-14}{2x + 4}$

## 4.7 EXERCISES

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***Concept Check*** *Fill in each blank with the correct response.*

**1.** In the statement $\frac{10x^2 + 8}{2} = 5x^2 + 4$, ______ is the dividend, ______ is the divisor, and ______ is the quotient.

**2.** The expression $\frac{3x + 13}{x}$ is undefined if $x =$ ______.

**3.** To check the division shown in **Exercise 1,** multiply ______ by ______ and show that the product is ______.

**4.** The expression $5x^2 - 4x + 6 + \frac{2}{x}$ ______________ a polynomial.
(is/is not)

**5.** Explain why the division problem $\frac{16m^3 - 12m^2}{4m}$ can be performed by using the methods of this section, while the division problem $\frac{4m}{16m^3 - 12m^2}$ cannot.

**6.** ***Concept Check*** A polynomial in the variable $x$ has degree 6 and is divided by a monomial in the variable $x$ having degree 4. What is the degree of the quotient?

*Perform each division.* ***See Examples 1–3.***

**7.** $\frac{60x^4 - 20x^2 + 10x}{2x}$

**8.** $\frac{120x^6 - 60x^3 + 80x^2}{2x}$

**9.** $\frac{20m^5 - 10m^4 + 5m^2}{5m^2}$

**10.** $\frac{12t^5 - 6t^3 + 6t^2}{6t^2}$

**11.** $\frac{8t^5 - 4t^3 + 4t^2}{2t}$

**12.** $\frac{8r^4 - 4r^3 + 6r^2}{2r}$

**13.** $\frac{4a^5 - 4a^2 + 8}{4a}$

**14.** $\frac{5t^8 + 5t^7 + 15}{5t}$

**15.** $\frac{18p^5 + 12p^3 - 6p^2}{-6p^3}$

**16.** $\frac{32x^8 + 24x^5 - 8x^2}{-8x^2}$

**17.** $\frac{-7r^7 + 6r^5 - r^4}{-r^5}$

**18.** $\frac{-13t^9 + 8t^6 - t^5}{-t^6}$

*Divide each polynomial by* $3x^2$. ***See Examples 1–3.***

**19.** $12x^5 - 9x^4 + 6x^3$

**20.** $24x^6 - 12x^5 + 30x^4$

**21.** $3x^2 + 15x^3 - 27x^4$

**22.** $3x^2 - 18x^4 + 30x^5$

**23.** $36x + 24x^2 + 6x^3$

**24.** $9x - 12x^2 + 9x^3$

**25.** $4x^4 + 3x^3 + 2x$

**26.** $5x^4 - 6x^3 + 8x$

**27.** $-81x^5 + 30x^4 + 12x^2$

**28.** *Concept Check* If $-60x^5 - 30x^4 + 20x^3$ is divided by $3x^2$, what is the sum of the coefficients of the third- and second-degree terms in the quotient?

*Perform each division.* ***See Examples 1–4.***

**29.** $\dfrac{-27r^4 + 36r^3 - 6r^2 - 26r + 2}{-3r}$

**30.** $\dfrac{-8k^4 + 12k^3 + 2k^2 - 7k + 3}{-2k}$

**31.** $\dfrac{2m^5 - 6m^4 + 8m^2}{-2m^3}$

**32.** $\dfrac{6r^5 - 8r^4 + 10r^2}{-2r^4}$

**33.** $(20a^4 - 15a^5 + 25a^3) \div (5a^4)$

**34.** $(36y^2 - 12y^3 + 20y) \div (4y^2)$

**35.** $(120x^{11} - 60x^{10} + 140x^9 - 100x^8) \div (10x^{12})$

**36.** $(120x^{12} - 84x^9 + 60x^8 - 36x^7) \div (12x^9)$

**37.** $(120x^5y^4 - 80x^2y^3 + 40x^2y^4 - 20x^5y^3) \div (20xy^2)$

**38.** $(200a^5b^6 - 160a^4b^7 - 120a^3b^9 + 40a^2b^2) \div (40a^2b)$

## RELATING CONCEPTS EXERCISES 39–42

### FOR INDIVIDUAL OR GROUP WORK

*Our system of numeration is called a decimal system. In a whole number such as* 2846, *each digit is understood to represent the number of powers of* 10 *for its place value. The* 2 *represents two thousands* $(2 \times 10^3)$, *the* 8 *represents eight hundreds* $(8 \times 10^2)$, *the* 4 *represents four tens* $(4 \times 10^1)$, *and the* 6 *represents six ones (or units)* $(6 \times 10^0)$.

$$2846 = (2 \times 10^3) + (8 \times 10^2) + (4 \times 10^1) + (6 \times 10^0) \qquad \text{Expanded form}$$

*Keeping this information in mind,* ***work Exercises 39–42 in order.***

**39.** Divide 2846 by 2, using paper-and-pencil methods: $2\overline{)2846}$.

**40.** Write your answer from **Exercise 39** in expanded form.

**41.** Divide the polynomial $2x^3 + 8x^2 + 4x + 6$ by 2.

**42.** How are your answers in **Exercises 40 and 41** similar? Different? For what value of $x$ does the answer in **Exercise 41** equal the answer in **Exercise 40?**

*Perform each division using the "long division" process.* ***See Examples 5 and 6.***

**43.** $\dfrac{x^2 - x - 6}{x - 3}$

**44.** $\dfrac{m^2 - 2m - 24}{m - 6}$

**45.** $\dfrac{2y^2 + 9y - 35}{y + 7}$

**46.** $\dfrac{2y^2 + 9y + 7}{y + 1}$

**47.** $\dfrac{p^2 + 2p + 20}{p + 6}$

**48.** $\dfrac{x^2 + 11x + 16}{x + 8}$

**49.** $\dfrac{12m^2 - 20m + 3}{2m - 3}$

**50.** $\dfrac{12y^2 + 20y + 7}{2y + 1}$

**51.** $\dfrac{4a^2 - 22a + 32}{2a + 3}$

**52.** $\dfrac{9w^2 + 6w + 10}{3w - 2}$

**53.** $\dfrac{8x^3 - 10x^2 - x + 3}{2x + 1}$

**54.** $\dfrac{12t^3 - 11t^2 + 9t + 18}{4t + 3}$

**55.** $\dfrac{8k^4 - 12k^3 - 2k^2 + 7k - 6}{2k - 3}$

**56.** $\dfrac{27r^4 - 36r^3 - 6r^2 + 26r - 24}{3r - 4}$

**57.** $\dfrac{5y^4 + 5y^3 + 2y^2 - y - 8}{y + 1}$

**58.** $\dfrac{2r^3 - 5r^2 - 6r + 15}{r - 3}$

**59.** $\dfrac{3k^3 - 4k^2 - 6k + 10}{k - 2}$

**60.** $\dfrac{5z^3 - z^2 + 10z + 2}{z + 2}$

**61.** $\dfrac{6p^4 - 16p^3 + 15p^2 - 5p + 10}{3p + 1}$

**62.** $\dfrac{6r^4 - 11r^3 - r^2 + 16r - 8}{2r - 3}$

*Perform each division.* ***See Examples 6–9.***

**63.** $(x^3 + 2x^2 - 3) \div (x - 1)$

**64.** $(x^3 - 2x^2 - 9) \div (x - 3)$

**65.** $(2x^3 + x + 2) \div (x + 1)$

**66.** $(3x^3 + x + 5) \div (x + 1)$

**67.** $\dfrac{5 - 2r^2 + r^4}{r^2 - 1}$

**68.** $\dfrac{4t^2 + t^4 + 7}{t^2 + 1}$

**69.** $\dfrac{-4x + 3x^3 + 2}{x - 1}$

**70.** $\dfrac{2x^3 - 8x^2 - 11x^4 - 4}{x + 1}$

**71.** $\dfrac{y^3 + 1}{y + 1}$

**72.** $\dfrac{y^3 - 1}{y - 1}$

**73.** $\dfrac{a^4 - 1}{a^2 - 1}$

**74.** $\dfrac{a^4 - 1}{a^2 + 1}$

**75.** $\dfrac{x^4 - 4x^3 + 5x^2 - 3x + 2}{x^2 + 3}$

**76.** $\dfrac{3t^4 + 5t^3 - 8t^2 - 13t + 2}{t^2 - 5}$

**77.** $\dfrac{2x^5 + 9x^4 + 8x^3 + 10x^2 + 14x + 5}{2x^2 + 3x + 1}$

**78.** $\dfrac{4t^5 - 11t^4 - 6t^3 + 5t^2 - t + 3}{4t^2 + t - 3}$

**79.** $(3a^2 - 11a + 17) \div (2a + 6)$

**80.** $(4x^2 + 11x - 8) \div (3x + 6)$

**81.** $\dfrac{3x^3 + 5x^2 - 9x + 5}{3x - 3}$

**82.** $\dfrac{5x^3 + 4x^2 + 10x + 20}{5x + 5}$

*In Exercises 83–88, if necessary, refer to the formulas on the inside covers.*

**83.** The area of the rectangle is given by the polynomial

$$5x^3 + 7x^2 - 13x - 6.$$

What polynomial expresses the length (in appropriate units)?

**84.** The area of the rectangle is given by the polynomial

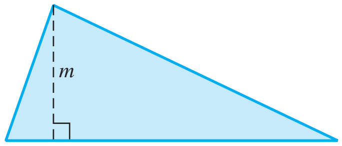
$$15x^3 + 12x^2 - 9x + 3.$$

What polynomial expresses the length (in appropriate units)?

**85.** The area of the triangle is given by the polynomial

$$24m^3 + 48m^2 + 12m.$$

What polynomial expresses the length of the base (in appropriate units)?



**86.** The area of the parallelogram is given by the polynomial

$$2x^3 + 2x^2 - 3x - 1.$$

What polynomial expresses the length of the base (in appropriate units)?

**87.** If the distance traveled is $(5x^3 - 6x^2 + 3x + 14)$ miles and the rate is $(x + 1)$ mph, write an expression, in hours, for the time traveled.

**88.** If it costs $(4x^5 + 3x^4 + 2x^3 + 9x^2 - 29x + 2)$ dollars to fertilize a garden, and fertilizer costs $(x + 2)$ dollars per square yard, write an expression, in square yards, for the area of the garden.

## PREVIEW EXERCISES

*List all positive integer factors of each number.* ***See Section 1.1.***

**89.** 18

**90.** 36

**91.** 48

**92.** 23

# CHAPTER 4 SUMMARY

## KEY TERMS

### 4.1

base
exponent
power
exponential expression

### 4.3

scientific notation

### 4.4

term
numerical coefficient
like terms
polynomial
descending powers
degree of a term
degree of a polynomial
monomial
binomial
trinomial
parabola
vertex
axis
line of symmetry

### 4.5

outer product
inner product
FOIL

## NEW SYMBOLS

| | |
|---|---|
| $x^{-n}$ | $x$ to the negative $n$ power |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **polynomial** is an algebraic expression made up of
**A.** a term or a finite product of terms with positive coefficients and exponents
**B.** a term or a finite sum of terms with real coefficients and whole number exponents
**C.** the product of two or more terms with positive exponents
**D.** the sum of two or more terms with whole number coefficients and exponents.

**2.** The **degree of a term** is
**A.** the number of variables in the term
**B.** the product of the exponents on the variables
**C.** the least exponent on the variables
**D.** the sum of the exponents on the variables.

**3.** **FOIL** is a method for
**A.** adding two binomials
**B.** adding two trinomials
**C.** multiplying two binomials
**D.** multiplying two trinomials.

**4.** A **binomial** is a polynomial with
**A.** only one term
**B.** exactly two terms
**C.** exactly three terms
**D.** more than three terms.

**5.** A **monomial** is a polynomial with
**A.** only one term
**B.** exactly two terms
**C.** exactly three terms
**D.** more than three terms.

**6.** A **trinomial** is a polynomial with
**A.** only one term
**B.** exactly two terms
**C.** exactly three terms
**D.** more than three terms.

### *ANSWERS*

**1.** B; *Example:* $5x^3 + 2x^2 - 7$ **2.** D; *Examples:* The term 6 has degree 0, $3x$ has degree 1, $-2x^8$ has degree 8, and $5x^2y^4$ has degree 6.
**3.** C; *Example:* $(m + 4)(m - 3) = \overset{F}{m(m)} - \overset{O}{3m} + \overset{I}{4m} + \overset{L}{4(-3)} = m^2 + m - 12$ **4.** B; *Example:* $3t^3 + 5t$ **5.** A; *Examples:* $-5$ and $4xy^5$
**6.** C; *Example:* $2a^2 - 3ab + b^2$

## QUICK REVIEW

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **4.1 The Product Rule and Power Rules for Exponents**<br>For any integers $m$ and $n$, the following are true.<br>**Product Rule** $a^m \cdot a^n = a^{m+n}$<br>**Power Rules (a)** $(a^m)^n = a^{mn}$<br>**(b)** $(ab)^m = a^m b^m$<br>**(c)** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$ | Perform the operations by using rules for exponents.<br>$2^4 \cdot 2^5 = 2^{4+5} = 2^9$<br>$(3^4)^2 = 3^{4 \cdot 2} = 3^8$<br>$(6a)^5 = 6^5 a^5$<br>$\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$ |
| **4.2 Integer Exponents and the Quotient Rule**<br>If $a \neq 0$, then for integers $m$ and $n$, the following are true.<br>**Zero Exponent** $a^0 = 1$<br>**Negative Exponent** $a^{-n} = \frac{1}{a^n}$<br>**Quotient Rule** $\frac{a^m}{a^n} = a^{m-n}$<br>**Negative-to-Positive Rules**<br>$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \quad (b \neq 0)$<br>$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m \quad (b \neq 0)$ | Simplify by using the rules for exponents.<br>$15^0 = 1$<br>$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$<br>$\frac{4^8}{4^3} = 4^{8-3} = 4^5$<br>$\frac{4^{-2}}{3^{-5}} = \frac{3^5}{4^2}$<br>$\left(\frac{6}{5}\right)^{-3} = \left(\frac{5}{6}\right)^3$ |
| **4.3 An Application of Exponents: Scientific Notation**<br>To write a number in scientific notation<br>$\boldsymbol{a \times 10^n}$, where $1 \le \lvert a \rvert < 10$,<br>move the decimal point to follow the first nonzero digit.<br>**1.** If moving the decimal point makes the number less, $n$ is positive.<br>**2.** If it makes the number greater, $n$ is negative.<br>**3.** If the decimal point is not moved, $n$ is 0. | Write in scientific notation.<br>$247 = 2.47 \times 10^2$<br>$0.0051 = 5.1 \times 10^{-3}$<br>$4.8 = 4.8 \times 10^0$<br>Write without exponents.<br>$3.25 \times 10^5 = 325{,}000$<br>$8.44 \times 10^{-6} = 0.00000844$ |
| **4.4 Adding and Subtracting Polynomials; Graphing Simple Polynomials**<br>**Adding Polynomials**<br>Add like terms.<br>**Subtracting Polynomials**<br>Change the signs of the terms in the second polynomial and add the second polynomial to the first. | Add. $\begin{array}{r} 2x^2 + 5x - 3 \\ 5x^2 - 2x + 7 \\ \hline 7x^2 + 3x + 4 \end{array}$<br>Subtract. $(2x^2 + 5x - 3) - (5x^2 - 2x + 7)$<br>$= (2x^2 + 5x - 3) + (-5x^2 + 2x - 7)$<br>$= -3x^2 + 7x - 10$ |

(continued)

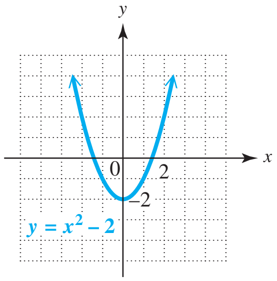
***CONCEPTS***

***EXAMPLES***

**Graphing Simple Polynomials**
To graph a simple polynomial equation such as $y = x^2 - 2$, plot points near the vertex. (In this chapter, all parabolas have a vertex on the $x$-axis or the $y$-axis.)

Graph $y = x^2 - 2$.

| $x$ | $y$ |
|---|---|
| $-2$ | $2$ |
| $-1$ | $-1$ |
| $0$ | $-2$ |
| $1$ | $-1$ |
| $2$ | $2$ |



## 4.5 Multiplying Polynomials

**General Method for Multiplying Polynomials**
Multiply each term of the first polynomial by each term of the second polynomial. Then add like terms.

Multiply.

$$\begin{array}{rrrrr}
 & 3x^3 & -\ 4x^2 & +\ 2x & -\ 7 \\
 & & & 4x & +\ 3 \\ \hline
 & 9x^3 & -\ 12x^2 & +\ 6x & -\ 21 \\
12x^4 & -\ 16x^3 & +\ 8x^2 & -\ 28x & \\ \hline
12x^4 & -\ 7x^3 & -\ 4x^2 & -\ 22x & -\ 21
\end{array}$$

**FOIL Method for Multiplying Binomials**

***Step 1*** Multiply the two **F**irst terms to get the first term of the product.

***Step 2*** Find the **O**uter product and the **I**nner product, and mentally add them, when possible, to get the middle term of the product.

***Step 3*** Multiply the two **L**ast terms to get the last term of the product.

Add the terms found in Steps 1–3.

Multiply. $(2x + 3)(5x - 4)$

$2x(5x) = 10x^2$ **F**

$2x(-4) + 3(5x) = 7x$ **O, I**

$3(-4) = -12$ **L**

The product is $10x^2 + 7x - 12$.

## 4.6 Special Products

**Square of a Binomial**

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

**Product of the Sum and Difference of Two Terms**

$$(x + y)(x - y) = x^2 - y^2$$

Multiply.

$$(3x + 1)^2 = 9x^2 + 6x + 1$$
$$(2m - 5n)^2 = 4m^2 - 20mn + 25n^2$$

$$(4a + 3)(4a - 3) = 16a^2 - 9$$

## 4.7 Dividing Polynomials

**Dividing a Polynomial by a Monomial**
Divide each term of the polynomial by the monomial.

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

**Dividing a Polynomial by a Polynomial**
Use "long division."

Divide.

$$\frac{4x^3 - 2x^2 + 6x - 9}{2x} = 2x^2 - x + 3 - \frac{9}{2x}$$

Divide each term in the numerator by $2x$.

$$\begin{array}{r}
2x - 5 \\
3x + 4\overline{)6x^2 - 7x - 21} \\
\underline{6x^2 + 8x}\phantom{{}-21} \\
-15x - 21 \\
\underline{-15x - 20} \\
-1 \leftarrow \text{Remainder}
\end{array}$$

The final answer is $2x - 5 + \frac{-1}{3x + 4}$.

# CHAPTER 4 REVIEW EXERCISES

**4.1** *Use the product rule, power rules, or both to simplify each expression. Write the answers in exponential form.*

**1.** $4^3 \cdot 4^8$

**2.** $(-5)^6(-5)^5$

**3.** $(-8x^4)(9x^3)$

**4.** $(2x^2)(5x^3)(x^9)$

**5.** $(19x)^5$

**6.** $(-4y)^7$

**7.** $5(pt)^4$

**8.** $\left(\frac{7}{5}\right)^6$

**9.** $(3x^2y^3)^3$

**10.** $(t^4)^8(t^2)^5$

**11.** $(6x^2z^4)^2(x^3yz^2)^4$

**12.** $\left(\frac{2m^3n}{p^2}\right)^3$

**13.** Why does the product rule for exponents not apply to the expression $7^2 + 7^4$?

**4.2** *Evaluate each expression.*

**14.** $6^0 + (-6)^0$

**15.** $-(-23)^0$

**16.** $-10^0$

*Simplify. Write each answer with only positive exponents. Assume that all variables represent nonzero real numbers.*

**17.** $-7^{-2}$

**18.** $\left(\frac{5}{8}\right)^{-2}$

**19.** $(5^{-2})^{-4}$

**20.** $9^3 \cdot 9^{-5}$

**21.** $2^{-1} + 4^{-1}$

**22.** $\frac{6^{-5}}{6^{-3}}$

**23.** $\frac{x^{-7}}{x^{-9}}$

**24.** $\frac{y^4 \cdot y^{-2}}{y^{-5}}$

**25.** $(3r^{-2})^{-4}$

**26.** $(3p)^4(3p^{-7})$

**27.** $\frac{ab^{-3}}{a^4b^2}$

**28.** $\frac{(6r^{-1})^2(2r^{-4})}{r^{-5}(r^2)^{-3}}$

**4.3** *Write each number in scientific notation.*

**29.** 48,000,000

**30.** 28,988,000,000

**31.** 0.0000000824

*Write each number without exponents.*

**32.** $2.4 \times 10^4$

**33.** $7.83 \times 10^7$

**34.** $8.97 \times 10^{-7}$

*Perform each indicated operation and write the answer without exponents.*

**35.** $(2 \times 10^{-3}) \times (4 \times 10^5)$

**36.** $\frac{8 \times 10^4}{2 \times 10^{-2}}$

**37.** $\frac{12 \times 10^{-5} \times 5 \times 10^4}{4 \times 10^3 \times 6 \times 10^{-2}}$

*Write each boldface italic number in the quote without exponents.*

**38.** The muon, a close relative of the electron produced by the bombardment of cosmic rays against the upper atmosphere, has a half-life of 2 millionths of a second (***$2 \times 10^{-6}$*** s). (Excerpt from *Conceptual Physics,* 6th edition, by Paul G. Hewitt. Copyright © by Paul G. Hewitt. Published by HarperCollins College Publishers.)

**39.** There are 13 red balls and 39 black balls in a box. Mix them up and draw 13 out one at a time without returning any ball . . . the probability that the 13 drawings each will produce a red ball is . . . ***$1.6 \times 10^{-12}$***. (Weaver, Warren, *Lady Luck.*)

*Write each boldface italic number in scientific notation.*

**40.** An electron and a positron attract each other in two ways: the electromagnetic attraction of their opposite electric charges, and the gravitational attraction of their two masses. The electromagnetic attraction is

***4,200,000,000,000,000,000,000,000,000,000,000,000,000,000,000***

times as strong as the gravitational. (Asimov, Isaac, *Isaac Asimov's Book of Facts.*)

**41.** The aircraft carrier USS John Stennis is a ***97,000***-ton nuclear powered floating city with a crew of ***5000.*** (*Source:* Seelye, Katharine Q., "Staunch Allies Hard to Beat: Defense Dept., Hollywood," *New York Times,* in *Plain Dealer*.)

**42.** A googol is

***10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,***
***000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000.***

The Web search engine Google is named after a googol. Sergey Brin, president and cofounder of Google, Inc., was a mathematics major. He chose the name Google to describe the vast reach of this search engine. (*Source: The Gazette*.)

**43.** According to Campbell, Mitchell, and Reece in *Biology Concepts and Connections* (Benjamin Cummings, 1994, p. 230), "The amount of DNA in a human cell is about ***1000*** times greater than the DNA in *E. coli.* Does this mean humans have 1000 times as many genes as the ***2000*** in *E. coli*? The answer is probably no; the human genome is thought to carry between ***50,000*** and ***100,000*** genes, which code for various proteins (as well as for tRNA and rRNA)."

**4.4** *In Exercises 44–48, combine like terms where possible in each polynomial. Write the answer in descending powers of the variable. Give the degree of the answer. Identify the polynomial as a* monomial, *a* binomial, *a* trinomial, *or* none of these.

**44.** $9m^2 + 11m^2 + 2m^2$

**45.** $-4p + p^3 - p^2 + 8p + 2$

**46.** $12a^5 - 9a^4 + 8a^3 + 2a^2 - a + 3$

**47.** $-7y^5 - 8y^4 - y^5 + y^4 + 9y$

**48.** $(12r^4 - 7r^3 + 2r^2) - (5r^4 - 3r^3 + 2r^2 - 1)$

**49.** Simplify. $(5x^3y^2 - 3xy^5 + 12x^2) - (-9x^2 - 8x^3y^2 + 2xy^5)$

*Add or subtract as indicated.*

**50.** Add.

$$\begin{array}{r} -2a^3 + 5a^2 \\ \underline{3a^3 - \phantom{5}a^2} \end{array}$$

**51.** Subtract.

$$\begin{array}{r} 6y^2 - 8y + 2 \\ \underline{5y^2 + 2y - 7} \end{array}$$

**52.** Subtract.

$$\begin{array}{r} -12k^4 - 8k^2 + \phantom{1}7k \\ \underline{k^4 + 7k^2 - 11k} \end{array}$$

*Graph each equation by completing the table of values.*

**53.** $y = -x^2 + 5$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | | | | | |

**54.** $y = 3x^2 - 2$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | | | | | |

**4.5** *Find each product.*

**55.** $(a + 2)(a^2 - 4a + 1)$

**56.** $(3r - 2)(2r^2 + 4r - 3)$

**57.** $(5p^2 + 3p)(p^3 - p^2 + 5)$

**58.** $(m - 9)(m + 2)$

**59.** $(3k - 6)(2k + 1)$

**60.** $(a + 3b)(2a - b)$

**61.** $(6k + 5q)(2k - 7q)$

**62.** $(s - 1)^3$

**4.6** *Find each product.*

**63.** $(a + 4)^2$

**64.** $(2r + 5t)^2$

**65.** $(6m - 5)(6m + 5)$

**66.** $(5a + 6b)(5a - 6b)$

**67.** $(r + 2)^3$

**68.** $t(5t - 3)^2$

**69.** Choose values for $x$ and $y$ to show that, in general, the following hold true.

**(a)** $(x + y)^2 \neq x^2 + y^2$

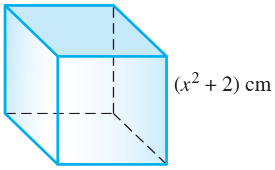
**(b)** $(x + y)^3 \neq x^3 + y^3$

**70.** Write an explanation on how to raise a binomial to the third power. Give an example.

**71.** Refer to **Exercise 69.** Suppose that you happened to let $x = 0$ and $y = 1$. Would your results be sufficient to illustrate the truth, in general, of the inequalities shown? If not, what would you need to do as your next step in working the exercise?

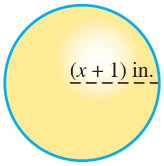
*In Exercises 72 and 73, if necessary, refer to the formulas on the inside covers.*

**72.** Find a polynomial that represents, in cubic centimeters, the volume of a cube with one side having length $(x^2 + 2)$ centimeters.



A cube

**73.** Find a polynomial that represents, in cubic inches, the volume of a sphere with radius $(x + 1)$ inches.



A sphere

**4.7** *Perform each division.*

**74.** $\dfrac{-15y^4}{9y^2}$

**75.** $\dfrac{6y^4 - 12y^2 + 18y}{6y}$

**76.** $(-10m^4n^2 + 5m^3n^2 + 6m^2n^4) \div (5m^2n)$

**77.** ***Concept Check*** What polynomial, when multiplied by $6m^2n$, gives the product

$$12m^3n^2 + 18m^6n^3 - 24m^2n^2?$$

**78.** ***Concept Check*** One of your friends in class simplified

$$\frac{6x^2 - 12x}{6} \quad \text{as} \quad x^2 - 12x.$$

***WHAT WENT WRONG?*** Give the correct answer.

*Perform each division.*

**79.** $\dfrac{2r^2 + 3r - 14}{r - 2}$

**80.** $\dfrac{10a^3 + 9a^2 - 14a + 9}{5a - 3}$

**81.** $\dfrac{x^4 - 5x^2 + 3x^3 - 3x + 4}{x^2 - 1}$

**82.** $\dfrac{m^4 + 4m^3 - 12m - 5m^2 + 6}{m^2 - 3}$

**83.** $\dfrac{16x^2 - 25}{4x + 5}$

**84.** $\dfrac{25y^2 - 100}{5y + 10}$

**85.** $\dfrac{y^3 - 8}{y - 2}$

**86.** $\dfrac{1000x^6 + 1}{10x^2 + 1}$

**87.** $\dfrac{6y^4 - 15y^3 + 14y^2 - 5y - 1}{3y^2 + 1}$

**88.** $\dfrac{4x^5 - 8x^4 - 3x^3 + 22x^2 - 15}{4x^2 - 3}$

## MIXED REVIEW EXERCISES

*Perform each indicated operation. Write answers with only positive exponents. Assume that all variables represent nonzero real numbers.*

**89.** $5^0 + 7^0$

**90.** $\left(\frac{6r^2p}{5}\right)^3$

**91.** $(12a + 1)(12a - 1)$

**92.** $2^{-4}$

**93.** $(8^{-3})^4$

**94.** $\frac{2p^3 - 6p^2 + 5p}{2p^2}$

**95.** $\frac{(2m^{-5})(3m^2)^{-1}}{m^{-2}(m^{-1})^2}$

**96.** $(3k - 6)(2k^2 + 4k + 1)$

**97.** $\frac{r^9 \cdot r^{-5}}{r^{-2} \cdot r^{-7}}$

**98.** $(2r + 5s)^2$

**99.** $(-5y^2 + 3y - 11) + (4y^2 - 7y + 15)$

**100.** $(2r + 5)(5r - 2)$

**101.** $\frac{2y^3 + 17y^2 + 37y + 7}{2y + 7}$

**102.** $(25x^2y^3 - 8xy^2 + 15x^3y) \div (10x^2y^3)$

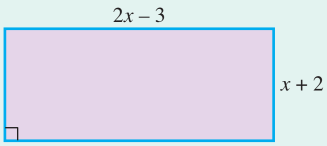
**103.** $(6p^2 - p - 8) - (-4p^2 + 2p - 3)$

**104.** $\frac{3x^3 - 2x + 5}{x - 3}$

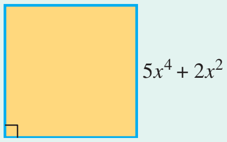
**105.** $(-7 + 2k)^2$

**106.** $\left(\frac{x}{y^{-3}}\right)^{-4}$

**107.** Find polynomials that represent, in appropriate units, the **(a)** perimeter and **(b)** area of the rectangle shown.



**108.** If the side of a square has a measure represented by $5x^4 + 2x^2$, what polynomials, in appropriate units, represent its **(a)** perimeter and **(b)** area?



## CHAPTER 4 TEST

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*Evaluate each expression.*

**1.** $5^{-4}$

**2.** $(-3)^0 + 4^0$

**3.** $4^{-1} + 3^{-1}$

**4.** Simplify $\frac{(3x^2y)^2(xy^3)^2}{(xy)^3}$. Assume that $x$ and $y$ represent nonzero numbers.

*Simplify, and write the answer using only positive exponents. Assume that all variables represent nonzero numbers.*

**5.** $\frac{8^{-1} \cdot 8^4}{8^{-2}}$

**6.** $\frac{(x^{-3})^{-2}(x^{-1}y)^2}{(xy^{-2})^2}$

**7.** Determine whether each expression represents a number that is *positive, negative,* or *zero.*

**(a)** $3^{-4}$ **(b)** $(-3)^4$ **(c)** $-3^4$ **(d)** $3^0$ **(e)** $(-3)^0 - 3^0$ **(f)** $(-3)^{-3}$

**8.** **(a)** Write 45,000,000,000 using scientific notation.

**(b)** Write $3.6 \times 10^{-6}$ without using exponents.

**(c)** Write the quotient without using exponents: $\dfrac{9.5 \times 10^{-1}}{5 \times 10^3}$.

**9.** A satellite galaxy of the Milky Way, known as the Large Magellanic Cloud, is ***1000*** light-years across. A *light-year* is equal to ***5,890,000,000,000*** mi. (*Source:* "Images of Brightest Nebula Unveiled," *USA Today*.)

**(a)** Write the two boldface italic numbers in scientific notation.

**(b)** How many miles across is the Large Magellanic Cloud?

*For each polynomial, combine like terms when possible and write the polynomial in descending powers of the variable. Give the degree of the simplified polynomial. Decide whether the simplified polynomial is a* monomial, *a* binomial, *a* trinomial, *or* none of these.

**10.** $5x^2 + 8x - 12x^2$

**11.** $13n^3 - n^2 + n^4 + 3n^4 - 9n^2$

**12.** Use the table to complete a set of ordered pairs that lie on the graph of $y = 2x^2 - 4$. Then graph the equation.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | | | | | |

*Perform each indicated operation.*

**13.** $(2y^2 - 8y + 8) + (-3y^2 + 2y + 3) - (y^2 + 3y - 6)$

**14.** $(-9a^3b^2 + 13ab^5 + 5a^2b^2) - (6ab^5 + 12a^3b^2 + 10a^2b^2)$

**15.** Subtract.

$$\begin{array}{r} 9t^3 - 4t^2 + 2t + 2 \\ \underline{9t^3 + 8t^2 - 3t - 6} \end{array}$$

**16.** $3x^2(-9x^3 + 6x^2 - 2x + 1)$

**17.** $(t - 8)(t + 3)$

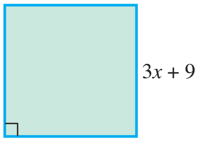
**18.** $(4x + 3y)(2x - y)$

**19.** $(5x - 2y)^2$

**20.** $(10v + 3w)(10v - 3w)$

**21.** $(2r - 3)(r^2 + 2r - 5)$

**22.** What polynomial expression represents, in appropriate units, the perimeter of this square? The area?



*Perform each division.*

**23.** $\dfrac{8y^3 - 6y^2 + 4y + 10}{2y}$

**24.** $(-9x^2y^3 + 6x^4y^3 + 12xy^3) \div (3xy)$

**25.** $\dfrac{5x^2 - x - 18}{5x + 9}$

**26.** $(3x^3 - x + 4) \div (x - 2)$

# CHAPTERS 1–4 CUMULATIVE REVIEW EXERCISES

*Write each fraction in lowest terms.*

**1.** $\frac{28}{16}$

**2.** $\frac{55}{11}$

**3.** A contractor installs sheds. Each requires $1\frac{1}{4}$ yd$^3$ of concrete. How much concrete would be needed for 25 sheds?

**4.** A retailer has \$34,000 invested in her business. She finds that last year she earned 5.4% on this investment. How much did she earn?

**5.** List all positive integer factors of 45.

**6.** If $x = -2$ and $y = 4$, find the value of $\frac{4x - 2y}{x + y}$.

*Perform each indicated operation.*

**7.** $\frac{(-13 + 15) - (3 + 2)}{6 - 12}$

**8.** $-7 - 3[2 + (5 - 8)]$

*Decide which property justifies each statement.*

**9.** $(9 + 2) + 3 = 9 + (2 + 3)$

**10.** $6(4 + 2) = 6(4) + 6(2)$

**11.** Simplify the expression $-3(2x^2 - 8x + 9) - (4x^2 + 3x + 2)$.

*Solve each equation.*

**12.** $2 - 3(t - 5) = 4 + t$

**13.** $2(5x + 1) = 10x + 4$

**14.** $d = rt$ for $r$

**15.** $\frac{x}{5} = \frac{x - 2}{7}$

**16.** $\frac{1}{3}p - \frac{1}{6}p = -2$

**17.** $0.05x + 0.15(50 - x) = 5.50$

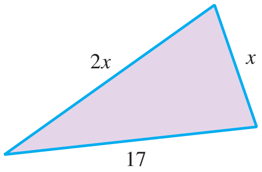
**18.** $4 - (3x + 12) = (2x - 9) - (5x - 1)$

*Solve each problem.*

**19.** A husky running the Iditarod burns $5\frac{3}{8}$ calories in exertion for every 1 calorie burned in thermoregulation in extreme cold. According to one scientific study, a husky in top condition burns an amazing total of 11,200 calories per day. How many calories are burned for exertion, and how many are burned for regulation of body temperature? Round answers to the nearest whole number.

**2008 XXXVI Iditarod**

**20.** One side of a triangle is twice as long as a second side. The third side of the triangle is 17 ft long. The perimeter of the triangle cannot be more than 50 ft. Find the longest possible values for the other two sides of the triangle.



*Solve each inequality.*

**21.** $-2(x + 4) > 3x + 6$

**22.** $-3 \le 2x + 5 < 9$

**23.** Graph $y = -3x + 6$.

**24.** Consider the two points $(-1, 5)$ and $(2, 8)$.

**(a)** Find the slope of the line joining them.

**(b)** Find the equation of the line joining them.

*Evaluate each expression.*

**25.** $4^{-1} + 3^0$

**26.** $\dfrac{8^{-5} \cdot 8^7}{8^2}$

**27.** Write with positive exponents only. $\dfrac{(a^{-3}b^2)^2}{(2a^{-4}b^{-3})^{-1}}$

**28.** It takes about $3.6 \times 10^1$ sec at a speed of $3.0 \times 10^5$ km per sec for light from the sun to reach Venus. How far is Venus from the sun? (*Source: World Almanac and Book of Facts*.)

**29.** Graph $y = (x + 4)^2$, using the $x$-values $-6, -5, -4, -3$, and $-2$ to obtain a set of points.

*Perform each indicated operation.*

**30.** $(7x^3 - 12x^2 - 3x + 8) + (6x^2 + 4) - (-4x^3 + 8x^2 - 2x - 2)$

**31.** $(7x + 4)(9x + 3)$

**32.** $\dfrac{y^3 - 3y^2 + 8y - 6}{y - 1}$

CHAPTER 5

# Factoring and Applications

Wireless communication uses radio waves to carry signals and messages across distances. Cellular phones, one of the most popular forms of wireless communication, have become an invaluable tool for people to stay connected to family, friends, and work while on the go. In 2007, there were about 243 million cell phone subscribers in the United States, with 81% of the population having cell phone service. Total revenue from this service was about $133 billion. (*Source:* CITA—The Wireless Association.)

In **Exercise 37** of **Section 5.6,** we use a *quadratic equation* to model the number of cell phone subscribers in the United States.

# 5.1 The Greatest Common Factor; Factoring by Grouping

**OBJECTIVES**

- **1** Find the greatest common factor of a list of terms.
- **2** Factor out the greatest common factor.
- **3** Factor by grouping.

Recall from **Section 1.1** that to **factor** means "to write a quantity as a product." That is, factoring is the opposite of multiplying.

*Multiplying*

$$\underset{\uparrow}{6} \cdot \underset{\uparrow}{2} = \underset{\uparrow}{12}$$

Factors Product

*Factoring*

$$\underset{\uparrow}{12} = \underset{\uparrow}{6} \cdot \underset{\uparrow}{2}$$

Product Factors

Other **factored forms** of 12 are

$$-6(-2), \quad 3 \cdot 4, \quad -3(-4), \quad 12 \cdot 1, \quad \text{and} \quad -12(-1).$$

More than two factors may be used, so another factored form of 12 is $2 \cdot 2 \cdot 3$.

**OBJECTIVE 1** **Find the greatest common factor of a list of terms.** An integer that is a factor of two or more integers is a **common factor** of those integers. For example, 6 is a common factor of 18 and 24, since 6 is a factor of both 18 and 24. Other common factors of 18 and 24 are 1, 2, and 3.

The **greatest common factor (GCF)** of a list of integers is the largest common factor of those integers. Thus, 6 is the greatest common factor of 18 and 24, since it is the largest of their common factors.

**NOTE** ***Factors*** of a number are also ***divisors*** of the number. The ***greatest common factor*** is actually the same as the ***greatest common divisor.*** Here are some useful divisibility rules for deciding what numbers divide into a given number.

| *A Whole Number Divisible by* | *Must Have the Following Property:* |
|---|---|
| 2 | Ends in 0, 2, 4, 6, or 8 |
| 3 | Sum of digits divisible by 3 |
| 4 | Last two digits form a number divisible by 4 |
| 5 | Ends in 0 or 5 |
| 6 | Divisible by both 2 and 3 |
| 8 | Last three digits form a number divisible by 8 |
| 9 | Sum of digits divisible by 9 |
| 10 | Ends in 0 |

**Finding the Greatest Common Factor (GCF)**

***Step 1*** **Factor.** Write each number in prime factored form.

***Step 2*** **List common factors.** List each prime number or each variable that is a factor of every term in the list. (If a prime does not appear in one of the prime factored forms, it cannot appear in the greatest common factor.)

***Step 3*** **Choose least exponents.** Use as exponents on the common prime factors the *least* exponents from the prime factored forms.

***Step 4*** **Multiply** the primes from Step 3. If there are no primes left after Step 3, the greatest common factor is 1.

**NOW TRY EXERCISE 1**

Find the greatest common factor for each list of numbers.

**(a)** 24, 36

**(b)** 54, 90, 108

**(c)** 15, 19, 25

## EXAMPLE 1 Finding the Greatest Common Factor for Numbers

Find the greatest common factor for each list of numbers.

**(a)** 30, 45

$$\begin{aligned} 30 &= 2 \cdot 3 \cdot 5 \\ 45 &= 3 \cdot 3 \cdot 5 \end{aligned}$$

Write the prime factored form of each number.

***Use each prime the least number of times it appears in all the factored forms.*** There is no 2 in the prime factored form of 45, so there will be no 2 in the greatest common factor. The least number of times 3 appears in all the factored forms is 1, and the least number of times 5 appears is also 1.

$$\text{GCF} = 3^1 \cdot 5^1 = 15$$

**(b)** 72, 120, 432

$$\begin{aligned} 72 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \\ 120 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \\ 432 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \end{aligned}$$

Write the prime factored form of each number.

The least number of times 2 appears in all the factored forms is 3, and the least number of times 3 appears is 1. There is no 5 in the prime factored form of either 72 or 432.

$$\text{GCF} = 2^3 \cdot 3^1 = 24$$

**(c)** 10, 11, 14

$$\begin{aligned} 10 &= 2 \cdot 5 \\ 11 &= 11 \\ 14 &= 2 \cdot 7 \end{aligned}$$

Write the prime factored form of each number.

There are no primes common to all three numbers, so the GCF is 1. NOW TRY

The greatest common factor can also be found for a list of variable terms. For example, the terms $x^4$, $x^5$, $x^6$, and $x^7$ have $x^4$ as the greatest common factor because each of these terms can be written with $x^4$ as a factor.

$$x^4 = 1 \cdot x^4, \quad x^5 = x \cdot x^4, \quad x^6 = x^2 \cdot x^4, \quad x^7 = x^3 \cdot x^4$$

**NOTE** ***The exponent on a variable in the GCF is the least exponent that appears in all the common factors.***

## EXAMPLE 2 Finding the Greatest Common Factor for Variable Terms

Find the greatest common factor for each list of terms.

**(a)** $21m^7$, $18m^6$, $45m^8$, $24m^5$

$$\begin{aligned} 21m^7 &= 3 \cdot 7 \cdot m^7 \\ 18m^6 &= 2 \cdot 3 \cdot 3 \cdot m^6 \\ 45m^8 &= 3 \cdot 3 \cdot 5 \cdot m^8 \\ 24m^5 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot m^5 \end{aligned}$$

Here, 3 is the greatest common factor of the coefficients 21, 18, 45, and 24. The least exponent on $m$ is 5.

$$\text{GCF} = 3m^5$$

**NOW TRY ANSWERS**

**1.** **(a)** 12 **(b)** 18 **(c)** 1

**NOW TRY EXERCISE 2**

Find the greatest common factor for each list of terms.

**(a)** $25k^3, 15k^2, 35k^5$

**(b)** $m^3n^5, m^4n^4, m^5n^2$

**(b)** $x^4y^2, \quad x^7y^5, \quad x^3y^7, \quad y^{15}$

$$\begin{aligned} x^4y^2 &= x^4 \cdot y^2 \\ x^7y^5 &= x^7 \cdot y^5 \\ x^3y^7 &= x^3 \cdot y^7 \\ y^{15} &= y^{15} \end{aligned}$$

There is no $x$ in the last term, $y^{15}$, so $x$ will not appear in the greatest common factor. There is a $y$ in each term, however, and 2 is the least exponent on $y$.

$$\text{GCF} = y^2$$

NOW TRY

**OBJECTIVE 2** **Factor out the greatest common factor.** Writing a polynomial (a sum) in factored form as a product is called **factoring.** For example, the polynomial

$$3m + 12$$

has two terms: $3m$ and 12. The greatest common factor of these two terms is 3. We can write $3m + 12$ so that each term is a product with 3 as one factor.

$$\begin{aligned} &3m + 12 \\ &= 3 \cdot m + 3 \cdot 4 && \text{GCF} = 3 \\ &= 3(m + 4) && \text{Distributive property} \end{aligned}$$

The factored form of $3m + 12$ is $3(m + 4)$. This process is called **factoring out the greatest common factor.**

**CAUTION** The polynomial $3m + 12$ is *not* in factored form when written as

$$3 \cdot m + 3 \cdot 4. \qquad \text{Not in factored form}$$

The *terms* are factored, but the polynomial is not. The factored form of $3m + 12$ is the *product*

$$3(m + 4). \qquad \text{In factored form}$$

### EXAMPLE 3 Factoring Out the Greatest Common Factor

Write in factored form by factoring out the greatest common factor.

**(a)** $5y^2 + 10y$

$$\begin{aligned} &= 5y(y) + 5y(2) && \text{GCF} = 5y \\ &= 5y(y + 2) && \text{Distributive property} \end{aligned}$$

*CHECK* Multiply the factored form.

$$\begin{aligned} &5y(y + 2) \\ &= 5y(y) + 5y(2) && \text{Distributive property} \\ &= 5y^2 + 10y \ \checkmark && \text{Original polynomial} \end{aligned}$$

**(b)** $20m^5 + 10m^4 + 15m^3$

$$\begin{aligned} &= 5m^3(4m^2) + 5m^3(2m) + 5m^3(3) && \text{GCF} = 5m^3 \\ &= 5m^3(4m^2 + 2m + 3) && \text{Factor out } 5m^3. \end{aligned}$$

*CHECK* $5m^3(4m^2 + 2m + 3)$

$$= 20m^5 + 10m^4 + 15m^3 \ \checkmark \qquad \text{Original polynomial}$$

**NOW TRY ANSWERS**

**2. (a)** $5k^2$ **(b)** $m^3n^2$

**NOW TRY EXERCISE 3**

Write in factored form by factoring out the greatest common factor.

**(a)** $7t^4 - 14t^3$

**(b)** $8x^6 - 20x^5 + 28x^4$

**(c)** $30m^4n^3 - 42m^2n^2$

**(c)** $x^5 + x^3$

$= x^3(x^2) + x^3(1)$ GCF $= x^3$

$= x^3(x^2 + 1)$ Don't forget the 1.

*Check* mentally by distributing $x^3$ over each term inside the parentheses.

**(d)** $20m^7p^2 - 36m^3p^4$

$= 4m^3p^2(5m^4) - 4m^3p^2(9p^2)$ GCF $= 4m^3p^2$

$= 4m^3p^2(5m^4 - 9p^2)$ Factor out $4m^3p^2$.

NOW TRY

**CAUTION** Be sure to include the 1 in a problem like **Example 3(c).** ***Check that the factored form can be multiplied out to give the original polynomial.***

**NOW TRY EXERCISE 4**

Write in factored form by factoring out the greatest common factor.

**(a)** $x(x + 2) + 5(x + 2)$

**(b)** $a(t + 10) - b(t + 10)$

## EXAMPLE 4 Factoring Out the Greatest Common Factor

Write in factored form by factoring out the greatest common factor.

Same

**(a)** $a(a + 3) + 4(a + 3)$ The binomial $a + 3$ is the greatest common factor.

$= (a + 3)(a + 4)$ Factor out $a + 3$.

**(b)** $x^2(x + 1) - 5(x + 1)$

$= (x + 1)(x^2 - 5)$ Factor out $x + 1$.

NOW TRY

**NOTE** In factored forms like those in **Example 4,** the order of the factors does not matter because of the commutative property of multiplication.

$$(a + 3)(a + 4) \quad \text{can also be written} \quad (a + 4)(a + 3).$$

**OBJECTIVE 3** **Factor by grouping.** ***When a polynomial has four terms, common factors can sometimes be used to factor by grouping.***

## EXAMPLE 5 Factoring by Grouping

Factor by grouping.

**(a)** $2x + 6 + ax + 3a$

Group the first two terms and the last two terms, since the first two terms have a common factor of 2 and the last two terms have a common factor of $a$.

$2x + 6 + ax + 3a$

$= (2x + 6) + (ax + 3a)$ Group the terms.

$= 2(x + 3) + a(x + 3)$ Factor each group.

The expression is still not in factored form because it is the *sum* of two terms. Now, however, $x + 3$ is a common factor and can be factored out.

$= 2(x + 3) + a(x + 3)$ $x + 3$ is a common factor.

$= (x + 3)(2 + a)$ Factor out $x + 3$.

$(2 + a)(x + 3)$ is also correct.

**NOW TRY ANSWERS**

3. **(a)** $7t^3(t - 2)$
   **(b)** $4x^4(2x^2 - 5x + 7)$
   **(c)** $6m^2n^2(5m^2n - 7)$
4. **(a)** $(x + 2)(x + 5)$
   **(b)** $(t + 10)(a - b)$

NOW TRY EXERCISE 5

Factor by grouping.

**(a)** $ab + 3a + 5b + 15$

**(b)** $12xy + 3x + 4y + 1$

**(c)** $x^3 + 5x^2 - 8x - 40$

The final result $(x + 3)(2 + a)$ is in factored form because it is a ***product.***

CHECK

$$\begin{aligned} &(x + 3)(2 + a) \\ &= 2x + ax + 6 + 3a && \text{FOIL (Section 5.5)} \\ &= 2x + 6 + ax + 3a \ \checkmark && \text{Rearrange terms to obtain the original polynomial.} \end{aligned}$$

**(b)** $6ax + 24x + a + 4$

$$\begin{aligned} &= (6ax + 24x) + (a + 4) && \text{Group the terms.} \\ &= 6x(a + 4) + 1(a + 4) && \text{Factor each group.} \end{aligned}$$

Remember the 1.

$$\begin{aligned} &= (a + 4)(6x + 1) && \text{Factor out } a + 4. \end{aligned}$$

CHECK

$$\begin{aligned} &(a + 4)(6x + 1) \\ &= 6ax + a + 24x + 4 && \text{FOIL} \\ &= 6ax + 24x + a + 4 \ \checkmark && \text{Rearrange terms to obtain the original polynomial.} \end{aligned}$$

**(c)** $2x^2 - 10x + 3xy - 15y$

$$\begin{aligned} &= (2x^2 - 10x) + (3xy - 15y) && \text{Group the terms.} \\ &= 2x(x - 5) + 3y(x - 5) && \text{Factor each group.} \\ &= (x - 5)(2x + 3y) && \text{Factor out } x - 5. \end{aligned}$$

CHECK

$$\begin{aligned} &(x - 5)(2x + 3y) \\ &= 2x^2 + 3xy - 10x - 15y && \text{FOIL} \\ &= 2x^2 - 10x + 3xy - 15y \ \checkmark && \text{Original polynomial} \end{aligned}$$

**(d)** $t^3 + 2t^2 - 3t - 6$

Write a + sign between the groups.

$$\begin{aligned} &= (t^3 + 2t^2) + (-3t - 6) && \text{Group the terms.} \\ &= t^2(t + 2) - 3(t + 2) && \text{Factor out } -3 \text{ so there is a common factor, } t + 2;\ -3(t + 2) = -3t - 6. \end{aligned}$$

Be careful with signs.

$$\begin{aligned} &= (t + 2)(t^2 - 3) && \text{Factor out } t + 2. \end{aligned}$$

*Check* by multiplying.

NOW TRY

---

**CAUTION** ***Be careful with signs when grouping*** in a problem like **Example 5(d).** It is wise to check the factoring in the second step, as shown in the side comment in that example, before continuing.

---

### Factoring a Polynomial with Four Terms by Grouping

***Step 1*** **Group terms.** Collect the terms into two groups so that each group has a common factor.

***Step 2*** **Factor within groups.** Factor out the greatest common factor from each group.

***Step 3*** **Factor the entire polynomial.** Factor out a common binomial factor from the results of Step 2.

***Step 4*** **If necessary, rearrange terms.** If Step 2 does not result in a common binomial factor, try a different grouping.

NOW TRY ANSWERS

**5. (a)** $(b + 3)(a + 5)$

**(b)** $(4y + 1)(3x + 1)$

**(c)** $(x + 5)(x^2 - 8)$

**NOW TRY EXERCISE 6**

Factor by grouping.

**(a)** $12p^2 - 28q - 16pq + 21p$

**(b)** $5xy - 6 - 15x + 2y$

## EXAMPLE 6 Rearranging Terms before Factoring by Grouping

Factor by grouping.

**(a)** $10x^2 - 12y + 15x - 8xy$

Factoring out the common factor of 2 from the first two terms and the common factor of $x$ from the last two terms gives the following.

$$\begin{aligned} &10x^2 - 12y + 15x - 8xy \\ &= 2(5x^2 - 6y) + x(15 - 8y) \end{aligned}$$

***This does not lead to a common factor, so we try rearranging the terms.***

$$\begin{aligned} &10x^2 - 12y + 15x - 8xy \\ &= 10x^2 - 8xy - 12y + 15x && \text{Commutative property} \\ &= (10x^2 - 8xy) + (-12y + 15x) && \text{Group the terms.} \\ &= 2x(5x - 4y) + 3(-4y + 5x) && \text{Factor each group.} \\ &= 2x(5x - 4y) + 3(5x - 4y) && \text{Rewrite } -4y + 5x. \\ &= (5x - 4y)(2x + 3) && \text{Factor out } 5x - 4y. \end{aligned}$$

*CHECK*

$$\begin{aligned} &(5x - 4y)(2x + 3) \\ &= 10x^2 + 15x - 8xy - 12y && \text{FOIL} \\ &= 10x^2 - 12y + 15x - 8xy \ \checkmark && \text{Original polynomial} \end{aligned}$$

**(b)** $2xy + 12 - 3y - 8x$

We need to rearrange these terms to get two groups that each have a common factor. Trial and error suggests the following grouping.

$$\begin{aligned} &2xy + 12 - 3y - 8x \\ &= (2xy - 3y) + (-8x + 12) && \text{Group the terms.} \\ &= y(2x - 3) - 4(2x - 3) && \text{Factor each group; } -4(2x - 3) = -8x + 12. \\ &= (2x - 3)(y - 4) && \text{Factor out } 2x - 3. \end{aligned}$$

Write a + sign between the groups.

Be careful with signs.

Since the quantities in parentheses in the second step must be the same, we factored out $-4$ rather than 4. *Check* by multiplying. NOW TRY

**NOW TRY ANSWERS**

**6. (a)** $(3p - 4q)(4p + 7)$
**(b)** $(5x + 2)(y - 3)$

## 5.1 EXERCISES

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*Find the greatest common factor for each list of numbers.* ***See Example 1.***

**1.** 40, 20, 4
**2.** 50, 30, 5
**3.** 18, 24, 36, 48
**4.** 15, 30, 45, 75
**5.** 6, 8, 9
**6.** 20, 22, 23

*Find the greatest common factor for each list of terms.* ***See Examples 1 and 2.***

**7.** $16y, 24$
**8.** $18w, 27$
**9.** $30x^3, 40x^6, 50x^7$
**10.** $60z^4, 70z^8, 90z^9$
**11.** $x^4y^3, xy^2$
**12.** $a^4b^5, a^3b$
**13.** $12m^3n^2, 18m^5n^4, 36m^8n^3$
**14.** $25p^5r^7, 30p^7r^8, 50p^5r^3$

***Concept Check*** *An expression is factored when it is written as a product, not a sum. Which of the following are not factored?*

**15.** $2k^2(5k)$

**16.** $2k^2(5k + 1)$

**17.** $2k^2 + (5k + 1)$

**18.** $(2k^2 + 5k) + 1$

*Complete each factoring by writing each polynomial as the product of two factors.*

**19.** $9m^4 = 3m^2(\underline{\qquad})$

**20.** $12p^5 = 6p^3(\underline{\qquad})$

**21.** $-8z^9 = -4z^5(\underline{\qquad})$

**22.** $-15k^{11} = -5k^8(\underline{\qquad})$

**23.** $6m^4n^5 = 3m^3n(\underline{\qquad})$

**24.** $27a^3b^2 = 9a^2b(\underline{\qquad})$

**25.** $12y + 24 = 12(\underline{\qquad})$

**26.** $18p + 36 = 18(\underline{\qquad})$

**27.** $10a^2 - 20a = 10a(\underline{\qquad})$

**28.** $15x^2 - 30x = 15x(\underline{\qquad})$

**29.** $8x^2y + 12x^3y^2 = 4x^2y(\underline{\qquad})$

**30.** $18s^3t^2 + 10st = 2st(\underline{\qquad})$

**31.** How can you check your answer when you factor a polynomial?

**32.** ***Concept Check*** A student factored $18x^3y^2 + 9xy$ as $9xy(2x^2y)$. **WHAT WENT WRONG?** Factor correctly.

*Write in factored form by factoring out the greatest common factor.* ***See Examples 3 and 4.***

**33.** $x^2 - 4x$

**34.** $m^2 - 7m$

**35.** $6t^2 + 15t$

**36.** $8x^2 + 6x$

**37.** $27m^3 - 9m$

**38.** $36p^3 - 24p$

**39.** $16z^4 + 24z^2$

**40.** $25k^4 + 15k^2$

**41.** $12x^3 + 6x^2$

**42.** $21b^3 + 7b^2$

**43.** $65y^{10} + 35y^6$

**44.** $100a^5 + 16a^3$

**45.** $11w^3 - 100$

**46.** $13z^5 - 80$

**47.** $8mn^3 + 24m^2n^3$

**48.** $19p^2y + 38p^2y^3$

**49.** $13y^8 + 26y^4 - 39y^2$

**50.** $5x^5 + 25x^4 - 20x^3$

**51.** $36p^6q + 45p^5q^4 + 81p^3q^2$

**52.** $125a^3z^5 + 60a^4z^4 + 85a^5z^2$

**53.** $a^5 + 2a^3b^2 - 3a^5b^2 + 4a^4b^3$

**54.** $x^6 + 5x^4y^3 - 6xy^4 + 10xy$

**55.** $c(x + 2) - d(x + 2)$

**56.** $r(x + 5) - t(x + 5)$

**57.** $m(m + 2n) + n(m + 2n)$

**58.** $q(q + 4p) + p(q + 4p)$

**59.** $q^2(p - 4) + 1(p - 4)$

**60.** $y^2(x - 9) + 1(x - 9)$

Students often have difficulty when factoring by grouping because they are not able to tell when the polynomial is completely factored. For example,

$$5y(2x - 3) + 8t(2x - 3) \qquad \text{Not in factored form}$$

is not in factored form, because it is the *sum* of two terms: $5y(2x - 3)$ and $8t(2x - 3)$. However, because $2x - 3$ is a common factor of these two terms, the expression can now be factored.

$$(2x - 3)(5y + 8t) \qquad \text{In factored form}$$

The factored form is a *product* of two factors: $2x - 3$ and $5y + 8t$.

***Concept Check*** *Determine whether each expression is in factored form or is not in factored form. If it is not in factored form, factor it if possible.*

**61.** $8(7t + 4) + x(7t + 4)$

**62.** $3r(5x - 1) + 7(5x - 1)$

**63.** $(8 + x)(7t + 4)$

**64.** $(3r + 7)(5x - 1)$

**65.** $18x^2(y + 4) + 7(y - 4)$

**66.** $12k^3(s - 3) + 7(s + 3)$

**67.** ***Concept Check*** Why is it not possible to factor the expression in **Exercise 65?**

**68.** ***Concept Check*** A student factored $x^3 + 4x^2 - 2x - 8$ as follows.

$$\begin{aligned} &x^3 + 4x^2 - 2x - 8 \\ &\quad = (x^3 + 4x^2) + (-2x - 8) \\ &\quad = x^2(x + 4) + 2(-x - 4) \end{aligned}$$

The student could not find a common factor of the two terms. ***WHAT WENT WRONG?*** Complete the factoring.

*Factor by grouping.* ***See Examples 5 and 6.***

**69.** $p^2 + 4p + pq + 4q$

**70.** $m^2 + 2m + mn + 2n$

**71.** $a^2 - 2a + ab - 2b$

**72.** $y^2 - 6y + yw - 6w$

**73.** $7z^2 + 14z - az - 2a$

**74.** $5m^2 + 15mp - 2mr - 6pr$

**75.** $18r^2 + 12ry - 3xr - 2xy$

**76.** $8s^2 - 4st + 6sy - 3yt$

**77.** $3a^3 + 3ab^2 + 2a^2b + 2b^3$

**78.** $4x^3 + 3x^2y + 4xy^2 + 3y^3$

**79.** $12 - 4a - 3b + ab$

**80.** $6 - 3x - 2y + xy$

**81.** $16m^3 - 4m^2p^2 - 4mp + p^3$

**82.** $10t^3 - 2t^2s^2 - 5ts + s^3$

**83.** $y^2 + 3x + 3y + xy$

**84.** $m^2 + 14p + 7m + 2mp$

**85.** $5m - 6p - 2mp + 15$

**86.** $7y - 9x - 3xy + 21$

**87.** $18r^2 - 2ty + 12ry - 3rt$

**88.** $12a^2 - 4bc + 16ac - 3ab$

**89.** $a^5 - 3 + 2a^5b - 6b$

**90.** $b^3 - 2 + 5ab^3 - 10a$

## *RELATING CONCEPTS* EXERCISES 91–94

### FOR INDIVIDUAL OR GROUP WORK

*In many cases, the choice of which pairs of terms to group when factoring by grouping can be made in different ways. To see this for* ***Example 6(b), work Exercises 91–94 in order.***

**91.** Start with the polynomial from **Example 6(b),** $2xy + 12 - 3y - 8x$, and rearrange the terms as follows:

$$2xy - 8x - 3y + 12.$$

What property from **Section 1.7** allows this?

**92.** Group the first two terms and the last two terms of the rearranged polynomial in **Exercise 91.** Then factor each group.

**93.** Is your result from **Exercise 92** in factored form? Explain your answer.

**94.** If your answer to **Exercise 93** is *no,* factor the polynomial. Is the result the same as that shown for **Example 6(b)?**

## *PREVIEW EXERCISES*

*Find each product.* ***See Section 4.5.***

**95.** $(x + 6)(x - 9)$

**96.** $(x - 3)(x - 6)$

**97.** $(x + 2)(x + 7)$

**98.** $2x(x + 5)(x - 1)$

**99.** $2x^2(x^2 + 3x + 5)$

**100.** $-5x^2(2x^2 - 4x - 9)$

# 5.2

# Factoring Trinomials

**OBJECTIVES**

1. Factor trinomials with a coefficient of 1 for the second-degree term.
2. Factor such trinomials after factoring out the greatest common factor.

Using the FOIL method, we can find the product of the binomials $k - 3$ and $k + 1$.

$$(k-3)(k+1) = k^2 - 2k - 3 \qquad \text{Multiplying}$$

Suppose instead that we are given the polynomial $k^2 - 2k - 3$ and want to rewrite it as the product $(k-3)(k+1)$.

$$k^2 - 2k - 3 = (k-3)(k+1) \qquad \text{Factoring}$$

Recall from **Section 5.1** that this process is called factoring the polynomial. Factoring reverses or "undoes" multiplying.

**OBJECTIVE 1 Factor trinomials with a coefficient of 1 for the second-degree term.** When factoring polynomials with integer coefficients, we use only integers in the factors. For example, we can factor $x^2 + 5x + 6$ by finding integers $m$ and $n$ such that

$$x^2 + 5x + 6 \quad \text{is written as} \quad (x+m)(x+n).$$

To find these integers $m$ and $n$, we multiply the two binomials on the right.

$$\begin{aligned} &(x+m)(x+n) \\ &= x^2 + nx + mx + mn && \text{FOIL} \\ &= x^2 + (n+m)x + mn && \text{Distributive property} \end{aligned}$$

Comparing this result with $x^2 + 5x + 6$ shows that we must find integers $m$ and $n$ having a sum of 5 and a product of 6.

Product of $m$ and $n$ is 6.

$$x^2 + 5x + 6 = x^2 + (n+m)x + mn$$

Sum of $m$ and $n$ is 5.

Since many pairs of integers have a sum of 5, it is best to begin by listing those pairs of integers whose product is 6. Both 5 and 6 are positive, so we consider only pairs in which both integers are positive.

| *Factors of 6* | *Sums of Factors* |
|---|---|
| 6, 1 | 6 + 1 = 7 |
| 3, 2 | 3 + 2 = 5 |

Sum is 5.

Both pairs have a product of 6, but only the pair 3 and 2 has a sum of 5. So 3 and 2 are the required integers.

$$x^2 + 5x + 6 \quad \text{factors as} \quad (x+3)(x+2).$$

Check by using the FOIL method to multiply the binomials. ***Make sure that the sum of the outer and inner products produces the correct middle term.***

*CHECK* $(x+3)(x+2) = x^2 + 5x + 6$ ✓ Correct

$$\begin{array}{r} 3x \\ \underline{2x} \\ 5x \end{array} \quad \text{Add.}$$

**NOW TRY EXERCISE 1**

Factor $p^2 + 7p + 10$.

### EXAMPLE 1 Factoring a Trinomial with All Positive Terms

Factor $m^2 + 9m + 14$.

Look for two integers whose product is 14 and whose sum is 9. List pairs of integers whose product is 14, and examine the sums. Again, only positive integers are needed because all signs in $m^2 + 9m + 14$ are positive.

| *Factors of 14* | *Sums of Factors* |
|---|---|
| 14, 1 | $14 + 1 = 15$ |
| 7, 2 | $7 + 2 = 9$ |

Sum is 9.

From the list, 7 and 2 are the required integers, since $7 \cdot 2 = 14$ and $7 + 2 = 9$.

$$m^2 + 9m + 14 \quad \text{factors as} \quad (m + 7)(m + 2).$$

$(m + 2)(m + 7)$ is also correct.

**CHECK** $(m + 7)(m + 2)$

$$= m^2 + 2m + 7m + 14 \qquad \text{FOIL}$$

$$= m^2 + 9m + 14 \ ✓ \qquad \text{Original polynomial}$$

NOW TRY

**NOW TRY EXERCISE 2**

Factor $t^2 - 9t + 18$.

### EXAMPLE 2 Factoring a Trinomial with a Negative Middle Term

Factor $x^2 - 9x + 20$.

We must find two integers whose product is 20 and whose sum is $-9$. Since the numbers we are looking for have a *positive product* and a *negative sum,* we consider only pairs of negative integers.

| *Factors of 20* | *Sums of Factors* |
|---|---|
| $-20, -1$ | $-20 + (-1) = -21$ |
| $-10, -2$ | $-10 + (-2) = -12$ |
| $-5, -4$ | $-5 + (-4) = -9$ |

Sum is $-9$.

The required integers are $-5$ and $-4$.

$$x^2 - 9x + 20 \quad \text{factors as} \quad (x - 5)(x - 4).$$

The order of the factors does not matter.

**CHECK** $(x - 5)(x - 4)$

$$= x^2 - 4x - 5x + 20 \qquad \text{FOIL}$$

$$= x^2 - 9x + 20 \ ✓ \qquad \text{Original polynomial}$$

NOW TRY

**NOW TRY EXERCISE 3**

Factor $x^2 + x - 42$.

### EXAMPLE 3 Factoring a Trinomial with a Negative Last (Constant) Term

Factor $x^2 + x - 6$.

We must find two integers whose product is $-6$ and whose sum is 1 (since the coefficient of $x$, or $1x$, is 1). To get a *negative product,* the pairs of integers must have different signs.

Once we find the required pair, we can stop listing factors.

| *Factors of $-6$* | *Sums of Factors* |
|---|---|
| $6, -1$ | $6 + (-1) = 5$ |
| $-6, 1$ | $-6 + 1 = -5$ |
| $3, -2$ | $3 + (-2) = 1$ |

Sum is 1.

The required integers are 3 and $-2$.

$$x^2 + x - 6 \quad \text{factors as} \quad (x + 3)(x - 2).$$

To check, multiply the factored form.

NOW TRY

**NOW TRY ANSWERS**

1. $(p + 2)(p + 5)$
2. $(t - 3)(t - 6)$
3. $(x + 7)(x - 6)$

NOW TRY EXERCISE 4

Factor $x^2 - 4x - 21$.

## EXAMPLE 4 Factoring a Trinomial with Two Negative Terms

Factor $p^2 - 2p - 15$.

Find two integers whose product is $-15$ and whose sum is $-2$. Because the constant term, $-15$, is negative, list pairs of integers with different signs.

| Factors of $-15$ | Sums of Factors |
|---|---|
| $15, -1$ | $15 + (-1) = 14$ |
| $-15, 1$ | $-15 + 1 = -14$ |
| $5, -3$ | $5 + (-3) = 2$ |
| $-5, 3$ | $-5 + 3 = -2$ |

Sum is $-2$.

The required integers are $-5$ and 3.

$$p^2 - 2p - 15 \quad \text{factors as} \quad (p - 5)(p + 3).$$

To check, multiply the factored form.

NOW TRY

**NOTE** In **Examples 1–4,** notice that we listed factors in descending order (disregarding their signs) when we were looking for the required pair of integers. This helps avoid skipping the correct combination.

Some trinomials cannot be factored by using only integers. We call such trinomials **prime polynomials.**

NOW TRY EXERCISE 5

Factor each trinomial if possible.

**(a)** $m^2 + 5m + 8$

**(b)** $t^2 + 11t - 24$

## EXAMPLE 5 Deciding Whether Polynomials Are Prime

Factor each trinomial if possible.

**(a)** $x^2 - 5x + 12$

As in **Example 2,** both factors must be negative to give a positive product and a negative sum. List pairs of negative integers whose product is 12, and examine the sums.

| Factors of 12 | Sums of Factors |
|---|---|
| $-12, -1$ | $-12 + (-1) = -13$ |
| $-6, -2$ | $-6 + (-2) = -8$ |
| $-4, -3$ | $-4 + (-3) = -7$ |

No sum is $-5$.

None of the pairs of integers has a sum of $-5$. Therefore, the trinomial $x^2 - 5x + 12$ *cannot be factored by using only integers.* It is a *prime polynomial.*

**(b)** $k^2 - 8k + 11$

There is no pair of integers whose product is 11 and whose sum is $-8$, so $k^2 - 8k + 11$ is a prime polynomial.

NOW TRY

### Guidelines for Factoring $x^2 + bx + c$

Find two integers whose product is $c$ and whose sum is $b$.

1. Both integers must be positive if $b$ and $c$ are positive. (See **Example 1.**)
2. Both integers must be negative if $c$ is positive and $b$ is negative. (See **Example 2.**)
3. One integer must be positive and one must be negative if $c$ is negative. (See **Examples 3 and 4.**)

NOW TRY ANSWERS

4. $(x + 3)(x - 7)$

5. **(a)** prime **(b)** prime

**NOW TRY EXERCISE 6**
Factor $a^2 + 2ab - 15b^2$.

### EXAMPLE 6 Factoring a Trinomial with Two Variables

Factor $z^2 - 2bz - 3b^2$.

Here, the coefficient of $z$ in the middle term is $-2b$, so we need to find two expressions whose product is $-3b^2$ and whose sum is $-2b$.

| Factors of $-3b^2$ | Sums of Factors |
|---|---|
| $3b, -b$ | $3b + (-b) = 2b$ |
| $-3b, b$ | $-3b + b = -2b$ |

Sum is $-2b$.

$$z^2 - 2bz - 3b^2 \quad \text{factors as} \quad (z - 3b)(z + b).$$

**CHECK** $(z - 3b)(z + b)$

$= z^2 + zb - 3bz - 3b^2$ FOIL

$= z^2 + 1bz - 3bz - 3b^2$ Identity and commutative properties

$= z^2 - 2bz - 3b^2$ ✓ Combine like terms.

NOW TRY

**OBJECTIVE 2 Factor such trinomials after factoring out the greatest common factor.** If a trinomial has a common factor, first factor it out.

**NOW TRY EXERCISE 7**
Factor $3y^4 - 27y^3 + 60y^2$.

### EXAMPLE 7 Factoring a Trinomial with a Common Factor

Factor $4x^5 - 28x^4 + 40x^3$.

$4x^5 - 28x^4 + 40x^3$

$= 4x^3(x^2 - 7x + 10)$ Factor out the greatest common factor, $4x^3$.

Factor $x^2 - 7x + 10$. The integers $-5$ and $-2$ have a product of 10 and a sum of $-7$.

Include $4x^3$. $= 4x^3(x - 5)(x - 2)$ Completely factored form

**CHECK** $4x^3(x - 5)(x - 2)$

$= 4x^3(x^2 - 7x + 10)$ FOIL; Combine like terms.

$= 4x^5 - 28x^4 + 40x^3$ ✓ Distributive property

NOW TRY

**NOW TRY ANSWERS**
6. $(a + 5b)(a - 3b)$
7. $3y^2(y - 5)(y - 4)$

⚠ **CAUTION** ***When factoring, always look for a common factor first.*** Remember to include the common factor as part of the answer. Always check by multiplying.

## 5.2 EXERCISES

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*In Exercises 1–4, list all pairs of integers with the given product. Then find the pair whose sum is given.* ***See the tables in Examples 1–4.***

**1.** Product: 48; Sum: $-19$

**2.** Product: 18; Sum: 9

**3.** Product: $-24$; Sum: $-5$

**4.** Product: $-36$; Sum: $-16$

**5.** ***Concept Check*** If a trinomial in $x$ is factored as $(x + a)(x + b)$, what must be true of $a$ and $b$ if the coefficient of the constant term of the trinomial is negative?

**6.** *Concept Check* In **Exercise 5,** what must be true of $a$ and $b$ if the coefficient of the constant term is positive?

**7.** What is meant by a *prime polynomial*?

**8.** How can you check your work when factoring a trinomial? Does the check ensure that the trinomial is completely factored?

**9.** *Concept Check* Which is the correct factored form of $x^2 - 12x + 32$?

**A.** $(x - 8)(x + 4)$ **B.** $(x + 8)(x - 4)$
**C.** $(x - 8)(x - 4)$ **D.** $(x + 8)(x + 4)$

**10.** *Concept Check* What is the suggested first step in factoring $2x^3 + 8x^2 - 10x$? (See **Example 7.**)

**11.** *Concept Check* What polynomial can be factored as $(a + 9)(a + 4)$?

**12.** *Concept Check* What polynomial can be factored as $(y - 7)(y + 3)$?

*Complete each factoring.* ***See Examples 1–4.***

**13.** $p^2 + 11p + 30 = (p + 5)(\underline{\qquad})$

**14.** $x^2 + 10x + 21 = (x + 7)(\underline{\qquad})$

**15.** $x^2 + 15x + 44 = (x + 4)(\underline{\qquad})$

**16.** $r^2 + 15r + 56 = (r + 7)(\underline{\qquad})$

**17.** $x^2 - 9x + 8 = (x - 1)(\underline{\qquad})$

**18.** $t^2 - 14t + 24 = (t - 2)(\underline{\qquad})$

**19.** $y^2 - 2y - 15 = (y + 3)(\underline{\qquad})$

**20.** $t^2 - t - 42 = (t + 6)(\underline{\qquad})$

**21.** $x^2 + 9x - 22 = (x - 2)(\underline{\qquad})$

**22.** $x^2 + 6x - 27 = (x - 3)(\underline{\qquad})$

**23.** $y^2 - 7y - 18 = (y + 2)(\underline{\qquad})$

**24.** $y^2 - 2y - 24 = (y + 4)(\underline{\qquad})$

*Factor completely. If the polynomial cannot be factored, write* prime. ***See Examples 1–5.*** (Hint: *In Exercises 43 and 44, first write the trinomial in descending powers and then factor.*)

**25.** $y^2 + 9y + 8$ **26.** $a^2 + 9a + 20$ **27.** $b^2 + 8b + 15$
**28.** $x^2 + 6x + 8$ **29.** $m^2 + m - 20$ **30.** $p^2 + 4p - 5$
**31.** $y^2 - 8y + 15$ **32.** $y^2 - 6y + 8$ **33.** $x^2 + 4x + 5$
**34.** $t^2 + 11t + 12$ **35.** $z^2 - 15z + 56$ **36.** $x^2 - 13x + 36$
**37.** $r^2 - r - 30$ **38.** $q^2 - q - 42$ **39.** $a^2 - 8a - 48$
**40.** $d^2 - 4d - 45$ **41.** $x^2 + 3x - 39$ **42.** $m^2 + 10m - 30$
**43.** $-32 + 14x + x^2$ **44.** $-39 + 10x + x^2$

*Factor completely.* ***See Example 6.***

**45.** $r^2 + 3ra + 2a^2$ **46.** $x^2 + 5xa + 4a^2$ **47.** $t^2 - tz - 6z^2$
**48.** $a^2 - ab - 12b^2$ **49.** $x^2 + 4xy + 3y^2$ **50.** $p^2 + 9pq + 8q^2$
**51.** $v^2 - 11vw + 30w^2$ **52.** $v^2 - 11vx + 24x^2$

*Factor completely.* ***See Example 7.***

**53.** $4x^2 + 12x - 40$ **54.** $5y^2 - 5y - 30$ **55.** $2t^3 + 8t^2 + 6t$

**56.** $3t^3 + 27t^2 + 24t$ **57.** $2x^6 + 8x^5 - 42x^4$ **58.** $4y^5 + 12y^4 - 40y^3$

**59.** $5m^5 + 25m^4 - 40m^2$ **60.** $12k^5 - 6k^3 + 10k^2$

**61.** $m^3n - 10m^2n^2 + 24mn^3$ **62.** $y^3z + 3y^2z^2 - 54yz^3$

***Brain Busters*** *Factor each polynomial.*

**63.** $a^5 + 3a^4b - 4a^3b^2$ **64.** $m^3n - 2m^2n^2 - 3mn^3$ **65.** $y^3z + y^2z^2 - 6yz^3$

**66.** $k^7 - 2k^6m - 15k^5m^2$ **67.** $z^{10} - 4z^9y - 21z^8y^2$ **68.** $x^9 + 5x^8w - 24x^7w^2$

**69.** $(a + b)x^2 + (a + b)x - 12(a + b)$

**70.** $(x + y)n^2 + (x + y)n - 20(x + y)$

**71.** $(2p + q)r^2 - 12(2p + q)r + 27(2p + q)$

**72.** $(3m - n)k^2 - 13(3m - n)k + 40(3m - n)$

## *PREVIEW EXERCISES*

*Find each product.* ***See Section 4.5.***

**73.** $(2y - 7)(y + 4)$ **74.** $(3a + 2)(2a + 1)$ **75.** $(5z + 2)(3z - 2)$

# 5.3 More on Factoring Trinomials

**OBJECTIVES**

1. Factor trinomials by grouping when the coefficient of the second-degree term is not 1.
2. Factor trinomials by using the FOIL method.

Trinomials such as $2x^2 + 7x + 6$, in which the coefficient of the second-degree term is *not* 1, are factored with extensions of the methods from the previous sections.

**OBJECTIVE 1 Factor trinomials by grouping when the coefficient of the second-degree term is not 1.** A trinomial such as $m^2 + 3m + 2$ is factored by finding two numbers whose product is 2 and whose sum is 3. To factor $2x^2 + 7x + 6$, we look for two integers whose product is $2 \cdot 6 = 12$ and whose sum is 7.

$$\underbrace{2x^2 + \overset{\substack{\text{Sum is 7.}\\ \downarrow}}{7x} + 6}_{\text{Product is } 2 \cdot 6 = 12.}$$

By considering pairs of positive integers whose product is 12, we find the required integers, 3 and 4. We use these integers to write the middle term, $7x$, as $7x = 3x + 4x$.

$$\begin{aligned} &2x^2 + 7x + 6 \\ &= 2x^2 + \underbrace{3x + 4x}_{7x} + 6 \\ &= (2x^2 + 3x) + (4x + 6) && \text{Group the terms.} \\ &= x(2x + 3) + 2(2x + 3) && \text{Factor each group.} \end{aligned}$$

Must be the same factor

$$= (2x + 3)(x + 2) \qquad \text{Factor out } 2x + 3.$$

*CHECK* Multiply $(2x + 3)(x + 2)$ to obtain $2x^2 + 7x + 6$. ✓

**NOTE** In the preceding example, we could have written $7x$ as $4x + 3x$, rather than as $3x + 4x$. Factoring by grouping would give the same answer. Try this.

**NOW TRY EXERCISE 1**

Factor.

**(a)** $2z^2 + 5z + 3$

**(b)** $15m^2 + m - 2$

**(c)** $8x^2 - 2xy - 3y^2$

### EXAMPLE 1 Factoring Trinomials by Grouping

Factor each trinomial.

**(a)** $6r^2 + r - 1$

We must find two integers with a product of $6(-1) = -6$ and a sum of 1.

$$\overset{\text{Sum is 1.}}{\downarrow}$$

$$\underbrace{6r^2 + 1r - 1}_{\text{Product is } 6(-1) = -6.}$$

The integers are $-2$ and 3. We write the middle term, $r$, as $-2r + 3r$.

$$\begin{aligned}
&6r^2 + r - 1 \\
&= 6r^2 - 2r + 3r - 1 && r = -2r + 3r \\
&= (6r^2 - 2r) + (3r - 1) && \text{Group the terms.} \\
&= 2r(3r - 1) + 1(3r - 1) && \text{The binomials must be the same.} \\
&= (3r - 1)(2r + 1) && \text{Factor out } 3r - 1.
\end{aligned}$$

Remember the 1.

*CHECK* Multiply $(3r - 1)(2r + 1)$ to obtain $6r^2 + r - 1$. ✓

**(b)** $12z^2 - 5z - 2$

Look for two integers whose product is $12(-2) = -24$ and whose sum is $-5$. The required integers are 3 and $-8$.

$$\begin{aligned}
&12z^2 - 5z - 2 \\
&= 12z^2 + 3z - 8z - 2 && -5z = 3z - 8z \\
&= (12z^2 + 3z) + (-8z - 2) && \text{Group the terms.} \\
&= 3z(4z + 1) - 2(4z + 1) && \text{Factor each group.} \\
&= (4z + 1)(3z - 2) && \text{Factor out } 4z + 1.
\end{aligned}$$

Be careful with signs.

*CHECK* Multiply $(4z + 1)(3z - 2)$ to obtain $12z^2 - 5z - 2$. ✓

**(c)** $10m^2 + mn - 3n^2$

Two integers whose product is $10(-3) = -30$ and whose sum is 1 are $-5$ and 6.

$$\begin{aligned}
&10m^2 + mn - 3n^2 \\
&= 10m^2 - 5mn + 6mn - 3n^2 && mn = -5mn + 6mn \\
&= (10m^2 - 5mn) + (6mn - 3n^2) && \text{Group the terms.} \\
&= 5m(2m - n) + 3n(2m - n) && \text{Factor each group.} \\
&= (2m - n)(5m + 3n) && \text{Factor out } 2m - n.
\end{aligned}$$

*CHECK* Multiply $(2m - n)(5m + 3n)$ to obtain $10m^2 + mn - 3n^2$. ✓

NOW TRY

**NOW TRY ANSWERS**

1. **(a)** $(2z + 3)(z + 1)$
   **(b)** $(3m - 1)(5m + 2)$
   **(c)** $(4x - 3y)(2x + y)$

**NOW TRY EXERCISE 2**

Factor $15z^6 + 18z^5 - 24z^4$.

**EXAMPLE 2** Factoring a Trinomial with a Common Factor by Grouping

Factor $28x^5 - 58x^4 - 30x^3$.

$$\begin{aligned} &28x^5 - 58x^4 - 30x^3 \\ &= 2x^3(14x^2 - 29x - 15) \qquad \text{Factor out the greatest common factor, } 2x^3. \end{aligned}$$

To factor $14x^2 - 29x - 15$, find two integers whose product is $14(-15) = -210$ and whose sum is $-29$. Factoring 210 into prime factors helps find these integers.

$$210 = 2 \cdot 3 \cdot 5 \cdot 7$$

Combine the prime factors of $210 = 2 \cdot 3 \cdot 5 \cdot 7$ into pairs in different ways, using one positive and one negative (to get $-210$). The factors 6 and $-35$ have the correct sum, $-29$.

$$\begin{aligned} &28x^5 - 58x^4 - 30x^3 \\ &= 2x^3(14x^2 - 29x - 15) \\ &= 2x^3(14x^2 + 6x - 35x - 15) && -29x = 6x - 35x \\ &= 2x^3[(14x^2 + 6x) + (-35x - 15)] && \text{Group the terms.} \\ &= 2x^3[2x(7x + 3) - 5(7x + 3)] && \text{Factor each group.} \\ &= 2x^3[(7x + 3)(2x - 5)] && \text{Factor out } 7x + 3. \\ &= 2x^3(7x + 3)(2x - 5) \end{aligned}$$

Remember the common factor.

Check by multiplying.

NOW TRY

**OBJECTIVE 2** **Factor trinomials by using the FOIL method.** There is an alternative method of factoring trinomials that uses trial and error.

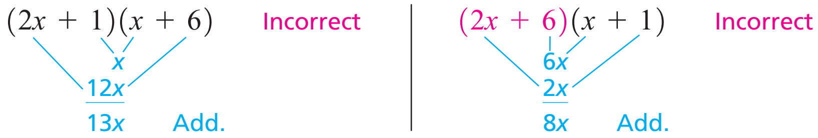
To factor $2x^2 + 7x + 6$ (the trinomial factored at the beginning of this section) by trial and error, we use the FOIL method in reverse. We want to write $2x^2 + 7x + 6$ as the product of two binomials.

$$\begin{aligned} &2x^2 + 7x + 6 \\ &= (\_\_\_\_\_\_)(\_\_\_\_\_\_) \end{aligned}$$

The product of the two first terms of the binomials is $2x^2$. The possible factors of $2x^2$ are $2x$ and $x$ or $-2x$ and $-x$. Since all terms of the trinomial are positive, we consider only positive factors. Thus, we have the following.

$$\begin{aligned} &2x^2 + 7x + 6 \\ &= (2x \_\_\_\_)(x \_\_\_\_) \end{aligned}$$

The product of the two last terms, 6, can be factored as $1 \cdot 6$, $6 \cdot 1$, $2 \cdot 3$, or $3 \cdot 2$. Try each pair to find the pair that gives the correct middle term, $7x$.



Since $2x + 6 = 2(x + 3)$, the binomial $2x + 6$ has a common factor of 2, while $2x^2 + 7x + 6$ has no common factor other than 1. The product $(2x + 6)(x + 1)$ cannot be correct.

**NOTE** If the terms of the original polynomial have greatest common factor 1, then each factor of that polynomial will also have terms with GCF 1.

**NOW TRY ANSWER**

**2.** $3z^4(5z - 4)(z + 2)$

Now try the numbers 2 and 3 as factors of 6. Because of the common factor 2 in $2x + 2$, the product $(2x + 2)(x + 3)$ will not work, so we try $(2x + 3)(x + 2)$.

$$(2x + 3)(x + 2) = 2x^2 + 7x + 6 \qquad \text{Correct}$$

$$\begin{array}{l} 3x \\ 4x \\ \hline 7x \quad \text{Add.} \end{array}$$

Thus, $2x^2 + 7x + 6$ factors as $(2x + 3)(x + 2)$.

**NOW TRY EXERCISE 3**

Factor $8y^2 + 22y + 5$.

### EXAMPLE 3 Factoring a Trinomial with All Positive Terms by Using FOIL

Factor $8p^2 + 14p + 5$.

The number 8 has several possible pairs of factors, but 5 has only 1 and 5 or $-1$ and $-5$, so begin by considering the factors of 5. Ignore the negative factors, since all coefficients in the trinomial are positive. The factors will have this form.

$$(\underline{\quad} + 5)(\underline{\quad} + 1)$$

The possible pairs of factors of $8p^2$ are $8p$ and $p$, or $4p$ and $2p$. Try various combinations, checking in each case to see if the middle term is $14p$.

$$(8p + 5)(p + 1) \quad \text{Incorrect} \qquad (p + 5)(8p + 1) \quad \text{Incorrect} \qquad (4p + 5)(2p + 1) \quad \text{Correct}$$

$$\begin{array}{l} 5p \\ 8p \\ \hline 13p \quad \text{Add.} \end{array} \qquad \begin{array}{l} 40p \\ p \\ \hline 41p \quad \text{Add.} \end{array} \qquad \begin{array}{l} 10p \\ 4p \\ \hline 14p \quad \text{Add.} \end{array}$$

Since the combination on the right produces $14p$, the correct middle term,

$$8p^2 + 14p + 5 \quad \text{factors as} \quad (4p + 5)(2p + 1).$$

*CHECK* Multiply $(4p + 5)(2p + 1)$ to obtain $8p^2 + 14p + 5$ ✓ NOW TRY

**NOW TRY EXERCISE 4**

Factor $10x^2 - 9x + 2$.

### EXAMPLE 4 Factoring a Trinomial with a Negative Middle Term by Using FOIL

Factor $6x^2 - 11x + 3$.

Since 3 has only 1 and 3 or $-1$ and $-3$ as factors, it is better here to begin by factoring 3. The last (constant) term of the trinomial $6x^2 - 11x + 3$ is positive and the middle term has a negative coefficient, so we consider only negative factors. We need two negative factors, because the *product* of two negative factors is positive and their *sum* is negative, as required. Try $-3$ and $-1$ as factors of 3.

$$(\underline{\quad} - 3)(\underline{\quad} - 1)$$

The factors of $6x^2$ may be either $6x$ and $x$ or $2x$ and $3x$.

$$(6x - 3)(x - 1) \quad \text{Incorrect} \qquad (2x - 3)(3x - 1) \quad \text{Correct}$$

$$\begin{array}{l} -3x \\ -6x \\ \hline -9x \quad \text{Add.} \end{array} \qquad \begin{array}{l} -9x \\ -2x \\ \hline -11x \quad \text{Add.} \end{array}$$

The factors $2x$ and $3x$ produce $-11x$, the correct middle term.

$$6x^2 - 11x + 3 \quad \text{factors as} \quad (2x - 3)(3x - 1).$$

*Check* by multiplying.

NOW TRY

**NOTE** In **Example 4,** we might also realize that our initial attempt to factor $6x^2 - 11x + 3$ as $(6x - 3)(x - 1)$ *cannot* be correct, since the terms of $6x - 3$ have a common factor of 3, while those of the original polynomial do not.

**NOW TRY ANSWERS**

**3.** $(4y + 1)(2y + 5)$

**4.** $(5x - 2)(2x - 1)$

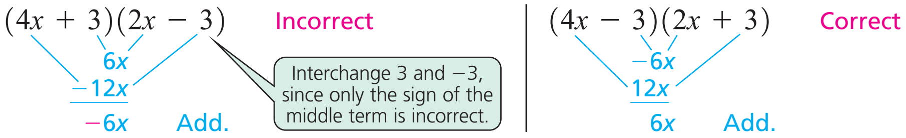
NOW TRY EXERCISE 5

Factor $10a^2 + 31a - 14$.

## EXAMPLE 5 Factoring a Trinomial with a Negative Constant Term by Using FOIL

Factor $8x^2 + 6x - 9$.

The integer 8 has several possible pairs of factors, as does $-9$. Since the constant term is negative, one positive factor and one negative factor of $-9$ are needed. Since the coefficient of the middle term is relatively small, it is wise to avoid large factors such as 8 or 9. We try $4x$ and $2x$ as factors of $8x^2$, and 3 and $-3$ as factors of $-9$.



Interchange 3 and $-3$, since only the sign of the middle term is incorrect.

The combination on the right produces the correct middle term.

$$8x^2 + 6x - 9 \quad \text{factors as} \quad (4x - 3)(2x + 3).$$

NOW TRY

NOW TRY EXERCISE 6

Factor $8z^2 + 2wz - 15w^2$.

## EXAMPLE 6 Factoring a Trinomial with Two Variables

Factor $12a^2 - ab - 20b^2$.

There are several pairs of factors of $12a^2$, including

$$12a \text{ and } a, \quad 6a \text{ and } 2a, \quad \text{and} \quad 3a \text{ and } 4a.$$

There are also many pairs of factors of $-20b^2$, including

$$20b \text{ and } -b, \quad -20b \text{ and } b, \quad 10b \text{ and } -2b, \quad -10b \text{ and } 2b,$$
$$4b \text{ and } -5b, \quad \text{and} \quad -4b \text{ and } 5b.$$

Once again, since the coefficient of the desired middle term is relatively small, avoid the larger factors. Try the factors $6a$ and $2a$, and $4b$ and $-5b$.

$$(6a + 4b)(2a - 5b)$$

This cannot be correct, since there is a factor of 2 in $6a + 4b$, while 2 is not a factor of the given trinomial. Try $3a$ and $4a$ with $4b$ and $-5b$.

$$(3a + 4b)(4a - 5b)$$
$$= 12a^2 + ab - 20b^2 \quad \text{Incorrect}$$

Here the middle term is $ab$ rather than $-ab$, so we interchange the signs of the last two terms in the factors.

$$12a^2 - ab - 20b^2 \quad \text{factors as} \quad (3a - 4b)(4a + 5b).$$

Check by multiplying.

NOW TRY

## EXAMPLE 7 Factoring Trinomials with Common Factors

Factor each trinomial.

**(a)** $15y^3 + 55y^2 + 30y$

$$= 5y(3y^2 + 11y + 6) \quad \text{Factor out the greatest common factor, } 5y.$$

To factor $3y^2 + 11y + 6$, try $3y$ and $y$ as factors of $3y^2$, and 2 and 3 as factors of 6.

$$(3y + 2)(y + 3)$$
$$= 3y^2 + 11y + 6 \quad \text{Correct}$$

NOW TRY ANSWERS

**5.** $(5a - 2)(2a + 7)$

**6.** $(4z - 5w)(2z + 3w)$

**NOW TRY EXERCISE 7**
Factor $-10x^3 - 45x^2 + 90x$.

This leads to the completely factored form.

$$\begin{aligned} &15y^3 + 55y^2 + 30y \\ &= 5y(3y + 2)(y + 3) \end{aligned}$$

Remember the common factor.

*CHECK* $5y(3y + 2)(y + 3)$

$= 5y(3y^2 + 11y + 6)$ FOIL; Combine like terms.

$= 15y^3 + 55y^2 + 30y$ ✓ Distributive property

**(b)** $-24a^3 - 42a^2 + 45a$

The common factor could be $3a$ or $-3a$. If we factor out $-3a$, the first term of the trinomial will be positive, which makes it easier to factor the remaining trinomial.

$$-24a^3 - 42a^2 + 45a$$

$= -3a(8a^2 + 14a - 15)$ Factor out $-3a$.

$= -3a(4a - 3)(2a + 5)$ Factor the trinomial.

*Check* by multiplying. NOW TRY

**NOW TRY ANSWER**
7. $-5x(2x - 3)(x + 6)$

**CAUTION** ***Include the common factor in the final factored form.***

## 5.3 EXERCISES

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***Concept Check*** *The middle term of each trinomial has been rewritten. Now factor by grouping.*

**1.** $10t^2 + 9t + 2$
$= 10t^2 + 5t + 4t + 2$

**2.** $6x^2 + 13x + 6$
$= 6x^2 + 9x + 4x + 6$

**3.** $15z^2 - 19z + 6$
$= 15z^2 - 10z - 9z + 6$

**4.** $12p^2 - 17p + 6$
$= 12p^2 - 9p - 8p + 6$

**5.** $8s^2 + 2st - 3t^2$
$= 8s^2 - 4st + 6st - 3t^2$

**6.** $3x^2 - xy - 14y^2$
$= 3x^2 - 7xy + 6xy - 14y^2$

***Concept Check*** *Complete the steps to factor each trinomial by grouping.*

**7.** $2m^2 + 11m + 12$

**(a)** Find two integers whose product is ________ · ________ = ________ and whose sum is ________.

**(b)** The required integers are ________ and ________.

**(c)** Write the middle term, $11m$, as ________ + ________.

**(d)** Rewrite the given trinomial as ________________________.

**(e)** Factor the polynomial in part (d) by grouping.

**(f)** Check by multiplying.

**8.** $6y^2 - 19y + 10$

**(a)** Find two integers whose product is ________ · ________ = ________ and whose sum is ________.

**(b)** The required integers are ________ and ________.

**(c)** Write the middle term, $-19y$, as ________ + ________.

**(d)** Rewrite the given trinomial as ________________________.

**(e)** Factor the polynomial in part (d) by grouping.

**(f)** Check by multiplying.

**9.** *Concept Check* Which pair of integers would be used to rewrite the middle term when one is factoring $12y^2 + 5y - 2$ by grouping?

**A.** $-8, 3$ **B.** $8, -3$

**C.** $-6, 4$ **D.** $6, -4$

**10.** *Concept Check* Which pair of integers would be used to rewrite the middle term when one is factoring $20b^2 - 13b + 2$ by grouping?

**A.** $10, 3$ **B.** $-10, -3$

**C.** $8, 5$ **D.** $-8, -5$

*Concept Check* *Which is the correct factored form of the given polynomial?*

**11.** $2x^2 - x - 1$

**A.** $(2x - 1)(x + 1)$

**B.** $(2x + 1)(x - 1)$

**12.** $3a^2 - 5a - 2$

**A.** $(3a + 1)(a - 2)$

**B.** $(3a - 1)(a + 2)$

**13.** $4y^2 + 17y - 15$

**A.** $(y + 5)(4y - 3)$

**B.** $(2y - 5)(2y + 3)$

**14.** $12c^2 - 7c - 12$

**A.** $(6c - 2)(2c + 6)$

**B.** $(4c + 3)(3c - 4)$

*Complete each factoring.* ***See Examples 1–7.***

**15.** $6a^2 + 7ab - 20b^2$
$= (3a - 4b)(\underline{\qquad\qquad})$

**16.** $9m^2 + 6mn - 8n^2$
$= (3m - 2n)(\underline{\qquad\qquad})$

**17.** $2x^2 + 6x - 8$
$= 2(\underline{\qquad\qquad\qquad})$
$= 2(\underline{\qquad})(\underline{\qquad})$

**18.** $3x^2 + 9x - 30$
$= 3(\underline{\qquad\qquad\qquad})$
$= 3(\underline{\qquad})(\underline{\qquad})$

**19.** $4z^3 - 10z^2 - 6z$
$= 2z(\underline{\qquad\qquad\qquad})$
$= 2z(\underline{\qquad})(\underline{\qquad})$

**20.** $15r^3 - 39r^2 - 18r$
$= 3r(\underline{\qquad\qquad\qquad})$
$= 3r(\underline{\qquad})(\underline{\qquad})$

**21.** The polynomial $12x^2 + 7x - 12$ does not have 2 as a factor. Explain why the binomial $2x - 6$, then, cannot be a factor of the polynomial.

**22.** *Concept Check* On a quiz, a student factored $3k^3 - 12k^2 - 15k$ by first factoring out the common factor $3k$ to get $3k(k^2 - 4k - 5)$. Then the student wrote the following.

$$\begin{aligned} &k^2 - 4k - 5 \\ &\quad = k^2 - 5k + k - 5 \\ &\quad = k(k - 5) + 1(k - 5) \\ &\quad = (k - 5)(k + 1) \qquad \text{Her answer} \end{aligned}$$

***WHAT WENT WRONG?*** What is the correct factored form?

*Factor each trinomial completely.* ***See Examples 1–7.*** (Hint: *In Exercises 55–58, first write the trinomial in descending powers and then factor.)*

**23.** $3a^2 + 10a + 7$

**24.** $7r^2 + 8r + 1$

**25.** $2y^2 + 7y + 6$

**26.** $5z^2 + 12z + 4$

**27.** $15m^2 + m - 2$

**28.** $6x^2 + x - 1$

**29.** $12s^2 + 11s - 5$

**30.** $20x^2 + 11x - 3$

**31.** $10m^2 - 23m + 12$

**32.** $6x^2 - 17x + 12$

**33.** $8w^2 - 14w + 3$

**34.** $9p^2 - 18p + 8$

**35.** $20y^2 - 39y - 11$

**36.** $10x^2 - 11x - 6$

**37.** $3x^2 - 15x + 16$

**38.** $2t^2 - 14t + 15$

**39.** $20x^2 + 22x + 6$

**40.** $36y^2 + 81y + 45$

**41.** $24x^2 - 42x + 9$

**42.** $48b^2 - 74b - 10$

**43.** $40m^2q + mq - 6q$

**44.** $15a^2b + 22ab + 8b$

**45.** $15n^4 - 39n^3 + 18n^2$

**46.** $24a^4 + 10a^3 - 4a^2$

**47.** $15x^2y^2 - 7xy^2 - 4y^2$

**48.** $14a^2b^3 + 15ab^3 - 9b^3$

**49.** $5a^2 - 7ab - 6b^2$

**50.** $6x^2 - 5xy - y^2$

**51.** $12s^2 + 11st - 5t^2$

**52.** $25a^2 + 25ab + 6b^2$

**53.** $6m^6n + 7m^5n^2 + 2m^4n^3$

**54.** $12k^3q^4 - 4k^2q^5 - kq^6$

**55.** $5 - 6x + x^2$

**56.** $7 - 8x + x^2$

**57.** $16 + 16x + 3x^2$

**58.** $18 + 65x + 7x^2$

**59.** $-10x^3 + 5x^2 + 140x$

**60.** $-18k^3 - 48k^2 + 66k$

**61.** $12x^2 - 47x - 4$

**62.** $12x^2 - 19x - 10$

**63.** $24y^2 - 41xy - 14x^2$

**64.** $24x^2 + 19xy - 5y^2$

**65.** $36x^4 - 64x^2y + 15y^2$

**66.** $36x^4 + 59x^2y + 24y^2$

**67.** $48a^2 - 94ab - 4b^2$

**68.** $48t^2 - 147ts + 9s^2$

**69.** $10x^4y^5 + 39x^3y^5 - 4x^2y^5$

**70.** $14x^7y^4 - 31x^6y^4 + 6x^5y^4$

**71.** $36a^3b^2 - 104a^2b^2 - 12ab^2$

**72.** $36p^4q + 129p^3q - 60p^2q$

**73.** $24x^2 - 46x + 15$

**74.** $24x^2 - 94x + 35$

**75.** $24x^4 + 55x^2 - 24$

**76.** $24x^4 + 17x^2 - 20$

**77.** $24x^2 + 38xy + 15y^2$

**78.** $24x^2 + 62xy + 33y^2$

If a trinomial has a negative coefficient for the squared term, as in $-2x^2 + 11x - 12$, it is usually easier to factor by first factoring out the common factor $-1$.

$$\begin{aligned} &-2x^2 + 11x - 12 \\ &= -1(2x^2 - 11x + 12) \\ &= -1(2x - 3)(x - 4) \end{aligned}$$

*Use this method to factor each trinomial.* ***See Example 7(b).***

**79.** $-x^2 - 4x + 21$

**80.** $-x^2 + x + 72$

**81.** $-3x^2 - x + 4$

**82.** $-5x^2 + 2x + 16$

**83.** $-2a^2 - 5ab - 2b^2$

**84.** $-3p^2 + 13pq - 4q^2$

***Brain Busters*** *Factor each polynomial. (*Hint: *As the first step, factor out the greatest common factor.)*

**85.** $25q^2(m + 1)^3 - 5q(m + 1)^3 - 2(m + 1)^3$

**86.** $18x^2(y - 3)^2 - 21x(y - 3)^2 - 4(y - 3)^2$

**87.** $9x^2(r + 3)^3 + 12xy(r + 3)^3 + 4y^2(r + 3)^3$

**88.** $4t^2(k + 9)^7 + 20ts(k + 9)^7 + 25s^2(k + 9)^7$

***Brain Busters*** *Find all integers k so that the trinomial can be factored by the methods of this section.*

**89.** $5x^2 + kx - 1$

**90.** $2x^2 + kx - 3$

**91.** $2m^2 + km + 5$

**92.** $3y^2 + ky + 4$

## PREVIEW EXERCISES

*Find each product.* ***See Section 4.6.***

**93.** $(7p + 3)(7p - 3)$

**94.** $(3h + 5k)(3h - 5k)$

**95.** $(x + 6)^2$

**96.** $(3t + 4)^2$

# 5.4 Special Factoring Techniques

**OBJECTIVES**

1. Factor a difference of squares.
2. Factor a perfect square trinomial.
3. Factor a difference of cubes.
4. Factor a sum of cubes.

By reversing the rules for multiplication of binomials from **Section 4.6,** we get rules for factoring polynomials in certain forms.

**OBJECTIVE 1** **Factor a difference of squares.** The formula for the product of the sum and difference of the same two terms is

$$(x+y)(x-y)=x^2-y^2.$$

Reversing this rule leads to the following special factoring rule.

**Factoring a Difference of Squares**

$$x^2-y^2=(x+y)(x-y)$$

For example,

$$\begin{aligned} &m^2-16\\ &=m^2-4^2\\ &=(m+4)(m-4).\end{aligned}$$

The following conditions must be true for a binomial to be a difference of squares.

1. Both terms of the binomial must be squares, such as

$$x^2,\quad 9y^2=(3y)^2,\quad 25=5^2,\quad 1=1^2,\quad m^4=(m^2)^2.$$

2. The terms of the binomial must have different signs (one positive and one negative).

**NOW TRY EXERCISE 1**

Factor each binomial if possible.

**(a)** $x^2-100$ **(b)** $x^2+49$

**EXAMPLE 1** Factoring Differences of Squares

Factor each binomial if possible.

$$\begin{array}{ccccccccc} & x^2 & - & y^2 & = & (x+y) & (x-y) \\ & \downarrow & & \downarrow & & \downarrow\ \ \downarrow & \downarrow\ \ \downarrow \\ \textbf{(a)}\ a^2-49= & a^2 & - & 7^2 & = & (a+7) & (a-7) \end{array}$$

**(b)** $y^2-m^2=(y+m)(y-m)$

**(c)** $x^2-8$

Because 8 is not the square of an integer, this binomial does not satisfy the conditions above. It is a prime polynomial.

**(d)** $p^2+16$

Since $p^2+16$ is a *sum* of squares, it is not equal to $(p+4)(p-4)$. Also, we use FOIL and try the following.

$$\begin{aligned} &(p-4)(p-4)\\ &=p^2-8p+16,\quad \text{not}\quad p^2+16.\\ &(p+4)(p+4)\\ &=p^2+8p+16,\quad \text{not}\quad p^2+16.\end{aligned}$$

Thus, $p^2+16$ is a prime polynomial. NOW TRY

**CAUTION** ***As* Example 1(d) *suggests, after any common factor is removed, a sum of squares cannot be factored.***

**NOW TRY ANSWERS**

1. **(a)** $(x+10)(x-10)$
   **(b)** prime

NOW TRY EXERCISE 2

Factor each difference of squares.

**(a)** $9t^2 - 100$

**(b)** $36a^2 - 49b^2$

## EXAMPLE 2 Factoring Differences of Squares

Factor each difference of squares.

**(a)** $25m^2 - 16 = \overset{\overset{x^2}{\downarrow}}{(5m)^2} - \overset{\overset{y^2}{\downarrow}}{4^2} = (\overset{\overset{(x}{\downarrow}}{5m} + \overset{\overset{y)}{\downarrow}}{4})(\overset{\overset{(x}{\downarrow}}{5m} - \overset{\overset{y)}{\downarrow}}{4})$

**(b)** $49z^2 - 64t^2$

$= (7z)^2 - (8t)^2$ Write each term as a square.

$= (7z + 8t)(7z - 8t)$ Factor the difference of squares.

NOW TRY

**NOTE** ***Always check a factored form by multiplying.***

NOW TRY EXERCISE 3

Factor completely.

**(a)** $16k^2 - 64$

**(b)** $m^4 - 144$

**(c)** $v^4 - 625$

## EXAMPLE 3 Factoring More Complex Differences of Squares

Factor completely.

**(a)** $81y^2 - 36$

$= 9(9y^2 - 4)$ Factor out the GCF, 9.

$= 9[(3y)^2 - 2^2]$ Write each term as a square.

$= 9(3y + 2)(3y - 2)$ Factor the difference of squares.

**(b)** $p^4 - 36$

$= (p^2)^2 - 6^2$ Write each term as a square.

Neither binomial can be factored further.

$= (p^2 + 6)(p^2 - 6)$ Factor the difference of squares.

**(c)** $m^4 - 16$

$= (m^2)^2 - 4^2$

$= (m^2 + 4)(m^2 - 4)$ Factor the difference of squares.

Don't stop here.

$= (m^2 + 4)(m + 2)(m - 2)$ Factor the difference of squares again.

NOW TRY

**CAUTION** ***Factor again when any of the factors is a difference of squares,*** as in **Example 3(c).** Check by multiplying.

**OBJECTIVE 2** **Factor a perfect square trinomial.** The expressions 144, $4x^2$, and $81m^6$ are called **perfect squares** because

$$144 = 12^2, \quad 4x^2 = (2x)^2, \quad \text{and} \quad 81m^6 = (9m^3)^2.$$

A **perfect square trinomial** is a trinomial that is the square of a binomial. For example, $x^2 + 8x + 16$ is a perfect square trinomial because it is the square of the binomial $x + 4$.

$$\begin{aligned} &x^2 + 8x + 16 \\ &= (x + 4)(x + 4) \\ &= (x + 4)^2 \end{aligned}$$

NOW TRY ANSWERS

2. **(a)** $(3t + 10)(3t - 10)$
   **(b)** $(6a + 7b)(6a - 7b)$
3. **(a)** $16(k + 2)(k - 2)$
   **(b)** $(m^2 + 12)(m^2 - 12)$
   **(c)** $(v^2 + 25)(v + 5)(v - 5)$

On the one hand, a necessary condition for a trinomial to be a perfect square is that *two of its terms be perfect squares.* For this reason, $16x^2 + 4x + 15$ is not a perfect square trinomial, because only the term $16x^2$ is a perfect square.

On the other hand, even if two of the terms are perfect squares, the trinomial may not be a perfect square trinomial. For example, $x^2 + 6x + 36$ has two perfect square terms, $x^2$ and 36, but it is not a perfect square trinomial.

**Factoring Perfect Square Trinomials**

$$\mathbf{x^2 + 2xy + y^2 = (x + y)^2}$$

$$\mathbf{x^2 - 2xy + y^2 = (x - y)^2}$$

***The middle term of a perfect square trinomial is always twice the product of the two terms in the squared binomial (as shown in Section 4.6). Use this rule to check any attempt to factor a trinomial that appears to be a perfect square.***

**NOW TRY EXERCISE 4**

Factor $y^2 + 14y + 49$.

### EXAMPLE 4 Factoring a Perfect Square Trinomial

Factor $x^2 + 10x + 25$.

The $x^2$-term is a perfect square, and so is 25.

$$\text{Try to factor} \quad x^2 + 10x + 25 \quad \text{as} \quad (x + 5)^2.$$

To check, take twice the product of the two terms in the squared binomial.

$$\underset{\text{Twice}}{2} \cdot \underset{\substack{\text{First term}\\ \text{of binomial}}}{x} \cdot \underset{\substack{\text{Last term}\\ \text{of binomial}}}{5} = 10x \longleftarrow \text{Middle term of } x^2 + 10x + 25$$

Since $10x$ is the middle term of the trinomial, the trinomial is a perfect square.

$$x^2 + 10x + 25 \quad \text{factors as} \quad (x + 5)^2.$$

NOW TRY

### EXAMPLE 5 Factoring Perfect Square Trinomials

Factor each trinomial.

**(a)** $x^2 - 22x + 121$

The first and last terms are perfect squares $(121 = 11^2 \text{ or } (-11)^2)$. Check to see whether the middle term of $x^2 - 22x + 121$ is twice the product of the first and last terms of the binomial $x - 11$.

$$\underset{\text{Twice}}{2} \cdot \underset{\substack{\text{First}\\ \text{term}}}{x} \cdot \underset{\substack{\text{Last}\\ \text{term}}}{(-11)} = -22x \longleftarrow \text{Middle term of } x^2 - 22x + 121$$

Thus, $x^2 - 22x + 121$ is a perfect square trinomial.

$$x^2 - 22x + 121 \quad \text{factors as} \quad (x - 11)^2.$$

Same sign

***Notice that the sign of the second term in the squared binomial is the same as the sign of the middle term in the trinomial.***

**NOW TRY ANSWER**

**4.** $(y + 7)^2$

**NOW TRY EXERCISE 5**

Factor each trinomial.

**(a)** $t^2 - 18t + 81$

**(b)** $4p^2 - 28p + 49$

**(c)** $9x^2 + 6x + 4$

**(d)** $80x^3 + 120x^2 + 45x$

**(b)** $9m^2 - 24m + 16 = (3m)^2 + \underset{\text{Twice}\nearrow}{2}(\underset{\uparrow\,\text{First term}}{3m})(\underset{\uparrow\,\text{Last term}}{-4}) + (-4)^2 = (3m - 4)^2$

**(c)** $25y^2 + 20y + 16$

The first and last terms are perfect squares.

$$25y^2 = (5y)^2 \quad \text{and} \quad 16 = 4^2$$

Twice the product of the first and last terms of the binomial $5y + 4$ is

$$2 \cdot 5y \cdot 4 = 40y,$$

which is *not* the middle term of

$$25y^2 + 20y + 16.$$

This trinomial is not a perfect square. In fact, the trinomial cannot be factored even with the methods of the previous sections. It is a prime polynomial.

**(d)** $12z^3 + 60z^2 + 75z$

$= 3z(4z^2 + 20z + 25)$ Factor out the common factor, $3z$.

$= 3z[(2z)^2 + 2(2z)(5) + 5^2]$ $4z^2 + 20z + 25$ is a perfect square trinomial.

$= 3z(2z + 5)^2$ Factor. NOW TRY

### NOTE

1. The sign of the second term in the squared binomial is always the same as the sign of the middle term in the trinomial.
2. The first and last terms of a perfect square trinomial must be *positive,* because they are squares. For example, the polynomial $x^2 - 2x - 1$ cannot be a perfect square, because the last term is negative.
3. Perfect square trinomials can also be factored by using grouping or the FOIL method, although using the method of this section is often easier.

**OBJECTIVE 3** **Factor a difference of cubes.** We can factor a **difference of cubes** by using the following pattern.

**Factoring a Difference of Cubes**

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

***This pattern for factoring a difference of cubes should be memorized.*** To see that the pattern is correct, multiply $(x - y)(x^2 + xy + y^2)$.

$$\begin{array}{rrrrl} & x^2 & + xy & + y^2 & \text{Multiply vertically.} \\ & & x & - y & \text{(Section 4.5)} \\ \hline & -x^2y & - xy^2 & - y^3 & -y(x^2 + xy + y^2) \\ x^3 & + x^2y & + xy^2 & & x(x^2 + xy + y^2) \\ \hline x^3 & & & - y^3 & \text{Add.} \end{array}$$

**NOW TRY ANSWERS**

5. **(a)** $(t - 9)^2$
   **(b)** $(2p - 7)^2$
   **(c)** prime
   **(d)** $5x(4x + 3)^2$

Notice the pattern of the terms in the factored form of $x^3 - y^3$.

- $x^3 - y^3 =$ (a binomial factor)(a trinomial factor)
- The binomial factor has the difference of the cube roots of the given terms.
- The terms in the trinomial factor are all positive.
- The terms in the binomial factor help to determine the trinomial factor.

$$x^3 - y^3 = (x - y)(\overset{\text{First term squared}}{x^2} + \overset{\text{positive product of the terms}}{xy} + \overset{\text{second term squared}}{y^2})$$

**CAUTION** The polynomial $x^3 - y^3$ is not equivalent to $(x - y)^3$.

| | |
|---|---|
| $x^3 - y^3$ | $(x - y)^3$ |
| $= (x - y)(x^2 + xy + y^2)$ | $= (x - y)(x - y)(x - y)$ |
| | $= (x - y)(x^2 - 2xy + y^2)$ |

## EXAMPLE 6 Factoring Differences of Cubes

Factor each polynomial.

**(a)** $m^3 - 125$

Let $x = m$ and $y = 5$ in the pattern for the difference of cubes.

$$\begin{array}{ccccccc} & & x^3 - y^3 & = & (x - y) & (x^2 + xy + y^2) & \\ & & \downarrow \quad \downarrow & & \downarrow \quad \downarrow & \downarrow \quad \downarrow \quad \downarrow & \\ m^3 - 125 & = & m^3 - 5^3 & = & (m - 5) & (m^2 + 5m + 5^2) & \text{Let } x = m,\ y = 5. \\ & & & = & (m - 5) & (m^2 + 5m + 25) & 5^2 = 25 \end{array}$$

**(b)** $8p^3 - 27$

$= (2p)^3 - 3^3$ $\quad$ $8p^3 = (2p)^3$ and $27 = 3^3$.

$= (2p - 3)[(2p)^2 + (2p)3 + 3^2]$ $\quad$ Let $x = 2p$, $y = 3$.

$= (2p - 3)(4p^2 + 6p + 9)$ $\quad$ Apply the exponents. Multiply.

$(2p)^2 = 2^2p^2 = 4p^2$, NOT $2p^2$.

**(c)** $4m^3 - 32$

$= 4(m^3 - 8)$ $\quad$ Factor out the common factor, 4.

$= 4(m^3 - 2^3)$ $\quad$ $8 = 2^3$

$= 4(m - 2)(m^2 + 2m + 4)$ $\quad$ Factor the difference of cubes.

**(d)** $125t^3 - 216s^6$

$= (5t)^3 - (6s^2)^3$ $\quad$ Write each term as a cube.

$= (5t - 6s^2)[(5t)^2 + 5t(6s^2) + (6s^2)^2]$ $\quad$ Factor the difference of cubes.

$= (5t - 6s^2)(25t^2 + 30ts^2 + 36s^4)$ $\quad$ Apply the exponents. Multiply.

NOW TRY

**NOW TRY EXERCISE 6**

Factor each polynomial.

**(a)** $a^3 - 27$

**(b)** $8t^3 - 125$

**(c)** $3k^3 - 192$

**(d)** $125x^3 - 343y^6$

**NOW TRY ANSWERS**

6. **(a)** $(a - 3)(a^2 + 3a + 9)$

**(b)** $(2t - 5)(4t^2 + 10t + 25)$

**(c)** $3(k - 4)(k^2 + 4k + 16)$

**(d)** $(5x - 7y^2) \cdot (25x^2 + 35xy^2 + 49y^4)$

**CAUTION** A common error in factoring a difference of cubes, such as $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, is to try to factor $x^2 + xy + y^2$. This is usually not possible.

**OBJECTIVE 4** **Factor a sum of cubes.** A sum of squares, such as $m^2 + 25$, cannot be factored by using real numbers, but a **sum of cubes** can.

### Factoring a Sum of Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Compare the pattern for the *sum* of cubes with that for the *difference* of cubes.

Positive

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Difference of cubes

Same sign

Opposite sign

The only difference between the patterns is the positive and negative signs.

Positive

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Sum of cubes

Same sign

Opposite sign

**EXAMPLE 7** Factoring Sums of Cubes

Factor each polynomial.

**(a)** $k^3 + 27$

$$= k^3 + 3^3 \qquad 27 = 3^3$$

$$= (k + 3)(k^2 - 3k + 3^2) \qquad \text{Factor the sum of cubes.}$$

$$= (k + 3)(k^2 - 3k + 9) \qquad \text{Apply the exponent.}$$

**(b)** $8m^3 + 125n^3$

$$= (2m)^3 + (5n)^3 \qquad 8m^3 = (2m)^3 \text{ and } 125n^3 = (5n)^3.$$

$$= (2m + 5n)[(2m)^2 - 2m(5n) + (5n)^2] \qquad \text{Factor the sum of cubes.}$$

$$= (2m + 5n)(4m^2 - 10mn + 25n^2)$$

Be careful: $(2m)^2 = 2^2m^2$ and $(5n)^2 = 5^2n^2$.

**(c)** $1000a^6 + 27b^3$

$$= (10a^2)^3 + (3b)^3$$

$$= (10a^2 + 3b)[(10a^2)^2 - (10a^2)(3b) + (3b)^2] \qquad \text{Factor the sum of cubes.}$$

$$= (10a^2 + 3b)(100a^4 - 30a^2b + 9b^2) \qquad (10a^2)^2 = 10^2(a^2)^2 = 100a^4$$

NOW TRY

NOW TRY EXERCISE 7

Factor each polynomial.

**(a)** $x^3 + 125$

**(b)** $27a^3 + 8b^3$

NOW TRY ANSWERS

7. **(a)** $(x + 5)(x^2 - 5x + 25)$

**(b)** $(3a + 2b)(9a^2 - 6ab + 4b^2)$

The methods of factoring discussed in this section are summarized here.

### Special Factorizations

| | |
|---|---|
| **Difference of squares** | $x^2 - y^2 = (x + y)(x - y)$ |
| **Perfect square trinomials** | $x^2 + 2xy + y^2 = (x + y)^2$ |
| | $x^2 - 2xy + y^2 = (x - y)^2$ |
| **Difference of cubes** | $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ |
| **Sum of cubes** | $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ |

***The sum of squares can be factored only if the terms have a common factor.***

## 5.4 EXERCISES

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**1.** *Concept Check* To help you factor the difference of squares, complete the following list of squares.

$1^2 =$ _____ $2^2 =$ _____ $3^2 =$ _____ $4^2 =$ _____ $5^2 =$ _____

$6^2 =$ _____ $7^2 =$ _____ $8^2 =$ _____ $9^2 =$ _____ $10^2 =$ _____

$11^2 =$ _____ $12^2 =$ _____ $13^2 =$ _____ $14^2 =$ _____ $15^2 =$ _____

$16^2 =$ _____ $17^2 =$ _____ $18^2 =$ _____ $19^2 =$ _____ $20^2 =$ _____

**2.** *Concept Check* The following powers of $x$ are all perfect squares: $x^2, x^4, x^6, x^8, x^{10}$. On the basis of this observation, we may make a conjecture (an educated guess) that if the power of a variable is divisible by _____ (with 0 remainder), then we have a perfect square.

**3.** *Concept Check* To help you factor the sum or difference of cubes, complete the following list of cubes.

$1^3 =$ _____ $2^3 =$ _____ $3^3 =$ _____ $4^3 =$ _____ $5^3 =$ _____

$6^3 =$ _____ $7^3 =$ _____ $8^3 =$ _____ $9^3 =$ _____ $10^3 =$ _____

**4.** *Concept Check* The following powers of $x$ are all perfect cubes: $x^3, x^6, x^9, x^{12}, x^{15}$. On the basis of this observation, we may make a conjecture that if the power of a variable is divisible by _____ (with 0 remainder), then we have a perfect cube.

**5.** *Concept Check* Identify each monomial as a *perfect square*, a *perfect cube*, *both of these*, or *neither of these*.

**(a)** $64x^6y^{12}$ **(b)** $125t^6$ **(c)** $49x^{12}$ **(d)** $81r^{10}$

**6.** *Concept Check* What must be true for $x^n$ to be both a perfect square and a perfect cube?

*Factor each binomial completely. If the binomial is prime, say so. Use your answers from* ***Exercises 1 and 2*** *as necessary.* ***See Examples 1–3.***

● **7.** $y^2 - 25$ **8.** $t^2 - 16$ **9.** $x^2 - 144$

**10.** $x^2 - 400$ **11.** $m^2 + 64$ **12.** $k^2 + 49$

**13.** $4m^2 + 16$ **14.** $9x^2 + 81$ ● **15.** $9r^2 - 4$

**16.** $4x^2 - 9$ ● **17.** $36x^2 - 16$ **18.** $32a^2 - 8$

**19.** $196p^2 - 225$
**20.** $361q^2 - 400$
**21.** $16r^2 - 25a^2$
**22.** $49m^2 - 100p^2$
**23.** $100x^2 + 49$
**24.** $81w^2 + 16$
**25.** $p^4 - 49$
**26.** $r^4 - 25$
**27.** $x^4 - 1$
**28.** $y^4 - 10{,}000$
**29.** $p^4 - 256$
**30.** $k^4 - 81$

**31.** ***Concept Check*** When a student was directed to factor $k^4 - 81$ from **Exercise 30** completely, his teacher did not give him full credit for the answer

$$(k^2 + 9)(k^2 - 9).$$

The student argued that since his answer does indeed give $k^4 - 81$ when multiplied out, he should be given full credit. ***WHAT WENT WRONG?*** Give the correct factored form.

**32.** ***Concept Check*** The binomial $4x^2 + 36$ is a sum of squares that *can* be factored. How is this binomial factored? When can the sum of squares be factored?

***Concept Check*** *Find the value of the indicated variable.*

**33.** Find $b$ so that $x^2 + bx + 25$ factors as $(x + 5)^2$.
**34.** Find $c$ so that $4m^2 - 12m + c$ factors as $(2m - 3)^2$.
**35.** Find $a$ so that $ay^2 - 12y + 4$ factors as $(3y - 2)^2$.
**36.** Find $b$ so that $100a^2 + ba + 9$ factors as $(10a + 3)^2$.

*Factor each trinomial completely.* ***See Examples 4 and 5.***

**37.** $w^2 + 2w + 1$
**38.** $p^2 + 4p + 4$
**39.** $x^2 - 8x + 16$
**40.** $x^2 - 10x + 25$
**41.** $2x^2 + 24x + 72$
**42.** $3y^2 + 48y + 192$
**43.** $16x^2 - 40x + 25$
**44.** $36y^2 - 60y + 25$
**45.** $49x^2 - 28xy + 4y^2$
**46.** $4z^2 - 12zw + 9w^2$
**47.** $64x^2 + 48xy + 9y^2$
**48.** $9t^2 + 24tr + 16r^2$
**49.** $50h^2 - 40hy + 8y^2$
**50.** $18x^2 - 48xy + 32y^2$
**51.** $4k^3 - 4k^2 + 9k$
**52.** $9r^3 - 6r^2 + 16r$
**53.** $25z^4 + 5z^3 + z^2$
**54.** $4x^4 + 2x^3 + x^2$

*Factor each binomial completely. Use your answers from* ***Exercises 3 and 4*** *as necessary.* ***See Examples 6 and 7.***

**55.** $a^3 - 1$
**56.** $m^3 - 8$
**57.** $m^3 + 8$
**58.** $b^3 + 1$
**59.** $k^3 + 1000$
**60.** $p^3 + 512$
**61.** $27x^3 - 64$
**62.** $64y^3 - 27$
**63.** $6p^3 + 6$
**64.** $81x^3 + 3$
**65.** $5x^3 + 40$
**66.** $128y^3 + 54$
**67.** $y^3 - 8x^3$
**68.** $w^3 - 216z^3$
**69.** $2x^3 - 16y^3$
**70.** $27w^3 - 216z^3$
**71.** $8p^3 + 729q^3$
**72.** $64x^3 + 125y^3$
**73.** $27a^3 + 64b^3$
**74.** $125m^3 + 8p^3$
**75.** $125t^3 + 8s^3$
**76.** $27r^3 + 1000s^3$
**77.** $8x^3 - 125y^6$
**78.** $27t^3 - 64s^6$
**79.** $27m^6 + 8n^3$
**80.** $1000r^6 + 27s^3$
**81.** $x^9 + y^9$
**82.** $x^9 - y^9$

*Although we usually factor polynomials using integers, we can apply the same concepts to factoring using fractions and decimals.*

$$z^2 - \frac{9}{16}$$

$$= z^2 - \left(\frac{3}{4}\right)^2 \qquad \frac{9}{16} = \left(\frac{3}{4}\right)^2$$

$$= \left(z + \frac{3}{4}\right)\left(z - \frac{3}{4}\right) \qquad \text{Factor the difference of squares.}$$

*Apply the special factoring rules of this section to factor each binomial or trinomial.*

**83.** $p^2 - \frac{1}{9}$

**84.** $q^2 - \frac{1}{4}$

**85.** $36m^2 - \frac{16}{25}$

**86.** $100b^2 - \frac{4}{49}$

**87.** $x^2 - 0.64$

**88.** $y^2 - 0.36$

**89.** $t^2 + t + \frac{1}{4}$

**90.** $m^2 + \frac{2}{3}m + \frac{1}{9}$

**91.** $x^2 - 1.0x + 0.25$

**92.** $y^2 - 1.4y + 0.49$

**93.** $x^3 + \frac{1}{8}$

**94.** $x^3 + \frac{1}{64}$

***Brain Busters*** *Factor each polynomial completely.*

**95.** $(m + n)^2 - (m - n)^2$

**96.** $(a - b)^3 - (a + b)^3$

**97.** $m^2 - p^2 + 2m + 2p$

**98.** $3r - 3k + 3r^2 - 3k^2$

## PREVIEW EXERCISES

*Solve each equation.* ***See Sections 2.1 and 2.2.***

**99.** $m - 4 = 0$

**100.** $3t + 2 = 0$

**101.** $2t + 10 = 0$

**102.** $7x = 0$

## SUMMARY EXERCISES on Factoring

As you factor a polynomial, ask yourself these questions to decide on a suitable factoring technique.

### Factoring a Polynomial

1. **Is there a common factor?** If so, factor it out.
2. **How many terms are in the polynomial?**

   ***Two terms:*** Check to see whether it is a difference of squares or a sum or difference of cubes. If so, factor as in **Section 5.4.**

   ***Three terms:*** Is it a perfect square trinomial? If the trinomial is not a perfect square, check to see whether the coefficient of the second-degree term is 1. If so, use the method of **Section 5.2.** If the coefficient of the second-degree term of the trinomial is not 1, use the general factoring methods of **Section 5.3.**

   ***Four terms:*** Try to factor the polynomial by grouping, as in **Section 5.1.**
3. **Can any factors be factored further?** If so, factor them.

(continued)

*Match each polynomial in Column I with the best choice for factoring it in Column II. The choices in Column II may be used once, more than once, or not at all.*

**I**

**1.** $12x^2 + 20x + 8$

**2.** $x^2 - 17x + 72$

**3.** $16m^2n + 24mn - 40mn^2$

**4.** $64a^2 - 121b^2$

**5.** $36p^2 - 60pq + 25q^2$

**6.** $z^2 - 4z + 6$

**7.** $8r^3 - 125$

**8.** $x^6 + 4x^4 - 3x^2 - 12$

**9.** $4w^2 + 49$

**10.** $z^2 - 24z + 144$

**II**

**A.** Factor out the GCF. No further factoring is possible.

**B.** Factor a difference of squares.

**C.** Factor a difference of cubes.

**D.** Factor a sum of cubes.

**E.** Factor a perfect square trinomial.

**F.** Factor by grouping.

**G.** Factor out the GCF. Then factor a trinomial by grouping or trial and error.

**H.** Factor into two binomials by finding two integers whose product is the constant in the trinomial and whose sum is the coefficient of the middle term.

**I.** The polynomial is prime.

*Factor each polynomial completely.*

**11.** $a^2 - 4a - 12$

**12.** $a^2 + 17a + 72$

**13.** $6y^2 - 6y - 12$

**14.** $7y^6 + 14y^5 - 168y^4$

**15.** $6a + 12b + 18c$

**16.** $m^2 - 3mn - 4n^2$

**17.** $p^2 - 17p + 66$

**18.** $z^2 - 6z + 7z - 42$

**19.** $10z^2 - 7z - 6$

**20.** $2m^2 - 10m - 48$

**21.** $17x^3y^2 + 51xy$

**22.** $15y + 5$

**23.** $8a^5 - 8a^4 - 48a^3$

**24.** $8k^2 - 10k - 3$

**25.** $z^2 - 3za - 10a^2$

**26.** $50z^2 - 100$

**27.** $x^2 - 4x - 5x + 20$

**28.** $100n^2r^2 + 30nr^3 - 50n^2r$

**29.** $6n^2 - 19n + 10$

**30.** $9y^2 + 12y - 5$

**31.** $16x + 20$

**32.** $m^2 + 2m - 15$

**33.** $6y^2 - 5y - 4$

**34.** $m^2 - 81$

**35.** $6z^2 + 31z + 5$

**36.** $12x^2 + 47x - 4$

**37.** $4k^2 - 12k + 9$

**38.** $8p^2 + 23p - 3$

**39.** $54m^2 - 24z^2$

**40.** $8m^2 - 2m - 3$

**41.** $3k^2 + 4k - 4$

**42.** $45a^3b^5 - 60a^4b^2 + 75a^6b^4$

**43.** $14k^3 + 7k^2 - 70k$

**44.** $5 + r - 5s - rs$

**45.** $y^4 - 16$

**46.** $20y^5 - 30y^4$

**47.** $8m - 16m^2$

**48.** $k^2 - 16$

**49.** $z^3 - 8$

**50.** $y^2 - y - 56$

**51.** $k^2 + 9$

**52.** $27p^{10} - 45p^9 - 252p^8$

**53.** $32m^9 + 16m^5 + 24m^3$

**54.** $8m^3 + 125$

**55.** $16r^2 + 24rm + 9m^2$

**56.** $z^2 - 12z + 36$

**57.** $15h^2 + 11hg - 14g^2$

**58.** $5z^3 - 45z^2 + 70z$

**59.** $k^2 - 11k + 30$

**60.** $64p^2 - 100m^2$

**61.** $3k^3 - 12k^2 - 15k$

**62.** $y^2 - 4yk - 12k^2$

**63.** $1000p^3 + 27$

**64.** $64r^3 - 343$

**65.** $6 + 3m + 2p + mp$

**66.** $2m^2 + 7mn - 15n^2$

**67.** $16z^2 - 8z + 1$

**68.** $125m^4 - 400m^3n + 195m^2n^2$

**69.** $108m^2 - 36m + 3$

**70.** $100a^2 - 81y^2$

**71.** $x^2 - xy + y^2$

**72.** $4y^2 - 25$

**73.** $32z^3 + 56z^2 - 16z$

**74.** $10m^2 + 25m - 60$

**75.** $20 + 5m + 12n + 3mn$

**76.** $4 - 2q - 6p + 3pq$

**77.** $6a^2 + 10a - 4$

**78.** $36y^6 - 42y^5 - 120y^4$

**79.** $a^3 - b^3 + 2a - 2b$

**80.** $16k^2 - 48k + 36$

**81.** $64m^2 - 80mn + 25n^2$

**82.** $72y^3z^2 + 12y^2 - 24y^4z^2$

**83.** $8k^2 - 2kh - 3h^2$

**84.** $2a^2 - 7a - 30$

**85.** $2x^3 + 128$

**86.** $8a^3 - 27$

**87.** $10y^2 - 7yz - 6z^2$

**88.** $m^2 - 4m + 4$

**89.** $8a^2 + 23ab - 3b^2$

**90.** $a^4 - 625$

## *RELATING CONCEPTS* EXERCISES 91–98

### FOR INDIVIDUAL OR GROUP WORK

*A binomial may be both a difference of squares and a difference of cubes. One example of such a binomial is $x^6 - 1$. With the techniques of* ***Section 5.4,*** *one factoring method will give the completely factored form, while the other will not.* ***Work Exercises 91–98 in order*** *to determine the method to use if you have to make such a decision.*

**91.** Factor $x^6 - 1$ as the difference of squares.

**92.** The factored form obtained in **Exercise 91** consists of a difference of cubes multiplied by a sum of cubes. Factor each binomial further.

**93.** Now start over and factor $x^6 - 1$ as the difference of cubes.

**94.** The factored form obtained in **Exercise 93** consists of a binomial that is a difference of squares and a trinomial. Factor the binomial further.

**95.** Compare your results in **Exercises 92 and 94.** Which one of these is factored completely?

**96.** Verify that the trinomial in the factored form in **Exercise 94** is the product of the two trinomials in the factored form in **Exercise 92.**

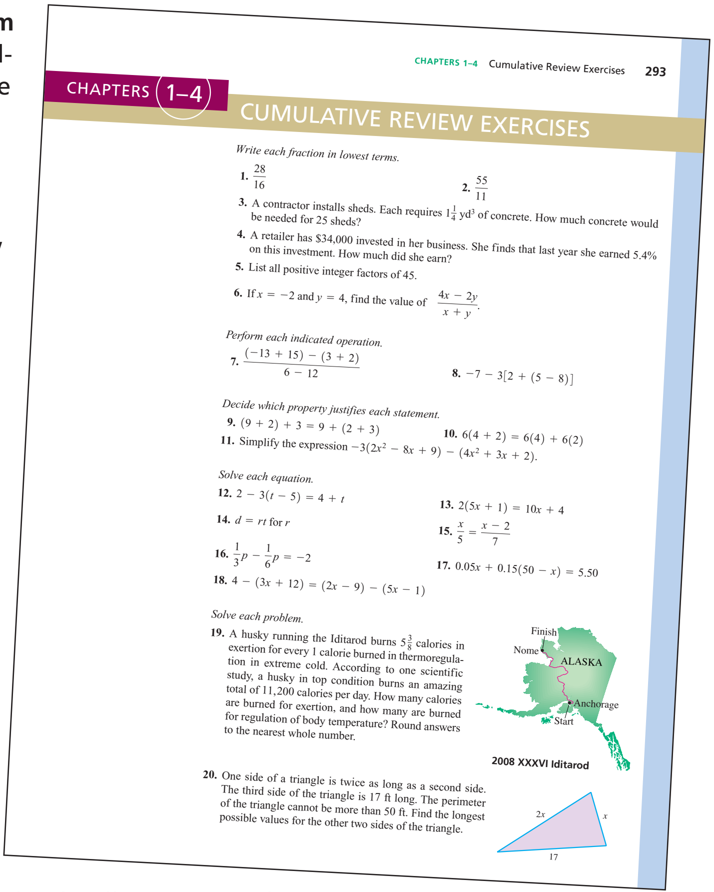
**97.** Use the results of **Exercises 91–96** to complete the following statement: In general, if I must choose between factoring first with the method for the difference of squares or the method for the difference of cubes, I should choose the ____________ method to eventually obtain the completely factored form.

**98.** Find the *completely* factored form of $x^6 - 729$ by using the knowledge you gained in **Exercises 91–97.**

STUDY SKILLS

# Preparing for Your Math Final Exam

Your math final exam is likely to be a comprehensive exam, which means it will cover material from the entire term.



CHAPTERS 1–4 Cumulative Review Exercises 293

CHAPTERS 1–4

CUMULATIVE REVIEW EXERCISES

*Write each fraction in lowest terms.*

1. $\frac{28}{16}$
2. $\frac{55}{11}$
3. A contractor installs sheds. Each requires $1\frac{1}{4}$ yd$^3$ of concrete. How much concrete would be needed for 25 sheds?
4. A retailer has \$34,000 invested in her business. She finds that last year she earned 5.4% on this investment. How much did she earn?
5. List all positive integer factors of 45.
6. If $x = -2$ and $y = 4$, find the value of $\frac{4x - 2y}{x + y}$.

*Perform each indicated operation.*

7. $\frac{(-13 + 15) - (3 + 2)}{6 - 12}$
8. $-7 - 3[2 + (5 - 8)]$

*Decide which property justifies each statement.*

9. $(9 + 2) + 3 = 9 + (2 + 3)$
10. $6(4 + 2) = 6(4) + 6(2)$
11. Simplify the expression $-3(2x^2 - 8x + 9) - (4x^2 + 3x + 2)$.

*Solve each equation.*

12. $2 - 3(t - 5) = 4 + t$
13. $2(5x + 1) = 10x + 4$
14. $d = rt$ for $r$
15. $\frac{x}{5} = \frac{x - 2}{7}$
16. $\frac{1}{3}p - \frac{1}{6}p = -2$
17. $0.05x + 0.15(50 - x) = 5.50$
18. $4 - (3x + 12) = (2x - 9) - (5x - 1)$

*Solve each problem.*

19. A husky running the Iditarod burns $5\frac{3}{8}$ calories in exertion for every 1 calorie burned in thermoregulation in extreme cold. According to one scientific study, a husky in top condition burns an amazing total of 11,200 calories per day. How many calories are burned for exertion, and how many are burned for regulation of body temperature? Round answers to the nearest whole number.

2008 XXXVI Iditarod

20. One side of a triangle is twice as long as a second side. The third side of the triangle is 17 ft long. The perimeter of the triangle cannot be more than 50 ft. Find the longest possible values for the other two sides of the triangle.

1. **Figure out the grade you need to earn on the final exam to get the course grade you want.** Check your course syllabus for grading policies, or ask your instructor if you are not sure.

   ***How many points do you need to earn on your math final exam to get the grade you want?***

2. **Create a final exam week plan.** Set priorities that allow you to spend extra time studying. This may mean making adjustments, in advance, in your work schedule or enlisting extra help with family responsibilities.

   ***What adjustments do you need to make for final exam week?***

3. **Use the following suggestions to guide your studying and reviewing.**

   - **Begin reviewing several days before the final exam.** DON'T wait until the last minute.
   - **Know exactly which chapters and sections will be covered on the exam.**
   - **Divide up the chapters.** Decide how much you will review each day.
   - **Use returned quizzes and tests to review earlier material.**
   - **Practice all types of problems. Use the Cumulative Reviews** that are at the end of each chapter in your textbook. All answers, with section references, are given in the answer section.
   - **Review or rewrite your notes** to create summaries of important information.
   - **Make study cards for all types of problems.** Carry the cards with you, and review them whenever you have a few minutes.
   - **Take plenty of short breaks to reduce physical and mental stress.** Exercising, listening to music, and enjoying a favorite activity are effective stress busters.

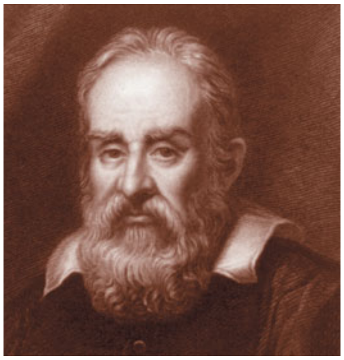
   **Finally, *DON'T* stay up all night the night before an exam—*get a good night's sleep.***

   ***Select several suggestions to use as you study for your math final exam.***

# 5.5 Solving Quadratic Equations by Factoring

**OBJECTIVES**

1. Solve quadratic equations by factoring.
2. Solve other equations by factoring.



**Galileo Galilei (1564–1642)**

Galileo Galilei developed theories to explain physical phenomena and set up experiments to test his ideas. According to legend, Galileo dropped objects of different weights from the Leaning Tower of Pisa to disprove the belief that heavier objects fall faster than lighter objects. He developed the formula

$$d = 16t^2$$

describing the motion of freely falling objects. In this formula, $d$ is the distance in feet that an object falls (disregarding air resistance) in $t$ seconds, regardless of weight.

The equation $d = 16t^2$ is a *quadratic equation*. A quadratic equation contains a second-degree term and no terms of greater degree.

### Quadratic Equation

A **quadratic equation** is an equation that can be written in the form

$$\boldsymbol{ax^2 + bx + c = 0,}$$

where $a$, $b$, and $c$ are real numbers, with $a \neq 0$.

The form $ax^2 + bx + c = 0$ is the **standard form** of a quadratic equation.

$x^2 + 5x + 6 = 0, \quad 2x^2 - 5x = 3, \quad x^2 = 4$ Quadratic equations

Of these quadratic equations, only $x^2 + 5x + 6 = 0$ is in standard form.

We have factored many quadratic *expressions* of the form $ax^2 + bx + c$. In this section, we use factored quadratic expressions to solve quadratic *equations.*

**OBJECTIVE 1** **Solve quadratic equations by factoring.** We use the **zero-factor property** to solve a quadratic equation by factoring.

### Zero-Factor Property

**If $a$ and $b$ are real numbers and if $ab = 0$, then $a = 0$ or $b = 0$.**

That is, if the product of two numbers is 0, then at least one of the numbers must be 0. One number *must* be 0, but both *may* be 0.

### EXAMPLE 1 Using the Zero-Factor Property

Solve each equation.

**(a)** $(x + 3)(2x - 1) = 0$

The product $(x + 3)(2x - 1)$ is equal to 0. By the zero-factor property, the only way that the product of these two factors can be 0 is if at least one of the factors equals 0. Therefore, either $x + 3 = 0$ or $2x - 1 = 0$.

$x + 3 = 0$ or $2x - 1 = 0$ Zero-factor property

$x = -3$ $\quad 2x = 1$ Solve each equation.

$x = \frac{1}{2}$ Divide each side by 2.

**NOW TRY EXERCISE 1**

Solve each equation.

**(a)** $(x-4)(3x+1)=0$

**(b)** $y(4y-5)=0$

The original equation, $(x+3)(2x-1)=0$, has two solutions, $-3$ and $\frac{1}{2}$. Check these solutions by substituting $-3$ for $x$ in this equation. ***Then start over*** and substitute $\frac{1}{2}$ for $x$.

CHECK Let $x=-3$.

$$(x+3)(2x-1)=0$$
$$(-3+3)[2(-3)-1]\overset{?}{=}0$$
$$0(-7)=0 \quad ✓ \quad \text{True}$$

Let $x=\frac{1}{2}$.

$$(x+3)(2x-1)=0$$
$$\left(\frac{1}{2}+3\right)\left(2\cdot\frac{1}{2}-1\right)\overset{?}{=}0$$
$$\frac{7}{2}(1-1)\overset{?}{=}0$$
$$\frac{7}{2}\cdot 0=0 \quad ✓ \quad \text{True}$$

Both $-3$ and $\frac{1}{2}$ result in true equations, so the solution set is $\left\{-3,\frac{1}{2}\right\}$.

**(b)**

$$y(3y-4)=0$$
$$y=0 \quad \text{or} \quad 3y-4=0 \qquad \text{Zero-factor property}$$
$$3y=4$$
$$y=\frac{4}{3}$$

Don't forget that 0 is a solution.

*Check* these solutions by substituting each one into the original equation. The solution set is $\left\{0,\frac{4}{3}\right\}$.

NOW TRY

**NOTE** The word *or* as used in **Example 1** means "one or the other or both."

If the polynomial in an equation is not already factored, first make sure that the equation is in standard form. Then factor.

## EXAMPLE 2 Solving Quadratic Equations

Solve each equation.

**(a)** $x^2-5x=-6$

First, rewrite the equation in standard form by adding 6 to each side.

Don't factor $x$ out at this step.

$$x^2-5x=-6$$
$$x^2-5x+6=0 \qquad \text{Add 6.}$$

Now factor $x^2-5x+6$. Find two numbers whose product is 6 and whose sum is $-5$. These two numbers are $-2$ and $-3$, so we factor as follows.

$$(x-2)(x-3)=0 \qquad \text{Factor.}$$
$$x-2=0 \quad \text{or} \quad x-3=0 \qquad \text{Zero-factor property}$$
$$x=2 \quad \text{or} \quad x=3 \qquad \text{Solve each equation.}$$

**NOW TRY ANSWERS**

1. **(a)** $\left\{-\frac{1}{3},4\right\}$ **(b)** $\left\{0,\frac{5}{4}\right\}$

**NOW TRY EXERCISE 2**

Solve $t^2 = -3t + 18$.

**CHECK** Let $x = 2$.

$$\begin{aligned} x^2 - 5x &= -6 \\ 2^2 - 5(2) &\stackrel{?}{=} -6 \\ 4 - 10 &\stackrel{?}{=} -6 \\ -6 &= -6 \ \checkmark \quad \text{True} \end{aligned}$$

Let $x = 3$.

$$\begin{aligned} x^2 - 5x &= -6 \\ 3^2 - 5(3) &\stackrel{?}{=} -6 \\ 9 - 15 &\stackrel{?}{=} -6 \\ -6 &= -6 \ \checkmark \quad \text{True} \end{aligned}$$

Both solutions check, so the solution set is $\{2, 3\}$.

**(b)**

$$\begin{aligned} y^2 &= y + 20 && \text{Write this equation in standard form.} \\ \text{Standard form} \rightarrow \quad y^2 - y - 20 &= 0 && \text{Subtract } y \text{ and } 20. \\ (y - 5)(y + 4) &= 0 && \text{Factor.} \\ y - 5 = 0 \quad \text{or} \quad y + 4 &= 0 && \text{Zero-factor property} \\ y = 5 \quad \text{or} \quad y &= -4 && \text{Solve each equation.} \end{aligned}$$

*Check* each solution to verify that the solution set is $\{-4, 5\}$. NOW TRY

### Solving a Quadratic Equation by Factoring

***Step 1*** **Write the equation in standard form**—that is, with all terms on one side of the equals symbol in descending powers of the variable and 0 on the other side.

***Step 2*** **Factor** completely.

***Step 3*** **Use the zero-factor property** to set each factor with a variable equal to 0.

***Step 4*** **Solve** the resulting equations.

***Step 5*** **Check** each solution in the original equation.

**NOTE** Not all quadratic equations can be solved by factoring. A more general method for solving such equations is given in **Chapter 11.**

**NOW TRY EXERCISE 3**

Solve $10p^2 + 65p = 35$.

## EXAMPLE 3 Solving a Quadratic Equation with a Common Factor

Solve $4p^2 + 40 = 26p$.

$$\begin{aligned} 4p^2 + 40 &= 26p \\ 4p^2 - 26p + 40 &= 0 && \text{Standard form} \\ \text{This 2 is } \textit{not} \text{ a solution of the equation.} \rightarrow \ 2(2p^2 - 13p + 20) &= 0 && \text{Factor out 2.} \\ 2p^2 - 13p + 20 &= 0 && \text{Divide each side by 2.} \\ (2p - 5)(p - 4) &= 0 && \text{Factor.} \\ 2p - 5 = 0 \quad \text{or} \quad p - 4 &= 0 && \text{Zero-factor property} \\ 2p = 5 \qquad\qquad p &= 4 && \text{Solve each equation.} \\ p = \frac{5}{2} \qquad\qquad & \end{aligned}$$

*Check* each solution to verify that the solution set is $\left\{\frac{5}{2}, 4\right\}$. NOW TRY

**NOW TRY ANSWERS**

**2.** $\{-6, 3\}$ **3.** $\left\{-7, \frac{1}{2}\right\}$

**CAUTION** A common error is to include the common factor 2 as a solution in **Example 3.** ***Only factors containing variables lead to solutions,*** such as the factor $y$ in the equation $y(3y-4)=0$ in **Example 1(b).**

**NOW TRY EXERCISE 4**

Solve each equation.

**(a)** $9x^2 - 64 = 0$

**(b)** $m^2 = 5m$

**(c)** $p(6p-1) = 2$

## EXAMPLE 4 Solving Quadratic Equations

Solve each equation.

**(a)**

$$16m^2 - 25 = 0$$

$$(4m+5)(4m-5) = 0 \qquad \text{Factor the difference of squares. } \textbf{(Section 6.4)}$$

$$4m+5=0 \quad \text{or} \quad 4m-5=0 \qquad \text{Zero-factor property}$$

$$4m=-5 \quad \text{or} \quad 4m=5 \qquad \text{Solve each equation.}$$

$$m=-\frac{5}{4} \quad \text{or} \quad m=\frac{5}{4}$$

*Check* the solutions, $-\frac{5}{4}$ and $\frac{5}{4}$, in the original equation. The solution set is $\left\{-\frac{5}{4}, \frac{5}{4}\right\}$.

**(b)**

$$y^2 = 2y$$

$$y^2 - 2y = 0 \qquad \text{Standard form}$$

$$y(y-2) = 0 \qquad \text{Factor.}$$

Don't forget to set the variable factor $y$ equal to 0.

$$y=0 \quad \text{or} \quad y-2=0 \qquad \text{Zero-factor property}$$

$$y=2 \qquad \text{Solve.}$$

The solution set is $\{0, 2\}$.

**(c)**

$$k(2k+1) = 3$$

To be in standard form, 0 must be on the right side.

$$2k^2 + k = 3 \qquad \text{Distributive property}$$

$$\text{Standard form} \longrightarrow 2k^2 + k - 3 = 0 \qquad \text{Subtract 3.}$$

$$(2k+3)(k-1) = 0 \qquad \text{Factor.}$$

$$2k+3=0 \quad \text{or} \quad k-1=0 \qquad \text{Zero-factor property}$$

$$2k=-3 \qquad k=1 \qquad \text{Solve each equation.}$$

$$k=-\frac{3}{2}$$

The solution set is $\left\{-\frac{3}{2}, 1\right\}$. NOW TRY

**CAUTION** In **Example 4(b),** it is tempting to begin by dividing both sides of

$$y^2 = 2y$$

by $y$ to get $y=2$. Note, however, that we do not get the other solution, 0, if we divide by a variable. (We *may* divide each side of an equation by a *nonzero* real number, however. For instance, in **Example 3** we divided each side by 2.)

In **Example 4(c),** we could not use the zero-factor property to solve the equation

$$k(2k+1) = 3$$

in its given form because of the 3 on the right. ***The zero-factor property applies only to a product that equals 0.***

**NOW TRY ANSWERS**

**4. (a)** $\left\{-\frac{8}{3}, \frac{8}{3}\right\}$ **(b)** $\{0, 5\}$

**(c)** $\left\{-\frac{1}{2}, \frac{2}{3}\right\}$

**NOW TRY EXERCISE 5**

Solve.

$$4x^2 - 4x + 1 = 0$$

### EXAMPLE 5 Solving Quadratic Equations with Double Solutions

Solve each equation.

**(a)**

$$\begin{aligned} z^2 - 22z + 121 &= 0 \\ (z-11)^2 &= 0 && \text{Factor the perfect square trinomial.} \\ (z-11)(z-11) &= 0 && a^2 = a \cdot a \\ z - 11 = 0 \quad \text{or} \quad z - 11 &= 0 && \text{Zero-factor property} \end{aligned}$$

Because the two factors are identical, they both lead to the same solution. (This is called a **double solution.**)

$$z = 11 \qquad \text{Add 11.}$$

*CHECK*

$$\begin{aligned} z^2 - 22z + 121 &= 0 \\ 11^2 - 22(11) + 121 &\stackrel{?}{=} 0 && \text{Let } z = 11. \\ 121 - 242 + 121 &\stackrel{?}{=} 0 \\ 0 &= 0 \ \checkmark && \text{True} \end{aligned}$$

The solution set is $\{11\}$.

**(b)**

$$\begin{aligned} 9t^2 - 30t &= -25 \\ 9t^2 - 30t + 25 &= 0 && \text{Standard form} \\ (3t-5)^2 &= 0 && \text{Factor the perfect square trinomial.} \\ 3t - 5 = 0 \quad \text{or} \quad 3t - 5 &= 0 && \text{Zero-factor property} \\ 3t &= 5 && \text{Solve the equation.} \\ t &= \frac{5}{3} && \tfrac{5}{3} \text{ is a double solution.} \end{aligned}$$

*Check* by substituting $\frac{5}{3}$ in the original equation. The solution set is $\left\{\frac{5}{3}\right\}$.

NOW TRY

---

**CAUTION** Each of the equations in **Example 5** has only *one* distinct solution. ***There is no need to write the same number more than once in a solution set.***

---

**OBJECTIVE 2** **Solve other equations by factoring.** We can also use the zero-factor property to solve equations that involve more than two factors with variables. (These equations are *not* quadratic equations. Why not?)

### EXAMPLE 6 Solving Equations with More Than Two Variable Factors

Solve each equation.

**(a)**

$$\begin{aligned} 6z^3 - 6z &= 0 \\ 6z(z^2 - 1) &= 0 && \text{Factor out } 6z. \\ 6z(z+1)(z-1) &= 0 && \text{Factor } z^2 - 1. \end{aligned}$$

By an extension of the zero-factor property, this product can equal 0 only if at least one of the factors is 0. Write and solve three equations, one for each factor with a variable.

**NOW TRY ANSWER**

5. $\left\{\frac{1}{2}\right\}$

**NOW TRY EXERCISE 6**

Solve each equation.

**(a)** $3x^3 - 27x = 0$

**(b)** $(3a - 1) \cdot (2a^2 - 5a - 12) = 0$

$$\begin{aligned} 6z &= 0 \quad \text{or} \quad z + 1 = 0 \quad \text{or} \quad z - 1 = 0 \\ z &= 0 \quad \text{or} \quad z = -1 \quad \text{or} \quad z = 1 \end{aligned}$$

*Check* by substituting, in turn, 0, $-1$, and 1 into the original equation. The solution set is $\{-1, 0, 1\}$.

**(b)**

$$\begin{aligned} (3x - 1)(x^2 - 9x + 20) &= 0 \\ (3x - 1)(x - 5)(x - 4) &= 0 && \text{Factor } x^2 - 9x + 20. \\ 3x - 1 = 0 \quad \text{or} \quad x - 5 = 0 \quad &\text{or} \quad x - 4 = 0 && \text{Zero-factor property} \\ x = \frac{1}{3} \quad \text{or} \quad x = 5 \quad &\text{or} \quad x = 4 && \text{Solve each equation.} \end{aligned}$$

*Check* each solution to verify that the solution set is $\left\{\frac{1}{3}, 4, 5\right\}$. NOW TRY

**CAUTION** In **Example 6(b),** it would be unproductive to begin by multiplying the two factors together. The zero-factor property requires the *product* of two or more factors to equal 0. ***Always consider first whether an equation is given in an appropriate form for the zero-factor property to apply.***

**NOW TRY EXERCISE 7**

Solve.

$x(4x - 9) = (x - 2)^2 + 24$

### EXAMPLE 7 Solving an Equation Requiring Multiplication before Factoring

Solve $(3x + 1)x = (x + 1)^2 + 5$.

The zero-factor property requires the *product* of two or more factors to equal 0.

$$\begin{aligned} (3x + 1)x &= (x + 1)^2 + 5 && (x + 1)^2 = (x + 1)(x + 1) \\ 3x^2 + x &= x^2 + 2x + 1 + 5 && \text{Multiply.} \\ 3x^2 + x &= x^2 + 2x + 6 && \text{Combine like terms.} \\ 2x^2 - x - 6 &= 0 && \text{Standard form} \\ (2x + 3)(x - 2) &= 0 && \text{Factor.} \\ 2x + 3 = 0 \quad &\text{or} \quad x - 2 = 0 && \text{Zero-factor property} \\ x = -\frac{3}{2} \quad &\text{or} \quad x = 2 && \text{Solve each equation.} \end{aligned}$$

*Check* that the solution set is $\left\{-\frac{3}{2}, 2\right\}$. NOW TRY

**NOW TRY ANSWERS**

6. **(a)** $\{-3, 0, 3\}$ **(b)** $\left\{-\frac{3}{2}, \frac{1}{3}, 4\right\}$

7. $\left\{-\frac{7}{3}, 4\right\}$

## 5.5 EXERCISES

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***Concept Check*** *In Exercises 1–5, fill in the blank with the correct response.*

**1.** A quadratic equation in $x$ is an equation that can be put into the form ________ $= 0$.

**2.** The form $ax^2 + bx + c = 0$ is called ________ form.

**3.** If a quadratic equation is in standard form, to solve the equation we should begin by attempting to ________ the polynomial.

**4.** The equation $x^3 + x^2 + x = 0$ is not a quadratic equation, because ________.

**5.** If a quadratic equation $ax^2 + bx + c = 0$ has $c = 0$, then ________ *must* be a solution because ________ is a factor of the polynomial.

**6.** ***Concept Check*** Identify each equation as *linear* or *quadratic*.

**(a)** $2x - 5 = 6$

**(b)** $x^2 - 5 = -4$

**(c)** $x^2 + 2x - 3 = 2x^2 - 2$

**(d)** $5^2x + 2 = 0$

**7.** Students often become confused as to how to handle a constant, such as 2 in the equation $2x(3x - 4) = 0$. How would you explain to someone how to solve this equation and how to handle the constant 2?

**8.** ***Concept Check*** The number 9 is a *double solution* of the equation $(x - 9)^2 = 0$. Why is this so?

**9.** ***Concept Check*** Look at this "solution." **WHAT WENT WRONG?**

$$x(7x - 1) = 0$$

$$7x - 1 = 0 \qquad \text{Zero-factor property}$$

$$x = \frac{1}{7}$$

The solution set is $\left\{\frac{1}{7}\right\}$.

**10.** ***Concept Check*** Look at this "solution." **WHAT WENT WRONG?**

$$3x(5x - 4) = 0$$

$$x = 3 \quad \text{or} \quad x = 0 \quad \text{or} \quad 5x - 4 = 0$$

$$x = \frac{4}{5}$$

The solution set is $\left\{3, 0, \frac{4}{5}\right\}$.

*Solve each equation, and check your solutions.* ***See Example 1.***

**11.** $(x + 5)(x - 2) = 0$

**12.** $(x - 1)(x + 8) = 0$

**13.** $(2m - 7)(m - 3) = 0$

**14.** $(6x + 5)(x + 4) = 0$

**15.** $(2x + 1)(6x - 1) = 0$

**16.** $(3x + 2)(10x - 1) = 0$

**17.** $t(6t + 5) = 0$

**18.** $w(4w + 1) = 0$

**19.** $2x(3x - 4) = 0$

**20.** $6y(4y + 9) = 0$

**21.** $(x - 6)(x - 6) = 0$

**22.** $(y + 1)(y + 1) = 0$

*Solve each equation, and check your solutions.* ***See Examples 2–7.***

**23.** $y^2 + 3y + 2 = 0$

**24.** $p^2 + 8p + 7 = 0$

**25.** $y^2 - 3y + 2 = 0$

**26.** $r^2 - 4r + 3 = 0$

**27.** $x^2 = 24 - 5x$

**28.** $t^2 = 2t + 15$

**29.** $x^2 = 3 + 2x$

**30.** $x^2 = 4 + 3x$

**31.** $z^2 + 3z = -2$

**32.** $p^2 - 2p = 3$

**33.** $m^2 + 8m + 16 = 0$

**34.** $x^2 - 6x + 9 = 0$

**35.** $3x^2 + 5x - 2 = 0$

**36.** $6r^2 - r - 2 = 0$

**37.** $12p^2 = 8 - 10p$

**38.** $18x^2 = 12 + 15x$

**39.** $9s^2 + 12s = -4$

**40.** $36x^2 + 60x = -25$

**41.** $y^2 - 9 = 0$

**42.** $m^2 - 100 = 0$

**43.** $16x^2 - 49 = 0$

**44.** $4w^2 - 9 = 0$

**45.** $n^2 = 121$

**46.** $x^2 = 400$

**47.** $x^2 = 7x$

**48.** $t^2 = 9t$

**49.** $6r^2 = 3r$

**50.** $10y^2 = -5y$

**51.** $x(x - 7) = -10$

**52.** $r(r - 5) = -6$

**53.** $3z(2z + 7) = 12$

**54.** $4x(2x + 3) = 36$

**55.** $2y(y + 13) = 136$

**56.** $t(3t - 20) = -12$

**57.** $(2r + 5)(3r^2 - 16r + 5) = 0$

**58.** $(3m + 4)(6m^2 + m - 2) = 0$

**59.** $(2x + 7)(x^2 + 2x - 3) = 0$

**60.** $(x + 1)(6x^2 + x - 12) = 0$

**61.** $9y^3 - 49y = 0$

**62.** $16r^3 - 9r = 0$

**63.** $r^3 - 2r^2 - 8r = 0$

**64.** $x^3 - x^2 - 6x = 0$

**65.** $x^3 + x^2 - 20x = 0$

**66.** $y^3 - 6y^2 + 8y = 0$

**67.** $r^4 = 2r^3 + 15r^2$

**68.** $x^3 = 3x + 2x^2$

**69.** $3x(x + 1) = (2x + 3)(x + 1)$

**70.** $2x(x + 3) = (3x + 1)(x + 3)$

**71.** $x^2 + (x + 1)^2 = (x + 2)^2$

**72.** $(x - 7)^2 + x^2 = (x + 1)^2$

***Brain Busters*** *Solve each equation, and check your solutions.*

**73.** $(2x)^2 = (2x + 4)^2 - (x + 5)^2$

**74.** $5 - (x - 1)^2 = (x - 2)^2$

**75.** $(x + 3)^2 - (2x - 1)^2 = 0$

**76.** $(4y - 3)^3 - 9(4y - 3) = 0$

**77.** $6p^2(p + 1) = 4(p + 1) - 5p(p + 1)$

**78.** $6x^2(2x + 3) = 4(2x + 3) + 5x(2x + 3)$

*Galileo's formula describing the motion of freely falling objects is*

$$d = 16t^2.$$

*The distance d in feet an object falls depends on the time t elapsed, in seconds. (This is an example of an important mathematical concept, the* ***function.****)*

**79. (a)** Use Galileo's formula and complete the following table. (*Hint:* Substitute each given value into the formula and solve for the unknown value.)

| *t in seconds* | 0 | 1 | 2 | 3 | | |
|---|---|---|---|---|---|---|
| *d in feet* | 0 | 16 | | | 256 | 576 |

**(b)** When $t = 0$, $d = 0$. Explain this in the context of the problem.

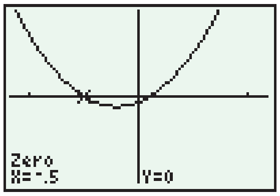
**80.** When you substituted 256 for $d$ and solved the formula for $t$ in **Exercise 79,** you should have found two solutions: 4 and $-4$. Why doesn't $-4$ make sense as an answer?

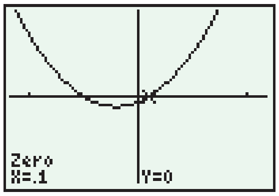
## TECHNOLOGY INSIGHTS EXERCISES 81–82

*In* ***Section 3.2,*** *we showed how an equation in one variable can be solved with a graphing calculator by getting 0 on one side and then replacing 0 with y to get a corresponding equation in two variables. The x-values of the x-intercepts of the graph of the two-variable equation then give the solutions of the original equation.*

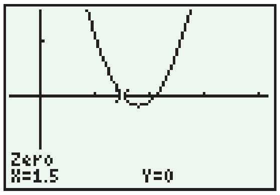
*Use the calculator screens to determine the solution set of each quadratic equation. Verify your answers by substitution.*

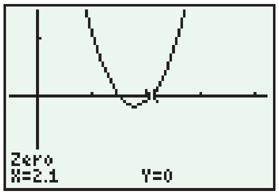
**81.** $x^2 + 0.4x - 0.05 = 0$





**82.** $2x^2 - 7.2x + 6.3 = 0$



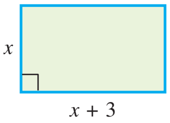


## PREVIEW EXERCISES

*Solve each problem.* ***See Sections 2.4 and 2.5.***

**83.** If a number is doubled and 6 is subtracted from this result, the answer is 3684. The unknown number is the year that Texas was admitted to the Union. What year was Texas admitted?

**84.** The length of the rectangle is 3 m more than its width. The perimeter of the rectangle is 34 m. Find the width of the rectangle.



**85.** Twice the sum of two consecutive integers is 28 more than the greater integer. Find the integers.

**86.** The area of a triangle with base 12 in. is 48 in.$^2$. Find the height of the triangle.

# 5.6

# Applications of Quadratic Equations

## OBJECTIVES

1. Solve problems involving geometric figures.
2. Solve problems involving consecutive integers.
3. Solve problems by applying the Pythagorean theorem.
4. Solve problems by using given quadratic models.

We use factoring to solve quadratic equations that arise in application problems. We follow the same six problem-solving steps given in **Section 2.4.**

### Solving an Applied Problem

***Step 1*** **Read** the problem carefully. What information is given? What are you asked to find?

***Step 2*** **Assign a variable** to represent the unknown value. Use a sketch, diagram, or table, as needed. If necessary, express any other unknown values in terms of the variable.

***Step 3*** **Write an equation,** using the variable expression(s).

***Step 4*** **Solve** the equation.

***Step 5*** **State the answer.** Label it appropriately. Does it seem reasonable?

***Step 6*** **Check** the answer in the words of the original problem.

**OBJECTIVE 1** **Solve problems involving geometric figures.** Refer to the formulas given on the inside covers of the text, if necessary.

### EXAMPLE 1 Solving an Area Problem

Abe Biggs wants to plant a triangular flower bed in a corner of his garden. One leg of the right-triangular flower bed will be 2 m shorter than the other leg. He wants the bed to have an area of 24 m$^2$. See **FIGURE 1**. Find the lengths of the legs.

***Step 1*** **Read** the problem. We need to find the lengths of the legs of a right triangle with area 24 m$^2$.

***Step 2*** **Assign a variable.**

Let $x$ = the length of one leg.

Then $x - 2$ = the length of the other leg.

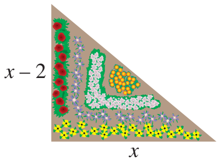


**FIGURE 1**

**NOW TRY EXERCISE 1**

A right triangle has one leg that is 4 ft shorter than the other leg. The area of the triangle is 6 ft$^2$. Determine the lengths of the legs.

***Step 3*** **Write an equation.** The area of a right triangle is given by the formula

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}.$$

In a right triangle, the legs are the base and height, so we substitute 24 for the area, $x$ for the base, and $x - 2$ for the height in the formula.

$$\mathcal{A} = \frac{1}{2}bh \qquad \text{Formula for the area of a triangle}$$

$$24 = \frac{1}{2}x(x-2) \qquad \text{Let } \mathcal{A} = 24,\ b = x,\ h = x - 2.$$

***Step 4*** **Solve.**

$$\begin{aligned} 48 &= x(x-2) && \text{Multiply by 2.} \\ 48 &= x^2 - 2x && \text{Distributive property} \\ x^2 - 2x - 48 &= 0 && \text{Standard form} \\ (x+6)(x-8) &= 0 && \text{Factor.} \\ x + 6 = 0 \quad &\text{or} \quad x - 8 = 0 && \text{Zero-factor property} \\ x = -6 \quad &\text{or} \quad x = 8 && \text{Solve each equation.} \end{aligned}$$

***Step 5*** **State the answer.** The solutions are $-6$ and 8. Because a triangle cannot have a side of negative length, we discard the solution $-6$. Then the lengths of the legs will be 8 m and $8 - 2 = 6$ m.

***Step 6*** **Check.** The length of one leg is 2 m less than the length of the other leg, and the area is

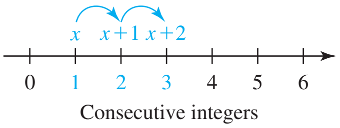
$$\frac{1}{2}(8)(6) = 24 \text{ m}^2, \quad \text{as required.}$$

NOW TRY

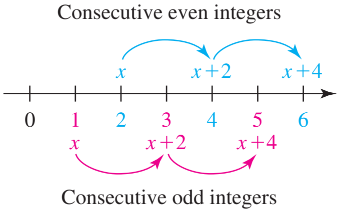
**CAUTION** ***In solving applied problems, always check solutions against physical facts*** and discard any answers that are not appropriate.

**OBJECTIVE 2** **Solve problems involving consecutive integers.** Recall from our work in **Section 2.4** that **consecutive integers** are integers that are next to each other on a number line, such as 3 and 4, or $-11$ and $-10$. See **FIGURE 2(a)**.

**Consecutive odd integers** are *odd* integers that are next to each other, such as 3 and 5, or $-13$ and $-11$. **Consecutive even integers** are defined similarly—for example, 4 and 6 are consecutive even integers, as are $-10$ and $-8$. See **FIGURE 2(b)**.



**(a)**



**(b)**

**FIGURE 2**

**PROBLEM-SOLVING HINT**

If $x$ represents the lesser integer, then, for any

| | |
|---|---|
| two consecutive integers, use | $x,\ x + 1;$ |
| three consecutive integers, use | $x,\ x + 1,\ x + 2;$ |
| two consecutive even or odd integers, use | $x,\ x + 2;$ |
| three consecutive even or odd integers, use | $x,\ x + 2,\ x + 4.$ |

As a general rule in this book, we list consecutive integers in increasing order when solving applications.

**NOW TRY ANSWER**

**1.** 2 ft, 6 ft

NOW TRY EXERCISE 2

The product of the first and second of three consecutive integers is 2 more than 8 times the third integer. Find the integers.

### EXAMPLE 2 Solving a Consecutive Integer Problem

The product of the second and third of three consecutive integers is 2 more than 7 times the first integer. Find the integers.

***Step 1*** **Read** the problem. Note that the integers are consecutive.

***Step 2*** **Assign a variable.**

Let $x =$ the first integer.

Then $x + 1 =$ the second integer,

and $x + 2 =$ the third integer.

***Step 3*** **Write an equation.**

$$\underset{\textstyle (x+1)(x+2)}{\overset{\text{The product of the second and third}}{\downarrow}} \quad \underset{\textstyle =}{\overset{\text{is}}{\downarrow}} \quad \underset{\textstyle 7x+2}{\overset{\text{2 more than 7 times the first.}}{\downarrow}}$$

***Step 4*** **Solve.**

$$\begin{aligned} x^2 + 3x + 2 &= 7x + 2 && \text{Multiply.} \\ x^2 - 4x &= 0 && \text{Standard form} \\ x(x-4) &= 0 && \text{Factor.} \\ x = 0 \quad \text{or} \quad x &= 4 && \text{Zero-factor property} \end{aligned}$$

***Step 5*** **State the answer.** The solutions 0 and 4 each lead to a correct answer.

$$0, 1, 2 \qquad \text{or} \qquad 4, 5, 6$$

***Step 6*** **Check.** The product of the second and third integers must equal 2 more than 7 times the first. Since $1 \cdot 2 = 7 \cdot 0 + 2$ and $5 \cdot 6 = 7 \cdot 4 + 2$, both sets of consecutive integers satisfy the statement of the problem.

NOW TRY

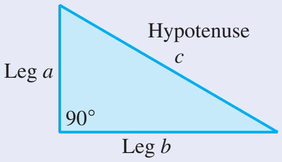
## OBJECTIVE 3 Solve problems by applying the Pythagorean theorem.

### Pythagorean Theorem

If a right triangle has longest side of length $c$ and two other sides of lengths $a$ and $b$, then

$$a^2 + b^2 = c^2.$$

The longest side, the **hypotenuse,** is opposite the right angle. The two shorter sides are the **legs** of the triangle.



### EXAMPLE 3 Applying the Pythagorean Theorem

Patricia Walker and Ali Ulku leave their office, Patricia traveling north and Ali traveling east. When Ali is 1 mi farther than Patricia from the office, the distance between them is 2 mi more than Patricia's distance from the office. Find their distances from the office and the distance between them.

***Step 1*** **Read** the problem again. There will be three answers to this problem.

NOW TRY ANSWER

**2.** 9, 10, 11 or −2, −1, 0

NOW TRY EXERCISE 3

The longer leg of a right triangle is 7 ft longer than the shorter leg and the hypotenuse is 8 ft longer than the shorter leg. Find the lengths of the sides of the triangle.

***Step 2*** **Assign a variable.**

Let $x =$ Patricia's distance from the office.

Then $x + 1 =$ Ali's distance from the office,

and $x + 2 =$ the distance between them.

Place these expressions on a right triangle, as in **FIGURE 3**.

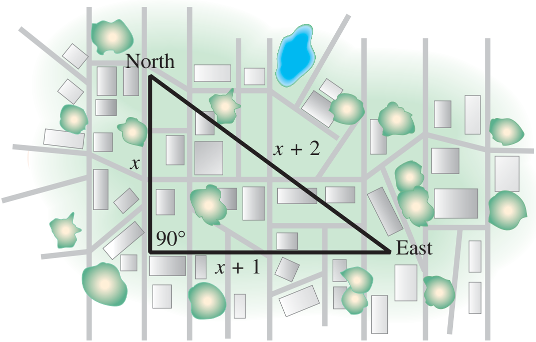


**FIGURE 3**

***Step 3*** **Write an equation.** Use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

$$x^2 + (x + 1)^2 = (x + 2)^2$$ Be careful to substitute properly.

***Step 4*** **Solve.**

$$x^2 + x^2 + 2x + 1 = x^2 + 4x + 4$$ Square each binomial.

$$x^2 - 2x - 3 = 0$$ Standard form

$$(x - 3)(x + 1) = 0$$ Factor.

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$ Zero-factor property

$$x = 3 \quad \text{or} \quad x = -1$$ Solve each equation.

***Step 5*** **State the answer.** Since $-1$ cannot represent a distance, 3 is the only possible answer. Patricia's distance is 3 mi, Ali's distance is $3 + 1 = 4$ mi, and the distance between them is $3 + 2 = 5$ mi.

***Step 6*** **Check.** Since $3^2 + 4^2 = 5^2$, the answers are correct. NOW TRY

**PROBLEM-SOLVING HINT**

In solving a problem involving the Pythagorean theorem, be sure that the expressions for the sides are properly placed.

$$(\textbf{one leg})^2 + (\textbf{other leg})^2 = \textbf{hypotenuse}^2$$

**OBJECTIVE 4** **Solve problems by using given quadratic models.** In **Examples 1–3,** we wrote quadratic equations to model, or mathematically describe, various situations and then solved the equations. In the last two examples of this section, we are given the quadratic models and must use them to determine data.

NOW TRY ANSWER

**3.** 5 ft, 12 ft, 13 ft

**NOW TRY EXERCISE 4**

Refer to **Example 4.** How long will it take for the ball to reach a height of 50 ft?

### EXAMPLE 4 Finding the Height of a Ball

A tennis player's serve travels 180 ft per sec (123 mph). If she hits the ball directly upward, the height $h$ of the ball in feet at time $t$ in seconds is modeled by the quadratic equation

$$h = -16t^2 + 180t + 6.$$

How long will it take for the ball to reach a height of 206 ft?

A height of 206 ft means that $h = 206$, so we substitute 206 for $h$ in the equation.

$$\begin{aligned}
h &= -16t^2 + 180t + 6 \\
206 &= -16t^2 + 180t + 6 && \text{Let } h = 206. \\
-16t^2 + 180t + 6 &= 206 && \text{Interchange sides.} \\
-16t^2 + 180t - 200 &= 0 && \text{Standard form} \\
4t^2 - 45t + 50 &= 0 && \text{Divide by } -4. \\
(4t - 5)(t - 10) &= 0 && \text{Factor.}
\end{aligned}$$

$$4t - 5 = 0 \quad \text{or} \quad t - 10 = 0 \qquad \text{Zero-factor property}$$

$$4t = 5 \quad \text{or} \quad t = 10 \qquad \text{Solve each equation.}$$

$$t = \frac{5}{4}$$

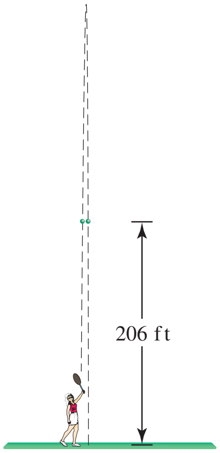


**FIGURE 4**

Since we found two acceptable answers, the ball will be 206 ft above the ground twice, once on its way up and once on its way down, at $\frac{5}{4}$ sec and at 10 sec. See **FIGURE 4**.

NOW TRY

### EXAMPLE 5 Modeling the Foreign-Born Population of the United States

The foreign-born population of the United States over the years 1930–2007 can be modeled by the quadratic equation

$$y = 0.01048x^2 - 0.5400x + 15.43,$$

where $x = 0$ represents 1930, $x = 10$ represents 1940, and so on, and $y$ is the number of people in millions. (*Source*: U.S. Census Bureau.)

**(a)** Use the model to find the foreign-born population in 1980 to the nearest tenth of a million.

Since $x = 0$ represents 1930, $x = 50$ represents 1980. Substitute 50 for $x$ in the equation.

$$\begin{aligned}
y &= 0.01048(50)^2 - 0.5400(50) + 15.43 && \text{Let } x = 50. \\
y &= 14.6 && \text{Round to the nearest tenth.}
\end{aligned}$$

In 1980, the foreign-born population of the United States was about 14.6 million.

**(b)** Repeat part (a) for 2007.

$$\begin{aligned}
y &= 0.01048(77)^2 - 0.5400(77) + 15.43 && \text{For 2007, let } x = 77. \\
y &= 36.0 && \text{Round to the nearest tenth.}
\end{aligned}$$

In 2007, the foreign-born population of the United States was about 36.0 million.

**NOW TRY ANSWER**

**4.** $\frac{1}{4}$ sec and 11 sec

**NOW TRY EXERCISE 5**

Use the model in **Example 5** to find the foreign-born population of the United States in the year 2000. Give your answer to the nearest tenth of a million. How does it compare to the actual value from the table?

**(c)** The model used in parts (a) and (b) was developed using the data in the table below. How do the results in parts (a) and (b) compare to the actual data from the table?

| Year | Foreign-Born Population (millions) |
|---|---|
| 1930 | 14.2 |
| 1940 | 11.6 |
| 1950 | 10.3 |
| 1960 | 9.7 |
| 1970 | 9.6 |
| 1980 | 14.1 |
| 1990 | 19.8 |
| 2000 | 28.4 |
| 2007 | 37.3 |

**NOW TRY ANSWER**

**5.** 29.0 million; The actual value is 28.4 million, so the answer using the model is slightly high.

From the table, the actual value for 1980 is 14.1 million. Our answer in part (a), 14.6 million, is slightly high. For 2007, the actual value is 37.3 million, so our answer of 36.0 million in part (b) is somewhat low. NOW TRY

## 5.6 EXERCISES

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**1.** *Concept Check* To review the six problem-solving steps first introduced in **Section 2.4,** complete each statement.

***Step 1:*** ________ the problem carefully.

***Step 2:*** Assign a ________ to represent the unknown value.

***Step 3:*** Write a(n) ________ using the variable expression(s).

***Step 4:*** ________ the equation.

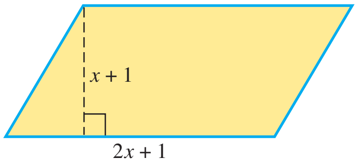
***Step 5:*** State the ________.

***Step 6:*** ________ the answer in the words of the ________ problem.

**2.** A student solves an applied problem and gets 6 or $-3$ for the length of the side of a square. Which of these answers is reasonable? Explain.

*In Exercises 3–6, a figure and a corresponding geometric formula are given. Using x as the variable, complete Steps 3–6 for each problem. (Refer to the steps in **Exercise 1** as needed.)*

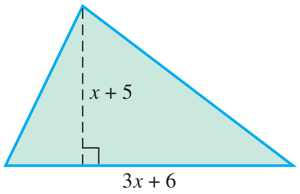
**3.**



Area of a parallelogram: $\mathcal{A} = bh$

The area of this parallelogram is 45 sq. units. Find its base and height.

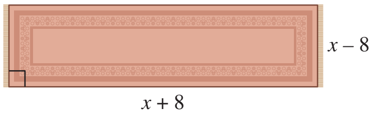
**4.**



Area of a triangle: $\mathcal{A} = \frac{1}{2}bh$

The area of this triangle is 60 sq. units. Find its base and height.

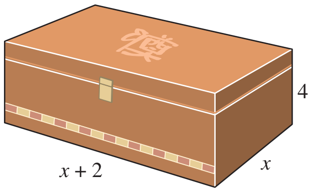
**5.**



Area of a rectangular rug: $\mathscr{A} = LW$

The area of this rug is 80 sq. units. Find its length and width.

**6.**



Volume of a rectangular Chinese box: $V = LWH$

The volume of this box is 192 cu. units. Find its length and width.

*Solve each problem. Check your answers to be sure that they are reasonable. Refer to the formulas on the inside covers.* ***See Example 1.***

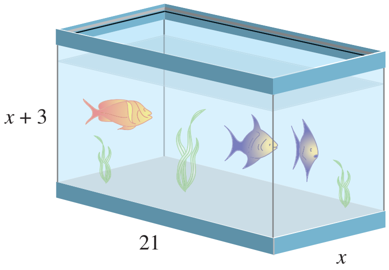
**7.** The length of a standard jewel case is 2 cm more than its width. The area of the rectangular top of the case is 168 $\text{cm}^2$. Find the length and width of the jewel case.

**8.** A standard DVD case is 6 cm longer than it is wide. The area of the rectangular top of the case is 247 $\text{cm}^2$. Find the length and width of the case.

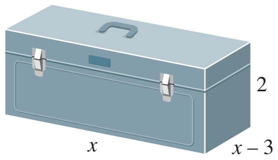
**9.** The area of a triangle is 30 $\text{in.}^2$. The base of the triangle measures 2 in. more than twice the height of the triangle. Find the measures of the base and the height.

**10.** A certain triangle has its base equal in measure to its height. The area of the triangle is 72 $\text{m}^2$. Find the equal base and height measure.

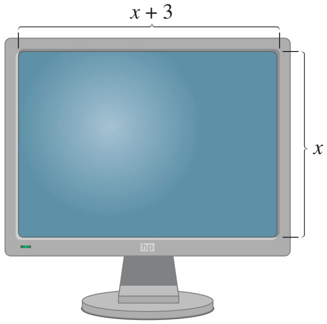
**11.** A 10-gal aquarium is 3 in. higher than it is wide. Its length is 21 in., and its volume is 2730 $\text{in.}^3$. What are the height and width of the aquarium?



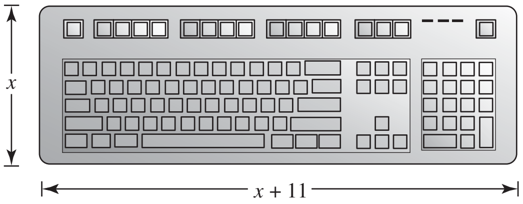
**12.** A toolbox is 2 ft high, and its width is 3 ft less than its length. If its volume is 80 $\text{ft}^3$, find the length and width of the box.



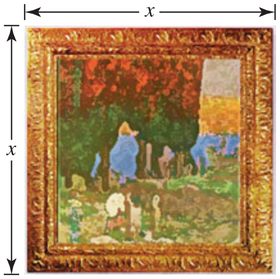
**13.** The dimensions of an HPf1905 flat-panel monitor are such that its length is 3 in. more than its width. If the length were doubled and if the width were decreased by 1 in., the area would be increased by 150 $\text{in.}^2$. What are the length and width of the flat panel?

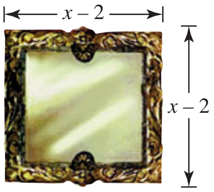


**14.** The keyboard that accompanies the monitor in **Exercise 13** is 11 in. longer than it is wide. If the length were doubled and if 2 in. were added to the width, the area would be increased by 198 $\text{in.}^2$. What are the length and width of the keyboard? (*Source:* Author's computer.)

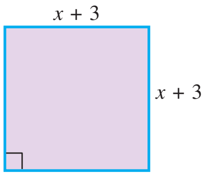


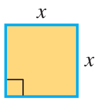
**15.** A square mirror has sides measuring 2 ft less than the sides of a square painting. If the difference between their areas is 32 $\text{ft}^2$, find the lengths of the sides of the mirror and the painting.





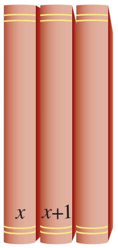
**16.** The sides of one square have length 3 m more than the sides of a second square. If the area of the larger square is subtracted from 4 times the area of the smaller square, the result is 36 $\text{m}^2$. What are the lengths of the sides of each square?





*Solve each problem.* ***See Example 2.***

**17.** The product of the numbers on two consecutive volumes of research data is 420. Find the volume numbers. See the figure.



**18.** The product of the page numbers on two facing pages of a book is 600. Find the page numbers.

**19.** The product of the second and third of three consecutive integers is 2 more than 10 times the first integer. Find the integers.

**20.** The product of the first and third of three consecutive integers is 3 more than 3 times the second integer. Find the integers.

**21.** Find three consecutive odd integers such that 3 times the sum of all three is 18 more than the product of the first and second integers.

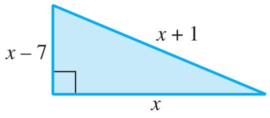
**22.** Find three consecutive odd integers such that the sum of all three is 42 less than the product of the second and third integers.

**23.** Find three consecutive even integers such that the sum of the squares of the first and second integers is equal to the square of the third integer.

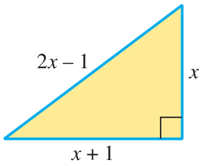
**24.** Find three consecutive even integers such that the square of the sum of the first and second integers is equal to twice the third integer.

*Solve each problem.* ***See Example 3.***

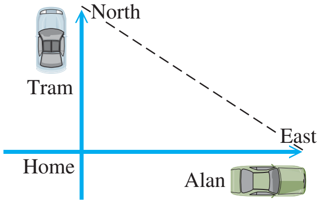
**25.** The hypotenuse of a right triangle is 1 cm longer than the longer leg. The shorter leg is 7 cm shorter than the longer leg. Find the length of the longer leg of the triangle.



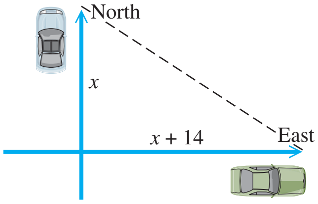
**26.** The longer leg of a right triangle is 1 m longer than the shorter leg. The hypotenuse is 1 m shorter than twice the shorter leg. Find the length of the shorter leg of the triangle.



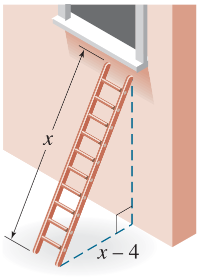
**27.** Tram works due north of home. Her husband Alan works due east. They leave for work at the same time. By the time Tram is 5 mi from home, the distance between them is 1 mi more than Alan's distance from home. How far from home is Alan?



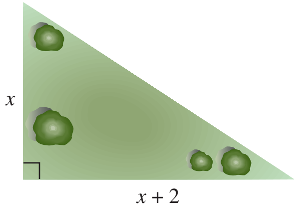
**28.** Two cars left an intersection at the same time. One traveled north. The other traveled 14 mi farther, but to the east. How far apart were they at that time if the distance between them was 4 mi more than the distance traveled east?



**29.** A ladder is leaning against a building. The distance from the bottom of the ladder to the building is 4 ft less than the length of the ladder. How high up the side of the building is the top of the ladder if that distance is 2 ft less than the length of the ladder?



**30.** A lot has the shape of a right triangle with one leg 2 m longer than the other. The hypotenuse is 2 m less than twice the length of the shorter leg. Find the length of the shorter leg.



*If an object is projected upward with an initial velocity of* 128 *ft per sec, its height h after t seconds is*

$$h = -16t^2 + 128t.$$

*Find the height of the object after each time listed.* ***See Example 4.***

**31.** 1 sec **32.** 2 sec **33.** 4 sec

**34.** How long does it take the object just described to return to the ground? (*Hint:* When the object hits the ground, $h = 0$.)

*Solve each problem.* ***See Examples 4 and 5.***

**35.** An object projected from a height of 48 ft with an initial velocity of 32 ft per sec after $t$ seconds has height

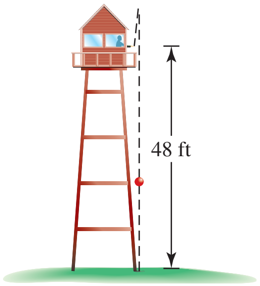
$$h = -16t^2 + 32t + 48.$$

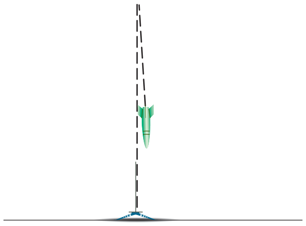
**(a)** After how many seconds is the height 64 ft? (*Hint:* Let $h = 64$ and solve.)

**(b)** After how many seconds is the height 60 ft?

**(c)** After how many seconds does the object hit the ground?

**(d)** The quadratic equation from part (c) has two solutions, yet only one of them is appropriate for answering the question. Why is this so?





**36.** If an object is projected upward from ground level with an initial velocity of 64 ft per sec, its height $h$ in feet $t$ seconds later is

$$h = -16t^2 + 64t.$$

**(a)** After how many seconds is the height 48 ft?

**(b)** The object reaches its maximum height 2 sec after it is projected. What is this maximum height?

**(c)** After how many seconds does the object hit the ground?

**(d)** Find the number of seconds after which the height is 60 ft.

**(e)** What is the physical interpretation of why part (d) has two answers?

**(f)** The quadratic equation from part (c) has two solutions, yet only one of them is appropriate for answering the question. Why is this so?

**37.** The table shows the number of cellular phone subscribers (in millions) in the United States.

| *Year* | *Subscribers (in millions)* |
|---|---|
| 1990 | 5 |
| 1992 | 11 |
| 1994 | 24 |
| 1996 | 44 |
| 1998 | 69 |
| 2000 | 109 |
| 2002 | 141 |
| 2004 | 182 |
| 2006 | 233 |
| 2008 | 263 |

*Source:* CTIA—The Wireless Association.

We used the preceding data to develop the quadratic equation

$$y = 0.590x^2 + 4.523x + 0.136,$$

which models the number $y$ of cellular phone subscribers (in millions) in the year $x$, where $x = 0$ represents 1990, $x = 2$ represents 1992, and so on.

**(a)** Use the model to find the number of subscribers in 2000, to the nearest tenth. How does the result compare with the actual data in the table?

**(b)** What value of $x$ corresponds to 2008?

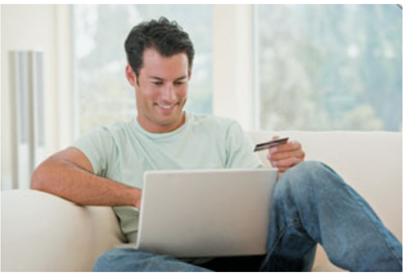
**(c)** Use the model to find the number of cellular phone subscribers in 2008, to the nearest tenth. How does the result compare with the actual data in the table?

**(d)** Assuming that the trend in the data continues, use the quadratic equation to estimate the number of cellular phone subscribers in 2010, to the nearest tenth.

**38.** Annual revenue in billions of dollars for eBay is shown in the table.

| *Year* | *Annual Revenue (in billions of dollars)* |
|---|---|
| 2002 | 1.21 |
| 2003 | 2.17 |
| 2004 | 3.27 |
| 2005 | 4.55 |
| 2006 | 5.97 |
| 2007 | 7.67 |

*Source*: eBay.



Using the data, we developed the quadratic equation

$$y = 0.089x^2 + 0.841x + 1.224$$

to model eBay revenues $y$ in year $x$, where $x = 0$ represents 2002, $x = 1$ represents 2003, and so on.

**(a)** Use the model to find annual revenue for eBay in 2005 and 2007, to the nearest hundredth. How do the results compare with the actual data in the table?

**(b)** Use the model to estimate annual revenue for eBay in 2009, to the nearest hundredth.

**(c)** Actual revenue for eBay in 2009 was \$8.73 billion. How does the result from part (b) compare with the actual revenue in 2009?

**(d)** Should the quadratic equation be used to estimate eBay revenue for years after 2007? Explain.

## PREVIEW EXERCISES

*Write each fraction in lowest terms.* ***See Section 1.1.***

**39.** $\frac{50}{72}$ **40.** $\frac{-26}{156}$ **41.** $\frac{48}{-27}$ **42.** $\frac{-35}{-21}$

# CHAPTER 5 SUMMARY

## KEY TERMS

**5.1**

factor
factored form
common factor
greatest common factor (GCF)

**5.2**

prime polynomial

**5.4**

perfect square
perfect square trinomial

**5.5**

quadratic equation
standard form
double solution

**5.6**

hypotenuse
legs

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** **Factoring** is
  **A.** a method of multiplying polynomials
  **B.** the process of writing a polynomial as a product
  **C.** the answer in a multiplication problem
  **D.** a way to add the terms of a polynomial.

**2.** A polynomial is in **factored form** when
  **A.** it is prime
  **B.** it is written as a sum
  **C.** the squared term has a coefficient of 1
  **D.** it is written as a product.

**3.** A **perfect square trinomial** is a trinomial
  **A.** that can be factored as the square of a binomial
  **B.** that cannot be factored
  **C.** that is multiplied by a binomial
  **D.** all of whose terms are perfect squares.

**4.** A **quadratic equation** is an equation that can be written in the form
  **A.** $y = mx + b$
  **B.** $ax^2 + bx + c = 0 \quad (a \neq 0)$
  **C.** $Ax + By = C$
  **D.** $x = k$.

**5.** A **hypotenuse** is
  **A.** either of the two shorter sides of a triangle
  **B.** the shortest side of a triangle
  **C.** the side opposite the right angle in a triangle
  **D.** the longest side in any triangle.

### *ANSWERS*

**1.** B; *Example:* $x^2 - 5x - 14$ factors as $(x - 7)(x + 2)$. **2.** D; *Example:* The factored form of $x^2 - 5x - 14$ is $(x - 7)(x + 2)$. **3.** A; *Example:* $a^2 + 2a + 1$ is a perfect square trinomial. Its factored form is $(a + 1)^2$. **4.** B; *Examples:* $y^2 - 3y + 2 = 0$, $x^2 - 9 = 0$, $2m^2 = 6m + 8$
**5.** C; *Example:* In **FIGURE 3** of **Section 5.6,** the hypotenuse is the side labeled $x + 2$.

## QUICK REVIEW

**CONCEPTS** | **EXAMPLES**

### 5.1 The Greatest Common Factor; Factoring by Grouping

**Finding the Greatest Common Factor (GCF)**

***Step 1*** Write each number in prime factored form.

***Step 2*** List each prime number or each variable that is a factor of every term in the list.

***Step 3*** Use as exponents on the common prime factors the *least* exponents from the prime factored forms.

***Step 4*** Multiply the primes from Step 3.

Find the greatest common factor of $4x^2y$, $6x^2y^3$, and $2xy^2$.

$$\begin{aligned} 4x^2y &= 2 \cdot 2 \cdot x^2 \cdot y \\ 6x^2y^3 &= 2 \cdot 3 \cdot x^2 \cdot y^3 \\ 2xy^2 &= 2 \cdot x \cdot y^2 \end{aligned}$$

The greatest common factor is $2xy$.

**Factoring by Grouping**

***Step 1*** Group the terms.

***Step 2*** Factor out the greatest common factor in each group.

***Step 3*** Factor out a common binomial factor from the results of Step 2.

***Step 4*** If necessary, try a different grouping.

Factor by grouping.

$$\begin{aligned} &3x^2 + 5x - 24xy - 40y \\ &= (3x^2 + 5x) + (-24xy - 40y) && \text{Group the terms.} \\ &= x(3x + 5) - 8y(3x + 5) && \text{Factor each group.} \\ &= (3x + 5)(x - 8y) && \text{Factor out } 3x + 5. \end{aligned}$$

### 5.2 Factoring Trinomials

To factor $x^2 + bx + c$, find $m$ and $n$ such that $mn = c$ and $m + n = b$.

$$x^2 + \underset{\substack{\uparrow \\ m + n = b}}{b}x + \overset{\substack{mn = c \\ \downarrow}}{c}$$

Then $x^2 + bx + c$ factors as $(x + m)(x + n)$.

Check by multiplying.

Factor $x^2 + 6x + 8$.

$$x^2 + \underset{\substack{\uparrow \\ m + n = 6}}{6}x + \overset{\substack{mn = 8 \\ \downarrow}}{8} \qquad m = 2 \text{ and } n = 4$$

$x^2 + 6x + 8$ factors as $(x + 2)(x + 4)$.

*CHECK* $(x + 2)(x + 4)$

$$\begin{aligned} &= x^2 + 4x + 2x + 8 \\ &= x^2 + 6x + 8 \ \checkmark \end{aligned}$$

### 5.3 More on Factoring Trinomials

To factor $ax^2 + bx + c$, use one of the following methods.

**Grouping**

Find $m$ and $n$.

$$\overbrace{\overset{\downarrow}{a}x^2 + \underset{\substack{\uparrow \\ m + n = b}}{b}x + \overset{\downarrow}{c}}^{mn = ac}$$

**Trial and Error**

Use FOIL in reverse.

Factor $\underbrace{\underset{\uparrow}{3}x^2 + 14x - \underset{\uparrow}{5}}_{-15}$.

$mn = -15$, $m + n = 14$

The required integers are $m = -1$ and $n = 15$.

By trial and error or by grouping,

$$3x^2 + 14x - 5 \quad \text{factors as} \quad (3x - 1)(x + 5).$$

(continued)

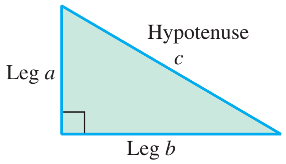
## 5.4 Special Factoring Techniques

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Difference of Squares**<br>$x^2 - y^2 = (x + y)(x - y)$ | Factor.<br>$4x^2 - 9$<br>$= (2x + 3)(2x - 3)$ |
| **Perfect Square Trinomials**<br>$x^2 + 2xy + y^2 = (x + y)^2$<br>$x^2 - 2xy + y^2 = (x - y)^2$ | $9x^2 + 6x + 1$<br>$= (3x + 1)^2$<br>$4x^2 - 20x + 25$<br>$= (2x - 5)^2$ |
| **Difference of Cubes**<br>$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ | $m^3 - 8$<br>$= m^3 - 2^3$<br>$= (m - 2)(m^2 + 2m + 4)$ |
| **Sum of Cubes**<br>$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ | $z^3 + 27$<br>$= z^3 + 3^3$<br>$= (z + 3)(z^2 - 3z + 9)$ |

## 5.5 Solving Quadratic Equations by Factoring

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Zero-Factor Property**<br>If $a$ and $b$ are real numbers and if $ab = 0$, then $a = 0$ or $b = 0$. | If $(x - 2)(x + 3) = 0$, then $x - 2 = 0$ or $x + 3 = 0$. |
| **Solving a Quadratic Equation by Factoring** | Solve $2x^2 = 7x + 15$. |
| ***Step 1*** Write the equation in standard form. | $2x^2 - 7x - 15 = 0$ |
| ***Step 2*** Factor. | $(2x + 3)(x - 5) = 0$ |
| ***Step 3*** Use the zero-factor property. | $2x + 3 = 0$ or $x - 5 = 0$ |
| ***Step 4*** Solve the resulting equations. | $2x = -3$ $\quad x = 5$<br>$x = -\frac{3}{2}$ |
| ***Step 5*** Check. | Both solutions satisfy the original equation. The solution set is $\left\{-\frac{3}{2}, 5\right\}$. |

## 5.6 Applications of Quadratic Equations

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Pythagorean Theorem**<br>In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.<br>$a^2 + b^2 = c^2$<br> | In a right triangle, one leg measures 2 ft longer than the other. The hypotenuse measures 4 ft longer than the shorter leg. Find the lengths of the three sides of the triangle.<br>Let $x =$ the length of the shorter leg. Then<br>$x^2 + (x + 2)^2 = (x + 4)^2.$<br>Verify that the solutions of this equation are $-2$ and 6. Discard $-2$ as a solution. Check that the sides have lengths<br>6 ft, $\quad 6 + 2 = 8$ ft, $\quad$ and $\quad 6 + 4 = 10$ ft. |

# CHAPTER 5

# REVIEW EXERCISES

**5.1** *Factor out the greatest common factor, or factor by grouping.*

**1.** $7t + 14$

**2.** $60z^3 + 30z$

**3.** $2xy - 8y + 3x - 12$

**4.** $6y^2 + 9y + 4xy + 6x$

**5.2** *Factor completely.*

**5.** $x^2 + 5x + 6$

**6.** $y^2 - 13y + 40$

**7.** $q^2 + 6q - 27$

**8.** $r^2 - r - 56$

**9.** $r^2 - 4rs - 96s^2$

**10.** $p^2 + 2pq - 120q^2$

**11.** $8p^3 - 24p^2 - 80p$

**12.** $3x^4 + 30x^3 + 48x^2$

**13.** $p^7 - p^6q - 2p^5q^2$

**14.** $3r^5 - 6r^4s - 45r^3s^2$

**15.** $9x^4y - 9x^3y - 54x^2y$

**16.** $2x^7 + 2x^6y - 12x^5y^2$

**5.3**

**17.** ***Concept Check*** To begin factoring $6r^2 - 5r - 6$, what are the possible first terms of the two binomial factors if we consider only positive integer coefficients?

**18.** ***Concept Check*** What is the first step you would use to factor $2z^3 + 9z^2 - 5z$?

*Factor completely.*

**19.** $2k^2 - 5k + 2$

**20.** $3r^2 + 11r - 4$

**21.** $6r^2 - 5r - 6$

**22.** $10z^2 - 3z - 1$

**23.** $8v^2 + 17v - 21$

**24.** $24x^5 - 20x^4 + 4x^3$

**25.** $-6x^2 + 3x + 30$

**26.** $10r^3s + 17r^2s^2 + 6rs^3$

**27.** $48x^4y + 4x^3y^2 - 4x^2y^3$

**28.** ***Concept Check*** On a quiz, a student factored $16x^2 - 24x + 5$ by grouping as follows.

$$\begin{aligned}&16x^2 - 24x + 5\\&\quad= 16x^2 - 4x - 20x + 5\\&\quad= 4x(4x - 1) - 5(4x - 1) \qquad \text{His answer}\end{aligned}$$

He thought his answer was correct, since it checked by multiplication. ***WHAT WENT WRONG?*** Give the correct factored form.

**5.4**

**29.** ***Concept Check*** Which one of the following is the difference of squares?

**A.** $32x^2 - 1$ **B.** $4x^2y^2 - 25z^2$ **C.** $x^2 + 36$ **D.** $25y^3 - 1$

**30.** ***Concept Check*** Which one of the following is a perfect square trinomial?

**A.** $x^2 + x + 1$ **B.** $y^2 - 4y + 9$ **C.** $4x^2 + 10x + 25$ **D.** $x^2 - 20x + 100$

*Factor completely.*

**31.** $n^2 - 49$

**32.** $25b^2 - 121$

**33.** $49y^2 - 25w^2$

**34.** $144p^2 - 36q^2$

**35.** $x^2 + 100$

**36.** $r^2 - 12r + 36$

**37.** $9t^2 - 42t + 49$

**38.** $m^3 + 1000$

**39.** $125k^3 + 64x^3$

**40.** $343x^3 - 64$

**41.** $1000 - 27x^6$

**42.** $x^6 - y^6$

**5.5** *Solve each equation, and check your solutions.*

**43.** $(4t + 3)(t - 1) = 0$

**44.** $(x + 7)(x - 4)(x + 3) = 0$

**45.** $x(2x - 5) = 0$

**46.** $z^2 + 4z + 3 = 0$

**47.** $m^2 - 5m + 4 = 0$

**48.** $x^2 = -15 + 8x$

**49.** $3z^2 - 11z - 20 = 0$

**50.** $81t^2 - 64 = 0$

**51.** $y^2 = 8y$

**52.** $n(n - 5) = 6$

**53.** $t^2 - 14t + 49 = 0$

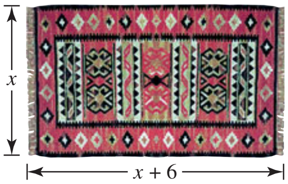
**54.** $t^2 = 12(t - 3)$

**55.** $(5z + 2)(z^2 + 3z + 2) = 0$

**56.** $x^2 = 9$

### 5.6 *Solve each problem.*

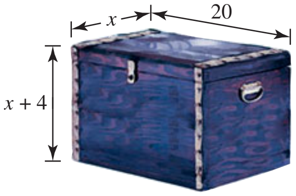
**57.** The length of a rug is 6 ft more than the width. The area is 40 $\text{ft}^2$. Find the length and width of the rug.



**58.** The surface area $S$ of a box is given by

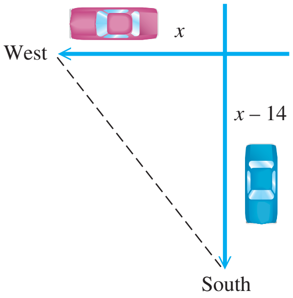
$$S = 2WH + 2WL + 2LH.$$

A treasure chest from a sunken galleon has the dimensions shown in the figure. Its surface area is 650 $\text{ft}^2$. Find its width.



**59.** The product of two consecutive integers is 29 more than their sum. What are the integers?

**60.** Two cars left an intersection at the same time. One traveled west, and the other traveled 14 mi less, but to the south. How far apart were they at that time, if the distance between them was 16 mi more than the distance traveled south?



**61.** If an object is dropped, the distance $d$ in feet it falls in $t$ seconds (disregarding air resistance) is given by the quadratic equation

$$d = 16t^2.$$

Find the distance an object would fall in **(a)** 4 sec and **(b)** 8 sec.

**62.** The numbers of alternative-fueled vehicles in use in the United States, in thousands, for the years 2001–2006 are given in the table.

| Year | Number (in thousands) |
|---|---|
| 2001 | 425 |
| 2002 | 471 |
| 2003 | 534 |
| 2004 | 565 |
| 2005 | 592 |
| 2006 | 635 |

*Source:* Energy Information Administration.

Using statistical methods, we developed the quadratic equation

$$y = -2.84x^2 + 61.1x + 366$$

to model the number of vehicles $y$ in year $x$. Here, we used $x = 1$ for 2001, $x = 2$ for 2002, and so on.

**(a)** Use the model to find the number of alternative-fueled vehicles in 2005, to the nearest thousand. How does the result compare with the actual data in the table?

**(b)** Use the model to estimate the number of alternative-fueled vehicles in 2007, to the nearest thousand.

**(c)** Why might the estimate for 2007 be unreliable?

## MIXED REVIEW EXERCISES

**63.** ***Concept Check*** Which of the following is *not* factored completely?

**A.** $3(7t)$ **B.** $3x(7t + 4)$ **C.** $(3 + x)(7t + 4)$ **D.** $3(7t + 4) + x(7t + 4)$

**64.** A student factored $6x^2 + 16x - 32$ as $(2x + 8)(3x - 4)$. Explain why the polynomial is not factored completely, and give the completely factored form.

*Factor completely.*

**65.** $3k^2 + 11k + 10$

**66.** $z^2 - 11zx + 10x^2$

**67.** $y^4 - 625$

**68.** $15m^2 + 20m - 12mp - 16p$

**69.** $24ab^3c^2 - 56a^2bc^3 + 72a^2b^2c$

**70.** $6m^3 - 21m^2 - 45m$

**71.** $12x^2yz^3 + 12xy^2z - 30x^3y^2z^4$

**72.** $25a^2 + 15ab + 9b^2$

**73.** $12r^2 + 18rq - 10r - 15q$

**74.** $2a^5 - 8a^4 - 24a^3$

**75.** $49t^2 + 56t + 16$

**76.** $1000a^3 + 27$

*Solve.*

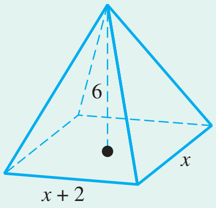
**77.** $t(t - 7) = 0$

**78.** $x^2 + 3x = 10$

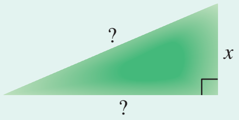
**79.** $25x^2 + 20x + 4 = 0$

**80.** The product of the first and second of three consecutive integers is equal to 23 plus the third. Find the integers.

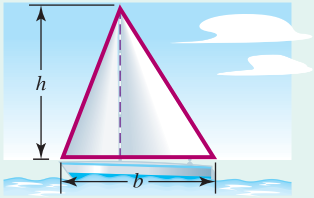
**81.** A pyramid has a rectangular base with a length that is 2 m more than its width. The height of the pyramid is 6 m, and its volume is 48 m$^3$. Find the length and width of the base.



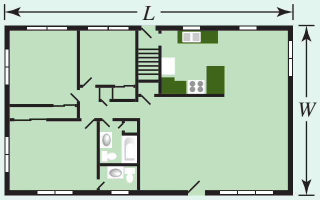
**82.** A lot is in the shape of a right triangle. The hypotenuse is 3 m longer than the longer leg. The longer leg is 6 m longer than twice the length of the shorter leg. Find the lengths of the sides of the lot.



**83.** The triangular sail of a schooner has an area of 30 m$^2$. The height of the sail is 4 m more than the base. Find the base of the sail.



**84.** The floor plan for a house is a rectangle with length 7 m more than its width. The area is 170 m$^2$. Find the width and length of the house.



# CHAPTER 5 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

**1.** ***Concept Check*** Which one of the following is the correct completely factored form of $2x^2 - 2x - 24$?

**A.** $(2x + 6)(x - 4)$ **B.** $(x + 3)(2x - 8)$

**C.** $2(x + 4)(x - 3)$ **D.** $2(x + 3)(x - 4)$

*Factor each polynomial completely. If the polynomial is prime, say so.*

**2.** $12x^2 - 30x$ **3.** $2m^3n^2 + 3m^3n - 5m^2n^2$ **4.** $2ax - 2bx + ay - by$

**5.** $x^2 - 5x - 24$ **6.** $2x^2 + x - 3$ **7.** $10z^2 - 17z + 3$

**8.** $t^2 + 2t + 3$ **9.** $x^2 + 36$ **10.** $12 - 6a + 2b - ab$

**11.** $9y^2 - 64$ **12.** $4x^2 - 28xy + 49y^2$ **13.** $-2x^2 - 4x - 2$

**14.** $6t^4 + 3t^3 - 108t^2$ **15.** $r^3 - 125$ **16.** $8k^3 + 64$

**17.** $x^4 - 81$ **18.** $81x^4 - 16y^4$ **19.** $9x^6y^4 + 12x^3y^2 + 4$

*Solve each equation.*

**20.** $2r^2 - 13r + 6 = 0$ **21.** $25x^2 - 4 = 0$ **22.** $t^2 = 9t$

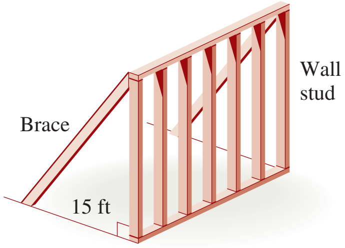
**23.** $x(x - 20) = -100$ **24.** $(s + 8)(6s^2 + 13s - 5) = 0$

*Solve each problem.*

**25.** The length of a rectangular flower bed is 3 ft less than twice its width. The area of the bed is 54 ft$^2$. Find the dimensions of the flower bed.

**26.** Find two consecutive integers such that the square of the sum of the two integers is 11 more than the first integer.

**27.** A carpenter needs to cut a brace to support a wall stud, as shown in the figure. The brace should be 7 ft less than three times the length of the stud. If the brace will be anchored on the floor 15 ft away from the stud, how long should the brace be?



**28.** The public debt $y$ (in billions of dollars) of the United States from 2000 through 2008 can be approximated by the quadratic equation

$$y = 29.92x^2 + 305.8x + 5581,$$

where $x = 0$ represents 2000, $x = 1$ represents 2001, and so on. (*Source:* Bureau of Public Debt.) Use the model to estimate the public debt, to the nearest billion dollars, in the year 2006.

# CHAPTERS 1–5 CUMULATIVE REVIEW EXERCISES

*Solve each equation.*

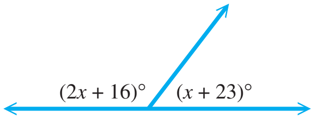
**1.** $3x + 2(x - 4) = 4(x - 2)$

**2.** $0.3x + 0.9x = 0.06$

**3.** $\frac{2}{3}m - \frac{1}{2}(m - 4) = 3$

**4.** Solve for $P$: $A = P + Prt$.

**5.** Find the measures of the marked angles.



*Solve each problem.*

**6.** At the 2006 Winter Olympics in Torino, Italy, the top medal winner was Germany, which won a total of 29 medals. Germany won 1 more silver medal than gold and 5 more gold medals than bronze. Find the number of each type of medal won. (*Source:* www.infoplease.com.)

**7.** From a list of "technology-related items," adults were recently surveyed as to those items they couldn't live without. Complete the results shown in the table if 500 adults were surveyed.

| *Item* | *Percent That Couldn't Live Without* | *Number That Couldn't Live Without* |
|---|---|---|
| Personal computer | 46% | |
| Cell phone | 41% | |
| High-speed Internet | | 190 |
| MP3 player | | 60 |

(Other items included digital cable, HDTV, and electronic gaming console.)
*Source:* Ipsos for AP.

**8.** Fill in each blank with *positive* or *negative*. The point with coordinates $(a, b)$ is in

**(a)** quadrant II if $a$ is ________ and $b$ is ________.

**(b)** quadrant III if $a$ is ________ and $b$ is ________.

**9.** Consider the equation $y = 12x + 3$. Find the following.

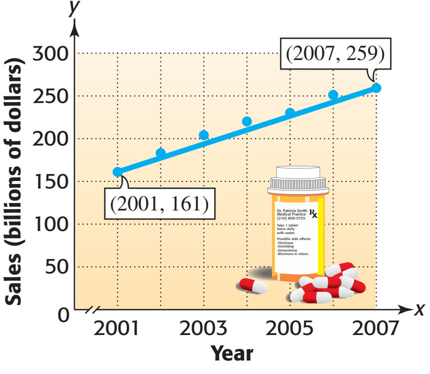
**(a)** The $x$- and $y$-intercepts **(b)** The slope **(c)** The graph

**10.** The points on the graph show the total retail sales of prescription drugs in the United States in the years 2001–2007, along with a graph of a linear equation that models the data.

**(a)** Use the ordered pairs shown on the graph to find the slope of the line to the nearest whole number. Interpret the slope.

**(b)** Use the graph to estimate sales in the year 2005. Write your answer as an ordered pair of the form (year, sales in billions of dollars).

**Retail Prescription Drug Sales**



***Source:*** National Association of Chain Drug Stores.

*Evaluate each expression.*

**11.** $\left(\frac{3}{4}\right)^{-2}$

**12.** $\left(\frac{4^{-3} \cdot 4^4}{4^5}\right)^{-1}$

*Simplify each expression, and write the answer with only positive exponents. Assume that no denominators are* 0.

**13.** $\frac{(p^2)^3 p^{-4}}{(p^{-3})^{-1} p}$

**14.** $\frac{(m^{-2})^3 m}{m^5 m^{-4}}$

*Perform each indicated operation.*

**15.** $(2k^2 + 4k) - (5k^2 - 2) - (k^2 + 8k - 6)$

**16.** $(9x + 6)(5x - 3)$

**17.** $(3p + 2)^2$

**18.** $\frac{8x^4 + 12x^3 - 6x^2 + 20x}{2x}$

**19.** To make a pound of honey, bees may travel 55,000 mi and visit more than 2,000,000 flowers. (*Source: Home & Garden.*) Write the two given numbers in scientific notation.

*Factor completely.*

**20.** $2a^2 + 7a - 4$

**21.** $10m^2 + 19m + 6$

**22.** $8t^2 + 10tv + 3v^2$

**23.** $4p^2 - 12p + 9$

**24.** $25r^2 - 81t^2$

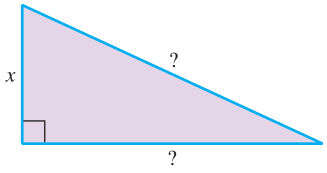
**25.** $2pq + 6p^3q + 8p^2q$

*Solve each equation.*

**26.** $6m^2 + m - 2 = 0$

**27.** $8x^2 = 64x$

**28.** The length of the hypotenuse of a right triangle is twice the length of the shorter leg, plus 3 m. The longer leg is 7 m longer than the shorter leg. Find the lengths of the sides.



## CHAPTER 6

# Rational Expressions and Applications

In 2006, Earth's temperature was within 1.8°F of its highest level in 12,000 years. This temperature increase is causing ocean levels to rise as ice fields in Greenland and elsewhere melt. To demonstrate the effects that such global warming is having, British adventurer and endurance swimmer Lewis Gordon Pugh swam at the North Pole in July 2007 in waters that were completely frozen 10 years ago. The swim, in a water hole where polar ice had melted, was in 28.8°F waters, the coldest ever endured by a human. Pugh, who has also swum in the waters of Antarctica, hopes his efforts will inspire world leaders to take climate change seriously. (*Source:* www.breitbart.com, IPCC.)

In **Exercise 11** of **Section 6.7,** we use a *rational expression* to determine the time Pugh swam at the North Pole.

# 6.1 The Fundamental Property of Rational Expressions

**OBJECTIVES**

1. Find the numerical value of a rational expression.
2. Find the values of the variable for which a rational expression is undefined.
3. Write rational expressions in lowest terms.
4. Recognize equivalent forms of rational expressions.

The quotient of two integers (with denominator not 0), such as $\frac{2}{3}$ or $-\frac{3}{4}$, is called a *rational number.* In the same way, the quotient of two polynomials with denominator not equal to 0 is called a *rational expression.*

**Rational Expression**

A **rational expression** is an expression of the form $\frac{P}{Q}$, where $P$ and $Q$ are polynomials and $Q \neq 0$.

$$\frac{-6x}{x^3+8}, \quad \frac{9x}{y+3}, \quad \text{and} \quad \frac{2m^3}{8} \qquad \text{Rational expressions}$$

Our work with rational expressions requires much of what we learned in **Chapters 4 and 5** on polynomials and factoring, as well as the rules for fractions from **Section 1.1.**

**OBJECTIVE 1 Find the numerical value of a rational expression.** We use substitution to evaluate a rational expression for a given value of the variable.

**EXAMPLE 1** Evaluating Rational Expressions

Find the numerical value of $\frac{3x+6}{2x-4}$ for the given value of $x$.

**(a)** $x = 1$

$$\begin{aligned} &\frac{3x+6}{2x-4} \\ &= \frac{3(1)+6}{2(1)-4} && \text{Let } x = 1. \\ &= \frac{9}{-2} && \text{Simplify.} \\ &= -\frac{9}{2} && \frac{a}{-b} = -\frac{a}{b} \end{aligned}$$

**(b)** $x = -2$

Use parentheses around negative numbers to avoid errors.

$$\begin{aligned} &\frac{3x+6}{2x-4} \\ &= \frac{3(-2)+6}{2(-2)-4} && \text{Let } x = -2. \\ &= \frac{0}{-8} && \text{Simplify.} \\ &= 0 && \frac{0}{b} = 0 \end{aligned}$$

NOW TRY

NOW TRY EXERCISE 1

Find the value of the following rational expression for $x = -3$.

$$\frac{2x-1}{x+4}$$

**OBJECTIVE 2 Find the values of the variable for which a rational expression is undefined.** In the definition of a rational expression $\frac{P}{Q}$, $Q$ cannot equal 0. ***The denominator of a rational expression cannot equal 0 because division by 0 is undefined.***

For instance, in the rational expression

$$\frac{3x+6}{2x-4} \leftarrow \text{Denominator cannot equal 0.}$$

from **Example 1,** the variable $x$ can take on any real number value except 2. If $x$ is 2, then the denominator becomes $2(2) - 4 = 0$, making the expression undefined. Thus, $x$ cannot equal 2. We indicate this restriction by writing $x \neq 2$.

NOW TRY ANSWER

**1.** $-7$

**NOTE** ***The numerator of a rational expression may be any real number.*** If the numerator equals 0 and the denominator does not equal 0, then the rational expression equals 0. See **Example 1(b).**

### Determining When a Rational Expression Is Undefined

***Step 1*** Set the denominator of the rational expression equal to 0.

***Step 2*** Solve this equation.

***Step 3*** The solutions of the equation are the values that make the rational expression undefined. The variable *cannot* equal these values.

**NOW TRY EXERCISE 2**

Find any values of the variable for which each rational expression is undefined.

**(a)** $\dfrac{k-4}{2k-1}$

**(b)** $\dfrac{2x}{x^2+5x-14}$

**(c)** $\dfrac{y+10}{y^2+10}$

### EXAMPLE 2 Finding Values That Make Rational Expressions Undefined

Find any values of the variable for which each rational expression is undefined.

**(a)** $\dfrac{x+5}{3x+2}$ We must find any value of $x$ that makes the *denominator* equal to 0, since division by 0 is undefined.

***Step 1*** Set the denominator equal to 0.

$$3x+2=0$$

***Step 2*** Solve.

$$\begin{aligned} 3x &= -2 && \text{Subtract 2.} \\ x &= -\frac{2}{3} && \text{Divide by 3.} \end{aligned}$$

***Step 3*** The given expression is undefined for $-\frac{2}{3}$, so $x \neq -\frac{2}{3}$.

**(b)** $\dfrac{8x^2+1}{x-3}$ The denominator $x-3=0$ when $x$ is 3. The given expression is undefined for 3, so $x \neq 3$.

**(c)** $\dfrac{9m^2}{m^2-5m+6}$

$$\begin{aligned} m^2-5m+6 &= 0 && \text{Set the denominator equal to 0.} \\ (m-2)(m-3) &= 0 && \text{Factor.} \\ m-2=0 \quad \text{or} \quad m-3 &= 0 && \text{Zero-factor property} \\ m=2 \quad \text{or} \quad m &= 3 && \text{Solve for } m. \end{aligned}$$

The given expression is undefined for 2 and 3, so $m \neq 2$, $m \neq 3$.

**(d)** $\dfrac{2r}{r^2+1}$ This denominator will not equal 0 for any value of $r$, because $r^2$ is always greater than or equal to 0, and adding 1 makes the sum greater than or equal to 1. There are no values for which this expression is undefined.

NOW TRY

**OBJECTIVE 3** **Write rational expressions in lowest terms.** A fraction such as $\frac{2}{3}$ is said to be in *lowest terms.*

### Lowest Terms

A rational expression $\frac{P}{Q}$ $(Q \neq 0)$ is in **lowest terms** if the greatest common factor of its numerator and denominator is 1.

**NOW TRY ANSWERS**

**2. (a)** $k \neq \frac{1}{2}$ **(b)** $x \neq -7, x \neq 2$
**(c)** never undefined

We use the **fundamental property of rational expressions** to write a rational expression in lowest terms.

### Fundamental Property of Rational Expressions

If $\frac{P}{Q}$ $(Q \neq 0)$ is a rational expression and if $K$ represents any polynomial, where $K \neq 0$, then the following is true.

$$\frac{PK}{QK} = \frac{P}{Q}$$

This property is based on the identity property of multiplication.

$$\frac{PK}{QK} = \frac{P}{Q} \cdot \frac{K}{K} = \frac{P}{Q} \cdot 1 = \frac{P}{Q}$$

**NOW TRY EXERCISE 3**

Write the rational expression in lowest terms.

$$\frac{21y^5}{7y^2}$$

## EXAMPLE 3 Writing in Lowest Terms

Write each rational expression in lowest terms.

**(a)** $\dfrac{30}{72}$

Begin by factoring.

$$\frac{30}{72} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}$$

**(b)** $\dfrac{14k^2}{2k^3}$

Write $k^2$ as $k \cdot k$ and $k^3$ as $k \cdot k \cdot k$.

$$\frac{14k^2}{2k^3} = \frac{2 \cdot 7 \cdot k \cdot k}{2 \cdot k \cdot k \cdot k}$$

Group any factors common to the numerator and denominator.

$$\frac{30}{72} = \frac{5 \cdot (2 \cdot 3)}{2 \cdot 2 \cdot 3 \cdot (2 \cdot 3)} \qquad \frac{14k^2}{2k^3} = \frac{7(2 \cdot k \cdot k)}{k(2 \cdot k \cdot k)}$$

Use the fundamental property.

$$\frac{30}{72} = \frac{5}{2 \cdot 2 \cdot 3} = \frac{5}{12} \qquad \frac{14k^2}{2k^3} = \frac{7}{k}$$

NOW TRY

### Writing a Rational Expression in Lowest Terms

***Step 1*** **Factor** the numerator and denominator completely.

***Step 2*** **Use the fundamental property** to divide out any common factors.

## EXAMPLE 4 Writing in Lowest Terms

Write each rational expression in lowest terms.

**(a)** $\dfrac{3x - 12}{5x - 20}$

$x \neq 4$, since the denominator is 0 for this value.

$$= \frac{3(x-4)}{5(x-4)} \quad \text{Factor. (Step 1)}$$

$$= \frac{3}{5} \quad \text{Fundamental property (Step 2)}$$

The given expression is equal to $\frac{3}{5}$ for all values of $x$, where $x \neq 4$ (since the denominator of the original rational expression is 0 when $x$ is 4).

**NOW TRY ANSWER**

**3.** $3y^3$

**NOW TRY EXERCISE 4**

Write each rational expression in lowest terms.

**(a)** $\dfrac{3x+15}{5x+25}$

**(b)** $\dfrac{k^2-36}{k^2+8k+12}$

**(b)** $\dfrac{2y^2-8}{2y+4}$

$y \neq -2$, since the denominator is 0 for this value.

$$
\begin{aligned}
&= \frac{2(y^2-4)}{2(y+2)} && \text{Factor. (Step 1)}\\
&= \frac{2(y+2)(y-2)}{2(y+2)} && \text{Factor the numerator completely.}\\
&= y-2 && \text{Fundamental property (Step 2)}
\end{aligned}
$$

**(c)** $\dfrac{m^2+2m-8}{2m^2-m-6}$

$m \neq -\frac{3}{2},\ m \neq 2$

$$
\begin{aligned}
&= \frac{(m+4)(m-2)}{(2m+3)(m-2)} && \text{Factor. (Step 1)}\\
&= \frac{m+4}{2m+3} && \text{Fundamental property (Step 2)}
\end{aligned}
$$

NOW TRY

***We write statements of equality of rational expressions with the understanding that they apply only to real numbers that make neither denominator equal to 0.***

**CAUTION** ***Rational expressions cannot be written in lowest terms until after the numerator and denominator have been factored.***

$$
\frac{6x+9}{4x+6} = \frac{3(2x+3)}{2\underset{\substack{\uparrow\\ \text{Divide out the}\\ \text{common factor.}}}{(2x+3)}} = \frac{3}{2}
\qquad\Bigg|\qquad
\frac{6+x}{\underbrace{4x}_{\text{Already in lowest terms}}} \leftarrow \text{Numerator cannot be factored.}
$$

**NOW TRY EXERCISE 5**

Write in lowest terms.

$$\frac{10-a^2}{a^2-10}$$

## EXAMPLE 5 Writing in Lowest Terms (Factors Are Opposites)

Write $\frac{x-y}{y-x}$ in lowest terms.

To get a common factor, the denominator $y-x$ can be factored as follows.

$$y-x$$

We are factoring out $-1$, **NOT** multiplying by it.

$$
\begin{aligned}
&= -1(-y+x) && \text{Factor out } -1.\\
&= -1(x-y) && \text{Commutative property}
\end{aligned}
$$

With this result in mind, we simplify.

$$
\begin{aligned}
&\frac{x-y}{y-x}\\
&= \frac{1(x-y)}{-1(x-y)} && y-x = -1(x-y) \text{ from above.}\\
&= \frac{1}{-1}, \quad \text{or} \quad -1 && \text{Fundamental property}
\end{aligned}
$$

NOW TRY

**NOTE** The numerator *or* the denominator could have been factored in the first step in **Example 5.** Factor $-1$ from the numerator, and confirm that the result is the same.

**NOW TRY ANSWERS**

**4.** **(a)** $\frac{3}{5}$ **(b)** $\frac{k-6}{k+2}$

**5.** $-1$

In **Example 5,** notice that $y - x$ is the **opposite** (or **additive inverse**) of $x - y$.

### Quotient of Opposites

If the numerator and the denominator of a rational expression are opposites, as in $\frac{x-y}{y-x}$, then the rational expression is equal to $-1$.

Based on this result, the following are true.

Numerator and denominator are opposites.

$$\frac{q-7}{7-q} = -1 \quad \text{and} \quad \frac{-5a+2b}{5a-2b} = -1$$

However, the following expression cannot be simplified further.

$$\frac{x-2}{x+2}$$

Numerator and denominator are *not* opposites.

**NOW TRY EXERCISE 6**

Write each rational expression in lowest terms.

**(a)** $\frac{p-4}{4-p}$ **(b)** $\frac{4m^2-n^2}{2n-4m}$

**(c)** $\frac{x+y}{x-y}$

### EXAMPLE 6 Writing in Lowest Terms (Factors Are Opposites)

Write each rational expression in lowest terms.

**(a)** $\frac{2-m}{m-2}$ Since $2 - m$ and $m - 2$ are opposites, this expression equals $-1$.

**(b)** $\frac{4x^2-9}{6-4x}$

$$= \frac{(2x+3)(2x-3)}{2(3-2x)} \qquad \text{Factor the numerator and denominator.}$$

$$= \frac{(2x+3)(2x-3)}{2(-1)(2x-3)} \qquad \text{Write } 3-2x \text{ in the denominator as } -1(2x-3).$$

$$= \frac{2x+3}{2(-1)} \qquad \text{Fundamental property}$$

$$= \frac{2x+3}{-2}, \quad \text{or} \quad -\frac{2x+3}{2} \qquad \frac{a}{-b} = -\frac{a}{b}$$

**(c)** $\frac{3+r}{3-r}$ $3 - r$ is *not* the opposite of $3 + r$.

This rational expression is already in lowest terms.

NOW TRY

**OBJECTIVE 4** **Recognize equivalent forms of rational expressions.** The common fraction $-\frac{5}{6}$ can also be written $\frac{-5}{6}$ and $\frac{5}{-6}$.

Consider the final rational expression from **Example 6(b).**

$$-\frac{2x+3}{2}$$

The $-$ sign representing the factor $-1$ is in front of the expression, even with the fraction bar. The factor $-1$ may instead be placed in the numerator or denominator.

Use parentheses.

$$\frac{-(2x+3)}{2} \quad \text{and} \quad \frac{2x+3}{-2}$$

**NOW TRY ANSWERS**

6. **(a)** $-1$
**(b)** $\frac{2m+n}{-2}$, or $-\frac{2m+n}{2}$
**(c)** already in lowest terms

The distributive property can also be applied.

$$\frac{-(2x+3)}{2} \quad \text{can also be written} \quad \frac{-2x-3}{2}.$$

Multiply *each* term in the numerator by −1.

---

**CAUTION** $\frac{-2x+3}{2}$ is *not* an equivalent form of $\frac{-(2x+3)}{2}$. The sign preceding 3 in the numerator of $\frac{-2x+3}{2}$ should be − rather than +. ***Be careful to apply the distributive property correctly.***

---

NOW TRY EXERCISE 7

Write four equivalent forms of the rational expression.

$$-\frac{4k-9}{k+3}$$

## EXAMPLE 7 Writing Equivalent Forms of a Rational Expression

Write four equivalent forms of the rational expression.

$$-\frac{3x+2}{x-6}$$

If we apply the negative sign to the numerator, we obtain these equivalent forms.

$$① \rightarrow \frac{-(3x+2)}{x-6} \quad \text{and, by the distributive property,} \quad \frac{-3x-2}{x-6} \leftarrow ②$$

If we apply the negative sign to the denominator, we obtain two more forms.

$$③ \rightarrow \frac{3x+2}{-(x-6)} \quad \text{or, distributing once again,} \quad \frac{3x+2}{-x+6} \leftarrow ④$$

NOW TRY

---

**CAUTION** Recall that $-\frac{5}{6} \neq \frac{-5}{-6}$. Thus, in **Example 7,** it would be incorrect to distribute the negative sign in $-\frac{3x+2}{x-6}$ to *both* the numerator *and* the denominator. (Doing this would actually lead to the *opposite* of the original expression.)

---

## CONNECTIONS

In **Section 4.7,** we used long division to find the quotient of two polynomials such as $(2x^2+5x-12) \div (2x-3)$, as shown on the left. The quotient is $x+4$. We get the same quotient by expressing the division problem as a rational expression (fraction) and writing this rational expression in lowest terms, as shown on the right.

$$\begin{array}{r} x+4 \\ 2x-3\overline{)2x^2+5x-12} \\ \underline{2x^2-3x\phantom{-12}} \\ 8x-12 \\ \underline{8x-12} \\ 0 \end{array}$$

$$\frac{2x^2+5x-12}{2x-3}$$

$$= \frac{(2x-3)(x+4)}{2x-3} \qquad \text{Factor.}$$

$$= x+4 \qquad \text{Fundamental property}$$

### For Discussion or Writing

What kind of division problem has a quotient that cannot be found by writing a fraction in lowest terms? Try using rational expressions to solve each division problem. Then use long division and compare.

**1.** $(3x^2+11x+8) \div (x+2)$

**2.** $(x^3-8) \div (x^2+2x+4)$

---

NOW TRY ANSWER

**7.** $\frac{-(4k-9)}{k+3}$, $\frac{-4k+9}{k+3}$, $\frac{4k-9}{-(k+3)}$, $\frac{4k-9}{-k-3}$

## 6.1 EXERCISES

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*Find the numerical value of each rational expression for* **(a)** $x = 2$ *and* **(b)** $x = -3$. ***See Example 1.***

◉ **1.** $\frac{3x + 1}{5x}$ **2.** $\frac{5x - 2}{4x}$ **3.** $\frac{x^2 - 4}{2x + 1}$

**4.** $\frac{2x^2 - 4x}{3x - 1}$ **5.** $\frac{(-2x)^3}{3x + 9}$ **6.** $\frac{(-3x)^2}{4x + 12}$

**7.** $\frac{7 - 3x}{3x^2 - 7x + 2}$ **8.** $\frac{5x + 2}{4x^2 - 5x - 6}$ **9.** $\frac{(x + 3)(x - 2)}{500x}$

**10.** $\frac{(x - 2)(x + 3)}{1000x}$ **11.** $\frac{x^2 - 4}{x^2 - 9}$ **12.** $\frac{x^2 - 9}{x^2 - 4}$

✎ **13.** Define *rational expression* in your own words, and give an example.

**14.** ***Concept Check*** Fill in each blank with the correct response: The rational expression $\frac{x + 5}{x - 3}$ is undefined when $x$ is _____, so $x \neq$ _____. This rational expression is equal to 0 when $x =$ _____.

✎ **15.** Why can't the denominator of a rational expression equal 0?

✎ **16.** If 2 is substituted for $x$ in the rational expression $\frac{x - 2}{x^2 - 4}$, the result is $\frac{0}{0}$. An often-heard statement is "Any number divided by itself is 1." Does this mean that this expression is equal to 1 for $x = 2$? If not, explain.

*Find any values of the variable for which each rational expression is undefined. Write answers with the symbol* $\neq$. ***See Example 2.***

**17.** $\frac{12}{5y}$ **18.** $\frac{-7}{3z}$ **19.** $\frac{x + 1}{x - 6}$ **20.** $\frac{m - 2}{m - 5}$

◉ **21.** $\frac{4x^2}{3x + 5}$ **22.** $\frac{2x^3}{3x + 4}$ **23.** $\frac{5m + 2}{m^2 + m - 6}$ **24.** $\frac{2r - 5}{r^2 - 5r + 4}$

**25.** $\frac{x^2 + 3x}{4}$ **26.** $\frac{x^2 - 4x}{6}$ **27.** $\frac{3x - 1}{x^2 + 2}$ **28.** $\frac{4q + 2}{q^2 + 9}$

**29. (a)** Identify the two *terms* in the numerator and the two *terms* in the denominator of the rational expression $\frac{x^2 + 4x}{x + 4}$.

✎ **(b)** Describe the steps you would use to write the rational expression in part (a) in lowest terms. (*Hint:* It simplifies to $x$.)

**30.** ***Concept Check*** Which one of these rational expressions can be simplified?

**A.** $\frac{x^2 + 2}{x^2}$ **B.** $\frac{x^2 + 2}{2}$ **C.** $\frac{x^2 + y^2}{y^2}$ **D.** $\frac{x^2 - 5x}{x}$

*Write each rational expression in lowest terms.* ***See Examples 3 and 4.***

◉ **31.** $\frac{18r^3}{6r}$ **32.** $\frac{27p^4}{3p}$ **33.** $\frac{4(y - 2)}{10(y - 2)}$

**34.** $\frac{15(m - 1)}{9(m - 1)}$ **35.** $\frac{(x + 1)(x - 1)}{(x + 1)^2}$ **36.** $\frac{(t + 5)(t - 3)}{(t + 5)^2}$

◉ **37.** $\frac{7m + 14}{5m + 10}$ **38.** $\frac{16x + 8}{14x + 7}$ **39.** $\frac{6m - 18}{7m - 21}$

**40.** $\frac{5r + 20}{3r + 12}$
**41.** $\frac{m^2 - n^2}{m + n}$
**42.** $\frac{a^2 - b^2}{a - b}$

**43.** $\frac{2t + 6}{t^2 - 9}$
**44.** $\frac{5s - 25}{s^2 - 25}$
**45.** $\frac{12m^2 - 3}{8m - 4}$

**46.** $\frac{20p^2 - 45}{6p - 9}$
**47.** $\frac{3m^2 - 3m}{5m - 5}$
**48.** $\frac{6t^2 - 6t}{5t - 5}$

**49.** $\frac{9r^2 - 4s^2}{9r + 6s}$
**50.** $\frac{16x^2 - 9y^2}{12x - 9y}$
**51.** $\frac{5k^2 - 13k - 6}{5k + 2}$

**52.** $\frac{7t^2 - 31t - 20}{7t + 4}$
**53.** $\frac{x^2 + 2x - 15}{x^2 + 6x + 5}$
**54.** $\frac{y^2 - 5y - 14}{y^2 + y - 2}$

**55.** $\frac{2x^2 - 3x - 5}{2x^2 - 7x + 5}$
**56.** $\frac{3x^2 + 8x + 4}{3x^2 - 4x - 4}$
**57.** $\frac{3x^3 + 13x^2 + 14x}{3x^3 - 5x^2 - 28x}$

**58.** $\frac{2x^3 + 7x^2 - 30x}{2x^3 - 11x^2 + 15x}$
**59.** $\frac{-3t + 6t^2 - 3t^3}{7t^2 - 14t^3 + 7t^4}$
**60.** $\frac{-20r - 20r^2 - 5r^3}{24r^2 + 24r^3 + 6r^4}$

*Exercises 61–82 involve factoring by grouping* ***(Section 5.1)*** *and factoring sums and differences of cubes* ***(Section 5.4).*** *Write each rational expression in lowest terms.*

**61.** $\frac{zw + 4z - 3w - 12}{zw + 4z + 5w + 20}$
**62.** $\frac{km + 4k - 4m - 16}{km + 4k + 5m + 20}$
**63.** $\frac{pr + qr + ps + qs}{pr + qr - ps - qs}$

**64.** $\frac{wt + ws + xt + xs}{wt - xs - xt + ws}$
**65.** $\frac{ac - ad + bc - bd}{ac - ad - bc + bd}$
**66.** $\frac{ac - bc - ad + bd}{ac - ad - bd + bc}$

**67.** $\frac{m^2 - n^2 - 4m - 4n}{2m - 2n - 8}$
**68.** $\frac{x^2 - y^2 - 7y - 7x}{3x - 3y - 21}$
**69.** $\frac{x^2y + y + x^2z + z}{xy + xz}$

**70.** $\frac{y^2k + pk - y^2z - pz}{yk - yz}$
**71.** $\frac{1 + p^3}{1 + p}$
**72.** $\frac{8 + x^3}{2 + x}$

**73.** $\frac{x^3 - 27}{x - 3}$
**74.** $\frac{r^3 - 1000}{r - 10}$
**75.** $\frac{b^3 - a^3}{a^2 - b^2}$

**76.** $\frac{8y^3 - 27z^3}{9z^2 - 4y^2}$
**77.** $\frac{k^3 + 8}{k^2 - 4}$
**78.** $\frac{r^3 + 27}{r^2 - 9}$

**79.** $\frac{z^3 + 27}{z^3 - 3z^2 + 9z}$
**80.** $\frac{t^3 + 64}{t^3 - 4t^2 + 16t}$

**81.** $\frac{1 - 8r^3}{8r^2 + 4r + 2}$
**82.** $\frac{8 - 27x^3}{27x^2 + 18x + 12}$

**83.** ***Concept Check*** Which two of the following rational expressions equal $-1$?

**A.** $\frac{2x + 3}{2x - 3}$ **B.** $\frac{2x - 3}{3 - 2x}$ **C.** $\frac{2x + 3}{3 + 2x}$ **D.** $\frac{2x + 3}{-2x - 3}$

**84.** ***Concept Check*** Make the correct choice for the blank: $\frac{4 - r^2}{4 + r^2}$ ______ equal to $-1$.
(is/is not)

*Write each rational expression in lowest terms.* ***See Examples 5 and 6.***

**85.** $\frac{6 - t}{t - 6}$
**86.** $\frac{2 - k}{k - 2}$
**87.** $\frac{m^2 - 1}{1 - m}$
**88.** $\frac{a^2 - b^2}{b - a}$

**89.** $\frac{q^2 - 4q}{4q - q^2}$
**90.** $\frac{z^2 - 5z}{5z - z^2}$
**91.** $\frac{p + 6}{p - 6}$
**92.** $\frac{5 - x}{5 + x}$

**93.** *Concept Check* Which one of these rational expressions is *not* equivalent to $\frac{x-3}{4-x}$?

**A.** $\frac{3-x}{x-4}$ **B.** $\frac{x+3}{4+x}$ **C.** $-\frac{3-x}{4-x}$ **D.** $-\frac{x-3}{x-4}$

**94.** *Concept Check* Make the correct choice for the blank: $\frac{5+2x}{3-x}$ and $\frac{-5-2x}{x-3}$ ________ equivalent rational expressions. (are/are not)

*Write four equivalent forms for each rational expression.* ***See Example 7.***

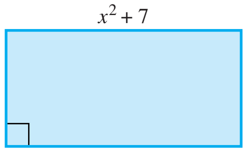
**95.** $-\frac{x+4}{x-3}$ **96.** $-\frac{x+6}{x-1}$ **97.** $-\frac{2x-3}{x+3}$

**98.** $-\frac{5x-6}{x+4}$ **99.** $-\frac{3x-1}{5x-6}$ **100.** $-\frac{2x-9}{7x-1}$

**101.** The area of the rectangle is represented by

$$x^4 + 10x^2 + 21.$$

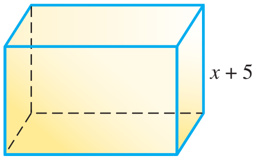
What is the width? $\left(\textit{Hint: } \text{Use } W = \frac{\mathcal{A}}{L}.\right)$



**102.** The volume of the box is represented by

$$(x^2 + 8x + 15)(x + 4).$$

Find the polynomial that represents the area of the bottom of the box.



*Solve each problem.*

**103.** The average number of vehicles waiting in line to enter a sports arena parking area is approximated by the rational expression

$$\frac{x^2}{2(1-x)},$$

where $x$ is a quantity between 0 and 1 known as the **traffic intensity.** (*Source:* Mannering, F., and W. Kilareski, *Principles of Highway Engineering and Traffic Control,* John Wiley and Sons.) To the nearest tenth, find the average number of vehicles waiting if the traffic intensity is the given number.

**(a)** 0.1 **(b)** 0.8 **(c)** 0.9

**(d)** What happens to waiting time as traffic intensity increases?

**104.** The percent of deaths caused by smoking is modeled by the rational expression

$$\frac{x-1}{x},$$

where $x$ is the number of times a smoker is more likely than a nonsmoker to die of lung cancer. This is called the **incidence rate.** (*Source:* Walker, A., *Observation and Inference: An Introduction to the Methods of Epidemiology,* Epidemiology Resources Inc.) For example, $x = 10$ means that a smoker is 10 times more likely than a nonsmoker to die of lung cancer. Find the percent of deaths if the incidence rate is the given number.

**(a)** 5 **(b)** 10 **(c)** 20

**(d)** Can the incidence rate equal 0? Explain.

## PREVIEW EXERCISES

*Multiply or divide as indicated. Write each answer in lowest terms.* ***See Section 1.1.***

**105.** $\frac{2}{3} \cdot \frac{5}{6}$ **106.** $\frac{3}{7} \cdot \frac{4}{5}$ **107.** $\frac{10}{3} \div \frac{5}{6}$ **108.** $\frac{7}{12} \div \frac{15}{4}$

# 6.2 Multiplying and Dividing Rational Expressions

**OBJECTIVES**

1. Multiply rational expressions.
2. Divide rational expressions.

**OBJECTIVE 1** **Multiply rational expressions.** The product of two fractions is found by multiplying the numerators and multiplying the denominators. Rational expressions are multiplied in the same way.

**Multiplying Rational Expressions**

The product of the rational expressions $\frac{P}{Q}$ and $\frac{R}{S}$ is defined as follows.

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

That is, to multiply rational expressions, multiply the numerators and multiply the denominators.

**NOW TRY EXERCISE 1**

Multiply. Write the answer in lowest terms.

$$\frac{4k^2}{7} \cdot \frac{14}{11k}$$

## EXAMPLE 1 Multiplying Rational Expressions

Multiply. Write each answer in lowest terms.

**(a)** $\frac{3}{10} \cdot \frac{5}{9}$

**(b)** $\frac{6}{x} \cdot \frac{x^2}{12}$

Indicate the product of the numerators and the product of the denominators.

(a) $= \frac{3 \cdot 5}{10 \cdot 9}$

(b) $= \frac{6 \cdot x^2}{x \cdot 12}$

Leave the products in factored form because common factors are needed to write the product in lowest terms. Factor the numerator and denominator to further identify any common factors. Then use the fundamental property to write each product in lowest terms.

(a)

$$= \frac{3 \cdot 5}{2 \cdot 5 \cdot 3 \cdot 3}$$

$$= \frac{1}{6}$$

Remember to write 1 in the numerator.

(b)

$$= \frac{6 \cdot x \cdot x}{2 \cdot 6 \cdot x}$$

$$= \frac{x}{2}$$

NOW TRY

**NOTE** It is also possible to divide out common factors in the numerator and denominator *before* multiplying the rational expressions. Consider this example.

$$\frac{6}{5} \cdot \frac{35}{22}$$

$$= \frac{2 \cdot 3}{5} \cdot \frac{5 \cdot 7}{2 \cdot 11} \quad \text{Identify the common factors.}$$

$$= \frac{3}{1} \cdot \frac{7}{11} \quad \text{Divide out the common factors.}$$

$$= \frac{21}{11} \quad \text{Multiply.}$$

**NOW TRY ANSWER**

1. $\frac{8k}{11}$

**NOW TRY EXERCISE 2**

Multiply. Write the answer in lowest terms.

$$\frac{m-3}{3m} \cdot \frac{9m^2}{8(m-3)^2}$$

## EXAMPLE 2 Multiplying Rational Expressions

Multiply. Write the answer in lowest terms.

$$\begin{aligned} &\frac{x+y}{2x} \cdot \frac{x^2}{(x+y)^2} \\ &= \frac{(x+y)x^2}{2x(x+y)^2} && \text{Multiply numerators.} \\ & && \text{Multiply denominators.} \\ &= \frac{(x+y)x \cdot x}{2x(x+y)(x+y)} && \text{Factor. Identify the common factors.} \\ &= \frac{x}{2(x+y)} && \frac{(x+y)x}{x(x+y)} = 1\text{; Write in lowest terms.} \end{aligned}$$

Use parentheses here around $x + y$.

NOW TRY

**NOW TRY EXERCISE 3**

Multiply. Write the answer in lowest terms.

$$\frac{y^2-3y-28}{y^2-9y+14} \cdot \frac{y^2-7y+10}{y^2+4y}$$

## EXAMPLE 3 Multiplying Rational Expressions

Multiply. Write the answer in lowest terms.

$$\begin{aligned} &\frac{x^2+3x}{x^2-3x-4} \cdot \frac{x^2-5x+4}{x^2+2x-3} \\ &= \frac{(x^2+3x)(x^2-5x+4)}{(x^2-3x-4)(x^2+2x-3)} && \text{Definition of multiplication} \\ &= \frac{x(x+3)(x-4)(x-1)}{(x-4)(x+1)(x+3)(x-1)} && \text{Factor.} \\ &= \frac{x}{x+1} && \text{Divide out the common factors.} \end{aligned}$$

The quotients $\frac{x+3}{x+3}$, $\frac{x-4}{x-4}$, and $\frac{x-1}{x-1}$ all equal 1, justifying the final product $\frac{x}{x+1}$.

NOW TRY

**OBJECTIVE 2** **Divide rational expressions.** Suppose we have $\frac{7}{8}$ gal of milk and want to find how many quarts we have. Since 1 qt is $\frac{1}{4}$ gal, we ask, "How many $\frac{1}{4}$s are there in $\frac{7}{8}$?" This would be interpreted as follows.

$$\frac{7}{8} \div \frac{1}{4}, \quad \text{or} \quad \frac{\frac{7}{8}}{\frac{1}{4}} \leftarrow \text{The fraction bar means division.}$$

The fundamental property of rational expressions discussed earlier can be applied to rational number values of $P$, $Q$, and $K$.

$$\frac{P}{Q} = \frac{P \cdot K}{Q \cdot K} = \frac{\frac{7}{8} \cdot 4}{\frac{1}{4} \cdot 4} = \frac{\frac{7}{8} \cdot 4}{1} = \frac{7}{8} \cdot \frac{4}{1} \qquad \text{Let } P = \tfrac{7}{8},\ Q = \tfrac{1}{4}, \text{ and } K = 4.\ \left(K \text{ is the reciprocal of } Q = \tfrac{1}{4}.\right)$$

So, to divide $\frac{7}{8}$ by $\frac{1}{4}$, we multiply $\frac{7}{8}$ by the reciprocal of $\frac{1}{4}$, namely, 4. Since $\frac{7}{8}(4) = \frac{7}{2}$, there are $\frac{7}{2}$, or $3\frac{1}{2}$, qt in $\frac{7}{8}$ gal.

**NOW TRY ANSWERS**

2. $\frac{3m}{8(m-3)}$

3. $\frac{y-5}{y}$

The preceding discussion illustrates dividing common fractions. Division of rational expressions is defined in the same way.

### Dividing Rational Expressions

If $\frac{P}{Q}$ and $\frac{R}{S}$ are any two rational expressions with $\frac{R}{S} \neq 0$, then their quotient is defined as follows.

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$$

That is, to divide one rational expression by another rational expression, multiply the first rational expression (dividend) by the reciprocal of the second rational expression (divisor).

**NOW TRY EXERCISE 4**

Divide. Write the answer in lowest terms.

$$\frac{2x-5}{3x^2} \div \frac{2x-5}{12x}$$

### EXAMPLE 4 Dividing Rational Expressions

Divide. Write each answer in lowest terms.

**(a)** $\frac{5}{8} \div \frac{7}{16}$

Multiply the dividend by the reciprocal of the divisor.

$$= \frac{5}{8} \cdot \frac{16}{7} \quad \leftarrow \text{Reciprocal of } \tfrac{7}{16}$$

$$= \frac{5 \cdot 16}{8 \cdot 7}$$

$$= \frac{5 \cdot 8 \cdot 2}{8 \cdot 7}$$

$$= \frac{10}{7}$$

**(b)** $\frac{y}{y+3} \div \frac{4y}{y+5}$

$$= \frac{y}{y+3} \cdot \frac{y+5}{4y} \quad \leftarrow \text{Reciprocal of } \tfrac{4y}{y+5}$$

$$= \frac{y(y+5)}{(y+3)(4y)}$$

$$= \frac{y+5}{4(y+3)}$$

NOW TRY

**NOW TRY EXERCISE 5**

Divide. Write the answer in lowest terms.

$$\frac{(3k)^3}{2j^4} \div \frac{9k^2}{6j}$$

### EXAMPLE 5 Dividing Rational Expressions

Divide. Write the answer in lowest terms.

$$\frac{(3m)^2}{(2p)^3} \div \frac{6m^3}{16p^2}$$

$$= \frac{(3m)^2}{(2p)^3} \cdot \frac{16p^2}{6m^3} \qquad \text{Multiply by the reciprocal.}$$

$(3m)^2 = 3^2m^2$; $(2p)^3 = 2^3p^3$

$$= \frac{9m^2}{8p^3} \cdot \frac{16p^2}{6m^3} \qquad \text{Power rule for exponents}$$

$$= \frac{9 \cdot 16m^2p^2}{8 \cdot 6p^3m^3} \qquad \begin{array}{l}\text{Multiply numerators.}\\ \text{Multiply denominators.}\end{array}$$

$$= \frac{3}{mp} \qquad \text{Lowest terms}$$

NOW TRY

**NOW TRY ANSWERS**

**4.** $\frac{4}{x}$ **5.** $\frac{9k}{j^3}$

NOW TRY EXERCISE 6

Divide. Write the answer in lowest terms.

$$\frac{\dfrac{(t+2)(t-5)}{-4t}}{} \div \frac{t^2-25}{(t+5)(t+2)}$$

### EXAMPLE 6 Dividing Rational Expressions

Divide. Write the answer in lowest terms.

$$\begin{aligned}
&\frac{x^2-4}{(x+3)(x-2)} \div \frac{(x+2)(x+3)}{-2x} \\
&= \frac{x^2-4}{(x+3)(x-2)} \cdot \frac{-2x}{(x+2)(x+3)} && \text{Multiply by the reciprocal.} \\
&= \frac{-2x(x^2-4)}{(x+3)(x-2)(x+2)(x+3)} && \text{Multiply numerators. Multiply denominators.} \\
&= \frac{-2x(x+2)(x-2)}{(x+3)(x-2)(x+2)(x+3)} && \text{Factor the numerator.} \\
&= \frac{-2x}{(x+3)^2}, \quad \text{or} \quad -\frac{2x}{(x+3)^2} && \text{Lowest terms; } \tfrac{-a}{b} = -\tfrac{a}{b}
\end{aligned}$$

NOW TRY

NOW TRY EXERCISE 7

Divide. Write the answer in lowest terms.

$$\frac{7-x}{2x+6} \div \frac{x^2-49}{x^2+6x+9}$$

### EXAMPLE 7 Dividing Rational Expressions (Factors Are Opposites)

Divide. Write the answer in lowest terms.

$$\begin{aligned}
&\frac{m^2-4}{m^2-1} \div \frac{2m^2+4m}{1-m} \\
&= \frac{m^2-4}{m^2-1} \cdot \frac{1-m}{2m^2+4m} && \text{Multiply by the reciprocal.} \\
&= \frac{(m^2-4)(1-m)}{(m^2-1)(2m^2+4m)} && \text{Multiply numerators. Multiply denominators.} \\
&= \frac{(m+2)(m-2)(1-m)}{(m+1)(m-1)(2m)(m+2)} && \text{Factor; } 1-m \text{ and } m-1 \text{ are opposites.} \\
&= \frac{-1(m-2)}{2m(m+1)} && \text{From Section 6.1, } \tfrac{1-m}{m-1} = -1. \\
&= \frac{-m+2}{2m(m+1)}, \quad \text{or} \quad \frac{2-m}{2m(m+1)} && \text{Distribute } -1 \text{ in the numerator. Rewrite } -m+2 \text{ as } 2-m.
\end{aligned}$$

NOW TRY

In summary, use the following steps to multiply or divide rational expressions.

### Multiplying or Dividing Rational Expressions

***Step 1*** **Note the operation.** If the operation is division, use the definition of division to rewrite it as multiplication.

***Step 2*** **Multiply** numerators and multiply denominators.

***Step 3*** **Factor** all numerators and denominators completely.

***Step 4*** **Write in lowest terms** using the fundamental property.

***Note:  Steps 2 and 3 may be interchanged based on personal preference.***

NOW TRY ANSWERS

6. $\frac{(t+2)^2}{-4t}$, or $-\frac{(t+2)^2}{4t}$

7. $\frac{-x-3}{2(x+7)}$, or $-\frac{x+3}{2(x+7)}$

## 6.2 EXERCISES

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**1.** ***Concept Check*** Match each multiplication problem in Column I with the correct product in Column II.

| I | II |
|---|---|
| **(a)** $\frac{5x^3}{10x^4} \cdot \frac{10x^7}{4x}$ | **A.** $\frac{4}{5x^5}$ |
| **(b)** $\frac{10x^4}{5x^3} \cdot \frac{10x^7}{4x}$ | **B.** $\frac{5x^5}{4}$ |
| **(c)** $\frac{5x^3}{10x^4} \cdot \frac{4x}{10x^7}$ | **C.** $\frac{1}{5x^7}$ |
| **(d)** $\frac{10x^4}{5x^3} \cdot \frac{4x}{10x^7}$ | **D.** $5x^7$ |

**2.** ***Concept Check*** Match each division problem in Column I with the correct quotient in Column II.

| I | II |
|---|---|
| **(a)** $\frac{5x^3}{10x^4} \div \frac{10x^7}{4x}$ | **A.** $\frac{5x^5}{4}$ |
| **(b)** $\frac{10x^4}{5x^3} \div \frac{10x^7}{4x}$ | **B.** $5x^7$ |
| **(c)** $\frac{5x^3}{10x^4} \div \frac{4x}{10x^7}$ | **C.** $\frac{4}{5x^5}$ |
| **(d)** $\frac{10x^4}{5x^3} \div \frac{4x}{10x^7}$ | **D.** $\frac{1}{5x^7}$ |

*Multiply. Write each answer in lowest terms.* ***See Examples 1 and 2.***

◉ **3.** $\frac{15a^2}{14} \cdot \frac{7}{5a}$

**4.** $\frac{21b^6}{18} \cdot \frac{9}{7b^4}$

**5.** $\frac{12x^4}{18x^3} \cdot \frac{-8x^5}{4x^2}$

**6.** $\frac{12m^5}{-2m^2} \cdot \frac{6m^6}{28m^3}$

**7.** $\frac{2(c+d)}{3} \cdot \frac{18}{6(c+d)^2}$

**8.** $\frac{4(y-2)}{x} \cdot \frac{3x}{6(y-2)^2}$

◉ **9.** $\frac{(x-y)^2}{2} \cdot \frac{24}{3(x-y)}$

**10.** $\frac{(a+b)^2}{5} \cdot \frac{30}{2(a+b)}$

**11.** $\frac{t-4}{8} \cdot \frac{4t^2}{t-4}$

**12.** $\frac{z+9}{12} \cdot \frac{3z^2}{z+9}$

**13.** $\frac{3x}{x+3} \cdot \frac{(x+3)^2}{6x^2}$

**14.** $\frac{(t-2)^2}{4t^2} \cdot \frac{2t}{t-2}$

*Divide. Write each answer in lowest terms.* ***See Examples 4 and 5.***

**15.** $\frac{9z^4}{3z^5} \div \frac{3z^2}{5z^3}$

**16.** $\frac{35x^8}{7x^9} \div \frac{5x^5}{9x^6}$

◉ **17.** $\frac{4t^4}{2t^5} \div \frac{(2t)^3}{-6}$

**18.** $\frac{-12a^6}{3a^2} \div \frac{(2a)^3}{27a}$

◉ **19.** $\frac{3}{2y-6} \div \frac{6}{y-3}$

**20.** $\frac{4m+16}{10} \div \frac{3m+12}{18}$

**21.** $\frac{7t+7}{-6} \div \frac{4t+4}{15}$

**22.** $\frac{8z-16}{-20} \div \frac{3z-6}{40}$

**23.** $\frac{2x}{x-1} \div \frac{x^2}{x+2}$

**24.** $\frac{y^2}{y+1} \div \frac{3y}{y-3}$

**25.** $\frac{(x-3)^2}{6x} \div \frac{x-3}{x^2}$

**26.** $\frac{2a}{a+4} \div \frac{a^2}{(a+4)^2}$

*Multiply or divide. Write each answer in lowest terms.* ***See Examples 3, 6, and 7.***

**27.** $\frac{5x-15}{3x+9} \cdot \frac{4x+12}{6x-18}$

**28.** $\frac{8r+16}{24r-24} \cdot \frac{6r-6}{3r+6}$

**29.** $\frac{2-t}{8} \div \frac{t-2}{6}$

**30.** $\frac{m-2}{4} \div \frac{2-m}{6}$

**31.** $\frac{27-3z}{4} \cdot \frac{12}{2z-18}$

**32.** $\frac{35-5x}{6} \cdot \frac{12}{3x-21}$

◉ **33.** $\frac{p^2+4p-5}{p^2+7p+10} \div \frac{p-1}{p+4}$

**34.** $\frac{z^2-3z+2}{z^2+4z+3} \div \frac{z-1}{z+1}$

◉ **35.** $\frac{m^2-4}{16-8m} \div \frac{m+2}{8}$

**36.** $\frac{r^2-36}{54-9r} \div \frac{r+6}{9}$

**37.** $\frac{2x^2-7x+3}{x-3} \cdot \frac{x+2}{x-1}$

**38.** $\frac{3x^2-5x-2}{x-2} \cdot \frac{x-3}{x+1}$

**39.** $\frac{2k^2-k-1}{2k^2+5k+3} \div \frac{4k^2-1}{2k^2+k-3}$

**40.** $\frac{3t^2-4t-4}{3t^2+10t+8} \div \frac{9t^2+21t+10}{3t^2+14t+15}$

**41.** $\dfrac{2k^2+3k-2}{6k^2-7k+2}\cdot\dfrac{4k^2-5k+1}{k^2+k-2}$

**42.** $\dfrac{2m^2-5m-12}{m^2-10m+24}\cdot\dfrac{m^2-9m+18}{4m^2-9}$

**43.** $\dfrac{m^2+2mp-3p^2}{m^2-3mp+2p^2}\div\dfrac{m^2+4mp+3p^2}{m^2+2mp-8p^2}$

**44.** $\dfrac{r^2+rs-12s^2}{r^2-rs-20s^2}\div\dfrac{r^2-2rs-3s^2}{r^2+rs-30s^2}$

**45.** $\dfrac{m^2+3m+2}{m^2+5m+4}\cdot\dfrac{m^2+10m+24}{m^2+5m+6}$

**46.** $\dfrac{z^2-z-6}{z^2-2z-8}\cdot\dfrac{z^2+7z+12}{z^2-9}$

**47.** $\dfrac{y^2+y-2}{y^2+3y-4}\div\dfrac{y+2}{y+3}$

**48.** $\dfrac{r^2+r-6}{r^2+4r-12}\div\dfrac{r+3}{r-1}$

**49.** $\dfrac{2m^2+7m+3}{m^2-9}\cdot\dfrac{m^2-3m}{2m^2+11m+5}$

**50.** $\dfrac{6s^2+17s+10}{s^2-4}\cdot\dfrac{s^2-2s}{6s^2+29s+20}$

**51.** $\dfrac{r^2+rs-12s^2}{r^2-rs-20s^2}\div\dfrac{r^2-2rs-3s^2}{r^2+rs-30s^2}$

**52.** $\dfrac{m^2+8mn+7n^2}{m^2+mn-42n^2}\div\dfrac{m^2-3mn-4n^2}{m^2-mn-30n^2}$

**53.** $\dfrac{(q-3)^4(q+2)}{q^2+3q+2}\div\dfrac{q^2-6q+9}{q^2+4q+4}$

**54.** $\dfrac{(x+4)^3(x-3)}{x^2-9}\div\dfrac{x^2+8x+16}{x^2+6x+9}$

***Brain Busters*** *Exercises 55–60 involve grouping symbols* ***(Section 1.2),*** *factoring by grouping* ***(Section 5.1),*** *and factoring sums and differences of cubes* ***(Section 5.4).*** *Multiply or divide as indicated. Write each answer in lowest terms.*

**55.** $\dfrac{x+5}{x+10}\div\left(\dfrac{x^2+10x+25}{x^2+10x}\cdot\dfrac{10x}{x^2+15x+50}\right)$

**56.** $\dfrac{m-8}{m-4}\div\left(\dfrac{m^2-12m+32}{8m}\cdot\dfrac{m^2-8m}{m^2-8m+16}\right)$

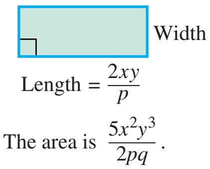
**57.** $\dfrac{3a-3b-a^2+b^2}{4a^2-4ab+b^2}\cdot\dfrac{4a^2-b^2}{2a^2-ab-b^2}$

**58.** $\dfrac{4r^2-t^2+10r-5t}{2r^2+rt+5r}\cdot\dfrac{4r^3+4r^2t+rt^2}{2r+t}$

**59.** $\dfrac{-x^3-y^3}{x^2-2xy+y^2}\div\dfrac{3y^2-3xy}{x^2-y^2}$

**60.** $\dfrac{b^3-8a^3}{4a^3+4a^2b+ab^2}\div\dfrac{4a^2+2ab+b^2}{-a^3-ab^3}$

**61.** If the rational expression $\frac{5x^2y^3}{2pq}$ represents the area of a rectangle and $\frac{2xy}{p}$ represents the length, what rational expression represents the width?



**62.** *Concept Check* If you are given the following problem, what must be the polynomial that is represented by the question mark?

$$\frac{4y+12}{2y-10}\div\frac{?}{y^2-y-20}=\frac{2(y+4)}{y-3}$$

## PREVIEW EXERCISES

*Write the prime factored form of each number.* ***See Section 1.1.***

**63.** 18 **64.** 48 **65.** 108 **66.** 60

*Find the greatest common factor of each group of terms.* ***See Section 5.1.***

**67.** $24m$, $18m^2$, 6 **68.** $14t^2$, $28t$, 7 **69.** $84q^3$, $90q^6$ **70.** $54k^3$, $36k^4$

# 6.3 Least Common Denominators

**OBJECTIVES**

1. Find the least common denominator for a group of fractions.
2. Write equivalent rational expressions.

**OBJECTIVE 1 Find the least common denominator for a group of fractions.** Adding or subtracting rational expressions often requires a **least common denominator (LCD).** The LCD is the simplest expression that is divisible by all of the denominators in all of the expressions. For example, the fractions

$$\frac{2}{9} \quad \text{and} \quad \frac{5}{12} \quad \text{have LCD 36,}$$

because 36 is the least positive number divisible by both 9 and 12.

We can often find least common denominators by inspection. In other cases, we find the LCD by a procedure similar to that used in **Section 5.1** for finding the greatest common factor.

**Finding the Least Common Denominator (LCD)**

***Step 1*** **Factor** each denominator into prime factors.

***Step 2*** **List each different denominator factor** the *greatest* number of times it appears in any of the denominators.

***Step 3*** **Multiply** the denominator factors from Step 2 to get the LCD.

When each denominator is factored into prime factors, every prime factor must be a factor of the least common denominator.

*NOW TRY EXERCISE 1*

Find the LCD for each pair of fractions.

**(a)** $\frac{5}{48}, \frac{1}{30}$ **(b)** $\frac{3}{10y}, \frac{1}{6y}$

**EXAMPLE 1** Finding the LCD

Find the LCD for each pair of fractions.

**(a)** $\frac{1}{24}, \frac{7}{15}$ **(b)** $\frac{1}{8x}, \frac{3}{10x}$

***Step 1*** Write each denominator in factored form with numerical coefficients in prime factored form.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$
$$15 = 3 \cdot 5$$

$$8x = 2 \cdot 2 \cdot 2 \cdot x = 2^3 \cdot x$$
$$10x = 2 \cdot 5 \cdot x$$

***Step 2*** Find the LCD by taking each different factor the *greatest* number of times it appears as a factor in any of the denominators.

The factor 2 appears three times in one product and not at all in the other, so the greatest number of times 2 appears is three. The greatest number of times both 3 and 5 appear is one.

Here, 2 appears three times in one product and once in the other, so the greatest number of times 2 appears is three. The greatest number of times 5 appears is one, and the greatest number of times $x$ appears in either product is one.

***Step 3***

$$\begin{aligned} \text{LCD} &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \\ &= 2^3 \cdot 3 \cdot 5 \\ &= 120 \end{aligned}$$

$$\begin{aligned} \text{LCD} &= 2 \cdot 2 \cdot 2 \cdot 5 \cdot x \\ &= 2^3 \cdot 5 \cdot x \\ &= 40x \end{aligned}$$

*NOW TRY*

*NOW TRY ANSWERS*

1. **(a)** 240 **(b)** $30y$

**NOW TRY EXERCISE 2**

Find the LCD for the pair of fractions.

$$\frac{5}{6x^4} \quad \text{and} \quad \frac{7}{8x^3}$$

## EXAMPLE 2 Finding the LCD

Find the LCD for $\frac{5}{6r^2}$ and $\frac{3}{4r^3}$.

***Step 1*** Factor each denominator.

$$6r^2 = 2 \cdot 3 \cdot r^2$$

$$4r^3 = 2 \cdot 2 \cdot r^3 = 2^2 \cdot r^3$$

***Step 2*** The greatest number of times 2 appears is two, the greatest number of times 3 appears is one, and the greatest number of times $r$ appears is three.

***Step 3*** $\text{LCD} = 2^2 \cdot 3 \cdot r^3 = 12r^3$

NOW TRY

**CAUTION** When finding the LCD, use each factor the *greatest* number of times it appears in any *single* denominator, not the *total* number of times it appears. For instance, the greatest number of times $r$ appears as a factor in one denominator in **Example 2** is 3, *not* 5.

**NOW TRY EXERCISE 3**

Find the LCD for the fractions in each list.

**(a)** $\frac{3t}{2t^2 - 10t}, \frac{t + 4}{t^2 - 25}$

**(b)** $\frac{1}{x^2 + 7x + 12}, \frac{2}{x^2 + 6x + 9}, \frac{5}{x^2 + 2x - 8}$

**(c)** $\frac{2}{a - 4}, \frac{1}{4 - a}$

## EXAMPLE 3 Finding LCDs

Find the LCD for the fractions in each list.

**(a)** $\frac{6}{5m}, \frac{4}{m^2 - 3m}$

$$\left.\begin{aligned} 5m &= 5 \cdot m \\ m^2 - 3m &= m(m - 3) \end{aligned}\right\} \text{Factor each denominator.}$$

Use each different factor the greatest number of times it appears.

$$\text{LCD} = 5 \cdot m \cdot (m - 3) = 5m(m - 3)$$

Be sure to include $m$ as a factor in the LCD.

Because $m$ is not a *factor* of $m - 3$, both $m$ and $m - 3$ must appear in the LCD.

**(b)** $\frac{1}{r^2 - 4r - 5}, \frac{3}{r^2 - r - 20}, \frac{1}{r^2 - 10r + 25}$

$$\left.\begin{aligned} r^2 - 4r - 5 &= (r - 5)(r + 1) \\ r^2 - r - 20 &= (r - 5)(r + 4) \\ r^2 - 10r + 25 &= (r - 5)^2 \end{aligned}\right\} \text{Factor each denominator.}$$

Use each different factor the greatest number of times it appears as a factor.

$$\text{LCD} = (r - 5)^2(r + 1)(r + 4)$$

Be sure to include the exponent 2.

**(c)** $\frac{1}{q - 5}, \frac{3}{5 - q}$

The expressions $q - 5$ and $5 - q$ are opposites of each other. This means that if we multiply $q - 5$ by $-1$, we will get $5 - q$.

$$-(q - 5) = -q + 5 = 5 - q$$

Therefore, either $q - 5$ or $5 - q$ can be used as the LCD.

NOW TRY

**NOW TRY ANSWERS**

**2.** $24x^4$

**3. (a)** $2t(t - 5)(t + 5)$

**(b)** $(x + 3)^2(x + 4)(x - 2)$

**(c)** either $a - 4$ or $4 - a$

**OBJECTIVE 2** **Write equivalent rational expressions.** Once the LCD has been found, the next step in preparing to add or subtract two rational expressions is to use the fundamental property to write equivalent rational expressions.

### Writing a Rational Expression with a Specified Denominator

***Step 1*** **Factor** both denominators.

***Step 2*** **Decide what factor(s) the denominator must be multiplied by** in order to equal the specified denominator.

***Step 3*** **Multiply** the rational expression by that factor divided by itself. (That is, multiply by 1.)

**NOW TRY EXERCISE 4**

Rewrite each rational expression with the indicated denominator.

**(a)** $\frac{2}{9} = \frac{?}{27}$ **(b)** $\frac{4t}{11} = \frac{?}{33t}$

### EXAMPLE 4 Writing Equivalent Rational Expressions

Rewrite each rational expression with the indicated denominator.

**(a)** $\frac{3}{8} = \frac{?}{40}$

**(b)** $\frac{9k}{25} = \frac{?}{50k}$

***Step 1*** For each example, first factor the denominator on the right. Then compare the denominator on the left with the one on the right to decide what factors are missing. (It may sometimes be necessary to factor both denominators.)

(a)

$$\frac{3}{8} = \frac{?}{5 \cdot 8}$$

***Step 2*** A factor of 5 is missing.

***Step 3*** Multiply $\frac{3}{8}$ by $\frac{5}{5}$.

$$\frac{3}{8} = \frac{3}{8} \cdot \frac{5}{5} = \frac{15}{40}$$

$\frac{5}{5} = 1$

(b)

$$\frac{9k}{25} = \frac{?}{25 \cdot 2k}$$

Factors of 2 and $k$ are missing.

Multiply $\frac{9k}{25}$ by $\frac{2k}{2k}$.

$$\frac{9k}{25} = \frac{9k}{25} \cdot \frac{2k}{2k} = \frac{18k^2}{50k}$$

$\frac{2k}{2k} = 1$

NOW TRY

### EXAMPLE 5 Writing Equivalent Rational Expressions

Rewrite each rational expression with the indicated denominator.

**(a)**

$$\frac{8}{3x+1} = \frac{?}{12x+4}$$

$$\frac{8}{3x+1} = \frac{?}{4(3x+1)}$$

Factor the denominator on the right.

The missing factor is 4, so multiply the fraction on the left by $\frac{4}{4}$.

$$\frac{8}{3x+1} \cdot \frac{4}{4} = \frac{32}{12x+4}$$

Fundamental property

**NOW TRY ANSWERS**

**4.** **(a)** $\frac{6}{27}$ **(b)** $\frac{12t^2}{33t}$

**NOW TRY EXERCISE 5**

Rewrite each rational expression with the indicated denominator.

**(a)** $\frac{8k}{5k-2} = \frac{?}{25k-10}$

**(b)** $\frac{2t-1}{t^2+4t} = \frac{?}{t^3+12t^2+32t}$

**(b)** $\frac{12p}{p^2+8p} = \frac{?}{p^3+4p^2-32p}$

Factor the denominator in each rational expression.

$$\frac{12p}{p(p+8)} = \frac{?}{p(p+8)(p-4)}$$

$p^3+4p^2-32p$
$= p(p^2+4p-32)$
$= p(p+8)(p-4)$

The factor $p-4$ is missing, so multiply $\frac{12p}{p(p+8)}$ by $\frac{p-4}{p-4}$.

$$\frac{12p}{p^2+8p} = \frac{12p}{p(p+8)} \cdot \frac{p-4}{p-4} \qquad \text{Fundamental property}$$

$$= \frac{12p(p-4)}{p(p+8)(p-4)} \qquad \text{Multiply numerators. Multiply denominators.}$$

$$= \frac{12p^2-48p}{p^3+4p^2-32p} \qquad \text{Multiply the factors.}$$

NOW TRY

**NOW TRY ANSWERS**

5. **(a)** $\frac{40k}{25k-10}$
**(b)** $\frac{2t^2+15t-8}{t^3+12t^2+32t}$

**NOTE** While it is beneficial to leave the denominator in factored form, we multiplied the factors in the denominator in **Example 5** to give the answer in the same form as the original problem.

## 6.3 EXERCISES

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***Concept Check*** *Choose the correct response in Exercises 1–4.*

**1.** Suppose that the greatest common factor of $x$ and $y$ is 1. What is the least common denominator for $\frac{1}{x}$ and $\frac{1}{y}$?

**A.** $x$ **B.** $y$ **C.** $xy$ **D.** 1

**2.** If $x$ is a factor of $y$, what is the least common denominator for $\frac{1}{x}$ and $\frac{1}{y}$?

**A.** $x$ **B.** $y$ **C.** $xy$ **D.** 1

**3.** What is the least common denominator for $\frac{9}{20}$ and $\frac{1}{2}$?

**A.** 40 **B.** 2 **C.** 20 **D.** none of these

**4.** Suppose that we wish to write the fraction $\frac{1}{(x-4)^2(y-3)}$ with denominator $(x-4)^3(y-3)^2$. By what must we multiply both the numerator and the denominator?

**A.** $(x-4)(y-3)$ **B.** $(x-4)^2$ **C.** $x-4$ **D.** $(x-4)^2(y-3)$

*Find the LCD for the fractions in each list.* ***See Examples 1–3.***

◉ **5.** $\frac{7}{15}, \frac{21}{20}$

**6.** $\frac{9}{10}, \frac{13}{25}$

**7.** $\frac{17}{100}, \frac{23}{120}, \frac{43}{180}$

**8.** $\frac{17}{250}, \frac{21}{300}, \frac{1}{360}$

**9.** $\frac{9}{x^2}, \frac{8}{x^5}$

**10.** $\frac{12}{m^7}, \frac{14}{m^8}$

**11.** $\frac{-2}{5p}, \frac{13}{6p}$

**12.** $\frac{-14}{15k}, \frac{11}{4k}$

◉ **13.** $\frac{17}{15y^2}, \frac{55}{36y^4}$

**14.** $\frac{4}{25m^3}, \frac{7}{10m^4}$

**15.** $\frac{5}{21r^3}, \frac{7}{12r^5}$

**16.** $\frac{6}{35t^2}, \frac{5}{49t^6}$

**17.** $\frac{13}{5a^2b^3}, \frac{29}{15a^5b}$

**18.** $\frac{7}{3r^4s^5}, \frac{23}{9r^6s^8}$

**19.** $\frac{7}{6p}, \frac{15}{4p - 8}$

**20.** $\frac{7}{8k}, \frac{28}{12k - 24}$

**21.** $\frac{9}{28m^2}, \frac{3}{12m - 20}$

**22.** $\frac{14}{27a^3}, \frac{7}{9a - 45}$

**23.** $\frac{7}{5b - 10}, \frac{11}{6b - 12}$

**24.** $\frac{3}{7x^2 + 21x}, \frac{2}{5x^2 + 15x}$

**25.** $\frac{37}{6r - 12}, \frac{25}{9r - 18}$

**26.** $\frac{14}{5p - 30}, \frac{11}{6p - 36}$

**27.** $\frac{5}{12p + 60}, \frac{-17}{p^2 + 5p}, \frac{-16}{p^2 + 10p + 25}$

**28.** $\frac{13}{r^2 + 7r}, \frac{-3}{5r + 35}, \frac{-4}{r^2 + 14r + 49}$

**29.** $\frac{-3}{8y + 16}, \frac{-22}{y^2 + 3y + 2}$

**30.** $\frac{-2}{9m - 18}, \frac{-6}{m^2 - 7m + 10}$

**31.** $\frac{5}{c - d}, \frac{8}{d - c}$

**32.** $\frac{4}{y - x}, \frac{8}{x - y}$

**33.** $\frac{12}{m - 3}, \frac{-4}{3 - m}$

**34.** $\frac{3}{a - 8}, \frac{-17}{8 - a}$

**35.** $\frac{29}{p - q}, \frac{18}{q - p}$

**36.** $\frac{16}{z - x}, \frac{9}{x - z}$

**37.** $\frac{3}{k^2 + 5k}, \frac{2}{k^2 + 3k - 10}$

**38.** $\frac{1}{z^2 - 4z}, \frac{9}{z^2 - 3z - 4}$

**39.** $\frac{6}{a^2 + 6a}, \frac{-5}{a^2 + 3a - 18}$

**40.** $\frac{8}{y^2 - 5y}, \frac{-5}{y^2 - 2y - 15}$

**41.** $\frac{5}{p^2 + 8p + 15}, \frac{3}{p^2 - 3p - 18}, \frac{12}{p^2 - p - 30}$

**42.** $\frac{10}{y^2 - 10y + 21}, \frac{2}{y^2 - 2y - 3}, \frac{15}{y^2 - 6y - 7}$

**43.** $\frac{-5}{k^2 + 2k - 35}, \frac{-8}{k^2 + 3k - 40}, \frac{19}{k^2 - 2k - 15}$

**44.** $\frac{-19}{z^2 + 4z - 12}, \frac{-16}{z^2 + z - 30}, \frac{16}{z^2 + 2z - 24}$

## RELATING CONCEPTS EXERCISES 45–50

### FOR INDIVIDUAL OR GROUP WORK

***Work Exercises 45–50 in order.***

**45.** Suppose that you want to write $\frac{3}{4}$ as an equivalent fraction with denominator 28. By what number must you multiply both the numerator and the denominator?

**46.** If you write $\frac{3}{4}$ as an equivalent fraction with denominator 28, by what number are you actually multiplying the fraction?

**47.** What property of multiplication is being used when we write a common fraction as an equivalent one with a larger denominator? (See **Section 1.7.**)

**48.** Suppose that you want to write $\frac{2x + 5}{x - 4}$ as an equivalent fraction with denominator $7x - 28$. By what number must you multiply both the numerator and the denominator?

**49.** If you write $\frac{2x + 5}{x - 4}$ as an equivalent fraction with denominator $7x - 28$, by what number are you actually multiplying the fraction?

**50.** Repeat **Exercise 47,** changing "a common" to "an algebraic."

*Rewrite each rational expression with the indicated denominator.* ***See Examples 4 and 5.***

**51.** $\frac{4}{11} = \frac{?}{55}$ **52.** $\frac{8}{7} = \frac{?}{42}$ **53.** $\frac{-5}{k} = \frac{?}{9k}$

**54.** $\frac{-4}{q} = \frac{?}{6q}$ **55.** $\frac{15m^2}{8k} = \frac{?}{32k^4}$ **56.** $\frac{7t^2}{3y} = \frac{?}{9y^2}$

**57.** $\frac{19z}{2z - 6} = \frac{?}{6z - 18}$ **58.** $\frac{3r}{5r - 5} = \frac{?}{15r - 15}$

**59.** $\frac{-2a}{9a - 18} = \frac{?}{18a - 36}$ **60.** $\frac{-7y}{6y + 18} = \frac{?}{24y + 72}$

**61.** $\frac{6}{k^2 - 4k} = \frac{?}{k(k - 4)(k + 1)}$ **62.** $\frac{25}{m^2 - 9m} = \frac{?}{m(m - 9)(m + 8)}$

**63.** $\frac{36r}{r^2 - r - 6} = \frac{?}{(r - 3)(r + 2)(r + 1)}$

**64.** $\frac{4m}{m^2 + m - 2} = \frac{?}{(m - 1)(m - 3)(m + 2)}$

**65.** $\frac{a + 2b}{2a^2 + ab - b^2} = \frac{?}{2a^3b + a^2b^2 - ab^3}$

**66.** $\frac{m - 4}{6m^2 + 7m - 3} = \frac{?}{12m^3 + 14m^2 - 6m}$

**67.** $\frac{4r - t}{r^2 + rt + t^2} = \frac{?}{t^3 - r^3}$ **68.** $\frac{3x - 1}{x^2 + 2x + 4} = \frac{?}{x^3 - 8}$

**69.** $\frac{2(z - y)}{y^2 + yz + z^2} = \frac{?}{y^4 - z^3y}$ **70.** $\frac{2p + 3q}{p^2 + 2pq + q^2} = \frac{?}{(p + q)(p^3 + q^3)}$

## PREVIEW EXERCISES

*Add or subtract as indicated. Write each answer in lowest terms.* ***See Section 1.1.***

**71.** $\frac{1}{2} + \frac{7}{8}$ **72.** $\frac{2}{3} + \frac{8}{27}$ **73.** $\frac{7}{5} - \frac{3}{4}$ **74.** $\frac{11}{6} - \frac{2}{5}$

# 6.4 Adding and Subtracting Rational Expressions

**OBJECTIVES**

1. Add rational expressions having the same denominator.
2. Add rational expressions having different denominators.
3. Subtract rational expressions.

**OBJECTIVE 1** **Add rational expressions having the same denominator.** We find the sum of two rational expressions with the same denominator using the same procedure that we used in **Section 1.1** for adding two common fractions.

**Adding Rational Expressions (Same Denominator)**

The rational expressions $\frac{P}{Q}$ and $\frac{R}{Q}$ $(Q \neq 0)$ are added as follows.

$$\frac{\mathbf{P}}{\mathbf{Q}} + \frac{\mathbf{R}}{\mathbf{Q}} = \frac{\mathbf{P + R}}{\mathbf{Q}}$$

That is, to add rational expressions with the same denominator, add the numerators and keep the same denominator.

**NOW TRY EXERCISE 1**

Add. Write each answer in lowest terms.

**(a)** $\dfrac{2}{7k}+\dfrac{4}{7k}$

**(b)** $\dfrac{4y}{y+3}+\dfrac{12}{y+3}$

### EXAMPLE 1 Adding Rational Expressions (Same Denominator)

Add. Write each answer in lowest terms.

**(a)** $\dfrac{4}{9}+\dfrac{2}{9}$ **(b)** $\dfrac{3x}{x+1}+\dfrac{3}{x+1}$

The denominators are the same, so the sum is found by adding the two numerators and keeping the same (common) denominator.

**(a)**

$$=\frac{4+2}{9} \qquad \text{Add.}$$

$$=\frac{6}{9}$$

$$=\frac{2\cdot 3}{3\cdot 3} \qquad \text{Factor.}$$

$$=\frac{2}{3} \qquad \text{Lowest terms}$$

**(b)**

$$=\frac{3x+3}{x+1} \qquad \text{Add.}$$

$$=\frac{3(x+1)}{x+1} \qquad \text{Factor.}$$

$$=3 \qquad \text{Lowest terms}$$

NOW TRY

## OBJECTIVE 2 Add rational expressions having different denominators.

As in **Section 1.1,** we use the following steps to add fractions having different denominators.

**Adding Rational Expressions (Different Denominators)**

***Step 1*** **Find the least common denominator (LCD).**

***Step 2*** **Rewrite each rational expression** as an equivalent rational expression with the LCD as the denominator.

***Step 3*** **Add** the numerators to get the numerator of the sum. The LCD is the denominator of the sum.

***Step 4*** **Write in lowest terms** using the fundamental property.

### EXAMPLE 2 Adding Rational Expressions (Different Denominators)

Add. Write each answer in lowest terms.

**(a)** $\dfrac{1}{12}+\dfrac{7}{15}$ **(b)** $\dfrac{2}{3y}+\dfrac{1}{4y}$

***Step 1*** First find the LCD, using the methods of the previous section.

**(a)**

$$12=2\cdot 2\cdot 3=2^2\cdot 3$$

$$15=3\cdot 5$$

$$\text{LCD}=2^2\cdot 3\cdot 5=60$$

**(b)**

$$3y=3\cdot y$$

$$4y=2\cdot 2\cdot y=2^2\cdot y$$

$$\text{LCD}=2^2\cdot 3\cdot y=12y$$

***Step 2*** Now rewrite each rational expression as a fraction with the LCD (60 and 12$y$, respectively) as the denominator.

**(a)**

$$\frac{1}{12}+\frac{7}{15}=\frac{1(5)}{12(5)}+\frac{7(4)}{15(4)}$$

$$=\frac{5}{60}+\frac{28}{60}$$

**(b)**

$$\frac{2}{3y}+\frac{1}{4y}=\frac{2(4)}{3y(4)}+\frac{1(3)}{4y(3)}$$

$$=\frac{8}{12y}+\frac{3}{12y}$$

**NOW TRY ANSWERS**

1\. **(a)** $\frac{6}{7k}$ **(b)** 4

**NOW TRY EXERCISE 2**

Add. Write each answer in lowest terms.

**(a)** $\frac{5}{12} + \frac{3}{20}$ **(b)** $\frac{3}{5x} + \frac{2}{7x}$

***Step 3*** Add the numerators. The LCD is the denominator.

***Step 4*** Write in lowest terms if necessary.

$$= \frac{5 + 28}{60} \qquad\qquad = \frac{8 + 3}{12y}$$

$$= \frac{33}{60}, \text{ or } \frac{11}{20} \qquad\qquad = \frac{11}{12y}$$

NOW TRY

**NOW TRY EXERCISE 3**

Add. Write the answer in lowest terms.

$$\frac{6t}{t^2 - 9} + \frac{-3}{t + 3}$$

## EXAMPLE 3 Adding Rational Expressions

Add. Write the answer in lowest terms.

$$\frac{2x}{x^2 - 1} + \frac{-1}{x + 1}$$

***Step 1*** Since the denominators are different, find the LCD.

$$\left.\begin{aligned} &x^2 - 1 = (x + 1)(x - 1) \\ &x + 1 \text{ is prime.} \end{aligned}\right\} \text{ The LCD is } (x + 1)(x - 1).$$

***Step 2*** Rewrite each rational expression with the LCD as the denominator.

$$\frac{2x}{x^2 - 1} + \frac{-1}{x + 1} \qquad \text{LCD} = (x + 1)(x - 1)$$

$$= \frac{2x}{(x + 1)(x - 1)} + \frac{-1(x - 1)}{(x + 1)(x - 1)} \qquad \text{Multiply the second fraction by } \tfrac{x - 1}{x - 1}.$$

$$= \frac{2x}{(x + 1)(x - 1)} + \frac{-x + 1}{(x + 1)(x - 1)} \qquad \text{Distributive property}$$

***Step 3***
$$= \frac{2x - x + 1}{(x + 1)(x - 1)} \qquad \text{Add numerators. Keep the same denominator.}$$

$$= \frac{x + 1}{(x + 1)(x - 1)} \qquad \text{Combine like terms.}$$

***Step 4***
$$= \frac{1(x + 1)}{(x + 1)(x - 1)} \qquad \text{Identity property of multiplication}$$

$$= \frac{1}{x - 1} \qquad \text{Divide out the common factors.}$$

Remember to write 1 in the numerator.

NOW TRY

## EXAMPLE 4 Adding Rational Expressions

Add. Write the answer in lowest terms.

$$\frac{2x}{x^2 + 5x + 6} + \frac{x + 1}{x^2 + 2x - 3}$$

$$= \frac{2x}{(x + 2)(x + 3)} + \frac{x + 1}{(x + 3)(x - 1)} \qquad \text{Factor the denominators.}$$

$$= \frac{2x(x - 1)}{(x + 2)(x + 3)(x - 1)} + \frac{(x + 1)(x + 2)}{(x + 2)(x + 3)(x - 1)} \qquad \text{The LCD is } (x + 2)(x + 3)(x - 1).$$

**NOW TRY ANSWERS**

**2.** **(a)** $\frac{17}{30}$ **(b)** $\frac{31}{35x}$

**3.** $\frac{3}{t - 3}$

**NOW TRY EXERCISE 4**

Add. Write the answer in lowest terms.

$$\frac{x-1}{x^2+6x+8}+\frac{4x}{x^2+x-12}$$

$$=\frac{2x(x-1)+(x+1)(x+2)}{(x+2)(x+3)(x-1)} \qquad \text{Add numerators. Keep the same denominator.}$$

$$=\frac{2x^2-2x+x^2+3x+2}{(x+2)(x+3)(x-1)} \qquad \text{Multiply.}$$

$$=\frac{3x^2+x+2}{(x+2)(x+3)(x-1)} \qquad \text{Combine like terms.}$$

The numerator cannot be factored here, so the expression is in lowest terms.

NOW TRY

**NOTE** If the final expression in **Example 4** could be written in lower terms, the numerator would have a factor of $x+2$, $x+3$, or $x-1$. Therefore, it is only necessary to check for possible factored forms of the numerator that would contain one of these binomials.

**NOW TRY EXERCISE 5**

Add. Write the answer in lowest terms.

$$\frac{2k}{k-7}+\frac{5}{7-k}$$

### EXAMPLE 5 Adding Rational Expressions (Denominators Are Opposites)

Add. Write the answer in lowest terms.

$$\frac{y}{y-2}+\frac{8}{2-y}$$

The denominators are opposites. Use the process of multiplying one of the fractions by 1 in the form $\frac{-1}{-1}$ to get the same denominator for both fractions.

$$=\frac{y}{y-2}+\frac{8(-1)}{(2-y)(-1)} \qquad \text{Multiply } \frac{8}{2-y} \text{ by } \frac{-1}{-1}.$$

$$=\frac{y}{y-2}+\frac{-8}{-2+y} \qquad \text{Distributive property}$$

$$=\frac{y}{y-2}+\frac{-8}{y-2} \qquad \text{Rewrite } -2+y \text{ as } y-2.$$

$$=\frac{y-8}{y-2} \qquad \text{Add numerators. Keep the same denominator.}$$

If we had chosen $2-y$ as the common denominator, the final answer would be $\frac{8-y}{2-y}$, which is equivalent to $\frac{y-8}{y-2}$.

NOW TRY

## OBJECTIVE 3 Subtract rational expressions.

**Subtracting Rational Expressions (Same Denominator)**

The rational expressions $\frac{P}{Q}$ and $\frac{R}{Q}$ $(Q \neq 0)$ are subtracted as follows.

$$\frac{P}{Q}-\frac{R}{Q}=\frac{P-R}{Q}$$

That is, to subtract rational expressions with the same denominator, subtract the numerators and keep the same denominator.

**NOW TRY ANSWERS**

4. $\frac{5x^2+4x+3}{(x+4)(x+2)(x-3)}$

5. $\frac{2k-5}{k-7}$, or $\frac{5-2k}{7-k}$

**NOW TRY EXERCISE 6**

Subtract. Write the answer in lowest terms.

$$\frac{2x}{x+5}-\frac{x+1}{x+5}$$

## EXAMPLE 6 Subtracting Rational Expressions (Same Denominator)

Subtract. Write the answer in lowest terms.

$$\frac{2m}{m-1}-\frac{m+3}{m-1}$$

Use parentheses around the numerator of the subtrahend.

$$=\frac{2m-(m+3)}{m-1}$$

Subtract numerators. Keep the same denominator.

Be careful with signs.

$$=\frac{2m-m-3}{m-1}$$

Distributive property

$$=\frac{m-3}{m-1}$$

Combine like terms.

NOW TRY

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**CAUTION** Sign errors often occur in subtraction problems like the one in **Example 6.** The numerator of the fraction being subtracted must be treated as a single quantity. ***Be sure to use parentheses after the subtraction symbol.***

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**NOW TRY EXERCISE 7**

Subtract. Write the answer in lowest terms.

$$\frac{6}{y-6}-\frac{2}{y}$$

## EXAMPLE 7 Subtracting Rational Expressions (Different Denominators)

Subtract. Write the answer in lowest terms.

$$\frac{9}{x-2}-\frac{3}{x}$$

The LCD is $x(x-2)$.

$$=\frac{9x}{x(x-2)}-\frac{3(x-2)}{x(x-2)}$$

Write each expression with the LCD.

$$=\frac{9x-3(x-2)}{x(x-2)}$$

Subtract numerators. Keep the same denominator.

Be careful with signs.

$$=\frac{9x-3x+6}{x(x-2)}$$

Distributive property

$$=\frac{6x+6}{x(x-2)},\quad \text{or}\quad \frac{6(x+1)}{x(x-2)}$$

Combine like terms. Factor the numerator.

NOW TRY

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**NOTE** We factored the final numerator in **Example 7** to get $\frac{6(x+1)}{x(x-2)}$. The fundamental property does not apply, however since there are no common factors to divide out. The answer is in lowest terms.

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## EXAMPLE 8 Subtracting Rational Expressions (Denominators Are Opposites)

Subtract. Write the answer in lowest terms.

$$\frac{3x}{x-5}-\frac{2x-25}{5-x}$$

The denominators are opposites. We choose $x-5$ as the common denominator.

$$=\frac{3x}{x-5}-\frac{(2x-25)(-1)}{(5-x)(-1)}$$

Multiply $\frac{2x-25}{5-x}$ by $\frac{-1}{-1}$ to get a common denominator.

$$=\frac{3x}{x-5}-\frac{-2x+25}{x-5}$$

$(5-x)(-1)=-5+x=x-5$

**NOW TRY ANSWERS**

6. $\frac{x-1}{x+5}$

7. $\frac{4(y+3)}{y(y-6)}$

**NOW TRY EXERCISE 8**

Subtract. Write the answer in lowest terms.

$$\frac{2m}{m-4}-\frac{m-12}{4-m}$$

Subtract the *entire* numerator. Use parentheses to show this.

$$=\frac{3x-(-2x+25)}{x-5} \qquad \text{Subtract numerators.}$$

Be careful with signs.

$$=\frac{3x+2x-25}{x-5} \qquad \text{Distributive property}$$

$$=\frac{5x-25}{x-5} \qquad \text{Combine like terms.}$$

$$=\frac{5(x-5)}{x-5} \qquad \text{Factor.}$$

$$=5 \qquad \text{Divide out the common factor.}$$

NOW TRY

**NOW TRY EXERCISE 9**

Subtract. Write the answer in lowest terms.

$$\frac{5}{t^2-6t+9}-\frac{2t}{t^2-9}$$

## EXAMPLE 9 Subtracting Rational Expressions

Subtract. Write the answer in lowest terms.

$$\frac{6x}{x^2-2x+1}-\frac{1}{x^2-1}$$

$$=\frac{6x}{(x-1)^2}-\frac{1}{(x-1)(x+1)}$$

Factor the denominators.
LCD $= (x-1)(x-1)(x+1)$,
or $(x-1)^2(x+1)$

$$=\frac{6x(x+1)}{(x-1)^2(x+1)}-\frac{1(x-1)}{(x-1)(x-1)(x+1)} \qquad \text{Fundamental property}$$

$$=\frac{6x(x+1)-1(x-1)}{(x-1)^2(x+1)} \qquad \text{Subtract numerators.}$$

$$=\frac{6x^2+6x-x+1}{(x-1)^2(x+1)} \qquad \text{Distributive property}$$

$$=\frac{6x^2+5x+1}{(x-1)^2(x+1)}, \quad \text{or} \quad \frac{(2x+1)(3x+1)}{(x-1)^2(x+1)}$$

Combine like terms.
Factor the numerator.

NOW TRY

**NOW TRY ANSWERS**

**8.** 3

**9.** $\frac{-2t^2+11t+15}{(t-3)^2(t+3)}$

## 6.4 EXERCISES

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***Concept Check*** *Match each expression in Column I with the correct sum or difference in Column II.*

| I | | II | |
|---|---|---|---|
| **1.** $\frac{x}{x+8}+\frac{8}{x+8}$ | **2.** $\frac{2x}{x-8}-\frac{16}{x-8}$ | **A.** 2 | **B.** $\frac{x-8}{x+8}$ |
| **3.** $\frac{8}{x-8}-\frac{x}{x-8}$ | **4.** $\frac{8}{x+8}-\frac{x}{x+8}$ | **C.** $-1$ | **D.** $\frac{8+x}{8x}$ |
| **5.** $\frac{x}{x+8}-\frac{8}{x+8}$ | **6.** $\frac{1}{x}+\frac{1}{8}$ | **E.** 1 | **F.** 0 |
| **7.** $\frac{1}{8}-\frac{1}{x}$ | **8.** $\frac{1}{8x}-\frac{1}{8x}$ | **G.** $\frac{x-8}{8x}$ | **H.** $\frac{8-x}{x+8}$ |

*Note: When adding and subtracting rational expressions, several different equivalent forms of the answer often exist. If your answer does not look exactly like the one given in the back of the book, check to see whether you have written an equivalent form.*

*Add or subtract. Write each answer in lowest terms.* ***See Examples 1 and 6.***

**9.** $\frac{4}{m}+\frac{7}{m}$

**10.** $\frac{5}{p}+\frac{12}{p}$

**11.** $\frac{5}{y+4}-\frac{1}{y+4}$

**12.** $\frac{6}{t+3}-\frac{3}{t+3}$

**13.** $\frac{x}{x+y}+\frac{y}{x+y}$

**14.** $\frac{a}{a+b}+\frac{b}{a+b}$

**15.** $\frac{5m}{m+1}-\frac{1+4m}{m+1}$

**16.** $\frac{4x}{x+2}-\frac{2+3x}{x+2}$

**17.** $\frac{a+b}{2}-\frac{a-b}{2}$

**18.** $\frac{x-y}{2}-\frac{x+y}{2}$

**19.** $\frac{x^2}{x+5}+\frac{5x}{x+5}$

**20.** $\frac{t^2}{t-3}+\frac{-3t}{t-3}$

**21.** $\frac{y^2-3y}{y+3}+\frac{-18}{y+3}$

**22.** $\frac{r^2-8r}{r-5}+\frac{15}{r-5}$

**23.** $\frac{x}{x^2-9}-\frac{-3}{x^2-9}$

**24.** $\frac{-4}{y^2-16}-\frac{-y}{y^2-16}$

*Add or subtract. Write each answer in lowest terms.* ***See Examples 2, 3, 4, and 7.***

**25.** $\frac{z}{5}+\frac{1}{3}$

**26.** $\frac{p}{8}+\frac{4}{5}$

**27.** $\frac{5}{7}-\frac{r}{2}$

**28.** $\frac{20}{9}-\frac{z}{3}$

**29.** $-\frac{3}{4}-\frac{1}{2x}$

**30.** $-\frac{7}{8}-\frac{3}{2a}$

**31.** $\frac{6}{5x}+\frac{9}{2x}$

**32.** $\frac{3}{2x}+\frac{3}{7x}$

**33.** $\frac{x+1}{6}+\frac{3x+3}{9}$

**34.** $\frac{2x-6}{4}+\frac{x+5}{6}$

**35.** $\frac{x+3}{3x}+\frac{2x+2}{4x}$

**36.** $\frac{x+2}{5x}+\frac{6x+3}{3x}$

**37.** $\frac{7}{3p^2}-\frac{2}{p}$

**38.** $\frac{12}{5m^2}-\frac{5}{m}$

**39.** $\frac{1}{k+4}-\frac{2}{k}$

**40.** $\frac{3}{m+1}-\frac{4}{m}$

**41.** $\frac{x}{x-2}+\frac{-8}{x^2-4}$

**42.** $\frac{2x}{x-1}+\frac{-4}{x^2-1}$

**43.** $\frac{4m}{m^2+3m+2}+\frac{2m-1}{m^2+6m+5}$

**44.** $\frac{a}{a^2+3a-4}+\frac{4a}{a^2+7a+12}$

**45.** $\frac{4y}{y^2-1}-\frac{5}{y^2+2y+1}$

**46.** $\frac{2x}{x^2-16}-\frac{3}{x^2+8x+16}$

**47.** $\frac{t}{t+2}+\frac{5-t}{t}-\frac{4}{t^2+2t}$

**48.** $\frac{2p}{p-3}+\frac{2+p}{p}-\frac{-6}{p^2-3p}$

**49.** ***Concept Check*** What are the two possible LCDs that could be used for the sum $\frac{10}{m-2}+\frac{5}{2-m}$?

**50.** ***Concept Check*** If one form of the correct answer to a sum or difference of rational expressions is $\frac{4}{k-3}$, what would an alternative form of the answer be if the denominator is $3-k$?

*Add or subtract. Write each answer in lowest terms.* ***See Examples 5 and 8.***

**51.** $\frac{4}{x-5}+\frac{6}{5-x}$

**52.** $\frac{10}{m-2}+\frac{5}{2-m}$

**53.** $\frac{-1}{1-y}-\frac{4y-3}{y-1}$

**54.** $\frac{-4}{p-3} - \frac{p+1}{3-p}$

**55.** $\frac{2}{x-y^2} + \frac{7}{y^2-x}$

**56.** $\frac{-8}{p-q^2} + \frac{3}{q^2-p}$

**57.** $\frac{x}{5x-3y} - \frac{y}{3y-5x}$

**58.** $\frac{t}{8t-9s} - \frac{s}{9s-8t}$

**59.** $\frac{3}{4p-5} + \frac{9}{5-4p}$

**60.** $\frac{8}{3-7y} - \frac{2}{7y-3}$

*In these subtraction problems, the rational expression that follows the subtraction sign has a numerator with more than one term.* ***Be careful with signs*** *and find each difference.* ***See Example 9.***

**61.** $\frac{2m}{m-n} - \frac{5m+n}{2m-2n}$

**62.** $\frac{5p}{p-q} - \frac{3p+1}{4p-4q}$

**63.** $\frac{5}{x^2-9} - \frac{x+2}{x^2+4x+3}$

**64.** $\frac{1}{a^2-1} - \frac{a-1}{a^2+3a-4}$

**65.** $\frac{2q+1}{3q^2+10q-8} - \frac{3q+5}{2q^2+5q-12}$

**66.** $\frac{4y-1}{2y^2+5y-3} - \frac{y+3}{6y^2+y-2}$

*Perform each indicated operation.* ***See Examples 1–9.***

**67.** $\frac{4}{r^2-r} + \frac{6}{r^2+2r} - \frac{1}{r^2+r-2}$

**68.** $\frac{6}{k^2+3k} - \frac{1}{k^2-k} + \frac{2}{k^2+2k-3}$

**69.** $\frac{x+3y}{x^2+2xy+y^2} + \frac{x-y}{x^2+4xy+3y^2}$

**70.** $\frac{m}{m^2-1} + \frac{m-1}{m^2+2m+1}$

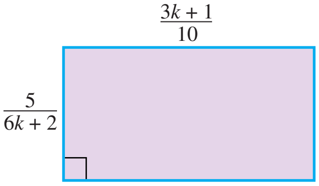
**71.** $\frac{r+y}{18r^2+9ry-2y^2} + \frac{3r-y}{36r^2-y^2}$

**72.** $\frac{2x-z}{2x^2+xz-10z^2} - \frac{x+z}{x^2-4z^2}$

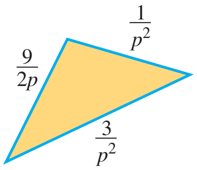
**73.** Refer to the rectangle in the figure.

**(a)** Find an expression that represents its perimeter. Give the simplified form.

**(b)** Find an expression that represents its area. Give the simplified form.



**74.** Refer to the triangle in the figure. Find an expression that represents its perimeter.



*A* ***concours d'elegance*** *is a competition in which a maximum of* 100 *points is awarded to a car based on its general attractiveness. The rational expression*

$$\frac{1010}{49(101-x)} - \frac{10}{49}$$

*approximates the cost, in thousands of dollars, of restoring a car so that it will win x points.*

*Use this information to work* ***Exercises 75 and 76.***

**75.** Simplify the given expression by performing the indicated subtraction.

**76.** Use the simplified expression from **Exercise 75** to determine how much it would cost to win 95 points.

## PREVIEW EXERCISES

*Perform the indicated operations, using the order of operations as necessary.* ***See Section 1.1.***

**77.** $\dfrac{\frac{5}{6}}{\frac{2}{3}}$

**78.** $\dfrac{\frac{3}{8}}{\frac{1}{4}}$

**79.** $\dfrac{\frac{3}{2}}{\frac{7}{4}}$

**80.** $\dfrac{\frac{5}{7}}{\frac{5}{3}}$

# 6.5 Complex Fractions

**OBJECTIVES**

1. Simplify a complex fraction by writing it as a division problem (Method 1).
2. Simplify a complex fraction by multiplying numerator and denominator by the least common denominator (Method 2).
3. Simplify rational expressions with negative exponents.

The quotient of two mixed numbers in arithmetic, such as $2\frac{1}{2} \div 3\frac{1}{4}$, can be written as a fraction.

$$2\frac{1}{2} \div 3\frac{1}{4} = \frac{2\frac{1}{2}}{3\frac{1}{4}} = \frac{2 + \frac{1}{2}}{3 + \frac{1}{4}}$$

We do this to illustrate a *complex fraction.*

In algebra, some rational expressions have fractions in the numerator, or denominator, or both.

### Complex Fraction

A quotient with one or more fractions in the numerator, or denominator, or both, is called a **complex fraction.**

$$\frac{2 + \frac{1}{2}}{3 + \frac{1}{4}}, \quad \frac{\frac{3x^2 - 5x}{6x^2}}{2x - \frac{1}{x}}, \quad \text{and} \quad \frac{3 + x}{5 - \frac{2}{x}} \qquad \text{Complex fractions}$$

The parts of a complex fraction are named as follows.

$$\frac{\frac{2}{p} - \frac{1}{q}}{\frac{3}{p} + \frac{5}{q}}$$

← Numerator of complex fraction

← Main fraction bar

← Denominator of complex fraction

**OBJECTIVE 1** **Simplify a complex fraction by writing it as a division problem (Method 1).** Since the main fraction bar represents division in a complex fraction, one method of simplifying a complex fraction involves division.

### Method 1 for Simplifying a Complex Fraction

***Step 1*** Write both the numerator and denominator as single fractions.

***Step 2*** Change the complex fraction to a division problem.

***Step 3*** Perform the indicated division.

NOW TRY EXERCISE 1

Simplify each complex fraction.

**(a)** $\dfrac{\frac{2}{5}+\frac{1}{4}}{\frac{1}{6}+\frac{3}{8}}$ **(b)** $\dfrac{2+\frac{4}{x}}{\frac{5}{6}+\frac{5x}{12}}$

## EXAMPLE 1 Simplifying Complex Fractions (Method 1)

Simplify each complex fraction.

**(a)** $\dfrac{\frac{2}{3}+\frac{5}{9}}{\frac{1}{4}+\frac{1}{12}}$ **(b)** $\dfrac{6+\frac{3}{x}}{\frac{x}{4}+\frac{1}{8}}$

***Step 1*** First, write each numerator as a single fraction.

$$\frac{2}{3}+\frac{5}{9}=\frac{2(3)}{3(3)}+\frac{5}{9}$$
$$=\frac{6}{9}+\frac{5}{9}=\frac{11}{9}$$

$$6+\frac{3}{x}=\frac{6}{1}+\frac{3}{x}$$
$$=\frac{6x}{x}+\frac{3}{x}=\frac{6x+3}{x}$$

Now, write each denominator as a single fraction.

$$\frac{1}{4}+\frac{1}{12}=\frac{1(3)}{4(3)}+\frac{1}{12}$$
$$=\frac{3}{12}+\frac{1}{12}=\frac{4}{12}$$

$$\frac{x}{4}+\frac{1}{8}=\frac{x(2)}{4(2)}+\frac{1}{8}$$
$$=\frac{2x}{8}+\frac{1}{8}=\frac{2x+1}{8}$$

***Step 2*** Write the equivalent complex fraction as a division problem.

$$\frac{\frac{11}{9}}{\frac{4}{12}}=\frac{11}{9}\div\frac{4}{12}$$

$$\frac{\frac{6x+3}{x}}{\frac{2x+1}{8}}=\frac{6x+3}{x}\div\frac{2x+1}{8}$$

***Step 3*** Use the rule for division and the fundamental property.

Multiply by the reciprocal.

$$\frac{11}{9}\div\frac{4}{12}=\frac{11}{9}\cdot\frac{12}{4}$$
$$=\frac{11\cdot 3\cdot 4}{3\cdot 3\cdot 4}$$
$$=\frac{11}{3}$$

Multiply by the reciprocal.

$$\frac{6x+3}{x}\div\frac{2x+1}{8}=\frac{6x+3}{x}\cdot\frac{8}{2x+1}$$
$$=\frac{3(2x+1)}{x}\cdot\frac{8}{2x+1}$$
$$=\frac{24}{x}$$

NOW TRY

NOW TRY EXERCISE 2

Simplify the complex fraction.

$$\frac{\frac{a^2b}{c}}{\frac{ab^2}{c^3}}$$

## EXAMPLE 2 Simplifying a Complex Fraction (Method 1)

Simplify the complex fraction.

$$\frac{\frac{xp}{q^3}}{\frac{p^2}{qx^2}}$$

The numerator and denominator are single fractions, so use the definition of division and then the fundamental property.

$$\frac{xp}{q^3}\div\frac{p^2}{qx^2}$$
$$=\frac{xp}{q^3}\cdot\frac{qx^2}{p^2}$$
$$=\frac{x^3}{q^2p}$$

NOW TRY

NOW TRY ANSWERS

1. (a) $\frac{6}{5}$ (b) $\frac{24}{5x}$

2. $\frac{ac^2}{b}$

NOW TRY EXERCISE 3

Simplify the complex fraction.

$$\frac{5+\frac{2}{a-3}}{\frac{1}{a-3}-2}$$

## EXAMPLE 3 Simplifying a Complex Fraction (Method 1)

Simplify the complex fraction.

$$\frac{\frac{3}{x+2}-4}{\frac{2}{x+2}+1}$$

$$=\frac{\frac{3}{x+2}-\frac{4(x+2)}{x+2}}{\frac{2}{x+2}+\frac{1(x+2)}{x+2}}$$

Write both second terms with a denominator of $x+2$.

$$=\frac{\frac{3-4(x+2)}{x+2}}{\frac{2+1(x+2)}{x+2}}$$

Subtract in the numerator.

Add in the denominator.

Be careful with signs.

$$=\frac{\frac{3-4x-8}{x+2}}{\frac{2+x+2}{x+2}}$$

Distributive property

$$=\frac{\frac{-5-4x}{x+2}}{\frac{4+x}{x+2}}$$

Combine like terms.

$$=\frac{-5-4x}{x+2}\cdot\frac{x+2}{4+x}$$

Multiply by the reciprocal of the denominator (divisor).

$$=\frac{-5-4x}{4+x}$$

Divide out the common factor.

NOW TRY

**OBJECTIVE 2 Simplify a complex fraction by multiplying numerator and denominator by the least common denominator (Method 2).** Any expression can be multiplied by a form of 1 to get an equivalent expression. Thus we can multiply both the numerator and the denominator of a complex fraction by the same nonzero expression to get an equivalent rational expression. If we choose the expression to be the LCD of all the fractions within the complex fraction, the complex fraction can then be simplified. This is Method 2.

### Method 2 for Simplifying a Complex Fraction

***Step 1*** Find the LCD of all fractions within the complex fraction.

***Step 2*** Multiply both the numerator and the denominator of the complex fraction by this LCD using the distributive property as necessary. Write in lowest terms.

NOW TRY ANSWER

**3.** $\frac{5a-13}{7-2a}$

**NOW TRY EXERCISE 4**

Simplify each complex fraction.

**(a)** $\dfrac{\frac{3}{5}-\frac{1}{4}}{\frac{1}{8}+\frac{3}{20}}$ **(b)** $\dfrac{\frac{2}{x}-3}{7+\frac{x}{5}}$

## EXAMPLE 4 Simplifying Complex Fractions (Method 2)

Simplify each complex fraction.

**(a)** $\dfrac{\frac{2}{3}+\frac{5}{9}}{\frac{1}{4}+\frac{1}{12}}$ **(b)** $\dfrac{6+\frac{3}{x}}{\frac{x}{4}+\frac{1}{8}}$ (In **Example 1,** we simplified these same fractions using Method 1.)

***Step 1*** Find the LCD for all denominators in the complex fraction.

The LCD for 3, 9, 4, and 12 is 36. | The LCD for $x$, 4, and 8 is $8x$.

***Step 2***

$$\frac{\frac{2}{3}+\frac{5}{9}}{\frac{1}{4}+\frac{1}{12}}$$

$$=\frac{36\left(\frac{2}{3}+\frac{5}{9}\right)}{36\left(\frac{1}{4}+\frac{1}{12}\right)}$$

Multiply each term by 36.

$$=\frac{36\left(\frac{2}{3}\right)+36\left(\frac{5}{9}\right)}{36\left(\frac{1}{4}\right)+36\left(\frac{1}{12}\right)}$$

$$=\frac{24+20}{9+3}$$

$$=\frac{44}{12}=\frac{4\cdot 11}{4\cdot 3}, \text{ or } \frac{11}{3}$$

$$\frac{6+\frac{3}{x}}{\frac{x}{4}+\frac{1}{8}}$$

Multiply numerator and denominator of the complex fraction by the LCD.

$$=\frac{8x\left(6+\frac{3}{x}\right)}{8x\left(\frac{x}{4}+\frac{1}{8}\right)}$$

Multiply each term by $8x$.

$$=\frac{8x(6)+8x\left(\frac{3}{x}\right)}{8x\left(\frac{x}{4}\right)+8x\left(\frac{1}{8}\right)}$$

Distributive property

$$=\frac{48x+24}{2x^2+x}$$

Multiply.

$$=\frac{24(2x+1)}{x(2x+1)}, \text{ or } \frac{24}{x}$$

NOW TRY

**NOW TRY EXERCISE 5**

Simplify the complex fraction.

$$\frac{\frac{1}{y}+\frac{2}{3y^2}}{\frac{5}{4y^2}-\frac{3}{2y^3}}$$

## EXAMPLE 5 Simplifying a Complex Fraction (Method 2)

Simplify the complex fraction.

$$\frac{\frac{3}{5m}-\frac{2}{m^2}}{\frac{9}{2m}+\frac{3}{4m^2}}$$

The LCD for $5m$, $m^2$, $2m$, and $4m^2$ is $20m^2$.

$$=\frac{20m^2\left(\frac{3}{5m}-\frac{2}{m^2}\right)}{20m^2\left(\frac{9}{2m}+\frac{3}{4m^2}\right)}$$

Multiply numerator and denominator by $20m^2$.

$$=\frac{20m^2\left(\frac{3}{5m}\right)-20m^2\left(\frac{2}{m^2}\right)}{20m^2\left(\frac{9}{2m}\right)+20m^2\left(\frac{3}{4m^2}\right)}$$

Distributive property.

$$=\frac{12m-40}{90m+15}$$

Multiply and simplify.

NOW TRY

**NOW TRY ANSWERS**

**4. (a)** $\frac{14}{11}$ **(b)** $\frac{10-15x}{x^2+35x}$

**5.** $\frac{12y^2+8y}{15y-18}$

Some students prefer Method 1 for problems like **Example 2,** which is the quotient of two fractions. They will use Method 2 for problems like **Examples 1, 3, 4, and 5,** which have sums or differences in the numerators, or denominators, or both.

### EXAMPLE 6 Deciding on a Method and Simplifying Complex Fractions

Simplify each complex fraction.

**(a)** $$\frac{\frac{1}{y}+\frac{2}{y+2}}{\frac{4}{y}-\frac{3}{y+2}}$$

There are sums and differences in the numerator and denominator. Use Method 2.

$$=\frac{\left(\frac{1}{y}+\frac{2}{y+2}\right)\cdot y(y+2)}{\left(\frac{4}{y}-\frac{3}{y+2}\right)\cdot y(y+2)}$$

Multiply numerator and denominator by the LCD, $y(y+2)$.

$$=\frac{\left(\frac{1}{y}\right)y(y+2)+\left(\frac{2}{y+2}\right)y(y+2)}{\left(\frac{4}{y}\right)y(y+2)-\left(\frac{3}{y+2}\right)y(y+2)}$$

Distributive property

$$=\frac{1(y+2)+2y}{4(y+2)-3y}$$

Fundamental property

$$=\frac{y+2+2y}{4y+8-3y}$$

Distributive property

$$=\frac{3y+2}{y+8}$$

Combine like terms.

Be careful not to use $y + 2$ as the LCD. Because $y$ appears in two denominators, it must be a factor in the LCD.

**(b)** $$\frac{1-\frac{2}{x}-\frac{3}{x^2}}{1-\frac{5}{x}+\frac{6}{x^2}}$$

There are sums and differences in the numerator and denominator. Use Method 2.

$$=\frac{\left(1-\frac{2}{x}-\frac{3}{x^2}\right)x^2}{\left(1-\frac{5}{x}+\frac{6}{x^2}\right)x^2}$$

Multiply numerator and denominator by the LCD, $x^2$.

$$=\frac{x^2-2x-3}{x^2-5x+6}$$

Distributive property

$$=\frac{(x-3)(x+1)}{(x-3)(x-2)}$$

Factor.

$$=\frac{x+1}{x-2}$$

Divide out the common factor.

**NOW TRY EXERCISE 6**

Simplify each complex fraction.

**(a)** $\dfrac{1-\frac{2}{x}-\frac{15}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}}$

**(b)** $\dfrac{\frac{9y^2-16}{y^2-100}}{\frac{3y-4}{y+10}}$

**(c)** $\dfrac{\frac{x+2}{x-3}}{\frac{x^2-4}{x^2-9}}$ This is a quotient of two rational expressions. Use Method 1.

$$= \frac{x+2}{x-3} \div \frac{x^2-4}{x^2-9} \qquad \text{Write as a division problem.}$$

$$= \frac{x+2}{x-3} \cdot \frac{x^2-9}{x^2-4} \qquad \text{Multiply by the reciprocal.}$$

$$= \frac{(x+2)(x+3)(x-3)}{(x-3)(x+2)(x-2)} \qquad \text{Multiply and then factor.}$$

$$= \frac{x+3}{x-2} \qquad \text{Divide out the common factors.}$$

NOW TRY

**OBJECTIVE 3** **Simplify rational expressions with negative exponents.** To simplify, we begin by rewriting the expressions with only positive exponents. Recall from **Section 4.2** that for any nonzero real number $a$ and any integer $n$,

$$\mathbf{a^{-n} = \frac{1}{a^n}}. \qquad \text{Definition of negative exponent}$$

**CAUTION** $a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b}$, ***not*** $\frac{1}{a+b}$. Avoid this common error.

## EXAMPLE 7 Simplifying Rational Expressions with Negative Exponents

Simplify each expression, using only positive exponents in the answer.

**(a)** $\dfrac{m^{-1}+p^{-2}}{2m^{-2}-p^{-1}}$ $\qquad a^{-n} = \frac{1}{a^n}$

$$= \frac{\frac{1}{m}+\frac{1}{p^2}}{\frac{2}{m^2}-\frac{1}{p}} \qquad \text{Write with positive exponents.} \quad 2m^{-2} = 2 \cdot m^{-2} = \frac{2}{1} \cdot \frac{1}{m^2} = \frac{2}{m^2}$$

The base of $2m^{-2}$ is $m$, not $2m$: $2m^{-2} = \frac{2}{m^2}$.

$$= \frac{m^2p^2\left(\frac{1}{m}+\frac{1}{p^2}\right)}{m^2p^2\left(\frac{2}{m^2}-\frac{1}{p}\right)} \qquad \text{Simplify by Method 2, multiplying the numerator and denominator by the LCD, } m^2p^2.$$

$$= \frac{m^2p^2 \cdot \frac{1}{m} + m^2p^2 \cdot \frac{1}{p^2}}{m^2p^2 \cdot \frac{2}{m^2} - m^2p^2 \cdot \frac{1}{p}} \qquad \text{Distributive property}$$

$$= \frac{mp^2+m^2}{2p^2-m^2p} \qquad \text{Lowest terms}$$

**NOW TRY ANSWERS**

**6. (a)** $\frac{x-5}{x+2}$ **(b)** $\frac{3y+4}{y-10}$

NOW TRY EXERCISE 7

Simplify the expression, using only positive exponents in the answer.

$$\frac{2y^{-1}-3y^{-2}}{y^{-2}+3x^{-1}}$$

**(b)** $\dfrac{x^{-2}-2y^{-1}}{y-2x^2}$

The 2 does *not* go in the denominator of this fraction.

$$=\frac{\frac{1}{x^2}-\frac{2}{y}}{y-2x^2}$$

Write with positive exponents.

$$=\frac{\left(\frac{1}{x^2}-\frac{2}{y}\right)x^2y}{(y-2x^2)x^2y}$$

Use Method 2. Multiply by the LCD, $x^2y$.

$$=\frac{y-2x^2}{(y-2x^2)x^2y}$$

Use the distributive property in the numerator.

$$=\frac{1}{x^2y}$$

Remember to write 1 in the numerator.

Lowest terms

NOW TRY

NOW TRY ANSWER

7. $\dfrac{2xy-3x}{x+3y^2}$

## 6.5 EXERCISES

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1. ***Concept Check*** Consider the complex fraction $\dfrac{\frac{3}{2}-\frac{4}{3}}{\frac{1}{6}-\frac{5}{12}}$. Answer each part, outlining Method 1 for simplifying this complex fraction.
   - **(a)** To combine the terms in the numerator, we must find the LCD of $\frac{3}{2}$ and $\frac{4}{3}$. What is this LCD? Determine the simplified form of the numerator of the complex fraction.
   - **(b)** To combine the terms in the denominator, we must find the LCD of $\frac{1}{6}$ and $\frac{5}{12}$. What is this LCD? Determine the simplified form of the denominator of the complex fraction.
   - **(c)** Now use the results from parts (a) and (b) to write the complex fraction as a division problem using the symbol $\div$.
   - **(d)** Perform the operation from part (c) to obtain the final simplification.

2. ***Concept Check*** Consider the complex fraction given in **Exercise 1:** $\dfrac{\frac{3}{2}-\frac{4}{3}}{\frac{1}{6}-\frac{5}{12}}$. Answer each part, outlining Method 2 for simplifying this complex fraction.
   - **(a)** We must determine the LCD of all the fractions within the complex fraction. What is this LCD?
   - **(b)** Multiply every term in the complex fraction by the LCD found in part (a), but do not yet combine the terms in the numerator and the denominator.
   - **(c)** Combine the terms from part (b) to obtain the simplified form of the complex fraction.

3. Which complex fraction is equivalent to $\dfrac{2-\frac{1}{4}}{3-\frac{1}{2}}$? Answer this question without showing any work, and explain your reasoning.

   **A.** $\dfrac{2+\frac{1}{4}}{3+\frac{1}{2}}$ **B.** $\dfrac{2-\frac{1}{4}}{-3+\frac{1}{2}}$ **C.** $\dfrac{-2-\frac{1}{4}}{-3-\frac{1}{2}}$ **D.** $\dfrac{-2+\frac{1}{4}}{-3+\frac{1}{2}}$

4. Only one of these choices is equal to $\dfrac{\frac{1}{3}+\frac{1}{12}}{\frac{1}{2}+\frac{1}{4}}$. Which one is it? Answer this question without showing any work, and explain your reasoning.

   **A.** $\dfrac{5}{9}$ **B.** $-\dfrac{5}{9}$ **C.** $-\dfrac{9}{5}$ **D.** $-\dfrac{1}{12}$

*Simplify each complex fraction. Use either method.* ***See Examples 1–6.***

**5.** $\dfrac{-\frac{4}{3}}{\frac{2}{9}}$

**6.** $\dfrac{-\frac{5}{6}}{\frac{5}{4}}$

**7.** $\dfrac{\frac{x}{y^2}}{\frac{x^2}{y}}$

**8.** $\dfrac{\frac{p^4}{r}}{\frac{p^2}{r^2}}$

**9.** $\dfrac{\frac{4a^4b^3}{3a}}{\frac{2ab^4}{b^2}}$

**10.** $\dfrac{\frac{2r^4t^2}{3t}}{\frac{5r^2t^5}{3r}}$

**11.** $\dfrac{\frac{m+2}{3}}{\frac{m-4}{m}}$

**12.** $\dfrac{\frac{q-5}{q}}{\frac{q+5}{3}}$

**13.** $\dfrac{\frac{2}{x}-3}{\frac{2-3x}{2}}$

**14.** $\dfrac{6+\frac{2}{r}}{\frac{3r+1}{4}}$

**15.** $\dfrac{\frac{1}{x}+x}{\frac{x^2+1}{8}}$

**16.** $\dfrac{\frac{3}{m}-m}{\frac{3-m^2}{4}}$

**17.** $\dfrac{a-\frac{5}{a}}{a+\frac{1}{a}}$

**18.** $\dfrac{q+\frac{1}{q}}{q+\frac{4}{q}}$

**19.** $\dfrac{\frac{5}{8}+\frac{2}{3}}{\frac{7}{3}-\frac{1}{4}}$

**20.** $\dfrac{\frac{6}{5}-\frac{1}{9}}{\frac{2}{5}+\frac{5}{3}}$

**21.** $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x}-\frac{1}{y}}$

**22.** $\dfrac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a}-\frac{1}{b}}$

**23.** $\dfrac{\frac{2}{p^2}-\frac{3}{5p}}{\frac{4}{p}+\frac{1}{4p}}$

**24.** $\dfrac{\frac{2}{m^2}-\frac{3}{m}}{\frac{2}{5m^2}+\frac{1}{3m}}$

**25.** $\dfrac{\frac{5}{x^2y}-\frac{2}{xy^2}}{\frac{3}{x^2y^2}+\frac{4}{xy}}$

**26.** $\dfrac{\frac{1}{m^3p}+\frac{2}{mp^2}}{\frac{4}{mp}+\frac{1}{m^2p}}$

**27.** $\dfrac{\frac{1}{4}-\frac{1}{a^2}}{\frac{1}{2}+\frac{1}{a}}$

**28.** $\dfrac{\frac{1}{9}-\frac{1}{m^2}}{\frac{1}{3}+\frac{1}{m}}$

**29.** $\dfrac{\frac{1}{z+5}}{\frac{4}{z^2-25}}$

**30.** $\dfrac{\frac{1}{a+1}}{\frac{2}{a^2-1}}$

**31.** $\dfrac{\frac{1}{m+1}-1}{\frac{1}{m+1}+1}$

**32.** $\dfrac{\frac{2}{x-1}+2}{\frac{2}{x-1}-2}$

**33.** $\dfrac{\frac{1}{m-1}+\frac{2}{m+2}}{\frac{2}{m+2}-\frac{1}{m-3}}$

**34.** $\dfrac{\frac{5}{r+3}-\frac{1}{r-1}}{\frac{2}{r+2}+\frac{3}{r+3}}$

**35.** $\dfrac{2+\frac{1}{x}-\frac{28}{x^2}}{3+\frac{13}{x}+\frac{4}{x^2}}$

**36.** $\dfrac{4-\frac{11}{x}-\frac{3}{x^2}}{2-\frac{1}{x}-\frac{15}{x^2}}$

**37.** $\dfrac{\frac{y+8}{y-4}}{\frac{y^2-64}{y^2-16}}$

**38.** $\dfrac{\frac{t+5}{t-8}}{\frac{t^2-25}{t^2-64}}$

*Simplify each expression, using only positive exponents in the answer.* ***See Example 7.***

**39.** $\dfrac{1}{x^{-2}+y^{-2}}$

**40.** $\dfrac{1}{p^{-2}-q^{-2}}$

**41.** $\dfrac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$

**42.** $\dfrac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$

**43.** $\dfrac{x^{-1}+2y^{-1}}{2y+4x}$

**44.** $\dfrac{a^{-2}-4b^{-2}}{3b-6a}$

***Brain Busters*** *Simplify each fraction.*

**45.** $\dfrac{1+x^{-1}-12x^{-2}}{1-x^{-1}-20x^{-2}}$

**46.** $\dfrac{1+t^{-1}-56t^{-2}}{1-t^{-1}-72t^{-2}}$

**47.** ***Concept Check*** In a fraction, what operation does the fraction bar represent?

**48.** ***Concept Check*** What property of real numbers justifies Method 2 of simplifying complex fractions?

## *RELATING CONCEPTS* EXERCISES 49–52

### FOR INDIVIDUAL OR GROUP WORK

*To find the average of two numbers, we add them and divide by 2. Suppose that we wish to find the average of* $\frac{3}{8}$ *and* $\frac{5}{6}$. ***Work Exercises 49–52 in order,*** *to see how a complex fraction occurs in a problem like this.*

**49.** Write in symbols: The sum of $\frac{3}{8}$ and $\frac{5}{6}$, divided by 2. Your result should be a complex fraction.

**50.** Use Method 1 to simplify the complex fraction from **Exercise 49.**

**51.** Use Method 2 to simplify the complex fraction from **Exercise 49.**

**52.** Your answers in **Exercises 50 and 51** should be the same. Which method did you prefer? Why?

***Brain Busters*** *The fractions in Exercises 53–58 are called* ***continued fractions.*** *Simplify by starting at "the bottom" and working upward.*

**53.** $1+\dfrac{1}{1+\dfrac{1}{1+1}}$

**54.** $5+\dfrac{5}{5+\dfrac{5}{5+5}}$

**55.** $7-\dfrac{3}{5+\dfrac{2}{4-2}}$

**56.** $3-\dfrac{2}{4+\dfrac{2}{4-2}}$

**57.** $r+\dfrac{r}{4-\dfrac{2}{6+2}}$

**58.** $\dfrac{2q}{7}-\dfrac{q}{6+\dfrac{8}{4+4}}$

## *PREVIEW EXERCISES*

*Simplify.* ***See Section 1.8.***

**59.** $9\left(\dfrac{4x}{3}+\dfrac{2}{9}\right)$

**60.** $8\left(\dfrac{3r}{4}+\dfrac{9}{8}\right)$

**61.** $-12\left(\dfrac{11p^2}{3}-\dfrac{9p}{4}\right)$

**62.** $6\left(\dfrac{5z^2}{2}-\dfrac{8z}{3}\right)$

*Solve each equation.* ***See Sections 2.3 and 5.5.***

**63.** $3x+5=7x+3$

**64.** $9z+2=7z+6$

**65.** $6(z-3)+5=8z-3$

**66.** $k^2+3k-4=0$

# 6.6 Solving Equations with Rational Expressions

**OBJECTIVES**

1. Distinguish between operations with rational expressions and equations with terms that are rational expressions.
2. Solve equations with rational expressions.
3. Solve a formula for a specified variable.

**OBJECTIVE 1 Distinguish between operations with rational expressions and equations with terms that are rational expressions.** Before solving equations with rational expressions, you must understand the difference between sums and differences of terms with rational coefficients, or rational *expressions,* and *equations* with terms that are rational expressions.

***Sums and differences are expressions to simplify. Equations are solved.***

## EXAMPLE 1 Distinguishing between Expressions and Equations

Identify each of the following as an *expression* or an *equation*. Then simplify the expression or solve the equation.

**(a)** $\frac{3}{4}x - \frac{2}{3}x$ This is a difference of two terms. It represents an *expression* to simplify since there is no equals symbol.

$= \frac{3 \cdot 3}{3 \cdot 4}x - \frac{4 \cdot 2}{4 \cdot 3}x$ The LCD is 12. Write each coefficient with this LCD.

$= \frac{9}{12}x - \frac{8}{12}x$ Multiply.

$= \frac{1}{12}x$ Combine like terms, using the distributive property: $\frac{9}{12}x - \frac{8}{12}x = \left(\frac{9}{12} - \frac{8}{12}\right)x$.

**(b)** $\frac{3}{4}x - \frac{2}{3}x = \frac{1}{2}$ Because there is an equals symbol, this is an *equation* to be solved.

$12\left(\frac{3}{4}x - \frac{2}{3}x\right) = 12\left(\frac{1}{2}\right)$ Use the multiplication property of equality to clear fractions. Multiply by 12, the LCD.

$12\left(\frac{3}{4}x\right) - 12\left(\frac{2}{3}x\right) = 12\left(\frac{1}{2}\right)$ Distributive property

Multiply *each* term by 12.

$9x - 8x = 6$ Multiply.

$x = 6$ Combine like terms.

*CHECK* $\frac{3}{4}x - \frac{2}{3}x = \frac{1}{2}$ Original equation

$\frac{3}{4}(6) - \frac{2}{3}(6) \stackrel{?}{=} \frac{1}{2}$ Let $x = 6$.

$\frac{9}{2} - 4 \stackrel{?}{=} \frac{1}{2}$ Multiply.

$\frac{1}{2} = \frac{1}{2}$ ✓ True

Since a true statement results, $\{6\}$ is the solution set of the equation. *NOW TRY*

*NOW TRY EXERCISE 1*

Identify each of the following as an *expression* or an *equation.* Then simplify the expression or solve the equation.

**(a)** $\frac{3}{2}t - \frac{5}{7}t = \frac{11}{7}$

**(b)** $\frac{3}{2}t - \frac{5}{7}t$

*NOW TRY ANSWERS*

**1. (a)** equation; $\{2\}$

**(b)** expression; $\frac{11}{14}t$

The ideas of **Example 1** can be summarized as follows.

**Uses of the LCD**

When adding or subtracting rational expressions, keep the LCD throughout the simplification. (See **Example 1(a).**)

When solving an equation, multiply each side by the LCD so that denominators are eliminated. (See **Example 1(b).**)

**OBJECTIVE 2** **Solve equations with rational expressions.** When an equation involves fractions, as in **Example 1(b),** we use the multiplication property of equality to clear the fractions. Choose as multiplier the LCD of all denominators in the fractions of the equation.

*NOW TRY EXERCISE 2*
Solve, and check the solution.

$$\frac{x+5}{5} - \frac{x}{7} = \frac{3}{7}$$

**EXAMPLE 2** Solving an Equation with Rational Expressions

Solve, and check the solution.

$$\begin{aligned}
\frac{p}{2} - \frac{p-1}{3} &= 1 \\
6\left(\frac{p}{2} - \frac{p-1}{3}\right) &= 6(1) && \text{Multiply each side by the LCD, 6.} \\
6\left(\frac{p}{2}\right) - 6\left(\frac{p-1}{3}\right) &= 6(1) && \text{Distributive property} \\
3p - 2(p-1) &= 6 && \text{Use parentheses around } p-1 \text{ to avoid errors.} \\
3p - 2(p) - 2(-1) &= 6 && \text{Distributive property} \\
3p - 2p + 2 &= 6 && \text{Multiply.} \\
p + 2 &= 6 && \text{Combine like terms.} \\
p &= 4 && \text{Subtract 2.}
\end{aligned}$$

Be careful with signs.

Check to see that $\{4\}$ is the solution set by replacing $p$ with 4 in the original equation.

*NOW TRY*

**CAUTION** In **Example 2,** we used the multiplication property of equality to multiply each side of an *equation* by the LCD. In **Section 6.5,** we used the fundamental property to multiply a *fraction* by another fraction that had the LCD as both its numerator and denominator. Be careful not to confuse these procedures.

Recall from **Section 6.1** that the denominator of a rational expression cannot equal 0, since division by 0 is undefined. ***Therefore, when solving an equation with rational expressions that have variables in the denominator, the solution cannot be a number that makes the denominator equal 0.***

A value of the variable that appears to be a solution after both sides of a rational equation are multiplied by a variable expression is called a **proposed solution.** ***All proposed solutions must be checked in the original equation.***

*NOW TRY ANSWER*
**2.** $\{-10\}$

**NOW TRY EXERCISE 3**

Solve, and check the proposed solution.

$$4 + \frac{6}{x-3} = \frac{2x}{x-3}$$

## EXAMPLE 3 Solving an Equation with Rational Expressions

Solve, and check the proposed solution.

$$\frac{x}{x-2} = \frac{2}{x-2} + 2$$

*x cannot* equal 2, since 2 causes both denominators to equal 0.

$$(x-2)\left(\frac{x}{x-2}\right) = (x-2)\left(\frac{2}{x-2} + 2\right)$$

Multiply each side by the LCD, $x - 2$.

$$(x-2)\left(\frac{x}{x-2}\right) = (x-2)\left(\frac{2}{x-2}\right) + (x-2)(2)$$

Distributive property

$$x = 2 + 2x - 4 \qquad \text{Simplify.}$$

$$x = -2 + 2x \qquad \text{Combine like terms.}$$

$$-x = -2 \qquad \text{Subtract } 2x.$$

$$x = 2 \qquad \text{Multiply by } -1.$$

As noted, $x$ cannot equal 2, since replacing $x$ with 2 in the original equation causes the denominators to equal 0.

**CHECK**

$$\frac{x}{x-2} = \frac{2}{x-2} + 2 \qquad \text{Original equation}$$

$$\frac{2}{2-2} \stackrel{?}{=} \frac{2}{2-2} + 2 \qquad \text{Let } x = 2.$$

$$\frac{2}{0} \stackrel{?}{=} \frac{2}{0} + 2 \qquad \text{Subtract in the denominators.}$$

Division by 0 is undefined.

Thus, 2 must be rejected as a solution, and the solution set is ∅. **NOW TRY**

A proposed solution that is not an actual solution of the original equation, such as 2 in **Example 3,** is called an **extraneous solution,** or **extraneous value.** Some students like to determine which numbers cannot be solutions *before* solving the equation, as we did in **Example 3.**

### Solving an Equation with Rational Expressions

***Step 1*** **Multiply each side of the equation by the LCD** to clear the equation of fractions. Be sure to distribute to *every* term on *both* sides.

***Step 2*** **Solve** the resulting equation.

***Step 3*** **Check** each proposed solution by substituting it into the original equation. Reject any that cause a denominator to equal 0.

## EXAMPLE 4 Solving an Equation with Rational Expressions

Solve, and check the proposed solution.

$$\frac{2}{x^2 - x} = \frac{1}{x^2 - 1}$$

***Step 1***

$$\frac{2}{x(x-1)} = \frac{1}{(x+1)(x-1)}$$

Factor the denominators to find the LCD, $x(x+1)(x-1)$.

**NOW TRY ANSWER**

**3.** ∅

**NOW TRY EXERCISE 4**

Solve, and check the proposed solution.

$$\frac{3}{2x^2-8x}=\frac{1}{x^2-16}$$

Notice that 0, 1, and $-1$ cannot be solutions. Otherwise a denominator will equal 0.

$$\frac{2}{x(x-1)}=\frac{1}{(x+1)(x-1)} \qquad \text{The LCD is } x(x+1)(x-1).$$

$$x(x+1)(x-1)\frac{2}{x(x-1)}=x(x+1)(x-1)\frac{1}{(x+1)(x-1)} \qquad \text{Multiply by the LCD.}$$

***Step 2***

$$2(x+1)=x \qquad \text{Divide out the common factors.}$$

$$2x+2=x \qquad \text{Distributive property}$$

$$x+2=0 \qquad \text{Subtract } x.$$

$$x=-2 \qquad \text{Subtract 2.}$$

***Step 3*** The proposed solution is $-2$, which does not make any denominator equal 0.

*CHECK*

$$\frac{2}{x^2-x}=\frac{1}{x^2-1} \qquad \text{Original equation}$$

$$\frac{2}{(-2)^2-(-2)}\stackrel{?}{=}\frac{1}{(-2)^2-1} \qquad \text{Let } x=-2.$$

$$\frac{2}{4+2}\stackrel{?}{=}\frac{1}{4-1} \qquad \text{Apply the exponents.}$$

$$\frac{1}{3}=\frac{1}{3}\ \checkmark \qquad \text{True}$$

The solution set is $\{-2\}$. NOW TRY

**NOW TRY EXERCISE 5**

Solve, and check the proposed solution.

$$\frac{2y}{y^2-25}=\frac{8}{y+5}-\frac{1}{y-5}$$

## EXAMPLE 5 Solving an Equation with Rational Expressions

Solve, and check the proposed solution.

$$\frac{2m}{m^2-4}+\frac{1}{m-2}=\frac{2}{m+2}$$

$$\frac{2m}{(m+2)(m-2)}+\frac{1}{m-2}=\frac{2}{m+2} \qquad \text{Factor the first denominator on the left to find the LCD, } (m+2)(m-2).$$

Notice that $-2$ and 2 cannot be solutions of this equation.

$$(m+2)(m-2)\left(\frac{2m}{(m+2)(m-2)}+\frac{1}{m-2}\right)=(m+2)(m-2)\frac{2}{m+2} \qquad \text{Multiply by the LCD.}$$

$$(m+2)(m-2)\frac{2m}{(m+2)(m-2)}+(m+2)(m-2)\frac{1}{m-2}=(m+2)(m-2)\frac{2}{m+2} \qquad \text{Distributive property}$$

$$2m+m+2=2(m-2) \qquad \text{Divide out the common factors.}$$

$$3m+2=2m-4 \qquad \text{Combine like terms; distributive property}$$

$$m+2=-4 \qquad \text{Subtract } 2m.$$

$$m=-6 \qquad \text{Subtract 2.}$$

A check verifies that $\{-6\}$ is the solution set. NOW TRY

*NOW TRY ANSWERS*

**4.** $\{-12\}$ **5.** $\{9\}$

**NOW TRY EXERCISE 6**

Solve, and check the proposed solution(s).

$$\frac{3}{m^2 - 9} = \frac{1}{2(m - 3)} - \frac{1}{4}$$

## EXAMPLE 6 Solving an Equation with Rational Expressions

Solve, and check the proposed solution(s).

$$\frac{1}{x - 1} + \frac{1}{2} = \frac{2}{x^2 - 1}$$

$x \neq 1, -1$ or a denominator is 0.

$$\frac{1}{x - 1} + \frac{1}{2} = \frac{2}{(x + 1)(x - 1)}$$

Factor the denominator on the right. The LCD is $2(x + 1)(x - 1)$.

$$2(x + 1)(x - 1)\left(\frac{1}{x - 1} + \frac{1}{2}\right) = 2(x + 1)(x - 1)\frac{2}{(x + 1)(x - 1)}$$

Multiply by the LCD.

$$2(x + 1)(x - 1)\frac{1}{x - 1} + 2(x + 1)(x - 1)\frac{1}{2} = 2(x + 1)(x - 1)\frac{2}{(x + 1)(x - 1)}$$

Distributive property

$$2(x + 1) + (x + 1)(x - 1) = 4$$

Divide out the common factors.

$$2x + 2 + x^2 - 1 = 4$$

Distributive property

Write in standard form.

$$x^2 + 2x - 3 = 0$$

Subtract 4. Combine like terms.

$$(x + 3)(x - 1) = 0$$

Factor.

$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$

Zero-factor property

$$x = -3 \quad \text{or} \quad x = 1 \leftarrow$$

Proposed solutions

Since 1 makes a denominator equal 0, 1 is *not* a solution. Check that $-3$ is a solution.

CHECK

$$\frac{1}{x - 1} + \frac{1}{2} = \frac{2}{x^2 - 1}$$

Original equation

$$\frac{1}{-3 - 1} + \frac{1}{2} \stackrel{?}{=} \frac{2}{(-3)^2 - 1}$$

Let $x = -3$.

$$\frac{1}{-4} + \frac{1}{2} \stackrel{?}{=} \frac{2}{9 - 1}$$

Simplify.

$$\frac{1}{4} = \frac{1}{4} \checkmark$$

True

The solution set is $\{-3\}$.

NOW TRY

## EXAMPLE 7 Solving an Equation with Rational Expressions

Solve, and check the proposed solution.

$$\frac{1}{k^2 + 4k + 3} + \frac{1}{2k + 2} = \frac{3}{4k + 12}$$

$$\frac{1}{(k + 1)(k + 3)} + \frac{1}{2(k + 1)} = \frac{3}{4(k + 3)}$$

Factor each denominator. The LCD is $4(k + 1)(k + 3)$.

$k \neq -1, -3$

$$4(k + 1)(k + 3)\left(\frac{1}{(k + 1)(k + 3)} + \frac{1}{2(k + 1)}\right) = 4(k + 1)(k + 3)\frac{3}{4(k + 3)}$$

Multiply by the LCD.

**NOW TRY ANSWER**

6. $\{-1\}$

**NOW TRY EXERCISE 7**

Solve, and check the proposed solution.

$$\frac{5}{k^2+k-2}=\frac{1}{3k-3}-\frac{1}{k+2}$$

$$4(k+1)(k+3)\frac{1}{(k+1)(k+3)}+2\cdot 2(k+1)(k+3)\frac{1}{2(k+1)}$$

$$=4(k+1)(k+3)\frac{3}{4(k+3)} \qquad \text{Distributive property}$$

Do *not* add 4 + 2 here.

$$4+2(k+3)=3(k+1) \qquad \text{Simplify.}$$

$$4+2k+6=3k+3 \qquad \text{Distributive property}$$

$$2k+10=3k+3 \qquad \text{Combine like terms.}$$

$$10=k+3 \qquad \text{Subtract } 2k.$$

$$7=k \qquad \text{Subtract 3.}$$

The proposed solution, 7, does not make an original denominator equal 0. A check shows that the algebra is correct (see **Exercise 78**), so $\{7\}$ is the solution set. NOW TRY

**OBJECTIVE 3** **Solve a formula for a specified variable.** When solving a formula for a specified variable, ***remember to treat the variable for which you are solving as if it were the only variable, and all others as if they were constants.***

**NOW TRY EXERCISE 8**

Solve each formula for the specified variable.

**(a)** $p=\frac{x-y}{z}$ for $x$

**(b)** $a=\frac{b}{c+d}$ for $d$

### EXAMPLE 8 Solving for a Specified Variable

Solve each formula for the specified variable.

**(a)** $a=\frac{v-w}{t}$ for $v$

$$a=\frac{v-w}{t}$$

Our goal is to isolate *v*.

$$at=v-w \qquad \text{Multiply by } t.$$

$$at+w=v, \quad \text{or} \quad v=at+w \qquad \text{Add } w.$$

**(b)** $F=\frac{k}{d-D}$ for $d$

We must isolate *d*.

$$F=\frac{k}{d-D} \qquad \text{Given equation}$$

$$F(d-D)=\frac{k}{d-D}(d-D) \qquad \text{Multiply by } d-D \text{ to clear the fraction.}$$

$$F(d-D)=k \qquad \text{Simplify.}$$

$$Fd-FD=k \qquad \text{Distributive property}$$

$$Fd=k+FD \qquad \text{Add } FD.$$

$$d=\frac{k+FD}{F} \qquad \text{Divide by } F.$$

We can write an equivalent form of this answer as follows.

$$d=\frac{k+FD}{F} \qquad \text{Answer from above}$$

$$d=\frac{k}{F}+\frac{FD}{F} \qquad \text{Definition of addition of fractions: } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

$$d=\frac{k}{F}+D \qquad \text{Divide out the common factor from } \frac{FD}{F}.$$

Either answer is correct. NOW TRY

**NOW TRY ANSWERS**

7. $\{-5\}$

8. **(a)** $x=pz+y$

   **(b)** $d=\frac{b-ac}{a}$

**NOW TRY EXERCISE 9**

Solve the following formula for $x$.

$$\frac{2}{w}=\frac{1}{x}-\frac{3}{y}$$

## EXAMPLE 9 Solving for a Specified Variable

Solve the following formula for $c$.

$$\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$$

Goal: Isolate $c$, the specified variable.

$$
\begin{aligned}
abc\left(\frac{1}{a}\right) &= abc\left(\frac{1}{b}+\frac{1}{c}\right) && \text{Multiply by the LCD, } abc.\\
abc\left(\frac{1}{a}\right) &= abc\left(\frac{1}{b}\right)+abc\left(\frac{1}{c}\right) && \text{Distributive property}\\
bc &= ac+ab && \text{Simplify.}\\
bc-ac &= ab && \text{Subtract } ac \text{ to get both terms with } c \text{ on the same side.}\\
c(b-a) &= ab && \text{Factor out } c.\\
c &= \frac{ab}{b-a} && \text{Divide by } b-a.
\end{aligned}
$$

Pay careful attention here.

NOW TRY

**CAUTION** Students often have trouble in the step that involves factoring out the variable for which they are solving. In **Example 9,** we needed to get both terms with $c$ on the same side of the equation. This allowed us to factor out $c$ on the left, and then isolate it by dividing each side by $b-a$.

***When solving an equation for a specified variable, be sure that the specified variable appears alone on only one side of the equals symbol in the final equation.***

**NOW TRY ANSWER**

**9.** $x=\frac{wy}{2y+3w}$

## 6.6 EXERCISES

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*Identify each of the following as an* expression *or an* equation. *Then simplify the expression or solve the equation.* ***See Example 1.***

◉ **1.** $\frac{7}{8}x+\frac{1}{5}x$

**2.** $\frac{4}{7}x+\frac{4}{5}x$

**3.** $\frac{7}{8}x+\frac{1}{5}x=1$

**4.** $\frac{4}{7}x+\frac{4}{5}x=1$

**5.** $\frac{3}{5}x-\frac{7}{10}x$

**6.** $\frac{2}{3}x-\frac{9}{4}x$

**7.** $\frac{3}{5}x-\frac{7}{10}x=1$

**8.** $\frac{2}{3}x-\frac{9}{4}x=-19$

**9.** $\frac{3}{4}x-\frac{1}{2}x=0$

✎ **10.** Explain why the equation in **Exercise 9** is easy to check.

*When solving an equation with variables in denominators, we must determine the values that cause these denominators to equal* 0, *so that we can reject these extraneous values if they appear as potential solutions. Find all values for which at least one denominator is equal to* 0. *Write answers using the symbol* $\neq$. *Do not solve.* ***See Examples 3–7.***

**11.** $\frac{3}{x+2}-\frac{5}{x}=1$

**12.** $\frac{7}{x}+\frac{9}{x-4}=5$

**13.** $\frac{-1}{(x+3)(x-4)}=\frac{1}{2x+1}$

**14.** $\frac{8}{(x-7)(x+3)}=\frac{7}{3x-10}$

**15.** $\dfrac{4}{x^2 + 8x - 9} + \dfrac{1}{x^2 - 4} = 0$

**16.** $\dfrac{-3}{x^2 + 9x - 10} - \dfrac{12}{x^2 - 49} = 0$

**17.** What is wrong with the following problem? "Solve $\frac{2}{3x} + \frac{1}{5x}$."

**18.** Explain how the LCD is used in a different way when adding and subtracting rational expressions as compared to solving equations with rational expressions.

*Solve each equation, and check your solutions.* ***See Examples 1(b), 2, and 3.***

**19.** $\dfrac{5}{m} - \dfrac{3}{m} = 8$

**20.** $\dfrac{4}{y} + \dfrac{1}{y} = 2$

**21.** $\dfrac{5}{y} + 4 = \dfrac{2}{y}$

**22.** $\dfrac{11}{q} - 3 = \dfrac{1}{q}$

**23.** $\dfrac{3x}{5} - 6 = x$

**24.** $\dfrac{5t}{4} + t = 9$

**25.** $\dfrac{4m}{7} + m = 11$

**26.** $a - \dfrac{3a}{2} = 1$

**27.** $\dfrac{z - 1}{4} = \dfrac{z + 3}{3}$

**28.** $\dfrac{r - 5}{2} = \dfrac{r + 2}{3}$

**29.** $\dfrac{3p + 6}{8} = \dfrac{3p - 3}{16}$

**30.** $\dfrac{2z + 1}{5} = \dfrac{7z + 5}{15}$

**31.** $\dfrac{2x + 3}{x} = \dfrac{3}{2}$

**32.** $\dfrac{7 - 2x}{x} = \dfrac{-17}{5}$

**33.** $\dfrac{k}{k - 4} - 5 = \dfrac{4}{k - 4}$

**34.** $\dfrac{-5}{a + 5} - 2 = \dfrac{a}{a + 5}$

**35.** $\dfrac{q + 2}{3} + \dfrac{q - 5}{5} = \dfrac{7}{3}$

**36.** $\dfrac{x - 6}{6} + \dfrac{x + 2}{8} = \dfrac{11}{4}$

**37.** $\dfrac{x}{2} = \dfrac{5}{4} + \dfrac{x - 1}{4}$

**38.** $\dfrac{8p}{5} = \dfrac{3p - 4}{2} + \dfrac{5}{2}$

**39.** $x + \dfrac{17}{2} = \dfrac{x}{2} + x + 6$

**40.** $t + \dfrac{8}{3} = \dfrac{t}{3} + t + \dfrac{14}{3}$

**41.** $\dfrac{9}{3x + 4} = \dfrac{36 - 27x}{16 - 9x^2}$

**42.** $\dfrac{25}{5x - 6} = \dfrac{-150 - 125x}{36 - 25x^2}$

*Solve each equation, and check your solutions.* ***Be careful with signs. See Example 2.***

**43.** $\dfrac{a + 7}{8} - \dfrac{a - 2}{3} = \dfrac{4}{3}$

**44.** $\dfrac{x + 3}{7} - \dfrac{x + 2}{6} = \dfrac{1}{6}$

**45.** $\dfrac{p}{2} - \dfrac{p - 1}{4} = \dfrac{5}{4}$

**46.** $\dfrac{r}{6} - \dfrac{r - 2}{3} = -\dfrac{4}{3}$

**47.** $\dfrac{3x}{5} - \dfrac{x - 5}{7} = 3$

**48.** $\dfrac{8k}{5} - \dfrac{3k - 4}{2} = \dfrac{5}{2}$

*Solve each equation, and check your solutions.* ***See Examples 3–7.***

**49.** $\dfrac{4}{x^2 - 3x} = \dfrac{1}{x^2 - 9}$

**50.** $\dfrac{2}{t^2 - 4} = \dfrac{3}{t^2 - 2t}$

**51.** $\dfrac{2}{m} = \dfrac{m}{5m + 12}$

**52.** $\dfrac{x}{4 - x} = \dfrac{2}{x}$

**53.** $\dfrac{-2}{z + 5} + \dfrac{3}{z - 5} = \dfrac{20}{z^2 - 25}$

**54.** $\dfrac{3}{r + 3} - \dfrac{2}{r - 3} = \dfrac{-12}{r^2 - 9}$

**55.** $\dfrac{3}{x - 1} + \dfrac{2}{4x - 4} = \dfrac{7}{4}$

**56.** $\dfrac{2}{p + 3} + \dfrac{3}{8} = \dfrac{5}{4p + 12}$

**57.** $\frac{x}{3x+3} = \frac{2x-3}{x+1} - \frac{2x}{3x+3}$

**58.** $\frac{2k+3}{k+1} - \frac{3k}{2k+2} = \frac{-2k}{2k+2}$

**59.** $\frac{2p}{p^2-1} = \frac{2}{p+1} - \frac{1}{p-1}$

**60.** $\frac{2x}{x^2-16} - \frac{2}{x-4} = \frac{4}{x+4}$

**61.** $\frac{5x}{14x+3} = \frac{1}{x}$

**62.** $\frac{m}{8m+3} = \frac{1}{3m}$

**63.** $\frac{2}{x-1} - \frac{2}{3} = \frac{-1}{x+1}$

**64.** $\frac{5}{p-2} = 7 - \frac{10}{p+2}$

**65.** $\frac{x}{2x+2} = \frac{-2x}{4x+4} + \frac{2x-3}{x+1}$

**66.** $\frac{5t+1}{3t+3} = \frac{5t-5}{5t+5} + \frac{3t-1}{t+1}$

**67.** $\frac{8x+3}{x} = 3x$

**68.** $\frac{10x-24}{x} = x$

**69.** $\frac{1}{x+4} + \frac{x}{x-4} = \frac{-8}{x^2-16}$

**70.** $\frac{x}{x-3} + \frac{4}{x+3} = \frac{18}{x^2-9}$

**71.** $\frac{4}{3x+6} - \frac{3}{x+3} = \frac{8}{x^2+5x+6}$

**72.** $\frac{-13}{t^2+6t+8} + \frac{4}{t+2} = \frac{3}{2t+8}$

**73.** $\frac{3x}{x^2+5x+6} = \frac{5x}{x^2+2x-3} - \frac{2}{x^2+x-2}$

**74.** $\frac{m}{m^2+m-2} + \frac{m}{m^2-1} = \frac{m}{m^2+3m+2}$

**75.** $\frac{x+4}{x^2-3x+2} - \frac{5}{x^2-4x+3} = \frac{x-4}{x^2-5x+6}$

**76.** $\frac{3}{r^2+r-2} - \frac{1}{r^2-1} = \frac{7}{2(r^2+3r+2)}$

**77.** ***Concept Check*** If you are solving a formula for the letter $k$, and your steps lead to the equation $kr - mr = km$, what would be your next step?

**78.** Refer to **Example 7,** and show that 7 is a solution.

*Solve each formula for the specified variable.* ***See Examples 8 and 9.***

**79.** $m = \frac{kF}{a}$ for $F$

**80.** $I = \frac{kE}{R}$ for $E$

**81.** $m = \frac{kF}{a}$ for $a$

**82.** $I = \frac{kE}{R}$ for $R$

**83.** $I = \frac{E}{R+r}$ for $R$

**84.** $I = \frac{E}{R+r}$ for $r$

**85.** $h = \frac{2\mathcal{A}}{B+b}$ for $\mathcal{A}$

**86.** $d = \frac{2S}{n(a+L)}$ for $S$

**87.** $d = \frac{2S}{n(a+L)}$ for $a$

**88.** $h = \frac{2\mathcal{A}}{B+b}$ for $B$

**89.** $\frac{1}{x} = \frac{1}{y} - \frac{1}{z}$ for $y$

**90.** $\frac{3}{k} = \frac{1}{p} + \frac{1}{q}$ for $q$

**91.** $\frac{2}{r} + \frac{3}{s} + \frac{1}{t} = 1$ for $t$

**92.** $\frac{5}{p} + \frac{2}{q} + \frac{3}{r} = 1$ for $r$

**93.** $9x + \frac{3}{z} = \frac{5}{y}$ for $z$

**94.** $-3t - \frac{4}{p} = \frac{6}{s}$ for $p$

**95.** $\frac{t}{x-1} - \frac{2}{x+1} = \frac{1}{x^2-1}$ for $t$

**96.** $\frac{5}{y+2} - \frac{r}{y-2} = \frac{3}{y^2-4}$ for $r$

## PREVIEW EXERCISES

*Write a mathematical expression for each exercise.* ***See Section 2.7.***

**97.** Andrew drives from Pittsburgh to Philadelphia, a distance of 288 mi, in $t$ hours. Find his rate in miles per hour.

**98.** Tyler drives for 20 hr, traveling from City $A$ to City $B$, a distance of $d$ kilometers. Find his rate in kilometers per hour.

**99.** Jack flies his small plane from St. Louis to Chicago, a distance of 289 mi, at $z$ miles per hour. Find his time in hours.

**100.** Joshua can do a job in $r$ hours. What portion of the job is done in 1 hr?

## *SUMMARY EXERCISES* on Rational Expressions and Equations

Students often confuse *simplifying expressions* with *solving equations*. We review the four operations to simplify the rational expressions $\frac{1}{x}$ and $\frac{1}{x-2}$ as follows.

***Add:*** $\frac{1}{x} + \frac{1}{x-2}$

$$= \frac{1(x-2)}{x(x-2)} + \frac{x(1)}{x(x-2)}$$ Write with a common denominator.

$$= \frac{x-2+x}{x(x-2)}$$ Add numerators. Keep the same denominator.

$$= \frac{2x-2}{x(x-2)}$$ Combine like terms.

***Subtract:*** $\frac{1}{x} - \frac{1}{x-2}$

$$= \frac{1(x-2)}{x(x-2)} - \frac{x(1)}{x(x-2)}$$ Write with a common denominator.

$$= \frac{x-2-x}{x(x-2)}$$ Subtract numerators. Keep the same denominator.

$$= \frac{-2}{x(x-2)}$$ Combine like terms.

***Multiply:*** $\frac{1}{x} \cdot \frac{1}{x-2}$

$$= \frac{1}{x(x-2)}$$ Multiply numerators and multiply denominators.

***Divide:*** $\frac{1}{x} \div \frac{1}{x-2}$

$$= \frac{1}{x} \cdot \frac{x-2}{1}$$ Multiply by the reciprocal of the divisor.

$$= \frac{x-2}{x}$$ Multiply numerators and multiply denominators.

(continued)

By contrast, consider the following *equation*.

$$\frac{1}{x} + \frac{1}{x-2} = \frac{3}{4}$$

$x \neq 0$ and $x \neq 2$ since a denominator is 0 for these values.

$$\begin{aligned}
4x(x-2)\frac{1}{x} + 4x(x-2)\frac{1}{x-2} &= 4x(x-2)\frac{3}{4} && \text{Multiply each side by the LCD, } 4x(x-2)\text{, to clear fractions.}\\
4(x-2) + 4x &= 3x(x-2) && \text{Divide out common factors.}\\
4x - 8 + 4x &= 3x^2 - 6x && \text{Distributive property}\\
3x^2 - 14x + 8 &= 0 && \text{Get 0 on one side.}\\
(3x-2)(x-4) &= 0 && \text{Factor.}\\
3x - 2 = 0 \quad \text{or} \quad x - 4 &= 0 && \text{Zero-factor property}\\
x = \frac{2}{3} \quad \text{or} \quad x &= 4 && \text{Solve for } x.
\end{aligned}$$

Both $\frac{2}{3}$ and 4 are solutions, since neither makes a denominator equal 0. Check to confirm that the solution set is $\left\{\frac{2}{3}, 4\right\}$.

## Points to Remember When Working with Rational Expressions and Equations

1. When simplifying rational expressions, the fundamental property is applied only after numerators and denominators have been *factored.*
2. When adding and subtracting rational expressions, the common denominator must be kept throughout the problem and in the final result.
3. When simplifying rational expressions, always check to see if the answer is in lowest terms. If it is not, use the fundamental property.
4. When solving equations with rational expressions, the LCD is used to clear the equation of fractions. Multiply each side by the LCD. (Notice how this use differs from that of the LCD in Point 2.)
5. When solving equations with rational expressions, reject any proposed solution that causes an original denominator to equal 0.

*For each exercise, indicate* "expression" *if an expression is to be simplified or* "equation" *if an equation is to be solved. Then simplify the expression or solve the equation.*

**1.** $\dfrac{4}{p} + \dfrac{6}{p}$

**2.** $\dfrac{x^3y^2}{x^2y^4} \cdot \dfrac{y^5}{x^4}$

**3.** $\dfrac{1}{x^2+x-2} \div \dfrac{4x^2}{2x-2}$

**4.** $\dfrac{8}{t-5} = 2$

**5.** $\dfrac{2y^2+y-6}{2y^2-9y+9} \cdot \dfrac{y^2-2y-3}{y^2-1}$

**6.** $\dfrac{2}{k^2-4k} + \dfrac{3}{k^2-16}$

**7.** $\dfrac{x-4}{5} = \dfrac{x+3}{6}$

**8.** $\dfrac{3t^2-t}{6t^2+15t} \div \dfrac{6t^2+t-1}{2t^2-5t-25}$

**9.** $\dfrac{4}{p+2} + \dfrac{1}{3p+6}$

**10.** $\dfrac{1}{x} + \dfrac{1}{x-3} = -\dfrac{5}{4}$

**11.** $\frac{3}{t-1} + \frac{1}{t} = \frac{7}{2}$

**12.** $\frac{6}{k} - \frac{2}{3k}$

**13.** $\frac{5}{4z} - \frac{2}{3z}$

**14.** $\frac{x+2}{3} = \frac{2x-1}{5}$

**15.** $\frac{1}{m^2+5m+6} + \frac{2}{m^2+4m+3}$

**16.** $\frac{2k^2-3k}{20k^2-5k} \div \frac{2k^2-5k+3}{4k^2+11k-3}$

**17.** $\frac{2}{x+1} + \frac{5}{x-1} = \frac{10}{x^2-1}$

**18.** $\frac{3}{x+3} + \frac{4}{x+6} = \frac{9}{x^2+9x+18}$

**19.** $\frac{4t^2-t}{6t^2+10t} \div \frac{8t^2+2t-1}{3t^2+11t+10}$

**20.** $\frac{x}{x-2} + \frac{3}{x+2} = \frac{8}{x^2-4}$

# 6.7 Applications of Rational Expressions

**OBJECTIVES**

1. Solve problems about numbers.
2. Solve problems about distance, rate, and time.
3. Solve problems about work.

For applications that lead to rational equations, the six-step problem-solving method of **Section 2.4** still applies.

## OBJECTIVE 1 Solve problems about numbers.

### EXAMPLE 1 Solving a Problem about an Unknown Number

If the same number is added to both the numerator and the denominator of the fraction $\frac{2}{5}$, the result is equivalent to $\frac{2}{3}$. Find the number.

***Step 1*** **Read** the problem carefully. We are trying to find a number.

***Step 2*** **Assign a variable.**

Let $x =$ the number added to the numerator and the denominator.

***Step 3*** **Write an equation.** The fraction $\frac{2+x}{5+x}$ represents the result of adding the same number to both the numerator and the denominator. Since this result is equivalent to $\frac{2}{3}$, the equation is written as follows.

$$\frac{2+x}{5+x} = \frac{2}{3}$$

***Step 4*** **Solve** this equation.

$$3(5+x)\frac{2+x}{5+x} = 3(5+x)\frac{2}{3} \qquad \text{Multiply by the LCD, } 3(5+x).$$

$$3(2+x) = 2(5+x) \qquad \text{Divide out common factors.}$$

$$6+3x = 10+2x \qquad \text{Distributive property}$$

$$x = 4 \qquad \text{Subtract } 2x\text{. Subtract 6.}$$

***Step 5*** **State the answer.** The number is 4.

***Step 6*** **Check** the solution in the words of the original problem. If 4 is added to both the numerator and the denominator of $\frac{2}{5}$, the result is $\frac{6}{9} = \frac{2}{3}$, as required.

NOW TRY

NOW TRY EXERCISE 1

In a certain fraction, the numerator is 4 less than the denominator. If 7 is added to both the numerator and denominator, the resulting fraction is equivalent to $\frac{7}{8}$. What is the original fraction?

NOW TRY ANSWER

1. $\frac{21}{25}$

**OBJECTIVE 2** **Solve problems about distance, rate, and time.** Recall from **Chapter 2** the following formulas relating distance, rate, and time. You may wish to refer to **Example 5** in **Section 2.7** to review the basic use of these formulas.

**Distance, Rate, and Time Relationship**

$$d = rt \qquad r = \frac{d}{t} \qquad t = \frac{d}{r}$$

### EXAMPLE 2 Solving a Problem about Distance, Rate, and Time

The Tickfaw River has a current of 3 mph. A motorboat takes as long to go 12 mi downstream as to go 8 mi upstream. What is the rate of the boat in still water?

***Step 1*** **Read** the problem again. We must find the rate (speed) of the boat in still water.

***Step 2*** **Assign a variable.** Let $x =$ the rate of the boat in still water.

Because the current pushes the boat when the boat is going downstream, the rate of the boat downstream will be the *sum* of the rate of the boat and the rate of the current, $(x + 3)$ mph.

Because the current slows down the boat when the boat is going upstream, the boat's rate going upstream is given by the *difference* between the rate of the boat and the rate of the current, $(x - 3)$ mph. See **FIGURE 1**.

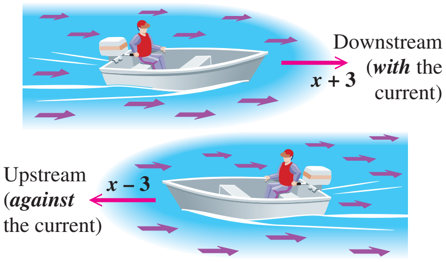

**FIGURE 1**

This information is summarized in the following table.

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Downstream* | 12 | $x + 3$ | |
| *Upstream* | 8 | $x - 3$ | |

Fill in the times by using the formula $t = \frac{d}{r}$.

The time downstream is the distance divided by the rate.

$$t = \frac{d}{r} = \frac{12}{x + 3} \qquad \text{Time downstream}$$

The time upstream is that distance divided by that rate.

$$t = \frac{d}{r} = \frac{8}{x - 3} \qquad \text{Time upstream}$$

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Downstream* | 12 | $x + 3$ | $\frac{12}{x + 3}$ |
| *Upstream* | 8 | $x - 3$ | $\frac{8}{x - 3}$ |

Times are equal.

NOW TRY EXERCISE 2

In her small boat, Jennifer can travel 12 mi downstream in the same amount of time that she can travel 4 mi upstream. The rate of the current is 2 mph. Find the rate of Jennifer's boat in still water.

***Step 3*** **Write an equation.**

$$\frac{12}{x+3} = \frac{8}{x-3}$$

The time downstream equals the time upstream, so the two times from the table must be equal.

***Step 4*** **Solve.**

$$(x+3)(x-3)\frac{12}{x+3} = (x+3)(x-3)\frac{8}{x-3}$$

Multiply by the LCD, $(x+3)(x-3)$.

$$12(x-3) = 8(x+3)$$

Divide out the common factors.

$$12x - 36 = 8x + 24$$

Distributive property

$$4x = 60$$

Subtract $8x$ and add 36.

$$x = 15$$

Divide by 4.

***Step 5*** **State the answer.** The rate of the boat in still water is 15 mph.

***Step 6*** **Check.** First we find the rate of the boat going downstream, which is $15 + 3 = 18$ mph. Divide 12 mi by 18 mph to find the time.

$$t = \frac{d}{r} = \frac{12}{18} = \frac{2}{3}\text{ hr}$$

The rate of the boat going upstream is $15 - 3 = 12$ mph. Divide 8 mi by 12 mph to find the time.

$$t = \frac{d}{r} = \frac{8}{12} = \frac{2}{3}\text{ hr}$$

The time upstream equals the time downstream, as required.

NOW TRY

**OBJECTIVE 3** **Solve problems about work.** Suppose that you can mow your lawn in 4 hr. Then after 1 hr, you will have mowed $\frac{1}{4}$ of the lawn. After 2 hr, you will have mowed $\frac{2}{4}$, or $\frac{1}{2}$, of the lawn, and so on. This idea is generalized as follows.

### Rate of Work

If a job can be completed in $t$ units of time, then the rate of work is

$$\frac{1}{t}\ \textbf{job per unit of time.}$$

### PROBLEM-SOLVING HINT

Recall that the formula $d = rt$ says that distance traveled is equal to rate of travel multiplied by time traveled. Similarly, the fractional part of a job accomplished is equal to the rate of work multiplied by the time worked. In the lawn-mowing example, after 3 hr, the fractional part of the job done is as follows.

$$\underbrace{\frac{1}{4}}_{\substack{\text{Rate of}\\ \text{work}}} \cdot \underbrace{3}_{\substack{\text{Time}\\ \text{worked}}} = \underbrace{\frac{3}{4}}_{\substack{\text{Fractional part}\\ \text{of job done}}}$$

After 4 hr, $\frac{1}{4}(4) = 1$ whole job has been done.

NOW TRY ANSWER

**2.** 4 mph

**NOW TRY EXERCISE 3**

Sarah can proofread a manuscript in 10 hr, while Joyce can proofread the same manuscript in 12 hr. How long will it take them to proofread the manuscript if they work together?

## EXAMPLE 3 Solving a Problem about Work Rates

"If Joe can paint a house in 3 hr and Sam can paint the same house in 5 hr, how long does it take for them to do it together?" (*Source*: The movie *Little Big League*.)

***Step 1*** **Read** the problem again. We are looking for time working together.

***Step 2*** **Assign a variable.** Let $x =$ the number of hours it takes Joe and Sam to paint the house, working together.

Certainly, $x$ will be less than 3, since Joe alone can complete the job in 3 hr. We begin by making a table. Based on the preceding discussion, Joe's rate alone is $\frac{1}{3}$ job per hour, and Sam's rate is $\frac{1}{5}$ job per hour.

| | *Rate* | *Time Working Together* | *Fractional Part of the Job Done When Working Together* |
|---|---|---|---|
| *Joe* | $\frac{1}{3}$ | $x$ | $\frac{1}{3}x$ |
| *Sam* | $\frac{1}{5}$ | $x$ | $\frac{1}{5}x$ |

Sum is 1 whole job.

***Step 3*** **Write an equation.**

$$\underbrace{\text{Fractional part done by Joe}}_{\frac{1}{3}x} + \underbrace{\text{Fractional part done by Sam}}_{\frac{1}{5}x} = \underbrace{\text{1 whole job.}}_{1}$$

Together, Joe and Sam complete 1 whole job. Add their individual fractional parts and set the sum equal to 1.

**From *Little Big League***

***Step 4*** **Solve.**

$$15\left(\frac{1}{3}x + \frac{1}{5}x\right) = 15(1) \quad \text{Multiply by the LCD, 15.}$$

$$15\left(\frac{1}{3}x\right) + 15\left(\frac{1}{5}x\right) = 15(1) \quad \text{Distributive property}$$

$$5x + 3x = 15$$

$$8x = 15 \quad \text{Combine like terms.}$$

$$x = \frac{15}{8} \quad \text{Divide by 8.}$$

***Step 5*** **State the answer.** Working together, Joe and Sam can paint the house in $\frac{15}{8}$ hr, or $1\frac{7}{8}$ hr.

***Step 6*** **Check** to be sure the answer is correct. NOW TRY

---

**NOTE** An alternative approach in work problems is to consider the part of the job that can be done in 1 hr. For instance, in **Example 3** Joe can do the entire job in 3 hr and Sam can do it in 5 hr. Thus, their work rates, as we saw in **Example 3,** are $\frac{1}{3}$ and $\frac{1}{5}$, respectively. Since it takes them $x$ hours to complete the job working together, in 1 hr they can paint $\frac{1}{x}$ of the house.

(continued)

**NOW TRY ANSWER**

**3.** $\frac{60}{11}$ hr, or $5\frac{5}{11}$ hr

The amount painted by Joe in 1 hr plus the amount painted by Sam in 1 hr must equal the amount they can do together. This relationship leads to the equation

$$\text{Amount by Joe} \rightarrow \frac{1}{3} + \overset{\substack{\text{Amount by Sam}\\ \downarrow}}{\frac{1}{5}} = \frac{1}{x}. \leftarrow \text{Amount together}$$

Compare this equation with the one in **Example 3.** Multiplying each side by $15x$ leads to

$$5x + 3x = 15,$$

the same equation found in the third line of Step 4 in the example. The same solution results.

**PROBLEM-SOLVING HINT**

A common error students make when solving a work problem like that in **Example 3** is to add the two times, 3 hr and 5 hr, to get an answer of 8 hr. We reason, however, that $x$, the time it will take Joe and Sam working together, must be *less than* 3 hr, since Joe can complete the job by himself in 3 hr.

Another common error students make is to try to split the job in half between the two workers so that Joe would work $\frac{1}{2}(3)$, or $1\frac{1}{2}$ hr, and Sam would work $\frac{1}{2}(5)$, or $2\frac{1}{2}$ hr. In this case, Joe finishes 1 hr before Sam and they have not worked together to get the entire job done as quickly as possible. If Joe, when he finishes, helps Sam, the job should actually be completed in a time between $1\frac{1}{2}$ hr and $2\frac{1}{2}$ hr.

Based on this reasoning, does our answer of $1\frac{7}{8}$ hr in **Example 3** hold up?

## 6.7 EXERCISES

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***Concept Check*** *Use Steps 2 and 3 of the six-step method to set up the equation you would use to solve each problem. (Remember that Step 1 is to read the problem carefully.) Do not actually solve the equation.* ***See Example 1.***

**1.** The numerator of the fraction $\frac{5}{6}$ is increased by an amount so that the value of the resulting fraction is equivalent to $\frac{13}{3}$. By what amount was the numerator increased?

**(a)** Let $x =$ ______________. (*Step 2*)

**(b)** Write an expression for "the numerator of the fraction $\frac{5}{6}$ is increased by an amount."

**(c)** Set up an equation to solve the problem. (*Step 3*)

**2.** If the same number is added to the numerator and subtracted from the denominator of $\frac{23}{12}$, the resulting fraction is equivalent to $\frac{3}{2}$. What is the number?

**(a)** Let $x =$ ______________. (*Step 2*)

**(b)** Write an expression for "a number is added to the numerator of $\frac{23}{12}$." Then write an expression for "the same number is subtracted from the denominator of $\frac{23}{12}$."

**(c)** Set up an equation to solve the problem. (*Step 3*)

*Solve each problem.* ***See Example 1.***

**3.** In a certain fraction, the denominator is 6 more than the numerator. If 3 is added to both the numerator and the denominator, the resulting fraction is equivalent to $\frac{5}{7}$. What was the original fraction (*not* written in lowest terms)?

**4.** In a certain fraction, the denominator is 4 less than the numerator. If 3 is added to both the numerator and the denominator, the resulting fraction is equivalent to $\frac{3}{2}$. What was the original fraction?

**5.** The numerator of a certain fraction is four times the denominator. If 6 is added to both the numerator and the denominator, the resulting fraction is equivalent to 2. What was the original fraction (*not* written in lowest terms)?

**6.** The denominator of a certain fraction is three times the numerator. If 2 is added to the numerator and subtracted from the denominator, the resulting fraction is equivalent to 1. What was the original fraction (*not* written in lowest terms)?

**7.** One-third of a number is 2 greater than one-sixth of the same number. What is the number?

**8.** One-seventh of a number is 6 greater than the same number. What is the number?

**9.** A quantity, $\frac{2}{3}$ of it, $\frac{1}{2}$ of it, and $\frac{1}{7}$ of it, added together, equals 33. What is the quantity? (*Source:* Rhind Mathematical Papyrus.)

**10.** A quantity, $\frac{3}{4}$ of it, $\frac{1}{2}$ of it, and $\frac{1}{3}$ of it, added together, equals 93. What is the quantity? (*Source:* Rhind Mathematical Papyrus.)

*Solve each problem.* ***See Example 5*** *in* ***Section 2.7 (pages 143 and 144).***

**11.** In 2007, British explorer and endurance swimmer Lewis Gordon Pugh became the first person to swim at the North Pole. He swam 0.6 mi at 0.0319 mi per min in waters created by melted sea ice. What was his time (to three decimal places)? (*Source: The Gazette*.)

**12.** In the 2008 Summer Olympics, Britta Steffen of Germany won the women's 100-m freestyle swimming event. Her rate was 1.8825 m per sec. What was her time (to two decimal places)? (*Source: World Almanac and Book of Facts*.)

**13.** Tirunesh Dibaba of Ethiopia won the women's 5000-m race in the 2008 Olympics with a time of 15.911 min. What was her rate (to three decimal places)? (*Source: World Almanac and Book of Facts*.)

**14.** The winner of the women's 1500-m run in the 2008 Olympics was Nancy Jebet Langat of Kenya with a time of 4.004 min. What was her rate (to three decimal places)? (*Source: World Almanac and Book of Facts*.)

**15.** The winner of the 2008 Daytona 500 (mile) race was Ryan Newman, who drove his Dodge to victory with a rate of 152.672 mph. What was his time (to the nearest thousandth of an hour)? (*Source: World Almanac and Book of Facts*.)

**16.** In 2008, Kasey Kahne drove his Dodge to victory in the Coca-Cola 600 (mile) race. His rate was 135.722 mph. What was his time (to the nearest thousandth of an hour)? (*Source: World Almanac and Book of Facts.*)

***Concept Check*** *Solve each problem.*

**17.** Suppose Stephanie walks $D$ miles at $R$ mph in the same time that Wally walks $d$ miles at $r$ mph. Give an equation relating $D$, $R$, $d$, and $r$.

**18.** If a migrating hawk travels $m$ mph in still air, what is its rate when it flies into a steady headwind of 6 mph? What is its rate with a tailwind of 6 mph?

*Set up the equation you would use to solve each problem. Do not actually solve the equation.* ***See Example 2.***

**19.** Mitch Levy flew his airplane 500 mi against the wind in the same time it took him to fly 600 mi with the wind. If the speed of the wind was 10 mph, what was the rate of his plane in still air? (Let $x$ = rate of the plane in still air.)

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Against the Wind* | 500 | $x - 10$ | |
| *With the Wind* | 600 | $x + 10$ | |

**20.** Janet Sturdy can row 4 mph in still water. She takes as long to row 8 mi upstream as 24 mi downstream. How fast is the current? (Let $x$ = rate of the current.)

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Upstream* | 8 | $4 - x$ | |
| *Downstream* | 24 | $4 + x$ | |

*Solve each problem.* ***See Example 2.***

**21.** A boat can go 20 mi against a current in the same time that it can go 60 mi with the current. The current is 4 mph. Find the rate of the boat in still water.

**22.** Vince Grosso can fly his plane 200 mi against the wind in the same time it takes him to fly 300 mi with the wind. The wind blows at 30 mph. Find the rate of his plane in still air.

**23.** The sanderling is a small shorebird about 6.5 in. long, with a thin, dark bill and a wide, white wing stripe. If a sanderling can fly 30 mi with the wind in the same time it can fly 18 mi against the wind when the wind speed is 8 mph, what is the rate of the bird in still air? (*Source:* U.S. Geological Survey.)

**24.** Airplanes usually fly faster from west to east than from east to west because the prevailing winds go from west to east. The air distance between Chicago and London is about 4000 mi, while the air distance between New York and London is about 3500 mi. If a jet can fly eastbound from Chicago to London in the same time it can fly westbound from London to New York in a 35-mph wind, what is the rate of the plane in still air? (*Source: Encyclopaedia Britannica.*)

**25.** An airplane maintaining a constant airspeed takes as long to go 450 mi with the wind as it does to go 375 mi against the wind. If the wind is blowing at 15 mph, what is the rate of the plane in still air?

**26.** A river has a current of 4 km per hr. Find the rate of Jai Singh's boat in still water if it goes 40 km downstream in the same time that it takes to go 24 km upstream.

**27.** Connie McNair's boat goes 12 mph. Find the rate of the current of the river if she can go 6 mi upstream in the same amount of time she can go 10 mi downstream.

**28.** Howie Sorkin can travel 8 mi upstream in the same time it takes him to go 12 mi downstream. His boat goes 15 mph in still water. What is the rate of the current?

**29.** The distance from Seattle, Washington, to Victoria, British Columbia, is about 148 mi by ferry. It takes about 4 hr less to travel by the same ferry from Victoria to Vancouver, British Columbia, a distance of about 74 mi. What is the average rate of the ferry?

**30.** Driving from Tulsa to Detroit, Dean Loring averaged 50 mph. He figured that if he had averaged 60 mph, his driving time would have decreased 3 hr. How far is it from Tulsa to Detroit?

***Concept Check*** *Solve each problem.*

**31.** If it takes Elayn 10 hr to do a job, what is her rate?

**32.** If it takes Clay 12 hr to do a job, how much of the job does he do in 8 hr?

*In Exercises 33 and 34, set up the equation you would use to solve each problem. Do not actually solve the equation.* ***See Example 3.***

**33.** Working alone, Edward Good can paint a room in 8 hr. Abdalla Elusta can paint the same room working alone in 6 hr. How long will it take them if they work together? (Let $t$ represent the time they work together.)

| | $r$ | $t$ | $w$ |
|---|---|---|---|
| *Edward* | | $t$ | |
| *Abdalla* | | $t$ | |

**34.** Donald Bridgewater can tune up his Chevy in 2 hr working alone. Jeff Bresner can do the job in 3 hr working alone. How long would it take them if they worked together? (Let $t$ represent the time they work together.)

| | $r$ | $t$ | $w$ |
|---|---|---|---|
| *Donald* | | $t$ | |
| *Jeff* | | $t$ | |

*Solve each problem.* ***See Example 3.***

**35.** Heather Schaefer, a high school mathematics teacher, gave a test on perimeter, area, and volume to her geometry classes. Working alone, it would take her 4 hr to grade the tests. Her student teacher, Courtney Slade, would take 6 hr to grade the same tests. How long would it take them to grade these tests if they work together?

**36.** Zachary and Samuel are brothers who share a bedroom. By himself, Zachary can completely mess up their room in 20 min, while it would take Samuel only 12 min to do the same thing. How long would it take them to mess up the room together?

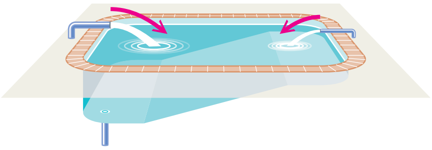
**37.** A pump can pump the water out of a flooded basement in 10 hr. A smaller pump takes 12 hr. How long would it take to pump the water from the basement with both pumps?

**38.** Lou Viggiano's copier can do a printing job in 7 hr. Nora Demosthenes' copier can do the same job in 12 hr. How long would it take to do the job with both copiers?

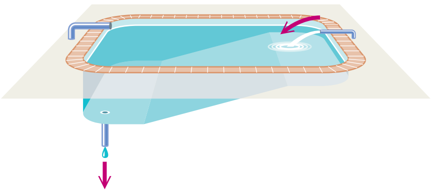
**39.** An experienced employee can enter tax data into a computer twice as fast as a new employee. Working together, it takes the employees 2 hr. How long would it take the experienced employee working alone?

**40.** One roofer can put a new roof on a house three times faster than another. Working together, they can roof a house in 4 days. How long would it take the faster roofer working alone?

**41.** One pipe can fill a swimming pool in 6 hr, and another pipe can do it in 9 hr. How long will it take the two pipes working together to fill the pool $\frac{3}{4}$ full?



**42.** An inlet pipe can fill a swimming pool in 9 hr, and an outlet pipe can empty the pool in 12 hr. Through an error, both pipes are left open. How long will it take to fill the pool?



***Brain Busters*** *Extend the concepts of* ***Example 3*** *to solve each problem.*

**43.** A cold-water faucet can fill a sink in 12 min, and a hot-water faucet can fill it in 15 min. The drain can empty the sink in 25 min. If both faucets are on and the drain is open, how long will it take to fill the sink?

**44.** Refer to **Exercise 42.** Assume that the error was discovered after both pipes had been running for 3 hr and the outlet pipe was then closed. How much more time would then be required to fill the pool? (*Hint:* Consider how much of the job had been done when the error was discovered.)

## PREVIEW EXERCISES

*Find each quotient.* ***See Section 1.6.***

**45.** $\frac{6-2}{5-3}$ **46.** $\frac{5-7}{-4-2}$ **47.** $\frac{4-(-1)}{-3-(-5)}$

**48.** $\frac{-6-0}{0-(-3)}$ **49.** $\frac{-5-(-5)}{3-2}$ **50.** $\frac{7-(-2)}{-3-(-3)}$

*Solve each equation for y.* ***See Section 2.5.***

**51.** $3x + 2y = 8$ **52.** $4x + 3y = 0$

# CHAPTER 6 SUMMARY

## KEY TERMS

| 6.1 | 6.3 | 6.5 | 6.6 |
|---|---|---|---|
| rational expression<br>lowest terms | least common denominator (LCD) | complex fraction | proposed solution<br>extraneous solution (value) |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **rational expression** is
- **A.** an algebraic expression made up of a term or the sum of a finite number of terms with real coefficients and whole number exponents
- **B.** a polynomial equation of degree 2
- **C.** an expression with one or more fractions in the numerator, or denominator, or both
- **D.** the quotient of two polynomials with denominator not 0.

**2.** In a given set of fractions, the **least common denominator** is
- **A.** the smallest denominator of all the denominators
- **B.** the smallest expression that is divisible by all the denominators
- **C.** the largest integer that evenly divides the numerator and denominator of all the fractions
- **D.** the largest denominator of all the denominators.

**3.** A **complex fraction** is
- **A.** an algebraic expression made up of a term or the sum of a finite number of terms with real coefficients and whole number exponents
- **B.** a polynomial equation of degree 2
- **C.** a quotient with one or more fractions in the numerator, or denominator, or both
- **D.** the quotient of two polynomials with denominator not 0.

### ANSWERS

**1.** D; *Examples:* $-\frac{3}{4y}, \frac{5x^3}{x+2}, \frac{a+3}{a^2-4a-5}$ **2.** B; *Example:* The LCD of $\frac{1}{x}, \frac{2}{3}$, and $\frac{5}{x+1}$ is $3x(x+1)$. **3.** C; *Examples:* $\frac{\frac{2}{3}}{\frac{4}{7}}, \frac{x-\frac{1}{y}}{x+\frac{1}{y}}, \frac{\frac{2}{a+1}}{a^2-1}$

## QUICK REVIEW

| CONCEPTS | EXAMPLES |
|---|---|
| **6.1 The Fundamental Property of Rational Expressions**<br>To find the value(s) for which a rational expression is undefined, set the denominator equal to 0 and solve the equation. | Find the values for which the expression $\frac{x-4}{x^2-16}$ is undefined.<br>$x^2-16=0$<br>$(x-4)(x+4)=0$ Factor.<br>$x-4=0$ or $x+4=0$ Zero-factor property<br>$x=4$ or $x=-4$ Solve for $x$.<br>The rational expression is undefined for 4 and $-4$, so $x \neq 4$ and $x \neq -4$. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Writing a Rational Expression in Lowest Terms**<br>***Step 1*** Factor the numerator and denominator.<br>***Step 2*** Use the fundamental property to divide out common factors. | Write in lowest terms. $\frac{x^2-1}{(x-1)^2}$<br>$=\frac{(x-1)(x+1)}{(x-1)(x-1)}$<br>$=\frac{x+1}{x-1}$ |
| **6.2 Multiplying and Dividing Rational Expressions**<br>**Multiplying or Dividing Rational Expressions**<br>***Step 1*** Note the operation. If the operation is division, use the definition of division to rewrite as multiplication.<br>***Step 2*** Multiply numerators and multiply denominators.<br>***Step 3*** Factor numerators and denominators completely.<br>***Step 4*** Write in lowest terms, using the fundamental property.<br>***Note: Steps 2 and 3 may be interchanged based on personal preference.*** | Multiply. $\frac{3x+9}{x-5}\cdot\frac{x^2-3x-10}{x^2-9}$<br>$=\frac{(3x+9)(x^2-3x-10)}{(x-5)(x^2-9)}$ Multiply numerators and denominators.<br>$=\frac{3(x+3)(x-5)(x+2)}{(x-5)(x+3)(x-3)}$ Factor.<br>$=\frac{3(x+2)}{x-3}$ Lowest terms<br>Divide. $\frac{2x+1}{x+5}\div\frac{6x^2-x-2}{x^2-25}$<br>$=\frac{2x+1}{x+5}\cdot\frac{x^2-25}{6x^2-x-2}$ Multiply by the reciprocal of the divisor.<br>$=\frac{(2x+1)(x^2-25)}{(x+5)(6x^2-x-2)}$ Multiply numerators and denominators.<br>$=\frac{(2x+1)(x+5)(x-5)}{(x+5)(2x+1)(3x-2)}$ Factor.<br>$=\frac{x-5}{3x-2}$ Lowest terms |
| **6.3 Least Common Denominators**<br>**Finding the LCD**<br>***Step 1*** Factor each denominator into prime factors.<br>***Step 2*** List each different factor the greatest number of times it appears.<br>***Step 3*** Multiply the factors from Step 2 to get the LCD.<br>**Writing a Rational Expression with a Specified Denominator**<br>***Step 1*** Factor both denominators.<br>***Step 2*** Decide what factor(s) the denominator must be multiplied by in order to equal the specified denominator.<br>***Step 3*** Multiply the rational expression by that factor divided by itself. (That is, multiply by 1.) | Find the LCD for $\frac{3}{k^2-8k+16}$ and $\frac{1}{4k^2-16k}$.<br>$\left.\begin{aligned}k^2-8k+16&=(k-4)^2\\4k^2-16k&=4k(k-4)\end{aligned}\right\}$ Factor each denominator.<br>$\text{LCD}=(k-4)^2\cdot 4\cdot k$<br>$=4k(k-4)^2$<br>Find the numerator. $\frac{5}{2z^2-6z}=\frac{?}{4z^3-12z^2}$<br>$\frac{5}{2z(z-3)}=\frac{?}{4z^2(z-3)}$<br>$2z(z-3)$ must be multiplied by $2z$ in order to obtain $4z^2(z-3)$.<br>$\frac{5}{2z(z-3)}\cdot\frac{2z}{2z}=\frac{10z}{4z^2(z-3)}=\frac{10z}{4z^3-12z^2}$ |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **6.4 Adding and Subtracting Rational Expressions** | |
| **Adding Rational Expressions** | Add. $\frac{2}{3m+6}+\frac{m}{m^2-4}$ |
| ***Step 1*** Find the LCD. | $\left.\begin{aligned}3m+6&=3(m+2)\\ m^2-4&=(m+2)(m-2)\end{aligned}\right\}$ The LCD is $3(m+2)(m-2)$. |
| ***Step 2*** Rewrite each rational expression with the LCD as denominator. | $=\frac{2(m-2)}{3(m+2)(m-2)}+\frac{3m}{3(m+2)(m-2)}$ Write with the LCD. |
| ***Step 3*** Add the numerators to get the numerator of the sum. The LCD is the denominator of the sum. | $=\frac{2m-4+3m}{3(m+2)(m-2)}$ Add numerators and keep the same denominator. |
| ***Step 4*** Write in lowest terms. | $=\frac{5m-4}{3(m+2)(m-2)}$ Combine like terms. |
| **Subtracting Rational Expressions**<br>Follow the same steps as for addition, but subtract in Step 3. | Subtract. $\frac{6}{k+4}-\frac{2}{k}$ The LCD is $k(k+4)$.<br>$=\frac{6k}{(k+4)k}-\frac{2(k+4)}{k(k+4)}$ Write with the LCD.<br>$=\frac{6k-2(k+4)}{k(k+4)}$ Subtract numerators and keep the same denominator.<br>$=\frac{6k-2k-8}{k(k+4)}$ Distributive property<br>$=\frac{4k-8}{k(k+4)}$ Combine like terms. |
| **6.5 Complex Fractions** | |
| **Simplifying Complex Fractions** | Simplify. |
| ***Method 1*** Simplify the numerator and denominator separately. Then divide the simplified numerator by the simplified denominator. | ***Method 1*** $\frac{\frac{1}{a}-a}{1-a}=\frac{\frac{1}{a}-\frac{a^2}{a}}{1-a}=\frac{\frac{1-a^2}{a}}{1-a}$<br>$=\frac{1-a^2}{a}\div(1-a)$<br>$=\frac{1-a^2}{a}\cdot\frac{1}{1-a}$ Multiply by the reciprocal of the divisor.<br>$=\frac{(1-a)(1+a)}{a(1-a)}=\frac{1+a}{a}$ |
| ***Method 2*** Multiply the numerator and denominator of the complex fraction by the LCD of all the denominators in the complex fraction. Write in lowest terms. | ***Method 2*** $\frac{\frac{1}{a}-a}{1-a}=\frac{\left(\frac{1}{a}-a\right)a}{(1-a)a}=\frac{\frac{a}{a}-a^2}{(1-a)a}$<br>$=\frac{1-a^2}{(1-a)a}=\frac{(1+a)(1-a)}{(1-a)a}$<br>$=\frac{1+a}{a}$ |

(continued)

**CONCEPTS**

**EXAMPLES**

## 6.6 Solving Equations with Rational Expressions

**Solving Equations with Rational Expressions**

***Step 1*** Multiply each side of the equation by the LCD to clear the equation of fractions. Be sure to distribute to *every* term on *both* sides.

***Step 2*** Solve the resulting equation.

***Step 3*** Check each proposed solution.

Solve. $\dfrac{x}{x-3} + \dfrac{4}{x+3} = \dfrac{18}{x^2-9}$

$$\frac{x}{x-3} + \frac{4}{x+3} = \frac{18}{(x-3)(x+3)} \qquad \text{Factor.}$$

The LCD is $(x-3)(x+3)$. Note that 3 and $-3$ cannot be solutions, as they cause a denominator to equal 0.

$$(x-3)(x+3)\left(\frac{x}{x-3} + \frac{4}{x+3}\right)$$
$$= (x-3)(x+3)\frac{18}{(x-3)(x+3)} \qquad \text{Multiply by the LCD.}$$

$$\begin{aligned} x(x+3) + 4(x-3) &= 18 && \text{Distributive property} \\ x^2 + 3x + 4x - 12 &= 18 && \text{Distributive property} \\ x^2 + 7x - 30 &= 0 && \text{Standard form} \\ (x-3)(x+10) &= 0 && \text{Factor.} \\ x - 3 = 0 \quad \text{or} \quad x + 10 &= 0 && \text{Zero-factor property} \\ \text{Reject} \rightarrow x = 3 \quad \text{or} \quad x &= -10 && \text{Solve for } x. \end{aligned}$$

Since 3 causes denominators to equal 0, the only solution is $-10$. Thus, $\{-10\}$ is the solution set.

## 6.7 Applications of Rational Expressions

**Solving Problems about Distance, Rate, and Time**
Use the formulas relating $d$, $r$, and $t$.

$$d = rt, \quad r = \frac{d}{t}, \quad t = \frac{d}{r}$$

**Solving Problems about Work**

***Step 1*** Read the problem carefully.

***Step 2*** Assign a variable. State what the variable represents. Put the information from the problem into a table. If a job is done in $t$ units of time, the rate is $\frac{1}{t}$.

***Step 3*** Write an equation. The sum of the fractional parts should equal 1 (whole job).

***Step 4*** Solve the equation.

***Steps 5 and 6*** State the answer and check the solution.

It takes the regular mail carrier 6 hr to cover her route. A substitute takes 8 hr to cover the same route. How long would it take them to cover the route together?

Let $x =$ the number of hours required to cover the route together.

| | Rate | Time | Part of the Job Done |
|---|---|---|---|
| Regular | $\frac{1}{6}$ | $x$ | $\frac{1}{6}x$ |
| Substitute | $\frac{1}{8}$ | $x$ | $\frac{1}{8}x$ |

$$\frac{1}{6}x + \frac{1}{8}x = 1$$

$$\begin{aligned} 24\left(\frac{1}{6}x + \frac{1}{8}x\right) &= 24(1) && \text{The LCD is 24.} \\ 4x + 3x &= 24 && \text{Distributive property} \\ 7x &= 24 && \text{Combine like terms.} \\ x &= \frac{24}{7} && \text{Divide by 7.} \end{aligned}$$

It would take them $\frac{24}{7}$ hr, or $3\frac{3}{7}$ hr, to cover the route together.

The solution checks because $\frac{1}{6}\left(\frac{24}{7}\right) + \frac{1}{8}\left(\frac{24}{7}\right) = 1$.

# CHAPTER 6

# REVIEW EXERCISES

**6.1** *Find the numerical value of each rational expression for* ***(a)*** *and* ***(b)*** *.*

**1.** $\dfrac{4x-3}{5x+2}$

**2.** $\dfrac{3x}{x^2-4}$

*Find any values of the variable for which each rational expression is undefined. Write answers with the symbol* $\neq$.

**3.** $\dfrac{4}{x-3}$

**4.** $\dfrac{y+3}{2y}$

**5.** $\dfrac{2k+1}{3k^2+17k+10}$

**6.** How do you determine the values of the variable for which a rational expression is undefined?

*Write each rational expression in lowest terms.*

**7.** $\dfrac{5a^3b^3}{15a^4b^2}$

**8.** $\dfrac{m-4}{4-m}$

**9.** $\dfrac{4x^2-9}{6-4x}$

**10.** $\dfrac{4p^2+8pq-5q^2}{10p^2-3pq-q^2}$

*Write four equivalent forms for each rational expression.*

**11.** $-\dfrac{4x-9}{2x+3}$

**12.** $-\dfrac{8-3x}{3-6x}$

**6.2** *Multiply or divide, and write each answer in lowest terms.*

**13.** $\dfrac{18p^3}{6}\cdot\dfrac{24}{p^4}$

**14.** $\dfrac{8x^2}{12x^5}\cdot\dfrac{6x^4}{2x}$

**15.** $\dfrac{x-3}{4}\cdot\dfrac{5}{2x-6}$

**16.** $\dfrac{2r+3}{r-4}\cdot\dfrac{r^2-16}{6r+9}$

**17.** $\dfrac{6a^2+7a-3}{2a^2-a-6}\div\dfrac{a+5}{a-2}$

**18.** $\dfrac{y^2-6y+8}{y^2+3y-18}\div\dfrac{y-4}{y+6}$

**19.** $\dfrac{2p^2+13p+20}{p^2+p-12}\cdot\dfrac{p^2+2p-15}{2p^2+7p+5}$

**20.** $\dfrac{3z^2+5z-2}{9z^2-1}\cdot\dfrac{9z^2+6z+1}{z^2+5z+6}$

**6.3** *Find the least common denominator for the fractions in each list.*

**21.** $\dfrac{4}{9y}, \dfrac{7}{12y^2}, \dfrac{5}{27y^4}$

**22.** $\dfrac{3}{x^2+4x+3}, \dfrac{5}{x^2+5x+4}$

*Rewrite each rational expression with the given denominator.*

**23.** $\dfrac{3}{2a^3}=\dfrac{?}{10a^4}$

**24.** $\dfrac{9}{x-3}=\dfrac{?}{18-6x}$

**25.** $\dfrac{-3y}{2y-10}=\dfrac{?}{50-10y}$

**26.** $\dfrac{4b}{b^2+2b-3}=\dfrac{?}{(b+3)(b-1)(b+2)}$

**6.4** *Add or subtract, and write each answer in lowest terms.*

**27.** $\frac{10}{x} + \frac{5}{x}$

**28.** $\frac{6}{3p} - \frac{12}{3p}$

**29.** $\frac{9}{k} - \frac{5}{k - 5}$

**30.** $\frac{4}{y} + \frac{7}{7 + y}$

**31.** $\frac{m}{3} - \frac{2 + 5m}{6}$

**32.** $\frac{12}{x^2} - \frac{3}{4x}$

**33.** $\frac{5}{a - 2b} + \frac{2}{a + 2b}$

**34.** $\frac{4}{k^2 - 9} - \frac{k + 3}{3k - 9}$

**35.** $\frac{8}{z^2 + 6z} - \frac{3}{z^2 + 4z - 12}$

**36.** $\frac{11}{2p - p^2} - \frac{2}{p^2 - 5p + 6}$

**6.5** *Simplify each complex fraction.*

**37.** $\dfrac{\frac{y - 3}{y}}{\frac{y + 3}{4y}}$

**38.** $\dfrac{\frac{2}{3} - \frac{1}{6}}{\frac{1}{4} + \frac{2}{5}}$

**39.** $\dfrac{x + \frac{1}{w}}{x - \frac{1}{w}}$

**40.** $\dfrac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{q - p}}$

**41.** $\dfrac{\frac{x^2 - 25}{x + 3}}{\frac{x + 5}{x^2 - 9}}$

**42.** $\dfrac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$

**6.6** *Solve each equation, and check your solutions.*

**43.** $\frac{3x - 1}{x - 2} = \frac{5}{x - 2} + 1$

**44.** $\frac{4 - z}{z} + \frac{3}{2} = \frac{-4}{z}$

**45.** $\frac{3}{x + 4} - \frac{2x}{5} = \frac{3}{x + 4}$

**46.** $\frac{3}{m - 2} + \frac{1}{m - 1} = \frac{7}{m^2 - 3m + 2}$

*Solve each formula for the specified variable.*

**47.** $m = \frac{Ry}{t}$ for $t$

**48.** $x = \frac{3y - 5}{4}$ for $y$

**49.** $p^2 = \frac{4}{3m - q}$ for $m$

**6.7** *Solve each problem.*

**50.** In a certain fraction, the denominator is 5 less than the numerator. If 5 is added to both the numerator and the denominator, the resulting fraction is equivalent to $\frac{5}{4}$. Find the original fraction (*not* written in lowest terms).

**51.** The denominator of a certain fraction is six times the numerator. If 3 is added to the numerator and subtracted from the denominator, the resulting fraction is equivalent to $\frac{2}{5}$. Find the original fraction (*not* written in lowest terms).

**52.** A plane flies 350 mi with the wind in the same time that it can fly 310 mi against the wind. The plane has a speed of 165 mph in still air. Find the speed of the wind.

**53.** Susan Costa can plant her garden in 5 hr working alone. A friend can do the same job in 8 hr. How long would it take them if they worked together?

**54.** The head gardener can mow the lawns in the city park twice as fast as his assistant. Working together, they can complete the job in $1\frac{1}{3}$ hr. How long would it take the head gardener working alone?

## MIXED REVIEW EXERCISES

*Perform each indicated operation.*

**55.** $\dfrac{4}{m-1}-\dfrac{3}{m+1}$

**56.** $\dfrac{8p^5}{5}\div\dfrac{2p^3}{10}$

**57.** $\dfrac{r-3}{8}\div\dfrac{3r-9}{4}$

**58.** $\dfrac{t^{-2}+s^{-2}}{t^{-1}-s^{-1}}$

**59.** $\dfrac{\frac{5}{x}-1}{\frac{5-x}{3x}}$

**60.** $\dfrac{4}{z^2-2z+1}-\dfrac{3}{z^2-1}$

**61.** $\dfrac{1}{t^2-4}+\dfrac{1}{2-t}$

*Solve.*

**62.** $\dfrac{2}{z}-\dfrac{z}{z+3}=\dfrac{1}{z+3}$

**63.** $a=\dfrac{v-w}{t}$ for $v$

**64.** Rob Fusco flew his plane 400 km with the wind in the same time it took him to go 200 km against the wind. The speed of the wind is 50 km per hr. Find the rate of the plane in still air.

**65.** With spraying equipment, Lizette Foley can paint the woodwork in a small house in 8 hr. Seyed Sadati needs 14 hr to complete the same job painting by hand. If Lizette and Seyed work together, how long will it take them to paint the woodwork?

## RELATING CONCEPTS EXERCISES 66–75

### FOR INDIVIDUAL OR GROUP WORK

*In these exercises, we summarize the various concepts involving rational expressions.* ***Work Exercises 66–75 in order.***

*Let P, Q, and R be rational expressions defined as follows:*

$$P=\frac{6}{x+3},\qquad Q=\frac{5}{x+1},\qquad R=\frac{4x}{x^2+4x+3}.$$

**66.** Find the value or values for which the expression is undefined.

**(a)** $P$ **(b)** $Q$ **(c)** $R$

**67.** Find and express $(P\cdot Q)\div R$ in lowest terms.

**68.** Why is $(P\cdot Q)\div R$ not defined if $x=0$?

**69.** Find the LCD for $P$, $Q$, and $R$.

**70.** Perform the operations and express $P+Q-R$ in lowest terms.

**71.** Simplify the complex fraction $\frac{P+Q}{R}$.

**72.** Solve the equation $P+Q=R$.

**73.** How does your answer to **Exercise 66** help you work **Exercise 72?**

**74.** Suppose that a car travels 6 miles in $(x+3)$ minutes. Explain why $P$ represents the rate of the car (in miles per minute).

**75.** For what value or values of $x$ is $R=\frac{40}{77}$?

# CHAPTER 6 TEST

CHAPTER Test Prep

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

**1.** Find the numerical value of $\dfrac{6r + 1}{2r^2 - 3r - 20}$ for **(a)** $r = -2$ and **(b)** $r = 4$.

**2.** Find any values for which $\dfrac{3x - 1}{x^2 - 2x - 8}$ is undefined. Write your answer with the symbol $\neq$.

**3.** Write four rational expressions equivalent to $-\dfrac{6x - 5}{2x + 3}$.

*Write each rational expression in lowest terms.*

**4.** $\dfrac{-15x^6y^4}{5x^4y}$

**5.** $\dfrac{6a^2 + a - 2}{2a^2 - 3a + 1}$

*Multiply or divide. Write each answer in lowest terms.*

**6.** $\dfrac{5(d - 2)}{9} \div \dfrac{3(d - 2)}{5}$

**7.** $\dfrac{6k^2 - k - 2}{8k^2 + 10k + 3} \cdot \dfrac{4k^2 + 7k + 3}{3k^2 + 5k + 2}$

**8.** $\dfrac{4a^2 + 9a + 2}{3a^2 + 11a + 10} \div \dfrac{4a^2 + 17a + 4}{3a^2 + 2a - 5}$

**9.** $\dfrac{x^2 - 10x + 25}{9 - 6x + x^2} \cdot \dfrac{x - 3}{5 - x}$

*Find the least common denominator for the fractions in each list.*

**10.** $\dfrac{-3}{10p^2}, \dfrac{21}{25p^3}, \dfrac{-7}{30p^5}$

**11.** $\dfrac{r + 1}{2r^2 + 7r + 6}, \dfrac{-2r + 1}{2r^2 - 7r - 15}$

*Rewrite each rational expression with the given denominator.*

**12.** $\dfrac{15}{4p} = \dfrac{?}{64p^3}$

**13.** $\dfrac{3}{6m - 12} = \dfrac{?}{42m - 84}$

*Add or subtract. Write each answer in lowest terms.*

**14.** $\dfrac{4x + 2}{x + 5} + \dfrac{-2x + 8}{x + 5}$

**15.** $\dfrac{-4}{y + 2} + \dfrac{6}{5y + 10}$

**16.** $\dfrac{x + 1}{3 - x} + \dfrac{x^2}{x - 3}$

**17.** $\dfrac{3}{2m^2 - 9m - 5} - \dfrac{m + 1}{2m^2 - m - 1}$

*Simplify each complex fraction.*

**18.** $\dfrac{\dfrac{2p}{k^2}}{\dfrac{3p^2}{k^3}}$

**19.** $\dfrac{\dfrac{1}{x + 3} - 1}{1 + \dfrac{1}{x + 3}}$

**20.** $\dfrac{2x^{-2} + y^{-2}}{x^{-1} - y^{-1}}$

*Solve.*

**21.** $\dfrac{3x}{x + 1} = \dfrac{3}{2x}$

**22.** $\dfrac{2x}{x - 3} + \dfrac{1}{x + 3} = \dfrac{-6}{x^2 - 9}$

**23.** $F = \dfrac{k}{d - D}$ for $D$

*Solve each problem.*

**24.** A boat goes 7 mph in still water. It takes as long to go 20 mi upstream as 50 mi downstream. Find the rate of the current.

**25.** Sanford Geraci can paint a room in his house, working alone, in 5 hr. His neighbor can do the job in 4 hr. How long will it take them to paint the room if they work together?

# CHAPTERS 1–6

# CUMULATIVE REVIEW EXERCISES

**1.** Use the order of operations to evaluate $3 + 4\left(\frac{1}{2} - \frac{3}{4}\right)$.

*Solve.*

**2.** $3(2y - 5) = 2 + 5y$

**3.** $\mathcal{A} = \frac{1}{2}bh$ for $b$

**4.** $\frac{2 + m}{2 - m} = \frac{3}{4}$

**5.** $5y \leq 6y + 8$

**6.** Consider the graph of $4x + 3y = -12$.

**(a)** What is the $x$-intercept? **(b)** What is the $y$-intercept?

*Sketch each graph.*

**7.** $y = -3x + 2$

**8.** $y = -x^2 + 1$

*Simplify each expression. Write with only positive exponents.*

**9.** $\frac{(2x^3)^{-1} \cdot x}{2^3x^5}$

**10.** $\frac{(m^{-2})^3m}{m^5m^{-4}}$

*Perform each indicated operation.*

**11.** $(2k^2 + 3k) - (k^2 + k - 1)$

**12.** $(2a - b)^2$

**13.** $(y^2 + 3y + 5)(3y - 1)$

**14.** $\frac{12p^3 + 2p^2 - 12p + 4}{2p - 2}$

*Factor completely.*

**15.** $8t^2 + 10tv + 3v^2$

**16.** $8r^2 - 9rs + 12s^2$

**17.** $16x^4 - 1$

*Solve each equation.*

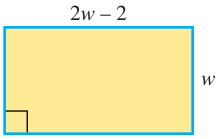
**18.** $r^2 = 2r + 15$

**19.** $(r - 5)(2r + 1)(3r - 2) = 0$

*Solve each problem.*

**20.** One number is 4 greater than another. The product of the numbers is 2 less than the lesser number. Find the lesser number.

**21.** The length of a rectangle is 2 m less than twice the width. The area is 60 m$^2$. Find the width of the rectangle.



**22.** Which one of the following is equal to 1 for *all* real numbers?

**A.** $\frac{k^2 + 2}{k^2 + 2}$ **B.** $\frac{4 - m}{4 - m}$ **C.** $\frac{2x + 9}{2x + 9}$ **D.** $\frac{x^2 - 1}{x^2 - 1}$

**23.** Which one of the following rational expressions is *not* equivalent to $\frac{4 - 3x}{7}$?

**A.** $-\frac{-4 + 3x}{7}$ **B.** $-\frac{4 - 3x}{-7}$ **C.** $\frac{-4 + 3x}{-7}$ **D.** $\frac{-(3x + 4)}{7}$

*Perform each operation and write the answer in lowest terms.*

**24.** $\frac{5}{q} - \frac{1}{q}$

**25.** $\frac{3}{7} + \frac{4}{r}$

**26.** $\frac{4}{5q - 20} - \frac{1}{3q - 12}$

**27.** $\frac{2}{k^2 + k} - \frac{3}{k^2 - k}$

**28.** $\frac{7z^2 + 49z + 70}{16z^2 + 72z - 40} \div \frac{3z + 6}{4z^2 - 1}$

**29.** $\dfrac{\frac{4}{a} + \frac{5}{2a}}{\frac{7}{6a} - \frac{1}{5a}}$

*Solve each equation. Check your solutions.*

**30.** $\frac{r + 2}{5} = \frac{r - 3}{3}$

**31.** $\frac{1}{x} = \frac{1}{x + 1} + \frac{1}{2}$

**32.** Jody Harris can weed the yard in 3 hr. Pat Tabler can weed the same yard in 2 hr. How long will it take them if they work together?

CHAPTER 7

# Graphs, Linear Equations, and Functions

The two most common measures of temperature are Fahrenheit (F) and Celsius (C). It is fairly common knowledge that water freezes at 32°F, or 0°C, and boils at 212°F, or 100°C. Because there is a *linear* relationship between the Fahrenheit and Celsius temperature scales, using these two equivalences we can derive the familiar formulas for converting from one temperature scale to the other, as seen in **Section 7.2, Exercises 93–100.**

Graphs are widely used in the media because they present a great deal of information in a concise form. In this chapter, we see how information such as the relationship between the two temperature scales can be depicted by graphs.

# 7.1 Review of Graphs and Slopes of Lines

**OBJECTIVES**

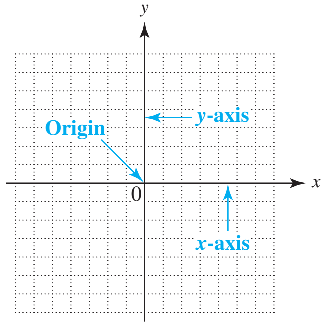
1. Plot ordered pairs.
2. Graph lines and find intercepts.
3. Recognize equations of horizontal and vertical lines and lines passing through the origin.
4. Use the midpoint formula.
5. Find the slope of a line.
6. Graph a line, given its slope and a point on the line.
7. Use slopes to determine whether two lines are parallel, perpendicular, or neither.
8. Solve problems involving average rate of change.

This section and the next review and extend some of the main topics of linear equations in two variables, first introduced in **Chapter 3.**

**OBJECTIVE 1 Plot ordered pairs.** Each of the pairs of numbers

$$(3, 2), \quad (-5, 6), \quad \text{and} \quad (4, -1)$$

is an example of an **ordered pair**—that is, a pair of numbers written within parentheses, consisting of a **first component** and a **second component.** We graph an ordered pair by using two perpendicular number lines that intersect at their 0 points, as shown in the plane in **FIGURE 1**. The common 0 point is called the **origin.**



Rectangular coordinate system

**FIGURE 1**

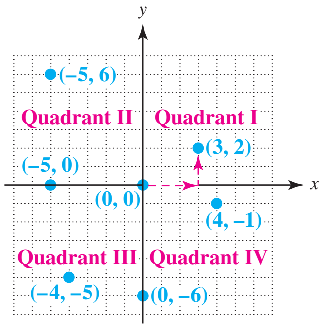


**FIGURE 2**

The position of any point in this plane is determined by referring to the horizontal number line, or ***x*-axis,** and the vertical number line, or ***y*-axis.** The $x$-axis and $y$-axis make up a **rectangular coordinate system,** also called a **Cartesian coordinate system** after René Descartes, the French mathematician credited with its invention.

In an ordered pair, the first component indicates position relative to the $x$-axis, and the second component indicates position relative to the $y$-axis. For example, to locate, or **plot,** the point on the graph that corresponds to the ordered pair $(3, 2)$, we move three units from 0 to the right along the $x$-axis and then two units up parallel to the $y$-axis. See **FIGURE 2**. The numbers in an ordered pair are called the **coordinates** of the corresponding point.

The four regions of the graph, shown in **FIGURE 2**, are called **quadrants I, II, III, and IV,** reading counterclockwise from the upper right quadrant. ***The points on the x-axis and y-axis do not belong to any quadrant.***

**OBJECTIVE 2 Graph lines and find intercepts.** Each solution of an equation with two variables, such as

$$2x + 3y = 6,$$

includes two numbers, one for each variable. To keep track of which number goes with which variable, we write the solutions as ordered pairs. ***(If x and y are used as the variables, the x-value is given first.)***

For example, we can show that $(6, -2)$ is a solution of the equation $2x + 3y = 6$ by substitution.

$$
\begin{aligned}
2x + 3y &= 6 \\
2(6) + 3(-2) &\overset{?}{=} 6 && \text{Let } x = 6,\ y = -2. \\
12 - 6 &\overset{?}{=} 6 && \text{Multiply.} \\
6 &= 6 \ \checkmark && \text{True}
\end{aligned}
$$

Use parentheses to avoid errors.

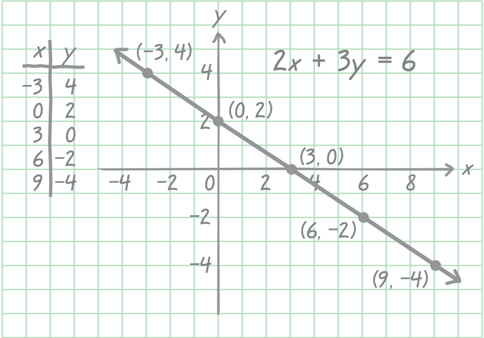
Because the ordered pair $(6, -2)$ makes the equation true, it is a solution. On the other hand, $(5, 1)$ is *not* a solution of the equation $2x + 3y = 6$.

$$
\begin{aligned}
2x + 3y &= 6 \\
2(5) + 3(1) &\overset{?}{=} 6 && \text{Let } x = 5,\ y = 1. \\
10 + 3 &\overset{?}{=} 6 && \text{Multiply.} \\
13 &= 6 && \text{False}
\end{aligned}
$$

To find ordered pairs that satisfy an equation, select a number for one of the variables, substitute it into the equation for that variable, and solve for the other variable. Two other ordered pairs satisfying $2x + 3y = 6$ are $(0, 2)$ and $(3, 0)$.

***Since any real number could be selected for one variable and would lead to a real number for the other variable, linear equations in two variables have an infinite number of solutions.***

The **graph of an equation** is the set of points corresponding to *all* ordered pairs that satisfy the equation. It gives a "picture" of the equation. The graph of the equation $2x + 3y = 6$ is shown in **FIGURE 3** along with a table of ordered pairs.



| x | y |
|---|---|
| −3 | 4 |
| 0 | 2 |
| 3 | 0 |
| 6 | −2 |
| 9 | −4 |

FIGURE 3

The equation $2x + 3y = 6$ is called a **first-degree equation,** because it has no term with a variable to a power greater than 1.

***The graph of any first-degree equation in two variables is a straight line.***

Since first-degree equations with two variables have straight-line graphs, they are called *linear equations in two variables.*

### Linear Equation in Two Variables

A **linear equation in two variables** can be written in the form

$$\boldsymbol{Ax + By = C,}$$

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0. This form is called **standard form.**

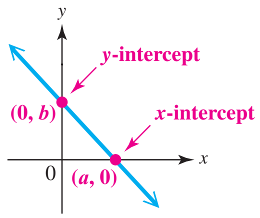


**FIGURE 4**

A straight line is determined if any two different points on the line are known. Two useful points for graphing are the $x$- and $y$-intercepts. The ***x*-intercept** is the point (if any) where the line intersects the $x$-axis. The ***y*-intercept** is the point (if any) where the line intersects the $y$-axis.* See **FIGURE 4**.

The $y$-value of the point where the line intersects the $x$-axis is 0. Similarly, the $x$-value of the point where the line intersects the $y$-axis is 0. This suggests a method for finding the $x$- and $y$-intercepts.

### Finding Intercepts

When graphing the equation of a line, find the intercepts as follows.

Let $y = 0$ to find the $x$-intercept.

Let $x = 0$ to find the $y$-intercept.

*NOW TRY EXERCISE 1*

Find the $x$- and $y$-intercepts, and graph the equation.

$$x - 2y = 4$$

### *EXAMPLE 1* Finding Intercepts

Find the $x$- and $y$-intercepts of $4x - y = -3$ and graph the equation.

To find the $x$-intercept, let $y = 0$.

$$\begin{aligned} 4x - y &= -3 \\ 4x - 0 &= -3 && \text{Let } y = 0. \\ 4x &= -3 \\ x &= -\frac{3}{4} && x\text{-intercept is } \left(-\tfrac{3}{4}, 0\right). \end{aligned}$$

To find the $y$-intercept, let $x = 0$.

$$\begin{aligned} 4x - y &= -3 \\ 4(0) - y &= -3 && \text{Let } x = 0. \\ -y &= -3 \\ y &= 3 && y\text{-intercept is } (0, 3). \end{aligned}$$

The intercepts of $4x - y = -3$ are the points $\left(-\frac{3}{4}, 0\right)$ and $(0, 3)$. Verify by substitution that $(-2, -5)$ also satisfies the equation. We use these ordered pairs to draw the graph in **FIGURE 5**.

| $x$ | $y$ |
|---|---|
| $-\frac{3}{4}$ | 0 |
| 0 | 3 |
| $-2$ | $-5$ |

Use a third point as a check.

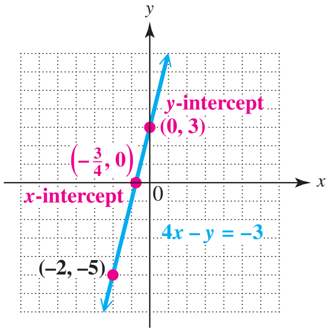


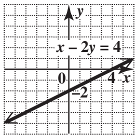
**FIGURE 5**

*NOW TRY*

**NOTE** While two points, such as the two intercepts in **FIGURE 5**, are sufficient to graph a straight line, ***it is a good idea to use a third point to guard against errors.***

*NOW TRY ANSWER*

**1.** $x$-intercept: $(4, 0)$; $y$-intercept: $(0, -2)$



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*Some texts define an intercept as a number, not a point. For example, "$y$-intercept $(0, 4)$" would be given as "$y$-intercept 4."

**OBJECTIVE 3 Recognize equations of horizontal and vertical lines and lines passing through the origin.** A line parallel to the $x$-axis will not have an $x$-intercept. Similarly, a line parallel to the $y$-axis will not have a $y$-intercept. We graph these types of lines in the next two examples.

*NOW TRY EXERCISE 2*
Graph $y = -2$.

### EXAMPLE 2 Graphing a Horizontal Line

Graph $y = 2$.

Writing $y = 2$ as $0x + 1y = 2$ shows that any value of $x$, including $x = 0$, gives $y = 2$. Thus, the $y$-intercept is $(0, 2)$. Since $y$ is always 2, there is no value of $x$ corresponding to $y = 0$, so the graph has no $x$-intercept. The graph is shown with a table of ordered pairs in **FIGURE 6**. It is a horizontal line.

| $x$ | $y$ |
|---|---|
| $-1$ | 2 |
| 0 | 2 |
| 3 | 2 |

To graph $y = 2$, do *not* simply graph the point $(0, 2)$. The graph is a *line*.

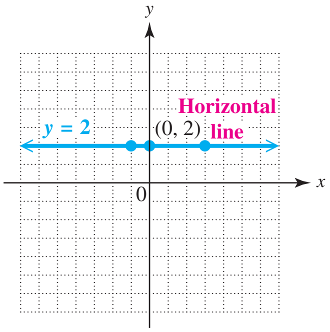


**FIGURE 6**

NOW TRY

**NOTE** The horizontal line $y = 0$ is the $x$-axis.

*NOW TRY EXERCISE 3*
Graph $x + 3 = 0$.

### EXAMPLE 3 Graphing a Vertical Line

Graph $x + 1 = 0$.

The form $1x + 0y = -1$ shows that every value of $y$ leads to $x = -1$, making the $x$-intercept $(-1, 0)$. No value of $y$ makes $x = 0$, so the graph has no $y$-intercept. A straight line that has no $y$-intercept is vertical. See **FIGURE 7**.

| $x$ | $y$ |
|---|---|
| $-1$ | $-4$ |
| $-1$ | 0 |
| $-1$ | 5 |

To graph $x + 1 = 0$ (or $x = -1$), do *not* simply graph the point $(-1, 0)$. The graph is a *line*.

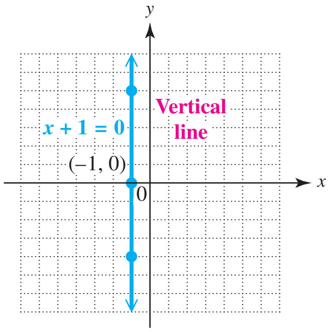
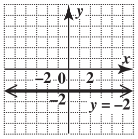


**FIGURE 7**

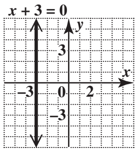
NOW TRY

*NOW TRY ANSWERS*

2.



3.



**NOTE** The vertical line $x = 0$ is the $y$-axis.

NOW TRY EXERCISE 4

Graph $2x + 3y = 0$.

## EXAMPLE 4 Graphing a Line That Passes through the Origin

Graph $x + 2y = 0$.

Find the $x$-intercept.

$$\begin{aligned} x + 2y &= 0 \\ x + 2(0) &= 0 && \text{Let } y = 0. \\ x + 0 &= 0 && \text{Multiply.} \\ x &= 0 && x\text{-intercept is } (0, 0). \end{aligned}$$

Find the $y$-intercept.

$$\begin{aligned} x + 2y &= 0 \\ 0 + 2y &= 0 && \text{Let } x = 0. \\ 2y &= 0 && \text{Add.} \\ y &= 0 && y\text{-intercept is } (0, 0). \end{aligned}$$

Both intercepts are the same point, $(0, 0)$, which means that the graph passes through the origin. To find another point, choose any nonzero number for $x$ or $y$ and solve for the other variable. We choose $x = 4$.

$$\begin{aligned} x + 2y &= 0 \\ 4 + 2y &= 0 && \text{Let } x = 4. \\ 2y &= -4 && \text{Subtract 4.} \\ y &= -2 && \text{Divide by 2.} \end{aligned}$$

This gives the ordered pair $(4, -2)$. As a check, verify that $(-2, 1)$ also lies on the line. The graph is shown in **FIGURE 8**.

| $x$ | $y$ |
|---|---|
| −2 | 1 |
| 0 | 0 |
| 4 | −2 |

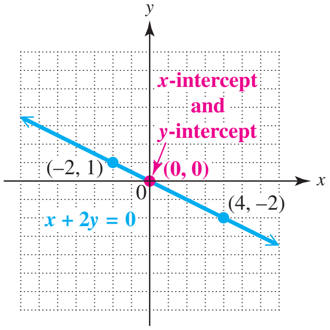


**FIGURE 8**

NOW TRY

**OBJECTIVE 4** **Use the midpoint formula.** If the coordinates of the endpoints of a line segment are known, then the coordinates of the *midpoint* of the segment can be found.

**FIGURE 9** shows a line segment $PQ$ with endpoints $P(-8, 4)$ and $Q(3, -2)$. $R$ is the point with the same $x$-coordinate as $P$ and the same $y$-coordinate as $Q$. So the coordinates of $R$ are $(-8, -2)$.

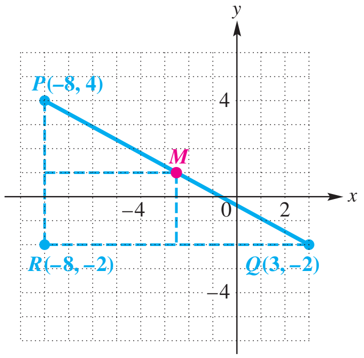
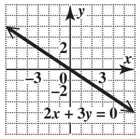


**FIGURE 9**

NOW TRY ANSWER

4.



The $x$-coordinate of the midpoint $M$ of $PQ$ is the same as the $x$-coordinate of the midpoint of $RQ$. Since $RQ$ is horizontal, the $x$-coordinate of its midpoint is the *average* of the $x$-coordinates of its endpoints.

$$\frac{1}{2}(-8+3) = -2.5$$

The $y$-coordinate of $M$ is the average of the $y$-coordinates of the midpoint of $PR$.

$$\frac{1}{2}(4+(-2)) = 1$$

The midpoint of $PQ$ is $M(-2.5, 1)$. This discussion leads to the *midpoint formula*.

### Midpoint Formula

If the endpoints of a line segment $PQ$ are $(x_1, y_1)$ and $(x_2, y_2)$, its midpoint $M$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Recall that the small numbers 1 and 2 in the ordered pairs above are called **subscripts.** Read $(x_1, y_1)$ as **"x-sub-one, y-sub-one."**

**NOW TRY EXERCISE 5**
Find the coordinates of the midpoint of the line segment $PQ$ with endpoints $P(2, -5)$ and $Q(-4, 7)$.

### EXAMPLE 5 Finding the Coordinates of a Midpoint

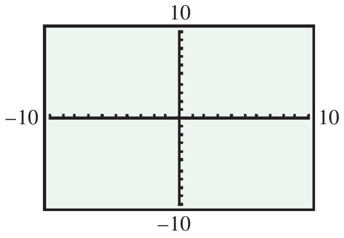
Find the coordinates of the midpoint of line segment $PQ$ with endpoints $P(4, -3)$ and $Q(6, -1)$.

Use the midpoint formula with $x_1 = 4$, $x_2 = 6$, $y_1 = -3$, and $y_2 = -1$.

$$\left(\frac{4+6}{2}, \frac{-3+(-1)}{2}\right) = \left(\frac{10}{2}, \frac{-4}{2}\right) = (5, -2) \longleftarrow \text{Midpoint}$$

NOW TRY

**NOTE** When finding the coordinates of the midpoint of a line segment, we are finding the *average* of the $x$-coordinates and the *average* of the $y$-coordinates of the endpoints of the segment. In both cases, add the corresponding coordinates and divide the sum by 2.



Standard viewing window

**FIGURE 10**

### CONNECTIONS

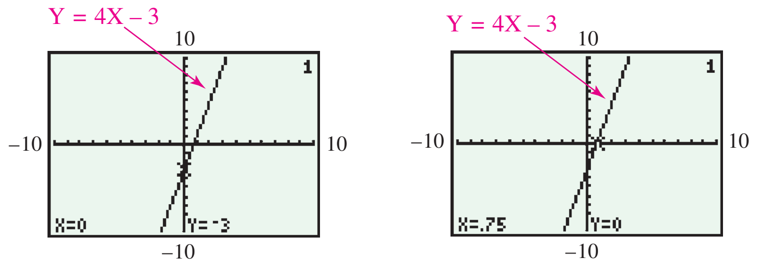
When graphing with a graphing calculator, we must tell the calculator how to set up a rectangular coordinate system. In the screen in **FIGURE 10**, we chose minimum $x$- and $y$-values of $-10$ and maximum $x$- and $y$-values of 10. The **scale** on each axis determines the distance between the tick marks. In the screen shown, the scale is 1 for both axes. We refer to this screen as the **standard viewing window.**

To graph an equation such as $4x - y = 3$, we must solve the equation for $y$ to enter it into the calculator.

$$\begin{aligned} 4x - y &= 3 \\ -y &= -4x + 3 && \text{Subtract } 4x. \\ y &= 4x - 3 && \text{Multiply by } -1. \end{aligned}$$

**NOW TRY ANSWER**
**5.** $(-1, 1)$

The graph of $y = 4x - 3$ in **FIGURE 11** also gives the intercepts at the bottoms of the screens. Some calculators have the capability of locating the $x$-intercept (called “Root” or “Zero”).



**(a)** **(b)**

**FIGURE 11**

## For Discussion or Writing

1. The graphing calculator screens in **Exercise 39** on **page 439** show the graph of a linear equation. What are the intercepts?

Graph each equation with a graphing calculator. Use the standard viewing window.

2. $4x - y = -3$ **(Example 1)**
3. $x + 2y = 0$ **(Example 4)**

**OBJECTIVE 5** **Find the slope of a line.** Slope (steepness) is used in many practical ways. The slope of a highway (sometimes called the *grade*) is often given as a percent. For example, a 10% $\left(\text{or } \frac{10}{100} = \frac{1}{10}\right)$ slope means that the highway rises 1 unit for every 10 horizontal units. Stairs and roofs have slopes too, as shown in **FIGURE 12**.

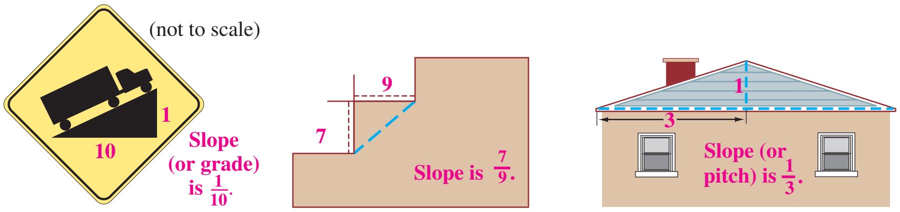


**FIGURE 12**

Slope is the ratio of vertical change, or **rise,** to horizontal change, or **run.** A simple way to remember this is to think, ***“Slope is rise over run.”***

To get a formal definition of the slope of a line, we designate two different points $(x_1, y_1)$ and $(x_2, y_2)$ on the line. See **FIGURE 13**. As we move along the line in **FIGURE 13** from $(x_1, y_1)$ to $(x_2, y_2)$, the $y$-value changes (vertically) from $y_1$ to $y_2$, an amount equal to $y_2 - y_1$. As $y$ changes from $y_1$ to $y_2$, the value of $x$ changes (horizontally) from $x_1$ to $x_2$ by the amount $x_2 - x_1$.

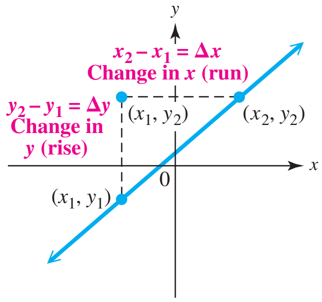


**FIGURE 13**

**NOTE** The Greek letter **delta, Δ,** is used in mathematics to denote “change in,” so $\Delta y$ and $\Delta x$ represent the change in $y$ and the change in $x$, respectively.

The ratio of the change in $y$ to the change in $x$ (the rise over the run) is called the *slope* of the line, with the letter $m$ traditionally used for slope.

**Slope Formula**

The **slope** $m$ of the line through the distinct points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2).$$

**NOW TRY EXERCISE 6**

Find the slope of the line through the points $(2, -6)$ and $(-3, 5)$.

### EXAMPLE 6 Finding the Slope of a Line

Find the slope of the line through the points $(2, -1)$ and $(-5, 3)$.

We let $(2, -1) = (x_1, y_1)$ and $(-5, 3) = (x_2, y_2)$ in the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{-5 - 2} = \frac{4}{-7} = -\frac{4}{7}$$

Thus, the slope is $-\frac{4}{7}$. See **FIGURE 14**.

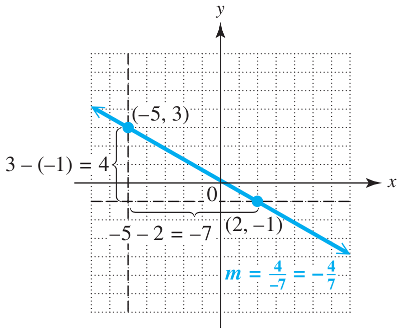


**FIGURE 14**

If we interchange the ordered pairs so that $(-5, 3) = (x_1, y_1)$ and $(2, -1) = (x_2, y_2)$ in the slope formula, the slope is the same.

*y*-values are in the *numerator*, *x*-values in the *denominator*.

$$m = \frac{-1 - 3}{2 - (-5)} = \frac{-4}{7} = -\frac{4}{7}$$

NOW TRY

**Example 6** suggests the following important ideas regarding slope:

1. The slope is the same no matter which point we consider first.
2. Using similar triangles from geometry, we can show that the slope is the same no matter which two different points on the line we choose.

**CAUTION** ***In calculating slope, be careful to subtract the y-values and the x-values in the same order.***

**Correct**

$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}$$

**Incorrect**

$$\cancel{\frac{y_2 - y_1}{x_1 - x_2} \quad \text{or} \quad \frac{y_1 - y_2}{x_2 - x_1}}$$

***The change in y is the numerator and the change in x is the denominator.***

NOW TRY ANSWER

**6.** $-\frac{11}{5}$

NOW TRY EXERCISE 7

Find the slope of the line $3x - 7y = 21$.

## EXAMPLE 7 Finding the Slope of a Line

Find the slope of the line $4x - y = -8$.

The intercepts can be used as the two different points needed to find the slope. Let $y = 0$ to find that the $x$-intercept is $(-2, 0)$. Then let $x = 0$ to find that the $y$-intercept is $(0, 8)$. Use these two points in the slope formula.

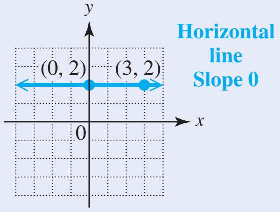
$$m = \frac{\text{rise}}{\text{run}} = \frac{8 - 0}{0 - (-2)} = \frac{8}{2} = 4$$

NOW TRY

We review the following special cases of slope.

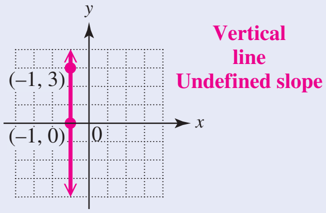
### Horizontal and Vertical Lines

- An equation of the form $y = b$ always intersects the $y$-axis at the point $(0, b)$. The line with that equation is horizontal and has slope 0. See **FIGURE 15**.
- An equation of the form $x = a$ always intersects the $x$-axis at the point $(a, 0)$. The line with that equation is vertical and has undefined slope. See **FIGURE 16**.



$m = \frac{2-2}{3-0} = \frac{0}{3} = 0$

FIGURE 15



$m = \frac{3-0}{-1-(-1)} = \frac{3}{0}$

FIGURE 16

The slope of a line can also be found directly from its equation. Look again at the equation $4x - y = -8$ from **Example 7.** Solve this equation for $y$.

$$\begin{aligned} 4x - y &= -8 && \text{Equation from } \textbf{Example 7} \\ -y &= -4x - 8 && \text{Subtract } 4x. \\ y &= 4x + 8 && \text{Multiply by } -1. \end{aligned}$$

Notice that the slope, 4, found with the slope formula in **Example 7** is the same number as the coefficient of $x$ in the equation $y = 4x + 8$. We will see in the next section that this always happens, *as long as the equation is solved for y.*

NOW TRY EXERCISE 8

Find the slope of the graph of $5x - 4y = 7$.

## EXAMPLE 8 Finding the Slope from an Equation

Find the slope of the graph of $3x - 5y = 8$.

Solve the equation for $y$.

$$\begin{aligned} 3x - 5y &= 8 \\ -5y &= -3x + 8 && \text{Subtract } 3x. \\ y &= \frac{3}{5}x - \frac{8}{5} && \text{Divide } \textit{each} \text{ term by } -5. \end{aligned}$$

$$\frac{-3x}{-5} = \frac{-3}{-5} \cdot \frac{x}{1} = \frac{3}{5}x$$

The slope is given by the coefficient of $x$, so the slope is $\frac{3}{5}$.

NOW TRY

NOW TRY ANSWERS

**7.** $\frac{3}{7}$ **8.** $\frac{5}{4}$

## OBJECTIVE 6 Graph a line, given its slope and a point on the line.

### EXAMPLE 9 Using the Slope and a Point to Graph Lines

**NOW TRY EXERCISE 9**

Graph the line passing through $(-4, 1)$ that has slope $-\frac{2}{3}$.

Graph each line described.

**(a)** With slope $\frac{2}{3}$ and $y$-intercept $(0, -4)$

Begin by plotting the point $P(0, -4)$, as shown in **FIGURE 17**. Then use the slope to find a second point.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{2}{3}$$

We move 2 units *up* from $(0, -4)$ and then 3 units to the *right* to locate another point on the graph, $R(3, -2)$. The line through $P(0, -4)$ and $R$ is the required graph.

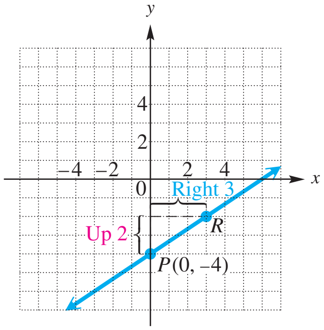


**FIGURE 17**

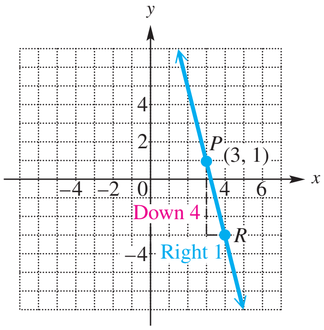


**FIGURE 18**

**(b)** Through $(3, 1)$ with slope $-4$

Start by locating the point $P(3, 1)$, as shown in **FIGURE 18**. Find a second point $R$ on the line by writing the slope $-4$ as $\frac{-4}{1}$ and using the slope formula.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-4}{1}$$

We move 4 units *down* from $(3, 1)$ and then 1 unit to the *right* to locate this second point $R(4, -3)$. The line through $P(3, 1)$ and $R$ is the required graph.

The slope also could be written as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{4}{-1}.$$

In this case, the second point $R$ is located 4 units *up* and 1 unit to the *left*. Verify that this approach also produces the line in **FIGURE 18**. NOW TRY

In **Example 9(a),** the slope of the line is the *positive* number $\frac{2}{3}$. The graph of the line in **FIGURE 17** slants up (rises) from left to right. The line in **Example 9(b)** has *negative* slope $-4$. As **FIGURE 18** shows, its graph slants down (falls) from left to right. These facts illustrate the following generalization.

**Orientation of a Line in the Plane**

A positive slope indicates that the line slants *up* (rises) from left to right.

A negative slope indicates that the line slants *down* (falls) from left to right.

**NOW TRY ANSWER**

9.

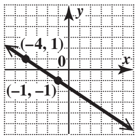


**FIGURE 19** shows lines of positive, 0, negative, and undefined slopes.

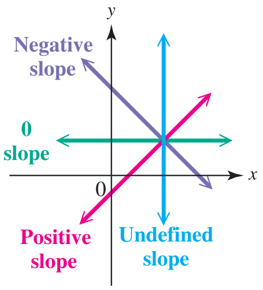


FIGURE 19

**OBJECTIVE 7 Use slopes to determine whether two lines are parallel, perpendicular, or neither.** Recall that the slopes of a pair of parallel or perpendicular lines are related in a special way.

### Slopes of Parallel Lines and Perpendicular Lines

- Two nonvertical lines with the same slope are parallel. Two nonvertical parallel lines have the same slope.
- Two perpendicular lines, neither of which is vertical, have slopes that are negative reciprocals— that is, their product is $-1$. Also, lines with slopes that are negative reciprocals are perpendicular.
- A line with 0 slope is perpendicular to a line with undefined slope.

### EXAMPLE 10 Determining Whether Two Lines Are Parallel, Perpendicular, or Neither

Determine whether the two lines described are *parallel, perpendicular,* or *neither*.

**(a)** Line $L_1$, through $(-2, 1)$ and $(4, 5)$, and line $L_2$, through $(3, 0)$ and $(0, -2)$

Find the slope of $L_1$.

$$m_1 = \frac{5-1}{4-(-2)} = \frac{4}{6} = \frac{2}{3}$$

Find the slope of $L_2$.

$$m_2 = \frac{-2-0}{0-3} = \frac{-2}{-3} = \frac{2}{3}$$

Because the slopes are equal, the two lines are parallel.

**(b)** The lines with equations $2y = 3x - 6$ and $2x + 3y = -6$

Find the slope of each line by solving each equation for $y$.

$$2y = 3x - 6$$

$$y = \underset{\substack{\uparrow \\ \text{Slope}}}{\frac{3}{2}}x - 3 \quad \text{Divide by 2.}$$

$$2x + 3y = -6$$

$$3y = -2x - 6 \quad \text{Subtract } 2x.$$

$$y = \underset{\substack{\uparrow \\ \text{Slope}}}{-\frac{2}{3}}x - 2 \quad \text{Divide by 3.}$$

Since the product of the slopes is $\frac{3}{2}\left(-\frac{2}{3}\right) = -1$, the lines are perpendicular.

**NOW TRY EXERCISE 10**

Determine whether the two lines described are *parallel, perpendicular,* or *neither*.

**(a)** Line $L_1$ through $(2, 5)$ and $(4, 8)$, and line $L_2$ through $(2, 0)$ and $(-1, -2)$

**(b)** The lines with equations

$$\begin{aligned} &x + 2y = 7\\ \text{and}\quad &2x = y - 4\end{aligned}$$

**(c)** The lines with equations

$$\begin{aligned} &2x - y = 4\\ \text{and}\quad &-2x + y = 6\end{aligned}$$

**(c)** The lines with equations $2x - 5y = 8$ and $2x + 5y = 8$

Find the slope of each line by solving each equation for $y$.

$$\begin{aligned} 2x - 5y &= 8 \\ -5y &= -2x + 8 && \text{Subtract } 2x. \\ y &= \underset{\substack{\uparrow \\ \text{Slope}}}{\frac{2}{5}}x - \frac{8}{5} && \text{Divide by } -5. \end{aligned}$$

$$\begin{aligned} 2x + 5y &= 8 \\ 5y &= -2x + 8 && \text{Subtract } 2x. \\ y &= \underset{\substack{\uparrow \\ \text{Slope}}}{-\frac{2}{5}}x + \frac{8}{5} && \text{Divide by } 5. \end{aligned}$$

The slopes, $\frac{2}{5}$ and $-\frac{2}{5}$, are not equal, and they are not negative reciprocals because their product is $-\frac{4}{25}$, not $-1$. Thus, the two lines are neither parallel nor perpendicular.

NOW TRY

**OBJECTIVE 8 Solve problems involving average rate of change.** The slope formula applied to any two points on a line gives the **average rate of change** in $y$ per unit change in $x$, where the value of $y$ depends on the value of $x$.

For example, suppose the height of a boy increased from 60 to 68 in. between the ages of 12 and 16, as shown in **FIGURE 20**.

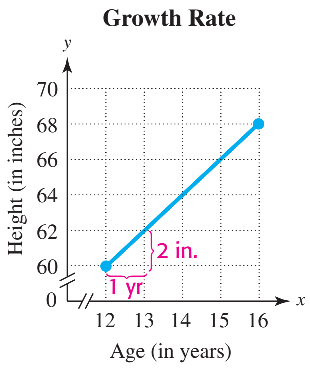


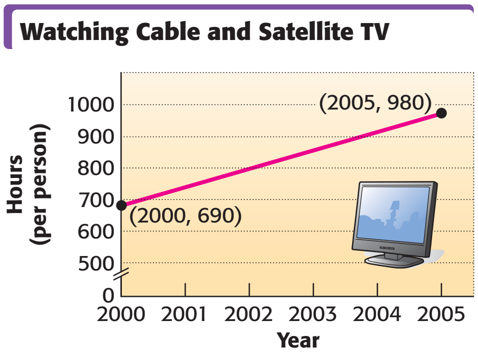
**FIGURE 20**

$$\frac{\text{Change in height } y \longrightarrow 68 - 60}{\text{Change in age } x \longrightarrow 16 - 12} = \frac{8}{4} = 2 \text{ in.} \qquad \text{Boy's average growth rate (or average change in height) } \textit{per year}$$

The boy may actually have grown more than 2 in. during some years and less than 2 in. during other years. If we plotted ordered pairs (age, height) for those years and drew a line connecting any two of the points, the average rate of change would likely be slightly different than that found above. However using the data for ages 12 and 16, the boy's *average* change in height was 2 in. per year over these years.

**EXAMPLE 11** Interpreting Slope as Average Rate of Change

The graph in **FIGURE 21** approximates the average number of hours per year spent watching cable and satellite TV for each person in the United States from 2000 to 2005. Find the average rate of change in number of hours per year.



*Source:* Veronis Suhler Stevenson.

**FIGURE 21**

**NOW TRY ANSWERS**

**10. (a)** neither **(b)** perpendicular **(c)** parallel

**NOW TRY EXERCISE 11**

Americans spent an average of 828 hr in 2002 watching cable and satellite TV. Using this number for 2002 and the number for 2000 from the graph in **FIGURE 21**, find the average rate of change from 2000 to 2002. How does it compare with the average rate of change found in **Example 11?**

**NOW TRY EXERCISE 12**

In 2000, sales of digital camcorders in the United States totaled \$2838 million. In 2008, sales totaled \$1885 million. Find the average rate of change in sales of digital camcorders per year, to the nearest million dollars. (*Source:* Consumer Electronics Association.)

To find the average rate of change, we need two pairs of data. From the graph, we have the ordered pairs $(2000, 690)$ and $(2005, 980)$. We use the slope formula.

$$\text{average rate of change} = \frac{980 - 690}{2005 - 2000} = \frac{290}{5} = 58$$

A positive slope indicates an increase.

This means that the average time per person spent watching cable and satellite TV *increased* by 58 hr per year from 2000 to 2005. **NOW TRY**

## EXAMPLE 12 Interpreting Slope as Average Rate of Change

During the year 2000, the average person in the United States spent 812 hr watching broadcast TV. In 2005, the average number of hours per person spent watching broadcast TV was 679. Find the average rate of change in number of hours per year. (*Source:* Veronis Suhler Stevenson.)

To use the slope formula, we let one ordered pair be $(2000, 812)$ and the other be $(2005, 679)$.

$$\text{average rate of change} = \frac{679 - 812}{2005 - 2000} = \frac{-133}{5} = -26.6$$

A negative slope indicates a decrease.

The graph in **FIGURE 22** confirms that the line through the ordered pairs falls from left to right and therefore has negative slope. Thus, the average time per person spent watching broadcast TV *decreased* by about 27 hr per year from 2000 to 2005.

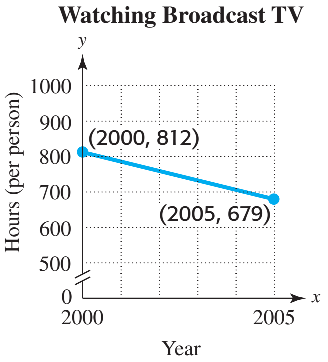


**FIGURE 22**

**NOW TRY**

**NOW TRY ANSWERS**

**11.** 69 hr per yr; It is greater than the average rate of change from 2000 to 2005.

**12.** −\$119 million per yr

## 7.1 EXERCISES

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**1.** *Concept Check* Name the quadrant, if any, in which each point is located.

**(a)** $(1, 6)$ **(b)** $(-4, -2)$ **(c)** $(-3, 6)$ **(d)** $(7, -5)$ **(e)** $(-3, 0)$ **(f)** $(0, -0.5)$

**2.** *Concept Check* Use the given information to determine the quadrants in which the point $(x, y)$ may lie.

**(a)** $xy > 0$ **(b)** $xy < 0$ **(c)** $\frac{x}{y} < 0$ **(d)** $\frac{x}{y} > 0$

**3.** *Concept Check* Plot each point in a rectangular coordinate system.

**(a)** $(2, 3)$ **(b)** $(-3, -2)$ **(c)** $(0, 5)$ **(d)** $(-2, 4)$ **(e)** $(-2, 0)$ **(f)** $(3, -3)$

**4.** *Concept Check* What must be true about the value of at least one of the coordinates of any point that lies along an axis?

*In Exercises 5–8,* ***(a)*** *complete the given table for each equation and then* ***(b)*** *graph the equation.* ***See*** **FIGURE 3**.

**5.** $x - y = 3$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| 5 | |
| 2 | |

**6.** $x + 2y = 5$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| 2 | |
| | 2 |

**7.** $y = -2x + 3$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 1 |
| 2 | |
| | −3 |

**8.** $4x - 5y = 20$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| 2 | |
| | −3 |

*Find the x- and y-intercepts. Then graph each equation.* ***See Examples 1–4.***

**9.** $2x + 3y = 12$
**10.** $5x + 2y = 10$
**11.** $x - 3y = 6$
**12.** $x - 2y = -4$
**13.** $5x + 6y = -10$
**14.** $3x - 7y = 9$
**15.** $y = 5$
**16.** $y = -3$
**17.** $x = 2$
**18.** $x = -3$
**19.** $x + 4 = 0$
**20.** $x - 4 = 0$
**21.** $y + 2 = 0$
**22.** $y - 5 = 0$
**23.** $x + 5y = 0$
**24.** $x - 3y = 0$
**25.** $2x = 3y$
**26.** $4y = 3x$

*Find the midpoint of each segment with the given endpoints.* ***See Example 5.***

**27.** $(-8, 4)$ and $(-2, -6)$
**28.** $(5, 2)$ and $(-1, 8)$
**29.** $(3, -6)$ and $(6, 3)$
**30.** $(-10, 4)$ and $(7, 1)$
**31.** $(-9, 3)$ and $(9, 8)$
**32.** $(4, -3)$ and $(-1, 3)$
**33.** $(2.5, 3.1)$ and $(1.7, -1.3)$
**34.** $(6.2, 5.8)$ and $(1.4, -0.6)$

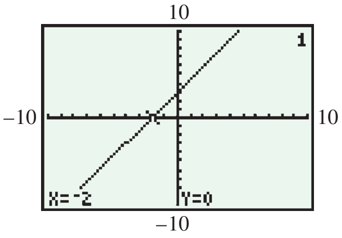
***Brain Busters*** *Find the midpoint of each segment with the given endpoints.*

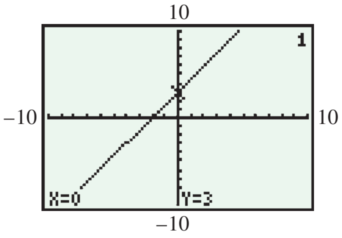
**35.** $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{3}{2}, \frac{5}{3}\right)$
**36.** $\left(\frac{21}{4}, \frac{2}{5}\right)$ and $\left(\frac{7}{4}, \frac{3}{5}\right)$
**37.** $\left(-\frac{1}{3}, \frac{2}{7}\right)$ and $\left(-\frac{1}{2}, \frac{1}{14}\right)$
**38.** $\left(\frac{3}{5}, -\frac{1}{3}\right)$ and $\left(\frac{1}{2}, -\frac{7}{2}\right)$

## TECHNOLOGY INSIGHTS EXERCISES 39 AND 40

**39.** The screens show the graph of one of the equations in A–D. Which equation is it?

**A.** $3x + 2y = 6$ **B.** $-3x + 2y = 6$ **C.** $-3x - 2y = 6$ **D.** $3x - 2y = 6$



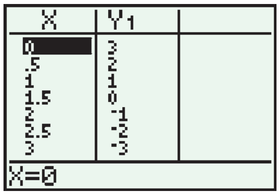


**40.** The table of ordered pairs was generated by a graphing calculator.

**(a)** What is the $x$-intercept? **(b)** What is the $y$-intercept?

**(c)** Which equation corresponds to this table of values?

**A.** $Y_1 = 2X - 3$ **B.** $Y_1 = -2X - 3$
**C.** $Y_1 = 2X + 3$ **D.** $Y_1 = -2X + 3$



| X | Y1 |
|---|---|
| 0 | 3 |
| .5 | 2 |
| 1 | 1 |
| 1.5 | 0 |
| 2 | -1 |
| 2.5 | -2 |
| 3 | -3 |

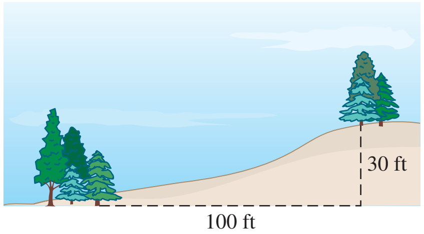
X=0

*Concept Check* *Answer each question about slope in Exercises 41 and 42.*

**41.** A hill rises 30 ft for every horizontal 100 ft. Which of the following express its slope (or grade)? (There are several correct choices.)

**A.** 0.3 **B.** $\frac{3}{10}$ **C.** $3\frac{1}{3}$ **D.** $\frac{30}{100}$

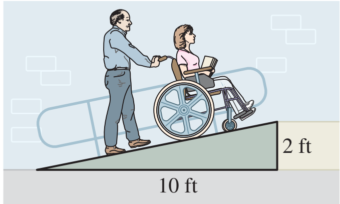
**E.** $\frac{10}{3}$ **F.** 30 **G.** 30% **H.** $-\frac{10}{3}$



**42.** If a walkway rises 2 ft for every 10 ft on the horizontal, which of the following express its slope (or grade)? (There are several correct choices.)

**A.** 0.2 **B.** $\frac{2}{10}$ **C.** $\frac{1}{5}$ **D.** 20%

**E.** 5 **F.** $\frac{20}{100}$ **G.** $\frac{10}{2}$ **H.** $-5$



**43.** *Concept Check* Match each situation in (a)–(d) with the most appropriate graph in A–D.

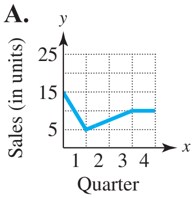
**(a)** Sales rose sharply during the first quarter, leveled off during the second quarter, and then rose slowly for the rest of the year.

**(b)** Sales fell sharply during the first quarter and then rose slowly during the second and third quarters before leveling off for the rest of the year.

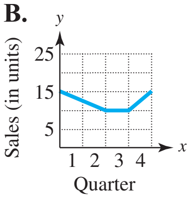
**(c)** Sales rose sharply during the first quarter and then fell to the original level during the second quarter before rising steadily for the rest of the year.

**(d)** Sales fell during the first two quarters of the year, leveled off during the third quarter, and rose during the fourth quarter.

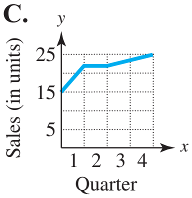
**A.**



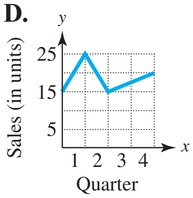
**B.**



**C.**

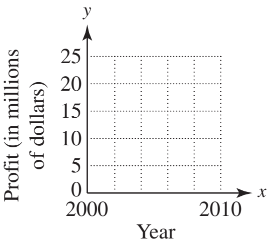


**D.**



**44.** *Concept Check* Using the given axes, draw a graph that illustrates the following description:

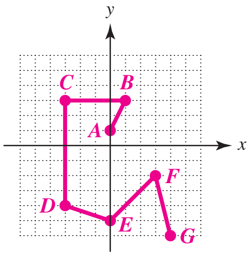
> Profits for a business were \$10 million in 2000. They rose sharply from 2000 through 2004, remained constant from 2004 through 2008, and then fell slowly from 2008 through 2010.



**45.** *Concept Check* Determine the slope of each line segment in the given figure.

**(a)** *AB* **(b)** *BC* **(c)** *CD*

**(d)** *DE* **(e)** *EF* **(f)** *FG*



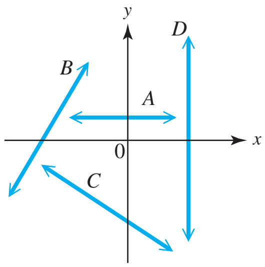
**46.** ***Concept Check*** On the basis of the figure shown here, determine which line satisfies the given description.

**(a)** The line has positive slope.

**(b)** The line has negative slope.

**(c)** The line has slope 0.

**(d)** The line has undefined slope.



*For Exercises 47–58,* ***(a)*** *find the slope of the line through each pair of points, if possible, and* ***(b)*** *based on the slope, indicate whether the line through the points* rises *from left to right,* falls *from left to right, is* horizontal, *or is* vertical. ***See Example 6 and*** **FIGURE 19**.

**47.** $(-2, -3)$ and $(-1, 5)$

**48.** $(-4, 1)$ and $(-3, 4)$

**49.** $(-4, 1)$ and $(2, 6)$

**50.** $(-3, -3)$ and $(5, 6)$

**51.** $(2, 4)$ and $(-4, 4)$

**52.** $(-6, 3)$ and $(2, 3)$

**53.** $(-2, 2)$ and $(4, -1)$

**54.** $(-3, 1)$ and $(6, -2)$

**55.** $(5, -3)$ and $(5, 2)$

**56.** $(4, -1)$ and $(4, 3)$

**57.** $(1.5, 2.6)$ and $(0.5, 3.6)$

**58.** $(3.4, 4.2)$ and $(1.4, 10.2)$

***Brain Busters*** *Find the slope of the line through each pair of points.* $\left(\textit{Hint: } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}\right)$

**59.** $\left(\frac{1}{6}, \frac{1}{2}\right)$ and $\left(\frac{5}{6}, \frac{9}{2}\right)$

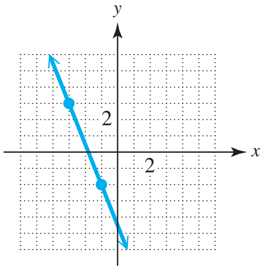
**60.** $\left(\frac{3}{4}, \frac{1}{3}\right)$ and $\left(\frac{5}{4}, \frac{10}{3}\right)$

**61.** $\left(-\frac{2}{9}, \frac{5}{18}\right)$ and $\left(\frac{1}{18}, -\frac{5}{9}\right)$

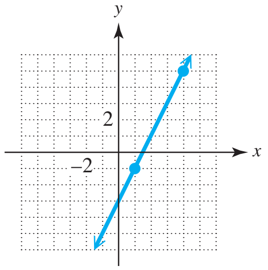
**62.** $\left(-\frac{4}{5}, \frac{9}{10}\right)$ and $\left(-\frac{3}{10}, \frac{1}{5}\right)$

*Find the slope of each line.*

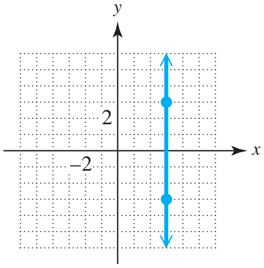
**63.**



**64.**



**65.**



**66.** ***Concept Check*** Let $k$ be the number of letters in your last name. Sketch the graph of $y = k$. What is the slope of this line?

*Find the slope of the line and sketch the graph.* ***See Examples 6–8.***

**67.** $x + 2y = 4$

**68.** $x + 3y = -6$

**69.** $5x - 2y = 10$

**70.** $4x - y = 4$

**71.** $y = 4x$

**72.** $y = -3x$

**73.** $x - 3 = 0$

**74.** $x + 2 = 0$

**75.** $y = -5$

**76.** $y = -4$

*Graph each line described.* ***See Example 9.***

**77.** Through $(-4, 2)$; $m = \frac{1}{2}$

**78.** Through $(-2, -3)$; $m = \frac{5}{4}$

**79.** $y$-intercept $(0, -2)$; $m = -\frac{2}{3}$

**80.** $y$-intercept $(0, -4)$; $m = -\frac{3}{2}$

**81.** Through $(-1, -2)$; $m = 3$

**82.** Through $(-2, -4)$; $m = 4$

**83.** $m = 0$; through $(2, -5)$

**84.** $m = 0$; through $(5, 3)$

**85.** Undefined slope; through $(-3, 1)$

**86.** Undefined slope; through $(-4, 1)$

**87.** *Concept Check* If a line has slope $-\frac{4}{9}$, then any line parallel to it has slope ______, and any line perpendicular to it has slope ______.

**88.** *Concept Check* If a line has slope 0.2, then any line parallel to it has slope ______, and any line perpendicular to it has slope ______.

*Decide whether each pair of lines is* parallel, perpendicular, *or* neither. ***See Example 10.***

**89.** The line through $(15, 9)$ and $(12, -7)$ and the line through $(8, -4)$ and $(5, -20)$

**90.** The line through $(4, 6)$ and $(-8, 7)$ and the line through $(-5, 5)$ and $(7, 4)$

**91.** $x + 4y = 7$ and $4x - y = 3$

**92.** $2x + 5y = -7$ and $5x - 2y = 1$

**93.** $4x - 3y = 6$ and $3x - 4y = 2$

**94.** $2x + y = 6$ and $x - y = 4$

**95.** $x = 6$ and $6 - x = 8$

**96.** $3x = y$ and $2y - 6x = 5$

**97.** $4x + y = 0$ and $5x - 8 = 2y$

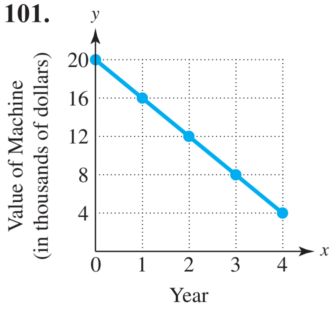
**98.** $2x + 5y = -8$ and $6 + 2x = 5y$

**99.** $2x = y + 3$ and $2y + x = 3$

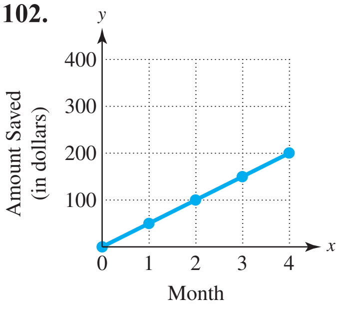
**100.** $4x - 3y = 8$ and $4y + 3x = 12$

*Concept Check* *Find and interpret the average rate of change illustrated in each graph.*

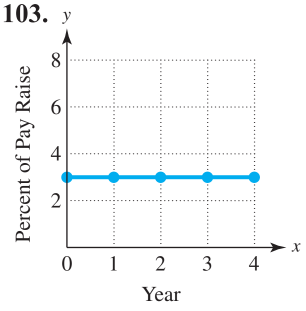
**101.**



**102.**



**103.**

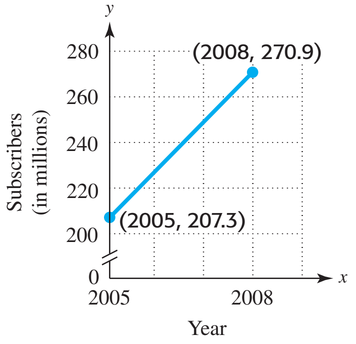


**104.** *Concept Check* If the graph of a linear equation rises from left to right, then the average rate of change is ________________ (positive/negative). If the graph of a linear equation falls from left to right, then the average rate of change is ________________ (positive/negative).

*Solve each problem.* ***See Examples 11 and 12.***

**105.** The graph shows the number of cellular phone subscribers (in millions) in the United States from 2005 to 2008.

**Cellular Phone Subscribers**



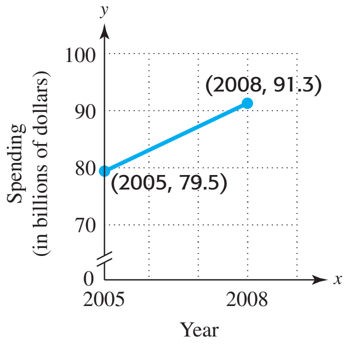
***Source:*** CTIA: The Wireless Association.

**(a)** Use the given ordered pairs to find the slope of the line.

**(b)** Interpret the slope in the context of this problem.

**106.** The graph shows spending on personal care products (in billions of dollars) in the United States from 2005 to 2008.

**Spending on Personal Care Products**



***Source:*** U.S. Department of Commerce.

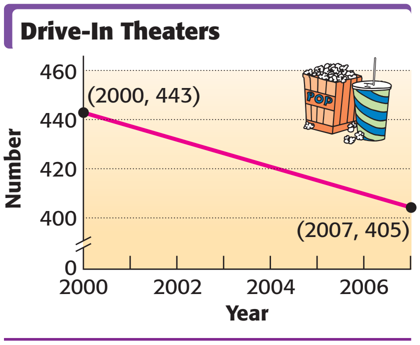
**(a)** Use the given ordered pairs to find the slope of the line to the nearest tenth.

**(b)** Interpret the slope in the context of this problem.

**107.** The graph provides a good approximation of the number of drive-in theaters in the United States from 2000 through 2007.

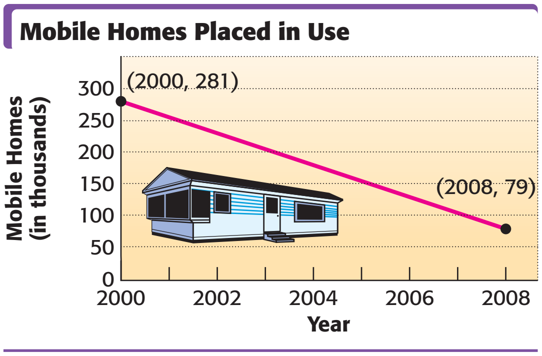
**(a)** Use the given ordered pairs to find the average rate of change in the number of drive-in theaters per year during this period. Round your answer to the nearest whole number.

✎ **(b)** Explain how a negative slope is interpreted in this situation.



***Source:*** www.drive-ins.com

**108.** The graph provides a good approximation of the number of mobile homes (in thousands) placed in use in the United States from 2000 through 2008.



***Source:*** U.S. Census Bureau.

**(a)** Use the given ordered pairs to find the average rate of change in the number of mobile homes per year during this period.

✎ **(b)** Explain how a negative slope is interpreted in this situation.

✎ **109.** The total amount spent on plasma TVs in the United States changed from \$1590 million in 2003 to \$5705 million in 2006. Find and interpret the average rate of change in sales, in millions of dollars per year. Round your answer to the nearest hundredth. (*Source:* Consumer Electronics Association.)

✎ **110.** The total amount spent on analog TVs in the United States changed from \$5836 million in 2003 to \$1424 million in 2006. Find and interpret the average rate of change in sales, in millions of dollars per year. Round your answer to the nearest hundredth. (*Source:* Consumer Electronics Association.)

## PREVIEW EXERCISES

*Write each equation in the form $Ax + By = C$.* ***See Section 3.4.***

**111.** $y - (-2) = \frac{3}{2}(x - 5)$

**112.** $y - (-1) = -\frac{1}{2}[x - (-2)]$

# 7.2 Review of Equations of Lines; Linear Models

**OBJECTIVES**

1. Write an equation of a line, given its slope and $y$-intercept.
2. Graph a line, using its slope and $y$-intercept.
3. Write an equation of a line, given its slope and a point on the line.
4. Write equations of horizontal and vertical lines.
5. Write an equation of a line, given two points on the line.
6. Write an equation of a line parallel or perpendicular to a given line.
7. Write an equation of a line that models real data.

**OBJECTIVE 1 Write an equation of a line, given its slope and $y$-intercept.** Recall that we can find the slope of a line from its equation by solving the equation for $y$. For example, we found that the slope of the line with equation

$$y = 4x + 8$$

is 4, the coefficient of $x$. What does the number 8 represent?

To find out, suppose a line has slope $m$ and $y$-intercept $(0, b)$. We can find an equation of this line by choosing another point $(x, y)$ on the line, as shown in **FIGURE 23**, and using the slope formula.

$$m = \frac{y - b}{x - 0} \quad \begin{array}{l} \leftarrow \text{Change in } y\text{-values} \\ \leftarrow \text{Change in } x\text{-values} \end{array}$$

$$m = \frac{y - b}{x} \qquad \text{Subtract in the denominator.}$$

$$mx = y - b \qquad \text{Multiply by } x.$$

$$mx + b = y \qquad \text{Add } b.$$

$$y = mx + b \qquad \text{Rewrite.}$$

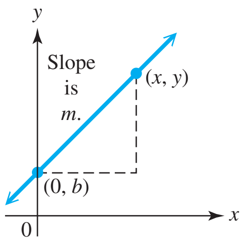


**FIGURE 23**

This last equation is called the *slope-intercept form* of the equation of a line, because we can identify the slope $m$ and $y$-intercept $(0, b)$ at a glance. Thus, in the line with equation $y = 4x + 8$, the number 8 indicates that the $y$-intercept is $(0, 8)$.

**Slope-Intercept Form**

The **slope-intercept form** of the equation of a line with slope $m$ and $y$-intercept $(0, b)$ is

$$\mathbf{y = \underset{\substack{\uparrow \\ \text{Slope}}}{m}x + \underset{\substack{\uparrow \\ y\text{-intercept is } (0,\, b).}}{b}.}$$

**NOW TRY EXERCISE 1**
Write an equation of the line with slope $\frac{2}{3}$ and $y$-intercept $(0, 1)$.

**EXAMPLE 1 Writing an Equation of a Line**

Write an equation of the line with slope $-\frac{4}{5}$ and $y$-intercept $(0, -2)$.

Here, $m = -\frac{4}{5}$ and $b = -2$. Substitute these values into the slope-intercept form.

$$y = mx + b \qquad \text{Slope-intercept form}$$

$$y = -\frac{4}{5}x - 2 \qquad m = -\tfrac{4}{5};\ b = -2$$

NOW TRY

**NOTE** Every linear equation (of a nonvertical line) has a *unique* (one and only one) slope-intercept form. In **Section 7.4,** we study *linear functions,* which are defined using slope-intercept form. Also, this is the form we use when graphing a line with a graphing calculator.

**OBJECTIVE 2 Graph a line, using its slope and $y$-intercept.** We first saw this approach in **Example 9(a)** of **Section 7.1.**

**NOW TRY ANSWER**
**1.** $y = \frac{2}{3}x + 1$

NOW TRY EXERCISE 2

Graph the line, using the slope and $y$-intercept.

$$4x + 3y = 6$$

### EXAMPLE 2 Graphing Lines Using Slope and $y$-Intercept

Graph each line, using the slope and $y$-intercept.

**(a)** $y = 3x - 6$

Here, $m = 3$ and $b = -6$. Plot the $y$-intercept $(0, -6)$. The slope 3 can be interpreted as

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{3}{1}.$$

From $(0, -6)$, move 3 units *up* and 1 unit to the *right*, and plot a second point at $(1, -3)$. Join the two points with a straight line. See **FIGURE 24**.

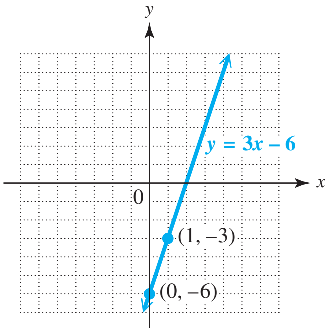


**FIGURE 24**

**(b)** $3y + 2x = 9$

Write the equation in slope-intercept form by solving for $y$.

$$3y + 2x = 9$$

$$3y = -2x + 9 \qquad \text{Subtract } 2x.$$

$$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x + \underbrace{3}_{y\text{-intercept is } (0,\, 3).} \qquad \text{Divide by 3.}$$

Plot the $y$-intercept $(0, 3)$. The slope can be interpreted as either $\frac{-2}{3}$ or $\frac{2}{-3}$. Using $\frac{-2}{3}$, begin at $(0, 3)$ and move 2 units *down* and 3 units to the *right* to locate the point $(3, 1)$. The line through these two points is the required graph. See **FIGURE 25**. (Verify that the point obtained with $\frac{2}{-3}$ as the slope is also on this line.)

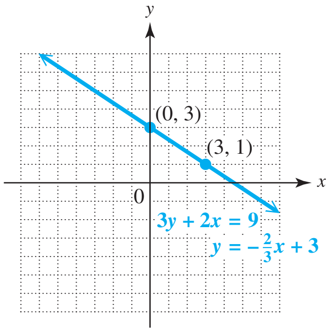


**FIGURE 25**

NOW TRY

### OBJECTIVE 3 Write an equation of a line, given its slope and a point on the line.

Let $m$ represent the slope of a line and $(x_1, y_1)$ represent a given point on the line. Let $(x, y)$ represent any other point on the line. See **FIGURE 26**.

$$m = \frac{y - y_1}{x - x_1} \qquad \text{Slope formula}$$

$$m(x - x_1) = y - y_1 \qquad \text{Multiply each side by } x - x_1.$$

$$y - y_1 = m(x - x_1) \qquad \text{Rewrite.}$$

This last equation is the *point-slope form* of the equation of a line.

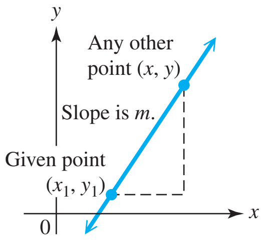


**FIGURE 26**

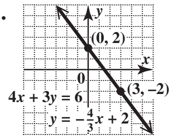
**Point-Slope Form**

The **point-slope form** of the equation of a line with slope $m$ passing through the point $(x_1, y_1)$ is

$$\boldsymbol{y - \underbrace{y_1}_{} = \overset{\text{Slope}}{m}(x - \underbrace{x_1}_{\text{Given point}}).}$$

NOW TRY ANSWER

2.



**NOW TRY EXERCISE 3**

Write an equation of the line with slope $-\frac{1}{5}$ and passing through the point $(5, -3)$.

### EXAMPLE 3 Writing an Equation of a Line, Given the Slope and a Point

Write an equation of the line with slope $\frac{1}{3}$ and passing through the point $(-2, 5)$.

***Method 1*** Use the point-slope form of the equation of a line, with $(x_1, y_1) = (-2, 5)$ and $m = \frac{1}{3}$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point-slope form}\\
y - 5 &= \frac{1}{3}[x - (-2)] && \text{Substitute for } y_1,\ m, \text{ and } x_1.\\
y - 5 &= \frac{1}{3}(x + 2) && \text{Definition of subtraction}\\
3y - 15 &= x + 2 && \text{(*) Multiply by 3.}\\
3y &= x + 17 && \text{Add 15.}\\
\text{Slope-intercept form} \longrightarrow y &= \frac{1}{3}x + \frac{17}{3} && \text{Divide by 3.}
\end{aligned}$$

***Method 2*** An alternative method for finding this equation uses slope-intercept form, with $(x, y) = (-2, 5)$ and $m = \frac{1}{3}$.

$$\begin{aligned}
y &= mx + b && \text{Slope-intercept form}\\
5 &= \frac{1}{3}(-2) + b && \text{Substitute for } y,\ m, \text{ and } x.\\
\text{Solve for } b. \quad 5 &= -\frac{2}{3} + b && \text{Multiply.}\\
\frac{17}{3} &= b, \quad \text{or} \quad b = \frac{17}{3} && 5 = \tfrac{15}{3};\ \text{Add } \tfrac{2}{3}.
\end{aligned}$$

Since $m = \frac{1}{3}$ and $b = \frac{17}{3}$, the equation is

$$y = \frac{1}{3}x + \frac{17}{3}. \quad \text{Same equation found in Method 1}$$

NOW TRY

**OBJECTIVE 4** **Write equations of horizontal and vertical lines.** A horizontal line has slope 0. Using point-slope form, we can find the equation of a horizontal line through the point $(a, b)$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point-slope form}\\
y - b &= 0(x - a) && y_1 = b,\ m = 0,\ x_1 = a\\
y - b &= 0 && \text{Multiplication property of 0}\\
y &= b && \text{Add } b.
\end{aligned}$$

Point-slope form does not apply to a vertical line, since the slope of a vertical line is undefined. A vertical line through the point $(a, b)$ has equation $x = a$.

**Equations of Horizontal and Vertical Lines**

The horizontal line through the point $(a, b)$ has equation $\boldsymbol{y = b}$.

The vertical line through the point $(a, b)$ has equation $\boldsymbol{x = a}$.

**NOW TRY ANSWER**

3. $y = -\frac{1}{5}x - 2$

**NOW TRY EXERCISE 4**

Write an equation of the line passing through the point $(4, -4)$ that satisfies the given condition.

**(a)** Undefined slope

**(b)** Slope 0

### EXAMPLE 4 Writing Equations of Horizontal and Vertical Lines

Write an equation of the line passing through the point $(-3, 3)$ that satisfies the given condition.

**(a)** Slope 0

Since the slope is 0, this is a horizontal line. A horizontal line through the point $(a, b)$ has equation $y = b$. Here the $y$-coordinate is 3, so the equation is $y = 3$.

**(b)** Undefined slope

This is a vertical line, since the slope is undefined. A vertical line through the point $(a, b)$ has equation $x = a$. Here the $x$-coordinate is $-3$, so the equation is $x = -3$.

Both lines are graphed in **FIGURE 27**.

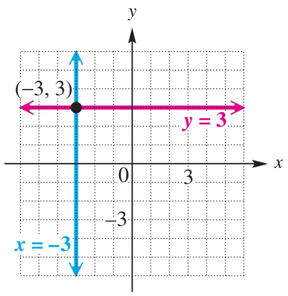


**FIGURE 27**

NOW TRY

## OBJECTIVE 5 Write an equation of a line, given two points on the line.

Recall that we defined *standard form* for a linear equation as

$$\boldsymbol{Ax + By = C}, \qquad \text{Standard form}$$

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0. (In most cases, $A$, $B$, and $C$ are rational numbers.) We give answers so that $A$, $B$, and $C$ are integers with greatest common factor 1, and $A \geq 0$. For example, the equation in **Example 3** is written in standard form as follows.

$$\begin{aligned} 3y - 15 &= x + 2 && \text{Equation (*) from } \textbf{Example 3} \\ -x + 3y &= 17 && \text{Subtract } x \text{ and add 15.} \\ \text{Standard form} \longrightarrow x - 3y &= -17 && \text{Multiply by } -1. \end{aligned}$$

**NOW TRY EXERCISE 5**

Write an equation of the line passing through the points $(3, -4)$ and $(-2, -1)$. Give the final answer in standard form.

### EXAMPLE 5 Writing an Equation of a Line, Given Two Points

Write an equation of the line passing through the points $(-4, 3)$ and $(5, -7)$. Give the final answer in standard form.

First find the slope by the slope formula.

$$m = \frac{-7 - 3}{5 - (-4)} = -\frac{10}{9}$$

Use either $(-4, 3)$ or $(5, -7)$ as $(x_1, y_1)$ in the point-slope form of the equation of a line. We choose $(-4, 3)$, so $-4 = x_1$ and $3 = y_1$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 3 &= -\frac{10}{9}[x - (-4)] && y_1 = 3,\ m = -\tfrac{10}{9},\ x_1 = -4 \\ y - 3 &= -\frac{10}{9}(x + 4) && \text{Definition of subtraction} \\ 9y - 27 &= -10(x + 4) && \text{Multiply by 9 to clear the fraction.} \\ 9y - 27 &= -10x - 40 && \text{Distributive property} \\ \text{Standard form} \longrightarrow 10x + 9y &= -13 && \text{Add } 10x. \text{ Add 27.} \end{aligned}$$

Verify that if $(5, -7)$ were used, the same equation would result.

NOW TRY

**NOW TRY ANSWERS**

**4.** **(a)** $x = 4$ **(b)** $y = -4$

**5.** $3x + 5y = -11$

**OBJECTIVE 6** **Write an equation of a line parallel or perpendicular to a given line.** Recall that parallel lines have the same slope and perpendicular lines have slopes that are negative reciprocals of each other.

**NOW TRY EXERCISE 6**

Write an equation of the line passing through the point $(6, -1)$ and

**(a)** parallel to the line $3x - 5y = 7$;

**(b)** perpendicular to the line $3x - 5y = 7$.

Give final answers in slope-intercept form.

## EXAMPLE 6 Writing Equations of Parallel or Perpendicular Lines

Write an equation of the line passing through the point $(-3, 6)$ and **(a)** parallel to the line $2x + 3y = 6$; **(b)** perpendicular to the line $2x + 3y = 6$. Give final answers in slope-intercept form.

**(a)** We find the slope of the line $2x + 3y = 6$ by solving for $y$.

$$2x + 3y = 6$$

$$3y = -2x + 6 \qquad \text{Subtract } 2x.$$

$$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x + 2 \qquad \text{Divide by 3.}$$

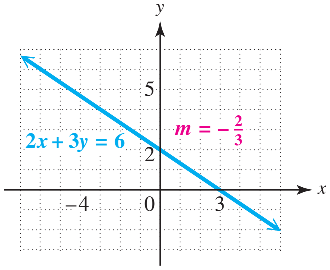


**FIGURE 28**

The slope of the line is given by the coefficient of $x$, so $m = -\frac{2}{3}$. See **FIGURE 28**.

The required equation of the line through $(-3, 6)$ and parallel to $2x + 3y = 6$ must also have slope $-\frac{2}{3}$. To find this equation, we use the point-slope form, with $(x_1, y_1) = (-3, 6)$ and $m = -\frac{2}{3}$.

$$y - 6 = -\frac{2}{3}[x - (-3)] \qquad y_1 = 6,\ m = -\tfrac{2}{3},\ x_1 = -3$$

$$y - 6 = -\frac{2}{3}(x + 3) \qquad \text{Definition of subtraction}$$

$$y - 6 = -\frac{2}{3}x - 2 \qquad \text{Distributive property}$$

$$y = -\frac{2}{3}x + 4 \qquad \text{Add 6.}$$

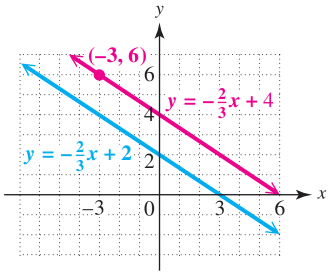


**FIGURE 29**

We did not clear the fraction here because we want the equation in slope-intercept form—that is, solved for $y$. Both lines are shown in **FIGURE 29**.

**(b)** To be perpendicular to the line $2x + 3y = 6$, a line must have a slope that is the negative reciprocal of $-\frac{2}{3}$, which is $\frac{3}{2}$. We use $(-3, 6)$ and slope $\frac{3}{2}$ in the point-slope form to find the equation of the perpendicular line shown in **FIGURE 30**.

$$y - 6 = \frac{3}{2}[x - (-3)] \qquad y_1 = 6,\ m = \tfrac{3}{2},\ x_1 = -3$$

$$y - 6 = \frac{3}{2}(x + 3) \qquad \text{Definition of subtraction}$$

$$y - 6 = \frac{3}{2}x + \frac{9}{2} \qquad \text{Distributive property}$$

$$y = \frac{3}{2}x + \frac{21}{2} \qquad \text{Add } 6 = \tfrac{12}{2}.$$

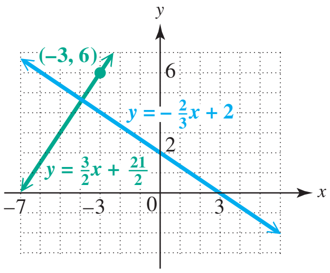


**FIGURE 30**

NOW TRY

**NOW TRY ANSWERS**

6. **(a)** $y = \frac{3}{5}x - \frac{23}{5}$

**(b)** $y = -\frac{5}{3}x + 9$

A summary of the various forms of linear equations follows.

**Forms of Linear Equations**

| *Equation* | *Description* | *When to Use* |
|---|---|---|
| $y = mx + b$ | **Slope-Intercept Form**<br>Slope is $m$.<br>$y$-intercept is $(0, b)$. | The slope and $y$-intercept can be easily identified and used to quickly graph the equation. |
| $y - y_1 = m(x - x_1)$ | **Point-Slope Form**<br>Slope is $m$.<br>Line passes through $(x_1, y_1)$. | This form is ideal for finding the equation of a line if the slope and a point on the line or two points on the line are known. |
| $Ax + By = C$ | **Standard Form**<br>($A$, $B$, and $C$ integers, $A \geq 0$)<br>Slope is $-\frac{A}{B}$ $(B \neq 0)$.<br>$x$-intercept is $\left(\frac{C}{A}, 0\right)$ $(A \neq 0)$.<br>$y$-intercept is $\left(0, \frac{C}{B}\right)$ $(B \neq 0)$. | The $x$- and $y$-intercepts can be found quickly and used to graph the equation. The slope must be calculated. |
| $y = b$ | **Horizontal Line**<br>Slope is 0.<br>$y$-intercept is $(0, b)$. | If the graph intersects only the $y$-axis, then $y$ is the only variable in the equation. |
| $x = a$ | **Vertical Line**<br>Slope is undefined.<br>$x$-intercept is $(a, 0)$. | If the graph intersects only the $x$-axis, then $x$ is the only variable in the equation. |

**OBJECTIVE 7** **Write an equation of a line that models real data.** If a given set of data changes at a fairly constant rate, the data may fit a linear pattern, where the rate of change is the slope of the line.

**EXAMPLE 7** Determining a Linear Equation to Describe Real Data

A local gasoline station is selling 89-octane gas for \$3.20 per gal.

**(a)** Write an equation that describes the cost $y$ to buy $x$ gallons of gas.

The total cost is determined by the number of gallons we buy multiplied by the price per gallon (in this case, \$3.20). As the gas is pumped, two sets of numbers spin by: the number of gallons pumped and the cost of that number of gallons. The table illustrates this situation.

| *Number of Gallons Pumped* | *Cost of This Number of Gallons* |
|---|---|
| 0 | 0(\$3.20) = \$ 0.00 |
| 1 | 1(\$3.20) = \$ 3.20 |
| 2 | 2(\$3.20) = \$ 6.40 |
| 3 | 3(\$3.20) = \$ 9.60 |
| 4 | 4(\$3.20) = \$12.80 |

If we let $x$ denote the number of gallons pumped, then the total cost $y$ in dollars can be found using the following linear equation.

$$\underset{\text{Total cost}}{y} = 3.20\underset{\text{Number of gallons}}{x}$$

Theoretically, there are infinitely many ordered pairs $(x, y)$ that satisfy this equation, but here we are limited to nonnegative values for $x$, since we cannot have a negative number of gallons. In this situation, there is also a practical maximum value for $x$ that varies from one car to another. What determines this maximum value?

**(b)** A car wash at this gas station costs an additional \$3.00. Write an equation that defines the cost of gas and a car wash.

The cost will be $3.20x + 3.00$ dollars for $x$ gallons of gas and a car wash.

$$y = 3.2x + 3 \qquad \text{Delete unnecessary zeros.}$$

*NOW TRY EXERCISE 7*

A cell phone plan costs \$100 for the telephone plus \$85 per month for service. Write an equation that gives the cost $y$ in dollars for $x$ months of cell phone service using this plan.

**(c)** Interpret the ordered pairs $(5, 19)$ and $(10, 35)$ in relation to the equation from part (b).

The ordered pair $(5, 19)$ indicates that 5 gal of gas and a car wash costs \$19.00. Similarly, $(10, 35)$ indicates that 10 gal of gas and a car wash costs \$35.00.

*NOW TRY*

**NOTE** In **Example 7(a),** the ordered pair $(0, 0)$ satisfied the equation, so the linear equation has the form $y = mx$, where $b = 0$. If a realistic situation involves an initial charge plus a charge per unit, as in **Example 7(b),** the equation has the form $y = mx + b$, where $b \neq 0$.

*NOW TRY EXERCISE 8*

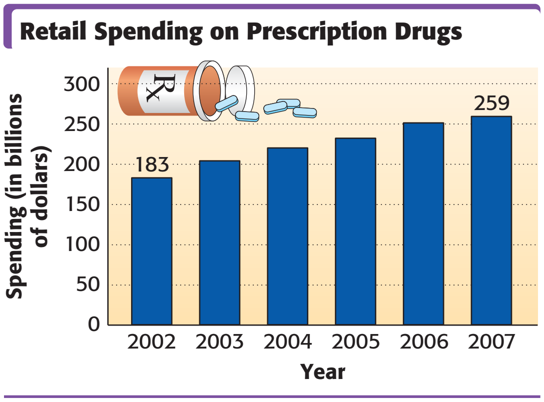
Refer to **Example 8.**

**(a)** Use the ordered pairs $(2, 183)$ and $(6, 251)$ to write an equation that models the data.

**(b)** Use the equation from part (a) to estimate retail spending on prescription drugs in 2011.

## *EXAMPLE 8* Writing an Equation of a Line That Models Data

Retail spending (in billions of dollars) on prescription drugs in the United States is shown in the graph in **FIGURE 31**.



*Source:* National Association of Chain Drug Stores.

**FIGURE 31**

**(a)** Write an equation that models the data.

The data increase linearly—that is, a straight line through the tops of any two bars in the graph would be close to the top of each bar. To model the relationship between year $x$ and spending on prescription drugs $y$, we let $x = 2$ represent 2002, $x = 3$ represent 2003, and so on. The given data for 2002 and 2007 can be written as the ordered pairs $(2, 183)$ and $(7, 259)$.

$$m = \frac{259 - 183}{7 - 2} = \frac{76}{5} = 15.2 \qquad \text{Find the slope of the line through } (2, 183) \text{ and } (7, 259).$$

Thus, spending increased by about \$15.2 billion per year. To write an equation, we substitute this slope and one of the points, say, $(2, 183)$, into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 183 &= 15.2(x - 2) && (x_1, y_1) = (2, 183);\ m = 15.2 \\ y - 183 &= 15.2x - 30.4 && \text{Distributive property} \\ y &= 15.2x + 152.6 && \text{Add 183.} \end{aligned}$$

Either point can be used here. (7, 259) provides the same answer.

Retail spending $y$ (in billions of dollars) on prescription drugs in the United States in year $x$ can be approximated by the equation $y = 15.2x + 152.6$.

*NOW TRY ANSWERS*

**7.** $y = 85x + 100$

**8. (a)** $y = 17x + 149$

**(b)** \$336 billion

**(b)** Use the equation from part (a) to estimate retail spending on prescription drugs in the United States in 2010. (Assume a constant rate of change.)

Since $x = 2$ represents 2002 and 2010 is 8 yr after 2002, $x = 10$ represents 2010.

$$y = 15.2x + 152.6 \qquad \text{Equation from part (a)}$$

$$y = 15.2(10) + 152.6 \qquad \text{Substitute 10 for } x.$$

$$y = 304.6 \qquad \text{Multiply, and then add.}$$

About $305 billion was spent on prescription drugs in 2010. NOW TRY

## 7.2 EXERCISES

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*Complete solution available on the Video Resources on DVD*

***Concept Check*** *In Exercises 1–6, provide the appropriate response.*

**1.** The following equations all represent the same line. Which one is in standard form as defined in the text?

**A.** $3x - 2y = 5$ **B.** $2y = 3x - 5$ **C.** $\frac{3}{5}x - \frac{2}{5}y = 1$ **D.** $3x = 2y + 5$

**2.** Which equation is in point-slope form?

**A.** $y = 6x + 2$ **B.** $4x + y = 9$ **C.** $y - 3 = 2(x - 1)$ **D.** $2y = 3x - 7$

**3.** Which equation in **Exercise 2** is in slope-intercept form?

**4.** Write the equation $y + 2 = -3(x - 4)$ in slope-intercept form.

**5.** Write the equation from **Exercise 4** in standard form.

**6.** Write the equation $10x - 7y = 70$ in slope-intercept form.

***Concept Check*** *Match each equation with the graph that it most closely resembles. (Hint: Determine the signs of m and b to help you make your decision.)*

**7.** $y = 2x + 3$

**8.** $y = -2x + 3$

**9.** $y = -2x - 3$

**10.** $y = 2x - 3$

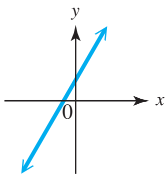
**11.** $y = 2x$

**12.** $y = -2x$

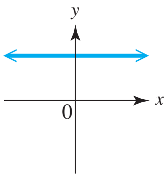
**13.** $y = 3$

**14.** $y = -3$

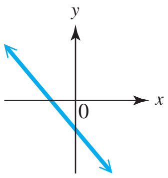
**A.**



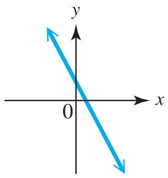
**B.**



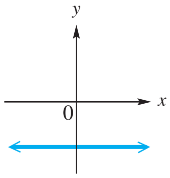
**C.**



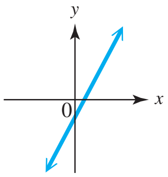
**D.**



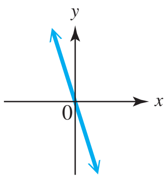
**E.**



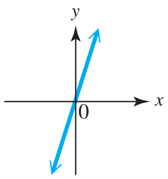
**F.**



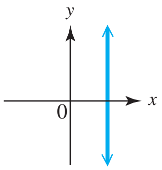
**G.**



**H.**



**I.**



*Write the equation in slope-intercept form of the line satisfying the given conditions.* ***See Example 1.***

**15.** $m = 5; b = 15$

**16.** $m = 2; b = 12$

**17.** $m = -\frac{2}{3}; b = \frac{4}{5}$

**18.** $m = -\frac{5}{8}; b = -\frac{1}{3}$

**19.** Slope 1; $y$-intercept $(0, -1)$

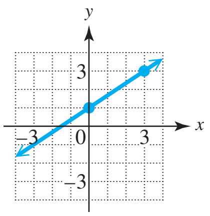
**20.** Slope $-1$; $y$-intercept $(0, -3)$

**21.** Slope $\frac{2}{5}$; $y$-intercept $(0, 5)$

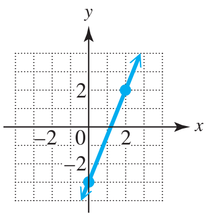
**22.** Slope $-\frac{3}{4}$; $y$-intercept $(0, 7)$

*Concept Check* *Write an equation in slope-intercept form of the line shown in each graph. (Hint: Use the indicated points to find the slope.)*

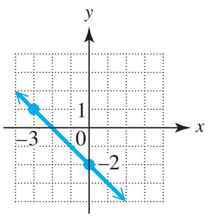
**23.**



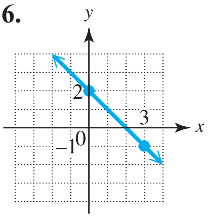
**24.**



**25.**



**26.**



*For each equation,* ***(a)*** *write it in slope-intercept form,* ***(b)*** *give the slope of the line,* ***(c)*** *give the y-intercept, and* ***(d)*** *graph the line.* ***See Example 2.***

**27.** $-x + y = 4$

**28.** $-x + y = 6$

**29.** $6x + 5y = 30$

**30.** $3x + 4y = 12$

**31.** $4x - 5y = 20$

**32.** $7x - 3y = 3$

**33.** $x + 2y = -4$

**34.** $x + 3y = -9$

*Find an equation of the line that satisfies the given conditions.* ***(a)*** *Write the equation in standard form.* ***(b)*** *Write the equation in slope-intercept form.* ***See Example 3.***

**35.** Through $(5, 8)$; slope $-2$

**36.** Through $(12, 10)$; slope 1

**37.** Through $(-2, 4)$; slope $-\frac{3}{4}$

**38.** Through $(-1, 6)$; slope $-\frac{5}{6}$

**39.** Through $(-5, 4)$; slope $\frac{1}{2}$

**40.** Through $(7, -2)$; slope $\frac{1}{4}$

**41.** $x$-intercept $(3, 0)$; slope 4

**42.** $x$-intercept $(-2, 0)$; slope $-5$

**43.** Through $(2, 6.8)$; slope 1.4

**44.** Through $(6, -1.2)$; slope 0.8

*Find an equation of the line that satisfies the given conditions.* ***See Example 4.***

**45.** Through $(9, 5)$; slope 0

**46.** Through $(-4, -2)$; slope 0

**47.** Through $(9, 10)$; undefined slope

**48.** Through $(-2, 8)$; undefined slope

**49.** Through $\left(-\frac{3}{4}, -\frac{3}{2}\right)$; slope 0

**50.** Through $\left(-\frac{5}{8}, -\frac{9}{2}\right)$; slope 0

**51.** Through $(-7, 8)$; horizontal

**52.** Through $(2, -7)$; horizontal

**53.** Through $(0.5, 0.2)$; vertical

**54.** Through $(0.1, 0.4)$; vertical

*Find an equation of the line passing through the given points.* ***(a)*** *Write the equation in standard form.* ***(b)*** *Write the equation in slope-intercept form if possible.* ***See Example 5.***

**55.** $(3, 4)$ and $(5, 8)$

**56.** $(5, -2)$ and $(-3, 14)$

**57.** $(6, 1)$ and $(-2, 5)$

**58.** $(-2, 5)$ and $(-8, 1)$

**59.** $(2, 5)$ and $(1, 5)$

**60.** $(-2, 2)$ and $(4, 2)$

**61.** $(7, 6)$ and $(7, -8)$

**62.** $(13, 5)$ and $(13, -1)$

**63.** $\left(\frac{1}{2}, -3\right)$ and $\left(-\frac{2}{3}, -3\right)$

**64.** $\left(-\frac{4}{9}, -6\right)$ and $\left(\frac{12}{7}, -6\right)$

**65.** $\left(-\frac{2}{5}, \frac{2}{5}\right)$ and $\left(\frac{4}{3}, \frac{2}{3}\right)$

**66.** $\left(\frac{3}{4}, \frac{8}{3}\right)$ and $\left(\frac{2}{5}, \frac{2}{3}\right)$

*Find an equation of the line that satisfies the given conditions.* ***(a)*** *Write the equation in slope-intercept form.* ***(b)*** *Write the equation in standard form.* ***See Example 6.***

**67.** Through $(7, 2)$; parallel to $3x - y = 8$

**68.** Through $(4, 1)$; parallel to $2x + 5y = 10$

**69.** Through $(-2, -2)$; parallel to $-x + 2y = 10$

**70.** Through $(-1, 3)$; parallel to $-x + 3y = 12$

**71.** Through $(8, 5)$; perpendicular to $2x - y = 7$

**72.** Through $(2, -7)$; perpendicular to $5x + 2y = 18$

**73.** Through $(-2, 7)$; perpendicular to $x = 9$

**74.** Through $(8, 4)$; perpendicular to $x = -3$

*Write an equation in the form $y = mx$ for each situation. Then give the three ordered pairs associated with the equation for x-values* 0, 5, *and* 10. ***See Example 7(a).***

**75.** $x$ represents the number of hours traveling at 45 mph, and $y$ represents the distance traveled (in miles).

**76.** $x$ represents the number of t-shirts sold at \$26 each, and $y$ represents the total cost of the t-shirts (in dollars).

**77.** $x$ represents the number of gallons of gas sold at \$3.10 per gal, and $y$ represents the total cost of the gasoline (in dollars).

**78.** $x$ represents the number of days a DVD movie is rented at \$4.50 per day, and $y$ represents the total charge for the rental (in dollars).

**79.** $x$ represents the number of credit hours taken at Kirkwood Community College at \$111 per credit hour, and $y$ represents the total tuition paid for the credit hours (in dollars). (*Source:* www.kirkwood.edu)

**80.** $x$ represents the number of tickets to a performance of *Jersey Boys* at the Des Moines Civic Center purchased at \$125 per ticket, and $y$ represents the total paid for the tickets (in dollars). (*Source:* Ticketmaster.)

*For each situation,* ***(a)*** *write an equation in the form $y = mx + b$,* ***(b)*** *find and interpret the ordered pair associated with the equation for $x = 5$, and* ***(c)*** *answer the question.* ***See Examples 7(b) and 7(c).***

**81.** A ticket for the 2010 Troubadour Reunion, featuring James Taylor and Carole King, costs \$112.50. A parking pass costs \$12. (*Source:* Ticketmaster.) Let $x$ represent the number of tickets and $y$ represent the cost. How much does it cost for 2 tickets and a parking pass?

**82.** Resident tuition at Broward College is \$87.95 per credit hour. There is also a \$20 health science application fee. (*Source:* www.broward.edu) Let $x$ represent the number of credit hours and $y$ represent the cost. How much does it cost for a student in health science to take 15 credit hours?

**83.** A membership in the Midwest Athletic Club costs \$99, plus \$41 per month. (*Source:* Midwest Athletic Club.) Let $x$ represent the number of months and $y$ represent the cost. How much does the first year's membership cost?

**84.** For a family membership, the athletic club in **Exercise 83** charges a membership fee of \$159, plus \$60 for each additional family member after the first. Let $x$ represent the number of additional family members and $y$ represent the cost. What is the membership fee for a four-person family?

**85.** A cell phone plan includes 900 anytime minutes for \$60 per month, plus a one-time activation fee of \$36. A Nokia 6650 cell phone is included at no additional charge. (*Source:* AT&T.) Let $x$ represent the number of months of service and $y$ represent the cost. If you sign a 1-yr contract, how much will this cell phone plan cost? (Assume that you never use more than the allotted number of minutes.)

**86.** Another cell phone plan includes 450 anytime minutes for \$40 per month, plus \$50 for a Nokia 2320 cell phone and \$36 for a one-time activation fee. (*Source:* AT&T.) Let $x$ represent the number of months of service and $y$ represent the cost. If you sign a 1-yr contract, how much will this cell phone plan cost? (Assume that you never use more than the allotted number of minutes.)

**87.** There is a \$30 fee to rent a chain saw, plus \$6 per day. Let $x$ represent the number of days the saw is rented and $y$ represent the charge to the user in dollars. If the total charge is \$138, for how many days is the saw rented?

**88.** A rental car costs \$50 plus \$0.20 per mile. Let $x$ represent the number of miles driven and $y$ represent the total charge to the renter. How many miles was the car driven if the renter paid \$84.60?

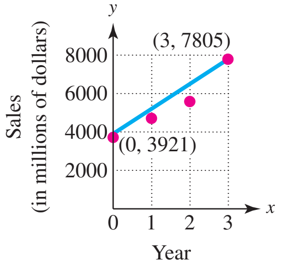
*Solve each problem. In part (a), give equations in slope-intercept form. (Round the slope to the nearest tenth.)* ***See Example 8.***

**89.** Total sales of digital cameras in the United States (in millions of dollars) are shown in the graph, where the year 2003 corresponds to $x = 0$.

**(a)** Use the ordered pairs from the graph to write an equation that models the data. What does the slope tell us in the context of this problem?

**(b)** Use the equation from part (a) to approximate the sales of digital cameras in the United States in 2007.

**Digital Camera Sales**



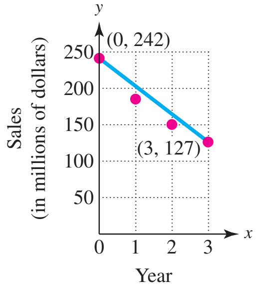
***Source:*** Consumer Electronics Association.

**90.** Total sales of fax machines in the United States (in millions of dollars) are shown in the graph, where the year 2003 corresponds to $x = 0$.

**(a)** Use the ordered pairs from the graph to write an equation that models the data. What does the slope tell us in the context of this problem?

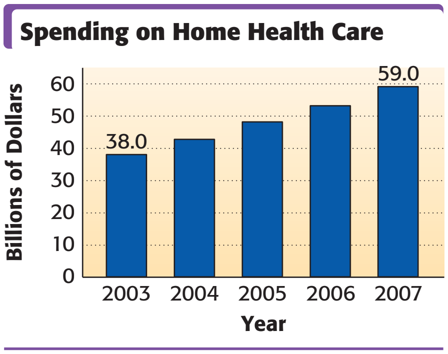
**(b)** Use the equation from part (a) to approximate the sales of fax machines in the United States in 2007.

**Fax Machine Sales**



***Source:*** Consumer Electronics Association.

**91.** Expenditures for home health care in the United States are shown in the graph.

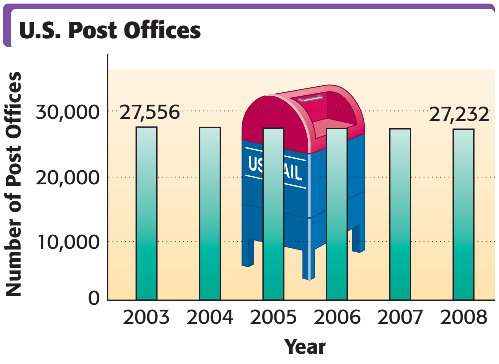


***Source:*** U.S. Centers for Medicare & Medicaid Services.

**(a)** Use the information given for the years 2003 and 2007, letting $x = 3$ represent 2003, $x = 7$ represent 2007, and $y$ represent the amount (in billions of dollars) to write an equation that models home health care spending.

**(b)** Use the equation from part (a) to approximate the amount spent on home health care in 2005. How does your result compare with the actual value, \$48.1 billion?

**92.** The number of post offices in the United States is shown in the graph.



***Source:*** U.S. Postal Service.

**(a)** Use the information given for the years 2003 and 2008, letting $x = 3$ represent 2003, $x = 8$ represent 2008, and $y$ represent the number of post offices, to write an equation that models the data.

**(b)** Use the equation to approximate the number of post offices in 2006. How does this result compare with the actual value, 27,318?

## *RELATING CONCEPTS* EXERCISES 93–100

### FOR INDIVIDUAL OR GROUP WORK

*In* ***Section 2.5,*** *we worked with formulas.* ***Work Exercises 93–100 in order,*** *to see how the formula that relates Celsius and Fahrenheit temperatures is derived.*

**93.** There is a linear relationship between Celsius and Fahrenheit temperatures. When $C = 0°$, $F =$ _____°, and when $C = 100°$, $F =$ _____°.

**94.** Think of ordered pairs of temperatures $(C, F)$, where $C$ and $F$ represent corresponding Celsius and Fahrenheit temperatures. The equation that relates the two scales has a straight-line graph that contains the two points determined in **Exercise 93.** What are these two points?

**95.** Find the slope of the line described in **Exercise 94.**

**96.** Use the slope found in **Exercise 95** and one of the two points determined earlier, and write an equation that gives $F$ in terms of $C$. (*Hint:* Use the point-slope form, with $C$ replacing $x$ and $F$ replacing $y$.)

**97.** To obtain another form of the formula, use the equation found in **Exercise 96** and solve for $C$ in terms of $F$.

**98.** Use the equation from **Exercise 96** to find the Fahrenheit temperature when $C = 30$.

**99.** Use the equation from **Exercise 97** to find the Celsius temperature when $F = 50$.

**100.** For what temperature is $F = C$? (Use the photo to confirm your answer.)

## *PREVIEW EXERCISES*

*Write each inequality using interval notation.* ***See Section 2.8.***

**101.** $x \geq 0$

**102.** $x \leq 0$

**103.** $-4 \leq x \leq 4$

**104.** Express the set of all real numbers using interval notation. **See Section 2.8.**

## *SUMMARY EXERCISES* on Slopes and Equations of Lines

*Find the slope of each line, if possible.*

**1.** $3x + 5y = 9$
**2.** $4x + 7y = 3$
**3.** $y = 2x - 5$
**4.** $5x - 2y = 4$
**5.** $x - 4 = 0$
**6.** $y = 0.5$

*For each line described, write an equation of the line **(a)** in slope-intercept form and **(b)** in standard form.*

**7.** Through the points $(-2, 6)$ and $(4, 1)$
**8.** Through $(-2, 5)$ and parallel to the graph of $3x - y = 4$
**9.** Through the origin and perpendicular to the graph of $2x - 5y = 6$
**10.** Through $(5, -8)$ and parallel to the graph of $y = 4$
**11.** Through $\left(\frac{3}{4}, -\frac{7}{9}\right)$ and perpendicular to the graph of $x = \frac{2}{3}$
**12.** Through $(4, -2)$ with slope $-3$
**13.** Through $(-4, 2)$ and parallel to the line through $(3, 9)$ and $(6, 11)$
**14.** Through $(4, -2)$ and perpendicular to the line through $(3, 7)$ and $(5, 6)$
**15.** Through the points $(4, -8)$ and $(-4, 12)$
**16.** Through $(-3, 6)$ with slope $\frac{2}{3}$
**17.** Through $(0, 3)$ and the midpoint of the segment with endpoints $(2, 8)$ and $(-4, 12)$

**18.** *Concept Check* Match the description in Column I with its equation in Column II.

| I | II |
|---|---|
| **(a)** Slope $-0.5$, $b = -2$ | **A.** $y = -\frac{1}{2}x$ |
| **(b)** $x$-intercept $(4, 0)$, $y$-intercept $(0, 2)$ | **B.** $y = -\frac{1}{2}x - 2$ |
| **(c)** Passes through $(4, -2)$ and $(0, 0)$ | **C.** $x - 2y = 2$ |
| **(d)** $m = \frac{1}{2}$, passes through $(-2, -2)$ | **D.** $x + 2y = 4$ |
| **(e)** $m = \frac{1}{2}$, passes through the origin | **E.** $x = 2y$ |

# 7.3 Introduction to Relations and Functions

**OBJECTIVES**

1. Distinguish between independent and dependent variables.
2. Define and identify relations and functions.
3. Find the domain and range.
4. Identify functions defined by graphs and equations.

### OBJECTIVE 1 Distinguish between independent and dependent variables.

We often describe one quantity in terms of another. Consider the following:

- The amount of a paycheck for an hourly employee depends on the number of hours worked.
- The cost at a gas station depends on the number of gallons of gas pumped.
- The distance traveled by a car moving at a constant rate depends on the time traveled.

We can use ordered pairs to represent these corresponding quantities. We indicate the relationship between hours worked and paycheck amount as follows.

$$(\underset{\substack{\uparrow \\ \text{Number of} \\ \text{hours worked}}}{5}, \underset{\substack{\uparrow \\ \text{Paycheck amount} \\ \text{in dollars}}}{40})$$

Working 5 hr results in a \$40 paycheck.

Similarly, the ordered pair $(10, 80)$ indicates that working 10 hr results in an \$80 paycheck. In this example, what would the ordered pair $(20, 160)$ indicate?

Since paycheck amount *depends* on number of hours worked, paycheck amount is called the *dependent variable,* and number of hours worked is called the *independent variable.* Generalizing, if the value of the variable $y$ depends on the value of the variable $x$, then $y$ is the **dependent variable** and $x$ is the **independent variable.**

$$(\overset{\substack{\text{Independent variable}\\ \downarrow}}{x}, \overset{\substack{\text{Dependent variable}\\ \downarrow}}{y})$$

**OBJECTIVE 2** **Define and identify relations and functions.** Since we can write related quantities as ordered pairs, a set of ordered pairs such as

$$\{(5, 40), (10, 80), (20, 160), (40, 320)\}$$

is called a *relation.*

**Relation**

A **relation** is any set of ordered pairs.

A *function* is a special kind of relation.

**Function**

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

*NOW TRY EXERCISE 1*

Determine whether each relation defines a function.

**(a)** $\{(1, 5), (3, 5), (5, 5)\}$

**(b)** $\{(-1, -3), (0, 2), (-1, 6)\}$

**EXAMPLE 1** Determining Whether Relations Are Functions

Determine whether each relation defines a function.

**(a)** $F = \{(1, 2), (-2, 4), (3, -1)\}$

For $x = 1$, there is only one value of $y$, $2$.

For $x = -2$, there is only one value of $y$, $4$.

For $x = 3$, there is only one value of $y$, $-1$.

Thus, relation $F$ is a function, because for each different $x$-value, there is exactly one $y$-value.

**(b)** $G = \{(-2, -1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

Relation $G$ is also a function. Although the last two ordered pairs have the same $y$-value (1 is paired with 2 and 2 is paired with 2), this does not violate the definition of a function. The first components ($x$-values) are different, and each is paired with only one second component ($y$-value).

**(c)** $H = \{(-4, 1), (-2, 1), (-2, 0)\}$

In relation $H$, the last two ordered pairs have the ***same*** $x$-value paired with ***two different*** $y$-values ($-2$ is paired with both 1 and 0), so $H$ is a relation, but *not* a function.

Different $y$-values

$$H = \{(-4, 1), (\underset{\uparrow}{-2}, \overset{\downarrow}{1}), (\underset{\uparrow}{-2}, \overset{\downarrow}{0})\} \quad \text{Not a function}$$

Same $x$-value

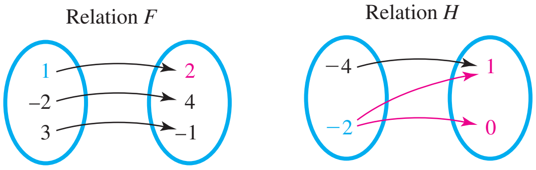
***In a function, no two ordered pairs can have the same first component and different second components.*** NOW TRY

*NOW TRY ANSWERS*

1. **(a)** yes **(b)** no

**Relations and functions can be defined in several different ways.**

- **As a set of ordered pairs (See Example 1.)**
- **As a correspondence or *mapping***



$F$ is a function. $H$ is not a function.

**FIGURE 32**

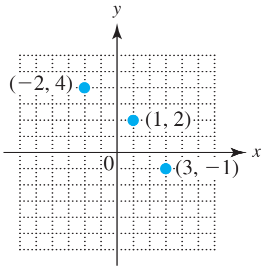
See **FIGURE 32**. In the mapping for relation $F$ from **Example 1(a),** 1 is mapped to 2, $-2$ is mapped to 4, and 3 is mapped to $-1$. Thus, $F$ is a function, since each first component is paired with exactly one second component. In the mapping for relation $H$ from **Example 1(c),** which is not a function, the first component $-2$ is paired with two different second components.

- **As a table**
- **As a graph**

  **FIGURE 33** includes a table and graph for relation $F$ from **Example 1(a).**

| $x$ | $y$ |
|---|---|
| 1 | 2 |
| −2 | 4 |
| 3 | −1 |

Table for relation $F$



Graph of relation $F$

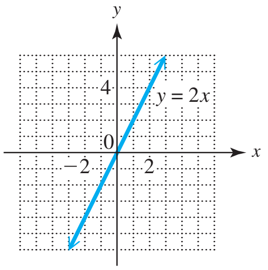
**FIGURE 33**

- **As an equation (or rule)**

  An equation (or rule) can tell how to determine the dependent variable for a specific value of the independent variable. For example, if the value of $y$ is twice the value of $x$, the equation is

$$\underset{\substack{\uparrow\\ \text{Dependent}\\ \text{variable}}}{y} = 2\underset{\substack{\uparrow\\ \text{Independent}\\ \text{variable}}}{x}.$$

The solutions of this equation define an infinite set of ordered pairs that can be represented by the graph in **FIGURE 34**.

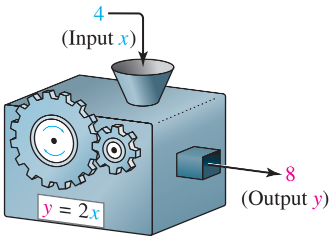


Graph of the relation defined by $y = 2x$

**FIGURE 34**

**NOTE** Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output. This is illustrated by the input-output (function) machine for the function defined by

$$y = 2x.$$



Function machine

***In a function, there is exactly one value of the dependent variable, the second component, for each value of the independent variable, the first component.***

**OBJECTIVE 3** **Find the domain and range.** For every relation, there are two important sets of elements called the *domain* and *range*.

### Domain and Range

In a relation, the set of all values of the independent variable $(x)$ is the **domain.** The set of all values of the dependent variable $(y)$ is the **range.**

**NOW TRY EXERCISE 2**

Give the domain and range of each relation. Tell whether the relation defines a function.

**(a)** $\{(2, 2), (2, 5), (4, 8), (6, 5)\}$

**(b)**

| $x$ | $y$ |
|---|---|
| 1 | \$ 1.39 |
| 10 | \$10.39 |
| 15 | \$20.85 |

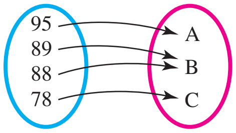
### EXAMPLE 2 Finding Domains and Ranges of Relations

Give the domain and range of each relation. Tell whether the relation defines a function.

**(a)** $\{(3, -1), (4, 2), (4, 5), (6, 8)\}$

The domain, the set of $x$-values, is $\{3, 4, 6\}$. The range, the set of $y$-values, is $\{-1, 2, 5, 8\}$. This relation is not a function because the same $x$-value 4 is paired with two different $y$-values, 2 and 5.

**(b)**



The domain of the relation represented by this mapping is $\{95, 89, 88, 78\}$, and the range is $\{\text{A, B, C}\}$. The mapping defines a function—each domain value corresponds to exactly one range value.

**(c)**

| $x$ | $y$ |
|---|---|
| −5 | 2 |
| 0 | 2 |
| 5 | 2 |

In this table, the domain is the set of $x$-values $\{-5, 0, 5\}$ and the range is the set of $y$-values $\{2\}$. The table defines a function—each $x$-value corresponds to exactly one $y$-value (even though it is the same $y$-value).

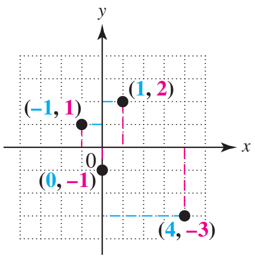
NOW TRY

A graph gives a "picture" of a relation and can be used to determine its domain and range.

### EXAMPLE 3 Finding Domains and Ranges from Graphs

Give the domain and range of each relation.

**(a)**



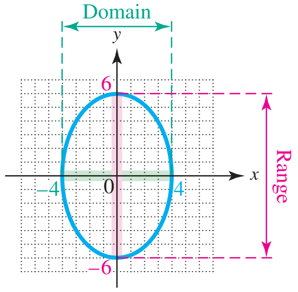
This relation includes the four ordered pairs that are graphed. The domain is the set of $x$-values,

$$\{-1, 0, 1, 4\}.$$

The range is the set of $y$-values,

$$\{-3, -1, 1, 2\}.$$

**(b)**



The $x$-values of the points on the graph include all numbers between $-4$ and 4, inclusive. The $y$-values include all numbers between $-6$ and 6, inclusive.

The domain is $[-4, 4]$.
The range is $[-6, 6]$.

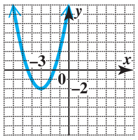
Use interval notation.

**NOW TRY ANSWERS**

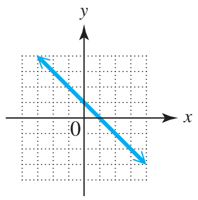
2. **(a)** domain: $\{2, 4, 6\}$; range: $\{2, 5, 8\}$; not a function
   **(b)** domain: $\{1, 10, 15\}$; range: $\{\$1.39, \$10.39, \$20.85\}$; function

**NOW TRY EXERCISE 3**

Give the domain and range of the relation.

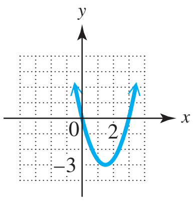


**(c)**



The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and the range include all real numbers, written $(-\infty, \infty)$.

**(d)**



The graph extends indefinitely left and right, as well as upward. The domain is $(-\infty, \infty)$. Because there is a least $y$-value, $-3$, the range includes all numbers greater than or equal to $-3$, written $[-3, \infty)$.

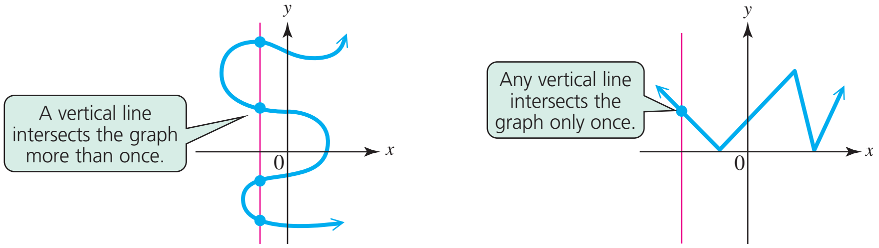
NOW TRY

**OBJECTIVE 4 Identify functions defined by graphs and equations.** Since each value of $x$ in a function corresponds to only one value of $y$, any vertical line drawn through the graph of a function must intersect the graph in at most one point.

### Vertical Line Test

If every vertical line intersects the graph of a relation in no more than one point, then the relation is a function.

**FIGURE 35** illustrates the vertical line test with the graphs of two relations.



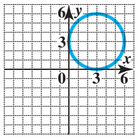
Not a function
The same $x$-value corresponds to four different $y$-values.

Function
Each $x$-value corresponds to only one $y$-value.

**FIGURE 35**

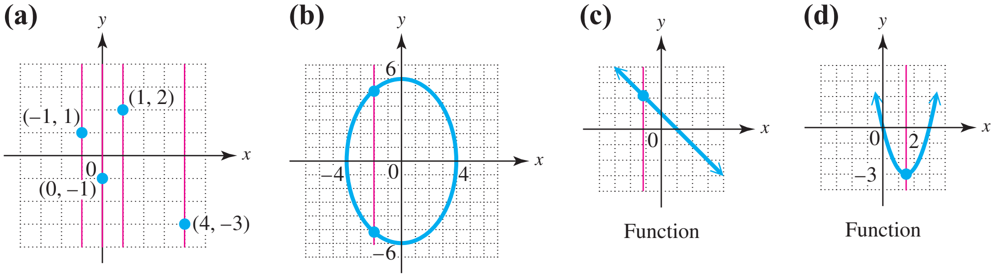
**NOW TRY EXERCISE 4**

Use the vertical line test to determine whether the relation is a function.



## EXAMPLE 4 Using the Vertical Line Test

Use the vertical line test to determine whether each relation graphed in **Example 3** is a function. (We repeat the graphs here.)



(a) Function (b) Not a function (c) Function (d) Function

The graphs in (a), (c), and (d) satisfy the vertical line test and represent functions. The graph in (b) fails the vertical line test, since the same $x$-value corresponds to two different $y$-values, and is not the graph of a function.

NOW TRY

**NOW TRY ANSWERS**

3. domain: $(-\infty, \infty)$; range: $[-2, \infty)$
4. not a function

**NOTE** Graphs that do not represent functions are still relations. ***All equations and graphs represent relations, and all relations have a domain and range.***

Relations are often defined by equations. If a relation is defined by an equation, keep the following in mind when finding its domain.

**Exclude from the domain any values that make the denominator of a fraction equal to 0.**

For example, the function defined by $y = \frac{1}{x}$ has all real numbers except 0 as its domain, since division by 0 is undefined.

**NOTE** As we will see in **Section 10.1,** we must also **exclude from the domain any values that result in an even root of a negative number.**

In this book, we assume the following agreement on the domain of a relation.

**Agreement on Domain**

Unless specified otherwise, the domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

**NOW TRY EXERCISE 5**

Decide whether each relation defines $y$ as a function of $x$, and give the domain.

**(a)** $y = 4x - 3$

**(b)** $y = \dfrac{1}{x - 8}$

### EXAMPLE 5 Identifying Functions from Their Equations

Decide whether each relation defines $y$ as a function of $x$, and give the domain.

**(a)** $y = x + 4$

In the defining equation (or rule) $y = x + 4$, $y$ is always found by adding 4 to $x$. Thus, each value of $x$ corresponds to just one value of $y$, and the relation defines a function. Since $x$ can be any real number, the domain is

$$\{x \mid x \text{ is a real number}\}, \quad \text{or} \quad (-\infty, \infty).$$

**(b)** $y^2 = x$

The ordered pairs $(16, 4)$ and $(16, -4)$ both satisfy this equation. Since one value of $x$, 16, corresponds to two values of $y$, 4 and $-4$, this equation does not define a function. Because $x$ is equal to the square of $y$, the values of $x$ must always be nonnegative. The domain of the relation is $[0, \infty)$.

**(c)** $y = \dfrac{5}{x - 1}$

Given any value of $x$ in the domain, we find $y$ by subtracting 1 and then dividing the result into 5. This process produces exactly one value of $y$ for each value in the domain, so the given equation defines a function.

The domain includes all real numbers except those which make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for $x$.

$$x - 1 = 0$$

$$x = 1 \quad \text{Add 1.}$$

The domain includes all real numbers *except* 1, written $(-\infty, 1) \cup (1, \infty)$.*

NOW TRY

**NOW TRY ANSWERS**

**5.** **(a)** yes; $(-\infty, \infty)$
**(b)** yes; $(-\infty, 8) \cup (8, \infty)$

---

*For any two sets $A$ and $B$, the union of $A$ and $B$, symbolized $A \cup B$, consists of the elements in either $A$ or $B$ (or both).

In summary, we give three variations of the definition of a function.

### Variations of the Definition of a Function

1. A **function** is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of distinct ordered pairs in which no first component is repeated.
3. A **function** is a correspondence or rule that assigns exactly one range value to each domain value.

## 7.3 EXERCISES

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**1.** In your own words, define a function and give an example.

**2.** In your own words, define the domain of a function and give an example.

**3.** *Concept Check* In an ordered pair of a relation, is the first element the independent or the dependent variable?

**4.** *Concept Check* Give an example of a relation that is not a function and that has domain $\{-3, 2, 6\}$ and range $\{4, 6\}$. (There are many possible correct answers.)

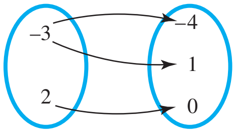
*Concept Check* *Express each relation using a different form. There is more than one correct way to do this.* ***See Objective 2.***

**5.** $\{(0, 2), (2, 4), (4, 6)\}$

**6.**

| $x$ | $y$ |
|---|---|
| $-1$ | $-3$ |
| $0$ | $-1$ |
| $1$ | $1$ |
| $3$ | $3$ |

**7.**



**8.** *Concept Check* Does the relation given in **Exercise 7** define a function? Why or why not?

*Decide whether each relation defines a function, and give the domain and range.* ***See Examples 1–4.***

**9.** $\{(5, 1), (3, 2), (4, 9), (7, 6)\}$

**10.** $\{(8, 0), (5, 4), (9, 3), (3, 8)\}$

**11.** $\{(2, 4), (0, 2), (2, 5)\}$

**12.** $\{(9, -2), (-3, 5), (9, 2)\}$

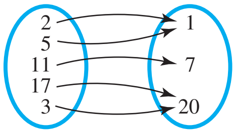
**13.** $\{(-3, 1), (4, 1), (-2, 7)\}$

**14.** $\{(-12, 5), (-10, 3), (8, 3)\}$

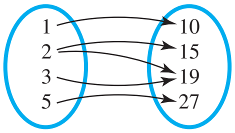
**15.** $\{(1, 1), (1, -1), (0, 0), (2, 4), (2, -4)\}$

**16.** $\{(2, 5), (3, 7), (4, 9), (5, 11)\}$

**17.**



**18.**



**19.**

| $x$ | $y$ |
|---|---|
| $1$ | $5$ |
| $1$ | $2$ |
| $1$ | $-1$ |
| $1$ | $-4$ |

**20.**

| $x$ | $y$ |
|---|---|
| $-4$ | $-4$ |
| $-4$ | $0$ |
| $-4$ | $4$ |
| $-4$ | $8$ |

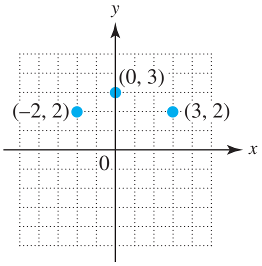
**21.**

| $x$ | $y$ |
|---|---|
| $4$ | $-3$ |
| $2$ | $-3$ |
| $0$ | $-3$ |
| $-2$ | $-3$ |

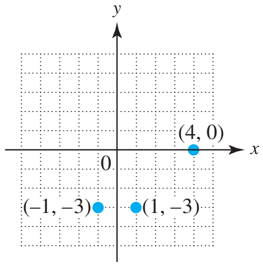
**22.**

| $x$ | $y$ |
|---|---|
| $-3$ | $-6$ |
| $-1$ | $-6$ |
| $1$ | $-6$ |
| $3$ | $-6$ |

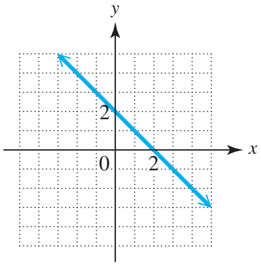
**23.**



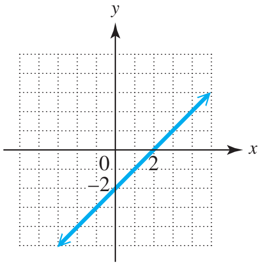
**24.**



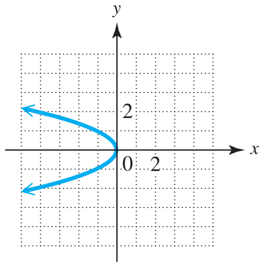
**25.**



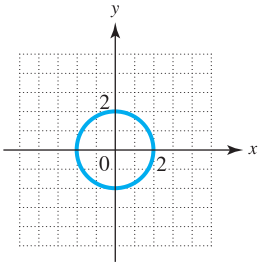
**26.**



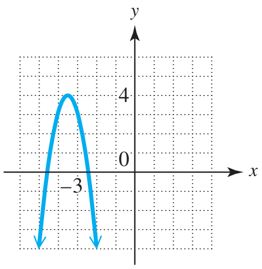
**27.**



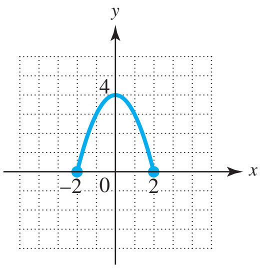
**28.**



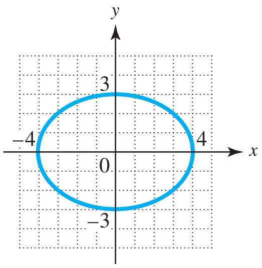
**29.**



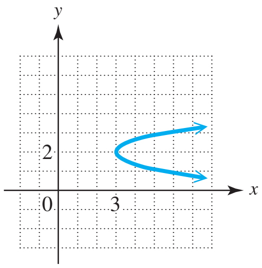
**30.**



**31.**



**32.**



*Decide whether each relation defines y as a function of x. Give the domain.* ***See Example 5.***

**33.** $y = -6x$

**34.** $y = -9x$

**35.** $y = 2x - 6$

**36.** $y = 6x + 8$

**37.** $y = x^2$

**38.** $y = x^3$

**39.** $x = y^6$

**40.** $x = y^4$

**41.** $y = \frac{x + 4}{5}$

**42.** $y = \frac{x - 3}{2}$

**43.** $y = -\frac{2}{x}$

**44.** $y = -\frac{6}{x}$

**45.** $y = \frac{2}{x - 4}$

**46.** $y = \frac{7}{x - 2}$

**47.** $y = \frac{1}{4x + 3}$

**48.** $y = \frac{1}{2x + 9}$

**49.** $x = y^2 + 1$

**50.** $x = y^2 - 3$

**51.** $xy = 1$

**52.** $xy = 3$

*Solve each problem.*

**53.** The table shows the percentage of students at 4-year public colleges who graduated within 5 years.

**(a)** Does the table define a function?

**(b)** What are the domain and range?

**(c)** Call this function $f$. Give two ordered pairs that belong to $f$.

| *Year* | *Percentage* |
|---|---|
| 2004 | 42.3 |
| 2005 | 42.3 |
| 2006 | 42.8 |
| 2007 | 43.7 |
| 2008 | 43.8 |

*Source:* ACT.

**54.** The table shows the percentage of full-time college freshmen who said they had discussed politics in election years.

**(a)** Does the table define a function?

**(b)** What are the domain and range?

**(c)** Call this function $g$. Give two ordered pairs that belong to $g$.

| *Year* | *Percentage* |
|---|---|
| 1992 | 83.7 |
| 1996 | 73.0 |
| 2000 | 69.6 |
| 2004 | 77.4 |
| 2008 | 85.9 |

*Source:* Cooperative Institutional Research Program.

## *PREVIEW EXERCISES*

*Evaluate $y$ for $x = 3$.* ***See Section 3.1.***

**55.** $y = -7x + 12$

**56.** $y = -5x - 4$

**57.** $y = 3x - 8$

*Solve for $y$.* ***See Section 2.5.***

**58.** $3x - 7y = 8$

**59.** $2x - 4y = 7$

**60.** $\frac{3}{4}x + 2y = 9$

# 7.4

# Function Notation and Linear Functions

**OBJECTIVES**

1. Use function notation.
2. Graph linear and constant functions.

**OBJECTIVE 1** **Use function notation.** When a function $f$ is defined with a rule or an equation using $x$ and $y$ for the independent and dependent variables, we say, "*$y$ is a function of $x$*" to emphasize that *$y$ depends on $x$*. We use the notation

$$y = f(x),$$

The parentheses here do *not* indicate multiplication.

called **function notation,** to express this and read $f(x)$ as **"$f$ of $x$."** The letter $f$ is a name for this particular function. For example, if $y = 9x - 5$, we can name this function $f$ and write

$$f(x) = 9x - 5.$$

$f$ is the name of the function.
$x$ is a value from the domain.
$f(x)$ is the function value (or $y$-value) that corresponds to $x$.

***$f(x)$ is just another name for the dependent variable $y$.***

We can evaluate a function at different values of $x$ by substituting $x$-values from the domain into the function.

**NOW TRY EXERCISE 1**

Let $f(x) = 4x + 3$. Find the value of the function $f$ for $x = -2$.

## EXAMPLE 1 Evaluating a Function

Let $f(x) = 9x - 5$. Find the value of the function $f$ for $x = 2$.

Read $f(2)$ as "$f$ of 2" or "$f$ at 2."

$$\begin{aligned} f(x) &= 9x - 5 \\ f(2) &= 9 \cdot 2 - 5 && \text{Replace } x \text{ with } 2. \\ f(2) &= 18 - 5 && \text{Multiply.} \\ f(2) &= 13 && \text{Add.} \end{aligned}$$

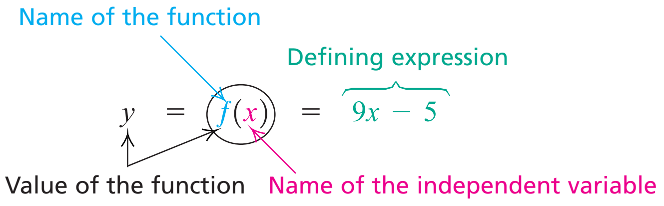
Thus, for $x = 2$, the corresponding function value (or $y$-value) is 13. $f(2) = 13$ is an abbreviation for the statement

"If $x = 2$ in the function $f$, then $y = 13$"

and is represented by the ordered pair $(2, 13)$. NOW TRY

**CAUTION** The symbol $f(x)$ *does not* indicate "$f$ times $x$," but represents the $y$-value associated with the indicated $x$-value. As just shown, $f(2)$ is the $y$-value that corresponds to the $x$-value 2 in the function.

These ideas can be illustrated as follows.



**NOW TRY EXERCISE 2**

Let $f(x) = 2x^2 - 4x + 1$. Find the following.

**(a)** $f(-2)$ **(b)** $f(a)$

## EXAMPLE 2 Evaluating a Function

Let $f(x) = -x^2 + 5x - 3$. Find the following.

**(a)** $f(4)$

Do *not* read this as "$f$ times 4." Read it as "$f$ of 4."

$$\begin{aligned} f(x) &= -x^2 + 5x - 3 && \text{The base in } -x^2 \text{ is } x, \textit{not } (-x). \\ f(4) &= -4^2 + 5 \cdot 4 - 3 && \text{Replace } x \text{ with } 4. \\ f(4) &= -16 + 20 - 3 && \text{Apply the exponent. Multiply.} \\ f(4) &= 1 && \text{Add and subtract.} \end{aligned}$$

Thus, $f(4) = 1$, and the ordered pair $(4, 1)$ belongs to $f$.

**(b)** $f(q)$

$$\begin{aligned} f(x) &= -x^2 + 5x - 3 \\ f(q) &= -q^2 + 5q - 3 && \text{Replace } x \text{ with } q. \end{aligned}$$

The replacement of one variable with another is important in later courses.

NOW TRY

Sometimes letters other than $f$, such as $g$, $h$, or capital letters $F$, $G$, and $H$ are used to name functions.

**NOW TRY ANSWERS**

**1.** $-5$

**2. (a)** 17 **(b)** $2a^2 - 4a + 1$

**NOW TRY EXERCISE 3**

Let $g(x) = 8x - 5$. Find and simplify $g(a - 2)$.

## EXAMPLE 3 Evaluating a Function

Let $g(x) = 2x + 3$. Find and simplify $g(a + 1)$.

$$
\begin{aligned}
g(x) &= 2x + 3 \\
g(a+1) &= 2(a+1) + 3 && \text{Replace } x \text{ with } a + 1. \\
g(a+1) &= 2a + 2 + 3 && \text{Distributive property} \\
g(a+1) &= 2a + 5 && \text{Add.}
\end{aligned}
$$

NOW TRY

**NOW TRY EXERCISE 4**

Find $f(-1)$ for each function.

**(a)** $f = \{(-5, -1), (-3, 2), (-1, 4)\}$

**(b)** $f(x) = x^2 - 12$

## EXAMPLE 4 Evaluating Functions

For each function, find $f(3)$.

**(a)** $f(x) = 3x - 7$

$$
\begin{aligned}
f(3) &= 3(3) - 7 && \text{Replace } x \text{ with 3.} \\
f(3) &= 9 - 7 && \text{Multiply.} \\
f(3) &= 2 && \text{Subtract.}
\end{aligned}
$$

**(b)**

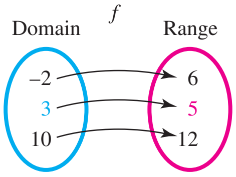
| $x$ | $y = f(x)$ |
|---|---|
| 6 | $-12$ |
| 3 | $-6$ |
| 0 | 0 |
| $-3$ | 6 |

← Here, $f(3) = -6$.

**(c)** $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$

We want $f(3)$, the $y$-value of the ordered pair whose first component is $x = 3$. As indicated by the ordered pair $(3, 1)$, for $x = 3$, $y = 1$. Thus, $f(3) = 1$.

**(d)**



The domain element 3 is paired with 5 in the range, so $f(3) = 5$.

NOW TRY

## EXAMPLE 5 Finding Function Values from a Graph

Refer to the function graphed in **FIGURE 36**.

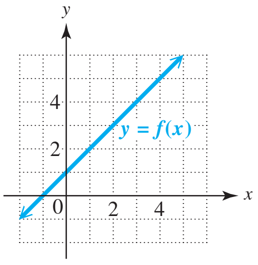


**FIGURE 36**

**(a)** Find $f(3)$.

Locate 3 on the $x$-axis. See **FIGURE 37**. Moving up to the graph of $f$ and over to the $y$-axis gives 4 for the corresponding $y$-value. Thus, $f(3) = 4$, which corresponds to the ordered pair $(3, 4)$.

**(b)** Find $f(0)$.

Refer to **FIGURE 37** to see that $f(0) = 1$.

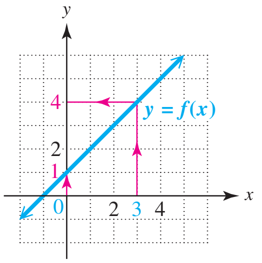


**FIGURE 37**

**NOW TRY ANSWERS**

3. $8a - 21$

4. **(a)** 4 **(b)** $-11$

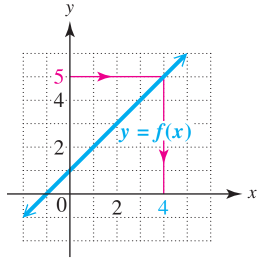


FIGURE 38

**(c)** For what value of $x$ is $f(x) = 5$?

Since $f(x) = y$, we want the value of $x$ that corresponds to $y = 5$. Locate 5 on the $y$-axis. See **FIGURE 38**. Moving across to the graph of $f$ and down to the $x$-axis gives $x = 4$. Thus, $f(4) = 5$, which corresponds to the ordered pair $(4, 5)$. NOW TRY

If a function $f$ is defined by an equation with $x$ and $y$, and $y$ is not solved for $x$, use the following steps to find $f(x)$.

**Finding an Expression for $f(x)$**

***Step 1*** Solve the equation for $y$.

***Step 2*** Replace $y$ with $f(x)$.

### EXAMPLE 6 Writing Equations Using Function Notation

Rewrite each equation using function notation $f(x)$. Then find $f(-2)$ and $f(a)$.

**(a)** $y = x^2 + 1$

This equation is already solved for $y$, so we replace $y$ with $f(x)$.

$$f(x) = x^2 + 1 \qquad y = f(x)$$

To find $f(-2)$, let $x = -2$.

$$\begin{aligned} f(x) &= x^2 + 1 \\ f(-2) &= (-2)^2 + 1 && \text{Let } x = -2. \\ f(-2) &= 4 + 1 && (-2)^2 = -2(-2) \\ f(-2) &= 5 && \text{Add.} \end{aligned}$$

Find $f(a)$ by letting $x = a$: $\quad f(a) = a^2 + 1$.

**(b)** $x - 4y = 5$ $\quad$ Solve for $y$. (Step 1)

$$x - 5 = 4y \qquad \text{Add } 4y. \text{ Subtract } 5.$$

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

$$y = \frac{x-5}{4}, \quad \text{so} \quad f(x) = \frac{1}{4}x - \frac{5}{4} \qquad y = f(x) \text{ (Step 2)}$$

Now find $f(-2)$ and $f(a)$.

$$f(-2) = \frac{1}{4}(-2) - \frac{5}{4} = -\frac{7}{4} \qquad \text{Let } x = -2.$$

$$f(a) = \frac{1}{4}a - \frac{5}{4} \qquad \text{Let } x = a.$$

NOW TRY

**OBJECTIVE 2** **Graph linear and constant functions.** Linear equations (except for vertical lines with equations $x = a$) define *linear functions.*

**Linear Function**

A function that can be defined by

$$\boldsymbol{f(x) = ax + b}$$

for real numbers $a$ and $b$ is a **linear function.** The value of $a$ is the slope $m$ of the graph of the function. The domain of any linear function is $(-\infty, \infty)$.

NOW TRY EXERCISE 5

Refer to the function graphed in **FIGURE 36** on the previous page.

**(a)** Find $f(-1)$.

**(b)** For what value of $x$ is $f(x) = 2$?

NOW TRY EXERCISE 6

Rewrite the equation using function notation $f(x)$. Then find $f(-3)$ and $f(h)$.

$$-4x^2 + y = 5$$

NOW TRY ANSWERS

**5.** **(a)** 0 **(b)** 1

**6.** $f(x) = 4x^2 + 5$; $f(-3) = 41$; $f(h) = 4h^2 + 5$

A linear function whose graph is a horizontal line is defined by

$$f(x) = b \qquad \text{Constant function}$$

and is sometimes called a **constant function.** While the range of any nonconstant linear function is $(-\infty, \infty)$, the range of a constant function defined by $f(x) = b$ is $\{b\}$.

**NOW TRY EXERCISE 7**

Graph the function. Give the domain and range.

$$g(x) = \frac{1}{3}x - 2$$

### EXAMPLE 7 Graphing Linear and Constant Functions

Graph each function. Give the domain and range.

**(a)** $f(x) = \underset{\text{Slope}}{\frac{1}{4}}x - \underset{y\text{-intercept is } \left(0, -\frac{5}{4}\right).}{\frac{5}{4}}$ (from **Example 6(b)**)

The graph of $y = \frac{1}{4}x - \frac{5}{4}$ has slope $m = \frac{1}{4}$ and $y$-intercept $\left(0, -\frac{5}{4}\right)$. To graph this function, plot the $y$-intercept $\left(0, -\frac{5}{4}\right)$ and use the definition of slope as $\frac{\text{rise}}{\text{run}}$ to find a second point on the line. Since the slope is $\frac{1}{4}$, move 1 unit up from $\left(0, -\frac{5}{4}\right)$ and 4 units to the right to find this second point. Draw the straight line through the points to obtain the graph shown in **FIGURE 39**. The domain and range are both $(-\infty, \infty)$.

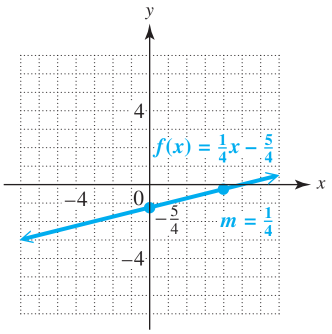


**FIGURE 39**

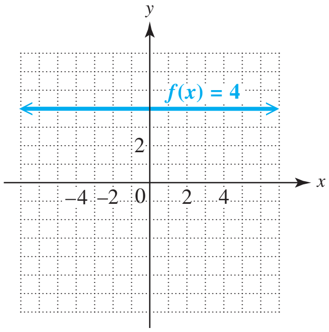


**FIGURE 40**

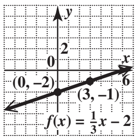
**(b)** $f(x) = 4$

The graph of this constant function is the horizontal line containing all points with $y$-coordinate 4. See **FIGURE 40**. The domain is $(-\infty, \infty)$ and the range is $\{4\}$.

NOW TRY

**NOW TRY ANSWER**

7.



domain: $(-\infty, \infty)$;
range: $(-\infty, \infty)$

## 7.4 EXERCISES

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1. *Concept Check* Choose the correct response: The notation $f(3)$ means
   - **A.** the variable $f$ times 3, or $3f$.
   - **B.** the value of the dependent variable when the independent variable is 3.
   - **C.** the value of the independent variable when the dependent variable is 3.
   - **D.** $f$ equals 3.

2. *Concept Check* Give an example of a function from everyday life. (*Hint:* Fill in the blanks: _____ depends on _____, so _____ is a function of _____.)

*Let $f(x) = -3x + 4$ and $g(x) = -x^2 + 4x + 1$. Find the following.* ***See Examples 1–3.***

**3.** $f(0)$ **4.** $f(-3)$ **5.** $g(-2)$ **6.** $g(10)$

**7.** $f\left(\frac{1}{3}\right)$ **8.** $f\left(\frac{7}{3}\right)$ **9.** $g(0.5)$ **10.** $g(1.5)$

**11.** $f(p)$ **12.** $g(k)$ **13.** $f(-x)$ **14.** $g(-x)$

**15.** $f(x + 2)$ **16.** $f(x - 2)$ **17.** $g(\pi)$ **18.** $g(e)$

**19.** $f(x + h)$ **20.** $f(x + h) - f(x)$ **21.** $f(4) - g(4)$ **22.** $f(10) - g(10)$

*For each function, find* ***(a)*** *$f(2)$ and* ***(b)*** *$f(-1)$.* ***See Examples 4 and 5.***

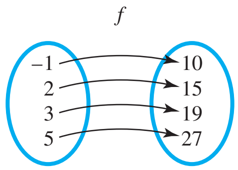
**23.** $f = \{(-2, 2), (-1, -1), (2, -1)\}$

**24.** $f = \{(-1, -5), (0, 5), (2, -5)\}$

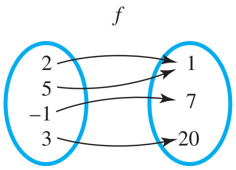
**25.** $f = \{(-1, 3), (4, 7), (0, 6), (2, 2)\}$

**26.** $f = \{(2, 5), (3, 9), (-1, 11), (5, 3)\}$

**27.**



**28.**



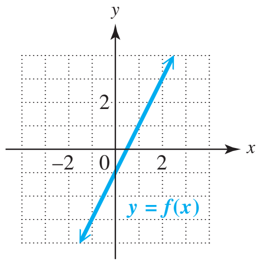
**29.**

| $x$ | $y = f(x)$ |
|---|---|
| 2 | 4 |
| 1 | 1 |
| 0 | 0 |
| −1 | 1 |
| −2 | 4 |

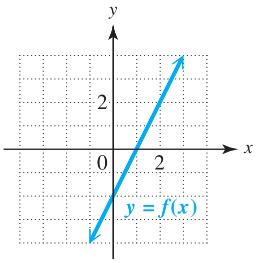
**30.**

| $x$ | $y = f(x)$ |
|---|---|
| 8 | 6 |
| 5 | 3 |
| 2 | 0 |
| −1 | −3 |
| −4 | −6 |

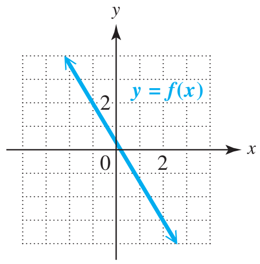
**31.**



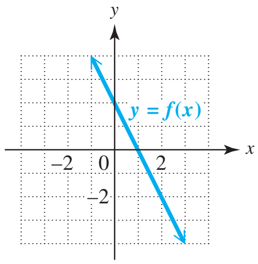
**32.**



**33.**



**34.**



**35.** Refer to **Exercise 31.** Find the value of $x$ for each value of $f(x)$. **See Example 5(c).**

**(a)** $f(x) = 3$ **(b)** $f(x) = -1$ **(c)** $f(x) = -3$

**36.** Refer to **Exercise 32.** Find the value of $x$ for each value of $f(x)$. **See Example 5(c).**

**(a)** $f(x) = 4$ **(b)** $f(x) = -2$ **(c)** $f(x) = 0$

*An equation that defines $y$ as a function $f$ of $x$ is given.* ***(a)*** *Solve for $y$ in terms of $x$, and replace $y$ with the function notation $f(x)$.* ***(b)*** *Find $f(3)$.* ***See Example 6.***

**37.** $x + 3y = 12$ **38.** $x - 4y = 8$ **39.** $y + 2x^2 = 3$

**40.** $y - 3x^2 = 2$ **41.** $4x - 3y = 8$ **42.** $-2x + 5y = 9$

**43.** *Concept Check* Fill in each blank with the correct response.

The equation $2x + y = 4$ has a straight _____ as its graph. One point that lies on the graph is $(3, \_\_\_\_\_)$. If we solve the equation for $y$ and use function notation, we obtain $f(x) = \_\_\_\_\_$. For this function, $f(3) = \_\_\_\_\_$, meaning that the point $(\_\_\_\_\_, \_\_\_\_\_)$ lies on the graph of the function.

**44.** *Concept Check* Which of the following defines $y$ as a linear function of $x$?

**A.** $y = \frac{1}{4}x - \frac{5}{4}$ **B.** $y = \frac{1}{x}$ **C.** $y = x^2$ **D.** $y = \sqrt{x}$

*Graph each linear function. Give the domain and range.* ***See Example 7.***

**45.** $f(x) = -2x + 5$ **46.** $g(x) = 4x - 1$ **47.** $h(x) = \frac{1}{2}x + 2$

**48.** $F(x) = -\frac{1}{4}x + 1$ **49.** $G(x) = 2x$ **50.** $H(x) = -3x$

**51.** $g(x) = -4$ **52.** $f(x) = 5$ **53.** $f(x) = 0$ **54.** $f(x) = -2.5$

**55.** *Concept Check* What is the name that is usually given to the graph in **Exercise 53?**

**56.** Can the graph of a linear function have an undefined slope? Explain.

*Solve each problem.*

**57.** A package weighing $x$ pounds costs $f(x)$ dollars to mail to a given location, where

$$f(x) = 3.75x.$$

**(a)** Evaluate $f(3)$.

**(b)** Describe what 3 and the value $f(3)$ mean in part (a), using the terminology *independent variable* and *dependent variable.*

**(c)** How much would it cost to mail a 5-lb package? Interpret this question and its answer, using function notation.

**58.** A taxicab driver charges \$2.50 per mile.

**(a)** Fill in the table with the correct response for the price $f(x)$ he charges for a trip of $x$ miles.

| $x$ | $f(x)$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |

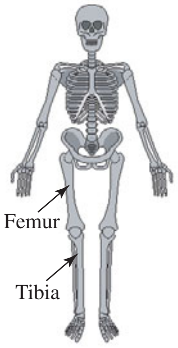
**(b)** The linear function that gives a rule for the amount charged is $f(x) = \_\_\_\_\_$.

**(c)** Graph this function for the domain $\{0, 1, 2, 3\}$.

**59.** Forensic scientists use the lengths of certain bones to calculate the height of a person. Two bones often used are the tibia $(t)$, the bone from the ankle to the knee, and the femur $(r)$, the bone from the knee to the hip socket. A person's height $(h)$ in centimeters is determined from the lengths of these bones by using functions defined by the following formulas.

*For men:* $h(r) = 69.09 + 2.24r$ or $h(t) = 81.69 + 2.39t$

*For women:* $h(r) = 61.41 + 2.32r$ or $h(t) = 72.57 + 2.53t$



**(a)** Find the height of a man with a femur measuring 56 cm.

**(b)** Find the height of a man with a tibia measuring 40 cm.

**(c)** Find the height of a woman with a femur measuring 50 cm.

**(d)** Find the height of a woman with a tibia measuring 36 cm.

**60.** Federal regulations set standards for the size of the quarters of marine mammals. A pool to house sea otters must have a volume of "the square of the sea otter's average adult length (in meters) multiplied by 3.14 and by 0.91 meter." If $x$ represents the sea otter's average adult length and $f(x)$ represents the volume (in cubic meters) of the corresponding pool size, this formula can be written as

$$f(x) = 0.91(3.14)x^2.$$

Find the volume of the pool for each adult sea otter length (in meters). Round answers to the nearest hundredth.

**(a)** 0.8 **(b)** 1.0 **(c)** 1.2 **(d)** 1.5

**61.** To print t-shirts, there is a \$100 set-up fee, plus a \$12 charge per t-shirt. Let $x$ represent the number of t-shirts printed and $f(x)$ represent the total charge.

**(a)** Write a linear function that models this situation.

**(b)** Find $f(125)$. Interpret your answer in the context of this problem.

**(c)** Find the value of $x$ if $f(x) = 1000$. Express this situation using function notation, and interpret it in the context of this problem.

**62.** Rental on a car is \$150, plus \$0.20 per mile. Let $x$ represent the number of miles driven and $f(x)$ represent the total cost to rent the car.

**(a)** Write a linear function that models this situation.

**(b)** How much would it cost to drive 250 mi? Interpret this question and answer, using function notation.

**(c)** Find the value of $x$ if $f(x) = 230$. Interpret your answer in the context of this problem.

**63.** The table represents a linear function.

| $x$ | $y = f(x)$ |
|---|---|
| 0 | 3.5 |
| 1 | 2.3 |
| 2 | 1.1 |
| 3 | −0.1 |
| 4 | −1.3 |
| 5 | −2.5 |

**(a)** What is $f(2)$?

**(b)** If $f(x) = -2.5$, what is the value of $x$?

**(c)** What is the slope of the line?

**(d)** What is the $y$-intercept of the line?

**(e)** Using your answers from parts (c) and (d), write an equation for $f(x)$.

**64.** The table represents a linear function.

| $x$ | $y = f(x)$ |
|---|---|
| −1 | −3.9 |
| 0 | −2.4 |
| 1 | −0.9 |
| 2 | 0.6 |
| 3 | 2.1 |

**(a)** What is $f(2)$?

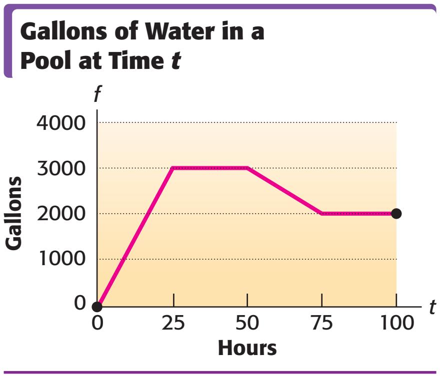
**(b)** If $f(x) = 2.1$, what is the value of $x$?

**(c)** What is the slope of the line?

**(d)** What is the $y$-intercept of the line?

**(e)** Using your answers from parts (c) and (d), write an equation for $f(x)$.

**65.** Refer to the graph to answer each of the questions.


**Gallons of Water in a Pool at Time *t***

**(a)** What numbers are possible values of the independent variable? The dependent variable?

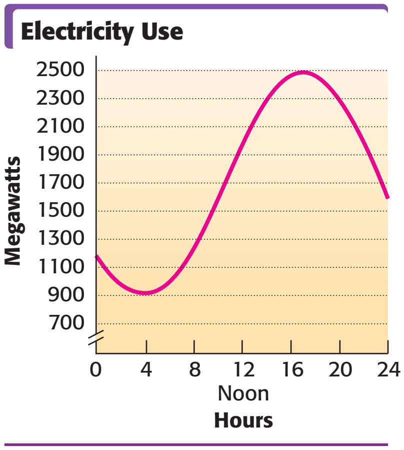
**(b)** For how long is the water level increasing? Decreasing?

**(c)** How many gallons of water are in the pool after 90 hr?

**(d)** Call this function $f$. What is $f(0)$? What does it mean?

**(e)** What is $f(25)$? What does it mean?

**66.** The graph shows megawatts of electricity used on a summer day.


**Electricity Use**

***Source:*** Sacramento Municipal Utility District.

**(a)** Why is this the graph of a function?

**(b)** What is the domain?

**(c)** Estimate the number of megawatts used at 8 A.M.

**(d)** At what time was the most electricity used? The least electricity?

**(e)** Call this function $f$. What is $f(12)$? What does it mean?

## PREVIEW EXERCISES

*Perform each indicated operation.* ***See Sections 4.4, 4.5, and 4.7.***

**67.** $(15x^2 - 2x) + (x - 4)$

**68.** $(3r + 8) - (2r - 5)$

**69.** $(4x - 5)(3x + 1)$

**70.** $(3x - 4)(2x^2 + x)$

**71.** $\dfrac{27x^3 - 18x^2}{9x}$

**72.** $\dfrac{q^2 + 2q - 35}{q - 5}$

# 7.5 Operations on Functions and Composition

**OBJECTIVES**

1. Recognize and evaluate polynomial functions.
2. Perform operations on polynomial functions.
3. Find the composition of functions.

**OBJECTIVE 1** **Recognize and evaluate polynomial functions.** In **Section 7.4,** we studied linear (first-degree polynomial) functions, defined as $f(x) = ax + b$. Now we consider more general polynomial functions.

### Polynomial Function

A **polynomial function of degree *n*** is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for real numbers $a_n, a_{n-1}, \ldots, a_1$, and $a_0$, where $a_n \neq 0$ and $n$ is a whole number.

Another way of describing a polynomial function is to say that it is a function defined by a polynomial in one variable, consisting of one or more terms. It is usually written in descending powers of the variable, and its degree is the degree of the polynomial that defines it.

We can evaluate a polynomial function $f(x)$ at different values of the variable $x$.

**NOW TRY EXERCISE 1**

Let $f(x) = x^3 - 2x^2 + 7$. Find $f(-3)$.

## EXAMPLE 1 Evaluating Polynomial Functions

Let $f(x) = 4x^3 - x^2 + 5$. Find each value.

**(a)** $f(3)$

Read this as "$f$ of 3," not "$f$ times 3."

$$\begin{aligned}
f(x) &= 4x^3 - x^2 + 5 && \text{Given function}\\
f(3) &= 4(3)^3 - 3^2 + 5 && \text{Substitute 3 for } x.\\
f(3) &= 4(27) - 9 + 5 && \text{Apply the exponents.}\\
f(3) &= 108 - 9 + 5 && \text{Multiply.}\\
f(3) &= 104 && \text{Subtract, and then add.}
\end{aligned}$$

Thus, $f(3) = 104$ and the ordered pair $(3, 104)$ belongs to $f$.

**(b)** $f(-4)$

$$\begin{aligned}
f(x) &= 4x^3 - x^2 + 5 && \text{Use parentheses.}\\
f(-4) &= 4 \cdot (-4)^3 - (-4)^2 + 5 && \text{Let } x = -4.\\
f(-4) &= 4 \cdot (-64) - 16 + 5 && \text{Be careful with signs.}\\
f(-4) &= -256 - 16 + 5 && \text{Multiply.}\\
f(-4) &= -267 && \text{Subtract, and then add.}
\end{aligned}$$

So, $f(-4) = -267$. The ordered pair $(-4, -267)$ belongs to $f$. NOW TRY

While $f$ is the most common letter used to represent functions, recall that other letters, such as $g$ and $h$, are also used. ***The capital letter P is often used for polynomial functions.*** The function defined as

$$P(x) = 4x^3 - x^2 + 5$$

yields the same ordered pairs as the function $f$ in **Example 1.**

**OBJECTIVE 2** **Perform operations on polynomial functions.** The operations of addition, subtraction, multiplication, and division are also defined for functions. For example, businesses use the equation "profit equals revenue minus cost," which can be written in function notation.

$$\underset{\substack{\uparrow\\ \text{Profit}\\ \text{function}}}{P(x)} = \underset{\substack{\uparrow\\ \text{Revenue}\\ \text{function}}}{R(x)} - \underset{\substack{\uparrow\\ \text{Cost}\\ \text{function}}}{C(x)}$$

$x$ is the number of items produced and sold.

The profit function is found by subtracting the cost function from the revenue function.

**NOW TRY ANSWER**

**1.** $-38$

We define the following **operations on functions.**

### Operations on Functions

If $f(x)$ and $g(x)$ define functions, then

$$(f+g)(x) = f(x) + g(x), \qquad \text{Sum function}$$

$$(f-g)(x) = f(x) - g(x), \qquad \text{Difference function}$$

$$(fg)(x) = f(x) \cdot g(x), \qquad \text{Product function}$$

and

$$\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \quad g(x) \neq 0. \qquad \text{Quotient function}$$

In each case, the domain of the new function is the intersection of the domains of $f(x)$ and $g(x)$. Additionally, the domain of the quotient function must exclude any values of $x$ for which $g(x) = 0$.

**NOW TRY EXERCISE 2**

For $f(x) = x^3 - 3x^2 + 4$ and $g(x) = -2x^3 + x^2 - 12$, find each of the following.

**(a)** $(f+g)(x)$

**(b)** $(f-g)(x)$

## EXAMPLE 2 Adding and Subtracting Polynomial Functions

Find each of the following for the polynomial functions defined by

$$f(x) = x^2 - 3x + 7 \quad \text{and} \quad g(x) = -3x^2 - 7x + 7.$$

**(a)** $(f+g)(x)$

This notation does *not* indicate the distributive property.

$$\begin{aligned} &= f(x) + g(x) && \text{Use the definition.}\\ &= (x^2 - 3x + 7) + (-3x^2 - 7x + 7) && \text{Substitute.}\\ &= -2x^2 - 10x + 14 && \text{Add the polynomials.} \end{aligned}$$

**(b)** $(f-g)(x)$

$$\begin{aligned} &= f(x) - g(x) && \text{Use the definition.}\\ &= (x^2 - 3x + 7) - (-3x^2 - 7x + 7) && \text{Substitute.}\\ &= (x^2 - 3x + 7) + (3x^2 + 7x - 7) && \text{Change subtraction to addition.}\\ &= 4x^2 + 4x && \text{Add.} \end{aligned}$$

NOW TRY

## EXAMPLE 3 Adding and Subtracting Polynomial Functions

Find each of the following for the functions defined by

$$f(x) = 10x^2 - 2x \quad \text{and} \quad g(x) = 2x.$$

**(a)** $(f+g)(2)$

$$\begin{aligned} &= f(2) + g(2) && \text{Use the definition.}\\ &= \overbrace{[10(2)^2 - 2(2)]}^{f(x) = 10x^2 - 2x} + \overbrace{2(2)}^{g(x) = 2x} && \text{Substitute.}\\ &= [40 - 4] + 4 && \text{Order of operations}\\ &= 40 && \text{Subtract, and then add.} \end{aligned}$$

This is a key step.

**NOW TRY ANSWERS**

**2. (a)** $-x^3 - 2x^2 - 8$

**(b)** $3x^3 - 4x^2 + 16$

NOW TRY EXERCISE 3

For $f(x) = x^2 - 4$
and $g(x) = -6x^2$,
find each of the following.

**(a)** $(f + g)(x)$

**(b)** $(f - g)(-4)$

Alternatively, we could first find $(f + g)(x)$.

$$\begin{aligned}
&(f + g)(x) \\
&= f(x) + g(x) && \text{Use the definition.} \\
&= (10x^2 - 2x) + 2x && \text{Substitute.} \\
&= 10x^2 && \text{Combine like terms.}
\end{aligned}$$

Then,

$$\begin{aligned}
&(f + g)(2) \\
&= 10(2)^2 && \text{Substitute.} \\
&= 40. && \text{The result is the same.}
\end{aligned}$$

**(b)** $(f - g)(x)$ and $(f - g)(1)$

$$\begin{aligned}
&(f - g)(x) \\
&= f(x) - g(x) && \text{Use the definition.} \\
&= (10x^2 - 2x) - 2x && \text{Substitute.} \\
&= 10x^2 - 4x && \text{Combine like terms.}
\end{aligned}$$

Then,

$$\begin{aligned}
&(f - g)(1) \\
&= 10(1)^2 - 4(1) && \text{Substitute.} \\
&= 6. && \text{Simplify.}
\end{aligned}$$

Confirm that $f(1) - g(1)$ gives the same result. NOW TRY

NOW TRY EXERCISE 4

For $f(x) = 3x^2 - 1$
and $g(x) = 8x + 7$,
find $(fg)(x)$ and $(fg)(-2)$.

## EXAMPLE 4 Multiplying Polynomial Functions

For $f(x) = 3x + 4$ and $g(x) = 2x^2 + x$, find $(fg)(x)$ and $(fg)(-1)$.

$$\begin{aligned}
&(fg)(x) \\
&= f(x) \cdot g(x) && \text{Use the definition.} \\
&= (3x + 4)(2x^2 + x) && \text{Substitute.} \\
&= 6x^3 + 3x^2 + 8x^2 + 4x && \text{FOIL} \\
&= 6x^3 + 11x^2 + 4x && \text{Combine like terms.}
\end{aligned}$$

$$\begin{aligned}
&(fg)(-1) \\
&= 6(-1)^3 + 11(-1)^2 + 4(-1) && \text{Let } x = -1 \text{ in } (fg)(x). \\
&= -6 + 11 - 4 \quad \longleftarrow \text{Be careful with signs.} \\
&= 1 && \text{Add and subtract.}
\end{aligned}$$

Confirm that $f(-1) \cdot g(-1)$ is equal to $(fg)(-1)$. NOW TRY

## EXAMPLE 5 Dividing Polynomial Functions

For $f(x) = 2x^2 + x - 10$ and $g(x) = x - 2$, find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(-3)$. What value of $x$ is not in the domain of the quotient function?

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2 + x - 10}{x - 2}$$

NOW TRY ANSWERS

3. **(a)** $-5x^2 - 4$ **(b)** 108

4. $24x^3 + 21x^2 - 8x - 7$; $-99$

**NOW TRY EXERCISE 5**

For $f(x) = 8x^2 + 2x - 3$
and $g(x) = 2x - 1$,
find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(8)$.

To find the quotient, divide as in **Section 4.7.**

$$\begin{array}{rl} 2x + \phantom{x}5 & \\ x-2\overline{)2x^2 + \phantom{4}x - 10} & \\ \underline{2x^2 - 4x\phantom{-10}} & 2x(x-2) \\ 5x - 10 & \text{Subtract.} \\ \underline{5x - 10} & 5(x-2) \\ 0 & \end{array}$$

To subtract, add the opposite.

The quotient here is $2x + 5$, so

$$\left(\frac{f}{g}\right)(x) = 2x + 5, \quad x \neq 2.$$

The number 2 is not in the domain because it causes the denominator $g(x) = x - 2$ to equal 0. Then

$$\left(\frac{f}{g}\right)(-3) = 2(-3) + 5 = -1. \quad \text{Let } x = -3.$$

Verify that the same value is found by evaluating $\frac{f(-3)}{g(-3)}$.

NOW TRY

**OBJECTIVE 3** **Find the composition of functions.** The diagram in **FIGURE 41** shows a function $f$ that assigns, to each element $x$ of set $X$, some element $y$ of set $Y$. Suppose that a function $g$ takes each element of set $Y$ and assigns a value $z$ of set $Z$. Then $f$ and $g$ together assign an element $x$ in $X$ to an element $z$ in $Z$. The result of this process is a new function $h$ that takes an element $x$ in $X$ and assigns it an element $z$ in $Z$.

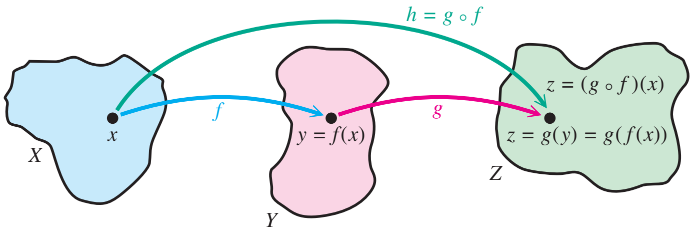


**FIGURE 41**

This function $h$ is called the *composition* of functions $g$ and $f$, written $\boldsymbol{g \circ f}$.

### Composition of Functions

If $f$ and $g$ are functions, then the **composite function,** or **composition,** of $g$ and $f$ is defined by

$$(\boldsymbol{g \circ f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$$

for all $x$ in the domain of $f$ such that $f(x)$ is in the domain of $g$.

**Read $g \circ f$ as "$g$ *of* $f$".**

As a real-life example of how composite functions occur, consider the following retail situation.

> *A* \$40 *pair of blue jeans is on sale for* 25% *off. If you purchase the jeans before noon, the retailer offers an additional* 10% *off. What is the final sale price of the blue jeans?*

**NOW TRY ANSWER**

**5.** $4x + 3, \quad x \neq \frac{1}{2}$; 35

You might be tempted to say that the blue jeans are $25\% + 10\% = 35\%$ off and calculate $\$40(0.35) = \$14$, giving a final sale price of

$\$40 - \$14 = \$26.$ This is not correct.

To find the correct final sale price, we must first find the price after taking 25% off, and then take an additional 10% off that price.

$\$40(0.25) = \$10$, giving a sale price of $\$40 - \$10 = \$30$. Take 25% off original price.

$\$30(0.10) = \$3$, giving a ***final sale price*** of $\$30 - \$3 = \$27$. Take additional 10% off.

This is the idea behind composition of functions.

NOW TRY EXERCISE 6

Let $f(x) = 3x + 7$

and $g(x) = x - 2$.

Find $(f \circ g)(7)$.

## EXAMPLE 6 Evaluating a Composite Function

Let $f(x) = x^2$ and $g(x) = x + 3$. Find $(f \circ g)(4)$.

$(f \circ g)(4)$ Evaluate the "inside" function value first.

$= f(g(4))$ Definition

$= f(4 + 3)$ Use the rule for $g(x)$; $g(4) = 4 + 3$.

$= f(7)$ Add.

Now evaluate the "outside" function.

$= 7^2$ Use the rule for $f(x)$; $f(7) = 7^2$.

$= 49$ Square 7.

NOW TRY

If we interchange the order of the functions in **Example 6,** the composition of $g$ and $f$ is defined by $g(f(x))$. To find $(g \circ f)(4)$, we let $x = 4$.

$(g \circ f)(4)$

$= g(f(4))$ Definition

$= g(4^2)$ Use the rule for $f(x)$; $f(4) = 4^2$.

$= g(16)$ Square 4.

$= 16 + 3$ Use the rule for $g(x)$; $g(16) = 16 + 3$.

$= 19$ Add.

Here we see that $(f \circ g)(4) \neq (g \circ f)(4)$ because $49 \neq 19$. In general,

$$\boldsymbol{(f \circ g)(x) \neq (g \circ f)(x).}$$

## EXAMPLE 7 Finding Composite Functions

Let $f(x) = 4x - 1$ and $g(x) = x^2 + 5$. Find the following.

**(a)** $(f \circ g)(2)$

$= f(g(2))$

$= f(2^2 + 5)$ $g(x) = x^2 + 5$

$= f(9)$ Work inside the parentheses.

$= 4(9) - 1$ $f(x) = 4x - 1$

$= 35$ Multiply, and then subtract.

NOW TRY ANSWER

**6.** 22

**(b)** $(f \circ g)(x)$

$= f(g(x))$ Use $g(x)$ as the input for the function $f$.

$= 4(g(x)) - 1$ Use the rule for $f(x)$; $f(x) = 4x - 1$.

$= 4(x^2 + 5) - 1$ $g(x) = x^2 + 5$

$= 4x^2 + 20 - 1$ Distributive property

$= 4x^2 + 19$ Combine like terms.

**(c)** Find $(f \circ g)(2)$ again, this time using the rule obtained in part (b).

$(f \circ g)(x) = 4x^2 + 19$ From part (b)

$(f \circ g)(2) = 4(2)^2 + 19$ Let $x = 2$.

$= 4(4) + 19$ Square 2.

$= 16 + 19$ Multiply.

Same result as in part (a) ⟶ $= 35$ Add.

NOW TRY

**NOW TRY EXERCISE 7**

Let $f(x) = x - 5$
and $g(x) = -x^2 + 2$.
Find the following.

**(a)** $(g \circ f)(-1)$

**(b)** $(f \circ g)(x)$

**NOW TRY ANSWERS**

**7.** **(a)** $-34$ **(b)** $-x^2 - 3$

## 7.5 EXERCISES

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*For each polynomial function, find* **(a)** $f(-1)$ *and* **(b)** $f(2)$. ***See Example 1.***

**1.** $f(x) = 6x - 4$
**2.** $f(x) = -2x + 5$
**3.** $f(x) = x^2 - 3x + 4$
**4.** $f(x) = 3x^2 + x - 5$
**5.** $f(x) = 5x^4 - 3x^2 + 6$
**6.** $f(x) = -4x^4 + 2x^2 - 1$
⊛ **7.** $f(x) = -x^2 + 2x^3 - 8$
**8.** $f(x) = -x^2 - x^3 + 11x$

*For each pair of functions, find* **(a)** $(f + g)(x)$ *and* **(b)** $(f - g)(x)$. ***See Example 2.***

**9.** $f(x) = 5x - 10,\quad g(x) = 3x + 7$

**10.** $f(x) = -4x + 1,\quad g(x) = 6x + 2$

⊛ **11.** $f(x) = 4x^2 + 8x - 3,\quad g(x) = -5x^2 + 4x - 9$

**12.** $f(x) = 3x^2 - 9x + 10,\quad g(x) = -4x^2 + 2x + 12$

*Let* $f(x) = x^2 - 9$, $g(x) = 2x$, *and* $h(x) = x - 3$. *Find each of the following.* ***See Example 3.***

⊛ **13.** $(f + g)(x)$
**14.** $(f - g)(x)$
**15.** $(f + g)(3)$
**16.** $(f - g)(-3)$
**17.** $(f - h)(x)$
**18.** $(f + h)(x)$
**19.** $(f - h)(-3)$
**20.** $(f + h)(-2)$
**21.** $(g + h)(-10)$
**22.** $(g - h)(10)$
**23.** $(g - h)(-3)$
**24.** $(g + h)(1)$
**25.** $(g + h)\left(\frac{1}{4}\right)$
**26.** $(g + h)\left(\frac{1}{3}\right)$
**27.** $(g + h)\left(-\frac{1}{2}\right)$
**28.** $(g + h)\left(-\frac{1}{4}\right)$

✎ **29.** Construct two functions defined by $f(x)$, a polynomial of degree 3, and $g(x)$, a polynomial of degree 4. Find $(f - g)(x)$ and $(g - f)(x)$. Use your answers to decide whether subtraction of functions is a commutative operation. Explain.

**30.** *Concept Check* Find two polynomial functions defined by $f(x)$ and $g(x)$ such that

$$(f + g)(x) = 3x^3 - x + 3.$$

*For each pair of functions, find the product $(fg)(x)$.* ***See Example 4.***

**31.** $f(x) = 2x, \quad g(x) = 5x - 1$

**32.** $f(x) = 3x, \quad g(x) = 6x - 8$

**33.** $f(x) = x + 1, \quad g(x) = 2x - 3$

**34.** $f(x) = x - 7, \quad g(x) = 4x + 5$

**35.** $f(x) = 2x - 3, \quad g(x) = 4x^2 + 6x + 9$

**36.** $f(x) = 3x + 4, \quad g(x) = 9x^2 - 12x + 16$

*Let $f(x) = x^2 - 9$, $g(x) = 2x$, and $h(x) = x - 3$. Find each of the following.* ***See Example 4.***

**37.** $(fg)(x)$

**38.** $(fh)(x)$

**39.** $(fg)(2)$

**40.** $(fh)(1)$

**41.** $(gh)(x)$

**42.** $(fh)(-1)$

**43.** $(gh)(-3)$

**44.** $(fg)(-2)$

**45.** $(fg)\left(-\frac{1}{2}\right)$

**46.** $(fg)\left(-\frac{1}{3}\right)$

**47.** $(fh)\left(-\frac{1}{4}\right)$

**48.** $(fh)\left(-\frac{1}{5}\right)$

*For each pair of functions, find the quotient $\left(\frac{f}{g}\right)(x)$ and give any x-values that are not in the domain of the quotient function.* ***See Example 5.***

**49.** $f(x) = 10x^2 - 2x, \quad g(x) = 2x$

**50.** $f(x) = 18x^2 - 24x, \quad g(x) = 3x$

**51.** $f(x) = 2x^2 - x - 3, \quad g(x) = x + 1$

**52.** $f(x) = 4x^2 - 23x - 35, \quad g(x) = x - 7$

**53.** $f(x) = 8x^3 - 27, \quad g(x) = 2x - 3$

**54.** $f(x) = 27x^3 + 64, \quad g(x) = 3x + 4$

*Let $f(x) = x^2 - 9$, $g(x) = 2x$, and $h(x) = x - 3$. Find each of the following.* ***See Example 5.***

**55.** $\left(\frac{f}{g}\right)(x)$

**56.** $\left(\frac{f}{h}\right)(x)$

**57.** $\left(\frac{f}{g}\right)(2)$

**58.** $\left(\frac{f}{h}\right)(1)$

**59.** $\left(\frac{h}{g}\right)(x)$

**60.** $\left(\frac{g}{h}\right)(x)$

**61.** $\left(\frac{h}{g}\right)(3)$

**62.** $\left(\frac{g}{h}\right)(-1)$

**63.** $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$

**64.** $\left(\frac{f}{g}\right)\left(\frac{3}{2}\right)$

**65.** $\left(\frac{h}{g}\right)\left(-\frac{1}{2}\right)$

**66.** $\left(\frac{h}{g}\right)\left(-\frac{3}{2}\right)$

*Let $f(x) = x^2 + 4$, $g(x) = 2x + 3$, and $h(x) = x - 5$. Find each value or expression.* ***See Examples 6 and 7.***

**67.** $(h \circ g)(4)$

**68.** $(f \circ g)(4)$

**69.** $(g \circ f)(6)$

**70.** $(h \circ f)(6)$

**71.** $(f \circ h)(-2)$

**72.** $(h \circ g)(-2)$

**73.** $(f \circ g)(0)$

**74.** $(f \circ h)(0)$

**75.** $(g \circ f)(x)$

**76.** $(g \circ h)(x)$

**77.** $(h \circ g)(x)$

**78.** $(h \circ f)(x)$

**79.** $(f \circ h)\left(\frac{1}{2}\right)$

**80.** $(h \circ f)\left(\frac{1}{2}\right)$

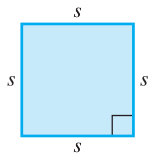
**81.** $(f \circ g)\left(-\frac{1}{2}\right)$

**82.** $(g \circ f)\left(-\frac{1}{2}\right)$

*Solve each problem.*

**83.** The function defined by $f(x) = 12x$ computes the number of inches in $x$ feet, and the function defined by $g(x) = 5280x$ computes the number of feet in $x$ miles. What is $(f \circ g)(x)$ and what does it compute?

**84.** The perimeter $x$ of a square with sides of length $s$ is given by the formula $x = 4s$.

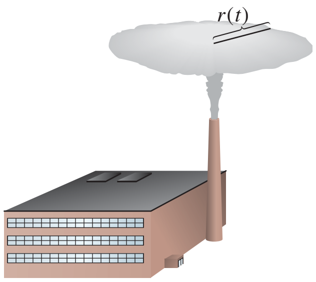


**(a)** Solve for $s$ in terms of $x$.

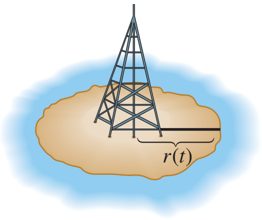
**(b)** If $y$ represents the area of this square, write $y$ as a function of the perimeter $x$.

**(c)** Use the composite function of part (b) to find the area of a square with perimeter 6.

**85.** When a thermal inversion layer is over a city (as happens often in Los Angeles), pollutants cannot rise vertically, but are trapped below the layer and must disperse horizontally. Assume that a factory smokestack begins emitting a pollutant at 8 A.M. Assume that the pollutant disperses horizontally over a circular area. Suppose that $t$ represents the time, in hours, since the factory began emitting pollutants ($t = 0$ represents 8 A.M.), and assume that the radius of the circle of pollution is $r(t) = 2t$ miles. Let $\mathcal{A}(r) = \pi r^2$ represent the area of a circle of radius $r$. Find and interpret $(\mathcal{A} \circ r)(t)$.



**86.** An oil well is leaking, with the leak spreading oil over the surface as a circle. At any time $t$, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ feet. Let $\mathcal{A}(r) = \pi r^2$ represent the area of a circle of radius $r$. Find and interpret $(\mathcal{A} \circ r)(t)$.



## PREVIEW EXERCISES

*Find k, given that $y = 1$ and $x = 3$.* ***See Sections 2.3 and 2.5.***

**87.** $y = kx$

**88.** $y = kx^2$

**89.** $y = \frac{k}{x}$

**90.** $y = \frac{k}{x^2}$

# 7.6 Variation

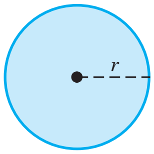
**OBJECTIVES**

1. Write an equation expressing direct variation.
2. Find the constant of variation, and solve direct variation problems.
3. Solve inverse variation problems.
4. Solve joint variation problems.
5. Solve combined variation problems.

Functions in which *y depends on a multiple of x* or *y depends on a number divided by x* are common in business and the physical sciences.

## OBJECTIVE 1 Write an equation expressing direct variation.

The circumference of a circle is given by the formula $C = 2\pi r$, where $r$ is the radius of the circle. See **FIGURE 42**. The circumference is always a constant multiple of the radius. ($C$ is always found by multiplying $r$ by the constant $2\pi$.)



$C = 2\pi r$

**FIGURE 42**

As the ***radius increases,*** the ***circumference increases.***

As the ***radius decreases,*** the ***circumference decreases.***

Because of these relationships, the circumference is said to *vary directly* as the radius.

**Direct Variation**

***y* varies directly as *x*** if there exists a real number $k$ such that

$$y = kx.$$

$y$ is said to be **proportional to** $x$. The number $k$ is called the **constant of variation.**

***In direct variation, for $k > 0$, as the value of x increases, the value of y increases. Similarly, as x decreases, y decreases.***

**OBJECTIVE 2** **Find the constant of variation, and solve direct variation problems.** ***The direct variation equation $y = kx$ defines a linear function, where the constant of variation $k$ is the slope of the line.*** For example, we wrote the following equation to describe the cost $y$ to buy $x$ gallons of gasoline.

$$y = 3.20x \qquad \text{See Section 7.2, Example 7.}$$

The cost varies directly as, or is proportional to, the number of gallons of gasoline purchased.

As the *number of gallons* of gasoline *increases,* the *cost increases*.

As the *number of gallons* of gasoline *decreases,* the *cost decreases*.

The constant of variation $k$ is 3.20, the cost of 1 gal of gasoline.

*NOW TRY EXERCISE 1*

One week Morgan sold 8 dozen eggs for $20. How much does she charge for one dozen eggs?

### EXAMPLE 1 Finding the Constant of Variation and the Variation Equation

Eva Lutchman is paid an hourly wage. One week she worked 43 hr and was paid $795.50. How much does she earn per hour?

Let $h$ represent the number of hours she works and $P$ represent her corresponding pay. Write the variation equation.

$$P = kh \qquad P \text{ varies directly as } h.$$

Here, $k$ represents Eva's hourly wage.

$$\begin{aligned} P &= kh && P \text{ varies directly as } h. \\ 795.50 &= 43k && \text{Substitute 795.50 for } P \text{ and 43 for } h. \\ k &= 18.50 && \text{Use a calculator.} \end{aligned}$$

This is the constant of variation.

Her hourly wage is $18.50, and $P$ and $h$ are related by

$$P = 18.50h.$$

We can use this equation to find her pay for any number of hours worked.

*NOW TRY*

### EXAMPLE 2 Solving a Direct Variation Problem

Hooke's law for an elastic spring states that the distance a spring stretches is directly proportional to the force applied. If a force of 150 newtons* stretches a certain spring 8 cm, how much will a force of 400 newtons stretch the spring? See **FIGURE 43**.

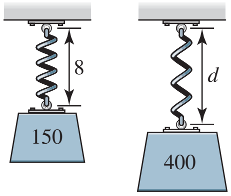


**FIGURE 43**

If $d$ is the distance the spring stretches and $f$ is the force applied, then $d = kf$ for some constant $k$. Since a force of 150 newtons stretches the spring 8 cm, we use these values to find $k$.

$$\begin{aligned} d &= kf && \text{Variation equation} \\ 8 &= k \cdot 150 && \text{Let } d = 8 \text{ and } f = 150. \\ k &= \frac{8}{150} && \text{Solve for } k. \\ k &= \frac{4}{75} && \text{Lowest terms} \end{aligned}$$

*NOW TRY ANSWER*

1. $2.50

---

*A newton is a unit of measure of force used in physics.

**NOW TRY EXERCISE 2**

For a constant height, the area of a parallelogram is directly proportional to its base. If the area is 20 cm$^2$ when the base is 4 cm, find the area when the base is 7 cm.

Substitute $\frac{4}{75}$ for $k$ in the variation equation $d = kf$.

$$d = \frac{4}{75}f \qquad \text{Here, } k = \tfrac{4}{75}.$$

For a force of 400 newtons, substitute 400 for $f$.

$$d = \frac{4}{75}(400) = \frac{64}{3} \qquad \text{Let } f = 400.$$

The spring will stretch $\frac{64}{3}$ cm, or $21\frac{1}{3}$ cm, if a force of 400 newtons is applied. NOW TRY

### Solving a Variation Problem

***Step 1*** Write the variation equation.

***Step 2*** Substitute the initial values and solve for $k$.

***Step 3*** Rewrite the variation equation with the value of $k$ from Step 2.

***Step 4*** Substitute the remaining values, solve for the unknown, and find the required answer.

One variable can be proportional to a power of another variable.

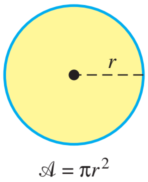


FIGURE 44

### Direct Variation as a Power

***y* varies directly as the *n*th power of *x*** if there exists a real number $k$ such that

$$y = kx^n.$$

The formula for the area of a circle, $\mathcal{A} = \pi r^2$, is an example. See FIGURE 44. Here, $\pi$ is the constant of variation, and the area varies directly as the *square* of the radius.

**NOW TRY EXERCISE 3**

Suppose $y$ varies directly as the square of $x$, and $y = 200$ when $x = 5$. Find $y$ when $x = 7$.

### EXAMPLE 3 Solving a Direct Variation Problem

The distance a body falls from rest varies directly as the square of the time it falls (disregarding air resistance). If a skydiver falls 64 ft in 2 sec, how far will she fall in 8 sec?

***Step 1*** If $d$ represents the distance the skydiver falls and $t$ the time it takes to fall, then $d$ is a function of $t$ for some constant $k$.

$$d = kt^2 \qquad d \text{ varies directly as the square of } t.$$

***Step 2*** To find the value of $k$, use the fact that the skydiver falls 64 ft in 2 sec.

$$d = kt^2 \qquad \text{Variation equation}$$
$$64 = k(2)^2 \qquad \text{Let } d = 64 \text{ and } t = 2.$$
$$k = 16 \qquad \text{Find } k.$$

***Step 3*** Now we rewrite the variation equation $d = kt^2$ using 16 for $k$.

$$d = 16t^2 \qquad \text{Here, } k = 16.$$

***Step 4*** Let $t = 8$ to find the number of feet the skydiver will fall in 8 sec.

$$d = 16(8)^2 = 1024 \qquad \text{Let } t = 8.$$

The skydiver will fall 1024 ft in 8 sec. NOW TRY

**NOW TRY ANSWERS**

**2.** 35 cm$^2$ **3.** 392

**OBJECTIVE 3** **Solve inverse variation problems.** Another type of variation is *inverse variation.* ***With inverse variation, where $k > 0$, as one variable increases, the other variable decreases.***

For example, in a closed space, volume decreases as pressure increases, which can be illustrated by a trash compactor. See **FIGURE 45**. As the compactor presses down, the pressure on the trash increases, and in turn, the trash occupies a smaller space.

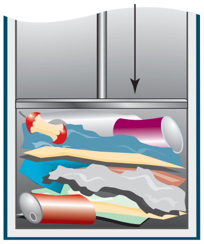


FIGURE 45

### Inverse Variation

***y* varies inversely as *x*** if there exists a real number $k$ such that

$$y = \frac{k}{x}.$$

Also, ***y* varies inversely as the *n*th power of *x*** if there exists a real number $k$ such that

$$y = \frac{k}{x^n}.$$

The inverse variation equation defines a rational function. Another example of inverse variation comes from the distance formula.

$$d = rt \qquad \text{Distance formula}$$

$$t = \frac{d}{r} \qquad \text{Divide each side by } r.$$

Here, $t$ (time) varies inversely as $r$ (rate or speed), with $d$ (distance) serving as the constant of variation. For example, if the distance between Chicago and Des Moines is 300 mi, then

$$t = \frac{300}{r},$$

and the values of $r$ and $t$ might be any of the following.

$$\left.\begin{aligned} r &= 50, t = 6 \\ r &= 60, t = 5 \\ r &= 75, t = 4 \end{aligned}\right\} \text{As } r \text{ increases, } t \text{ decreases.} \qquad \left.\begin{aligned} r &= 30, t = 10 \\ r &= 25, t = 12 \\ r &= 20, t = 15 \end{aligned}\right\} \text{As } r \text{ decreases, } t \text{ increases.}$$

If we *increase* the rate (speed) at which we drive, time *decreases.* If we *decrease* the rate (speed) at which we drive, time *increases*.

### EXAMPLE 4 Solving an Inverse Variation Problem

In the manufacture of a certain medical syringe, the cost of producing the syringe varies inversely as the number produced. If 10,000 syringes are produced, the cost is \$2 per syringe. Find the cost per syringe of producing 25,000 syringes.

Let $x$ = the number of syringes produced,

and $c$ = the cost per syringe.

Here, as production increases, cost decreases, and as production decreases, cost increases. We write a variation equation using the variables $c$ and $x$ and the constant $k$.

$$c = \frac{k}{x} \qquad c \text{ varies inversely as } x.$$

**NOW TRY EXERCISE 4**

For a constant area, the height of a triangle varies inversely as the base. If the height is 7 cm when the base is 8 cm, find the height when the base is 14 cm.

To find $k$, we replace $c$ with 2 and $x$ with 10,000 in the variation equation $c = \frac{k}{x}$.

$$2 = \frac{k}{10{,}000} \qquad \text{Substitute in the variation equation.}$$

$$20{,}000 = k \qquad \text{Multiply by 10,000.}$$

Since $c = \frac{k}{x}$,

$$c = \frac{20{,}000}{25{,}000} = 0.80. \qquad \text{Here, } k = 20{,}000. \text{ Let } x = 25{,}000.$$

The cost per syringe to make 25,000 syringes is \$0.80. NOW TRY

**NOW TRY EXERCISE 5**

The weight of an object above Earth varies inversely as the square of its distance from the center of Earth. If an object weighs 150 lb on the surface of Earth, and the radius of Earth is about 3960 mi, how much does it weigh when it is 1000 mi above Earth's surface?

### EXAMPLE 5 Solving an Inverse Variation Problem

The weight of an object above Earth varies inversely as the square of its distance from the center of Earth. A space shuttle in an elliptical orbit has a maximum distance from the center of Earth (***apogee***) of 6700 mi. Its minimum distance from the center of Earth (***perigee***) is 4090 mi. See **FIGURE 46**. If an astronaut in the shuttle weighs 57 lb at its apogee, what does the astronaut weigh at its perigee?

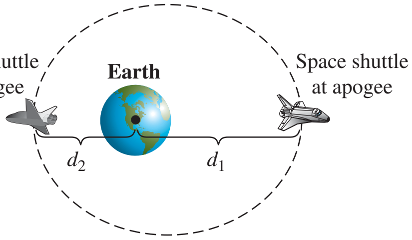


**FIGURE 46**

Let $w$ = the weight and $d$ = the distance from the center of Earth, for some constant $k$.

$$w = \frac{k}{d^2} \qquad w \text{ varies inversely as the square of } d.$$

At the apogee, the astronaut weighs 57 lb, and the distance from the center of Earth is 6700 mi. Use these values to find $k$.

$$57 = \frac{k}{(6700)^2} \qquad \text{Let } w = 57 \text{ and } d = 6700.$$

$$k = 57(6700)^2 \qquad \text{Solve for } k.$$

Substitute $k = 57(6700)^2$ and $d = 4090$ to find the weight at the perigee.

$$w = \frac{57(6700)^2}{(4090)^2} \approx 153 \text{ lb} \qquad \text{Use a calculator.}$$

NOW TRY

**OBJECTIVE 4** **Solve joint variation problems.** If one variable varies directly as the *product* of several other variables (perhaps raised to powers), the first variable is said to *vary jointly* as the others.

**Joint Variation**

***y* varies jointly as *x* and *z*** if there exists a real number $k$ such that

$$y = kxz.$$

**NOW TRY ANSWERS**

**4.** 4 cm **5.** about 96 lb

**NOW TRY EXERCISE 6**

The volume of a right pyramid varies jointly as the height and the area of the base. If the volume is 100 $\text{ft}^3$ when the area of the base is 30 $\text{ft}^2$ and the height is 10 ft, find the volume when the area of the base is 90 $\text{ft}^2$ and the height is 20 ft.

### EXAMPLE 6 Solving a Joint Variation Problem

The interest on a loan or an investment is given by the formula $I = prt$. Here, for a given principal $p$, the interest earned, $I$, varies jointly as the interest rate $r$ and the time $t$ the principal is left earning interest. If an investment earns $100 interest at 5% for 2 yr, how much interest will the same principal earn at 4.5% for 3 yr?

We use the formula $I = prt$, where $p$ is the constant of variation because it is the same for both investments.

$$
\begin{aligned}
I &= prt \\
100 &= p(0.05)(2) && \text{Let } I = 100,\ r = 0.05, \text{ and } t = 2. \\
100 &= 0.1p \\
p &= 1000 && \text{Divide by 0.1. Rewrite.}
\end{aligned}
$$

Now we find $I$ when $p = 1000$, $r = 0.045$, and $t = 3$.

$$I = 1000(0.045)(3) = 135 \qquad \text{Here, } p = 1000. \text{ Let } r = 0.045 \text{ and } t = 3.$$

The interest will be $135.

NOW TRY

**CAUTION** Note that *and* in the expression "$y$ varies directly as $x$ *and* $z$" translates as a product in $y = kxz$. The word *and* does not indicate addition here.

**OBJECTIVE 5** **Solve combined variation problems.** There are many combinations of direct and inverse variation, typically called **combined variation.**

**NOW TRY EXERCISE 7**

In statistics, the sample size used to estimate a population mean varies directly as the variance and inversely as the square of the maximum error of the estimate. If the sample size is 200 when the variance is 25 $\text{m}^2$ and the maximum error of the estimate is 0.5 m, find the sample size when the variance is 25 $\text{m}^2$ and the maximum error of the estimate is 0.1 m.

### EXAMPLE 7 Solving a Combined Variation Problem

Body mass index, or BMI, is used to assess a person's level of fatness. A BMI from 19 through 25 is considered desirable. BMI varies directly as an individual's weight in pounds and inversely as the square of the individual's height in inches.

A person who weighs 116.5 lb and is 64 in. tall has a BMI of 20. (The BMI is rounded to the nearest whole number.) Find the BMI of a man who weighs 165 lb and is 70 in. tall. (*Source: Washington Post.*)

Let $B$ represent the BMI, $w$ the weight, and $h$ the height.

$$B = \frac{kw}{h^2} \qquad \begin{aligned} &\longleftarrow \text{BMI varies directly as the weight.} \\ &\longleftarrow \text{BMI varies inversely as the square of the height.} \end{aligned}$$

To find $k$, let $B = 20$, $w = 116.5$, and $h = 64$.

$$
\begin{aligned}
20 &= \frac{k(116.5)}{64^2} && B = \frac{kw}{h^2} \\
k &= \frac{20(64^2)}{116.5} && \text{Multiply by } 64^2. \text{ Divide by 116.5.} \\
k &\approx 703 && \text{Use a calculator.}
\end{aligned}
$$

Now find $B$ when $k = 703$, $w = 165$, and $h = 70$.

$$B = \frac{703(165)}{70^2} \approx 24 \qquad \text{Nearest whole number}$$

The man's BMI is 24.

NOW TRY

**NOW TRY ANSWERS**

**6.** 600 $\text{ft}^3$ **7.** 5000

## 7.6 EXERCISES

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***Concept Check*** *Use personal experience or intuition to determine whether the situation suggests* direct *or* inverse *variation.*

**1.** The number of lottery tickets you buy and your probability of winning that lottery

**2.** The rate and the distance traveled by a pickup truck in 3 hr

**3.** The amount of pressure put on the accelerator of a car and the speed of the car

**4.** The number of days from now until December 25 and the magnitude of the frenzy of Christmas shopping

**5.** Your age and the probability that you believe in Santa Claus

**6.** The surface area of a balloon and its diameter

**7.** The number of days until the end of the baseball season and the number of home runs that Albert Pujols has

**8.** The amount of gasoline you pump and the amount you pay

***Concept Check*** *Determine whether each equation represents* direct, inverse, joint, *or* combined *variation.*

**9.** $y = \frac{3}{x}$

**10.** $y = \frac{8}{x}$

**11.** $y = 10x^2$

**12.** $y = 2x^3$

**13.** $y = 3xz^4$

**14.** $y = 6x^3z^2$

**15.** $y = \frac{4x}{wz}$

**16.** $y = \frac{6x}{st}$

**17.** ***Concept Check*** For $k > 0$, if $y$ varies directly as $x$, then when $x$ increases, $y$ ________, and when $x$ decreases, $y$ ________.

**18.** ***Concept Check*** For $k > 0$, if $y$ varies inversely as $x$, then when $x$ increases, $y$ ________, and when $x$ decreases, $y$ ________.

***Concept Check*** *Write each formula using the "language" of variation. For example, the formula for the circumference of a circle, $C = 2\pi r$, can be written as*

*"The circumference of a circle varies directly as the length of its radius."*

**19.** $P = 4s$, where $P$ is the perimeter of a square with side of length $s$

**20.** $d = 2r$, where $d$ is the diameter of a circle with radius $r$

**21.** $S = 4\pi r^2$, where $S$ is the surface area of a sphere with radius $r$

**22.** $V = \frac{4}{3}\pi r^3$, where $V$ is the volume of a sphere with radius $r$

**23.** $\mathcal{A} = \frac{1}{2}bh$, where $\mathcal{A}$ is the area of a triangle with base $b$ and height $h$

**24.** $V = \frac{1}{3}\pi r^2 h$, where $V$ is the volume of a cone with radius $r$ and height $h$

**25.** ***Concept Check*** What is the constant of variation in each of the variation equations in **Exercises 19–24?**

**26.** What is meant by the constant of variation in a direct variation problem? If you were to graph the linear equation $y = kx$ for some nonnegative constant $k$, what role would $k$ play in the graph?

*Solve each problem.* ***See Examples 1–7.***

**27.** If $x$ varies directly as $y$, and $x = 9$ when $y = 3$, find $x$ when $y = 12$.

**28.** If $x$ varies directly as $y$, and $x = 10$ when $y = 7$, find $y$ when $x = 50$.

**29.** If $a$ varies directly as the square of $b$, and $a = 4$ when $b = 3$, find $a$ when $b = 2$.

**30.** If $h$ varies directly as the square of $m$, and $h = 15$ when $m = 5$, find $h$ when $m = 7$.

**31.** If $z$ varies inversely as $w$, and $z = 10$ when $w = 0.5$, find $z$ when $w = 8$.

**32.** If $t$ varies inversely as $s$, and $t = 3$ when $s = 5$, find $s$ when $t = 5$.

**33.** If $m$ varies inversely as $p^2$, and $m = 20$ when $p = 2$, find $m$ when $p = 5$.

**34.** If $a$ varies inversely as $b^2$, and $a = 48$ when $b = 4$, find $a$ when $b = 7$.

**35.** $p$ varies jointly as $q$ and $r^2$, and $p = 200$ when $q = 2$ and $r = 3$. Find $p$ when $q = 5$ and $r = 2$.

**36.** $f$ varies jointly as $g^2$ and $h$, and $f = 50$ when $g = 4$ and $h = 2$. Find $f$ when $g = 3$ and $h = 6$.

*Solve each problem.* ***See Examples 1–7.***

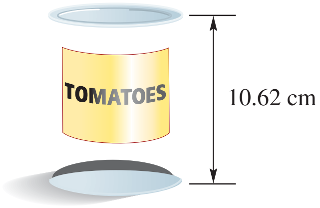
**37.** Ben bought 15 gal of gasoline and paid \$43.79. To the nearest tenth of a cent, what is the price of gasoline per gallon?

**38.** Sara gives horseback rides at Shadow Mountain Ranch. A 2.5-hr ride costs \$50.00. What is the price per hour?

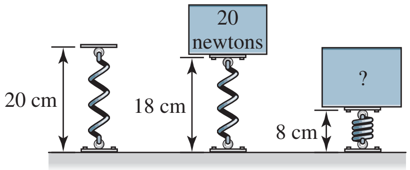
**39.** The weight of an object on Earth is directly proportional to the weight of that same object on the moon. A 200-lb astronaut would weigh 32 lb on the moon. How much would a 50-lb dog weigh on the moon?

**40.** The pressure exerted by a certain liquid at a given point is directly proportional to the depth of the point beneath the surface of the liquid. The pressure at 30 m is 80 newtons. What pressure is exerted at 50 m?

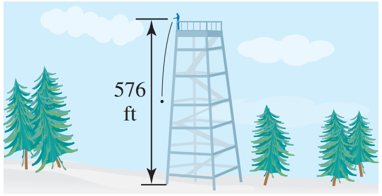
**41.** The volume of a can of tomatoes is directly proportional to the height of the can. If the volume of the can is 300 $\text{cm}^3$ when its height is 10.62 cm, find the volume of a can with height 15.92 cm.



**42.** The force required to compress a spring is directly proportional to the change in length of the spring. If a force of 20 newtons is required to compress a certain spring 2 cm, how much force is required to compress the spring from 20 cm to 8 cm?



**43.** For a body falling freely from rest (disregarding air resistance), the distance the body falls varies directly as the square of the time. If an object is dropped from the top of a tower 576 ft high and hits the ground in 6 sec, how far did it fall in the first 4 sec?



**44.** The amount of water emptied by a pipe varies directly as the square of the diameter of the pipe. For a certain constant water flow, a pipe emptying into a canal will allow 200 gal of water to escape in an hour. The diameter of the pipe is 6 in. How much water would a 12-in. pipe empty into the canal in an hour, assuming the same water flow?

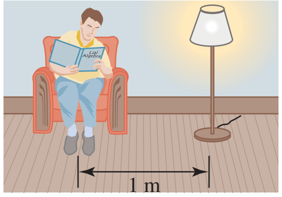
**45.** Over a specified distance, rate varies inversely with time. If a Dodge Viper on a test track goes a certain distance in one-half minute at 160 mph, what rate is needed to go the same distance in three-fourths minute?

**46.** For a constant area, the length of a rectangle varies inversely as the width. The length of a rectangle is 27 ft when the width is 10 ft. Find the width of a rectangle with the same area if the length is 18 ft.

**47.** The frequency of a vibrating string varies inversely as its length. That is, a longer string vibrates fewer times in a second than a shorter string. Suppose a piano string 2 ft long vibrates 250 cycles per sec. What frequency would a string 5 ft long have?

**48.** The current in a simple electrical circuit varies inversely as the resistance. If the current is 20 amps when the resistance is 5 ohms, find the current when the resistance is 7.5 ohms.

**49.** The amount of light (measured in foot-candles) produced by a light source varies inversely as the square of the distance from the source. If the illumination produced 1 m from a light source is 768 foot-candles, find the illumination produced 6 m from the same source.



**50.** The force with which Earth attracts an object above Earth's surface varies inversely as the square of the distance of the object from the center of Earth. If an object 4000 mi from the center of Earth is attracted with a force of 160 lb, find the force of attraction if the object were 6000 mi from the center of Earth.

**51.** For a given interest rate, simple interest varies jointly as principal and time. If $2000 left in an account for 4 yr earned interest of $280, how much interest would be earned in 6 yr?

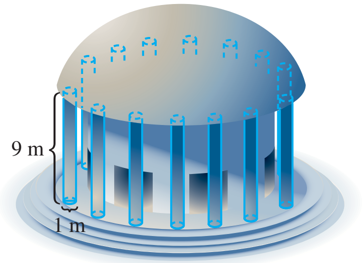
**52.** The collision impact of an automobile varies jointly as its mass and the square of its speed. Suppose a 2000-lb car traveling at 55 mph has a collision impact of 6.1. What is the collision impact of the same car at 65 mph?

**53.** The weight of a bass varies jointly as its girth and the square of its length. (**Girth** is the distance around the body of the fish.) A prize-winning bass weighed in at 22.7 lb and measured 36 in. long with a 21-in. girth. How much would a bass 28 in. long with an 18-in. girth weigh?

**54.** The weight of a trout varies jointly as its length and the square of its girth. One angler caught a trout that weighed 10.5 lb and measured 26 in. long with an 18-in. girth. Find the weight of a trout that is 22 in. long with a 15-in. girth.

**55.** The force needed to keep a car from skidding on a curve varies inversely as the radius of the curve and jointly as the weight of the car and the square of the speed. If 242 lb of force keeps a 2000-lb car from skidding on a curve of radius 500 ft at 30 mph, what force would keep the same car from skidding on a curve of radius 750 ft at 50 mph?

**56.** The maximum load that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter of the cross section and inversely as the square of the height. A 9-m column 1 m in diameter will support 8 metric tons. How many metric tons can be supported by a column 12 m high and $\frac{2}{3}$ m in diameter?



Load = 8 metric tons

**57.** The number of long-distance phone calls between two cities during a certain period varies jointly as the populations of the cities, $p_1$ and $p_2$, and inversely as the distance between them. If 80,000 calls are made between two cities 400 mi apart, with populations of 70,000 and 100,000, how many calls are made between cities with populations of 50,000 and 75,000 that are 250 mi apart?

**58.** In 2007, 51.2% of the homes in the United States used natural gas as the primary heating fuel. (*Source:* U.S. Census Bureau.) The volume of gas varies inversely as the pressure and directly as the temperature. (Temperature must be measured in *Kelvin* (K), a unit of measurement used in physics.) If a certain gas occupies a volume of 1.3 L at 300 K and a pressure of 18 newtons, find the volume at 340 K and a pressure of 24 newtons.

**59.** A body mass index from 27 through 29 carries a slight risk of weight-related health problems, while one of 30 or more indicates a great increase in risk. Use your own height and weight and the information in **Example 7** to determine your BMI and whether you are at risk.

**60.** The maximum load of a horizontal beam that is supported at both ends varies directly as the width and the square of the height and inversely as the length between the supports. A beam 6 m long, 0.1 m wide, and 0.06 m high supports a load of 360 kg. What is the maximum load supported by a beam 16 m long, 0.2 m wide, and 0.08 m high?

## PREVIEW EXERCISES

*Graph each pair of equations on the same coordinate axes.* ***See Sections 3.2 and 7.1.***

**61.** $2x + 3y = 12$
$4x - 2y = 8$

**62.** $x + y = 4$
$2x = 8 - 2y$

**63.** $-5x + 2y = 10$
$2y = -4 + 5x$

**64.** $2y = -3x$
$-2x + 3y = 0$

# CHAPTER 7
# SUMMARY

## KEY TERMS

### 7.1

ordered pair
components
origin
$x$-axis
$y$-axis
rectangular (Cartesian) coordinate system
plot
coordinate
quadrant
graph of an equation
first-degree equation
linear equation in two variables
standard form
$x$-intercept
$y$-intercept
rise
run
slope

### 7.2

slope-intercept form
point-slope form

### 7.3

dependent variable
independent variable
relation
function
domain
range

### 7.4

function notation
linear function
constant function

### 7.5

polynomial function
composition of functions

### 7.6

vary directly
proportional
constant of variation
vary inversely
vary jointly
combined variation

## NEW SYMBOLS

| | |
|---|---|
| $(a, b)$ | ordered pair |
| $x_1$ | a specific value of $x$ (read "$x$-sub-one") |
| $\Delta$ | Greek letter delta |
| $m$ | slope |
| $f(x)$ | function of $x$ (read "$f$ of $x$") |
| $(f \circ g)(x) = f(g(x))$ | composite function |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **linear equation in two variables** is an equation that can be written in the form
**A.** $Ax + By < C$
**B.** $ax = b$
**C.** $y = x^2$
**D.** $Ax + By = C$.

**2.** The **slope** of a line is
**A.** the measure of the run over the rise of the line
**B.** the distance between two points on the line
**C.** the ratio of the change in $y$ to the change in $x$ along the line
**D.** the horizontal change compared with the vertical change between two points on the line.

**3.** A **relation** is
**A.** a set of ordered pairs
**B.** the ratio of the change in $y$ to the change in $x$ along a line
**C.** the set of all possible values of the independent variable
**D.** all the second components of a set of ordered pairs.

**4.** A **function** is
**A.** the pair of numbers in an ordered pair
**B.** a set of ordered pairs in which each $x$-value corresponds to exactly one $y$-value
**C.** a pair of numbers written between parentheses
**D.** the set of all ordered pairs that satisfy an equation.

**5.** The **domain** of a function is
**A.** the set of all possible values of the dependent variable $y$
**B.** a set of ordered pairs
**C.** the difference between the $x$-values
**D.** the set of all possible values of the independent variable $x$.

**6.** The **range** of a function is
**A.** the set of all possible values of the dependent variable $y$
**B.** a set of ordered pairs
**C.** the difference between the $y$-values
**D.** the set of all possible values of the independent variable $x$.

### ANSWERS

**1.** D; *Examples:* $3x + 2y = 6$, $x = y - 7$ **2.** C; *Example:* The line through $(3, 6)$ and $(5, 4)$ has slope $\frac{4-6}{5-3} = \frac{-2}{2} = -1$. **3.** A; *Example:* The set $\{(2, 0), (4, 3), (6, 6)\}$ defines a relation. **4.** B; The relation given in Answer 3 is a function. **5.** D; *Example:* In the function in Answer 3, the domain is the set of $x$-values, $\{2, 4, 6\}$. **6.** A; *Example:* In the function in Answer 3, the range is the set of $y$-values, $\{0, 3, 6\}$.

## QUICK REVIEW

| CONCEPTS | EXAMPLES |
|---|---|

### 7.1 Review of Graphs and Slopes of Lines

**Finding Intercepts**

To find the $x$-intercept, let $y = 0$ and solve for $x$.

To find the $y$-intercept, let $x = 0$ and solve for $y$.

Find the intercepts of the graph of $2x + 3y = 12$.

$$2x + 3(0) = 12 \qquad 2(0) + 3y = 12$$
$$2x = 12 \qquad 3y = 12$$
$$x = 6 \qquad y = 4$$

The $x$-intercept is $(6, 0)$. The $y$-intercept is $(0, 4)$.

**Midpoint Formula**

If the endpoints of a line segment $PQ$ are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then its midpoint $M$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Find the midpoint of the segment with endpoints $(4, -7)$ and $(-10, -13)$.

$$\left(\frac{4 + (-10)}{2}, \frac{-7 + (-13)}{2}\right) = (-3, -10)$$

**Slope Formula**

If $x_2 \neq x_1$, then

$$\text{slope } m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Find the slope of the graph of $2x + 3y = 12$.

Use the intercepts $(6, 0)$ and $(0, 4)$ and the slope formula.

$$m = \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3} \qquad x_1 = 6,\ y_1 = 0,\ x_2 = 0,\ y_2 = 4$$

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| A vertical line has undefined slope. | The graph of the line $x = 3$ has undefined slope. |
| A horizontal line has 0 slope. | The graph of the line $y = -5$ has slope $m = 0$. |
| Parallel lines have equal slopes. | The lines $y = 2x + 3$ and $4x - 2y = 6$ are parallel—both have $m = 2$.<br>$\begin{array}{c\|c} y = 2x + 3 & 4x - 2y = 6 \\ m = 2 & -2y = -4x + 6 \\ & y = 2x - 3 \\ & m = 2 \end{array}$ |
| The slopes of perpendicular lines, neither of which is vertical, are negative reciprocals with a product of $-1$. | The lines $y = 3x - 1$ and $x + 3y = 4$ are perpendicular—their slopes are negative reciprocals.<br>$\begin{array}{c\|c} y = 3x - 1 & x + 3y = 4 \\ m = 3 & 3y = -x + 4 \\ & y = -\frac{1}{3}x + \frac{4}{3} \\ & m = -\frac{1}{3} \end{array}$ |

## 7.2 Review of Equations of Lines; Linear Models

| | |
|---|---|
| **Slope-Intercept Form**<br>$y = mx + b$ | $y = 2x + 3$   $m = 2$, $y$-intercept is $(0, 3)$. |
| **Point-Slope Form**<br>$y - y_1 = m(x - x_1)$ | $y - 3 = 4(x - 5)$   $(5, 3)$ is on the line, $m = 4$. |
| **Standard Form**<br>$Ax + By = C$ | $2x - 5y = 8$   Standard form |
| **Horizontal Line**<br>$y = b$ | $y = 4$   Horizontal line |
| **Vertical Line**<br>$x = a$ | $x = -1$   Vertical line |

## 7.3 Introduction to Relations and Functions

| | |
|---|---|
| A **function** is a set of ordered pairs such that, for each first component, there is one and only one second component. The set of first components is called the **domain,** and the set of second components is called the **range.** | $f = \{(-1, 4), (0, 6), (1, 4)\}$ defines a function $f$ with domain, the set of $x$-values, $\{-1, 0, 1\}$ and range, the set of $y$-values, $\{4, 6\}$.<br>$y = x^2$ defines a function with domain $(-\infty, \infty)$ and range $[0, \infty)$. |

## 7.4 Function Notation and Linear Functions

| | |
|---|---|
| To evaluate a function $f$, where $f(x)$ defines the range value for a given value of $x$ in the domain, substitute the value wherever $x$ appears. | If $f(x) = x^2 - 7x + 12$, then<br>$f(1) = 1^2 - 7(1) + 12 = 6$. |

(continued)

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| To write an equation that defines a function $f$ in function notation, follow these steps.<br>***Step 1*** Solve the equation for $y$.<br>***Step 2*** Replace $y$ with $f(x)$. | Write $2x + 3y = 12$ using notation for a function $f$.<br>$3y = -2x + 12$ Subtract $2x$.<br>$y = -\frac{2}{3}x + 4$ Divide by 3.<br>$f(x) = -\frac{2}{3}x + 4$ $y = f(x)$ |
| **7.5 Operations on Functions and Composition**<br>**Operations on Functions**<br>If $f(x)$ and $g(x)$ define functions, then<br>$(f + g)(x) = f(x) + g(x)$,<br>$(f - g)(x) = f(x) - g(x)$,<br>$(fg)(x) = f(x) \cdot g(x)$,<br>and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0.$ | Let $f(x) = x^2$ and $g(x) = 2x + 1$.<br>$(f + g)(x)$<br>$= f(x) + g(x)$<br>$= x^2 + 2x + 1$<br>$(f - g)(x)$<br>$= f(x) - g(x)$<br>$= x^2 - (2x + 1)$<br>$= x^2 - 2x - 1$<br>$(fg)(x)$<br>$= f(x) \cdot g(x)$<br>$= x^2(2x + 1)$<br>$= 2x^3 + x^2$<br>$\left(\frac{f}{g}\right)(x)$<br>$= \frac{f(x)}{g(x)}$<br>$= \frac{x^2}{2x + 1}, \quad x \neq -\frac{1}{2}$ |
| **Composition of $f$ and $g$**<br>$(f \circ g)(x) = f(g(x))$ | Let $f(x) = x^2$ and $g(x) = 2x + 1$.<br>$(f \circ g)(x) = f(g(x))$<br>$= f(2x + 1)$<br>$= (2x + 1)^2$<br>$(g \circ f)(x) = g(f(x))$<br>$= g(x^2)$<br>$= 2x^2 + 1$ |
| **7.6 Variation**<br>Let $k$ be a real number.<br>If $y = kx^n$, then $y$ varies directly as $x^n$.<br>If $y = \frac{k}{x^n}$, then $y$ varies inversely as $x^n$.<br>If $y = kxz$, then $y$ varies jointly as $x$ and $z$. | The area of a circle varies directly as the square of the radius.<br>$\mathscr{A} = kr^2$ Here, $k = \pi$.<br>Pressure varies inversely as volume.<br>$p = \frac{k}{V}$<br>For a given principal, interest varies jointly as interest rate and time.<br>$I = krt$ $k$ is the given principal. |

## CHAPTER 7

# REVIEW EXERCISES

**7.1** *Complete the table of ordered pairs for each equation. Then graph the equation.*

**1.** $3x + 2y = 10$

| $x$ | $y$ |
|---|---|
| 0 | |
| | 0 |
| 2 | |
| | −2 |

**2.** $x - y = 8$

| $x$ | $y$ |
|---|---|
| 2 | |
| | −3 |
| 3 | |
| | −2 |

*Find the x- and y-intercepts and then graph each equation.*

**3.** $4x - 3y = 12$

**4.** $5x + 7y = 28$

**5.** $2x + 5y = 20$

**6.** $x - 4y = 8$

*Use the midpoint formula to find the midpoint of each segment with the given endpoints.*

**7.** $(-8, -12)$ and $(8, 16)$

**8.** $(0, -5)$ and $(-9, 8)$

*Find the slope of each line.*

**9.** Through $(-1, 2)$ and $(4, -5)$

**10.** Through $(0, 3)$ and $(-2, 4)$

**11.** $y = 2x + 3$

**12.** $3x - 4y = 5$

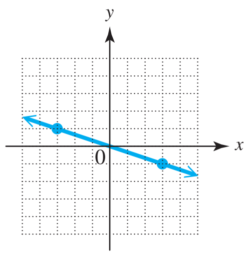
**13.** $x = 5$

**14.** Parallel to $3y = 2x + 5$

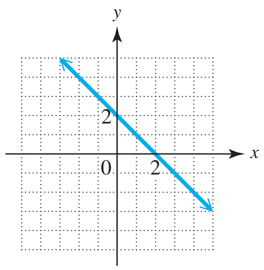
**15.** Perpendicular to $3x - y = 4$

**16.** Through $(-1, 5)$ and $(-1, -4)$

**17.**

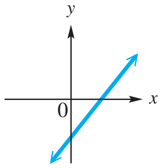


**18.**

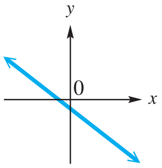


*Tell whether each line has* positive, negative, 0, *or* undefined *slope.*

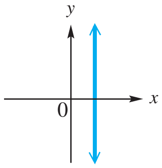
**19.**



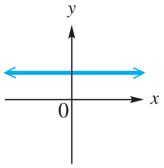
**20.**



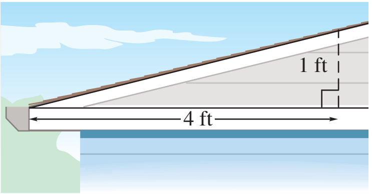
**21.**



**22.**



**23.** ***Concept Check*** If the pitch of a roof is $\frac{1}{4}$, how many feet in the horizontal direction correspond to a rise of 3 ft?



**24.** Family income in the United States has increased steadily for many years (primarily due to inflation). In 1980, the median family income was about \$21,000 per year. In 2007, it was about \$61,400 per year. Find the average rate of change of median family income to the nearest dollar over that period. (*Source:* U.S. Census Bureau.)

**7.2** *Find an equation for each line.* ***(a)*** *Write the equation in slope-intercept form.* ***(b)*** *Write the equation in standard form.*

**25.** Slope $-\frac{1}{3}$; $y$-intercept $(0, -1)$

**26.** Slope 0; $y$-intercept $(0, -2)$

**27.** Slope $-\frac{4}{3}$; through $(2, 7)$

**28.** Slope 3; through $(-1, 4)$

**29.** Vertical; through $(2, 5)$

**30.** Through $(2, -5)$ and $(1, 4)$

**31.** Through $(-3, -1)$ and $(2, 6)$

**32.** The line pictured in **Exercise 18**

**33.** Parallel to $4x - y = 3$ and through $(7, -1)$

**34.** Perpendicular to $2x - 5y = 7$ and through $(4, 3)$

**35.** The Midwest Athletic Club offers two special membership plans. (*Source:* Midwest Athletic Club.) For each plan, write a linear equation in slope-intercept form and give the cost $y$ in dollars of a 1-yr membership. Let $x$ represent the number of months.

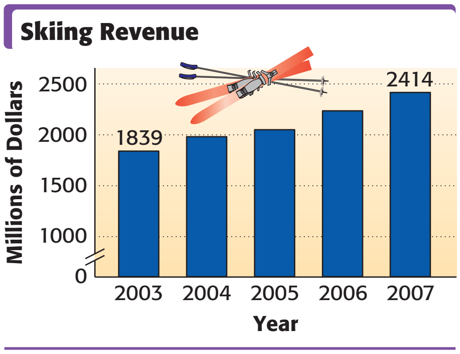
**(a)** Executive VIP/Gold membership: $159 fee, plus $57 per month

**(b)** Executive Regular/Silver membership: $159 fee, plus $47 per month

**36.** Revenue for skiing facilities in the United States is shown in the graph.

**(a)** Use the information given for the years 2003 and 2007, letting $x = 3$ represent 2003, $x = 7$ represent 2007, and $y$ represent revenue (in millions of dollars) to find a linear equation that models the data. Write the equation in slope-intercept form. Interpret the slope.

**(b)** Use your equation from part (a) to estimate revenue for skiing facilities in 2008, to the nearest million.

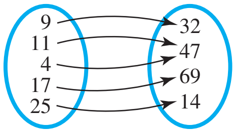


***Source:*** U.S. Census Bureau.

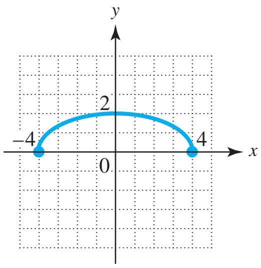
**7.3** *In Exercises 37–40, give the domain and range of each relation. Identify any functions.*

**37.** $\{(-4, 2), (-4, -2), (1, 5), (1, -5)\}$

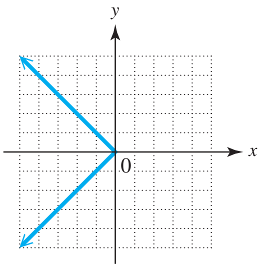
**38.**



**39.**



**40.**



**7.3–7.4** *Determine whether each equation defines y as a function of x. Give the domain in each case. Identify any linear functions.*

**41.** $y = 3x - 3$

**42.** $x = y^2$

**43.** $y = \dfrac{7}{x - 6}$

**7.4** *Given* $f(x) = -2x^2 + 3x - 6$, *find each function value or expression.*

**44.** $f(0)$

**45.** $f(2.1)$

**46.** $f\left(-\frac{1}{2}\right)$

**47.** $f(k)$

**48.** The equation $2x^2 - y = 0$ defines $y$ as a function $f$ of $x$. Write it using function notation, and find $f(3)$.

**49.** *Concept Check* Suppose that $2x - 5y = 7$ defines $y$ as a function $f$ of $x$. If $y = f(x)$, which one of the following defines the same function?

**A.** $f(x) = -\frac{2}{5}x + \frac{7}{5}$ **B.** $f(x) = -\frac{2}{5}x - \frac{7}{5}$

**C.** $f(x) = \frac{2}{5}x - \frac{7}{5}$ **D.** $f(x) = \frac{2}{5}x + \frac{7}{5}$

**50.** The table shows life expectancy at birth in the United States for selected years.

**(a)** Does the table define a function?

**(b)** What are the domain and range?

**(c)** Call this function $f$. Give two ordered pairs that belong to $f$.

**(d)** Find $f(1980)$. What does this mean?

**(e)** If $f(x) = 76.8$, what does $x$ equal?

| *Year* | *Life Expectancy at Birth (years)* |
|---|---|
| 1960 | 69.7 |
| 1970 | 70.8 |
| 1980 | 73.7 |
| 1990 | 75.4 |
| 2000 | 76.8 |
| 2009 | 78.1 |

*Source:* National Center for Health Statistics.

## 7.5

**51.** Find each of the following for the polynomial function defined by

$$f(x) = -2x^2 + 5x + 7.$$

**(a)** $f(-2)$ **(b)** $f(3)$

**52.** Find each of the following for the polynomial functions defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 5x^2 - 3x + 2.$$

**(a)** $(f + g)(x)$ **(b)** $(f - g)(x)$ **(c)** $(f + g)(-1)$ **(d)** $(f - g)(-1)$

**53.** Find each of the following for the polynomial functions defined by

$$f(x) = 12x^2 - 3x \quad \text{and} \quad g(x) = 3x.$$

**(a)** $(fg)(x)$ **(b)** $\left(\frac{f}{g}\right)(x)$ **(c)** $(fg)(-1)$ **(d)** $\left(\frac{f}{g}\right)(2)$

**54.** Find each of the following for the polynomial functions defined by

$$f(x) = 3x^2 + 2x - 1 \quad \text{and} \quad g(x) = 5x + 7.$$

**(a)** $(g \circ f)(3)$ **(b)** $(f \circ g)(3)$ **(c)** $(f \circ g)(-2)$

**(d)** $(g \circ f)(-2)$ **(e)** $(f \circ g)(x)$ **(f)** $(g \circ f)(x)$

## 7.6

**55.** *Concept Check* In which one of the following does $y$ vary inversely as $x$?

**A.** $y = 2x$ **B.** $y = \frac{x}{3}$ **C.** $y = \frac{3}{x}$ **D.** $y = x^2$

*Solve each problem.*

**56.** For the subject in a photograph to appear in the same perspective in the photograph as in real life, the viewing distance must be properly related to the amount of enlargement. For a particular camera, the viewing distance varies directly as the amount of enlargement. A picture that is taken with this camera and enlarged 5 times should be viewed from a distance of 250 mm. Suppose a print 8.6 times the size of the negative is made. From what distance should it be viewed?

**57.** The frequency (number of vibrations per second) of a vibrating guitar string varies inversely as its length. That is, a longer string vibrates fewer times in a second than a shorter string. Suppose a guitar string 0.65 m long vibrates 4.3 times per sec. What frequency would a string 0.5 m long have?

**58.** The volume of a rectangular box of a given height is proportional to its width and length. A box with width 2 ft and length 4 ft has volume 12 $\text{ft}^3$. Find the volume of a box with the same height, but that is 3 ft wide and 5 ft long.

# CHAPTER 7 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

**1.** Complete the table of ordered pairs for the equation $2x - 3y = 12$.

| $x$ | $y$ |
|---|---|
| 1 | |
| 3 | |
| | −4 |

*Find the x- and y-intercepts, and graph each equation.*

**2.** $3x - 2y = 20$

**3.** $y = 5$

**4.** $x = 2$

**5.** Find the slope of the line through the points $(6, 4)$ and $(-4, -1)$.

**6.** Describe how the graph of a line with undefined slope is situated in a rectangular coordinate system.

*Determine whether each pair of lines is* parallel, perpendicular, *or* neither.

**7.** $5x - y = 8$ and $5y = -x + 3$

**8.** $2y = 3x + 12$ and $3y = 2x - 5$

**9.** In 1980, there were 119,000 farms in Iowa. As of 2008, there were 93,000. Find and interpret the average rate of change in the number of farms per year, to the nearest whole number. (*Source:* U.S. Department of Agriculture.)

*Find an equation of each line, and write it in* **(a)** *slope-intercept form if possible and* **(b)** *standard form.*

**10.** Through $(4, -1)$; $m = -5$

**11.** Through $(-3, 14)$; horizontal

**12.** Through $(-2, 3)$ and $(6, -1)$

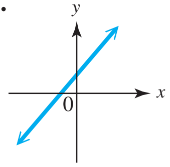
**13.** Through $(5, -6)$; vertical

**14.** Through $(-7, 2)$ and parallel to $3x + 5y = 6$

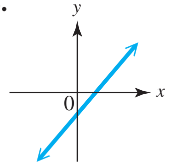
**15.** Through $(-7, 2)$ and perpendicular to $y = 2x$

**16.** ***Concept Check*** Which line has positive slope and negative $y$-coordinate for its $y$-intercept?

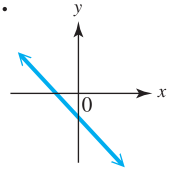
**A.**



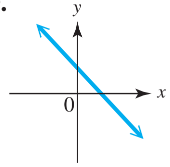
**B.**



**C.**



**D.**

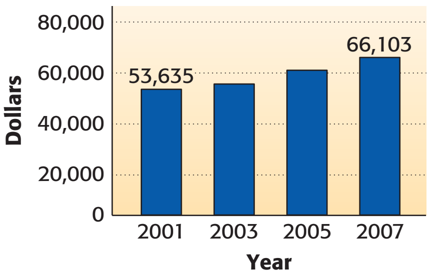


**17.** The bar graph shows median household income for Asians and Pacific Islanders in the United States.

**(a)** Use the information for the years 2001 and 2007 to find an equation that models the data. Let $x = 1$ represent 2001, $x = 7$ represent 2007, and $y$ represent the median income. Write the equation in slope-intercept form.

**(b)** Use the equation from part (a) to approximate median household income for 2005 to the nearest dollar. How does your result compare against the actual value, \$61,094?

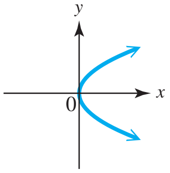
**Median Household Income for Asians and Pacific Islanders**



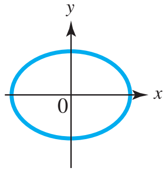
*Source:* U.S. Census Bureau.

**18.** Which one of the following is the graph of a function?

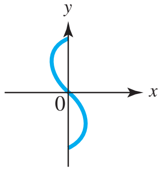
**A.**



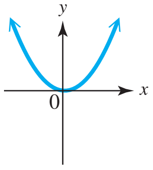
**B.**



**C.**



**D.**



**19.** Which of the following does not define $y$ as a function of $x$?

**A.** $\{(0, 1), (-2, 3), (4, 8)\}$ **B.** $y = 2x - 6$ **C.** $y = \sqrt{x + 2}$ **D.**

| x | y |
|---|---|
| 0 | 1 |
| 3 | 2 |
| 0 | 2 |
| 6 | 3 |

**20.** Give the domain and range of the relation shown in each of the following.

**(a)** Choice A of **Exercise 18** **(b)** Choice A of **Exercise 19**

**21.** For $f(x) = -x^2 + 2x - 1$, find **(a)** $f(1)$, and **(b)** $f(a)$.

**22.** Graph the linear function defined by $f(x) = \frac{2}{3}x - 1$. What is its domain and range?

**23.** Find each of the following for the functions defined by

$$f(x) = -2x^2 + 5x - 6 \quad \text{and} \quad g(x) = 7x - 3.$$

**(a)** $f(4)$ **(b)** $(f + g)(x)$ **(c)** $(f - g)(x)$ **(d)** $(f - g)(-2)$

**24.** If $f(x) = x^2 + 3x + 2$ and $g(x) = x + 1$, find each of the following.

**(a)** $(fg)(x)$ **(b)** $(fg)(-2)$

**25.** Use $f(x)$ and $g(x)$ from **Exercise 24** to find each of the following.

**(a)** $\left(\frac{f}{g}\right)(x)$ **(b)** $\left(\frac{f}{g}\right)(-2)$

**26.** Find each of the following for the functions defined by

$$f(x) = 3x + 5 \quad \text{and} \quad g(x) = x^2 + 2.$$

**(a)** $(f \circ g)(-2)$ **(b)** $(f \circ g)(x)$ **(c)** $(g \circ f)(x)$

**27.** The current in a simple electrical circuit is inversely proportional to the resistance. If the current is 80 amps when the resistance is 30 ohms, find the current when the resistance is 12 ohms.

**28.** The force of the wind blowing on a vertical surface varies jointly as the area of the surface and the square of the velocity. If a wind blowing at 40 mph exerts a force of 50 lb on a surface of 500 ft$^2$, how much force will a wind of 80 mph place on a surface of 2 ft$^2$?

## CHAPTERS 1–7 CUMULATIVE REVIEW EXERCISES

*Decide whether each statement is* always true, sometimes true, *or* never true. *If the statement is* sometimes true, *give examples in which it is true and in which it is false.*

**1.** The absolute value of a negative number equals the additive inverse of the number.

**2.** The sum of two negative numbers is positive.

**3.** The sum of a positive number and a negative number is 0.

*Simplify.*

**4.** $-|-2| - 4 + |-3| + 7$ **5.** $-(-4m + 3)$ **6.** $\frac{(4^2 - 4) - (-1)7}{4 + (-6)}$

*Evaluate each expression for* $p = -4$, $q = \frac{1}{2}$, *and* $r = 16$.

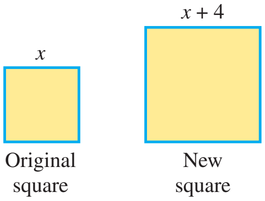
**7.** $-3(2q - 3p)$ **8.** $\frac{r}{8p + 2r}$

*Solve.*

**9.** $2z - 5 + 3z = 2 - z$ **10.** $\frac{3x - 1}{5} + \frac{x + 2}{2} = -\frac{3}{10}$

*Solve each problem.*

**11.** If each side of a square were increased by 4 in., the perimeter would be 8 in. less than twice the perimeter of the original square. Find the length of a side of the original square.



**12.** Two planes leave the Dallas-Fort Worth airport at the same time. One travels east at 550 mph, and the other travels west at 500 mph. Assuming no wind, how long will it take for the planes to be 2100 mi apart?

*Solve. Write each solution set in interval notation and graph it.*

**13.** $-4 < 3 - 2k < 9$ **14.** $-0.3x + 2.1(x - 4) \le -6.6$

**15.** Find the $x$- and $y$-intercepts of the line with equation $3x + 5y = 12$, and graph the line.

**16.** Consider the points $A(-2, 1)$ and $B(3, -5)$.

**(a)** Find the slope of the line $AB$.

**(b)** Find the slope of a line perpendicular to line $AB$.

*Write an equation for each line. Express the equation* **(a)** *in slope-intercept form if possible and* **(b)** *in standard form.*

**17.** Slope $-\frac{3}{4}$; $y$-intercept $(0, -1)$ **18.** Through $(4, -3)$ and $(1, 1)$

*Perform the indicated operations. In Exercise 22, assume that variables represent nonzero real numbers.*

**19.** $(3x^2y^{-1})^{-2}(2x^{-3}y)^{-1}$ **20.** $(7x + 3y)^2$

**21.** $(3x^3 + 4x^2 - 7) - (2x^3 - 8x^2 + 3x)$ **22.** $\frac{m^3 - 3m^2 + 5m - 3}{m - 1}$

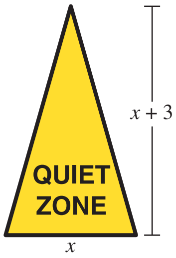
*Factor.*

**23.** $16w^2 + 50wz - 21z^2$ **24.** $4x^2 - 4x + 1 - y^2$ **25.** $8p^3 + 27$

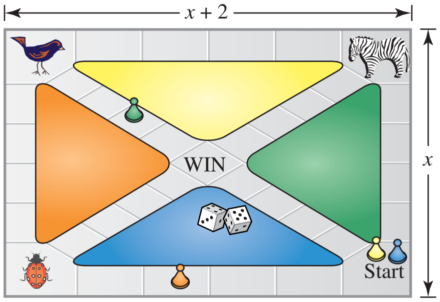
**26.** Solve $9x^2 = 6x - 1$.

*Solve each problem.*

**27.** A sign is to have the shape of a triangle with a height 3 ft greater than the length of the base. How long should the base be if the area is to be 14 ft$^2$?



**28.** A game board has the shape of a rectangle. The longer sides are each 2 in. longer than the distance between them. The area of the board is 288 in.$^2$. Find the length of the longer sides and the distance between them.



*Perform each indicated operation. Write the answer in lowest terms.*

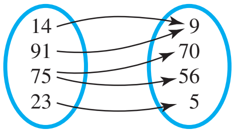
**29.** $\dfrac{8}{x+1} - \dfrac{2}{x+3}$

**30.** $\dfrac{x^2+5x+6}{3x} \div \dfrac{x^2-4}{x^2+x-6}$

**31.** Simplify $\dfrac{\dfrac{12}{x+6}}{\dfrac{4}{2x+12}}$.

**32.** Solve $\dfrac{2}{x-1} = \dfrac{5}{x-1} - \dfrac{3}{4}$.

**33.** Give the domain and range of the relation. Does it define a function? Explain.



**34.** Consider the function defined by

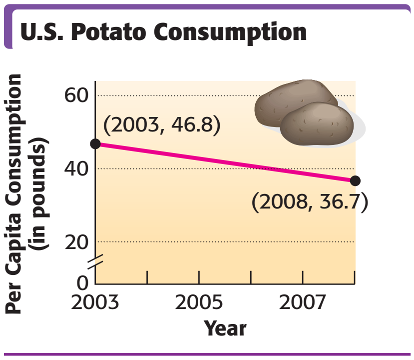
$$f(x) = -4x + 10.$$

**(a)** Find the domain and range.

**(b)** Evaluate $f(-3)$.

**(c)** If $f(x) = 6$, find the value of $x$.

**35.** Use the information in the graph to find and interpret the average rate of change in the per capita consumption of potatoes in the United States from 2003 to 2008.



***Source:*** U.S. Department of Agriculture.

## CHAPTER 8

# Systems of Linear Equations

In the early 1970s, the NBC television network presented *The Bill Cosby Show,* in which the popular comedian played Chet Kincaid, a Los Angeles high school physical education teacher. In the episode "Let *x* Equal a Lousy Weekend," Chet must substitute for the algebra teacher. He and the entire class are stumped by the following problem:

> *How many pounds of candy that sells for \$0.75 per lb must be mixed with candy that sells for \$1.25 per lb to obtain 9 lb of a mixture that should sell for \$0.96 per lb?*

The smartest student in the class eventually helps Chet solve this problem. In **Exercise 31** of **Section 8.5,** we ask you to use a *system of linear equations,* the topic of this chapter, to do so.

## 8.1

# Solving Systems of Linear Equations by Graphing

**OBJECTIVES**

1. Decide whether a given ordered pair is a solution of a system.
2. Solve linear systems by graphing.
3. Solve special systems by graphing.
4. Identify special systems without graphing.

A **system of linear equations,** often called a **linear system,** consists of two or more linear equations with the same variables.

$$\begin{aligned} 2x + 3y &= 4 \\ 3x - y &= -5 \end{aligned} \qquad \begin{aligned} x + 3y &= 1 \\ -y &= 4 - 2x \end{aligned} \qquad \begin{aligned} x - y &= 1 \\ y &= 3 \end{aligned} \qquad \text{Linear systems}$$

**NOTE** In the system on the right, think of $y = 3$ as an equation in two variables by writing it as $0x + y = 3$.

**OBJECTIVE 1** **Decide whether a given ordered pair is a solution of a system.** A **solution of a system** of linear equations is an ordered pair that makes both equations true at the same time. A solution of an equation is said to *satisfy* the equation.

**NOW TRY EXERCISE 1**

Decide whether the ordered pair $(5, 2)$ is a solution of each system.

**(a)** $\begin{aligned} 2x + 5y &= 20 \\ x - y &= 7 \end{aligned}$

**(b)** $\begin{aligned} 3x - y &= 13 \\ 2x + y &= 12 \end{aligned}$

**EXAMPLE 1** Determining Whether an Ordered Pair Is a Solution

Decide whether the ordered pair $(4, -3)$ is a solution of each system.

**(a)** $\begin{aligned} x + 4y &= -8 \\ 3x + 2y &= 6 \end{aligned}$

To decide whether $(4, -3)$ is a solution of the system, substitute 4 for $x$ and $-3$ for $y$ in each equation.

$$\begin{aligned} x + 4y &= -8 & \\ 4 + 4(-3) &\overset{?}{=} -8 & \text{Substitute.} \\ 4 + (-12) &\overset{?}{=} -8 & \text{Multiply.} \\ -8 &= -8 \ \checkmark & \text{True} \end{aligned} \qquad \begin{aligned} 3x + 2y &= 6 & \\ 3(4) + 2(-3) &\overset{?}{=} 6 & \text{Substitute.} \\ 12 + (-6) &\overset{?}{=} 6 & \text{Multiply.} \\ 6 &= 6 \ \checkmark & \text{True} \end{aligned}$$

Because $(4, -3)$ satisfies both equations, it is a solution of the system.

**(b)** $\begin{aligned} 2x + 5y &= -7 \\ 3x + 4y &= 2 \end{aligned}$

Again, substitute 4 for $x$ and $-3$ for $y$ in both equations.

$$\begin{aligned} 2x + 5y &= -7 & \\ 2(4) + 5(-3) &\overset{?}{=} -7 & \text{Substitute.} \\ 8 + (-15) &\overset{?}{=} -7 & \text{Multiply.} \\ -7 &= -7 \ \checkmark & \text{True} \end{aligned} \qquad \begin{aligned} 3x + 4y &= 2 & \\ 3(4) + 4(-3) &\overset{?}{=} 2 & \text{Substitute.} \\ 12 + (-12) &\overset{?}{=} 2 & \text{Multiply.} \\ 0 &= 2 & \text{False} \end{aligned}$$

The ordered pair $(4, -3)$ is not a solution of this system because it does not satisfy the second equation.

NOW TRY

**OBJECTIVE 2** **Solve linear systems by graphing.** The set of all ordered pairs that are solutions of a system is its **solution set.** One way to find the solution set of a system of two linear equations is to graph both equations on the same axes.

**NOW TRY ANSWERS**

1. **(a)** no **(b)** yes

Any intersection point would be on both lines and would therefore be a solution of *both* equations. ***Thus, the coordinates of any point at which the lines intersect give a solution of the system.***

The graph in **FIGURE 1** shows that the solution of the system in **Example 1(a)** is the intersection point $(4, -3)$. Because two *different* straight lines can intersect at no more than one point, there can never be more than one solution for such a system.

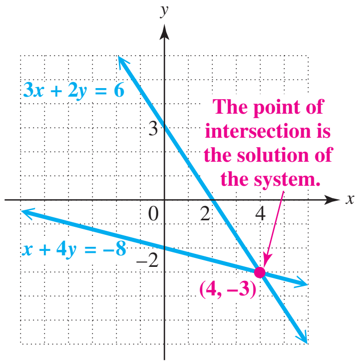


**FIGURE 1**

**NOW TRY EXERCISE 2**

Solve the system by graphing.

$$x - 2y = 4$$
$$2x + y = 3$$

## EXAMPLE 2 Solving a System by Graphing

Solve the system of equations by graphing both equations on the same axes.

$$2x + 3y = 4$$
$$3x - y = -5$$

We graph these two lines by plotting several points for each line. Recall from **Section 3.2** that the intercepts are often convenient choices.

$2x + 3y = 4$

| $x$ | $y$ |
|---|---|
| 0 | $\frac{4}{3}$ |
| 2 | 0 |
| −2 | $\frac{8}{3}$ |

Find a third ordered pair as a check.

$3x - y = -5$

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| $-\frac{5}{3}$ | 0 |
| −2 | −1 |

The lines in **FIGURE 2** suggest that the graphs intersect at the point $(-1, 2)$. We check this by substituting $-1$ for $x$ and 2 for $y$ in both equations.

*CHECK*

| | |
|---|---|
| $2x + 3y = 4$ | First equation |
| $2(-1) + 3(2) \stackrel{?}{=} 4$ | Substitute. |
| $4 = 4$ ✓ | True |
| $3x - y = -5$ | Second equation |
| $3(-1) - 2 \stackrel{?}{=} -5$ | Substitute. |
| $-5 = -5$ ✓ | True |

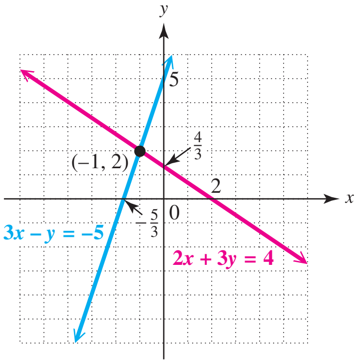


**FIGURE 2**

Because $(-1, 2)$ satisfies both equations, the solution set of this system is $\{(-1, 2)\}$.

NOW TRY

*NOW TRY ANSWER*

**2.** $\{(2, -1)\}$

**Solving a Linear System by Graphing**

***Step 1*** **Graph each equation** of the system on the same coordinate axes.

***Step 2*** **Find the coordinates of the point of intersection** of the graphs if possible. This is the solution of the system.

***Step 3*** **Check** the solution in *both* of the original equations. Then write the solution set.

---

**CAUTION** With the graphing method, it may not be possible to determine the exact coordinates of the point that represents the solution, particularly if those coordinates are not integers. The graphing method does, however, show geometrically how solutions are found and is useful when approximate answers will do.

---

**OBJECTIVE 3** **Solve special systems by graphing.** Sometimes the graphs of the two equations in a system either do not intersect at all or are the same line.

*NOW TRY EXERCISE 3*

Solve each system by graphing.

**(a)** $5x - 3y = 2$

$10x - 6y = 4$

**(b)** $4x + y = 7$

$12x + 3y = 10$

**EXAMPLE 3** Solving Special Systems by Graphing

Solve each system by graphing.

**(a)** $2x + y = 2$

$2x + y = 8$

The graphs of these lines are shown in **FIGURE 3**. The two lines are parallel and have no points in common. For such a system, there is no solution. We write the solution set as $\emptyset$.

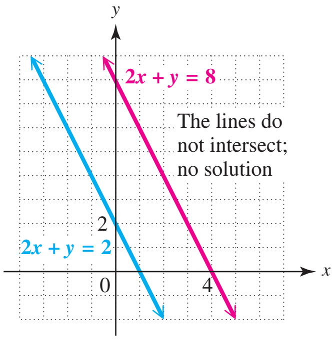


FIGURE 3

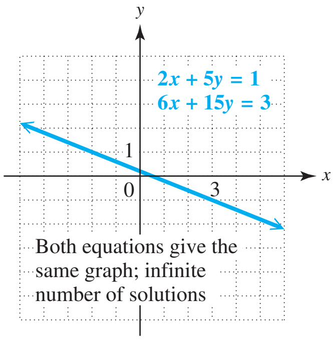


FIGURE 4

**(b)** $2x + 5y = 1$

$6x + 15y = 3$

The graphs of these two equations are the same line. See **FIGURE 4**. We can obtain the second equation by multiplying each side of the first equation by 3. In this case, every point on the line is a solution of the system, and the solution set contains an infinite number of ordered pairs, each of which satisfies both equations of the system. We write the solution set as

$$\{(x, y) \mid 2x + 5y = 1\},$$

This is the first equation in the system. See the Note on the next page.

read "the set of ordered pairs $(x, y)$ such that $2x + 5y = 1$." Recall from **Section 1.4** that this notation is called **set-builder notation.** *NOW TRY*

*NOW TRY ANSWERS*

**3. (a)** $\{(x, y) \mid 5x - 3y = 2\}$

**(b)** $\emptyset$

---

**NOTE** When a system has an infinite number of solutions, as in **Example 3(b),** either equation of the system could be used to write the solution set. ***We prefer to use the equation in standard form with integer coefficients that have greatest common factor 1.*** If neither of the given equations is in this form, we will use an *equivalent* equation that is in standard form with integer coefficients that have greatest common factor 1.

---

The system in **Example 2** has exactly one solution. A system with at least one solution is called a **consistent system.** A system with no solution, such as the one in **Example 3(a),** is called an **inconsistent system.**

The equations in **Example 2** are **independent equations** with different graphs. The equations of the system in **Example 3(b)** have the same graph and are equivalent. Because they are different forms of the same equation, these equations are called **dependent equations.**

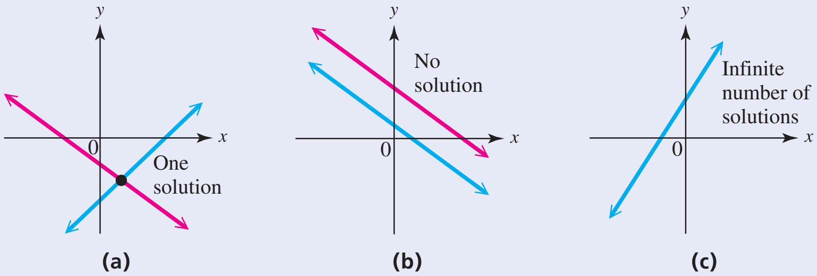
**Examples 2 and 3** show the three cases that may occur when solving a system of equations with two variables.

### Three Cases for Solutions of Systems

1. The graphs intersect at exactly one point, which gives the (single) ordered-pair solution of the system. The **system is consistent** and the **equations are independent.** See **FIGURE 5(a)**.
2. The graphs are parallel lines, so there is no solution and the solution set is $\emptyset$. The **system is inconsistent** and the **equations are independent.** See **FIGURE 5(b)**.
3. The graphs are the same line. There is an infinite number of solutions, and the solution set is written in set-builder notation as

$$\{(x, y) \mid \underline{\hspace{3cm}}\},$$

where one of the equations is written after the $\mid$ symbol. The **system is consistent** and the **equations are dependent.** See **FIGURE 5(c)**.



(a) (b) (c)

FIGURE 5

**OBJECTIVE 4** **Identify special systems without graphing.** **Example 3** showed that the graphs of an inconsistent system are parallel lines and the graphs of a system of dependent equations are the same line. We can recognize these special kinds of systems without graphing by using slopes.

**NOW TRY EXERCISE 4**

Describe each system without graphing. State the number of solutions.

**(a)** $5x - 8y = 4$
$x - \frac{8}{5}y = \frac{4}{5}$

**(b)** $2x + y = 7$
$3y = -6x - 12$

**(c)** $y - 3x = 7$
$3y - x = 0$

## EXAMPLE 4 Identifying the Three Cases by Using Slopes

Describe each system without graphing. State the number of solutions.

**(a)** $3x + 2y = 6$
$-2y = 3x - 5$

Write each equation in slope-intercept form, $y = mx + b$, by solving for $y$.

$$\begin{aligned} 3x + 2y &= 6 \\ 2y &= -3x + 6 && \text{Subtract } 3x. \\ y &= -\frac{3}{2}x + 3 && \text{Divide by 2.} \end{aligned}$$

$$\begin{aligned} -2y &= 3x - 5 \\ y &= -\frac{3}{2}x + \frac{5}{2} && \text{Divide by } -2. \end{aligned}$$

Both equations have slope $-\frac{3}{2}$ but they have different $y$-intercepts, $3$ and $\frac{5}{2}$. Recall that lines with the same slope are parallel, so these equations have graphs that are parallel lines. Thus, the system has no solution.

**(b)** $2x - y = 4$
$x = \frac{y}{2} + 2$

Again, write the equations in slope-intercept form.

$$\begin{aligned} 2x - y &= 4 \\ -y &= -2x + 4 && \text{Subtract } 2x. \\ y &= 2x - 4 && \text{Multiply by } -1. \end{aligned}$$

$$\begin{aligned} x &= \frac{y}{2} + 2 \\ \frac{y}{2} + 2 &= x && \text{Interchange sides.} \\ \frac{y}{2} &= x - 2 && \text{Subtract 2.} \\ y &= 2x - 4 && \text{Multiply by 2.} \end{aligned}$$

The equations are exactly the same—their graphs are the same line. Thus, the system has an infinite number of solutions.

**(c)** $x - 3y = 5$
$2x + y = 8$

In slope-intercept form, the equations are as follows.

$$\begin{aligned} x - 3y &= 5 \\ -3y &= -x + 5 && \text{Subtract } x. \\ y &= \frac{1}{3}x - \frac{5}{3} && \text{Divide by } -3. \end{aligned}$$

$$\begin{aligned} 2x + y &= 8 \\ y &= -2x + 8 && \text{Subtract } 2x. \end{aligned}$$

The graphs of these equations are neither parallel nor the same line, since the slopes are different. This system has exactly one solution. NOW TRY

**NOTE** The solution set of the system in **Example 4(a)** is $\emptyset$, since the graphs of the equations of the system are parallel lines. The solution set of the system in **Example 4(b),** written using set-builder notation and the first equation, is

$$\{(x, y) \mid 2x - y = 4\}.$$

If we try to solve the system in **Example 4(c)** by graphing, we will have difficulty identifying the point of intersection of the graphs. We introduce an algebraic method for solving systems like this in **Section 8.2.**

**NOW TRY ANSWERS**

4. **(a)** The equations represent the same line. The system has an infinite number of solutions.
   **(b)** The equations represent parallel lines. The system has no solution.
   **(c)** The equations represent lines that are neither parallel nor the same line. The system has exactly one solution.

## CONNECTIONS

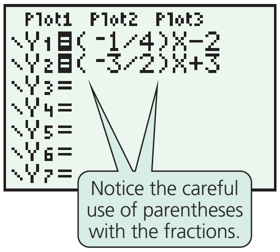
We can solve the system from **Example 1(a)** by graphing with a calculator.

$$\begin{aligned} x + 4y &= -8 \\ 3x + 2y &= 6 \end{aligned}$$

To enter the equations in a graphing calculator, first solve each equation for $y$.

$$\begin{aligned} x + 4y &= -8 \\ 4y &= -x - 8 && \text{Subtract } x. \\ y &= -\frac{1}{4}x - 2 && \text{Divide by 4.} \end{aligned} \qquad \begin{aligned} 3x + 2y &= 6 \\ 2y &= -3x + 6 && \text{Subtract } 3x. \\ y &= -\frac{3}{2}x + 3 && \text{Divide by 2.} \end{aligned}$$

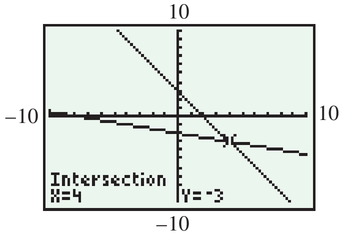
We designate the first equation $Y_1$ and the second equation $Y_2$. See **FIGURE 6(a)**. We graph the two equations using a standard window and then use the capability of the calculator to find the coordinates of the point of intersection of the graphs. See **FIGURE 6(b)**.



```
Plot1 Plot2 Plot3
\Y1=(-1/4)X-2
\Y2=(-3/2)X+3
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Notice the careful use of parentheses with the fractions.

**(a)**



The display at the bottom of the screen indicates that the solution set is $\{(4, -3)\}$.

**(b)**

**FIGURE 6**

### For Discussion or Writing

Use a graphing calculator to solve each system.

**1.** $3x + y = 2$
$2x - y = -7$

**2.** $8x + 4y = 0$
$4x - 2y = 2$

**3.** $3x + 3y = 0$
$4x + 2y = 3$

## 8.1 EXERCISES

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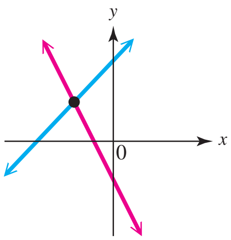
*Complete solution available on the Video Resources on DVD*

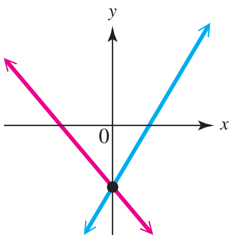
**1.** ***Concept Check*** Which ordered pair could not be a solution of the system graphed? Why is it the only valid choice?

**A.** $(-4, -4)$ **B.** $(-2, 2)$
**C.** $(-4, 4)$ **D.** $(-3, 3)$



**2.** ***Concept Check*** Which ordered pair could be a solution of the system graphed? Why is it the only valid choice?

**A.** $(2, 0)$ **B.** $(0, 2)$
**C.** $(-2, 0)$ **D.** $(0, -2)$



*Decide whether the given ordered pair is a solution of the given system.* ***See Example 1.***

**3.** $(2, -3)$
$x + y = -1$
$2x + 5y = 19$

**4.** $(4, 3)$
$x + 2y = 10$
$3x + 5y = 3$

**5.** $(-1, -3)$
$3x + 5y = -18$
$4x + 2y = -10$

**6.** $(-9, -2)$
$2x - 5y = -8$
$3x + 6y = -39$

**7.** $(7, -2)$
$4x = 26 - y$
$3x = 29 + 4y$

**8.** $(9, 1)$
$2x = 23 - 5y$
$3x = 24 + 3y$

**9.** $(6, -8)$
$$\begin{aligned} -2y &= x + 10 \\ 3y &= 2x + 30 \end{aligned}$$

**10.** $(-5, 2)$
$$\begin{aligned} 5y &= 3x + 20 \\ 3y &= -2x - 4 \end{aligned}$$

**11.** $(0, 0)$
$$\begin{aligned} 4x + 2y &= 0 \\ x + y &= 0 \end{aligned}$$

**12.** *Concept Check* When a student was asked to determine whether the ordered pair $(1, -2)$ is a solution of the following system, he answered "yes." His reasoning was that the ordered pair satisfies the equation $x + y = -1$, since $1 + (-2) = -1$. ***WHAT WENT WRONG?***

$$\begin{aligned} x + y &= -1 \\ 2x + y &= 4 \end{aligned}$$

**13.** *Concept Check* Each ordered pair in (a)–(d) is a solution of one of the systems graphed in A–D. Because of the location of the point of intersection, you should be able to determine the correct system for each solution. Match each system from A–D with its solution from (a)–(d).

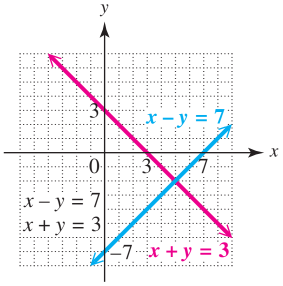
**(a)** $(3, 4)$

**(b)** $(-2, 3)$

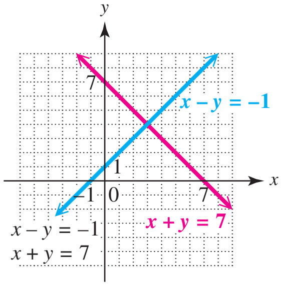
**(c)** $(-3, 2)$

**(d)** $(5, -2)$

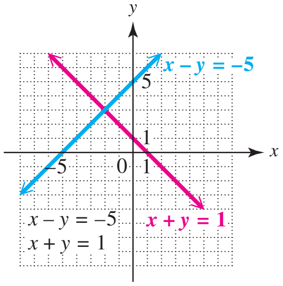
**A.**



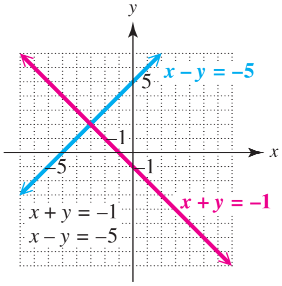
**B.**



**C.**



**D.**



**14.** *Concept Check* The following system has infinitely many solutions. Write its solution set, using set-builder notation as described in **Example 3(b).**

$$\begin{aligned} 6x - 4y &= 8 \\ 3x - 2y &= 4 \end{aligned}$$

*Solve each system of equations by graphing. If the system is inconsistent or the equations are dependent, say so.* ***See Examples 2 and 3.***

**15.** $\begin{aligned} x - y &= 2 \\ x + y &= 6 \end{aligned}$

**16.** $\begin{aligned} x - y &= 3 \\ x + y &= -1 \end{aligned}$

**17.** $\begin{aligned} x + y &= 4 \\ y - x &= 4 \end{aligned}$

**18.** $\begin{aligned} x + y &= -5 \\ y - x &= -5 \end{aligned}$

**19.** $\begin{aligned} x - 2y &= 6 \\ x + 2y &= 2 \end{aligned}$

**20.** $\begin{aligned} 2x - y &= 4 \\ 4x + y &= 2 \end{aligned}$

**21.** $\begin{aligned} 3x - 2y &= -3 \\ -3x - y &= -6 \end{aligned}$

**22.** $\begin{aligned} 2x - y &= 4 \\ 2x + 3y &= 12 \end{aligned}$

**23.** $\begin{aligned} 2x - 3y &= -6 \\ y &= -3x + 2 \end{aligned}$

**24.** $\begin{aligned} -3x + y &= -3 \\ y &= x - 3 \end{aligned}$

**25.** $\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 8 \end{aligned}$

**26.** $\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 12 \end{aligned}$

**27.** $\begin{aligned} 3x + y &= 5 \\ 6x + 2y &= 10 \end{aligned}$

**28.** $\begin{aligned} 2x - y &= 4 \\ 4x - 2y &= 8 \end{aligned}$

**29.** $\begin{aligned} 3x - 4y &= 24 \\ y &= -\frac{3}{2}x + 3 \end{aligned}$

**30.** $4x + y = 5$
$y = \frac{3}{2}x - 6$

**31.** $2x = y - 4$
$4x + 4 = 2y$

**32.** $3x = y + 5$
$6x - 5 = 2y$

**33.** Solve the system by graphing. Can you check your solution? Why or why not?

$$\begin{aligned} 2x + 3y &= 6 \\ x - 3y &= 5 \end{aligned}$$

**34.** Explain one of the drawbacks of solving a system of equations graphically.

*Without graphing, answer the following questions for each linear system.* ***See Example 4.***

***(a)*** *Is the system inconsistent, are the equations dependent, or neither?*
***(b)*** *Is the graph a pair of intersecting lines, a pair of parallel lines, or one line?*
***(c)*** *Does the system have one solution, no solution, or an infinite number of solutions?*

**35.** $y - x = -5$
$x + y = 1$

**36.** $y + 2x = 6$
$x - 3y = -4$

**37.** $x + 2y = 0$
$4y = -2x$

**38.** $2x - y = 4$
$y + 4 = 2x$

**39.** $x - 3y = 5$
$2x + y = 8$

**40.** $2x + 3y = 12$
$2x - y = 4$

**41.** $5x + 4y = 7$
$10x + 8y = 4$

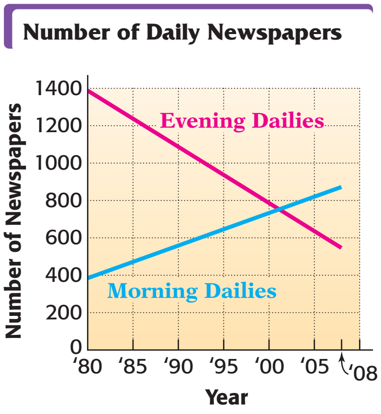
**42.** $3x + 2y = 5$
$6x + 4y = 3$

*Work each problem using the graph provided.*

**43.** The numbers of daily morning and evening newspapers in the United States in selected years over the period 1980–2008 are shown in the graph.

**(a)** For which years were there more evening dailies than morning dailies?

**(b)** Estimate the year in which the number of evening and morning dailies was closest to the same. About how many newspapers of each type were there in that year?



***Source:*** Editor & Publisher International Year Book.

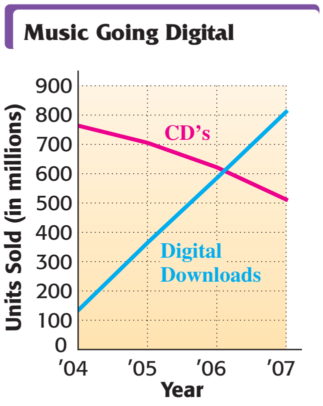
**44.** The graph shows how sales of music CDs and digital downloads of single songs (in millions) in the United States have changed over the years 2004 through 2007.

**(a)** In what year did Americans purchase about the same number of CDs as single digital downloads? How many units was this?

**(b)** Express the point of intersection of the two graphs as an ordered pair of the form (year, units in millions).

**(c)** Describe the trend in sales of music CDs over the years 2004 to 2007. If a straight line were used to approximate its graph, would the line have positive, negative, or zero slope? Explain.

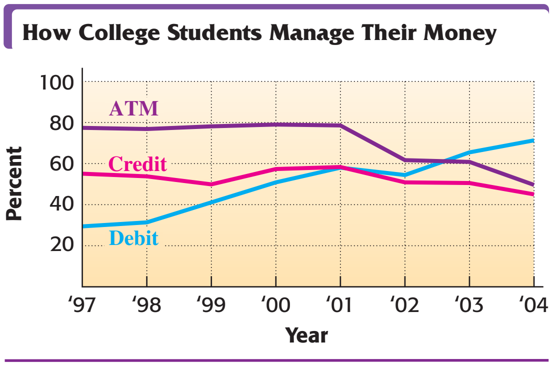
**(d)** If a straight line were used to approximate the graph of sales of digital downloads over the years 2004 to 2007, would the line have positive, negative, or zero slope? Explain.



***Source:*** Recording Industry Association of America.

**45.** The graph shows how college students managed their money during the years 1997 through 2004.

- **(a)** During what period did ATM use dominate both credit card *and* debit card use?
- **(b)** In what year did debit card use overtake credit card use?
- **(c)** In what year did debit card use overtake ATM use?
- **(d)** Write an ordered pair for the debit card use data in the year 1998.



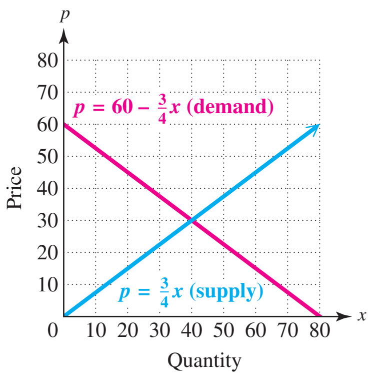
***Source:*** Georgetown University Credit Research Center.

*An application of mathematics in economics deals with* ***supply and demand.*** *Typically, as the price of an item increases, the demand for the item decreases while the supply increases. If supply and demand can be described by straight-line equations, the point at which the lines intersect determines the* ***equilibrium supply*** *and* ***equilibrium demand.***

*The price per unit, p, and the demand, x, for a particular aluminum siding are related by the linear equation* $p = 60 - \frac{3}{4}x$*, while the supply is given by the linear equation* $p = \frac{3}{4}x$*, as shown in the figure.*

*Use the graph to answer the questions in Exercises 46–48.*

SUPPLY AND DEMAND



**46.** At what value of $x$ does supply equal demand?

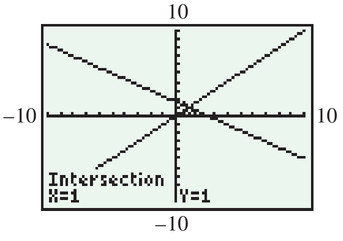
**47.** At what value of $p$ does supply equal demand?

**48.** When $x > 40$, does demand exceed supply or does supply exceed demand?

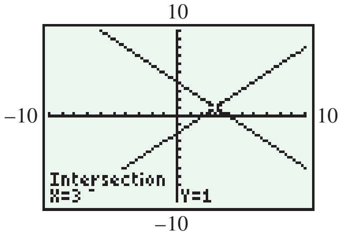
## TECHNOLOGY INSIGHTS EXERCISES 49–52

*Match the graphing calculator screens in choices A–D with the appropriate system in Exercises 49–52.* ***See the Connections box.***

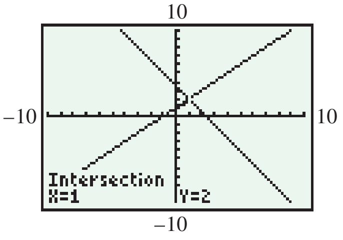
**A.**



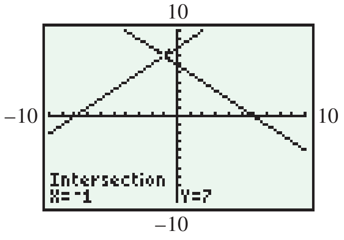
**B.**



**C.**



**D.**



**49.** $x + y = 4$
$x - y = 2$

**50.** $x + y = 6$
$x - y = -8$

**51.** $2x + 3y = 5$
$x - y = 0$

**52.** $3x + 2y = 7$
$-x + y = 1$

## *PREVIEW EXERCISES*

*Solve each equation for y.* ***See Sections 2.5, 3.3, and 7.1.***

**53.** $3x + y = 4$

**54.** $-2x + y = 9$

**55.** $9x - 2y = 4$

**56.** $5x - 3y = 12$

*Solve each equation. Check the solution.* ***See Section 2.3.***

**57.** $-2(x - 2) + 5x = 10$

**58.** $4(3 - 2k) + 3k = 12$

**59.** $4x - 2(1 - 3x) = 6$

**60.** $t + 3(2t - 4) = -13$

# 8.2

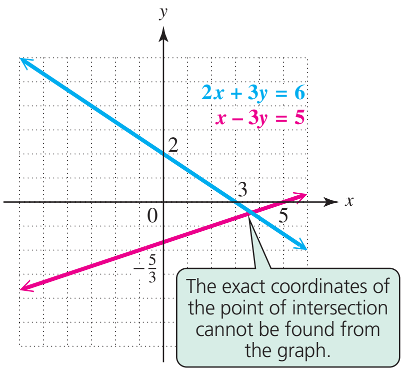
# Solving Systems of Linear Equations by Substitution

**OBJECTIVES**

1. Solve linear systems by substitution.
2. Solve special systems by substitution.
3. Solve linear systems with fractions and decimals by substitution.

**OBJECTIVE 1 Solve linear systems by substitution.** Graphing to solve a system of equations has a serious drawback. For example, consider the system graphed in **FIGURE 7**. It is difficult to determine an accurate solution of the system from the graph.

As a result, there are algebraic methods for solving systems of equations. The **substitution method,** which gets its name from the fact that an expression in one variable is *substituted* for the other variable, is one such method.



The exact coordinates of the point of intersection cannot be found from the graph.

**FIGURE 7**

### EXAMPLE 1 Using the Substitution Method

Solve the system by the substitution method.

$$
\begin{aligned}
3x + 5y &= 26 && (1)\\
y &= 2x && (2)
\end{aligned}
$$

We number the equations for reference in our discussion.

Equation (2), $y = 2x$, is already solved for $y$, so we substitute $2x$ for $y$ in equation (1).

$$
\begin{aligned}
3x + 5y &= 26 && (1)\\
3x + 5(2x) &= 26 && \text{Let } y = 2x.\\
3x + 10x &= 26 && \text{Multiply.}\\
13x &= 26 && \text{Combine like terms.}\\
x &= 2 && \text{Divide by 13.}
\end{aligned}
$$

Don't stop here.

Now we can find the value of $y$ by substituting 2 for $x$ in either equation. We choose equation (2).

$$
\begin{aligned}
y &= 2x && (2)\\
y &= 2(2) && \text{Let } x = 2.\\
y &= 4 && \text{Multiply.}
\end{aligned}
$$

**NOW TRY EXERCISE 1**

Solve the system by the substitution method.

$$\begin{aligned} 2x - 4y &= 28 \\ y &= -3x \end{aligned}$$

We check the solution $(2, 4)$ by substituting 2 for $x$ and 4 for $y$ in *both* equations.

CHECK

$$\begin{array}{rll|rll} 3x + 5y &= 26 & \text{(1)} & y &= 2x & \text{(2)} \\ 3(2) + 5(4) &\stackrel{?}{=} 26 & \text{Substitute.} & 4 &\stackrel{?}{=} 2(2) & \text{Substitute.} \\ 6 + 20 &\stackrel{?}{=} 26 & \text{Multiply.} & 4 &= 4 \ \checkmark & \text{True} \\ 26 &= 26 \ \checkmark & \text{True} & & & \end{array}$$

Since $(2, 4)$ satisfies both equations, the solution set is $\{(2, 4)\}$. NOW TRY

---

**CAUTION** ***A system is not completely solved until values for both x and y are found.*** Write the solution set as a set containing an ordered pair.

---

**NOW TRY EXERCISE 2**

Solve the system by the substitution method.

$$\begin{aligned} 4x + 9y &= 1 \\ x &= y - 3 \end{aligned}$$

## EXAMPLE 2 Using the Substitution Method

Solve the system by the substitution method.

$$\begin{aligned} 2x + 5y &= 7 && \text{(1)} \\ x &= -1 - y && \text{(2)} \end{aligned}$$

Equation (2) gives $x$ in terms of $y$. Substitute $-1 - y$ for $x$ in equation (1).

$$\begin{aligned} 2x + 5y &= 7 && \text{(1)} \\ 2(-1 - y) + 5y &= 7 && \text{Let } x = -1 - y. \\ -2 - 2y + 5y &= 7 && \text{Distributive property} \\ -2 + 3y &= 7 && \text{Combine like terms.} \\ 3y &= 9 && \text{Add 2.} \\ y &= 3 && \text{Divide by 3.} \end{aligned}$$

Distribute 2 to *both* −1 and −*y*.

To find $x$, substitute 3 for $y$ in equation (2), $x = -1 - y$, to get

$$x = -1 - 3 = -4.$$

Write the $x$-coordinate first.

Check that the solution set of the given system is $\{(-4, 3)\}$. NOW TRY

---

**CAUTION** Even though we found $y$ first in **Example 2,** ***the x-coordinate is always written first in the ordered-pair solution of a system.***

---

### Solving a Linear System by Substitution

***Step 1*** **Solve one equation for either variable.** If one of the variables has coefficient 1 or −1, choose it, since it usually makes the substitution method easier.

***Step 2*** **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

***Step 3*** **Solve** the equation from Step 2.

***Step 4*** **Substitute** the result from Step 3 into the equation from Step 1 to find the value of the other variable.

***Step 5*** **Check** the solution in both of the original equations. Then write the solution set.

NOW TRY ANSWERS

**1.** $\{(2, -6)\}$ **2.** $\{(-2, 1)\}$

**NOW TRY EXERCISE 3**

Use substitution to solve the system.

$$\begin{aligned} 2y &= x - 2 \\ 4x - 5y &= -4 \end{aligned}$$

## EXAMPLE 3 Using the Substitution Method

Use substitution to solve the system.

$$\begin{aligned} 2x &= 4 - y && \text{(1)} \\ 5x + 3y &= 10 && \text{(2)} \end{aligned}$$

***Step 1*** We must solve one of the equations for either $x$ or $y$. Because the coefficient of $y$ in equation (1) is $-1$, we avoid fractions by solving this equation for $y$.

$$\begin{aligned} 2x &= 4 - y && \text{(1)} \\ y + 2x &= 4 && \text{Add } y. \\ y &= -2x + 4 && \text{Subtract } 2x. \end{aligned}$$

***Step 2*** Now substitute $-2x + 4$ for $y$ in equation (2).

$$\begin{aligned} 5x + 3y &= 10 && \text{(2)} \\ 5x + 3(-2x + 4) &= 10 && \text{Let } y = -2x + 4. \end{aligned}$$

***Step 3*** Solve the equation from Step 2.

$$\begin{aligned} 5x - 6x + 12 &= 10 && \text{Distributive property} \\ -x + 12 &= 10 && \text{Combine like terms.} \\ -x &= -2 && \text{Subtract 12.} \\ x &= 2 && \text{Multiply by } -1. \end{aligned}$$

Distribute 3 to *both* $-2x$ and 4.

***Step 4*** Equation (1) solved for $y$ is $y = -2x + 4$. Since $x = 2$,

$$y = -2(2) + 4 = 0.$$

***Step 5*** Check that $(2, 0)$ is the solution.

CHECK

$$\begin{aligned} 2x &= 4 - y && \text{(1)} \\ 2(2) &\stackrel{?}{=} 4 - 0 && \text{Substitute.} \\ 4 &= 4 \;\checkmark && \text{True} \end{aligned}$$

$$\begin{aligned} 5x + 3y &= 10 && \text{(2)} \\ 5(2) + 3(0) &\stackrel{?}{=} 10 && \text{Substitute.} \\ 10 &= 10 \;\checkmark && \text{True} \end{aligned}$$

Since both results are true, the solution set of the system is $\{(2, 0)\}$. NOW TRY

**OBJECTIVE 2** **Solve special systems by substitution.** Recall from **Section 8.1** that systems of equations with graphs that are parallel lines have no solution. Systems of equations with graphs that are the same line have an infinite number of solutions.

## EXAMPLE 4 Solving an Inconsistent System by Substitution

Use substitution to solve the system.

$$\begin{aligned} x &= 5 - 2y && \text{(1)} \\ 2x + 4y &= 6 && \text{(2)} \end{aligned}$$

Equation (1) is already solved for $x$, so substitute $5 - 2y$ for $x$ in equation (2).

$$\begin{aligned} 2x + 4y &= 6 && \text{(2)} \\ 2(5 - 2y) + 4y &= 6 && \text{Let } x = 5 - 2y \text{ from equation (1).} \\ 10 - 4y + 4y &= 6 && \text{Distributive property} \\ 10 &= 6 && \text{False} \end{aligned}$$

**NOW TRY ANSWER**

**3.** $\{(-6, -4)\}$

NOW TRY EXERCISE 4

Use substitution to solve the system.

$$\begin{aligned} 8x - 2y &= 1 \\ y &= 4x - 8 \end{aligned}$$

The false result $10 = 6$ means that the equations in the system have graphs that are parallel lines. The system is inconsistent and has no solution, so the solution set is $\emptyset$. See **FIGURE 8**.

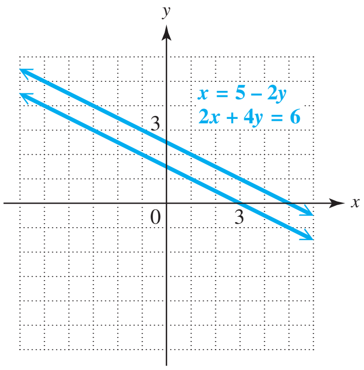


**FIGURE 8**

NOW TRY

**CAUTION** It is a common error to give "false" as the solution of an inconsistent system. The correct response is $\emptyset$.

NOW TRY EXERCISE 5

Solve the system by the substitution method.

$$\begin{aligned} 5x - \phantom{2}y &= 6 \\ -10x + 2y &= -12 \end{aligned}$$

### EXAMPLE 5 Solving a System with Dependent Equations by Substitution

Solve the system by the substitution method.

$$\begin{aligned} 3x - \phantom{3}y &= 4 && (1) \\ -9x + 3y &= -12 && (2) \end{aligned}$$

Begin by solving equation (1) for $y$ to get $y = 3x - 4$. Substitute $3x - 4$ for $y$ in equation (2) and solve the resulting equation.

$$\begin{aligned} -9x + 3y &= -12 && (2) \\ -9x + 3(3x - 4) &= -12 && \text{Let } y = 3x - 4 \text{ from equation (1).} \\ -9x + 9x - 12 &= -12 && \text{Distributive property} \\ 0 &= 0 && \text{Add 12. Combine like terms.} \end{aligned}$$

This true result means that every solution of one equation is also a solution of the other, so the system has an infinite number of solutions. The solution set is

$$\{(x, y) \mid 3x - y = 4\}.$$

A graph of the equations of this system is shown in **FIGURE 9**.

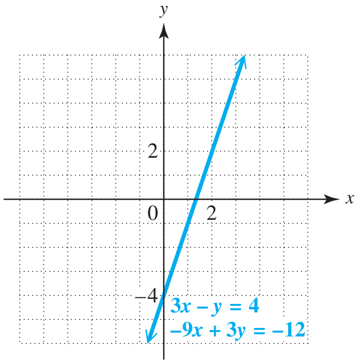


**FIGURE 9**

NOW TRY

**CAUTION** It is a common error to give "true" as the solution of a system of dependent equations. Write the solution set in set-builder notation using the equation in the system (or an equivalent equation) that is in standard form, with integer coefficients that have greatest common factor 1.

NOW TRY ANSWERS

**4.** $\emptyset$ **5.** $\{(x, y) \mid 5x - y = 6\}$

## OBJECTIVE 3 Solve linear systems with fractions and decimals by substitution.

**NOW TRY EXERCISE 6**

Solve the system by the substitution method.

$$x + \frac{1}{2}y = \frac{1}{2}$$
$$\frac{1}{6}x - \frac{1}{3}y = \frac{4}{3}$$

### EXAMPLE 6 Using the Substitution Method with Fractions as Coefficients

Solve the system by the substitution method.

$$3x + \frac{1}{4}y = 2 \qquad (1)$$
$$\frac{1}{2}x + \frac{3}{4}y = -\frac{5}{2} \qquad (2)$$

Clear equation (1) of fractions by multiplying each side by 4.

$$\begin{aligned} 4\left(3x + \frac{1}{4}y\right) &= 4(2) && \text{Multiply by 4.} \\ 4(3x) + 4\left(\frac{1}{4}y\right) &= 4(2) && \text{Distributive property} \\ 12x + y &= 8 && (3) \end{aligned}$$

Now clear equation (2) of fractions by multiplying each side by 4.

$$\begin{aligned} 4\left(\frac{1}{2}x + \frac{3}{4}y\right) &= 4\left(-\frac{5}{2}\right) && \text{Multiply by 4, the common denominator.} \\ 4\left(\frac{1}{2}x\right) + 4\left(\frac{3}{4}y\right) &= 4\left(-\frac{5}{2}\right) && \text{Distributive property} \\ 2x + 3y &= -10 && (4) \end{aligned}$$

The given system of equations has been simplified to an equivalent system.

$$12x + y = 8 \qquad (3)$$
$$2x + 3y = -10 \qquad (4)$$

To solve this system by substitution, solve equation (3) for $y$.

$$\begin{aligned} 12x + y &= 8 && (3) \\ y &= -12x + 8 && \text{Subtract } 12x. \end{aligned}$$

Now substitute this result for $y$ in equation (4).

$$\begin{aligned} 2x + 3y &= -10 && (4) \\ 2x + 3(-12x + 8) &= -10 && \text{Let } y = -12x + 8. \\ 2x - 36x + 24 &= -10 && \text{Distributive property} \\ -34x &= -34 && \text{Combine like terms. Subtract 24.} \\ x &= 1 && \text{Divide by } -34. \end{aligned}$$

Distribute 3 to *both* $-12x$ and 8.

Since equation (3) solved for $y$ is $y = -12x + 8$, substitute 1 for $x$ to get

$$y = -12(1) + 8 = -4.$$

Check by substituting 1 for $x$ and $-4$ for $y$ in both of the original equations. The solution set is $\{(1, -4)\}$.

NOW TRY

**NOW TRY ANSWER**

**6.** $\{(2, -3)\}$

**NOW TRY EXERCISE 7**

Solve the system by the substitution method.

$$\begin{aligned} 0.2x + 0.3y &= 0.5 \\ 0.3x - 0.1y &= 1.3 \end{aligned}$$

### EXAMPLE 7 Using the Substitution Method with Decimals as Coefficients

Solve the system by the substitution method.

$$\begin{aligned} 0.5x + 2.4y &= 4.2 && (1) \\ -0.1x + 1.5y &= 5.1 && (2) \end{aligned}$$

Clear each equation of decimals by multiplying by 10.

$$\begin{aligned} 10(0.5x + 2.4y) &= 10(4.2) && \text{Multiply equation (1) by 10.} \\ 10(0.5x) + 10(2.4y) &= 10(4.2) && \text{Distributive property} \\ 5x + 24y &= 42 && (3) \end{aligned}$$

$$\begin{aligned} 10(-0.1x + 1.5y) &= 10(5.1) && \text{Multiply equation (2) by 10.} \\ 10(-0.1x) + 10(1.5y) &= 10(5.1) && \text{Distributive property} \\ -x + 15y &= 51 && (4) \end{aligned}$$

$10(-0.1x) = -1x = -x$

Now solve the equivalent system of equations by substitution.

$$\begin{aligned} 5x + 24y &= 42 && (3) \\ -x + 15y &= 51 && (4) \end{aligned}$$

Equation (4) can be solved for $x$.

$$x = 15y - 51 \qquad \text{Equation (4) solved for } x$$

Substitute this result for $x$ in equation (3).

$$\begin{aligned} 5x + 24y &= 42 && (3) \\ 5(15y - 51) + 24y &= 42 && \text{Let } x = 15y - 51. \\ 75y - 255 + 24y &= 42 && \text{Distributive property} \\ 99y &= 297 && \text{Combine like terms. Add 255.} \\ y &= 3 && \text{Divide by 99.} \end{aligned}$$

Since equation (4) solved for $x$ is $x = 15y - 51$, substitute 3 for $y$ to get

$$x = 15(3) - 51 = -6.$$

Check $(-6, 3)$ in both of the original equations. The solution set is $\{(-6, 3)\}$.

NOW TRY

**NOW TRY ANSWER**

7. $\{(4, -1)\}$

## 8.2 EXERCISES

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*Complete solution available on the Video Resources on DVD*

1. ***Concept Check*** A student solves the following system and finds that $x = 3$, which is correct. The student gives the solution set as $\{3\}$. ***WHAT WENT WRONG?***

$$\begin{aligned} 5x - y &= 15 \\ 7x + y &= 21 \end{aligned}$$

2. ***Concept Check*** A student solves the following system and obtains the equation $0 = 0$. The student gives the solution set as $\{(0, 0)\}$. ***WHAT WENT WRONG?***

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 8 \end{aligned}$$

*Solve each system by the substitution method. Check each solution.* ***See Examples 1–5.***

**3.** $\begin{aligned} x + y &= 12 \\ y &= 3x \end{aligned}$

**4.** $\begin{aligned} x + 3y &= -28 \\ y &= -5x \end{aligned}$

**5.** $\begin{aligned} 3x + 2y &= 27 \\ x &= y + 4 \end{aligned}$

**6.** $\begin{aligned} 4x + 3y &= -5 \\ x &= y - 3 \end{aligned}$

**7.** $\begin{aligned} 3x + 4 &= -y \\ 2x + y &= 0 \end{aligned}$

**8.** $\begin{aligned} 2x - 5 &= -y \\ x + 3y &= 0 \end{aligned}$

**9.** $\begin{aligned} 7x + 4y &= 13 \\ x + y &= 1 \end{aligned}$

**10.** $\begin{aligned} 3x - 2y &= 19 \\ x + y &= 8 \end{aligned}$

**11.** $\begin{aligned} 3x + 5y &= 25 \\ x - 2y &= -10 \end{aligned}$

**12.** $\begin{aligned} 5x + 2y &= -15 \\ 2x - y &= -6 \end{aligned}$

**13.** $\begin{aligned} 3x - y &= 5 \\ y &= 3x - 5 \end{aligned}$

**14.** $\begin{aligned} 4x - y &= -3 \\ y &= 4x + 3 \end{aligned}$

**15.** $\begin{aligned} 2x + y &= 0 \\ 4x - 2y &= 2 \end{aligned}$

**16.** $\begin{aligned} x + y &= 0 \\ 4x + 2y &= 3 \end{aligned}$

**17.** $\begin{aligned} 2x + 8y &= 3 \\ x &= 8 - 4y \end{aligned}$

**18.** $\begin{aligned} 2x + 10y &= 3 \\ x &= 1 - 5y \end{aligned}$

**19.** $\begin{aligned} 2y &= 4x + 24 \\ 2x - y &= -12 \end{aligned}$

**20.** $\begin{aligned} 2y &= 14 - 6x \\ 3x + y &= 7 \end{aligned}$

*Solve each system by the substitution method. Check each solution.* ***See Examples 6 and 7.***

**21.** $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 3 \\ y &= 3x \end{aligned}$

**22.** $\begin{aligned} \frac{1}{4}x - \frac{1}{5}y &= 9 \\ y &= 5x \end{aligned}$

**23.** $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= -\frac{1}{3} \\ \frac{1}{2}x + 2y &= -7 \end{aligned}$

**24.** $\begin{aligned} \frac{1}{6}x + \frac{1}{6}y &= 1 \\ -\frac{1}{2}x - \frac{1}{3}y &= -5 \end{aligned}$

**25.** $\begin{aligned} \frac{x}{5} + 2y &= \frac{8}{5} \\ \frac{3x}{5} + \frac{y}{2} &= -\frac{7}{10} \end{aligned}$

**26.** $\begin{aligned} \frac{x}{2} + \frac{y}{3} &= \frac{7}{6} \\ \frac{x}{4} - \frac{3y}{2} &= \frac{9}{4} \end{aligned}$

**27.** $\begin{aligned} \frac{1}{6}x + \frac{1}{3}y &= 8 \\ \frac{1}{4}x + \frac{1}{2}y &= 12 \end{aligned}$

**28.** $\begin{aligned} \frac{1}{2}x - \frac{1}{8}y &= -\frac{1}{4} \\ \frac{1}{3}x - \frac{1}{12}y &= -\frac{1}{6} \end{aligned}$

**29.** $\begin{aligned} 0.2x - 1.3y &= -3.2 \\ -0.1x + 2.7y &= 9.8 \end{aligned}$

**30.** $\begin{aligned} 0.1x + 0.9y &= -2 \\ 0.5x - 0.2y &= 4.1 \end{aligned}$

**31.** $\begin{aligned} 0.3x - 0.1y &= 2.1 \\ 0.6x + 0.3y &= -0.3 \end{aligned}$

**32.** $\begin{aligned} 0.8x - 0.1y &= 1.3 \\ 2.2x + 1.5y &= 8.9 \end{aligned}$

## RELATING CONCEPTS EXERCISES 33–36

### FOR INDIVIDUAL OR GROUP WORK

*A system of linear equations can be used to model the cost and the revenue of a business.* ***Work Exercises 33–36 in order.***

**33.** Suppose that you start a business manufacturing and selling bicycles, and it costs you \$5000 to get started. Each bicycle will cost \$400 to manufacture. Explain why the linear equation

$$y_1 = 400x + 5000 \quad (y_1 \text{ in dollars})$$

gives your *total* cost of manufacturing $x$ bicycles.

**34.** You decide to sell each bike for \$600. Write an equation using $y_2$ (in dollars) to express your revenue when you sell $x$ bikes.

**35.** Form a system from the two equations in **Exercises 33 and 34.** Solve the system.

**36.** The value of $x$ from **Exercise 35** is the number of bikes it takes to *break even.* Fill in the blanks: When ______ bikes are sold, the break-even point is reached. At that point, you have spent ______ dollars and taken in ______ dollars.

*Solve each system by substitution. Then graph both lines in the standard viewing window of a graphing calculator, and use the intersection feature to support your answer. See the Connections box in* ***Section 8.1.*** *(In Exercises 41 and 42, solve each equation for y before graphing.)*

**37.** $\begin{aligned} y &= 6 - x \\ y &= 2x \end{aligned}$

**38.** $\begin{aligned} y &= 4x - 4 \\ y &= -3x - 11 \end{aligned}$

**39.** $\begin{aligned} y &= -\frac{4}{3}x + \frac{19}{3} \\ y &= \frac{15}{2}x - \frac{5}{2} \end{aligned}$

**40.** $\begin{aligned} y &= -\frac{15}{2}x + 10 \\ y &= \frac{25}{3}x - \frac{65}{3} \end{aligned}$

**41.** $\begin{aligned} 4x + 5y &= 5 \\ 2x + 3y &= 1 \end{aligned}$

**42.** $\begin{aligned} 6x + 5y &= 13 \\ 3x + 3y &= 4 \end{aligned}$

## PREVIEW EXERCISES

*Simplify.* ***See Section 1.8.***

**43.** $(14x - 3y) + (2x + 3y)$

**44.** $(-6x + 8y) + (6x + 2y)$

**45.** $(-x + 7y) + (3y + x)$

**46.** $(3x - 4y) + (4y - 3x)$

**47.** What must be added to $-4x$ to get a sum of 0?

**48.** What must be added to $6y$ to get a sum of 0?

**49.** What must $4y$ be multiplied by so that when the product is added to $8y$, the sum is 0?

**50.** What must $-3x$ be multiplied by so that when the product is added to $-12x$, the result is 0?

# 8.3 Solving Systems of Linear Equations by Elimination

**OBJECTIVES**

1. Solve linear systems by elimination.
2. Multiply when using the elimination method.
3. Use an alternative method to find the second value in a solution.
4. Solve special systems by elimination.

**OBJECTIVE 1** **Solve linear systems by elimination.** Recall that adding the same quantity to each side of an equation results in equal sums.

$$\textbf{If} \quad A = B, \quad \textbf{then} \quad A + C = B + C.$$

We can take this addition a step further. Adding *equal* quantities, rather than the *same* quantity, to each side of an equation also results in equal sums.

$$\textbf{If} \quad A = B \quad \textbf{and} \quad C = D, \quad \textbf{then} \quad A + C = B + D.$$

Using the addition property of equality to solve systems is called the **elimination method.**

### EXAMPLE 1 Using the Elimination Method

Use the elimination method to solve the system.

$$\begin{aligned} x + y &= 5 \quad &(1) \\ x - y &= 3 \quad &(2) \end{aligned}$$

Each equation in this system is a statement of equality, so the sum of the right sides equals the sum of the left sides. Adding vertically in this way gives the following.

$$\begin{array}{rcll} x + y & = & 5 & \text{(1)} \\ x - y & = & 3 & \text{(2)} \\ \hline 2x \phantom{{}+y} & = & 8 & \text{Add left sides and add right sides.} \\ x & = & 4 & \text{Divide by 2.} \end{array}$$

**NOW TRY EXERCISE 1**

Use the elimination method to solve the system.

$$\begin{aligned} x - y &= 4 \\ 3x + y &= 8 \end{aligned}$$

Notice that $y$ has been eliminated. The result, $x = 4$, gives the $x$-value of the solution of the given system. To find the $y$-value of the solution, substitute 4 for $x$ in either of the two equations of the system. We choose equation (1).

$$\begin{aligned} x + y &= 5 && \text{(1)} \\ 4 + y &= 5 && \text{Let } x = 4. \\ y &= 1 && \text{Subtract 4.} \end{aligned}$$

Check the solution, $(4, 1)$, in both equations of the given system.

CHECK

$$\begin{array}{rl|rl} x + y = 5 & \text{(1)} & x - y = 3 & \text{(2)} \\ 4 + 1 \overset{?}{=} 5 & \text{Substitute.} & 4 - 1 \overset{?}{=} 3 & \text{Substitute.} \\ 5 = 5 \ \checkmark & \text{True} & 3 = 3 \ \checkmark & \text{True} \end{array}$$

Since both results are true, the solution set of the system is $\{(4, 1)\}$. NOW TRY

With the elimination method, the idea is to *eliminate* one of the variables. ***To do this, one pair of variable terms in the two equations must have coefficients that are opposites (additive inverses).***

### Solving a Linear System by Elimination

***Step 1*** **Write both equations in standard form,** $Ax + By = C$.

***Step 2*** **Transform the equations as needed so that the coefficients of one pair of variable terms are opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the $x$- or $y$-terms is 0.

***Step 3*** **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

***Step 4*** **Solve** the equation from Step 3 for the remaining variable.

***Step 5*** **Substitute** the result from Step 4 into either of the original equations, and solve for the other variable.

***Step 6*** **Check** the solution in both of the original equations. Then write the solution set.

***It does not matter which variable is eliminated first. Usually, we choose the one that is more convenient to work with.***

## EXAMPLE 2 Using the Elimination Method

Solve the system.

$$\begin{aligned} y + 11 &= 2x && \text{(1)} \\ 5x &= y + 26 && \text{(2)} \end{aligned}$$

***Step 1*** Write both equations in standard form, $Ax + By = C$.

$$\begin{aligned} -2x + y &= -11 && \text{Subtract } 2x \text{ and 11 in equation (1).} \\ 5x - y &= 26 && \text{Subtract } y \text{ in equation (2).} \end{aligned}$$

***Step 2*** Because the coefficients of $y$ are 1 and $-1$, adding will eliminate $y$. It is not necessary to multiply either equation by a number.

**NOW TRY ANSWER**

**1.** $\{(3, -1)\}$

**NOW TRY EXERCISE 2**

Solve the system.

$$\begin{aligned} 2x - 6 &= -3y \\ 5x - 3y &= -27 \end{aligned}$$

***Step 3*** Add the two equations.

$$\begin{array}{rcll} -2x + y &=& -11 & \\ 5x - y &=& 26 & \\ \hline 3x \qquad &=& 15 & \text{Add in columns.} \end{array}$$

***Step 4*** Solve. $\quad x = 5 \quad$ Divide by 3.

***Step 5*** Find the value of $y$ by substituting 5 for $x$ in either of the original equations.

$$\begin{aligned} y + 11 &= 2x && (1) \\ y + 11 &= 2(5) && \text{Let } x = 5. \\ y + 11 &= 10 && \text{Multiply.} \\ y &= -1 && \text{Subtract 11.} \end{aligned}$$

***Step 6*** Check by substituting $x = 5$ and $y = -1$ into both of the original equations.

CHECK

$$\begin{aligned} y + 11 &= 2x && (1) \\ (-1) + 11 &\overset{?}{=} 2(5) && \text{Substitute.} \\ 10 &= 10 \; ✓ && \text{True} \end{aligned}$$

$$\begin{aligned} 5x &= y + 26 && (2) \\ 5(5) &= -1 + 26 && \text{Substitute.} \\ 25 &= 25 \; ✓ && \text{True} \end{aligned}$$

Since $(5, -1)$ is a solution of *both* equations, the solution set is $\{(5, -1)\}$.

NOW TRY

**OBJECTIVE 2** **Multiply when using the elimination method.** Sometimes we need to multiply each side of one or both equations in a system by some number before adding will eliminate a variable.

**NOW TRY EXERCISE 3**

Solve the system.

$$\begin{aligned} 3x - 5y &= 25 \\ 2x + 8y &= -6 \end{aligned}$$

### EXAMPLE 3 Using the Elimination Method

Solve the system.

$$\begin{aligned} 2x + 3y &= -15 && (1) \\ 5x + 2y &= 1 && (2) \end{aligned}$$

Adding the two equations gives $7x + 5y = -14$, which does not eliminate either variable. However, we can multiply each equation by a suitable number so that the coefficients of one of the two variables are opposites. For example, to eliminate $x$, we multiply each side of $2x + 3y = -15$ (equation (1)) by 5 and each side of $5x + 2y = 1$ (equation (2)) by $-2$.

$$\begin{array}{rcll} 10x + 15y &=& -75 & \text{Multiply equation (1) by 5.} \\ -10x - \;\; 4y &=& -2 & \text{Multiply equation (2) by } -2. \\ \hline 11y &=& -77 & \text{Add.} \\ y &=& -7 & \text{Divide by 11.} \end{array}$$

The coefficients of $x$ are opposites.

Find the value of $x$ by substituting $-7$ for $y$ in either equation (1) or (2).

$$\begin{aligned} 5x + 2y &= 1 && (2) \\ 5x + 2(-7) &= 1 && \text{Let } y = -7. \\ 5x - 14 &= 1 && \text{Multiply.} \\ 5x &= 15 && \text{Add 14.} \\ x &= 3 && \text{Divide by 5.} \end{aligned}$$

Check that the solution set of the system is $\{(3, -7)\}$.

NOW TRY

**NOW TRY ANSWERS**

**2.** $\{(-3, 4)\}$ **3.** $\{(5, -2)\}$

**NOTE** In **Example 3,** we eliminated the variable $x$. Alternatively, we could multiply each equation of the system by a suitable number so that the variable $y$ is eliminated.

$$\begin{aligned} 2x + 3y &= -15 \quad (1) \xrightarrow{\text{Multiply by 2.}} & 4x + 6y &= -30 \\ 5x + 2y &= 1 \quad (2) \xrightarrow{\text{Multiply by } -3.} & -15x - 6y &= -3 \end{aligned}$$

Complete this approach and confirm that the same solution results.

**CAUTION** When using the elimination method, remember to *multiply both sides* of an equation by the same nonzero number.

**OBJECTIVE 3** **Use an alternative method to find the second value in a solution.** Sometimes it is easier to find the value of the second variable in a solution by using the elimination method twice.

**NOW TRY EXERCISE 4**

Solve the system.

$$\begin{aligned} 4x + 9y &= 3 \\ 5y &= 6 - 3x \end{aligned}$$

### EXAMPLE 4 Finding the Second Value by Using an Alternative Method

Solve the system.

$$\begin{aligned} 4x &= 9 - 3y \quad (1) \\ 5x - 2y &= 8 \quad (2) \end{aligned}$$

Write equation (1) in standard form by adding $3y$ to each side.

$$\begin{aligned} 4x + 3y &= 9 \quad (3) \\ 5x - 2y &= 8 \quad (2) \end{aligned}$$

One way to proceed is to eliminate $y$ by multiplying each side of equation (3) by 2 and each side of equation (2) by 3 and then adding.

$$\begin{aligned} 8x + 6y &= 18 && \text{Multiply equation (3) by 2.} \\ 15x - 6y &= 24 && \text{Multiply equation (2) by 3.} \\ \hline 23x \phantom{{}+6y} &= 42 && \text{Add.} \\ x &= \frac{42}{23} && \text{Divide by 23.} \end{aligned}$$

The coefficients of $y$ are opposites.

Substituting $\frac{42}{23}$ for $x$ in one of the given equations would give $y$, but the arithmetic would be messy. Instead, solve for $y$ by starting again with the original equations written in standard form (equations (3) and (2)) and eliminating $x$.

$$\begin{aligned} 20x + 15y &= 45 && \text{Multiply equation (3) by 5.} \\ -20x + 8y &= -32 && \text{Multiply equation (2) by } -4. \\ \hline 23y &= 13 && \text{Add.} \\ y &= \frac{13}{23} && \text{Divide by 23.} \end{aligned}$$

The coefficients of $x$ are opposites.

Check that the solution set is $\left\{\left(\frac{42}{23}, \frac{13}{23}\right)\right\}$. NOW TRY

**NOTE** When the value of the first variable is a fraction, the method used in **Example 4** helps avoid arithmetic errors. This method could be used to solve any system.

**NOW TRY ANSWER**

**4.** $\left\{\left(\frac{39}{7}, -\frac{15}{7}\right)\right\}$

**NOW TRY EXERCISE 5**

Solve each system by the elimination method.

**(a)** $$\begin{aligned} x - y &= 2 \\ 5x - 5y &= 10 \end{aligned}$$

**(b)** $$\begin{aligned} 4x + 3y &= 0 \\ -4x - 3y &= -1 \end{aligned}$$

## OBJECTIVE 4 Solve special systems by elimination.

### EXAMPLE 5 Solving Special Systems Using the Elimination Method

Solve each system by the elimination method.

**(a)** $$\begin{aligned} 2x + 4y &= 5 && (1) \\ 4x + 8y &= -9 && (2) \end{aligned}$$

Multiply each side of equation (1) by $-2$. Then add the two equations.

$$\begin{aligned} -4x - 8y &= -10 && \text{Multiply equation (1) by } -2. \\ 4x + 8y &= -9 && (2) \\ \hline 0 &= -19 && \text{False} \end{aligned}$$

The false statement $0 = -19$ indicates that the given system has solution set $\emptyset$.

**(b)** $$\begin{aligned} 3x - y &= 4 && (1) \\ -9x + 3y &= -12 && (2) \end{aligned}$$

Multiply each side of equation (1) by 3. Then add the two equations.

$$\begin{aligned} 9x - 3y &= 12 && \text{Multiply equation (1) by 3.} \\ -9x + 3y &= -12 && (2) \\ \hline 0 &= 0 && \text{True} \end{aligned}$$

A true statement occurs when the equations are equivalent. This indicates that every solution of one equation is also a solution of the other. The solution set is

$$\{(x, y) \mid 3x - y = 4\}.$$

NOW TRY

**NOW TRY ANSWERS**

5\. **(a)** $\{(x, y) \mid x - y = 2\}$
**(b)** $\emptyset$

## 8.3 EXERCISES

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*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Answer* true *or* false *for each statement. If false, tell why.*

1\. If the elimination method leads to $0 = -1$, the solution set of the system is $\{(0, -1)\}$.

2\. A system that includes the equation $5x - 4y = 0$ cannot have $(4, -5)$ as a solution.

*Solve each system by the elimination method. Check each solution.* ***See Examples 1 and 2.***

**3.** $x - y = -2$
$x + y = 10$

**4.** $x + y = 10$
$x - y = -6$

**5.** $2x + y = -5$
$x - y = 2$

**6.** $2x + y = -15$
$-x - y = 10$

**7.** $2y = -3x$
$-3x - y = 3$

**8.** $5x = y + 5$
$-5x + 2y = 0$

**9.** $6x - y = -1$
$5y = 17 + 6x$

**10.** $y = 9 - 6x$
$-6x + 3y = 15$

*Solve each system by the elimination method. (Hint: In Exercises 29–34, first clear all fractions or decimals.) Check each solution.* ***See Examples 3–5.****

**11.** $2x - y = 12$
$3x + 2y = -3$

**12.** $x + y = 3$
$-3x + 2y = -19$

**13.** $x + 4y = 16$
$3x + 5y = 20$

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*The authors thank Mitchel Levy of Broward College for his suggestions for this group of exercises.

**14.** $\begin{aligned} 2x + y &= 8 \\ 5x - 2y &= -16 \end{aligned}$

**15.** $\begin{aligned} 2x - 8y &= 0 \\ 4x + 5y &= 0 \end{aligned}$

**16.** $\begin{aligned} 3x - 15y &= 0 \\ 6x + 10y &= 0 \end{aligned}$

**17.** $\begin{aligned} 3x + 3y &= 33 \\ 5x - 2y &= 27 \end{aligned}$

**18.** $\begin{aligned} 4x - 3y &= -19 \\ 3x + 2y &= 24 \end{aligned}$

**19.** $\begin{aligned} 5x + 4y &= 12 \\ 3x + 5y &= 15 \end{aligned}$

**20.** $\begin{aligned} 2x + 3y &= 21 \\ 5x - 2y &= -14 \end{aligned}$

**21.** $\begin{aligned} 5x - 4y &= 15 \\ -3x + 6y &= -9 \end{aligned}$

**22.** $\begin{aligned} 4x + 5y &= -16 \\ 5x - 6y &= -20 \end{aligned}$

**23.** $\begin{aligned} -x + 3y &= 4 \\ -2x + 6y &= 8 \end{aligned}$

**24.** $\begin{aligned} 6x - 2y &= 24 \\ -3x + y &= -12 \end{aligned}$

**25.** $\begin{aligned} 5x - 2y &= 3 \\ 10x - 4y &= 5 \end{aligned}$

**26.** $\begin{aligned} 3x - 5y &= 1 \\ 6x - 10y &= 4 \end{aligned}$

**27.** $\begin{aligned} 6x - 2y &= -22 \\ -3x + 4y &= 17 \end{aligned}$

**28.** $\begin{aligned} 5x - 4y &= -1 \\ x + 8y &= -9 \end{aligned}$

**29.** $\begin{aligned} 3x &= 3 + 2y \\ -\frac{4}{3}x + y &= \frac{1}{3} \end{aligned}$

**30.** $\begin{aligned} 3x &= 27 + 2y \\ x - \frac{7}{2}y &= -25 \end{aligned}$

**31.** $\begin{aligned} \frac{1}{5}x + y &= \frac{6}{5} \\ \frac{1}{10}x + \frac{1}{3}y &= \frac{5}{6} \end{aligned}$

**32.** $\begin{aligned} \frac{1}{3}x + \frac{1}{2}y &= \frac{13}{6} \\ \frac{1}{2}x - \frac{1}{4}y &= -\frac{3}{4} \end{aligned}$

**33.** $\begin{aligned} 2.4x + 1.7y &= 7.6 \\ 1.2x - 0.5y &= 9.2 \end{aligned}$

**34.** $\begin{aligned} 0.5x + 3.4y &= 13 \\ 1.5x - 2.6y &= -25 \end{aligned}$

**35.** $\begin{aligned} x + 3y &= 6 \\ -2x + 12 &= 6y \end{aligned}$

**36.** $\begin{aligned} 7x + 2y &= 0 \\ 4y &= -14x \end{aligned}$

**37.** $\begin{aligned} 4x - 3y &= 1 \\ 8x &= 3 + 6y \end{aligned}$

**38.** $\begin{aligned} 5x + 8y &= 10 \\ 24y &= -15x - 10 \end{aligned}$

**39.** $\begin{aligned} 4x &= 3y - 2 \\ 5x + 3 &= 2y \end{aligned}$

**40.** $\begin{aligned} 2x + 3y &= 0 \\ 4x + 12 &= 9y \end{aligned}$

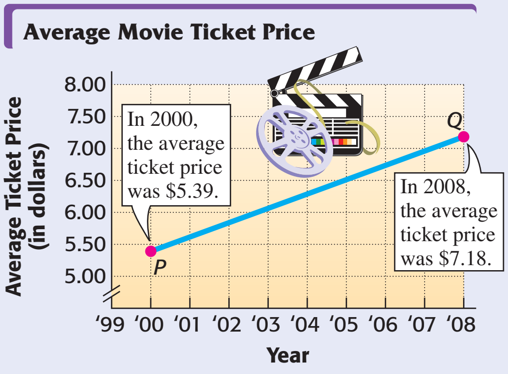
**41.** $\begin{aligned} 24x + 12y &= -7 \\ 16x - 18y &= 17 \end{aligned}$

**42.** $\begin{aligned} 9x + 4y &= -3 \\ 6x + 6y &= -7 \end{aligned}$

## RELATING CONCEPTS EXERCISES 43–48

### FOR INDIVIDUAL OR GROUP WORK

*The graph shows average U.S. movie theater ticket prices from 2000 through 2008. In 2000, the average price was \$5.39, as represented by the point $P(2000, 5.39)$. In 2008, the average price was \$7.18, as represented by the point $Q(2008, 7.18)$.* ***Work Exercises 43–48 in order.***



***Source:*** Motion Picture Association of America.

*(continued)*

**43.** Line segment $PQ$ has an equation that can be written in the form $y = ax + b$. Using the coordinates of point $P$ with $x = 2000$ and $y = 5.39$, write an equation in the variables $a$ and $b$.

**44.** Using the coordinates of point $Q$ with $x = 2008$ and $y = 7.18$, write a second equation in the variables $a$ and $b$.

**45.** Write the system of equations formed from the two equations in **Exercises 43 and 44,** and solve the system by using the elimination method.

**46. (a)** What is the equation of the line on which segment $PQ$ lies?

**(b)** Let $x = 2007$ in the equation from part (a), and solve for $y$ (to two decimal places). How does the result compare with the actual figure of $6.88?

## *PREVIEW EXERCISES*

*Multiply both sides of each equation by the given number.* ***See Section 2.3.***

**47.** $2x - 3y + z = 5$ by 4

**48.** $-3x + 8y - z = 0$ by $-3$

*Solve for z if* $x = 1$ *and* $y = -2$. ***See Section 2.3.***

**49.** $x + 2y + 3z = 9$

**50.** $-3x - y + z = 1$

*By what number must the first equation be multiplied so that x is eliminated when the two equations are added?*

**51.**
$$\begin{aligned} x + 2y - z &= 0 \\ 3x - 4y + 2z &= 6 \end{aligned}$$

**52.**
$$\begin{aligned} x - 2y + 5z &= -7 \\ -2x - 3y + 4z &= -14 \end{aligned}$$

## *SUMMARY EXERCISES* on Solving Systems of Linear Equations

### Guidelines for Choosing a Method to Solve a System of Linear Equations

**1.** If one of the equations of the system is already solved for one of the variables, as in the following systems, the substitution method is the better choice.

$$\begin{aligned} 3x + 4y &= 9 \\ y &= 2x - 6 \end{aligned} \quad \text{and} \quad \begin{aligned} -5x + 3y &= 9 \\ x &= 3y - 7 \end{aligned}$$

**2.** If both equations are in standard $Ax + By = C$ form and none of the variables has coefficient $-1$ or 1, as in the following system, the elimination method is the better choice.

$$\begin{aligned} 4x - 11y &= 3 \\ -2x + 3y &= 4 \end{aligned}$$

**3.** If one or both of the equations are in standard form and the coefficient of one of the variables is $-1$ or 1, as in the following systems, either method is appropriate.

$$\begin{aligned} 3x + y &= -2 \\ -5x + 2y &= 4 \end{aligned} \quad \text{and} \quad \begin{aligned} -x + 3y &= -4 \\ 3x - 2y &= 8 \end{aligned}$$

**Concept Check** *Use the preceding guidelines to solve each problem.*

**1.** To minimize the amount of work required, tell whether you would use the substitution or elimination method to solve each system, and why. *Do not actually solve.*

**(a)** $\begin{aligned} 3x + 5y &= 69 \\ y &= 4x \end{aligned}$ **(b)** $\begin{aligned} 3x + y &= -7 \\ x - y &= -5 \end{aligned}$ **(c)** $\begin{aligned} 3x - 2y &= 0 \\ 9x + 8y &= 7 \end{aligned}$

**2.** Which system would be easier to solve with the substitution method? Why?

*System A:* $\begin{aligned} 5x - 3y &= 7 \\ 2x + 8y &= 3 \end{aligned}$ *System B:* $\begin{aligned} 7x + 2y &= 4 \\ y &= -3x + 1 \end{aligned}$

*In Exercises 3 and 4,* ***(a)*** *solve the system by the elimination method,* ***(b)*** *solve the system by the substitution method, and* ***(c)*** *tell which method you prefer for that particular system and why.*

**3.** $\begin{aligned} 4x - 3y &= -8 \\ x + 3y &= 13 \end{aligned}$

**4.** $\begin{aligned} 2x + 5y &= 0 \\ x &= -3y + 1 \end{aligned}$

*Solve each system by the method of your choice. (For Exercises 5–7, see your answers to* ***Exercise 1.****)*

**5.** $\begin{aligned} 3x + 5y &= 69 \\ y &= 4x \end{aligned}$

**6.** $\begin{aligned} 3x + y &= -7 \\ x - y &= -5 \end{aligned}$

**7.** $\begin{aligned} 3x - 2y &= 0 \\ 9x + 8y &= 7 \end{aligned}$

**8.** $\begin{aligned} x + y &= 7 \\ x &= -3 - y \end{aligned}$

**9.** $\begin{aligned} 6x + 7y &= 4 \\ 5x + 8y &= -1 \end{aligned}$

**10.** $\begin{aligned} 6x - y &= 5 \\ y &= 11x \end{aligned}$

**11.** $\begin{aligned} 4x - 6y &= 10 \\ -10x + 15y &= -25 \end{aligned}$

**12.** $\begin{aligned} 3x - 5y &= 7 \\ 2x + 3y &= 30 \end{aligned}$

**13.** $\begin{aligned} 5x &= 7 + 2y \\ 5y &= 5 - 3x \end{aligned}$

**14.** $\begin{aligned} 4x + 3y &= 1 \\ 3x + 2y &= 2 \end{aligned}$

**15.** $\begin{aligned} 2x - 3y &= 7 \\ -4x + 6y &= 14 \end{aligned}$

**16.** $\begin{aligned} 2x + 3y &= 10 \\ -3x + y &= 18 \end{aligned}$

**17.** $\begin{aligned} 2x + 5y &= 4 \\ x + y &= -1 \end{aligned}$

**18.** $\begin{aligned} x - 3y &= 7 \\ 4x + y &= 5 \end{aligned}$

**19.** $\begin{aligned} 7x - 4y &= 0 \\ 3x &= 2y \end{aligned}$

*Solve each system by any method. First clear all fractions or decimals.*

**20.** $\begin{aligned} \frac{1}{5}x + \frac{2}{3}y &= -\frac{8}{5} \\ 3x - y &= 9 \end{aligned}$

**21.** $\begin{aligned} \frac{1}{6}x + \frac{1}{6}y &= 2 \\ -\frac{1}{2}x - \frac{1}{3}y &= -8 \end{aligned}$

**22.** $\begin{aligned} \frac{x}{3} - \frac{3y}{4} &= -\frac{1}{2} \\ \frac{x}{6} + \frac{y}{8} &= \frac{3}{4} \end{aligned}$

**23.** $\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 9 \\ \frac{x}{5} - \frac{y}{4} &= 5 \end{aligned}$

**24.** $\begin{aligned} 0.1x + y &= 1.6 \\ 0.6x + 0.5y &= -1.4 \end{aligned}$

**25.** $\begin{aligned} 0.2x - 0.3y &= 0.1 \\ 0.3x - 0.2y &= 0.9 \end{aligned}$

# 8.4 Solving Systems of Linear Equations in Three Variables

**OBJECTIVES**

1. Understand the geometry of systems of three equations in three variables.
2. Solve linear systems (with three equations and three variables) by elimination.
3. Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.
4. Solve special systems.

A solution of an equation in three variables, such as

$$2x + 3y - z = 4,$$

Linear equation in three variables

is called an **ordered triple** and is written $(x, y, z)$. For example, the ordered triple $(0, 1, -1)$ is a solution of the preceding equation, because

$$2(0) + 3(1) - (-1) = 4$$

is a true statement. Verify that another solution of this equation is $(10, -3, 7)$.

We now extend the term *linear equation* to equations of the form

$$Ax + By + Cz + \cdots + Dw = K,$$

where not all the coefficients $A, B, C, \ldots, D$ equal 0. For example,

$$2x + 3y - 5z = 7 \quad \text{and} \quad x - 2y - z + 3w = 8$$

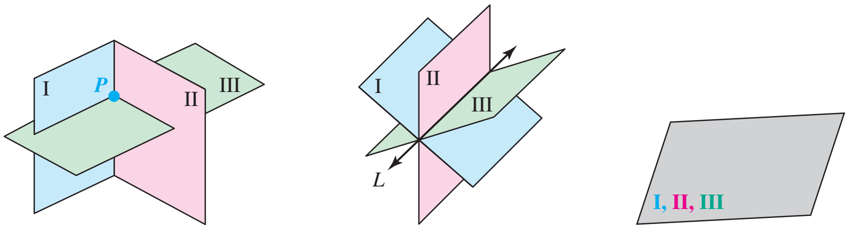
are linear equations, the first with three variables and the second with four.

**OBJECTIVE 1 Understand the geometry of systems of three equations in three variables.** Consider the solution of a system such as the following.

$$\begin{aligned} 4x + 8y + \phantom{3}z &= 2 \\ x + 7y - 3z &= -14 \\ 2x - 3y + 2z &= 3 \end{aligned}$$

System of linear equations in three variables

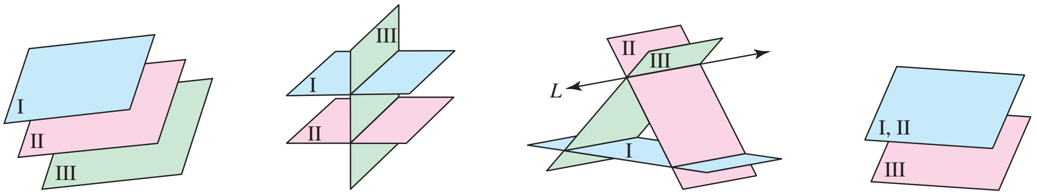
Theoretically, a system of this type can be solved by graphing. However, the graph of a linear equation with three variables is a *plane,* not a line. Since visualizing a plane requires three-dimensional graphing, the method of graphing is not practical with these systems. However, it does illustrate the number of solutions possible for such systems, as shown in **FIGURE 10**.



A single solution
**(a)**

Points of a line in common
**(b)**

All points in common
**(c)**



No points in common
**(d)**

No points in common
**(e)**

No points in common
**(f)**

No points in common
**(g)**

**FIGURE 10**

**FIGURE 10** illustrates the following cases.

### Graphs of Linear Systems in Three Variables

*Case 1* **The three planes may meet at a single, common point** that is the solution of the system. See **FIGURE 10(a)**.

*Case 2* **The three planes may have the points of a line in common,** so that the infinite set of points that satisfy the equation of the line is the solution of the system. See **FIGURE 10(b)**.

*Case 3* **The three planes may coincide,** so that the solution of the system is the set of all points on a plane. See **FIGURE 10(c)**.

*Case 4* **The planes may have no points common to all three,** so that there is no solution of the system. See **FIGURES 10(d)–(g)**.

**OBJECTIVE 2** **Solve linear systems (with three equations and three variables) by elimination.** Since graphing to find the solution set of a system of three equations in three variables is impractical, these systems are solved with an extension of the elimination method from **Section 8.3.**

In the steps that follow, we use the term **focus variable** to identify the first variable to be eliminated in the process. The focus variable will always be present in the **working equation,** which will be used twice to eliminate this variable.

### Solving a Linear System in Three Variables*

*Step 1* **Select a variable and an equation.** A good choice for the variable, which we call the *focus variable,* is one that has coefficient 1 or $-1$. Then select an equation, one that contains the focus variable, as the *working equation*.

*Step 2* **Eliminate the focus variable.** Use the working equation and one of the other two equations of the original system. The result is an equation in two variables.

*Step 3* **Eliminate the focus variable again.** Use the working equation and the remaining equation of the original system. The result is another equation in two variables.

*Step 4* **Write the equations in two variables that result from Steps 2 and 3 as a system, and solve it.** Doing this gives the values of two of the variables.

*Step 5* **Find the value of the remaining variable.** Substitute the values of the two variables found in Step 4 into the working equation to obtain the value of the focus variable.

*Step 6* **Check** the ordered-triple solution in *each* of the *original* equations of the system. Then write the solution set.

---

*The authors wish to thank Christine Heinecke Lehmann of Purdue University North Central for her suggestions here.

## EXAMPLE 1 Solving a System in Three Variables

Solve the system.

$$\begin{array}{rcll} 4x + 8y + z &=& 2 & \text{(1)} \\ x + 7y - 3z &=& -14 & \text{(2)} \\ 2x - 3y + 2z &=& 3 & \text{(3)} \end{array}$$

***Step 1*** Since $z$ in equation (1) has coefficient 1, we choose $z$ as the focus variable and (1) as the working equation. (Another option would be to choose $x$ as the focus variable, since it also has coefficient 1, and use (2) as the working equation.)

$$4x + 8y + \overset{\text{Focus variable}}{\overset{\downarrow}{z}} = 2 \qquad \text{(1)} \longleftarrow \text{Working equation}$$

***Step 2*** Multiply working equation (1) by 3 and add the result to equation (2) to eliminate focus variable $z$.

$$\begin{array}{rcrl} 12x + 24y + 3z &=& 6 & \text{Multiply each side of (1) by 3.} \\ x + 7y - 3z &=& -14 & \text{(2)} \\ \hline 13x + 31y \phantom{{}+3z} &=& -8 & \text{Add.} \quad \text{(4)} \end{array}$$

***Step 3*** Multiply working equation (1) by $-2$ and add the result to remaining equation (3) to again eliminate focus variable $z$.

$$\begin{array}{rcrl} -8x - 16y - 2z &=& -4 & \text{Multiply each side of (1) by } -2. \\ 2x - 3y + 2z &=& 3 & \text{(3)} \\ \hline -6x - 19y \phantom{{}+2z} &=& -1 & \text{Add.} \quad \text{(5)} \end{array}$$

***Step 4*** Write the equations in two variables that result in Steps 2 and 3 as a system.

Make sure these equations have the same variables.

$$\begin{array}{rcrll} 13x + 31y &=& -8 & \text{(4)} & \text{The result from Step 2} \\ -6x - 19y &=& -1 & \text{(5)} & \text{The result from Step 3} \end{array}$$

Now solve this system. We choose to eliminate $x$.

$$\begin{array}{rcrl} 78x + 186y &=& -48 & \text{Multiply each side of (4) by 6.} \\ -78x - 247y &=& -13 & \text{Multiply each side of (5) by 13.} \\ \hline -61y &=& -61 & \text{Add.} \\ y &=& 1 & \text{Divide by } -61. \end{array}$$

Substitute 1 for $y$ in either equation (4) or (5) to find $x$.

$$\begin{array}{rcrl} -6x - 19y &=& -1 & \text{(5)} \\ -6x - 19(1) &=& -1 & \text{Let } y = 1. \\ -6x - 19 &=& -1 & \text{Multiply.} \\ -6x &=& 18 & \text{Add 19.} \\ x &=& -3 & \text{Divide by } -6. \end{array}$$

***Step 5*** Now substitute the two values we found in Step 4 in working equation (1) to find the value of the remaining variable, focus variable $z$.

$$\begin{array}{rcrl} 4x + 8y + z &=& 2 & \text{(1)} \\ 4(-3) + 8(1) + z &=& 2 & \text{Let } x = -3 \text{ and } y = 1. \\ -4 + z &=& 2 & \text{Multiply, and then add.} \\ z &=& 6 & \text{Add 4.} \end{array}$$

**NOW TRY EXERCISE 1**

Solve the system.

$$\begin{aligned} x - y + 2z &= 1 \\ 3x + 2y + 7z &= 8 \\ -3x - 4y + 9z &= -10 \end{aligned}$$

Write the values of $x$, $y$, and $z$ in the correct order.

***Step 6*** It appears that the ordered triple $(-3, 1, 6)$ is the only solution of the system. We must check that the solution satisfies all three original equations of the system. We begin with equation (1).

**CHECK**

$$\begin{aligned} 4x + 8y + z &= 2 && (1) \\ 4(-3) + 8(1) + 6 &\stackrel{?}{=} 2 && \text{Substitute.} \\ -12 + 8 + 6 &\stackrel{?}{=} 2 && \text{Multiply.} \\ 2 &= 2 \checkmark && \text{True} \end{aligned}$$

In **Exercise 2** you are asked to show that $(-3, 1, 6)$ also satisfies equations (2) and (3). The solution set is $\{(-3, 1, 6)\}$. NOW TRY

**OBJECTIVE 3** **Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.** If a linear system has an equation missing a term or terms, one elimination step can be omitted.

### EXAMPLE 2 Solving a System of Equations with Missing Terms

Solve the system.

$$\begin{aligned} 6x - 12y &= -5 && (1) \quad \text{Missing } z \\ 8y + z &= 0 && (2) \quad \text{Missing } x \\ 9x - z &= 12 && (3) \quad \text{Missing } y \end{aligned}$$

Since equation (3) is missing the variable $y$, one way to begin is to eliminate $y$ again, using equations (1) and (2).

$$\begin{aligned} 12x - 24y \phantom{{}+3z} &= -10 && \text{Multiply each side of (1) by 2.} \\ 24y + 3z &= 0 && \text{Multiply each side of (2) by 3.} \\ \hline 12x \phantom{{}-24y} + 3z &= -10 && \text{Add.} \quad (4) \end{aligned}$$

Leave space for the missing term.

Use the resulting equation (4) in $x$ and $z$, together with equation (3), $9x - z = 12$, to eliminate $z$. Multiply equation (3) by 3.

$$\begin{aligned} 27x - 3z &= 36 && \text{Multiply each side of (3) by 3.} \\ 12x + 3z &= -10 && (4) \\ \hline 39x &= 26 && \text{Add.} \\ x &= \frac{26}{39}, \text{ or } \frac{2}{3} && \text{Divide by 39; lowest terms} \end{aligned}$$

We can find $z$ by substituting this value for $x$ in equation (3).

$$\begin{aligned} 9x - z &= 12 && (3) \\ 9\left(\frac{2}{3}\right) - z &= 12 && \text{Let } x = \tfrac{2}{3}. \\ 6 - z &= 12 && \text{Multiply.} \\ z &= -6 && \text{Subtract 6. Multiply by } -1. \end{aligned}$$

**NOW TRY ANSWER**

**1.** $\{(2, 1, 0)\}$

**NOW TRY EXERCISE 2**

Solve the system.

$$\begin{aligned} 3x - z &= -10 \\ 4y + 5z &= 24 \\ x - 6y &= -8 \end{aligned}$$

We can find $y$ by substituting $-6$ for $z$ in equation (2).

$$\begin{aligned} 8y + z &= 0 && \text{(2)} \\ 8y - 6 &= 0 && \text{Let } z = -6. \\ 8y &= 6 && \text{Add 6.} \\ y &= \frac{6}{8}, \text{ or } \frac{3}{4} && \text{Divide by 8; lowest terms} \end{aligned}$$

Check to verify that the solution set is $\left\{\left(\frac{2}{3}, \frac{3}{4}, -6\right)\right\}$. NOW TRY

## OBJECTIVE 4 Solve special systems.

**NOW TRY EXERCISE 3**

Solve the system.

$$\begin{aligned} x - 5y + 2z &= 4 \\ 3x + y - z &= 6 \\ -2x + 10y - 4z &= 7 \end{aligned}$$

### EXAMPLE 3 Solving an Inconsistent System with Three Variables

Solve the system.

$$\begin{aligned} 2x - 4y + 6z &= 5 && \text{(1)} \\ -x + 3y - 2z &= -1 && \text{(2)} \\ x - 2y + 3z &= 1 && \text{(3)} \end{aligned}$$

Use as the working equation, with focus variable *x*.

Eliminate the focus variable, $x$, using equations (1) and (3).

$$\begin{aligned} -2x + 4y - 6z &= -2 && \text{Multiply each side of (3) by } -2. \\ 2x - 4y + 6z &= 5 && \text{(1)} \\ \hline 0 &= 3 && \text{Add; false} \end{aligned}$$

The resulting false statement indicates that equations (1) and (3) have no common solution. Thus, the system is inconsistent and the solution set is $\emptyset$. The graph of this system would show the two planes parallel to one another. NOW TRY

**NOTE** If a false statement results when adding as in **Example 3,** it is not necessary to go any further with the solution. Since two of the three planes are parallel, it is not possible for the three planes to have any points in common.

**NOW TRY EXERCISE 4**

Solve the system.

$$\begin{aligned} x - 3y + 2z &= 10 \\ -2x + 6y - 4z &= -20 \\ \frac{1}{2}x - \frac{3}{2}y + z &= 5 \end{aligned}$$

### EXAMPLE 4 Solving a System of Dependent Equations with Three Variables

Solve the system.

$$\begin{aligned} 2x - 3y + 4z &= 8 && \text{(1)} \\ -x + \frac{3}{2}y - 2z &= -4 && \text{(2)} \\ 6x - 9y + 12z &= 24 && \text{(3)} \end{aligned}$$

Multiplying each side of equation (1) by 3 gives equation (3). Multiplying each side of equation (2) by $-6$ also gives equation (3). Because of this, the equations are dependent. All three equations have the same graph, as illustrated in **FIGURE 10(c)**. The solution set is written as follows.

$$\{(x, y, z) \mid 2x - 3y + 4z = 8\} \qquad \text{Set-builder notation}$$

Although any one of the three equations could be used to write the solution set, we use the equation in standard form with coefficients that are integers with greatest common factor 1, as we did in **Section 8.1.** NOW TRY

**NOW TRY ANSWERS**

**2.** $\{(-2, 1, 4)\}$
**3.** $\emptyset$
**4.** $\{(x, y, z) \mid x - 3y + 2z = 10\}$

**NOW TRY EXERCISE 5**

Solve the system.

$$\begin{aligned} x - 3y + 2z &= 4 \\ \frac{1}{3}x - y + \frac{2}{3}z &= 7 \\ \frac{1}{2}x - \frac{3}{2}y + z &= 2 \end{aligned}$$

## EXAMPLE 5 Solving Another Special System

Solve the system.

$$\begin{aligned} 2x - y + 3z &= 6 && (1) \\ x - \frac{1}{2}y + \frac{3}{2}z &= 3 && (2) \\ 4x - 2y + 6z &= 1 && (3) \end{aligned}$$

Multiplying each side of equation (2) by 2 gives equation (1), so these two equations are dependent. Equations (1) and (3) are not equivalent, however. Multiplying equation (3) by $\frac{1}{2}$ does *not* give equation (1). Instead, we obtain two equations with the same coefficients, but with different constant terms.

The graphs of equations (1) and (3) have no points in common (that is, the planes are parallel). Thus, the system is inconsistent and the solution set is $\emptyset$, as illustrated in **FIGURE 10(g)**.

NOW TRY

**NOW TRY ANSWER**

**5.** $\emptyset$

## 8.4 EXERCISES

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**1.** ***Concept Check*** The two equations $\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \end{aligned}$ have a common solution of $(1, 2, 3)$. Which equation would complete a system of three linear equations in three variables having solution set $\{(1, 2, 3)\}$?

**A.** $3x + 2y - z = 1$ **B.** $3x + 2y - z = 4$

**C.** $3x + 2y - z = 5$ **D.** $3x + 2y - z = 6$

**2.** Complete the work of **Example 1** and show that the ordered triple $(-3, 1, 6)$ is also a solution of equations (2) and (3).

$$\begin{aligned} x + 7y - 3z &= -14 && \text{Equation (2)} \\ 2x - 3y + 2z &= 3 && \text{Equation (3)} \end{aligned}$$

*Solve each system of equations.* ***See Example 1.***

**3.** $\begin{aligned} 2x - 5y + 3z &= -1 \\ x + 4y - 2z &= 9 \\ x - 2y - 4z &= -5 \end{aligned}$

**4.** $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + z &= 1 \\ x + 2y + 2z &= -1 \end{aligned}$

**5.** $\begin{aligned} 3x + 2y + z &= 8 \\ 2x - 3y + 2z &= -16 \\ x + 4y - z &= 20 \end{aligned}$

**6.** $\begin{aligned} -3x + y - z &= -10 \\ -4x + 2y + 3z &= -1 \\ 2x + 3y - 2z &= -5 \end{aligned}$

**7.** $\begin{aligned} 2x + 5y + 2z &= 0 \\ 4x - 7y - 3z &= 1 \\ 3x - 8y - 2z &= -6 \end{aligned}$

**8.** $\begin{aligned} 5x - 2y + 3z &= -9 \\ 4x + 3y + 5z &= 4 \\ 2x + 4y - 2z &= 14 \end{aligned}$

**9.** $\begin{aligned} x + 2y + z &= 4 \\ 2x + y - z &= -1 \\ x - y - z &= -2 \end{aligned}$

**10.** $\begin{aligned} x - 2y + 5z &= -7 \\ -2x - 3y + 4z &= -14 \\ -3x + 5y - z &= -7 \end{aligned}$

**11.** $\begin{aligned} -x + 2y + 6z &= 2 \\ 3x + 2y + 6z &= 6 \\ x + 4y - 3z &= 1 \end{aligned}$

**12.** $\begin{aligned} 2x + y + 2z &= 1 \\ x + 2y + z &= 2 \\ x - y - z &= 0 \end{aligned}$

**13.** $\begin{aligned} x + y - z &= -2 \\ 2x - y + z &= -5 \\ -x + 2y - 3z &= -4 \end{aligned}$

**14.** $\begin{aligned} x + 2y + 3z &= 1 \\ -x - y + 3z &= 2 \\ -6x + y + z &= -2 \end{aligned}$

**15.**
$$\begin{aligned}\frac{1}{3}x+\frac{1}{6}y-\frac{2}{3}z&=-1\\ -\frac{3}{4}x-\frac{1}{3}y-\frac{1}{4}z&=3\\ \frac{1}{2}x+\frac{3}{2}y+\frac{3}{4}z&=21\end{aligned}$$

**16.**
$$\begin{aligned}\frac{2}{3}x-\frac{1}{4}y+\frac{5}{8}z&=0\\ \frac{1}{5}x+\frac{2}{3}y-\frac{1}{4}z&=-7\\ -\frac{3}{5}x+\frac{4}{3}y-\frac{7}{8}z&=-5\end{aligned}$$

**17.**
$$\begin{aligned}5.5x-2.5y+1.6z&=11.83\\ 2.2x+5.0y-0.1z&=-5.97\\ 3.3x-7.5y+3.2z&=21.25\end{aligned}$$

**18.**
$$\begin{aligned}6.2x-1.4y+2.4z&=-1.80\\ 3.1x+2.8y-0.2z&=5.68\\ 9.3x-8.4y-4.8z&=-34.20\end{aligned}$$

*Solve each system of equations.* ***See Example 2.***

**19.**
$$\begin{aligned}2x-3y+2z&=-1\\ x+2y+z&=17\\ 2y-z&=7\end{aligned}$$

**20.**
$$\begin{aligned}2x-y+3z&=6\\ x+2y-z&=8\\ 2y+z&=1\end{aligned}$$

**21.**
$$\begin{aligned}4x+2y-3z&=6\\ x-4y+z&=-4\\ -x+2z&=2\end{aligned}$$

**22.**
$$\begin{aligned}2x+3y-4z&=4\\ x-6y+z&=-16\\ -x+3z&=8\end{aligned}$$

**23.**
$$\begin{aligned}2x+y&=6\\ 3y-2z&=-4\\ 3x-5z&=-7\end{aligned}$$

**24.**
$$\begin{aligned}4x-8y&=-7\\ 4y+z&=7\\ -8x+z&=-4\end{aligned}$$

**25.**
$$\begin{aligned}-5x+2y+z&=5\\ -3x-2y-z&=3\\ -x+6y&=1\end{aligned}$$

**26.**
$$\begin{aligned}-4x+3y-z&=4\\ -5x-3y+z&=-4\\ -2x-3z&=12\end{aligned}$$

**27.**
$$\begin{aligned}7x-3z&=-34\\ 2y+4z&=20\\ \frac{3}{4}x+\frac{1}{6}y&=-2\end{aligned}$$

**28.**
$$\begin{aligned}5x-2z&=8\\ 4y+3z&=-9\\ \frac{1}{2}x+\frac{2}{3}y&=-1\end{aligned}$$

**29.**
$$\begin{aligned}4x-z&=-6\\ \frac{3}{5}y+\frac{1}{2}z&=0\\ \frac{1}{3}x+\frac{2}{3}z&=-5\end{aligned}$$

**30.**
$$\begin{aligned}5x-z&=38\\ \frac{2}{3}y+\frac{1}{4}z&=-17\\ \frac{1}{5}y+\frac{5}{6}z&=4\end{aligned}$$

*Solve each system of equations. If the system is inconsistent or has dependent equations, say so.* ***See Examples 1, 3, 4, and 5.***

**31.**
$$\begin{aligned}2x+2y-6z&=5\\ -3x+y-z&=-2\\ -x-y+3z&=4\end{aligned}$$

**32.**
$$\begin{aligned}-2x+5y+z&=-3\\ 5x+14y-z&=-11\\ 7x+9y-2z&=-5\end{aligned}$$

**33.**
$$\begin{aligned}-5x+5y-20z&=-40\\ x-y+4z&=8\\ 3x-3y+12z&=24\end{aligned}$$

**34.**
$$\begin{aligned}x+4y-z&=3\\ -2x-8y+2z&=-6\\ 3x+12y-3z&=9\end{aligned}$$

**35.**
$$\begin{aligned}x+5y-2z&=-1\\ -2x+8y+z&=-4\\ 3x-y+5z&=19\end{aligned}$$

**36.**
$$\begin{aligned}x+3y+z&=2\\ 4x+y+2z&=-4\\ 5x+2y+3z&=-2\end{aligned}$$

**37.**
$$\begin{aligned}2x+y-z&=6\\ 4x+2y-2z&=12\\ -x-\frac{1}{2}y+\frac{1}{2}z&=-3\end{aligned}$$

**38.**
$$\begin{aligned}2x-8y+2z&=-10\\ -x+4y-z&=5\\ \frac{1}{8}x-\frac{1}{2}y+\frac{1}{8}z&=-\frac{5}{8}\end{aligned}$$

**39.**
$$\begin{aligned}x+y-2z&=0\\ 3x-y+z&=0\\ 4x+2y-z&=0\end{aligned}$$

**40.**
$$\begin{aligned}2x+3y-z&=0\\ x-4y+2z&=0\\ 3x-5y-z&=0\end{aligned}$$

**41.**
$$\begin{aligned}x-2y+\frac{1}{3}z&=4\\ 3x-6y+z&=12\\ -6x+12y-2z&=-3\end{aligned}$$

**42.**
$$\begin{aligned}4x+y-2z&=3\\ x+\frac{1}{4}y-\frac{1}{2}z&=\frac{3}{4}\\ 2x+\frac{1}{2}y-z&=1\end{aligned}$$

***Brain Busters*** *Extend the method of this section to solve each system. Express the solution in the form $(x, y, z, w)$.*

**43.**
$$\begin{aligned} x + y + z - w &= 5 \\ 2x + y - z + w &= 3 \\ x - 2y + 3z + w &= 18 \\ -x - y + z + 2w &= 8 \end{aligned}$$

**44.**
$$\begin{aligned} 3x + y - z + 2w &= 9 \\ x + y + 2z - w &= 10 \\ x - y - z + 3w &= -2 \\ -x + y - z + w &= -6 \end{aligned}$$

**45.**
$$\begin{aligned} 3x + y - z + w &= -3 \\ 2x + 4y + z - w &= -7 \\ -2x + 3y - 5z + w &= 3 \\ 5x + 4y - 5z + 2w &= -7 \end{aligned}$$

**46.**
$$\begin{aligned} x - 3y + 7z + w &= 11 \\ 2x + 4y + 6z - 3w &= -3 \\ 3x + 2y + z + 2w &= 19 \\ 4x + y - 3z + w &= 22 \end{aligned}$$

## PREVIEW EXERCISES

*Solve each problem.* ***See Sections 2.4 and 2.7.***

**47.** The perimeter of a triangle is 323 in. The shortest side measures five-sixths the length of the longest side, and the medium side measures 17 in. less than the longest side. Find the lengths of the sides of the triangle.

**48.** The sum of the three angles of a triangle is 180°. The largest angle is twice the measure of the smallest, and the third angle measures 10° less than the largest. Find the measures of the three angles.

**49.** The sum of three numbers is 16. The greatest number is $-3$ times the least, while the middle number is four less than the greatest. Find the three numbers.

**50.** Witny Librun has a collection of pennies, dimes, and quarters. The number of dimes is one less than twice the number of pennies. If there are 27 coins in all worth a total of \$4.20, how many of each denomination of coin is in the collection?

# 8.5 Applications of Systems of Linear Equations

**OBJECTIVES**

1. Solve geometry problems by using two variables.
2. Solve money problems by using two variables.
3. Solve mixture problems by using two variables.
4. Solve distance-rate-time problems by using two variables.
5. Solve problems with three variables by using a system of three equations.

Although some problems with two unknowns can be solved by using just one variable, it is often easier to use two variables and a system of equations. The following problem, which can be solved with a system, appeared in a Hindu work that dates back to about A.D. 850. (See **Exercise 35.**)

*The mixed price of 9 citrons (a lemonlike fruit) and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and the price of a wood apple here, having distinctly separated those prices well.*

**PROBLEM-SOLVING HINT**

When solving an applied problem using two variables, it is a good idea to pick letters that correspond to the descriptions of the unknown quantities. In the example above, we could choose $c$ to represent the number of citrons, and $w$ to represent the number of wood apples.

The following steps are based on the problem-solving method of **Section 2.4.**

### Solving an Applied Problem by Writing a System of Equations

***Step 1*** **Read** the problem, several times if necessary. What information is given? What is to be found? This is often stated in the last sentence.

***Step 2*** **Assign variables** to represent the unknown values. Use a sketch, diagram, or table, as needed.

***Step 3*** **Write a system of equations** using the variable expressions.

***Step 4*** **Solve** the system of equations.

***Step 5*** **State the answer** to the problem. Label it appropriately. Does it seem reasonable?

***Step 6*** **Check** the answer in the words of the *original* problem.

## OBJECTIVE 1 Solve geometry problems by using two variables.

### *EXAMPLE 1* Finding the Dimensions of a Soccer Field

A rectangular soccer field may have a width between 50 and 100 yd and a length between 100 and 130 yd. One particular soccer field has a perimeter of 320 yd. Its length measures 40 yd more than its width. What are the dimensions of this field? (*Source:* www.soccer-training-guide.com)

***Step 1*** **Read** the problem again. We are asked to find the dimensions of the field.

***Step 2*** **Assign variables.** Let $L$ = the length and $W$ = the width. See **FIGURE 11**.

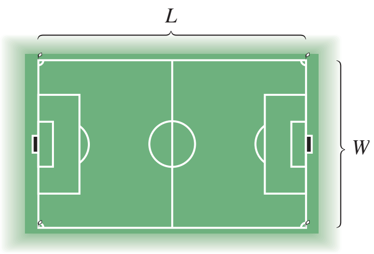

**FIGURE 11**

***Step 3*** **Write a system of equations.** Because the perimeter is 320 yd, we find one equation by using the perimeter formula.

$$2L + 2W = 320 \qquad 2L + 2W = P$$

We write a second equation using the fact that the length is 40 yd more than the width.

$$L = W + 40$$

These two equations form a system of equations.

$$2L + 2W = 320 \qquad (1)$$
$$L = W + 40 \qquad (2)$$

**NOW TRY EXERCISE 1**

A rectangular parking lot has a length that is 10 ft more than twice its width. The perimeter of the parking lot is 620 ft. What are the dimensions of the parking lot?

***Step 4*** **Solve** the system of equations. Since equation (2), $L = W + 40$, is solved for $L$, we can substitute $W + 40$ for $L$ in equation (1) and solve for $W$.

$$\begin{aligned} 2L + 2W &= 320 && \text{(1)} \\ 2(W + 40) + 2W &= 320 && \text{Let } L = W + 40. \\ 2W + 80 + 2W &= 320 && \text{Distributive property} \\ 4W + 80 &= 320 && \text{Combine like terms.} \\ 4W &= 240 && \text{Subtract 80.} \\ W &= 60 && \text{Divide by 4.} \end{aligned}$$

Be sure to use parentheses around $W + 40$.

Don't stop here.

Let $W = 60$ in the equation $L = W + 40$ to find $L$.

$$L = 60 + 40 = 100$$

***Step 5*** **State the answer.** The length is 100 yd, and the width is 60 yd. Both dimensions are within the ranges given in the problem.

***Step 6*** **Check.** Calculate the perimeter and the difference between the length and the width.

$$\begin{aligned} 2(100) + 2(60) &= 320 && \text{The perimeter is 320 yd, as required.} \\ 100 - 60 &= 40 && \text{Length is 40 yd more than width, as required.} \end{aligned}$$

The answer is correct.

NOW TRY

## OBJECTIVE 2 Solve money problems by using two variables.

### EXAMPLE 2 Solving a Problem about Ticket Prices

For the 2008–2009 National Hockey League and National Basketball Association seasons, two hockey tickets and one basketball ticket purchased at their average prices would have cost \$148.79. One hockey ticket and two basketball tickets would have cost \$148.60. What were the average ticket prices for the two sports? (*Source:* Team Marketing Report.)

***Step 1*** **Read** the problem again. There are two unknowns.

***Step 2*** **Assign variables.**

Let $h$ = the average price for a hockey ticket

and $b$ = the average price for a basketball ticket.

***Step 3*** **Write a system of equations.** Because two hockey tickets and one basketball ticket cost a total of \$148.79, one equation for the system is

$$2h + b = 148.79.$$

By similar reasoning, the second equation is

$$h + 2b = 148.60.$$

These two equations form a system of equations.

$$\begin{aligned} 2h + b &= 148.79 && \text{(1)} \\ h + 2b &= 148.60 && \text{(2)} \end{aligned}$$

**NOW TRY ANSWER**

**1.** length: 210 ft; width: 100 ft

**NOW TRY EXERCISE 2**

For the 2009–2010 season at Six Flags St. Louis, two general admission tickets and three tickets for children under 48 in. tall cost \$172.98. One general admission ticket and four tickets for children under 48 in. tall cost \$163.99. Determine the ticket prices for general admission and for children under 48 in. tall. (*Source:* www.sixflags.com)

***Step 4*** **Solve** the system. To eliminate $h$, multiply equation (2), $h + 2b = 148.60$, by $-2$ and add.

$$\begin{aligned} 2h + b &= 148.79 && \text{(1)} \\ -2h - 4b &= -297.20 && \text{Multiply each side of (2) by } -2. \\ \hline -3b &= -148.41 && \text{Add.} \\ b &= 49.47 && \text{Divide by } -3. \end{aligned}$$

To find the value of $h$, let $b = 49.47$ in equation (2).

$$\begin{aligned} h + 2b &= 148.60 && \text{(2)} \\ h + 2(49.47) &= 148.60 && \text{Let } b = 49.47. \\ h + 98.94 &= 148.60 && \text{Multiply.} \\ h &= 49.66 && \text{Subtract 98.94.} \end{aligned}$$

***Step 5*** **State the answer.** The average price for one basketball ticket was \$49.47. For one hockey ticket, the average price was \$49.66.

***Step 6*** **Check** that these values satisfy the conditions stated in the problem.

NOW TRY

**OBJECTIVE 3** **Solve mixture problems by using two variables.** We solved mixture problems in **Section 2.7** using one variable. Many mixture problems can also be solved using more than one variable and a system of equations.

### EXAMPLE 3 Solving a Mixture Problem

How many ounces each of 5% hydrochloric acid and 20% hydrochloric acid must be combined to get 10 oz of solution that is 12.5% hydrochloric acid?

***Step 1*** **Read** the problem. Two solutions of different strengths are being mixed together to get a specific amount of a solution with an "in-between" strength.

***Step 2*** **Assign variables.**

Let $x$ = the number of ounces of 5% solution

and $y$ = the number of ounces of 20% solution.

Use a table to summarize the information from the problem.

| *Ounces of Solution* | *Percent (as a decimal)* | *Ounces of Pure Acid* |
|---|---|---|
| $x$ | $5\% = 0.05$ | $0.05x$ |
| $y$ | $20\% = 0.20$ | $0.20y$ |
| 10 | $12.5\% = 0.125$ | $(0.125)10$ |

Gives equation (1)

Gives equation (2)

Multiply the amount of each solution (given in the first column) by its concentration of acid (given in the second column) to find the amount of acid in that solution (given in the third column).

**FIGURE 12** illustrates what is happening in the problem.

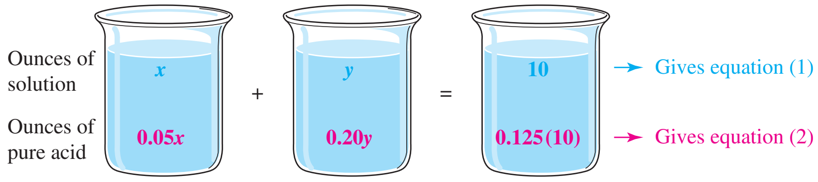

**FIGURE 12**

**NOW TRY ANSWER**

2. general admission: \$39.99; children under 48 in. tall: \$31.00

**NOW TRY EXERCISE 3**

How many liters each of a 15% acid solution and a 25% acid solution should be mixed to get 30 L of an 18% acid solution?

***Step 3*** **Write a system of equations.** When the $x$ ounces of 5% solution and the $y$ ounces of 20% solution are combined, the total number of ounces is 10, giving the following equation.

$$x + y = 10 \qquad (1)$$

The number of ounces of acid in the 5% solution $(0.05x)$ plus the number of ounces of acid in the 20% solution $(0.20y)$ should equal the total number of ounces of acid in the mixture, which is $(0.125)10$, or 1.25.

$$0.05x + 0.20y = 1.25 \qquad (2)$$

***Notice that these equations can be quickly determined by reading down the table or using the labels in* FIGURE 12.**

***Step 4*** **Solve** the system of equations (1) and (2) by eliminating $x$.

$$\begin{array}{rcrl} -5x - 5y & = & -50 & \text{Multiply each side of (1) by } -5. \\ 5x + 20y & = & 125 & \text{Multiply each side of (2) by 100.} \\ \hline 15y & = & 75 & \text{Add.} \\ y & = & 5 & \text{Divide by 15.} \end{array}$$

Substitute $y = 5$ in equation (1) to find that $x$ is also 5.

***Step 5*** **State the answer.** The desired mixture will require 5 oz of the 5% solution and 5 oz of the 20% solution.

***Step 6*** **Check.**

*Total amount of solution:*
$$\begin{aligned} x + y &= 5 \text{ oz} + 5 \text{ oz} \\ &= 10 \text{ oz}, \quad \text{as required.} \end{aligned}$$

*Total amount of acid:*
$$\begin{aligned} 5\% \text{ of } 5 \text{ oz} &+ 20\% \text{ of } 5 \text{ oz} \\ &= 0.05(5) + 0.20(5) \\ &= 1.25 \text{ oz} \end{aligned}$$

*Percent of acid in solution:*

$$\begin{array}{r} \text{Total acid} \longrightarrow \\ \text{Total solution} \longrightarrow \end{array} \frac{1.25}{10} = 0.125, \quad \text{or} \quad 12.5\%, \quad \text{as required.}$$

NOW TRY

## OBJECTIVE 4 Solve distance-rate-time problems by using two variables.

Motion problems require the distance formula $\boldsymbol{d = rt}$, where $d$ is distance, $r$ is rate (or speed), and $t$ is time.

### EXAMPLE 4 Solving a Motion Problem

A car travels 250 km in the same time that a truck travels 225 km. If the rate of the car is 8 km per hr faster than the rate of the truck, find both rates.

***Step 1*** **Read** the problem again. Given the distances traveled, we need to find the rate of each vehicle.

***Step 2*** **Assign variables.**

Let $x$ = the rate of the car,
and $y$ = the rate of the truck.

**NOW TRY ANSWER**

**3.** 9 L of the 25% acid solution; 21 L of the 15% acid solution

**NOW TRY EXERCISE 4**

Vann and Ivy Sample are planning a bicycle ride to raise money for cancer research. Vann can travel 50 mi in the same amount of time that Ivy can travel 40 mi. Determine both bicyclists' rates, if Vann's rate is 2 mph faster than Ivy's.

As in **Example 3,** a table helps organize the information. Fill in the distance for each vehicle, and the variables for the unknown rates.

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Car* | 250 | $x$ | $\frac{250}{x}$ |
| *Truck* | 225 | $y$ | $\frac{225}{y}$ |

To find the expressions for time, we solved the distance formula $d = rt$ for $t$. Thus, $\frac{d}{r} = t$.

***Step 3*** **Write a system of equations.** The car travels 8 km per hr faster than the truck. Since the two rates are $x$ and $y$,

$$x = y + 8. \qquad (1)$$

Both vehicles travel for the *same* time, so the times must be equal.

$$\text{Time for car} \longrightarrow \frac{250}{x} = \frac{225}{y} \longleftarrow \text{Time for truck}$$

This is not a linear equation. However, multiplying each side by $xy$ gives

$$250y = 225x, \qquad (2)$$

which is linear. The system to solve consists of equations (1) and (2).

$$\begin{aligned} x &= y + 8 && (1) \\ 250y &= 225x && (2) \end{aligned}$$

***Step 4*** **Solve** the system by substitution. Replace $x$ with $y + 8$ in equation (2).

$$\begin{aligned} 250y &= 225x && (2) \\ 250y &= 225(y + 8) && \text{Let } x = y + 8. \\ 250y &= 225y + 1800 && \text{Distributive property} \\ 25y &= 1800 && \text{Subtract } 225y. \\ y &= 72 && \text{Divide by 25.} \end{aligned}$$

Be sure to use parentheses around $y + 8$.

Because $x = y + 8$, the value of $x$ is $72 + 8 = 80$.

***Step 5*** **State the answer.** The rate of the car is 80 km per hr, and the rate of the truck is 72 km per hr.

***Step 6*** **Check.**

$$\begin{aligned} \textit{Car:} \quad t &= \frac{d}{r} = \frac{250}{80} = 3.125 \\ \textit{Truck:} \quad t &= \frac{d}{r} = \frac{225}{72} = 3.125 \end{aligned}$$

Times are equal.

Since 80 is 8 greater than 72, the conditions of the problem are satisfied.

NOW TRY

## OBJECTIVE 5 Solve problems with three variables by using a system of three equations.

**PROBLEM-SOLVING HINT**

If an application requires finding *three* unknown quantities, we can use a system of *three* equations to solve it. We extend the method used for two unknowns.

**NOW TRY ANSWER**

**4.** Vann: 10 mph; Ivy: 8 mph

NOW TRY EXERCISE 5

At a concession stand, a bottle of Gatorade costs \$2.00, a pretzel costs \$1.50, and candy items cost \$1.00. Workers sold four times as many pretzels as candy items. The number of Gatorades sold exceeded the number of pretzels sold by 310. Sales for these items totaled \$1520. How many of each item was sold?

## EXAMPLE 5 Solving a Problem Involving Prices

At Panera Bread, a loaf of honey wheat bread costs \$2.95, a loaf of sunflower bread costs \$2.99, and a loaf of French bread costs \$5.79. On a recent day, three times as many loaves of honey wheat bread were sold as sunflower bread. The number of loaves of French bread sold was 5 less than the number of loaves of honey wheat bread sold. Total receipts for these breads were \$87.89. How many loaves of each type of bread were sold? (*Source:* Panera Bread menu.)

***Step 1*** **Read** the problem again. There are three unknowns in this problem.

***Step 2*** **Assign variables** to represent the three unknowns.

Let $x$ = the number of loaves of honey wheat bread,
$y$ = the number of loaves of sunflower bread,
and $z$ = the number of loaves of French bread.

***Step 3*** **Write a system of three equations.** Since three times as many loaves of honey wheat bread were sold as sunflower bread,

$$x = 3y, \quad \text{or} \quad x - 3y = 0. \qquad \text{Subtract } 3y. \quad (1)$$

Also, we have the information needed for another equation.

$$\begin{array}{ccc} \text{Number of loaves of French} & \text{equals} & \text{5 less than the number of loaves of honey wheat.} \\ \downarrow & \downarrow & \downarrow \\ z & = & x - 5 \end{array}$$

$$-x + z = -5 \qquad \text{Subtract } x.$$

$$x - z = 5 \qquad \text{Multiply by } -1. \quad (2)$$

Multiplying the cost of a loaf of each kind of bread by the number of loaves of that kind sold and adding gives the total receipts.

$$2.95x + 2.99y + 5.79z = 87.89$$

Multiply each side of this equation by 100 to clear it of decimals.

$$295x + 299y + 579z = 8789 \qquad (3)$$

***Step 4*** **Solve** the system of three equations.

$$\begin{aligned} x - 3y &= 0 && (1) \\ x - z &= 5 && (2) \\ 295x + 299y + 579z &= 8789 && (3) \end{aligned}$$

Using the method shown in **Section 8.4,** we will find that $x = 12$, $y = 4$, and $z = 7$.

***Step 5*** **State the answer.** The solution set is $\{(12, 4, 7)\}$, meaning that 12 loaves of honey wheat bread, 4 loaves of sunflower bread, and 7 loaves of French bread were sold.

***Step 6*** **Check.** Since $12 = 3 \cdot 4$, the number of loaves of honey wheat bread is three times the number of loaves of sunflower bread. Also, $12 - 7 = 5$, so the number of loaves of French bread is 5 less than the number of loaves of honey wheat bread. Multiply the appropriate cost per loaf by the number of loaves sold and add the results to check that total receipts were \$87.89. NOW TRY

NOW TRY ANSWER

**5.** 60 candy items, 240 pretzels, 550 bottles of Gatorade

**NOW TRY EXERCISE 6**

Katherine has a quilting shop and makes three kinds of quilts: the lone star quilt, the bandana quilt, and the log cabin quilt.

- Each lone star quilt requires 8 hr of piecework, 4 hr of machine quilting, and 2 hr of finishing.
- Each bandana quilt requires 2 hr of piecework, 2 hr of machine quilting, and 2 hr of finishing.
- Each log cabin quilt requires 10 hr of piecework, 5 hr of machine quilting, and 2 hr of finishing.

Katherine allocates 74 hr for piecework, 42 hr for machine quilting, and 24 hr for finishing quilts each month. How many of each type of quilt should be made each month if all available time must be used?

## EXAMPLE 6 Solving a Business Production Problem

A company produces three flat screen television sets: models X, Y, and Z.

- Each model X set requires 2 hr of electronics work, 2 hr of assembly time, and 1 hr of finishing time.
- Each model Y requires 1 hr of electronics work, 3 hr of assembly time, and 1 hr of finishing time.
- Each model Z requires 3 hr of electronics work, 2 hr of assembly time, and 2 hr of finishing time.

There are 100 hr available for electronics, 100 hr available for assembly, and 65 hr available for finishing per week. How many of each model should be produced each week if all available time must be used?

***Step 1*** **Read** the problem again. There are three unknowns.

***Step 2*** **Assign variables.** Then organize the information in a table.

Let $x =$ the number of model X produced per week,

$y =$ the number of model Y produced per week,

and $z =$ the number of model Z produced per week.

| | *Each Model X* | *Each Model Y* | *Each Model Z* | *Totals* |
|---|---|---|---|---|
| *Hours of Electronics Work* | 2 | 1 | 3 | 100 |
| *Hours of Assembly Time* | 2 | 3 | 2 | 100 |
| *Hours of Finishing Time* | 1 | 1 | 2 | 65 |

***Step 3*** **Write a system of three equations.** The $x$ model X sets require $2x$ hours of electronics, the $y$ model Y sets require $1y$ (or $y$) hours of electronics, and the $z$ model Z sets require $3z$ hours of electronics. Since 100 hr are available for electronics, we write the first equation.

$$2x + y + 3z = 100 \qquad (1)$$

Since 100 hr are available for assembly, we can write another equation.

$$2x + 3y + 2z = 100 \qquad (2)$$

The fact that 65 hr are available for finishing leads to this equation.

$$x + y + 2z = 65 \qquad (3)$$

Notice that by reading across the table, we can easily determine the coefficients and constants in the equations of the system.

***Step 4*** **Solve** the system of equations (1), (2), and (3).

$$\begin{aligned} 2x + y + 3z &= 100 \qquad (1)\\ 2x + 3y + 2z &= 100 \qquad (2)\\ x + y + 2z &= 65 \qquad (3) \end{aligned}$$

We find that $x = 15$, $y = 10$, and $z = 20$.

***Step 5*** **State the answer.** The company should produce 15 model X, 10 model Y, and 20 model Z sets per week.

***Step 6*** **Check** that these values satisfy the conditions of the problem. NOW TRY

**NOW TRY ANSWER**

**6.** lone star quilts: 3;
bandana quilts: 5;
log cabin quilts: 4

## 8.5 EXERCISES

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*Solve each problem.* ***See Example 1.***

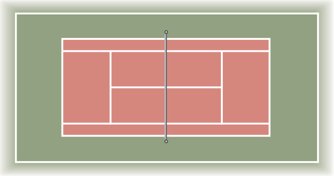
| Team | W | L |
|---|---|---|
| L.A. Dodgers | —— | —— |
| Colorado | 92 | 70 |
| San Francisco | 88 | 74 |
| San Diego | 75 | 87 |
| Arizona | —— | —— |

*Source: World Almanac and Book of Facts.*

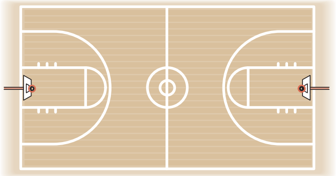
**1.** During the 2009 Major League Baseball season, the Los Angeles Dodgers played 162 games. They won 28 more games than they lost. What was their win-loss record that year?

**2.** Refer to **Exercise 1.** During the same 162-game season, the Arizona Diamondbacks lost 22 more games than they won. What was the team's win-loss record?

**3.** Venus and Serena measured a tennis court and found that it was 42 ft longer than it was wide and had a perimeter of 228 ft. What were the length and the width of the tennis court?



**4.** LeBron and Shaq measured a basketball court and found that the width of the court was 44 ft less than the length. If the perimeter was 288 ft, what were the length and the width of the basketball court?

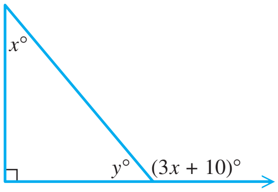


**5.** In 2009, the two American telecommunication companies with the greatest revenues were AT&T and Verizon. The two companies had combined revenues of \$221.4 billion. AT&T's revenue was \$26.6 more than that of Verizon. What was the revenue for each company? (*Source: Fortune* magazine.)

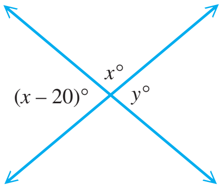
**6.** In 2008, U.S. exports to Canada were \$110 billion more than exports to Mexico. Together, exports to these two countries totaled \$412 billion. How much were exports to each country? (*Source:* U.S. Census Bureau.)

*In Exercises 7 and 8, find the measures of the angles marked x and y. Remember that* **(1)** *the sum of the measures of the angles of a triangle is* 180°, **(2)** *supplementary angles have a sum of* 180°, *and* **(3)** *vertical angles have equal measures.*

**7.**



**8.**



*The Fan Cost Index (FCI) represents the cost of four average-price tickets, four small soft drinks, two small beers, four hot dogs, parking for one car, two game programs, and two souvenir caps to a sporting event.* (*Source:* Team Marketing Report.)

*Use the concept of FCI in Exercises 9 and 10.* ***See Example 2.***

**9.** For the 2008–2009 season, the FCI prices for the National Hockey League and the National Basketball Association totaled \$580.16. The hockey FCI was \$3.70 less than that of basketball. What were the FCIs for these sports?

**10.** In 2009, the FCI prices for Major League Baseball and the National Football League totaled \$609.53. The football FCI was \$215.75 more than that of baseball. What were the FCIs for these sports?

*Solve each problem.* ***See Example 2.***

**11.** Andrew McGinnis works at Arby's. During one particular day he sold 15 Junior Roast Beef sandwiches and 10 Big Montana sandwiches, totaling \$75.25. Another day he sold 30 Junior Roast Beef sandwiches and 5 Big Montana sandwiches, totaling \$84.65. How much did each type of sandwich cost? (*Source:* Arby's menu.)

**12.** New York City and Washington, D.C., were the two most expensive cities for business travel in 2009. On the basis of the average total costs per day for each city (which include a hotel room, car rental, and three meals), 2 days in New York and 3 days in Washington cost \$2772, while 4 days in New York and 2 days in Washington cost \$3488. What was the average cost per day in each city? (*Source: Business Travel News.*)

***Concept Check*** *The formulas* $p = br$ *(percentage = base × rate) and* $I = prt$ *(simple interest = principal × rate × time) are used in the applications in* ***Exercises 17–24.*** *To prepare to use these formulas, answer the questions in Exercises 13 and 14.*

**13.** If a container of liquid contains 60 oz of solution, what is the number of ounces of pure acid if the given solution contains the following acid concentrations?

**(a)** 10% **(b)** 25% **(c)** 40% **(d)** 50%

**14.** If \$5000 is invested in an account paying simple annual interest, how much interest will be earned during the first year at the following rates?

**(a)** 2% **(b)** 3% **(c)** 4% **(d)** 3.5%

**15.** ***Concept Check*** If 1 pound of turkey costs \$2.29, give an expression for the cost of $x$ pounds.

**16.** ***Concept Check*** If 1 ticket to the movie *Avatar* costs \$9 and $y$ tickets are sold, give an expression for the amount collected from the sale.

*Solve each problem.* ***See Example 3.***

**17.** How many gallons each of 25% alcohol and 35% alcohol should be mixed to get 20 gal of 32% alcohol?

| *Gallons of Solution* | *Percent (as a decimal)* | *Gallons of Pure Alcohol* |
|---|---|---|
| $x$ | 25% = 0.25 | |
| $y$ | 35% = 0.35 | |
| 20 | 32% = | |

**18.** How many liters each of 15% acid and 33% acid should be mixed to get 120 L of 21% acid?

| *Liters of Solution* | *Percent (as a decimal)* | *Liters of Pure Acid* |
|---|---|---|
| $x$ | 15% = 0.15 | |
| $y$ | 33% = | |
| 120 | 21% = | |

**19.** Pure acid is to be added to a 10% acid solution to obtain 54 L of a 20% acid solution. What amounts of each should be used?

**20.** A truck radiator holds 36 L of fluid. How much pure antifreeze must be added to a mixture that is 4% antifreeze to fill the radiator with a mixture that is 20% antifreeze?

**21.** A party mix is made by adding nuts that sell for \$2.50 per kg to a cereal mixture that sells for \$1 per kg. How much of each should be added to get 30 kg of a mix that will sell for \$1.70 per kg?

| | Number of Kilograms | Price per Kilogram | Value |
|---|---|---|---|
| Nuts | $x$ | 2.50 | |
| Cereal | $y$ | 1.00 | |
| Mixture | | 1.70 | |

**22.** A fruit drink is made by mixing fruit juices. Such a drink with 50% juice is to be mixed with another drink that is 30% juice to get 200 L of a drink that is 45% juice. How much of each should be used?

| | Liters of Drink | Percent (as a decimal) | Liters of Pure Juice |
|---|---|---|---|
| 50% Juice | $x$ | 0.50 | |
| 30% Juice | $y$ | 0.30 | |
| Mixture | | 0.45 | |

**23.** A total of \$3000 is invested, part at 2% simple interest and part at 4%. If the total annual return from the two investments is \$100, how much is invested at each rate?

| Principal | Rate (as a decimal) | Interest |
|---|---|---|
| $x$ | 0.02 | $0.02x$ |
| $y$ | 0.04 | $0.04y$ |
| 3000 | | 100 |

**24.** An investor will invest a total of \$15,000 in two accounts, one paying 4% annual simple interest and the other 3%. If he wants to earn \$550 annual interest, how much should he invest at each rate?

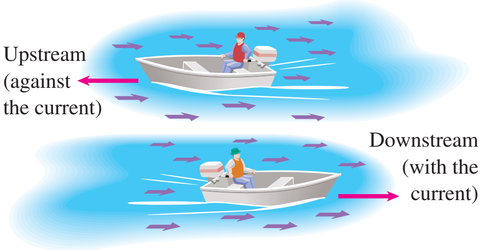
| Principal | Rate (as a decimal) | Interest |
|---|---|---|
| $x$ | 0.04 | |
| $y$ | 0.03 | |
| 15,000 | | |

***Concept Check*** *The formula $d = rt$ (distance $=$ rate $\times$ time) is used in the applications in* ***Exercises 27–30.*** *To prepare to use this formula, work Exercises 25 and 26.*

**25.** If the rate of a boat in still water is 10 mph, and the rate of the current of a river is $x$ mph, what is the rate of the boat

**(a)** going upstream (that is, against the current, which slows the boat down);

**(b)** going downstream (that is, with the current, which speeds the boat up)?



**26.** If the rate of a killer whale is 25 mph and the whale swims for $y$ hours, give an expression for the number of miles the whale travels.

*Solve each problem.* ***See Example 4.***

**27.** A train travels 150 km in the same time that a plane covers 400 km. If the rate of the plane is 20 km per hr less than 3 times the rate of the train, find both rates.

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| Train | $x$ | | 150 |
| Plane | $y$ | | 400 |

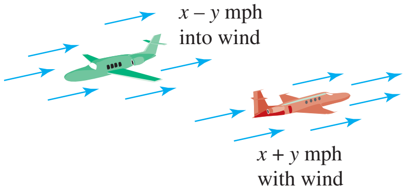
**28.** A freight train and an express train leave towns 390 km apart, traveling toward one another. The freight train travels 30 km per hr slower than the express train. They pass one another 3 hr later. What are their rates?

| | $r$ | $t$ | $d$ |
|---|---|---|---|
| Freight Train | $x$ | 3 | |
| Express Train | $y$ | 3 | |

**29.** In his motorboat, Bill Ruhberg travels upstream at top speed to his favorite fishing spot, a distance of 36 mi, in 2 hr. Returning, he finds that the trip downstream, still at top speed, takes only 1.5 hr. Find the rate of Bill's boat and the rate of the current. Let $x =$ the rate of the boat and $y =$ the rate of the current.

| | *r* | *t* | *d* |
|---|---|---|---|
| *Upstream* | $x - y$ | 2 | |
| *Downstream* | $x + y$ | | |

**30.** Traveling for 3 hr into a steady head wind, a plane flies 1650 mi. The pilot determines that flying *with* the same wind for 2 hr, he could make a trip of 1300 mi. Find the rate of the plane and the wind speed.



*Solve each problem by using two variables.* ***See Examples 1–4.***

**31.** (See the Chapter Introduction.) How many pounds of candy that sells for \$0.75 per lb must be mixed with candy that sells for \$1.25 per lb to obtain 9 lb of a mixture that should sell for \$0.96 per lb?

**32.** The top-grossing tour on the North American concert circuit for 2009 was U2, followed in second place by Bruce Springsteen and the E Street Band. Together, they took in \$217.5 million from ticket sales. If Springsteen took in \$28.5 million less than U2, how much did each band generate? (*Source:* Pollstar.)

**33.** Tickets to a production of *A Midsummer Night's Dream* at Broward College cost \$5 for general admission or \$4 with a student ID. If 184 people paid to see a performance and \$812 was collected, how many of each type of ticket were sold?

**34.** At a business meeting at Panera Bread, the bill for two cappuccinos and three house lattes was \$14.55. At another table, the bill for one cappuccino and two house lattes was \$8.77. How much did each type of beverage cost? (*Source:* Panera Bread menu.)

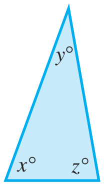
**35.** The mixed price of 9 citrons and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and the price of a wood apple here, having distinctly separated those prices well. (*Source:* Hindu work, A.D. 850.)

**36.** Braving blizzard conditions on the planet Hoth, Luke Skywalker sets out in his snow speeder for a rebel base 4800 mi away. He travels into a steady head wind and makes the trip in 3 hr. Returning, he finds that the trip back, now with a tailwind, takes only 2 hr. Find the rate of Luke's snow speeder and the speed of the wind.

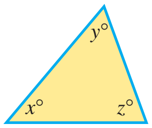
| | *r* | *t* | *d* |
|---|---|---|---|
| *Into Head Wind* | | | |
| *With Tailwind* | | | |

*Solve each problem by using three variables.* ***See Examples 5 and 6.*** *(In Exercises 37–40, remember that the sum of the measures of the angles of a triangle is* 180°.*)*

**37.** In the figure, $z = x + 10$ and $x + y = 100$. Determine a third equation involving $x$, $y$, and $z$, and then find the measures of the three angles.



**38.** In the figure, $x$ is 10 less than $y$ and 20 less than $z$. Write a system of equations and find the measures of the three angles.



**39.** In a certain triangle, the measure of the second angle is 10° greater than three times the first. The third angle measure is equal to the sum of the measures of the other two. Find the measures of the three angles.

**40.** The measure of the largest angle of a triangle is 12° less than the sum of the measures of the other two. The smallest angle measures 58° less than the largest. Find the measures of the angles.

**41.** The perimeter of a triangle is 70 cm. The longest side is 4 cm less than the sum of the other two sides. Twice the shortest side is 9 cm less than the longest side. Find the length of each side of the triangle.

**42.** The perimeter of a triangle is 56 in. The longest side measures 4 in. less than the sum of the other two sides. Three times the shortest side is 4 in. more than the longest side. Find the lengths of the three sides.

**43.** In the 2008 Summer Olympics in Beijing, China, Russia earned 5 fewer gold medals than bronze. The number of silver medals earned was 35 less than twice the number of bronze medals. Russia earned a total of 72 medals. How many of each kind of medal did Russia earn? (*Source: World Almanac and Book of Facts.*)

**44.** In a random sample of Americans of voting age conducted in 2010, 8% more people identified themselves as Independents than as Republicans, while 6% fewer people identified themselves as Republicans than as Democrats. Of those sampled, 2% did not identify with any of the three categories. What percent of the people in the sample identified themselves with each of the three political affiliations? (*Source:* Gallup, Inc.)

**45.** Tickets for the Harlem Globetrotters show at Michigan State University in 2010 cost \$16, \$23, or, for VIP seats, \$40. If nine times as many \$16 tickets were sold as VIP tickets, and the number of \$16 tickets sold was 55 more than the sum of the number of \$23 tickets and VIP tickets, sales of all three kinds of tickets would total \$46,575. How many of each kind of ticket would have been sold? (*Source:* Breslin Student Events Center.)

**46.** Three kinds of tickets are available for a *Cowboy Mouth* concert: “up close,” “in the middle,” and “far out.” “Up close” tickets cost \$10 more than “in the middle” tickets, while “in the middle” tickets cost \$10 more than “far out” tickets. Twice the cost of an “up close” ticket is \$20 more than 3 times the cost of a “far out” ticket. Find the price of each kind of ticket.

**47.** A wholesaler supplies college T-shirts to three college bookstores: A, B, and C. The wholesaler recently shipped a total of 800 T-shirts to the three bookstores. In order to meet student demand at the three colleges, twice as many T-shirts were shipped to bookstore B as to bookstore A, and the number shipped to bookstore C was 40 less than the sum of the numbers shipped to the other two bookstores. How many T-shirts were shipped to each bookstore?

**48.** An office supply store sells three models of computer desks: A, B, and C. In January, the store sold a total of 85 computer desks. The number of model B desks was five more than the number of model C desks, and the number of model A desks was four more than twice the number of model C desks. How many of each model did the store sell in January?

**49.** A plant food is to be made from three chemicals. The mix must include 60% of the first and second chemicals. The second and third chemicals must be in the ratio of 4 to 3 by weight. How much of each chemical is needed to make 750 kg of the plant food?

**50.** How many ounces of 5% hydrochloric acid, 20% hydrochloric acid, and water must be combined to get 10 oz of solution that is 8.5% hydrochloric acid if the amount of water used must equal the total amount of the other two solutions?

*Starting with the 2005–2006 season, the National Hockey League adopted a new system for awarding points used to determine team standings. A team is awarded* 2 *points for a win* (W), 0 *points for a loss in regulation play* (L), *and* 1 *point for an overtime loss* (OTL). *Use this information in Exercises 51 and 52.*

**51.** During the 2008–2009 NHL regular season, the Boston Bruins played 82 games. Their wins and overtime losses resulted in a total of 116 points. They had 9 more losses in regulation play than overtime losses. How many wins, losses, and overtime losses did they have that year?

| *Team* | *GP* | *W* | *L* | *OTL* | *Points* |
|---|---|---|---|---|---|
| Boston | 82 | ____ | ____ | ____ | 116 |
| Montreal | 82 | 41 | 30 | 11 | 93 |
| Buffalo | 82 | 41 | 32 | 9 | 91 |
| Ottawa | 82 | 36 | 35 | 11 | 83 |
| Toronto | 82 | 34 | 35 | 13 | 81 |

*Source: World Almanac and Book of Facts.*

**52.** During the 2008–2009 NHL regular season, the Los Angeles Kings played 82 games. Their wins and overtime losses resulted in a total of 79 points. They had 14 more total losses (in regulation play and overtime) than wins. How many wins, losses, and overtime losses did they have that year?

| *Team* | *GP* | *W* | *L* | *OTL* | *Points* |
|---|---|---|---|---|---|
| San Jose | 82 | 53 | 18 | 11 | 117 |
| Anaheim | 82 | 42 | 33 | 7 | 91 |
| Dallas | 82 | 36 | 35 | 11 | 83 |
| Phoenix | 82 | 36 | 39 | 7 | 79 |
| Los Angeles | 82 | ____ | ____ | ____ | 79 |

*Source: World Almanac and Book of Facts.*

## *PREVIEW EXERCISES*

*Give* ***(a)*** *the additive inverse and* ***(b)*** *the multiplicative inverse (reciprocal) of each number.* ***See Sections 1.4 and 1.6.***

**53.** $-6$

**54.** 0.2

**55.** $\frac{7}{8}$

**56.** 2.25

# 8.6

# Solving Systems of Linear Equations by Matrix Methods

**OBJECTIVES**

1. Define a matrix.
2. Write the augmented matrix of a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.

**OBJECTIVE 1** **Define a matrix.** An ordered array of numbers such as

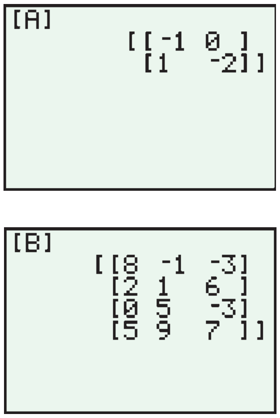
$$\begin{array}{cc} & \text{Columns} \\ \text{Rows} & \begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix} \end{array} \qquad \text{Matrix}$$

is called a **matrix.** The numbers are called **elements** of the matrix. *Matrices* (the plural of *matrix*) are named according to the number of **rows** and **columns** they contain. The rows are read horizontally, and the columns are read vertically. This matrix is a $2 \times 3$ (read "two by three") matrix, because it has 2 rows and 3 columns. The number of rows followed by the number of columns gives the **dimensions** of the matrix.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \quad \begin{matrix} 2 \times 2 \\ \text{matrix} \end{matrix} \qquad \begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix} \quad \begin{matrix} 4 \times 3 \\ \text{matrix} \end{matrix}$$

A **square matrix** is a matrix that has the same number of rows as columns. The $2 \times 2$ matrix above is a square matrix.

**FIGURE 13** shows how a graphing calculator displays the preceding two matrices. Consult your owner's manual for details for using matrices.



```
[A]
        [[-1  0 ]
         [1   -2]]
```

```
[B]
     [[8  -1  -3]
      [2  1   6 ]
      [0  5   -3]
      [5  9   7 ]]
```

FIGURE 13

In this section, we discuss a matrix method of solving linear systems that is a structured way of using the elimination method. The advantage of this new method is that it can be done by a graphing calculator or a computer.

**OBJECTIVE 2** **Write the augmented matrix of a system.** To solve a linear system using matrices, we begin by writing an *augmented matrix* of the system. An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups. For example, to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + \phantom{3}y &= -5, \end{aligned}$$

start by writing the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]. \qquad \text{Augmented matrix}$$

Place the coefficients of the variables to the left of the bar, and the constants to the right. The bar separates the coefficients from the constants. ***The matrix is just a shorthand way of writing the system of equations, so the rows of the augmented matrix can be treated the same as the equations of a system of equations.***

Exchanging the positions of two equations in a system does not change the system. Also, multiplying any equation in a system by a nonzero number does not change the system. Comparable changes to the augmented matrix of a system of equations produce new matrices that correspond to systems with the same solutions as the original system.

The following **row operations** produce new matrices that lead to systems having the same solutions as the original system.

### Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements of any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.

Examples of these row operations follow.

**Row operation 1**

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}$$

Interchange row 1 and row 3.

**Row operation 2**

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply the numbers in row 1 by 3.

**Row operation 3**

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply the numbers in row 3 by $-2$. Add them to the corresponding numbers in row 1.

The third row operation corresponds to the way we eliminated a variable from a pair of equations in previous sections.

## OBJECTIVE 3 Use row operations to solve a system with two equations.

Row operations can be used to rewrite a matrix until it is the matrix of a system whose solution is easy to find. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two and three equations, respectively. Notice that there are 1's down the diagonal from upper left to lower right and 0's below the 1's. A matrix written this way is said to be in **row echelon form.**

### EXAMPLE 1 Using Row Operations to Solve a System with Two Variables

Use row operations to solve the system.

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5 \end{aligned}$$

We start by writing the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Write the augmented matrix.

**NOW TRY EXERCISE 1**

Use row operations to solve the system.

$$\begin{aligned} x + 3y &= 3 \\ 2x - 3y &= -12 \end{aligned}$$

***Our goal is to use the various row operations to change this matrix into one that leads to a system that is easier to solve. It is best to work by columns.***

We start with the first column and make sure that there is a 1 in the first row, first column, position. There already is a 1 in this position.

Next, we get 0 in every position below the first. To get a 0 in row two, column one, we add to the numbers in row two the result of multiplying each number in row one by $-2$. (We abbreviate this as $-2R_1 + R_2$.) Row one remains unchanged.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ \underset{\uparrow}{2} + \underset{\uparrow}{1(-2)} & 1 + (-3)(-2) & -5 + 1(-2) \end{array}\right]$$

Original number from row two

$-2$ times number from row one

$$\begin{array}{r} \text{1 in the first position of column one} \longrightarrow \\ \text{0 in every position below the first} \longrightarrow \end{array} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right] \quad -2R_1 + R_2$$

Now we go to column two. The number 1 is needed in row two, column two. We use the second row operation, multiplying each number of row two by $\frac{1}{7}$.

Stop here—this matrix is in row echelon form.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -1 \end{array}\right] \quad \tfrac{1}{7}R_2$$

This augmented matrix leads to the system of equations

$$\begin{aligned} 1x - 3y &= 1 \\ 0x + 1y &= -1, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x - 3y &= 1 \\ y &= -1. \end{aligned}$$

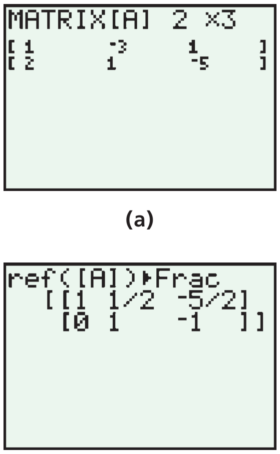
From the second equation, $y = -1$, we substitute $-1$ for $y$ in the first equation to find $x$.

$$\begin{aligned} x - 3y &= 1 & & \\ x - 3(-1) &= 1 & & \text{Let } y = -1. \\ x + 3 &= 1 & & \text{Multiply.} \\ x &= -2 & & \text{Subtract 3.} \end{aligned}$$

The solution set of the system is $\{(-2, -1)\}$. Check this solution by substitution in both equations of the system.

Write the values of $x$ and $y$ in the correct order.

**NOW TRY**



```
MATRIX[A] 2 ×3
[1    -3    1    ]
[2    1     -5   ]
```

(a)

```
ref([A])▸Frac
  [[1  1/2  -5/2]
   [0  1    -1  ]]
```

(b)

FIGURE 14

**NOTE** If the augmented matrix of the system in **Example 1** is entered as matrix [A] in a graphing calculator (**FIGURE 14(a)**) and the row echelon form of the matrix is found (**FIGURE 14(b)**), then the system becomes the following.

$$\begin{aligned} x + \frac{1}{2}y &= -\frac{5}{2} \\ y &= -1 \end{aligned}$$

While this system looks different from the one we obtained in **Example 1,** it is equivalent, since its solution set is also $\{(-2, -1)\}$.

**NOW TRY ANSWER**

1. $\{(-3, 2)\}$

## OBJECTIVE 4 Use row operations to solve a system with three equations.

### EXAMPLE 2 Using Row Operations to Solve a System with Three Variables

Use row operations to solve the system.

$$\begin{aligned} x - \phantom{3}y + 5z &= -6 \\ 3x + 3y - \phantom{5}z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Start by writing the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \qquad \text{Write the augmented matrix.}$$

This matrix already has 1 in row one, column one. Next get 0’s in the rest of column one. First, add to row two the results of multiplying each number of row one by $-3$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 1 & 3 & 2 & 5 \end{array}\right] \qquad -3R_1 + R_2$$

Now add to the numbers in row three the results of multiplying each number of row one by $-1$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \qquad -1R_1 + R_3$$

Introduce 1 in row two, column two, by multiplying each number in row two by $\frac{1}{6}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \qquad \tfrac{1}{6}R_2$$

To obtain 0 in row three, column two, add to row three the results of multiplying each number in row two by $-4$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \qquad -4R_2 + R_3$$

Obtain 1 in row three, column three, by multiplying each number in row three by $\frac{3}{23}$.

This matrix is in row echelon form.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \tfrac{3}{23}R_3$$

The final matrix gives this system of equations.

$$\begin{aligned} x - y + 5z &= -6 \\ y - \frac{8}{3}z &= \frac{14}{3} \\ z &= -1 \end{aligned}$$

**NOW TRY EXERCISE 2**

Use row operations to solve the system.

$$\begin{aligned} x + y - 2z &= -5 \\ -x + 2y + z &= -1 \\ 2x - y + 3z &= 14 \end{aligned}$$

Substitute $-1$ for $z$ in the second equation, $y - \frac{8}{3}z = \frac{14}{3}$, to find that $y = 2$. Finally, substitute 2 for $y$ and $-1$ for $z$ in the first equation,

$$x - y + 5z = -6,$$

to determine that $x = 1$. The solution set of the original system is $\{(1, 2, -1)\}$. Check by substitution. NOW TRY

## OBJECTIVE 5 Use row operations to solve special systems.

**NOW TRY EXERCISE 3**

Use row operations to solve each system.

**(a)** $\begin{aligned} 3x - y &= 8 \\ -6x + 2y &= 4 \end{aligned}$

**(b)** $\begin{aligned} x + 2y &= 7 \\ -x - 2y &= -7 \end{aligned}$

### EXAMPLE 3 Recognizing Inconsistent Systems or Dependent Equations

Use row operations to solve each system.

**(a)** $\begin{aligned} 2x - 3y &= 8 \\ -6x + 9y &= 4 \end{aligned}$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -6 & 9 & 4 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 28 \end{array}\right] \quad 6R_1 + R_2$$

The corresponding system of equations is

$$x - \frac{3}{2}y = 4$$

$$0 = 28, \quad \text{False}$$

which has no solution and is inconsistent. The solution set is $\emptyset$.

**(b)** $\begin{aligned} -10x + 12y &= 30 \\ 5x - 6y &= -15 \end{aligned}$

$$\left[\begin{array}{cc|c} -10 & 12 & 30 \\ 5 & -6 & -15 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{6}{5} & -3 \\ 5 & -6 & -15 \end{array}\right] \quad -\tfrac{1}{10}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{6}{5} & -3 \\ 0 & 0 & 0 \end{array}\right] \quad -5R_1 + R_2$$

The corresponding system is

$$x - \frac{6}{5}y = -3$$

$$0 = 0, \quad \text{True}$$

which has dependent equations. We use the second equation of the given system, which is in standard form, to express the solution set.

$$\{(x, y) \mid 5x - 6y = -15\}$$

NOW TRY

**NOW TRY ANSWERS**

**2.** $\{(2, -1, 3)\}$

**3. (a)** $\emptyset$

**(b)** $\{(x, y) \mid x + 2y = 7\}$

## 8.6 EXERCISES

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◉ *Complete solution available on the Video Resources on DVD*

**1.** *Concept Check* Consider the matrix $\begin{bmatrix} -2 & 3 & 1 \\ 0 & 5 & -3 \\ 1 & 4 & 8 \end{bmatrix}$ and answer the following.

**(a)** What are the elements of the second row?

**(b)** What are the elements of the third column?

✎ **(c)** Is this a square matrix? Explain why or why not.

**(d)** Give the matrix obtained by interchanging the first and third rows.

**(e)** Give the matrix obtained by multiplying the first row by $-\frac{1}{2}$.

**(f)** Give the matrix obtained by multiplying the third row by 3 and adding to the first row.

**2.** Repeat **Exercise 1** for the matrix $\begin{bmatrix} -7 & 0 & 1 \\ 3 & 2 & -2 \\ 0 & 1 & 6 \end{bmatrix}$.

*Concept Check* *Give the dimensions of each matrix.*

**3.** $\begin{bmatrix} 3 & -7 \\ 4 & 5 \\ -1 & 0 \end{bmatrix}$

**4.** $\begin{bmatrix} 4 & 9 & 0 \\ -1 & 2 & -4 \end{bmatrix}$

**5.** $\begin{bmatrix} 6 & 3 \\ -2 & 5 \\ 4 & 10 \\ 1 & -1 \end{bmatrix}$

**6.** $\begin{bmatrix} 8 & 4 & 3 & 2 \end{bmatrix}$

*Use row operations to solve each system.* ***See Examples 1 and 3.***

**7.** $\begin{aligned} x + y &= 5 \\ x - y &= 3 \end{aligned}$

**8.** $\begin{aligned} x + 2y &= 7 \\ x - y &= -2 \end{aligned}$

◉ **9.** $\begin{aligned} 2x + 4y &= 6 \\ 3x - y &= 2 \end{aligned}$

**10.** $\begin{aligned} 4x + 5y &= -7 \\ x - y &= 5 \end{aligned}$

**11.** $\begin{aligned} 3x + 4y &= 13 \\ 2x - 3y &= -14 \end{aligned}$

**12.** $\begin{aligned} 5x + 2y &= 8 \\ 3x - y &= 7 \end{aligned}$

◉ **13.** $\begin{aligned} -4x + 12y &= 36 \\ x - 3y &= 9 \end{aligned}$

**14.** $\begin{aligned} 2x - 4y &= 8 \\ -3x + 6y &= 5 \end{aligned}$

**15.** $\begin{aligned} 2x + y &= 4 \\ 4x + 2y &= 8 \end{aligned}$

**16.** $\begin{aligned} -3x - 4y &= 1 \\ 6x + 8y &= -2 \end{aligned}$

**17.** $\begin{aligned} -3x + 2y &= 0 \\ x - y &= 0 \end{aligned}$

**18.** $\begin{aligned} -5x + 3y &= 0 \\ 7x + 2y &= 0 \end{aligned}$

*Use row operations to solve each system.* ***See Examples 2 and 3.***

◉ **19.** $\begin{aligned} x + y - 3z &= 1 \\ 2x - y + z &= 9 \\ 3x + y - 4z &= 8 \end{aligned}$

**20.** $\begin{aligned} 2x + 4y - 3z &= -18 \\ 3x + y - z &= -5 \\ x - 2y + 4z &= 14 \end{aligned}$

**21.** $\begin{aligned} x + y - z &= 6 \\ 2x - y + z &= -9 \\ x - 2y + 3z &= 1 \end{aligned}$

**22.** $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + 2z &= 0 \\ x + y + 2z &= -1 \end{aligned}$

**23.** $\begin{aligned} x - y &= 1 \\ y - z &= 6 \\ x + z &= -1 \end{aligned}$

**24.** $\begin{aligned} x + y &= 1 \\ 2x - z &= 0 \\ y + 2z &= -2 \end{aligned}$

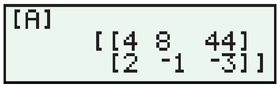
**25.** $\begin{aligned} x - 2y + z &= 4 \\ 3x - 6y + 3z &= 12 \\ -2x + 4y - 2z &= -8 \end{aligned}$

**26.** $\begin{aligned} x + 3y + z &= 1 \\ 2x + 6y + 2z &= 2 \\ 3x + 9y + 3z &= 3 \end{aligned}$

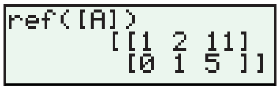
**27.** $\begin{aligned} x + 2y + 3z &= -2 \\ 2x + 4y + 6z &= -5 \\ x - y + 2z &= 6 \end{aligned}$

**28.** $\begin{aligned} 4x + 8y + 4z &= 9 \\ x + 3y + 4z &= 10 \\ 5x + 10y + 5z &= 12 \end{aligned}$

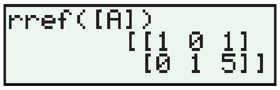
*The augmented matrix of the system* $\begin{aligned} 4x + 8y &= 44 \\ 2x - y &= -3 \end{aligned}$ *is shown in the graphing calculator screen on the left as matrix* [A]. *The screen in the middle shows the row echelon form for* [A]. *The screen on the right shows the "reduced" row echelon form, and from this it can be determined by inspection that the solution set of the system is* $\{(1, 5)\}$.



```
[A]
       [[4 8  44]
        [2 -1  -3]]
```



```
ref([A])
        [[1 2 11]
         [0 1 5 ]]
```



```
rref([A])
         [[1 0 1]
          [0 1 5]]
```

*Use a graphing calculator and either matrix method illustrated to solve each system.*

**29.** $\begin{aligned} 4x + y &= 5 \\ 2x + y &= 3 \end{aligned}$

**30.** $\begin{aligned} 5x + 3y &= 7 \\ 7x - 3y &= -19 \end{aligned}$

**31.** $\begin{aligned} 5x + y - 3z &= -6 \\ 2x + 3y + z &= 5 \\ -3x - 2y + 4z &= 3 \end{aligned}$

**32.** $\begin{aligned} x + y + z &= 3 \\ 3x - 3y - 4z &= -1 \\ x + y + 3z &= 11 \end{aligned}$

**33.** $\begin{aligned} x + z &= -3 \\ y + z &= 3 \\ x + y &= 8 \end{aligned}$

**34.** $\begin{aligned} x - y &= -1 \\ -y + z &= -2 \\ x + z &= -2 \end{aligned}$

## PREVIEW EXERCISES

*Solve each inequality. Write the solution set in interval notation and graph it.* ***See Section 2.8.***

**35.** $x - 4 \geq 12$

**36.** $3x + 1 > 22$

**37.** $-5z + 18 > -2$

**38.** Which one of the following inequalities is equivalent to $x < -3$?

**A.** $-3x < 9$ **B.** $-3x > -9$ **C.** $-3x > 9$ **D.** $-3x < -9$

# CHAPTER 8 SUMMARY

## KEY TERMS

**8.1**

system of linear equations (linear system)
solution of a system
solution set of a system
set-builder notation
consistent system
inconsistent system
independent equations
dependent equations

**8.4**

ordered triple
focus variable
working equation

**8.6**

matrix
element of a matrix
row
column
square matrix
augmented matrix
row operations
row echelon form

## NEW SYMBOLS

| | |
|---|---|
| $(x, y, z)$ | ordered triple |
| $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ | matrix with two rows, three columns |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **system of equations** consists of
- **A.** at least two equations with different variables
- **B.** two or more equations that have an infinite number of solutions
- **C.** two or more equations that are to be solved at the same time
- **D.** two or more inequalities that are to be solved.

**2.** The **solution set of a system of equations** is
- **A.** all ordered pairs that satisfy one equation of the system
- **B.** all ordered pairs that satisfy all the equations of the system at the same time
- **C.** any ordered pair that satisfies one or more equations of the system
- **D.** the set of values that make all the equations of the system false.

**3.** A **consistent system** is a system of equations
- **A.** with one solution
- **B.** with no solution
- **C.** with an infinite number of solutions
- **D.** that have the same graph.

**4.** An **inconsistent system** is a system of equations
- **A.** with one solution
- **B.** with no solution
- **C.** with an infinite number of solutions
- **D.** that have the same graph.

**5.** **Dependent equations**
- **A.** have different graphs
- **B.** have no solution
- **C.** have one solution
- **D.** are different forms of the same equation.

**6.** A **matrix** is
- **A.** an ordered pair of numbers
- **B.** an array of numbers with the same number of rows and columns
- **C.** a pair of numbers written between brackets
- **D.** a rectangular array of numbers.

### ANSWERS

**1.** C; *Example:* $\begin{aligned} 3x - y &= 3 \\ 2x + y &= 7 \end{aligned}$ **2.** B; *Example:* The ordered pair $(2, 3)$ satisfies both equations of the system in Answer 1, so $\{(2, 3)\}$ is the solution set of the system. **3.** A; *Example:* The system in Answer 1 is consistent. The graphs of the equations intersect at exactly one point—in this case, the solution $(2, 3)$. **4.** B; *Example:* The equations of two parallel lines form an inconsistent system. Their graphs never intersect, so the system has no solution. **5.** D; *Example:* The equations $4x - y = 8$ and $8x - 2y = 16$ are dependent because their graphs are the same line. **6.** D; *Examples:* $\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

## QUICK REVIEW

**CONCEPTS** | **EXAMPLES**

### 8.1 Solving Systems of Linear Equations by Graphing

An ordered pair is a solution of a system if it makes all equations of the system true at the same time.

Is $(4, -1)$ a solution of the following system?
$$\begin{aligned} x + y &= 3 \\ 2x - y &= 9 \end{aligned}$$

Yes, because $4 + (-1) = 3$ and $2(4) - (-1) = 9$ are both true, $(4, -1)$ is a solution.

To solve a linear system by graphing, follow these steps.

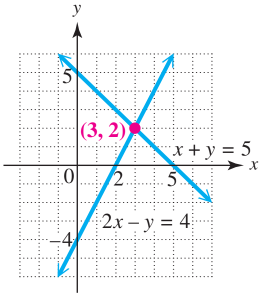
***Step 1*** Graph each equation of the system on the same axes.

***Step 2*** Find the coordinates of the point of intersection.

***Step 3*** Check. Write the solution set.

Solve the system by graphing.
$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

The solution $(3, 2)$ checks, so $\{(3, 2)\}$ is the solution set.



### 8.2 Solving Systems of Linear Equations by Substitution

***Step 1*** Solve one equation for either variable.

Solve by substitution.
$$\begin{aligned} x + 2y &= -5 \quad (1) \\ y &= -2x - 1 \quad (2) \end{aligned}$$

Equation (2) is already solved for $y$.

(continued)

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| ***Step 2*** Substitute for that variable in the other equation to get an equation in one variable.<br>***Step 3*** Solve the equation from Step 2. | Substitute $-2x - 1$ for $y$ in equation (1).<br>$\begin{aligned} x + 2(-2x - 1) &= -5 && \text{Let } y = -2x - 1 \text{ in (1).} \\ x - 4x - 2 &= -5 && \text{Distributive property} \\ -3x - 2 &= -5 && \text{Combine like terms.} \\ -3x &= -3 && \text{Add 2.} \\ x &= 1 && \text{Divide by } -3. \end{aligned}$ |
| ***Step 4*** Substitute the result into the equation from Step 1 to get the value of the other variable. | To find $y$, let $x = 1$ in equation (2).<br>$y = -2(1) - 1 = -3$ |
| ***Step 5*** Check. Write the solution set. | The solution, $(1, -3)$, checks, so $\{(1, -3)\}$ is the solution set. |
| **8.3 Solving Systems of Linear Equations by Elimination** | |
| ***Step 1*** Write both equations in standard form, $Ax + By = C$. | Solve by elimination.<br>$\begin{aligned} x + 3y &= 7 && (1) \\ 3x - y &= 1 && (2) \end{aligned}$ |
| ***Step 2*** Multiply to transform the equations so that the coefficients of one pair of variable terms are opposites.<br>***Step 3*** Add the equations to get an equation with only one variable.<br>***Step 4*** Solve the equation from Step 3. | Multiply equation (1) by $-3$ to eliminate the $x$-terms.<br>$\begin{aligned} -3x - 9y &= -21 && \text{Multiply equation (1) by } -3. \\ 3x - y &= 1 && (2) \\ \hline -10y &= -20 && \text{Add.} \\ y &= 2 && \text{Divide by } -10. \end{aligned}$ |
| ***Step 5*** Substitute the solution from Step 4 into either of the original equations to find the value of the remaining variable. | Substitute to get the value of $x$.<br>$\begin{aligned} x + 3y &= 7 && (1) \\ x + 3(2) &= 7 && \text{Let } y = 2. \\ x + 6 &= 7 && \text{Multiply.} \\ x &= 1 && \text{Subtract 6.} \end{aligned}$ |
| ***Step 6*** Check. Write the solution set. | Since $1 + 3(2) = 7$ and $3(1) - 2 = 1$, the solution set is $\{(1, 2)\}$. |
| If the result of the addition step (Step 3) is a false statement, such as $0 = 4$, the graphs are parallel lines and *there is no solution. The solution set is* $\emptyset$. | $\begin{aligned} x - 2y &= 6 \\ -x + 2y &= -2 \\ \hline 0 &= 4 && \text{Solution set: } \emptyset \end{aligned}$ |
| If the result is a true statement, such as $0 = 0$, the graphs are the same line, and an *infinite number of ordered pairs are solutions. The solution set is written in set-builder notation as* $\{(x, y) \mid \underline{\hspace{3cm}}\}$, *where a form of the equation is written in the blank.* | $\begin{aligned} x - 2y &= 6 \\ -x + 2y &= -6 \\ \hline 0 &= 0 && \text{Solution set: } \{(x, y) \mid x - 2y = 6\} \end{aligned}$ |
| **8.4 Solving Systems of Linear Equations in Three Variables** | |
| **Solving a Linear System in Three Variables** | Solve the system. |
| ***Step 1*** Select a focus variable, preferably one with coefficient 1 or $-1$, and a working equation. | $\begin{aligned} x + 2y - z &= 6 && (1) \\ x + y + z &= 6 && (2) \\ 2x + y - z &= 7 && (3) \end{aligned}$<br>We choose $z$ as the focus variable and (2) as the working equation. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| ***Step 2*** Eliminate the focus variable, using the working equation and one of the equations of the system. | Add equations (1) and (2).<br>$2x + 3y = 12 \quad (4)$ |
| ***Step 3*** Eliminate the focus variable again, using the working equation and the remaining equation of the system. | Add equations (2) and (3).<br>$3x + 2y = 13 \quad (5)$ |
| ***Step 4*** Solve the system of two equations in two variables formed by the equations from Steps 2 and 3. | Use equations (4) and (5) to eliminate $x$.<br>$-6x - 9y = -36$ Multiply (4) by $-3$.<br>$6x + 4y = 26$ Multiply (5) by 2.<br>$-5y = -10$ Add.<br>$y = 2$ Divide by $-5$.<br>To find $x$, substitute 2 for $y$ in equation (4).<br>$2x + 3(2) = 12$ Let $y = 2$ in (4).<br>$2x + 6 = 12$ Multiply.<br>$2x = 6$ Subtract 6.<br>$x = 3$ Divide by 2. |
| ***Step 5*** Find the value of the remaining variable. | Substitute 3 for $x$ and 2 for $y$ in working equation (2).<br>$x + y + z = 6 \quad (2)$<br>$3 + 2 + z = 6$<br>$z = 1$ |
| ***Step 6*** Check the ordered-triple solution in each of the original equations of the system. Then write the solution set. | A check of the solution $(3, 2, 1)$ confirms that the solution set is $\{(3, 2, 1)\}$. |
| **8.5 Applications of Systems of Linear Equations**<br>Use the six-step problem-solving method.<br>***Step 1*** Read the problem carefully.<br>***Step 2*** Assign variables.<br>***Step 3*** Write a system of equations that relates the unknowns.<br>***Step 4*** Solve the system.<br>***Step 5*** State the answer.<br>***Step 6*** Check. | The perimeter of a rectangle is 18 ft. The length is 3 ft more than twice the width. What are the dimensions of the rectangle?<br>Let $x$ represent the length and $y$ represent the width. From the perimeter formula, one equation is $2x + 2y = 18$. From the problem, another equation is $x = 2y + 3$. Solve the system<br>$2x + 2y = 18$<br>$x = 2y + 3$<br>to get $x = 7$ and $y = 2$. The length is 7 ft, and the width is 2 ft. Since the perimeter is<br>$2(7) + 2(2) = 18, \quad \text{and} \quad 2(2) + 3 = 7,$<br>the solution checks. |
| **8.6 Solving Systems of Linear Equations by Matrix Methods**<br>**Matrix Row Operations**<br>1. Any two rows of the matrix may be interchanged. | $\begin{bmatrix} 1 & 5 & 7 \\ 3 & 9 & -2 \\ 0 & 6 & 4 \end{bmatrix}$ becomes $\begin{bmatrix} 3 & 9 & -2 \\ 1 & 5 & 7 \\ 0 & 6 & 4 \end{bmatrix}$ Interchange $R_1$ and $R_2$. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **2.** The elements of any row may be multiplied by any nonzero real number. | $\begin{bmatrix} 1 & 5 & 7 \\ 3 & 9 & -2 \\ 0 & 6 & 4 \end{bmatrix}$ becomes $\begin{bmatrix} 1 & 5 & 7 \\ 1 & 3 & -\frac{2}{3} \\ 0 & 6 & 4 \end{bmatrix}$ $\frac{1}{3}R_2$ |
| **3.** Any row may be changed by adding to the elements of the row the product of a real number and the elements of another row. | $\begin{bmatrix} 1 & 5 & 7 \\ 3 & 9 & -2 \\ 0 & 6 & 4 \end{bmatrix}$ becomes $\begin{bmatrix} 1 & 5 & 7 \\ 0 & -6 & -23 \\ 0 & 6 & 4 \end{bmatrix}$ $-3R_1 + R_2$ |
| A system can be solved by matrix methods. Write the augmented matrix and use row operations to obtain a matrix in row echelon form. | Solve using row operations. $\begin{aligned} x + 3y &= 7 \\ 2x + y &= 4 \end{aligned}$<br>$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 2 & 1 & 4 \end{array}\right]$ Write the augmented matrix.<br>$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right]$ $-2R_1 + R_2$<br>$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right]$ $-\frac{1}{5}R_2$ $\xrightarrow{\text{implies}}$ $\begin{aligned} x + 3y &= 7 \\ y &= 2 \end{aligned}$<br>When $y = 2$, $x + 3(2) = 7$, so $x = 1$. The solution set is $\{(1, 2)\}$. |

## CHAPTER 8 REVIEW EXERCISES

**8.1** *Decide whether the given ordered pair is a solution of the given system.*

**1.** $(3, 4)$
$$\begin{aligned} 4x - 2y &= 4 \\ 5x + y &= 19 \end{aligned}$$

**2.** $(-5, 2)$
$$\begin{aligned} x - 4y &= -13 \\ 2x + 3y &= 4 \end{aligned}$$

*Solve each system by graphing.*

**3.** $$\begin{aligned} x + y &= 4 \\ 2x - y &= 5 \end{aligned}$$

**4.** $$\begin{aligned} x - 2y &= 4 \\ 2x + y &= -2 \end{aligned}$$

**5.** $$\begin{aligned} 2x + 4 &= 2y \\ y - x &= -3 \end{aligned}$$

**6.** $$\begin{aligned} x - 2 &= 2y \\ 2x - 4y &= 4 \end{aligned}$$

**8.2**

**7.** ***Concept Check*** Suppose that you were asked to solve the following system by substitution. Which variable in which equation would be easiest to solve for in your first step?
$$\begin{aligned} 5x - 3y &= 7 \\ -x + 2y &= 4 \end{aligned}$$

**8.** ***Concept Check*** After solving a system of linear equations by the substitution method, a student obtained the equation "$0 = 0$." He gave the solution set of the system as $\{(0, 0)\}$. ***WHAT WENT WRONG?***

*Solve each system by the substitution method.*

**9.** $$\begin{aligned} 3x + y &= 7 \\ x &= 2y \end{aligned}$$

**10.** $$\begin{aligned} 2x - 5y &= -19 \\ y &= x + 2 \end{aligned}$$

**11.** $$\begin{aligned} 4x + 5y &= 44 \\ x + 2 &= 2y \end{aligned}$$

**12.** $$\begin{aligned} 5x + 15y &= 30 \\ x + 3y &= 6 \end{aligned}$$

**8.3**

**13.** *Concept Check* Which system does not require that we multiply one or both equations by a constant to solve the system by the elimination method?

**A.** $\begin{aligned} -4x + 3y &= 7 \\ 3x - 4y &= 4 \end{aligned}$ **B.** $\begin{aligned} 5x + 8y &= 13 \\ 12x + 24y &= 36 \end{aligned}$ **C.** $\begin{aligned} 2x + 3y &= 5 \\ x - 3y &= 12 \end{aligned}$ **D.** $\begin{aligned} x + 2y &= 9 \\ 3x - y &= 6 \end{aligned}$

**14.** *Concept Check* For the system

$$\begin{aligned} 2x + 12y &= 7 \qquad (1) \\ 3x + 4y &= 1, \qquad (2) \end{aligned}$$

if we were to multiply equation (1) by −3, by what number would we have to multiply equation (2) in order to

**(a)** eliminate the $x$-terms when solving by the elimination method?

**(b)** eliminate the $y$-terms when solving by the elimination method?

*Solve each system by the elimination method.*

**15.** $\begin{aligned} 2x - y &= 13 \\ x + y &= 8 \end{aligned}$ **16.** $\begin{aligned} -4x + 3y &= 25 \\ 6x - 5y &= -39 \end{aligned}$ **17.** $\begin{aligned} 3x - 4y &= 9 \\ 6x - 8y &= 18 \end{aligned}$ **18.** $\begin{aligned} 2x + y &= 3 \\ -4x - 2y &= 6 \end{aligned}$

**8.1–8.3** *Solve each system by any method.*

**19.** $\begin{aligned} 2x + 3y &= -5 \\ 3x + 4y &= -8 \end{aligned}$ **20.** $\begin{aligned} 6x - 9y &= 0 \\ 2x - 3y &= 0 \end{aligned}$ **21.** $\begin{aligned} x - 2y &= 5 \\ y &= x - 7 \end{aligned}$

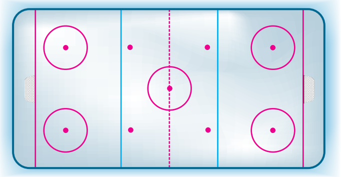
**22.** $\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 7 \\ \frac{x}{4} + \frac{2y}{3} &= 8 \end{aligned}$ **23.** $\begin{aligned} \frac{3}{4}x - \frac{1}{3}y &= \frac{7}{6} \\ \frac{1}{2}x + \frac{2}{3}y &= \frac{5}{3} \end{aligned}$ **24.** $\begin{aligned} 0.4x - 0.5y &= -2.2 \\ 0.3x + 0.2y &= -0.5 \end{aligned}$

**8.4** *Solve each system. If a system is inconsistent or has dependent equations, say so.*

**25.** $\begin{aligned} 2x + 3y - z &= -16 \\ x + 2y + 2z &= -3 \\ -3x + y + z &= -5 \end{aligned}$ **26.** $\begin{aligned} 4x - y &= 2 \\ 3y + z &= 9 \\ x + 2z &= 7 \end{aligned}$ **27.** $\begin{aligned} 3x - y - z &= -8 \\ 4x + 2y + 3z &= 15 \\ -6x + 2y + 2z &= 10 \end{aligned}$

**8.5** *Solve each problem by using a system of equations.*

**28.** A regulation National Hockey League ice rink has perimeter 570 ft. The length of the rink is 30 ft longer than twice the width. What are the dimensions of an NHL ice rink? (*Source:* www.nhl.com)



**29.** In 2009, the New York Yankees and the Boston Red Sox had the most expensive ticket prices in Major League Baseball. Two Yankees tickets and three Red Sox tickets purchased at their average prices cost \$296.66, while three Yankees tickets and two Red Sox tickets cost \$319.39. Find the average ticket price for a Yankees ticket and a Red Sox ticket. (*Source:* Team Marketing Report.)

**30.** A plane flies 560 mi in 1.75 hr traveling with the wind. The return trip later against the same wind takes the plane 2 hr. Find the speed of the plane and the speed of the wind. Let $x$ = the speed of the plane and $y$ = the speed of the wind.

| | *r* | *t* | *d* |
|---|---|---|---|
| *With Wind* | $x + y$ | 1.75 | |
| *Against Wind* | | 2 | |

**31.** For Valentine's Day, Ms. Sweet will mix some \$2-per-lb nuts with some \$1-per-lb chocolate candy to get 100 lb of mix, which she will sell at \$1.30 per lb. How many pounds of each should she use?

| | *Number of Pounds* | *Price per Pound* | *Value* |
|---|---|---|---|
| *Nuts* | $x$ | | |
| *Chocolate* | $y$ | | |
| *Mixture* | 100 | | |

**32.** The sum of the measures of the angles of a triangle is 180°. The largest angle measures 10° less than the sum of the other two. The measure of the middle-sized angle is the average of the other two. Find the measures of the three angles.

**33.** Noemi Alfonso-Triana sells real estate. On three recent sales, she made 10% commission, 6% commission, and 5% commission. Her total commissions on these sales were \$17,000, and she sold property worth \$280,000. If the 5% sale amounted to the sum of the other two, what were the three sales prices?

**34.** How many liters each of 8%, 10%, and 20% hydrogen peroxide should be mixed together to get 8 L of 12.5% solution if the amount of 8% solution used must be 2 L more than the amount of 20% solution used?

**35.** In the great baseball year of 1961, Yankee teammates Mickey Mantle, Roger Maris, and Yogi Berra combined for 137 home runs. Mantle hit 7 fewer than Maris. Maris hit 39 more than Berra. What were the home run totals for each player? (*Source:* Neft, David S., Richard M. Cohen, and Michael Lo Neft, *The Sports Encyclopedia: Baseball 2006.*)

**8.6** *Solve each system of equations by using row operations.*

**36.** $\begin{aligned} 2x + 5y &= -4 \\ 4x - y &= 14 \end{aligned}$

**37.** $\begin{aligned} 6x + 3y &= 9 \\ -7x + 2y &= 17 \end{aligned}$

**38.** $\begin{aligned} x + 2y - z &= 1 \\ 3x + 4y + 2z &= -2 \\ -2x - y + z &= -1 \end{aligned}$

**39.** $\begin{aligned} x + 3y &= 7 \\ 3x + z &= 2 \\ y - 2z &= 4 \end{aligned}$

## MIXED REVIEW EXERCISES

**40.** ***Concept Check*** Which system, A or B, would be easier to solve using the substitution method? Why?

**A.** $\begin{aligned} 5x - 3y &= 7 \\ 2x + 8y &= 3 \end{aligned}$

**B.** $\begin{aligned} 7x + 2y &= 4 \\ y &= -3x + 1 \end{aligned}$

*Solve by any method.*

**41.** $\begin{aligned} \frac{2}{3}x + \frac{1}{6}y &= \frac{19}{2} \\ \frac{1}{3}x - \frac{2}{9}y &= 2 \end{aligned}$

**42.** $\begin{aligned} 2x + 5y - z &= 12 \\ -x + y - 4z &= -10 \\ -8x - 20y + 4z &= 31 \end{aligned}$

**43.** $\begin{aligned} x &= 7y + 10 \\ 2x + 3y &= 3 \end{aligned}$

**44.** $\begin{aligned} x + 4y &= 17 \\ -3x + 2y &= -9 \end{aligned}$

**45.** $\begin{aligned} -7x + 3y &= 12 \\ 5x + 2y &= 8 \end{aligned}$

**46.** $\begin{aligned} 2x - 5y &= 8 \\ 3x + 4y &= 10 \end{aligned}$

**47.** To make a 10% acid solution, Jeffrey Guild wants to mix some 5% solution with 10 L of 20% solution. How many liters of 5% solution should he use?

**48.** In the 2010 Winter Olympics, Germany, the United States, and Canada won a combined total of 93 medals. Germany won seven fewer medals than the United States, while Canada won 11 fewer medals than the United States. How many medals did each country win? (*Source:* www.vancouver2010.com/olympic-medals)

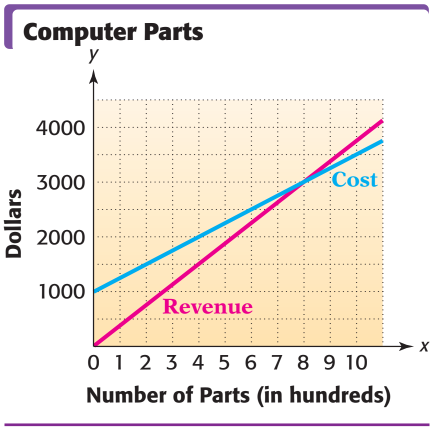
# CHAPTER 8 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*The graph shows a company's costs to produce computer parts and the revenue from the sale of computer parts.*



**1.** At what production level does the cost equal the revenue? What is the revenue at that point?

**2.** Profit is revenue less cost. Estimate the profit on the sale of 1100 parts.

**3.** Decide whether each ordered pair is a solution of the system.

$$\begin{aligned} 2x + y &= -3 \\ x - y &= -9 \end{aligned}$$

**(a)** $(1, -5)$ **(b)** $(1, 10)$ **(c)** $(-4, 5)$

**4.** Use a graph to solve the system. $\begin{aligned} x + y &= 7 \\ x - y &= 5 \end{aligned}$

*Solve each system by substitution or elimination. If a system is inconsistent or has dependent equations, say so.*

**5.** $\begin{aligned} 2x - 3y &= 24 \\ y &= -\frac{2}{3}x \end{aligned}$

**6.** $\begin{aligned} 3x - y &= -8 \\ 2x + 6y &= 3 \end{aligned}$

**7.** $\begin{aligned} 12x - 5y &= 8 \\ 3x &= \frac{5}{4}y + 2 \end{aligned}$

**8.** $\begin{aligned} 3x + y &= 12 \\ 2x - y &= 3 \end{aligned}$

**9.** $\begin{aligned} -5x + 2y &= -4 \\ 6x + 3y &= -6 \end{aligned}$

**10.** $\begin{aligned} 3x + 4y &= 8 \\ 8y &= 7 - 6x \end{aligned}$

**11.** $\begin{aligned} \frac{6}{5}x - \frac{1}{3}y &= -20 \\ -\frac{2}{3}x + \frac{1}{6}y &= 11 \end{aligned}$

**12.** $\begin{aligned} 3x + 5y + 3z &= 2 \\ 6x + 5y + z &= 0 \\ 3x + 10y - 2z &= 6 \end{aligned}$

**13.** $\begin{aligned} 4x + y + z &= 11 \\ x - y - z &= 4 \\ y + 2z &= 0 \end{aligned}$

*Solve each problem using a system of equations.*

**14.** Harrison Ford is a box-office star. As of January 2010, his two top-grossing domestic films, *Star Wars Episode IV: A New Hope* and *Indiana Jones and the Kingdom of the Crystal Skull,* earned \$778.0 million together. If *Indiana Jones and the Kingdom of the Crystal Skull* grossed \$144.0 million less than *Star Wars Episode IV: A New Hope,* how much did each film gross? (*Source:* www.the-numbers.com)

**15.** Two cars start from points 420 mi apart and travel toward each other. They meet after 3.5 hr. Find the average rate of each car if one travels 30 mph slower than the other.

**16.** A chemist needs 12 L of a 40% alcohol solution. She must mix a 20% solution and a 50% solution. How many liters of each will be required to obtain what she needs?

**17.** A local electronics store will sell seven AC adaptors and two rechargeable flashlights for \$86, or three AC adaptors and four rechargeable flashlights for \$84. What is the price of a single AC adaptor and a single rechargeable flashlight?

**18.** The owner of a tea shop wants to mix three kinds of tea to make 100 oz of a mixture that will sell for \$0.83 per oz. He uses Orange Pekoe, which sells for \$0.80 per oz, Irish Breakfast, for \$0.85 per oz, and Earl Grey, for \$0.95 per oz. If he wants to use twice as much Orange Pekoe as Irish Breakfast, how much of each kind of tea should he use?

*Solve each system using row operations.*

**19.** $$\begin{aligned} 3x + 2y &= 4 \\ 5x + 5y &= 9 \end{aligned}$$

**20.** $$\begin{aligned} x + 3y + 2z &= 11 \\ 3x + 7y + 4z &= 23 \\ 5x + 3y - 5z &= -14 \end{aligned}$$

# CHAPTERS 1–8 CUMULATIVE REVIEW EXERCISES

**1.** List all integer factors of 40.

**2.** Evaluate $-2 + 6[3 - (4 - 9)]$.

**3.** Find the value of the expression $\dfrac{3x^2 + 2y^2}{10y + 3}$ for $x = 1$ and $y = 5$.

**4.** Name the property that justifies the statement: $r(s - k) = rs - rk$.

*Solve each linear equation.*

**5.** $2 - 3(6x + 2) = 4(x + 1) + 18$

**6.** $\dfrac{3}{2}\left(\dfrac{1}{3}x + 4\right) = 6\left(\dfrac{1}{4} + x\right)$

**7.** Solve the formula $P = \dfrac{kT}{V}$ for $T$.

*Solve each linear inequality.*

**8.** $-\dfrac{5}{6}x < 15$

**9.** $-8 < 2x + 3$

**10.** A survey measured public recognition of some classic advertising slogans. Complete the results shown in the table if 2500 people were surveyed.

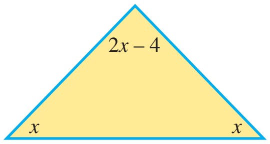
| *Slogan (product or company)* | *Percent Recognition (nearest tenth of a percent)* | *Actual Number That Recognized Slogan (nearest whole number)* |
|---|---|---|
| Please Don't Squeeze the . . . (Charmin®) | 80.4% | |
| The Breakfast of Champions (Wheaties) | 72.5% | |
| The King of Beers (Budweiser®) | | 1570 |
| Like a Good Neighbor (State Farm) | | 1430 |

(Other slogans included "You're in Good Hands" (Allstate), "Snap, Crackle, Pop" (Rice Krispies®), and "The Un-Cola" (7-Up).)
*Source:* Department of Integrated Marketing Communications, Northwestern University.

*Solve each problem.*

**11.** On August 6, 2009, the U.S. Senate confirmed Sonia Sotomayor, as the 111th Justice of the United States Supreme Court. With 99 senators voting, 37 more voted in favor of her confirmation than voted against it. How many senators voted each way? (*Source: The New York Times*.)

**12.** Two angles of a triangle have the same measure. The measure of the third angle is 4° less than twice the measure of each of the equal angles. Find the measures of the three angles.



Measures are in degrees.

*Graph each linear equation.*

**13.** $x - y = 4$

**14.** $3x + y = 6$

*Find the slope of each line.*

**15.** Through $(-5, 6)$ and $(1, -2)$

**16.** Perpendicular to the line $y = 4x - 3$

*Find an equation for each line. Write it in slope-intercept form.*

**17.** Through $(-4, 1)$ with slope $\frac{1}{2}$

**18.** Through the points $(1, 3)$ and $(-2, -3)$

**19.** **(a)** Write an equation of the vertical line through $(9, -2)$.

**(b)** Write an equation of the horizontal line through $(4, -1)$.

**20.** Simplify $\left(\dfrac{m^{-4}n^2}{m^2n^{-3}}\right) \cdot \left(\dfrac{m^5n^{-1}}{m^{-2}n^5}\right)$. Write the answer with only positive exponents. Assume that all variables represent nonzero real numbers.

*Perform the indicated operations.*

**21.** $(3y^2 - 2y + 6) - (-y^2 + 5y + 12)$

**22.** $(4f + 3)(3f - 1)$

**23.** $\left(\dfrac{1}{4}x + 5\right)^2$

**24.** $(3x^3 + 13x^2 - 17x - 7) \div (3x + 1)$

*Factor each polynomial completely.*

**25.** $2x^2 - 13x - 45$

**26.** $100t^4 - 25$

**27.** $8p^3 + 125$

**28.** Solve the equation $3x^2 + 4x = 7$.

**29.** Write $\dfrac{y^2 - 16}{y^2 - 8y + 16}$ in lowest terms.

*Perform the indicated operations. Express the answer in lowest terms.*

**30.** $\dfrac{2a^2}{a + b} \cdot \dfrac{a - b}{4a}$

**31.** $\dfrac{x + 4}{x - 2} + \dfrac{2x - 10}{x - 2}$

**32.** Solve the equation $\dfrac{-3x}{x + 1} + \dfrac{4x + 1}{x} = \dfrac{-3}{x^2 + x}$.

**33.** Suppose that $y = f(x)$ and $5x - 3y = 8$.

**(a)** Find the equation that defines $f(x)$. That is, $f(x) =$ ________.

**(b)** Find $f(1)$.

**34.** For the polynomial functions defined by

$$f(x) = x^2 + 2x - 3, \quad g(x) = 2x^3 - 3x^2 + 4x - 1, \quad \text{and} \quad h(x) = x^2,$$

find **(a)** $(f + g)(x)$, **(b)** $(g - f)(x)$, **(c)** $(f + g)(-1)$, and **(d)** $(f \circ h)(x)$.

*Solve by any method.*

**35.** $\begin{aligned} -2x + 3y &= -15 \\ 4x - y &= 15 \end{aligned}$

**36.** $\begin{aligned} x - 3y &= 7 \\ 2x - 6y &= 14 \end{aligned}$

**37.** $\begin{aligned} x + y + z &= 10 \\ x - y - z &= 0 \\ -x + y - z &= -4 \end{aligned}$

**38.** Ten years after the original Tickle Me Elmo became a must-have toy, a new version, called T.M.X., was released in the fall of 2006. The original Tickle Me Elmo's average cost was \$12.37 less than the recommended cost of T.M.X., and one of each cost \$67.63. Find the average cost of Tickle Me Elmo and the recommended cost of T.M.X. (*Source:* NPD Group, Inc.; *USA Today*.)

CHAPTER 9

# Inequalities and Absolute Value

During the past 30 years, the cost of a college education in the United States has increased more rapidly than average prices of other goods and services in the economy in general. For four-year public colleges and universities, the average cost of tuition and fees, adjusted for inflation, increased 62% from the 1999–2000 school year to the 2009–2010 school year. For two-year public colleges, this increase was a more affordable 19% during the same period. Yet a college degree remains of major importance to individual long-term financial stability. (*Source:* The College Board.)

In **Exercises 63–66** of **Section 9.1,** we apply the concepts of this chapter to college student expenses.

# 9.1 Set Operations and Compound Inequalities

**OBJECTIVES**

1. Find the intersection of two sets.
2. Solve compound inequalities with the word *and*.
3. Find the union of two sets.
4. Solve compound inequalities with the word *or*.

Consider the two sets $A$ and $B$ defined as follows.

$$A = \{1, 2, 3\}, \qquad B = \{2, 3, 4\}$$

The set of all elements that belong to both $A$ **and** $B$, called their *intersection* and symbolized $\boldsymbol{A \cap B}$, is given by

$$A \cap B = \{2, 3\}. \qquad \text{Intersection}$$

The set of all elements that belong to either $A$ **or** $B$, or both, called their *union* and symbolized $\boldsymbol{A \cup B}$, is given by

$$A \cup B = \{1, 2, 3, 4\}. \qquad \text{Union}$$

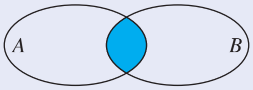
We discuss the use of the words *and* and *or* as they relate to sets and inequalities.

**OBJECTIVE 1** **Find the intersection of two sets.** The intersection of two sets is defined with the word *and*.

**Intersection of Sets**

For any two sets $A$ and $B$, the **intersection** of $A$ and $B$, symbolized $A \cap B$, is defined as follows.

$$A \cap B = \{x \mid x \text{ is an element of } A \textbf{ and } x \text{ is an element of } B\}$$



**NOW TRY EXERCISE 1**
Let $A = \{2, 4, 6, 8\}$ and $B = \{0, 2, 6, 8\}$. Find $A \cap B$.

**EXAMPLE 1** Finding the Intersection of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cap B$.

The set $A \cap B$ contains those elements that belong to both $A$ *and* $B$: the numbers 2 and 4. Therefore,

$$\begin{aligned} A \cap B &= \{1, 2, 3, 4\} \cap \{2, 4, 6\} \\ &= \{2, 4\}. \end{aligned}$$

NOW TRY

A **compound inequality** consists of two inequalities linked by a connective word.

$$x + 1 \le 9 \quad \text{and} \quad x - 2 \ge 3$$
$$2x > 4 \quad \text{or} \quad 3x - 6 < 5$$

Examples of compound inequalities linked by *and* or *or*

**OBJECTIVE 2** **Solve compound inequalities with the word *and*.** We use the following steps to solve a compound inequality such as "$x + 1 \le 9$ *and* $x - 2 \ge 3$."

**Solving a Compound Inequality with *and***

***Step 1*** Solve each inequality individually.

***Step 2*** Since the inequalities are joined with *and,* the solution set of the compound inequality will include all numbers that satisfy both inequalities in Step 1 (the intersection of the solution sets).

**NOW TRY ANSWER**
1. $\{2, 6, 8\}$

**NOW TRY EXERCISE 2**

Solve the compound inequality, and graph the solution set.

$$x - 2 \le 5 \quad \text{and} \quad x + 5 \ge 9$$

## EXAMPLE 2 Solving a Compound Inequality with *and*

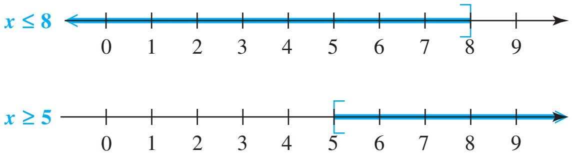
Solve the compound inequality, and graph the solution set.

$$x + 1 \le 9 \quad \text{and} \quad x - 2 \ge 3$$

***Step 1*** Solve each inequality individually using the addition property of inequality.

$$\begin{array}{rlcrl} x + 1 \le 9 & & \text{and} & x - 2 \ge 3 & \\ x + 1 - 1 \le 9 - 1 & \text{Subtract 1.} & \text{and} & x - 2 + 2 \ge 3 + 2 & \text{Add 2.} \\ x \le 8 & & \text{and} & x \ge 5 & \end{array}$$

***Step 2*** Because of the word *and,* the solution set will include all numbers that satisfy both inequalities in Step 1 at the same time. The compound inequality is true whenever $x \le 8$ and $x \ge 5$ are both true. See the graphs in **FIGURE 1**.



The set of points where the graphs "overlap" represents the intersection.

**FIGURE 1**

The intersection of the two graphs is the solution set of the compound inequality. **FIGURE 2** shows that the solution set, in interval notation, is $[5, 8]$.

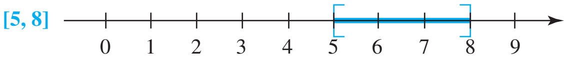


**FIGURE 2**

NOW TRY

**NOW TRY EXERCISE 3**

Solve and graph.

$$-4x - 1 < 7 \quad \text{and} \quad 3x + 4 \ge -5$$

## EXAMPLE 3 Solving a Compound Inequality with *and*

Solve the compound inequality, and graph the solution set.

$$-3x - 2 > 5 \quad \text{and} \quad 5x - 1 \le -21$$

***Step 1*** Solve each inequality individually.

$$\begin{array}{rlcrl} -3x - 2 > 5 & & \text{and} & 5x - 1 \le -21 & \\ -3x > 7 & \text{Add 2.} & \text{and} & 5x \le -20 & \text{Add 1.} \\ x < -\frac{7}{3} & \text{Divide by } -3. & \text{and} & x \le -4 & \text{Divide by 5.} \end{array}$$

Reverse the inequality symbol when dividing by a negative number.

The graphs of $x < -\frac{7}{3}$ and $x \le -4$ are shown in **FIGURE 3**.

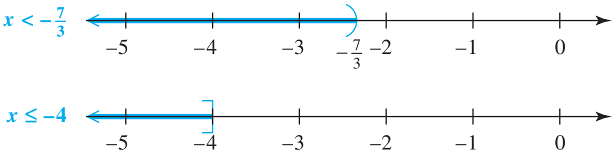


**FIGURE 3**

***Step 2*** Now find all values of $x$ that are less than $-\frac{7}{3}$ and also less than or equal to $-4$. As shown in **FIGURE 4**, the solution set is $(-\infty, -4]$.

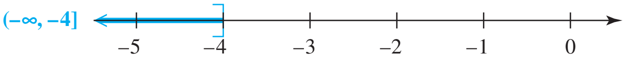


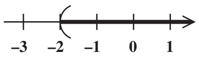
**FIGURE 4**

NOW TRY

**NOW TRY ANSWERS**

**2.** $[4, 7]$

**3.** $(-2, \infty)$



### EXAMPLE 4 Solving a Compound Inequality with *and*

Solve the compound inequality, and graph the solution set.

$$x + 2 < 5 \quad \text{and} \quad x - 10 > 2$$

***Step 1*** Solve each inequality individually.

$$\begin{aligned} x + 2 &< 5 & &\text{and} & x - 10 &> 2 \\ x &< 3 \quad \text{Subtract 2.} & &\text{and} & x &> 12 \quad \text{Add 10.} \end{aligned}$$

The graphs of $x < 3$ and $x > 12$ are shown in **FIGURE 5**.

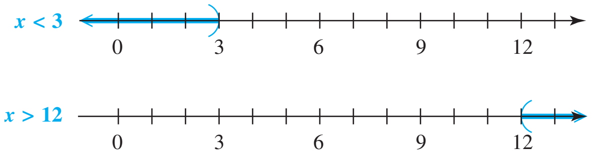


**FIGURE 5**

***Step 2*** There is no number that is both less than 3 *and* greater than 12, so the given compound inequality has no solution. The solution set is $\emptyset$. See **FIGURE 6**.

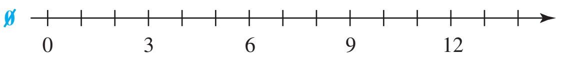


**FIGURE 6**

NOW TRY

**NOW TRY EXERCISE 4**

Solve and graph.

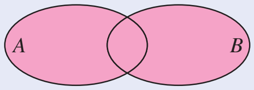
$$x - 7 < -12 \quad \text{and} \quad 2x + 1 > 5$$

**OBJECTIVE 3** **Find the union of two sets.** The union of two sets is defined with the word *or*.

### Union of Sets

For any two sets $A$ and $B$, the **union** of $A$ and $B$, symbolized $A \cup B$, is defined as follows.

$$A \cup B = \{x \mid x \text{ is an element of } A \textbf{ or } x \text{ is an element of } B\}$$



### EXAMPLE 5 Finding the Union of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cup B$.

Begin by listing all the elements of set $A$: 1, 2, 3, 4. Then list any additional elements from set $B$. In this case the elements 2 and 4 are already listed, so the only additional element is 6.

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4\} \cup \{2, 4, 6\} \\ &= \{1, 2, 3, 4, 6\} \end{aligned}$$

The union consists of all elements in either $A$ *or* $B$ (or both). NOW TRY

**NOW TRY EXERCISE 5**

Let $A = \{5, 10, 15, 20\}$ and $B = \{5, 15, 25\}$. Find $A \cup B$.

**NOTE** In **Example 5,** notice that although the elements 2 and 4 appeared in both sets $A$ and $B$, they are written only once in $A \cup B$.

**NOW TRY ANSWERS**

**4.** $\emptyset$ **5.** $\{5, 10, 15, 20, 25\}$

**OBJECTIVE 4** **Solve compound inequalities with the word *or*.** Use the following steps to solve a compound inequality such as “$6x - 4 < 2x$ *or* $-3x \le -9$.”

### Solving a Compound Inequality with *or*

***Step 1*** Solve each inequality individually.

***Step 2*** Since the inequalities are joined with *or,* the solution set of the compound inequality includes all numbers that satisfy either one of the two inequalities in Step 1 (the union of the solution sets).

*NOW TRY EXERCISE 6*

Solve and graph.

$-12x \le -24$ or $x + 9 < 8$

### *EXAMPLE 6* Solving a Compound Inequality with *or*

Solve the compound inequality, and graph the solution set.

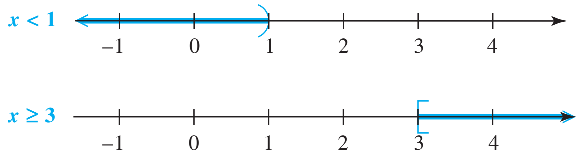
$$6x - 4 < 2x \quad \text{or} \quad -3x \le -9$$

***Step 1*** Solve each inequality individually.

$$\begin{aligned} 6x - 4 &< 2x & &\text{or} & -3x &\le -9 \\ 4x &< 4 & & & & \\ x &< 1 \quad \text{Divide by 4.} & &\text{or} & x &\ge 3 \quad \text{Divide by } -3. \end{aligned}$$

Remember to reverse the inequality symbol.

The graphs of these two inequalities are shown in **FIGURE 7**.



The set of points in either of the graphs represents the union.

**FIGURE 7**

***Step 2*** Since the inequalities are joined with *or,* find the union of the two solution sets. The union is shown in **FIGURE 8** and is written

$$(-\infty, 1) \cup [3, \infty).$$

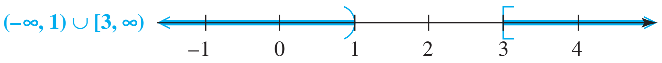


**FIGURE 8**

*NOW TRY*

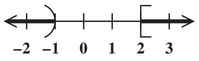
**CAUTION** When inequalities are used to write the solution set in **Example 6,** it *must* be written as

$$x < 1 \quad \text{or} \quad x \ge 3,$$

which keeps the numbers 1 and 3 in their order on the number line. Writing $3 \le x < 1$, which translates using *and*, would imply that $3 \le 1$, which is ***FALSE.*** There is no other way to write the solution set of such a union.

*NOW TRY ANSWER*

**6.** $(-\infty, -1) \cup [2, \infty)$



*NOW TRY EXERCISE 7*

Solve and graph.

$-x + 2 < 6$ or $6x - 8 \geq 10$

## EXAMPLE 7 Solving a Compound Inequality with *or*

Solve the compound inequality, and graph the solution set.

$$-4x + 1 \geq 9 \quad \text{or} \quad 5x + 3 \leq -12$$

***Step 1*** Solve each inequality individually.

$-4x + 1 \geq 9$ or $5x + 3 \leq -12$

$-4x \geq 8$ Subtract 1. or $5x \leq -15$ Subtract 3.

$x \leq -2$ Divide by $-4$. or $x \leq -3$ Divide by 5.

The graphs of these two inequalities are shown in **FIGURE 9**.

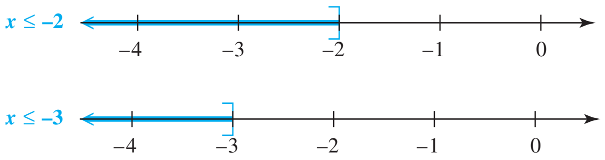


**FIGURE 9**

***Step 2*** By taking the union, we obtain the interval $(-\infty, -2]$. See **FIGURE 10**.

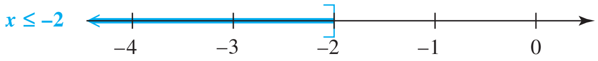


**FIGURE 10**

*NOW TRY*

*NOW TRY EXERCISE 8*

Solve and graph.

$8x - 4 \geq 20$ or $-2x + 1 > -9$

## EXAMPLE 8 Solving a Compound Inequality with *or*

Solve the compound inequality, and graph the solution set.

$$-2x + 5 \geq 11 \quad \text{or} \quad 4x - 7 \geq -27$$

***Step 1*** Solve each inequality separately.

$-2x + 5 \geq 11$ or $4x - 7 \geq -27$

$-2x \geq 6$ Subtract 5. or $4x \geq -20$ Add 7.

$x \leq -3$ Divide by $-2$. or $x \geq -5$ Divide by 4.

The graphs of these two inequalities are shown in **FIGURE 11**.

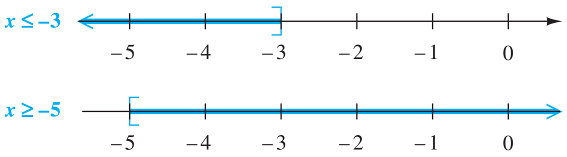


**FIGURE 11**

***Step 2*** By taking the union, we obtain every real number as a solution, since every real number satisfies at least one of the two inequalities. The set of all real numbers is written in interval notation as $(-\infty, \infty)$ and graphed as in **FIGURE 12**.

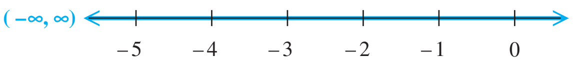
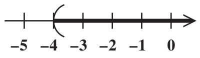


**FIGURE 12**

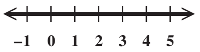
*NOW TRY*

*NOW TRY ANSWERS*

7. $(-4, \infty)$



8. $(-\infty, \infty)$



NOW TRY EXERCISE 9

In **Example 9,** list the elements that satisfy each set.

**(a)** The set of films with admissions greater than 140,000,000 and gross income less than \$1,200,000,000

**(b)** The set of films with admissions less than 200,000,000 or gross income less than \$1,200,000,000

## EXAMPLE 9 Applying Intersection and Union

The five highest-grossing domestic films (adjusted for inflation) as of 2009 are listed in the table.

Five All-Time Highest-Grossing Domestic Films

| *Film* | *Admissions* | *Gross Income* |
|---|---|---|
| *Gone with the Wind* | 202,044,600 | \$1,450,680,400 |
| *Star Wars* | 178,119,600 | \$1,278,898,700 |
| *The Sound of Music* | 142,415,400 | \$1,022,542,400 |
| *E.T.* | 141,854,300 | \$1,018,514,100 |
| *The Ten Commandments* | 131,000,000 | \$ 940,580,000 |

*Source:* boxofficemojo.com.

List the elements of the following sets.

**(a)** The set of the top five films with admissions greater than 180,000,000 *and* gross income greater than \$1,000,000,000

The only film that satisfies both conditions is *Gone with the Wind,* so the set is

$$\{\textit{Gone with the Wind}\}.$$

**(b)** The set of the top five films with admissions less than 140,000,000 *or* gross income greater than \$1,000,000,000

Here, any film that satisfies at least one of the conditions is in the set. This set includes all five films:

{*Gone with the Wind*, *Star Wars*, *The Sound of Music*, *E.T.*, *The Ten Commandments*}.

NOW TRY

NOW TRY ANSWERS

**9. (a)** {*The Sound of Music, E.T.*}

**(b)** {*Star Wars, The Sound of Music, E.T., The Ten Commandments*}

## 9.1 EXERCISES

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***Concept Check*** *Decide whether each statement is* true *or* false. *If it is false, explain why.*

**1.** The union of the solution sets of $x + 1 = 6$, $x + 1 < 6$, and $x + 1 > 6$ is $(-\infty, \infty)$.

**2.** The intersection of the sets $\{x \mid x \geq 9\}$ and $\{x \mid x \leq 9\}$ is $\emptyset$.

**3.** The union of the sets $(-\infty, 7)$ and $(7, \infty)$ is $\{7\}$.

**4.** The intersection of the sets $(-\infty, 7]$ and $[7, \infty)$ is $\{7\}$.

**5.** The intersection of the set of rational numbers and the set of irrational numbers is $\{0\}$.

**6.** The union of the set of rational numbers and the set of irrational numbers is the set of real numbers.

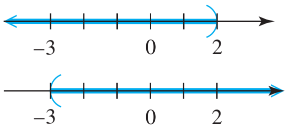
*Let* $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5\}$, $C = \{1, 6\}$ *and* $D = \{4\}$. *Specify each set.* ***See Examples 1 and 5.***

◉ **7.** $B \cap A$ **8.** $A \cap B$ **9.** $A \cap D$ **10.** $B \cap C$

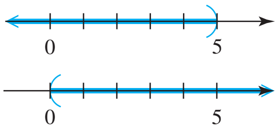
**11.** $B \cap \emptyset$ **12.** $A \cap \emptyset$ ◉ **13.** $A \cup B$ **14.** $B \cup D$

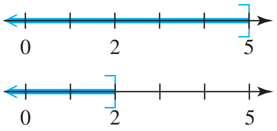
***Concept Check*** *Two sets are specified by graphs. Graph the intersection of the two sets.*

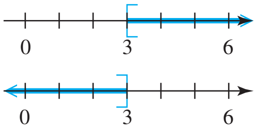
**15.**



**16.**



**17.**


**18.**


*For each compound inequality, give the solution set in both interval and graph form.* ***See Examples 2–4.***

**19.** $x < 2$ and $x > -3$

**20.** $x < 5$ and $x > 0$

**21.** $x \leq 2$ and $x \leq 5$

**22.** $x \geq 3$ and $x \geq 6$

**23.** $x \leq 3$ and $x \geq 6$

**24.** $x \leq -1$ and $x \geq 3$

**25.** $x - 3 \leq 6$ and $x + 2 \geq 7$

**26.** $x + 5 \leq 11$ and $x - 3 \geq -1$

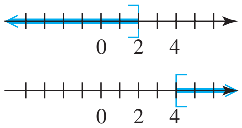
**27.** $-3x > 3$ and $x + 3 > 0$

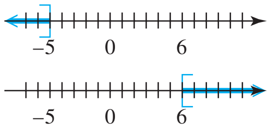
**28.** $-3x < 3$ and $x + 2 < 6$

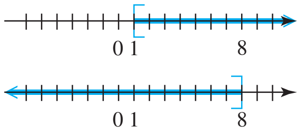
**29.** $3x - 4 \leq 8$ and $-4x + 1 \geq -15$

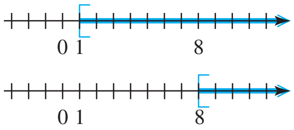
**30.** $7x + 6 \leq 48$ and $-4x \geq -24$

***Concept Check*** *Two sets are specified by graphs. Graph the union of the two sets.*

**31.**


**32.**


**33.**


**34.**


*For each compound inequality, give the solution set in both interval and graph form.* ***See Examples 6–8.***

**35.** $x \leq 1$ or $x \leq 8$

**36.** $x \geq 1$ or $x \geq 8$

**37.** $x \geq -2$ or $x \geq 5$

**38.** $x \leq -2$ or $x \leq 6$

**39.** $x \geq -2$ or $x \leq 4$

**40.** $x \geq 5$ or $x \leq 7$

**41.** $x + 2 > 7$ or $1 - x > 6$

**42.** $x + 1 > 3$ or $x + 4 < 2$

**43.** $x + 1 > 3$ or $-4x + 1 > 5$

**44.** $3x < x + 12$ or $x + 1 > 10$

**45.** $4x + 1 \geq -7$ or $-2x + 3 \geq 5$

**46.** $3x + 2 \leq -7$ or $-2x + 1 \leq 9$

***Concept Check*** *Express each set in the simplest interval form. (Hint: Graph each set and look for the intersection or union.)*

**47.** $(-\infty, -1] \cap [-4, \infty)$

**48.** $[-1, \infty) \cap (-\infty, 9]$

**49.** $(-\infty, -6] \cap [-9, \infty)$

**50.** $(5, 11] \cap [6, \infty)$

**51.** $(-\infty, 3) \cup (-\infty, -2)$

**52.** $[-9, 1] \cup (-\infty, -3)$

**53.** $[3, 6] \cup (4, 9)$

**54.** $[-1, 2] \cup (0, 5)$

*For each compound inequality, decide whether* intersection *or* union *should be used. Then give the solution set in both interval and graph form.* ***See Examples 2–4 and 6–8.***

**55.** $x < -1$ and $x > -5$

**56.** $x > -1$ and $x < 7$

**57.** $x < 4$ or $x < -2$

**58.** $x < 5$ or $x < -3$

**59.** $-3x \leq -6$ or $-3x \geq 0$

**60.** $2x - 6 \leq -18$ and $2x \geq -18$

**61.** $x + 1 \geq 5$ and $x - 2 \leq 10$

**62.** $-8x \leq -24$ or $-5x \geq 15$

*Average expenses for full-time resident college students at 4-year institutions during the 2007–2008 academic year are shown in the table.*

College Expenses (in Dollars), 4-Year Institutions

| *Type of Expense* | *Public Schools (in-state)* | *Private Schools* |
|---|---|---|
| Tuition and fees | 5950 | 21,588 |
| Board rates | 3402 | 3993 |
| Dormitory charges | 4072 | 4812 |

*Source:* National Center for Education Statistics.

*Refer to the table on college expenses. List the elements of each set.* ***See Example 9.***

**63.** The set of expenses that are less than $6500 for public schools *and* are greater than $10,000 for private schools

**64.** The set of expenses that are greater than $3000 for public schools *and* are less than $4000 for private schools

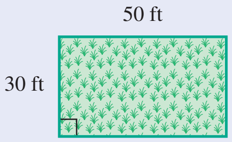
**65.** The set of expenses that are less than $6500 for public schools *or* are greater than $10,000 for private schools

**66.** The set of expenses that are greater than $12,000 *or* are between $5000 and $6000

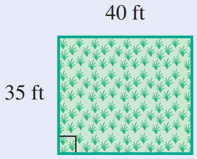
## *RELATING CONCEPTS* EXERCISES 67–72

### FOR INDIVIDUAL OR GROUP WORK

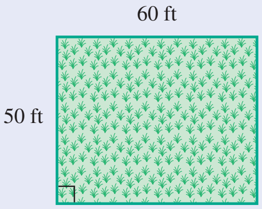
*The figures represent the backyards of neighbors Luigi, Maria, Than, and Joe. Find the area and the perimeter of each yard. Suppose that each resident has* 150 *ft of fencing and enough sod to cover* 1400 *ft*$^2$ *of lawn. Give the name or names of the residents whose yards satisfy each description.* ***Work Exercises 67–72 in order.***



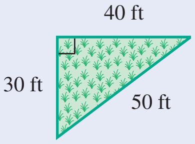
Luigi's yard



Maria's yard



Than's yard



Joe's yard

**67.** The yard can be fenced *and* the yard can be sodded.

**68.** The yard can be fenced *and* the yard cannot be sodded.

**69.** The yard cannot be fenced *and* the yard can be sodded.

**70.** The yard cannot be fenced *and* the yard cannot be sodded.

**71.** The yard can be fenced *or* the yard can be sodded.

**72.** The yard cannot be fenced *or* the yard can be sodded.

## *PREVIEW EXERCISES*

*Solve each three-part inequality.* ***See Section 2.8.***

**73.** $-5 < 2x + 1 < 5$

**74.** $-7 \leq 3x - 2 < 7$

*Evaluate.* ***See Sections 1.4 and 1.5.***

**75.** $-|6| - |-11| + (-4)$

**76.** $(-5) - |-9| + |5 - 4|$

**77.** *True* or *false*? The absolute value of a number is always positive.

**78.** *True* or *false*? If $a < 0$, then $|a| = -a$.

## 9.2 Absolute Value Equations and Inequalities

**OBJECTIVES**

1. Use the distance definition of absolute value.
2. Solve equations of the form $|ax + b| = k$, for $k > 0$.
3. Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.
4. Solve absolute value equations that involve rewriting.
5. Solve equations of the form $|ax + b| = |cx + d|$.
6. Solve special cases of absolute value equations and inequalities.

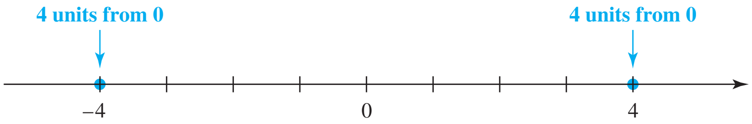
Suppose that the government of a country decides that it will comply with a certain restriction on greenhouse gas emissions *within* 3 years of 2020. This means that the *difference* between the year it will comply and 2020 is less than 3, *without regard to sign*. We state this mathematically as

$$|x - 2020| < 3, \quad \text{Absolute value inequality}$$

where $x$ represents the year in which it complies.

Reasoning tells us that the year must be between 2017 and 2023, and thus $2017 < x < 2023$ makes this inequality true. But what general procedure is used to solve such an inequality? We now investigate how to solve absolute value equations and inequalities.

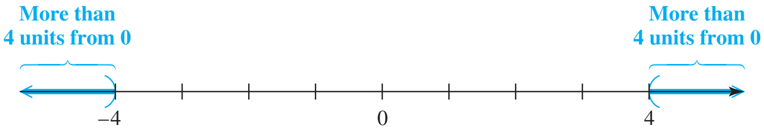
**OBJECTIVE 1** **Use the distance definition of absolute value.** In **Section 1.4,** we saw that the absolute value of a number $x$, written $\mathbf{|x|}$, represents the distance from $x$ to 0 on the number line. For example, the solutions of $|x| = 4$ are 4 and $-4$, as shown in **FIGURE 13**.



$x = -4$ or $x = 4$

FIGURE 13

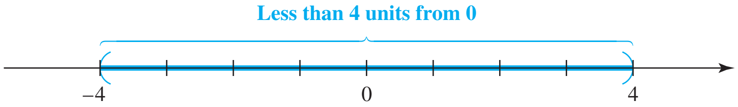
Because absolute value represents distance from 0, we interpret the solutions of $|x| > 4$ to be all numbers that are *more* than four units from 0. The set $(-\infty, -4) \cup (4, \infty)$ fits this description. **FIGURE 14** shows the graph of the solution set of $|x| > 4$. Because the graph consists of two separate intervals, the solution set is described using the word *or*: $x < -4$ or $x > 4$.



$x < -4$ or $x > 4$

FIGURE 14

The solution set of $|x| < 4$ consists of all numbers that are *less* than 4 units from 0 on the number line. This is represented by all numbers *between* $-4$ and 4. This set of numbers is given by $(-4, 4)$, as shown in **FIGURE 15**. Here, the graph shows that $-4 < x < 4$, which means $x > -4$ *and* $x < 4$.



$-4 < x < 4$

FIGURE 15

The equation and inequalities just described are examples of **absolute value equations and inequalities.** They involve the absolute value of a variable expression and generally take the form

$$|ax + b| = k, \qquad |ax + b| > k, \qquad \text{or} \qquad |ax + b| < k,$$

where $k$ is a positive number. From **FIGURES 13–15**, we see that

$|x| = 4$ has the same solution set as $x = -4$ or $x = 4$,
$|x| > 4$ has the same solution set as $x < -4$ or $x > 4$,
$|x| < 4$ has the same solution set as $x > -4$ and $x < 4$.

Thus, we solve an absolute value equation or inequality by solving the appropriate compound equation or inequality.

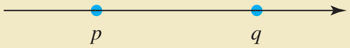
### Solving Absolute Value Equations and Inequalities

Let $k$ be a positive real number and $p$ and $q$ be real numbers.

***Case 1*** To solve $|ax + b| = k$, solve the following compound equation.

$$ax + b = k \quad \text{or} \quad ax + b = -k$$

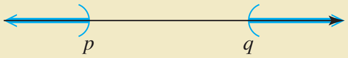
The solution set is usually of the form $\{p, q\}$, which includes two numbers.



***Case 2*** To solve $|ax + b| > k$, solve the following compound inequality.

$$ax + b > k \quad \text{or} \quad ax + b < -k$$

The solution set is of the form $(-\infty, p) \cup (q, \infty)$, which is a disjoint interval.



***Case 3*** To solve $|ax + b| < k$, solve the following three-part inequality.

$$-k < ax + b < k$$

The solution set is of the form $(p, q)$, a single interval.

**NOTE** Some people prefer to write the compound statements in Cases 1 and 2 of the preceding box as follows.

$ax + b = k$ or $-(ax + b) = k$ Alternative for Case 1
and $ax + b > k$ or $-(ax + b) > k$ Alternative for Case 2

These forms produce the same results.

### OBJECTIVE 2 Solve equations of the form $|ax + b| = k$, for $k > 0$.

***Remember that because absolute value refers to distance from the origin, an absolute value equation will have two parts.***

*NOW TRY EXERCISE 1*

Solve $|4x - 1| = 11$.

**EXAMPLE 1** Solving an Absolute Value Equation

Solve $|2x + 1| = 7$. Graph the solution set.

For $|2x + 1|$ to equal 7, $2x + 1$ must be 7 units from 0 on the number line. This can happen only when $2x + 1 = 7$ or $2x + 1 = -7$. This is Case 1 in the preceding box. Solve this compound equation as follows.

$$\begin{aligned} 2x + 1 &= 7 \quad \text{or} \quad 2x + 1 = -7 \\ 2x &= 6 \quad \text{or} \quad \quad 2x = -8 \qquad \text{Subtract 1.} \\ x &= 3 \quad \text{or} \quad \quad \; x = -4 \qquad \text{Divide by 2.} \end{aligned}$$

Check by substituting 3 and then $-4$ into the original absolute value equation to verify that the solution set is $\{-4, 3\}$. The graph is shown in **FIGURE 16**.

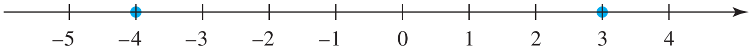


**FIGURE 16**

*NOW TRY*

## OBJECTIVE 3 Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.

*NOW TRY EXERCISE 2*

Solve $|4x - 1| > 11$.

**EXAMPLE 2** Solving an Absolute Value Inequality with $>$

Solve $|2x + 1| > 7$. Graph the solution set.

By Case 2 described in the previous box, this absolute value inequality is rewritten as

$$2x + 1 > 7 \quad \text{or} \quad 2x + 1 < -7,$$

because $2x + 1$ must represent a number that is *more* than 7 units from 0 on either side of the number line. Now, solve the compound inequality.

$$\begin{aligned} 2x + 1 &> 7 \quad \text{or} \quad 2x + 1 < -7 \\ 2x &> 6 \quad \text{or} \quad \quad 2x < -8 \qquad \text{Subtract 1.} \\ x &> 3 \quad \text{or} \quad \quad \; x < -4 \qquad \text{Divide by 2.} \end{aligned}$$

Check these solutions. The solution set is $(-\infty, -4) \cup (3, \infty)$. See **FIGURE 17**. Notice that the graph is a disjoint interval.

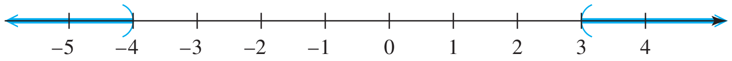


**FIGURE 17**

*NOW TRY*

**EXAMPLE 3** Solving an Absolute Value Inequality with $<$

Solve $|2x + 1| < 7$. Graph the solution set.

The expression $2x + 1$ must represent a number that is less than 7 units from 0 on either side of the number line. That is, $2x + 1$ must be between $-7$ and 7. As Case 3 in the previous box shows, that relationship is written as a three-part inequality.

$$\begin{aligned} -7 < {} & 2x + 1 < 7 \\ -8 < {} & 2x < 6 \qquad \text{Subtract 1 from each part.} \\ -4 < {} & x < 3 \qquad \text{Divide each part by 2.} \end{aligned}$$

*NOW TRY ANSWERS*

**1.** $\left\{-\frac{5}{2}, 3\right\}$

**2.** $\left(-\infty, -\frac{5}{2}\right) \cup (3, \infty)$

**NOW TRY EXERCISE 3**
Solve $|4x - 1| < 11$.

Check that the solution set is $(-4, 3)$. The graph consists of the single interval shown in **FIGURE 18**.

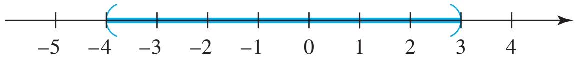


**FIGURE 18**

NOW TRY

Look back at **FIGURES 16, 17, AND 18**, with the graphs of

$$|2x + 1| = 7, \quad |2x + 1| > 7, \quad \text{and} \quad |2x + 1| < 7,$$

respectively. If we find the union of the three sets, we get the set of all real numbers. This is because, for any value of $x$, $|2x + 1|$ will satisfy one and only one of the following: It is equal to 7, greater than 7, or less than 7.

---

**CAUTION** When solving absolute value equations and inequalities of the types in **Examples 1, 2, and 3,** remember the following.

1. The methods described apply when the constant is alone on one side of the equation or inequality and is *positive.*
2. Absolute value equations and absolute value inequalities of the form $|ax + b| > k$ translate into "or" compound statements.
3. Absolute value inequalities of the form $|ax + b| < k$ translate into "and" compound statements, which may be written as three-part inequalities.
4. An "or" statement *cannot* be written in three parts. It would be incorrect to write $-7 > 2x + 1 > 7$ in **Example 2,** because this would imply that $-7 > 7$, which is *false.*

---

## OBJECTIVE 4 Solve absolute value equations that involve rewriting.

**NOW TRY EXERCISE 4**
Solve $|10x - 2| - 2 = 12$.

### EXAMPLE 4 Solving an Absolute Value Equation That Requires Rewriting

Solve $|x + 3| + 5 = 12$.

First isolate the absolute value expression on one side of the equals symbol.

$$|x + 3| + 5 = 12$$

$$|x + 3| + 5 - 5 = 12 - 5 \qquad \text{Subtract 5.}$$

$$|x + 3| = 7 \qquad \text{Combine like terms.}$$

Now use the method shown in **Example 1** to solve $|x + 3| = 7$.

$$x + 3 = 7 \quad \text{or} \quad x + 3 = -7$$

$$x = 4 \quad \text{or} \quad x = -10 \qquad \text{Subtract 3.}$$

Check these solutions by substituting each one in the original equation.

*CHECK* $\quad |x + 3| + 5 = 12$

$$|4 + 3| + 5 \overset{?}{=} 12 \qquad \text{Let } x = 4.$$

$$|7| + 5 \overset{?}{=} 12$$

$$12 = 12 \; ✓ \qquad \text{True}$$

$$|-10 + 3| + 5 \overset{?}{=} 12 \qquad \text{Let } x = -10.$$

$$|-7| + 5 \overset{?}{=} 12$$

$$12 = 12 \; ✓ \qquad \text{True}$$

The check confirms that the solution set is $\{-10, 4\}$. NOW TRY

**NOW TRY ANSWERS**
**3.** $\left(-\frac{5}{2}, 3\right)$ **4.** $\left\{-\frac{6}{5}, \frac{8}{5}\right\}$

NOW TRY EXERCISE 5

Solve each inequality.

**(a)** $|x-1|-4 \le 2$

**(b)** $|x-1|-4 \ge 2$

### EXAMPLE 5 Solving Absolute Value Inequalities That Require Rewriting

Solve each inequality.

**(a)**

$$\begin{aligned} |x+3|+5 &\ge 12 \\ |x+3| &\ge 7 \\ x+3 \ge 7 \quad \text{or} \quad x+3 &\le -7 \\ x \ge 4 \quad \text{or} \quad x &\le -10 \end{aligned}$$

Solution set: $(-\infty, -10] \cup [4, \infty)$

**(b)**

$$\begin{aligned} |x+3|+5 &\le 12 \\ |x+3| &\le 7 \\ -7 \le x+3 &\le 7 \\ -10 \le \quad x \quad &\le 4 \end{aligned}$$

Solution set: $[-10, 4]$

NOW TRY

**OBJECTIVE 5** **Solve equations of the form $|ax+b| = |cx+d|$.** ***If two expressions have the same absolute value, they must either be equal or be negatives of each other.***

### Solving $|ax+b| = |cx+d|$

To solve an absolute value equation of the form

$$|ax+b| = |cx+d|,$$

solve the following compound equation.

$$\boldsymbol{ax+b = cx+d \quad \text{or} \quad ax+b = -(cx+d)}$$

NOW TRY EXERCISE 6

Solve

$$|3x-4| = |5x+12|.$$

### EXAMPLE 6 Solving an Equation with Two Absolute Values

Solve $|x+6| = |2x-3|$.

This equation is satisfied either if $x+6$ and $2x-3$ are equal to each other or if $x+6$ and $2x-3$ are negatives of each other.

$$\begin{aligned} x+6 &= 2x-3 & & \text{or} & x+6 &= -(2x-3) & & \\ x+9 &= 2x & &\text{Add 3.} \quad \text{or} & x+6 &= -2x+3 & &\text{Distributive property} \\ 9 &= x & &\text{Subtract } x. \quad \text{or} & 3x &= -3 & &\text{Subtract 6.} \\ & & & & x &= -1 & &\text{Divide by 3.} \end{aligned}$$

Check that the solution set is $\{-1, 9\}$.

NOW TRY

**OBJECTIVE 6** **Solve special cases of absolute value equations and inequalities.** When an absolute value equation or inequality involves a *negative constant or 0* alone on one side, use the properties of absolute value to solve the equation or inequality.

### Special Cases of Absolute Value

*Case 1* The absolute value of an expression can never be negative. That is, $|a| \ge 0$ for all real numbers $a$.

*Case 2* The absolute value of an expression equals 0 only when the expression is equal to 0.

NOW TRY ANSWERS

5. **(a)** $[-5, 7]$
   **(b)** $(-\infty, -5] \cup [7, \infty)$
6. $\{-8, -1\}$

**NOW TRY EXERCISE 7**

Solve each equation.

**(a)** $|3x - 8| = -2$

**(b)** $|7x + 12| = 0$

## EXAMPLE 7 Solving Special Cases of Absolute Value Equations

Solve each equation.

**(a)** $|5x - 3| = -4$

See Case 1 in the preceding box. ***The absolute value of an expression can never be negative,*** so there are no solutions for this equation. The solution set is $\emptyset$.

**(b)** $|7x - 3| = 0$

See Case 2 in the preceding box. The expression $|7x - 3|$ will equal 0 *only* if

$$7x - 3 = 0$$

$$7x = 3 \qquad \text{Add 3.}$$

$$x = \frac{3}{7}. \qquad \text{Divide by 7.}$$

Check by substituting in the original equation.

The solution of this equation is $\frac{3}{7}$. Thus, the solution set is $\left\{\frac{3}{7}\right\}$, with just one element.

NOW TRY

**NOW TRY EXERCISE 8**

Solve each inequality.

**(a)** $|x| > -10$

**(b)** $|4x + 1| + 5 < 4$

**(c)** $|x - 2| - 3 \leq -3$

## EXAMPLE 8 Solving Special Cases of Absolute Value Inequalities

Solve each inequality.

**(a)** $|x| \geq -4$

***The absolute value of a number is always greater than or equal to 0.*** Thus, $|x| \geq -4$ is true for *all* real numbers. The solution set is $(-\infty, \infty)$.

**(b)**

$$|x + 6| - 3 < -5$$

$$|x + 6| < -2 \qquad \text{Add 3 to each side.}$$

There is no number whose absolute value is less than $-2$, so this inequality has no solution. The solution set is $\emptyset$.

**(c)**

$$|x - 7| + 4 \leq 4$$

$$|x - 7| \leq 0 \qquad \text{Subtract 4 from each side.}$$

The value of $|x - 7|$ will never be less than 0. However, $|x - 7|$ will equal 0 when $x = 7$. Therefore, the solution set is $\{7\}$.

NOW TRY

## CONNECTIONS

Absolute value is used to find the *relative error* of a measurement. If $x_t$ represents the expected measurement and $x$ represents the actual measurement, then the relative error in $x$ equals the absolute value of the difference between $x_t$ and $x$, divided by $x_t$.

$$\text{relative error in } x = \left|\frac{x_t - x}{x_t}\right|$$

In quality control situations, the relative error often must be less than some predetermined amount. For example, suppose a machine filling *quart* milk cartons is set for a relative error *no greater than* 0.05. Here $x_t = 32$ oz, the relative error $= 0.05$ oz, and we must find $x$, given the following condition.

$$\left|\frac{32 - x}{32}\right| \leq 0.05 \qquad \textit{No greater than} \text{ translates as } \leq.$$

### For Discussion or Writing

With this tolerance level, how many *ounces* may a carton contain?

**NOW TRY ANSWERS**

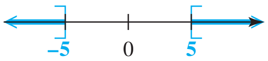
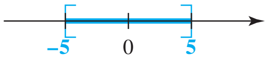
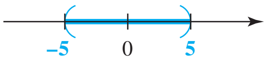
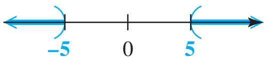
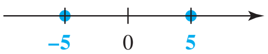
7. (a) $\emptyset$ (b) $\left\{-\frac{12}{7}\right\}$

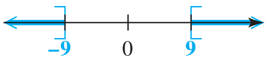
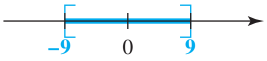
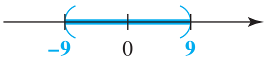
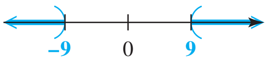
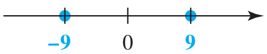
8. (a) $(-\infty, \infty)$ (b) $\emptyset$ (c) $\{2\}$

## 9.2 EXERCISES

*Complete solution available on the Video Resources on DVD*

*Concept Check* *Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.*

| I | II |
|---|---|
| **1.** $\lvert x\rvert = 5$ | **A.**  |
| $\lvert x\rvert < 5$ | **B.**  |
| $\lvert x\rvert > 5$ | **C.**  |
| $\lvert x\rvert \le 5$ | **D.**  |
| $\lvert x\rvert \ge 5$ | **E.**  |

| I | II |
|---|---|
| **2.** $\lvert x\rvert = 9$ | **A.**  |
| $\lvert x\rvert > 9$ | **B.**  |
| $\lvert x\rvert \ge 9$ | **C.**  |
| $\lvert x\rvert < 9$ | **D.**  |
| $\lvert x\rvert \le 9$ | **E.**  |

**3.** *Concept Check* How many solutions will $|ax + b| = k$ have for each situation?

**(a)** $k = 0$ **(b)** $k > 0$ **(c)** $k < 0$

**4.** Explain when to use *and* and when to use *or* if you are solving an absolute value equation or inequality of the form $|ax + b| = k$, $|ax + b| < k$, or $|ax + b| > k$, where $k$ is a positive number.

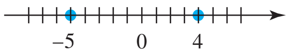
*Solve each equation.* ***See Example 1.***

**5.** $|x| = 12$
**6.** $|x| = 14$
**7.** $|4x| = 20$
**8.** $|5x| = 30$
**9.** $|x - 3| = 9$
**10.** $|x - 5| = 13$
**11.** $|2x - 1| = 11$
**12.** $|2x + 3| = 19$
**13.** $|4x - 5| = 17$
**14.** $|5x - 1| = 21$
**15.** $|2x + 5| = 14$
**16.** $|2x - 9| = 18$
**17.** $\left|\frac{1}{2}x + 3\right| = 2$
**18.** $\left|\frac{2}{3}x - 1\right| = 5$
**19.** $\left|1 + \frac{3}{4}x\right| = 7$
**20.** $\left|2 - \frac{5}{2}x\right| = 14$
**21.** $|0.02x - 1| = 2.50$
**22.** $|0.04x - 3| = 5.96$

*Solve each inequality, and graph the solution set.* ***See Example 2.***

**23.** $|x| > 3$
**24.** $|x| > 5$
**25.** $|x| \ge 4$
**26.** $|x| \ge 6$
**27.** $|r + 5| \ge 20$
**28.** $|3r - 1| \ge 8$
**29.** $|x + 2| > 10$
**30.** $|4x + 1| \ge 21$
**31.** $|3 - x| > 5$
**32.** $|5 - x| > 3$
**33.** $|-5x + 3| \ge 12$
**34.** $|-2x - 4| \ge 5$

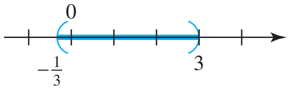
**35.** *Concept Check* The graph of the solution set of $|2x + 1| = 9$ is given here.



Without actually doing the algebraic work, graph the solution set of each inequality, referring to the graph shown.

**(a)** $|2x + 1| < 9$ **(b)** $|2x + 1| > 9$

**36.** ***Concept Check*** The graph of the solution set of $|3x - 4| < 5$ is given here.



Without actually doing the algebraic work, graph the solution set of the following, referring to the graph shown.

**(a)** $|3x - 4| = 5$ **(b)** $|3x - 4| > 5$

*Solve each inequality, and graph the solution set.* ***See Example 3.*** *(Hint: Compare your answers with those in* ***Exercises 23–34.****)*

**37.** $|x| \leq 3$
**38.** $|x| \leq 5$
**39.** $|x| < 4$
**40.** $|x| < 6$
**41.** $|r + 5| < 20$
**42.** $|3r - 1| < 8$
**43.** $|x + 2| \leq 10$
**44.** $|4x + 1| < 21$
**45.** $|3 - x| \leq 5$
**46.** $|5 - x| \leq 3$
**47.** $|-5x + 3| < 12$
**48.** $|-2x - 4| < 5$

*In Exercises 49–66, decide which method of solution applies, and find the solution set. In Exercises 49–60, graph the solution set.* ***See Examples 1–3.***

**49.** $|-4 + x| > 9$
**50.** $|-3 + x| > 8$
**51.** $|x + 5| > 20$
**52.** $|2x - 1| < 7$
**53.** $|7 + 2x| = 5$
**54.** $|9 - 3x| = 3$
**55.** $|3x - 1| \leq 11$
**56.** $|2x - 6| \leq 6$
**57.** $|-6x - 6| \leq 1$
**58.** $|-2x - 6| \leq 5$
**59.** $|2x - 1| \geq 7$
**60.** $|-4 + x| \leq 9$
**61.** $|x + 2| = 3$
**62.** $|x + 3| = 10$
**63.** $|x - 6| = 3$
**64.** $|x - 4| = 1$
**65.** $|2 - 0.2x| = 2$
**66.** $|5 - 0.5x| = 4$

*Solve each equation or inequality.* ***See Examples 4 and 5.***

**67.** $|x| - 1 = 4$
**68.** $|x| + 3 = 10$
**69.** $|x + 4| + 1 = 2$
**70.** $|x + 5| - 2 = 12$
**71.** $|2x + 1| + 3 > 8$
**72.** $|6x - 1| - 2 > 6$
**73.** $|x + 5| - 6 \leq -1$
**74.** $|x - 2| - 3 \leq 4$
**75.** $\left|\frac{1}{2}x + \frac{1}{3}\right| + \frac{1}{4} = \frac{3}{4}$
**76.** $\left|\frac{2}{3}x + \frac{1}{6}\right| + \frac{1}{2} = \frac{5}{2}$
**77.** $|0.1x - 2.5| + 0.3 \geq 0.8$
**78.** $|0.5x - 3.5| + 0.2 \geq 0.6$

*Solve each equation.* ***See Example 6.***

**79.** $|3x + 1| = |2x + 4|$
**80.** $|7x + 12| = |x - 8|$
**81.** $\left|x - \frac{1}{2}\right| = \left|\frac{1}{2}x - 2\right|$
**82.** $\left|\frac{2}{3}x - 2\right| = \left|\frac{1}{3}x + 3\right|$
**83.** $|6x| = |9x + 1|$
**84.** $|13x| = |2x + 1|$
**85.** $|2x - 6| = |2x + 11|$
**86.** $|3x - 1| = |3x + 9|$

*Solve each equation or inequality.* ***See Examples 7 and 8.***

**87.** $|x| \geq -10$
**88.** $|x| \geq -15$
**89.** $|12t - 3| = -8$
**90.** $|13x + 1| = -3$
**91.** $|4x + 1| = 0$
**92.** $|6x - 2| = 0$
**93.** $|2x - 1| = -6$
**94.** $|8x + 4| = -4$
**95.** $|x + 5| > -9$
**96.** $|x + 9| > -3$
**97.** $|7x + 3| \leq 0$
**98.** $|4x - 1| \leq 0$
**99.** $|5x - 2| = 0$
**100.** $|7x + 4| = 0$
**101.** $|x - 2| + 3 \geq 2$
**102.** $|x - 4| + 5 \geq 4$
**103.** $|10x + 7| + 3 < 1$
**104.** $|4x + 1| - 2 < -5$

**105.** The recommended daily intake (RDI) of calcium for females aged 19–50 is 1000 mg. Actual needs vary from person to person. Write this statement as an absolute value inequality, with $x$ representing the RDI, to express the RDI plus or minus 100 mg, and solve the inequality. (*Source:* National Academy of Sciences—Institute of Medicine.)

**106.** The average clotting time of blood is 7.45 sec, with a variation of plus or minus 3.6 sec. Write this statement as an absolute value inequality, with $x$ representing the time, and solve the inequality.

## *RELATING CONCEPTS* EXERCISES 107–110

### FOR INDIVIDUAL OR GROUP WORK

*The 10 tallest buildings in Houston, Texas, as of 2009 are listed, along with their heights.*

| *Building* | *Height (in feet)* |
|---|---|
| JPMorgan Chase Tower | 1002 |
| Wells Fargo Plaza | 992 |
| Williams Tower | 901 |
| Bank of America Center | 780 |
| Texaco Heritage Plaza | 762 |
| Enterprise Plaza | 756 |
| Centerpoint Energy Plaza | 741 |
| Continental Center I | 732 |
| Fulbright Tower | 725 |
| One Shell Plaza | 714 |

*Source: World Almanac and Book of Facts.*

*Use this information to* ***work Exercises 107–110 in order.***

**107.** To find the average of a group of numbers, we add the numbers and then divide by the number of numbers added. Use a calculator to find the average of the heights.

**108.** Let $k$ represent the average height of these buildings. If a height $x$ satisfies the inequality

$$|x - k| < t,$$

then the height is said to be within $t$ feet of the average. Using your result from **Exercise 107,** list the buildings that are within 50 ft of the average.

**109.** Repeat **Exercise 108,** but list the buildings that are within 95 ft of the average.

**110.** **(a)** Write an absolute value inequality that describes the height of a building that is *not* within 95 ft of the average. Solve this inequality.

**(b)** Use the result of part (a) to list the buildings that are not within 95 ft of the average. Does your answer makes sense compared with your answer to **Exercise 109.**

## *PREVIEW EXERCISES*

*Graph each equation.* ***See Sections 3.2 and 7.1.***

**111.** $x - y = 5$

**112.** $x = -5y$

*Decide whether each ordered pair is a solution of the equation.* ***See Section 3.1.***

**113.** $3x - 4y = 12;\quad (-4, 3)$

**114.** $x + 2y = 0;\quad (2, -1)$

## *SUMMARY EXERCISES* on Solving Linear and Absolute Value Equations and Inequalities

*Solve each equation or inequality. Give the solution set in set notation for equations and in interval notation for inequalities.*

**1.** $4x + 1 = 49$

**2.** $|x - 1| = 6$

**3.** $6x - 9 = 12 + 3x$

**4.** $3x + 7 = 9 + 8x$

**5.** $|x + 3| = -4$

**6.** $2x + 1 \leq x$

**7.** $8x + 2 \geq 5x$

**8.** $4(x - 11) + 3x = 20x - 31$

**9.** $2x - 1 = -7$

**10.** $|3x - 7| - 4 = 0$

**11.** $6x - 5 \leq 3x + 10$

**12.** $|5x - 8| + 9 \geq 7$

**13.** $9x - 3(x + 1) = 8x - 7$

**14.** $|x| \geq 8$

**15.** $9x - 5 \geq 9x + 3$

**16.** $13x - 5 > 13x - 8$

**17.** $|x| < 5.5$

**18.** $4x - 1 = 12 + x$

**19.** $\frac{2}{3}x + 8 = \frac{1}{4}x$

**20.** $-\frac{5}{8}x \geq -20$

**21.** $\frac{1}{4}x < -6$

**22.** $7x - 3 + 2x = 9x - 8x$

**23.** $\frac{3}{5}x - \frac{1}{10} = 2$

**24.** $|x - 1| < 7$

**25.** $x + 9 + 7x = 4(3 + 2x) - 3$

**26.** $6 - 3(2 - x) < 2(1 + x) + 3$

**27.** $|2x - 3| > 11$

**28.** $\frac{x}{4} - \frac{2x}{3} = -10$

**29.** $|5x + 1| \leq 0$

**30.** $5x - (3 + x) \geq 2(3x + 1)$

**31.** $-2 \leq 3x - 1 \leq 8$

**32.** $-1 \leq 6 - x \leq 5$

**33.** $|7x - 1| = |5x + 3|$

**34.** $|x + 2| = |x + 4|$

**35.** $|1 - 3x| \geq 4$

**36.** $\frac{1}{2} \leq \frac{2}{3}x \leq \frac{5}{4}$

**37.** $-(x + 4) + 2 = 3x + 8$

**38.** $\frac{x}{6} - \frac{3x}{5} = x - 86$

**39.** $-6 \leq \frac{3}{2} - x \leq 6$

**40.** $|5 - x| < 4$

**41.** $|x - 1| \geq -6$

**42.** $|2x - 5| = |x + 4|$

**43.** $8x - (1 - x) = 3(1 + 3x) - 4$

**44.** $8x - (x + 3) = -(2x + 1) - 12$

**45.** $|x - 5| = |x + 9|$

**46.** $|x + 2| < -3$

**47.** $2x + 1 > 5$ or $3x + 4 < 1$

**48.** $1 - 2x \geq 5$ and $7 + 3x \geq -2$

## 9.3

# Linear Inequalities in Two Variables

**OBJECTIVES**

1. Graph linear inequalities in two variables.
2. Graph an inequality with a boundary line through the origin.
3. Graph the intersection of two linear inequalities.
4. Graph the union of two linear inequalities.

**OBJECTIVE 1** **Graph linear inequalities in two variables.** In **Chapter 2,** we graphed linear inequalities in one variable on the number line. In this section, we graph linear inequalities in two variables on a rectangular coordinate system.

### Linear Inequality in Two Variables

An inequality that can be written as

$$Ax + By < C, \quad Ax + By \leq C, \quad Ax + By > C, \quad \text{or} \quad Ax + By \geq C,$$

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0, is a **linear inequality in two variables.**

Consider the graph in FIGURE 19. The graph of the line $x + y = 5$ divides the points in the rectangular coordinate system into three sets:

1. Those points that lie on the line itself and satisfy the equation $\boldsymbol{x + y = 5}$ [like $(0, 5)$, $(2, 3)$, and $(5, 0)$];
2. Those that lie in the half-plane above the line and satisfy the inequality $\boldsymbol{x + y > 5}$ [like $(5, 3)$ and $(2, 4)$];
3. Those that lie in the half-plane below the line and satisfy the inequality $\boldsymbol{x + y < 5}$ [like $(0, 0)$ and $(-3, -1)$].

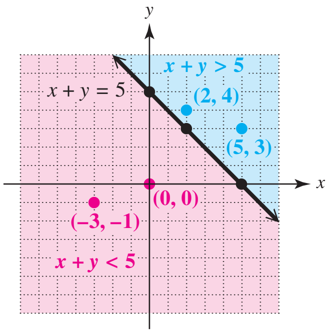


FIGURE 19

The graph of the line $x + y = 5$ is called the **boundary line** for the inequalities $x + y > 5$ and $x + y < 5$. Graphs of linear inequalities in two variables are *regions* in the real number plane that may or may not include boundary lines.

To graph a linear inequality in two variables, follow these steps.

### Graphing a Linear Inequality

***Step 1*** **Draw the graph of the straight line that is the boundary.** Make the line solid if the inequality involves $\leq$ or $\geq$. Make the line dashed if the inequality involves $<$ or $>$.

***Step 2*** **Choose a test point.** Choose any point not on the line, and substitute the coordinates of that point in the inequality.

***Step 3*** **Shade the appropriate region.** Shade the region that includes the test point if it satisfies the original inequality. Otherwise, shade the region on the other side of the boundary line.

**CAUTION** When drawing the boundary line in Step 1, be careful to draw a solid line if the inequality includes equality $(\leq, \geq)$ or a dashed line if equality is not included $(<, >)$.

NOW TRY EXERCISE 1

Graph $-x + 2y \geq 4$.

## EXAMPLE 1 Graphing a Linear Inequality

Graph $3x + 2y \geq 6$.

***Step 1*** First graph the boundary line $3x + 2y = 6$, as shown in **FIGURE 20**.

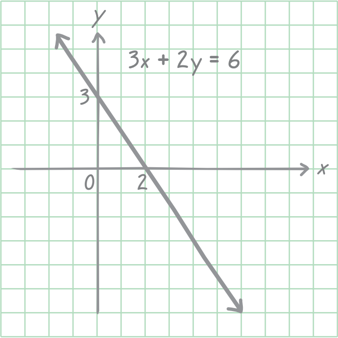


**FIGURE 20**

***Step 2*** The graph of the inequality $3x + 2y \geq 6$ includes the points of the line $3x + 2y = 6$ and either the points *above* that line or the points *below* it. To decide which, select any point not on the boundary line to use as a test point. Substitute the values from the test point, here $(0, 0)$, for $x$ and $y$ in the inequality.

$$\begin{aligned} 3x + 2y &\geq 6 && \text{Original inequality} \\ 3(0) + 2(0) &\overset{?}{\geq} 6 && \text{Let } x = 0 \text{ and } y = 0. \\ 0 &\geq 6 && \text{False} \end{aligned}$$

(0, 0) is a convenient test point.

***Step 3*** Because the result is false, $(0, 0)$ does *not* satisfy the inequality. The solution set includes all points on the other side of the line. See **FIGURE 21**.

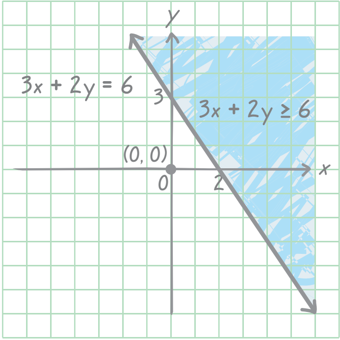


**FIGURE 21**

NOW TRY

If the inequality is written in the form $y > mx + b$ or $y < mx + b$, then the inequality symbol indicates which half-plane to shade.

If $y > mx + b$, then shade above the boundary line.

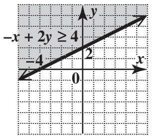
If $y < mx + b$, then shade below the boundary line.

***This method works only if the inequality is solved for y.***

**CAUTION** A common error in using the method just described is to use the original inequality symbol when deciding which half-plane to shade. Be sure to use the inequality symbol found in the inequality *after* it is solved for $y$.

NOW TRY ANSWER

**1.**



**NOW TRY EXERCISE 2**

Graph $3x - y < 6$.

### EXAMPLE 2 Graphing a Linear Inequality

Graph $x - 3y < 4$.

First graph the boundary line, shown in **FIGURE 22**. The points of the boundary line do not belong to the inequality $x - 3y < 4$ (because the inequality symbol is $<$, not $\leq$). For this reason, the line is dashed. Now solve the inequality for $y$.

$$x - 3y < 4$$

$$-3y < -x + 4 \quad \text{Subtract } x.$$

$$y > \frac{1}{3}x - \frac{4}{3} \quad \text{Multiply by } -\tfrac{1}{3}. \text{ Change } < \text{ to } >.$$

Because of the *is greater than* symbol that occurs ***when the inequality is solved for y***, shade *above* the line.

*CHECK* Choose a test point not on the line, say, $(0, 0)$.

$$x - 3y < 4$$

$$0 - 3(0) \overset{?}{<} 4 \quad \text{Let } x = 0 \text{ and } y = 0.$$

$$0 < 4 \ ✓ \quad \text{True}$$

This result agrees with the decision to shade above the line. The solution set, graphed in **FIGURE 22**, includes only those points in the shaded half-plane (not those on the line).

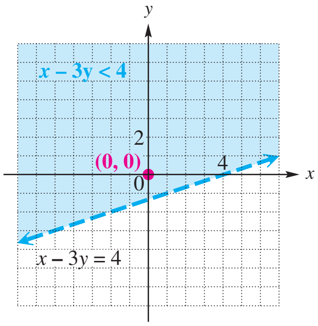


**FIGURE 22**

NOW TRY

**NOW TRY EXERCISE 3**

Graph $x > 2$.

### EXAMPLE 3 Graphing a Linear Inequality with a Vertical Boundary Line

Graph $x < 3$.

First, we graph $x = 3$, a vertical line passing through the point $(3, 0)$. We use a dashed line (why?) and choose $(0, 0)$ as a test point.

$$x < 3 \quad \text{Original inequality}$$

$$0 \overset{?}{<} 3 \quad \text{Let } x = 0.$$

$$0 < 3 \quad \text{True}$$

Since $0 < 3$ is true, we shade the region containing $(0, 0)$, as in **FIGURE 23**.

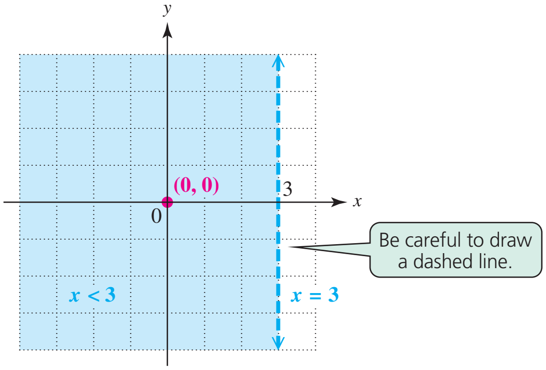
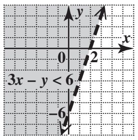


**FIGURE 23**

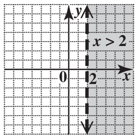
NOW TRY

**NOW TRY ANSWERS**

2.



3.



**OBJECTIVE 2** **Graph an inequality with a boundary line through the origin.** ***If the graph of an inequality has a boundary line that goes through the origin, (0, 0) cannot be used as a test point.***

**NOW TRY EXERCISE 4**

Graph $y \leq -2x$.

## EXAMPLE 4 Graphing a Linear Inequality with a Boundary Line through the Origin

Graph $x \leq 2y$.

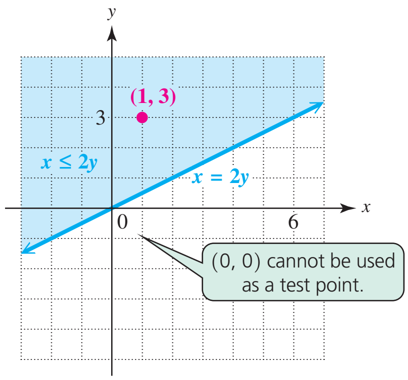
Graph $x = 2y$, using a solid line. Some ordered pairs that can be used to graph this line are $(0, 0)$, $(6, 3)$, and $(4, 2)$. Since $(0, 0)$ is *on* the line $x = 2y$, it cannot be used as a test point. Instead, we choose a test point *off* the line, say $(1, 3)$.

$$x \leq 2y \qquad \text{Original inequality}$$

$$1 \overset{?}{\leq} 2(3) \qquad \text{Let } x = 1 \text{ and } y = 3.$$

$$1 \leq 6 \qquad \text{True}$$

Since $1 \leq 6$ is true, shade the region containing the test point $(1, 3)$. See **FIGURE 24**.



(0, 0) cannot be used as a test point.

**FIGURE 24**

**NOW TRY**

**OBJECTIVE 3** **Graph the intersection of two linear inequalities.** A pair of inequalities joined with the word ***and*** is interpreted as the intersection of the solution sets of the inequalities. ***The graph of the intersection of two or more inequalities is the region of the plane where all points satisfy all of the inequalities at the same time.***

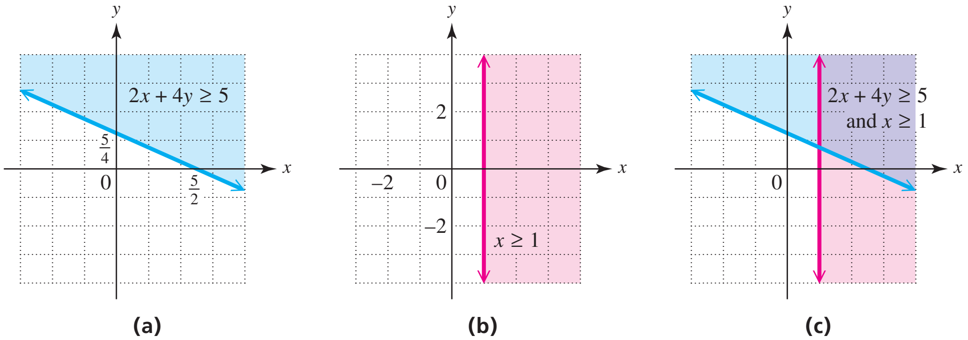
**NOW TRY EXERCISE 5**

Graph $x + y < 3$ and $y \leq 2$.

## EXAMPLE 5 Graphing the Intersection of Two Inequalities

Graph $2x + 4y \geq 5$ and $x \geq 1$.

To begin, we graph each of the two inequalities $2x + 4y \geq 5$ and $x \geq 1$ separately, as shown in **FIGURES 25(a) AND (b)**. Then we use heavy shading to identify the intersection of the graphs, as shown in **FIGURE 25(c)**.



(a) (b) (c)

**FIGURE 25**

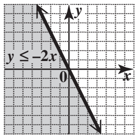
In practice, the graphs in **FIGURES 25(a) AND (b)** are graphed on the same axes.

*CHECK* Using **FIGURE 25(c)**, choose a test point from each of the four regions formed by the intersection of the boundary lines. Verify that only ordered pairs in the heavily shaded region satisfy *both* inequalities.

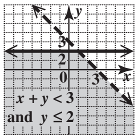
**NOW TRY**

**NOW TRY ANSWERS**

**4.**



**5.**



**OBJECTIVE 4 Graph the union of two linear inequalities.** When two inequalities are joined by the word ***or,*** we must find the union of the graphs of the inequalities. ***The graph of the union of two inequalities includes all of the points that satisfy either inequality.***

NOW TRY EXERCISE 6

Graph

$3x - 5y < 15$ or $x > 4$.

## EXAMPLE 6 Graphing the Union of Two Inequalities

Graph $2x + 4y \geq 5$ or $x \geq 1$.

The graphs of the two inequalities are shown in **FIGURES 25(a) AND (b)** in **Example 5** on the preceding page. The graph of the union is shown in **FIGURE 26**.

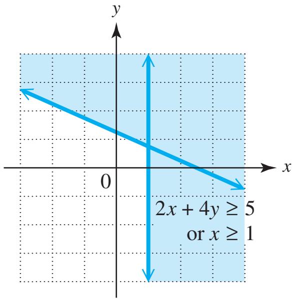


**FIGURE 26**

NOW TRY

## CONNECTIONS

In **Section 3.2,** we saw that the $x$-intercept of the graph of the line $y = mx + b$ indicates the solution of the equation $mx + b = 0$. We can extend this observation to find solutions of the associated inequalities $mx + b > 0$ and $mx + b < 0$.

For example, to solve the equation

$$-2(3x + 1) = -2x + 18$$

and the associated inequalities

$$-2(3x + 1) > -2x + 18 \quad \text{and} \quad -2(3x + 1) < -2x + 18,$$

we rewrite the equation so that the right side equals 0.

$$-2(3x + 1) + 2x - 18 = 0$$

We graph

$$\text{Y} = -2(3\text{X} + 1) + 2\text{X} - 18$$

to find the $x$-intercept $(-5, 0)$, as shown in **FIGURE 27**.

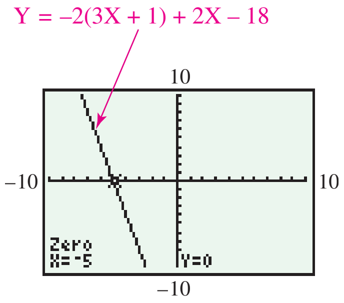


**FIGURE 27**

Thus, the solution set of $-2(3x + 1) = -2x + 18$ is $\{-5\}$.

The graph of Y lies *above* the $x$-axis for $x$-values less than $-5$.

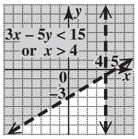
Thus, the solution set of $-2(3x + 1) > -2x + 18$ is $(-\infty, -5)$.

The graph of Y lies *below* the $x$-axis for $x$-values greater than $-5$.

Thus, the solution set of $-2(3x + 1) < -2x + 18$ is $(-5, \infty)$.

NOW TRY ANSWER

6.



### For Discussion or Writing

Solve the equation in part (a) and the associated inequalities in parts (b) and (c), by graphing the left side as $y$ in the standard viewing window of a graphing calculator. Explain your answers using the graph.

**1.** **(a)** $5x + 3 = 0$ **(b)** $5x + 3 > 0$ **(c)** $5x + 3 < 0$

**2.** **(a)** $6x + 3 = 0$ **(b)** $6x + 3 > 0$ **(c)** $6x + 3 < 0$

**3.** **(a)** $-8x - (2x + 12) = 0$ **(b)** $-8x - (2x + 12) \geq 0$
**(c)** $-8x - (2x + 12) \leq 0$

**4.** **(a)** $-4x - (2x + 18) = 0$ **(b)** $-4x - (2x + 18) \geq 0$
**(c)** $-4x - (2x + 18) \leq 0$

## 9.3 EXERCISES

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*Complete solution available on the Video Resources on DVD*

***Concept Check*** *In Exercises 1–4, fill in the first blank with either* solid *or* dashed. *Fill in the second blank with either* above *or* below.

**1.** The boundary of the graph of $y \leq -x + 2$ will be a ______ line, and the shading will be ______ the line.

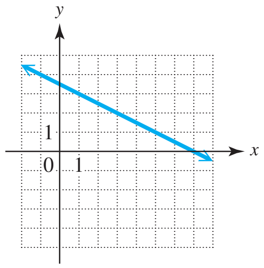
**2.** The boundary of the graph of $y < -x + 2$ will be a ______ line, and the shading will be ______ the line.

**3.** The boundary of the graph of $y > -x + 2$ will be a ______ line, and the shading will be ______ the line.

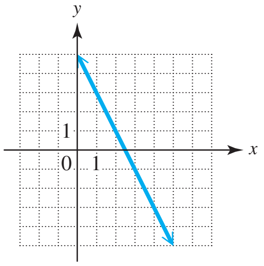
**4.** The boundary of the graph of $y \geq -x + 2$ will be a ______ line, and the shading will be ______ the line.

*In Exercises 5–10, the straight-line boundary has been drawn. Complete the graph by shading the correct region.* ***See Examples 1–4.***

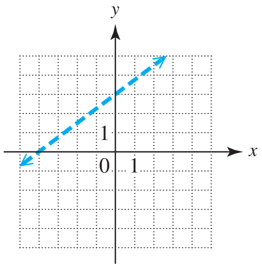
**5.** $x + 2y \geq 7$



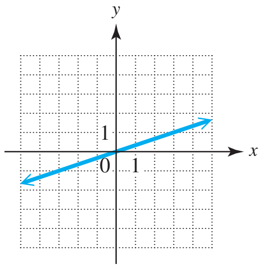
**6.** $2x + y \geq 5$



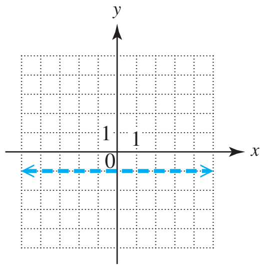
**7.** $-3x + 4y > 12$



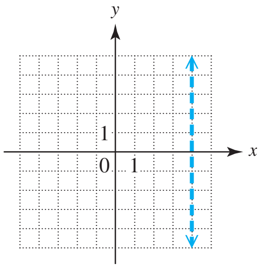
**8.** $x \leq 3y$



**9.** $y < -1$



**10.** $x > 4$



**11.** Explain how to determine whether to use a dashed line or a solid line when graphing a linear inequality in two variables.

**12.** Explain why the point $(0, 0)$ is not an appropriate choice for a test point when graphing an inequality whose boundary goes through the origin.

*Graph each linear inequality in two variables.* ***See Examples 1–4.***

**13.** $x + y \leq 2$

**14.** $x + y \leq -3$

**15.** $4x - y < 4$

**16.** $3x - y < 3$

**17.** $x + 3y \geq -2$

**18.** $x + 4y \geq -3$

**19.** $2x + 3y \geq 6$

**20.** $3x + 4y \geq 12$

**21.** $5x - 3y > 15$

**22.** $4x - 5y > 20$

**23.** $x < -2$

**24.** $x > 1$

**25.** $y \leq 5$

**26.** $y \leq -3$

**27.** $x + y > 0$

**28.** $x + 2y > 0$

**29.** $x - 3y \leq 0$

**30.** $x - 5y \leq 0$

**31.** $y < x$

**32.** $y \leq 4x$

*Graph each compound inequality.* ***See Example 5.***

**33.** $x + y \leq 1$ and $x \geq 1$

**34.** $x - y \geq 2$ and $x \geq 3$

**35.** $2x - y \geq 2$ and $y < 4$

**36.** $3x - y \geq 3$ and $y < 3$

**37.** $x + y > -5$ and $y < -2$

**38.** $6x - 4y < 10$ and $y > 2$

*Use the method described in* ***Section 9.2*** *to write each inequality as a compound inequality, and graph its solution set in the rectangular coordinate plane.*

**39.** $|x| < 3$

**40.** $|y| < 5$

**41.** $|x + 1| < 2$

**42.** $|y - 3| < 2$

*Graph each compound inequality.* ***See Example 6.***

**43.** $x - y \geq 1$ or $y \geq 2$

**44.** $x + y \leq 2$ or $y \geq 3$

**45.** $x - 2 > y$ or $x < 1$

**46.** $x + 3 < y$ or $x > 3$

**47.** $3x + 2y < 6$ or $x - 2y > 2$

**48.** $x - y \geq 1$ or $x + y \leq 4$

## TECHNOLOGY INSIGHTS EXERCISES 49–56

*Match each inequality in Exercises 49–52 with its calculator graph in choices A–D at the top of the next page. (Hint: Use the slope, y-intercept, and inequality symbol in making your choice.)*

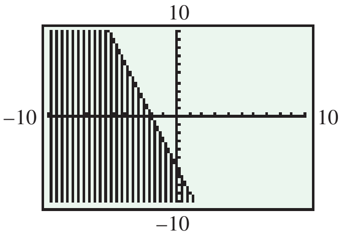
**49.** $y \leq 3x - 6$

**50.** $y \geq 3x - 6$

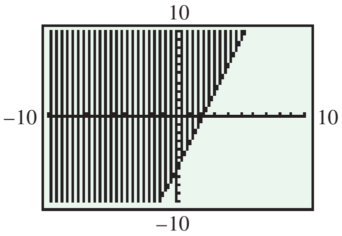
**51.** $y \leq -3x - 6$

**52.** $y \geq -3x - 6$

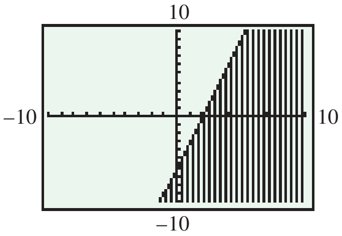
**A.**



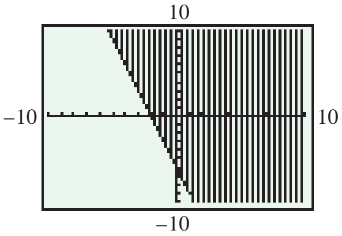
**B.**



**C.**

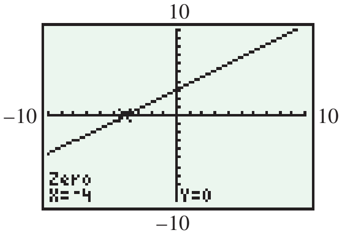


**D.**

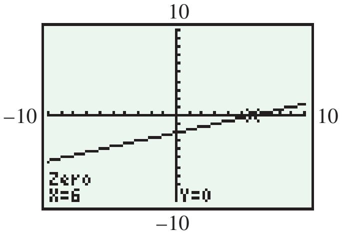


*The graph of a linear equation* $y = mx + b$ *is shown on a graphing calculator screen, along with the x-value of the x-intercept of the line. Use the screen to solve* ***(a)*** $y = 0$, ***(b)*** $y < 0$, *and* ***(c)*** $y > 0$. ***See the Connections box.***

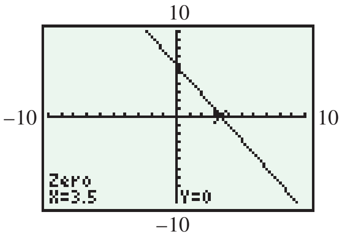
**53.**



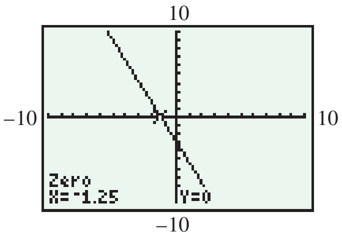
**54.**



**55.**



**56.**



## RELATING CONCEPTS EXERCISES 57–62

### FOR INDIVIDUAL OR GROUP WORK

*Suppose a factory can have* no more than 200 *workers on a shift, but must have* at least 100 *and must manufacture* at least 3000 *units at minimum cost. The managers need to know how many workers should be on a shift in order to produce the required units at minimal cost.* ***Linear programming*** *is a method for finding the optimal (best possible) solution that meets all the conditions for such problems.*

*Let x represent the number of workers and y represent the number of units manufactured.* ***Work Exercises 57–62 in order.***

**57.** Write three inequalities expressing the conditions given in the problem.

**58.** Graph the inequalities from **Exercise 57** and shade the intersection.

(continued)

**59.** The cost per worker is \$50 per day and the cost to manufacture 1 unit is \$100. Write an equation in $x$, $y$, and $C$ representing the total daily cost $C$.

**60.** Find values of $x$ and $y$ for several points in or on the boundary of the shaded region. Include any "corner points." These are the points that maximize or minimize $C$.

**61.** Of the values of $x$ and $y$ that you chose in **Exercise 60,** which gives the least value when substituted in the cost equation from **Exercise 59?**

**62.** What does your answer in **Exercise 61** mean in terms of the given problem?

## PREVIEW EXERCISES

*Find each power.* ***See Sections 1.2 and 4.1.***

**63.** $8^2$ **64.** $(-4)^2$ **65.** $-12^2$ **66.** $1.5^2$

# CHAPTER 9 SUMMARY

## KEY TERMS

**9.1**
intersection
compound inequality
union

**9.2**
absolute value equation
absolute value inequality

**9.3**
linear inequality in two variables
boundary line

## NEW SYMBOLS

$\cap$ set intersection

$\cup$ set union

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** The **intersection** of two sets $A$ and $B$ is the set of elements that belong
**A.** to both $A$ and $B$
**B.** to either $A$ or $B$, or both
**C.** to either $A$ or $B$, but not both
**D.** to just $A$.

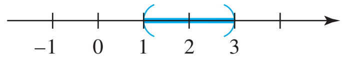
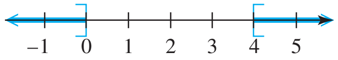
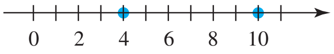
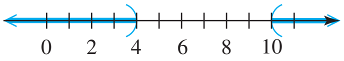
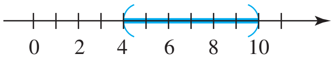
**2.** The **union** of two sets $A$ and $B$ is the set of elements that belong
**A.** to both $A$ and $B$
**B.** to either $A$ or $B$, or both
**C.** to either $A$ or $B$, but not both
**D.** to just $B$.

**3.** A **linear inequality in two variables** is an inequality that can be written in the form
**A.** $Ax + By < C$ or $Ax + By > C$ ($\leq$ or $\geq$ can be used)
**B.** $ax < b$
**C.** $y \geq x^2$
**D.** $Ax + By = C$.

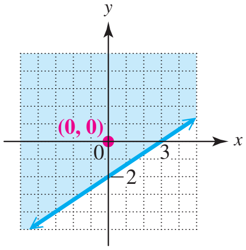
***ANSWERS***

**1.** A; *Example:* If $A = \{2, 4, 6, 8\}$ and $B = \{1, 2, 3\}$, then $A \cap B = \{2\}$. **2.** B; *Example:* Using the sets $A$ and $B$ from Answer 1, $A \cup B = \{1, 2, 3, 4, 6, 8\}$. **3.** A; *Examples:* $4x + 3y < 12$, $x > 6y$, $2x \geq 4y + 5$

## QUICK REVIEW

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| **9.1 Set Operations and Compound Inequalities** | |
| **Solving a Compound Inequality**<br>***Step 1*** Solve each inequality in the compound inequality individually.<br>***Step 2*** If the inequalities are joined with *and,* then the solution set is the intersection of the two individual solution sets. | Solve $x + 1 > 2$ and $2x < 6$.<br>$x + 1 > 2$ and $2x < 6$<br>$x > 1$ and $x < 3$<br>The solution set is $(1, 3)$.<br> |
| If the inequalities are joined with *or*, then the solution set is the union of the two individual solution sets. | Solve $x \geq 4$ or $x \leq 0$.<br>The solution set is $(-\infty, 0] \cup [4, \infty)$.<br> |
| **9.2 Absolute Value Equations and Inequalities** | |
| **Solving Absolute Value Equations and Inequalities**<br>Let $k$ be a positive number.<br>To solve $\lvert ax + b \rvert = k$, solve the following compound equation.<br>$ax + b = k$ **or** $ax + b = -k$ | Solve $\lvert x - 7 \rvert = 3$.<br>$x - 7 = 3$ or $x - 7 = -3$<br>$x = 10$ or $x = 4$ Add 7.<br>The solution set is $\{4, 10\}$.<br> |
| To solve $\lvert ax + b \rvert > k$, solve the following compound inequality.<br>$ax + b > k$ **or** $ax + b < -k$ | Solve $\lvert x - 7 \rvert > 3$.<br>$x - 7 > 3$ or $x - 7 < -3$<br>$x > 10$ or $x < 4$ Add 7.<br>The solution set is $(-\infty, 4) \cup (10, \infty)$.<br> |
| To solve $\lvert ax + b \rvert < k$, solve the following compound inequality.<br>$-k < ax + b < k$ | Solve $\lvert x - 7 \rvert < 3$.<br>$-3 < x - 7 < 3$<br>$4 < x < 10$ Add 7.<br>The solution set is $(4, 10)$.<br> |
| To solve an absolute value equation of the form<br>$\lvert ax + b \rvert = \lvert cx + d \rvert$,<br>solve the following compound equation.<br>$ax + b = cx + d$ **or** $ax + b = -(cx + d)$ | Solve $\lvert x + 2 \rvert = \lvert 2x - 6 \rvert$.<br>$x + 2 = 2x - 6$ or $x + 2 = -(2x - 6)$<br>$x = 8$ $\quad$ $x + 2 = -2x + 6$<br>$3x = 4$<br>$x = \frac{4}{3}$<br>The solution set is $\left\{\frac{4}{3}, 8\right\}$. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **9.3 Linear Inequalities in Two Variables** | |
| **Graphing a Linear Inequality** | Graph $2x - 3y \le 6$. |
| ***Step 1*** Draw the graph of the line that is the boundary. Make the line solid if the inequality involves $\le$ or $\ge$. Make the line dashed if the inequality involves $<$ or $>$. | Draw the graph of $2x - 3y = 6$. Use a solid line because of the inclusion of equality in the symbol $\le$. |
| ***Step 2*** Choose any point not on the line as a test point. Substitute the coordinates into the inequality. | Choose $(0, 0)$ as a test point.<br>$2(0) - 3(0) \overset{?}{\le} 6$<br>$0 \le 6$ True |
| ***Step 3*** Shade the region that includes the test point if the test point satisfies the original inequality. Otherwise, shade the region on the other side of the boundary line. | Shade the side of the line that includes $(0, 0)$.  |

# CHAPTER 9

## REVIEW EXERCISES

**9.1** *Let* $A = \{a, b, c, d\}$, $B = \{a, c, e, f\}$, *and* $C = \{a, e, f, g\}$. *Find each set.*

**1.** $A \cap B$ **2.** $A \cap C$ **3.** $B \cup C$ **4.** $A \cup C$

*Solve each compound inequality. Give the solution set in both interval and graph form.*

**5.** $x > 6$ and $x < 9$

**6.** $x + 4 > 12$ and $x - 2 < 12$

**7.** $x > 5$ or $x \le -3$

**8.** $x \ge -2$ or $x < 2$

**9.** $x - 4 > 6$ and $x + 3 \le 10$

**10.** $-5x + 1 \ge 11$ or $3x + 5 \ge 26$

*Express each union or intersection in simplest interval form.*

**11.** $(-3, \infty) \cap (-\infty, 4)$

**12.** $(-\infty, 6) \cap (-\infty, 2)$

**13.** $(4, \infty) \cup (9, \infty)$

**14.** $(1, 2) \cup (1, \infty)$

**9.2** *Solve each absolute value equation.*

**15.** $|x| = 7$

**16.** $|x + 2| = 9$

**17.** $|3x - 7| = 8$

**18.** $|x - 4| = -12$

**19.** $|2x - 7| + 4 = 11$

**20.** $|4x + 2| - 7 = -3$

**21.** $|3x + 1| = |x + 2|$

**22.** $|2x - 1| = |2x + 3|$

*Solve each absolute value inequality. Give the solution set in interval form.*

**23.** $|x| < 14$ **24.** $|-x + 6| \le 7$ **25.** $|2x + 5| \le 1$

**26.** $|x + 1| \ge -3$ **27.** $|5x - 1| > 9$ **28.** $|11x - 3| \le -2$

**29.** $|11x - 3| \ge -2$

**30.** $|11x - 3| \le 0$

**9.3** *Graph the solution set of each inequality or compound inequality.*

**31.** $3x - 2y \le 12$

**32.** $5x - y > 6$

**33.** $3x + 2y < 0$

**34.** $2x + y \le 1$ and $x \ge 2y$

**35.** $x \ge 2$ or $y \ge 2$

**36.** ***Concept Check*** Which one of the following has as its graph a dashed boundary line and shading below the line?

**A.** $y \ge 4x + 3$ **B.** $y > 4x + 3$ **C.** $y \le 4x + 3$ **D.** $y < 4x + 3$

## MIXED REVIEW EXERCISES

*Solve.*

**37.** $x < 3$ and $x \ge -2$

**38.** $|3x + 6| \ge 0$

**39.** $|3x + 2| + 4 = 9$

**40.** $|x + 3| \le 13$

**41.** $|5x - 1| > 14$

**42.** $x \ge -2$ or $x < 4$

**43.** $|x - 1| = |2x + 3|$

**44.** $|x + 3| \le 1$

**45.** $|3x - 7| = 4$

**46.** ***Concept Check*** If $k < 0$, what is the solution set of each of the following?

**(a)** $|2x - 5| = k$ **(b)** $|2x - 5| < k$ **(c)** $|2x - 5| > k$

*Solve. Give the solution set in both interval and graph form.*

**47.** $x > 6$ and $x < 8$

**48.** $-5x + 1 \ge 11$ or $3x + 5 \ge 26$

*Graph the solution set of each inequality.*

**49.** $2x - 3y > -6$

**50.** $3x + 5y > 9$

**51.** The numbers of civilian workers (to the nearest thousand) for several states in 2008 are shown in the table.

**Number of Workers**

| *State* | *Female* | *Male* |
|---|---|---|
| Illinois | 2,918,000 | 3,345,000 |
| Maine | 320,000 | 349,000 |
| North Carolina | 2,016,000 | 2,242,000 |
| Oregon | 869,000 | 976,000 |
| Utah | 571,000 | 755,000 |
| Wisconsin | 1,427,000 | 1,526,000 |

*Source:* U.S. Bureau of Labor Statistics.

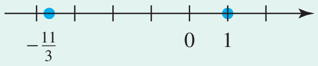
List the elements of each set.

**(a)** The set of states with less than 3 million female workers *and* more than 3 million male workers

**(b)** The set of states with less than 1 million female workers *or* more than 2 million male workers

**(c)** The set of states with a total of more than 7 million civilian workers

**52.** ***Concept Check*** The solution set of $|3x + 4| = 7$ is shown on the number line.



**(a)** What is the solution set of $|3x + 4| \ge 7$?

**(b)** What is the solution set of $|3x + 4| \le 7$?

## CHAPTER 9 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*The median weekly earnings of full-time workers by occupation for a recent year were as shown in the table.*

**Weekly Earnings of Full-Time Workers (in dollars)**

| *Occupation* | *Men* | *Women* |
|---|---|---|
| Managerial/Professional | 994 | 709 |
| Technical/Sales/Administrative Support | 655 | 452 |
| Service | 357 | 316 |
| Operators/Fabricators/Laborers | 487 | 351 |

*Source:* U.S. Bureau of Labor Statistics.

*List the elements of each set.*

**1.** The set of occupations with median earnings for men less than \$900 and for women greater than \$500

**2.** The set of occupations with median earnings for men greater than \$600 or for women less than \$400

*Let* $A = \{1, 2, 5, 7\}$ *and* $B = \{1, 5, 9, 12\}$. *Write each of the following sets.*

**3.** $A \cap B$

**4.** $A \cup B$

*Solve each compound inequality. Give the solution set in both interval and graph form.*

**5.** $3k \geq 6$ and $k - 4 < 5$

**6.** $-4x \leq -24$ or $4x - 2 < 10$

*Solve each absolute value equation or inequality. Give the solution set in interval form.*

**7.** $|4x - 3| = 7$

**8.** $|5 - 6x| > 12$

**9.** $|7 - x| \leq -1$

**10.** $|3 - 5x| = |2x + 8|$

**11.** $|-3x + 4| - 4 < -1$

**12.** $|12t + 7| \geq 0$

**13.** ***Concept Check*** If $k < 0$, what is the solution set of each of the following?

**(a)** $|5x + 3| < k$ **(b)** $|5x + 3| > k$ **(c)** $|5x + 3| = k$

*Graph the solution set of each inequality or compound inequality.*

**14.** $3x - 2y > 6$

**15.** $3x - y > 0$

**16.** $y < 2x - 1$ and $x - y < 3$

**17.** $x - 2 \geq y$ or $y \geq 3$

## CHAPTERS 1–9 CUMULATIVE REVIEW EXERCISES

**1.** Match each number in Column I with the choice or choices of sets of numbers in Column II to which the number belongs.

**I**

**(a)** 34
**(b)** 0
**(c)** 2.16
**(d)** $-\sqrt{36}$
**(e)** $\sqrt{13}$
**(f)** $-\frac{4}{5}$

**II**

**A.** Natural numbers
**B.** Whole numbers
**C.** Integers
**D.** Rational numbers
**E.** Irrational numbers
**F.** Real numbers

*Evaluate.*

**2.** $9 \cdot 4 - 16 \div 4$

**3.** $-|8 - 13| - |-4| + |-9|$

*Solve.*

**4.** $-5(8 - 2z) + 4(7 - z) = 7(8 + z) - 3$

**5.** $3(x + 2) - 5(x + 2) = -2x - 4$

**6.** $A = p + prt$ for $t$

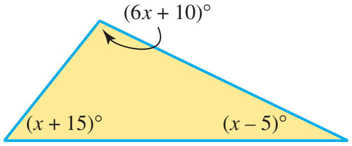
**7.** $2(m + 5) - 3m + 1 > 5$

**8.** A recent survey polled teens about the most important inventions of the 20th century. Complete the results shown in the table if 1500 teens were surveyed.

| *Most Important Invention* | *Percent* | *Actual Number* |
|---|---|---|
| Personal computer | | 480 |
| Pacemaker | 26% | |
| Wireless communication | 18% | |
| Television | | 150 |

*Source:* Lemelson–MIT Program.

**9.** Find the measure of each angle of the triangle.



*Find the slope of each line described.*

**10.** Through $(-4, 5)$ and $(2, -3)$

**11.** Horizontal, through $(4, 5)$

*Find an equation of each line. Write the equation in* ***(a)*** *slope-intercept form and in* ***(b)*** *standard form.*

**12.** Through $(4, -1)$, $m = -4$

**13.** Through $(0, 0)$ and $(1, 4)$

**14.** Graph $-3x + 4y = 12$.

*Simplify. Write answers with only positive exponents. Assume that all variables represent positive real numbers.*

**15.** $\left(\frac{2m^3n}{p^2}\right)^3$

**16.** $\frac{x^{-6}y^3z^{-1}}{x^7y^{-4}z}$

*Perform the indicated operations.*

**17.** $(3x^2 - 8x + 1) - (x^2 - 3x - 9)$

**18.** $(x + 2y)(x^2 - 2xy + 4y^2)$

**19.** $(3x + 2y)(5x - y)$

**20.** $\frac{16x^3y^5 - 8x^2y^2 + 4}{4x^2y}$

*Factor each polynomial completely.*

**21.** $m^2 + 12m + 32$

**22.** $25t^4 - 36$

**23.** $81z^2 + 72z + 16$

**24.** Solve the equation $(x + 4)(x - 1) = -6$.

**25.** For what real number(s) is the expression $\frac{3}{x^2 + 5x - 14}$ undefined?

*Perform each indicated operation. Express answers in lowest terms.*

**26.** $\frac{x^2 - 3x - 4}{x^2 + 3x} \cdot \frac{x^2 + 2x - 3}{x^2 - 5x + 4}$

**27.** $\frac{t^2 + 4t - 5}{t + 5} \div \frac{t - 1}{t^2 + 8t + 15}$

**28.** $\frac{2}{x + 3} - \frac{4}{x - 1}$

**29.** $\frac{\frac{2}{3} + \frac{1}{2}}{\frac{1}{9} - \frac{1}{6}}$

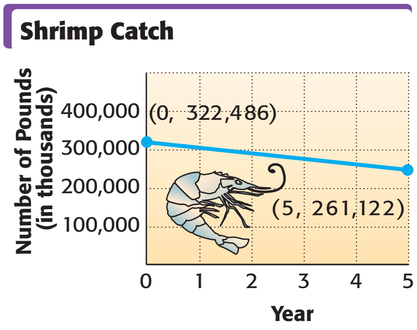
**30.** Solve the equation $\dfrac{x}{x+8} - \dfrac{3}{x-8} = \dfrac{128}{x^2-64}$.

**31.** The graph shows the number of pounds of shrimp caught in the United States (in thousands of pounds) in selected years.

**(a)** Use the information given in the graph to find and interpret the average rate of change in the number of pounds of shrimp caught per year.

**(b)** If $x = 0$ represents the year 2000, $x = 1$ represents 2001, and so on, use your answer from part (a) to write an equation of the line in slope-intercept form that models the annual amount of shrimp caught (in thousands of pounds, to the nearest whole number) for the years 2000 through 2005.

**(c)** Use the equation from part (b) to approximate the amount of shrimp caught in 2003.



*Source:* National Oceanic and Atmospheric Administration.

**32.** Give the domain and range of the relation

$$\{(-4,-2), (-1,0), (2,0), (5,2)\}.$$

Does this relation define a function?

**33.** If $g(x) = -x^2 - 2x + 6$, find $g(3)$.

*Solve each system.*

**34.** $\begin{aligned} 3x - 4y &= 1 \\ 2x + 3y &= 12 \end{aligned}$

**35.** $\begin{aligned} 3x - 2y &= 4 \\ -6x + 4y &= 7 \end{aligned}$

**36.** $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + z &= 1 \\ x + 2y + 2z &= -1 \end{aligned}$

*Use a system of equations to solve each problem.*

**37.** The Star-Spangled Banner that flew over Fort McHenry during the War of 1812 had a perimeter of 144 ft. Its length measured 12 ft more than its width. Find the dimensions of this flag, which is displayed in the Smithsonian Institution's Museum of American History in Washington, DC. (*Source:* National Park Service brochure.)

**38.** Agbe Asiamigbe needs 9 L of a 20% solution of alcohol. Agbe has a 15% solution on hand, as well as a 30% solution. How many liters of the 15% solution and the 30% solution should Agbe mix to get the 20% solution needed?

*Solve each equation or inequality.*

**39.** $x > -4$ and $x < 4$

**40.** $2x + 1 > 5$ or $2 - x \geq 2$

**41.** $|3x - 1| = 2$

**42.** $|3z + 1| \geq 7$

*Graph each solution set.*

**43.** $y \leq 2x - 6$

**44.** $x - y \geq 3$ and $3x + 4y \leq 12$

## CHAPTER 10

# Roots, Radicals, and Root Functions

The Pythagorean theorem states that in any right triangle with perpendicular sides $a$ and $b$ and longest side $c$,

$$a^2 + b^2 = c^2.$$

It is used in surveying, drafting, engineering, navigation, and many other fields. Although attributed to Pythagoras, it was known to surveyors from Egypt to China for a thousand years before Pythagoras.

In the 1939 movie *The Wizard of Oz,* the Scarecrow asks the Wizard for a brain. When the Wizard presents him with a diploma granting him a Th.D. (Doctor of Thinkology), the Scarecrow declares the following.

> *The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. . . . Oh joy! Rapture! I've got a brain.*

Did the Scarecrow state the Pythagorean theorem correctly? We will investigate this in **Exercises 133–134** of **Section 10.3.**

# 10.1 Radical Expressions and Graphs

**OBJECTIVES**

1. Find square roots.
2. Decide whether a given root is rational, irrational, or not a real number.
3. Find cube, fourth, and other roots.
4. Graph functions defined by radical expressions.
5. Find *n*th roots of *n*th powers.
6. Use a calculator to find roots.

**OBJECTIVE 1** **Find square roots.** In **Section 1.2,** we discussed the idea of the *square* of a number. Recall that squaring a number means multiplying the number by itself.

$$7^2 = 7 \cdot 7 = 49 \qquad \text{The square of 7 is 49.}$$

The opposite (inverse) of squaring a number is taking its *square root*. This is equivalent to asking

"What number when multiplied by itself equals 49?"

From the example above, one answer is 7, since $7 \cdot 7 = 49$.

This discussion can be generalized.

### Square Root

A number $b$ is a **square root** of $a$ if $\boldsymbol{b^2 = a}$.

### EXAMPLE 1 Finding All Square Roots of a Number

Find all square roots of 49.

We ask, "What number when multiplied by itself equals 49?" As mentioned above, one square root is 7, because $7 \cdot 7 = 49$. Another square root of 49 is $-7$, because

$$(-7)(-7) = 49.$$

Thus, the number 49 has *two* square roots: 7 and $-7$. One square root is positive, and one is negative. NOW TRY

NOW TRY EXERCISE 1

Find all square roots of 81.

The **positive** or **principal square root** of a number is written with the symbol $\sqrt{\phantom{x}}$. For example, the positive square root of 121 is 11.

$$\sqrt{121} = 11$$

The symbol $-\sqrt{\phantom{x}}$ is used for the **negative square root** of a number. For example, the negative square root of 121 is $-11$.

$$-\sqrt{121} = -11$$

The symbol $\sqrt{\phantom{x}}$, called a **radical symbol,** always represents the positive square root $\left(\text{except that } \sqrt{0} = 0\right)$. The number inside the radical symbol is called the **radicand,** and the entire expression—radical symbol and radicand—is called a **radical.**

Radical symbol Radicand

$$\underbrace{\sqrt{a}}_{\text{Radical}}$$

An algebraic expression containing a radical is called a **radical expression.**

The radical symbol $\sqrt{\phantom{x}}$ has been used since 16th-century Germany and was probably derived from the letter *R*. The radical symbol in the margin comes from the Latin word *radix,* for *root*. It was first used by Leonardo of Pisa (Fibonacci) in 1220.

**Early radical symbol**

NOW TRY ANSWER

**1.** 9, −9

We summarize our discussion of square roots as follows.

**Square Roots of *a***

If $a$ is a positive real number, then

$\sqrt{a}$ is the positive or principal square root of $a$,

and $-\sqrt{a}$ is the negative square root of $a$.

For nonnegative $a$,

$$\sqrt{a} \cdot \sqrt{a} = \left(\sqrt{a}\right)^2 = a \quad \text{and} \quad -\sqrt{a} \cdot \left(-\sqrt{a}\right) = \left(-\sqrt{a}\right)^2 = a.$$

Also, $\sqrt{0} = 0$.

NOW TRY EXERCISE 2

Find each square root.

**(a)** $\sqrt{400}$ **(b)** $-\sqrt{169}$

**(c)** $\sqrt{\dfrac{100}{121}}$

**EXAMPLE 2** Finding Square Roots

Find each square root.

**(a)** $\sqrt{144}$

The radical $\sqrt{144}$ represents the positive or principal square root of 144. Think of a positive number whose square is 144.

$$12^2 = 144, \quad \text{so} \quad \sqrt{144} = 12.$$

**(b)** $-\sqrt{1024}$

This symbol represents the negative square root of 1024. A calculator with a square root key can be used to find $\sqrt{1024} = 32$. Therefore,

$$-\sqrt{1024} = -32.$$

**(c)** $\sqrt{\dfrac{4}{9}} = \dfrac{2}{3}$ **(d)** $-\sqrt{\dfrac{16}{49}} = -\dfrac{4}{7}$ **(e)** $\sqrt{0.81} = 0.9$ NOW TRY

As shown in the preceding definition, when the square root of a positive real number is squared, the result is that positive real number. $\left(\text{Also, } \left(\sqrt{0}\right)^2 = 0.\right)$

NOW TRY EXERCISE 3

Find the square of each radical expression.

**(a)** $\sqrt{15}$ **(b)** $-\sqrt{23}$

**(c)** $\sqrt{2k^2 + 5}$

**EXAMPLE 3** Squaring Radical Expressions

Find the *square* of each radical expression.

**(a)** $\sqrt{13}$

$\left(\sqrt{13}\right)^2 = 13$ Definition of square root

**(b)** $-\sqrt{29}$

$\left(-\sqrt{29}\right)^2 = 29$ The square of a *negative* number is positive.

**(c)** $\sqrt{p^2 + 1}$

$\left(\sqrt{p^2 + 1}\right)^2 = p^2 + 1$ NOW TRY

NOW TRY ANSWERS

2. **(a)** 20 **(b)** $-13$ **(c)** $\frac{10}{11}$

3. **(a)** 15 **(b)** 23 **(c)** $2k^2 + 5$

**OBJECTIVE 2** **Decide whether a given root is rational, irrational, or not a real number.** Numbers with square roots that are rational are called **perfect squares.**

| Perfect squares<br>↓ | | Rational square roots<br>↓ |
|---|---|---|
| 25 | | $\sqrt{25} = 5$ |
| 144 | are perfect squares since | $\sqrt{144} = 12$ |
| $\frac{4}{9}$ | | $\sqrt{\frac{4}{9}} = \frac{2}{3}$ |

A number that is not a perfect square has a square root that is not a rational number. For example, $\sqrt{5}$ is not a rational number because it cannot be written as the ratio of two integers. Its decimal equivalent neither terminates nor repeats. However, $\sqrt{5}$ is a real number and corresponds to a point on the number line.

A real number that is not rational is called an **irrational number.** The number $\sqrt{5}$ is irrational. ***Many square roots of integers are irrational.***

If $a$ is a positive real number that is *not* a perfect square, then $\sqrt{a}$ is irrational.

***Not every number has a real number square root.*** For example, there is no real number that can be squared to get $-36$. (The square of a real number can never be negative.) Because of this, $\sqrt{-36}$ ***is not a real number.***

If $a$ is a *negative* real number, then $\sqrt{a}$ is *not* a real number.

---

**CAUTION** $\sqrt{-36}$ is ***not*** a real number, since there is no real number that can be squared to obtain $-36$. However, $-\sqrt{36}$ is the negative square root of 36, or $-6$.

---

*NOW TRY EXERCISE 4*

Tell whether each square root is *rational, irrational,* or *not a real number.*

**(a)** $\sqrt{31}$ **(b)** $\sqrt{900}$

**(c)** $\sqrt{-16}$

### *EXAMPLE 4* Identifying Types of Square Roots

Tell whether each square root is *rational, irrational,* or *not a real number.*

**(a)** $\sqrt{17}$ Because 17 is not a perfect square, $\sqrt{17}$ is irrational.

**(b)** $\sqrt{64}$ The number 64 is a perfect square, $8^2$, so $\sqrt{64} = 8$, a rational number.

**(c)** $\sqrt{-25}$ There is no real number whose square is $-25$. Therefore, $\sqrt{-25}$ is not a real number.

*NOW TRY*

---

**NOTE** Not all irrational numbers are square roots of integers. For example, $\pi$ (approximately 3.14159) is an irrational number that is not a square root of any integer.

---

**OBJECTIVE 3** **Find cube, fourth, and other roots.** Finding the square root of a number is the opposite (inverse) of squaring a number. In a similar way, there are inverses to finding the cube of a number or to finding the fourth or greater power of a number. These inverses are, respectively, the **cube root,** written $\sqrt[3]{a}$**,** and the **fourth root,** written $\sqrt[4]{a}$**.** Similar symbols are used for other roots.

*NOW TRY ANSWERS*

4. **(a)** irrational **(b)** rational
   **(c)** not a real number

### $\sqrt[n]{a}$

The $n$th root of $a$, written $\sqrt[n]{a}$, is a number whose $n$th power equals $a$. That is,

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.$$

In $\sqrt[n]{a}$, the number $n$ is the **index** or **order** of the radical.

Radical symbol Index Radicand

$$\underbrace{\sqrt[n]{a}}_{\text{Radical}}$$

We could write $\sqrt[2]{a}$ instead of $\sqrt{a}$, but the simpler symbol $\sqrt{a}$ is customary, since the square root is the most commonly used root.

---

**NOTE** When working with cube roots or fourth roots, it is helpful to memorize the first few **perfect cubes** ($2^3 = 8$, $3^3 = 27$, and so on) and the first few **perfect fourth powers** ($2^4 = 16$, $3^4 = 81$, and so on). See **Exercises 63 and 64.**

---

NOW TRY EXERCISE 5

Find each cube root.

**(a)** $\sqrt[3]{343}$ **(b)** $\sqrt[3]{-1000}$

**(c)** $\sqrt[3]{27}$

### EXAMPLE 5 Finding Cube Roots

Find each cube root.

**(a)** $\sqrt[3]{8}$ What number can be cubed to give 8? Because $2^3 = 8$, $\sqrt[3]{8} = 2$.

**(b)** $\sqrt[3]{-8} = -2$, because $(-2)^3 = -8$.

**(c)** $\sqrt[3]{216} = 6$, because $6^3 = 216$. NOW TRY

Notice in **Example 5(b)** that we can find the cube root of a negative number. (Contrast this with the square root of a negative number, which is not real.) In fact, the cube root of a positive number is positive, and the cube root of a negative number is negative. ***There is only one real number cube root for each real number.***

When a radical has an ***even index*** (square root, fourth root, and so on), ***the radicand must be nonnegative*** to yield a real number root. Also, for $a > 0$,

$$\sqrt{a},\ \sqrt[4]{a},\ \sqrt[6]{a}, \text{ and so on are positive (principal) roots.}$$

$$-\sqrt{a},\ -\sqrt[4]{a},\ -\sqrt[6]{a}, \text{ and so on are negative roots.}$$

NOW TRY EXERCISE 6

Find each root.

**(a)** $\sqrt[4]{625}$ **(b)** $\sqrt[4]{-625}$

**(c)** $\sqrt[5]{3125}$ **(d)** $\sqrt[5]{-3125}$

### EXAMPLE 6 Finding Other Roots

Find each root.

**(a)** $\sqrt[4]{16} = 2$, because 2 is positive and $2^4 = 16$.

**(b)** $-\sqrt[4]{16}$ From part (a), $\sqrt[4]{16} = 2$, so the negative root is $-\sqrt[4]{16} = -2$.

**(c)** $\sqrt[4]{-16}$ For a fourth root to be a real number, the radicand must be nonnegative. There is no real number that equals $\sqrt[4]{-16}$.

**(d)** $-\sqrt[5]{32}$

First find $\sqrt[5]{32}$. Because 2 is the number whose fifth power is 32, $\sqrt[5]{32} = 2$. Since $\sqrt[5]{32} = 2$, it follows that

$$-\sqrt[5]{32} = -2.$$

**(e)** $\sqrt[5]{-32} = -2$, because $(-2)^5 = -32$. NOW TRY

NOW TRY ANSWERS

**5.** **(a)** 7 **(b)** $-10$ **(c)** 3

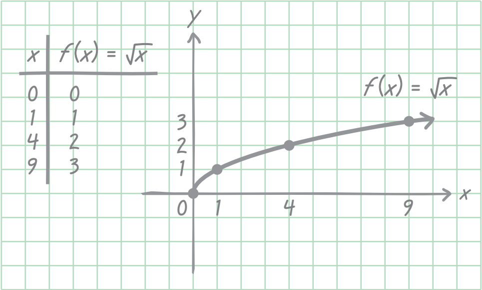
**6.** **(a)** 5 **(b)** not a real number
**(c)** 5 **(d)** $-5$

**OBJECTIVE 4** **Graph functions defined by radical expressions.** A **radical expression** is an algebraic expression that contains radicals.

$$3-\sqrt{x},\quad \sqrt[3]{x},\quad \text{and}\quad \sqrt{2x-1}$$ Examples of radical expressions

In earlier chapters, we graphed functions defined by polynomial and rational expressions. Now we examine the graphs of functions defined by the basic radical expressions $f(x)=\sqrt{x}$ and $f(x)=\sqrt[3]{x}$.

**FIGURE 1** shows the graph of the **square root function,** together with a table of selected points. Only nonnegative values can be used for $x$, so the domain is $[0, \infty)$. Because $\sqrt{x}$ is the principal square root of $x$, it always has a nonnegative value, so the range is also $[0, \infty)$.



| $x$ | $f(x)=\sqrt{x}$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |

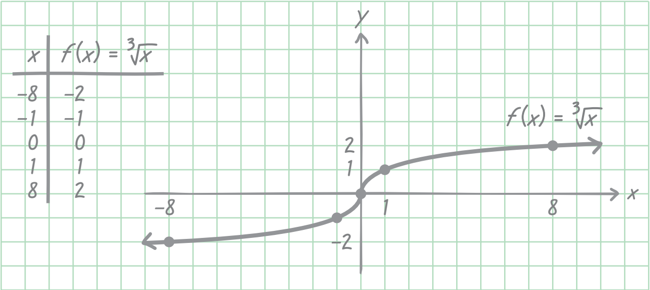
**Square root function**

$$f(x)=\sqrt{x}$$

Domain: $[0, \infty)$
Range: $[0, \infty)$

FIGURE 1

**FIGURE 2** shows the graph of the **cube root function.** Since any real number (positive, negative, or 0) can be used for $x$ in the cube root function, $\sqrt[3]{x}$ can be positive, negative, or 0. Thus, both the domain and the range of the cube root function are $(-\infty, \infty)$.



| $x$ | $f(x)=\sqrt[3]{x}$ |
|---|---|
| -8 | -2 |
| -1 | -1 |
| 0 | 0 |
| 1 | 1 |
| 8 | 2 |

**Cube root function**

$$f(x)=\sqrt[3]{x}$$

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

FIGURE 2

### EXAMPLE 7 Graphing Functions Defined with Radicals

Graph each function by creating a table of values. Give the domain and range.

**(a)** $f(x)=\sqrt{x-3}$

A table of values is given with the graph in **FIGURE 3** on the next page. The $x$-values were chosen in such a way that the function values are all integers. For the radicand to be nonnegative, we must have

$$x-3 \geq 0, \quad \text{or} \quad x \geq 3.$$

Therefore, the domain of this function is $[3, \infty)$. Function values are positive or 0, so the range is $[0, \infty)$.

**NOW TRY EXERCISE 7**

Graph each function. Give the domain and range.

**(a)** $f(x) = \sqrt{x+1}$

**(b)** $f(x) = \sqrt[3]{x} - 1$

| $x$ | $f(x) = \sqrt{x-3}$ |
|---|---|
| 3 | $\sqrt{3-3} = 0$ |
| 4 | $\sqrt{4-3} = 1$ |
| 7 | $\sqrt{7-3} = 2$ |

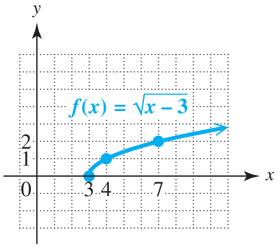


FIGURE 3

**(b)** $f(x) = \sqrt[3]{x} + 2$

See **FIGURE 4**. Both the domain and range are $(-\infty, \infty)$.

| $x$ | $f(x) = \sqrt[3]{x} + 2$ |
|---|---|
| $-8$ | $\sqrt[3]{-8} + 2 = 0$ |
| $-1$ | $\sqrt[3]{-1} + 2 = 1$ |
| 0 | $\sqrt[3]{0} + 2 = 2$ |
| 1 | $\sqrt[3]{1} + 2 = 3$ |
| 8 | $\sqrt[3]{8} + 2 = 4$ |

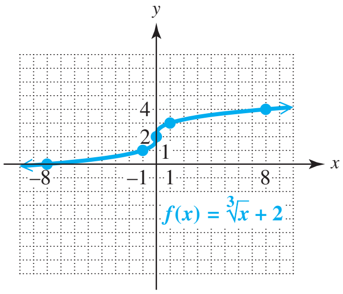


FIGURE 4

NOW TRY

**NOW TRY EXERCISE 8**

Find each square root.

**(a)** $\sqrt{11^2}$ **(b)** $\sqrt{(-11)^2}$

**(c)** $\sqrt{z^2}$ **(d)** $\sqrt{(-z)^2}$

**OBJECTIVE 5** **Find *n*th roots of *n*th powers.** Consider the expression $\sqrt{a^2}$. At first glance, you may think that it is equivalent to $a$. However, this is not necessarily true. For example, consider the following.

If $a = 6$, then $\sqrt{a^2} = \sqrt{6^2} = \sqrt{36} = 6$.

If $a = -6$, then $\sqrt{a^2} = \sqrt{(-6)^2} = \sqrt{36} = 6$. ← Instead of $-6$, we get 6, the *absolute value* of $-6$.

Since the symbol $\sqrt{a^2}$ represents the *nonnegative* square root, we express $\sqrt{a^2}$ with absolute value bars, as $|a|$, because $a$ may be a negative number.

### $\sqrt{a^2}$

For any real number $a$, $\boldsymbol{\sqrt{a^2} = |a|}$.

That is, the principal square root of $a^2$ is the absolute value of $a$.

## EXAMPLE 8 Simplifying Square Roots by Using Absolute Value

Find each square root.

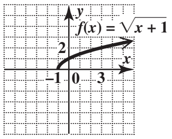
**(a)** $\sqrt{7^2} = |7| = 7$ **(b)** $\sqrt{(-7)^2} = |-7| = 7$

**(c)** $\sqrt{k^2} = |k|$ **(d)** $\sqrt{(-k)^2} = |-k| = |k|$

NOW TRY

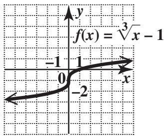
**NOW TRY ANSWERS**

**7. (a)**



domain: $[-1, \infty)$;
range: $[0, \infty)$

**(b)**



domain: $(-\infty, \infty)$;
range: $(-\infty, \infty)$

**8. (a)** 11 **(b)** 11 **(c)** $|z|$ **(d)** $|z|$

We can generalize this idea to any $n$th root.

### $\sqrt[n]{a^n}$

If $n$ is an ***even*** positive integer, then $\sqrt[n]{a^n} = |a|$.

If $n$ is an ***odd*** positive integer, then $\sqrt[n]{a^n} = a$.

That is, use the absolute value symbol when $n$ is even. Absolute value is not used when $n$ is odd.

**NOW TRY EXERCISE 9**

Simplify each root.

(a) $\sqrt[8]{(-2)^8}$ (b) $\sqrt[3]{(-9)^3}$

(c) $-\sqrt[4]{(-10)^4}$ (d) $-\sqrt{m^8}$

(e) $\sqrt[3]{x^{18}}$ (f) $\sqrt[4]{t^{20}}$

### EXAMPLE 9 Simplifying Higher Roots by Using Absolute Value

Simplify each root.

**(a)** $\sqrt[6]{(-3)^6} = |-3| = 3$ $n$ is even. Use absolute value.

**(b)** $\sqrt[5]{(-4)^5} = -4$ $n$ is odd.

**(c)** $-\sqrt[4]{(-9)^4} = -|-9| = -9$ $n$ is even. Use absolute value.

**(d)** $-\sqrt{m^4} = -|m^2| = -m^2$ For all $m$, $|m^2| = m^2$.

No absolute value bars are needed here, because $m^2$ is nonnegative for any real number value of $m$.

**(e)** $\sqrt[3]{a^{12}} = a^4$, because $a^{12} = (a^4)^3$.

**(f)** $\sqrt[4]{x^{12}} = |x^3|$

We use absolute value to guarantee that the result is not negative (because $x^3$ is negative when $x$ is negative). If desired $|x^3|$ can be written as $x^2 \cdot |x|$. NOW TRY

**OBJECTIVE 6** **Use a calculator to find roots.** While numbers such as $\sqrt{9}$ and $\sqrt[3]{-8}$ are rational, radicals are often irrational numbers. To find approximations of such radicals, we usually use a scientific or graphing calculator. For example,

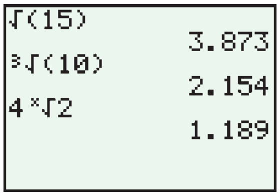
$$\sqrt{15} \approx 3.872983346, \quad \sqrt[3]{10} \approx 2.15443469, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189207115,$$

where the symbol $\approx$ means "is approximately equal to." In this book, we often show approximations rounded to three decimal places. Thus,

$$\sqrt{15} \approx 3.873, \quad \sqrt[3]{10} \approx 2.154, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189.$$

**FIGURE 5** shows how the preceding approximations are displayed on a TI-83/84 Plus graphing calculator.

There is a simple way to check that a calculator approximation is "in the ballpark." For example, because 16 is a little larger than 15, $\sqrt{16} = 4$ should be a little larger than $\sqrt{15}$. Thus, 3.873 is reasonable as an approximation for $\sqrt{15}$.



```
√(15)
                3.873
³√(10)
                2.154
4 ˣ√2
                1.189
```

FIGURE 5

**NOTE** The methods for finding approximations differ among makes and models of calculators. ***You should always consult your owner's manual for keystroke instructions.*** Be aware that graphing calculators often differ from scientific calculators in the order in which keystrokes are made.

**NOW TRY ANSWERS**

9. (a) 2 (b) $-9$ (c) $-10$
(d) $-m^4$ (e) $x^6$ (f) $|t^5|$

NOW TRY EXERCISE 10

Use a calculator to approximate each radical to three decimal places.

**(a)** $-\sqrt{92}$ **(b)** $\sqrt[4]{39}$

**(c)** $\sqrt[5]{33}$

## EXAMPLE 10 Finding Approximations for Roots

Use a calculator to verify that each approximation is correct.

**(a)** $\sqrt{39} \approx 6.245$

**(b)** $-\sqrt{72} \approx -8.485$

**(c)** $\sqrt[3]{93} \approx 4.531$

**(d)** $\sqrt[4]{39} \approx 2.499$

NOW TRY

NOW TRY EXERCISE 11

Use the formula in **Example 11** to approximate $f$ to the nearest thousand if

$$L = 7 \times 10^{-5}$$

and $$C = 3 \times 10^{-9}.$$

## EXAMPLE 11 Using Roots to Calculate Resonant Frequency

In electronics, the resonant frequency $f$ of a circuit may be found by the formula

$$f = \frac{1}{2\pi\sqrt{LC}},$$

where $f$ is in cycles per second, $L$ is in henrys, and $C$ is in farads. (Henrys and farads are units of measure in electronics.) Find the resonant frequency $f$ if $L = 5 \times 10^{-4}$ henry and $C = 3 \times 10^{-10}$ farad. Give your answer to the nearest thousand.

Find the value of $f$ when $L = 5 \times 10^{-4}$ and $C = 3 \times 10^{-10}$.

$$f = \frac{1}{2\pi\sqrt{LC}} \quad \text{Given formula}$$

$$= \frac{1}{2\pi\sqrt{(5 \times 10^{-4})(3 \times 10^{-10})}} \quad \text{Substitute for } L \text{ and } C.$$

$$\approx 411{,}000 \quad \text{Use a calculator.}$$

The resonant frequency $f$ is approximately 411,000 cycles per sec. NOW TRY

NOW TRY ANSWERS

**10. (a)** $-9.592$ **(b)** 2.499 **(c)** 2.012

**11.** 347,000 cycles per sec

## 10.1 EXERCISES

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***Concept Check*** *Decide whether each statement is* true *or* false. *If false, tell why.*

**1.** Every positive number has two real square roots.

**2.** A negative number has negative real square roots.

**3.** Every nonnegative number has two real square roots.

**4.** The positive square root of a positive number is its principal square root.

**5.** The cube root of every nonzero real number has the same sign as the number itself.

**6.** Every positive number has three real cube roots.

*Find all square roots of each number.* ***See Example 1.***

**7.** 9 **8.** 16 **9.** 64 **10.** 100 **11.** 169

**12.** 225 **13.** $\frac{25}{196}$ **14.** $\frac{81}{400}$ **15.** 900 **16.** 1600

*Find each square root.* ***See Examples 2 and 4(c).***

**17.** $\sqrt{1}$ **18.** $\sqrt{4}$ **19.** $\sqrt{49}$ **20.** $\sqrt{81}$ **21.** $-\sqrt{256}$

**22.** $-\sqrt{196}$ **23.** $-\sqrt{\frac{144}{121}}$ **24.** $-\sqrt{\frac{49}{36}}$ **25.** $\sqrt{0.64}$ **26.** $\sqrt{0.16}$

**27.** $\sqrt{-121}$ **28.** $\sqrt{-64}$ **29.** $-\sqrt{-49}$ **30.** $-\sqrt{-100}$

*Find the square of each radical expression.* ***See Example 3.***

**31.** $\sqrt{19}$ **32.** $\sqrt{59}$ **33.** $-\sqrt{19}$ **34.** $-\sqrt{59}$

**35.** $\sqrt{\frac{2}{3}}$ **36.** $\sqrt{\frac{5}{7}}$ **37.** $\sqrt{3x^2 + 4}$ **38.** $\sqrt{9y^2 + 3}$

***Concept Check*** *What must be true about the variable a for each statement to be true?*

**39.** $\sqrt{a}$ represents a positive number. **40.** $-\sqrt{a}$ represents a negative number.

**41.** $\sqrt{a}$ is not a real number. **42.** $-\sqrt{a}$ is not a real number.

*Determine whether each number is* rational, irrational, *or* not a real number. *If a number is rational, give its exact value. If a number is irrational, give a decimal approximation to the nearest thousandth. Use a calculator as necessary.* ***See Examples 4 and 10.***

**43.** $\sqrt{25}$ **44.** $\sqrt{169}$ **45.** $\sqrt{29}$ **46.** $\sqrt{33}$

**47.** $-\sqrt{64}$ **48.** $-\sqrt{81}$ **49.** $-\sqrt{300}$ **50.** $-\sqrt{500}$

**51.** $\sqrt{-29}$ **52.** $\sqrt{-47}$ **53.** $\sqrt{1200}$ **54.** $\sqrt{1500}$

***Concept Check*** *Without using a calculator, determine between which two consecutive integers each square root lies. For example,*

$$\sqrt{75} \text{ is between } 8 \text{ and } 9 \text{, because } \sqrt{64} = 8,\ \sqrt{81} = 9, \text{ and } 64 < 75 < 81.$$

**55.** $\sqrt{94}$ **56.** $\sqrt{43}$ **57.** $\sqrt{51}$ **58.** $\sqrt{30}$

**59.** $-\sqrt{40}$ **60.** $-\sqrt{63}$ **61.** $\sqrt{23.2}$ **62.** $\sqrt{10.3}$

**63.** ***Concept Check*** To help find cube roots, complete this list of perfect cubes.

$1^3 =$ _____ $2^3 =$ _____ $3^3 =$ _____ $4^3 =$ _____ $5^3 =$ _____

$6^3 =$ _____ $7^3 =$ _____ $8^3 =$ _____ $9^3 =$ _____ $10^3 =$ _____

**64.** ***Concept Check*** To help find fourth roots, complete this list of perfect fourth powers.

$1^4 =$ _____ $2^4 =$ _____ $3^4 =$ _____ $4^4 =$ _____ $5^4 =$ _____

$6^4 =$ _____ $7^4 =$ _____ $8^4 =$ _____ $9^4 =$ _____ $10^4 =$ _____

**65.** Match each expression from Column I with the equivalent choice from Column II. Answers may be used more than once. ***See Examples 5 and 6.***

| **I** | | **II** | |
|---|---|---|---|
| **(a)** $-\sqrt{25}$ | **(b)** $\sqrt{-25}$ | **A.** 3 | **B.** $-2$ |
| **(c)** $\sqrt[3]{-27}$ | **(d)** $\sqrt[5]{-32}$ | **C.** 2 | **D.** $-3$ |
| **(e)** $\sqrt[4]{81}$ | **(f)** $\sqrt[3]{8}$ | **E.** $-5$ | **F.** Not a real number |

**66.** ***Concept Check*** Consider the expression $-\sqrt{-a}$. Decide whether it is *positive, negative,* 0, or *not a real number* if

**(a)** $a > 0$, **(b)** $a < 0$, **(c)** $a = 0$.

*Find each root that is a real number.* ***See Examples 5 and 6.***

**67.** $-\sqrt{81}$
**68.** $-\sqrt{121}$
**69.** $\sqrt[3]{216}$
**70.** $\sqrt[3]{343}$
**71.** $\sqrt[3]{-64}$
**72.** $\sqrt[3]{-125}$
**73.** $-\sqrt[3]{512}$
**74.** $-\sqrt[3]{1000}$
**75.** $\sqrt[4]{1296}$
**76.** $\sqrt[4]{625}$
**77.** $-\sqrt[4]{16}$
**78.** $-\sqrt[4]{256}$
**79.** $\sqrt[4]{-625}$
**80.** $\sqrt[4]{-256}$
**81.** $\sqrt[6]{64}$
**82.** $\sqrt[6]{729}$
**83.** $\sqrt[6]{-32}$
**84.** $\sqrt[8]{-1}$
**85.** $\sqrt{\frac{64}{81}}$
**86.** $\sqrt{\frac{100}{9}}$
**87.** $\sqrt[3]{\frac{64}{27}}$
**88.** $\sqrt[4]{\frac{81}{16}}$
**89.** $-\sqrt[6]{\frac{1}{64}}$
**90.** $-\sqrt[5]{\frac{1}{32}}$
**91.** $-\sqrt[3]{-27}$
**92.** $-\sqrt[3]{-64}$
**93.** $\sqrt{0.25}$
**94.** $\sqrt{0.36}$
**95.** $-\sqrt{0.49}$
**96.** $-\sqrt{0.81}$
**97.** $\sqrt[3]{0.001}$
**98.** $\sqrt[3]{0.125}$

*Graph each function and give its domain and range.* ***See Example 7.***

**99.** $f(x) = \sqrt{x + 3}$
**100.** $f(x) = \sqrt{x - 5}$
**101.** $f(x) = \sqrt{x} - 2$
**102.** $f(x) = \sqrt{x} + 4$
**103.** $f(x) = \sqrt[3]{x} - 3$
**104.** $f(x) = \sqrt[3]{x} + 1$
**105.** $f(x) = \sqrt[3]{x - 3}$
**106.** $f(x) = \sqrt[3]{x + 1}$

*Simplify each root.* ***See Examples 8 and 9.***

**107.** $\sqrt{12^2}$
**108.** $\sqrt{19^2}$
**109.** $\sqrt{(-10)^2}$
**110.** $\sqrt{(-13)^2}$
**111.** $\sqrt[6]{(-2)^6}$
**112.** $\sqrt[6]{(-4)^6}$
**113.** $\sqrt[5]{(-9)^5}$
**114.** $\sqrt[5]{(-8)^5}$
**115.** $-\sqrt[6]{(-5)^6}$
**116.** $-\sqrt[6]{(-7)^6}$
**117.** $\sqrt{x^2}$
**118.** $-\sqrt{x^2}$
**119.** $\sqrt{(-z)^2}$
**120.** $\sqrt{(-q)^2}$
**121.** $\sqrt[3]{x^3}$
**122.** $-\sqrt[3]{x^3}$
**123.** $\sqrt[3]{x^{15}}$
**124.** $\sqrt[3]{m^9}$
**125.** $\sqrt[6]{x^{30}}$
**126.** $\sqrt[4]{k^{20}}$

***Concept Check*** *Choose the closest approximation of each square root.*

**127.** $\sqrt{123.5}$

**A.** 9 **B.** 10 **C.** 11 **D.** 12

**128.** $\sqrt{67.8}$

**A.** 7 **B.** 8 **C.** 9 **D.** 10

*Find a decimal approximation for each radical. Round the answer to three decimal places.* ***See Example 10.***

**129.** $\sqrt{9483}$
**130.** $\sqrt{6825}$
**131.** $\sqrt{284.361}$
**132.** $\sqrt{846.104}$
**133.** $-\sqrt{82}$
**134.** $-\sqrt{91}$
**135.** $\sqrt[3]{423}$
**136.** $\sqrt[3]{555}$
**137.** $\sqrt[4]{100}$
**138.** $\sqrt[4]{250}$
**139.** $\sqrt[5]{23.8}$
**140.** $\sqrt[5]{98.4}$

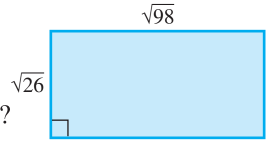
*Refer to the rectangle to answer the questions in Exercises 141 and 142.*

**141.** Which one of the following is the best estimate of its area?

**A.** 50 **B.** 250 **C.** 2500 **D.** 100

**142.** Which one of the following is the best estimate of its perimeter?

**A.** 15 **B.** 250 **C.** 100 **D.** 30



*Solve each problem.* ***See Example 11.***

**143.** Use the formula in **Example 11** to calculate the resonant frequency of a circuit to the nearest thousand if $L = 7.237 \times 10^{-5}$ henry and $C = 2.5 \times 10^{-10}$ farad.

**144.** The threshold weight $T$ for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to height $h$ in inches by the formula

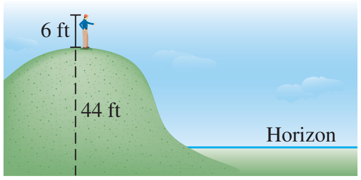
$$h = 12.3\sqrt[3]{T}.$$

What height corresponds to a threshold weight of 216 lb for a 43-year-old man? Round your answer to the nearest inch and then to the nearest tenth of a foot.

**145.** According to an article in *The World Scanner Report,* the distance $D$, in miles, to the horizon from an observer's point of view over water or "flat" earth is given by

$$D = \sqrt{2H},$$

where $H$ is the height of the point of view, in feet. If a person whose eyes are 6 ft above ground level is standing at the top of a hill 44 ft above "flat" earth, approximately how far to the horizon will she be able to see?



**146.** The time $t$ in seconds for one complete swing of a simple pendulum, where $L$ is the length of the pendulum in feet, and $g$, the acceleration due to gravity, is about 32 ft per $\text{sec}^2$, is

$$t = 2\pi\sqrt{\frac{L}{g}}.$$

Find the time of a complete swing of a 2-ft pendulum to the nearest tenth of a second.

**147. Heron's formula** gives a method of finding the area of a triangle if the lengths of its sides are known. Suppose that $a$, $b$, and $c$ are the lengths of the sides. Let $s$ denote one-half of the perimeter of the triangle (called the *semiperimeter*); that is, $s = \frac{1}{2}(a + b + c)$. Then the area of the triangle is

$$\mathcal{A} = \sqrt{s(s - a)(s - b)(s - c)}.$$

Find the area of the Bermuda Triangle, to the nearest thousand square miles, if the "sides" of this triangle measure approximately 850 mi, 925 mi, and 1300 mi.

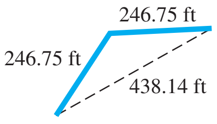
**148.** Use Heron's formula from **Exercise 147** to find the area of a triangle with sides of lengths $a = 11$ m, $b = 60$ m, and $c = 61$ m.

**149.** The coefficient of self-induction $L$ (in henrys), the energy $P$ stored in an electronic circuit (in joules), and the current $I$ (in amps) are related by this formula.

$$I = \sqrt{\frac{2P}{L}}$$

**(a)** Find $I$ if $P = 120$ and $L = 80$. **(b)** Find $I$ if $P = 100$ and $L = 40$.

**150.** The Vietnam Veterans Memorial in Washington, DC, is in the shape of an unenclosed isosceles triangle with equal sides of length 246.75 ft. If the triangle were enclosed, the third side would have length 438.14 ft. Use Heron's formula from **Exercise 147** to find the area of this enclosure to the nearest hundred square feet. (*Source:* Information pamphlet obtained at the Vietnam Veterans Memorial.)



Not to scale

## PREVIEW EXERCISES

*Apply the rules for exponents. Write each result with only positive exponents. Assume that all variables represent nonzero real numbers.* ***See Sections 4.1 and 4.2.***

**151.** $x^5 \cdot x^{-1} \cdot x^{-3}$

**152.** $(4x^2y^3)(2^3x^5y)$

**153.** $\left(\frac{2}{3}\right)^{-3}$

**154.** $\frac{5}{5^{-1}}$

# 10.2

# Rational Exponents

**OBJECTIVES**

1. Use exponential notation for *n*th roots.
2. Define and use expressions of the form $a^{m/n}$.
3. Convert between radicals and rational exponents.
4. Use the rules for exponents with rational exponents.

**OBJECTIVE 1** **Use exponential notation for *n*th roots.** Consider the product $(3^{1/2})^2 = 3^{1/2} \cdot 3^{1/2}$. Using the rules of exponents from **Section 4.1,** we can simplify this product as follows.

$$\begin{aligned} (3^{1/2})^2 &= 3^{1/2} \cdot 3^{1/2} & \\ &= 3^{1/2+1/2} & \text{Product rule: } a^m \cdot a^n = a^{m+n} \\ &= 3^1 & \text{Add exponents.} \\ &= 3 & a^1 = a \end{aligned}$$

Also, by definition,

$$\left(\sqrt{3}\right)^2 = \sqrt{3} \cdot \sqrt{3} = 3.$$

Since both $(3^{1/2})^2$ and $\left(\sqrt{3}\right)^2$ are equal to 3, it seems reasonable to define

$$3^{1/2} = \sqrt{3}.$$

This suggests the following generalization.

**$a^{1/n}$**

If $\sqrt[n]{a}$ is a real number, then $\quad \boldsymbol{a^{1/n} = \sqrt[n]{a}}.$

$$4^{1/2} = \sqrt{4}, \quad 8^{1/3} = \sqrt[3]{8}, \quad \text{and} \quad 16^{1/4} = \sqrt[4]{16} \qquad \text{Examples of } a^{1/n}$$

***Notice that the denominator of the rational exponent is the index of the radical.***

NOW TRY EXERCISE 1

Evaluate each exponential.

**(a)** $81^{1/2}$ **(b)** $125^{1/3}$

**(c)** $-625^{1/4}$ **(d)** $(-625)^{1/4}$

**(e)** $(-125)^{1/3}$ **(f)** $\left(\frac{1}{16}\right)^{1/4}$

## EXAMPLE 1 Evaluating Exponentials of the Form $a^{1/n}$

Evaluate each exponential.

The denominator is the index.

The denominator is the index. $\sqrt{\phantom{x}}$ means $\sqrt[2]{\phantom{x}}$.

**(a)** $64^{1/3} = \sqrt[3]{64} = 4$ **(b)** $100^{1/2} = \sqrt{100} = 10$

**(c)** $-256^{1/4} = -\sqrt[4]{256} = -4$

**(d)** $(-256)^{1/4} = \sqrt[4]{-256}$ is not a real number, because the radicand, $-256$, is negative and the index is even.

**(e)** $(-32)^{1/5} = \sqrt[5]{-32} = -2$ **(f)** $\left(\frac{1}{8}\right)^{1/3} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ NOW TRY

---

**CAUTION** Notice the difference between **Examples 1(c) and (d).** The radical in part (c) is the ***negative fourth root of a positive number,*** while the radical in part (d) is the ***principal fourth root of a negative number, which is not a real number.***

---

**OBJECTIVE 2** **Define and use expressions of the form $a^{m/n}$.** We know that $8^{1/3} = \sqrt[3]{8}$. We can define a number like $8^{2/3}$, where the numerator of the exponent is not 1. For past rules of exponents to be valid,

$$8^{2/3} = 8^{(1/3)2} = (8^{1/3})^2.$$

Since $8^{1/3} = \sqrt[3]{8}$,

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$$

Generalizing from this example, we define $a^{m/n}$ as follows.

### $a^{m/n}$

If $m$ and $n$ are positive integers with $m/n$ in lowest terms, then

$$\boldsymbol{a^{m/n} = (a^{1/n})^m},$$

provided that $a^{1/n}$ is a real number. If $a^{1/n}$ is not a real number, then $a^{m/n}$ is not a real number.

## EXAMPLE 2 Evaluating Exponentials of the Form $a^{m/n}$

Evaluate each exponential.

Think: $36^{1/2} = \sqrt{36} = 6$

Think: $125^{1/3} = \sqrt[3]{125} = 5$

**(a)** $36^{3/2} = (36^{1/2})^3 = 6^3 = 216$ **(b)** $125^{2/3} = (125^{1/3})^2 = 5^2 = 25$

Be careful. The base is 4.

**(c)** $-4^{5/2} = -(4^{5/2}) = -(4^{1/2})^5 = -(2)^5 = -32$

Because the base here is 4, the negative sign is *not* affected by the exponent.

NOW TRY ANSWERS

1. **(a)** 9 **(b)** 5 **(c)** $-5$
   **(d)** It is not a real number.
   **(e)** $-5$ **(f)** $\frac{1}{2}$

**NOW TRY EXERCISE 2**

Evaluate each exponential.

**(a)** $32^{2/5}$ **(b)** $8^{5/3}$

**(c)** $-100^{3/2}$ **(d)** $(-121)^{3/2}$

**(e)** $(-125)^{4/3}$

**(d)** $(-27)^{2/3} = [(-27)^{1/3}]^2 = (-3)^2 = 9$

Notice in part (c) that we first evaluate the exponential and then find its negative. In part (d), the − sign is part of the base, −27.

**(e)** $(-100)^{3/2} = [(-100)^{1/2}]^3$, which is not a real number, since $(-100)^{1/2}$, or $\sqrt{-100}$, is not a real number. NOW TRY

When a rational exponent is negative, the earlier interpretation of negative exponents is applied.

**$a^{-m/n}$**

If $a^{m/n}$ is a real number, then

$$a^{-m/n} = \frac{1}{a^{m/n}} \quad (a \neq 0).$$

**NOW TRY EXERCISE 3**

Evaluate each exponential.

**(a)** $243^{-3/5}$ **(b)** $4^{-5/2}$

**(c)** $\left(\frac{216}{125}\right)^{-2/3}$

## EXAMPLE 3 Evaluating Exponentials with Negative Rational Exponents

Evaluate each exponential.

**(a)** $16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{(16^{1/4})^3} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}$

The denominator of 3/4 is the index and the numerator is the exponent.

**(b)** $25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{(25^{1/2})^3} = \frac{1}{(\sqrt{25})^3} = \frac{1}{5^3} = \frac{1}{125}$

**(c)** $\left(\frac{8}{27}\right)^{-2/3} = \frac{1}{\left(\frac{8}{27}\right)^{2/3}} = \frac{1}{\left(\sqrt[3]{\frac{8}{27}}\right)^2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$

$\frac{1}{\frac{4}{9}} = 1 \div \frac{4}{9} = 1 \cdot \frac{9}{4}$

We can also use the rule $\left(\frac{b}{a}\right)^{-m} = \left(\frac{a}{b}\right)^m$ here, as follows.

$$\left(\frac{8}{27}\right)^{-2/3} = \left(\frac{27}{8}\right)^{2/3} = \left(\sqrt[3]{\frac{27}{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Take the reciprocal only of the base, *not* the exponent.

NOW TRY

⚠ **CAUTION** Be careful to distinguish between exponential expressions like the following.

$16^{-1/4}$, which equals $\frac{1}{2}$, $-16^{1/4}$, which equals $-2$, and $-16^{-1/4}$, which equals $-\frac{1}{2}$.

***A negative exponent does not necessarily lead to a negative result. Negative exponents lead to reciprocals, which may be positive.***

**NOW TRY ANSWERS**

2. **(a)** 4 **(b)** 32 **(c)** −1000
   **(d)** It is not a real number.
   **(e)** 625

3. **(a)** $\frac{1}{27}$ **(b)** $\frac{1}{32}$ **(c)** $\frac{25}{36}$

We obtain an alternative definition of $a^{m/n}$ by using the power rule for exponents differently than in the earlier definition. If all indicated roots are real numbers,

then $$a^{m/n} = a^{m(1/n)} = (a^m)^{1/n}, \quad \text{so} \quad a^{m/n} = (a^m)^{1/n}.$$

### $a^{m/n}$

If all indicated roots are real numbers, then

$$\mathbf{a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.}$$

We can now evaluate an expression such as $27^{2/3}$ in two ways.

$$27^{2/3} = (27^{1/3})^2 = 3^2 = 9$$

or $$27^{2/3} = (27^2)^{1/3} = 729^{1/3} = 9$$

The result is the same.

***In most cases, it is easier to use*** $(a^{1/n})^m$.

### Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$\mathbf{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.}$$

That is, raise $a$ to the $m$th power and then take the $n$th root, or take the $n$th root of $a$ and then raise to the $m$th power.

For example,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4, \quad \text{and} \quad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4,$$

so $$8^{2/3} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2.$$

**OBJECTIVE 3** **Convert between radicals and rational exponents.** Using the definition of rational exponents, we can simplify many problems involving radicals by converting the radicals to numbers with rational exponents. After simplifying, we can convert the answer back to radical form if required.

### EXAMPLE 4 Converting between Rational Exponents and Radicals

Write each exponential as a radical. Assume that all variables represent positive real numbers. Use the definition that takes the root first.

**(a)** $13^{1/2} = \sqrt{13}$ **(b)** $6^{3/4} = \left(\sqrt[4]{6}\right)^3$ **(c)** $9m^{5/8} = 9\left(\sqrt[8]{m}\right)^5$

**(d)** $6x^{2/3} - (4x)^{3/5} = 6\left(\sqrt[3]{x}\right)^2 - \left(\sqrt[5]{4x}\right)^3$

**(e)** $r^{-2/3} = \dfrac{1}{r^{2/3}} = \dfrac{1}{\left(\sqrt[3]{r}\right)^2}$

**(f)** $(a^2 + b^2)^{1/2} = \sqrt{a^2 + b^2}$

$\sqrt{a^2 + b^2} \neq a + b$

NOW TRY EXERCISE 4

Write each exponential as a radical. Assume that all variables represent positive real numbers.

**(a)** $21^{1/2}$ **(b)** $17^{5/4}$

**(c)** $4t^{3/5} + (4t)^{2/3}$

**(d)** $w^{-2/5}$ **(e)** $(a^2 - b^2)^{1/4}$

In parts (f)–(h), write each radical as an exponential. Simplify. Assume that all variables represent positive real numbers.

**(f)** $\sqrt[3]{15}$ **(g)** $\sqrt[4]{4^2}$

**(h)** $\sqrt[4]{x^4}$

In parts (g)–(i), write each radical as an exponential. Simplify. Assume that all variables represent positive real numbers.

**(g)** $\sqrt{10} = 10^{1/2}$ **(h)** $\sqrt[4]{3^8} = 3^{8/4} = 3^2 = 9$

**(i)** $\sqrt[6]{z^6} = z$, since $z$ is positive.

NOW TRY

---

**NOTE** In **Example 4(i),** it is not necessary to use absolute value bars, since the directions specifically state that the variable represents a positive real number. Because the absolute value of the positive real number $z$ is $z$ itself, the answer is simply $z$.

---

**OBJECTIVE 4** **Use the rules for exponents with rational exponents.** The definition of rational exponents allows us to apply the rules for exponents from **Sections 4.1 and 4.2.**

### Rules for Rational Exponents

Let $r$ and $s$ be rational numbers. For all real numbers $a$ and $b$ for which the indicated expressions exist, the following are true.

$$a^r \cdot a^s = a^{r+s} \qquad a^{-r} = \frac{1}{a^r} \qquad \frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^{-r} = \frac{b^r}{a^r}$$

$$(a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad a^{-r} = \left(\frac{1}{a}\right)^r$$

### EXAMPLE 5 Applying Rules for Rational Exponents

Write with only positive exponents. Assume that all variables represent positive real numbers.

**(a)** $2^{1/2} \cdot 2^{1/4}$

$= 2^{1/2+1/4}$ Product rule

$= 2^{3/4}$ Add exponents.

**(b)** $\dfrac{5^{2/3}}{5^{7/3}}$

$= 5^{2/3-7/3}$ Quotient rule

$= 5^{-5/3}$ Subtract exponents.

$= \dfrac{1}{5^{5/3}}$ $a^{-r} = \frac{1}{a^r}$

**(c)** $\dfrac{(x^{1/2}y^{2/3})^4}{y}$

$= \dfrac{(x^{1/2})^4(y^{2/3})^4}{y}$ Power rule

$= \dfrac{x^2y^{8/3}}{y^1}$ Power rule

$= x^2y^{8/3-1}$ Quotient rule

$= x^2y^{5/3}$ $\frac{8}{3} - 1 = \frac{8}{3} - \frac{3}{3} = \frac{5}{3}$

NOW TRY ANSWERS

4. **(a)** $\sqrt{21}$ **(b)** $\left(\sqrt[4]{17}\right)^5$
**(c)** $4\left(\sqrt[5]{t}\right)^3 + \left(\sqrt[3]{4t}\right)^2$
**(d)** $\dfrac{1}{\left(\sqrt[5]{w}\right)^2}$ **(e)** $\sqrt[4]{a^2 - b^2}$
**(f)** $15^{1/3}$ **(g)** 2 **(h)** $x$

**NOW TRY EXERCISE 5**

Write with only positive exponents. Assume that all variables represent positive real numbers.

**(a)** $5^{1/4} \cdot 5^{2/3}$ **(b)** $\dfrac{9^{3/5}}{9^{7/5}}$

**(c)** $\dfrac{(r^{2/3}t^{1/4})^8}{t}$

**(d)** $\left(\dfrac{2x^{1/2}y^{-2/3}}{x^{-3/5}y^{-1/5}}\right)^{-3}$

**(e)** $y^{2/3}(y^{1/3} + y^{5/3})$

**(d)** $\left(\dfrac{x^4y^{-6}}{x^{-2}y^{1/3}}\right)^{-2/3}$

$$= \frac{(x^4)^{-2/3}(y^{-6})^{-2/3}}{(x^{-2})^{-2/3}(y^{1/3})^{-2/3}} \quad \text{Power rule}$$

$$= \frac{x^{-8/3}y^4}{x^{4/3}y^{-2/9}} \quad \text{Power rule}$$

$$= x^{-8/3-4/3}y^{4-(-2/9)} \quad \text{Quotient rule}$$

Use parentheses to avoid errors.

$$= x^{-4}y^{38/9} \quad 4 - \left(-\tfrac{2}{9}\right) = \tfrac{36}{9} + \tfrac{2}{9} = \tfrac{38}{9}$$

$$= \frac{y^{38/9}}{x^4} \quad \text{Definition of negative exponent}$$

The same result is obtained if we simplify within the parentheses first.

$$\left(\frac{x^4y^{-6}}{x^{-2}y^{1/3}}\right)^{-2/3}$$

$$= (x^{4-(-2)}y^{-6-1/3})^{-2/3} \quad \text{Quotient rule}$$

$$= (x^6y^{-19/3})^{-2/3} \quad -6 - \tfrac{1}{3} = -\tfrac{18}{3} - \tfrac{1}{3} = -\tfrac{19}{3}$$

$$= (x^6)^{-2/3}(y^{-19/3})^{-2/3} \quad \text{Power rule}$$

$$= x^{-4}y^{38/9} \quad \text{Power rule}$$

$$= \frac{y^{38/9}}{x^4} \quad \text{Definition of negative exponent}$$

**(e)** $m^{3/4}(m^{5/4} - m^{1/4})$

$$= m^{3/4}(m^{5/4}) - m^{3/4}(m^{1/4}) \quad \text{Distributive property}$$

Do not make the common mistake of multiplying exponents in the first step.

$$= m^{3/4+5/4} - m^{3/4+1/4} \quad \text{Product rule}$$

$$= m^{8/4} - m^{4/4} \quad \text{Add exponents.}$$

$$= m^2 - m \quad \text{Lowest terms in exponents}$$

NOW TRY

---

**CAUTION** Use the rules of exponents in problems like those in **Example 5.** Do not convert the expressions to radical form.

---

## EXAMPLE 6 Applying Rules for Rational Exponents

Write all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$

$$= x^{2/3} \cdot x^{1/4} \quad \text{Convert to rational exponents.}$$

$$= x^{2/3+1/4} \quad \text{Product rule}$$

$$= x^{8/12+3/12} \quad \text{Write exponents with a common denominator.}$$

$$= x^{11/12} \quad \text{Add exponents.}$$

**NOW TRY ANSWERS**

**5.** **(a)** $5^{11/12}$ **(b)** $\dfrac{1}{9^{4/5}}$

**(c)** $r^{16/3}t$ **(d)** $\dfrac{y^{7/5}}{8x^{33/10}}$

**(e)** $y + y^{7/3}$

**(b)** $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}}$

$= \dfrac{x^{3/2}}{x^{2/3}}$ Convert to rational exponents.

$= x^{3/2-2/3}$ Quotient rule

$= x^{5/6}$ $\frac{3}{2} - \frac{2}{3} = \frac{9}{6} - \frac{4}{6} = \frac{5}{6}$

**(c)** $\sqrt{\sqrt[4]{z}}$

$= \sqrt{z^{1/4}}$ Convert the inside radical to rational exponents.

$= (z^{1/4})^{1/2}$ Convert to rational exponents.

$= z^{1/8}$ Power rule

NOW TRY

**NOTE** The ability to convert between radicals and rational exponents is important in the study of exponential and logarithmic functions in **Chapter 12.**

NOW TRY EXERCISE 6

Write all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[5]{y^3} \cdot \sqrt[3]{y}$ **(b)** $\dfrac{\sqrt[4]{y^3}}{\sqrt{y^5}}$

**(c)** $\sqrt{\sqrt[3]{y}}$

NOW TRY ANSWERS

**6. (a)** $y^{14/15}$ **(b)** $\dfrac{1}{y^{7/4}}$ **(c)** $y^{1/6}$

## 10.2 EXERCISES

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***Concept Check*** *Match each expression from Column I with the equivalent choice from Column II.*

| I | | II | |
|---|---|---|---|
| **1.** $3^{1/2}$ | **2.** $(-27)^{1/3}$ | **A.** $-4$ | **B.** $8$ |
| **3.** $-16^{1/2}$ | **4.** $(-25)^{1/2}$ | **C.** $\sqrt{3}$ | **D.** $-\sqrt{6}$ |
| **5.** $(-32)^{1/5}$ | **6.** $(-32)^{2/5}$ | **E.** $-3$ | **F.** $\sqrt{6}$ |
| **7.** $4^{3/2}$ | **8.** $6^{2/4}$ | **G.** $4$ | **H.** $-2$ |
| **9.** $-6^{2/4}$ | **10.** $36^{0.5}$ | **I.** $6$ | **J.** Not a real number |

*Evaluate each exponential.* ***See Examples 1–3.***

◉ **11.** $169^{1/2}$ **12.** $121^{1/2}$ **13.** $729^{1/3}$ **14.** $512^{1/3}$

**15.** $16^{1/4}$ **16.** $625^{1/4}$ **17.** $\left(\dfrac{64}{81}\right)^{1/2}$ **18.** $\left(\dfrac{8}{27}\right)^{1/3}$

◉ **19.** $(-27)^{1/3}$ **20.** $(-32)^{1/5}$ ◉ **21.** $(-144)^{1/2}$ **22.** $(-36)^{1/2}$

◉ **23.** $100^{3/2}$ **24.** $64^{3/2}$ **25.** $81^{3/4}$ **26.** $216^{2/3}$

**27.** $-16^{5/2}$ **28.** $-32^{3/5}$ **29.** $(-8)^{4/3}$ **30.** $(-243)^{2/5}$

◉ **31.** $32^{-3/5}$ **32.** $27^{-4/3}$ **33.** $64^{-3/2}$ **34.** $81^{-3/2}$

**35.** $\left(\dfrac{125}{27}\right)^{-2/3}$ **36.** $\left(\dfrac{64}{125}\right)^{-2/3}$ **37.** $\left(\dfrac{16}{81}\right)^{-3/4}$ **38.** $\left(\dfrac{729}{64}\right)^{-5/6}$

*Write with radicals. Assume that all variables represent positive real numbers.* ***See Example 4.***

◉ **39.** $10^{1/2}$ **40.** $3^{1/2}$ **41.** $8^{3/4}$

**42.** $7^{2/3}$ ◉ **43.** $(9q)^{5/8} - (2x)^{2/3}$ **44.** $(3p)^{3/4} + (4x)^{1/3}$

**45.** $(2m)^{-3/2}$ **46.** $(5y)^{-3/5}$ **47.** $(2y + x)^{2/3}$

**48.** $(r + 2z)^{3/2}$ **49.** $(3m^4 + 2k^2)^{-2/3}$ **50.** $(5x^2 + 3z^3)^{-5/6}$

*Simplify by first converting to rational exponents. Assume that all variables represent positive real numbers.* ***See Example 4.***

**51.** $\sqrt{2^{12}}$ **52.** $\sqrt{5^{10}}$ **53.** $\sqrt[3]{4^9}$ **54.** $\sqrt[4]{6^8}$ ◉ **55.** $\sqrt{x^{20}}$

**56.** $\sqrt{r^{50}}$ **57.** $\sqrt[3]{x} \cdot \sqrt{x}$ **58.** $\sqrt[4]{y} \cdot \sqrt[5]{y^2}$ **59.** $\dfrac{\sqrt[3]{t^4}}{\sqrt[5]{t^4}}$ **60.** $\dfrac{\sqrt[4]{w^3}}{\sqrt[6]{w}}$

*Simplify each expression. Write all answers with positive exponents. Assume that all variables represent positive real numbers.* ***See Example 5.***

**61.** $3^{1/2} \cdot 3^{3/2}$
**62.** $6^{4/3} \cdot 6^{2/3}$
**63.** $\dfrac{64^{5/3}}{64^{4/3}}$

**64.** $\dfrac{125^{7/3}}{125^{5/3}}$
**65.** $y^{7/3} \cdot y^{-4/3}$
**66.** $r^{-8/9} \cdot r^{17/9}$

**67.** $x^{2/3} \cdot x^{-1/4}$
**68.** $x^{2/5} \cdot x^{-1/3}$
**69.** $\dfrac{k^{1/3}}{k^{2/3} \cdot k^{-1}}$

**70.** $\dfrac{z^{3/4}}{z^{5/4} \cdot z^{-2}}$
**71.** $\dfrac{(x^{1/4}y^{2/5})^{20}}{x^2}$
**72.** $\dfrac{(r^{1/5}s^{2/3})^{15}}{r^2}$

**73.** $\dfrac{(x^{2/3})^2}{(x^2)^{7/3}}$
**74.** $\dfrac{(p^3)^{1/4}}{(p^{5/4})^2}$
**75.** $\dfrac{m^{3/4}n^{-1/4}}{(m^2n)^{1/2}}$

**76.** $\dfrac{(a^2b^5)^{-1/4}}{(a^{-3}b^2)^{1/6}}$
**77.** $\dfrac{p^{1/5}p^{7/10}p^{1/2}}{(p^3)^{-1/5}}$
**78.** $\dfrac{z^{1/3}z^{-2/3}z^{1/6}}{(z^{-1/6})^3}$

**79.** $\left(\dfrac{b^{-3/2}}{c^{-5/3}}\right)^2(b^{-1/4}c^{-1/3})^{-1}$
**80.** $\left(\dfrac{m^{-2/3}}{a^{-3/4}}\right)^4(m^{-3/8}a^{1/4})^{-2}$
**81.** $\left(\dfrac{p^{-1/4}q^{-3/2}}{3^{-1}p^{-2}q^{-2/3}}\right)^{-2}$

**82.** $\left(\dfrac{2^{-2}w^{-3/4}x^{-5/8}}{w^{3/4}x^{-1/2}}\right)^{-3}$
**83.** $p^{2/3}(p^{1/3} + 2p^{4/3})$
**84.** $z^{5/8}(3z^{5/8} + 5z^{11/8})$

**85.** $k^{1/4}(k^{3/2} - k^{1/2})$
**86.** $r^{3/5}(r^{1/2} + r^{3/4})$
**87.** $6a^{7/4}(a^{-7/4} + 3a^{-3/4})$

**88.** $4m^{5/3}(m^{-2/3} - 4m^{-5/3})$
**89.** $-5x^{7/6}(x^{5/6} - x^{-1/6})$
**90.** $-8y^{11/7}(y^{3/7} - y^{-4/7})$

*Write with rational exponents, and then apply the properties of exponents. Assume that all radicands represent positive real numbers. Give answers in exponential form.* ***See Example 6.***

**91.** $\sqrt[5]{x^3} \cdot \sqrt[4]{x}$
**92.** $\sqrt[6]{y^5} \cdot \sqrt[3]{y^2}$
**93.** $\dfrac{\sqrt{x^5}}{\sqrt{x^8}}$
**94.** $\dfrac{\sqrt[3]{k^5}}{\sqrt[3]{k^7}}$

**95.** $\sqrt{y} \cdot \sqrt[3]{yz}$
**96.** $\sqrt[3]{xz} \cdot \sqrt{z}$
**97.** $\sqrt[4]{\sqrt[3]{m}}$
**98.** $\sqrt[3]{\sqrt{k}}$

**99.** $\sqrt{\sqrt{\sqrt{x}}}$
**100.** $\sqrt{\sqrt{\sqrt{\sqrt{x}}}}$
**101.** $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$
**102.** $\sqrt[3]{\sqrt[5]{\sqrt{y}}}$

**103.** Show that, in general, $\sqrt{a^2 + b^2} \neq a + b$ by replacing $a$ with 3 and $b$ with 4.

**104.** Suppose someone claims that $\sqrt[n]{a^n + b^n}$ must equal $a + b$, since, when $a = 1$ and $b = 0$, a true statement results:

$$\sqrt[n]{a^n + b^n} = \sqrt[n]{1^n + 0^n} = \sqrt[n]{1^n} = 1 = 1 + 0 = a + b.$$

Explain why this is faulty reasoning.

*Solve each problem.*

**105.** Meteorologists can determine the duration of a storm by using the function defined by

$$T(D) = 0.07D^{3/2},$$

where $D$ is the diameter of the storm in miles and $T$ is the time in hours. Find the duration of a storm with a diameter of 16 mi. Round your answer to the nearest tenth of an hour.

**106.** The threshold weight $T$, in pounds, for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to height $h$ in inches by the function defined by

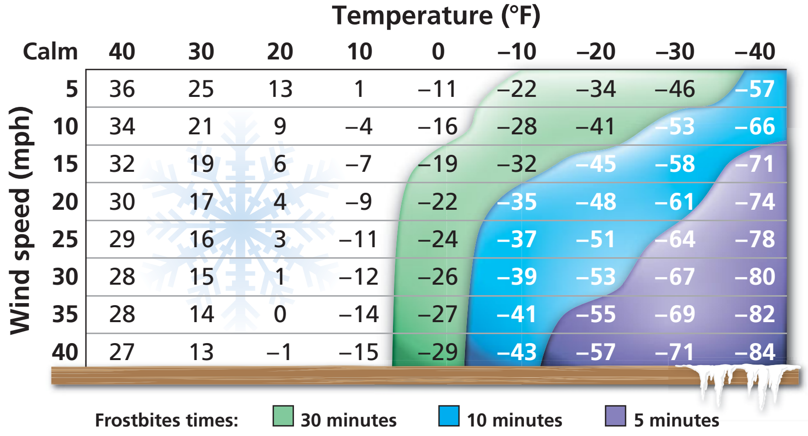
$$h(T) = (1860.867T)^{1/3}.$$

What height corresponds to a threshold weight of 200 lb for a 46-yr-old man? Round your answer to the nearest inch and then to the nearest tenth of a foot.

*The **windchill factor** is a measure of the cooling effect that the wind has on a person's skin. It calculates the equivalent cooling temperature if there were no wind. The National Weather Service uses the formula*

$$\text{Windchill temperature} = 35.74 + 0.6215T - 35.75V^{4/25} + 0.4275TV^{4/25},$$

*where $T$ is the temperature in °F and $V$ is the wind speed in miles per hour, to calculate windchill. The chart gives the windchill factor for various wind speeds and temperatures at which frostbite is a risk, and how quickly it may occur.*



| Wind speed (mph) | Temperature (°F) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Calm | 40 | 30 | 20 | 10 | 0 | –10 | –20 | –30 | –40 |
| 5 | 36 | 25 | 13 | 1 | –11 | –22 | –34 | –46 | –57 |
| 10 | 34 | 21 | 9 | –4 | –16 | –28 | –41 | –53 | –66 |
| 15 | 32 | 19 | 6 | –7 | –19 | –32 | –45 | –58 | –71 |
| 20 | 30 | 17 | 4 | –9 | –22 | –35 | –48 | –61 | –74 |
| 25 | 29 | 16 | 3 | –11 | –24 | –37 | –51 | –64 | –78 |
| 30 | 28 | 15 | 1 | –12 | –26 | –39 | –53 | –67 | –80 |
| 35 | 28 | 14 | 0 | –14 | –27 | –41 | –55 | –69 | –82 |
| 40 | 27 | 13 | –1 | –15 | –29 | –43 | –57 | –71 | –84 |

Frostbites times: 🟩 30 minutes 🟦 10 minutes 🟪 5 minutes

*Source*: National Oceanic and Atmospheric Administration, National Weather Service.

*Use the formula and a calculator to determine the windchill to the nearest tenth of a degree, given the following conditions. Compare your answers with the appropriate entries in the table.*

**107.** 30°F, 15-mph wind

**108.** 10°F, 30-mph wind

**109.** 20°F, 20-mph wind

**110.** 40°F, 10-mph wind

## PREVIEW EXERCISES

*Simplify each pair of expressions, and then compare the results.* ***See Section 10.1.***

**111.** $\sqrt{25} \cdot \sqrt{36}, \quad \sqrt{25 \cdot 36}$

**112.** $\dfrac{\sqrt[3]{27}}{\sqrt[3]{729}}, \quad \sqrt[3]{\dfrac{27}{729}}$

# 10.3 Simplifying Radical Expressions

**OBJECTIVES**

1. Use the product rule for radicals.
2. Use the quotient rule for radicals.
3. Simplify radicals.
4. Simplify products and quotients of radicals with different indexes.
5. Use the Pythagorean theorem.
6. Use the distance formula.

**OBJECTIVE 1 Use the product rule for radicals.** Consider the expressions $\sqrt{36 \cdot 4}$ and $\sqrt{36} \cdot \sqrt{4}$. Are they equal?

$$\sqrt{36 \cdot 4} = \sqrt{144} = 12$$
$$\sqrt{36} \cdot \sqrt{4} = 6 \cdot 2 = 12$$

The result is the same.

This is an example of the **product rule for radicals.**

### Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $n$ is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

That is, the product of two $n$th roots is the $n$th root of the product.

We justify the product rule by using the rules for rational exponents. Since $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{b} = b^{1/n}$,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}.$$

**CAUTION** ***Use the product rule only when the radicals have the same index.***

**NOW TRY EXERCISE 1**

Multiply. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{7} \cdot \sqrt{11}$

**(b)** $\sqrt{2mn} \cdot \sqrt{15}$

### EXAMPLE 1 Using the Product Rule

Multiply. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{5} \cdot \sqrt{7}$
$= \sqrt{5 \cdot 7}$
$= \sqrt{35}$

**(b)** $\sqrt{11} \cdot \sqrt{p}$
$= \sqrt{11p}$

**(c)** $\sqrt{7} \cdot \sqrt{11xyz}$
$= \sqrt{77xyz}$

NOW TRY

**NOW TRY EXERCISE 2**

Multiply. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[3]{4} \cdot \sqrt[3]{5}$

**(b)** $\sqrt[4]{5t} \cdot \sqrt[4]{6r^3}$

**(c)** $\sqrt[7]{20x} \cdot \sqrt[7]{3xy^3}$

**(d)** $\sqrt[3]{5} \cdot \sqrt[4]{9}$

### EXAMPLE 2 Using the Product Rule

Multiply. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[3]{3} \cdot \sqrt[3]{12}$
$= \sqrt[3]{3 \cdot 12}$
$= \sqrt[3]{36}$

**(b)** $\sqrt[4]{8y} \cdot \sqrt[4]{3r^2}$
$= \sqrt[4]{24yr^2}$

Remember to write the index.

**(c)** $\sqrt[6]{10m^4} \cdot \sqrt[6]{5m}$
$= \sqrt[6]{50m^5}$

**(d)** $\sqrt[4]{2} \cdot \sqrt[5]{2}$ cannot be simplified using the product rule for radicals, because the indexes $(4$ and $5)$ are different.

NOW TRY

**OBJECTIVE 2** **Use the quotient rule for radicals.** The **quotient rule for radicals** is similar to the product rule.

**Quotient Rule for Radicals**

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and $n$ is a natural number, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

That is, the $n$th root of a quotient is the quotient of the $n$th roots.

### EXAMPLE 3 Using the Quotient Rule

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$

**(b)** $\sqrt{\frac{7}{36}} = \frac{\sqrt{7}}{\sqrt{36}} = \frac{\sqrt{7}}{6}$

**(c)** $\sqrt[3]{-\frac{8}{125}} = \sqrt[3]{\frac{-8}{125}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{125}} = \frac{-2}{5} = -\frac{2}{5}$

$\frac{-a}{b} = -\frac{a}{b}$

**NOW TRY ANSWERS**

**1. (a)** $\sqrt{77}$ **(b)** $\sqrt{30mn}$

**2. (a)** $\sqrt[3]{20}$ **(b)** $\sqrt[4]{30tr^3}$

**(c)** $\sqrt[7]{60x^2y^3}$

**(d)** This expression cannot be simplified by the product rule.

**NOW TRY EXERCISE 3**

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{\frac{49}{36}}$ **(b)** $\sqrt{\frac{5}{144}}$

**(c)** $\sqrt[3]{-\frac{27}{1000}}$ **(d)** $\sqrt[4]{\frac{t}{16}}$

**(e)** $-\sqrt[5]{\frac{m^{15}}{243}}$

**(d)** $\sqrt[3]{\frac{7}{216}} = \frac{\sqrt[3]{7}}{\sqrt[3]{216}} = \frac{\sqrt[3]{7}}{6}$ **(e)** $\sqrt[5]{\frac{x}{32}} = \frac{\sqrt[5]{x}}{\sqrt[5]{32}} = \frac{\sqrt[5]{x}}{2}$

**(f)** $-\sqrt[3]{\frac{m^6}{125}} = -\frac{\sqrt[3]{m^6}}{\sqrt[3]{125}} = -\frac{m^2}{5}$ Think: $\sqrt[3]{m^6} = m^{6/3} = m^2$

NOW TRY

**OBJECTIVE 3** **Simplify radicals.** We use the product and quotient rules to simplify radicals. A radical is **simplified** if the following four conditions are met.

### Conditions for a Simplified Radical

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.

### EXAMPLE 4 Simplifying Roots of Numbers

Simplify.

**(a)** $\sqrt{24}$

Check to see whether 24 is divisible by a perfect square (the square of a natural number) such as 4, 9, 16, . . . . The greatest perfect square that divides into 24 is 4.

$$\begin{aligned} &\sqrt{24} \\ &= \sqrt{4 \cdot 6} && \text{Factor; 4 is a perfect square.} \\ &= \sqrt{4} \cdot \sqrt{6} && \text{Product rule} \\ &= 2\sqrt{6} && \sqrt{4} = 2 \end{aligned}$$

**(b)** $\sqrt{108}$

As shown on the left, the number 108 is divisible by the perfect square 36. If this perfect square is not immediately clear, try factoring 108 into its prime factors, as shown on the right.

$$\begin{aligned} &\sqrt{108} \\ &= \sqrt{36 \cdot 3} && \text{Factor.} \\ &= \sqrt{36} \cdot \sqrt{3} && \text{Product rule} \\ &= 6\sqrt{3} && \sqrt{36} = 6 \end{aligned}$$

$$\begin{aligned} &\sqrt{108} \\ &= \sqrt{2^2 \cdot 3^3} \\ &= \sqrt{2^2 \cdot 3^2 \cdot 3} && a^3 = a^2 \cdot a \\ &= \sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3} && \text{Product rule} \\ &= 2 \cdot 3 \cdot \sqrt{3} && \sqrt{2^2} = 2,\ \sqrt{3^2} = 3 \\ &= 6\sqrt{3} && \text{Multiply.} \end{aligned}$$

**(c)** $\sqrt{10}$

No perfect square (other than 1) divides into 10, so $\sqrt{10}$ cannot be simplified further.

**NOW TRY ANSWERS**

**3.** **(a)** $\frac{7}{6}$ **(b)** $\frac{\sqrt{5}}{12}$ **(c)** $-\frac{3}{10}$

**(d)** $\frac{\sqrt[4]{t}}{2}$ **(e)** $-\frac{m^3}{3}$

**NOW TRY EXERCISE 4**

Simplify.

**(a)** $\sqrt{50}$ **(b)** $\sqrt{192}$

**(c)** $\sqrt{42}$ **(d)** $\sqrt[3]{108}$

**(e)** $-\sqrt[4]{80}$

**(d)** $\sqrt[3]{16}$

The greatest perfect *cube* that divides into 16 is 8, so factor 16 as $8 \cdot 2$.

$$\begin{aligned} &\sqrt[3]{16} \\ &= \sqrt[3]{8 \cdot 2} && \text{8 is a perfect cube.} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{2} && \text{Product rule} \\ &= 2\sqrt[3]{2} && \sqrt[3]{8} = 2 \end{aligned}$$

Remember to write the index.

**(e)**

$$\begin{aligned} &-\sqrt[4]{162} \\ &= -\sqrt[4]{81 \cdot 2} && \text{81 is a perfect 4th power.} \\ &= -\sqrt[4]{81} \cdot \sqrt[4]{2} && \text{Product rule} \\ &= -3\sqrt[4]{2} && \sqrt[4]{81} = 3 \end{aligned}$$

Remember the negative sign in each line.

NOW TRY

---

**CAUTION** ***Be careful with which factors belong outside the radical sign and which belong inside.*** Note in **Example 4(b)** how $2 \cdot 3$ is written outside because $\sqrt{2^2} = 2$ and $\sqrt{3^2} = 3$, while the remaining 3 is left inside the radical.

---

**NOW TRY EXERCISE 5**

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{36x^5}$ **(b)** $\sqrt{32m^5n^4}$

**(c)** $\sqrt[3]{-125k^3p^7}$

**(d)** $-\sqrt[4]{162x^7y^8}$

## EXAMPLE 5 Simplifying Radicals Involving Variables

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt{16m^3}$

$$\begin{aligned} &= \sqrt{16m^2 \cdot m} && \text{Factor.} \\ &= \sqrt{16m^2} \cdot \sqrt{m} && \text{Product rule} \\ &= 4m\sqrt{m} && \text{Take the square root.} \end{aligned}$$

Absolute value bars are not needed around the $m$ in color because all the variables represent *positive* real numbers.

**(b)** $\sqrt{200k^7q^8}$

$$\begin{aligned} &= \sqrt{10^2 \cdot 2 \cdot (k^3)^2 \cdot k \cdot (q^4)^2} && \text{Factor.} \\ &= 10k^3q^4\sqrt{2k} && \text{Remove perfect square factors.} \end{aligned}$$

**(c)** $\sqrt[3]{-8x^4y^5}$

$$\begin{aligned} &= \sqrt[3]{(-8x^3y^3)(xy^2)} && \text{Choose } -8x^3y^3 \text{ as the perfect cube that divides into } -8x^4y^5. \\ &= \sqrt[3]{-8x^3y^3} \cdot \sqrt[3]{xy^2} && \text{Product rule} \\ &= -2xy\sqrt[3]{xy^2} && \text{Take the cube root.} \end{aligned}$$

**(d)** $-\sqrt[4]{32y^9}$

$$\begin{aligned} &= -\sqrt[4]{(16y^8)(2y)} && 16y^8 \text{ is the greatest 4th power that divides } 32y^9. \\ &= -\sqrt[4]{16y^8} \cdot \sqrt[4]{2y} && \text{Product rule} \\ &= -2y^2\sqrt[4]{2y} && \text{Take the fourth root.} \end{aligned}$$

NOW TRY

**NOW TRY ANSWERS**

**4.** **(a)** $5\sqrt{2}$ **(b)** $8\sqrt{3}$
**(c)** $\sqrt{42}$ cannot be simplified further.
**(d)** $3\sqrt[3]{4}$ **(e)** $-2\sqrt[4]{5}$

**5.** **(a)** $6x^2\sqrt{x}$ **(b)** $4m^2n^2\sqrt{2m}$
**(c)** $-5kp^2\sqrt[3]{p}$ **(d)** $-3xy^2\sqrt[4]{2x^3}$

**NOTE** From **Example 5,** we see that if a variable is raised to a power with an exponent divisible by 2, it is a perfect square. If it is raised to a power with an exponent divisible by 3, it is a perfect cube. ***In general, if it is raised to a power with an exponent divisible by n, it is a perfect nth power.***

The conditions for a simplified radical given earlier state that an exponent in the radicand and the index of the radical should have greatest common factor 1.

**NOW TRY EXERCISE 6**

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[6]{7^2}$ **(b)** $\sqrt[6]{y^4}$

### EXAMPLE 6 Simplifying Radicals by Using Smaller Indexes

Simplify. Assume that all variables represent positive real numbers.

**(a)** $\sqrt[9]{5^6}$

We write this radical by using rational exponents and then write the exponent in lowest terms. We then express the answer as a radical.

$$\sqrt[9]{5^6} = (5^6)^{1/9} = 5^{6/9} = 5^{2/3} = \sqrt[3]{5^2}, \quad \text{or} \quad \sqrt[3]{25}$$

**(b)** $\sqrt[4]{p^2} = (p^2)^{1/4} = p^{2/4} = p^{1/2} = \sqrt{p}$ (Recall the assumption that $p > 0$.)

NOW TRY

These examples suggest the following rule.

**$\sqrt[kn]{a^{km}}$**

If $m$ is an integer, $n$ and $k$ are natural numbers, and all indicated roots exist, then

$$\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}.$$

**OBJECTIVE 4 Simplify products and quotients of radicals with different indexes.** We multiply and divide radicals with different indexes by using rational exponents.

**NOW TRY EXERCISE 7**

Simplify $\sqrt[3]{3} \cdot \sqrt{6}$.

### EXAMPLE 7 Multiplying Radicals with Different Indexes

Simplify $\sqrt{7} \cdot \sqrt[3]{2}$.

Because the different indexes, 2 and 3, have a least common multiple of 6, use rational exponents to write each radical as a sixth root.

$$\sqrt{7} = 7^{1/2} = 7^{3/6} = \sqrt[6]{7^3} = \sqrt[6]{343}$$

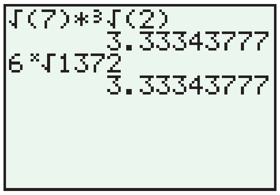
$$\sqrt[3]{2} = 2^{1/3} = 2^{2/6} = \sqrt[6]{2^2} = \sqrt[6]{4}$$

Now we can multiply.

$$\sqrt{7} \cdot \sqrt[3]{2} = \sqrt[6]{343} \cdot \sqrt[6]{4} \qquad \text{Substitute; } \sqrt{7} = \sqrt[6]{343},\ \sqrt[3]{2} = \sqrt[6]{4}$$

$$= \sqrt[6]{1372} \qquad \text{Product rule}$$

NOW TRY



```
√(7)*³√(2)
          3.33343777
6 ˣ√1372
          3.33343777
```

FIGURE 6

Results such as the one in **Example 7** can be supported with a calculator, as shown in FIGURE 6. Notice that the calculator gives the same approximation for the initial product and the final radical that we obtained.

**CAUTION** The computation in FIGURE 6 is not *proof* that the two expressions are equal. The algebra in **Example 7,** however, is valid proof of their equality.

**NOW TRY ANSWERS**

6. **(a)** $\sqrt[3]{7}$ **(b)** $\sqrt[3]{y^2}$

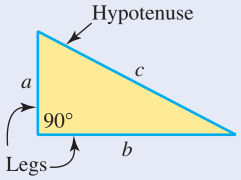
7. **(a)** $\sqrt[6]{1944}$

**OBJECTIVE 5** **Use the Pythagorean theorem.** The **Pythagorean theorem** provides an equation that relates the lengths of the three sides of a right triangle.

### Pythagorean Theorem

If $a$ and $b$ are the lengths of the shorter sides of a right triangle and $c$ is the length of the longest side, then

$$a^2 + b^2 = c^2.$$



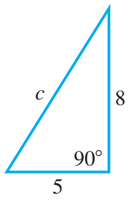
The two shorter sides are the **legs** of the triangle, and the longest side is the **hypotenuse.** The hypotenuse is the side opposite the right angle.

In **Section 11.1** we will see that an equation such as $x^2 = 7$ has two solutions: $\sqrt{7}$ (the principal, or positive, square root of 7) and $-\sqrt{7}$. Similarly, $c^2 = 52$ has two solutions, $\pm\sqrt{52} = \pm 2\sqrt{13}$. In applications we often choose only the principal square root.

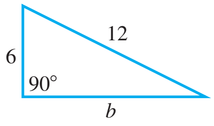
*NOW TRY EXERCISE 8*

Find the length of the unknown side in each triangle.

**(a)**



**(b)**



### *EXAMPLE 8* Using the Pythagorean Theorem

Use the Pythagorean theorem to find the length of the unknown side of the triangle in **FIGURE 7**.

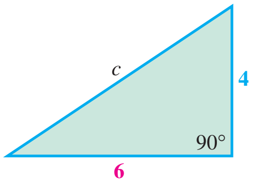


**FIGURE 7**

Substitute carefully.

$$a^2 + b^2 = c^2 \quad \text{Pythagorean theorem}$$

$$4^2 + 6^2 = c^2 \quad \text{Let } a = 4 \text{ and } b = 6.$$

$$16 + 36 = c^2 \quad \text{Apply the exponents.}$$

$$c^2 = 52 \quad \text{Add. Interchange sides.}$$

$$c = \sqrt{52} \quad \text{Choose the principal root.}$$

$$c = \sqrt{4 \cdot 13} \quad \text{Factor.}$$

$$c = \sqrt{4} \cdot \sqrt{13} \quad \text{Product rule}$$

$$c = 2\sqrt{13} \quad \text{Simplify.}$$

The length of the hypotenuse is $2\sqrt{13}$.

*NOW TRY*

**CAUTION** When substituting in the equation $a^2 + b^2 = c^2$, of the Pythagorean theorem, be sure that the length of the hypotenuse is substituted for $c$ and that the lengths of the legs are substituted for $a$ and $b$.

**OBJECTIVE 6** **Use the distance formula.** The *distance formula* allows us to find the distance between two points in the coordinate plane, or the length of the line segment joining those two points.

**FIGURE 8** on the next page shows the points $(3, -4)$ and $(-5, 3)$. The vertical line through $(-5, 3)$ and the horizontal line through $(3, -4)$ intersect at the point $(-5, -4)$. Thus, the point $(-5, -4)$ becomes the vertex of the right angle in a right triangle.

*NOW TRY ANSWERS*

**8.** **(a)** $\sqrt{89}$ **(b)** $6\sqrt{3}$

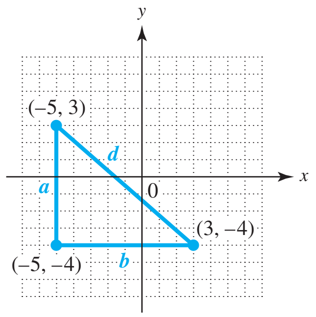


FIGURE 8

By the Pythagorean theorem, the square of the length of the hypotenuse $d$ of the right triangle in **FIGURE 8** is equal to the sum of the squares of the lengths of the two legs $a$ and $b$.

$$a^2 + b^2 = d^2$$

The length $a$ is the difference between the $y$-coordinates of the endpoints. Since the $x$-coordinate of both points in **FIGURE 8** is $-5$, the side is vertical, and we can find $a$ by finding the difference between the $y$-coordinates. We subtract $-4$ from 3 to get a positive value for $a$.

$$a = 3 - (-4) = 7$$

Similarly, we find $b$ by subtracting $-5$ from 3.

$$b = 3 - (-5) = 8$$

Now substitute these values into the equation.

$$d^2 = a^2 + b^2$$

$$d^2 = 7^2 + 8^2 \quad \text{Let } a = 7 \text{ and } b = 8.$$

$$d^2 = 49 + 64 \quad \text{Apply the exponents.}$$

$$d^2 = 113 \quad \text{Add.}$$

$$d = \sqrt{113} \quad \text{Choose the principal root.}$$

We choose the principal root, since distance cannot be negative. Therefore, the distance between $(-5, 3)$ and $(3, -4)$ is $\sqrt{113}$.

**NOTE** It is customary to leave the distance in simplified radical form. Do not use a calculator to get an approximation, unless you are specifically directed to do so.

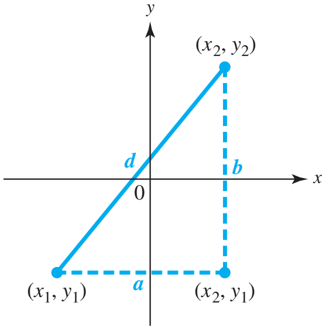


FIGURE 9

This result can be generalized. **FIGURE 9** shows the two points $(x_1, y_1)$ and $(x_2, y_2)$. The distance $a$ between $(x_1, y_1)$ and $(x_2, y_1)$ is given by

$$a = |x_2 - x_1|,$$

and the distance $b$ between $(x_2, y_2)$ and $(x_2, y_1)$ is given by

$$b = |y_2 - y_1|.$$

From the Pythagorean theorem, we obtain the following.

$$d^2 = a^2 + b^2$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Choosing the principal square root gives the **distance formula.**

**Distance Formula**

The distance $d$ between the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

NOW TRY EXERCISE 9

Find the distance between the points $(-4, -3)$ and $(-8, 6)$.

**EXAMPLE 9** Using the Distance Formula

Find the distance between the points $(-3, 5)$ and $(6, 4)$.

Designating the points as $(x_1, y_1)$ and $(x_2, y_2)$ is arbitrary. We choose $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (6, 4)$.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[6 - (-3)]^2 + (4 - 5)^2} \qquad x_2 = 6,\ y_2 = 4,\ x_1 = -3,\ y_1 = 5 \\ &= \sqrt{9^2 + (-1)^2} \\ &= \sqrt{82} \end{aligned}$$

Substitute carefully.

Leave in radical form.

NOW TRY

NOW TRY ANSWER

**9.** $\sqrt{97}$

## 10.3 EXERCISES

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*Multiply, if possible, using the product rule. Assume that all variables represent positive real numbers.* ***See Examples 1 and 2.***

**1.** $\sqrt{3} \cdot \sqrt{3}$ **2.** $\sqrt{5} \cdot \sqrt{5}$ **3.** $\sqrt{18} \cdot \sqrt{2}$ **4.** $\sqrt{12} \cdot \sqrt{3}$

**5.** $\sqrt{5} \cdot \sqrt{6}$ **6.** $\sqrt{10} \cdot \sqrt{3}$ **7.** $\sqrt{14} \cdot \sqrt{x}$ **8.** $\sqrt{23} \cdot \sqrt{t}$

**9.** $\sqrt{14} \cdot \sqrt{3pqr}$ **10.** $\sqrt{7} \cdot \sqrt{5xt}$ **11.** $\sqrt[3]{2} \cdot \sqrt[3]{5}$ **12.** $\sqrt[3]{3} \cdot \sqrt[3]{6}$

**13.** $\sqrt[3]{7x} \cdot \sqrt[3]{2y}$ **14.** $\sqrt[3]{9x} \cdot \sqrt[3]{4y}$ **15.** $\sqrt[4]{11} \cdot \sqrt[4]{3}$ **16.** $\sqrt[4]{6} \cdot \sqrt[4]{9}$

**17.** $\sqrt[4]{2x} \cdot \sqrt[4]{3x^2}$ **18.** $\sqrt[4]{3y^2} \cdot \sqrt[4]{6y}$ **19.** $\sqrt[3]{7} \cdot \sqrt[4]{3}$ **20.** $\sqrt[5]{8} \cdot \sqrt[6]{12}$

*Simplify each radical. Assume that all variables represent positive real numbers.* ***See Example 3.***

**21.** $\sqrt{\frac{64}{121}}$ **22.** $\sqrt{\frac{16}{49}}$ **23.** $\sqrt{\frac{3}{25}}$ **24.** $\sqrt{\frac{13}{49}}$

**25.** $\sqrt{\frac{x}{25}}$ **26.** $\sqrt{\frac{k}{100}}$ **27.** $\sqrt{\frac{p^6}{81}}$ **28.** $\sqrt{\frac{w^{10}}{36}}$

**29.** $\sqrt[3]{-\frac{27}{64}}$ **30.** $\sqrt[3]{-\frac{216}{125}}$ **31.** $\sqrt[3]{\frac{r^2}{8}}$ **32.** $\sqrt[3]{\frac{t}{125}}$

**33.** $-\sqrt[4]{\frac{81}{x^4}}$ **34.** $-\sqrt[4]{\frac{625}{y^4}}$ **35.** $\sqrt[5]{\frac{1}{x^{15}}}$ **36.** $\sqrt[5]{\frac{32}{y^{20}}}$

*Express each radical in simplified form.* ***See Example 4.***

**37.** $\sqrt{12}$ **38.** $\sqrt{18}$ **39.** $\sqrt{288}$ **40.** $\sqrt{72}$ **41.** $-\sqrt{32}$

**42.** $-\sqrt{48}$ **43.** $-\sqrt{28}$ **44.** $-\sqrt{24}$ **45.** $\sqrt{30}$ **46.** $\sqrt{46}$

**47.** $\sqrt[3]{128}$ **48.** $\sqrt[3]{24}$ **49.** $\sqrt[3]{-16}$ **50.** $\sqrt[3]{-250}$ **51.** $\sqrt[3]{40}$

**52.** $\sqrt[3]{375}$ **53.** $-\sqrt[4]{512}$ **54.** $-\sqrt[4]{1250}$ **55.** $\sqrt[5]{64}$ **56.** $\sqrt[5]{128}$

**57.** $-\sqrt[5]{486}$ **58.** $-\sqrt[5]{2048}$ **59.** $\sqrt[6]{128}$ **60.** $\sqrt[6]{1458}$

**61.** A student claimed that $\sqrt[3]{14}$ is not in simplified form, since $14 = 8 + 6$, and 8 is a perfect cube. Was his reasoning correct? Why or why not?

**62.** Explain in your own words why $\sqrt[3]{k^4}$ is not a simplified radical.

*Express each radical in simplified form. Assume that all variables represent positive real numbers.* ***See Example 5.***

**63.** $\sqrt{72k^2}$ **64.** $\sqrt{18m^2}$ **65.** $\sqrt{144x^3y^9}$

**66.** $\sqrt{169s^5t^{10}}$ **67.** $\sqrt{121x^6}$ **68.** $\sqrt{256z^{12}}$

**69.** $-\sqrt[3]{27t^{12}}$ **70.** $-\sqrt[3]{64y^{18}}$ **71.** $-\sqrt{100m^8z^4}$

**72.** $-\sqrt{25t^6s^{20}}$ **73.** $-\sqrt[3]{-125a^6b^9c^{12}}$ **74.** $-\sqrt[3]{-216y^{15}x^6z^3}$

**75.** $\sqrt[4]{\frac{1}{16}r^8t^{20}}$ **76.** $\sqrt[4]{\frac{81}{256}t^{12}u^8}$ **77.** $\sqrt{50x^3}$ **78.** $\sqrt{300z^3}$

**79.** $-\sqrt{500r^{11}}$ **80.** $-\sqrt{200p^{13}}$ **81.** $\sqrt{13x^7y^8}$ **82.** $\sqrt{23k^9p^{14}}$

**83.** $\sqrt[3]{8z^6w^9}$ **84.** $\sqrt[3]{64a^{15}b^{12}}$ **85.** $\sqrt[3]{-16z^5t^7}$ **86.** $\sqrt[3]{-81m^4n^{10}}$

**87.** $\sqrt[4]{81x^{12}y^{16}}$ **88.** $\sqrt[4]{81t^8u^{28}}$ **89.** $-\sqrt[4]{162r^{15}s^{10}}$ **90.** $-\sqrt[4]{32k^5m^{10}}$

**91.** $\sqrt{\frac{y^{11}}{36}}$ **92.** $\sqrt{\frac{v^{13}}{49}}$ **93.** $\sqrt[3]{\frac{x^{16}}{27}}$ **94.** $\sqrt[3]{\frac{y^{17}}{125}}$

*Simplify each radical. Assume that* $x \geq 0$. ***See Example 6.***

**95.** $\sqrt[4]{48^2}$ **96.** $\sqrt[4]{50^2}$ **97.** $\sqrt[4]{25}$

**98.** $\sqrt[6]{8}$ **99.** $\sqrt[10]{x^{25}}$ **100.** $\sqrt[12]{x^{44}}$

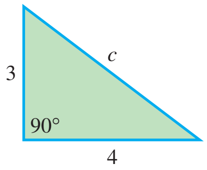
*Simplify by first writing the radicals as radicals with the same index. Then multiply. Assume that all variables represent positive real numbers.* ***See Example 7.***

**101.** $\sqrt[3]{4} \cdot \sqrt{3}$ **102.** $\sqrt[3]{5} \cdot \sqrt{6}$ **103.** $\sqrt[4]{3} \cdot \sqrt[3]{4}$

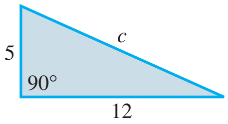
**104.** $\sqrt[5]{7} \cdot \sqrt[7]{5}$ **105.** $\sqrt{x} \cdot \sqrt[3]{x}$ **106.** $\sqrt[3]{y} \cdot \sqrt[4]{y}$

*Find the unknown length in each right triangle. Simplify the answer if possible.* ***See Example 8.***

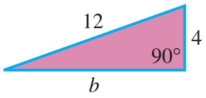
**107.**



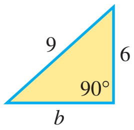
**108.**



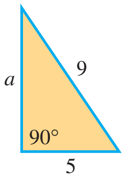
**109.**



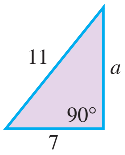
**110.**



**111.**



**112.**



*Find the distance between each pair of points.* ***See Example 9.***

**113.** $(6, 13)$ and $(1, 1)$ **114.** $(8, 13)$ and $(2, 5)$

**115.** $(-6, 5)$ and $(3, -4)$ **116.** $(-1, 5)$ and $(-7, 7)$

**117.** $(-8, 2)$ and $(-4, 1)$ **118.** $(-1, 2)$ and $(5, 3)$

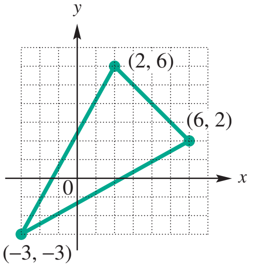
**119.** $(4.7, 2.3)$ and $(1.7, -1.7)$ **120.** $(-2.9, 18.2)$ and $(2.1, 6.2)$

**121.** $(\sqrt{2}, \sqrt{6})$ and $(-2\sqrt{2}, 4\sqrt{6})$ **122.** $(\sqrt{7}, 9\sqrt{3})$ and $(-\sqrt{7}, 4\sqrt{3})$

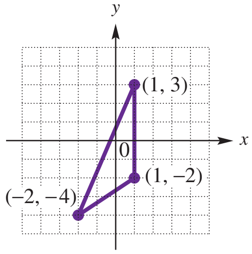
**123.** $(x + y, y)$ and $(x - y, x)$ **124.** $(c, c - d)$ and $(d, c + d)$

*Find the perimeter of each triangle.* $\left(\textit{Hint: For Exercise 125, use } \sqrt{k} + \sqrt{k} = 2\sqrt{k}.\right)$

**125.**



**126.**



*Solve each problem.*

**127.** The following letter appeared in the column "Ask Tom Why," written by Tom Skilling of the *Chicago Tribune*:

> *Dear Tom,*
>
> *I cannot remember the formula to calculate the distance to the horizon. I have a stunning view from my 14th-floor condo, 150 ft above the ground. How far can I see?*
>
> *Ted Fleischaker; Indianapolis, Ind.*

Skilling's answer was as follows:

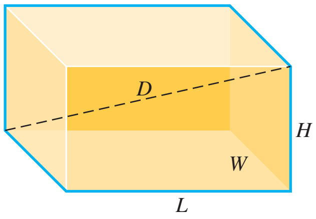
> To find the distance to the horizon in miles, take the square root of the height of your view in feet and multiply that result by 1.224. Your answer will be the number of miles to the horizon. (*Source: Chicago Tribune*.)

Assuming that Ted's eyes are 6 ft above the ground, the total height from the ground is $150 + 6 = 156$ ft. To the nearest tenth of a mile, how far can he see to the horizon?

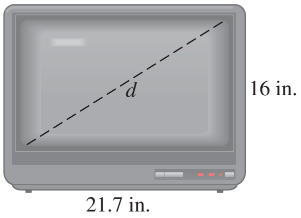
**128.** The length of the diagonal of a box is given by

$$D = \sqrt{L^2 + W^2 + H^2},$$

where $L$, $W$, and $H$ are, respectively, the length, width, and height of the box. Find the length of the diagonal $D$ of a box that is 4 ft long, 2 ft wide, and 3 ft high. Give the exact value, and then round to the nearest tenth of a foot.



**129.** A Sanyo color television, model AVM-2755, has a rectangular screen with a 21.7-in. width. Its height is 16 in. What is the measure of the diagonal of the screen, to the nearest tenth of an inch? (*Source:* Actual measurements of the author's television.)



**130.** A formula from electronics dealing with the impedance of parallel resonant circuits is

$$I = \frac{E}{\sqrt{R^2 + \omega^2 L^2}},$$

where the variables are in appropriate units. Find $I$ if $E = 282$, $R = 100$, $L = 264$, and $\omega = 120\pi$. Give your answer to the nearest thousandth.

**131.** In the study of sound, one version of the law of tensions is

$$f_1 = f_2\sqrt{\frac{F_1}{F_2}}.$$

If $F_1 = 300$, $F_2 = 60$, and $f_2 = 260$, find $f_1$ to the nearest unit.

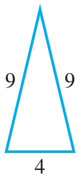
**132.** The illumination $I$, in foot-candles, produced by a light source is related to the distance $d$, in feet, from the light source by the equation

$$d = \sqrt{\frac{k}{I}},$$

where $k$ is a constant. If $k = 640$, how far from the light source will the illumination be 2 foot-candles? Give the exact value, and then round to the nearest tenth of a foot.

*Refer to the Chapter Opener on* ***page 599.*** *Recall that an* ***isosceles triangle*** *is a triangle that has two sides of equal length.*

**133.** The statement made by the Scarecrow in *The Wizard of Oz* can be proved false by providing at least one situation in which it leads to a false statement. Use the isosceles triangle shown here to prove that the statement is false.



**134.** Use the same style of wording as the Scarecrow to state the Pythagorean theorem correctly.

*The table gives data on three different solar modules available for roofing.*

| *Model* | *Watts* | *Volts* | *Amps* | *Size (in inches)* | *Cost (in dollars)* |
|---|---|---|---|---|---|
| MSX-77 | 77 | 16.9 | 4.56 | 44 × 26 | 475 |
| MSX-83 | 83 | 17.1 | 4.85 | 44 × 24 | 490 |
| MSX-60 | 60 | 17.1 | 3.5 | 44 × 20 | 382 |

*Source:* Solarex table in Jade Mountain catalog.

*You must determine the size of frame needed to support each panel on a roof. (Note: The sides of each frame will form a right triangle, and the hypotenuse of the triangle will be the width of the panel.) In Exercises 135–136, use the Pythagorean theorem to find the dimensions of the legs for each frame under the given conditions. Round answers to the nearest tenth.*

**135.** The legs have equal length.

**136.** One leg is twice the length of the other.

## *PREVIEW EXERCISES*

*Combine like terms.* ***See Section 4.4.***

**137.** $13x^4 - 12x^3 + 9x^4 + 2x^3$

**138.** $-15z^3 - z^2 + 4z^4 + 12z^8$

**139.** $9q^2 + 2q - 5q - q^2$

**140.** $7m^5 - 2m^3 + 8m^5 - m^3$

# 10.4 Adding and Subtracting Radical Expressions

**OBJECTIVE**

**1** Simplify radical expressions involving addition and subtraction.

**OBJECTIVE 1 Simplify radical expressions involving addition and subtraction.** Expressions such as $4\sqrt{2} + 3\sqrt{2}$ and $2\sqrt{3} - 5\sqrt{3}$ can be simplified using the distributive property.

$$4\sqrt{2} + 3\sqrt{2}$$
$$= (4 + 3)\sqrt{2} = 7\sqrt{2}$$

This is similar to simplifying $4x + 3x$ to $7x$.

$$2\sqrt{3} - 5\sqrt{3}$$
$$= (2 - 5)\sqrt{3} = -3\sqrt{3}$$

This is similar to simplifying $2x - 5x$ to $-3x$.

**NOW TRY EXERCISE 1**

Add or subtract to simplify each radical expression.

**(a)** $\sqrt{12} + \sqrt{75}$

**(b)** $-\sqrt{63t} + 3\sqrt{28t}, \quad t \geq 0$

**(c)** $6\sqrt{7} - 2\sqrt{3}$

**CAUTION** ***Only radical expressions with the same index and the same radicand may be combined.***

### EXAMPLE 1 Adding and Subtracting Radicals

Add or subtract to simplify each radical expression.

**(a)** $3\sqrt{24} + \sqrt{54}$

$= 3\sqrt{4} \cdot \sqrt{6} + \sqrt{9} \cdot \sqrt{6}$ Product rule

$= 3 \cdot 2\sqrt{6} + 3\sqrt{6}$ $\sqrt{4} = 2$; $\sqrt{9} = 3$

$= 6\sqrt{6} + 3\sqrt{6}$ Multiply.

$= 9\sqrt{6}$ $6\sqrt{6} + 3\sqrt{6} = (6 + 3)\sqrt{6}$

**(b)** $2\sqrt{20x} - \sqrt{45x}, \quad x \geq 0$

$= 2\sqrt{4} \cdot \sqrt{5x} - \sqrt{9} \cdot \sqrt{5x}$ Product rule

$= 2 \cdot 2\sqrt{5x} - 3\sqrt{5x}$ $\sqrt{4} = 2$; $\sqrt{9} = 3$

$= 4\sqrt{5x} - 3\sqrt{5x}$ Multiply.

$= \sqrt{5x}$ Combine like terms.

**(c)** $2\sqrt{3} - 4\sqrt{5}$ The radicands differ and are already simplified, so $2\sqrt{3} - 4\sqrt{5}$ cannot be simplified further.

NOW TRY

**CAUTION** ***The root of a sum does not equal the sum of the roots.*** For example,

$$\sqrt{9 + 16} \neq \sqrt{9} + \sqrt{16}$$

since $\sqrt{9 + 16} = \sqrt{25} = 5$, but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

### EXAMPLE 2 Adding and Subtracting Radicals with Higher Indexes

Add or subtract to simplify each radical expression. Assume that all variables represent positive real numbers.

**(a)** $2\sqrt[3]{16} - 5\sqrt[3]{54}$

Remember to write the index with each radical.

$= 2\sqrt[3]{8 \cdot 2} - 5\sqrt[3]{27 \cdot 2}$ Factor.

$= 2\sqrt[3]{8} \cdot \sqrt[3]{2} - 5\sqrt[3]{27} \cdot \sqrt[3]{2}$ Product rule

$= 2 \cdot 2 \cdot \sqrt[3]{2} - 5 \cdot 3 \cdot \sqrt[3]{2}$ Find the cube roots.

$= 4\sqrt[3]{2} - 15\sqrt[3]{2}$ Multiply.

$= (4 - 15)\sqrt[3]{2}$ Distributive property

$= -11\sqrt[3]{2}$ Combine like terms.

**(b)** $2\sqrt[3]{x^2y} + \sqrt[3]{8x^5y^4}$

$= 2\sqrt[3]{x^2y} + \sqrt[3]{(8x^3y^3)x^2y}$ Factor.

$= 2\sqrt[3]{x^2y} + \sqrt[3]{8x^3y^3} \cdot \sqrt[3]{x^2y}$ Product rule

$= 2\sqrt[3]{x^2y} + 2xy\sqrt[3]{x^2y}$ Find the cube root.

$= (2 + 2xy)\sqrt[3]{x^2y}$ Distributive property

This result cannot be simplified further.

**NOW TRY ANSWERS**

1. **(a)** $7\sqrt{3}$ **(b)** $3\sqrt{7t}$
   **(c)** The expression cannot be simplified further.

**NOW TRY EXERCISE 2**

Add or subtract to simplify each radical expression. Assume that all variables represent positive real numbers.

**(a)** $3\sqrt[3]{2000} - 4\sqrt[3]{128}$

**(b)** $5\sqrt[4]{a^5b^3} + \sqrt[4]{81ab^7}$

**(c)** $\sqrt[3]{128t^4} - 2\sqrt{72t^3}$

Be careful. The indexes are different.

**(c)** $5\sqrt{4x^3} + 3\sqrt[3]{64x^4}$

$$
\begin{aligned}
&= 5\sqrt{4x^2 \cdot x} + 3\sqrt[3]{64x^3 \cdot x} && \text{Factor.}\\
&= 5\sqrt{4x^2} \cdot \sqrt{x} + 3\sqrt[3]{64x^3} \cdot \sqrt[3]{x} && \text{Product rule}\\
&= 5 \cdot 2x\sqrt{x} + 3 \cdot 4x\sqrt[3]{x}\\
&= 10x\sqrt{x} + 12x\sqrt[3]{x}
\end{aligned}
$$

Keep track of the indexes.

The radicands are both $x$, but since the indexes are different, this expression cannot be simplified further.

NOW TRY

**NOW TRY EXERCISE 3**

Perform the indicated operations. Assume that all variables represent positive real numbers.

**(a)** $5\dfrac{\sqrt{5}}{\sqrt{45}} - 4\sqrt{\dfrac{28}{9}}$

**(b)** $6\sqrt[3]{\dfrac{16}{x^{12}}} + 7\sqrt[3]{\dfrac{9}{x^9}}$

## EXAMPLE 3 Adding and Subtracting Radicals with Fractions

Perform the indicated operations. Assume that all variables represent positive real numbers.

**(a)** $2\sqrt{\dfrac{75}{16}} + 4\dfrac{\sqrt{8}}{\sqrt{32}}$

$$
\begin{aligned}
&= 2\frac{\sqrt{25 \cdot 3}}{\sqrt{16}} + 4\frac{\sqrt{4 \cdot 2}}{\sqrt{16 \cdot 2}} && \text{Quotient rule; factor.}\\
&= 2\left(\frac{5\sqrt{3}}{4}\right) + 4\left(\frac{2\sqrt{2}}{4\sqrt{2}}\right) && \text{Product rule; find the square roots.}\\
&= \frac{5\sqrt{3}}{2} + 2 && \text{Multiply; } \frac{\sqrt{2}}{\sqrt{2}} = 1.\\
&= \frac{5\sqrt{3}}{2} + \frac{4}{2} && \text{Write with a common denominator.}\\
&= \frac{5\sqrt{3} + 4}{2} && \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}
\end{aligned}
$$

**(b)** $10\sqrt[3]{\dfrac{5}{x^6}} - 3\sqrt[3]{\dfrac{4}{x^9}}$

$$
\begin{aligned}
&= 10\frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - 3\frac{\sqrt[3]{4}}{\sqrt[3]{x^9}} && \text{Quotient rule}\\
&= \frac{10\sqrt[3]{5}}{x^2} - \frac{3\sqrt[3]{4}}{x^3} && \text{Simplify denominators.}\\
&= \frac{10\sqrt[3]{5} \cdot x}{x^2 \cdot x} - \frac{3\sqrt[3]{4}}{x^3} && \text{Write with a common denominator.}\\
&= \frac{10x\sqrt[3]{5} - 3\sqrt[3]{4}}{x^3} && \text{Subtract fractions.}
\end{aligned}
$$

NOW TRY

**NOW TRY ANSWERS**

**2. (a)** $14\sqrt[3]{2}$

**(b)** $(5a + 3b)\sqrt[4]{ab^3}$

**(c)** $4t\sqrt[3]{2t} - 12t\sqrt{2t}$

**3. (a)** $\dfrac{5 - 8\sqrt{7}}{3}$

**(b)** $\dfrac{12\sqrt[3]{2} + 7x\sqrt[3]{9}}{x^4}$

## 10.4 EXERCISES

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*Simplify. Assume that all variables represent positive real numbers.* ***See Examples 1 and 2.***

**1.** $\sqrt{36} - \sqrt{100}$

**2.** $\sqrt{25} - \sqrt{81}$

💿 **3.** $-2\sqrt{48} + 3\sqrt{75}$

**4.** $4\sqrt{32} - 2\sqrt{8}$

**5.** $\sqrt[3]{16} + 4\sqrt[3]{54}$

**6.** $3\sqrt[3]{24} - 2\sqrt[3]{192}$

**7.** $\sqrt[4]{32} + 3\sqrt[4]{2}$

**8.** $\sqrt[4]{405} - 2\sqrt[4]{5}$

**9.** $6\sqrt{18} - \sqrt{32} + 2\sqrt{50}$

**10.** $5\sqrt{8} + 3\sqrt{72} - 3\sqrt{50}$

**11.** $5\sqrt{6} + 2\sqrt{10}$

**12.** $3\sqrt{11} - 5\sqrt{13}$

**13.** $2\sqrt{5} + 3\sqrt{20} + 4\sqrt{45}$

**14.** $5\sqrt{54} - 2\sqrt{24} - 2\sqrt{96}$

**15.** $\sqrt{72x} - \sqrt{8x}$

**16.** $\sqrt{18k} - \sqrt{72k}$

**17.** $3\sqrt{72m^2} - 5\sqrt{32m^2} - 3\sqrt{18m^2}$

**18.** $9\sqrt{27p^2} - 14\sqrt{108p^2} + 2\sqrt{48p^2}$

**19.** $2\sqrt[3]{16} + \sqrt[3]{54}$

**20.** $15\sqrt[3]{81} + 4\sqrt[3]{24}$

💿 **21.** $2\sqrt[3]{27x} - 2\sqrt[3]{8x}$

**22.** $6\sqrt[3]{128m} - 3\sqrt[3]{16m}$

**23.** $3\sqrt[3]{x^2y} - 5\sqrt[3]{8x^2y}$

**24.** $3\sqrt[3]{x^2y^2} - 2\sqrt[3]{64x^2y^2}$

**25.** $3x\sqrt[3]{xy^2} - 2\sqrt[3]{8x^4y^2}$

**26.** $6q^2\sqrt[3]{5q} - 2q\sqrt[3]{40q^4}$

**27.** $5\sqrt[4]{32} + 3\sqrt[4]{162}$

**28.** $2\sqrt[4]{512} + 4\sqrt[4]{32}$

**29.** $3\sqrt[4]{x^5y} - 2x\sqrt[4]{xy}$

**30.** $2\sqrt[4]{m^9p^6} - 3m^2p\sqrt[4]{mp^2}$

**31.** $2\sqrt[4]{32a^3} + 5\sqrt[4]{2a^3}$

**32.** $5\sqrt[4]{243x^3} + 2\sqrt[4]{3x^3}$

**33.** $\sqrt[3]{64xy^2} + \sqrt[3]{27x^4y^5}$

**34.** $\sqrt[4]{625s^3t} + \sqrt[4]{81s^7t^5}$

**35.** $\sqrt[3]{192st^4} - \sqrt{27s^3t}$

**36.** $\sqrt{125a^5b^5} + \sqrt[3]{125a^4b^4}$

**37.** $2\sqrt[3]{8x^4} + 3\sqrt[4]{16x^5}$

**38.** $3\sqrt[3]{64m^4} + 5\sqrt[4]{81m^5}$

*Simplify. Assume that all variables represent positive real numbers.* ***See Example 3.***

**39.** $\sqrt{8} - \dfrac{\sqrt{64}}{\sqrt{16}}$

**40.** $\sqrt{48} - \dfrac{\sqrt{81}}{\sqrt{9}}$

**41.** $\dfrac{2\sqrt{5}}{3} + \dfrac{\sqrt{5}}{6}$

**42.** $\dfrac{4\sqrt{3}}{3} + \dfrac{2\sqrt{3}}{9}$

**43.** $\sqrt{\dfrac{8}{9}} + \sqrt{\dfrac{18}{36}}$

**44.** $\sqrt{\dfrac{12}{16}} + \sqrt{\dfrac{48}{64}}$

**45.** $\dfrac{\sqrt{32}}{3} + \dfrac{2\sqrt{2}}{3} - \dfrac{\sqrt{2}}{\sqrt{9}}$

**46.** $\dfrac{\sqrt{27}}{2} - \dfrac{3\sqrt{3}}{2} + \dfrac{\sqrt{3}}{\sqrt{4}}$

💿 **47.** $3\sqrt{\dfrac{50}{9}} + 8\dfrac{\sqrt{2}}{\sqrt{8}}$

**48.** $5\sqrt{\dfrac{288}{25}} + 21\dfrac{\sqrt{2}}{\sqrt{18}}$

**49.** $\sqrt{\dfrac{25}{x^8}} + \sqrt{\dfrac{9}{x^6}}$

**50.** $\sqrt{\dfrac{100}{y^4}} + \sqrt{\dfrac{81}{y^{10}}}$

**51.** $3\sqrt[3]{\dfrac{m^5}{27}} - 2m\sqrt[3]{\dfrac{m^2}{64}}$

**52.** $2a\sqrt[4]{\dfrac{a}{16}} - 5a\sqrt[4]{\dfrac{a}{81}}$

**53.** $3\sqrt[3]{\dfrac{2}{x^6}} - 4\sqrt[3]{\dfrac{5}{x^9}}$

**54.** $-4\sqrt[3]{\dfrac{4}{t^9}} + 3\sqrt[3]{\dfrac{9}{t^{12}}}$

**55.** *Concept Check* Which sum could be simplified without first simplifying the individual radical expressions?

**A.** $\sqrt{50} + \sqrt{32}$ **B.** $3\sqrt{6} + 9\sqrt{6}$ **C.** $\sqrt[3]{32} + \sqrt[3]{108}$ **D.** $\sqrt[5]{6} + \sqrt[5]{192}$

**56.** *Concept Check* Let $a = 1$ and let $b = 64$.

**(a)** Evaluate $\sqrt{a} + \sqrt{b}$. Then find $\sqrt{a + b}$. Are they equal?

**(b)** Evaluate $\sqrt[3]{a} + \sqrt[3]{b}$. Then find $\sqrt[3]{a + b}$. Are they equal?

**(c)** Complete the following: In general, $\sqrt[n]{a} + \sqrt[n]{b} \neq$ ____________________, based on the observations in parts (a) and (b) of this exercise.

**57.** Even though the root indexes of the terms are not equal, the sum $\sqrt{64} + \sqrt[3]{125} + \sqrt[4]{16}$ can be simplified quite easily. What is this sum? Why can we add these terms so easily?

**58.** Explain why $28 - 4\sqrt{2}$ is not equal to $24\sqrt{2}$. (This is a common error among algebra students.)

*Solve each problem.*

**59.** A rectangular yard has a length of $\sqrt{192}$ m and a width of $\sqrt{48}$ m. Choose the best estimate of its dimensions. Then estimate the perimeter.

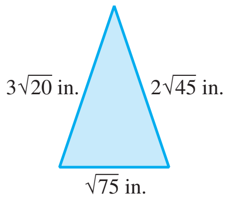
**A.** 14 m by 7 m **B.** 5 m by 7 m **C.** 14 m by 8 m **D.** 15 m by 8 m

**60.** If the sides of a triangle are $\sqrt{65}$ in., $\sqrt{35}$ in., and $\sqrt{26}$ in., which one of the following is the best estimate of its perimeter?

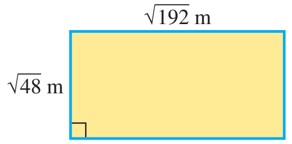
**A.** 20 in. **B.** 26 in. **C.** 19 in. **D.** 24 in.

*Solve each problem. Give answers as simplified radical expressions.*

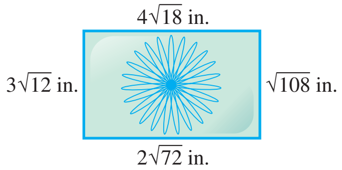
**61.** Find the perimeter of the triangle.



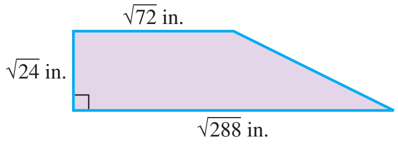
**62.** Find the perimeter of the rectangle.



**63.** What is the perimeter of the computer graphic?



**64.** Find the area of the trapezoid.



## PREVIEW EXERCISES

*Find each product.* ***See Sections 4.5 and 4.6.***

**65.** $5xy(2x^2y^3 - 4x)$

**66.** $(3x + 7)(2x - 6)$

**67.** $(a^2 + b)(a^2 - b)$

**68.** $(2p - 7)^2$

**69.** $(4x^3 + 3)^3$

**70.** $(2 + 3y)(2 - 3y)$

*Write in lowest terms.* ***See Section 6.1.***

**71.** $\dfrac{8x^2 - 10x}{6x^2}$

**72.** $\dfrac{15y^3 - 9y^2}{6y}$

# 10.5 Multiplying and Dividing Radical Expressions

**OBJECTIVES**

1. Multiply radical expressions.
2. Rationalize denominators with one radical term.
3. Rationalize denominators with binomials involving radicals.
4. Write radical quotients in lowest terms.

**OBJECTIVE 1** **Multiply radical expressions.** We multiply binomial expressions involving radicals by using the FOIL method from **Section 4.5.** Recall that the acronym **FOIL** refers to multiplying the **F**irst terms, **O**uter terms, **I**nner terms, and **L**ast terms of the binomials.

**EXAMPLE 1** Multiplying Binomials Involving Radical Expressions

Multiply, using the FOIL method.

**(a)** $\left(\sqrt{5}+3\right)\left(\sqrt{6}+1\right)$

$$= \overbrace{\sqrt{5}\cdot\sqrt{6}}^{\text{First}} + \overbrace{\sqrt{5}\cdot 1}^{\text{Outer}} + \overbrace{3\cdot\sqrt{6}}^{\text{Inner}} + \overbrace{3\cdot 1}^{\text{Last}}$$

$$= \sqrt{30}+\sqrt{5}+3\sqrt{6}+3$$

This result cannot be simplified further.

**(b)** $\left(7-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{2}\right)$

$$= \overset{\text{F}}{7\sqrt{5}} + \overset{\text{O}}{7\sqrt{2}} - \overset{\text{I}}{\sqrt{3}\cdot\sqrt{5}} - \overset{\text{L}}{\sqrt{3}\cdot\sqrt{2}}$$

$$= 7\sqrt{5}+7\sqrt{2}-\sqrt{15}-\sqrt{6}$$

**(c)** $\left(\sqrt{10}+\sqrt{3}\right)\left(\sqrt{10}-\sqrt{3}\right)$

$$= \sqrt{10}\cdot\sqrt{10}-\sqrt{10}\cdot\sqrt{3}+\sqrt{10}\cdot\sqrt{3}-\sqrt{3}\cdot\sqrt{3} \quad \text{FOIL}$$

$$= 10-3$$

$$= 7$$

The product $\left(\sqrt{10}+\sqrt{3}\right)\left(\sqrt{10}-\sqrt{3}\right)=\left(\sqrt{10}\right)^2-\left(\sqrt{3}\right)^2$ is the difference of squares.

$$(x+y)(x-y)=x^2-y^2 \qquad \text{Here, } x=\sqrt{10} \text{ and } y=\sqrt{3}.$$

**(d)** $\left(\sqrt{7}-3\right)^2$

$$= \left(\sqrt{7}-3\right)\left(\sqrt{7}-3\right)$$

$$= \sqrt{7}\cdot\sqrt{7}-3\sqrt{7}-3\sqrt{7}+3\cdot 3$$

$$= 7-6\sqrt{7}+9$$

$$= 16-6\sqrt{7}$$

Be careful. These terms cannot be combined.

**(e)** $\left(5-\sqrt[3]{3}\right)\left(5+\sqrt[3]{3}\right)$

$$= 5\cdot 5+5\sqrt[3]{3}-5\sqrt[3]{3}-\sqrt[3]{3}\cdot\sqrt[3]{3}$$

$$= 25-\sqrt[3]{3^2}$$

$$= 25-\sqrt[3]{9}$$

Remember to write the index 3 in each radical.

*NOW TRY EXERCISE 1*

Multiply, using the FOIL method.

**(a)** $\left(8-\sqrt{5}\right)\left(9-\sqrt{2}\right)$

**(b)** $\left(\sqrt{7}+\sqrt{5}\right)\left(\sqrt{7}-\sqrt{5}\right)$

**(c)** $\left(\sqrt{15}-4\right)^2$

**(d)** $\left(8+\sqrt[3]{5}\right)\left(8-\sqrt[3]{5}\right)$

**(e)** $\left(\sqrt{m}-\sqrt{n}\right)\left(\sqrt{m}+\sqrt{n}\right)$, $m \geq 0$ and $n \geq 0$

**(f)** $\left(\sqrt{k}+\sqrt{y}\right)\left(\sqrt{k}-\sqrt{y}\right)$

$$= \left(\sqrt{k}\right)^2 - \left(\sqrt{y}\right)^2 \qquad \text{Difference of squares}$$

$$= k - y, \quad k \geq 0 \text{ and } y \geq 0$$

*NOW TRY*

**NOTE** In **Example 1(d),** we could have used the formula for the square of a binomial to obtain the same result.

$$\left(\sqrt{7}-3\right)^2$$

$$= \left(\sqrt{7}\right)^2 - 2\left(\sqrt{7}\right)(3) + 3^2 \qquad (x-y)^2 = x^2 - 2xy + y^2$$

$$= 7 - 6\sqrt{7} + 9 \qquad \text{Apply the exponents. Multiply.}$$

$$= 16 - 6\sqrt{7} \qquad \text{Add.}$$

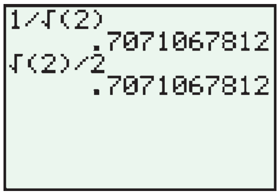
**OBJECTIVE 2** **Rationalize denominators with one radical term.** As defined earlier, a simplified radical expression has no radical in the denominator. The origin of this agreement no doubt occurred before the days of high-speed calculation, when computation was a tedious process performed by hand.

For example, consider the radical expression $\frac{1}{\sqrt{2}}$. To find a decimal approximation by hand, it is necessary to divide 1 by a decimal approximation for $\sqrt{2}$, such as 1.414. It is much easier if the divisor is a whole number. This can be accomplished by multiplying $\frac{1}{\sqrt{2}}$ by 1 in the form $\frac{\sqrt{2}}{\sqrt{2}}$. ***Multiplying by 1 in any form does not change the value of the original expression.***

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \text{Multiply by 1; } \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Now the computation requires dividing 1.414 by 2 to obtain 0.707, a much easier task.

With current technology, either form of this fraction can be approximated with the same number of keystrokes. See **FIGURE 10**, which shows how a calculator gives the same approximation for both forms of the expression.



```
1/√(2)
          .7071067812
√(2)/2
          .7071067812
```

**FIGURE 10**

### Rationalizing the Denominator

A common way of "standardizing" the form of a radical expression is to have the denominator contain no radicals. The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalizing the denominator.** This is done by multiplying by a form of 1.

### *EXAMPLE 2* Rationalizing Denominators with Square Roots

Rationalize each denominator.

**(a)** $\frac{3}{\sqrt{7}}$

Multiply the numerator and denominator by $\sqrt{7}$. This is, in effect, multiplying by 1.

$$\frac{3}{\sqrt{7}} = \frac{3 \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} = \frac{3\sqrt{7}}{7}$$

In the denominator, $\sqrt{7} \cdot \sqrt{7} = \sqrt{7 \cdot 7} = \sqrt{49} = 7$. The final denominator is now a rational number.

*NOW TRY ANSWERS*

1. **(a)** $72 - 8\sqrt{2} - 9\sqrt{5} + \sqrt{10}$ **(b)** 2 **(c)** $31 - 8\sqrt{15}$ **(d)** $64 - \sqrt[3]{25}$ **(e)** $m - n$

NOW TRY EXERCISE 2

Rationalize each denominator.

**(a)** $\dfrac{8}{\sqrt{13}}$ **(b)** $\dfrac{9\sqrt{7}}{\sqrt{3}}$

**(c)** $\dfrac{-10}{\sqrt{20}}$

**(b)** $$\frac{5\sqrt{2}}{\sqrt{5}} = \frac{5\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{5\sqrt{10}}{5} = \sqrt{10}$$

**(c)** $\dfrac{-6}{\sqrt{12}}$

Less work is involved if we simplify the radical in the denominator first.

$$\frac{-6}{\sqrt{12}} = \frac{-6}{\sqrt{4\cdot 3}} = \frac{-6}{2\sqrt{3}} = \frac{-3}{\sqrt{3}}$$

Now we rationalize the denominator.

$$\frac{-3}{\sqrt{3}} = \frac{-3\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{-3\sqrt{3}}{3} = -\sqrt{3}$$

NOW TRY

NOW TRY EXERCISE 3

Simplify each radical.

**(a)** $-\sqrt{\dfrac{27}{80}}$

**(b)** $\sqrt{\dfrac{48x^8}{y^3}},\quad y > 0$

## EXAMPLE 3 Rationalizing Denominators in Roots of Fractions

Simplify each radical. In part (b), $p > 0$.

**(a)** $-\sqrt{\dfrac{18}{125}}$

$$= -\frac{\sqrt{18}}{\sqrt{125}} \quad \text{Quotient rule}$$

$$= -\frac{\sqrt{9\cdot 2}}{\sqrt{25\cdot 5}} \quad \text{Factor.}$$

$$= -\frac{3\sqrt{2}}{5\sqrt{5}} \quad \text{Product rule}$$

$$= -\frac{3\sqrt{2}\cdot\sqrt{5}}{5\sqrt{5}\cdot\sqrt{5}} \quad \text{Multiply by } \frac{\sqrt{5}}{\sqrt{5}}.$$

$$= -\frac{3\sqrt{10}}{5\cdot 5} \quad \text{Product rule}$$

$$= -\frac{3\sqrt{10}}{25} \quad \text{Multiply.}$$

**(b)** $\sqrt{\dfrac{50m^4}{p^5}}$

$$= \frac{\sqrt{50m^4}}{\sqrt{p^5}} \quad \text{Quotient rule}$$

$$= \frac{5m^2\sqrt{2}}{p^2\sqrt{p}} \quad \text{Product rule}$$

$$= \frac{5m^2\sqrt{2}\cdot\sqrt{p}}{p^2\sqrt{p}\cdot\sqrt{p}} \quad \text{Multiply by } \frac{\sqrt{p}}{\sqrt{p}}.$$

$$= \frac{5m^2\sqrt{2p}}{p^2\cdot p} \quad \text{Product rule}$$

$$= \frac{5m^2\sqrt{2p}}{p^3} \quad \text{Multiply.}$$

NOW TRY

## EXAMPLE 4 Rationalizing Denominators with Cube and Fourth Roots

Simplify.

**(a)** $\sqrt[3]{\dfrac{27}{16}}$

Use the quotient rule, and simplify the numerator and denominator.

$$\sqrt[3]{\frac{27}{16}} = \frac{\sqrt[3]{27}}{\sqrt[3]{16}} = \frac{3}{\sqrt[3]{8}\cdot\sqrt[3]{2}} = \frac{3}{2\sqrt[3]{2}}$$

Since $2\cdot 4 = 8$, a perfect cube, multiply the numerator and denominator by $\sqrt[3]{4}$.

NOW TRY ANSWERS

2. (a) $\dfrac{8\sqrt{13}}{13}$ (b) $3\sqrt{21}$
   (c) $-\sqrt{5}$

3. (a) $-\dfrac{3\sqrt{15}}{20}$ (b) $\dfrac{4x^4\sqrt{3y}}{y^2}$

**NOW TRY EXERCISE 4**

Simplify.

**(a)** $\sqrt[3]{\dfrac{8}{81}}$

**(b)** $\sqrt[4]{\dfrac{7x}{y}}, \quad x \geq 0, y > 0$

$$\frac{3}{2\sqrt[3]{2}} \qquad \sqrt[3]{\frac{27}{16}} = \frac{3}{2\sqrt[3]{2}} \text{ from page 636}$$

$$= \frac{3 \cdot \sqrt[3]{4}}{2\sqrt[3]{2} \cdot \sqrt[3]{4}} \qquad \text{Multiply by } \sqrt[3]{4} \text{ in numerator and denominator. This will give } \sqrt[3]{8} = 2 \text{ in the denominator.}$$

$$= \frac{3\sqrt[3]{4}}{2\sqrt[3]{8}} \qquad \text{Multiply.}$$

$$= \frac{3\sqrt[3]{4}}{2 \cdot 2} \qquad \sqrt[3]{8} = 2$$

$$= \frac{3\sqrt[3]{4}}{4} \qquad \text{Multiply.}$$

**(b)**

$$\sqrt[4]{\frac{5x}{z}}$$

$$= \frac{\sqrt[4]{5x}}{\sqrt[4]{z}} \qquad \text{Quotient rule}$$

$$= \frac{\sqrt[4]{5x}}{\sqrt[4]{z}} \cdot \frac{\sqrt[4]{z^3}}{\sqrt[4]{z^3}} \qquad \text{Multiply by 1.}$$

$\sqrt[4]{z} \cdot \sqrt[4]{z^3}$ will give $\sqrt[4]{z^4}$.

$$= \frac{\sqrt[4]{5xz^3}}{\sqrt[4]{z^4}} \qquad \text{Product rule}$$

$$= \frac{\sqrt[4]{5xz^3}}{z}, \quad x \geq 0, z > 0$$

NOW TRY

**CAUTION** In **Example 4(a),** a typical error is to multiply the numerator and denominator by $\sqrt[3]{2}$, forgetting that $\sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{2^2}$, which does ***not*** equal 2. We need *three* factors of 2 to obtain $2^3$ under the radical.

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{2^3} \quad \text{which does equal} \quad 2.$$

## OBJECTIVE 3 Rationalize denominators with binomials involving radicals.

Recall the special product $(x + y)(x - y) = x^2 - y^2$. To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

$$\frac{3}{1 + \sqrt{2}},$$

we must use *conjugates*. The conjugate of $1 + \sqrt{2}$ is $1 - \sqrt{2}$. In general, $x + y$ and $x - y$ are **conjugates.**

### Rationalizing a Binomial Denominator

Whenever a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.

**NOW TRY ANSWERS**

4. **(a)** $\dfrac{2\sqrt[3]{9}}{9}$ **(b)** $\dfrac{\sqrt[4]{7xy^3}}{y}$

NOW TRY EXERCISE 5

Rationalize each denominator.

**(a)** $\dfrac{4}{1+\sqrt{3}}$ **(b)** $\dfrac{4}{5+\sqrt{7}}$

**(c)** $\dfrac{\sqrt{3}+\sqrt{7}}{\sqrt{5}-\sqrt{2}}$

**(d)** $\dfrac{8}{\sqrt{3x}-\sqrt{y}}$, $3x \neq y, x > 0, y > 0$

## EXAMPLE 5 Rationalizing Binomial Denominators

Rationalize each denominator.

**(a)** $$\frac{3}{1+\sqrt{2}}$$

Again, we are multiplying by a form of 1.

$$=\frac{3(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$$

Multiply the numerator and denominator by $1-\sqrt{2}$, the conjugate of the denominator.

$$(1+\sqrt{2})(1-\sqrt{2})$$
$$= 1^2-(\sqrt{2})^2$$
$$= 1-2, \text{ or } -1$$

$$=\frac{3(1-\sqrt{2})}{-1}$$

The denominator is now a rational number.

$$=\frac{3}{-1}(1-\sqrt{2})$$

$$=-3(1-\sqrt{2}), \quad \text{or} \quad -3+3\sqrt{2}$$

Distributive property

**(b)** $$\frac{5}{4-\sqrt{3}}$$

$$=\frac{5(4+\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})}$$

Multiply the numerator and denominator by $4+\sqrt{3}$.

$$=\frac{5(4+\sqrt{3})}{16-3}$$

Multiply in the denominator.

$$=\frac{5(4+\sqrt{3})}{13}$$

Subtract in the denominator.

Notice that the numerator is left in factored form. This makes it easier to determine whether the expression is written in lowest terms.

**(c)** $$\frac{\sqrt{2}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{(\sqrt{2}-\sqrt{3})(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$$

Multiply the numerator and denominator by $\sqrt{5}-\sqrt{3}$.

$$=\frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{5-3}$$

Multiply.

$$=\frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{2}$$

Subtract in the denominator.

**(d)** $$\frac{3}{\sqrt{5m}-\sqrt{p}}, \quad 5m \neq p, m > 0, p > 0$$

$$=\frac{3(\sqrt{5m}+\sqrt{p})}{(\sqrt{5m}-\sqrt{p})(\sqrt{5m}+\sqrt{p})}$$

Multiply the numerator and denominator by $\sqrt{5m}+\sqrt{p}$.

$$=\frac{3(\sqrt{5m}+\sqrt{p})}{5m-p}$$

Multiply in the denominator.

NOW TRY

NOW TRY ANSWERS

5. **(a)** $-2(1-\sqrt{3})$, or $-2+2\sqrt{3}$

**(b)** $\dfrac{2(5-\sqrt{7})}{9}$

**(c)** $\dfrac{\sqrt{15}+\sqrt{6}+\sqrt{35}+\sqrt{14}}{3}$

**(d)** $\dfrac{8(\sqrt{3x}+\sqrt{y})}{3x-y}$

## OBJECTIVE 4 Write radical quotients in lowest terms.

**NOW TRY EXERCISE 6**

Write each quotient in lowest terms.

**(a)** $\dfrac{15 - 6\sqrt{2}}{18}$

**(b)** $\dfrac{15k + \sqrt{50k^2}}{20k}, \quad k > 0$

### EXAMPLE 6 Writing Radical Quotients in Lowest Terms

Write each quotient in lowest terms.

**(a)** $\dfrac{6 + 2\sqrt{5}}{4}$

$= \dfrac{2(3 + \sqrt{5})}{2 \cdot 2}$ Factor the numerator and denominator.

This is a key step.

$= \dfrac{3 + \sqrt{5}}{2}$ Divide out the common factor.

Here is an alternative method for writing this expression in lowest terms.

$$\frac{6 + 2\sqrt{5}}{4} = \frac{6}{4} + \frac{2\sqrt{5}}{4} = \frac{3}{2} + \frac{\sqrt{5}}{2} = \frac{3 + \sqrt{5}}{2}$$

**(b)** $\dfrac{5y - \sqrt{8y^2}}{6y}, \quad y > 0$

$= \dfrac{5y - 2y\sqrt{2}}{6y}$ $\sqrt{8y^2} = \sqrt{4y^2 \cdot 2} = 2y\sqrt{2}$

$= \dfrac{y(5 - 2\sqrt{2})}{6y}$ Factor the numerator.

$= \dfrac{5 - 2\sqrt{2}}{6}$ Divide out the common factor.

NOW TRY

**CAUTION** ***Be careful to factor before writing a quotient in lowest terms.***

## CONNECTIONS

In calculus, it is sometimes desirable to **rationalize the numerator.** For example, to rationalize the numerator of

$$\frac{6 - \sqrt{2}}{4},$$

we multiply the numerator and the denominator by the conjugate of the *numerator*.

$$\frac{6 - \sqrt{2}}{4} = \frac{(6 - \sqrt{2})(6 + \sqrt{2})}{4(6 + \sqrt{2})} = \frac{36 - 2}{4(6 + \sqrt{2})} = \frac{34}{4(6 + \sqrt{2})} = \frac{17}{2(6 + \sqrt{2})}$$

### For Discussion or Writing

Rationalize the numerator of each expression. ($a$ and $b$ are nonnegative real numbers.)

**1.** $\dfrac{8\sqrt{5} - 1}{6}$ **2.** $\dfrac{3\sqrt{a} + \sqrt{b}}{b}$ **3.** $\dfrac{3\sqrt{a} + \sqrt{b}}{\sqrt{b} - \sqrt{a}} \quad (b \neq a)$

**4.** Rationalize the denominator of the expression in **Exercise 3,** and then describe the difference in the procedure you used from what you did in **Exercise 3.**

**NOW TRY ANSWERS**

**6. (a)** $\dfrac{5 - 2\sqrt{2}}{6}$ **(b)** $\dfrac{3 + \sqrt{2}}{4}$

## 10.5 EXERCISES

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***Concept Check*** *Match each part of a rule for a special product in Column I with the other part in Column II. Assume that A and B represent positive real numbers.*

| I | II |
|---|---|
| **1.** $\left(A+\sqrt{B}\right)\left(A-\sqrt{B}\right)$ | **A.** $A-B$ |
| **2.** $\left(\sqrt{A}+B\right)\left(\sqrt{A}-B\right)$ | **B.** $A+2B\sqrt{A}+B^2$ |
| **3.** $\left(\sqrt{A}+\sqrt{B}\right)\left(\sqrt{A}-\sqrt{B}\right)$ | **C.** $A-B^2$ |
| **4.** $\left(\sqrt{A}+\sqrt{B}\right)^2$ | **D.** $A-2\sqrt{AB}+B$ |
| **5.** $\left(\sqrt{A}-\sqrt{B}\right)^2$ | **E.** $A^2-B$ |
| **6.** $\left(\sqrt{A}+B\right)^2$ | **F.** $A+2\sqrt{AB}+B$ |

*Multiply, and then simplify each product. Assume that all variables represent positive real numbers.* ***See Example 1.***

**7.** $\sqrt{6}\left(3+\sqrt{2}\right)$
**8.** $\sqrt{2}\left(\sqrt{32}-\sqrt{9}\right)$
**9.** $5\left(\sqrt{72}-\sqrt{8}\right)$
**10.** $7\left(\sqrt{50}-\sqrt{18}\right)$
**11.** $\left(\sqrt{7}+3\right)\left(\sqrt{7}-3\right)$
**12.** $\left(\sqrt{3}-5\right)\left(\sqrt{3}+5\right)$
**13.** $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)$
**14.** $\left(\sqrt{7}+\sqrt{14}\right)\left(\sqrt{7}-\sqrt{14}\right)$
**15.** $\left(\sqrt{8}-\sqrt{2}\right)\left(\sqrt{8}+\sqrt{2}\right)$
**16.** $\left(\sqrt{20}-\sqrt{5}\right)\left(\sqrt{20}+\sqrt{5}\right)$
**17.** $\left(\sqrt{2}+1\right)\left(\sqrt{3}-1\right)$
**18.** $\left(\sqrt{3}+3\right)\left(\sqrt{5}-2\right)$
**19.** $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{2}+\sqrt{5}\right)$
**20.** $\left(\sqrt{13}-\sqrt{7}\right)\left(\sqrt{3}+\sqrt{11}\right)$
**21.** $\left(2\sqrt{3}+\sqrt{5}\right)\left(3\sqrt{3}-2\sqrt{5}\right)$
**22.** $\left(\sqrt{7}-\sqrt{11}\right)\left(2\sqrt{7}+3\sqrt{11}\right)$
**23.** $\left(\sqrt{5}+2\right)^2$
**24.** $\left(\sqrt{11}-1\right)^2$
**25.** $\left(\sqrt{21}-\sqrt{5}\right)^2$
**26.** $\left(\sqrt{6}-\sqrt{2}\right)^2$
**27.** $\left(2+\sqrt[3]{6}\right)\left(2-\sqrt[3]{6}\right)$
**28.** $\left(\sqrt[3]{3}+6\right)\left(\sqrt[3]{3}-6\right)$
**29.** $\left(2+\sqrt[3]{2}\right)\left(4-2\sqrt[3]{2}+\sqrt[3]{4}\right)$
**30.** $\left(\sqrt[3]{3}-1\right)\left(\sqrt[3]{9}+\sqrt[3]{3}+1\right)$
**31.** $\left(3\sqrt{x}-\sqrt{5}\right)\left(2\sqrt{x}+1\right)$
**32.** $\left(4\sqrt{p}+\sqrt{7}\right)\left(\sqrt{p}-9\right)$
**33.** $\left(3\sqrt{r}-\sqrt{s}\right)\left(3\sqrt{r}+\sqrt{s}\right)$
**34.** $\left(\sqrt{k}+4\sqrt{m}\right)\left(\sqrt{k}-4\sqrt{m}\right)$
**35.** $\left(\sqrt[3]{2y}-5\right)\left(4\sqrt[3]{2y}+1\right)$
**36.** $\left(\sqrt[3]{9z}-2\right)\left(5\sqrt[3]{9z}+7\right)$
**37.** $\left(\sqrt{3x}+2\right)\left(\sqrt{3x}-2\right)$
**38.** $\left(\sqrt{6y}-4\right)\left(\sqrt{6y}+4\right)$
**39.** $\left(2\sqrt{x}+\sqrt{y}\right)\left(2\sqrt{x}-\sqrt{y}\right)$
**40.** $\left(\sqrt{p}+5\sqrt{s}\right)\left(\sqrt{p}-5\sqrt{s}\right)$
**41.** $\left[\left(\sqrt{2}+\sqrt{3}\right)-\sqrt{6}\right]\left[\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right]$
**42.** $\left[\left(\sqrt{5}-\sqrt{2}\right)-\sqrt{3}\right]\left[\left(\sqrt{5}-\sqrt{2}\right)+\sqrt{3}\right]$

*Rationalize the denominator in each expression. Assume that all variables represent positive real numbers.* ***See Examples 2 and 3.***

**43.** $\frac{7}{\sqrt{7}}$
**44.** $\frac{11}{\sqrt{11}}$
**45.** $\frac{15}{\sqrt{3}}$
**46.** $\frac{12}{\sqrt{6}}$
**47.** $\frac{\sqrt{3}}{\sqrt{2}}$

**48.** $\frac{\sqrt{7}}{\sqrt{6}}$
**49.** $\frac{9\sqrt{3}}{\sqrt{5}}$
**50.** $\frac{3\sqrt{2}}{\sqrt{11}}$
**51.** $\frac{-7}{\sqrt{48}}$
**52.** $\frac{-5}{\sqrt{24}}$

**53.** $\sqrt{\frac{7}{2}}$
**54.** $\sqrt{\frac{10}{3}}$
**55.** $-\sqrt{\frac{7}{50}}$
**56.** $-\sqrt{\frac{13}{75}}$
**57.** $\sqrt{\frac{24}{x}}$

**58.** $\sqrt{\frac{52}{y}}$
**59.** $\frac{-8\sqrt{3}}{\sqrt{k}}$
**60.** $\frac{-4\sqrt{13}}{\sqrt{m}}$
**61.** $-\sqrt{\frac{150m^5}{n^3}}$
**62.** $-\sqrt{\frac{98r^3}{s^5}}$

**63.** $\sqrt{\frac{288x^7}{y^9}}$
**64.** $\sqrt{\frac{242t^9}{u^{11}}}$
**65.** $\frac{5\sqrt{2m}}{\sqrt{y^3}}$

**66.** $\frac{2\sqrt{5r}}{\sqrt{m^3}}$
**67.** $-\sqrt{\frac{48k^2}{z}}$
**68.** $-\sqrt{\frac{75m^3}{p}}$

*Simplify. Assume that all variables represent positive real numbers.* ***See Example 4.***

**69.** $\sqrt[3]{\frac{2}{3}}$
**70.** $\sqrt[3]{\frac{4}{5}}$
**71.** $\sqrt[3]{\frac{4}{9}}$
**72.** $\sqrt[3]{\frac{5}{16}}$
**73.** $\sqrt[3]{\frac{9}{32}}$

**74.** $\sqrt[3]{\frac{10}{9}}$
**75.** $-\sqrt[3]{\frac{2p}{r^2}}$
**76.** $-\sqrt[3]{\frac{6x}{y^2}}$
**77.** $\sqrt[3]{\frac{x^6}{y}}$
**78.** $\sqrt[3]{\frac{m^9}{q}}$

**79.** $\sqrt[4]{\frac{16}{x}}$
**80.** $\sqrt[4]{\frac{81}{y}}$
**81.** $\sqrt[4]{\frac{2y}{z}}$
**82.** $\sqrt[4]{\frac{7t}{s^2}}$

*Rationalize the denominator in each expression. Assume that all variables represent positive real numbers and no denominators are* 0. ***See Example 5.***

**83.** $\frac{3}{4 + \sqrt{5}}$
**84.** $\frac{4}{5 + \sqrt{6}}$
**85.** $\frac{\sqrt{8}}{3 - \sqrt{2}}$

**86.** $\frac{\sqrt{27}}{3 - \sqrt{3}}$
**87.** $\frac{2}{3\sqrt{5} + 2\sqrt{3}}$
**88.** $\frac{-1}{3\sqrt{2} - 2\sqrt{7}}$

**89.** $\frac{\sqrt{2} - \sqrt{3}}{\sqrt{6} - \sqrt{5}}$
**90.** $\frac{\sqrt{5} + \sqrt{6}}{\sqrt{3} - \sqrt{2}}$
**91.** $\frac{m - 4}{\sqrt{m} + 2}$

**92.** $\frac{r - 9}{\sqrt{r} - 3}$
**93.** $\frac{4}{\sqrt{x} - 2\sqrt{y}}$
**94.** $\frac{5}{3\sqrt{r} + \sqrt{s}}$

**95.** $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$
**96.** $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$
**97.** $\frac{5\sqrt{k}}{2\sqrt{k} + \sqrt{q}}$
**98.** $\frac{3\sqrt{x}}{\sqrt{x} - 2\sqrt{y}}$

*Write each expression in lowest terms. Assume that all variables represent positive real numbers.* ***See Example 6.***

**99.** $\frac{30 - 20\sqrt{6}}{10}$
**100.** $\frac{24 + 12\sqrt{5}}{12}$
**101.** $\frac{3 - 3\sqrt{5}}{3}$
**102.** $\frac{-5 + 5\sqrt{2}}{5}$

**103.** $\frac{16 - 4\sqrt{8}}{12}$
**104.** $\frac{12 - 9\sqrt{72}}{18}$
**105.** $\frac{6p + \sqrt{24p^3}}{3p}$
**106.** $\frac{11y - \sqrt{242y^5}}{22y}$

***Brain Busters*** *Rationalize each denominator. Assume that all radicals represent real numbers and no denominators are* 0.

**107.** $\dfrac{3}{\sqrt{x+y}}$ **108.** $\dfrac{5}{\sqrt{m-n}}$ **109.** $\dfrac{p}{\sqrt{p+2}}$ **110.** $\dfrac{q}{\sqrt{5+q}}$

**111.** The following expression occurs in a certain standard problem in trigonometry.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

Show that it simplifies to $\frac{\sqrt{6}-\sqrt{2}}{4}$. Then verify, using a calculator approximation.

**112.** The following expression occurs in a certain standard problem in trigonometry.

$$\frac{\sqrt{3}+1}{1-\sqrt{3}}$$

Show that it simplifies to $-2-\sqrt{3}$. Then verify, using a calculator approximation.

*Rationalize the numerator in each expression. Assume that all variables represent positive real numbers. (Hint: See the* ***Connections box*** *following* ***Example 6.****)*

**113.** $\dfrac{6-\sqrt{3}}{8}$ **114.** $\dfrac{2\sqrt{5}-3}{2}$ **115.** $\dfrac{2\sqrt{x}-\sqrt{y}}{3x}$ **116.** $\dfrac{\sqrt{p}-3\sqrt{q}}{4q}$

## PREVIEW EXERCISES

*Solve each equation.* ***See Sections 2.3 and 5.5.***

**117.** $-8x+7=4$ **118.** $3x-7=12$

**119.** $6x^2-7x=3$ **120.** $x(15x-11)=-2$

## SUMMARY EXERCISES on Operations with Radicals and Rational Exponents

### Conditions for a Simplified Radical

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.

*Perform all indicated operations, and express each answer in simplest form with positive exponents. Assume that all variables represent positive real numbers.*

**1.** $6\sqrt{10}-12\sqrt{10}$ **2.** $\sqrt{7}\left(\sqrt{7}-\sqrt{2}\right)$ **3.** $\left(1-\sqrt{3}\right)\left(2+\sqrt{6}\right)$

**4.** $\sqrt{50}-\sqrt{98}+\sqrt{72}$ **5.** $\left(3\sqrt{5}+2\sqrt{7}\right)^2$ **6.** $\dfrac{-3}{\sqrt{6}}$

**7.** $\dfrac{8}{\sqrt{7}+\sqrt{5}}$

**8.** $\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$

**9.** $\left(\sqrt{5}+7\right)\left(\sqrt{5}-7\right)$

**10.** $\dfrac{1}{\sqrt{x}-\sqrt{5}}, \quad x \neq 5$

**11.** $\sqrt[3]{8a^3b^5c^9}$

**12.** $\dfrac{15}{\sqrt[3]{9}}$

**13.** $\dfrac{3}{\sqrt{5}+2}$

**14.** $\sqrt{\dfrac{3}{5x}}$

**15.** $\dfrac{16\sqrt{3}}{5\sqrt{12}}$

**16.** $\dfrac{2\sqrt{25}}{8\sqrt{50}}$

**17.** $\dfrac{-10}{\sqrt[3]{10}}$

**18.** $\dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{6}-\sqrt{5}}$

**19.** $\sqrt{12x}-\sqrt{75x}$

**20.** $\left(5-3\sqrt{3}\right)^2$

**21.** $\sqrt[3]{\dfrac{13}{81}}$

**22.** $\dfrac{\sqrt{3}+\sqrt{7}}{\sqrt{6}-\sqrt{5}}$

**23.** $\dfrac{6}{\sqrt[4]{3}}$

**24.** $\dfrac{1}{1-\sqrt[3]{3}}$

**25.** $\sqrt[3]{\dfrac{x^2y}{x^{-3}y^4}}$

**26.** $\sqrt{12}-\sqrt{108}-\sqrt[3]{27}$

**27.** $\dfrac{x^{-2/3}y^{4/5}}{x^{-5/3}y^{-2/5}}$

**28.** $\left(\dfrac{x^{3/4}y^{2/3}}{x^{1/3}y^{5/8}}\right)^{24}$

**29.** $(125x^3)^{-2/3}$

**30.** $\dfrac{4^{1/2}+3^{1/2}}{4^{1/2}-3^{1/2}}$

**31.** $\sqrt[3]{16x^2}-\sqrt[3]{54x^2}+\sqrt[3]{128x^2}$

**32.** $\left(1-\sqrt[3]{3}\right)\left(1+\sqrt[3]{3}+\sqrt[3]{9}\right)$

Students often have trouble distinguishing between the following two types of problems:

| **Simplifying a Radical Involving a Square Root** | **Solving an Equation Using Square Roots** |
|---|---|
| *Exercise:* Simplify $\sqrt{25}$. | *Exercise:* Solve $x^2 = 25$. |
| *Answer:* 5 | *Answer:* $\{-5, 5\}$ |
| In this situation, $\sqrt{25}$ represents the positive square root of 25, namely 5. | In this situation, $x^2 = 25$ has two solutions, the negative square root of 25 or the positive square root of 25: $-5$, 5. |

*In Exercises 33–40, provide the appropriate responses.*

**33.** **(a)** Simplify $\sqrt{64}$.
**(b)** Solve $x^2 = 64$.

**34.** **(a)** Simplify $\sqrt{100}$.
**(b)** Solve $x^2 = 100$.

**35.** **(a)** Solve $x^2 = 16$.
**(b)** Simplify $-\sqrt{16}$.

**36.** **(a)** Solve $x^2 = 25$.
**(b)** Simplify $-\sqrt{25}$.

**37.** **(a)** Simplify $-\sqrt{\frac{81}{121}}$.
**(b)** Solve $x^2 = \frac{81}{121}$.

**38.** **(a)** Simplify $-\sqrt{\frac{49}{100}}$.
**(b)** Solve $x^2 = \frac{49}{100}$.

**39.** **(a)** Solve $x^2 = 0.04$.
**(b)** Simplify $\sqrt{0.04}$.

**40.** **(a)** Solve $x^2 = 0.09$.
**(b)** Simplify $\sqrt{0.09}$.

# 10.6 Solving Equations with Radicals

**OBJECTIVES**

1. Solve radical equations by using the power rule.
2. Solve radical equations that require additional steps.
3. Solve radical equations with indexes greater than 2.
4. Use the power rule to solve a formula for a specified variable.

An equation that includes one or more radical expressions with a variable is called a **radical equation.**

$$\sqrt{x-4}=8,\quad \sqrt{5x+12}=3\sqrt{2x-1},\quad \text{and}\quad \sqrt[3]{6+x}=27$$

Examples of radical equations

**OBJECTIVE 1 Solve radical equations by using the power rule.** The equation $x = 1$ has only one solution. Its solution set is $\{1\}$. If we square both sides of this equation, we get $x^2 = 1$. This new equation has *two* solutions: $-1$ and $1$. Notice that the solution of the original equation is also a solution of the equation following squaring. However, that equation has another solution, $-1$, that is *not* a solution of the original equation.

When solving equations with radicals, we use this idea of raising both sides to a power. It is an application of the **power rule.**

**Power Rule for Solving an Equation with Radicals**

If both sides of an equation are raised to the same power, all solutions of the original equation are also solutions of the new equation.

***The power rule does not say that all solutions of the new equation are solutions of the original equation. They may or may not be.*** Solutions that do not satisfy the original equation are called **extraneous solutions.** They must be rejected.

**CAUTION** When the power rule is used to solve an equation, ***every solution of the new equation must be checked in the original equation.***

*NOW TRY EXERCISE 1*

Solve $\sqrt{9x+7}=5$.

**EXAMPLE 1** Using the Power Rule

Solve $\sqrt{3x+4}=8$.

$$(\sqrt{3x+4})^2 = 8^2 \qquad \text{Use the power rule and square each side.}$$

$(\sqrt{a})^2 = \sqrt{a}\cdot\sqrt{a} = a$

$$3x+4=64 \qquad \text{Apply the exponents.}$$

$$3x=60 \qquad \text{Subtract 4.}$$

$$x=20 \qquad \text{Divide by 3.}$$

*CHECK*

$$\sqrt{3x+4}=8 \qquad \text{Original equation}$$

$$\sqrt{3\cdot 20+4}\stackrel{?}{=}8 \qquad \text{Let } x = 20.$$

$$\sqrt{64}\stackrel{?}{=}8 \qquad \text{Simplify.}$$

$$8=8\ \checkmark \qquad \text{True}$$

Since 20 satisfies the *original* equation, the solution set is $\{20\}$. *NOW TRY*

*NOW TRY ANSWER*

**1.** $\{2\}$

Use the following steps to solve equations with radicals.

### Solving an Equation with Radicals

***Step 1*** **Isolate the radical.** Make sure that one radical term is alone on one side of the equation.

***Step 2*** **Apply the power rule.** Raise each side of the equation to a power that is the same as the index of the radical.

***Step 3*** **Solve** the resulting equation. If it still contains a radical, repeat Steps 1 and 2.

***Step 4*** **Check** all proposed solutions in the original equation.

### EXAMPLE 2 Using the Power Rule

Solve $\sqrt{5x-1}+3=0$.

***Step 1*** $\sqrt{5x-1}=-3$ To isolate the radical on one side, subtract 3 from each side.

***Step 2*** $\left(\sqrt{5x-1}\right)^2=(-3)^2$ Square each side.

***Step 3*** $5x-1=9$ Apply the exponents.

$5x=10$ Add 1.

$x=2$ Divide by 5.

***Step 4*** *CHECK* $\sqrt{5x-1}+3=0$ Original equation

$\sqrt{5\cdot 2-1}+3\overset{?}{=}0$ Let $x=2$.

$3+3=0$ False

Be sure to check the proposed solution.

This false result shows that the *proposed* solution 2 is *not* a solution of the original equation. It is extraneous. The solution set is $\emptyset$. *NOW TRY*

**NOTE** We could have determined after Step 1 that the equation in **Example 2** has no solution because the expression on the left cannot be negative. (Why?)

*NOW TRY EXERCISE 2*

Solve $\sqrt{3x+4}+5=0$.

**OBJECTIVE 2** **Solve radical equations that require additional steps.** The next examples involve finding the square of a binomial. Recall the rule from **Section 4.6.**

$$(x+y)^2=x^2+2xy+y^2$$

### EXAMPLE 3 Using the Power Rule (Squaring a Binomial)

Solve $\sqrt{4-x}=x+2$.

***Step 1*** The radical is alone on the left side of the equation.

***Step 2*** Square each side. The square of $x+2$ is $(x+2)^2=x^2+2(x)(2)+4$.

$$\left(\sqrt{4-x}\right)^2=(x+2)^2$$

Remember the middle term.

$$4-x=x^2+4x+4$$

Twice the product of 2 and $x$

*NOW TRY ANSWER*

2. $\emptyset$

NOW TRY EXERCISE 3

Solve $\sqrt{16-x}=x+4$.

***Step 3*** The new equation is quadratic, so write it in standard form.

$$\begin{aligned} 4-x &= x^2+4x+4 && \text{Equation from Step 2} \\ x^2+5x &= 0 && \text{Subtract 4. Add } x. \\ x(x+5) &= 0 && \text{Factor.} \\ x=0 \quad \text{or} \quad x+5 &= 0 && \text{Zero-factor property} \\ x &= -5 && \text{Solve for } x. \end{aligned}$$

Set *each* factor equal to 0.

***Step 4*** Check each proposed solution in the original equation.

CHECK

$$\begin{aligned} \sqrt{4-x} &= x+2 \\ \sqrt{4-0} &\stackrel{?}{=} 0+2 && \text{Let } x=0. \\ \sqrt{4} &\stackrel{?}{=} 2 \\ 2 &= 2 \checkmark && \text{True} \end{aligned}$$

$$\begin{aligned} \sqrt{4-x} &= x+2 \\ \sqrt{4-(-5)} &\stackrel{?}{=} -5+2 && \text{Let } x=-5. \\ \sqrt{9} &\stackrel{?}{=} -3 \\ 3 &= -3 && \text{False} \end{aligned}$$

The solution set is $\{0\}$. The other proposed solution, $-5$, is extraneous. NOW TRY

NOW TRY EXERCISE 4

Solve

$\sqrt{x^2-3x+18}=x+3$.

## EXAMPLE 4 Using the Power Rule (Squaring a Binomial)

Solve $\sqrt{x^2-4x+9}=x-1$.

Squaring gives $(x-1)^2=x^2-2(x)(1)+1^2$ on the right.

$$\begin{aligned} \left(\sqrt{x^2-4x+9}\right)^2 &= (x-1)^2 \\ x^2-4x+9 &= x^2-2x+1 \\ -2x &= -8 && \text{Subtract } x^2 \text{ and 9. Add } 2x. \\ x &= 4 && \text{Divide by } -2. \end{aligned}$$

Remember the middle term.

Twice the product of $x$ and $-1$

CHECK

$$\begin{aligned} \sqrt{x^2-4x+9} &= x-1 && \text{Original equation} \\ \sqrt{4^2-4\cdot 4+9} &\stackrel{?}{=} 4-1 && \text{Let } x=4. \\ 3 &= 3 \checkmark && \text{True} \end{aligned}$$

The solution set is $\{4\}$. NOW TRY

## EXAMPLE 5 Using the Power Rule (Squaring Twice)

Solve $\sqrt{5x+6}+\sqrt{3x+4}=2$.

Isolate one radical on one side of the equation by subtracting $\sqrt{3x+4}$ from each side.

$$\begin{aligned} \sqrt{5x+6} &= 2-\sqrt{3x+4} && \text{Subtract } \sqrt{3x+4}. \\ \left(\sqrt{5x+6}\right)^2 &= \left(2-\sqrt{3x+4}\right)^2 && \text{Square each side.} \\ 5x+6 &= 4-4\sqrt{3x+4}+(3x+4) \end{aligned}$$

Be careful here.

Remember the middle term.

Twice the product of 2 and $-\sqrt{3x+4}$

NOW TRY ANSWERS

**3.** $\{0\}$ **4.** $\{1\}$

**NOW TRY EXERCISE 5**

Solve

$\sqrt{3x+1} - \sqrt{x+4} = 1.$

The equation still contains a radical, so isolate the radical term on the right and square both sides again.

$$5x + 6 = 4 - 4\sqrt{3x+4} + 3x + 4 \quad \text{Result after squaring}$$

$$5x + 6 = 8 - 4\sqrt{3x+4} + 3x \quad \text{Combine like terms.}$$

$$2x - 2 = -4\sqrt{3x+4} \quad \text{Subtract 8 and } 3x.$$

Divide each term by 2.

$$x - 1 = -2\sqrt{3x+4} \quad \text{Divide by 2.}$$

$$(x-1)^2 = \left(-2\sqrt{3x+4}\right)^2 \quad \text{Square each side again.}$$

$$x^2 - 2x + 1 = (-2)^2\left(\sqrt{3x+4}\right)^2 \quad \text{On the right, } (ab)^2 = a^2b^2.$$

$$x^2 - 2x + 1 = 4(3x+4) \quad \text{Apply the exponents.}$$

$$x^2 - 2x + 1 = 12x + 16 \quad \text{Distributive property}$$

$$x^2 - 14x - 15 = 0 \quad \text{Standard form}$$

$$(x-15)(x+1) = 0 \quad \text{Factor.}$$

$$x - 15 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Zero-factor property}$$

$$x = 15 \quad \text{or} \quad x = -1 \quad \text{Solve each equation.}$$

*CHECK*

$$\sqrt{5x+6} + \sqrt{3x+4} = 2 \quad \text{Original equation}$$

$$\sqrt{5(15)+6} + \sqrt{3(15)+4} \stackrel{?}{=} 2 \quad \text{Let } x = 15.$$

$$\sqrt{81} + \sqrt{49} \stackrel{?}{=} 2 \quad \text{Simplify.}$$

$$9 + 7 \stackrel{?}{=} 2 \quad \text{Take square roots.}$$

$$16 = 2 \quad \text{False}$$

Thus, 15 is an extraneous solution and must be rejected. Confirm that the proposed solution $-1$ checks, so the solution set is $\{-1\}$. NOW TRY

## OBJECTIVE 3 Solve radical equations with indexes greater than 2.

**NOW TRY EXERCISE 6**

Solve $\sqrt[3]{4x-5} = \sqrt[3]{3x+2}.$

### EXAMPLE 6 Using the Power Rule for a Power Greater Than 2

Solve $\sqrt[3]{z+5} = \sqrt[3]{2z-6}.$

$$\left(\sqrt[3]{z+5}\right)^3 = \left(\sqrt[3]{2z-6}\right)^3 \quad \text{Cube each side.}$$

$$z + 5 = 2z - 6$$

$$11 = z \quad \text{Subtract } z\text{. Add 6.}$$

*CHECK*

$$\sqrt[3]{z+5} = \sqrt[3]{2z-6} \quad \text{Original equation}$$

$$\sqrt[3]{11+5} \stackrel{?}{=} \sqrt[3]{2 \cdot 11 - 6} \quad \text{Let } z = 11.$$

$$\sqrt[3]{16} = \sqrt[3]{16} \; ✓ \quad \text{True}$$

The solution set is $\{11\}$. NOW TRY

**NOW TRY ANSWERS**

**5.** $\{5\}$ **6.** $\{7\}$

## OBJECTIVE 4 Use the power rule to solve a formula for a specified variable.

### EXAMPLE 7 Solving a Formula from Electronics for a Variable

An important property of a radio-frequency transmission line is its **characteristic impedance,** represented by $Z$ and measured in ohms. If $L$ and $C$ are the inductance and capacitance, respectively, per unit of length of the line, then these quantities are related by the formula $Z = \sqrt{\frac{L}{C}}$. Solve this formula for $C$.

$$Z = \sqrt{\frac{L}{C}}$$ Given formula

Our goal is to isolate $C$ on one side of the equals symbol.

$$Z^2 = \left(\sqrt{\frac{L}{C}}\right)^2$$ Square each side.

$$Z^2 = \frac{L}{C}$$ $\left(\sqrt{a}\right)^2 = a$

$$CZ^2 = L$$ Multiply by $C$.

$$C = \frac{L}{Z^2}$$ Divide by $Z^2$.

NOW TRY

**NOW TRY EXERCISE 7**

Solve the formula for $a$.

$$x = \sqrt{\frac{y+2}{a}}$$

**NOW TRY ANSWER**

7. $a = \frac{y+2}{x^2}$

## 10.6 EXERCISES

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***Concept Check*** *Check each equation to see if the given value for x is a solution.*

**1.** $\sqrt{3x + 18} - x = 0$

**(a)** $6$ **(b)** $-3$

**2.** $\sqrt{3x - 3} - x + 1 = 0$

**(a)** $1$ **(b)** $4$

**3.** $\sqrt{x + 2} - \sqrt{9x - 2} = -2\sqrt{x - 1}$

**(a)** $2$ **(b)** $7$

**4.** $\sqrt{8x - 3} - 2x = 0$

**(a)** $\frac{3}{2}$ **(b)** $\frac{1}{2}$

✎ **5.** Is 9 a solution of the equation $\sqrt{x} = -3$? If not, what is the solution of this equation? Explain.

✎ **6.** Before even attempting to solve $\sqrt{3x + 18} = x$, how can you be sure that the equation cannot have a negative solution?

*Solve each equation.* ***See Examples 1–4.***

**7.** $\sqrt{x - 2} = 3$

**8.** $\sqrt{x + 1} = 7$

◉ **9.** $\sqrt{6k - 1} = 1$

**10.** $\sqrt{7x - 3} = 6$

◉ **11.** $\sqrt{4r + 3} + 1 = 0$

**12.** $\sqrt{5k - 3} + 2 = 0$

**13.** $\sqrt{3x + 1} - 4 = 0$

**14.** $\sqrt{5x + 1} - 11 = 0$

**15.** $4 - \sqrt{x - 2} = 0$

**16.** $9 - \sqrt{4x + 1} = 0$

**17.** $\sqrt{9x - 4} = \sqrt{8x + 1}$

**18.** $\sqrt{4x - 2} = \sqrt{3x + 5}$

**19.** $2\sqrt{x} = \sqrt{3x + 4}$

**20.** $2\sqrt{x} = \sqrt{5x - 16}$

**21.** $3\sqrt{x - 1} = 2\sqrt{2x + 2}$

**22.** $5\sqrt{4x + 1} = 3\sqrt{10x + 25}$

**23.** $x = \sqrt{x^2 + 4x - 20}$

**24.** $x = \sqrt{x^2 - 3x + 18}$

**25.** $x = \sqrt{x^2 + 3x + 9}$

**26.** $x = \sqrt{x^2 - 4x - 8}$

**27.** $\sqrt{9 - x} = x + 3$

**28.** $\sqrt{5 - x} = x + 1$

**29.** $\sqrt{k^2 + 2k + 9} = k + 3$

**30.** $\sqrt{x^2 - 3x + 3} = x - 1$

**31.** $\sqrt{x^2 + 12x - 4} = x - 4$

**32.** $\sqrt{x^2 - 15x + 15} = x - 5$

**33.** $\sqrt{r^2 + 9r + 15} - r - 4 = 0$

**34.** $\sqrt{m^2 + 3m + 12} - m - 2 = 0$

**35.** ***Concept Check*** In solving the equation $\sqrt{3x + 4} = 8 - x$, a student wrote the following for her first step. ***WHAT WENT WRONG?*** Solve the given equation correctly.

$$3x + 4 = 64 + x^2$$

**36.** ***Concept Check*** In solving the equation $\sqrt{5x + 6} - \sqrt{x + 3} = 3$, a student wrote the following for his first step. ***WHAT WENT WRONG?*** Solve the given equation correctly.

$$(5x + 6) + (x + 3) = 9$$

*Solve each equation.* ***See Examples 5 and 6.***

**37.** $\sqrt[3]{2x + 5} = \sqrt[3]{6x + 1}$

**38.** $\sqrt[3]{p + 5} = \sqrt[3]{2p - 4}$

**39.** $\sqrt[3]{x^2 + 5x + 1} = \sqrt[3]{x^2 + 4x}$

**40.** $\sqrt[3]{r^2 + 2r + 8} = \sqrt[3]{r^2 + 3r + 12}$

**41.** $\sqrt[3]{2m - 1} = \sqrt[3]{m + 13}$

**42.** $\sqrt[3]{2k - 11} = \sqrt[3]{5k + 1}$

**43.** $\sqrt[4]{x + 12} = \sqrt[4]{3x - 4}$

**44.** $\sqrt[4]{z + 11} = \sqrt[4]{2z + 6}$

**45.** $\sqrt[3]{x - 8} + 2 = 0$

**46.** $\sqrt[3]{r + 1} + 1 = 0$

**47.** $\sqrt[4]{2k - 5} + 4 = 0$

**48.** $\sqrt[4]{8z - 3} + 2 = 0$

**49.** $\sqrt{k + 2} - \sqrt{k - 3} = 1$

**50.** $\sqrt{r + 6} - \sqrt{r - 2} = 2$

**51.** $\sqrt{2r + 11} - \sqrt{5r + 1} = -1$

**52.** $\sqrt{3x - 2} - \sqrt{x + 3} = 1$

**53.** $\sqrt{3p + 4} - \sqrt{2p - 4} = 2$

**54.** $\sqrt{4x + 5} - \sqrt{2x + 2} = 1$

**55.** $\sqrt{3 - 3p} - 3 = \sqrt{3p + 2}$

**56.** $\sqrt{4x + 7} - 4 = \sqrt{4x - 1}$

**57.** $\sqrt{2\sqrt{x + 11}} = \sqrt{4x + 2}$

**58.** $\sqrt{1 + \sqrt{24 - 10x}} = \sqrt{3x + 5}$

*For each equation, write the expressions with rational exponents as radical expressions, and then solve, using the procedures explained in this section.*

**59.** $(2x - 9)^{1/2} = 2 + (x - 8)^{1/2}$

**60.** $(3w + 7)^{1/2} = 1 + (w + 2)^{1/2}$

**61.** $(2w - 1)^{2/3} - w^{1/3} = 0$

**62.** $(x^2 - 2x)^{1/3} - x^{1/3} = 0$

*Solve each formula for the indicated variable.* ***See Example 7.*** (Source: *Cooke, Nelson M., and Joseph B. Orleans,* Mathematics Essential to Electricity and Radio, *McGraw-Hill.*)

**63.** $Z = \sqrt{\frac{L}{C}}$ for $L$

**64.** $r = \sqrt{\frac{\mathcal{A}}{\pi}}$ for $\mathcal{A}$

**65.** $V = \sqrt{\frac{2K}{m}}$ for $K$

**66.** $V = \sqrt{\frac{2K}{m}}$ for $m$

**67.** $r = \sqrt{\frac{Mm}{F}}$ for $M$

**68.** $r = \sqrt{\frac{Mm}{F}}$ for $F$

*The formula*

$$N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$$

*is used to find the rotational rate N of a space station. Here, a is the acceleration and r represents the radius of the space station, in meters. To find the value of r that will make N simulate the effect of gravity on Earth, the equation must be solved for r, using the required value of N. (Source: Kastner, Bernice,* Space Mathematics, *NASA.)*

**69.** Solve the equation for $r$.

**70. (a)** Approximate the value of $r$ so that $N = 0.063$ rotation per sec if $a = 9.8$ m per sec$^2$.

**(b)** Approximate the value of $r$ so that $N = 0.04$ rotation per sec if $a = 9.8$ m per sec$^2$.

## PREVIEW EXERCISES

*Perform the indicated operations.* ***See Sections 4.4 and 4.5.***

**71.** $(5 + 9x) + (-4 - 8x)$

**72.** $(12 + 7y) - (-3 + 2y)$

**73.** $(x + 3)(2x - 5)$

*Simplify each radical.* ***See Section 10.5.***

**74.** $\frac{2}{4 + \sqrt{3}}$

**75.** $\frac{-7}{5 - \sqrt{2}}$

**76.** $\frac{\sqrt{2} + \sqrt{7}}{\sqrt{5} + \sqrt{3}}$

# 10.7 Complex Numbers

**OBJECTIVES**

1. Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.
2. Recognize subsets of the complex numbers.
3. Add and subtract complex numbers.
4. Multiply complex numbers.
5. Divide complex numbers.
6. Find powers of $i$.

**OBJECTIVE 1** **Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.** The equation $x^2 + 1 = 0$ has no real number solution, since any solution must be a number whose square is $-1$. In the set of real numbers, all squares are nonnegative numbers because the product of two positive numbers or two negative numbers is positive and $0^2 = 0$. To provide a solution of the equation $x^2 + 1 = 0$, we introduce a new number $i$.

**Imaginary Unit *i***

The **imaginary unit *i*** is defined as

$$\boldsymbol{i = \sqrt{-1}, \quad \text{where} \quad i^2 = -1.}$$

That is, $i$ is the principal square root of $-1$.

We can use this definition to define any square root of a negative real number.

**$\sqrt{-b}$**

For any positive real number $b$, $\quad \sqrt{-b} = i\sqrt{b}$.

**NOW TRY EXERCISE 1**

Write each number as a product of a real number and $i$.

**(a)** $\sqrt{-49}$ **(b)** $-\sqrt{-121}$

**(c)** $\sqrt{-3}$ **(d)** $\sqrt{-32}$

## EXAMPLE 1 Simplifying Square Roots of Negative Numbers

Write each number as a product of a real number and $i$.

**(a)** $\sqrt{-100} = i\sqrt{100} = 10i$ **(b)** $-\sqrt{-36} = -i\sqrt{36} = -6i$

**(c)** $\sqrt{-2} = i\sqrt{2}$ **(d)** $\sqrt{-54} = i\sqrt{54} = i\sqrt{9 \cdot 6} = 3i\sqrt{6}$

NOW TRY

**CAUTION** It is easy to mistake $\sqrt{2}i$ for $\sqrt{2i}$, with the $i$ under the radical. For this reason, we usually write $\sqrt{2}i$ as $i\sqrt{2}$, as in the definition of $\sqrt{-b}$.

When finding a product such as $\sqrt{-4} \cdot \sqrt{-9}$, we cannot use the product rule for radicals because it applies only to *nonnegative* radicands. ***For this reason, we change $\sqrt{-b}$ to the form $i\sqrt{b}$ before performing any multiplications or divisions.***

**NOW TRY EXERCISE 2**

Multiply.

**(a)** $\sqrt{-4} \cdot \sqrt{-16}$

**(b)** $\sqrt{-5} \cdot \sqrt{-11}$

**(c)** $\sqrt{-3} \cdot \sqrt{-12}$

**(d)** $\sqrt{13} \cdot \sqrt{-2}$

## EXAMPLE 2 Multiplying Square Roots of Negative Numbers

Multiply.

**(a)**
$$\begin{aligned} &\sqrt{-4} \cdot \sqrt{-9} \\ &= i\sqrt{4} \cdot i\sqrt{9} && \sqrt{-b} = i\sqrt{b} \\ &= i \cdot 2 \cdot i \cdot 3 && \text{Take square roots.} \\ &= 6i^2 && \text{Multiply.} \\ &= 6(-1) && \text{Substitute } -1 \text{ for } i^2. \\ &= -6 \end{aligned}$$

First write all square roots in terms of $i$.

**(b)**
$$\begin{aligned} &\sqrt{-3} \cdot \sqrt{-7} \\ &= i\sqrt{3} \cdot i\sqrt{7} && \sqrt{-b} = i\sqrt{b} \\ &= i^2\sqrt{3 \cdot 7} && \text{Product rule} \\ &= (-1)\sqrt{21} && \text{Substitute } -1 \text{ for } i^2. \\ &= -\sqrt{21} \end{aligned}$$

First write all square roots in terms of $i$.

**(c)**
$$\begin{aligned} &\sqrt{-2} \cdot \sqrt{-8} \\ &= i\sqrt{2} \cdot i\sqrt{8} && \sqrt{-b} = i\sqrt{b} \\ &= i^2\sqrt{2 \cdot 8} && \text{Product rule} \\ &= (-1)\sqrt{16} && i^2 = -1 \\ &= -4 && \text{Take the square root.} \end{aligned}$$

**(d)**
$$\begin{aligned} &\sqrt{-5} \cdot \sqrt{6} \\ &= i\sqrt{5} \cdot \sqrt{6} \\ &= i\sqrt{30} \end{aligned}$$

NOW TRY

**NOW TRY ANSWERS**

1. (a) $7i$ (b) $-11i$ (c) $i\sqrt{3}$ (d) $4i\sqrt{2}$
2. (a) $-8$ (b) $-\sqrt{55}$ (c) $-6$ (d) $i\sqrt{26}$

**CAUTION** Using the product rule for radicals *before* using the definition of $\sqrt{-b}$ gives a *wrong* answer. **Example 2(a)** shows that

$$\sqrt{-4} \cdot \sqrt{-9} = -6, \qquad \text{Correct}$$

but

$$\sqrt{-4(-9)} = \sqrt{36} = 6, \qquad \text{Incorrect}$$

so

$$\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{-4(-9)}.$$

**NOW TRY EXERCISE 3**

Divide.

**(a)** $\dfrac{\sqrt{-72}}{\sqrt{-8}}$ **(b)** $\dfrac{\sqrt{-48}}{\sqrt{3}}$

## EXAMPLE 3 Dividing Square Roots of Negative Numbers

Divide.

**(a)** $\dfrac{\sqrt{-75}}{\sqrt{-3}}$

$$= \frac{i\sqrt{75}}{i\sqrt{3}}$$ First write all square roots in terms of $i$.

$$= \sqrt{\frac{75}{3}} \qquad \text{Quotient rule}$$

$$= \sqrt{25} \qquad \text{Divide.}$$

$$= 5$$

**(b)** $\dfrac{\sqrt{-32}}{\sqrt{8}}$

$$= \frac{i\sqrt{32}}{\sqrt{8}} \qquad \sqrt{-32} = i\sqrt{32}$$

$$= i\sqrt{\frac{32}{8}} \qquad \text{Quotient rule}$$

$$= i\sqrt{4} \qquad \text{Divide.}$$

$$= 2i$$

NOW TRY

**OBJECTIVE 2** **Recognize subsets of the complex numbers.** A new set of numbers, the *complex numbers,* are defined as follows.

### Complex Number

If $a$ and $b$ are real numbers, then any number of the form $\boldsymbol{a + bi}$ is called a **complex number.** In the complex number $a + bi$, the number $a$ is called the **real part** and $b$ is called the **imaginary part.***

For a complex number $a + bi$, if $b = 0$, then $a + bi = a$, which is a real number. ***Thus, the set of real numbers is a subset of the set of complex numbers.*** If $a = 0$ and $b \neq 0$, the complex number is a **pure imaginary number.** For example, $3i$ is a pure imaginary number. A number such as $7 + 2i$ is a **nonreal complex number.** A complex number written in the form $a + bi$ is in **standard form.**

The relationships among the various sets of numbers are shown in **FIGURE 11**.

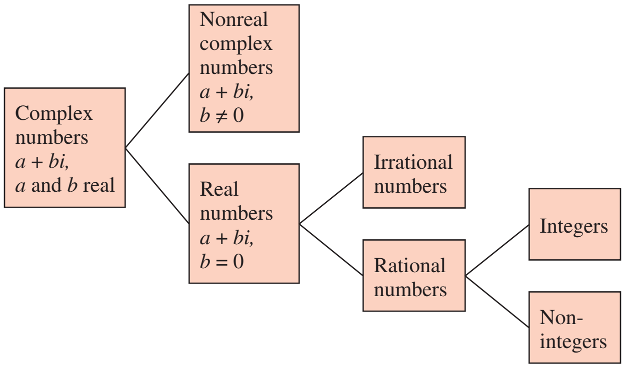

**FIGURE 11**

NOW TRY ANSWERS

3. **(a)** 3 **(b)** $4i$

---

*Some texts define $bi$ as the imaginary part of the complex number $a + bi$.

**OBJECTIVE 3** **Add and subtract complex numbers.** The commutative, associative, and distributive properties for real numbers are also valid for complex numbers. ***Thus, to add complex numbers, we add their real parts and add their imaginary parts.***

**NOW TRY EXERCISE 4**

Add.

**(a)** $(-3 + 2i) + (4 + 7i)$

**(b)** $(5 - i) + (-3 + 3i) + (6 - 4i)$

### EXAMPLE 4 Adding Complex Numbers

Add.

**(a)** $(2 + 3i) + (6 + 4i)$

$= (2 + 6) + (3 + 4)i$ Properties of real numbers

$= 8 + 7i$ Add real parts. Add imaginary parts.

**(b)** $(4 + 2i) + (3 - i) + (-6 + 3i)$

$= [4 + 3 + (-6)] + [2 + (-1) + 3]i$ Associative property

$= 1 + 4i$ Add real parts. Add imaginary parts.

NOW TRY

***To subtract complex numbers, we subtract their real parts and subtract their imaginary parts.***

**NOW TRY EXERCISE 5**

Subtract.

**(a)** $(7 + 10i) - (3 + 5i)$

**(b)** $(5 - 2i) - (9 - 7i)$

**(c)** $(-1 + 12i) - (-1 - i)$

### EXAMPLE 5 Subtracting Complex Numbers

Subtract.

**(a)** $(6 + 5i) - (3 + 2i)$

$= (6 - 3) + (5 - 2)i$ Properties of real numbers

$= 3 + 3i$ Subtract real parts. Subtract imaginary parts.

**(b)** $(7 - 3i) - (8 - 6i)$

$= (7 - 8) + [-3 - (-6)]i$

$= -1 + 3i$

**(c)** $(-9 + 4i) - (-9 + 8i)$

$= (-9 + 9) + (4 - 8)i$

$= 0 - 4i$

$= -4i$

NOW TRY

**OBJECTIVE 4** **Multiply complex numbers.**

### EXAMPLE 6 Multiplying Complex Numbers

Multiply.

**(a)** $4i(2 + 3i)$

$= 4i(2) + 4i(3i)$ Distributive property

$= 8i + 12i^2$ Multiply.

$= 8i + 12(-1)$ Substitute $-1$ for $i^2$.

$= -12 + 8i$ Standard form

**(b)** $(3 + 5i)(4 - 2i)$

$$= \underbrace{3(4)}_{\text{First}} + \underbrace{3(-2i)}_{\text{Outer}} + \underbrace{5i(4)}_{\text{Inner}} + \underbrace{5i(-2i)}_{\text{Last}}$$ Use the FOIL method.

$= 12 - 6i + 20i - 10i^2$ Multiply.

$= 12 + 14i - 10(-1)$ Combine imaginary terms; $i^2 = -1$.

$= 12 + 14i + 10$ Multiply.

$= 22 + 14i$ Combine real terms.

**NOW TRY ANSWERS**

**4.** **(a)** $1 + 9i$ **(b)** $8 - 2i$

**5.** **(a)** $4 + 5i$ **(b)** $-4 + 5i$ **(c)** $13i$

**NOW TRY EXERCISE 6**

Multiply.

**(a)** $8i(3-5i)$

**(b)** $(7-2i)(4+3i)$

**(c)** $(2+3i)(1-5i)$

$$\begin{aligned}&= 2(1)+2(-5i)+3i(1)+3i(-5i) && \text{FOIL}\\ &= 2-10i+3i-15i^2 && \text{Multiply.}\\ &= 2-7i-15(-1) && \text{Use parentheses around } -1 \text{ to avoid errors.}\\ &= 2-7i+15 &&\\ &= 17-7i &&\end{aligned}$$

NOW TRY

The two complex numbers $a+bi$ and $a-bi$ are called **complex conjugates**, or simply *conjugates,* of each other. ***The product of a complex number and its conjugate is always a real number,*** as shown here.

$$\begin{aligned}(a+bi)(a-bi) &= a^2-abi+abi-b^2i^2\\ &= a^2-b^2(-1)\\ \boldsymbol{(a+bi)(a-bi)} &\boldsymbol{= a^2+b^2}\end{aligned}$$

The product eliminates $i$.

For example, $(3+7i)(3-7i)=3^2+7^2=9+49=58.$

**OBJECTIVE 5** **Divide complex numbers.** The quotient of two complex numbers should be a complex number. To write the quotient as a complex number, we need to eliminate $i$ in the denominator. We use conjugates and a process similar to that for rationalizing a denominator to do this.

## EXAMPLE 7 Dividing Complex Numbers

Find each quotient.

**(a)** $\dfrac{8+9i}{5+2i}$

Multiply both the numerator and denominator by the conjugate of the denominator. The conjugate of $5+2i$ is $5-2i$.

$$\begin{aligned}&\frac{8+9i}{5+2i}\\ &= \frac{(8+9i)(5-2i)}{(5+2i)(5-2i)} && \frac{5-2i}{5-2i}=1\\ &= \frac{40-16i+45i-18i^2}{5^2+2^2} && \text{In the denominator, } (a+bi)(a-bi)=a^2+b^2.\\ &= \frac{58+29i}{29} && -18i^2=-18(-1)=18; \text{ Combine like terms.}\\ &= \frac{29(2+i)}{29} && \text{Factor the numerator.}\\ &= 2+i && \text{Lowest terms}\end{aligned}$$

Factor first. Then divide out the common factor.

**NOW TRY ANSWERS**

**6. (a)** $40+24i$ **(b)** $34+13i$

**NOW TRY EXERCISE 7**

Find each quotient.

**(a)** $\frac{4+2i}{1+3i}$ **(b)** $\frac{5-4i}{i}$

**(b)** $\frac{1+i}{i}$

$$= \frac{(1+i)(-i)}{i(-i)}$$

Multiply numerator and denominator by $-i$, the conjugate of $i$.

$$= \frac{-i-i^2}{-i^2}$$

Distributive property; multiply.

$$= \frac{-i-(-1)}{-(-1)}$$

Substitute $-1$ for $i^2$.

Use parentheses to avoid errors.

$$= \frac{-i+1}{1}$$

$$= 1 - i$$

NOW TRY

**OBJECTIVE 6** **Find powers of *i*.** Because $i^2$ is defined to be $-1$, we can find greater powers of $i$ as shown in the following examples.

$$\begin{array}{ll} i^3 = i \cdot i^2 = i(-1) = -i & i^6 = i^2 \cdot i^4 = (-1) \cdot 1 = -1 \\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 & i^7 = i^3 \cdot i^4 = (-i) \cdot 1 = -i \\ i^5 = i \cdot i^4 = i \cdot 1 = i & i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1 \end{array}$$

Notice that the powers of $i$ rotate through the four numbers $i$, $-1$, $-i$, and 1. Greater powers of $i$ can be simplified by using the fact that $i^4 = 1$.

**NOW TRY EXERCISE 8**

Find each power of $i$.

**(a)** $i^{16}$ **(b)** $i^{21}$

**(c)** $i^{-6}$ **(d)** $i^{-13}$

### EXAMPLE 8 Simplifying Powers of *i*

Find each power of $i$.

**(a)** $i^{12} = (i^4)^3 = 1^3 = 1$

**(b)** $i^{39} = i^{36} \cdot i^3 = (i^4)^9 \cdot i^3 = 1^9 \cdot (-i) = -i$

**(c)** $i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$

**(d)** $i^{-1} = \frac{1}{i} = \frac{1(-i)}{i(-i)} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = \frac{-i}{1} = -i$

NOW TRY

**NOW TRY ANSWERS**

**7.** **(a)** $1-i$ **(b)** $-4-5i$

**8.** **(a)** 1 **(b)** $i$ **(c)** $-1$ **(d)** $-i$

## 10.7 EXERCISES

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*Complete solution available on the Video Resources on DVD*

***Concept Check*** *Decide whether each expression is equal to 1, −1, i, or −i.*

**1.** $\sqrt{-1}$ **2.** $-\sqrt{-1}$ **3.** $i^2$ **4.** $-i^2$ **5.** $\frac{1}{i}$ **6.** $(-i)^2$

*Write each number as a product of a real number and i. Simplify all radical expressions.* ***See Example 1.***

**7.** $\sqrt{-169}$ **8.** $\sqrt{-225}$ **9.** $-\sqrt{-144}$ **10.** $-\sqrt{-196}$

**11.** $\sqrt{-5}$ **12.** $\sqrt{-21}$ **13.** $\sqrt{-48}$ **14.** $\sqrt{-96}$

*Multiply or divide as indicated.* ***See Examples 2 and 3.***

**15.** $\sqrt{-7} \cdot \sqrt{-15}$
**16.** $\sqrt{-3} \cdot \sqrt{-19}$
**17.** $\sqrt{-4} \cdot \sqrt{-25}$
**18.** $\sqrt{-9} \cdot \sqrt{-81}$

**19.** $\sqrt{-3} \cdot \sqrt{11}$
**20.** $\sqrt{-10} \cdot \sqrt{2}$
**21.** $\dfrac{\sqrt{-300}}{\sqrt{-100}}$
**22.** $\dfrac{\sqrt{-40}}{\sqrt{-10}}$

**23.** $\dfrac{\sqrt{-75}}{\sqrt{3}}$
**24.** $\dfrac{\sqrt{-160}}{\sqrt{10}}$
**25.** $\dfrac{-\sqrt{-64}}{\sqrt{-16}}$
**26.** $\dfrac{-\sqrt{-100}}{\sqrt{-25}}$

**27.** Every real number is a complex number. Explain why this is so.

**28.** Not every complex number is a real number. Give an example, and explain why this statement is true.

*Add or subtract as indicated. Write your answers in the form* $a + bi$. ***See Examples 4 and 5.***

**29.** $(3 + 2i) + (-4 + 5i)$
**30.** $(7 + 15i) + (-11 + 14i)$

**31.** $(5 - i) + (-5 + i)$
**32.** $(-2 + 6i) + (2 - 6i)$

**33.** $(4 + i) - (-3 - 2i)$
**34.** $(9 + i) - (3 + 2i)$

**35.** $(-3 - 4i) - (-1 - 4i)$
**36.** $(-2 - 3i) - (-5 - 3i)$

**37.** $(-4 + 11i) + (-2 - 4i) + (7 + 6i)$
**38.** $(-1 + i) + (2 + 5i) + (3 + 2i)$

**39.** $[(7 + 3i) - (4 - 2i)] + (3 + i)$
**40.** $[(7 + 2i) + (-4 - i)] - (2 + 5i)$

**41.** *Concept Check* Fill in the blank with the correct response:

Because $(4 + 2i) - (3 + i) = 1 + i$, using the definition of subtraction, we can check this to find that $(1 + i) + (3 + i) =$ ________________.

**42.** *Concept Check* Fill in the blank with the correct response:

Because $\frac{-5}{2 - i} = -2 - i$, using the definition of division, we can check this to find that $(-2 - i)(2 - i) =$ ________________.

*Multiply.* ***See Example 6.***

**43.** $(3i)(27i)$
**44.** $(5i)(125i)$
**45.** $(-8i)(-2i)$

**46.** $(-32i)(-2i)$
**47.** $5i(-6 + 2i)$
**48.** $3i(4 + 9i)$

**49.** $(4 + 3i)(1 - 2i)$
**50.** $(7 - 2i)(3 + i)$
**51.** $(4 + 5i)^2$

**52.** $(3 + 2i)^2$
**53.** $2i(-4 - i)^2$
**54.** $3i(-3 - i)^2$

**55.** $(12 + 3i)(12 - 3i)$
**56.** $(6 + 7i)(6 - 7i)$
**57.** $(4 + 9i)(4 - 9i)$

**58.** $(7 + 2i)(7 - 2i)$
**59.** $(1 + i)^2(1 - i)^2$
**60.** $(2 - i)^2(2 + i)^2$

**61.** *Concept Check* What is the conjugate of $a + bi$?

**62.** *Concept Check* If we multiply $a + bi$ by its conjugate, we get ________________, which is always a real number.

*Find each quotient.* ***See Example 7.***

**63.** $\dfrac{2}{1 - i}$
**64.** $\dfrac{2}{1 + i}$
**65.** $\dfrac{8i}{2 + 2i}$
**66.** $\dfrac{-8i}{1 + i}$

**67.** $\dfrac{-7 + 4i}{3 + 2i}$
**68.** $\dfrac{-38 - 8i}{7 + 3i}$
**69.** $\dfrac{2 - 3i}{2 + 3i}$
**70.** $\dfrac{-1 + 5i}{3 + 2i}$

**71.** $\dfrac{3 + i}{i}$
**72.** $\dfrac{5 - i}{i}$
**73.** $\dfrac{3 - i}{-i}$
**74.** $\dfrac{5 + i}{-i}$

*Find each power of i.* ***See Example 8.***

**75.** $i^{18}$ **76.** $i^{26}$ **77.** $i^{89}$ **78.** $i^{48}$ **79.** $i^{38}$

**80.** $i^{102}$ **81.** $i^{43}$ **82.** $i^{83}$ **83.** $i^{-5}$ **84.** $i^{-17}$

**85.** A student simplified $i^{-18}$ as follows:

$$i^{-18} = i^{-18} \cdot i^{20} = i^{-18+20} = i^2 = -1.$$

Explain the mathematical justification for this correct work.

**86.** Explain why

$$(46 + 25i)(3 - 6i) \quad \text{and} \quad (46 + 25i)(3 - 6i)i^{12}$$

must be equal. (Do not actually perform the computation.)

***Ohm's law*** *for the current I in a circuit with voltage E, resistance R, capacitive reactance* $X_c$*, and inductive reactance* $X_L$ *is*

$$I = \frac{E}{R + (X_L - X_c)i}.$$

*Use this law to work Exercises 87 and 88.*

**87.** Find $I$ if $E = 2 + 3i$, $R = 5$, $X_L = 4$, and $X_c = 3$.

**88.** Find $E$ if $I = 1 - i$, $R = 2$, $X_L = 3$, and $X_c = 1$.

*Complex numbers will appear again in this book in* ***Chapter 11,*** *when we study quadratic equations. The following exercises examine how a complex number can be a solution of a quadratic equation.*

**89.** Show that $1 + 5i$ is a solution of

$$x^2 - 2x + 26 = 0.$$

Then show that its conjugate is also a solution.

**90.** Show that $3 + 2i$ is a solution of

$$x^2 - 6x + 13 = 0.$$

Then show that its conjugate is also a solution.

***Brain Busters*** *Perform the indicated operations. Give answers in standard form.*

**91.** $\dfrac{3}{2 - i} + \dfrac{5}{1 + i}$

**92.** $\dfrac{2}{3 + 4i} + \dfrac{4}{1 - i}$

**93.** $\left(\dfrac{2 + i}{2 - i} + \dfrac{i}{1 + i}\right)i$

**94.** $\left(\dfrac{4 - i}{1 + i} - \dfrac{2i}{2 + i}\right)4i$

## PREVIEW EXERCISES

*Solve each equation.* ***See Sections 2.3 and 5.5.***

**95.** $6x + 13 = 0$

**96.** $4x - 7 = 0$

**97.** $x(x + 3) = 40$

**98.** $2x^2 - 5x - 7 = 0$

**99.** $5x^2 - 3x = 2$

**100.** $-6x^2 + 7x = -10$

# CHAPTER 10 SUMMARY

## KEY TERMS

### 10.1

square root
principal square root
negative square root
radicand
radical
perfect square
cube root
fourth root
index (order)
radical expression
square root function
cube root function

### 10.3

simplified radical

### 10.5

rationalizing the denominator
conjugates

### 10.6

radical equation
extraneous solution

### 10.7

complex number
real part
imaginary part
pure imaginary number
standard form
complex conjugates

## NEW SYMBOLS

| | |
|---|---|
| $\sqrt{\phantom{a}}$ | radical symbol |
| $\sqrt[3]{a}$ | cube root of $a$ |
| $\sqrt[4]{a}$ | fourth root of $a$ |
| $\sqrt[n]{a}$ | principal $n$th root of $a$ |
| $\approx$ | is approximately equal to |
| $a^{1/n}$ | $a$ to the power $\frac{1}{n}$ |
| $a^{m/n}$ | $a$ to the power $\frac{m}{n}$ |
| $i$ | imaginary unit |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** A **radicand** is
- **A.** the index of a radical
- **B.** the number or expression under the radical sign
- **C.** the positive root of a number
- **D.** the radical sign.

**2.** The **Pythagorean theorem** states that, in a right triangle,
- **A.** the sum of the measures of the angles is 180°
- **B.** the sum of the lengths of the two shorter sides equals the length of the longest side
- **C.** the longest side is opposite the right angle
- **D.** the square of the length of the longest side equals the sum of the squares of the lengths of the two shorter sides.

**3.** A **hypotenuse** is
- **A.** either of the two shorter sides of a triangle
- **B.** the shortest side of a triangle
- **C.** the side opposite the right angle in a triangle
- **D.** the longest side in any triangle.

**4.** **Rationalizing the denominator** is the process of
- **A.** eliminating fractions from a radical expression
- **B.** changing the denominator of a fraction from a radical to a rational number
- **C.** clearing a radical expression of radicals
- **D.** multiplying radical expressions.

**5.** An **extraneous solution** is a solution
- **A.** that does not satisfy the original equation
- **B.** that makes an equation true
- **C.** that makes an expression equal 0
- **D.** that checks in the original equation.

**6.** A **complex number** is
- **A.** a real number that includes a complex fraction
- **B.** a zero multiple of $i$
- **C.** a number of the form $a + bi$, where $a$ and $b$ are real numbers
- **D.** the square root of $-1$.

### ANSWERS

**1.** B; *Example:* In $\sqrt{3xy}$, $3xy$ is the radicand. **2.** D; *Example:* In a right triangle where $a = 6$, $b = 8$, and $c = 10$, $6^2 + 8^2 = 10^2$.
**3.** C; *Example:* In a right triangle where the sides measure 9, 12, and 15 units, the hypotenuse is the side opposite the right angle, with measure 15 units. **4.** B; *Example:* To rationalize the denominator of $\frac{5}{\sqrt{3}+1}$, multiply both the numerator and denominator by $\sqrt{3} - 1$ to get $\frac{5\left(\sqrt{3}-1\right)}{2}$.
**5.** A; *Example:* The proposed solution 2 is extraneous in $\sqrt{5x-1} + 3 = 0$. **6.** C; *Examples:* $-5$ (or $-5 + 0i$), $7i$ (or $0 + 7i$), $\sqrt{2} - 4i$

## *QUICK REVIEW*

***CONCEPTS*** | ***EXAMPLES***

### 10.1 Radical Expressions and Graphs

$\sqrt[n]{a} = b$ **means** $b^n = a$.

$\sqrt[n]{a}$ is the principal $n$th root of $a$.

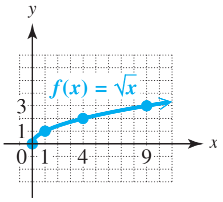
$\sqrt[n]{a^n} = |a|$ if $n$ is even $\qquad \sqrt[n]{a^n} = a$ if $n$ is odd.

**Functions Defined by Radical Expressions**

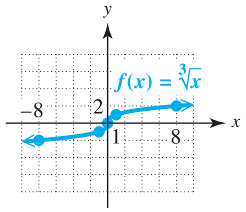
The square root function defined by $f(x) = \sqrt{x}$ and the cube root function defined by $f(x) = \sqrt[3]{x}$ are two important functions defined by radical expressions.

The two square roots of 64 are $\sqrt{64} = 8$ (the principal square root) and $-\sqrt{64} = -8$.

$$\sqrt[4]{(-2)^4} = |-2| = 2 \qquad \sqrt[3]{-27} = -3$$



Square root function



Cube root function

### 10.2 Rational Exponents

$a^{1/n} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ exists.

If $m$ and $n$ are positive integers with $\frac{m}{n}$ in lowest terms, then $a^{m/n} = (a^{1/n})^m$, provided that $a^{1/n}$ is a real number.

All of the usual definitions and rules for exponents are valid for rational exponents.

$$81^{1/2} = \sqrt{81} = 9 \qquad -64^{1/3} = -\sqrt[3]{64} = -4$$

$$8^{5/3} = (8^{1/3})^5 = 2^5 = 32 \qquad (y^{2/5})^{10} = y^4$$

$$\begin{aligned} 5^{-1/2} \cdot 5^{1/4} &= 5^{-1/2+1/4} \\ &= 5^{-1/4} \\ &= \frac{1}{5^{1/4}} \end{aligned} \qquad \begin{aligned} \frac{x^{-1/3}}{x^{-1/2}} &= x^{-1/3-(-1/2)} \\ &= x^{-1/3+1/2} \\ &= x^{1/6}, \quad x > 0 \end{aligned}$$

### 10.3 Simplifying Radical Expressions

**Product and Quotient Rules for Radicals**

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $n$ is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \quad b \neq 0.$$

**Conditions for a Simplified Radical**

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.

$$\sqrt{3} \cdot \sqrt{7} = \sqrt{21} \qquad \sqrt[5]{x^3y} \cdot \sqrt[5]{xy^2} = \sqrt[5]{x^4y^3}$$

$$\frac{\sqrt{x^5}}{\sqrt{x^4}} = \sqrt{\frac{x^5}{x^4}} = \sqrt{x}, \quad x > 0$$

$$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

$$\sqrt[3]{54x^5y^3} = \sqrt[3]{27x^3y^3 \cdot 2x^2} = 3xy\sqrt[3]{2x^2}$$

$$\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$$

$$\sqrt[9]{x^3} = x^{3/9} = x^{1/3}, \quad \text{or} \quad \sqrt[3]{x}$$

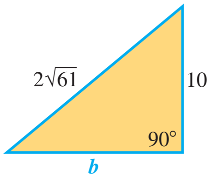
**Pythagorean Theorem**

If $a$ and $b$ are the lengths of the shorter sides of a right triangle and $c$ is the length of the longest side, then

$$a^2 + b^2 = c^2.$$

Find $b$ for the triangle in the figure.

$$\begin{aligned} 10^2 + b^2 &= \left(2\sqrt{61}\right)^2 \\ b^2 &= 4(61) - 100 \\ b^2 &= 144 \\ b &= 12 \end{aligned}$$



(continued)

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| **Distance Formula**<br>The distance $d$ between $(x_1, y_1)$ and $(x_2, y_2)$ is<br>$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. | Find the distance between $(3, -2)$ and $(-1, 1)$.<br>$\sqrt{(-1 - 3)^2 + [1 - (-2)]^2}$<br>$= \sqrt{(-4)^2 + 3^2}$<br>$= \sqrt{16 + 9}$<br>$= \sqrt{25}$<br>$= 5$ |
| **10.4 Adding and Subtracting Radical Expressions**<br>***Only radical expressions with the same index and the same radicand may be combined.*** | $2\sqrt{28} - 3\sqrt{63} + 8\sqrt{112}$<br>$= 2\sqrt{4 \cdot 7} - 3\sqrt{9 \cdot 7} + 8\sqrt{16 \cdot 7}$<br>$= 2 \cdot 2\sqrt{7} - 3 \cdot 3\sqrt{7} + 8 \cdot 4\sqrt{7}$<br>$= 4\sqrt{7} - 9\sqrt{7} + 32\sqrt{7}$<br>$= (4 - 9 + 32)\sqrt{7}$<br>$= 27\sqrt{7}$<br>$\left.\begin{array}{l}\sqrt{15} + \sqrt{30}\\ \sqrt{3} + \sqrt[3]{9}\end{array}\right\}$ cannot be simplified further |
| **10.5 Multiplying and Dividing Radical Expressions**<br>Multiply binomial radical expressions by using the FOIL method. Special products from **Section 4.6** may apply. | $\left(\sqrt{2} + \sqrt{7}\right)\left(\sqrt{3} - \sqrt{6}\right)$<br>$= \sqrt{6} - 2\sqrt{3} + \sqrt{21} - \sqrt{42}$ $\quad \sqrt{12} = 2\sqrt{3}$<br>$\left(\sqrt{5} - \sqrt{10}\right)\left(\sqrt{5} + \sqrt{10}\right)$<br>$= 5 - 10$, or $-5$<br>$\left(\sqrt{3} - \sqrt{2}\right)^2$<br>$= 3 - 2\sqrt{3} \cdot \sqrt{2} + 2$<br>$= 5 - 2\sqrt{6}$ |
| Rationalize the denominator by multiplying both the numerator and the denominator by the same expression, one that will yield a rational number in the final denominator. | $\dfrac{\sqrt{7}}{\sqrt{5}} = \dfrac{\sqrt{7} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \dfrac{\sqrt{35}}{5}$<br>$\dfrac{4}{\sqrt{5} - \sqrt{2}} = \dfrac{4\left(\sqrt{5} + \sqrt{2}\right)}{\left(\sqrt{5} - \sqrt{2}\right)\left(\sqrt{5} + \sqrt{2}\right)}$<br>$= \dfrac{4\left(\sqrt{5} + \sqrt{2}\right)}{5 - 2} = \dfrac{4\left(\sqrt{5} + \sqrt{2}\right)}{3}$ |
| To write a radical quotient in lowest terms, factor the numerator and denominator and then divide out any common factor(s). | $\dfrac{5 + 15\sqrt{6}}{10} = \dfrac{5\left(1 + 3\sqrt{6}\right)}{5 \cdot 2} = \dfrac{1 + 3\sqrt{6}}{2}$ |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **10.6 Solving Equations with Radicals**<br>**Solving an Equation with Radicals**<br>***Step 1*** Isolate one radical on one side of the equation.<br>***Step 2*** Raise both sides of the equation to a power that is the same as the index of the radical.<br>***Step 3*** Solve the resulting equation. If it still contains a radical, repeat Steps 1 and 2.<br>***Step 4*** Check all proposed solutions in the *original* equation. | Solve $\sqrt{2x+3}-x=0$.<br>$\sqrt{2x+3}=x$ Subtract $x$.<br>$\left(\sqrt{2x+3}\right)^2=x^2$ Square each side.<br>$2x+3=x^2$ Apply the exponents.<br>$x^2-2x-3=0$ Standard form<br>$(x-3)(x+1)=0$ Factor.<br>$x-3=0$ or $x+1=0$ Zero-factor property<br>$x=3$ or $x=-1$ Solve each equation.<br>A check shows that 3 is a solution, but $-1$ is extraneous. The solution set is $\{3\}$. |
| **10.7 Complex Numbers**<br>$i=\sqrt{-1}$, where $i^2=-1$.<br>For any positive number $b$, $\sqrt{-b}=i\sqrt{b}$.<br>***To multiply radicals with negative radicands, first change each factor to the form $i\sqrt{b}$ and then multiply. The same procedure applies to quotients.*** | $\sqrt{-25}=i\sqrt{25}=5i$<br>$\sqrt{-3}\cdot\sqrt{-27}$<br>$=i\sqrt{3}\cdot i\sqrt{27}$ $\sqrt{-b}=i\sqrt{b}$<br>$=i^2\sqrt{81}$<br>$=-1\cdot 9$ $i^2=-1$<br>$=-9$<br>$\dfrac{\sqrt{-18}}{\sqrt{-2}}=\dfrac{i\sqrt{18}}{i\sqrt{2}}=\sqrt{\dfrac{18}{2}}=\sqrt{9}=3$ |
| **Adding and Subtracting Complex Numbers**<br>Add (or subtract) the real parts and add (or subtract) the imaginary parts. | $(5+3i)+(8-7i)$<br>$=13-4i$<br>$(5+3i)-(8-7i)$<br>$=-3+10i$ |
| **Multiplying Complex Numbers**<br>Multiply complex numbers by using the FOIL method. | $(2+i)(5-3i)$<br>$=10-6i+5i-3i^2$ FOIL<br>$=10-i-3(-1)$ $i^2=-1$<br>$=10-i+3$ Multiply.<br>$=13-i$ Combine real terms. |
| **Dividing Complex Numbers**<br>Divide complex numbers by multiplying the numerator and the denominator by the conjugate of the denominator. | $\dfrac{20}{3+i}$<br>$=\dfrac{20(3-i)}{(3+i)(3-i)}$ Multiply by the conjugate.<br>$=\dfrac{20(3-i)}{9-i^2}$ $(a+b)(a-b)=a^2-b^2$<br>$=\dfrac{20(3-i)}{10}$ $i^2=-1$<br>$=2(3-i)$, or $6-2i$ |

# CHAPTER 10 REVIEW EXERCISES

**10.1** *Find each root.*

**1.** $\sqrt{1764}$ **2.** $-\sqrt{289}$ **3.** $\sqrt[3]{216}$

**4.** $\sqrt[3]{-125}$ **5.** $-\sqrt[3]{27}$ **6.** $\sqrt[5]{-32}$

**7.** ***Concept Check*** Under what conditions is $\sqrt[n]{a}$ not a real number?

**8.** Simplify each radical so that no radicals appear. Assume that $x$ represents any real number.

**(a)** $\sqrt{x^2}$ **(b)** $-\sqrt{x^2}$ **(c)** $\sqrt[3]{x^3}$

*Use a calculator to find a decimal approximation for each number. Give the answer to the nearest thousandth.*

**9.** $-\sqrt{47}$ **10.** $\sqrt[3]{-129}$ **11.** $\sqrt[4]{605}$

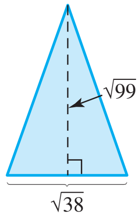
**12.** $\sqrt[4]{500^{-3}}$ **13.** $-\sqrt[3]{500^4}$ **14.** $-\sqrt{28^{-1}}$

*Graph each function. Give the domain and range.*

**15.** $f(x) = \sqrt{x-1}$ **16.** $f(x) = \sqrt[3]{x} + 4$

**17.** What is the best estimate of the area of the triangle shown here?

**A.** 3600 **B.** 30 **C.** 60 **D.** 360



**10.2**

**18.** ***Concept Check*** Fill in the blanks with the correct responses: One way to evaluate $8^{2/3}$ is to first find the ______ root of ______, which is ______. Then raise that result to the ______ power, to get an answer of ______. Therefore, $8^{2/3} =$ ______.

**19.** ***Concept Check*** Which one of the following is a positive number?

**A.** $(-27)^{2/3}$ **B.** $(-64)^{5/3}$ **C.** $(-100)^{1/2}$ **D.** $(-32)^{1/5}$

**20.** ***Concept Check*** If $a$ is a negative number and $n$ is odd, then what must be true about $m$ for $a^{m/n}$ to be **(a)** positive **(b)** negative?

**21.** ***Concept Check*** If $a$ is negative and $n$ is even, is $a^{1/n}$ a real number?

*Simplify. If the expression does not represent a real number, say so.*

**22.** $49^{1/2}$ **23.** $-121^{1/2}$ **24.** $16^{5/4}$ **25.** $-8^{2/3}$

**26.** $-\left(\frac{36}{25}\right)^{3/2}$ **27.** $\left(-\frac{1}{8}\right)^{-5/3}$ **28.** $\left(\frac{81}{10{,}000}\right)^{-3/4}$ **29.** $(-16)^{3/4}$

**30.** ***Concept Check*** Illustrate two different ways of writing $8^{2/3}$ as a radical expression.

**31.** Explain the relationship between the expressions $a^{m/n}$ and $\sqrt[n]{a^m}$. Give an example.

*Write each expression as a radical.*

**32.** $(m + 3n)^{1/2}$ **33.** $(3a + b)^{-5/3}$

*Write each expression with a rational exponent.*

**34.** $\sqrt{7^9}$ **35.** $\sqrt[5]{p^4}$

*Use the rules for exponents to simplify each expression. Write the answer with only positive exponents. Assume that all variables represent positive real numbers.*

**36.** $5^{1/4} \cdot 5^{7/4}$

**37.** $\dfrac{96^{2/3}}{96^{-1/3}}$

**38.** $\dfrac{(a^{1/3})^4}{a^{2/3}}$

**39.** $\dfrac{y^{-1/3} \cdot y^{5/6}}{y}$

**40.** $\left(\dfrac{z^{-1}x^{-3/5}}{2^{-2}z^{-1/2}x}\right)^{-1}$

**41.** $r^{-1/2}(r + r^{3/2})$

*Simplify by first writing each radical in exponential form. Leave the answer in exponential form. Assume that all variables represent positive real numbers.*

**42.** $\sqrt[8]{s^4}$

**43.** $\sqrt[6]{r^9}$

**44.** $\dfrac{\sqrt{p^5}}{p^2}$

**45.** $\sqrt[4]{k^3} \cdot \sqrt{k^3}$

**46.** $\sqrt[3]{m^5} \cdot \sqrt[3]{m^8}$

**47.** $\sqrt[4]{\sqrt[3]{z}}$

**48.** $\sqrt{\sqrt{\sqrt{x}}}$

**49.** $\sqrt[3]{\sqrt[5]{x}}$

**50.** $\sqrt{\sqrt[6]{\sqrt[3]{x}}}$

**51.** The product rule does not apply to $3^{1/4} \cdot 2^{1/5}$. Why?

**10.3** *Simplify each radical. Assume that all variables represent positive real numbers.*

**52.** $\sqrt{6} \cdot \sqrt{11}$

**53.** $\sqrt{5} \cdot \sqrt{r}$

**54.** $\sqrt[3]{6} \cdot \sqrt[3]{5}$

**55.** $\sqrt[4]{7} \cdot \sqrt[4]{3}$

**56.** $\sqrt{20}$

**57.** $\sqrt{75}$

**58.** $-\sqrt{125}$

**59.** $\sqrt[3]{-108}$

**60.** $\sqrt{100y^7}$

**61.** $\sqrt[3]{64p^4q^6}$

**62.** $\sqrt[3]{108a^8b^5}$

**63.** $\sqrt[3]{632r^8t^4}$

**64.** $\sqrt{\dfrac{y^3}{144}}$

**65.** $\sqrt[3]{\dfrac{m^{15}}{27}}$

**66.** $\sqrt[3]{\dfrac{r^2}{8}}$

**67.** $\sqrt[4]{\dfrac{a^9}{81}}$

*Simplify each radical expression.*

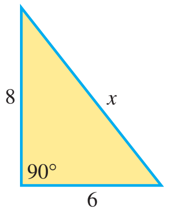
**68.** $\sqrt[6]{15^3}$

**69.** $\sqrt[4]{p^6}$

**70.** $\sqrt[3]{2} \cdot \sqrt[4]{5}$

**71.** $\sqrt{x} \cdot \sqrt[5]{x}$

**72.** Find the unknown length in the right triangle. Simplify the answer if applicable.



**73.** Find the distance between the points $(-4, 7)$ and $(10, 6)$.

**10.4** *Perform the indicated operations. Assume that all variables represent positive real numbers.*

**74.** $2\sqrt{8} - 3\sqrt{50}$

**75.** $8\sqrt{80} - 3\sqrt{45}$

**76.** $-\sqrt{27y} + 2\sqrt{75y}$

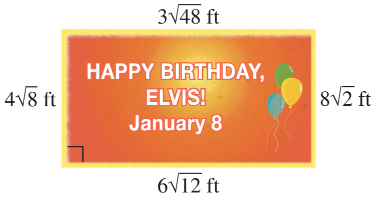
**77.** $2\sqrt{54m^3} + 5\sqrt{96m^3}$

**78.** $3\sqrt[3]{54} + 5\sqrt[3]{16}$

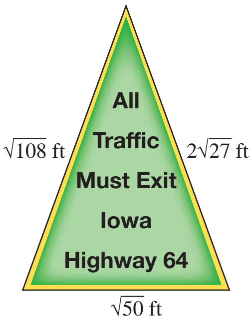
**79.** $-6\sqrt[4]{32} + \sqrt[4]{512}$

*In Exercises 80 and 81, leave answers as simplified radicals.*

**80.** Find the perimeter of a rectangular electronic billboard having sides of lengths shown in the figure.



**81.** Find the perimeter of a triangular electronic highway road sign having the dimensions shown in the figure.



**10.5** *Multiply.*

**82.** $\left(\sqrt{3}+1\right)\left(\sqrt{3}-2\right)$

**83.** $\left(\sqrt{7}+\sqrt{5}\right)\left(\sqrt{7}-\sqrt{5}\right)$

**84.** $\left(3\sqrt{2}+1\right)\left(2\sqrt{2}-3\right)$

**85.** $\left(\sqrt{13}-\sqrt{2}\right)^2$

**86.** $\left(\sqrt[3]{2}+3\right)\left(\sqrt[3]{4}-3\sqrt[3]{2}+9\right)$

**87.** $\left(\sqrt[3]{4y}-1\right)\left(\sqrt[3]{4y}+3\right)$

**88.** Use a calculator to show that the answer to **Exercise 85,** $15-2\sqrt{26}$, is not equal to $13\sqrt{26}$.

**89.** ***Concept Check*** A friend tried to rationalize the denominator of $\frac{5}{\sqrt[3]{6}}$, by multiplying the numerator and denominator by $\sqrt[3]{6}$. ***WHAT WENT WRONG?***

*Rationalize each denominator. Assume that all variables represent positive real numbers.*

**90.** $\frac{\sqrt{6}}{\sqrt{5}}$

**91.** $\frac{-6\sqrt{3}}{\sqrt{2}}$

**92.** $\frac{3\sqrt{7p}}{\sqrt{y}}$

**93.** $\sqrt{\frac{11}{8}}$

**94.** $-\sqrt[3]{\frac{9}{25}}$

**95.** $\sqrt[3]{\frac{108m^3}{n^5}}$

**96.** $\frac{1}{\sqrt{2}+\sqrt{7}}$

**97.** $\frac{-5}{\sqrt{6}-3}$

*Write in lowest terms.*

**98.** $\frac{2-2\sqrt{5}}{8}$

**99.** $\frac{4-8\sqrt{8}}{12}$

**100.** $\frac{-18+\sqrt{27}}{6}$

**10.6** *Solve each equation.*

**101.** $\sqrt{8x+9}=5$

**102.** $\sqrt{2x-3}-3=0$

**103.** $\sqrt{3x+1}-2=-3$

**104.** $\sqrt{7x+1}=x+1$

**105.** $3\sqrt{x}=\sqrt{10x-9}$

**106.** $\sqrt{x^2+3x+7}=x+2$

**107.** $\sqrt{x+2}-\sqrt{x-3}=1$

**108.** $\sqrt[3]{5x-1}=\sqrt[3]{3x-2}$

**109.** $\sqrt[3]{2x^2+3x-7}=\sqrt[3]{2x^2+4x+6}$

**110.** $\sqrt[3]{3x^2-4x+6}=\sqrt[3]{3x^2-2x+8}$

**111.** $\sqrt[3]{1-2x}-\sqrt[3]{-x-13}=0$

**112.** $\sqrt[3]{11-2x}-\sqrt[3]{-1-5x}=0$

**113.** $\sqrt[4]{x-1}+2=0$

**114.** $\sqrt[4]{2x+3}+1=0$

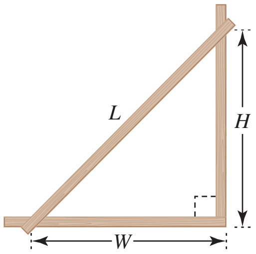
**115.** $\sqrt[4]{x+7}=\sqrt[4]{2x}$

**116.** $\sqrt[4]{x+8}=\sqrt[4]{3x}$

**117.** Carpenters stabilize wall frames with a diagonal brace, as shown in the figure. The length of the brace is given by $L=\sqrt{H^2+W^2}$.

**(a)** Solve this formula for $H$.

**(b)** If the bottom of the brace is attached 9 ft from the corner and the brace is 12 ft long, how far up the corner post should it be nailed? Give your answer to the nearest tenth of a foot.



**10.7** *Write each expression as a product of a real number and i.*

**118.** $\sqrt{-25}$

**119.** $\sqrt{-200}$

**120.** ***Concept Check*** If $a$ is a positive real number, is $-\sqrt{-a}$ a real number?

*Perform the indicated operations. Give answers in standard form.*

**121.** $(-2 + 5i) + (-8 - 7i)$ **122.** $(5 + 4i) - (-9 - 3i)$ **123.** $\sqrt{-5} \cdot \sqrt{-7}$

**124.** $\sqrt{-25} \cdot \sqrt{-81}$ **125.** $\dfrac{\sqrt{-72}}{\sqrt{-8}}$ **126.** $(2 + 3i)(1 - i)$

**127.** $(6 - 2i)^2$ **128.** $\dfrac{3 - i}{2 + i}$ **129.** $\dfrac{5 + 14i}{2 + 3i}$

*Find each power of i.*

**130.** $i^{11}$ **131.** $i^{36}$ **132.** $i^{-10}$ **133.** $i^{-8}$

## MIXED REVIEW EXERCISES

*Simplify. Assume that all variables represent positive real numbers.*

**134.** $-\sqrt[4]{256}$ **135.** $1000^{-2/3}$ **136.** $\dfrac{z^{-1/5} \cdot z^{3/10}}{z^{7/10}}$

**137.** $\sqrt[4]{k^{24}}$ **138.** $\sqrt[3]{54z^9t^8}$ **139.** $-5\sqrt{18} + 12\sqrt{72}$

**140.** $\dfrac{-1}{\sqrt{12}}$ **141.** $\sqrt[3]{\dfrac{12}{25}}$ **142.** $i^{-1000}$

**143.** $\sqrt{-49}$ **144.** $(4 - 9i) + (-1 + 2i)$ **145.** $\dfrac{\sqrt{50}}{\sqrt{-2}}$

**146.** $\dfrac{3 + \sqrt{54}}{6}$ **147.** $(3 + 2i)^2$ **148.** $8\sqrt[3]{x^3y^2} - 2x\sqrt[3]{y^2}$

**149.** $9\sqrt{5} - 4\sqrt{15}$ **150.** $\left(\sqrt{5} - \sqrt{3}\right)\left(\sqrt{7} + \sqrt{3}\right)$

*Solve each equation.*

**151.** $\sqrt{x + 4} = x - 2$ **152.** $\sqrt[3]{2x - 9} = \sqrt[3]{5x + 3}$

**153.** $\sqrt{6 + 2x} - 1 = \sqrt{7 - 2x}$ **154.** $\sqrt{7x + 11} - 5 = 0$

**155.** $\sqrt{6x + 2} - \sqrt{5x + 3} = 0$ **156.** $\sqrt{3 + 5x} - \sqrt{x + 11} = 0$

**157.** $3\sqrt{x} = \sqrt{8x + 9}$ **158.** $6\sqrt{x} = \sqrt{30x + 24}$

**159.** $\sqrt{11 + 2x} + 1 = \sqrt{5x + 1}$ **160.** $\sqrt{5x + 6} - \sqrt{x + 3} = 3$

# CHAPTER 10 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*Evaluate.*

**1.** $-\sqrt{841}$ **2.** $\sqrt[3]{-512}$ **3.** $125^{1/3}$

**4.** ***Concept Check*** For $\sqrt{146.25}$, which choice gives the best estimate?

**A.** 10 **B.** 11 **C.** 12 **D.** 13

*Use a calculator to approximate each root to the nearest thousandth.*

**5.** $\sqrt{478}$ **6.** $\sqrt[3]{-832}$

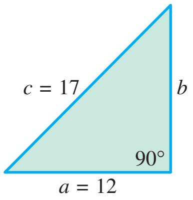
**7.** Graph the function defined by $f(x) = \sqrt{x + 6}$, and give the domain and range.

*Simplify each expression. Assume that all variables represent positive real numbers.*

**8.** $\left(\dfrac{16}{25}\right)^{-3/2}$ **9.** $(-64)^{-4/3}$ **10.** $\dfrac{3^{2/5}x^{-1/4}y^{2/5}}{3^{-8/5}x^{7/4}y^{1/10}}$

**11.** $\left(\dfrac{x^{-4}y^{-6}}{x^{-2}y^{3}}\right)^{-2/3}$ **12.** $7^{3/4} \cdot 7^{-1/4}$ **13.** $\sqrt[3]{a^4} \cdot \sqrt[3]{a^7}$

**14.** Use the Pythagorean theorem to find the exact length of side $b$ in the figure.



**15.** Find the distance between the points $(-4, 2)$ and $(2, 10)$.

*Simplify each expression. Assume that all variables represent positive real numbers.*

**16.** $\sqrt{54x^5y^6}$ **17.** $\sqrt[4]{32a^7b^{13}}$

**18.** $\sqrt{2} \cdot \sqrt[3]{5}$ (Express as a radical.) **19.** $3\sqrt{20} - 5\sqrt{80} + 4\sqrt{500}$

**20.** $\sqrt[3]{16t^3s^5} - \sqrt[3]{54t^6s^2}$ **21.** $(7\sqrt{5} + 4)(2\sqrt{5} - 1)$

**22.** $(\sqrt{3} - 2\sqrt{5})^2$ **23.** $\dfrac{-5}{\sqrt{40}}$

**24.** $\dfrac{2}{\sqrt[3]{5}}$ **25.** $\dfrac{-4}{\sqrt{7} + \sqrt{5}}$

**26.** Write $\dfrac{6 + \sqrt{24}}{2}$ in lowest terms.

**27.** The following formula is used in physics, relating the velocity $V$ of sound to the temperature $T$.

$$V = \frac{V_0}{\sqrt{1 - kT}}$$

**(a)** Find an approximation of $V$ to the nearest tenth if $V_0 = 50$, $k = 0.01$, and $T = 30$. Use a calculator.

**(b)** Solve the formula for $T$.

*Solve each equation.*

**28.** $\sqrt[3]{5x} = \sqrt[3]{2x - 3}$ **29.** $x + \sqrt{x + 6} = 9 - x$

**30.** $\sqrt{x + 4} - \sqrt{1 - x} = -1$

*In Exercises 31–33, perform the indicated operations. Give the answers in standard form.*

**31.** $(-2 + 5i) - (3 + 6i) - 7i$ **32.** $(1 + 5i)(3 + i)$

**33.** $\dfrac{7 + i}{1 - i}$ **34.** Simplify $i^{37}$.

**35.** ***Concept Check*** Answer *true* or *false* to each of the following.

**(a)** $i^2 = -1$ **(b)** $i = \sqrt{-1}$ **(c)** $i = -1$ **(d)** $\sqrt{-3} = i\sqrt{3}$

# CHAPTERS 1–10
# CUMULATIVE REVIEW EXERCISES

*Evaluate each expression for $a = -3$, $b = 5$, and $c = -4$.*

**1.** $|2a^2 - 3b + c|$

**2.** $\dfrac{(a + b)(a + c)}{3b - 6}$

*Solve each equation or inequality.*

**3.** $3(x + 2) - 4(2x + 3) = -3x + 2$

**4.** $\dfrac{1}{3}x + \dfrac{1}{4}(x + 8) = x + 7$

**5.** $0.04x + 0.06(100 - x) = 5.88$

**6.** $-5 - 3(x - 2) < 11 - 2(x + 2)$

*Solve each problem.*

**7.** A piggy bank has 100 coins, all of which are nickels and quarters. The total value of the money is \$17.80. How many of each denomination are there in the bank?

**8.** How many liters of pure alcohol must be mixed with 40 L of 18% alcohol to obtain a 22% alcohol solution?

**9.** Graph the equation $4x - 3y = 12$.

**10.** Find the slope of the line passing through the points $(-4, 6)$ and $(2, -3)$. Then find the equation of the line and write it in the form $y = mx + b$.

*Perform the indicated operations.*

**11.** $(3k^3 - 5k^2 + 8k - 2) - (4k^3 + 11k + 7) + (2k^2 - 5k)$

**12.** $(8x - 7)(x + 3)$

**13.** $\dfrac{6y^4 - 3y^3 + 5y^2 + 6y - 9}{2y + 1}$

*Factor each polynomial completely.*

**14.** $2p^2 - 5pq + 3q^2$

**15.** $3k^4 + k^2 - 4$

**16.** $x^3 + 512$

*Solve by factoring.*

**17.** $2x^2 + 11x + 15 = 0$

**18.** $5x(x - 1) = 2(1 - x)$

**19.** For what values of the variable is the rational expression $\dfrac{4}{x^2 - 9}$ undefined?

*Perform each operation and express the answer in lowest terms.*

**20.** $\dfrac{y^2 + y - 12}{y^3 + 9y^2 + 20y} \div \dfrac{y^2 - 9}{y^3 + 3y^2}$

**21.** $\dfrac{1}{x + y} + \dfrac{3}{x - y}$

*Simplify each complex fraction.*

**22.** $\dfrac{\dfrac{-6}{x - 2}}{\dfrac{8}{3x - 6}}$

**23.** $\dfrac{x^{-1}}{y - x^{-1}}$

**24.** Solve the equation $\dfrac{x + 1}{x - 3} = \dfrac{4}{x - 3} + 6$.

**25.** Danielle can ride her bike 4 mph faster than her husband, Richard. If Danielle can ride 48 mi in the same time that Richard can ride 24 mi, what are their speeds?

**26.** If $f(x) = 3x - 7$, find $f(-10)$.

**27.** The cost of a pizza varies directly as the square of its radius. If a pizza with a 7-in. radius costs \$6.00, how much should a pizza with a 9-in. radius cost, to the nearest cent?

*Solve.*

**28.** $3x - y = 23$
$2x + 3y = 8$

**29.** $x + y + z = 1$
$x - y - z = -3$
$x + y - z = -1$

**30.** In 2010, if you had sent five 2-oz letters and three 3-oz letters by first-class mail, it would have cost you \$5.39. Sending three 2-oz letters and five 3-oz letters would have cost \$5.73. What was the 2010 postage rate for one 2-oz letter and for one 3-oz letter? (*Source:* U.S. Postal Service.)

*Solve each equation or inequality.*

**31.** $2x + 4 < 10 \quad \text{and} \quad 3x - 1 > 5$

**32.** $2x + 4 > 10 \quad \text{or} \quad 3x - 1 < 5$

**33.** $|6x + 7| = 13$

**34.** $|2p - 5| \geq 9$

*Write each expression in simplest form, using only positive exponents. Assume that all variables represent positive real numbers.*

**35.** $\sqrt[3]{16x^2y} \cdot \sqrt[3]{3x^3y}$

**36.** $\sqrt{50} + \sqrt{8}$

**37.** $\dfrac{1}{\sqrt{10} - \sqrt{8}}$

**38.** Find the distance between the points $(-4, 4)$ and $(-2, 9)$.

**39.** Solve the equation $\sqrt{3x - 8} = x - 2$.

**40.** Express $\frac{6 - 2i}{1 - i}$ in standard form.

# CHAPTER 11

# Quadratic Equations, Inequalities, and Functions

The prices of food, gasoline, and other products have increased throughout the world. In particular, escalating oil prices in recent years have caused increases in transportation and shipping costs, which trickled down to affect prices of a variety of goods and services.

Although prices tend to go up over time, the rate at which they increase (the inflation rate) varies considerably. The Consumer Price Index (CPI) used by the U.S. government measures changes in prices for goods purchased by typical American families over time. In **Example 6** of **Section 11.5,** we use a *quadratic function* to model the CPI.

# 11.1 Solving Quadratic Equations by the Square Root Property

**OBJECTIVES**

1. Review the zero-factor property.
2. Solve equations of the form $x^2 = k$, where $k > 0$.
3. Solve equations of the form $(ax + b)^2 = k$, where $k > 0$.
4. Solve quadratic equations with solutions that are not real numbers.

Recall from **Section 5.5** that a *quadratic equation* is defined as follows.

**Quadratic Equation**

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where $a$, $b$, and $c$ are real numbers, with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

A quadratic equation is a *second-degree equation,* that is, an equation with a squared variable term and no terms of greater degree.

$$4x^2 + 4x - 5 = 0 \quad \text{and} \quad 3x^2 = 4x - 8$$

Quadratic equations (The first equation is in standard form.)

**OBJECTIVE 1** **Review the zero-factor property.** In **Section 5.5** we used factoring and the zero-factor property to solve quadratic equations.

**Zero-Factor Property**

If two numbers have a product of 0, then at least one of the numbers must be 0. **That is, if $ab = 0$, then $a = 0$ or $b = 0$.**

*NOW TRY EXERCISE 1*

Solve each equation by the zero-factor property.

**(a)** $x^2 - x - 20 = 0$

**(b)** $x^2 = 36$

**EXAMPLE 1** Solving Quadratic Equations by the Zero-Factor Property

Solve each equation by the zero-factor property.

**(a)**

$$\begin{aligned} x^2 + 4x + 3 &= 0 && \\ (x + 3)(x + 1) &= 0 && \text{Factor.} \\ x + 3 = 0 \quad \text{or} \quad x + 1 &= 0 && \text{Zero-factor property} \\ x = -3 \quad \text{or} \quad x &= -1 && \text{Solve each equation.} \end{aligned}$$

The solution set is $\{-3, -1\}$.

**(b)**

$$\begin{aligned} x^2 &= 9 && \\ x^2 - 9 &= 0 && \text{Subtract 9.} \\ (x + 3)(x - 3) &= 0 && \text{Factor.} \\ x + 3 = 0 \quad \text{or} \quad x - 3 &= 0 && \text{Zero-factor property} \\ x = -3 \quad \text{or} \quad x &= 3 && \text{Solve each equation.} \end{aligned}$$

The solution set is $\{-3, 3\}$. *NOW TRY*

**OBJECTIVE 2** **Solve equations of the form $x^2 = k$, where $k > 0$.** In **Example 1(b),** we might also have solved $x^2 = 9$ by noticing that $x$ must be a number whose square is 9. Thus, $x = \sqrt{9} = 3$ or $x = -\sqrt{9} = -3$. This is generalized as the **square root property.**

*NOW TRY ANSWERS*

1. **(a)** $\{-4, 5\}$ **(b)** $\{-6, 6\}$

**Square Root Property**

If $k$ is a positive number and if $x^2 = k$, then

$$\boldsymbol{x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.}$$

The solution set is $\left\{-\sqrt{k}, \sqrt{k}\right\}$, which can be written $\left\{\pm\sqrt{k}\right\}$. (± is read "positive or negative" or "plus or minus.")

**NOTE** When we solve an equation, we must find *all* values of the variable that satisfy the equation. Therefore, we want both the positive and negative square roots of $k$.

**NOW TRY EXERCISE 2**

Solve each equation. Write radicals in simplified form.

**(a)** $t^2 = 25$

**(b)** $x^2 = 13$

**(c)** $3x^2 - 54 = 0$

**(d)** $2x^2 - 5 = 35$

### EXAMPLE 2 Solving Quadratic Equations of the Form $x^2 = k$

Solve each equation. Write radicals in simplified form.

**(a)** $x^2 = 16$

By the square root property, if $x^2 = 16$, then

$$x = \sqrt{16} = 4 \quad \text{or} \quad x = -\sqrt{16} = -4.$$

*Check* each solution by substituting it for $x$ in the original equation. The solution set is

$$\{-4, 4\}, \quad \text{or} \quad \{\pm 4\}.$$

This notation indicates *two* solutions, one positive and one negative.

**(b)** $x^2 = 5$

By the square root property, if $x^2 = 5$, then

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5}.$$

Don't forget the negative solution.

The solution set is $\left\{\sqrt{5}, -\sqrt{5}\right\}$, or $\left\{\pm\sqrt{5}\right\}$.

**(c)**

$$\begin{aligned} 4x^2 - 48 &= 0 && \\ 4x^2 &= 48 && \text{Add 48.} \\ x^2 &= 12 && \text{Divide by 4.} \\ x = \sqrt{12} \quad \text{or} \quad x &= -\sqrt{12} && \text{Square root property} \\ x = 2\sqrt{3} \quad \text{or} \quad x &= -2\sqrt{3} && \sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3} \end{aligned}$$

Don't stop here. Simplify the radicals.

The solutions are $2\sqrt{3}$ and $-2\sqrt{3}$. Check each in the original equation.

**CHECK** $4x^2 - 48 = 0$ Original equation

$$\begin{aligned} 4\left(2\sqrt{3}\right)^2 - 48 &\overset{?}{=} 0 && \text{Let } x = 2\sqrt{3}. \\ 4(12) - 48 &\overset{?}{=} 0 && \\ 48 - 48 &\overset{?}{=} 0 && \\ 0 &= 0 \ \checkmark && \text{True} \end{aligned}$$

$$\left(2\sqrt{3}\right)^2 = 2^2 \cdot \left(\sqrt{3}\right)^2$$

$$\begin{aligned} 4\left(-2\sqrt{3}\right)^2 - 48 &\overset{?}{=} 0 && \text{Let } x = -2\sqrt{3}. \\ 4(12) - 48 &\overset{?}{=} 0 && \\ 48 - 48 &\overset{?}{=} 0 && \\ 0 &= 0 \ \checkmark && \text{True} \end{aligned}$$

The solution set is $\left\{2\sqrt{3}, -2\sqrt{3}\right\}$, or $\left\{\pm 2\sqrt{3}\right\}$.

**(d)**

$$\begin{aligned} 3x^2 + 5 &= 11 && \\ 3x^2 &= 6 && \text{Subtract 5.} \\ x^2 &= 2 && \text{Divide by 3.} \\ x = \sqrt{2} \quad \text{or} \quad x &= -\sqrt{2} && \text{Square root property} \end{aligned}$$

The solution set is $\left\{\sqrt{2}, -\sqrt{2}\right\}$, or $\left\{\pm\sqrt{2}\right\}$.

NOW TRY

**NOW TRY ANSWERS**

**2.** **(a)** $\{\pm 5\}$ **(b)** $\left\{\pm\sqrt{13}\right\}$

**(c)** $\left\{\pm 3\sqrt{2}\right\}$ **(d)** $\left\{\pm 2\sqrt{5}\right\}$

**NOW TRY EXERCISE 3**

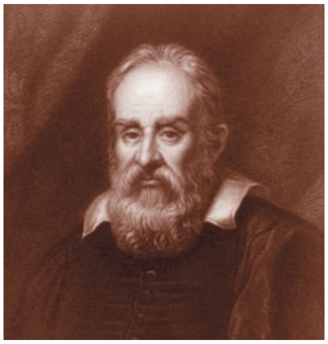
Tim is dropping roofing nails from the top of a roof 25 ft high into a large bucket on the ground. Use the formula in **Example 3** to determine how long it will take a nail dropped from 25 ft to hit the bottom of the bucket.

### EXAMPLE 3 Using the Square Root Property in an Application

Galileo Galilei developed a formula for freely falling objects described by

$$d = 16t^2,$$

where $d$ is the distance in feet that an object falls (disregarding air resistance) in $t$ seconds, regardless of weight. Galileo dropped objects from the Leaning Tower of Pisa. If the Leaning Tower is about 180 ft tall, use Galileo's formula to determine how long it would take an object dropped from the top of the tower to fall to the ground. (*Source:* www.brittanica.com)



**Galileo Galilei (1564–1642)**

$$\begin{aligned} d &= 16t^2 && \text{Galileo's formula} \\ 180 &= 16t^2 && \text{Let } d = 180. \\ 11.25 &= t^2 && \text{Divide by 16.} \end{aligned}$$

$$t = \sqrt{11.25} \quad \text{or} \quad t = -\sqrt{11.25} \qquad \text{Square root property}$$

Time cannot be negative, so we discard the negative solution. Since $\sqrt{11.25} \approx 3.4$, $t \approx 3.4$. The object would fall to the ground in about 3.4 sec. NOW TRY

## OBJECTIVE 3 Solve equations of the form $(ax + b)^2 = k$, where $k > 0$.

In each equation in **Example 2,** the exponent 2 had a single variable as its base. We can extend the square root property to solve equations in which the base is a binomial.

**NOW TRY EXERCISE 4**

Solve $(x - 2)^2 = 32$.

### EXAMPLE 4 Solving Quadratic Equations of the Form $(x + b)^2 = k$

Solve each equation.

**(a)** Use $(x - 3)$ as the base.

$$(x - 3)^2 = 16$$

$$\begin{aligned} x - 3 &= \sqrt{16} \quad \text{or} \quad x - 3 = -\sqrt{16} && \text{Square root property} \\ x - 3 &= 4 \quad \text{or} \quad x - 3 = -4 && \sqrt{16} = 4 \\ x &= 7 \quad \text{or} \quad x = -1 && \text{Add 3.} \end{aligned}$$

*CHECK* Substitute each solution in the original equation.

$$\begin{aligned} (x - 3)^2 &= 16 \\ (7 - 3)^2 &\overset{?}{=} 16 && \text{Let } x = 7. \\ 4^2 &\overset{?}{=} 16 && \text{Subtract.} \\ 16 &= 16 \ \checkmark && \text{True} \end{aligned}$$

$$\begin{aligned} (x - 3)^2 &= 16 \\ (-1 - 3)^2 &\overset{?}{=} 16 && \text{Let } x = -1. \\ (-4)^2 &\overset{?}{=} 16 && \text{Subtract.} \\ 16 &= 16 \ \checkmark && \text{True} \end{aligned}$$

The solution set is $\{-1, 7\}$.

**(b)**

$$(x - 1)^2 = 6$$

$$\begin{aligned} x - 1 &= \sqrt{6} \quad \text{or} \quad x - 1 = -\sqrt{6} && \text{Square root property} \\ x &= 1 + \sqrt{6} \quad \text{or} \quad x = 1 - \sqrt{6} && \text{Add 1.} \end{aligned}$$

*CHECK*

$$\begin{aligned} &\left(1 + \sqrt{6} - 1\right)^2 = \left(\sqrt{6}\right)^2 = 6 \ \checkmark && \text{Let } x = 1 + \sqrt{6}. \\ &\left(1 - \sqrt{6} - 1\right)^2 = \left(-\sqrt{6}\right)^2 = 6 \ \checkmark && \text{Let } x = 1 - \sqrt{6}. \end{aligned}$$

The solution set is $\left\{1 + \sqrt{6}, 1 - \sqrt{6}\right\}$, or $\left\{1 \pm \sqrt{6}\right\}$. NOW TRY

**NOW TRY ANSWERS**

**3.** 1.25 sec **4.** $\left\{2 \pm 4\sqrt{2}\right\}$

**NOW TRY EXERCISE 5**

Solve $(2t - 4)^2 = 50$.

### EXAMPLE 5 Solving a Quadratic Equation of the Form $(ax + b)^2 = k$

Solve $(3r - 2)^2 = 27$.

$$(3r - 2)^2 = 27$$

$$\begin{array}{rclcrcll}
3r - 2 &=& \sqrt{27} & \text{or} & 3r - 2 &=& -\sqrt{27} & \text{Square root property} \\
3r - 2 &=& 3\sqrt{3} & \text{or} & 3r - 2 &=& -3\sqrt{3} & \sqrt{27} = \sqrt{9} \cdot \sqrt{3} = 3\sqrt{3} \\
3r &=& 2 + 3\sqrt{3} & \text{or} & 3r &=& 2 - 3\sqrt{3} & \text{Add 2.} \\
r &=& \dfrac{2 + 3\sqrt{3}}{3} & \text{or} & r &=& \dfrac{2 - 3\sqrt{3}}{3} & \text{Divide by 3.}
\end{array}$$

**CHECK**

$$\begin{array}{rcll}
\left(3 \cdot \dfrac{2 + 3\sqrt{3}}{3} - 2\right)^2 &\stackrel{?}{=}& 27 & \text{Let } r = \frac{2 + 3\sqrt{3}}{3}. \\
\left(2 + 3\sqrt{3} - 2\right)^2 &\stackrel{?}{=}& 27 & \text{Multiply.} \\
\left(3\sqrt{3}\right)^2 &\stackrel{?}{=}& 27 & \text{Subtract.} \\
27 &=& 27 \;\checkmark & \text{True}
\end{array}$$

$(ab)^2 = a^2b^2$

The check of the other solution is similar. The solution set is $\left\{\frac{2 \pm 3\sqrt{3}}{3}\right\}$.

NOW TRY

**CAUTION** The solutions in **Example 5** are fractions that cannot be simplified, since 3 is *not* a common factor in the numerator.

**OBJECTIVE 4 Solve quadratic equations with solutions that are not real numbers.** In $x^2 = k$, if $k < 0$, there will be two nonreal complex solutions.

**NOW TRY EXERCISE 6**

Solve each equation.

**(a)** $t^2 = -24$

**(b)** $(x + 4)^2 = -36$

### EXAMPLE 6 Solving for Nonreal Complex Solutions

Solve each equation.

**(a)**

$$x^2 = -15$$

$$\begin{array}{rclcrcll}
x &=& \sqrt{-15} & \text{or} & x &=& -\sqrt{-15} & \text{Square root property} \\
x &=& i\sqrt{15} & \text{or} & x &=& -i\sqrt{15} & \sqrt{-1} = i \quad \text{(See Section 10.7.)}
\end{array}$$

The solution set is $\{i\sqrt{15}, -i\sqrt{15}\}$, or $\{\pm i\sqrt{15}\}$.

**(b)**

$$(x + 2)^2 = -16$$

$$\begin{array}{rclcrcll}
x + 2 &=& \sqrt{-16} & \text{or} & x + 2 &=& -\sqrt{-16} & \text{Square root property} \\
x + 2 &=& 4i & \text{or} & x + 2 &=& -4i & \sqrt{-16} = 4i \\
x &=& -2 + 4i & \text{or} & x &=& -2 - 4i & \text{Add } -2.
\end{array}$$

The solution set is $\{-2 + 4i, -2 - 4i\}$, or $\{-2 \pm 4i\}$.

NOW TRY

**NOW TRY ANSWERS**

5. $\left\{\frac{4 \pm 5\sqrt{2}}{2}\right\}$
6. **(a)** $\{\pm 2i\sqrt{6}\}$
   **(b)** $\{-4 \pm 6i\}$

## 11.1 EXERCISES

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◉ *Complete solution available on the Video Resources on DVD*

**1.** *Concept Check* Which of the following are quadratic equations?

**A.** $x + 2y = 0$ **B.** $x^2 - 8x + 16 = 0$ **C.** $2t^2 - 5t = 3$ **D.** $x^3 + x^2 + 4 = 0$

**2.** *Concept Check* Which quadratic equation identified in **Exercise 1** is in standard form?

**3.** *Concept Check* A student incorrectly solved the equation $x^2 - x - 2 = 5$ as follows. **WHAT WENT WRONG?**

$$\begin{aligned} x^2 - x - 2 &= 5 & & \\ (x-2)(x+1) &= 5 & & \text{Factor.} \\ x - 2 = 5 \quad \text{or} \quad x + 1 &= 5 & & \text{Zero-factor property} \\ x = 7 \quad \text{or} \quad x &= 4 & & \text{Solve each equation.} \end{aligned}$$

**4.** *Concept Check* A student was asked to solve the quadratic equation $x^2 = 16$ and did not get full credit for the solution set $\{4\}$. **WHAT WENT WRONG?**

*Solve each equation by the zero-factor property.* ***See Example 1.***

**5.** $x^2 - x - 56 = 0$ **6.** $x^2 - 2x - 99 = 0$ **7.** $x^2 = 121$

**8.** $x^2 = 144$ **9.** $3x^2 - 13x = 30$ **10.** $5x^2 - 14x = 3$

*Solve each equation by using the square root property. Simplify all radicals.* ***See Example 2.***

◉ **11.** $x^2 = 81$ **12.** $z^2 = 169$ **13.** $x^2 = 14$

**14.** $m^2 = 22$ **15.** $t^2 = 48$ **16.** $x^2 = 54$

**17.** $x^2 = \frac{25}{4}$ **18.** $m^2 = \frac{36}{121}$ **19.** $x^2 = 2.25$

**20.** $w^2 = 56.25$ **21.** $r^2 - 3 = 0$ **22.** $x^2 - 13 = 0$

**23.** $x^2 - 20 = 0$ **24.** $p^2 - 50 = 0$ **25.** $7x^2 = 4$

**26.** $3p^2 = 10$ **27.** $3n^2 - 72 = 0$ **28.** $5z^2 - 200 = 0$

**29.** $5x^2 + 4 = 8$ **30.** $4p^2 - 3 = 7$ **31.** $2t^2 + 7 = 61$

**32.** $3x^2 + 8 = 80$ **33.** $-8x^2 = -64$ **34.** $-12x^2 = -144$

*Solve each equation by using the square root property. Simplify all radicals.* ***See Examples 4 and 5.***

◉ **35.** $(x - 3)^2 = 25$ **36.** $(x - 7)^2 = 16$ **37.** $(x - 4)^2 = 3$

**38.** $(x + 3)^2 = 11$ **39.** $(x - 8)^2 = 27$ **40.** $(p - 5)^2 = 40$

**41.** $(3x + 2)^2 = 49$ **42.** $(5t + 3)^2 = 36$ **43.** $(4x - 3)^2 = 9$

**44.** $(7z - 5)^2 = 25$ **45.** $(3x - 1)^2 = 7$ **46.** $(2x - 5)^2 = 10$

◉ **47.** $(3k + 1)^2 = 18$ **48.** $(5z + 6)^2 = 75$ **49.** $(5 - 2x)^2 = 30$

**50.** $(3 - 2x)^2 = 70$ **51.** $\left(\frac{1}{2}x + 5\right)^2 = 12$ **52.** $\left(\frac{1}{3}m + 4\right)^2 = 27$

**53.** $(4x - 1)^2 - 48 = 0$ **54.** $(2x - 5)^2 - 180 = 0$

*Use a calculator with a square root key to solve each equation. Round your answers to the nearest hundredth.*

**55.** $(k + 2.14)^2 = 5.46$ **56.** $(r - 3.91)^2 = 9.28$

**57.** $(2.11p + 3.42)^2 = 9.58$ **58.** $(1.71m - 6.20)^2 = 5.41$

*Find the nonreal complex solutions of each equation.* ***See Example 6.***

**59.** $x^2 = -12$
**60.** $x^2 = -18$
**61.** $(r - 5)^2 = -4$
**62.** $(t + 6)^2 = -9$
**63.** $(6x - 1)^2 = -8$
**64.** $(4m - 7)^2 = -27$

*In Exercises 65 and 66, round answers to the nearest tenth.* ***See Example 3.***

**65.** The sculpture of American presidents at Mount Rushmore National Memorial is 500 ft above the valley floor. How long would it take a rock dropped from the top of the sculpture to fall to the ground? (*Source:* www.travelsd.com)

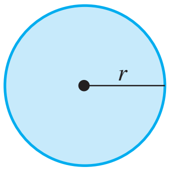
**66.** The Gateway Arch in St. Louis, Missouri, is 630 ft tall. How long would it take an object dropped from the top of the arch to fall to the ground? (*Source:* www.gatewayarch.com)

*Solve each problem.* ***See Example 3.***

**67.** The area $\mathcal{A}$ of a circle with radius $r$ is given by the formula

$$\mathcal{A} = \pi r^2.$$

If a circle has area $81\pi$ in.$^2$, what is its radius?

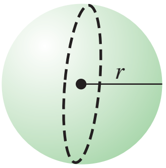


$\mathcal{A} = \pi r^2$

**68.** The surface area $S$ of a sphere with radius $r$ is given by the formula

$$S = 4\pi r^2.$$

If a sphere has surface area $36\pi$ ft$^2$, what is its radius?



$S = 4\pi r^2$

*The amount $A$ that $P$ dollars invested at an annual rate of interest $r$ will grow to in 2 yr is*

$$A = P(1 + r)^2.$$

**69.** At what interest rate will \$100 grow to \$104.04 in 2 yr?

**70.** At what interest rate will \$500 grow to \$530.45 in 2 yr?

## PREVIEW EXERCISES

*Simplify all radicals, and combine like terms. Express fractions in lowest terms.* ***See Sections 10.3–10.5.***

**71.** $\frac{4}{5} + \sqrt{\frac{48}{25}}$
**72.** $\frac{12 - \sqrt{27}}{9}$
**73.** $\frac{6 + \sqrt{24}}{8}$

*Factor each perfect square trinomial.* ***See Section 5.4.***

**74.** $z^2 + 4z + 4$
**75.** $x^2 - 10x + 25$
**76.** $z^2 + z + \frac{1}{4}$

# 11.2

# Solving Quadratic Equations by Completing the Square

**OBJECTIVES**

1. Solve quadratic equations by completing the square when the coefficient of the second-degree term is 1.
2. Solve quadratic equations by completing the square when the coefficient of the second-degree term is not 1.
3. Simplify the terms of an equation before solving.

**OBJECTIVE 1 Solve quadratic equations by completing the square when the coefficient of the second-degree term is 1.** The methods we have studied so far are not enough to solve an equation such as

$$x^2 + 6x + 7 = 0.$$

If we could write the equation in the form $(x + 3)^2$ equals a constant, we could solve it with the square root property discussed in **Section 11.1.** To do that, we need to have a perfect square trinomial on one side of the equation.

Recall from **Section 5.4** that the perfect square trinomial

$$x^2 + 6x + 9 \quad \text{can be factored as} \quad (x + 3)^2.$$

If we take half of 6, the coefficient of $x$ (the first-degree term), and square it, we get the constant term, 9.

$$\left[\frac{1}{2}\Big(\overset{\substack{\text{Coefficient of } x\\ \downarrow}}{6}\Big)\right]^2 = 3^2 = \overset{\substack{\text{Constant}\\ \downarrow}}{9}$$

Similarly, in $x^2 + 12x + 36$, $\left[\frac{1}{2}(12)\right]^2 = 6^2 = 36$,

and in $m^2 - 6m + 9$, $\left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9$.

This relationship is true in general and is the idea behind writing a quadratic equation so that the square root property can be applied.

## EXAMPLE 1 Rewriting an Equation to Use the Square Root Property

Solve $x^2 + 6x + 7 = 0$.

This quadratic equation cannot be solved by factoring, and it is not in the correct form to solve using the square root property. To obtain this form, we need a perfect square trinomial on the left side of the equation.

$$\begin{aligned} x^2 + 6x + 7 &= 0 && \text{Original equation} \\ x^2 + 6x &= -7 && \text{Subtract 7.} \end{aligned}$$

We must add a constant to get a perfect square trinomial on the left.

$$\underbrace{x^2 + 6x + \underline{\ \ ?\ \ }}_{\substack{\text{Needs to be a perfect}\\ \text{square trinomial}}}$$

As above, take half the coefficient of the first-degree term, $6x$, and square the result.

$$\left[\frac{1}{2}(6)\right]^2 = 3^2 = 9 \longleftarrow \text{Desired constant}$$

If we add 9 to each side of $x^2 + 6x = -7$, the equation will have a perfect square trinomial on the left side, as needed.

$$\begin{aligned} x^2 + 6x &= -7 && \\ \text{This is a key step.} \longrightarrow \quad x^2 + 6x + 9 &= -7 + 9 && \text{Add 9.} \\ (x + 3)^2 &= 2 && \text{Factor. Add.} \end{aligned}$$

**NOW TRY EXERCISE 1**
Solve $x^2 + 10x + 8 = 0$.

Now use the square root property to complete the solution.

$$\begin{aligned} x + 3 &= \sqrt{2} &\quad \text{or} \quad x + 3 &= -\sqrt{2} \\ x &= -3 + \sqrt{2} &\quad \text{or} \quad x &= -3 - \sqrt{2} \end{aligned}$$

*Check* by substituting $-3 + \sqrt{2}$ and $-3 - \sqrt{2}$ for $x$ in the original equation. The solution set is $\{-3 \pm \sqrt{2}\}$. NOW TRY

The process of changing the form of the equation in **Example 1** from

$$x^2 + 6x + 7 = 0 \qquad \text{to} \qquad (x + 3)^2 = 2$$

is called **completing the square.** Completing the square changes only the form of the equation. To see this, multiply out the left side of $(x + 3)^2 = 2$ and combine like terms. Then subtract 2 from each side to see that the result is $x^2 + 6x + 7 = 0$.

**NOW TRY EXERCISE 2**
Solve $x^2 - 6x = 9$.

## EXAMPLE 2 Completing the Square to Solve a Quadratic Equation

Solve $x^2 - 8x = 5$.

To complete the square on $x^2 - 8x$, take half the coefficient of $x$ and square it.

$$\frac{1}{2}(\underset{\uparrow}{-8}) = -4 \qquad \text{and} \qquad (-4)^2 = 16$$

Coefficient of $x$

Add the result, 16, to each side of the equation.

$$\begin{aligned} x^2 - 8x &= 5 && \text{Given equation} \\ x^2 - 8x + 16 &= 5 + 16 && \text{Add 16.} \\ (x - 4)^2 &= 21 && \text{Factor on the left. Add on the right.} \\ x - 4 = \sqrt{21} \quad &\text{or} \quad x - 4 = -\sqrt{21} && \text{Square root property} \\ x = 4 + \sqrt{21} \quad &\text{or} \quad x = 4 - \sqrt{21} && \text{Add 4.} \end{aligned}$$

A check indicates that the solution set is $\{4 \pm \sqrt{21}\}$. NOW TRY

### Completing the Square

To solve $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square, use these steps.

***Step 1*** **Be sure the second-degree (squared) term has coefficient 1.** If the coefficient of the second-degree term is 1, proceed to Step 2. If the coefficient of the second-degree term is not 1 but some other nonzero number $a$, divide each side of the equation by $a$.

***Step 2*** **Write the equation in correct form** so that terms with variables are on one side of the equals symbol and the constant is on the other side.

***Step 3*** **Square half the coefficient of the first-degree (linear) term.**

***Step 4*** **Add the square to each side.**

***Step 5*** **Factor the perfect square trinomial.** One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

***Step 6*** **Solve the equation.** Apply the square root property to complete the solution.

**NOW TRY ANSWERS**

1. $\{-5 \pm \sqrt{17}\}$
2. $\{3 \pm 3\sqrt{2}\}$

**NOW TRY EXERCISE 3**

Solve $x^2 + x - 3 = 0$.

**EXAMPLE 3** Solving a Quadratic Equation by Completing the Square $(a = 1)$

Solve $x^2 + 5x - 1 = 0$.

Since the coefficient of the squared term is 1, begin with Step 2.

***Step 2***

$$x^2 + 5x = 1 \qquad \text{Add 1 to each side.}$$

***Step 3*** Take half the coefficient of the first-degree term and square the result.

$$\left[\frac{1}{2}(5)\right]^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

***Step 4***

$$x^2 + 5x + \frac{25}{4} = 1 + \frac{25}{4}$$

Add the square to each side of the equation.

***Step 5***

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

Factor on the left.
Add on the right.

***Step 6***

$$x + \frac{5}{2} = \sqrt{\frac{29}{4}} \quad \text{or} \quad x + \frac{5}{2} = -\sqrt{\frac{29}{4}} \qquad \text{Square root property}$$

$$x + \frac{5}{2} = \frac{\sqrt{29}}{2} \quad \text{or} \quad x + \frac{5}{2} = -\frac{\sqrt{29}}{2} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$x = -\frac{5}{2} + \frac{\sqrt{29}}{2} \quad \text{or} \quad x = -\frac{5}{2} - \frac{\sqrt{29}}{2} \qquad \text{Add } -\tfrac{5}{2}.$$

$$x = \frac{-5 + \sqrt{29}}{2} \quad \text{or} \quad x = \frac{-5 - \sqrt{29}}{2} \qquad \frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

Check that the solution set is $\left\{\frac{-5 \pm \sqrt{29}}{2}\right\}$.

NOW TRY

**OBJECTIVE 2** **Solve quadratic equations by completing the square when the coefficient of the second-degree term is not 1.** If a quadratic equation has the form

$$ax^2 + bx + c = 0, \quad \text{where} \quad a \neq 1,$$

we obtain 1 as the coefficient of $x^2$ by dividing each side of the equation by $a$.

**EXAMPLE 4** Solving a Quadratic Equation by Completing the Square $(a \neq 1)$

Solve $4x^2 + 16x - 9 = 0$.

***Step 1*** ***Before completing the square, the coefficient of $x^2$ must be 1,*** not 4. We get 1 as the coefficient of $x^2$ here by dividing each side by 4.

$$4x^2 + 16x - 9 = 0 \qquad \text{Given equation}$$

$$x^2 + 4x - \frac{9}{4} = 0 \qquad \text{Divide by 4.}$$

The coefficient of $x^2$ must be 1.

***Step 2*** Write the equation so that all variable terms are on one side of the equation and all constant terms are on the other side.

$$x^2 + 4x = \frac{9}{4} \qquad \text{Add } \tfrac{9}{4}.$$

**NOW TRY ANSWER**

3. $\left\{\frac{-1 \pm \sqrt{13}}{2}\right\}$

**NOW TRY EXERCISE 4**

Solve $4t^2 - 4t - 3 = 0$.

***Step 3*** Complete the square by taking half the coefficient of $x$, and squaring it.

$$\frac{1}{2}(4) = 2 \qquad \text{and} \qquad 2^2 = 4$$

***Step 4*** We add the result, 4, to each side of the equation.

$$x^2 + 4x + 4 = \frac{9}{4} + 4 \qquad \text{Add 4.}$$

***Step 5***

$$(x + 2)^2 = \frac{25}{4} \qquad \text{Factor; } \tfrac{9}{4} + 4 = \tfrac{9}{4} + \tfrac{16}{4} = \tfrac{25}{4}.$$

***Step 6*** Solve the equation by using the square root property.

$$\begin{aligned}
x + 2 &= \sqrt{\frac{25}{4}} \quad \text{or} \quad x + 2 = -\sqrt{\frac{25}{4}} && \text{Square root property} \\
x + 2 &= \frac{5}{2} \quad \text{or} \quad x + 2 = -\frac{5}{2} && \text{Take square roots.} \\
x &= -2 + \frac{5}{2} \quad \text{or} \quad x = -2 - \frac{5}{2} && \text{Add } -2. \\
x &= \frac{1}{2} \quad \text{or} \quad x = -\frac{9}{2} && -2 = -\tfrac{4}{2}
\end{aligned}$$

CHECK

$$\begin{aligned}
4x^2 + 16x - 9 &= 0 \\
4\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) - 9 &\stackrel{?}{=} 0 \qquad \text{Let } x = \tfrac{1}{2}. \\
4\left(\frac{1}{4}\right) + 8 - 9 &\stackrel{?}{=} 0 \\
1 + 8 - 9 &\stackrel{?}{=} 0 \\
0 &= 0 \ \checkmark \quad \text{True}
\end{aligned}$$

$$\begin{aligned}
4x^2 + 16x - 9 &= 0 \\
4\left(-\frac{9}{2}\right)^2 + 16\left(-\frac{9}{2}\right) - 9 &\stackrel{?}{=} 0 \qquad \text{Let } x = -\tfrac{9}{2}. \\
4\left(\frac{81}{4}\right) - 72 - 9 &\stackrel{?}{=} 0 \\
81 - 72 - 9 &\stackrel{?}{=} 0 \\
0 &= 0 \ \checkmark \quad \text{True}
\end{aligned}$$

The two solutions, $\frac{1}{2}$ and $-\frac{9}{2}$, check, so the solution set is $\left\{-\frac{9}{2}, \frac{1}{2}\right\}$. NOW TRY

### EXAMPLE 5 Solving a Quadratic Equation by Completing the Square ($a \neq 1$)

Solve $2x^2 - 4x - 5 = 0$.

Divide each side by 2 to get 1 as the coefficient of the second-degree term.

$$\begin{aligned}
x^2 - 2x - \frac{5}{2} &= 0 && \text{Step 1} \\
x^2 - 2x &= \frac{5}{2} && \text{Step 2} \\
\left[\frac{1}{2}(-2)\right]^2 &= (-1)^2 = 1 && \text{Step 3} \\
x^2 - 2x + 1 &= \frac{5}{2} + 1 && \text{Step 4}
\end{aligned}$$

**NOW TRY ANSWER**

4. $\left\{-\frac{1}{2}, \frac{3}{2}\right\}$

NOW TRY EXERCISE 5

Solve $3x^2 + 12x - 5 = 0$.

$$(x-1)^2 = \frac{7}{2} \qquad \text{Step 5}$$

$$x - 1 = \sqrt{\frac{7}{2}} \quad \text{or} \quad x - 1 = -\sqrt{\frac{7}{2}} \qquad \text{Step 6}$$

$$x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}} \qquad \text{Add 1.}$$

$$x = 1 + \frac{\sqrt{14}}{2} \quad \text{or} \quad x = 1 - \frac{\sqrt{14}}{2} \qquad \sqrt{\frac{7}{2}} = \frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14}}{2}$$

Add the two terms in each solution as follows.

$$1 + \frac{\sqrt{14}}{2} = \frac{2}{2} + \frac{\sqrt{14}}{2} = \frac{2 + \sqrt{14}}{2} \qquad 1 = \frac{2}{2}$$

$$1 - \frac{\sqrt{14}}{2} = \frac{2}{2} - \frac{\sqrt{14}}{2} = \frac{2 - \sqrt{14}}{2}$$

Check that the solution set is $\left\{\frac{2 \pm \sqrt{14}}{2}\right\}$. NOW TRY

NOW TRY EXERCISE 6

Solve $x^2 + 8x + 21 = 0$.

## EXAMPLE 6 Solving a Quadratic Equation with Nonreal Complex Solutions

Solve $4p^2 + 8p + 5 = 0$.

$$4p^2 + 8p + 5 = 0$$

The coefficient of the second-degree term must be 1.

$$p^2 + 2p + \frac{5}{4} = 0 \qquad \text{Divide by 4.}$$

$$p^2 + 2p = -\frac{5}{4} \qquad \text{Add } -\tfrac{5}{4} \text{ to each side.}$$

The coefficient of $p$ is 2. Take half of 2, square the result, and add it to each side.

$$p^2 + 2p + 1 = -\frac{5}{4} + 1 \qquad \left[\tfrac{1}{2}(2)\right]^2 = 1^2 = 1;\ \text{Add 1.}$$

$$(p+1)^2 = -\frac{1}{4} \qquad \begin{array}{l}\text{Factor on the left.}\\ \text{Add on the right.}\end{array}$$

$$p + 1 = \sqrt{-\frac{1}{4}} \quad \text{or} \quad p + 1 = -\sqrt{-\frac{1}{4}} \qquad \text{Square root property}$$

$$p + 1 = \frac{1}{2}i \quad \text{or} \quad p + 1 = -\frac{1}{2}i \qquad \sqrt{-\tfrac{1}{4}} = \tfrac{1}{2}i$$

$$p = -1 + \frac{1}{2}i \quad \text{or} \quad p = -1 - \frac{1}{2}i \qquad \text{Add } -1.$$

The solution set is $\left\{-1 \pm \frac{1}{2}i\right\}$. NOW TRY

NOW TRY ANSWERS

**5.** $\left\{\frac{-6 \pm \sqrt{51}}{3}\right\}$

**6.** $\left\{-4 \pm i\sqrt{5}\right\}$

## OBJECTIVE 3 Simplify the terms of an equation before solving.

NOW TRY EXERCISE 7

Solve $(x - 5)(x + 1) = 2$.

### EXAMPLE 7 Simplifying the Terms of an Equation before Solving

Solve $(x + 3)(x - 1) = 2$.

$$\begin{aligned} (x + 3)(x - 1) &= 2 && \\ x^2 + 2x - 3 &= 2 && \text{Multiply by using the FOIL method.} \\ x^2 + 2x &= 5 && \text{Add 3.} \\ x^2 + 2x + 1 &= 5 + 1 && \text{Complete the square. Add } \left[\tfrac{1}{2}(2)\right]^2 = 1^2 = 1. \\ (x + 1)^2 &= 6 && \text{Factor on the left. Add on the right.} \end{aligned}$$

$$x + 1 = \sqrt{6} \quad \text{or} \quad x + 1 = -\sqrt{6} \qquad \text{Square root property}$$

$$x = -1 + \sqrt{6} \quad \text{or} \quad x = -1 - \sqrt{6} \qquad \text{Subtract 1.}$$

The solution set is $\{-1 \pm \sqrt{6}\}$.

NOW TRY

NOW TRY ANSWER

7. $\{2 \pm \sqrt{11}\}$

## 11.2 EXERCISES

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1. ***Concept Check*** Which one of the two equations

$$(2x + 1)^2 = 5 \quad \text{and} \quad x^2 + 4x = 12,$$

is more suitable for solving by the square root property? Which one is more suitable for solving by completing the square?

✎ 2. Why would most students find the equation $x^2 + 4x = 20$ easier to solve by completing the square than the equation $5x^2 + 2x = 3$?

***Concept Check*** *Decide what number must be added to make each expression a perfect square trinomial. Then factor the trinomial.*

3. $x^2 + 6x +$ _____
4. $x^2 + 14x +$ _____
5. $p^2 - 12p +$ _____
6. $x^2 - 20x +$ _____
7. $q^2 + 9q +$ _____
8. $t^2 + 13t +$ _____
9. $x^2 + \frac{1}{4}x +$ _____
10. $x^2 + \frac{1}{2}x +$ _____
11. $x^2 - 0.8x +$ _____

12. ***Concept Check*** What would be the first step in solving $2x^2 + 8x = 9$ by completing the square?

*Determine the number that will complete the square to solve each equation, after the constant term has been written on the right side and the coefficient of the second-degree term is 1. Do not actually solve.* ***See Examples 1–5.***

13. $x^2 + 4x - 2 = 0$
14. $t^2 + 2t - 1 = 0$
15. $x^2 + 10x + 18 = 0$
16. $x^2 + 8x + 11 = 0$
17. $3w^2 - w - 24 = 0$
18. $4z^2 - z - 39 = 0$

*Solve each equation by completing the square. Use the results of* ***Exercises 13–16*** *to solve Exercises 23–26.* ***See Examples 1–3.***

◉ 19. $x^2 - 4x = -3$
20. $p^2 - 2p = 8$
◉ 21. $x^2 + 2x - 5 = 0$
22. $r^2 + 4r + 1 = 0$
◉ 23. $x^2 + 4x - 2 = 0$
24. $t^2 + 2t - 1 = 0$

**25.** $x^2 + 10x + 18 = 0$
**26.** $x^2 + 8x + 11 = 0$
**27.** $x^2 - 8x = -4$
**28.** $m^2 - 4m = 14$
**29.** $x^2 + 7x - 1 = 0$
**30.** $x^2 + 13x - 3 = 0$

*Solve each equation by completing the square. Use the results of* ***Exercises 17 and 18*** *to solve Exercises 33 and 34.* ***See Examples 4, 5, and 7.***

**31.** $4x^2 + 4x = 3$
**32.** $9x^2 + 3x = 2$
**33.** $3w^2 - w = 24$
**34.** $4z^2 - z = 39$
**35.** $2k^2 + 5k - 2 = 0$
**36.** $3r^2 + 2r - 2 = 0$
**37.** $5x^2 - 10x + 2 = 0$
**38.** $2x^2 - 16x + 25 = 0$
**39.** $9x^2 - 24x = -13$
**40.** $25n^2 - 20n = 1$
**41.** $(x + 3)(x - 1) = 5$
**42.** $(x - 8)(x + 2) = 24$
**43.** $(r - 3)(r - 5) = 2$
**44.** $(x - 1)(x - 7) = 1$
**45.** $-x^2 + 2x = -5$
**46.** $-x^2 + 4x = 1$
**47.** $z^2 - \frac{4}{3}z = -\frac{1}{9}$
**48.** $p^2 - \frac{8}{3}p = -1$
**49.** $0.1x^2 - 0.2x - 0.1 = 0$ (*Hint*: First clear the decimals.)
**50.** $0.1p^2 - 0.4p + 0.1 = 0$ (*Hint*: First clear the decimals.)

*Solve each equation by completing the square. Give* **(a)** *exact solutions and* **(b)** *solutions rounded to the nearest thousandth.*

**51.** $3r^2 - 2 = 6r + 3$
**52.** $4p + 3 = 2p^2 + 2p$
**53.** $(x + 1)(x + 3) = 2$
**54.** $(x - 3)(x + 1) = 1$

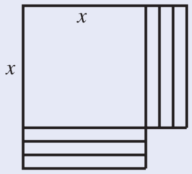
*Find the nonreal complex solutions of each equation.* ***See Example 6.***

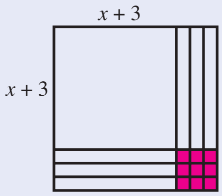
**55.** $m^2 + 4m + 13 = 0$
**56.** $t^2 + 6t + 10 = 0$
**57.** $3r^2 + 4r + 4 = 0$
**58.** $4x^2 + 5x + 5 = 0$
**59.** $-m^2 - 6m - 12 = 0$
**60.** $-x^2 - 5x - 10 = 0$

## *RELATING CONCEPTS* EXERCISES 61–66

### FOR INDIVIDUAL OR GROUP WORK

*The Greeks had a method of completing the square geometrically in which they literally changed a figure into a square. For example, to complete the square for $x^2 + 6x$, we begin with a square of side $x$, as in the figure on the left. We add three rectangles of width 1 to the right side and the bottom to get a region with area $x^2 + 6x$. To fill in the corner (complete the square), we must add nine 1-by-1 squares as shown.*





***Work Exercises 61–66 in order.***

**61.** What is the area of the original square?

**62.** What is the area of each strip?

**63.** What is the total area of the six strips?

**64.** What is the area of each small square in the corner of the second figure?

**65.** What is the total area of the small squares?

**66.** What is the area of the new “complete” square?

***Brain Busters*** *Solve for x. Assume that a and b represent positive real numbers.*

**67.** $x^2 - b = 0$

**68.** $x^2 = 4b$

**69.** $4x^2 = b^2 + 16$

**70.** $9x^2 - 25a = 0$

**71.** $(5x - 2b)^2 = 3a$

**72.** $x^2 - a^2 - 36 = 0$

## *PREVIEW EXERCISES*

*Evaluate* $\sqrt{b^2 - 4ac}$ *for the given values of a, b, and c.* ***See Sections 1.3 and 10.1.***

**73.** $a = 3, b = 1, c = -1$

**74.** $a = 4, b = 11, c = -3$

**75.** $a = 6, b = 7, c = 2$

**76.** $a = 1, b = -6, c = 9$

## 11.3

# Solving Quadratic Equations by the Quadratic Formula

**OBJECTIVES**

1. Derive the quadratic formula.
2. Solve quadratic equations by using the quadratic formula.
3. Use the discriminant to determine the number and type of solutions.

In this section, we complete the square to solve the general quadratic equation

$$ax^2 + bx + c = 0,$$

where $a$, $b$, and $c$ are complex numbers and $a \neq 0$. The solution of this general equation gives a formula for finding the solution of *any* specific quadratic equation.

**OBJECTIVE 1** **Derive the quadratic formula.** To solve $ax^2 + bx + c = 0$ by completing the square (assuming $a > 0$), we follow the steps given in **Section 11.2.**

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide by $a$. (Step 1)

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Subtract $\frac{c}{a}$. (Step 2)

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

(Step 3)

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Add $\frac{b^2}{4a^2}$ to each side. (Step 4)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a}$$

Write the left side as a perfect square. Rearrange the right side. (Step 5)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2}$$

Write with a common denominator.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Add fractions.

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Square root property (Step 6)

We can simplify $\sqrt{\frac{b^2 - 4ac}{4a^2}}$ as $\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$, or $\frac{\sqrt{b^2 - 4ac}}{2a}$.

The right side of each equation can be expressed as follows.

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

If $a < 0$, the same two solutions are obtained.

$$\boldsymbol{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \text{or} \quad \boldsymbol{x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

The result is the **quadratic formula,** which is abbreviated as follows.

### Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ (with $a \neq 0$) are given by

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

**CAUTION** In the quadratic formula, ***the square root is added to or subtracted from the value of −b before dividing by 2a.***

## OBJECTIVE 2 Solve quadratic equations by using the quadratic formula.

NOW TRY EXERCISE 1

Solve $2x^2 + 3x - 20 = 0$.

### EXAMPLE 1 Using the Quadratic Formula (Rational Solutions)

Solve $6x^2 - 5x - 4 = 0$.

This equation is in standard form, so we identify the values of $a$, $b$, and $c$. Here $a$, the coefficient of the second-degree term, is 6, and $b$, the coefficient of the first-degree term, is $-5$. The constant $c$ is $-4$. Now substitute into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)} \qquad a = 6, b = -5, c = -4$$

Use parentheses and substitute carefully to avoid errors.

$$x = \frac{5 \pm \sqrt{25 + 96}}{12}$$

$$x = \frac{5 \pm \sqrt{121}}{12} \qquad \text{Simplify the radical.}$$

$$x = \frac{5 \pm 11}{12} \qquad \text{Take the square root.}$$

There are two solutions, one from the + sign and one from the − sign.

$$x = \frac{5 + 11}{12} = \frac{16}{12} = \frac{4}{3} \quad \text{or} \quad x = \frac{5 - 11}{12} = \frac{-6}{12} = -\frac{1}{2}$$

Check each solution in the original equation. The solution set is $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$.

NOW TRY

NOW TRY ANSWER

1. $\left\{-4, \frac{5}{2}\right\}$

**NOTE** We could have used factoring to solve the equation in **Example 1.**

$$\begin{aligned} 6x^2 - 5x - 4 &= 0 && \\ (3x-4)(2x+1) &= 0 && \text{Factor.} \\ 3x - 4 = 0 \quad \text{or} \quad 2x + 1 &= 0 && \text{Zero-factor property} \\ 3x = 4 \quad \text{or} \quad 2x &= -1 && \text{Solve each equation.} \\ x = \frac{4}{3} \quad \text{or} \quad x &= -\frac{1}{2} && \text{Same solutions as in } \textbf{Example 1} \end{aligned}$$

When solving quadratic equations, it is a good idea to try factoring first. If the polynomial cannot be factored or if factoring is difficult, then use the quadratic formula.

NOW TRY EXERCISE 2

Solve $3x^2 + 1 = -5x$.

## EXAMPLE 2 Using the Quadratic Formula (Irrational Solutions)

Solve $4x^2 = 8x - 1$.

Write the equation in standard form as $4x^2 - 8x + 1 = 0$. This is a key step.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\ x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} && a = 4, b = -8, c = 1 \\ x &= \frac{8 \pm \sqrt{64 - 16}}{8} && \text{Simplify.} \\ x &= \frac{8 \pm \sqrt{48}}{8} && \\ x &= \frac{8 \pm 4\sqrt{3}}{8} && \sqrt{48} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3} \\ x &= \frac{4(2 \pm \sqrt{3})}{4(2)} && \text{Factor.} \\ x &= \frac{2 \pm \sqrt{3}}{2} && \text{Lowest terms} \end{aligned}$$

Factor first. Then divide out the common factor.

The solution set is $\left\{\frac{2 \pm \sqrt{3}}{2}\right\}$.

NOW TRY

**CAUTION**

1. ***Every quadratic equation must be expressed in standard form $ax^2 + bx + c = 0$ before we begin to solve it,*** whether we use factoring or the quadratic formula.
2. ***When writing solutions in lowest terms, be sure to FACTOR FIRST. Then divide out the common factor,*** as shown in the last two steps in **Example 2.**

NOW TRY ANSWER

**2.** $\left\{\frac{-5 \pm \sqrt{13}}{6}\right\}$

NOW TRY EXERCISE 3

Solve $(x + 5)(x - 1) = -18$.

### EXAMPLE 3 Using the Quadratic Formula (Nonreal Complex Solutions)

Solve $(9x + 3)(x - 1) = -8$.

$$\begin{aligned} (9x + 3)(x - 1) &= -8 & & \\ 9x^2 - 6x - 3 &= -8 & & \text{Multiply.} \\ \text{Standard form} \rightarrow 9x^2 - 6x + 5 &= 0 & & \text{Add 8.} \end{aligned}$$

From the equation $9x^2 - 6x + 5 = 0$, we identify $a = 9$, $b = -6$, and $c = 5$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} & & \text{Quadratic formula} \\ x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)} & & \text{Substitute.} \\ x &= \frac{6 \pm \sqrt{-144}}{18} & & \text{Simplify.} \\ x &= \frac{6 \pm 12i}{18} & & \sqrt{-144} = 12i \\ x &= \frac{6(1 \pm 2i)}{6(3)} & & \text{Factor.} \\ x &= \frac{1 \pm 2i}{3} & & \text{Lowest terms} \\ x &= \frac{1}{3} \pm \frac{2}{3}i & & \text{Standard form } a + bi \text{ for a complex number} \end{aligned}$$

The solution set is $\left\{\frac{1}{3} \pm \frac{2}{3}i\right\}$. NOW TRY

**OBJECTIVE 3 Use the discriminant to determine the number and type of solutions.** The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac \leftarrow \text{Discriminant}}}{2a}.$$

If $a$, $b$, and $c$ are integers, the type of solutions of a quadratic equation—that is, rational, irrational, or nonreal complex—is determined by the expression under the radical symbol, $b^2 - 4ac$, called the *discriminant* (because it distinguishes among the three types of solutions). By calculating the discriminant, we can predict the number and type of solutions of a quadratic equation.

### Discriminant

The **discriminant** of $ax^2 + bx + c = 0$ is $\boldsymbol{b^2 - 4ac}$**.** If $a$, $b$, and $c$ are integers, then the number and type of solutions are determined as follows.

| *Discriminant* | *Number and Type of Solutions* |
|---|---|
| Positive, and the square of an integer | Two rational solutions |
| Positive, but not the square of an integer | Two irrational solutions |
| Zero | One rational solution |
| Negative | Two nonreal complex solutions |

NOW TRY ANSWER

**3.** $\{-2 \pm 3i\}$

Calculating the discriminant can also help you decide how to solve a quadratic equation. ***If the discriminant is a perfect square (including 0), then the equation can be solved by factoring. Otherwise, the quadratic formula should be used.***

**NOW TRY EXERCISE 4**

Find each discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

**(a)** $8x^2 - 6x - 5 = 0$

**(b)** $9x^2 = 24x - 16$

**(c)** $3x^2 + 2x = -1$

## EXAMPLE 4 Using the Discriminant

Find the discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

**(a)** $6x^2 - x - 15 = 0$

We find the discriminant by evaluating $b^2 - 4ac$. Because $-x = -1x$, the value of $b$ in this equation is $-1$.

Use parentheses and substitute carefully.

$$\begin{aligned}
&b^2 - 4ac\\
&= (-1)^2 - 4(6)(-15) && a = 6, b = -1, c = -15\\
&= 1 + 360 && \text{Apply the exponent. Multiply.}\\
&= 361, \quad \text{or} \quad 19^2, \quad \text{which is a perfect square.}
\end{aligned}$$

Since $a$, $b$, and $c$ are integers and the discriminant 361 is a perfect square, there will be two rational solutions. The equation can be solved by factoring.

**(b)** $3x^2 - 4x = 5$ Write in standard form as $3x^2 - 4x - 5 = 0$.

$$\begin{aligned}
&b^2 - 4ac\\
&= (-4)^2 - 4(3)(-5) && a = 3, b = -4, c = -5\\
&= 16 + 60 && \text{Apply the exponent. Multiply.}\\
&= 76 && \text{Add.}
\end{aligned}$$

Because 76 is positive but *not* the square of an integer and $a$, $b$, and $c$ are integers, the equation will have two irrational solutions and is best solved using the quadratic formula.

**(c)** $4x^2 + x + 1 = 0$

$x = 1x$, so $b = 1$.

$$\begin{aligned}
&b^2 - 4ac\\
&= 1^2 - 4(4)(1) && a = 4, b = 1, c = 1\\
&= 1 - 16 && \text{Apply the exponent. Multiply.}\\
&= -15 && \text{Subtract.}
\end{aligned}$$

Because the discriminant is negative and $a$, $b$, and $c$ are integers, this equation will have two nonreal complex solutions. The quadratic formula should be used to solve it.

**(d)** $4x^2 + 9 = 12x$ Write in standard form as $4x^2 - 12x + 9 = 0$.

$$\begin{aligned}
&b^2 - 4ac\\
&= (-12)^2 - 4(4)(9) && a = 4, b = -12, c = 9\\
&= 144 - 144 && \text{Apply the exponent. Multiply.}\\
&= 0 && \text{Subtract.}
\end{aligned}$$

The discriminant is 0, so the quantity under the radical in the quadratic formula is 0, and there is only one rational solution. The equation can be solved by factoring.

NOW TRY

**NOW TRY ANSWERS**

**4.** **(a)** 196; two rational solutions; factoring
  **(b)** 0; one rational solution; factoring
  **(c)** −8; two nonreal complex solutions; quadratic formula

**NOW TRY EXERCISE 5**

Find $k$ so that the equation will have exactly one rational solution.

$$4x^2 + kx + 25 = 0$$

### EXAMPLE 5 Using the Discriminant

Find $k$ so that $9x^2 + kx + 4 = 0$ will have exactly one rational solution.

The equation will have only one rational solution if the discriminant is 0.

$$\begin{aligned} & b^2 - 4ac \\ &= k^2 - 4(9)(4) && \text{Here, } a = 9, b = k, \text{ and } c = 4. \\ &= k^2 - 144 \longleftarrow && \text{Value of the discriminant} \end{aligned}$$

Set the discriminant equal to 0 and solve for $k$.

$$\begin{aligned} k^2 - 144 &= 0 \\ k^2 &= 144 && \text{Add 144.} \\ k = 12 \quad \text{or} \quad k &= -12 && \text{Square root property} \end{aligned}$$

The equation will have only one rational solution if $k = 12$ or $k = -12$.

NOW TRY

**NOW TRY ANSWER**

**5.** 20, $-20$

## 11.3 EXERCISES

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***Concept Check*** *Answer each question in Exercises 1–4.*

**1.** An early version of Microsoft *Word* for Windows included the 1.0 edition of *Equation Editor*. The documentation used the following for the quadratic formula. Was this correct? If not, correct it.

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**2.** The Cadillac Bar in Houston, Texas, encourages patrons to write (tasteful) messages on the walls. One person wrote the quadratic formula, as shown here. Was this correct? If not, correct it.

$$x = \frac{-b\sqrt{b^2 - 4ac}}{2a}$$

**3.** A student incorrectly solved $5x^2 - 5x + 1 = 0$ as follows. ***WHAT WENT WRONG?***

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(5)(1)}}{2(5)}$$

$$x = \frac{5 \pm \sqrt{5}}{10}$$

$$x = \frac{1}{2} \pm \sqrt{5}$$

$$\text{Solution set: } \left\{\frac{1}{2} \pm \sqrt{5}\right\}$$

**4.** A student claimed that the equation $2x^2 - 5 = 0$ cannot be solved using the quadratic formula because there is no first-degree $x$-term. Was the student correct? If not, give the values of $a$, $b$, and $c$.

*Use the quadratic formula to solve each equation. (All solutions for these equations are real numbers.)* ***See Examples 1 and 2.***

**5.** $x^2 - 8x + 15 = 0$

**6.** $x^2 + 3x - 28 = 0$

**7.** $2x^2 + 4x + 1 = 0$

**8.** $2x^2 + 3x - 1 = 0$

**9.** $2x^2 - 2x = 1$

**10.** $9x^2 + 6x = 1$

**11.** $x^2 + 18 = 10x$

**12.** $x^2 - 4 = 2x$

**13.** $4x^2 + 4x - 1 = 0$

**14.** $4r^2 - 4r - 19 = 0$

**15.** $2 - 2x = 3x^2$

**16.** $26r - 2 = 3r^2$

**17.** $\frac{x^2}{4} - \frac{x}{2} = 1$

**18.** $p^2 + \frac{p}{3} = \frac{1}{6}$

**19.** $-2t(t + 2) = -3$

**20.** $-3x(x + 2) = -4$

**21.** $(r - 3)(r + 5) = 2$

**22.** $(x + 1)(x - 7) = 1$

**23.** $(x + 2)(x - 3) = 1$

**24.** $(x - 5)(x + 2) = 6$

**25.** $p = \frac{5(5 - p)}{3(p + 1)}$

**26.** $x = \frac{2(x + 3)}{x + 5}$

**27.** $(2x + 1)^2 = x + 4$

**28.** $(2x - 1)^2 = x + 2$

*Use the quadratic formula to solve each equation. (All solutions for these equations are nonreal complex numbers.)* ***See Example 3.***

**29.** $x^2 - 3x + 6 = 0$

**30.** $x^2 - 5x + 20 = 0$

**31.** $r^2 - 6r + 14 = 0$

**32.** $t^2 + 4t + 11 = 0$

**33.** $4x^2 - 4x = -7$

**34.** $9x^2 - 6x = -7$

**35.** $x(3x + 4) = -2$

**36.** $z(2z + 3) = -2$

**37.** $(2x - 1)(8x - 4) = -1$

**38.** $(x - 1)(9x - 3) = -2$

*Use the discriminant to determine whether the solutions for each equation are*

**A.** *two rational numbers*

**B.** *one rational number*

**C.** *two irrational numbers*

**D.** *two nonreal complex numbers.*

*Tell whether the equation can be solved by factoring or whether the quadratic formula should be used. Do not actually solve.* ***See Example 4.***

**39.** $25x^2 + 70x + 49 = 0$

**40.** $4x^2 - 28x + 49 = 0$

**41.** $x^2 + 4x + 2 = 0$

**42.** $9x^2 - 12x - 1 = 0$

**43.** $3x^2 = 5x + 2$

**44.** $4x^2 = 4x + 3$

**45.** $3m^2 - 10m + 15 = 0$

**46.** $18x^2 + 60x + 82 = 0$

*Based on your answers in* ***Exercises 39–46,*** *solve the equation given in each exercise.*

**47. Exercise 39**

**48. Exercise 40**

**49. Exercise 43**

**50. Exercise 44**

**51.** Find the discriminant for each quadratic equation. Use it to tell whether the equation can be solved by factoring or whether the quadratic formula should be used. Then solve each equation.

**(a)** $3x^2 + 13x = -12$ **(b)** $2x^2 + 19 = 14x$

**52.** ***Concept Check*** Is it possible for the solution of a quadratic equation with integer coefficients to include just one irrational number? Why or why not?

*Find the value of a, b, or c so that each equation will have exactly one rational solution.* ***See Example 5.***

**53.** $p^2 + bp + 25 = 0$

**54.** $r^2 - br + 49 = 0$

**55.** $am^2 + 8m + 1 = 0$

**56.** $at^2 + 24t + 16 = 0$

**57.** $9x^2 - 30x + c = 0$

**58.** $4m^2 + 12m + c = 0$

**59.** One solution of $4x^2 + bx - 3 = 0$ is $-\frac{5}{2}$. Find $b$ and the other solution.

**60.** One solution of $3x^2 - 7x + c = 0$ is $\frac{1}{3}$. Find $c$ and the other solution.

### PREVIEW EXERCISES

*Solve each equation.* ***See Section 2.3.***

**61.** $\frac{3}{4}x + \frac{1}{2}x = -10$

**62.** $\frac{x}{5} + \frac{3x}{4} = -19$

*Solve each equation.* ***See Section 10.6.***

**63.** $\sqrt{2x + 6} = x - 1$

**64.** $\sqrt{2x + 1} + \sqrt{x + 3} = 0$

# 11.4 Equations Quadratic in Form

**OBJECTIVES**

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.

**OBJECTIVE 1 Solve an equation with fractions by writing it in quadratic form.** A variety of nonquadratic equations can be written in the form of a quadratic equation and solved by using the methods of this chapter.

### EXAMPLE 1 Solving an Equation with Fractions that Leads to a Quadratic Equation

Solve $\frac{1}{x} + \frac{1}{x-1} = \frac{7}{12}$.

Clear fractions by multiplying each term by the least common denominator, $12x(x - 1)$. (Note that the domain must be restricted to $x \neq 0, x \neq 1$.)

$$
\begin{array}{rcll}
12x(x-1)\left(\frac{1}{x} + \frac{1}{x-1}\right) &=& 12x(x-1)\left(\frac{7}{12}\right) & \text{Multiply by the LCD.} \\
12x(x-1)\frac{1}{x} + 12x(x-1)\frac{1}{x-1} &=& 12x(x-1)\frac{7}{12} & \text{Distributive property} \\
12(x-1) + 12x &=& 7x(x-1) & \\
12x - 12 + 12x &=& 7x^2 - 7x & \text{Distributive property} \\
24x - 12 &=& 7x^2 - 7x & \text{Combine like terms.} \\
7x^2 - 31x + 12 &=& 0 & \text{Standard form} \\
(7x-3)(x-4) &=& 0 & \text{Factor.} \\
7x - 3 = 0 \quad \text{or} \quad x - 4 &=& 0 & \text{Zero-factor property} \\
7x = 3 \quad \text{or} \quad x &=& 4 & \text{Solve for } x. \\
x = \frac{3}{7} && &
\end{array}
$$

The solution set is $\left\{\frac{3}{7}, 4\right\}$. NOW TRY

NOW TRY EXERCISE 1

Solve $\frac{2}{x} + \frac{3}{x+2} = 1$.

**OBJECTIVE 2 Use quadratic equations to solve applied problems.** Some distance-rate-time (or motion) problems lead to quadratic equations. We continue to use the six-step problem-solving method from **Section 2.4.**

NOW TRY ANSWER

**1.** $\{-1, 4\}$

*NOW TRY EXERCISE 2*

A small fishing boat averages 18 mph in still water. It takes the boat $\frac{9}{10}$ hr to travel 8 mi upstream and return. Find the rate of the current.

## EXAMPLE 2 Solving a Motion Problem

A riverboat for tourists averages 12 mph in still water. It takes the boat 1 hr, 4 min to go 6 mi upstream and return. Find the rate of the current.

***Step 1*** **Read** the problem carefully.

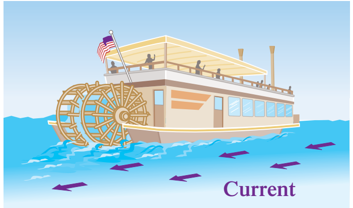
***Step 2*** **Assign a variable.** Let $x =$ the rate of the current.

The current slows down the boat when it is going upstream, so the rate of the boat going upstream is its rate in still water *less* the rate of the current, or $(12 - x)$ mph. See **FIGURE 1**.

Similarly, the current speeds up the boat as it travels downstream, so its rate downstream is $(12 + x)$ mph. Thus,

$12 - x =$ the rate upstream in miles per hour,

and $12 + x =$ the rate downstream in miles per hour.



Riverboat traveling upstream—the current slows it down.

**FIGURE 1**

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| *Upstream* | 6 | $12 - x$ | $\frac{6}{12 - x}$ |
| *Downstream* | 6 | $12 + x$ | $\frac{6}{12 + x}$ |

Times in hours

Complete a table. Use the distance formula, $d = rt$, solved for time $t$, $t = \frac{d}{r}$, to write expressions for $t$.

***Step 3*** **Write an equation.** We use the total time of 1 hr, 4 min written as a fraction.

$$1 + \frac{4}{60} = 1 + \frac{1}{15} = \frac{16}{15}\text{ hr} \qquad \text{Total time}$$

The time upstream plus the time downstream equals $\frac{16}{15}$ hr.

$$\underset{\displaystyle\frac{6}{12 - x}}{\overset{\text{Time upstream}}{\downarrow}} + \underset{\displaystyle\frac{6}{12 + x}}{\overset{\text{Time downstream}}{\downarrow}} = \underset{\displaystyle\frac{16}{15}}{\overset{\text{Total time}}{\downarrow}}$$

***Step 4*** **Solve** the equation. The LCD is $15(12 - x)(12 + x)$.

$$15(12 - x)(12 + x)\left(\frac{6}{12 - x} + \frac{6}{12 + x}\right) = 15(12 - x)(12 + x)\left(\frac{16}{15}\right) \qquad \text{Multiply by the LCD.}$$

$$15(12 + x) \cdot 6 + 15(12 - x) \cdot 6 = 16(12 - x)(12 + x) \qquad \text{Distributive property; multiply.}$$

$$90(12 + x) + 90(12 - x) = 16(144 - x^2) \qquad \text{Multiply.}$$

$$1080 + 90x + 1080 - 90x = 2304 - 16x^2 \qquad \text{Distributive property}$$

$$2160 = 2304 - 16x^2 \qquad \text{Combine like terms.}$$

$$16x^2 = 144 \qquad \text{Add } 16x^2\text{. Subtract 2160.}$$

$$x^2 = 9 \qquad \text{Divide by 16.}$$

$$x = 3 \quad \text{or} \quad x = -3 \qquad \text{Square root property}$$

***Step 5*** **State the answer.** The current rate cannot be $-3$, so the answer is 3 mph.

***Step 6*** **Check** that this value satisfies the original problem. *NOW TRY*

*NOW TRY ANSWER*

**2.** 2 mph

**PROBLEM-SOLVING HINT**

Recall from **Section 6.7** that a person's work rate is $\frac{1}{t}$ part of the job per hour, where $t$ is the time in hours required to do the complete job. Thus, the part of the job the person will do in $x$ hours is $\frac{1}{t}x$.

## EXAMPLE 3 Solving a Work Problem

It takes two carpet layers 4 hr to carpet a room. If each worked alone, one of them could do the job in 1 hr less time than the other. How long would it take each carpet layer to complete the job alone?

***Step 1*** **Read** the problem again. There will be two answers.

***Step 2*** **Assign a variable.** Let $x =$ the number of hours for the slower carpet layer to complete the job alone. Then the faster carpet layer could do the entire job in $(x - 1)$ hours. The slower person's rate is $\frac{1}{x}$, and the faster person's rate is $\frac{1}{x-1}$. Together, they do the job in 4 hr.

| | *Rate* | *Time Working Together* | *Fractional Part of the Job Done* |
|---|---|---|---|
| *Slower Worker* | $\frac{1}{x}$ | 4 | $\frac{1}{x}(4)$ |
| *Faster Worker* | $\frac{1}{x-1}$ | 4 | $\frac{1}{x-1}(4)$ |

Complete a table.

Sum is 1 whole job.

***Step 3*** **Write an equation.**

$$\underset{\frac{4}{x}}{\underset{\downarrow}{\text{Part done by slower worker}}} + \underset{\frac{4}{x-1}}{\underset{\downarrow}{\text{Part done by faster worker}}} = \underset{1}{\underset{\downarrow}{\text{1 whole job}}}$$

***Step 4*** **Solve** the equation from Step 3.

$$x(x-1)\left(\frac{4}{x} + \frac{4}{x-1}\right) = x(x-1)(1)$$ Multiply by the LCD, $x(x - 1)$.

$$4(x-1) + 4x = x(x-1)$$ Distributive property

$$4x - 4 + 4x = x^2 - x$$ Distributive property

$$x^2 - 9x + 4 = 0$$ Standard form

This equation cannot be solved by factoring, so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(4)}}{2(1)}$$ $a = 1, b = -9, c = 4$

$$x = \frac{9 \pm \sqrt{65}}{2}$$ Simplify.

$$x = \frac{9 + \sqrt{65}}{2} \approx 8.5 \quad \text{or} \quad x = \frac{9 - \sqrt{65}}{2} \approx 0.5$$ Use a calculator.

**NOW TRY EXERCISE 3**

Two electricians are running wire to finish a basement. One electrician could finish the job in 2 hr less time than the other. Together, they complete the job in 6 hr. How long (to the nearest tenth) would it take the slower electrician to complete the job alone?

**NOW TRY EXERCISE 4**

Solve each equation.

**(a)** $x = \sqrt{9x - 20}$

**(b)** $x + \sqrt{x} = 20$

***Step 5*** **State the answer.** Only the solution 8.5 makes sense in the original problem, because if $x = 0.5$, then

$$x - 1 = 0.5 - 1 = -0.5,$$

which cannot represent the time for the faster worker. The slower worker could do the job in about 8.5 hr and the faster in about $8.5 - 1 = 7.5$ hr.

***Step 6*** **Check** that these results satisfy the original problem. NOW TRY

## OBJECTIVE 3 Solve an equation with radicals by writing it in quadratic form.

### EXAMPLE 4 Solving Radical Equations That Lead to Quadratic Equations

Solve each equation.

**(a)** $x = \sqrt{6x - 8}$

This equation is not quadratic. However, squaring each side of the equation gives a quadratic equation that can be solved by factoring.

$$\begin{aligned}
x^2 &= \left(\sqrt{6x - 8}\right)^2 && \text{Square each side.}\\
x^2 &= 6x - 8 && \left(\sqrt{a}\right)^2 = a\\
x^2 - 6x + 8 &= 0 && \text{Standard form}\\
(x - 4)(x - 2) &= 0 && \text{Factor.}\\
x - 4 = 0 \quad \text{or} \quad & x - 2 = 0 && \text{Zero-factor property}\\
x = 4 \quad \text{or} \quad & x = 2 && \text{Proposed solutions}
\end{aligned}$$

Squaring each side of an equation can introduce extraneous solutions. ***All proposed solutions must be checked in the original (not the squared) equation.***

CHECK

$$\begin{aligned}
x &= \sqrt{6x - 8}\\
4 &\overset{?}{=} \sqrt{6(4) - 8} && \text{Let } x = 4.\\
4 &\overset{?}{=} \sqrt{16}\\
4 &= 4 \ \checkmark && \text{True}
\end{aligned}$$

$$\begin{aligned}
x &= \sqrt{6x - 8}\\
2 &\overset{?}{=} \sqrt{6(2) - 8} && \text{Let } x = 2.\\
2 &\overset{?}{=} \sqrt{4}\\
2 &= 2 \ \checkmark && \text{True}
\end{aligned}$$

Both solutions check, so the solution set is $\{2, 4\}$.

**(b)**

$$\begin{aligned}
x + \sqrt{x} &= 6 && (a - b)^2 = a^2 - 2ab + b^2\\
\sqrt{x} &= 6 - x && \text{Isolate the radical on one side.}\\
x &= 36 - 12x + x^2 && \text{Square each side.}\\
x^2 - 13x + 36 &= 0 && \text{Standard form}\\
(x - 4)(x - 9) &= 0 && \text{Factor.}\\
x - 4 = 0 \quad \text{or} \quad & x - 9 = 0 && \text{Zero-factor property}\\
x = 4 \quad \text{or} \quad & x = 9 && \text{Proposed solutions}
\end{aligned}$$

CHECK

$$\begin{aligned}
x + \sqrt{x} &= 6\\
4 + \sqrt{4} &\overset{?}{=} 6 && \text{Let } x = 4.\\
6 &= 6 \ \checkmark && \text{True}
\end{aligned}$$

$$\begin{aligned}
x + \sqrt{x} &= 6\\
9 + \sqrt{9} &\overset{?}{=} 6 && \text{Let } x = 9.\\
12 &= 6 && \text{False}
\end{aligned}$$

Only the solution 4 checks, so the solution set is $\{4\}$. NOW TRY

**NOW TRY ANSWERS**

**3.** 13.1 hr

**4. (a)** $\{4, 5\}$ **(b)** $\{16\}$

## OBJECTIVE 4 Solve an equation that is quadratic in form by substitution.

A nonquadratic equation that can be written in the form

$$au^2 + bu + c = 0,$$

for $a \neq 0$ and an algebraic expression $u$, is called **quadratic in form.**

Many equations that are quadratic in form can be solved more easily by defining and substituting a "temporary" variable $u$ for an expression involving the variable in the original equation.

**NOW TRY EXERCISE 5**

Define a variable $u$, and write each equation in the form $au^2 + bu + c = 0$.

**(a)** $x^4 - 10x^2 + 9 = 0$

**(b)** $6(x + 2)^2 - 11(x + 2) + 4 = 0$

### EXAMPLE 5 Defining Substitution Variables

Define a variable $u$, and write each equation in the form $au^2 + bu + c = 0$.

**(a)** $x^4 - 13x^2 + 36 = 0$

Look at the two terms involving the variable $x$, ignoring their coefficients. Try to find one variable expression that is the square of the other. Since $x^4 = (x^2)^2$, we can define $u = x^2$, and rewrite the original equation as a quadratic equation.

$$u^2 - 13u + 36 = 0 \qquad \text{Here, } u = x^2.$$

**(b)** $2(4x - 3)^2 + 7(4x - 3) + 5 = 0$

Because this equation involves both $(4x - 3)^2$ and $(4x - 3)$, we choose $u = 4x - 3$. Substituting $u$ for $4x - 3$ gives the quadratic equation

$$2u^2 + 7u + 5 = 0. \qquad \text{Here, } u = 4x - 3.$$

**(c)** $2x^{2/3} - 11x^{1/3} + 12 = 0$

We apply a power rule for exponents **(Section 4.1)**, $(a^m)^n = a^{mn}$. Because $(x^{1/3})^2 = x^{2/3}$, we define $u = x^{1/3}$. The original equation becomes

$$2u^2 - 11u + 12 = 0. \qquad \text{Here, } u = x^{1/3}.$$

NOW TRY

### EXAMPLE 6 Solving Equations That Are Quadratic in Form

Solve each equation.

**(a)** $x^4 - 13x^2 + 36 = 0$

We can write this equation in quadratic form by substituting $u$ for $x^2$. (See **Example 5(a).**)

$$\begin{array}{rclcrcll}
 & & x^4 - 13x^2 + 36 = 0 & & & & & \\
 & & (x^2)^2 - 13x^2 + 36 = 0 & & & & & x^4 = (x^2)^2 \\
 & & u^2 - 13u + 36 = 0 & & & & & \text{Let } u = x^2. \\
 & & (u - 4)(u - 9) = 0 & & & & & \text{Factor.} \\
u - 4 & = & 0 & \text{or} & u - 9 & = & 0 & \text{Zero-factor property} \\
\text{Don't stop here.} \quad u & = & 4 & \text{or} & u & = & 9 & \text{Solve.} \\
x^2 & = & 4 & \text{or} & x^2 & = & 9 & \text{Substitute } x^2 \text{ for } u. \\
x & = & \pm 2 & \text{or} & x & = & \pm 3 & \text{Square root property}
\end{array}$$

The equation $x^4 - 13x^2 + 36 = 0$, a fourth-degree equation, has four solutions, $-3, -2, 2, 3$.* The solution set is abbreviated $\{\pm 2, \pm 3\}$. Each solution can be verified by substituting it into the original equation for $x$.

**NOW TRY ANSWERS**

5. **(a)** $u = x^2$; $u^2 - 10u + 9 = 0$
   **(b)** $u = x + 2$; $6u^2 - 11u + 4 = 0$

---

*In general, an equation in which an $n$th-degree polynomial equals 0 has $n$ complex solutions, although some of them may be repeated.

**NOW TRY EXERCISE 6**

Solve each equation.

**(a)** $x^4 - 17x^2 + 16 = 0$

**(b)** $x^4 + 4 = 8x^2$

**(b)**

$$\begin{aligned}
4x^4 + 1 &= 5x^2 \\
4(x^2)^2 + 1 &= 5x^2 && x^4 = (x^2)^2 \\
4u^2 + 1 &= 5u && \text{Let } u = x^2. \\
4u^2 - 5u + 1 &= 0 && \text{Standard form} \\
(4u - 1)(u - 1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
4u - 1 = 0 \quad &\text{or} \quad u - 1 = 0 && \text{Zero-factor property} \\
u = \frac{1}{4} \quad &\text{or} \quad u = 1 && \text{Solve.} \\
x^2 = \frac{1}{4} \quad &\text{or} \quad x^2 = 1 && \text{Substitute } x^2 \text{ for } u. \\
x = \pm\frac{1}{2} \quad &\text{or} \quad x = \pm 1 && \text{Square root property}
\end{aligned}$$

This is a key step.

Check that the solution set is $\left\{\pm\frac{1}{2}, \pm 1\right\}$.

**(c)**

$$\begin{aligned}
x^4 &= 6x^2 - 3 \\
x^4 - 6x^2 + 3 &= 0 && \text{Standard form} \\
(x^2)^2 - 6x^2 + 3 &= 0 && x^4 = (x^2)^2 \\
u^2 - 6u + 3 &= 0 && \text{Let } u = x^2.
\end{aligned}$$

Since this equation cannot be solved by factoring, use the quadratic formula.

$$\begin{aligned}
u &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} && a = 1, b = -6, c = 3 \\
u &= \frac{6 \pm \sqrt{24}}{2} && \text{Simplify.} \\
u &= \frac{6 \pm 2\sqrt{6}}{2} && \sqrt{24} = \sqrt{4} \cdot \sqrt{6} = 2\sqrt{6} \\
u &= \frac{2\left(3 \pm \sqrt{6}\right)}{2} && \text{Factor.} \\
u &= 3 \pm \sqrt{6} && \text{Lowest terms}
\end{aligned}$$

$$\begin{aligned}
x^2 = 3 + \sqrt{6} \quad &\text{or} \quad x^2 = 3 - \sqrt{6} && u = x^2 \\
x = \pm\sqrt{3 + \sqrt{6}} \quad &\text{or} \quad x = \pm\sqrt{3 - \sqrt{6}}
\end{aligned}$$

Find *both* square roots in each case.

The solution set $\left\{\pm\sqrt{3 + \sqrt{6}}, \pm\sqrt{3 - \sqrt{6}}\right\}$ contains four numbers. NOW TRY

---

**NOTE** Equations like those in **Examples 6(a) and (b)** can be solved by factoring.

$$\begin{aligned}
x^4 - 13x^2 + 36 &= 0 && \text{Example 6(a) equation} \\
(x^2 - 9)(x^2 - 4) &= 0 && \text{Factor.} \\
(x + 3)(x - 3)(x + 2)(x - 2) &= 0 && \text{Factor again.}
\end{aligned}$$

Using the zero-factor property gives the same solutions obtained in **Example 6(a).** Equations that cannot be solved by factoring (as in **Example 6(c)**) must be solved by substitution and the quadratic formula.

---

**NOW TRY ANSWERS**

**6.** **(a)** $\{\pm 1, \pm 4\}$

**(b)** $\left\{\pm\sqrt{4 + 2\sqrt{3}}, \pm\sqrt{4 - 2\sqrt{3}}\right\}$

### Solving an Equation That Is Quadratic in Form by Substitution

***Step 1*** **Define a temporary variable $u$,** based on the relationship between the variable expressions in the given equation. Substitute $u$ in the original equation and rewrite the equation in the form $au^2 + bu + c = 0$.

***Step 2*** **Solve the quadratic equation obtained in Step 1** by factoring or the quadratic formula.

***Step 3*** **Replace $u$ with the expression it defined in Step 1.**

***Step 4*** **Solve the resulting equations for the original variable.**

***Step 5*** **Check** all solutions by substituting them in the original equation.

**NOW TRY EXERCISE 7**

Solve each equation.

**(a)** $6(x-4)^2 + 11(x-4) - 10 = 0$

**(b)** $2x^{2/3} - 7x^{1/3} + 3 = 0$

## EXAMPLE 7 Solving Equations That Are Quadratic in Form

Solve each equation.

**(a)** $2(4x-3)^2 + 7(4x-3) + 5 = 0$

***Step 1*** Because of the repeated quantity $4x - 3$, substitute $u$ for $4x - 3$. (See **Example 5(b).**)

$$2(4x-3)^2 + 7(4x-3) + 5 = 0$$

$$2u^2 + 7u + 5 = 0 \qquad \text{Let } u = 4x - 3.$$

***Step 2***

$$(2u+5)(u+1) = 0 \qquad \text{Factor.}$$

$$2u + 5 = 0 \quad \text{or} \quad u + 1 = 0 \qquad \text{Zero-factor property}$$

Don't stop here.

$$u = -\frac{5}{2} \quad \text{or} \quad u = -1 \qquad \text{Solve for } u.$$

***Step 3***

$$4x - 3 = -\frac{5}{2} \quad \text{or} \quad 4x - 3 = -1 \qquad \text{Substitute } 4x - 3 \text{ for } u.$$

***Step 4***

$$4x = \frac{1}{2} \quad \text{or} \quad 4x = 2 \qquad \text{Solve for } x.$$

$$x = \frac{1}{8} \quad \text{or} \quad x = \frac{1}{2}$$

***Step 5*** Check that the solution set of the original equation is $\left\{\frac{1}{8}, \frac{1}{2}\right\}$.

**(b)** $2x^{2/3} - 11x^{1/3} + 12 = 0$

Substitute $u$ for $x^{1/3}$. (See **Example 5(c).**)

$$2u^2 - 11u + 12 = 0 \qquad \text{Let } x^{1/3} = u;\, x^{2/3} = u^2.$$

$$(2u-3)(u-4) = 0 \qquad \text{Factor.}$$

$$2u - 3 = 0 \quad \text{or} \quad u - 4 = 0 \qquad \text{Zero-factor property}$$

$$u = \frac{3}{2} \quad \text{or} \quad u = 4 \qquad \text{Solve for } u.$$

$$x^{1/3} = \frac{3}{2} \quad \text{or} \quad x^{1/3} = 4 \qquad u = x^{1/3}$$

$$(x^{1/3})^3 = \left(\frac{3}{2}\right)^3 \quad \text{or} \quad (x^{1/3})^3 = 4^3 \qquad \text{Cube each side.}$$

$$x = \frac{27}{8} \quad \text{or} \quad x = 64$$

Check that the solution set is $\left\{\frac{27}{8}, 64\right\}$.

NOW TRY

**NOW TRY ANSWERS**

**7. (a)** $\left\{\frac{3}{2}, \frac{14}{3}\right\}$ **(b)** $\left\{\frac{1}{8}, 27\right\}$

**CAUTION** A common error when solving problems like those in **Examples 6 and 7** is to stop too soon. ***Once you have solved for u, remember to substitute and solve for the values of the original variable.***

## 11.4 EXERCISES

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*Concept Check Write a sentence describing the first step you would take to solve each equation. Do not actually solve.*

**1.** $\frac{14}{x} = x - 5$

**2.** $\sqrt{1 + x} + x = 5$

**3.** $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$

**4.** $3x = \sqrt{16 - 10x}$

**5.** *Concept Check* Study this incorrect "solution." ***WHAT WENT WRONG?***

$$x = \sqrt{3x + 4}$$

$$x^2 = 3x + 4 \quad \text{Square each side.}$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 4 \quad \text{or} \quad x = -1$$

Solution set: $\{4, -1\}$

**6.** *Concept Check* Study this incorrect "solution." ***WHAT WENT WRONG?***

$$2(x - 1)^2 - 3(x - 1) + 1 = 0$$

$$2u^2 - 3u + 1 = 0 \quad \text{Let } u = x - 1.$$

$$(2u - 1)(u - 1) = 0$$

$$2u - 1 = 0 \quad \text{or} \quad u - 1 = 0$$

$$u = \frac{1}{2} \quad \text{or} \quad u = 1$$

Solution set: $\left\{\frac{1}{2}, 1\right\}$

*Solve each equation. Check your solutions.* ***See Example 1.***

**7.** $\frac{14}{x} = x - 5$

**8.** $\frac{-12}{x} = x + 8$

**9.** $1 - \frac{3}{x} - \frac{28}{x^2} = 0$

**10.** $4 - \frac{7}{r} - \frac{2}{r^2} = 0$

**11.** $3 - \frac{1}{t} = \frac{2}{t^2}$

**12.** $1 + \frac{2}{x} = \frac{3}{x^2}$

◉ **13.** $\frac{1}{x} + \frac{2}{x + 2} = \frac{17}{35}$

**14.** $\frac{2}{m} + \frac{3}{m + 9} = \frac{11}{4}$

**15.** $\frac{2}{x + 1} + \frac{3}{x + 2} = \frac{7}{2}$

**16.** $\frac{4}{3 - p} + \frac{2}{5 - p} = \frac{26}{15}$

**17.** $\frac{3}{2x} - \frac{1}{2(x + 2)} = 1$

**18.** $\frac{4}{3x} - \frac{1}{2(x + 1)} = 1$

**19.** $3 = \frac{1}{t + 2} + \frac{2}{(t + 2)^2}$

**20.** $1 + \frac{2}{3z + 2} = \frac{15}{(3z + 2)^2}$

**21.** $\frac{6}{p} = 2 + \frac{p}{p + 1}$

**22.** $\frac{x}{2 - x} + \frac{2}{x} = 5$

**23.** $1 - \frac{1}{2x + 1} - \frac{1}{(2x + 1)^2} = 0$

**24.** $1 - \frac{1}{3x - 2} - \frac{1}{(3x - 2)^2} = 0$

***Concept Check*** *Answer each question.*

**25.** A boat goes 20 mph in still water, and the rate of the current is $t$ mph.

**(a)** What is the rate of the boat when it travels upstream?

**(b)** What is the rate of the boat when it travels downstream?

**26.** **(a)** If it takes $m$ hours to grade a set of papers, what is the grader's rate (in job per hour)?

**(b)** How much of the job will the grader do in 2 hr?

*Solve each problem.* ***See Examples 2 and 3.***

**27.** On a windy day William Kunz found that he could go 16 mi downstream and then 4 mi back upstream at top speed in a total of 48 min. What was the top speed of William's boat if the rate of the current was 15 mph?

| | *d* | *r* | *t* |
|---|---|---|---|
| *Upstream* | 4 | $x - 15$ | |
| *Downstream* | 16 | | |

**28.** Vera Koutsoyannis flew her plane for 6 hr at a constant rate. She traveled 810 mi with the wind, then turned around and traveled 720 mi against the wind. The wind speed was a constant 15 mph. Find the rate of the plane.

| | *d* | *r* | *t* |
|---|---|---|---|
| *With Wind* | 810 | | |
| *Against Wind* | 720 | | |

**29.** The distance from Jackson to Lodi is about 40 mi, as is the distance from Lodi to Manteca. Adrian Iorgoni drove from Jackson to Lodi, stopped in Lodi for a high-energy drink, and then drove on to Manteca at 10 mph faster. Driving time for the entire trip was 88 min. Find the rate from Jackson to Lodi. (*Source: State Farm Road Atlas.*)

**30.** Medicine Hat and Cranbrook are 300 km apart. Steve Roig-Watnik rides his Harley 20 km per hr faster than Mohammad Shakil rides his Yamaha. Find Steve's average rate if he travels from Cranbrook to Medicine Hat in $1\frac{1}{4}$ hr less time than Mohammad. (*Source: State Farm Road Atlas.*)

**31.** Working together, two people can cut a large lawn in 2 hr. One person can do the job alone in 1 hr less time than the other. How long (to the nearest tenth) would it take the faster worker to do the job? (*Hint: x* is the time of the faster worker.)

| | *Rate* | *Time Working Together* | *Fractional Part of the Job Done* |
|---|---|---|---|
| *Faster Worker* | $\frac{1}{x}$ | 2 | |
| *Slower Worker* | | 2 | |

**32.** Working together, two people can clean an office building in 5 hr. One person is new to the job and would take 2 hr longer than the other person to clean the building alone. How long (to the nearest tenth) would it take the new worker to clean the building alone?

| | *Rate* | *Time Working Together* | *Fractional Part of the Job Done* |
|---|---|---|---|
| *Faster Worker* | | | |
| *Slower Worker* | | | |

**33.** Rusty and Nancy Brauner are planting flats of spring flowers. Working alone, Rusty would take 2 hr longer than Nancy to plant the flowers. Working together, they do the job in 12 hr. How long (to the nearest tenth) would it have taken each person working alone?

**34.** Joel Spring can work through a stack of invoices in 1 hr less time than Noel White can. Working together they take $1\frac{1}{2}$ hr. How long (to the nearest tenth) would it take each person working alone?

**35.** Two pipes together can fill a tank in 2 hr. One of the pipes, used alone, takes 3 hr longer than the other to fill the tank. How long would each pipe take to fill the tank alone?

**36.** A washing machine can be filled in 6 min if both the hot and cold water taps are fully opened. Filling the washer with hot water alone takes 9 min longer than filling it with cold water alone. How long does it take to fill the washer with cold water?

*Solve each equation. Check your solutions.* ***See Example 4.***

**37.** $x = \sqrt{7x - 10}$

**38.** $z = \sqrt{5z - 4}$

**39.** $2x = \sqrt{11x + 3}$

**40.** $4x = \sqrt{6x + 1}$

**41.** $3x = \sqrt{16 - 10x}$

**42.** $4t = \sqrt{8t + 3}$

**43.** $t + \sqrt{t} = 12$

**44.** $p - 2\sqrt{p} = 8$

**45.** $x = \sqrt{\dfrac{6 - 13x}{5}}$

**46.** $r = \sqrt{\dfrac{20 - 19r}{6}}$

**47.** $-x = \sqrt{\dfrac{8 - 2x}{3}}$

**48.** $-x = \sqrt{\dfrac{3x + 7}{4}}$

*Solve each equation. Check your solutions.* ***See Examples 5–7.***

**49.** $x^4 - 29x^2 + 100 = 0$

**50.** $x^4 - 37x^2 + 36 = 0$

**51.** $4q^4 - 13q^2 + 9 = 0$

**52.** $9x^4 - 25x^2 + 16 = 0$

**53.** $x^4 + 48 = 16x^2$

**54.** $z^4 + 72 = 17z^2$

**55.** $(x + 3)^2 + 5(x + 3) + 6 = 0$

**56.** $(x - 4)^2 + (x - 4) - 20 = 0$

**57.** $3(m + 4)^2 - 8 = 2(m + 4)$

**58.** $(t + 5)^2 + 6 = 7(t + 5)$

**59.** $x^{2/3} + x^{1/3} - 2 = 0$

**60.** $x^{2/3} - 2x^{1/3} - 3 = 0$

**61.** $r^{2/3} + r^{1/3} - 12 = 0$

**62.** $3x^{2/3} - x^{1/3} - 24 = 0$

**63.** $4x^{4/3} - 13x^{2/3} + 9 = 0$

**64.** $9t^{4/3} - 25t^{2/3} + 16 = 0$

**65.** $2 + \dfrac{5}{3x - 1} = \dfrac{-2}{(3x - 1)^2}$

**66.** $3 - \dfrac{7}{2p + 2} = \dfrac{6}{(2p + 2)^2}$

**67.** $2 - 6(z - 1)^{-2} = (z - 1)^{-1}$

**68.** $3 - 2(x - 1)^{-1} = (x - 1)^{-2}$

*The equations in Exercises 69–82 are not grouped by type. Solve each equation. Exercises 81 and 82 require knowledge of complex numbers.* ***See Examples 1 and 4–7.***

**69.** $12x^4 - 11x^2 + 2 = 0$

**70.** $\left(x - \dfrac{1}{2}\right)^2 + 5\left(x - \dfrac{1}{2}\right) - 4 = 0$

**71.** $\sqrt{2x + 3} = 2 + \sqrt{x - 2}$

**72.** $\sqrt{m + 1} = -1 + \sqrt{2m}$

**73.** $2\left(1 + \sqrt{r}\right)^2 = 13\left(1 + \sqrt{r}\right) - 6$

**74.** $(x^2 + x)^2 + 12 = 8(x^2 + x)$

**75.** $2m^6 + 11m^3 + 5 = 0$

**76.** $8x^6 + 513x^3 + 64 = 0$

**77.** $6 = 7(2w - 3)^{-1} + 3(2w - 3)^{-2}$

**78.** $x^6 - 10x^3 = -9$

**79.** $2x^4 - 9x^2 = -2$

**80.** $8x^4 + 1 = 11x^2$

**81.** $2x^4 + x^2 - 3 = 0$

**82.** $4x^4 + 5x^2 + 1 = 0$

## PREVIEW EXERCISES

*Solve each equation for the specified variable.* ***See Section 2.5.***

**83.** $P = 2L + 2W$ for $W$ **84.** $\mathcal{A} = \frac{1}{2}bh$ for $h$ **85.** $F = \frac{9}{5}C + 32$ for $C$

## *SUMMARY EXERCISES* on Solving Quadratic Equations

*We have introduced four methods for solving quadratic equations written in standard form* $ax^2 + bx + c = 0$.

| *Method* | *Advantages* | *Disadvantages* |
|---|---|---|
| **Factoring** | This is usually the fastest method. | Not all polynomials are factorable. Some factorable polynomials are difficult to factor. |
| **Square root property** | This is the simplest method for solving equations of the form $(ax + b)^2 = c$. | Few equations are given in this form. |
| **Completing the square** | This method can always be used, although most people prefer the quadratic formula. | It requires more steps than other methods. |
| **Quadratic formula** | This method can always be used. | Sign errors are common when evaluating $\sqrt{b^2 - 4ac}$. |

***Concept Check*** *Decide whether* factoring, *the* square root property, *or the* quadratic formula *is most appropriate for solving each quadratic equation. Do not actually solve.*

**1.** $(2x + 3)^2 = 4$ **2.** $4x^2 - 3x = 1$ **3.** $x^2 + 5x - 8 = 0$

**4.** $2x^2 + 3x = 1$ **5.** $3x^2 = 2 - 5x$ **6.** $x^2 = 5$

*Solve each quadratic equation by the method of your choice.*

**7.** $p^2 = 7$ **8.** $6x^2 - x - 15 = 0$ **9.** $n^2 + 6n + 4 = 0$

**10.** $(x - 3)^2 = 25$ **11.** $\frac{5}{x} + \frac{12}{x^2} = 2$ **12.** $3x^2 = 3 - 8x$

**13.** $2r^2 - 4r + 1 = 0$ ***14.** $x^2 = -12$ **15.** $x\sqrt{2} = \sqrt{5x - 2}$

**16.** $x^4 - 10x^2 + 9 = 0$ **17.** $(2x + 3)^2 = 8$ **18.** $\frac{2}{x} + \frac{1}{x - 2} = \frac{5}{3}$

**19.** $t^4 + 14 = 9t^2$ **20.** $8x^2 - 4x = 2$ ***21.** $z^2 + z + 1 = 0$

**22.** $5x^6 + 2x^3 - 7 = 0$ **23.** $4t^2 - 12t + 9 = 0$ **24.** $x\sqrt{3} = \sqrt{2 - x}$

**25.** $r^2 - 72 = 0$ **26.** $-3x^2 + 4x = -4$ **27.** $x^2 - 5x - 36 = 0$

**28.** $w^2 = 169$ ***29.** $3p^2 = 6p - 4$ **30.** $z = \sqrt{\frac{5z + 3}{2}}$

***31.** $\frac{4}{r^2} + 3 = \frac{1}{r}$ **32.** $2(3x - 1)^2 + 5(3x - 1) = -2$

---

*This exercise requires knowledge of complex numbers.

# 11.5 Formulas and Further Applications

## OBJECTIVES

1. Solve formulas for variables involving squares and square roots.
2. Solve applied problems using the Pythagorean theorem.
3. Solve applied problems using area formulas.
4. Solve applied problems using quadratic functions as models.

## OBJECTIVE 1 Solve formulas for variables involving squares and square roots.

### EXAMPLE 1 Solving for Variables Involving Squares or Square Roots

Solve each formula for the given variable. Keep ± in the answer in part (a).

**(a)** $w = \frac{kFr}{v^2}$ for $v$

The goal is to isolate $v$ on one side.

$$\begin{aligned}
w &= \frac{kFr}{v^2} \\
v^2 w &= kFr && \text{Multiply by } v^2. \\
v^2 &= \frac{kFr}{w} && \text{Divide by } w. \\
v &= \pm\sqrt{\frac{kFr}{w}} && \text{Square root property} \\
v &= \frac{\pm\sqrt{kFr}}{\sqrt{w}} \cdot \frac{\sqrt{w}}{\sqrt{w}} && \text{Rationalize the denominator.} \\
v &= \frac{\pm\sqrt{kFrw}}{w} && \sqrt{a}\cdot\sqrt{b} = \sqrt{ab};\ \sqrt{a}\cdot\sqrt{a} = a
\end{aligned}$$

**(b)** $d = \sqrt{\frac{4\mathcal{A}}{\pi}}$ for $\mathcal{A}$

The goal is to isolate $\mathcal{A}$ on one side.

$$\begin{aligned}
d &= \sqrt{\frac{4\mathcal{A}}{\pi}} \\
d^2 &= \frac{4\mathcal{A}}{\pi} && \text{Square both sides.} \\
\pi d^2 &= 4\mathcal{A} && \text{Multiply by } \pi. \\
\frac{\pi d^2}{4} &= \mathcal{A}, \quad \text{or} \quad \mathcal{A} = \frac{\pi d^2}{4} && \text{Divide by 4.}
\end{aligned}$$

NOW TRY

NOW TRY EXERCISE 1

Solve each formula for the given variable. Keep ± in the answer in part (a).

**(a)** $n = \frac{ab}{E^2}$ for $E$

**(b)** $S = \sqrt{\frac{pq}{n}}$ for $p$

---

**NOTE** In formulas like $v = \frac{\pm\sqrt{kFrw}}{w}$ in **Example 1(a),** we include both positive and negative values.

---

### EXAMPLE 2 Solving for a Variable That Appears in First- and Second-Degree Terms

Solve $s = 2t^2 + kt$ for $t$.

Since the given equation has terms with $t^2$ and $t$, write it in standard form $ax^2 + bx + c = 0$, with $t$ as the variable instead of $x$.

$$\begin{aligned}
s &= 2t^2 + kt \\
0 &= 2t^2 + kt - s && \text{Subtract } s. \\
2t^2 + kt - s &= 0 && \text{Standard form}
\end{aligned}$$

NOW TRY ANSWERS

1. (a) $E = \frac{\pm\sqrt{abn}}{n}$

   (b) $p = \frac{nS^2}{q}$

NOW TRY EXERCISE 2

Solve for $r$.

$$r^2 + 9r = -c$$

To solve $2t^2 + kt - s = 0$, use the quadratic formula with $a = 2$, $b = k$, and $c = -s$.

$$t = \frac{-k \pm \sqrt{k^2 - 4(2)(-s)}}{2(2)} \qquad \text{Substitute.}$$

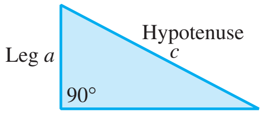
$$t = \frac{-k \pm \sqrt{k^2 + 8s}}{4} \qquad \text{Solve for } t.$$

The solutions are $t = \dfrac{-k + \sqrt{k^2 + 8s}}{4}$ and $t = \dfrac{-k - \sqrt{k^2 + 8s}}{4}$. NOW TRY

**OBJECTIVE 2** **Solve applied problems using the Pythagorean theorem.** The Pythagorean theorem, represented by the equation

$$a^2 + b^2 = c^2,$$

is illustrated in **FIGURE 2** and was introduced in **Sections 5.6 and 10.3.** It is used to solve applications involving right triangles.



Leg $b$

$a^2 + b^2 = c^2$

Pythagorean theorem

**FIGURE 2**

NOW TRY EXERCISE 3

Matt Porter is building a new barn, with length 10 ft more than width. While determining the footprint of the barn, he measured the diagonal as 50 ft. What will be the dimensions of the barn?

## EXAMPLE 3 Using the Pythagorean Theorem

Two cars left an intersection at the same time, one heading due north, the other due west. Some time later, they were exactly 100 mi apart. The car headed north had gone 20 mi farther than the car headed west. How far had each car traveled?

***Step 1*** **Read** the problem carefully.

***Step 2*** **Assign a variable.**

Let $x =$ the distance traveled by the car headed west.

Then $x + 20 =$ the distance traveled by the car headed north.

See **FIGURE 3**. The cars are 100 mi apart, so the hypotenuse of the right triangle equals 100.

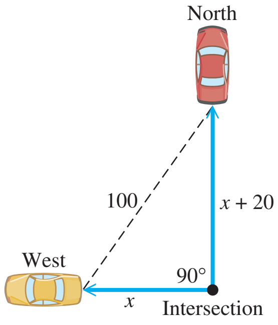


**FIGURE 3**

***Step 3*** **Write an equation.** Use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

$$x^2 + (x + 20)^2 = 100^2$$

$(x + y)^2 = x^2 + 2xy + y^2$

***Step 4*** **Solve.**

$$x^2 + x^2 + 40x + 400 = 10{,}000 \qquad \text{Square the binomial.}$$

$$2x^2 + 40x - 9600 = 0 \qquad \text{Standard form}$$

$$x^2 + 20x - 4800 = 0 \qquad \text{Divide by 2.}$$

$$(x + 80)(x - 60) = 0 \qquad \text{Factor.}$$

$$x + 80 = 0 \quad \text{or} \quad x - 60 = 0 \qquad \text{Zero-factor property}$$

$$x = -80 \quad \text{or} \quad x = 60 \qquad \text{Solve for } x.$$

***Step 5*** **State the answer.** Since distance cannot be negative, discard the negative solution. The required distances are 60 mi and $60 + 20 = 80$ mi.

***Step 6*** **Check.** Since $60^2 + 80^2 = 100^2$, the answer is correct. NOW TRY

NOW TRY ANSWERS

**2.** $r = \dfrac{-9 \pm \sqrt{81 - 4c}}{2}$

**3.** 30 ft by 40 ft

## OBJECTIVE 3 Solve applied problems using area formulas.

**NOW TRY EXERCISE 4**

A football practice field is 30 yd wide and 40 yd long. A strip of grass sod of uniform width is to be placed around the perimeter of the practice field. There is enough money budgeted for 296 sq yd of sod. How wide will the strip be?

### EXAMPLE 4 Solving an Area Problem

A rectangular reflecting pool in a park is 20 ft wide and 30 ft long. The gardener wants to plant a strip of grass of uniform width around the edge of the pool. She has enough seed to cover 336 $\text{ft}^2$. How wide will the strip be?

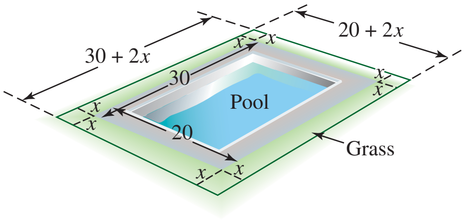


**FIGURE 4**

***Step 1*** **Read** the problem carefully.

***Step 2*** **Assign a variable.** The pool is shown in **FIGURE 4**.

Let $x$ = the unknown width of the grass strip.

Then $20 + 2x$ = the width of the large rectangle (the width of the pool plus two grass strips),

and $30 + 2x$ = the length of the large rectangle.

***Step 3*** **Write an equation.** Refer to **FIGURE 4**.

$$(30 + 2x)(20 + 2x) \qquad \text{Area of large rectangle (length} \cdot \text{width)}$$

$$30 \cdot 20, \quad \text{or} \quad 600 \qquad \text{Area of pool (in square feet)}$$

The area of the large rectangle minus the area of the pool should equal 336 $\text{ft}^2$, the area of the grass strip.

$$\overset{\substack{\text{Area}\\ \text{of large}\\ \text{rectangle}\\ \downarrow}}{(30 + 2x)(20 + 2x)} - \overset{\substack{\text{Area}\\ \text{of}\\ \text{pool}\\ \downarrow}}{600} = \overset{\substack{\text{Area}\\ \text{of}\\ \text{grass}\\ \downarrow}}{336}$$

***Step 4*** **Solve.**

$$\begin{aligned} 600 + 100x + 4x^2 - 600 &= 336 && \text{Multiply.}\\ 4x^2 + 100x - 336 &= 0 && \text{Standard form}\\ x^2 + 25x - 84 &= 0 && \text{Divide by 4.}\\ (x + 28)(x - 3) &= 0 && \text{Factor.}\\ x + 28 = 0 \quad \text{or} \quad x - 3 &= 0 && \text{Zero-factor property}\\ x = -28 \quad \text{or} \quad x &= 3 && \text{Solve for } x. \end{aligned}$$

***Step 5*** **State the answer.** The width cannot be $-28$ ft, so the grass strip should be 3 ft wide.

***Step 6*** **Check.** If $x = 3$, we can find the area of the large rectangle (which includes the grass strip).

$$(30 + 2 \cdot 3)(20 + 2 \cdot 3) = 36 \cdot 26 = 936 \text{ ft}^2 \qquad \text{Area of pool and strip}$$

The area of the pool is $30 \cdot 20 = 600 \text{ ft}^2$. So, the area of the grass strip is $936 - 600 = 336 \text{ ft}^2$, as required. The answer is correct. NOW TRY

## OBJECTIVE 4 Solve applied problems using quadratic functions as models.

Some applied problems can be modeled by *quadratic functions,* which for real numbers $a$, $b$, and $c$, can be written in the form

$$f(x) = ax^2 + bx + c, \quad \text{with } a \neq 0.$$

**NOW TRY ANSWER**

**4.** 2 yd

**NOW TRY EXERCISE 5**

If an object is projected upward from the top of a 120-ft building at 60 ft per sec, its position (in feet above the ground) is given by

$$s(t) = -16t^2 + 60t + 120,$$

where $t$ is time in seconds after it was projected. When does it hit the ground (to the nearest tenth)?

### EXAMPLE 5 Solving an Applied Problem Using a Quadratic Function

If an object is projected upward from the top of a 144-ft building at 112 ft per sec, its position (in feet above the ground) is given by

$$s(t) = -16t^2 + 112t + 144,$$

where $t$ is time in seconds after it was projected. When does it hit the ground?

When the object hits the ground, its distance above the ground is 0. We must find the value of $t$ that makes $s(t) = 0$.

$$\begin{aligned} 0 &= -16t^2 + 112t + 144 && \text{Let } s(t) = 0. \\ 0 &= t^2 - 7t - 9 && \text{Divide by } -16. \\ t &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-9)}}{2(1)} && \text{Substitute into the quadratic formula.} \\ t &= \frac{7 \pm \sqrt{85}}{2} \approx \frac{7 \pm 9.2}{2} && \text{Use a calculator.} \end{aligned}$$

The solutions are $t \approx 8.1$ or $t \approx -1.1$. Time cannot be negative, so we discard the negative solution. The object hits the ground about 8.1 sec after it is projected. NOW TRY

**NOW TRY EXERCISE 6**

Refer to **Example 6.**

**(a)** Use the model to approximate the CPI for 2005, to the nearest whole number.

**(b)** In what year did the CPI reach 500? (Round down for the year.)

### EXAMPLE 6 Using a Quadratic Function to Model the CPI

The Consumer Price Index (CPI) is used to measure trends in prices for a "basket" of goods purchased by typical American families. This index uses a base year of 1967, which means that the index number for 1967 is 100. The quadratic function defined by

$$f(x) = -0.065x^2 + 14.8x + 249$$

approximates the CPI for the years 1980–2005, where $x$ is the number of years that have elapsed since 1980. (*Source:* Bureau of Labor Statistics.)

**(a)** Use the model to approximate the CPI for 1995.

For 1995, $x = 1995 - 1980 = 15$, so find $f(15)$.

$$\begin{aligned} f(x) &= -0.065x^2 + 14.8x + 249 && \text{Given model} \\ f(15) &= -0.065(15)^2 + 14.8(15) + 249 && \text{Let } x = 15. \\ f(15) &\approx 456 && \text{Nearest whole number} \end{aligned}$$

The CPI for 1995 was about 456.

**(b)** In what year did the CPI reach 550?

Find the value of $x$ that makes $f(x) = 550$.

$$\begin{aligned} f(x) &= -0.065x^2 + 14.8x + 249 && \text{Given model} \\ 550 &= -0.065x^2 + 14.8x + 249 && \text{Let } f(x) = 550. \\ 0 &= -0.065x^2 + 14.8x - 301 && \text{Standard form} \\ x &= \frac{-14.8 \pm \sqrt{14.8^2 - 4(-0.065)(-301)}}{2(-0.065)} && \text{Use } a = -0.065, b = 14.8, \text{ and } c = -301 \text{ in the quadratic formula.} \\ x &\approx 22.6 \quad \text{or} \quad x \approx 205.1 \end{aligned}$$

Rounding the first solution 22.6 down, the CPI first reached 550 in $1980 + 22 = 2002$. (Reject the solution $x \approx 205.1$, as this corresponds to a year far beyond the period covered by the model.) NOW TRY

**NOW TRY ANSWERS**

**5.** 5.2 sec after it is projected

**6.** **(a)** 578 **(b)** 1998

## 11.5 EXERCISES

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***Concept Check*** *Answer each question in Exercises 1–4.*

**1.** In solving a formula that has the specified variable in the denominator, what is the first step?

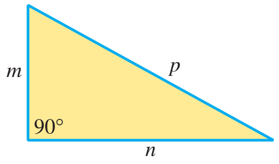
**2.** What is the first step in solving a formula like $gw^2 = 2r$ for $w$?

**3.** What is the first step in solving a formula like $gw^2 = kw + 24$ for $w$?

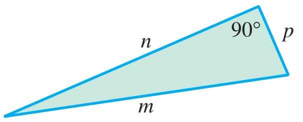
**4.** Why is it particularly important to check all proposed solutions to an applied problem against the information in the original problem?

*In Exercises 5 and 6, solve for m in terms of the other variables* $(m > 0)$.

**5.**



**6.**



*Solve each equation for the indicated variable. (Leave ± in your answers.)* ***See Examples 1 and 2.***

**7.** $d = kt^2$ for $t$

**8.** $S = 6e^2$ for $e$

◉ **9.** $I = \frac{ks}{d^2}$ for $d$

**10.** $R = \frac{k}{d^2}$ for $d$

**11.** $F = \frac{kA}{v^2}$ for $v$

**12.** $L = \frac{kd^4}{h^2}$ for $h$

**13.** $V = \frac{1}{3}\pi r^2 h$ for $r$

**14.** $V = \pi(r^2 + R^2)h$ for $r$

◉ **15.** $At^2 + Bt = -C$ for $t$

**16.** $S = 2\pi rh + \pi r^2$ for $r$

**17.** $D = \sqrt{kh}$ for $h$

**18.** $F = \frac{k}{\sqrt{d}}$ for $d$

**19.** $p = \sqrt{\frac{k\ell}{g}}$ for $\ell$

**20.** $p = \sqrt{\frac{k\ell}{g}}$ for $g$

**21.** $S = 4\pi r^2$ for $r$

**22.** $s = kwd^2$ for $d$

***Brain Busters*** *Solve each equation for the indicated variable. (Leave ± in your answers.)*

**23.** $p = \frac{E^2R}{(r + R)^2}$ for $R$ $(E > 0)$

**24.** $S(6S - t) = t^2$ for $S$

**25.** $10p^2c^2 + 7pcr = 12r^2$ for $r$

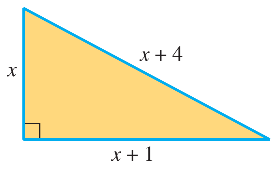
**26.** $S = vt + \frac{1}{2}gt^2$ for $t$

**27.** $LI^2 + RI + \frac{1}{c} = 0$ for $I$

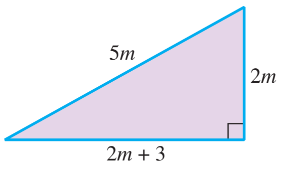
**28.** $P = EI - RI^2$ for $I$

*Solve each problem. When appropriate, round answers to the nearest tenth.* ***See Example 3.***

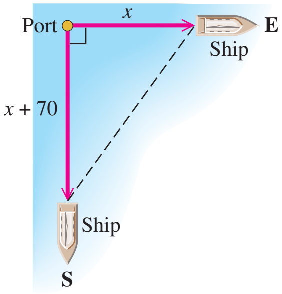
**29.** Find the lengths of the sides of the triangle.



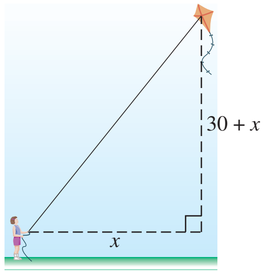
**30.** Find the lengths of the sides of the triangle.



**31.** Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 mi apart. If the ship traveling south traveled 70 mi farther than the other ship, how many miles did they each travel?



**32.** Deborah Israel is flying a kite that is 30 ft farther above her hand than its horizontal distance from her. The string from her hand to the kite is 150 ft long. How high is the kite?

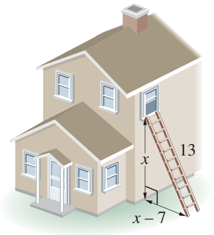


**33.** A game board is in the shape of a right triangle. The hypotenuse is 2 inches longer than the longer leg, and the longer leg is 1 inch less than twice as long as the shorter leg. How long is each side of the game board?

**34.** Manuel Bovi is planting a vegetable garden in the shape of a right triangle. The longer leg is 3 ft longer than the shorter leg, and the hypotenuse is 3 ft longer than the longer leg. Find the lengths of the three sides of the garden.

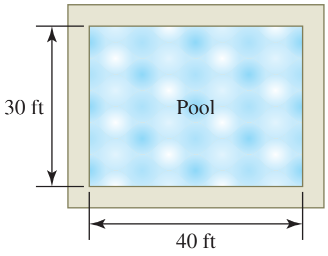
**35.** The diagonal of a rectangular rug measures 26 ft, and the length is 4 ft more than twice the width. Find the length and width of the rug.

**36.** A 13-ft ladder is leaning against a house. The distance from the bottom of the ladder to the house is 7 ft less than the distance from the top of the ladder to the ground. How far is the bottom of the ladder from the house?

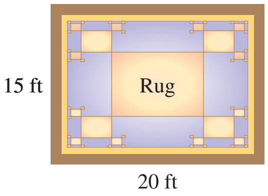


*Solve each problem.* ***See Example 4.***

**37.** A club swimming pool is 30 ft wide and 40 ft long. The club members want an exposed aggregate border in a strip of uniform width around the pool. They have enough material for 296 ft$^2$. How wide can the strip be?



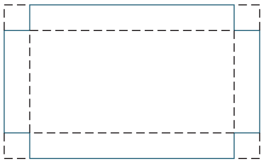
**38.** Lyudmila Slavina wants to buy a rug for a room that is 20 ft long and 15 ft wide. She wants to leave an even strip of flooring uncovered around the edges of the room. How wide a strip will she have if she buys a rug with an area of 234 ft$^2$?



**39.** A rectangle has a length 2 m less than twice its width. When 5 m are added to the width, the resulting figure is a square with an area of 144 m$^2$. Find the dimensions of the original rectangle.

**40.** Mariana Coanda's backyard measures 20 m by 30 m. She wants to put a flower garden in the middle of the yard, leaving a strip of grass of uniform width around the flower garden. Mariana must have 184 m$^2$ of grass. Under these conditions, what will the length and width of the garden be?

**41.** A rectangular piece of sheet metal has a length that is 4 in. less than twice the width. A square piece 2 in. on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume 256 in.$^3$. Find the length and width of the original piece of metal.



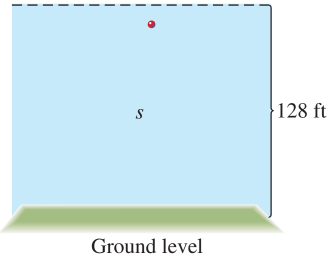
**42.** Another rectangular piece of sheet metal is 2 in. longer than it is wide. A square piece 3 in. on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume 765 in.$^3$. Find the dimensions of the original piece of metal.

*Solve each problem. When appropriate, round answers to the nearest tenth.* ***See Example 5.***

**43.** An object is projected directly upward from the ground. After $t$ seconds its distance in feet above the ground is

$$s(t) = 144t - 16t^2.$$

After how many seconds will the object be 128 ft above the ground? (*Hint:* Look for a common factor before solving the equation.)



**44.** When does the object in **Exercise 43** strike the ground?

**45.** A ball is projected upward from the ground. Its distance in feet from the ground in $t$ seconds is given by

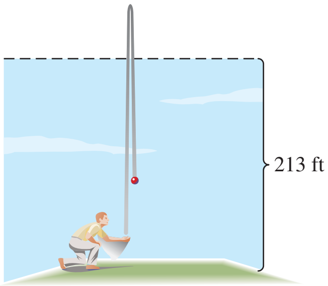
$$s(t) = -16t^2 + 128t.$$

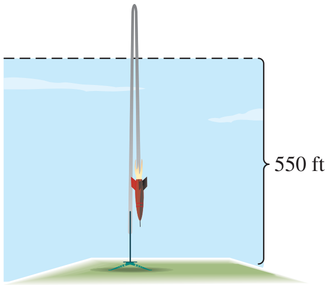
At what times will the ball be 213 ft from the ground?

**46.** A toy rocket is launched from ground level. Its distance in feet from the ground in $t$ seconds is given by

$$s(t) = -16t^2 + 208t.$$

At what times will the rocket be 550 ft from the ground?

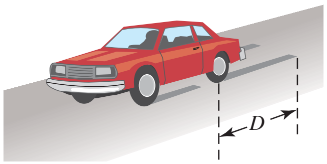




**47.** The function defined by

$$D(t) = 13t^2 - 100t$$

gives the distance in feet a car going approximately 68 mph will skid in $t$ seconds. Find the time it would take for the car to skid 180 ft.



**48.** The function given in **Exercise 47** becomes $D(t) = 13t^2 - 73t$ for a car going 50 mph. Find the time it takes for this car to skid 218 ft.

*A ball is projected upward from ground level, and its distance in feet from the ground in t seconds is given by* $s(t) = -16t^2 + 160t$.

**49.** After how many seconds does the ball reach a height of 400 ft? How would you describe in words its position at this height?

**50.** After how many seconds does the ball reach a height of 425 ft? How would you interpret the mathematical result here?

*Solve each problem using a quadratic equation.*

**51.** A certain bakery has found that the daily demand for blueberry muffins is $\frac{3200}{p}$, where $p$ is the price of a muffin in cents. The daily supply is $3p - 200$. Find the price at which supply and demand are equal.

**52.** In one area the demand for compact discs is $\frac{700}{P}$ per day, where $P$ is the price in dollars per disc. The supply is $5P - 1$ per day. At what price, to the nearest cent, does supply equal demand?

**53.** The formula

$$A = P(1 + r)^2$$

gives the amount $A$ in dollars that $P$ dollars will grow to in 2 yr at interest rate $r$ (where $r$ is given as a decimal), using compound interest. What interest rate will cause \$2000 to grow to \$2142.45 in 2 yr?

**54.** Use the formula $A = P(1 + r)^2$ to find the interest rate $r$ at which a principal $P$ of \$10,000 will increase to \$10,920.25 in 2 yr.

*William Froude was a 19th century naval architect who used the expression*

$$\frac{v^2}{g\ell}$$

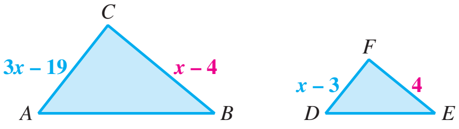
*in shipbuilding. This expression, known as the* ***Froude number,*** *was also used by R. McNeill Alexander in his research on dinosaurs. (Source: "How Dinosaurs Ran,"* Scientific American, *April 1991.) In Exercises 55 and 56, find the value of v (in meters per second), given* $g = 9.8$ m per sec$^2$. (Round to the nearest tenth.)

**55.** Rhinoceros: $\ell = 1.2$;
Froude number $= 2.57$

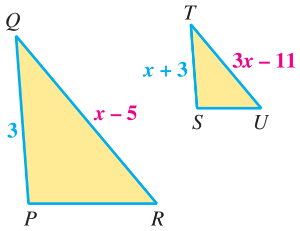
**56.** Triceratops: $\ell = 2.8$;
Froude number $= 0.16$

*Recall that corresponding sides of similar triangles are proportional. Use this fact to find the lengths of the indicated sides of each pair of similar triangles. Check all possible solutions in both triangles. Sides of a triangle cannot be negative (and are not drawn to scale here).*

**57.** Side $AC$



**58.** Side $RQ$

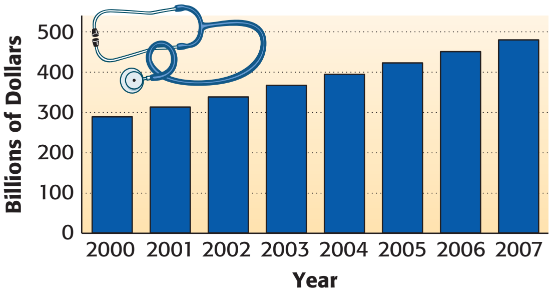


*Total spending (in billions of dollars) in the United States from all sources on physician and clinical services for the years 2000–2007 are shown in the bar graph on the next page and can be modeled by the quadratic function defined by*

$$f(x) = 0.3214x^2 + 25.06x + 288.2.$$

*Here,* $x = 0$ *represents 2000,* $x = 1$ *represents 2001, and so on. Use the graph and the model to work Exercises 59–62.* ***See Example 6.***

**Spending on Physician and Clinical Services**



***Source:*** U.S. Centers for Medicare and Medicaid Services.

**59.** **(a)** Use the graph to estimate spending on physician and clinical services in 2005 to the nearest \$10 billion.

**(b)** Use the model to approximate spending to the nearest \$10 billion. How does this result compare to your estimate in part (a)?

**60.** Based on the model, in what year did spending on physician and clinical services first exceed \$350 billion? (Round down for the year.) How does this result compare to the amount of spending shown in the graph?

**61.** Based on the model, in what year did spending on physician and clinical services first exceed \$400 billion? (Round down for the year.) How does this result compare to the amount of spending shown in the graph?

**62.** If these data were modeled by a *linear* function defined by $f(x) = ax + b$, would the value of $a$ be positive or negative? Explain.

## PREVIEW EXERCISES

*Find each function value.* ***See Section 7.4.***

**63.** $f(x) = x^2 + 4x - 3$. Find $f(2)$.

**64.** $f(x) = 2(x - 3)^2 + 5$. Find $f(3)$.

**65.** Graph $f(x) = 2x^2$. Give the domain and range. **See Sections 4.4 and 7.3.**

# 11.6 Graphs of Quadratic Functions

## OBJECTIVES

1. Graph a quadratic function.
2. Graph parabolas with horizontal and vertical shifts.
3. Use the coefficient of $x^2$ to predict the shape and direction in which a parabola opens.
4. Find a quadratic function to model data.

**OBJECTIVE 1** **Graph a quadratic function.** FIGURE 5 gives a graph of the simplest *quadratic function,* defined by $y = x^2$. This graph is called a **parabola.** (See **Section 4.4.**) The point $(0, 0)$, the lowest point on the curve, is the **vertex** of this parabola. The vertical line through the vertex is the **axis** of the parabola, here $x = 0$. A parabola is **symmetric about its axis**—if the graph were folded along the axis, the two portions of the curve would coincide.

| $x$ | $y$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

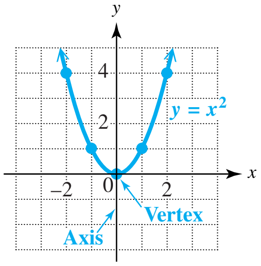


FIGURE 5

As FIGURE 5 suggests, $x$ can be any real number, so the domain of the function defined by $y = x^2$ is $(-\infty, \infty)$. Since $y$ is always nonnegative, the range is $[0, \infty)$.

### Quadratic Function

A function that can be written in the form

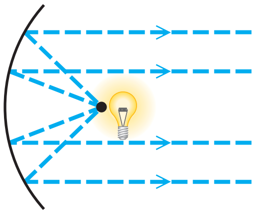
$$f(x) = ax^2 + bx + c$$

for real numbers $a$, $b$, and $c$, with $a \neq 0$, is a **quadratic function.**

***The graph of any quadratic function is a parabola with a vertical axis.***

**NOTE** We use the variable $y$ and function notation $f(x)$ interchangeably. Although we use the letter $f$ most often to name quadratic functions, other letters can be used. We use the capital letter $F$ to distinguish between different parabolas graphed on the same coordinate axes.

Parabolas have a special reflecting property that makes them useful in the design of telescopes, radar equipment, solar furnaces, and automobile headlights. (See the figure.)



Headlight

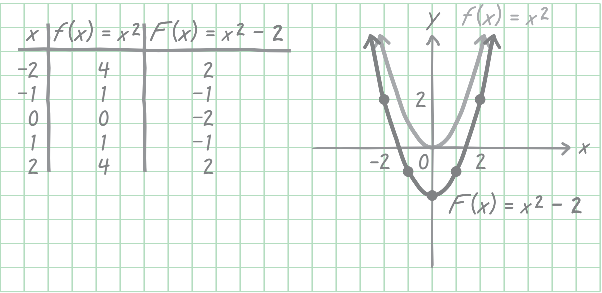
**OBJECTIVE 2** **Graph parabolas with horizontal and vertical shifts.** Parabolas need not have their vertices at the origin, as does the graph of $f(x) = x^2$.

**NOW TRY EXERCISE 1**
Graph $f(x) = x^2 - 3$. Give the vertex, axis, domain, and range.

### EXAMPLE 1 Graphing a Parabola (Vertical Shift)

Graph $F(x) = x^2 - 2$.

The graph of $F(x) = x^2 - 2$ has the same shape as that of $f(x) = x^2$ but is *shifted*, or *translated*, 2 units down, with vertex $(0, -2)$. Every function value is 2 less than the corresponding function value of $f(x) = x^2$. Plotting points on both sides of the vertex gives the graph in **FIGURE 6**.



| $x$ | $f(x) = x^2$ | $F(x) = x^2 - 2$ |
|---|---|---|
| −2 | 4 | 2 |
| −1 | 1 | −1 |
| 0 | 0 | −2 |
| 1 | 1 | −1 |
| 2 | 4 | 2 |

$F(x) = x^2 - 2$
Vertex: $(0, -2)$
Axis: $x = 0$
Domain: $(-\infty, \infty)$
Range: $[-2, \infty)$
The graph of $f(x) = x^2$ is shown for comparison.

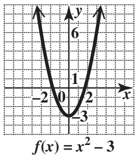
**FIGURE 6**

This parabola is symmetric about its axis $x = 0$, so the plotted points are "mirror images" of each other. Since $x$ can be any real number, the domain is still $(-\infty, \infty)$. The value of $y$ (or $F(x)$) is always greater than or equal to $-2$, so the range is $[-2, \infty)$.

NOW TRY

**NOW TRY ANSWER**

1.



vertex: $(0, -3)$; axis: $x = 0$;
domain: $(-\infty, \infty)$; range: $[-3, \infty)$

## Vertical Shift

The graph of $\boldsymbol{F(x) = x^2 + k}$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The parabola is shifted $k$ units up if $k > 0$, and $|k|$ units down if $k < 0$.
- The vertex of the parabola is $(0, k)$.

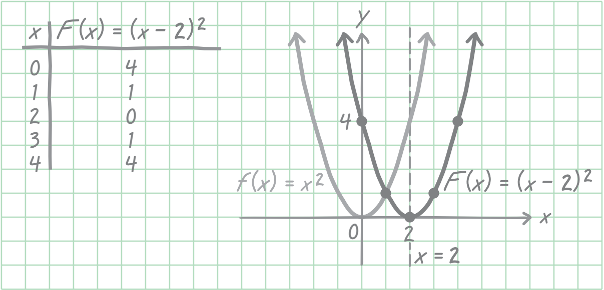
**NOW TRY EXERCISE 2**

Graph $f(x) = (x + 1)^2$. Give the vertex, axis, domain, and range.

## EXAMPLE 2 Graphing a Parabola (Horizontal Shift)

Graph $F(x) = (x - 2)^2$.

If $x = 2$, then $F(x) = 0$, giving the vertex $(2, 0)$. The graph of $F(x) = (x - 2)^2$ has the same shape as that of $f(x) = x^2$ but is shifted 2 units to the right. Plotting points on one side of the vertex, and using symmetry about the axis $x = 2$ to find corresponding points on the other side, gives the graph in **FIGURE 7**.



| $x$ | $F(x) = (x - 2)^2$ |
|---|---|
| 0 | 4 |
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 4 |

$F(x) = (x - 2)^2$
Vertex: $(2, 0)$
Axis: $x = 2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

**FIGURE 7**

NOW TRY

## Horizontal Shift

The graph of $\boldsymbol{F(x) = (x - h)^2}$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The parabola is shifted $h$ units to the right if $h > 0$, and $|h|$ units to the left if $h < 0$.
- The vertex of the parabola is $(h, 0)$.

**CAUTION** ***Errors frequently occur when horizontal shifts are involved.*** To determine the direction and magnitude of a horizontal shift, find the value that causes the expression $x - h$ to equal 0, as shown below.

| $F(x) = (x - 5)^2$ | $F(x) = (x + 5)^2$ |
|---|---|
| Shift the graph of $F(x)$ **5 units to the right,** because $+5$ causes $x - 5$ to equal 0. | Shift the graph of $F(x)$ **5 units to the left,** because $-5$ causes $x + 5$ to equal 0. |

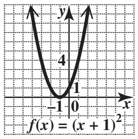
## EXAMPLE 3 Graphing a Parabola (Horizontal and Vertical Shifts)

Graph $F(x) = (x + 3)^2 - 2$.

This graph has the same shape as that of $f(x) = x^2$, but is shifted 3 units to the left (since $x + 3 = 0$ if $x = -3$) and 2 units down (because of the $-2$). See **FIGURE 8** on the next page.

**NOW TRY ANSWER**

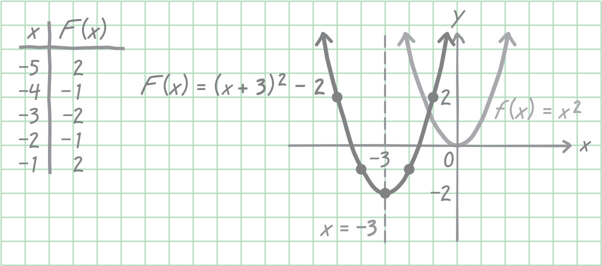
2.



vertex: $(-1, 0)$; axis: $x = -1$; domain: $(-\infty, \infty)$; range: $[0, \infty)$

**NOW TRY EXERCISE 3**

Graph $f(x) = (x + 1)^2 - 2$. Give the vertex, axis, domain, and range.



$F(x) = (x + 3)^2 - 2$

Vertex: $(-3, -2)$

Axis: $x = -3$

Domain: $(-\infty, \infty)$

Range: $[-2, \infty)$

NOW TRY

**FIGURE 8**

## Vertex and Axis of a Parabola

The graph of $\boldsymbol{F(x) = (x - h)^2 + k}$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The vertex of the parabola is $(h, k)$.
- The axis is the vertical line $x = h$.

**OBJECTIVE 3 Use the coefficient of $x^2$ to predict the shape and direction in which a parabola opens.** Not all parabolas open up, and not all parabolas have the same shape as the graph of $f(x) = x^2$.

**NOW TRY EXERCISE 4**

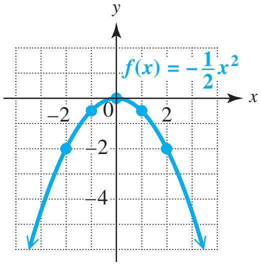
Graph $f(x) = -3x^2$. Give the vertex, axis, domain, and range.

### EXAMPLE 4 Graphing a Parabola That Opens Down

Graph $f(x) = -\frac{1}{2}x^2$.

This parabola is shown in **FIGURE 9**. The coefficient $-\frac{1}{2}$ affects the shape of the graph—the $\frac{1}{2}$ makes the parabola wider $\left(\text{since the values of } \frac{1}{2}x^2 \text{ increase more slowly than those of } x^2\right)$, and the negative sign makes the parabola open down. The graph is not shifted in any direction. Unlike the parabolas graphed in **Examples 1–3,** the vertex here has the *greatest* function value of any point on the graph.

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $-2$ |
| $-1$ | $-\frac{1}{2}$ |
| $0$ | $0$ |
| $1$ | $-\frac{1}{2}$ |
| $2$ | $-2$ |



$f(x) = -\frac{1}{2}x^2$

Vertex: $(0, 0)$

Axis: $x = 0$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0]$

**FIGURE 9**

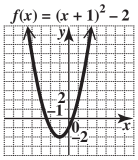
NOW TRY

## General Characteristics of $F(x) = a(x - h)^2 + k \quad (a \neq 0)$

1. The graph of the quadratic function defined by
$$\boldsymbol{F(x) = a(x - h)^2 + k, \quad \text{with } a \neq 0,}$$
is a parabola with vertex $(h, k)$ and the vertical line $x = h$ as axis.
2. The graph opens up if $a$ is positive and down if $a$ is negative.
3. The graph is wider than that of $f(x) = x^2$ if $0 < |a| < 1$.
The graph is narrower than that of $f(x) = x^2$ if $|a| > 1$.

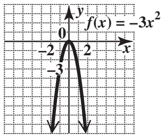
**NOW TRY ANSWERS**

**3.**



vertex: $(-1, -2)$; axis: $x = -1$; domain: $(-\infty, \infty)$; range: $[-2, \infty)$

**4.**



vertex: $(0, 0)$; axis: $x = 0$; domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

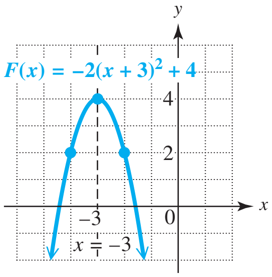
*NOW TRY EXERCISE 5*

Graph $f(x) = 2(x - 1)^2 + 2$.

## EXAMPLE 5 Using the General Characteristics to Graph a Parabola

Graph $F(x) = -2(x + 3)^2 + 4$.

The parabola opens down (because $a < 0$) and is narrower than the graph of $f(x) = x^2$, since $|-2| = 2$ and $2 > 1$. This causes values of $F(x)$ to decrease more quickly than those of $f(x) = -x^2$. This parabola has vertex $(-3, 4)$, as shown in **FIGURE 10**. To complete the graph, we plotted the ordered pairs $(-4, 2)$ and, by symmetry, $(-2, 2)$. Symmetry can be used to find additional ordered pairs that satisfy the equation.



$F(x) = -2(x + 3)^2 + 4$
Vertex: $(-3, 4)$
Axis: $x = -3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$

**FIGURE 10**

*NOW TRY*

## OBJECTIVE 4 Find a quadratic function to model data.

## EXAMPLE 6 Modeling the Number of Multiple Births

The number of higher-order multiple births (triplets or more) in the United States has declined in recent years, as shown by the data in the table. Here, $x$ represents the number of years since 1995 and $y$ represents the number of higher-order multiple births.

| Year | $x$ | $y$ |
|---|---|---|
| 1995 | 0 | 4973 |
| 1996 | 1 | 5939 |
| 1997 | 2 | 6737 |
| 1999 | 4 | 7321 |
| 2001 | 6 | 7471 |
| 2003 | 8 | 7663 |
| 2004 | 9 | 7275 |
| 2005 | 10 | 6694 |

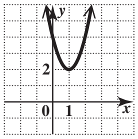
*Source:* National Center for Health Statistics.

Find a quadratic function that models the data.

A scatter diagram of the ordered pairs $(x, y)$ is shown in **FIGURE 11** on the next page. The general shape suggested by the scatter diagram indicates that a parabola should approximate these points, as shown by the dashed curve in **FIGURE 12**. The equation for such a parabola would have a negative coefficient for $x^2$ since the graph opens down.

*NOW TRY ANSWER*

5.



$f(x) = 2(x - 1)^2 + 2$

**NOW TRY EXERCISE 6**

Using the points $(0, 4973)$, $(4, 7321)$, and $(8, 7663)$, find another quadratic model for the data on higher-order multiple births in **Example 6.**

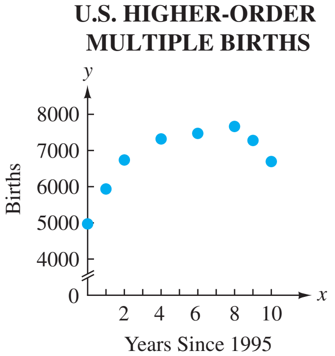


**FIGURE 11**

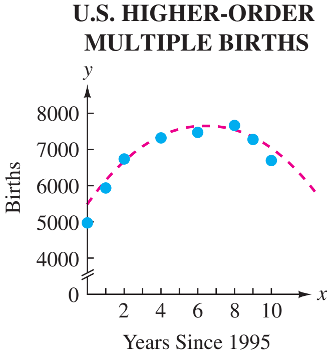


**FIGURE 12**

To find a quadratic function of the form

$$y = ax^2 + bx + c$$

that models, or *fits*, these data, we choose three representative ordered pairs and use them to write a system of three equations. Using

$$(0, 4973), \quad (4, 7321), \quad \text{and} \quad (10, 6694),$$

we substitute the $x$- and $y$-values from the ordered pairs into the quadratic form $y = ax^2 + bx + c$ to get three equations.

$$\begin{array}{rclr} a(0)^2 + b(0) + c = 4973 & \text{or} & c = 4973 & (1) \\ a(4)^2 + b(4) + c = 7321 & \text{or} & 16a + 4b + c = 7321 & (2) \\ a(10)^2 + b(10) + c = 6694 & \text{or} & 100a + 10b + c = 6694 & (3) \end{array}$$

We can find the values of $a$, $b$, and $c$ by solving this system of three equations in three variables using the methods of **Section 8.4.** From equation (1), $c = 4973$. Substitute 4973 for $c$ in equations (2) and (3) to obtain two equations.

$$\begin{array}{rclr} 16a + 4b + 4973 = 7321, & \text{or} & 16a + 4b = 2348 & (4) \\ 100a + 10b + 4973 = 6694, & \text{or} & 100a + 10b = 1721 & (5) \end{array}$$

We can eliminate $b$ from this system of equations in two variables by multiplying equation (4) by $-5$ and equation (5) by 2, and adding the results.

$$120a = -8298$$

$$a = -69.15 \qquad \text{Divide by 120. Use a calculator.}$$

We substitute $-69.15$ for $a$ in equation (4) or (5) to find that $b = 863.6$. Using the values we have found for $a$, $b$, and $c$, our model is defined by

$$y = -69.15x^2 + 863.6x + 4973.$$

NOW TRY

---

**NOTE** In **Example 6,** if we had chosen three different ordered pairs of data, a slightly different model would result. The *quadratic regression* feature on a graphing calculator can also be used to generate the quadratic model that best fits given data. See your owner's manual for details.

---

**NOW TRY ANSWER**

**6.** $y = -62.69x^2 + 837.75x + 4973$

## 11.6 EXERCISES

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

*Complete solution available on the Video Resources on DVD*

1. ***Concept Check*** Match each quadratic function with its graph from choices A–D.

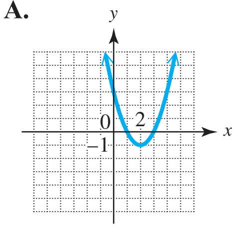
**(a)** $f(x) = (x + 2)^2 - 1$

**(b)** $f(x) = (x + 2)^2 + 1$

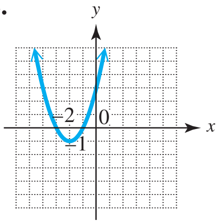
**(c)** $f(x) = (x - 2)^2 - 1$

**(d)** $f(x) = (x - 2)^2 + 1$

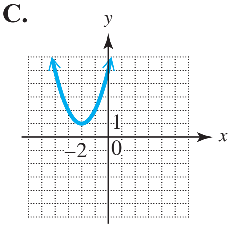
**A.**



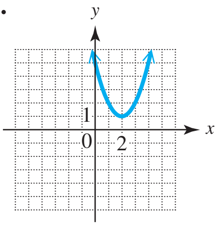
**B.**



**C.**



**D.**



2. ***Concept Check*** Match each quadratic function with its graph from choices A–D.

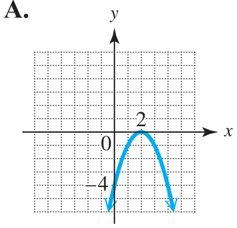
**(a)** $f(x) = -x^2 + 2$

**(b)** $f(x) = -x^2 - 2$

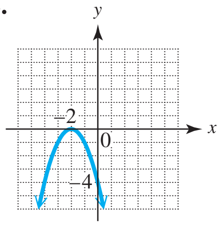
**(c)** $f(x) = -(x + 2)^2$

**(d)** $f(x) = -(x - 2)^2$

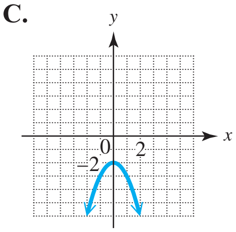
**A.**



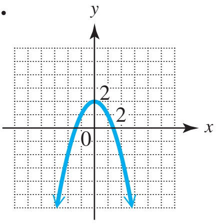
**B.**



**C.**



**D.**



*Identify the vertex of each parabola.* ***See Examples 1–4.***

3. $f(x) = -3x^2$

4. $f(x) = \frac{1}{2}x^2$

5. $f(x) = x^2 + 4$

6. $f(x) = x^2 - 4$

7. $f(x) = (x - 1)^2$

8. $f(x) = (x + 3)^2$

9. $f(x) = (x + 3)^2 - 4$

10. $f(x) = (x + 5)^2 - 8$

11. $f(x) = -(x - 5)^2 + 6$

12. $f(x) = -(x - 2)^2 + 1$

*For each quadratic function, tell whether the graph opens up or down and whether the graph is wider, narrower, or the same shape as the graph of* $f(x) = x^2$. ***See Examples 4 and 5.***

13. $f(x) = -\frac{2}{5}x^2$

14. $f(x) = -2x^2$

15. $f(x) = 3x^2 + 1$

16. $f(x) = \frac{2}{3}x^2 - 4$

17. $f(x) = -4(x + 2)^2 + 5$

18. $f(x) = -\frac{1}{3}(x + 6)^2 + 3$

**19.** ***Concept Check*** Match each quadratic function with the description of the parabola that is its graph.

| | |
|---|---|
| **(a)** $f(x) = (x - 4)^2 - 2$ | **A.** Vertex $(2, -4)$, opens down |
| **(b)** $f(x) = (x - 2)^2 - 4$ | **B.** Vertex $(2, -4)$, opens up |
| **(c)** $f(x) = -(x - 4)^2 - 2$ | **C.** Vertex $(4, -2)$, opens down |
| **(d)** $f(x) = -(x - 2)^2 - 4$ | **D.** Vertex $(4, -2)$, opens up |

**20.** ***Concept Check*** For $f(x) = a(x - h)^2 + k$, in what quadrant is the vertex if

**(a)** $h > 0, k > 0$ **(b)** $h > 0, k < 0$ **(c)** $h < 0, k > 0$ **(d)** $h < 0, k < 0$?

*Graph each parabola. Plot at least two points as well as the vertex. Give the vertex, axis, domain, and range in Exercises 27–36.* ***See Examples 1–5.***

**21.** $f(x) = -2x^2$

**22.** $f(x) = -\frac{1}{3}x^2$

**23.** $f(x) = x^2 - 1$

**24.** $f(x) = x^2 + 3$

**25.** $f(x) = -x^2 + 2$

**26.** $f(x) = -x^2 - 2$

**27.** $f(x) = (x - 4)^2$

**28.** $f(x) = (x + 1)^2$

**29.** $f(x) = (x + 2)^2 - 1$

**30.** $f(x) = (x - 1)^2 + 2$

**31.** $f(x) = 2(x - 2)^2 - 4$

**32.** $f(x) = 3(x - 2)^2 + 1$

**33.** $f(x) = -\frac{1}{2}(x + 1)^2 + 2$

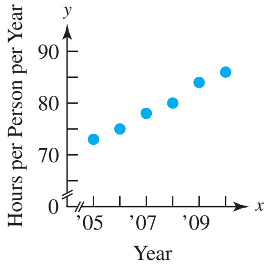
**34.** $f(x) = -\frac{2}{3}(x + 2)^2 + 1$

**35.** $f(x) = 2(x - 2)^2 - 3$

**36.** $f(x) = \frac{4}{3}(x - 3)^2 - 2$

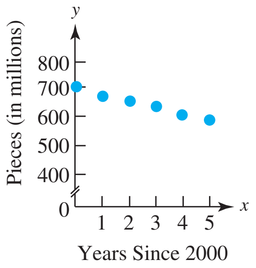
***Concept Check*** *In Exercises 37–42, tell whether a linear or quadratic function would be a more appropriate model for each set of graphed data. If linear, tell whether the slope should be positive or negative. If quadratic, tell whether the coefficient a of $x^2$ should be positive or negative.* ***See Example 6.***

**37.** TIME SPENT PLAYING VIDEO GAMES



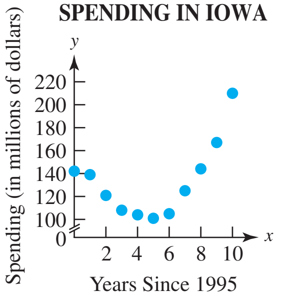
***Source:*** Veronis Suhler Stevenson.

**38.** AVERAGE DAILY VOLUME OF FIRST-CLASS MAIL



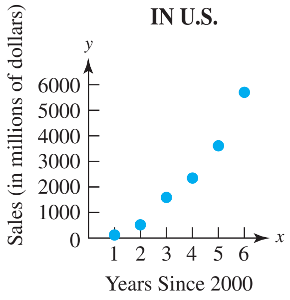
***Source:*** General Accounting Office.

**39.** FOOD ASSISTANCE SPENDING IN IOWA



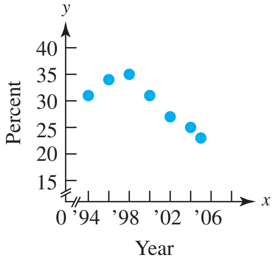
***Source:*** Iowa Department of Human Services.

**40.** PLASMA TV SALES IN U.S.



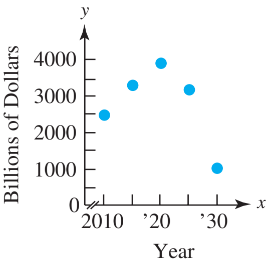
***Source:*** Consumer Electronics Association.

**41.** HIGH SCHOOL STUDENTS WHO SMOKE



***Source:*** www.cdc.gov

**42.** SOCIAL SECURITY ASSETS*



*Projected

***Source:*** Social Security Administration.

*Solve each problem.* ***See Example 6.***

**43.** Sales of digital cameras in the United States (in millions of dollars) between 2000 and 2006 are shown in the table. In the year column, 0 represents 2000, 1 represents 2001, and so on.

| *Year* | *Sales* |
|---|---|
| 0 | 1825 |
| 1 | 1972 |
| 2 | 2794 |
| 3 | 3921 |
| 4 | 4739 |
| 5 | 5611 |
| 6 | 7805 |

*Source:* Consumer Electronics Association.

**(a)** Use the ordered pairs (year, sales) to make a scatter diagram of the data.

**(b)** Use the scatter diagram to decide whether a linear or quadratic function would better model the data. If quadratic, should the coefficient $a$ of $x^2$ be positive or negative?

**(c)** Use the ordered pairs $(0, 1825)$, $(3, 3921)$, and $(6, 7805)$ to find a quadratic function that models the data. Round the values of $a$, $b$, and $c$ in your model to the nearest tenth, as necessary.

**(d)** Use your model from part (c) to approximate the sales of digital cameras in the United States in 2007. Round your answer to the nearest whole number (of millions).

**(e)** Sales of digital cameras were \$6517 million in 2007. Based on this, is the model valid for 2007? Explain.

**44.** The number (in thousands) of new, privately owned housing units started in the United States is shown in the table for the years 2002–2008. In the year column, 2 represents 2002, 3 represents 2003, and so on.

| *Year* | *Housing Starts (thousands)* |
|---|---|
| 2 | 1700 |
| 3 | 1850 |
| 4 | 1960 |
| 5 | 2070 |
| 6 | 1800 |
| 7 | 1360 |
| 8 | 910 |

*Source:* U.S. Census Bureau.

**(a)** Use the ordered pairs (year, housing starts) to make a scatter diagram of the data.

**(b)** Would a linear or quadratic function better model the data?

**(c)** Should the coefficient $a$ of $x^2$ in a quadratic model be positive or negative?

**(d)** Use the ordered pairs $(2, 1700)$, $(4, 1960)$, and $(7, 1360)$ to find a quadratic function that models the data. Round the values of $a$, $b$, and $c$ in your model to the nearest whole number, as necessary.

**(e)** Use your model from part (d) to approximate the number of housing starts during 2003 and 2008 to the nearest thousand. How well does the model approximate the actual data from the table?

**45.** In **Example 6**, we determined that the quadratic function defined by

$$y = -69.15x^2 + 863.6x + 4973$$

modeled the number of higher-order multiple births, where $x$ represents the number of years since 1995.

**(a)** Use this model to approximate the number of higher-order births in 2006 to the nearest whole number.

**(b)** The actual number of higher-order births in 2006 was 6540. (*Source:* National Center for Health Statistics.) How does the approximation using the model compare to the actual number for 2006?

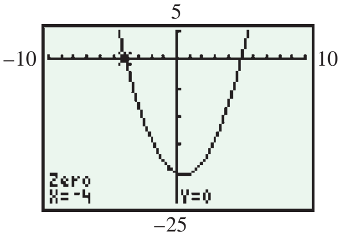
**46.** Should the model from **Exercise 45** be used to approximate the rate of higher-order multiple births in years after 2006? Explain.

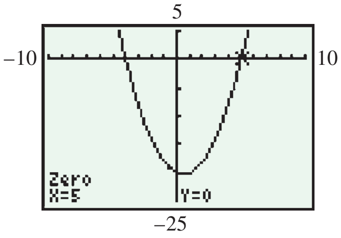
## TECHNOLOGY INSIGHTS EXERCISES 47–48

*Recall from **Sections 3.2 and 7.1** that the x-value of the x-intercept of the graph of the line $y = mx + b$ is the solution of the linear equation $mx + b = 0$. In the same way, the x-values of the x-intercepts of the graph of the parabola $y = ax^2 + bx + c$ are the real solutions of the quadratic equation $ax^2 + bx + c = 0$.*

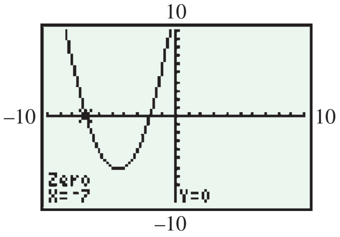
*In Exercises 47–48, the calculator graphs show the x-values of the x-intercepts of the graph of the polynomial in the equation. Use the graphs to solve each equation.*

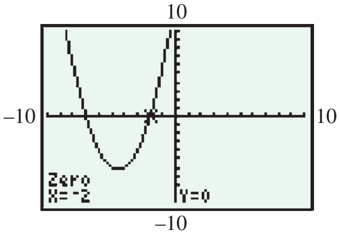
**47.** $x^2 - x - 20 = 0$





**48.** $x^2 + 9x + 14 = 0$





## PREVIEW EXERCISES

*Complete each factoring. **See Section 5.1.***

**49.** $-2x^2 + 6x =$ _____ $(x^2 - 3x)$

**50.** $-3x^2 - 15x =$ _____ $(x^2 + 5x)$

*Solve each quadratic equation by factoring or by completing the square. **See Sections 11.1 and 11.2.***

**51.** $x^2 + 3x - 4 = 0$

**52.** $x^2 - x - 6 = 0$

**53.** $x^2 + 6x - 3 = 0$

**54.** $x^2 + 8x - 4 = 0$

## 11.7

# More About Parabolas and Their Applications

**OBJECTIVES**

1. Find the vertex of a vertical parabola.
2. Graph a quadratic function.
3. Use the discriminant to find the number of *x*-intercepts of a parabola with a vertical axis.
4. Use quadratic functions to solve problems involving maximum or minimum value.
5. Graph parabolas with horizontal axes.

**OBJECTIVE 1** **Find the vertex of a vertical parabola.** When the equation of a parabola is given in the form $f(x) = ax^2 + bx + c$, there are two ways to locate the vertex.

1. Complete the square, as shown in **Examples 1 and 2,** or
2. Use a formula derived by completing the square, as shown in **Example 3.**

### EXAMPLE 1 Completing the Square to Find the Vertex $(a = 1)$

Find the vertex of the graph of $f(x) = x^2 - 4x + 5$.

We can express $x^2 - 4x + 5$ in the form $(x - h)^2 + k$ by completing the square on $x^2 - 4x$, as in **Section 11.2.** The process is slightly different here because we want to keep $f(x)$ alone on one side of the equation. Instead of adding the appropriate number to each side, we *add and subtract* it on the right.

$$f(x) = x^2 - 4x + 5$$

$$= (x^2 - 4x \qquad) + 5 \qquad \text{Group the variable terms.}$$

$$\left[\tfrac{1}{2}(-4)\right]^2 = (-2)^2 = 4$$

This is equivalent to adding 0.

$$= (x^2 - 4x + 4 - 4) + 5 \qquad \text{Add and subtract 4.}$$

$$= (x^2 - 4x + 4) - 4 + 5 \qquad \text{Bring } -4 \text{ outside the parentheses.}$$

$$f(x) = (x - 2)^2 + 1 \qquad \text{Factor. Combine like terms.}$$

The vertex of this parabola is $(2, 1)$. NOW TRY

**NOW TRY EXERCISE 1**
Find the vertex of the graph of
$f(x) = x^2 + 2x - 8$.

### EXAMPLE 2 Completing the Square to Find the Vertex $(a \neq 1)$

Find the vertex of the graph of $f(x) = -3x^2 + 6x - 1$.

Because the $x^2$-term has a coefficient other than 1, we factor that coefficient out of the first two terms before completing the square.

$$f(x) = -3x^2 + 6x - 1$$

$$= -3(x^2 - 2x) - 1 \qquad \text{Factor out } -3.$$

$$\left[\tfrac{1}{2}(-2)\right]^2 = (-1)^2 = 1$$

$$= -3(x^2 - 2x + 1 - 1) - 1 \qquad \text{Add and subtract 1 within the parentheses.}$$

***Now bring −1 outside the parentheses. Be sure to multiply it by −3.***

$$= -3(x^2 - 2x + 1) + (-3)(-1) - 1 \qquad \text{Distributive property}$$

$$= -3(x^2 - 2x + 1) + 3 - 1$$

This is a key step.

$$f(x) = -3(x - 1)^2 + 2 \qquad \text{Factor. Combine like terms.}$$

The vertex is $(1, 2)$. NOW TRY

**NOW TRY EXERCISE 2**
Find the vertex of the graph of
$f(x) = -4x^2 + 16x - 10$.

*NOW TRY ANSWERS*
**1.** $(-1, -9)$ **2.** $(2, 6)$

To derive a formula for the vertex of the graph of the quadratic function defined by $f(x) = ax^2 + bx + c$ (with $a \neq 0$), complete the square.

$$f(x) = ax^2 + bx + c$$ Standard form

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$ Factor $a$ from the first two terms.

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$ Add and subtract $\frac{b^2}{4a^2}$.

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c$$ Distributive property

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$ $-\frac{ab^2}{4a^2} = -\frac{b^2}{4a}$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$ Factor. Rewrite terms with a common denominator.

$$f(x) = a\left[x - \underbrace{\left(\frac{-b}{2a}\right)}_{h}\right]^2 + \underbrace{\frac{4ac - b^2}{4a}}_{k}$$ $f(x) = a(x - h)^2 + k$ The vertex $(h, k)$ can be expressed in terms of $a$, $b$, and $c$.

The expression for $k$ can be found by replacing $x$ with $\frac{-b}{2a}$. Using function notation, if $y = f(x)$, then the $y$-value of the vertex is $f\left(\frac{-b}{2a}\right)$.

### Vertex Formula

The graph of the quadratic function defined by $f(x) = ax^2 + bx + c$ (with $a \neq 0$) has vertex

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right),$$

and the axis of the parabola is the line

$$x = \frac{-b}{2a}.$$

**NOW TRY EXERCISE 3**
Use the vertex formula to find the vertex of the graph of

$$f(x) = 3x^2 - 2x + 8.$$

### EXAMPLE 3 Using the Formula to Find the Vertex

Use the vertex formula to find the vertex of the graph of $f(x) = x^2 - x - 6$.

The $x$-coordinate of the vertex of the parabola is given by $\frac{-b}{2a}$.

$$\frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2} \longleftarrow$$ $a = 1, b = -1,$ and $c = -6$. $x$-coordinate of vertex

The $y$-coordinate is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right)$.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 6 = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4} \longleftarrow$$ $y$-coordinate of vertex

The vertex is $\left(\frac{1}{2}, -\frac{25}{4}\right)$. NOW TRY

**NOW TRY ANSWER**
**3.** $\left(\frac{1}{3}, \frac{23}{3}\right)$

**OBJECTIVE 2** **Graph a quadratic function.** We give a general approach.

### Graphing a Quadratic Function $y = f(x)$

***Step 1*** **Determine whether the graph opens up or down.** If $a > 0$, the parabola opens up. If $a < 0$, it opens down.

***Step 2*** **Find the vertex.** Use the vertex formula or completing the square.

***Step 3*** **Find any intercepts.** To find the $x$-intercepts (if any), solve $f(x) = 0$. To find the $y$-intercept, evaluate $f(0)$.

***Step 4*** **Complete the graph.** Plot the points found so far. Find and plot additional points as needed, using symmetry about the axis.

*NOW TRY EXERCISE 4*

Graph the quadratic function defined by

$$f(x) = x^2 + 2x - 3.$$

Give the vertex, axis, domain, and range.

## EXAMPLE 4 Graphing a Quadratic Function

Graph the quadratic function defined by $f(x) = x^2 - x - 6$.

***Step 1*** From the equation, $a = 1$, so the graph of the function opens up.

***Step 2*** The vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, was found in **Example 3** by using the vertex formula.

***Step 3*** Find any intercepts. Since the vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, is in quadrant IV and the graph opens up, there will be two $x$-intercepts. Let $f(x) = 0$ and solve.

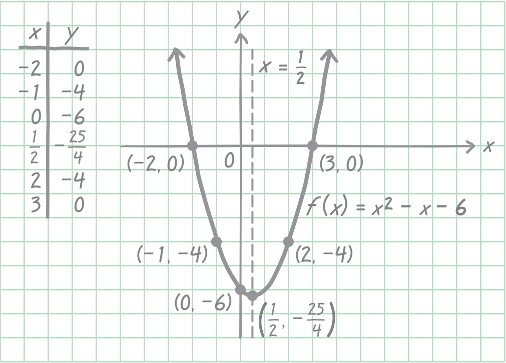
$$\begin{aligned} f(x) &= x^2 - x - 6 & \\ 0 &= x^2 - x - 6 & \text{Let } f(x) = 0. \\ 0 &= (x-3)(x+2) & \text{Factor.} \\ x - 3 = 0 \quad &\text{or} \quad x + 2 = 0 & \text{Zero-factor property} \\ x = 3 \quad &\text{or} \quad x = -2 & \text{Solve each equation.} \end{aligned}$$

The $x$-intercepts are $(3, 0)$ and $(-2, 0)$. Find the $y$-intercept by evaluating $f(0)$.

$$\begin{aligned} f(x) &= x^2 - x - 6 & \\ f(0) &= 0^2 - 0 - 6 & \text{Let } x = 0. \\ f(0) &= -6 & \end{aligned}$$

The $y$-intercept is $(0, -6)$.

***Step 4*** Plot the points found so far and additional points as needed using symmetry about the axis, $x = \frac{1}{2}$. The graph is shown in **FIGURE 13**.



$f(x) = x^2 - x - 6$

Vertex: $\left(\frac{1}{2}, -\frac{25}{4}\right)$

Axis: $x = \frac{1}{2}$

Domain: $(-\infty, \infty)$

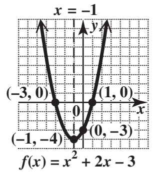
Range: $\left[-\frac{25}{4}, \infty\right)$

**FIGURE 13**

*NOW TRY*

*NOW TRY ANSWER*

4.



vertex: $(-1, -4)$; axis: $x = -1$; domain: $(-\infty, \infty)$; range: $[-4, \infty)$

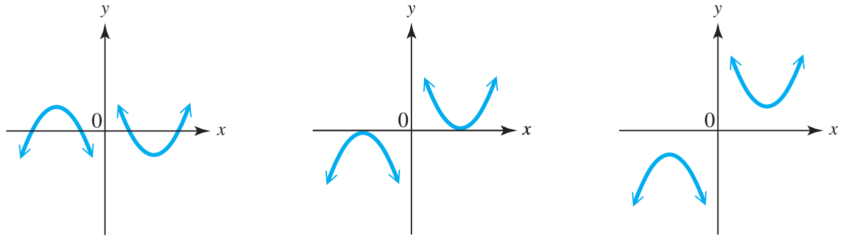
**OBJECTIVE 3** **Use the discriminant to find the number of $x$-intercepts of a parabola with a vertical axis.** Recall from **Section 11.3** that

$$b^2 - 4ac \qquad \text{Discriminant}$$

is called the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$ and that we can use it to determine the number of real solutions of a quadratic equation.

In a similar way, we can use the discriminant of a quadratic *function* to determine the number of $x$-intercepts of its graph. The three possibilities are shown in **FIGURE 14**.

1. If the discriminant is positive, the parabola will have two $x$-intercepts.
2. If the discriminant is 0, there will be only one $x$-intercept, and it will be the vertex of the parabola.
3. If the discriminant is negative, the graph will have no $x$-intercepts.



$b^2 - 4ac > 0$
Two $x$-intercepts

$b^2 - 4ac = 0$
One $x$-intercept

$b^2 - 4ac < 0$
No $x$-intercepts

**FIGURE 14**

### EXAMPLE 5 Using the Discriminant to Determine the Number of $x$-Intercepts

Find the discriminant and use it to determine the number of $x$-intercepts of the graph of each quadratic function.

**(a)** $f(x) = 2x^2 + 3x - 5$

$$\begin{aligned} & b^2 - 4ac && \text{Discriminant} \\ &= 3^2 - 4(2)(-5) && a = 2, b = 3, c = -5 \\ &= 9 - (-40) && \text{Apply the exponent. Multiply.} \\ &= 49 && \text{Subtract.} \end{aligned}$$

Since the discriminant is positive, the parabola has two $x$-intercepts.

**(b)** $f(x) = -3x^2 - 1$

$$\begin{aligned} & b^2 - 4ac && \\ &= 0^2 - 4(-3)(-1) && a = -3, b = 0, c = -1 \\ &= -12 && \end{aligned}$$

The discriminant is negative, so the graph has no $x$-intercepts.

**(c)** $f(x) = 9x^2 + 6x + 1$

$$\begin{aligned} & b^2 - 4ac && \\ &= 6^2 - 4(9)(1) && a = 9, b = 6, c = 1 \\ &= 0 && \end{aligned}$$

The parabola has only one $x$-intercept (its vertex). NOW TRY

NOW TRY EXERCISE 5

Find the discriminant and use it to determine the number of $x$-intercepts of the graph of each quadratic function.

**(a)** $f(x) = -2x^2 + 3x - 2$

**(b)** $f(x) = 3x^2 + 2x - 1$

**(c)** $f(x) = 4x^2 - 12x + 9$

NOW TRY ANSWERS

**5.** **(a)** $-7$; none **(b)** 16; two
**(c)** 0; one

**OBJECTIVE 4** **Use quadratic functions to solve problems involving maximum or minimum value.** The vertex of the graph of a quadratic function is either the highest or the lowest point on the parabola. It provides the following information.

1. The $y$-value of the vertex gives the maximum or minimum value of $y$.
2. The $x$-value tells where the maximum or minimum occurs.

**PROBLEM-SOLVING HINT**

In many applied problems we must find the greatest or least value of some quantity. When we can express that quantity in terms of a quadratic function, the value of $k$ in the vertex $(h, k)$ gives that optimum value.

*NOW TRY EXERCISE 6*

Solve the problem in **Example 6** if the farmer has only 80 ft of fencing.

**EXAMPLE 6** Finding the Maximum Area of a Rectangular Region

A farmer has 120 ft of fencing to enclose a rectangular area next to a building. (See **FIGURE 15**.) Find the maximum area he can enclose and the dimensions of the field when the area is maximized.

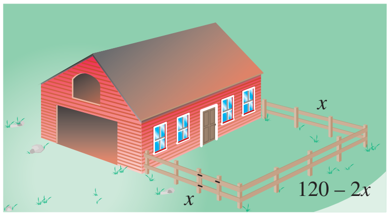


**FIGURE 15**

Let $x$ = the width of the field.

$$x + x + \text{length} = 120 \qquad \text{Sum of the sides is 120 ft.}$$

$$2x + \text{length} = 120 \qquad \text{Combine like terms.}$$

$$\text{length} = 120 - 2x \qquad \text{Subtract } 2x.$$

The area $\mathcal{A}(x)$ is given by the product of the length and width.

$$\mathcal{A}(x) = (120 - 2x)x \qquad \text{Area} = \text{length} \cdot \text{width}$$

$$\mathcal{A}(x) = 120x - 2x^2 \qquad \text{Distributive property}$$

To determine the maximum area, use the vertex formula to find the vertex of the parabola given by $\mathcal{A}(x) = 120x - 2x^2$. Write the equation in standard form.

$$\mathcal{A}(x) = -2x^2 + 120x \qquad a = -2, b = 120, c = 0$$

Then $$x = \frac{-b}{2a} = \frac{-120}{2(-2)} = \frac{-120}{-4} = 30,$$

and $$\mathcal{A}(30) = -2(30)^2 + 120(30) = -2(900) + 3600 = 1800.$$

The graph is a parabola that opens down, and its vertex is $(30, 1800)$. Thus, the maximum area will be 1800 ft$^2$. This area will occur if $x$, the width of the field, is 30 ft and the length is

$$120 - 2(30) = 60 \text{ ft.}$$

*NOW TRY*

*NOW TRY ANSWER*

6. The field should be 20 ft by 40 ft with maximum area 800 ft$^2$.

**CAUTION** ***Be careful when interpreting the meanings of the coordinates of the vertex.*** The first coordinate, $x$, gives the value for which the *function value*, $y$ or $f(x)$, is a maximum or a minimum. Be sure to read the problem carefully to determine whether you are asked to find the value of the independent variable, the function value, or both.

**NOW TRY EXERCISE 7**

A stomp rocket is launched from the ground with an initial velocity of 48 ft per sec so that its distance in feet above the ground after $t$ seconds is

$$s(t) = -16t^2 + 48t.$$

Find the maximum height attained by the rocket and the number of seconds it takes to reach that height.

### EXAMPLE 7 Finding the Maximum Height Attained by a Projectile

If air resistance is neglected, a projectile on Earth shot straight upward with an initial velocity of 40 m per sec will be at a height $s$ in meters given by

$$s(t) = -4.9t^2 + 40t,$$

where $t$ is the number of seconds elapsed after projection. After how many seconds will it reach its maximum height, and what is this maximum height?

For this function, $a = -4.9$, $b = 40$, and $c = 0$. Use the vertex formula.

$$t = \frac{-b}{2a} = \frac{-40}{2(-4.9)} \approx 4.1 \qquad \text{Use a calculator.}$$

This indicates that the maximum height is attained at 4.1 sec. To find this maximum height, calculate $s(4.1)$.

$$\begin{aligned} s(t) &= -4.9t^2 + 40t \\ s(4.1) &= -4.9(4.1)^2 + 40(4.1) && \text{Let } t = 4.1. \\ s(4.1) &\approx 81.6 && \text{Use a calculator.} \end{aligned}$$

The projectile will attain a maximum height of approximately 81.6 m at 4.1 sec.

NOW TRY

**OBJECTIVE 5** **Graph parabolas with horizontal axes.** If $x$ and $y$ are interchanged in the equation

$$y = ax^2 + bx + c,$$

the equation becomes

$$x = ay^2 + by + c.$$

Because of the interchange of the roles of $x$ and $y$, these parabolas are horizontal (with horizontal lines as axes).

**Graph of a Horizontal Parabola**

The graph of $\boldsymbol{x = ay^2 + by + c}$ or $\boldsymbol{x = a(y - k)^2 + h}$ is a parabola.

- The vertex of the parabola is $(h, k)$.
- The axis is the horizontal line $y = k$.
- The graph opens to the right if $a > 0$ and to the left if $a < 0$.

**NOW TRY ANSWER**

**7.** 36 ft; 1.5 sec

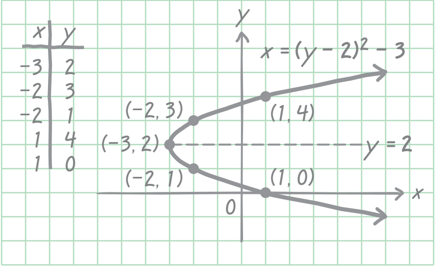
**NOW TRY EXERCISE 8**

Graph $x = (y + 2)^2 - 1$. Give the vertex, axis, domain, and range.

## EXAMPLE 8 Graphing a Horizontal Parabola ($a = 1$)

Graph $x = (y - 2)^2 - 3$. Give the vertex, axis, domain, and range.

This graph has its vertex at $(-3, 2)$, since the roles of $x$ and $y$ are interchanged. It opens to the right (the positive $x$-direction) because $a = 1$ and $1 > 0$, and has the same shape as $y = x^2$. Plotting a few additional points gives the graph shown in **FIGURE 16**.



$x = (y - 2)^2 - 3$
Vertex: $(-3, 2)$
Axis: $y = 2$
Domain: $[-3, \infty)$
Range: $(-\infty, \infty)$

**FIGURE 16**

NOW TRY

**NOW TRY EXERCISE 9**

Graph $x = -3y^2 - 6y - 5$. Give the vertex, axis, domain, and range.

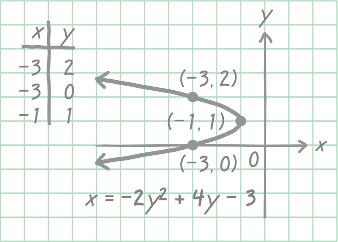
## EXAMPLE 9 Completing the Square to Graph a Horizontal Parabola ($a \neq 1$)

Graph $x = -2y^2 + 4y - 3$. Give the vertex, axis, domain, and range of the relation.

$$\begin{aligned} x &= -2y^2 + 4y - 3 \\ &= -2(y^2 - 2y) - 3 && \text{Factor out } -2. \\ &= -2(y^2 - 2y + 1 - 1) - 3 && \text{Complete the square within the parentheses. Add and subtract 1.} \\ &= -2(y^2 - 2y + 1) + (-2)(-1) - 3 && \text{Distributive property} \\ x &= -2(y - 1)^2 - 1 && \text{Factor. Simplify.} \end{aligned}$$

Be careful here.

Because of the negative coefficient $-2$ in $x = -2(y - 1)^2 - 1$, the graph opens to the left (the negative $x$-direction). The graph is narrower than the graph of $y = x^2$ because $|-2| > 1$. See **FIGURE 17**.



$x = -2y^2 + 4y - 3$
Vertex: $(-1, 1)$
Axis: $y = 1$
Domain: $(-\infty, -1]$
Range: $(-\infty, \infty)$

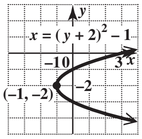
**FIGURE 17**

NOW TRY

**CAUTION** ***Only quadratic equations solved for y (whose graphs are vertical parabolas) are examples of functions.*** The horizontal parabolas in **Examples 8 and 9** are ***not*** graphs of functions, because they do not satisfy the conditions of the vertical line test.

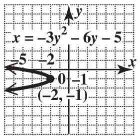
**NOW TRY ANSWERS**

8.



vertex: $(-1, -2)$; axis: $y = -2$; domain: $[-1, \infty)$; range: $(-\infty, \infty)$

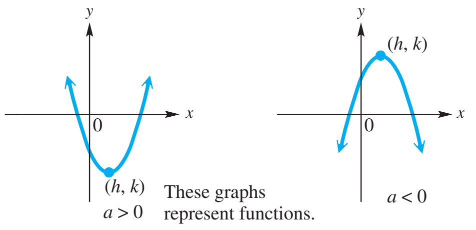
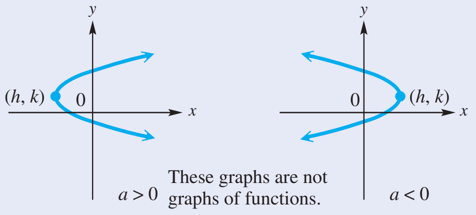
9.



vertex: $(-2, -1)$; axis: $y = -1$; domain: $(-\infty, -2]$; range: $(-\infty, \infty)$

In summary, the graphs of parabolas fall into the following categories.

### Graphs of Parabolas

| *Equation* | *Graph* |
|---|---|
| $y = ax^2 + bx + c$<br>or<br>$y = a(x - h)^2 + k$ |  <br>These graphs represent functions. |
| $x = ay^2 + by + c$<br>or<br>$x = a(y - k)^2 + h$ |  <br>These graphs are not graphs of functions. |

## 11.7 EXERCISES

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***Concept Check*** *In Exercises 1–4, answer each question.*

**1.** How can you determine just by looking at the equation of a parabola whether it has a vertical or a horizontal axis?

**2.** Why can't the graph of a quadratic function be a parabola with a horizontal axis?

**3.** How can you determine the number of $x$-intercepts of the graph of a quadratic function without graphing the function?

**4.** If the vertex of the graph of a quadratic function is $(1, -3)$, and the graph opens down, how many $x$-intercepts does the graph have?

*Find the vertex of each parabola.* ***See Examples 1–3.***

💿 **5.** $f(x) = x^2 + 8x + 10$

**6.** $f(x) = x^2 + 10x + 23$

💿 **7.** $f(x) = -2x^2 + 4x - 5$

**8.** $f(x) = -3x^2 + 12x - 8$

💿 **9.** $f(x) = x^2 + x - 7$

**10.** $f(x) = x^2 - x + 5$

*Find the vertex of each parabola. For each equation, decide whether the graph opens up, down, to the left, or to the right, and whether it is wider, narrower, or the same shape as the graph of $y = x^2$. If it is a parabola with vertical axis, find the discriminant and use it to determine the number of x-intercepts.* ***See Examples 1–3, 5, 8, and 9.***

💿 **11.** $f(x) = 2x^2 + 4x + 5$

**12.** $f(x) = 3x^2 - 6x + 4$

**13.** $f(x) = -x^2 + 5x + 3$

**14.** $f(x) = -x^2 + 7x + 2$

**15.** $x = \frac{1}{3}y^2 + 6y + 24$

**16.** $x = \frac{1}{2}y^2 + 10y - 5$

***Concept Check*** *Match each equation in Exercises 17–22 with its graph in choices A–F.*

**17.** $y = 2x^2 + 4x - 3$

**18.** $y = -x^2 + 3x + 5$

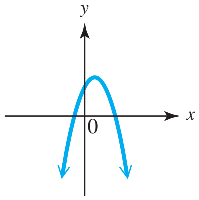
**19.** $y = -\frac{1}{2}x^2 - x + 1$

**20.** $x = y^2 + 6y + 3$

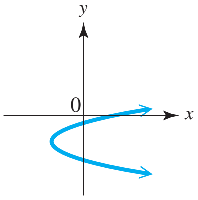
**21.** $x = -y^2 - 2y + 4$

**22.** $x = 3y^2 + 6y + 5$

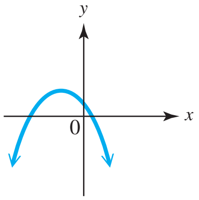
**A.**



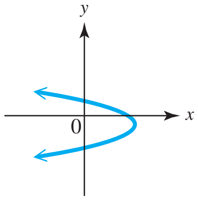
**B.**



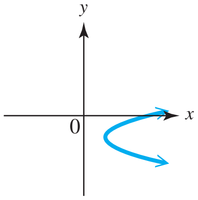
**C.**



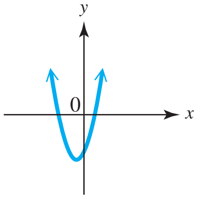
**D.**



**E.**



**F.**



*Graph each parabola. (Use the results of* ***Exercises 5–8*** *to help graph the parabolas in Exercises 23–26.) Give the vertex, axis, domain, and range.* ***See Examples 4, 8, and 9.***

**23.** $f(x) = x^2 + 8x + 10$

**24.** $f(x) = x^2 + 10x + 23$

**25.** $f(x) = -2x^2 + 4x - 5$

**26.** $f(x) = -3x^2 + 12x - 8$

**27.** $x = (y + 2)^2 + 1$

**28.** $x = (y + 3)^2 - 2$

**29.** $x = -\frac{1}{5}y^2 + 2y - 4$

**30.** $x = -\frac{1}{2}y^2 - 4y - 6$

**31.** $x = 3y^2 + 12y + 5$

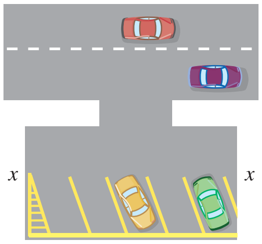
**32.** $x = 4y^2 + 16y + 11$

*Solve each problem.* ***See Examples 6 and 7.***

**33.** Find the pair of numbers whose sum is 40 and whose product is a maximum. (*Hint:* Let $x$ and $40 - x$ represent the two numbers.)

**34.** Find the pair of numbers whose sum is 60 and whose product is a maximum.

**35.** Polk Community College wants to construct a rectangular parking lot on land bordered on one side by a highway. It has 280 ft of fencing that is to be used to fence off the other three sides. What should be the dimensions of the lot if the enclosed area is to be a maximum? What is the maximum area?



**36.** Bonnie Wolansky has 100 ft of fencing material to enclose a rectangular exercise run for her dog. One side of the run will border her house, so she will only need to fence three sides. What dimensions will give the enclosure the maximum area? What is the maximum area?

**37.** If an object on Earth is projected upward with an initial velocity of 32 ft per sec, then its height after $t$ seconds is given by

$$s(t) = -16t^2 + 32t.$$

Find the maximum height attained by the object and the number of seconds it takes to hit the ground.

**38.** A projectile on Earth is fired straight upward so that its distance (in feet) above the ground $t$ seconds after firing is given by

$$s(t) = -16t^2 + 400t.$$

Find the maximum height it reaches and the number of seconds it takes to reach that height.

**39.** After experimentation, two physics students from American River College find that when a bottle of California wine is shaken several times, held upright, and uncorked, its cork travels according to the function defined by

$$s(t) = -16t^2 + 64t + 1,$$

where $s$ is its height in feet above the ground $t$ seconds after being released. After how many seconds will it reach its maximum height? What is the maximum height?

**40.** Professor Barbu has found that the number of students attending his intermediate algebra class is approximated by

$$S(x) = -x^2 + 20x + 80,$$

where $x$ is the number of hours that the Campus Center is open daily. Find the number of hours that the center should be open so that the number of students attending class is a maximum. What is this maximum number of students?

**41.** Klaus Loewy has a taco stand. He has found that his daily costs are approximated by

$$C(x) = x^2 - 40x + 610,$$

where $C(x)$ is the cost, in dollars, to sell $x$ units of tacos. Find the number of units of tacos he should sell to minimize his costs. What is the minimum cost?

**42.** Mohammad Asghar has a frozen yogurt cart. His daily costs are approximated by

$$C(x) = x^2 - 70x + 1500,$$

where $C(x)$ is the cost, in dollars, to sell $x$ units of frozen yogurt. Find the number of units of frozen yogurt he must sell to minimize his costs. What is the minimum cost?

**43.** The total receipts from individual income taxes by the U.S. Treasury in the years 2000–2007 can be modeled by the quadratic function defined by

$$f(x) = 22.88x^2 - 141.3x + 1044,$$

where $x = 0$ represents 2000, $x = 1$ represents 2001, and so on, and $f(x)$ is in billions of dollars. (*Source: World Almanac and Book of Facts*.)

**(a)** Since the coefficient of $x^2$ given in the model is positive, the graph of this quadratic function is a parabola that opens up. Will the $y$-value of the vertex of this graph be a maximum or minimum?

**(b)** In what year during this period were total receipts from individual taxes a minimum? (Round down for the year.) Use the actual $x$-value of the vertex, to the nearest tenth, to find this amount.

**44.** The percent of births in the United States to teenage mothers in the years 1990–2005 can be modeled by the quadratic function defined by

$$f(x) = -0.0198x^2 + 0.1054x + 12.87,$$

where $x = 0$ represents 1990, $x = 1$ represents 1991, and so on. (*Source:* U.S. National Center for Health Statistics.)

**(a)** Since the coefficient of $x^2$ in the model is negative, the graph of this quadratic function is a parabola that opens down. Will the $y$-value of the vertex of this graph be a maximum or a minimum?

**(b)** In what year during this period was the percent of births in the U.S. to teenage mothers a maximum? (Round down for the year.) Use the actual $x$-value of the vertex, to the nearest tenth, to find this percent.

**45.** The graph on the next page shows how Social Security trust fund assets are expected to change, and suggests that a quadratic function would be a good fit to the data. The data are approximated by the function defined by

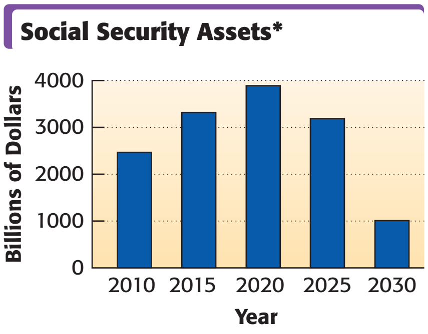
$$f(x) = -20.57x^2 + 758.9x - 3140.$$

In the model, $x = 10$ represents 2010, $x = 15$ represents 2015, and so on, and $f(x)$ is in billions of dollars.

**(a)** *Concept Check* How could you have predicted this quadratic model would have a negative coefficient for $x^2$, based only on the graph shown?

**(b)** Algebraically determine the vertex of the graph, with coordinates to four significant digits.

**(c)** Interpret the answer to part (b) as it applies to this application.



*Projected

***Source:*** Social Security Administration.

**46.** The graph shows the performance of investment portfolios with different mixtures of U.S. and foreign investments over a 25-yr period.

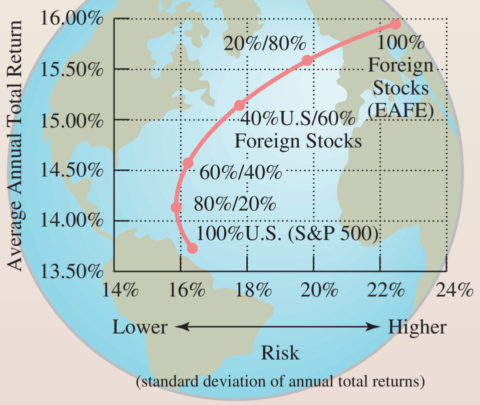
**(a)** Is this the graph of a function? Explain.

**(b)** What investment mixture shown on the graph appears to represent the vertex? What relative amount of risk does this point represent? What return on investment does it provide?

**(c)** Which point on the graph represents the riskiest investment mixture? What return on investment does it provide?

LOOKING FOR THE RIGHT MIX

A portfolio that includes international investments potentially can produce higher returns with lower risk than a 100% U.S. stock portfolio.



*Source*: *Financial Ink* Newsletter, Investment Management and Research, Inc. Thanks to David Van Geffen for this information.

**47.** A charter flight charges a fare of \$200 per person, plus \$4 per person for each unsold seat on the plane. If the plane holds 100 passengers and if $x$ represents the number of unsold seats, find the following.

**(a)** A function defined by $R(x)$ that describes the total revenue received for the flight (*Hint:* Multiply the number of people flying, $100 - x$, by the price per ticket, $200 + 4x$.)

**(b)** The graph of the function from part (a)

**(c)** The number of unsold seats that will produce the maximum revenue

**(d)** The maximum revenue

**48.** For a trip to a resort, a charter bus company charges a fare of \$48 per person, plus \$2 per person for each unsold seat on the bus. If the bus has 42 seats and $x$ represents the number of unsold seats, find the following.

**(a)** A function defined by $R(x)$ that describes the total revenue from the trip (*Hint:* Multiply the total number riding, $42 - x$, by the price per ticket, $48 + 2x$.)

**(b)** The graph of the function from part (a)

**(c)** The number of unsold seats that produces the maximum revenue

**(d)** The maximum revenue

## PREVIEW EXERCISES

*Graph each interval on a number line.* ***See Section 2.8.***

**49.** $[1, 5]$ **50.** $(-6, 1]$ **51.** $(-\infty, 1] \cup [5, \infty)$

*Solve each inequality.* ***See Section 2.8.***

**52.** $3 - x \le 5$ **53.** $-2x + 1 < 4$ **54.** $-\frac{1}{2}x - 3 > 5$

# 11.8 Polynomial and Rational Inequalities

**OBJECTIVES**

1. Solve quadratic inequalities.
2. Solve polynomial inequalities of degree 3 or greater.
3. Solve rational inequalities.

**OBJECTIVE 1** **Solve quadratic inequalities.** Now we combine the methods of solving linear inequalities with the methods of solving quadratic equations to solve *quadratic inequalities.*

### Quadratic Inequality

A **quadratic inequality** can be written in the form

$$\mathbf{ax^2 + bx + c < 0,} \qquad \mathbf{ax^2 + bx + c > 0,}$$

$$\mathbf{ax^2 + bx + c \le 0,} \qquad \text{or} \qquad \mathbf{ax^2 + bx + c \ge 0,}$$

where $a$, $b$, and $c$ are real numbers, with $a \neq 0$.

One way to solve a quadratic inequality is by graphing the related quadratic function.

### EXAMPLE 1 Solving Quadratic Inequalities by Graphing

Solve each inequality.

**(a)** $x^2 - x - 12 > 0$

To solve the inequality, we graph the related quadratic function defined by $f(x) = x^2 - x - 12$. We are particularly interested in the $x$-intercepts, which are found as in **Section 11.7** by letting $f(x) = 0$ and solving the following quadratic equation.

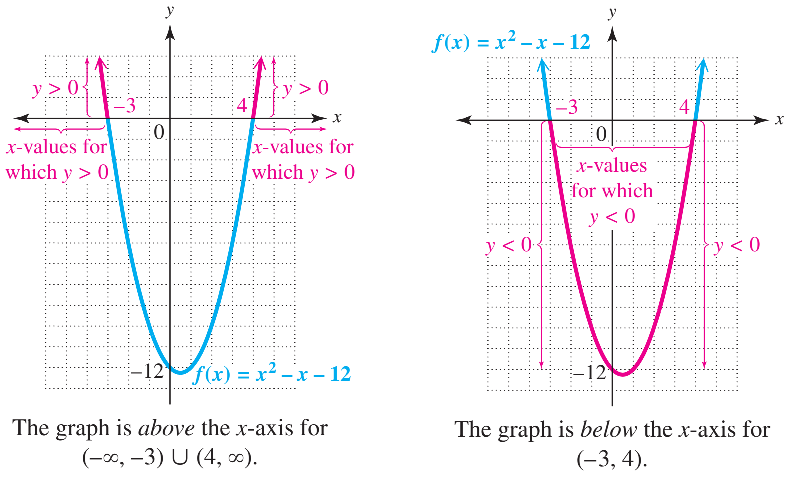
$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0 \qquad \text{Factor.}$$

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-factor property}$$

$$x = 4 \quad \text{or} \quad x = -3 \leftarrow \text{The } x\text{-intercepts are } (4, 0) \text{ and } (-3, 0).$$

The graph, which opens up since the coefficient of $x^2$ is positive, is shown in **FIGURE 18(a)**. Notice from this graph that $x$-values less than $-3$ or greater than 4 result in $y$-values *greater than* 0. Thus, the solution set of $x^2 - x - 12 > 0$, written in interval notation, is $(-\infty, -3) \cup (4, \infty)$.



The graph is *above* the $x$-axis for $(-\infty, -3) \cup (4, \infty)$.

**(a)**

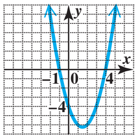
The graph is *below* the $x$-axis for $(-3, 4)$.

**(b)**

**FIGURE 18**

**NOW TRY EXERCISE 1**

Use the graph to solve each quadratic inequality.



$f(x) = x^2 - 3x - 4$

**(a)** $x^2 - 3x - 4 > 0$

**(b)** $x^2 - 3x - 4 < 0$

**(b)** $x^2 - x - 12 < 0$

We want values of $y$ that are *less than* 0. Referring to **FIGURE 18(b)**, we notice from the graph that $x$-values between $-3$ and 4 result in $y$-values less than 0. Thus, the solution set of $x^2 - x - 12 < 0$, written in interval notation, is $(-3, 4)$. NOW TRY

---

**NOTE** If the inequalities in **Example 1** had used $\geq$ and $\leq$, the solution sets would have included the $x$-values of the intercepts, which make the quadratic expression equal to 0. They would have been written in interval notation as

$$(-\infty, -3] \cup [4, \infty) \quad \text{and} \quad [-3, 4].$$

Square brackets would indicate that the endpoints $-3$ and 4 are *included* in the solution sets.

---

Another method for solving a quadratic inequality uses the basic ideas of **Example 1** without actually graphing the related quadratic function.

## EXAMPLE 2 Solving a Quadratic Inequality Using Test Numbers

Solve and graph the solution set of $x^2 - x - 12 > 0$.

Solve the quadratic equation $x^2 - x - 12 = 0$ by factoring, as in **Example 1(a).**

$$\begin{aligned} (x-4)(x+3) &= 0 \\ x - 4 = 0 \quad \text{or} \quad x + 3 &= 0 \\ x = 4 \quad \text{or} \qquad x &= -3 \end{aligned}$$

The numbers 4 and $-3$ divide a number line into Intervals A, B, and C, as shown in **FIGURE 19**. ***Be careful to put the lesser number on the left.***

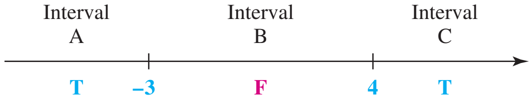


**FIGURE 19**

Notice the similarity between **FIGURE 19** and the $x$-axis with intercepts $(-3, 0)$ and $(4, 0)$ in **FIGURE 18(a)**.

The numbers 4 and $-3$ are the only numbers that make the quadratic expression $x^2 - x - 12$ equal to 0. All other numbers make the expression either positive or negative. The sign of the expression can change from positive to negative or from negative to positive only at a number that makes it 0. Therefore, if one number in an interval satisfies the inequality, then all the numbers in that interval will satisfy the inequality.

To see if the numbers in Interval A satisfy the inequality, choose any number from Interval A in **FIGURE 19** (that is, any number less than $-3$). We choose $-5$. Substitute this test number for $x$ in the original inequality $x^2 - x - 12 > 0$.

$$\begin{aligned} x^2 - x - 12 &> 0 && \text{Original inequality} \\ (-5)^2 - (-5) - 12 &\overset{?}{>} 0 && \text{Let } x = -5. \\ 25 + 5 - 12 &\overset{?}{>} 0 && \text{Simplify.} \\ 18 &> 0 \ ✓ && \text{True} \end{aligned}$$

Use parentheses to avoid sign errors.

Because $-5$ satisfies the inequality, *all* numbers from Interval A are solutions.

**NOW TRY ANSWERS**

1. **(a)** $(-\infty, -1) \cup (4, \infty)$
   **(b)** $(-1, 4)$

**NOW TRY EXERCISE 2**

Solve and graph the solution set.

$$x^2 + 2x - 8 > 0$$

Now try 0 from Interval B.

$$x^2 - x - 12 > 0 \quad \text{Original inequality}$$
$$0^2 - 0 - 12 \overset{?}{>} 0 \quad \text{Let } x = 0.$$
$$-12 > 0 \quad \text{False}$$

The numbers in Interval B are *not* solutions. Verify that the test number 5 from Interval C satisfies the inequality, so all numbers there are also solutions.

Based on these results (shown by the colored letters in **FIGURE 19**), the solution set includes the numbers in Intervals A and C, as shown on the graph in **FIGURE 20**. The solution set is written in interval notation as

$$(-\infty, -3) \cup (4, \infty).$$

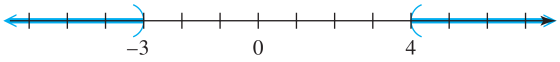


**FIGURE 20**

This agrees with the solution set found in **Example 1(a).** NOW TRY

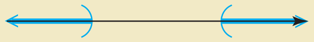
In summary, follow these steps to solve a quadratic inequality.

### Solving a Quadratic Inequality

***Step 1*** **Write the inequality as an equation and solve it.**

***Step 2*** **Use the solutions from Step 1 to determine intervals.** Graph the numbers found in Step 1 on a number line. These numbers divide the number line into intervals.

***Step 3*** **Find the intervals that satisfy the inequality.** Substitute a test number from each interval into the original inequality to determine the intervals that satisfy the inequality. All numbers in those intervals are in the solution set. A graph of the solution set will usually look like one of these. (Square brackets might be used instead of parentheses.)



or

***Step 4*** **Consider the endpoints separately.** The numbers from Step 1 are included in the solution set if the inequality symbol is $\leq$ or $\geq$. They are not included if it is $<$ or $>$.

**NOW TRY EXERCISE 3**

Solve each inequality.

**(a)** $(4x - 1)^2 > -3$

**(b)** $(4x - 1)^2 < -3$

## EXAMPLE 3 Solving Special Cases

Solve each inequality.

**(a)** $(2x - 3)^2 > -1$

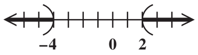
Because $(2x - 3)^2$ is never negative, it is always greater than $-1$. Thus, the solution set for $(2x - 3)^2 > -1$ is the set of all real numbers, $(-\infty, \infty)$.

**(b)** $(2x - 3)^2 < -1$

Using the same reasoning as in part (a), there is no solution for this inequality. The solution set is $\emptyset$. NOW TRY

**NOW TRY ANSWERS**

**2.** $(-\infty, -4) \cup (2, \infty)$



**3.** **(a)** $(-\infty, \infty)$ **(b)** $\emptyset$

## OBJECTIVE 2 Solve polynomial inequalities of degree 3 or greater.

**NOW TRY EXERCISE 4**

Solve and graph the solution set.

$(x + 4)(x - 3)(2x + 1) \leq 0$

### EXAMPLE 4 Solving a Third-Degree Polynomial Inequality

Solve and graph the solution set of $(x - 1)(x + 2)(x - 4) \leq 0$.

This is a *cubic* (third-degree) inequality rather than a quadratic inequality, but it can be solved using the preceding method by extending the zero-factor property to more than two factors. (Step 1)

$$(x - 1)(x + 2)(x - 4) = 0$$

Set the factored polynomial *equal* to 0.

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$

Zero-factor property

$$x = 1 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 4$$

Solve each equation.

Locate the numbers $-2$, 1, and 4 on a number line, as in **FIGURE 21**, to determine the Intervals A, B, C, and D. (Step 2)

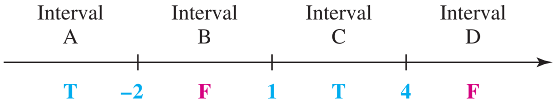


**FIGURE 21**

Substitute a test number from each interval in the *original* inequality to determine which intervals satisfy the inequality. (Step 3)

| *Interval* | *Test Number* | *Test of Inequality* | *True or False?* |
|---|---|---|---|
| A | $-3$ | $-28 \leq 0$ | T |
| B | 0 | $8 \leq 0$ | F |
| C | 2 | $-8 \leq 0$ | T |
| D | 5 | $28 \leq 0$ | F |

We use a table to organize this information. (Verify it.)

The numbers in Intervals A and C are in the solution set, which is written in interval notation as $(-\infty, -2] \cup [1, 4]$, and graphed in **FIGURE 22**. The three endpoints are included since the inequality symbol, $\leq$, includes equality. (Step 4)

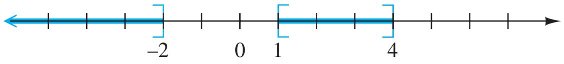


**FIGURE 22**

NOW TRY

## OBJECTIVE 3 Solve rational inequalities.

Inequalities that involve rational expressions, called **rational inequalities,** are solved similarly using the following steps.

### Solving a Rational Inequality

*Step 1* **Write the inequality so that 0 is on one side** and there is a single fraction on the other side.

*Step 2* **Determine the numbers that make the numerator or denominator equal to 0.**

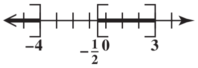
*Step 3* **Divide a number line into intervals.** Use the numbers from Step 2.

*Step 4* **Find the intervals that satisfy the inequality.** Test a number from each interval by substituting it into the *original* inequality.

*Step 5* **Consider the endpoints separately.** Exclude any values that make the denominator 0.

NOW TRY ANSWER

4. $(-\infty, -4] \cup \left[-\frac{1}{2}, 3\right]$



**CAUTION** ***When solving a rational inequality, any number that makes the denominator 0 must be excluded from the solution set.***

*NOW TRY EXERCISE 5*

Solve and graph the solution set.

$$\frac{3}{x+1} > 4$$

## EXAMPLE 5 Solving a Rational Inequality

Solve and graph the solution set of $\frac{-1}{x-3} > 1$.

Write the inequality so that 0 is on one side. (Step 1)

$$\frac{-1}{x-3} - 1 > 0 \qquad \text{Subtract 1.}$$

$$\frac{-1}{x-3} - \frac{x-3}{x-3} > 0 \qquad \text{Use } x-3 \text{ as the common denominator.}$$

Be careful with signs.

$$\frac{-1-x+3}{x-3} > 0 \qquad \text{Write the left side as a single fraction.}$$

$$\frac{-x+2}{x-3} > 0 \qquad \text{Combine like terms in the numerator.}$$

The sign of $\frac{-x+2}{x-3}$ will change from positive to negative or negative to positive only at those numbers that make the numerator or denominator 0. The number 2 makes the numerator 0, and 3 makes the denominator 0. (Step 2) These two numbers, 2 and 3, divide a number line into three intervals. See **FIGURE 23**. (Step 3)

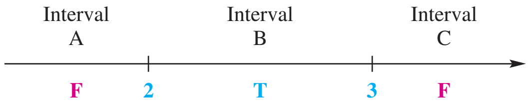


**FIGURE 23**

Testing a number from each interval in the *original* inequality, $\frac{-1}{x-3} > 1$, gives the results shown in the table. (Step 4)

| *Interval* | *Test Number* | *Test of Inequality* | *True or False?* |
|---|---|---|---|
| A | 0 | $\frac{1}{3} > 1$ | F |
| B | 2.5 | $2 > 1$ | T |
| C | 4 | $-1 > 1$ | F |

The solution set is the interval $(2, 3)$. This interval does not include 3 since it would make the denominator of the original equality 0. The number 2 is not included either since the inequality symbol, $>$, does not include equality. (Step 5) See **FIGURE 24**.

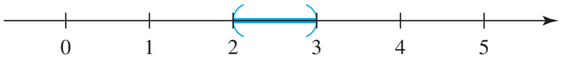
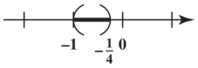


**FIGURE 24**

NOW TRY

*NOW TRY ANSWER*

**5.** $\left(-1, -\frac{1}{4}\right)$



NOW TRY EXERCISE 6

Solve and graph the solution set.

$$\frac{x-3}{x+3} \le 2$$

## EXAMPLE 6 Solving a Rational Inequality

Solve and graph the solution set of $\frac{x-2}{x+2} \le 2$.

Write the inequality so that 0 is on one side. (Step 1)

$$\frac{x-2}{x+2} - 2 \le 0 \qquad \text{Subtract 2.}$$

$$\frac{x-2}{x+2} - \frac{2(x+2)}{x+2} \le 0 \qquad \text{Use } x+2 \text{ as the common denominator.}$$

Be careful with signs.

$$\frac{x-2-2x-4}{x+2} \le 0 \qquad \text{Write as a single fraction.}$$

$$\frac{-x-6}{x+2} \le 0 \qquad \text{Combine like terms in the numerator.}$$

The number $-6$ makes the numerator 0, and $-2$ makes the denominator 0. (Step 2) These two numbers determine three intervals. (Step 3) Test one number from each interval (Step 4) to see that the solution set is

$$(-\infty, -6] \cup (-2, \infty).$$

The number $-6$ satisfies the original inequality, but $-2$ does not since it makes the denominator 0. (Step 5) **FIGURE 25** shows a graph of the solution set.

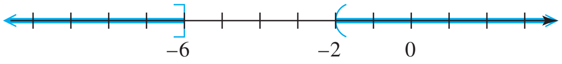
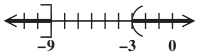


**FIGURE 25**

NOW TRY

NOW TRY ANSWER

6. $(-\infty, -9] \cup (-3, \infty)$



## 11.8 EXERCISES

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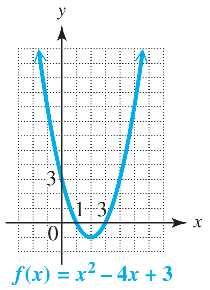
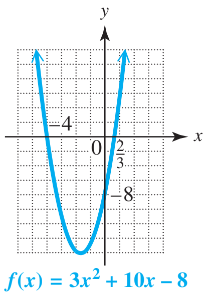
*In Exercises 1–3, the graph of a quadratic function f is given. Use the graph to find the solution set of each equation or inequality.* ***See Example 1.***

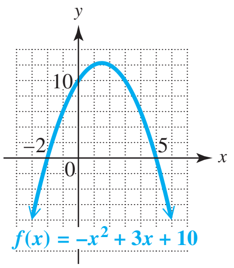
1. **(a)** $x^2 - 4x + 3 = 0$
   **(b)** $x^2 - 4x + 3 > 0$
   **(c)** $x^2 - 4x + 3 < 0$



2. **(a)** $3x^2 + 10x - 8 = 0$
   **(b)** $3x^2 + 10x - 8 \ge 0$
   **(c)** $3x^2 + 10x - 8 < 0$



3. **(a)** $-x^2 + 3x + 10 = 0$
   **(b)** $-x^2 + 3x + 10 \ge 0$
   **(c)** $-x^2 + 3x + 10 \le 0$



4. *Concept Check* The solution set of the inequality $x^2 + x - 12 < 0$ is the interval $(-4, 3)$. Without actually performing any work, give the solution set of the inequality $x^2 + x - 12 \ge 0$.

*Solve each inequality, and graph the solution set.* **See Example 2.** *(Hint: In Exercises 21 and 22, use the quadratic formula.)*

**5.** $(x + 1)(x - 5) > 0$

**6.** $(x + 6)(x - 2) > 0$

**7.** $(x + 4)(x - 6) < 0$

**8.** $(x + 4)(x - 8) < 0$

**9.** $x^2 - 4x + 3 \geq 0$

**10.** $x^2 - 3x - 10 \geq 0$

**11.** $10x^2 + 9x \geq 9$

**12.** $3x^2 + 10x \geq 8$

**13.** $4x^2 - 9 \leq 0$

**14.** $9x^2 - 25 \leq 0$

**15.** $6x^2 + x \geq 1$

**16.** $4x^2 + 7x \geq -3$

**17.** $z^2 - 4z \geq 0$

**18.** $x^2 + 2x < 0$

**19.** $3x^2 - 5x \leq 0$

**20.** $2z^2 + 3z > 0$

**21.** $x^2 - 6x + 6 \geq 0$

**22.** $3x^2 - 6x + 2 \leq 0$

*Solve each inequality.* **See Example 3.**

**23.** $(4 - 3x)^2 \geq -2$

**24.** $(7 - 6x)^2 \geq -1$

**25.** $(3x + 5)^2 \leq -4$

**26.** $(8x + 5)^2 \leq -5$

*Solve each inequality, and graph the solution set.* **See Example 4.**

**27.** $(x - 1)(x - 2)(x - 4) < 0$

**28.** $(2x + 1)(3x - 2)(4x + 7) < 0$

**29.** $(x - 4)(2x + 3)(3x - 1) \geq 0$

**30.** $(x + 2)(4x - 3)(2x + 7) \geq 0$

*Solve each inequality, and graph the solution set.* **See Examples 5 and 6.**

**31.** $\dfrac{x - 1}{x - 4} > 0$

**32.** $\dfrac{x + 1}{x - 5} > 0$

**33.** $\dfrac{2x + 3}{x - 5} \leq 0$

**34.** $\dfrac{3x + 7}{x - 3} \leq 0$

**35.** $\dfrac{8}{x - 2} \geq 2$

**36.** $\dfrac{20}{x - 1} \geq 1$

**37.** $\dfrac{3}{2x - 1} < 2$

**38.** $\dfrac{6}{x - 1} < 1$

**39.** $\dfrac{x - 3}{x + 2} \geq 2$

**40.** $\dfrac{m + 4}{m + 5} \geq 2$

**41.** $\dfrac{x - 8}{x - 4} < 3$

**42.** $\dfrac{2t - 3}{t + 1} > 4$

**43.** $\dfrac{4k}{2k - 1} < k$

**44.** $\dfrac{r}{r + 2} < 2r$

**45.** $\dfrac{2x - 3}{x^2 + 1} \geq 0$

**46.** $\dfrac{9x - 8}{4x^2 + 25} < 0$

**47.** $\dfrac{(3x - 5)^2}{x + 2} > 0$

**48.** $\dfrac{(5x - 3)^2}{2x + 1} \leq 0$

## PREVIEW EXERCISES

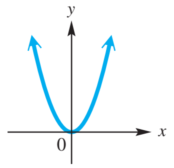
*Give the domain and the range of each function.* **See Section 7.3.**

**49.** $\{(0, 1), (1, 2), (2, 4), (3, 8)\}$

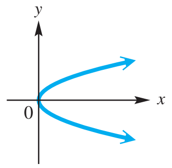
**50.** $f(x) = x^2$

*Decide whether each graph is that of a function.* **See Section 7.3.**

**51.**



**52.**



# CHAPTER 11 SUMMARY

## KEY TERMS

**11.1**

quadratic equation

**11.3**

quadratic formula
discriminant

**11.4**

quadratic in form

**11.6**

parabola
vertex
axis
quadratic function

**11.8**

quadratic inequality
rational inequality

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

1. The **quadratic formula** is
   - **A.** a formula to find the number of solutions of a quadratic equation
   - **B.** a formula to find the type of solutions of a quadratic equation
   - **C.** the standard form of a quadratic equation
   - **D.** a general formula for solving any quadratic equation.
2. A **quadratic function** is a function that can be written in the form
   - **A.** $f(x) = mx + b$ for real numbers $m$ and $b$
   - **B.** $f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$
   - **C.** $f(x) = ax^2 + bx + c$ for real numbers $a$, $b$, and $c$ $(a \neq 0)$
   - **D.** $f(x) = \sqrt{x}$ for $x \geq 0$.
3. A **parabola** is the graph of
   - **A.** any equation in two variables
   - **B.** a linear equation
   - **C.** an equation of degree 3
   - **D.** a quadratic equation in two variables, where one is first-degree.
4. The **vertex** of a parabola is
   - **A.** the point where the graph intersects the $y$-axis
   - **B.** the point where the graph intersects the $x$-axis
   - **C.** the lowest point on a parabola that opens up or the highest point on a parabola that opens down
   - **D.** the origin.
5. The **axis** of a parabola is
   - **A.** either the $x$-axis or the $y$-axis
   - **B.** the vertical line (of a vertical parabola) or the horizontal line (of a horizontal parabola) through the vertex
   - **C.** the lowest or highest point on the graph of a parabola
   - **D.** a line through the origin.
6. A parabola is **symmetric about its axis** since
   - **A.** its graph is near the axis
   - **B.** its graph is identical on each side of the axis
   - **C.** its graph looks different on each side of the axis
   - **D.** its graph intersects the axis.

### ANSWERS

**1.** D; *Example:* The solutions of $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. **2.** C; *Examples:* $f(x) = x^2 - 2$, $f(x) = (x + 4)^2 + 1$, $f(x) = x^2 - 4x + 5$ **3.** D; *Examples:* See the figures in the Quick Review for **Sections 11.6 and 11.7.** **4.** C; *Example:* The graph of $y = (x + 3)^2$ has vertex $(-3, 0)$, which is the lowest point on the graph. **5.** B; *Example:* The axis of $y = (x + 3)^2$ is the vertical line $x = -3$. **6.** B; *Example:* Since the graph of $y = (x + 3)^2$ is symmetric about its axis $x = -3$, the points $(-2, 1)$ and $(-4, 1)$ are on the graph.

## QUICK REVIEW

| CONCEPTS | EXAMPLES |
|---|---|
| **11.1 Solving Quadratic Equations by the Square Root Property** | |
| **Square Root Property**<br>If $x$ and $k$ are complex numbers and $x^2 = k$, then<br>$\boldsymbol{x = \sqrt{k}}$ or $\boldsymbol{x = -\sqrt{k}}$. | Solve $(x - 1)^2 = 8$.<br>$x - 1 = \sqrt{8}$ or $x - 1 = -\sqrt{8}$<br>$x = 1 + 2\sqrt{2}$ or $x = 1 - 2\sqrt{2}$<br>The solution set is $\{1 + 2\sqrt{2}, 1 - 2\sqrt{2}\}$, or $\{1 \pm 2\sqrt{2}\}$. |

(continued)

**CONCEPTS** | **EXAMPLES**

## 11.2 Solving Quadratic Equations by Completing the Square

**Completing the Square**

To solve $ax^2 + bx + c = 0$ (with $a \neq 0$):

***Step 1*** If $a \neq 1$, divide each side by $a$.

***Step 2*** Write the equation with the variable terms on one side and the constant on the other.

***Step 3*** Take half the coefficient of $x$ and square it.

***Step 4*** Add the square to each side.

***Step 5*** Factor the perfect square trinomial, and write it as the square of a binomial. Simplify the other side.

***Step 6*** Use the square root property to complete the solution.

Solve $2x^2 - 4x - 18 = 0$.

$$\begin{aligned} x^2 - 2x - 9 &= 0 && \text{Divide by 2.} \\ x^2 - 2x &= 9 && \text{Add 9.} \\ \left[\tfrac{1}{2}(-2)\right]^2 &= (-1)^2 = 1 \\ x^2 - 2x + 1 &= 9 + 1 && \text{Add 1.} \\ (x - 1)^2 &= 10 && \text{Factor. Add.} \end{aligned}$$

$$\begin{aligned} x - 1 &= \sqrt{10} \quad \text{or} \quad x - 1 = -\sqrt{10} && \text{Square root} \\ x &= 1 + \sqrt{10} \quad \text{or} \quad x = 1 - \sqrt{10} && \text{property} \end{aligned}$$

The solution set is $\left\{1 + \sqrt{10}, 1 - \sqrt{10}\right\}$, or $\left\{1 \pm \sqrt{10}\right\}$

## 11.3 Solving Quadratic Equations by the Quadratic Formula

**Quadratic Formula**

The solutions of $ax^2 + bx + c = 0$ (with $a \neq 0$) are given by

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

**The Discriminant**

If $a$, $b$, and $c$ are integers, then the discriminant, $\boldsymbol{b^2 - 4ac}$, of $ax^2 + bx + c = 0$ determines the number and type of solutions as follows.

| *Discriminant* | *Number and Type of Solutions* |
|---|---|
| Positive, the square of an integer | Two rational solutions |
| Positive, not the square of an integer | Two irrational solutions |
| Zero | One rational solution |
| Negative | Two nonreal complex solutions |

Solve $3x^2 + 5x + 2 = 0$.

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(2)}}{2(3)} = \frac{-5 \pm 1}{6}$$

$$x = \frac{-5 + 1}{6} = -\frac{2}{3} \quad \text{or} \quad x = \frac{-5 - 1}{6} = -1$$

The solution set is $\left\{-1, -\frac{2}{3}\right\}$.

For $x^2 + 3x - 10 = 0$, the discriminant is

$3^2 - 4(1)(-10) = 49.$ Two rational solutions

For $4x^2 + x + 1 = 0$, the discriminant is

$1^2 - 4(4)(1) = -15.$ Two nonreal complex solutions

## 11.4 Equations Quadratic in Form

A nonquadratic equation that can be written in the form

$$\boldsymbol{au^2 + bu + c = 0,}$$

for $a \neq 0$ and an algebraic expression $u$, is called quadratic in form. Substitute $u$ for the expression, solve for $u$, and then solve for the variable in the expression.

Solve $3(x + 5)^2 + 7(x + 5) + 2 = 0$.

$$\begin{aligned} 3u^2 + 7u + 2 &= 0 && \text{Let } u = x + 5. \\ (3u + 1)(u + 2) &= 0 && \text{Factor.} \end{aligned}$$

$$\begin{aligned} u = -\frac{1}{3} \quad &\text{or} \quad u = -2 \\ x + 5 = -\frac{1}{3} \quad &\text{or} \quad x + 5 = -2 && x + 5 = u \\ x = -\frac{16}{3} \quad &\text{or} \quad x = -7 && \text{Subtract 5.} \end{aligned}$$

The solution set is $\left\{-7, -\frac{16}{3}\right\}$.

(continued)

***CONCEPTS*** ***EXAMPLES***

## 11.5 Formulas and Further Applications

To solve a formula for a squared variable, proceed as follows.

**(a) If the variable appears only to the second power:** Isolate the squared variable on one side of the equation, and then use the square root property.

**(b) If the variable appears to the first and second powers:** Write the equation in standard form, and then use the quadratic formula.

Solve $A = \frac{2mp}{r^2}$ for $r$.

$r^2 A = 2mp$ Multiply by $r^2$.

$r^2 = \frac{2mp}{A}$ Divide by $A$.

$r = \pm\sqrt{\frac{2mp}{A}}$ Square root property

$r = \frac{\pm\sqrt{2mpA}}{A}$ Rationalize denominator.

Solve $x^2 + rx = t$ for $x$.

$x^2 + rx - t = 0$ Standard form

$x = \frac{-r \pm \sqrt{r^2 - 4(1)(-t)}}{2(1)}$

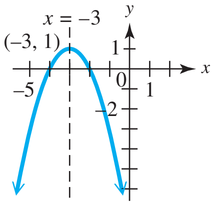
$a = 1, b = r, c = -t$

$x = \frac{-r \pm \sqrt{r^2 + 4t}}{2}$

## 11.6 Graphs of Quadratic Functions

1. The graph of the quadratic function defined by $\boldsymbol{F(x) = a(x - h)^2 + k}$, $a \neq 0$, is a parabola with vertex at $(h, k)$ and the vertical line $x = h$ as axis.
2. The graph opens up if $a$ is positive and down if $a$ is negative.
3. The graph is wider than the graph of $f(x) = x^2$ if $0 < |a| < 1$ and narrower if $|a| > 1$.

Graph $f(x) = -(x + 3)^2 + 1$.



The graph opens down since $a < 0$.

Vertex: $(-3, 1)$

Axis: $x = -3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1]$

## 11.7 More about Parabolas and Their Applications

The vertex of the graph of $\boldsymbol{f(x) = ax^2 + bx + c}$, $a \neq 0$, may be found by completing the square.

The vertex has coordinates $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

**Graphing a Quadratic Function**

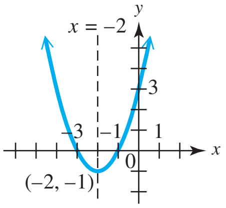
***Step 1*** Determine whether the graph opens up or down.

***Step 2*** Find the vertex.

***Step 3*** Find the $x$-intercepts (if any). Find the $y$-intercept.

***Step 4*** Find and plot additional points as needed.

Graph $f(x) = x^2 + 4x + 3$.



The graph opens up since $a > 0$.

Vertex: $(-2, -1)$

The solutions of $x^2 + 4x + 3 = 0$ are $-1$ and $-3$, so the $x$-intercepts are $(-1, 0)$ and $(-3, 0)$.

$f(0) = 3$, so the $y$-intercept is $(0, 3)$.

Domain: $(-\infty, \infty)$

Range: $[-1, \infty)$

**Horizontal Parabolas**

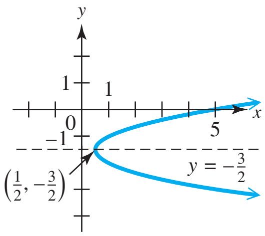
The graph of

$$\boldsymbol{x = ay^2 + by + c \quad \text{or} \quad x = a(y - k)^2 + h}$$

is a horizontal parabola with vertex $(h, k)$ and the horizontal line $y = k$ as axis. The graph opens to the right if $a > 0$ and to the left if $a < 0$.

***Horizontal parabolas do not represent functions.***

Graph $x = 2y^2 + 6y + 5$.



The graph opens to the right since $a > 0$.

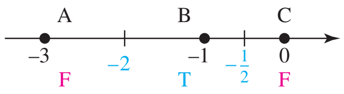
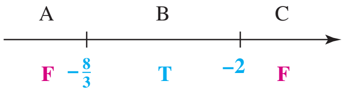
Vertex: $\left(\frac{1}{2}, -\frac{3}{2}\right)$

Axis: $y = -\frac{3}{2}$

Domain: $\left[\frac{1}{2}, \infty\right)$

Range: $(-\infty, \infty)$

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **11.8 Polynomial and Rational Inequalities** | |
| **Solving a Quadratic (or Higher-Degree Polynomial) Inequality**<br>***Step 1*** Write the inequality as an equation and solve. | Solve $2x^2 + 5x + 2 < 0$.<br>$2x^2 + 5x + 2 = 0$<br>$(2x + 1)(x + 2) = 0$<br>$x = -\frac{1}{2}$ or $x = -2$ |
| ***Step 2*** Use the numbers found in Step 1 to divide a number line into intervals. |  <br>Intervals: $(-\infty, -2)$, $\left(-2, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \infty\right)$ |
| ***Step 3*** Substitute a test number from each interval into the original inequality to determine the intervals that belong to the solution set.<br>***Step 4*** Consider the endpoints separately. | Test values: $-3, -1, 0$<br>$x = -3$ makes the original inequality false, $x = -1$ makes it true, and $x = 0$ makes it false. Choose the interval(s) which yield(s) a true statement. The solution set is the interval $\left(-2, -\frac{1}{2}\right)$. |
| **Solving a Rational Inequality**<br>***Step 1*** Write the inequality so that 0 is on one side and there is a single fraction on the other side. | Solve $\frac{x}{x + 2} \geq 4$.<br>$\frac{x}{x + 2} - 4 \geq 0$ Subtract 4.<br>$\frac{x}{x + 2} - \frac{4(x + 2)}{x + 2} \geq 0$ Write with a common denominator.<br>$\frac{-3x - 8}{x + 2} \geq 0$ Subtract fractions. |
| ***Step 2*** Determine the numbers that make the numerator or denominator 0. | $-\frac{8}{3}$ makes the numerator 0, and $-2$ makes the denominator 0. |
| ***Step 3*** Use the numbers from Step 2 to divide a number line into intervals. |  |
| ***Step 4*** Substitute a test number from each interval into the original inequality to determine the intervals that belong to the solution set. | $-4$ from A makes the original inequality false, $-\frac{7}{3}$ from B makes it true, and 0 from C makes it false. |
| ***Step 5*** Consider the endpoints separately. | The solution set is the interval $\left[-\frac{8}{3}, -2\right)$. The endpoint $-2$ is not included since it makes the denominator 0. |

## CHAPTER 11 REVIEW EXERCISES

**11.1** *Solve each equation by using the square root property.*

**1.** $t^2 = 121$

**2.** $p^2 = 3$

**3.** $(2x + 5)^2 = 100$

***4.** $(3x - 2)^2 = -25$

---

*This exercise requires knowledge of complex numbers.

**5.** ***Concept Check*** A student gave the following "solution." ***WHAT WENT WRONG?***

$$\begin{aligned} x^2 &= 12 & \\ x &= \sqrt{12} & \text{Square root property} \\ x &= 2\sqrt{3} & \text{Simplify.} \end{aligned}$$

Solution set: $\{2\sqrt{3}\}$

**6.** The Singapore Flyer, the world's largest Ferris wheel as of 2008, has a height of 165 m. To find how long it would take a wallet dropped from the top of the Singapore Flyer to reach the ground, use the metric version of Galileo's formula,

$$d = 4.9t^2 \quad \text{(where } d \text{ is in meters).}$$

Round your answer to the nearest tenth of a second. (*Source:* www.singaporeflyer.com)

## 11.2 *Solve each equation by completing the square.*

**7.** $x^2 + 4x = 15$

**8.** $2x^2 - 3x = -1$

**9.** $2z^2 + 8z - 3 = 0$

***10.** $4x^2 - 3x + 6 = 0$

## 11.3 *Solve each equation by using the quadratic formula.*

**11.** $2x^2 + x - 21 = 0$

**12.** $x^2 + 5x = 7$

**13.** $(t + 3)(t - 4) = -2$

***14.** $2x^2 + 3x + 4 = 0$

***15.** $3p^2 = 2(2p - 1)$

**16.** $x(2x - 7) = 3x^2 + 3$

*Use the discriminant to predict whether the solutions to each equation are*

**A.** *two rational numbers*

**B.** *one rational number*

**C.** *two irrational numbers*

**D.** *two nonreal complex numbers.*

**17. (a)** $x^2 + 5x + 2 = 0$ **(b)** $4t^2 = 3 - 4t$

**18. (a)** $4x^2 = 6x - 8$ **(b)** $9z^2 + 30z + 25 = 0$

## 11.4 *Solve each equation. Check your solutions.*

**19.** $\frac{15}{x} = 2x - 1$

**20.** $\frac{1}{n} + \frac{2}{n + 1} = 2$

**21.** $-2r = \sqrt{\frac{48 - 20r}{2}}$

**22.** $8(3x + 5)^2 + 2(3x + 5) - 1 = 0$

**23.** $2x^{2/3} - x^{1/3} - 28 = 0$

**24.** $p^4 - 10p^2 + 9 = 0$

*Solve each problem. Round answers to the nearest tenth, as necessary.*

**25.** Bahaa Mourad paddled a canoe 20 mi upstream, then paddled back. If the rate of the current was 3 mph and the total trip took 7 hr, what was Bahaa's rate?

**26.** Carol-Ann Vassell drove 8 mi to pick up a friend, and then drove 11 mi to a mall at a rate 15 mph faster. If Carol-Ann's total travel time was 24 min, what was her rate on the trip to pick up her friend?

**27.** An old machine processes a batch of checks in 1 hr more time than a new one. How long would it take the old machine to process a batch of checks that the two machines together process in 2 hr?

**28.** Zoran Pantic can process a stack of invoices 1 hr faster than Claude Sassine can. Working together, they take 1.5 hr. How long would it take each person working alone?

## 11.5 *Solve each formula for the indicated variable. (Give answers with ±.)*

**29.** $k = \frac{rF}{wv^2}$ for $v$

**30.** $p = \sqrt{\frac{yz}{6}}$ for $y$

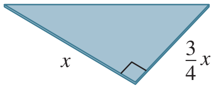
**31.** $mt^2 = 3mt + 6$ for $t$

---

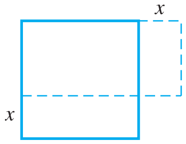
*This exercise requires knowledge of complex numbers.

*Solve each problem. Round answers to the nearest tenth, as necessary.*

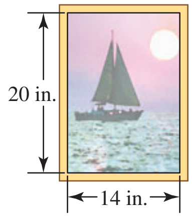
**32.** A large machine requires a part in the shape of a right triangle with a hypotenuse 9 ft less than twice the length of the longer leg. The shorter leg must be $\frac{3}{4}$ the length of the longer leg. Find the lengths of the three sides of the part.



**33.** A square has an area of 256 cm$^2$. If the same amount is removed from one dimension and added to the other, the resulting rectangle has an area 16 cm$^2$ less. Find the dimensions of the rectangle.



**34.** Allen Moser wants to buy a mat for a photograph that measures 14 in. by 20 in. He wants to have an even border around the picture when it is mounted on the mat. If the area of the mat he chooses is 352 in.$^2$, how wide will the border be?



**35.** If a square piece of cardboard has 3-in. squares cut from its corners and then has the flaps folded up to form an open-top box, the volume of the box is given by the formula $V = 3(x - 6)^2$, where $x$ is the length of each side of the original piece of cardboard in inches. What original length would yield a box with volume 432 in.$^3$?

**36.** Wachovia Center Tower in Raleigh, North Carolina, is 400 ft high. Suppose that a ball is projected upward from the top of the tower, and its position in feet above the ground is given by the quadratic function defined by

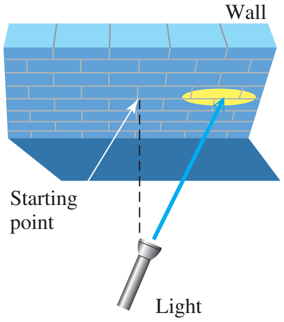
$$f(t) = -16t^2 + 45t + 400,$$

where $t$ is the number of seconds elapsed. How long will it take for the ball to reach a height of 200 ft above the ground? (*Source: World Almanac and Book of Facts.*)

**37.** A searchlight moves horizontally back and forth along a wall with the distance of the light from a starting point at $t$ minutes given by the quadratic function defined by

$$f(t) = 100t^2 - 300t.$$

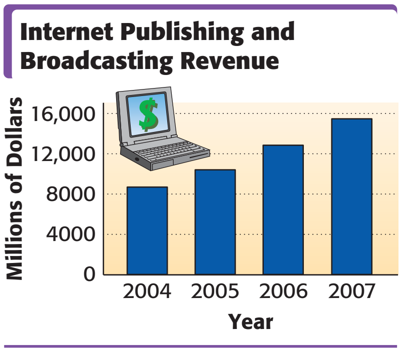
How long will it take before the light returns to the starting point?



**38.** Internet publishing and broadcasting revenue in the United States (in millions of dollars) for the years 2004–2007 is shown in the graph and can be modeled by the quadratic function defined by

$$f(x) = 230.5x^2 - 252.9x + 5987.$$

In the model, $x = 4$ represents 2004, $x = 5$ represents 2005, and so on.



***Source:*** U.S. Census Bureau.

**(a)** Use the model to approximate revenue from Internet publishing and broadcasting in 2007 to the nearest million dollars. How does this result compare to the number suggested by the graph?

**(b)** Based on the model, in what year did the revenue from Internet publishing and broadcasting reach \$14,000 million (\$14 billion)? (Round down for the year.) How does this result compare to the number shown in the graph?

**11.6–11.7** *Identify the vertex of each parabola.*

**39.** $f(x) = -(x - 1)^2$

**40.** $f(x) = (x - 3)^2 + 7$

**41.** $x = (y - 3)^2 - 4$

**42.** $y = -3x^2 + 4x - 2$

*Graph each parabola. Give the vertex, axis, domain, and range.*

**43.** $y = 2(x - 2)^2 - 3$

**44.** $f(x) = -2x^2 + 8x - 5$

**45.** $x = 2(y + 3)^2 - 4$

**46.** $x = -\frac{1}{2}y^2 + 6y - 14$

*Solve each problem.*

**47.** Total consumer spending on computers, peripherals, and software in the United States for selected years is given in the table. Let $x = 0$ represent 1985, $x = 5$ represent 1990, and so on.

**(a)** Use the data for 1985, 1995, and 2005 in the quadratic form $ax^2 + bx + c = y$ to write a system of three equations.

**(b)** Solve the system from part (a) to get a quadratic function $f$ that models the data.

**(c)** Use the model found in part (b) to approximate consumer spending for computers, peripherals, and software games in 2006 to the nearest tenth. How does your answer compare to the actual data from the table?

CONSUMER SPENDING ON COMPUTERS, PERIPHERALS, AND SOFTWARE

| Year | Spending (billions of dollars) |
|---|---|
| 1985 | 2.9 |
| 1990 | 8.9 |
| 1995 | 24.3 |
| 2000 | 43.8 |
| 2004 | 51.6 |
| 2005 | 56.5 |
| 2006 | 61.4 |

*Source:* Bureau of Economic Analysis.

**48.** The height (in feet) of a projectile $t$ seconds after being fired from Earth into the air is given by

$$f(t) = -16t^2 + 160t.$$

Find the number of seconds required for the projectile to reach maximum height. What is the maximum height?

**49.** Find the length and width of a rectangle having a perimeter of 200 m if the area is to be a maximum. What is the maximum area?

**11.8** *Solve each inequality, and graph the solution set.*

**50.** $(x - 4)(2x + 3) > 0$

**51.** $x^2 + x \leq 12$

**52.** $(x + 2)(x - 3)(x + 5) \leq 0$

**53.** $(4x + 3)^2 \leq -4$

**54.** $\frac{6}{2z - 1} < 2$

**55.** $\frac{3t + 4}{t - 2} \leq 1$

## MIXED REVIEW EXERCISES

*Solve.*

**56.** $V = r^2 + R^2h$ for $R$

***57.** $3t^2 - 6t = -4$

**58.** $(3x + 11)^2 = 7$

**59.** $S = \frac{Id^2}{k}$ for $d$

**60.** $(8x - 7)^2 \geq -1$

**61.** $2x - \sqrt{x} = 6$

---

*This exercise requires knowledge of complex numbers.

(continued)

**62.** $x^4 - 8x^2 = -1$ **63.** $\frac{-2}{x+5} \leq -5$ **64.** $6 + \frac{15}{s^2} = -\frac{19}{s}$

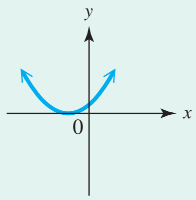
**65.** $(x^2 - 2x)^2 = 11(x^2 - 2x) - 24$ **66.** $(r - 1)(2r + 3)(r + 6) < 0$

**67.** ***Concept Check*** Match each equation in parts (a)–(f) with the figure that most closely resembles its graph in choices A–F.

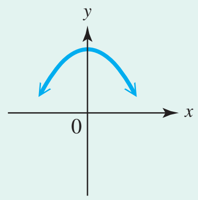
**(a)** $g(x) = x^2 - 5$ **(b)** $h(x) = -x^2 + 4$ **(c)** $F(x) = (x - 1)^2$

**(d)** $G(x) = (x + 1)^2$ **(e)** $H(x) = (x - 1)^2 + 1$ **(f)** $K(x) = (x + 1)^2 + 1$

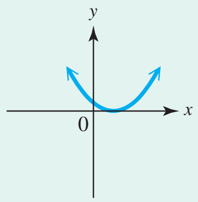
**A.**



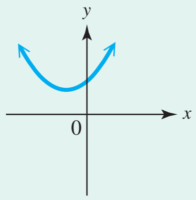
**B.**



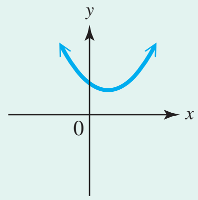
**C.**



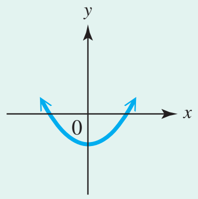
**D.**



**E.**



**F.**



**68.** Graph $f(x) = 4x^2 + 4x - 2$. Give the vertex, axis, domain, and range.

**69.** In 4 hr, Rajeed Carriman can go 15 mi upriver and come back. The rate of the current is 5 mph. Find the rate of the boat in still water.

**70.** Two pieces of a large wooden puzzle fit together to form a rectangle with length 1 cm less than twice the width. The diagonal, where the two pieces meet, is 2.5 cm in length. Find the length and width of the rectangle.

## CHAPTER 11 TEST

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*Solve each equation by using the square root property.*

**1.** $t^2 = 54$ **2.** $(7x + 3)^2 = 25$

**3.** Solve $x^2 + 2x = 4$ by completing the square.

*Solve each equation by using the quadratic formula.*

**4.** $2x^2 - 3x - 1 = 0$ ***5.** $3t^2 - 4t = -5$

***6.** ***Concept Check*** If $k$ is a negative number, then which one of the following equations will have two nonreal complex solutions?

**A.** $x^2 = 4k$ **B.** $x^2 = -4k$ **C.** $(x + 2)^2 = -k$ **D.** $x^2 + k = 0$

**7.** What is the discriminant for $2x^2 - 8x - 3 = 0$? How many and what type of solutions does this equation have? (Do not actually solve.)

---

*This exercise requires knowledge of complex numbers.

*Solve by any method.*

**8.** $3x = \sqrt{\dfrac{9x + 2}{2}}$

**9.** $3 - \dfrac{16}{x} - \dfrac{12}{x^2} = 0$

**10.** $4x^2 + 7x - 3 = 0$

**11.** $9x^4 + 4 = 37x^2$

**12.** $12 = (2n + 1)^2 + (2n + 1)$

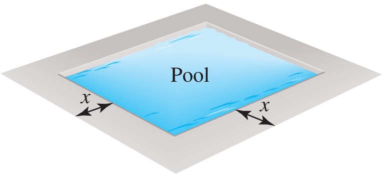
**13.** Solve $S = 4\pi r^2$ for $r$. (Leave ± in your answer.)

*Solve each problem.*

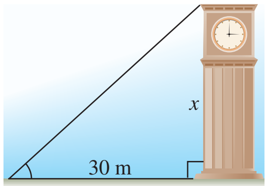
**14.** Terry and Callie do word processing. For a certain prospectus, Callie can prepare it 2 hr faster than Terry can. If they work together, they can do the entire prospectus in 5 hr. How long will it take each of them working alone to prepare the prospectus? Round your answers to the nearest tenth of an hour.

**15.** Qihong Shen paddled a canoe 10 mi upstream and then paddled back to the starting point. If the rate of the current was 3 mph and the entire trip took $3\frac{1}{2}$ hr, what was Qihong's rate?

**16.** Endre Borsos has a pool 24 ft long and 10 ft wide. He wants to construct a concrete walk around the pool. If he plans for the walk to be of uniform width and cover 152 $\text{ft}^2$, what will the width of the walk be?

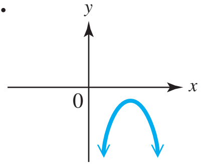


**17.** At a point 30 m from the base of a tower, the distance to the top of the tower is 2 m more than twice the height of the tower. Find the height of the tower.

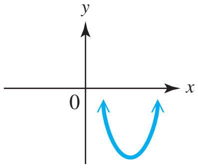


**18.** ***Concept Check*** Which one of the following figures most closely resembles the graph of $f(x) = a(x - h)^2 + k$ if $a < 0$, $h > 0$, and $k < 0$?

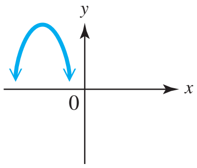
**A.**



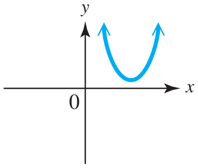
**B.**



**C.**



**D.**



*Graph each parabola. Identify the vertex, axis, domain, and range.*

**19.** $f(x) = \frac{1}{2}x^2 - 2$

**20.** $f(x) = -x^2 + 4x - 1$

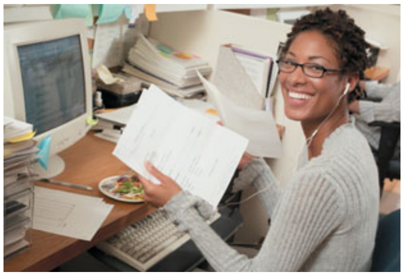
**21.** $x = -(y - 2)^2 + 2$

*Solve each problem.*

**22.** The total number (in millions) of civilians employed in the United States during the years 2004–2008 can be modeled by the quadratic function defined by

$$f(x) = -0.529x^2 + 8.00x + 115$$

where $x = 4$ represents 2004, $x = 5$ represents 2005, and so on. (*Source:* U.S. Bureau of Labor Statistics.)



**(a)** Based on this model, how many civilians, to the nearest million, were employed in the United States in 2004?

**(b)** In what year during this period was the maximum civilian employment? (Round down for the year.) To the nearest million, what was the total civilian employment in that year? Use the actual $x$-value, to the nearest tenth, to find this number.

**23.** Houston Community College is planning to construct a rectangular parking lot on land bordered on one side by a highway. The plan is to use 640 ft of fencing to fence off the other three sides. What should the dimensions of the lot be if the enclosed area is to be a maximum?

*Solve each inequality, and graph the solution set.*

**24.** $2x^2 + 7x > 15$

**25.** $\frac{5}{t-4} \le 1$

## CHAPTERS 1–11

## CUMULATIVE REVIEW EXERCISES

**1.** Let $S = \left\{-\frac{7}{3}, -2, -\sqrt{3}, 0, 0.7, \sqrt{12}, \sqrt{-8}, 7, \frac{32}{3}\right\}$. List the elements of $S$ that are elements of each set.

**(a)** Integers **(b)** Rational numbers **(c)** Real numbers **(d)** Complex numbers

*Solve each equation or inequality.*

**2.** $7 - (4 + 3t) + 2t = -6(t - 2) - 5$

**3.** $|6x - 9| = |-4x + 2|$

**4.** $2x = \sqrt{\frac{5x + 2}{3}}$

**5.** $\frac{3}{x-3} - \frac{2}{x-2} = \frac{3}{x^2 - 5x + 6}$

**6.** $(r - 5)(2r + 3) = 1$

**7.** $x^4 - 5x^2 + 4 = 0$

**8.** $-2x + 4 \le -x + 3$

**9.** $|3x - 7| \le 1$

**10.** $x^2 - 4x + 3 < 0$

**11.** $\frac{3}{p+2} > 1$

*Graph each relation. Tell whether or not y can be expressed as a function f of x, and if so, give its domain and range, and write using function notation.*

**12.** $4x - 5y = 15$

**13.** $4x - 5y < 15$

**14.** $y = -2(x - 1)^2 + 3$

**15.** Find the slope and intercepts of the line with equation

$$-2x + 7y = 16.$$

**16.** Write an equation for the specified line. Express each equation in slope-intercept form.

**(a)** Through $(2, -3)$ and parallel to the line with equation $5x + 2y = 6$

**(b)** Through $(-4, 1)$ and perpendicular to the line with equation $5x + 2y = 6$

*Write with positive exponents only. Assume that variables represent positive real numbers.*

**17.** $\left(\frac{x^{-3}y^2}{x^5y^{-2}}\right)^{-1}$

**18.** $\frac{(4x^{-2})^2(2y^3)}{8x^{-3}y^5}$

*Perform the indicated operations.*

**19.** $\left(\frac{2}{3}t + 9\right)^2$

**20.** Divide $4x^3 + 2x^2 - x + 26$ by $x + 2$.

*Factor completely.*

**21.** $24m^2 + 2m - 15$

**22.** $8x^3 + 27y^3$

**23.** $9x^2 - 30xy + 25y^2$

*Perform the indicated operations or simplify the complex fraction, and express each answer in lowest terms. Assume denominators are nonzero.*

**24.** $\dfrac{5x+2}{-6} \div \dfrac{15x+6}{5}$

**25.** $\dfrac{3}{2-x} - \dfrac{5}{x} + \dfrac{6}{x^2-2x}$

**26.** $\dfrac{\dfrac{r}{s} - \dfrac{s}{r}}{\dfrac{r}{s} + 1}$

*Solve each system of equations.*

**27.** $2x - 4y = 10$
$9x + 3y = 3$

**28.** $x + y + 2z = 3$
$-x + y + z = -5$
$2x + 3y - z = -8$

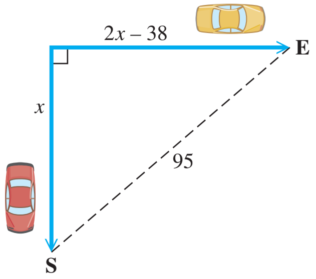
**29.** In 2009, the two American computer software companies with the greatest revenues were Microsoft and Oracle. The two companies had combined revenues of \$82.8 billion. Revenues for Microsoft were \$6.8 billion less than three times those of Oracle. What were the 2009 revenues for each company? (*Source: Fortune*.)

*Simplify each radical expression.*

**30.** $\sqrt[3]{\dfrac{27}{16}}$

**31.** $\dfrac{2}{\sqrt{7} - \sqrt{5}}$

**32.** Two cars left an intersection at the same time, one heading due south and the other due east. Later they were exactly 95 mi apart. The car heading east had gone 38 mi less than twice as far as the car heading south. How far had each car traveled?



CHAPTER 12

# Inverse, Exponential, and Logarithmic Functions

In 2001, Apple Computer Inc., introduced the iPod. By mid-2009, the company had sold over 220 million of the popular music players, in spite of warnings by experts that listening to the devices at high volumes may put people at increased risk of hearing loss.

In **Example 4** of **Section 12.5,** we use a *logarithmic function* to calculate the volume level, in *decibels,* of an iPod.

# 12.1 Inverse Functions

**OBJECTIVES**

1. Decide whether a function is one-to-one and, if it is, find its inverse.
2. Use the horizontal line test to determine whether a function is one-to-one.
3. Find the equation of the inverse of a function.
4. Graph $f^{-1}$ given the graph of $f$.

In this chapter we study two important types of functions, *exponential* and *logarithmic*. These functions are related: They are *inverses* of one another.

**OBJECTIVE 1 Decide whether a function is one-to-one and, if it is, find its inverse.** Suppose we define the function

$$G = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}.$$

We can form another set of ordered pairs from $G$ by interchanging the $x$- and $y$-values of each pair in $G$. We can call this set $F$, so

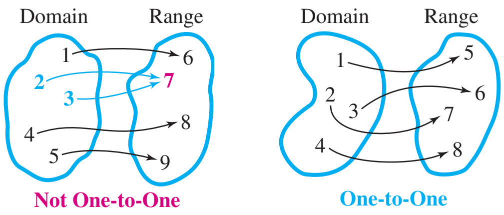
$$F = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}.$$

To show that these two sets are related as just described, $F$ is called the *inverse* of $G$. For a function $f$ to have an inverse, $f$ must be a *one-to-one function.*

### One-to-One Function

In a **one-to-one function,** each $x$-value corresponds to only one $y$-value, and each $y$-value corresponds to only one $x$-value.

The function shown in **FIGURE 1(a)** is not one-to-one because the $y$-value 7 corresponds to *two* $x$-values, 2 and 3. That is, the ordered pairs $(2, 7)$ and $(3, 7)$ both belong to the function. The function in **FIGURE 1(b)** is one-to-one.



**(a)** **(b)**

**FIGURE 1**

The *inverse* of any one-to-one function $f$ is found by interchanging the components of the ordered pairs of $f$. The inverse of $f$ is written $\boldsymbol{f^{-1}}$**.** Read $f^{-1}$ as **"the inverse of *f*"** or **"*f*-inverse."**

**CAUTION** The symbol $f^{-1}(x)$ does ***not*** represent $\dfrac{1}{f(x)}$.

The definition of the inverse of a function follows.

### Inverse of a Function

The **inverse** of a one-to-one function $f$, written $f^{-1}$, is the set of all ordered pairs of the form $(y, x)$, where $(x, y)$ belongs to $f$. Since the inverse is formed by interchanging $x$ and $y$, the domain of $f$ becomes the range of $f^{-1}$ and the range of $f$ becomes the domain of $f^{-1}$.

For inverses $f$ and $f^{-1}$, it follows that for all $x$ in their domains,

$$(\boldsymbol{f} \circ \boldsymbol{f}^{-1})(\boldsymbol{x}) = \boldsymbol{x} \quad \text{and} \quad (\boldsymbol{f}^{-1} \circ \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{x}.$$

### EXAMPLE 1 Finding Inverses of One-to-One Functions

Decide whether each function is one-to-one. If it is, find the inverse.

**(a)** $F = \{(-2, 1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

Each $x$-value in $F$ corresponds to just one $y$-value. However, the $y$-value 1 corresponds to two $x$-values, $-2$ and 0. Also, the $y$-value 2 corresponds to both 1 and 2. Because some $y$-values correspond to more than one $x$-value, $F$ is not one-to-one and does not have an inverse.

**(b)** $G = \{(3, 1), (0, 2), (2, 3), (4, 0)\}$

Every $x$-value in $G$ corresponds to only one $y$-value, and every $y$-value corresponds to only one $x$-value, so $G$ is a one-to-one function. The inverse function is found by interchanging the $x$- and $y$-values in each ordered pair.

$$G^{-1} = \{(1, 3), (2, 0), (3, 2), (0, 4)\}$$

The domain and range of $G$ become the range and domain, respectively, of $G^{-1}$.

**(c)** The table shows the number of days in which the air in Connecticut exceeded the 8-hour average ground-level ozone standard for the years 1997–2006.

| Year | *Number of Days Exceeding Standard* | Year | *Number of Days Exceeding Standard* |
|---|---|---|---|
| 1997 | 27 | 2002 | 36 |
| 1998 | 25 | 2003 | 14 |
| 1999 | 33 | 2004 | 6 |
| 2000 | 13 | 2005 | 20 |
| 2001 | 26 | 2006 | 13 |

*Source:* U.S. Environmental Protection Agency.

Let $f$ be the function defined in the table, with the years forming the domain and the numbers of days exceeding the ozone standard forming the range. Then $f$ is not one-to-one, because in two different years (2000 and 2006), the number of days with unacceptable ozone levels was the same, 13. NOW TRY

NOW TRY EXERCISE 1

Decide whether each function is one-to-one. If it is, find the inverse.

**(a)** $F = \{(-1, -2), (0, 0), (1, -2), (2, -8)\}$

**(b)** $G = \{(0, 0), (1, 1), (4, 2), (9, 3)\}$

**(c)** The number of stories and height of several tall buildings are given in the table.

| *Stories* | *Height* |
|---|---|
| 31 | 639 |
| 35 | 582 |
| 40 | 620 |
| 41 | 639 |
| 64 | 810 |

**OBJECTIVE 2 Use the horizontal line test to determine whether a function is one-to-one.** By graphing a function and observing the graph, we can use the *horizontal line test* to tell whether the function is one-to-one.

**Horizontal Line Test**

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

The horizontal line test follows from the definition of a one-to-one function. Any two points that lie on the same horizontal line have the same $y$-coordinate. No two ordered pairs that belong to a one-to-one function may have the same $y$-coordinate. Therefore, no horizontal line will intersect the graph of a one-to-one function more than once.

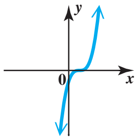
NOW TRY ANSWERS

1. **(a)** not one-to-one
   **(b)** one-to-one; $G^{-1} = \{(0, 0), (1, 1), (2, 4), (3, 9)\}$
   **(c)** not one-to-one

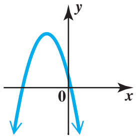
**NOW TRY EXERCISE 2**

Use the horizontal line test to determine whether each graph is the graph of a one-to-one function.

**(a)**



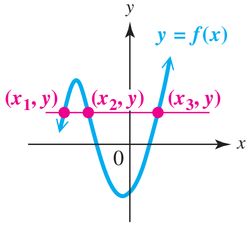
**(b)**



## EXAMPLE 2 Using the Horizontal Line Test

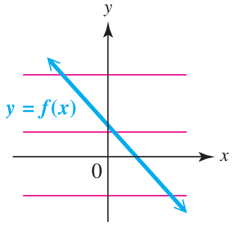
Use the horizontal line test to determine whether each graph is the graph of a one-to-one function.

**(a)**



Because a horizontal line intersects the graph in more than one point (actually three points), the function is not one-to-one.

**(b)**



Every horizontal line will intersect the graph in exactly one point. This function is one-to-one.

NOW TRY

**OBJECTIVE 3** **Find the equation of the inverse of a function.** The inverse of a one-to-one function is found by interchanging the $x$- and $y$-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

### Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function $f$ defined by an equation $y = f(x)$, find the defining equation of the inverse as follows.

***Step 1*** Interchange $x$ and $y$.

***Step 2*** Solve for $y$.

***Step 3*** Replace $y$ with $f^{-1}(x)$.

## EXAMPLE 3 Finding Equations of Inverses

Decide whether each equation defines a one-to-one function. If so, find the equation that defines the inverse.

**(a)** $f(x) = 2x + 5$

The graph of $y = 2x + 5$ is a nonvertical line, so by the horizontal line test, $f$ is a one-to-one function. To find the inverse, let $y = f(x)$ and follow the steps.

$$\begin{aligned} y &= 2x + 5 \\ x &= 2y + 5 && \text{Interchange } x \text{ and } y. \text{ (Step 1)} \\ 2y &= x - 5 && \text{Solve for } y. \text{ (Step 2)} \\ y &= \frac{x-5}{2} \\ f^{-1}(x) &= \frac{x-5}{2} && \text{Replace } y \text{ with } f^{-1}(x). \text{ (Step 3)} \end{aligned}$$

This equation can be written as follows.

$$f^{-1}(x) = \frac{x}{2} - \frac{5}{2}, \quad \text{or} \quad f^{-1}(x) = \frac{1}{2}x - \frac{5}{2} \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

**NOW TRY ANSWERS**

2. **(a)** one-to-one
   **(b)** not one-to-one

NOW TRY EXERCISE 3

Decide whether each equation defines a one-to-one function. If so, find the equation that defines the inverse.

**(a)** $f(x) = 5x - 7$

**(b)** $f(x) = (x + 1)^2$

**(c)** $f(x) = x^3 - 4$

Thus, $f^{-1}$ is a linear function. In the function defined by

$$y = 2x + 5,$$

the value of $y$ is found by starting with a value of $x$, multiplying by 2, and adding 5. The equation

$$f^{-1}(x) = \frac{x - 5}{2}$$

for the inverse has us *subtract* 5, and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable $x$.

**(b)** $y = x^2 + 2$

This equation has a vertical parabola as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the $x^2$-term, there are many pairs of $x$-values that correspond to the same $y$-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

Alternatively, applying the steps for finding the equation of an inverse leads to the following.

$$\begin{aligned} y &= x^2 + 2 & & \\ x &= y^2 + 2 & & \text{Interchange } x \text{ and } y. \\ y^2 &= x - 2 & & \text{Solve for } y. \\ y &= \pm\sqrt{x - 2} & & \text{Square root property} \end{aligned}$$

The last step shows that there are two $y$-values for each choice of $x > 2$, so we again see that the given function is not one-to-one. It does not have an inverse.

**(c)** $f(x) = (x - 2)^3$

Because of the cube, each value of $x$ produces a different value of $y$, so this is a one-to-one function.

$$\begin{aligned} f(x) &= (x - 2)^3 & & \\ y &= (x - 2)^3 & & \text{Replace } f(x) \text{ with } y. \\ x &= (y - 2)^3 & & \text{Interchange } x \text{ and } y. \\ \sqrt[3]{x} &= \sqrt[3]{(y - 2)^3} & & \text{Take the cube root on each side.} \\ \sqrt[3]{x} &= y - 2 & & \sqrt[3]{a^3} = a \\ y &= \sqrt[3]{x} + 2 & & \text{Solve for } y. \\ f^{-1}(x) &= \sqrt[3]{x} + 2 & & \text{Replace } y \text{ with } f^{-1}(x). \end{aligned}$$

NOW TRY

**OBJECTIVE 4** **Graph $f^{-1}$, given the graph of $f$.** One way to graph the inverse of a function $f$ whose equation is given is as follows.

1. Find several ordered pairs that belong to $f$.
2. Interchange $x$ and $y$ to obtain ordered pairs that belong to $f^{-1}$.
3. Plot those points, and sketch the graph of $f^{-1}$ through them.

A simpler way is to select points on the graph of $f$ and use symmetry to find corresponding points on the graph of $f^{-1}$.

NOW TRY ANSWERS

3. **(a)** one-to-one function; $f^{-1}(x) = \frac{x + 7}{5}$, or $f^{-1}(x) = \frac{1}{5}x + \frac{7}{5}$
   **(b)** not a one-to-one function
   **(c)** one-to-one function; $f^{-1}(x) = \sqrt[3]{x + 4}$

For example, suppose the point $(a, b)$ shown in **FIGURE 2** belongs to a one-to-one function $f$. Then the point $(b, a)$ belongs to $f^{-1}$. The line segment connecting $(a, b)$ and $(b, a)$ is perpendicular to, and cut in half by, the line $y = x$. The points $(a, b)$ and $(b, a)$ are "mirror images" of each other with respect to $y = x$.

***We can find the graph of $f^{-1}$ from the graph of $f$ by locating the mirror image of each point in $f$ with respect to the line $y = x$.***

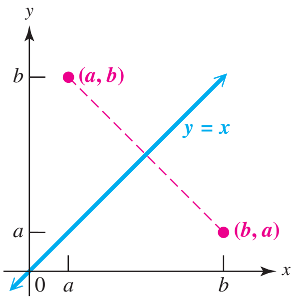
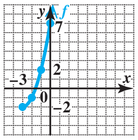


**FIGURE 2**

NOW TRY EXERCISE 4

Use the given graph to graph the inverse of $f$.



## EXAMPLE 4 Graphing the Inverse

Graph the inverses of the functions $f$ (shown in blue) in **FIGURE 3**.

In **FIGURE 3** the graphs of two functions $f$ are shown in blue. Their inverses are shown in red. In each case, the graph of $f^{-1}$ is a reflection of the graph of $f$ with respect to the line $y = x$.

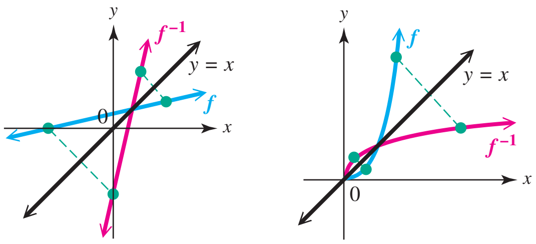


**FIGURE 3**

NOW TRY

## CONNECTIONS

In **Example 3** we showed that the inverse of the one-to-one function defined by $f(x) = 2x + 5$ is given by $f^{-1}(x) = \frac{x-5}{2}$. If we use a square viewing window of a graphing calculator and graph

$$y_1 = f(x) = 2x + 5, \quad y_2 = f^{-1}(x) = \frac{x-5}{2}, \quad \text{and} \quad y_3 = x,$$

we can see how this reflection appears on the screen. See **FIGURE 4**.

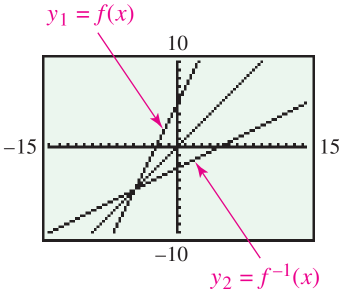


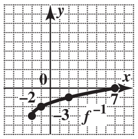
**FIGURE 4**

### For Discussion or Writing

Some graphing calculators have the capability to "draw" the inverse of a function. Use a graphing calculator to draw the graphs of $f(x) = x^3 + 2$ and its inverse in a square viewing window.

NOW TRY ANSWER

4.



## 12.1 EXERCISES

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*Complete solution available on the Video Resources on DVD*

*In Exercises 1–4, write a few sentences of explanation.* ***See Example 1.***

**1.** A study found that the trans fat content in fast-food products varied widely around the world, based on the type of frying oil used, as shown in the table. If the set of countries is the domain and the set of trans fat percentages is the range of the function consisting of the six pairs listed, is it a one-to-one function? Why or why not?

| *Country* | *Percentage of Trans Fat in McDonald's Chicken* |
|---|---|
| Scotland | 14 |
| France | 11 |
| United States | 11 |
| Peru | 9 |
| Russia | 5 |
| Denmark | 1 |

*Source:* New England Journal of Medicine.

**2.** The table shows the number of uncontrolled hazardous waste sites in 2008 that require further investigation to determine whether remedies are needed under the Superfund program. The eight states listed are ranked in the top ten on the EPA's National Priority List.

If this correspondence is considered to be a function that pairs each state with its number of uncontrolled waste sites, is it one-to-one? If not, explain why.

| *State* | *Number of Sites* |
|---|---|
| New Jersey | 116 |
| California | 97 |
| Pennsylvania | 96 |
| New York | 86 |
| Michigan | 67 |
| Florida | 52 |
| Illinois | 49 |
| Texas | 49 |

*Source:* U.S. Environmental Protection Agency.

**3.** The road mileage between Denver, Colorado, and several selected U.S. cities is shown in the table. If we consider this as a function that pairs each city with a distance, is it a one-to-one function? How could we change the answer to this question by adding 1 mile to one of the distances shown?

| *City* | *Distance to Denver (in miles)* |
|---|---|
| Atlanta | 1398 |
| Dallas | 781 |
| Indianapolis | 1058 |
| Kansas City, MO | 600 |
| Los Angeles | 1059 |
| San Francisco | 1235 |

**4.** Suppose you consider the set of ordered pairs $(x, y)$ such that $x$ represents a person in your mathematics class and $y$ represents that person's father. Explain how this function might not be a one-to-one function.

*In Exercises 5–8, choose the correct response from the given list.*

**5.** ***Concept Check*** If a function is made up of ordered pairs in such a way that the same $y$-value appears in a correspondence with two different $x$-values, then

**A.** the function is one-to-one **B.** the function is not one-to-one

**C.** its graph does not pass the vertical line test

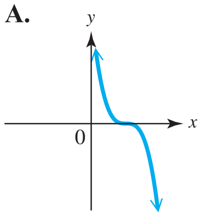
**D.** it has an inverse function associated with it.

**6.** Which equation defines a one-to-one function? Explain why the others are not, using specific examples.

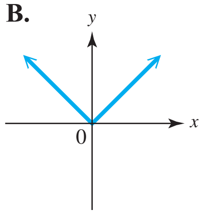
**A.** $f(x) = x$ **B.** $f(x) = x^2$ **C.** $f(x) = |x|$ **D.** $f(x) = -x^2 + 2x - 1$

**7.** Only one of the graphs illustrates a one-to-one function. Which one is it? **(See Example 2.)**

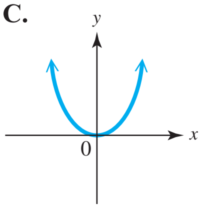
**A.**



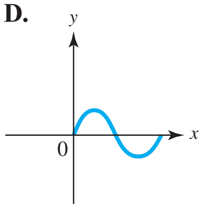
**B.**



**C.**



**D.**



**8.** ***Concept Check*** If a function $f$ is one-to-one and the point $(p, q)$ lies on the graph of $f$, then which point *must* lie on the graph of $f^{-1}$?

**A.** $(-p, q)$ **B.** $(-q, -p)$ **C.** $(p, -q)$ **D.** $(q, p)$

*If the function is one-to-one, find its inverse.* ***See Examples 1–3.***

**9.** $\{(3, 6), (2, 10), (5, 12)\}$

**10.** $\left\{(-1, 3), (0, 5), (5, 0), \left(7, -\frac{1}{2}\right)\right\}$

**11.** $\{(-1, 3), (2, 7), (4, 3), (5, 8)\}$

**12.** $\{(-8, 6), (-4, 3), (0, 6), (5, 10)\}$

**13.** $f(x) = 2x + 4$

**14.** $f(x) = 3x + 1$

**15.** $g(x) = \sqrt{x - 3}, \quad x \geq 3$

**16.** $g(x) = \sqrt{x + 2}, \quad x \geq -2$

**17.** $f(x) = 3x^2 + 2$

**18.** $f(x) = 4x^2 - 1$

**19.** $f(x) = x^3 - 4$

**20.** $f(x) = x^3 + 5$

***Concept Check*** *Let $f(x) = 2^x$. We will see in the next section that this function is one-to-one. Find each value, always working part (a) before part (b).*

**21. (a)** $f(3)$
**(b)** $f^{-1}(8)$

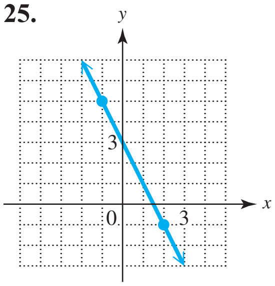
**22. (a)** $f(4)$
**(b)** $f^{-1}(16)$

**23. (a)** $f(0)$
**(b)** $f^{-1}(1)$

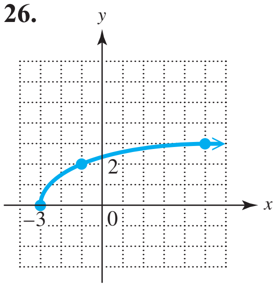
**24. (a)** $f(-2)$
**(b)** $f^{-1}\left(\frac{1}{4}\right)$

*The graphs of some functions are given in Exercises 25–30.* ***(a)*** *Use the horizontal line test to determine whether the function is one-to-one.* ***(b)*** *If the function is one-to-one, then graph the inverse of the function. (Remember that if $f$ is one-to-one and $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$.)* ***See Example 4.***

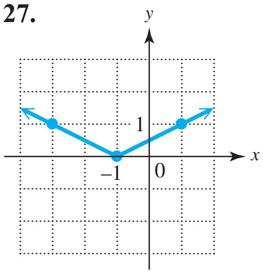
**25.**



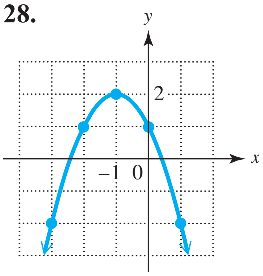
**26.**



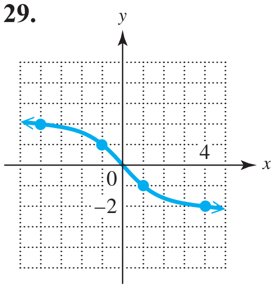
**27.**



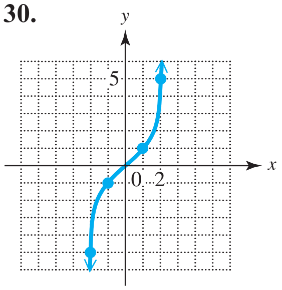
**28.**



**29.**



**30.**



*Each function defined in Exercises 31–38 is a one-to-one function. Graph the function as a solid line (or curve) and then graph its inverse on the same set of axes as a dashed line (or curve). In Exercises 35–38 complete the table so that graphing the function will be easier.* ***See Example 4.***

**31.** $f(x) = 2x - 1$

**32.** $f(x) = 2x + 3$

**33.** $g(x) = -4x$

**34.** $g(x) = -2x$

**35.** $f(x) = \sqrt{x}, \quad x \geq 0$

| $x$ | $f(x)$ |
|---|---|
| 0 | |
| 1 | |
| 4 | |

**36.** $f(x) = -\sqrt{x}, \quad x \geq 0$

| $x$ | $f(x)$ |
|---|---|
| 0 | |
| 1 | |
| 4 | |

**37.** $f(x) = x^3 - 2$

| $x$ | $f(x)$ |
|---|---|
| −1 | |
| 0 | |
| 1 | |
| 2 | |

**38.** $f(x) = x^3 + 3$

| $x$ | $f(x)$ |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |

## RELATING CONCEPTS EXERCISES 39–42

### FOR INDIVIDUAL OR GROUP WORK

*Inverse functions can be used to send and receive coded information. A simple example might use the function defined by* $f(x) = 2x + 5$. *(Note that it is one-to-one.) Suppose that each letter of the alphabet is assigned a numerical value according to its position, as follows.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | 1 | G | 7 | L | 12 | Q | 17 | V | 22 |
| B | 2 | H | 8 | M | 13 | R | 18 | W | 23 |
| C | 3 | I | 9 | N | 14 | S | 19 | X | 24 |
| D | 4 | J | 10 | O | 15 | T | 20 | Y | 25 |
| E | 5 | K | 11 | P | 16 | U | 21 | Z | 26 |
| F | 6 | | | | | | | | |

**This is an Enigma machine, used by the Germans in World War II to send coded messages.**

*Using the function, the word* ALGEBRA *would be encoded as*

$$7 \quad 29 \quad 19 \quad 15 \quad 9 \quad 41 \quad 7,$$

*because*

$$f(\text{A}) = f(1) = 2(1) + 5 = 7, \quad f(\text{L}) = f(12) = 2(12) + 5 = 29, \quad \textit{and so on.}$$

*The message would then be decoded by using the inverse of* $f$, *defined by* $f^{-1}(x) = \frac{x-5}{2}$ $\left(\textit{or } f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}\right)$. *For example,*

$$f^{-1}(7) = \frac{7-5}{2} = 1 = \text{A}, \quad f^{-1}(29) = \frac{29-5}{2} = 12 = \text{L}, \quad \textit{and so on.}$$

***Work Exercises 39–42 in order.***

**39.** Suppose that you are an agent for a detective agency. Today's function for your code is defined by $f(x) = 4x - 5$. Find the rule for $f^{-1}$ algebraically.

**40.** You receive the following coded message today. (Read across from left to right.)

47 95 23 67 −1 59 27 31 51 23 7 −1 43 7 79 43 −1 75 55 67
31 71 75 27 15 23 67 15 −1 75 15 71 75 75 27 31 51
23 71 31 51 7 15 71 43 31 7 15 11 3 67 15 −1 11

Use the letter/number assignment described earlier to decode the message.

**41.** Why is a one-to-one function essential in this encoding/decoding process?

**42.** Use $f(x) = x^3 + 4$ to encode your name, using the letter/number assignment described earlier.

*Each function defined is one-to-one. Find the inverse algebraically, and then graph both the function and its inverse on the same graphing calculator screen. Use a square viewing window.* ***See the Connections box.***

**43.** $f(x) = 2x - 7$

**44.** $f(x) = -3x + 2$

**45.** $f(x) = x^3 + 5$

**46.** $f(x) = \sqrt[3]{x + 2}$

## PREVIEW EXERCISES

*If* $f(x) = 4^x$, *find each value indicated. In Exercise 50, use a calculator, and give the answer to the nearest hundredth.* ***See Sections 7.4 and 10.2.***

**47.** $f(3)$

**48.** $f\left(\frac{1}{2}\right)$

**49.** $f\left(-\frac{1}{2}\right)$

**50.** $f(2.73)$

# 12.2 Exponential Functions

**OBJECTIVES**

1. Define an exponential function.
2. Graph an exponential function.
3. Solve exponential equations of the form $a^x = a^k$ for $x$.
4. Use exponential functions in applications involving growth or decay.

**OBJECTIVE 1 Define an exponential function.** In **Section 10.2** we showed how to evaluate $2^x$ for rational values of $x$.

$$2^3 = 8, \quad 2^{-1} = \frac{1}{2}, \quad 2^{1/2} = \sqrt{2}, \quad \text{and} \quad 2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}$$

Examples of $2^x$ for rational $x$

In more advanced courses it is shown that $2^x$ exists for all real number values of $x$, both rational and irrational. The following definition of an exponential function assumes that $a^x$ exists for all real numbers $x$.

**Exponential Function**

For $a > 0$, $a \neq 1$, and all real numbers $x$,

$$f(x) = a^x$$

defines the **exponential function with base *a*.**

**NOTE** ***The two restrictions on the value of a in the definition of an exponential function f(x) = $a^x$ are important.***

1. The restriction $a > 0$ is necessary so that the function can be defined for all real numbers $x$. Letting $a$ be negative ($a = -2$, for instance) and letting $x = \frac{1}{2}$ would give the expression $(-2)^{1/2}$, which is not real.
2. The restriction $a \neq 1$ is necessary because 1 raised to any power is equal to 1, resulting in the linear function defined by $f(x) = 1$.

**OBJECTIVE 2 Graph an exponential function.** When graphing an exponential function of the form $f(x) = a^x$, pay particular attention to whether $a > 1$ or $0 < a < 1$.

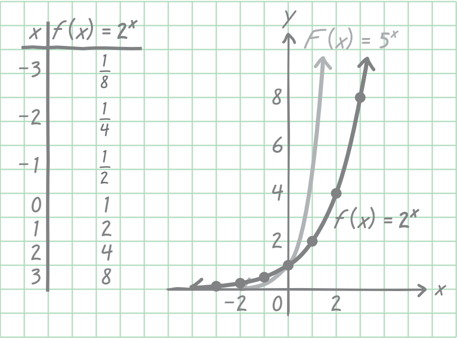
NOW TRY EXERCISE 1

Graph $y = 4^x$.

## EXAMPLE 1 Graphing an Exponential Function $(a > 1)$

Graph $f(x) = 2^x$. Then compare it to the graph of $F(x) = 5^x$.

Choose some values of $x$, and find the corresponding values of $f(x)$. Plotting these points and drawing a smooth curve through them gives the darker graph shown in **FIGURE 5**. This graph is typical of the graphs of exponential functions of the form $F(x) = a^x$, where $a > 1$. ***The larger the value of a, the faster the graph rises.*** To see this, compare the graph of $F(x) = 5^x$ with the graph of $f(x) = 2^x$ in **FIGURE 5**.



| $x$ | $f(x) = 2^x$ |
|---|---|
| $-3$ | $\frac{1}{8}$ |
| $-2$ | $\frac{1}{4}$ |
| $-1$ | $\frac{1}{2}$ |
| $0$ | $1$ |
| $1$ | $2$ |
| $2$ | $4$ |
| $3$ | $8$ |

**Exponential function with base $a > 1$**

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

The function is one-to-one, and its graph rises from left to right.

**FIGURE 5**

The vertical line test assures us that the graphs in **FIGURE 5** represent functions. **FIGURE 5** also shows an important characteristic of exponential functions with $a > 1$: ***As x gets larger, y increases at a faster and faster rate.*** NOW TRY

**CAUTION** The graph of an exponential function *approaches* the $x$-axis, but does ***not*** touch it.

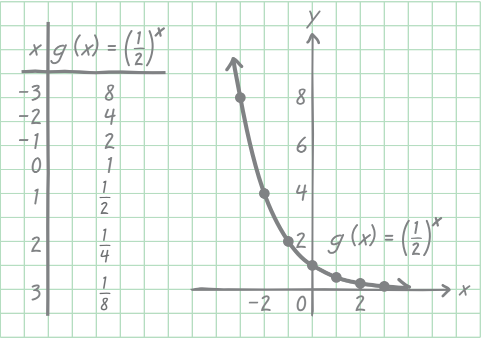
NOW TRY EXERCISE 2

Graph $g(x) = \left(\frac{1}{10}\right)^x$.

## EXAMPLE 2 Graphing an Exponential Function $(0 < a < 1)$

Graph $g(x) = \left(\frac{1}{2}\right)^x$.

Again, find some points on the graph. The graph, shown in **FIGURE 6**, is very similar to that of $f(x) = 2^x$ (**FIGURE 5**) except that here ***as x gets larger, y decreases.*** This graph is typical of the graph of a function of the form $f(x) = a^x$, where $0 < a < 1$.



| $x$ | $g(x) = \left(\frac{1}{2}\right)^x$ |
|---|---|
| $-3$ | $8$ |
| $-2$ | $4$ |
| $-1$ | $2$ |
| $0$ | $1$ |
| $1$ | $\frac{1}{2}$ |
| $2$ | $\frac{1}{4}$ |
| $3$ | $\frac{1}{8}$ |

**Exponential function with base $0 < a < 1$**

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

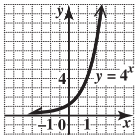
The function is one-to-one, and its graph falls from left to right.

**FIGURE 6**

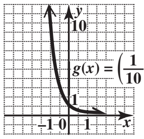
NOW TRY

NOW TRY ANSWERS

1.



2.



### Characteristics of the Graph of $f(x) = a^x$

1. The graph contains the point $(0, 1)$.
2. The function is one-to-one. When $a > 1$, the graph will *rise* from left to right. (See **FIGURE 5**.) When $0 < a < 1$, the graph will *fall* from left to right. (See **FIGURE 6**.) In both cases, the graph goes from the second quadrant to the first.
3. The graph will approach the $x$-axis, but never touch it. (Such a line is called an **asymptote.**)
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

NOW TRY EXERCISE 3

Graph $f(x) = 4^{2x-1}$.

### EXAMPLE 3 Graphing a More Complicated Exponential Function

Graph $f(x) = 3^{2x-4}$.

Find some ordered pairs. We let $x = 0$ and $x = 2$ and find values of $f(x)$, or $y$.

$y = 3^{2(0)-4}$ Let $x = 0$.

$y = 3^{-4}$, or $\frac{1}{81}$

$y = 3^{2(2)-4}$ Let $x = 2$.

$y = 3^{0}$, or $1$

These ordered pairs, $\left(0, \frac{1}{81}\right)$ and $(2, 1)$, along with the other ordered pairs shown in the table, lead to the graph in **FIGURE 7**. The graph is similar to the graph of $f(x) = 3^x$ except that it is shifted to the right and rises more rapidly.

| $x$ | $y$ |
|---|---|
| 0 | $\frac{1}{81}$ |
| 1 | $\frac{1}{9}$ |
| 2 | 1 |
| 3 | 9 |

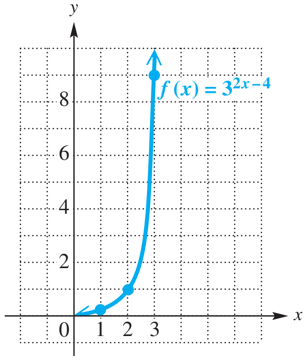


**FIGURE 7**

NOW TRY

**OBJECTIVE 3** Solve exponential equations of the form $a^x = a^k$ for $x$.

Until this chapter, we have solved only equations that had the variable as a base, like $x^2 = 8$. In these equations, all exponents have been constants. An **exponential equation** is an equation that has a variable in an exponent, such as

$$9^x = 27.$$

We can use the following property to solve certain exponential equations.

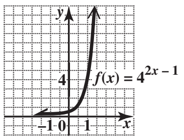
### Property for Solving an Exponential Equation

For $a > 0$ and $a \neq 1$, if $\boldsymbol{a^x = a^y}$ then $\boldsymbol{x = y}$.

This property would not necessarily be true if $a = 1$.

NOW TRY ANSWER

3.



**Solving an Exponential Equation**

***Step 1*** **Each side must have the same base.** If the two sides of the equation do not have the same base, express each as a power of the same base if possible.

***Step 2*** **Simplify exponents** if necessary, using the rules of exponents.

***Step 3*** **Set exponents equal** using the property given in this section.

***Step 4*** **Solve** the equation obtained in Step 3.

*NOW TRY EXERCISE 4*

Solve the equation.

$$8^x = 16$$

## EXAMPLE 4 Solving an Exponential Equation

Solve the equation $9^x = 27$.

$$9^x = 27$$

$$(3^2)^x = 3^3 \quad \text{Write with the same base; } 9 = 3^2 \text{ and } 27 = 3^3. \text{ (Step 1)}$$

$$3^{2x} = 3^3 \quad \text{Power rule for exponents (Step 2)}$$

$$2x = 3 \quad \text{If } a^x = a^y, \text{ then } x = y. \text{ (Step 3)}$$

$$x = \frac{3}{2} \quad \text{Solve for } x. \text{ (Step 4)}$$

*CHECK* Substitute $\frac{3}{2}$ for $x$.

$$9^x = 9^{3/2} = (9^{1/2})^3 = 3^3 = 27 \checkmark \quad \text{True}$$

The solution set is $\left\{\frac{3}{2}\right\}$.

*NOW TRY*

## EXAMPLE 5 Solving Exponential Equations

Solve each equation.

**(a)** $4^{3x-1} = 16^{x+2}$

Be careful multiplying the exponents.

$$4^{3x-1} = (4^2)^{x+2} \quad \text{Write with the same base; } 16 = 4^2.$$

$$4^{3x-1} = 4^{2x+4} \quad \text{Power rule for exponents}$$

$$3x - 1 = 2x + 4 \quad \text{Set exponents equal.}$$

$$x = 5 \quad \text{Subtract } 2x. \text{ Add 1.}$$

Verify that the solution set is $\{5\}$.

**(b)** $6^x = \frac{1}{216}$

$$6^x = \frac{1}{6^3} \quad 216 = 6^3$$

$$6^x = 6^{-3} \quad \text{Write with the same base; } \frac{1}{6^3} = 6^{-3}.$$

$$x = -3 \quad \text{Set exponents equal.}$$

*CHECK* $6^x = 6^{-3} = \frac{1}{6^3} = \frac{1}{216} \checkmark$ Substitute $-3$ for $x$; true

The solution set is $\{-3\}$.

*NOW TRY ANSWER*

**4.** $\left\{\frac{4}{3}\right\}$

NOW TRY EXERCISE 5

Solve each equation.

**(a)** $3^{2x-1} = 27^{x+4}$

**(b)** $5^x = \frac{1}{625}$

**(c)** $\left(\frac{2}{7}\right)^x = \frac{343}{8}$

**(c)** $$\left(\frac{2}{3}\right)^x = \frac{9}{4}$$

$$\left(\frac{2}{3}\right)^x = \left(\frac{4}{9}\right)^{-1} \qquad \tfrac{9}{4} = \left(\tfrac{4}{9}\right)^{-1}$$

$$\left(\frac{2}{3}\right)^x = \left[\left(\frac{2}{3}\right)^2\right]^{-1} \qquad \text{Write with the same base.}$$

$$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2} \qquad \text{Power rule for exponents}$$

$$x = -2 \qquad \text{Set exponents equal.}$$

Check that the solution set is $\{-2\}$. NOW TRY

**NOTE** The steps used in **Examples 4 and 5** cannot be applied to an equation like

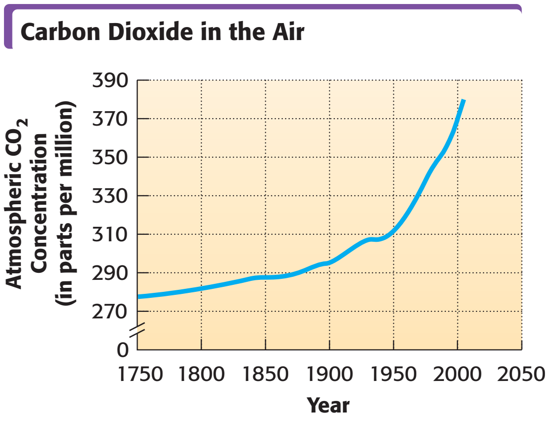
$$3^x = 12$$

because Step 1 cannot easily be done. A method for solving such exponential equations is given in **Section 12.6.**

## OBJECTIVE 4 Use exponential functions in applications involving growth or decay.

### EXAMPLE 6 Solving an Application Involving Exponential Growth

The graph in **FIGURE 8** shows the concentration of carbon dioxide (in parts per million) in the air. This concentration is increasing exponentially.



***Source:*** *Sacramento Bee*; National Oceanic and Atmospheric Administration.

**FIGURE 8**

The data are approximated by the function defined by

$$f(x) = 266(1.001)^x,$$

where $x$ is the number of years since 1750. Use this function and a calculator to approximate the concentration of carbon dioxide in parts per million, to the nearest unit, for each year.

NOW TRY ANSWERS

5. **(a)** $\{-13\}$ **(b)** $\{-4\}$
**(c)** $\{-3\}$

NOW TRY EXERCISE 6

Use the function in **Example 6** to approximate, to the nearest unit, the carbon dioxide concentration in 2000.

**(a)** 1900

Because $x$ represents the number of years since 1750, $x = 1900 - 1750 = 150$.

$$\begin{aligned} f(x) &= 266(1.001)^x && \text{Given function} \\ f(150) &= 266(1.001)^{150} && \text{Let } x = 150. \\ f(150) &\approx 309 \text{ parts per million} && \text{Use a calculator.} \end{aligned}$$

The concentration in 1900 was about 309 parts per million.

**(b)** 1950

$$\begin{aligned} f(200) &= 266(1.001)^{200} && x = 1950 - 1750 = 200 \\ f(200) &\approx 325 \text{ parts per million} && \text{Use a calculator.} \end{aligned}$$

The concentration in 1950 was about 325 parts per million. NOW TRY

NOW TRY EXERCISE 7

Use the function in **Example 7** to approximate the pressure at 6000 m. Round to the nearest unit.

## EXAMPLE 7 Applying an Exponential Decay Function

The atmospheric pressure (in millibars) at a given altitude $x$, in meters, can be approximated by the function defined by

$$f(x) = 1038(1.000134)^{-x}, \quad \text{for values of } x \text{ between 0 and 10,000.}$$

Because the base is greater than 1 and the coefficient of $x$ in the exponent is negative, function values decrease as $x$ increases. This means that as altitude increases, atmospheric pressure decreases. (*Source:* Miller, A. and J. Thompson, *Elements of Meteorology,* Fourth Edition, Charles E. Merrill Publishing Company.)

**(a)** According to this function, what is the pressure at ground level?

$$\begin{aligned} f(0) &= 1038(1.000134)^{-0} && \text{Let } x = 0. \\ &= 1038(1) && a^0 = 1 \\ &= 1038 \end{aligned}$$

At ground level, $x = 0$.

The pressure is 1038 millibars.

**(b)** Approximate the pressure at 5000 m. Round to the nearest unit.

$$\begin{aligned} f(5000) &= 1038(1.000134)^{-5000} && \text{Let } x = 5000. \\ f(5000) &\approx 531 && \text{Use a calculator.} \end{aligned}$$

The pressure is approximately 531 millibars. NOW TRY

NOW TRY ANSWERS

6. 342 parts per million
7. approximately 465 millibars

## 12.2 EXERCISES

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***Concept Check*** *Choose the correct response in Exercises 1–3.*

**1.** Which point lies on the graph of $f(x) = 3^x$?

**A.** $(1, 0)$ **B.** $(3, 1)$ **C.** $(0, 1)$ **D.** $\left(\sqrt{3}, \frac{1}{3}\right)$

**2.** Which statement is true?

**A.** The point $\left(\frac{1}{2}, \sqrt{5}\right)$ lies on the graph of $f(x) = 5^x$.

**B.** For any $a > 1$, the graph of $f(x) = a^x$ falls from left to right.

**C.** The $y$-intercept of the graph of $f(x) = 10^x$ is $(0, 10)$.

**D.** The graph of $y = 4^x$ rises at a faster rate than the graph of $y = 10^x$.

**3.** The asymptote of the graph of $f(x) = a^x$

**A.** is the $x$-axis  **B.** is the $y$-axis

**C.** has equation $x = 1$  **D.** has equation $y = 1$.

**4.** In your own words, describe the characteristics of the graph of an exponential function. Use the exponential function defined by $f(x) = 3^x$ **(Exercise 5)** and the words *asymptote, domain,* and *range* in your explanation.

*Graph each exponential function.* ***See Examples 1–3.***

**5.** $f(x) = 3^x$  **6.** $f(x) = 5^x$  **7.** $g(x) = \left(\frac{1}{3}\right)^x$

**8.** $g(x) = \left(\frac{1}{5}\right)^x$  **9.** $y = 4^{-x}$  **10.** $y = 6^{-x}$

**11.** $y = 2^{2x-2}$  **12.** $y = 2^{2x+1}$

**13.** ***Concept Check*** For an exponential function defined by $f(x) = a^x$, if $a > 1$, the graph $\underset{\text{(rises/falls)}}{\underline{\hspace{6em}}}$ from left to right. If $0 < a < 1$, the graph $\underset{\text{(rises/falls)}}{\underline{\hspace{6em}}}$ from left to right.

**14.** ***Concept Check*** Based on your answers in **Exercise 13,** make a conjecture (an educated guess) concerning whether an exponential function defined by $f(x) = a^x$ is one-to-one. Then decide whether it has an inverse based on the concepts of **Section 12.1.**

*Solve each equation.* ***See Examples 4 and 5.***

**15.** $6^x = 36$  **16.** $8^x = 64$  **17.** $100^x = 1000$

**18.** $8^x = 4$  **19.** $16^{2x+1} = 64^{x+3}$  **20.** $9^{2x-8} = 27^{x-4}$

**21.** $5^x = \frac{1}{125}$  **22.** $3^x = \frac{1}{81}$  **23.** $5^x = 0.2$

**24.** $10^x = 0.1$  **25.** $\left(\frac{3}{2}\right)^x = \frac{8}{27}$  **26.** $\left(\frac{4}{3}\right)^x = \frac{27}{64}$

*Use the exponential key of a calculator to find an approximation to the nearest thousandth.*

**27.** $12^{2.6}$  **28.** $13^{1.8}$  **29.** $0.5^{3.921}$

**30.** $0.6^{4.917}$  **31.** $2.718^{2.5}$  **32.** $2.718^{-3.1}$

*A major scientific periodical published an article in 1990 dealing with the problem of global warming. The article was accompanied by a graph that illustrated two possible scenarios.*

***(a)*** *The warming might be modeled by an exponential function of the form*

$$y = (1.046 \times 10^{-38})(1.0444^x).$$

***(b)*** *The warming might be modeled by a linear function of the form*

$$y = 0.009x - 17.67.$$

*In both cases, x represents the year, and y represents the increase in degrees Celsius due to the warming. Use these functions to approximate the increase in temperature for each of the following years.*

**33.** 2000  **34.** 2010  **35.** 2020  **36.** 2040

*Solve each problem.* ***See Examples 6 and 7.***

**37.** Based on figures from 1970 through 2005, the worldwide carbon dioxide emissions in millions of metric tons are approximated by the exponential function defined by

$$f(x) = 4231(1.0174)^x,$$

where $x = 0$ corresponds to 1970, $x = 5$ corresponds to 1975, and so on. (*Source:* Carbon Dioxide Information Analysis Center.) Give answers to the nearest unit.

**(a)** Use this model to approximate the emissions in 1980.

**(b)** Use this model to approximate the emissions in 1995.

**(c)** In 2000, the actual amount of emissions was 6735 million tons. How does this compare to the number that the model provides?

**38.** Based on figures from 1980 through 2007, the municipal solid waste generated in millions of tons can be approximated by the exponential function defined by

$$f(x) = 159.51(1.0186)^x,$$

where $x = 0$ corresponds to 1980, $x = 5$ corresponds to 1985, and so on. (*Source:* U.S. Environmental Protection Agency.) Give answers to the nearest hundredth.

**(a)** Use the model to approximate the number of tons of this waste in 1980.

**(b)** Use the model to approximate the number of tons of this waste in 1995.

**(c)** In 2007, the actual number of millions of tons of this waste was 254.1. How does this compare to the number that the model provides?

**39.** A small business estimates that the value $V(t)$ of a copy machine is decreasing according to the function defined by

$$V(t) = 5000(2)^{-0.15t},$$

where $t$ is the number of years that have elapsed since the machine was purchased, and $V(t)$ is in dollars.

**(a)** What was the original value of the machine?

**(b)** What is the value of the machine 5 yr after purchase, to the nearest dollar?

**(c)** What is the value of the machine 10 yr after purchase, to the nearest dollar?

**(d)** Graph the function.

**40.** The amount of radioactive material in an ore sample is given by the function defined by

$$A(t) = 100(3.2)^{-0.5t},$$

where $A(t)$ is the amount present, in grams, of the sample $t$ months after the initial measurement.

**(a)** How much was present at the initial measurement? (*Hint:* $t = 0$.)

**(b)** How much was present 2 months later?

**(c)** How much was present 10 months later?

**(d)** Graph the function.

**41.** Refer to the function in **Exercise 39.** When will the value of the machine be \$2500? (*Hint:* Let $V(t) = 2500$, divide both sides by 5000, and use the method of **Example 4.**)

**42.** Refer to the function in **Exercise 39.** When will the value of the machine be \$1250?

## *PREVIEW EXERCISES*

*Determine what number would have to be placed in each box for the statement to be true. See* ***Sections 4.1, 4.2, and 10.2.***

**43.** $2^{\square} = 16$

**44.** $2^{\square} = \frac{1}{16}$

**45.** $2^{\square} = 1$

**46.** $2^{\square} = \sqrt{2}$

# 12.3 Logarithmic Functions

## OBJECTIVES

1. Define a logarithm.
2. Convert between exponential and logarithmic forms.
3. Solve logarithmic equations of the form $\log_a b = k$ for $a$, $b$, or $k$.
4. Define and graph logarithmic functions.
5. Use logarithmic functions in applications involving growth or decay.

The graph of $y = 2^x$ is the curve shown in blue in **FIGURE 9**. Because $y = 2^x$ defines a one-to-one function, it has an inverse. Interchanging $x$ and $y$ gives

$x = 2^y$, the inverse of $y = 2^x$. Roles of $x$ and $y$ are interchanged.

As we saw in **Section 12.1,** the graph of the inverse is found by reflecting the graph of $y = 2^x$ about the line $y = x$. The graph of $x = 2^y$ is shown as a red curve in **FIGURE 9**.

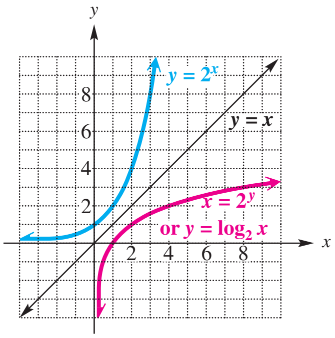


FIGURE 9

**OBJECTIVE 1** **Define a logarithm.** We cannot solve the equation $x = 2^y$ for the dependent variable $y$ with the methods presented up to now. The following definition is used to solve $x = 2^y$ for $y$.

### Logarithm

For all positive numbers $a$, with $a \neq 1$, and all positive numbers $x$,

$$\boldsymbol{y = \log_a x} \quad \text{means the same as} \quad \boldsymbol{x = a^y}.$$

***This key statement should be memorized.*** The abbreviation **log** is used for the word **logarithm.** Read **log**$_a$ ***x*** as "**the logarithm of *x* with base *a***" or "**the base *a* logarithm of *x*.**" To remember the location of the base and the exponent in each form, refer to the following diagrams.

Logarithmic form: $\overset{\text{Exponent}}{\overset{\downarrow}{y}} = \log_{\underset{\text{Base}}{\underset{\uparrow}{a}}} x$

Exponential form: $x = \overset{\text{Exponent}}{\overset{\downarrow}{\phantom{a}}}\underset{\text{Base}}{\underset{\uparrow}{a}}{}^{y}$

In work with logarithmic form and exponential form, remember the following.

### Meaning of $\log_a x$

A logarithm is an exponent. ***The expression* log$_a$ *x represents the exponent to which the base a must be raised to obtain x.***

**OBJECTIVE 2** **Convert between exponential and logarithmic forms.** We can use the definition of logarithm to convert between exponential and logarithmic forms.

**NOW TRY EXERCISE 1**

**(a)** Write $6^3 = 216$ in logarithmic form.

**(b)** Write $\log_{64} 4 = \frac{1}{3}$ in exponential form.

### EXAMPLE 1 Converting Between Exponential and Logarithmic Forms

The table shows several pairs of equivalent forms.

| *Exponential Form* | *Logarithmic Form* |
|---|---|
| $3^2 = 9$ | $\log_3 9 = 2$ |
| $\left(\frac{1}{5}\right)^{-2} = 25$ | $\log_{1/5} 25 = -2$ |
| $10^5 = 100{,}000$ | $\log_{10} 100{,}000 = 5$ |
| $4^{-3} = \frac{1}{64}$ | $\log_4 \frac{1}{64} = -3$ |

$y = \log_a x$ means $x = a^y$.

NOW TRY

**OBJECTIVE 3** **Solve logarithmic equations of the form $\log_a b = k$ for $a$, $b$, or $k$.** A **logarithmic equation** is an equation with a logarithm in at least one term.

### EXAMPLE 2 Solving Logarithmic Equations

Solve each equation.

**(a)** $\log_4 x = -2$

By the definition of logarithm, $\log_4 x = -2$ is equivalent to $x = 4^{-2}$.

$$x = 4^{-2} = \frac{1}{16}$$

The solution set is $\left\{\frac{1}{16}\right\}$.

**(b)**

$$\log_{1/2}(3x + 1) = 2$$

This is a key step.

$$3x + 1 = \left(\frac{1}{2}\right)^2 \quad \text{Write in exponential form.}$$

$$3x + 1 = \frac{1}{4} \quad \text{Apply the exponent.}$$

$$12x + 4 = 1 \quad \text{Multiply each term by 4.}$$

$$12x = -3 \quad \text{Subtract 4.}$$

$$x = -\frac{1}{4} \quad \text{Divide by 12. Write in lowest terms.}$$

*CHECK*

$$\log_{1/2}\left(3\left(-\frac{1}{4}\right) + 1\right) \stackrel{?}{=} 2 \quad \text{Let } x = -\tfrac{1}{4}.$$

$$\log_{1/2}\frac{1}{4} \stackrel{?}{=} 2 \quad \text{Simplify within parentheses.}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ ✓ \quad \text{Exponential form; true}$$

The solution set is $\left\{-\frac{1}{4}\right\}$.

**(c)**

$$\log_x 3 = 2$$

$$x^2 = 3 \quad \text{Write in exponential form.}$$

$$x = \pm\sqrt{3} \quad \text{Take square roots.}$$

Be careful here. $-\sqrt{3}$ is extraneous.

Only the *principal* square root satisfies the equation since the base must be a positive number. The solution set is $\left\{\sqrt{3}\right\}$.

**NOW TRY ANSWERS**

1. **(a)** $\log_6 216 = 3$
   **(b)** $64^{1/3} = 4$

**NOW TRY EXERCISE 2**

Solve each equation.

**(a)** $\log_2 x = -5$

**(b)** $\log_{3/2}(2x - 1) = 3$

**(c)** $\log_x 10 = 2$

**(d)** $\log_{125}\sqrt[3]{5} = x$

**(d)**

$$
\begin{aligned}
\log_{49}\sqrt[3]{7} &= x \\
49^x &= \sqrt[3]{7} && \text{Write in exponential form.} \\
(7^2)^x &= 7^{1/3} && \text{Write with the same base.} \\
7^{2x} &= 7^{1/3} && \text{Power rule for exponents} \\
2x &= \frac{1}{3} && \text{Set exponents equal.} \\
x &= \frac{1}{6} && \text{Divide by 2 (which is the same as multiplying by } \tfrac{1}{2}\text{).}
\end{aligned}
$$

The solution set is $\left\{\frac{1}{6}\right\}$.

NOW TRY

For any real number $b$, we know that $b^1 = b$ and for $b \neq 0$, $b^0 = 1$. Writing these statements in logarithmic form gives the following properties of logarithms.

### Properties of Logarithms

For any positive real number $b$, with $b \neq 1$, the following are true.

$$\log_b b = 1 \quad \text{and} \quad \log_b 1 = 0$$

**NOW TRY EXERCISE 3**

Evaluate each logarithm.

**(a)** $\log_{10} 10$

**(b)** $\log_8 1$

**(c)** $\log_{0.1} 1$

### EXAMPLE 3 Using Properties of Logarithms

Evaluate each logarithm.

**(a)** $\log_7 7 = 1$ $\quad \log_b b = 1$

**(b)** $\log_{\sqrt{2}}\sqrt{2} = 1$

**(c)** $\log_9 1 = 0$ $\quad \log_b 1 = 0$

**(d)** $\log_{0.2} 1 = 0$

NOW TRY

**OBJECTIVE 4** **Define and graph logarithmic functions.** Now we define the logarithmic function with base $a$.

### Logarithmic Function

If $a$ and $x$ are positive numbers, with $a \neq 1$, then

$$g(x) = \log_a x$$

defines the **logarithmic function with base *a*.**

### EXAMPLE 4 Graphing a Logarithmic Function $(a > 1)$

Graph $f(x) = \log_2 x$.

By writing $y = f(x) = \log_2 x$ in exponential form as $x = 2^y$, we can identify ordered pairs that satisfy the equation. It is easier to choose values for $y$ and find the corresponding values of $x$. Plotting the points in the table of ordered pairs and connecting them with a smooth curve gives the graph in **FIGURE 10** on the next page. This graph is typical of logarithmic functions with base $a > 1$.

**NOW TRY ANSWERS**

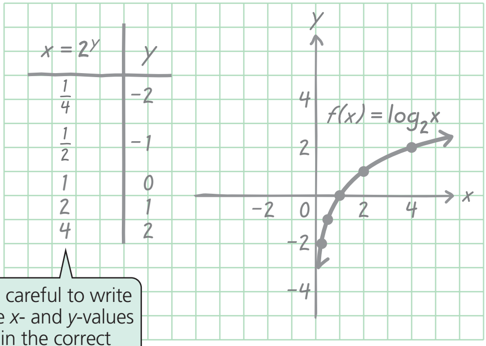
**2. (a)** $\left\{\frac{1}{32}\right\}$ **(b)** $\left\{\frac{35}{16}\right\}$

**(c)** $\left\{\sqrt{10}\right\}$ **(d)** $\left\{\frac{1}{9}\right\}$

**3. (a)** 1 **(b)** 0 **(c)** 0

NOW TRY EXERCISE 4

Graph $f(x) = \log_6 x$.



| $x = 2^y$ | $y$ |
|---|---|
| $\frac{1}{4}$ | $-2$ |
| $\frac{1}{2}$ | $-1$ |
| $1$ | $0$ |
| $2$ | $1$ |
| $4$ | $2$ |

Be careful to write the *x*- and *y*-values in the correct order.

FIGURE 10

**Logarithmic function with base $a > 1$**

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

The function is one-to-one, and its graph rises from left to right.

NOW TRY

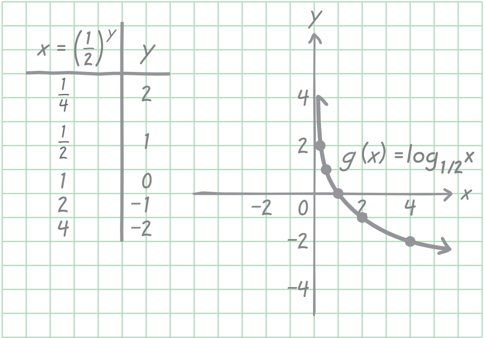
NOW TRY EXERCISE 5

Graph $g(x) = \log_{1/4} x$.

## EXAMPLE 5 Graphing a Logarithmic Function ($0 < a < 1$)

Graph $g(x) = \log_{1/2} x$.

We write $y = g(x) = \log_{1/2} x$ in exponential form as $x = \left(\frac{1}{2}\right)^y$, then choose values for $y$ and find the corresponding values of $x$. Plotting these points and connecting them with a smooth curve gives the graph in **FIGURE 11**. This graph is typical of logarithmic functions with base $0 < a < 1$.



| $x = \left(\frac{1}{2}\right)^y$ | $y$ |
|---|---|
| $\frac{1}{4}$ | $2$ |
| $\frac{1}{2}$ | $1$ |
| $1$ | $0$ |
| $2$ | $-1$ |
| $4$ | $-2$ |

FIGURE 11

**Logarithmic function with base $0 < a < 1$**

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

The function is one-to-one, and its graph falls from left to right.

NOW TRY

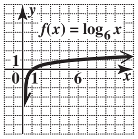
**NOTE** See the box titled "Characteristics of the Graph of $f(x) = a^x$" on **page 760.** Below we give a similar set of characteristics for the graph of $g(x) = \log_a x$. Compare the four characteristics one by one to see how the concepts of inverse functions, introduced in **Section 12.1,** are illustrated by these two classes of functions.

### Characteristics of the Graph of $g(x) = \log_a x$

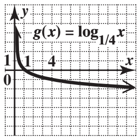
1. The graph contains the point $(1, 0)$.
2. The function is one-to-one. When $a > 1$, the graph will *rise* from left to right, from the fourth quadrant to the first. (See **FIGURE 10**.) When $0 < a < 1$, the graph will *fall* from left to right, from the first quadrant to the fourth. (See **FIGURE 11**.)
3. The graph will approach the $y$-axis, but never touch it. (The $y$-axis is an asymptote.)
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

NOW TRY ANSWERS

4.



5.



## OBJECTIVE 5 Use logarithmic functions in applications involving growth or decay.

*NOW TRY EXERCISE 6*

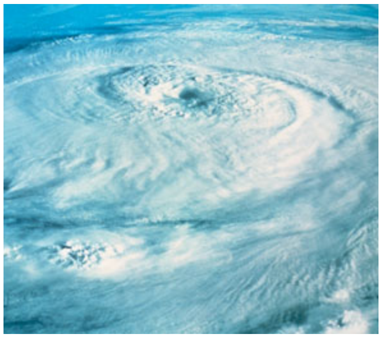
Suppose the gross national product (GNP) of a small country (in millions of dollars) is approximated by

$$G(t) = 15.0 + 2.00 \log_{10} t,$$

where $t$ is time in years since 2003. Approximate to the nearest tenth the GNP for each value of $t$.

**(a)** $t = 1$ **(b)** $t = 10$

### EXAMPLE 6 Solving an Application of a Logarithmic Function



The function defined by

$$f(x) = 27 + 1.105 \log_{10} (x + 1)$$

approximates the barometric pressure in inches of mercury at a distance of $x$ miles from the eye of a typical hurricane. (*Source:* Miller, A. and R. Anthes, *Meteorology,* Fifth Edition, Charles E. Merrill Publishing Company.) Approximate the pressure 9 mi from the eye of the hurricane.

Let $x = 9$, and find $f(9)$.

$$\begin{aligned} f(9) &= 27 + 1.105 \log_{10} (9 + 1) && \text{Let } x = 9. \\ f(9) &= 27 + 1.105 \log_{10} 10 && \text{Add inside parentheses.} \\ f(9) &= 27 + 1.105(1) && \log_{10} 10 = 1 \\ f(9) &= 28.105 && \text{Add.} \end{aligned}$$

The pressure 9 mi from the eye of the hurricane is 28.105 in. *NOW TRY*

*NOW TRY ANSWERS*

6. **(a)** \$15.0 million
**(b)** \$17.0 million

## 12.3 EXERCISES

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**1.** *Concept Check* Match the logarithmic equation in Column I with the corresponding exponential equation from Column II. **See Example 1.**

| I | II |
|---|---|
| **(a)** $\log_{1/3} 3 = -1$ | **A.** $8^{1/3} = \sqrt[3]{8}$ |
| **(b)** $\log_5 1 = 0$ | **B.** $\left(\frac{1}{3}\right)^{-1} = 3$ |
| **(c)** $\log_2 \sqrt{2} = \frac{1}{2}$ | **C.** $4^1 = 4$ |
| **(d)** $\log_{10} 1000 = 3$ | **D.** $2^{1/2} = \sqrt{2}$ |
| **(e)** $\log_8 \sqrt[3]{8} = \frac{1}{3}$ | **E.** $5^0 = 1$ |
| **(f)** $\log_4 4 = 1$ | **F.** $10^3 = 1000$ |

**2.** *Concept Check* Match the logarithm in Column I with its value in Column II. (*Example:* $\log_3 9 = 2$ because 2 is the exponent to which 3 must be raised in order to obtain 9.)

| I | II |
|---|---|
| **(a)** $\log_4 16$ | **A.** $-2$ |
| **(b)** $\log_3 81$ | **B.** $-1$ |
| **(c)** $\log_3 \left(\frac{1}{3}\right)$ | **C.** 2 |
| **(d)** $\log_{10} 0.01$ | **D.** 0 |
| **(e)** $\log_5 \sqrt{5}$ | **E.** $\frac{1}{2}$ |
| **(f)** $\log_{13} 1$ | **F.** 4 |

*Write in logarithmic form.* ***See Example 1.***

**3.** $4^5 = 1024$
**4.** $3^6 = 729$
**5.** $\left(\frac{1}{2}\right)^{-3} = 8$
**6.** $\left(\frac{1}{6}\right)^{-3} = 216$
**7.** $10^{-3} = 0.001$
**8.** $36^{1/2} = 6$
**9.** $\sqrt[4]{625} = 5$
**10.** $\sqrt[3]{343} = 7$
**11.** $8^{-2/3} = \frac{1}{4}$
**12.** $16^{-3/4} = \frac{1}{8}$
**13.** $5^0 = 1$
**14.** $7^0 = 1$

*Write in exponential form.* ***See Example 1.***

**15.** $\log_4 64 = 3$

**16.** $\log_2 512 = 9$

**17.** $\log_{10} \frac{1}{10{,}000} = -4$

**18.** $\log_{100} 100 = 1$

**19.** $\log_6 1 = 0$

**20.** $\log_\pi 1 = 0$

**21.** $\log_9 3 = \frac{1}{2}$

**22.** $\log_{64} 2 = \frac{1}{6}$

**23.** $\log_{1/4} \frac{1}{2} = \frac{1}{2}$

**24.** $\log_{1/8} \frac{1}{2} = \frac{1}{3}$

**25.** $\log_5 5^{-1} = -1$

**26.** $\log_{10} 10^{-2} = -2$

**27.** Match each logarithm in Column I with its value in Column II. **See Example 3.**

| I | II |
|---|---|
| **(a)** $\log_8 8$ | **A.** $-1$ |
| **(b)** $\log_{16} 1$ | **B.** $0$ |
| **(c)** $\log_{0.3} 1$ | **C.** $1$ |
| **(d)** $\log_{\sqrt{7}} \sqrt{7}$ | **D.** $0.1$ |

**28.** When a student asked his teacher to explain how to evaluate

$$\log_9 3$$

without showing any work, his teacher told him, "Think radically." Explain what the teacher meant by this hint.

*Solve each equation.* ***See Examples 2 and 3.***

**29.** $x = \log_{27} 3$

**30.** $x = \log_{125} 5$

**31.** $\log_x 9 = \frac{1}{2}$

**32.** $\log_x 5 = \frac{1}{2}$

**33.** $\log_x 125 = -3$

**34.** $\log_x 64 = -6$

**35.** $\log_{12} x = 0$

**36.** $\log_4 x = 0$

**37.** $\log_x x = 1$

**38.** $\log_x 1 = 0$

**39.** $\log_x \frac{1}{25} = -2$

**40.** $\log_x \frac{1}{10} = -1$

**41.** $\log_8 32 = x$

**42.** $\log_{81} 27 = x$

**43.** $\log_\pi \pi^4 = x$

**44.** $\log_{\sqrt{2}} \left(\sqrt{2}\right)^9 = x$

**45.** $\log_6 \sqrt{216} = x$

**46.** $\log_4 \sqrt{64} = x$

**47.** $\log_4 (2x + 4) = 3$

**48.** $\log_3 (2x + 7) = 4$

*If $(p, q)$ is on the graph of $f(x) = a^x$ (for $a > 0$ and $a \neq 1$), then $(q, p)$ is on the graph of $f^{-1}(x) = \log_a x$. Use this fact, and refer to the graphs required in* ***Exercises 5–8*** *in* ***Section 12.2*** *to graph each logarithmic function.* ***See Examples 4 and 5.***

**49.** $y = \log_3 x$

**50.** $y = \log_5 x$

**51.** $y = \log_{1/3} x$

**52.** $y = \log_{1/5} x$

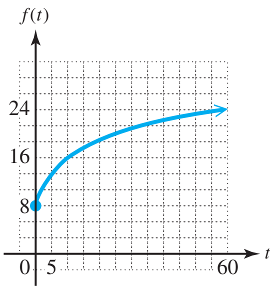
**53.** Explain why 1 is not allowed as a base for a logarithmic function.

**54.** Compare the summary of facts about the graph of $f(x) = a^x$ in **Section 12.2** with the similar summary of facts about the graph of $g(x) = \log_a x$ in this section. Make a list of the facts that reinforce the concept that $f$ and $g$ are inverse functions.

**55.** ***Concept Check*** The domain of $f(x) = a^x$ is $(-\infty, \infty)$, while the range is $(0, \infty)$. Therefore, since $g(x) = \log_a x$ defines the inverse of $f$, the domain of $g$ is ____________, while the range of $g$ is ____________.

**56.** ***Concept Check*** The graphs of both $f(x) = 3^x$ and $g(x) = \log_3 x$ rise from left to right. Which one rises at a faster rate?

***Concept Check*** *Use the graph at the right to predict the value of $f(t)$ for the given value of $t$.*



**57.** $t = 0$

**58.** $t = 10$

**59.** $t = 60$

**60.** Show that the points determined in **Exercises 57–59** lie on the graph of $f(t) = 8 \log_5 (2t + 5)$.

*Solve each problem.* ***See Example 6.***

**61.** For 1981–2003, the number of billion cubic feet of natural gas gross withdrawals from crude oil wells in the United States can be approximated by the function defined by

$$f(x) = 3800 + 585 \log_2 x,$$

where $x = 1$ corresponds to 1981, $x = 2$ to 1982, and so on. (*Source:* Energy Information Administration.) Use this function to approximate, to the nearest unit, the number of cubic feet withdrawn in each of the following years.

**(a)** 1982 **(b)** 1988 **(c)** 1996

**62.** According to selected figures from the last two decades of the 20th century, the number of trillion cubic feet of dry natural gas consumed worldwide can be approximated by the function defined by

$$f(x) = 51.47 + 6.044 \log_2 x,$$

where $x = 1$ corresponds to 1980, $x = 2$ to 1981, and so on. (*Source:* Energy Information Administration.) Use the function to approximate, to the nearest hundredth, consumption in each year.

**(a)** 1980 **(b)** 1987 **(c)** 1995

**63.** Sales (in thousands of units) of a new product are approximated by the function defined by

$$S(t) = 100 + 30 \log_3 (2t + 1),$$

where $t$ is the number of years after the product is introduced.

**(a)** What were the sales, to the nearest unit, after 1 yr?

**(b)** What were the sales, to the nearest unit, after 13 yr?

**(c)** Graph $y = S(t)$.

**64.** A study showed that the number of mice in an old abandoned house was approximated by the function defined by

$$M(t) = 6 \log_4 (2t + 4),$$

where $t$ is measured in months and $t = 0$ corresponds to January 2008. Find the number of mice in the house in

**(a)** January 2008 **(b)** July 2008 **(c)** July 2010.

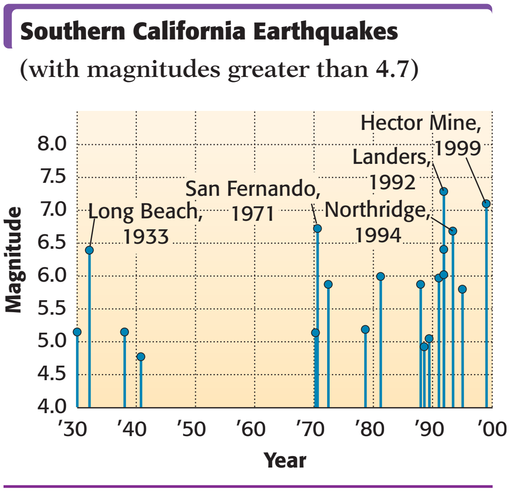
**(d)** Graph the function.

*The* ***Richter scale*** *is used to measure the intensity of earthquakes. The Richter scale rating of an earthquake of intensity x is given by*

$$R = \log_{10} \frac{x}{x_0},$$

*where $x_0$ is the intensity of an earthquake of a certain (small) size. The figure here shows Richter scale ratings for Southern California earthquakes from 1930 to 2000 with magnitudes greater than 4.7.*

**65.** The 1994 Northridge earthquake had a Richter scale rating of 6.7. The 1992 Landers earthquake had a rating of 7.3. How much more powerful was the Landers earthquake than the Northridge earthquake?



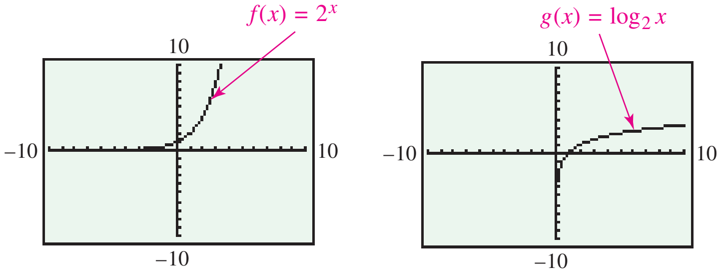
**Southern California Earthquakes**

(with magnitudes greater than 4.7)

***Source:*** Caltech; U.S. Geological Survey.

**66.** Compare the smallest rated earthquake in the figure (at 4.8) with the Landers quake. How much more powerful was the Landers quake?

*Some graphing calculators have the capability of drawing the inverse of a function. For example, the two screens that follow show the graphs of* $f(x) = 2^x$ *and* $g(x) = \log_2 x$. *The graph of g was obtained by drawing the graph of* $f^{-1}$, *since* $g(x) = f^{-1}(x)$. *(Compare to* **FIGURE 9** *in this section.)*



*Use a graphing calculator with the capability of drawing the inverse of a function to draw the graph of each logarithmic function. Use the standard viewing window.*

**67.** $g(x) = \log_3 x$
(Compare to **Exercise 49.**)

**68.** $g(x) = \log_5 x$
(Compare to **Exercise 50.**)

**69.** $g(x) = \log_{1/3} x$
(Compare to **Exercise 51.**)

**70.** $g(x) = \log_{1/5} x$
(Compare to **Exercise 52.**)

## *PREVIEW EXERCISES*

*Simplify each expression. Write answers using only positive exponents.* ***See Sections 4.1 and 4.2.***

**71.** $4^7 \cdot 4^2$

**72.** $\frac{5^{-3}}{5^8}$

**73.** $\frac{7^8}{7^{-4}}$

**74.** $(9^3)^{-2}$

# 12.4 Properties of Logarithms

**OBJECTIVES**

1. Use the product rule for logarithms.
2. Use the quotient rule for logarithms.
3. Use the power rule for logarithms.
4. Use properties to write alternative forms of logarithmic expressions.

Logarithms were used as an aid to numerical calculation for several hundred years. Today the widespread use of calculators has made the use of logarithms for calculation obsolete. However, logarithms are still very important in applications and in further work in mathematics.

**OBJECTIVE 1** **Use the product rule for logarithms.** One way in which logarithms simplify problems is by changing a problem of multiplication into one of addition. We know that $\log_2 4 = 2$, $\log_2 8 = 3$, and $\log_2 32 = 5$.

$$\log_2 32 = \log_2 4 + \log_2 8 \qquad 5 = 2 + 3$$

$$\log_2 (4 \cdot 8) = \log_2 4 + \log_2 8 \qquad 32 = 4 \cdot 8$$

This is an example of the following rule.

**Product Rule for Logarithms**

If $x$, $y$, and $b$ are positive real numbers, where $b \neq 1$, then the following is true.

$$\mathbf{\log_b xy = \log_b x + \log_b y}$$

That is, the logarithm of a product is the sum of the logarithms of the factors.

**NOTE** The word statement of the product rule can be restated by replacing "logarithm" with "exponent." The rule then becomes the familiar rule for multiplying exponential expressions: The *exponent* of a product is the sum of the *exponents* of the factors.

To prove this rule, let $m = \log_b x$ and $n = \log_b y$, and recall that

$$\log_b x = m \quad \text{means} \quad b^m = x \qquad \text{and} \qquad \log_b y = n \quad \text{means} \quad b^n = y.$$

Now consider the product $xy$.

$$\begin{aligned} xy &= b^m \cdot b^n && \text{Substitute.} \\ xy &= b^{m+n} && \text{Product rule for exponents} \\ \log_b xy &= m + n && \text{Convert to logarithmic form.} \\ \log_b xy &= \log_b x + \log_b y && \text{Substitute.} \end{aligned}$$

The last statement is the result we wished to prove.

NOW TRY EXERCISE 1

Use the product rule to rewrite each logarithm.

**(a)** $\log_{10} (7 \cdot 9)$

**(b)** $\log_5 11 + \log_5 8$

**(c)** $\log_5 (5x), \quad x > 0$

**(d)** $\log_2 t^3, \quad t > 0$

### EXAMPLE 1 Using the Product Rule

Use the product rule to rewrite each logarithm. Assume $x > 0$.

**(a)** $\log_5 (6 \cdot 9)$

$$= \log_5 6 + \log_5 9 \qquad \text{Product rule}$$

**(b)** $\log_7 8 + \log_7 12$

$$\begin{aligned} &= \log_7 (8 \cdot 12) && \text{Product rule} \\ &= \log_7 96 && \text{Multiply.} \end{aligned}$$

**(c)** $\log_3 (3x)$

$$\begin{aligned} &= \log_3 3 + \log_3 x && \text{Product rule} \\ &= 1 + \log_3 x && \log_3 3 = 1 \end{aligned}$$

**(d)** $\log_4 x^3$

$$\begin{aligned} &= \log_4 (x \cdot x \cdot x) && x^3 = x \cdot x \cdot x \\ &= \log_4 x + \log_4 x + \log_4 x && \text{Product rule} \\ &= 3 \log_4 x && \text{Combine like terms.} \end{aligned}$$

NOW TRY

**OBJECTIVE 2** **Use the quotient rule for logarithms.** The rule for division is similar to the rule for multiplication.

**Quotient Rule for Logarithms**

If $x$, $y$, and $b$ are positive real numbers, where $b \neq 1$, then the following is true.

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

That is, the logarithm of a quotient is the difference between the logarithm of the numerator and the logarithm of the denominator.

The proof of this rule is similar to the proof of the product rule.

NOW TRY ANSWERS

1. **(a)** $\log_{10} 7 + \log_{10} 9$
   **(b)** $\log_5 88$
   **(c)** $1 + \log_5 x$
   **(d)** $3 \log_2 t$

**NOW TRY EXERCISE 2**

Use the quotient rule to rewrite each logarithm.

**(a)** $\log_{10} \frac{7}{9}$

**(b)** $\log_4 x - \log_4 12, \quad x > 0$

**(c)** $\log_5 \frac{25}{27}$

## EXAMPLE 2 Using the Quotient Rule

Use the quotient rule to rewrite each logarithm. Assume $x > 0$.

**(a)** $\log_4 \frac{7}{9}$

$= \log_4 7 - \log_4 9$ Quotient rule

**(b)** $\log_5 6 - \log_5 x$

$= \log_5 \frac{6}{x}$ Quotient rule

**(c)** $\log_3 \frac{27}{5}$

$= \log_3 27 - \log_3 5$ Quotient rule

$= 3 - \log_3 5$ $\log_3 27 = 3$

NOW TRY

**CAUTION** ***There is no property of logarithms to rewrite the logarithm of a sum or difference.*** For example, we ***cannot*** write $\log_b (x + y)$ in terms of $\log_b x$ and $\log_b y$. Also,

$$\log_b \frac{x}{y} \neq \frac{\log_b x}{\log_b y}.$$

**OBJECTIVE 3** **Use the power rule for logarithms.** An exponential expression such as $2^3$ means $2 \cdot 2 \cdot 2$. The base is used as a factor 3 times. Similarly, the product rule can be extended to rewrite the logarithm of a power as the product of the exponent and the logarithm of the base.

$\log_5 2^3$

$= \log_5 (2 \cdot 2 \cdot 2)$

$= \log_5 2 + \log_5 2 + \log_5 2$

$= 3 \log_5 2$

$\log_2 7^4$

$= \log_2 (7 \cdot 7 \cdot 7 \cdot 7)$

$= \log_2 7 + \log_2 7 + \log_2 7 + \log_2 7$

$= 4 \log_2 7$

Furthermore, we saw in **Example 1(d)** that $\log_4 x^3 = 3 \log_4 x$. These examples suggest the following rule.

### Power Rule for Logarithms

If $x$ and $b$ are positive real numbers, where $b \neq 1$, and if $r$ is any real number, then the following is true.

$$\boldsymbol{\log_b x^r = r \log_b x}$$

That is, the logarithm of a number to a power equals the exponent times the logarithm of the number.

As further examples of this rule,

$$\log_b m^5 = 5 \log_b m \quad \text{and} \quad \log_3 5^4 = 4 \log_3 5.$$

**NOW TRY ANSWERS**

2. **(a)** $\log_{10} 7 - \log_{10} 9$

**(b)** $\log_4 \frac{x}{12}$

**(c)** $2 - \log_5 27$

To prove the power rule, let $\log_b x = m$.

$$\begin{aligned}
b^m &= x && \text{Convert to exponential form.}\\
(b^m)^r &= x^r && \text{Raise to the power } r.\\
b^{mr} &= x^r && \text{Power rule for exponents}\\
\log_b x^r &= rm && \text{Convert to logarithmic form; commutative property}\\
\log_b x^r &= r \log_b x && m = \log_b x \text{ from above}
\end{aligned}$$

This is the statement to be proved.

As a special case of the power rule, let $r = \frac{1}{p}$, so

$$\log_b \sqrt[p]{x} = \log_b x^{1/p} = \frac{1}{p} \log_b x.$$

For example, using this result, with $x > 0$,

$$\log_b \sqrt[5]{x} = \log_b x^{1/5} = \frac{1}{5} \log_b x \quad \text{and} \quad \log_b \sqrt[3]{x^4} = \log_b x^{4/3} = \frac{4}{3} \log_b x.$$

Another special case is

$$\log_b \frac{1}{x} = \log_b x^{-1} = -\log_b x.$$

**NOW TRY EXERCISE 3**

Use the power rule to rewrite each logarithm. Assume $a > 0$, $x > 0$, and $a \neq 1$.

**(a)** $\log_7 5^3$ **(b)** $\log_a \sqrt{10}$

**(c)** $\log_3 \sqrt[4]{x^3}$

## EXAMPLE 3 Using the Power Rule

Use the power rule to rewrite each logarithm. Assume $b > 0$, $x > 0$, and $b \neq 1$.

**(a)** $\log_5 4^2$

$$= 2 \log_5 4 \qquad \text{Power rule}$$

**(b)** $\log_b x^5$

$$= 5 \log_b x \qquad \text{Power rule}$$

**(c)** $\log_b \sqrt{7}$

$$\begin{aligned}
&= \log_b 7^{1/2} && \sqrt{x} = x^{1/2}\\
&= \frac{1}{2} \log_b 7 && \text{Power rule}
\end{aligned}$$

**(d)** $\log_2 \sqrt[5]{x^2}$

$$\begin{aligned}
&= \log_2 x^{2/5} && \sqrt[5]{x^2} = x^{2/5}\\
&= \frac{2}{5} \log_2 x && \text{Power rule}
\end{aligned}$$

NOW TRY

Two special properties involving both exponential and logarithmic expressions come directly from the fact that logarithmic and exponential functions are inverses of each other.

### Special Properties

If $b > 0$ and $b \neq 1$, then the following are true.

$$\boldsymbol{b^{\log_b x} = x, \quad x > 0} \quad \text{and} \quad \boldsymbol{\log_b b^x = x}$$

To prove the first statement, let $y = \log_b x$.

$$\begin{aligned}
y &= \log_b x\\
b^y &= x && \text{Convert to exponential form.}\\
b^{\log_b x} &= x && \text{Replace } y \text{ with } \log_b x.
\end{aligned}$$

The proof of the second statement is similar.

**NOW TRY ANSWERS**

**3. (a)** $3 \log_7 5$ **(b)** $\frac{1}{2} \log_a 10$

**(c)** $\frac{3}{4} \log_3 x$

NOW TRY EXERCISE 4

Find each value.

**(a)** $\log_4 4^7$

**(b)** $\log_{10} 10{,}000$

**(c)** $8^{\log_8 5}$

### EXAMPLE 4 Using the Special Properties

Find each value.

**(a)** $\log_5 5^4 = 4$, since $\log_b b^x = x$. **(b)** $\log_3 9 = \log_3 3^2 = 2$

**(c)** $4^{\log_4 10} = 10$

NOW TRY

We summarize the properties of logarithms.

### Properties of Logarithms

If $x$, $y$, and $b$ are positive real numbers, where $b \neq 1$, and $r$ is any real number, then the following are true.

**Product Rule** $\quad \log_b xy = \log_b x + \log_b y$

**Quotient Rule** $\quad \log_b \dfrac{x}{y} = \log_b x - \log_b y$

**Power Rule** $\quad \log_b x^r = r \log_b x$

**Special Properties** $\quad b^{\log_b x} = x \quad$ and $\quad \log_b b^x = x$

### OBJECTIVE 4 Use properties to write alternative forms of logarithmic expressions.

### EXAMPLE 5 Writing Logarithms in Alternative Forms

Use the properties of logarithms to rewrite each expression if possible. Assume that all variables represent positive real numbers.

**(a)** $\log_4 4x^3$

$= \log_4 4 + \log_4 x^3$ Product rule

$= 1 + 3 \log_4 x$ $\log_4 4 = 1$; power rule

**(b)** $\log_7 \sqrt{\dfrac{m}{n}}$

$= \log_7 \left(\dfrac{m}{n}\right)^{1/2}$ Write the radical expression with a rational exponent.

$= \dfrac{1}{2} \log_7 \dfrac{m}{n}$ Power rule

$= \dfrac{1}{2}(\log_7 m - \log_7 n)$ Quotient rule

**(c)** $\log_5 \dfrac{a^2}{bc}$

$= \log_5 a^2 - \log_5 bc$ Quotient rule

$= 2 \log_5 a - \log_5 bc$ Power rule

$= 2 \log_5 a - (\log_5 b + \log_5 c)$ Product rule

$= 2 \log_5 a - \log_5 b - \log_5 c$

Parentheses are necessary here.

NOW TRY ANSWERS

**4.** **(a)** 7 **(b)** 4 **(c)** 5

NOW TRY EXERCISE 5

Use properties of logarithms to rewrite each expression if possible. Assume that all variables represent positive real numbers.

**(a)** $\log_3 9z^4$

**(b)** $\log_6 \sqrt{\dfrac{n}{3m}}$

**(c)** $\log_2 x + 3\log_2 y - \log_2 z$

**(d)** $\log_5 (x + 10) + \log_5 (x - 10) - \dfrac{3}{5}\log_5 x, \quad x > 10$

**(e)** $\log_7 (49 + 2x)$

**(d)** $4\log_b m - \log_b n, \quad b \neq 1$

$$= \log_b m^4 - \log_b n \qquad \text{Power rule}$$

$$= \log_b \frac{m^4}{n} \qquad \text{Quotient rule}$$

**(e)** $\log_b (x + 1) + \log_b (2x + 1) - \dfrac{2}{3}\log_b x, \quad b \neq 1$

$$= \log_b (x + 1) + \log_b (2x + 1) - \log_b x^{2/3} \qquad \text{Power rule}$$

$$= \log_b \frac{(x + 1)(2x + 1)}{x^{2/3}} \qquad \text{Product and quotient rules}$$

$$= \log_b \frac{2x^2 + 3x + 1}{x^{2/3}} \qquad \text{Multiply in the numerator.}$$

**(f)** $\log_8 (2p + 3r)$ cannot be rewritten using the properties of logarithms. There is no property of logarithms to rewrite the logarithm of a sum. NOW TRY

In the next example, we use numerical values for $\log_2 5$ and $\log_2 3$. While we use the equals symbol to give these values, they are actually just approximations since most logarithms of this type are irrational numbers. ***We use = with the understanding that the values are correct to four decimal places.***

NOW TRY EXERCISE 6

Given that $\log_2 7 = 2.8074$ and $\log_2 10 = 3.3219$, evaluate the following.

**(a)** $\log_2 70$ **(b)** $\log_2 0.7$

**(c)** $\log_2 49$

### EXAMPLE 6 Using the Properties of Logarithms with Numerical Values

Given that $\log_2 5 = 2.3219$ and $\log_2 3 = 1.5850$, evaluate the following.

**(a)** $\log_2 15$

$$\begin{aligned} &= \log_2(3 \cdot 5) && \text{Factor 15.}\\ &= \log_2 3 + \log_2 5 && \text{Product rule}\\ &= 1.5850 + 2.3219 && \text{Substitute the given values.}\\ &= 3.9069 && \text{Add.} \end{aligned}$$

**(b)** $\log_2 0.6$

$$\begin{aligned} &= \log_2 \frac{3}{5} && 0.6 = \tfrac{6}{10} = \tfrac{3}{5}\\ &= \log_2 3 - \log_2 5 && \text{Quotient rule}\\ &= 1.5850 - 2.3219 && \text{Substitute the given values.}\\ &= -0.7369 && \text{Subtract.} \end{aligned}$$

**(c)** $\log_2 27$

$$\begin{aligned} &= \log_2 3^3 && \text{Write 27 as a power of 3.}\\ &= 3\log_2 3 && \text{Power rule}\\ &= 3(1.5850) && \text{Substitute the given value.}\\ &= 4.7550 && \text{Multiply.} \end{aligned}$$

NOW TRY

NOW TRY ANSWERS

5. **(a)** $2 + 4\log_3 z$
**(b)** $\frac{1}{2}(\log_6 n - \log_6 3 - \log_6 m)$
**(c)** $\log_2 \dfrac{xy^3}{z}$ **(d)** $\log_5 \dfrac{x^2 - 100}{x^{3/5}}$
**(e)** cannot be rewritten

6. **(a)** 6.1293 **(b)** $-0.5145$
**(c)** 5.6148

NOW TRY EXERCISE 7

Decide whether each statement is *true* or *false*.

**(a)** $\log_2 16 + \log_2 16 = \log_2 32$

**(b)** $(\log_2 4)(\log_3 9) = \log_6 36$

## EXAMPLE 7 Deciding Whether Statements about Logarithms Are True

Decide whether each statement is *true* or *false*.

**(a)** $\log_2 8 - \log_2 4 = \log_2 4$

Evaluate each side.

$$\begin{aligned} &\log_2 8 - \log_2 4 && \text{Left side} \\ &= \log_2 2^3 - \log_2 2^2 && \text{Write 8 and 4 as powers of 2.} \\ &= 3 - 2 && \log_a a^x = x \\ &= 1 && \text{Subtract.} \end{aligned}$$

$$\begin{aligned} &\log_2 4 && \text{Right side} \\ &= \log_2 2^2 && \text{Write 4 as a power of 2.} \\ &= 2 && \log_a a^x = x \end{aligned}$$

The statement is false because $1 \neq 2$.

**(b)** $\log_3 (\log_2 8) = \dfrac{\log_7 49}{\log_8 64}$

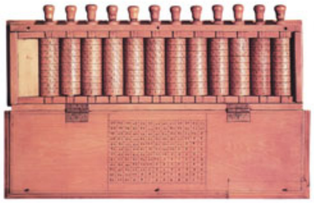
Evaluate each side.

$$\begin{aligned} &\log_3 (\log_2 8) && \text{Left side} \\ &= \log_3 (\log_2 2^3) && \text{Write 8 as a power of 2.} \\ &= \log_3 3 && \log_a a^x = x \\ &= 1 && 3 = 3^1 \end{aligned}$$

$$\begin{aligned} &\frac{\log_7 49}{\log_8 64} && \text{Right side} \\ &= \frac{\log_7 7^2}{\log_8 8^2} && \text{Write 49 and 64 using exponents.} \\ &= \frac{2}{2} && \log_a a^x = x \\ &= 1 && \text{Simplify.} \end{aligned}$$

The statement is true because $1 = 1$.

NOW TRY



**Napier's Rods**

*Source:* IBM Corporate Archives.

## CONNECTIONS

Long before the days of calculators and computers, the search for making calculations easier was an ongoing process. Machines built by Charles Babbage and Blaise Pascal, a system of "rods" used by John Napier, and slide rules were the forerunners of today's electronic marvels. The invention of logarithms by John Napier in the sixteenth century was a great breakthrough in the search for easier methods of calculation.

Since logarithms are exponents, their properties allowed users of tables of common logarithms to multiply by adding, divide by subtracting, raise to powers by multiplying, and take roots by dividing. Although logarithms are no longer used for computations, they play an important part in higher mathematics.

### For Discussion or Writing

1. To multiply 458.3 by 294.6 using logarithms, we add $\log_{10} 458.3$ and $\log_{10} 294.6$, and then find 10 to this power. Perform this multiplication using the `log x` key* and the `10^x` key on your calculator. Check your answer by multiplying directly with your calculator.
2. Try division, raising to a power, and taking a root by this method.

NOW TRY ANSWERS

7. **(a)** false **(b)** false

---

*In this text, the notation $\log x$ is used to mean $\log_{10} x$. This is also the meaning of the log key on calculators.

## 12.4 EXERCISES

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*Use the indicated rule of logarithms to complete each equation.* ***See Examples 1–4.***

**1.** $\log_{10}(7 \cdot 8) =$ ____________ (product rule)

**2.** $\log_{10}\frac{7}{8} =$ ____________ (quotient rule)

◉ **3.** $3^{\log_3 4} =$ ____________ (special property)

**4.** $\log_{10} 3^6 =$ ____________ (power rule)

**5.** $\log_3 3^9 =$ ____________ (special property)

✎ **6.** Evaluate $\log_2(8 + 8)$. Then evaluate $\log_2 8 + \log_2 8$. Are the results the same? How could you change the operation in the first expression to make the two expressions equal?

*Use the properties of logarithms to express each logarithm as a sum or difference of logarithms, or as a single number if possible. Assume that all variables represent positive real numbers.* ***See Examples 1–5.***

◉ **7.** $\log_7(4 \cdot 5)$

**8.** $\log_8(9 \cdot 11)$

◉ **9.** $\log_5\frac{8}{3}$

**10.** $\log_3\frac{7}{5}$

◉ **11.** $\log_4 6^2$

**12.** $\log_5 7^4$

◉ **13.** $\log_3\frac{\sqrt[3]{4}}{x^2y}$

**14.** $\log_7\frac{\sqrt[3]{13}}{pq^2}$

**15.** $\log_3\sqrt{\frac{xy}{5}}$

**16.** $\log_6\sqrt{\frac{pq}{7}}$

**17.** $\log_2\frac{\sqrt[3]{x} \cdot \sqrt[5]{y}}{r^2}$

**18.** $\log_4\frac{\sqrt[4]{z} \cdot \sqrt[5]{w}}{s^2}$

**19.** *Concept Check* A student erroneously wrote $\log_a(x + y) = \log_a x + \log_a y$. When his teacher explained that this was indeed wrong, the student claimed that he had used the distributive property. **WHAT WENT WRONG?**

✎ **20.** Write a few sentences explaining how the rules for multiplying and dividing powers of the same base are similar to the rules for finding logarithms of products and quotients.

*Use the properties of logarithms to write each expression as a single logarithm. Assume that all variables are defined in such a way that the variable expressions are positive, and bases are positive numbers not equal to 1.* ***See Examples 1–5.***

**21.** $\log_b x + \log_b y$

**22.** $\log_b w + \log_b z$

**23.** $\log_a m - \log_a n$

**24.** $\log_b x - \log_b y$

**25.** $(\log_a r - \log_a s) + 3\log_a t$

**26.** $(\log_a p - \log_a q) + 2\log_a r$

**27.** $3\log_a 5 - 4\log_a 3$

**28.** $3\log_a 5 - \frac{1}{2}\log_a 9$

**29.** $\log_{10}(x + 3) + \log_{10}(x - 3)$

**30.** $\log_{10}(x + 4) + \log_{10}(x - 4)$

**31.** $3\log_p x + \frac{1}{2}\log_p y - \frac{3}{2}\log_p z - 3\log_p a$

**32.** $\frac{1}{3}\log_b x + \frac{2}{3}\log_b y - \frac{3}{4}\log_b s - \frac{2}{3}\log_b t$

*To four decimal places, the values of* $\log_{10} 2$ *and* $\log_{10} 9$ *are*

$$\log_{10} 2 = 0.3010 \quad \text{and} \quad \log_{10} 9 = 0.9542.$$

*Evaluate each logarithm by applying the appropriate rule or rules from this section. DO NOT USE A CALCULATOR.* ***See Example 6.***

**33.** $\log_{10} 18$

**34.** $\log_{10} 4$

**35.** $\log_{10} \frac{2}{9}$

**36.** $\log_{10} \frac{9}{2}$

**37.** $\log_{10} 36$

**38.** $\log_{10} 162$

**39.** $\log_{10} \sqrt[4]{9}$

**40.** $\log_{10} \sqrt[5]{2}$

**41.** $\log_{10} 3$

**42.** $\log_{10} \frac{1}{9}$

**43.** $\log_{10} 9^5$

**44.** $\log_{10} 2^{19}$

*Decide whether each statement is* true *or* false. ***See Example 7.***

**45.** $\log_2 (8 + 32) = \log_2 8 + \log_2 32$

**46.** $\log_2 (64 - 16) = \log_2 64 - \log_2 16$

**47.** $\log_3 7 + \log_3 7^{-1} = 0$

**48.** $\log_3 49 + \log_3 49^{-1} = 0$

**49.** $\log_6 60 - \log_6 10 = 1$

**50.** $\log_3 8 + \log_3 \frac{1}{8} = 0$

**51.** $\dfrac{\log_{10} 7}{\log_{10} 14} = \dfrac{1}{2}$

**52.** $\dfrac{\log_{10} 10}{\log_{10} 100} = \dfrac{1}{10}$

**53.** ***Concept Check*** Refer to the Note following the word statement of the product rule for logarithms in this section. Now, state the quotient rule in words, replacing "logarithm" with "exponent."

**54.** Explain why the statement for the power rule for logarithms requires that $x$ be a positive real number.

**55.** ***Concept Check*** Why can't we determine a logarithm of 0? (*Hint*: Think of the definition of logarithm.)

**56.** ***Concept Check*** Consider the following "proof" that $\log_2 16$ does not exist.

$$\begin{aligned} &\log_2 16 \\ &= \log_2 (-4)(-4) \\ &= \log_2 (-4) + \log_2 (-4) \end{aligned}$$

Since the logarithm of a negative number is not defined, the final step cannot be evaluated, and so $\log_2 16$ does not exist. ***WHAT WENT WRONG?***

## *PREVIEW EXERCISES*

*Write each exponential statement in logarithmic form.* ***See Section 12.3.***

**57.** $10^4 = 10{,}000$

**58.** $10^{1/2} = \sqrt{10}$

**59.** $10^{-2} = 0.01$

*Write each logarithmic statement in exponential form.* ***See Section 12.3.***

**60.** $\log_{10} 0.001 = -3$

**61.** $\log_{10} 1 = 0$

**62.** $\log_{10} \sqrt[3]{10} = \frac{1}{3}$

# 12.5 Common and Natural Logarithms

**OBJECTIVES**

1. Evaluate common logarithms using a calculator.
2. Use common logarithms in applications.
3. Evaluate natural logarithms using a calculator.
4. Use natural logarithms in applications.
5. Use the change-of-base rule.

Logarithms are important in many applications in biology, engineering, economics, and social science. In this section we find numerical approximations for logarithms. Traditionally, base 10 logarithms were used most often because our number system is base 10. Logarithms to base 10 are called **common logarithms,** and

$$\log_{10} x \text{ is abbreviated as } \log x,$$

where the base is understood to be 10.

**OBJECTIVE 1** **Evaluate common logarithms using a calculator.** In the first example, we give the results of evaluating some common logarithms using a calculator with a LOG key. Consult your calculator manual to see how to use this key.

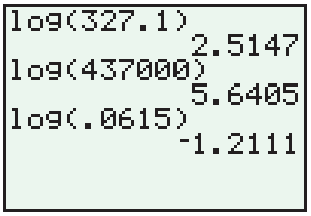
### EXAMPLE 1 Evaluating Common Logarithms

Using a calculator, evaluate each logarithm to four decimal places.

**(a)** $\log 327.1 \approx 2.5147$ **(b)** $\log 437{,}000 \approx 5.6405$

**(c)** $\log 0.0615 \approx -1.2111$

**FIGURE 12** shows how a graphing calculator displays these common logarithms to four decimal places.



```
log(327.1)
          2.5147
log(437000)
          5.6405
log(.0615)
         -1.2111
```

FIGURE 12

NOW TRY

*NOW TRY EXERCISE 1*

Using a calculator, evaluate each logarithm to four decimal places.

**(a)** $\log 115$ **(b)** $\log 0.25$

In **Example 1(c),** $\log 0.0615 \approx -1.2111$, a negative result. ***The common logarithm of a number between 0 and 1 is always negative*** because the logarithm is the exponent on 10 that produces the number. In this case, we have

$$10^{-1.2111} \approx 0.0615.$$

If the exponent (the logarithm) were positive, the result would be greater than 1 because $10^0 = 1$. The graph in **FIGURE 13** illustrates these concepts.

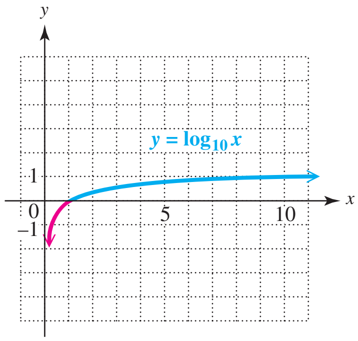


FIGURE 13

**OBJECTIVE 2** **Use common logarithms in applications.** In chemistry, pH is a measure of the acidity or alkalinity of a solution. Pure water, for example, has pH 7. In general, acids have pH numbers less than 7, and alkaline solutions have pH values greater than 7, as shown in **FIGURE 14** on the next page.

*NOW TRY ANSWERS*

1. **(a)** 2.0607 **(b)** $-0.6021$

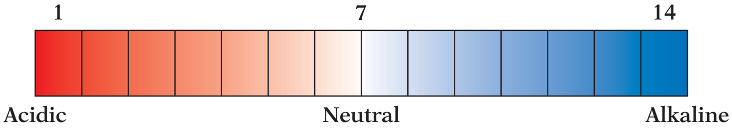


**FIGURE 14** pH Scale

The **pH** of a solution is defined as

$$\mathbf{pH = -\log\,[H_3O^+]},$$

where $[H_3O^+]$ is the hydronium ion concentration in moles per liter. ***It is customary to round pH values to the nearest tenth.***

**NOW TRY EXERCISE 2**

Water taken from a wetland has a hydronium ion concentration of

$$3.4 \times 10^{-5}.$$

Find the pH value for the water and classify the wetland as a rich fen, a poor fen, or a bog.

## EXAMPLE 2 Using pH in an Application

Wetlands are classified as *bogs, fens, marshes,* and *swamps,* on the basis of pH values. A pH value between 6.0 and 7.5, such as that of Summerby Swamp in Michigan's Hiawatha National Forest, indicates that the wetland is a "rich fen." When the pH is between 3.0 and 6.0, the wetland is a "poor fen," and if the pH falls to 3.0 or less, it is a "bog." (*Source:* Mohlenbrock, R., "Summerby Swamp, Michigan," *Natural History.*)

Suppose that the hydronium ion concentration of a sample of water from a wetland is $6.3 \times 10^{-3}$. How would this wetland be classified?

$$\begin{aligned} \text{pH} &= -\log\,(6.3 \times 10^{-3}) && \text{Definition of pH} \\ \text{pH} &= -(\log 6.3 + \log 10^{-3}) && \text{Product rule} \\ \text{pH} &= -[0.7993 - 3(1)] && \text{Use a calculator to find log 6.3.} \\ \text{pH} &= -0.7993 + 3 && \text{Distributive property} \\ \text{pH} &\approx 2.2 && \text{Add.} \end{aligned}$$

Since the pH is less than 3.0, the wetland is a bog. NOW TRY

**NOW TRY EXERCISE 3**

Find the hydronium ion concentration of a solution with pH 2.6.

## EXAMPLE 3 Finding Hydronium Ion Concentration

Find the hydronium ion concentration of drinking water with pH 6.5.

$$\begin{aligned} \text{pH} &= -\log\,[H_3O^+] \\ 6.5 &= -\log\,[H_3O^+] && \text{Let pH} = 6.5. \\ \log\,[H_3O^+] &= -6.5 && \text{Multiply by } -1. \end{aligned}$$

Solve for $[H_3O^+]$ by writing the equation in exponential form using base 10.

$$\begin{aligned} [H_3O^+] &= 10^{-6.5} \\ [H_3O^+] &\approx 3.2 \times 10^{-7} && \text{Use a calculator.} \end{aligned}$$

NOW TRY

**NOW TRY ANSWERS**

**2.** 4.5; poor fen

**3.** $2.5 \times 10^{-3}$

The loudness of sound is measured in a unit called a **decibel,** abbreviated **dB.** To measure with this unit, we first assign an intensity of $I_0$ to a very faint sound, called the **threshold sound.** If a particular sound has intensity $I$, then the decibel level of this louder sound is

$$\mathbf{D = 10 \log \left(\frac{I}{I_0}\right).}$$

The table gives average decibel levels for some common sounds. Any sound over 85 dB exceeds what hearing experts consider safe. Permanent hearing damage can be suffered at levels above 150 dB.

| *Decibel Level* | *Example* |
|---|---|
| 60 | Normal conversation |
| 90 | Rush hour traffic, lawn mower |
| 100 | Garbage truck, chain saw, pneumatic drill |
| 120 | Rock concert, thunderclap |
| 140 | Gunshot blast, jet engine |
| 180 | Rocket launching pad |

*Source:* Deafness Research Foundation.

*NOW TRY EXERCISE 4*

Find the decibel level to the nearest whole number of the sound from a jet engine with intensity $I$ of

$$6.312 \times 10^{13} I_0.$$

## EXAMPLE 4 Measuring the Loudness of Sound

If music delivered through the earphones of an iPod has intensity $I$ of $3.162 \times 10^9 I_0$, find the average decibel level.

$$D = 10 \log \left(\frac{I}{I_0}\right)$$

Substitute the given value for *I*.

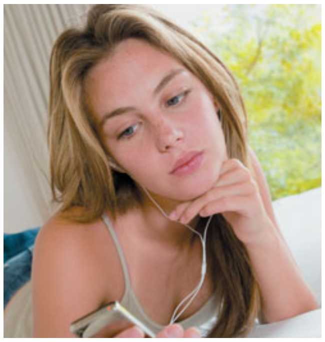
$$D = 10 \log \left(\frac{3.162 \times 10^9 I_0}{I_0}\right)$$

Substitute the given value for *I*.

$$D = 10 \log (3.162 \times 10^9)$$

$$D \approx 95$$

Use a calculator. Round to the nearest unit.



*NOW TRY*

**OBJECTIVE 3** **Evaluate natural logarithms using a calculator.** Logarithms used in applications are often **natural logarithms,** which have as base the number ***e*.** The number $e$, like $\pi$, is a **universal constant.** The letter $e$ was chosen to honor Leonhard Euler, who published extensive results on the number in 1748. Since it is an irrational number, its decimal expansion never terminates and never repeats.

***e***

$$e \approx 2.718281828$$

A calculator with an $\boxed{e^x}$ key can approximate powers of $e$.

$$e^2 \approx 7.389056099, \quad e^3 \approx 20.08553692, \quad e^{0.6} \approx 1.8221188$$

Powers of e

Logarithms with base $e$ are called natural logarithms because they occur in natural situations that involve growth or decay. The base $e$ logarithm of $x$ is written $\mathbf{\ln x}$ (read "**el en *x***"). The graph of $y = \ln x$ is given in **FIGURE 15** on the next page.

*NOW TRY ANSWER*

**4.** 138 dB

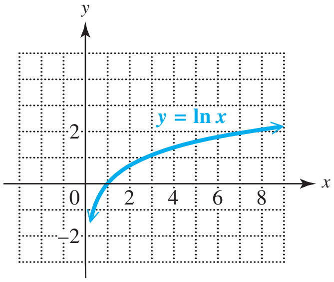


FIGURE 15

A calculator key labeled (ln *x*) is used to evaluate natural logarithms. Consult your calculator manual to see how to use this key.

NOW TRY EXERCISE 5

Using a calculator, evaluate each logarithm to four decimal places.

**(a)** ln 0.26 **(b)** ln 12

**(c)** ln 150

## EXAMPLE 5 Evaluating Natural Logarithms

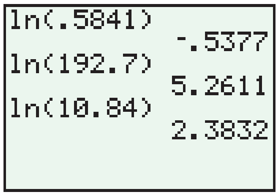
Using a calculator, evaluate each logarithm to four decimal places.

**(a)** $\ln 0.5841 \approx -0.5377$

As with common logarithms, ***a number between 0 and 1 has a negative natural logarithm.***

**(b)** $\ln 192.7 \approx 5.2611$ **(c)** $\ln 10.84 \approx 2.3832$

See **FIGURE 16**.



```
ln(.5841)
              -.5377
ln(192.7)
              5.2611
ln(10.84)
              2.3832
```

FIGURE 16

NOW TRY

## OBJECTIVE 4 Use natural logarithms in applications.

NOW TRY EXERCISE 6

Use the logarithmic function in **Example 6** to approximate the altitude when atmospheric pressure is 600 millibars. Round to the nearest hundred.

## EXAMPLE 6 Applying a Natural Logarithmic Function

The altitude in meters that corresponds to an atmospheric pressure of $x$ millibars is given by the logarithmic function defined by

$$f(x) = 51{,}600 - 7457 \ln x.$$

(*Source:* Miller, A. and J. Thompson, *Elements of Meteorology,* Fourth Edition, Charles E. Merrill Publishing Company.) Use this function to find the altitude when atmospheric pressure is 400 millibars. Round to the nearest hundred.

Let $x = 400$ and substitute in the expression for $f(x)$.

$$f(400) = 51{,}600 - 7457 \ln 400 \qquad \text{Let } x = 400.$$

$$f(400) \approx 6900 \qquad \text{Use a calculator.}$$

Atmospheric pressure is 400 millibars at approximately 6900 m. NOW TRY

**NOTE** In **Example 6,** the final answer was obtained using a calculator *without* rounding the intermediate values. In general, it is best to wait until the final step to round the answer. Round-offs in intermediate steps can lead to a buildup of round-off error, which may cause the final answer to have an incorrect final decimal place digit or digits.

NOW TRY ANSWERS

**5.** **(a)** $-1.3471$ **(b)** 2.4849 **(c)** 5.0106

**6.** approximately 3900 m

**Leonhard Euler (1707–1783)**

**The number e is named after Euler.**

**OBJECTIVE 5 Use the change-of-base rule.** We have used a calculator to approximate the values of common logarithms (base 10) and natural logarithms (base $e$). However, some applications involve logarithms with other bases. For example, the amount of crude oil (in millions of barrels) imported into the United States during the years 1990–2008 can be approximated by the function

$$f(x) = 2014 + 384.7 \log_2 x,$$

where $x = 1$ represents 1990, $x = 2$ represents 1991, and so on. (*Source:* U.S. Energy Information Administration.) To use this function, we need to find a base 2 logarithm. The following rule is used to convert logarithms from one base to another.

### Change-of-Base Rule

If $a > 0, a \neq 1, b > 0, b \neq 1$, and $x > 0$, then the following is true.

$$\boldsymbol{\log_a x = \frac{\log_b x}{\log_b a}}$$

**NOTE** Any positive number other than 1 can be used for base $b$ in the change-of-base rule. Usually the only practical bases are $e$ and 10 because calculators generally give logarithms only for these two bases.

To derive the change-of-base rule, let $\log_a x = m$.

$$\log_a x = m$$

$$a^m = x \qquad \text{Change to exponential form.}$$

Since logarithmic functions are one-to-one, if all variables are positive and if $x = y$, then $\log_b x = \log_b y$.

$$\begin{aligned} \log_b (a^m) &= \log_b x && \text{Take the logarithm on each side.} \\ m \log_b a &= \log_b x && \text{Power rule} \\ (\log_a x)(\log_b a) &= \log_b x && \text{Substitute for } m. \\ \log_a x &= \frac{\log_b x}{\log_b a} && \text{Divide by } \log_b a. \end{aligned}$$

**NOW TRY EXERCISE 7**

Evaluate $\log_8 60$ to four decimal places.

### EXAMPLE 7 Using the Change-of-Base Rule

Evaluate $\log_5 12$ to four decimal places.

Use common logarithms and the change-of-base rule.

$$\log_5 12 = \frac{\log 12}{\log 5} \approx 1.5440 \qquad \text{Use a calculator.}$$

NOW TRY

**NOTE** Either common or natural logarithms can be used when applying the change-of-base rule. Verify that the same value is found in **Example 7** if natural logarithms are used.

**NOW TRY ANSWER**

**7.** 1.9690

### EXAMPLE 8 Using the Change-of-Base Rule in an Application

Use natural logarithms in the change-of-base rule and the function defined by

$$f(x) = 2014 + 384.7 \log_2 x$$

(given earlier) to estimate total crude oil imports (in millions of barrels) into the United States in 2006. Compare this to the actual amount of 3685 million barrels. In the equation, $x = 1$ represents 1990.

$$\begin{aligned} f(x) &= 2014 + 384.7 \log_2 x \\ f(17) &= 2014 + 384.7 \log_2 17 && \text{For 2006, } x = 17. \\ &= 2014 + 384.7\left(\frac{\ln 17}{\ln 2}\right) && \text{Change-of-base rule} \\ &\approx 3586 && \text{Use a calculator.} \end{aligned}$$

The model gives about 3586 million barrels for 2006, which is less than the actual amount.

NOW TRY

NOW TRY EXERCISE 8

Use the model in **Example 8** to estimate total crude oil imports into the United States in 2002. Compare this to the actual amount of 3336 million barrels.

NOW TRY ANSWER

**8.** 3438 million barrels; This is greater than the actual amount.

## 12.5 EXERCISES

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◉ *Complete solution available on the Video Resources on DVD*

*Concept Check* *Choose the correct response in Exercises 1–4.*

**1.** What is the base in the expression $\log x$?

**A.** $e$ **B.** 1 **C.** 10 **D.** $x$

**2.** What is the base in the expression $\ln x$?

**A.** $e$ **B.** 1 **C.** 10 **D.** $x$

**3.** Since $10^0 = 1$ and $10^1 = 10$, between what two consecutive integers is the value of $\log 6.3$?

**A.** 6 and 7 **B.** 10 and 11 **C.** 0 and 1 **D.** $-1$ and 0

**4.** Since $e^1 \approx 2.718$ and $e^2 \approx 7.389$, between what two consecutive integers is the value of $\ln 6.3$?

**A.** 6 and 7 **B.** 2 and 3 **C.** 1 and 2 **D.** 0 and 1

**5.** *Concept Check* Without using a calculator, give the value of $\log 10^{31.6}$.

**6.** *Concept Check* Without using a calculator, give the value of $\ln e^{\sqrt{3}}$.

***You will need a calculator for the remaining exercises in this set.***

*Find each logarithm. Give approximations to four decimal places.* ***See Examples 1 and 5.***

◉ **7.** $\log 43$ **8.** $\log 98$ **9.** $\log 328.4$
**10.** $\log 457.2$ **11.** $\log 0.0326$ **12.** $\log 0.1741$
**13.** $\log (4.76 \times 10^9)$ **14.** $\log (2.13 \times 10^4)$ ◉ **15.** $\ln 7.84$
**16.** $\ln 8.32$ **17.** $\ln 0.0556$ **18.** $\ln 0.0217$
**19.** $\ln 388.1$ **20.** $\ln 942.6$ **21.** $\ln (8.59 \times e^2)$
**22.** $\ln (7.46 \times e^3)$ **23.** $\ln 10$ **24.** $\log e$

**25.** Use your calculator to find approximations of the following logarithms.

**(a)** log 356.8 **(b)** log 35.68 **(c)** log 3.568

**(d)** Observe your answers and make a conjecture concerning the decimal values of the common logarithms of numbers greater than 1 that have the same digits.

**26.** Let $k$ represent the number of letters in your last name.

**(a)** Use your calculator to find log $k$.

**(b)** Raise 10 to the power indicated by the number in part (a). What is your result?

**(c)** Use the concepts of **Section 12.1** to explain why you obtained the answer you found in part (b). Would it matter what number you used for $k$ to observe the same result?

*Suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration. Is the wetland a* rich fen, *a* poor fen, *or a* bog? ***See Example 2.***

**27.** $3.1 \times 10^{-5}$
**28.** $2.5 \times 10^{-5}$
**29.** $2.5 \times 10^{-2}$
**30.** $3.6 \times 10^{-2}$
**31.** $2.7 \times 10^{-7}$
**32.** $2.5 \times 10^{-7}$

*Find the* pH *of the substance with the given hydronium ion concentration.* ***See Example 2.***

**33.** Ammonia, $2.5 \times 10^{-12}$
**34.** Sodium bicarbonate, $4.0 \times 10^{-9}$
**35.** Grapes, $5.0 \times 10^{-5}$
**36.** Tuna, $1.3 \times 10^{-6}$

*Find the hydronium ion concentration of the substance with the given* pH. ***See Example 3.***

**37.** Human blood plasma, 7.4
**38.** Human gastric contents, 2.0
**39.** Spinach, 5.4
**40.** Bananas, 4.6

*Solve each problem.* ***See Examples 4 and 6.***

**41.** Consumers can now enjoy movies at home in elaborate home-theater systems. Find the average decibel level

$$D = 10 \log \left( \frac{I}{I_0} \right)$$

for each movie with the given intensity $I$.

**(a)** *Avatar;* $5.012 \times 10^{10} I_0$

**(b)** *Iron Man 2;* $10^{10} I_0$

**(c)** *Clash of the Titans;* $6{,}310{,}000{,}000\, I_0$

**42.** The time $t$ in years for an amount increasing at a rate of $r$ (in decimal form) to double is given by

$$t(r) = \frac{\ln 2}{\ln (1 + r)}.$$

This is called **doubling time.** Find the doubling time to the nearest tenth for an investment at each interest rate.

**(a)** 2% (or 0.02) **(b)** 5% (or 0.05) **(c)** 8% (or 0.08)

**43.** The number of years, $N(r)$, since two independently evolving languages split off from a common ancestral language is approximated by

$$N(r) = -5000 \ln r,$$

where $r$ is the percent of words (in decimal form) from the ancestral language common to both languages now. Find the number of years (to the nearest hundred years) since the split for each percent of common words.

**(a)** 85% (or 0.85) **(b)** 35% (or 0.35) **(c)** 10% (or 0.10)

**44.** The concentration of a drug injected into the bloodstream decreases with time. The intervals of time $T$ when the drug should be administered are given by

$$T = \frac{1}{k} \ln \frac{C_2}{C_1},$$

where $k$ is a constant determined by the drug in use, $C_2$ is the concentration at which the drug is harmful, and $C_1$ is the concentration below which the drug is ineffective. (*Source:* Horelick, Brindell and Sinan Koont, "Applications of Calculus to Medicine: Prescribing Safe and Effective Dosage," *UMAP Module 202.*) Thus, if $T = 4$, the drug should be administered every 4 hr. For a certain drug, $k = \frac{1}{3}$, $C_2 = 5$, and $C_1 = 2$. How often should the drug be administered? (*Hint:* Round down.)

**45.** The growth of outpatient surgeries as a percent of total surgeries at hospitals is approximated by

$$f(x) = -1317 + 304 \ln x,$$

where $x$ is the number of years since 1900. (*Source:* American Hospital Association.)

**(a)** What does this function predict for the percent of outpatient surgeries in 1998?

**(b)** When did outpatient surgeries reach 50%? (*Hint:* Substitute for $y$, then write the equation in exponential form to solve it.)

**46.** In the central Sierra Nevada of California, the percent of moisture that falls as snow rather than rain is approximated reasonably well by

$$f(x) = 86.3 \ln x - 680,$$

where $x$ is the altitude in feet.

**(a)** What percent of the moisture at 5000 ft falls as snow?

**(b)** What percent at 7500 ft falls as snow?

**47.** The **cost-benefit equation**

$$T = -0.642 - 189 \ln (1 - p)$$

describes the approximate tax $T$, in dollars per ton, that would result in a $p\%$ (in decimal form) reduction in carbon dioxide emissions.

**(a)** What tax will reduce emissions 25%?

**(b)** Explain why the equation is not valid for $p = 0$ or $p = 1$.

**48.** The age in years of a female blue whale of length $L$ in feet is approximated by

$$t = -2.57 \ln \left( \frac{87 - L}{63} \right).$$

**(a)** How old is a female blue whale that measures 80 ft?

**(b)** The equation that defines $t$ has domain $24 < L < 87$. Explain why.

*Use the change-of-base rule (with either common or natural logarithms) to find each logarithm to four decimal places.* ***See Example 7.***

**49.** $\log_3 12$

**50.** $\log_4 18$

**51.** $\log_5 3$

**52.** $\log_7 4$

**53.** $\log_3 \sqrt{2}$

**54.** $\log_6 \sqrt[3]{5}$

**55.** $\log_\pi e$

**56.** $\log_\pi 10$

**57.** $\log_e 12$

**58.** To solve the equation $5^x = 7$, we must find the exponent to which 5 must be raised in order to obtain 7. This is $\log_5 7$.

**(a)** Use the change-of-base rule and your calculator to find $\log_5 7$.

**(b)** Raise 5 to the number you found in part (a). What is your result?

**(c)** Using as many decimal places as your calculator gives, write the solution set of $5^x = 7$. (Equations of this type will be studied in more detail in **Section 12.6.**)

**59.** Let $m$ be the number of letters in your first name, and let $n$ be the number of letters in your last name.

**(a)** In your own words, explain what $\log_m n$ means.

**(b)** Use your calculator to find $\log_m n$.

**(c)** Raise $m$ to the power indicated by the number found in part (b). What is your result?

**60.** The value of $e$ can be expressed as

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

Approximate $e$ using two terms of this expression, then three terms, four terms, five terms, and six terms. How close is the approximation to the value of $e \approx 2.718281828$ with six terms? Does this infinite sum approach the value of $e$ very quickly?

*Solve each application of a logarithmic function (from **Exercises 61 and 62 of Section 12.3**).*

**61.** For 1981–2003, the number of billion cubic feet of natural gas gross withdrawals from crude oil wells in the United States can be approximated by the function defined by

$$f(x) = 3800 + 585 \log_2 x,$$

where $x = 1$ represents 1981, $x = 2$ represents 1982, and so on. (*Source:* Energy Information Administration.) Use this function to approximate the number of cubic feet withdrawn in 2003, to the nearest unit.

**62.** According to selected figures from the last two decades of the 20th century, the number of trillion cubic feet of dry natural gas consumed worldwide can be approximated by the function defined by

$$f(x) = 51.47 + 6.044 \log_2 x,$$

where $x = 1$ represents 1980, $x = 2$ represents 1981, and so on. (*Source:* Energy Information Administration.) Use this function to approximate consumption in 2003, to the nearest hundredth.

## PREVIEW EXERCISES

*Solve each equation. **See Sections 12.2 and 12.3.***

**63.** $4^{2x} = 8^{3x+1}$

**64.** $2^{5x} = \left(\frac{1}{16}\right)^{x+3}$

**65.** $\log_3 (x + 4) = 2$

**66.** $\log_x 64 = 2$

**67.** $\log_{1/2} 8 = x$

**68.** $\log_a 1 = 0$

*Write as a single logarithm. Assume $x > 0$. **See Section 12.4.***

**69.** $\log (x + 2) + \log (x + 3)$

**70.** $\log_4 (x + 4) - 2 \log_4 (3x + 1)$

# 12.6 Exponential and Logarithmic Equations; Further Applications

**OBJECTIVES**

1. Solve equations involving variables in the exponents.
2. Solve equations involving logarithms.
3. Solve applications of compound interest.
4. Solve applications involving base $e$ exponential growth and decay.

We solved exponential and logarithmic equations in **Sections 12.2 and 12.3.** General methods for solving these equations depend on the following properties.

**Properties for Solving Exponential and Logarithmic Equations**

For all real numbers $b > 0$, $b \neq 1$, and any real numbers $x$ and $y$, the following are true.

1. If $x = y$, then $b^x = b^y$.
2. If $b^x = b^y$, then $x = y$.
3. If $x = y$, and $x > 0$, $y > 0$, then $\log_b x = \log_b y$.
4. If $x > 0$, $y > 0$, and $\log_b x = \log_b y$, then $x = y$.

We used Property 2 to solve exponential equations in **Section 12.2.**

**OBJECTIVE 1** **Solve equations involving variables in the exponents.** In **Examples 1 and 2,** we use Property 3.

*NOW TRY EXERCISE 1*
Solve the equation. Approximate the solution to three decimal places.

$$5^x = 20$$

**EXAMPLE 1** Solving an Exponential Equation

Solve $3^x = 12$. Approximate the solution to three decimal places.

$$\begin{aligned} 3^x &= 12 && \\ \log 3^x &= \log 12 && \text{Property 3 (common logs)} \\ x \log 3 &= \log 12 && \text{Power rule} \\ \text{Exact solution} \longrightarrow x &= \frac{\log 12}{\log 3} && \text{Divide by log 3.} \\ \text{Decimal approximation} \longrightarrow x &\approx 2.262 && \text{Use a calculator.} \end{aligned}$$

*CHECK* $3^x = 3^{2.262} \approx 12$ ✓ Use a calculator; true

The solution set is $\{2.262\}$. *NOW TRY*

**CAUTION** Be careful: $\frac{\log 12}{\log 3}$ is ***not*** equal to $\log 4$. Check to see that

$$\log 4 \approx 0.6021, \quad \text{but} \quad \frac{\log 12}{\log 3} \approx 2.262.$$

When an exponential equation has $e$ as the base, as in the next example, it is easiest to use base $e$ logarithms.

*NOW TRY ANSWER*
1. $\{1.861\}$

NOW TRY EXERCISE 2

Solve $e^{0.12x} = 10$. Approximate the solution to three decimal places.

## EXAMPLE 2 Solving an Exponential Equation with Base e

Solve $e^{0.003x} = 40$. Approximate the solution to three decimal places.

$$\begin{aligned} \ln e^{0.003x} &= \ln 40 && \text{Property 3 (natural logs)} \\ 0.003x \ln e &= \ln 40 && \text{Power rule} \\ 0.003x &= \ln 40 && \ln e = \ln e^1 = 1 \\ x &= \frac{\ln 40}{0.003} && \text{Divide by 0.003.} \\ x &\approx 1229.626 && \text{Use a calculator.} \end{aligned}$$

The solution set is $\{1229.626\}$. Check that $e^{0.003(1229.626)} \approx 40$. NOW TRY

### General Method for Solving an Exponential Equation

Take logarithms to the same base on both sides and then use the power rule of logarithms or the special property $\log_b b^x = x$. (See **Examples 1 and 2.**)

As a special case, if both sides can be written as exponentials with the same base, do so, and set the exponents equal. (See **Section 12.2.**)

**OBJECTIVE 2 Solve equations involving logarithms.** We use the definition of logarithm and the properties of logarithms to change equations to exponential form.

NOW TRY EXERCISE 3

Solve $\log_5 (x - 1)^3 = 2$. Give the exact solution.

## EXAMPLE 3 Solving a Logarithmic Equation

Solve $\log_2 (x + 5)^3 = 4$. Give the exact solution.

$$\begin{aligned} \log_2 (x + 5)^3 &= 4 && \\ (x + 5)^3 &= 2^4 && \text{Convert to exponential form.} \\ (x + 5)^3 &= 16 && 2^4 = 16 \\ x + 5 &= \sqrt[3]{16} && \text{Take the cube root on each side.} \\ x &= -5 + \sqrt[3]{16} && \text{Add } -5. \\ x &= -5 + 2\sqrt[3]{2} && \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2} \end{aligned}$$

CHECK

$$\begin{aligned} \log_2 (x + 5)^3 &= 4 && \text{Original equation} \\ \log_2 (-5 + 2\sqrt[3]{2} + 5)^3 &\stackrel{?}{=} 4 && \text{Let } x = -5 + 2\sqrt[3]{2}. \\ \log_2 \left(2\sqrt[3]{2}\right)^3 &\stackrel{?}{=} 4 && \text{Work inside the parentheses.} \\ \log_2 16 &\stackrel{?}{=} 4 && \left(2\sqrt[3]{2}\right)^3 = 2^3\left(\sqrt[3]{2}\right)^3 = 8 \cdot 2 = 16 \\ 2^4 &\stackrel{?}{=} 16 && \text{Write in exponential form.} \\ 16 &= 16 \ ✓ && \text{True} \end{aligned}$$

A true statement results, so the solution set is $\left\{-5 + 2\sqrt[3]{2}\right\}$. NOW TRY

**CAUTION** Recall that the domain of $y = \log_b x$ is $(0, \infty)$. ***For this reason, always check that each proposed solution of an equation with logarithms yields only logarithms of positive numbers in the original equation.***

NOW TRY ANSWERS

**2.** $\{19.188\}$ **3.** $\left\{1 + \sqrt[3]{25}\right\}$

**NOW TRY EXERCISE 4**

Solve.

$$\log_4 (2x + 13) - \log_4 (x + 1) = \log_4 10$$

## EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log_2 (x + 1) - \log_2 x = \log_2 7$.

$$\log_2 (x + 1) - \log_2 x = \log_2 7$$

Transform the left side to an expression with only *one* logarithm.

$$\log_2 \frac{x + 1}{x} = \log_2 7 \qquad \text{Quotient rule}$$

$$\frac{x + 1}{x} = 7 \qquad \text{Property 4}$$

$$x + 1 = 7x \qquad \text{Multiply by } x.$$

$$1 = 6x \qquad \text{Subtract } x.$$

$$\frac{1}{6} = x \qquad \text{Divide by 6.}$$

This proposed solution must be checked.

Since we cannot take the logarithm of a *nonpositive* number, both $x + 1$ and $x$ must be positive here. If $x = \frac{1}{6}$, then this condition is satisfied.

*CHECK*

$$\log_2 (x + 1) - \log_2 x = \log_2 7 \qquad \text{Original equation}$$

$$\log_2 \left(\frac{1}{6} + 1\right) - \log_2 \frac{1}{6} \stackrel{?}{=} \log_2 7 \qquad \text{Let } x = \tfrac{1}{6}.$$

$$\log_2 \frac{7}{6} - \log_2 \frac{1}{6} \stackrel{?}{=} \log_2 7 \qquad \text{Add.}$$

$$\log_2 \frac{\frac{7}{6}}{\frac{1}{6}} \stackrel{?}{=} \log_2 7 \qquad \text{Quotient rule}$$

$$\frac{\frac{7}{6}}{\frac{1}{6}} = \frac{7}{6} \div \frac{1}{6} = \frac{7}{6} \cdot \frac{6}{1} = 7$$

$$\log_2 7 = \log_2 7 \quad ✓ \quad \text{True}$$

A true statement results, so the solution set is $\left\{\frac{1}{6}\right\}$. *NOW TRY*

**NOW TRY EXERCISE 5**

Solve.

$$\log_4 (x + 2) + \log_4 2x = 2$$

## EXAMPLE 5 Solving a Logarithmic Equation

Solve $\log x + \log (x - 21) = 2$.

$$\log x + \log (x - 21) = 2$$

$$\log x(x - 21) = 2 \qquad \text{Product rule}$$

The base is 10.

$$x(x - 21) = 10^2 \qquad \text{Write in exponential form.}$$

$$x^2 - 21x = 100 \qquad \text{Distributive property; multiply.}$$

$$x^2 - 21x - 100 = 0 \qquad \text{Standard form}$$

$$(x - 25)(x + 4) = 0 \qquad \text{Factor.}$$

$$x - 25 = 0 \quad \text{or} \quad x + 4 = 0 \qquad \text{Zero-factor property}$$

$$x = 25 \quad \text{or} \quad x = -4 \qquad \text{Proposed solutions}$$

The value $-4$ must be rejected as a solution since it leads to the logarithm of a negative number in the original equation.

$$\log (-4) + \log (-4 - 21) = 2 \qquad \text{The left side is undefined.}$$

Check that the only solution is 25, so the solution set is $\{25\}$. *NOW TRY*

**NOW TRY ANSWERS**

**4.** $\left\{\frac{3}{8}\right\}$ **5.** $\{2\}$

**CAUTION** ***Do not reject a potential solution just because it is nonpositive. Reject any value that leads to the logarithm of a nonpositive number.***

### Solving a Logarithmic Equation

*Step 1* **Transform the equation so that a single logarithm appears on one side.** Use the product rule or quotient rule of logarithms to do this.

*Step 2* (a) **Use Property 4.** If $\log_b x = \log_b y$, then $x = y$. (See **Example 4.**)

(b) **Write the equation in exponential form.** If $\log_b x = k$, then $x = b^k$. (See **Examples 3 and 5.**)

**OBJECTIVE 3** **Solve applications of compound interest.** We have solved simple interest problems using the formula

$$I = prt. \qquad \text{Simple interest formula}$$

In most cases, interest paid or charged is **compound interest** (interest paid on both principal and interest). The formula for compound interest is an application of exponential functions. In this book, monetary amounts are given to the nearest cent.

### Compound Interest Formula (for a Finite Number of Periods)

If a principal of $P$ dollars is deposited at an annual rate of interest $r$ compounded (paid) $n$ times per year, then the account will contain

$$\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

dollars after $t$ years. (In this formula, $r$ is expressed as a decimal.)

NOW TRY EXERCISE 6

How much money will there be in an account at the end of 10 yr if $10,000 is deposited at 2.5% compounded monthly?

### EXAMPLE 6 Solving a Compound Interest Problem for $A$

How much money will there be in an account at the end of 5 yr if $1000 is deposited at 3% compounded quarterly? (Assume no withdrawals are made.)

Because interest is compounded quarterly, $n = 4$. The other given values are $P = 1000$, $r = 0.03$ (because $3\% = 0.03$), and $t = 5$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Compound interest formula}$$

$$A = 1000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 5} \qquad \text{Substitute the given values.}$$

$$A = 1000(1.0075)^{20} \qquad \text{Simplify.}$$

$$A = 1161.18 \qquad \text{Use a calculator. Round to the nearest cent.}$$

The account will contain $1161.18. (The actual amount of interest earned is $1161.18 − $1000 = $161.18. Why?) NOW TRY

NOW TRY ANSWER

6. $12,836.92

NOW TRY EXERCISE 7

Approximate the time it would take for money deposited in an account paying 4% interest compounded quarterly to double. Round to the nearest hundredth.

### EXAMPLE 7 Solving a Compound Interest Problem for $t$

Suppose inflation is averaging 3% per year. Approximate the time it will take for prices to double. Round to the nearest hundredth.

We want the number of years $t$ for $P$ dollars to grow to $2P$ dollars at a rate of 3% per year. In the compound interest formula, we substitute $2P$ for $A$, and let $r = 0.03$ and $n = 1$.

$$2P = P\left(1 + \frac{0.03}{1}\right)^{1t}$$ Substitute in the compound interest formula.

$$2 = (1.03)^t$$ Divide by $P$. Simplify.

$$\log 2 = \log (1.03)^t$$ Property 3

$$\log 2 = t \log (1.03)$$ Power rule

$$t = \frac{\log 2}{\log 1.03}$$ Interchange sides. Divide by log 1.03.

$$t \approx 23.45$$ Use a calculator.

Prices will double in about 23.45 yr. (This is called the **doubling time** of the money.) To check, verify that $1.03^{23.45} \approx 2$. NOW TRY

Interest can be compounded annually, semiannually, quarterly, daily, and so on. The number of compounding periods can get larger and larger. If the value of $n$ is allowed to approach infinity, we have an example of **continuous compounding.** The formula for continuous compounding is derived in advanced courses, and is an example of exponential growth involving the number $e$.

### Continuous Compound Interest Formula

If a principal of $P$ dollars is deposited at an annual rate of interest $r$ compounded continuously for $t$ years, the final amount $A$ on deposit is given by

$$\boldsymbol{A = Pe^{rt}}.$$

### EXAMPLE 8 Solving a Continuous Compound Interest Problem

In **Example 6** we found that \$1000 invested for 5 yr at 3% interest compounded quarterly would grow to \$1161.18.

**(a)** How much would this same investment grow to if interest were compounded continuously?

$$A = Pe^{rt}$$ Continuous compounding formula

$$A = 1000e^{0.03(5)}$$ Let $P = 1000$, $r = 0.03$, and $t = 5$.

$$A = 1000e^{0.15}$$ Multiply in the exponent.

$$A = 1161.83$$ Use a calculator. Round to the nearest cent.

Continuous compounding would cause the investment to grow to \$1161.83. This is \$0.65 more than the amount the investment grew to in **Example 6,** when interest was compounded quarterly.

NOW TRY ANSWER

**7.** 17.42 yr

NOW TRY EXERCISE 8

Suppose that \$4000 is invested at 3% interest for 2 yr.

**(a)** How much will the investment grow to if it is compounded continuously?

**(b)** Approximate the time it would take for the amount to double. Round to the nearest tenth.

**(b)** Approximate the time it would take for the initial investment to triple its original amount. Round to the nearest tenth.

We must find the value of $t$ that will cause $A$ to be $3(\$1000) = \$3000$.

$$\begin{aligned}
A &= Pe^{rt} && \text{Continuous compounding formula}\\
3000 &= 1000e^{0.03t} && \text{Let } A = 3P = 3000,\ P = 1000,\ r = 0.03.\\
3 &= e^{0.03t} && \text{Divide by 1000.}\\
\ln 3 &= \ln e^{0.03t} && \text{Take natural logarithms.}\\
\ln 3 &= 0.03t && \ln e^k = k\\
t &= \frac{\ln 3}{0.03} && \text{Divide by 0.03.}\\
t &\approx 36.6 && \text{Use a calculator.}
\end{aligned}$$

It would take about 36.6 yr for the original investment to triple. NOW TRY

**OBJECTIVE 4** **Solve applications involving base *e* exponential growth and decay.** When situations involve growth or decay of a population, the amount or number of some quantity present at time $t$ can be approximated by

$$y = y_0e^{kt}.$$

In this equation, $y_0$ is the amount or number present at time $t = 0$ and $k$ is a constant.

The continuous compounding of money is an example of exponential growth. In **Example 9,** we investigate exponential decay.

NOW TRY EXERCISE 9

Radium 226 decays according to the function defined by

$$y = y_0e^{-0.00043t},$$

where $t$ is time in years.

**(a)** If an initial sample contains $y_0 = 4.5$ g of radium 226, how many grams, to the nearest tenth, will be present after 150 yr?

**(b)** Approximate the half-life of radium 226. Round to the nearest unit.

### EXAMPLE 9 Solving an Application Involving Exponential Decay

Carbon 14 is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radioactive carbon 14 disintegrates according to the function defined by

$$y = y_0e^{-0.000121t},$$

where $t$ is time in years, $y$ is the amount of the sample at time $t$, and $y_0$ is the initial amount present at $t = 0$.

**(a)** If an initial sample contains $y_0 = 10$ g of carbon 14, how many grams, to the nearest tenth, will be present after 3000 yr?

Let $y_0 = 10$ and $t = 3000$ in the formula, and use a calculator.

$$y = 10e^{-0.000121(3000)} \approx 6.96 \text{ g}$$

**(b)** About how long would it take for the initial sample to decay to half of its original amount? (This is called the **half-life.**) Round to the nearest unit.

Let $y = \frac{1}{2}(10) = 5$, and solve for $t$.

$$\begin{aligned}
5 &= 10e^{-0.000121t} && \text{Substitute in } y = y_0e^{kt}.\\
\frac{1}{2} &= e^{-0.000121t} && \text{Divide by 10.}\\
\ln\frac{1}{2} &= -0.000121t && \text{Take natural logarithms; } \ln e^k = k.\\
t &= \frac{\ln\frac{1}{2}}{-0.000121} && \text{Interchange sides. Divide by } -0.000121.\\
t &\approx 5728 && \text{Use a calculator.}
\end{aligned}$$

The half-life is about 5728 yr. NOW TRY

NOW TRY ANSWERS

**8.** **(a)** \$4247.35 **(b)** 23.1 yr

**9.** **(a)** 4.2 g **(b)** 1612 yr

## CONNECTIONS

Recall that the $x$-intercepts of the graph of a function $f$ correspond to the real solutions of the equation $f(x) = 0$. In **Example 1,** we solved the equation $3^x = 12$ algebraically using rules for logarithms and found the solution set to be $\{2.262\}$. This can be supported graphically by showing that the $x$-intercept of the graph of the function defined by $y = 3^x - 12$ corresponds to this solution. See **FIGURE 17**.

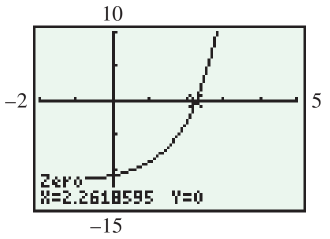

**FIGURE 17**

### For Discussion or Writing

In **Example 5,** we solved $\log x + \log(x - 21) = 2$ to find the solution set $\{25\}$. (We rejected the proposed solution $-4$ since it led to the logarithm of a negative number.) Show that the $x$-intercept of the graph of the function defined by $y = \log x + \log(x - 21) - 2$ supports this result.

## 12.6 EXERCISES

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***Many of the problems in these exercises require a scientific calculator.***

*Solve each equation. Give solutions to three decimal places. See* ***Example 1.***

**1.** $7^x = 5$

**2.** $4^x = 3$

**3.** $9^{-x+2} = 13$

**4.** $6^{-x+1} = 22$

**5.** $3^{2x} = 14$

**6.** $5^{0.3x} = 11$

**7.** $2^{x+3} = 5^x$

**8.** $6^{x+3} = 4^x$

**9.** $2^{x+3} = 3^{x-4}$

**10.** $4^{x-2} = 5^{3x+2}$

**11.** $4^{2x+3} = 6^{x-1}$

**12.** $3^{2x+1} = 5^{x-1}$

*Solve each equation. Use natural logarithms. When appropriate, give solutions to three decimal places.* ***See Example 2.***

**13.** $e^{0.012x} = 23$

**14.** $e^{0.006x} = 30$

**15.** $e^{-0.205x} = 9$

**16.** $e^{-0.103x} = 7$

**17.** $\ln e^{3x} = 9$

**18.** $\ln e^{2x} = 4$

**19.** $\ln e^{0.45x} = \sqrt{7}$

**20.** $\ln e^{0.04x} = \sqrt{3}$

**21.** $\ln e^{-x} = \pi$

**22.** $\ln e^{2x} = \pi$

**23.** $e^{\ln 2x} = e^{\ln(x+1)}$

**24.** $e^{\ln(6-x)} = e^{\ln(4+2x)}$

**25.** Solve one of the equations in **Exercises 13–16** using common logarithms rather than natural logarithms. (You should get the same solution.) Explain why using natural logarithms is a better choice.

**26.** *Concept Check* If you were asked to solve

$$10^{0.0025x} = 75,$$

would natural or common logarithms be a better choice? Why?

*Solve each equation. Give the exact solution.* ***See Example 3.***

**27.** $\log_3 (6x + 5) = 2$

**28.** $\log_5 (12x - 8) = 3$

**29.** $\log_2 (2x - 1) = 5$

**30.** $\log_6 (4x + 2) = 2$

**31.** $\log_7 (x + 1)^3 = 2$

**32.** $\log_4 (x - 3)^3 = 4$

**33.** ***Concept Check*** Suppose that in solving a logarithmic equation having the term $\log (x - 3)$, you obtain a proposed solution of 2. All algebraic work is correct. Why must you reject 2 as a solution of the equation?

**34.** ***Concept Check*** Suppose that in solving a logarithmic equation having the term $\log (3 - x)$, you obtain a proposed solution of $-4$. All algebraic work is correct. Should you reject $-4$ as a solution of the equation? Why or why not?

*Solve each equation. Give exact solutions.* ***See Examples 4 and 5.***

**35.** $\log (6x + 1) = \log 3$

**36.** $\log (7 - 2x) = \log 4$

**37.** $\log_5 (3t + 2) - \log_5 t = \log_5 4$

**38.** $\log_2 (x + 5) - \log_2 (x - 1) = \log_2 3$

**39.** $\log 4x - \log (x - 3) = \log 2$

**40.** $\log (-x) + \log 3 = \log (2x - 15)$

**41.** $\log_2 x + \log_2 (x - 7) = 3$

**42.** $\log (2x - 1) + \log 10x = \log 10$

**43.** $\log 5x - \log (2x - 1) = \log 4$

**44.** $\log_3 x + \log_3 (2x + 5) = 1$

**45.** $\log_2 x + \log_2 (x - 6) = 4$

**46.** $\log_2 x + \log_2 (x + 4) = 5$

*Solve each problem.* ***See Examples 6–8.***

**47.** **(a)** How much money will there be in an account at the end of 6 yr if \$2000 is deposited at 4% compounded quarterly? (Assume no withdrawals are made.)

**(b)** To one decimal place, how long will it take for the account to grow to \$3000?

**48.** **(a)** How much money will there be in an account at the end of 7 yr if \$3000 is deposited at 3.5% compounded quarterly? (Assume no withdrawals are made.)

**(b)** To one decimal place, when will the account grow to \$5000?

**49.** **(a)** What will be the amount $A$ in an account with initial principal \$4000 if interest is compounded continuously at an annual rate of 3.5% for 6 yr?

**(b)** To one decimal place, how long will it take for the initial amount to double?

**50.** Refer to **Exercise 48(a).** Does the money grow to a greater value under those conditions, or when invested for 7 yr at 3% compounded continuously?

**51.** Find the amount of money in an account after 12 yr if \$5000 is deposited at 7% annual interest compounded as follows.

**(a)** Annually **(b)** Semiannually **(c)** Quarterly

**(d)** Daily (Use $n = 365$.) **(e)** Continuously

**52.** How much money will be in an account at the end of 8 yr if \$4500 is deposited at 6% annual interest compounded as follows?

**(a)** Annually **(b)** Semiannually **(c)** Quarterly

**(d)** Daily (Use $n = 365$.) **(e)** Continuously

**53.** How much money must be deposited today to amount to \$1850 in 40 yr at 6.5% compounded continuously?

**54.** How much money must be deposited today to amount to \$1000 in 10 yr at 5% compounded continuously?

*Solve each problem.* ***See Example 9.***

**55.** The total volume in millions of tons of materials recovered from municipal solid waste collections in the United States during the period 1980–2007 can be approximated by the function defined by

$$f(x) = 15.94e^{0.0656x},$$

where $x = 0$ corresponds to 1980, $x = 1$ to 1981, and so on. Approximate, to the nearest tenth, the volume recovered each year. (*Source:* U.S. Environmental Protection Agency.)

**(a)** 1980 **(b)** 1990 **(c)** 2000 **(d)** 2007

**56.** Worldwide emissions in millions of metric tons of the greenhouse gas carbon dioxide from fossil fuel consumption during the period 1990–2006 can be modeled by the function defined by

$$f(x) = 20{,}761e^{0.01882x},$$

where $x = 0$ corresponds to 1990, $x = 1$ to 1991, and so on. Approximate, to the nearest unit, the emissions for each year. (*Source:* U.S. Department of Energy.)

**(a)** 1990 **(b)** 1995 **(c)** 2000 **(d)** 2006

**57.** Revenues of software publishers in the United States for the years 2004–2007 can be modeled by the function defined by

$$S(x) = 112{,}047e^{0.0827x},$$

where $x = 0$ represents 2004, $x = 1$ represents 2005, and so on, and $S(x)$ is in millions of dollars. Approximate, to the nearest unit, consumer expenditures for 2007. (*Source:* U.S. Census Bureau.)

**58.** Based on selected figures obtained during the years 1980–2007, the total number of bachelor's degrees earned in the United States can be modeled by the function defined by

$$D(x) = 900{,}584e^{0.0185x},$$

where $x = 0$ corresponds to 1980, $x = 10$ corresponds to 1990, and so on. Approximate, to the nearest unit, the number of bachelor's degrees earned in 2005. (*Source:* U.S. National Center for Education Statistics.)

**59.** Suppose that the amount, in grams, of plutonium 241 present in a given sample is determined by the function defined by

$$A(t) = 2.00e^{-0.053t},$$

where $t$ is measured in years. Approximate the amount present, to the nearest hundredth, in the sample after the given number of years.

**(a)** 4 **(b)** 10 **(c)** 20 **(d)** What was the initial amount present?

**60.** Suppose that the amount, in grams, of radium 226 present in a given sample is determined by the function defined by

$$A(t) = 3.25e^{-0.00043t},$$

where $t$ is measured in years. Approximate the amount present, to the nearest hundredth, in the sample after the given number of years.

**(a)** 20 **(b)** 100 **(c)** 500 **(d)** What was the initial amount present?

**61.** A sample of 400 g of lead 210 decays to polonium 210 according to the function defined by

$$A(t) = 400e^{-0.032t},$$

where $t$ is time in years. Approximate answers to the nearest hundredth.

**(a)** How much lead will be left in the sample after 25 yr?

**(b)** How long will it take the initial sample to decay to half of its original amount?

**62.** The concentration of a drug in a person's system decreases according to the function defined by

$$C(t) = 2e^{-0.125t},$$

where $C(t)$ is in appropriate units, and $t$ is in hours. Approximate answers to the nearest hundredth.

**(a)** How much of the drug will be in the system after 1 hr?

**(b)** Approximate the time it will take for the concentration to be half of its original amount.

**63.** Refer to **Exercise 55.** Assuming that the function continued to apply past 2007, in what year can we expect the volume of materials recovered to reach 130 million tons? (*Source:* Environmental Protection Agency.)

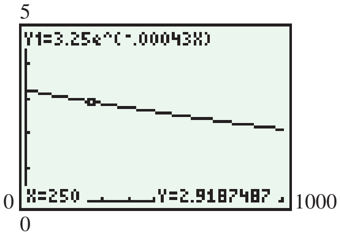
**64.** Refer to **Exercise 56.** Assuming that the function continued to apply past 2006, in what year can we expect worldwide carbon dioxide emissions from fossil fuel consumption to reach 34,000 million metric tons? (*Source:* U.S. Department of Energy.)

## *TECHNOLOGY INSIGHTS* EXERCISES 65–66

**65.** The function defined by

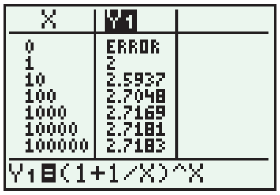
$$A(x) = 3.25e^{-0.00043x},$$

with $x = t$, described in **Exercise 60,** is graphed on the screen at the right. Interpret the meanings of X and Y in the display at the bottom of the screen in the context of **Exercise 60.**


```
Y1=3.25e^(-.00043X)

X=250               Y=2.9187487
```

**66.** The screen shows a table of selected values for the function defined by $Y_1 = \left(1 + \frac{1}{X}\right)^X$.


```
   X    | Y1     |
--------+--------+----------
 0      | ERROR  |
 1      | 2      |
 10     | 2.5937 |
 100    | 2.7048 |
 1000   | 2.7169 |
 10000  | 2.7181 |
 100000 | 2.7183 |
Y1=(1+1/X)^X
```

**(a)** Why is there an error message for X = 0?

**(b)** What number does the function value seem to approach as X takes on larger and larger values?

**(c)** Use a calculator to evaluate this function for X = 1,000,000. What value do you get? Now evaluate $e = e^1$. How close are these two values?

**(d)** Make a conjecture: As the values of $x$ approach infinity, the value of $\left(1 + \frac{1}{x}\right)^x$ approaches ______.

## *PREVIEW EXERCISES*

*Graph each function.* ***See Section 11.6.***

**67.** $f(x) = 2x^2$

**68.** $f(x) = x^2 - 1$

**69.** $f(x) = (x + 1)^2$

**70.** $f(x) = (x - 1)^2 + 2$

# CHAPTER 12 SUMMARY

## KEY TERMS

**12.1**
one-to-one function
inverse of a function

**12.2**
exponential function
asymptote
exponential equation

**12.3**
logarithm
logarithmic equation
logarithmic function with base *a*

**12.5**
common logarithm
natural logarithm
universal constant

**12.6**
compound interest
continuous compounding

## NEW SYMBOLS

| | |
|---|---|
| $f^{-1}(x)$ | the inverse of $f(x)$ |
| $\log_a x$ | the logarithm of $x$ with base $a$ |
| $\log x$ | common (base 10) logarithm of $x$ |
| $\ln x$ | natural (base $e$) logarithm of $x$ |
| $e$ | a constant, approximately 2.718281828 |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

1. In a **one-to-one function**
   **A.** each $x$-value corresponds to only one $y$-value
   **B.** each $x$-value corresponds to one or more $y$-values
   **C.** each $x$-value is the same as each $y$-value
   **D.** each $x$-value corresponds to only one $y$-value and each $y$-value corresponds to only one $x$-value.

2. If $f$ is a one-to-one function, then the **inverse** of $f$ is
   **A.** the set of all solutions of $f$
   **B.** the set of all ordered pairs formed by interchanging the coordinates of the ordered pairs of $f$
   **C.** the set of all ordered pairs that are the opposite (negative) of the coordinates of the ordered pairs of $f$
   **D.** an equation involving an exponential expression.

3. An **exponential function** is a function defined by an expression of the form
   **A.** $f(x) = ax^2 + bx + c$ for real numbers $a, b, c$ $(a \neq 0)$
   **B.** $f(x) = \log_a x$ for positive numbers $a$ and $x$ $(a \neq 1)$
   **C.** $f(x) = a^x$ for all real numbers $x$ $(a > 0, a \neq 1)$
   **D.** $f(x) = \sqrt{x}$ for $x \geq 0$.

4. An **asymptote** is
   **A.** a line that a graph intersects just once
   **B.** a line that the graph of a function more and more closely approaches as the $x$-values increase or decrease
   **C.** the $x$-axis or $y$-axis
   **D.** a line about which a graph is symmetric.

5. A **logarithm** is
   **A.** an exponent
   **B.** a base
   **C.** an equation
   **D.** a polynomial.

6. A **logarithmic function** is a function that is defined by an expression of the form
   **A.** $f(x) = ax^2 + bx + c$ for real numbers $a, b, c$ $(a \neq 0)$
   **B.** $f(x) = \log_a x$ for positive numbers $a$ and $x$ $(a \neq 1)$
   **C.** $f(x) = a^x$ for all real numbers $x$ $(a > 0, a \neq 1)$
   **D.** $f(x) = \sqrt{x}$ for $x \geq 0$.

### ANSWERS

**1.** D; *Example:* The function $f = \{(0, 2), (1, -1), (3, 5), (-2, 3)\}$ is one-to-one. **2.** B; *Example:* The inverse of the one-to-one function $f$ defined in Answer 1 is $f^{-1} = \{(2, 0), (-1, 1), (5, 3), (3, -2)\}$. **3.** C; *Examples:* $f(x) = 4^x$, $g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = 2^{-x+3}$ **4.** B; *Example:* The graph of $f(x) = 2^x$ has the $x$-axis $(y = 0)$ as an asymptote. **5.** A; *Example:* $\log_a x$ is the exponent to which $a$ must be raised to obtain $x$; $\log_3 9 = 2$ since $3^2 = 9$. **6.** B; *Examples:* $y = \log_3 x$, $y = \log_{1/3} x$

## QUICK REVIEW

***CONCEPTS*** ***EXAMPLES***

### 12.1 Inverse Functions

**Horizontal Line Test**

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

Find $f^{-1}$ if $f(x) = 2x - 3$.

The graph of $f$ is a non-horizontal straight line, so $f$ is one-to-one by the horizontal line test.

**Inverse Functions**

For a one-to-one function $f$ defined by an equation $y = f(x)$, the equation that defines the inverse function $f^{-1}$ is found by interchanging $x$ and $y$, solving for $y$, and replacing $y$ with $f^{-1}(x)$.

To find $f^{-1}(x)$, interchange $x$ and $y$ in the equation $y = 2x - 3$.

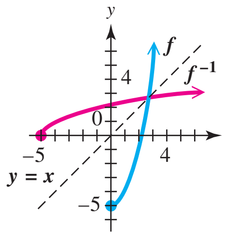
$$x = 2y - 3$$

Solve for $y$ to get $y = \frac{x+3}{2}$.

Therefore, $f^{-1}(x) = \frac{x+3}{2}$, or $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$.

In general, the graph of $f^{-1}$ is the mirror image of the graph of $f$ with respect to the line $y = x$.

The graphs of a function $f$ and its inverse $f^{-1}$ are shown here.



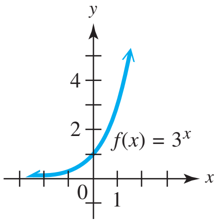
### 12.2 Exponential Functions

For $a > 0$, $a \neq 1$, $\boldsymbol{f(x) = a^x}$ defines the exponential function with base $a$.

**Graph of $f(x) = a^x$**

1. The graph contains the point $(0, 1)$.
2. When $a > 1$, the graph rises from left to right. When $0 < a < 1$, the graph falls from left to right.
3. The $x$-axis is an asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

$f(x) = 3^x$ defines the exponential function with base 3.



### 12.3 Logarithmic Functions

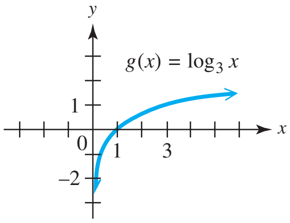
$\boldsymbol{y = \log_a x}$ means $\boldsymbol{x = a^y}$.

For $b > 0, b \neq 1,$ $\boldsymbol{\log_b b = 1}$ and $\boldsymbol{\log_b 1 = 0}$.

$y = \log_2 x$ means $x = 2^y$.

$$\log_3 3 = 1 \qquad \log_5 1 = 0$$

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| For $a > 0, a \neq 1, x > 0$, $g(x) = \log_a x$ defines the logarithmic function with base $a$.<br>**Graph of $g(x) = \log_a x$**<br>**1.** The graph contains the point $(1, 0)$.<br>**2.** When $a > 1$, the graph rises from left to right. When $0 < a < 1$, the graph falls from left to right.<br>**3.** The $y$-axis is an asymptote.<br>**4.** The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$. | $g(x) = \log_3 x$ defines the logarithmic function with base 3.<br> |

## 12.4 Properties of Logarithms

| | |
|---|---|
| **Product Rule** $\log_a xy = \log_a x + \log_a y$ | $\log_2 3m = \log_2 3 + \log_2 m$ Product rule |
| **Quotient Rule** $\log_a \frac{x}{y} = \log_a x - \log_a y$ | $\log_5 \frac{9}{4} = \log_5 9 - \log_5 4$ Quotient rule |
| **Power Rule** $\log_a x^r = r \log_a x$ | $\log_{10} 2^3 = 3 \log_{10} 2$ Power rule |
| **Special Properties**<br>$b^{\log_b x} = x$ and $\log_b b^x = x$ | $6^{\log_6 10} = 10$ $\log_3 3^4 = 4$ Special properties |

## 12.5 Common and Natural Logarithms

| | |
|---|---|
| **Common logarithms (base 10)** are used in applications such as pH, sound level, and intensity of an earthquake. | Use the formula $\text{pH} = -\log [\text{H}_3\text{O}^+]$ to find the pH (to one decimal place) of grapes with hydronium ion concentration $5.0 \times 10^{-5}$.<br>$\text{pH} = -\log(5.0 \times 10^{-5})$ Substitute.<br>$= -(\log 5.0 + \log 10^{-5})$ Property of logarithms<br>$\approx 4.3$ Evaluate with a calculator. |
| **Natural logarithms (base *e*)** are often found in formulas for applications of growth and decay, such as time for money invested to double, decay of chemical compounds, and biological growth. | Use the formula for doubling time (in years) $t(r) = \frac{\ln 2}{\ln(1 + r)}$ to find the doubling time to the nearest tenth at an interest rate of 4%.<br>$t(0.04) = \frac{\ln 2}{\ln(1 + 0.04)}$ Substitute.<br>$\approx 17.7$ Evaluate with a calculator.<br>The doubling time is about 17.7 yr. |
| **Change-of-Base Rule**<br>If $a > 0, a \neq 1, b > 0, b \neq 1, x > 0$, then<br>$\log_a x = \frac{\log_b x}{\log_b a}$. | $\log_3 17 = \frac{\ln 17}{\ln 3} = \frac{\log 17}{\log 3} \approx 2.5789$ |

## 12.6 Exponential and Logarithmic Equations; Further Applications

| | |
|---|---|
| To solve exponential equations, use these properties $(b > 0, b \neq 1)$.<br>**1.** If $b^x = b^y$, then $x = y$. | Solve. $2^{3x} = 2^5$<br>$3x = 5$ Set exponents equal.<br>$x = \frac{5}{3}$ Divide by 3.<br>The solution set is $\left\{\frac{5}{3}\right\}$. |

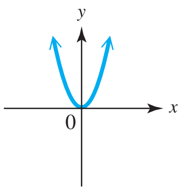
(continued)

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| **2.** If $x = y, x > 0, y > 0$, then $\log_b x = \log_b y$. | Solve. $5^m = 8$<br>$\log 5^m = \log 8$ Take common logarithms.<br>$m \log 5 = \log 8$ Power rule<br>$m = \frac{\log 8}{\log 5} \approx 1.2920$ Divide by log 5.<br>The solution set is $\{1.2920\}$. |
| To solve logarithmic equations, use these properties, where $b > 0, b \neq 1, x > 0, y > 0$. First use the properties of **Section 12.4,** if necessary, to write the equation in the proper form. | |
| **1.** If $\log_b x = \log_b y$, then $x = y$. | Solve. $\log_3 2x = \log_3 (x + 1)$<br>$2x = x + 1$<br>$x = 1$ Subtract $x$.<br>This value checks, so the solution set is $\{1\}$. |
| **2.** If $\log_b x = y$, then $b^y = x$. | Solve. $\log_2 (3x - 1) = 4$<br>$3x - 1 = 2^4$ Exponential form<br>$3x - 1 = 16$ Apply the exponent.<br>$3x = 17$ Add 1.<br>$x = \frac{17}{3}$ Divide by 3. |
| Always check proposed solutions in logarithmic equations. | This value checks, so the solution set is $\{\frac{17}{3}\}$. |

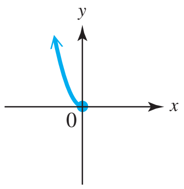
# CHAPTER 12 REVIEW EXERCISES

**12.1** *Determine whether each graph is the graph of a one-to-one function.*

**1.**



**2.**



*Determine whether each function is one-to-one. If it is, find its inverse.*

**3.** $f(x) = -3x + 7$

**4.** $f(x) = \sqrt[3]{6x - 4}$

**5.** $f(x) = -x^2 + 3$

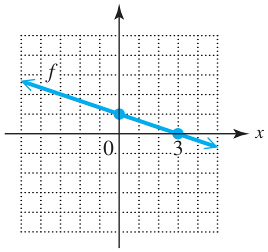
**6.** The table lists caffeine amounts in several popular 12-oz sodas. If the set of sodas is the domain and the set of caffeine amounts is the range of the function consisting of the six pairs listed, is it a one-to-one function? Why or why not?

| *Soda* | *Caffeine (mg)* |
|---|---|
| Mountain Dew | 55 |
| Diet Coke | 45 |
| Dr. Pepper | 41 |
| Sunkist Orange Soda | 41 |
| Diet Pepsi-Cola | 36 |
| Coca-Cola Classic | 34 |

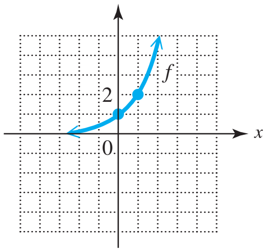
*Source:* National Soft Drink Association.

*Each function graphed is one-to-one. Graph its inverse.*

**7.**



**8.**



**12.2** *Graph each function.*

**9.** $f(x) = 3^x$

**10.** $f(x) = \left(\frac{1}{3}\right)^x$

**11.** $y = 2^{2x+3}$

*Solve each equation.*

**12.** $5^{2x+1} = 25$

**13.** $4^{3x} = 8^{x+4}$

**14.** $\left(\frac{1}{27}\right)^{x-1} = 9^{2x}$

**15.** Sulfur dioxide emissions in the United States, in millions of tons, from 1970 through 2007 can be approximated by the exponential function defined by

$$S(x) = 33.07(1.0241)^{-x},$$

where $x = 0$ corresponds to 1970, $x = 5$ to 1975, and so on. Use this function to approximate, to the nearest tenth, the amounts for each year. (*Source:* U.S. Environmental Protection Agency.)

**(a)** 1975 **(b)** 1995 **(c)** 2005

**12.3** *Graph each function.*

**16.** $g(x) = \log_3 x$ (*Hint:* See **Exercise 9.**)

**17.** $g(x) = \log_{1/3} x$ (*Hint:* See **Exercise 10.**)

*Solve each equation.*

**18.** $\log_8 64 = x$

**19.** $\log_2 \sqrt{8} = x$

**20.** $\log_x \left(\frac{1}{49}\right) = -2$

**21.** $\log_4 x = \frac{3}{2}$

**22.** $\log_k 4 = 1$

**23.** $\log_b b^2 = 2$

**24.** In your own words, explain the meaning of $\log_b a$.

**25.** *Concept Check* Based on the meaning of $\log_b a$, what is the simplest form of $b^{\log_b a}$?

**26.** A company has found that total sales, in thousands of dollars, are given by the function defined by

$$S(x) = 100 \log_2 (x + 2),$$

where $x$ is the number of weeks after a major advertising campaign was introduced.

**(a)** What were the total sales 6 weeks after the campaign was introduced?

**(b)** Graph the function.

**12.4** *Apply the properties of logarithms to express each logarithm as a sum or difference of logarithms. Assume that all variables represent positive real numbers.*

**27.** $\log_2 3xy^2$

**28.** $\log_4 \dfrac{\sqrt{x} \cdot w^2}{z}$

*Apply the properties of logarithms to write each expression as a single logarithm. Assume that all variables represent positive real numbers,* $b \neq 1$.

**29.** $\log_b 3 + \log_b x - 2 \log_b y$

**30.** $\log_3 (x + 7) - \log_3 (4x + 6)$

**12.5** *Evaluate each logarithm. Give approximations to four decimal places.*

**31.** $\log 28.9$

**32.** $\log 0.257$

**33.** $\ln 28.9$

**34.** $\ln 0.257$

*Use the change-of-base rule (with either common or natural logarithms) to find each logarithm. Give approximations to four decimal places.*

**35.** $\log_{16} 13$

**36.** $\log_4 12$

*Use the formula* $\text{pH} = -\log [\text{H}_3\text{O}^+]$ *to find the* pH *of each substance with the given hydronium ion concentration.*

**37.** Milk, $4.0 \times 10^{-7}$

**38.** Crackers, $3.8 \times 10^{-9}$

**39.** If orange juice has pH 4.6, what is its hydronium ion concentration?

**40.** The magnitude of a star is defined by the equation

$$M = 6 - 2.5 \log \frac{I}{I_0},$$

where $I_0$ is the measure of the faintest star and $I$ is the actual intensity of the star being measured. The dimmest stars are of magnitude 6, and the brightest are of magnitude 1. Determine the ratio of intensities between stars of magnitude 1 and 3.

**41. Section 12.5, Exercise 42** introduced the doubling function defined by

$$t(r) = \frac{\ln 2}{\ln (1 + r)},$$

that gives the number of years required to double your money when it is invested at interest rate $r$ (in decimal form) compounded annually. How long does it take to double your money at each rate? Round answers to the nearest year.

**(a)** 4% **(b)** 6% **(c)** 10% **(d)** 12%

**(e)** Compare each answer in parts (a)–(d) with the following numbers. What do you find?

$$\frac{72}{4}, \frac{72}{6}, \frac{72}{10}, \frac{72}{12}$$

**12.6** *Solve each equation. Give solutions to three decimal places.*

**42.** $3^x = 9.42$ **43.** $2^{x-1} = 15$ **44.** $e^{0.06x} = 3$

*Solve each equation. Give exact solutions.*

**45.** $\log_3 (9x + 8) = 2$

**46.** $\log_5 (x + 6)^3 = 2$

**47.** $\log_3 (x + 2) - \log_3 x = \log_3 2$

**48.** $\log (2x + 3) = 1 + \log x$

**49.** $\log_4 x + \log_4 (8 - x) = 2$

**50.** $\log_2 x + \log_2 (x + 15) = \log_2 16$

**51.** ***Concept Check*** Consider the following "solution" of the equation $\log x^2 = 2$. ***WHAT WENT WRONG?*** Give the correct solution set.

$$\begin{aligned} \log x^2 &= 2 && \text{Original equation} \\ 2 \log x &= 2 && \text{Power rule for logarithms} \\ \log x &= 1 && \text{Divide each side by 2.} \\ x &= 10^1 && \text{Write in exponential form.} \\ x &= 10 && 10^1 = 10 \end{aligned}$$

Solution set: $\{10\}$

*Solve each problem. Use a calculator as necessary.*

**52.** If \$20,000 is deposited at 4% annual interest compounded quarterly, how much will be in the account after 5 yr, assuming no withdrawals are made?

**53.** How much will \$10,000 compounded continuously at 3.75% annual interest amount to in 3 yr?

**54.** Which is a better plan?

*Plan A:* Invest \$1000 at 4% compounded quarterly for 3 yr
*Plan B:* Invest \$1000 at 3.9% compounded monthly for 3 yr

**55.** What is the half-life of a radioactive substance that decays according to the function

$$Q(t) = A_0 e^{-0.05t}, \quad \text{where } t \text{ is in days?}$$

**56.** A machine purchased for business use **depreciates,** or loses value, over a period of years. The value of the machine at the end of its useful life is called its **scrap value.** By one method of depreciation (where it is assumed a constant percentage of the value depreciates annually), the scrap value, $S$, is given by

$$S = C(1 - r)^n,$$

where $C$ is the original cost, $n$ is the useful life in years, and $r$ is the constant percent of depreciation.

**(a)** Find the scrap value of a machine costing \$30,000, having a useful life of 12 yr and a constant annual rate of depreciation of 15%.

**(b)** A machine has a "half-life" of 6 yr. Find the constant annual rate of depreciation.

**57.** Recall from **Exercise 43** in **Section 12.5** that the number of years, $N(r)$, since two independently evolving languages split off from a common ancestral language is approximated by

$$N(r) = -5000 \ln r,$$

where $r$ is the percent of words from the ancestral language common to both languages now. Find $r$ if the split occurred 2000 yr ago.

**58.** ***Concept Check*** Which one is *not* a representation of the solution of $7^x = 23$?

**A.** $\dfrac{\log 23}{\log 7}$ **B.** $\dfrac{\ln 23}{\ln 7}$ **C.** $\log_7 23$ **D.** $\log_{23} 7$

## MIXED REVIEW EXERCISES

*Evaluate.*

**59.** $\log_2 128$ **60.** $5^{\log_5 36}$ **61.** $e^{\ln 4}$

**62.** $10^{\log e}$ **63.** $\log_3 3^{-5}$ **64.** $\ln e^{5.4}$

*Solve.*

**65.** $\log_3(x + 9) = 4$ **66.** $\ln e^x = 3$ **67.** $\log_x \frac{1}{81} = 2$

**68.** $27^x = 81$ **69.** $2^{2x-3} = 8$

**70.** $5^{x+2} = 25^{2x+1}$ **71.** $\log_3 (x + 1) - \log_3 x = 2$

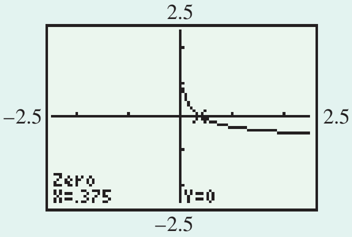
**72.** $\log (3x - 1) = \log 10$ **73.** $\ln (x^2 + 3x + 4) = \ln 2$

**74.** Consider the logarithmic equation

$$\log (2x + 3) = \log x + 1.$$

**(a)** Solve the equation using properties of logarithms.

**(b)** If $Y_1 = \log (2X + 3)$ and $Y_2 = \log X + 1$, then the graph of $Y_1 - Y_2$ looks like that shown. Explain how the display at the bottom of the screen confirms the solution set found in part (a).



**75.** Based on selected figures from 1980 through 2007, the fractional part (as a decimal) of the generation of municipal solid waste recovered can be approximated by the function defined by

$$R(x) = 0.0997(e^{0.0470x}),$$

where $x = 0$ corresponds to 1980, $x = 10$ to 1990, and so on. Based on this model, approximate the percent, to the nearest hundredth, of municipal solid waste recovered in 2005. (*Source:* U.S. Environmental Protection Agency.)

**76.** One measure of the diversity of the species in an ecological community is the **index of diversity,** the logarithmic expression

$$-(p_1 \ln p_1 + p_2 \ln p_2 + \cdots + p_n \ln p_n),$$

where $p_1, p_2, \ldots, p_n$ are the proportions of a sample belonging to each of $n$ species in the sample. (*Source:* Ludwig, John and James Reynolds, *Statistical Ecology: A Primer on Methods and Computing,* New York, John Wiley and Sons.) Approximate the index of diversity to the nearest thousandth if a sample of 100 from a community produces the following numbers.

**(a)** 90 of one species, 10 of another **(b)** 60 of one species, 40 of another

## CHAPTER 12 TEST

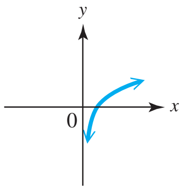
CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

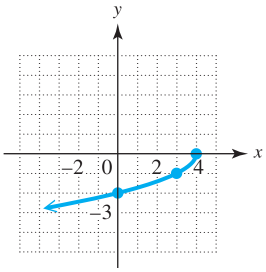
**1.** Decide whether each function is one-to-one.

**(a)** $f(x) = x^2 + 9$ **(b)**



**2.** Find $f^{-1}(x)$ for the one-to-one function defined by $f(x) = \sqrt[3]{x + 7}$.

**3.** Graph the inverse of $f$, given the graph of $f$.



*Graph each function.*

**4.** $f(x) = 6^x$

**5.** $g(x) = \log_6 x$

**6.** Explain how the graph of the function in **Exercise 5** can be obtained from the graph of the function in **Exercise 4.**

*Solve each equation. Give the exact solution.*

**7.** $5^x = \frac{1}{625}$

**8.** $2^{3x-7} = 8^{2x+2}$

**9.** A 2008 report predicted that the U.S. Hispanic population will increase from 46.9 million in 2008 to 132.8 million in 2050. (*Source:* U.S. Census Bureau.) Assuming an exponential growth pattern, the population is approximated by

$$f(x) = 46.9e^{0.0247x},$$

where $x$ represents the number of years since 2008. Use this function to approximate, to the nearest tenth, the Hispanic population in each year.

**(a)** 2015 **(b)** 2030

**10.** Write in logarithmic form: $4^{-2} = 0.0625$.

**11.** Write in exponential form: $\log_7 49 = 2$.

*Solve each equation.*

**12.** $\log_{1/2} x = -5$

**13.** $x = \log_9 3$

**14.** $\log_x 16 = 4$

**15.** ***Concept Check*** Fill in the blanks with the correct responses: The value of $\log_2 32$ is ______. This means that if we raise ______ to the ______ power, the result is ______.

*Use properties of logarithms to write each expression as a sum or difference of logarithms. Assume that variables represent positive real numbers.*

**16.** $\log_3 x^2y$

**17.** $\log_5 \left(\frac{\sqrt{x}}{yz}\right)$

*Use properties of logarithms to write each expression as a single logarithm. Assume that variables represent positive real numbers, $b \neq 1$.*

**18.** $3 \log_b s - \log_b t$

**19.** $\frac{1}{4} \log_b r + 2 \log_b s - \frac{2}{3} \log_b t$

**20.** Use a calculator to approximate each logarithm to four decimal places.

**(a)** log 23.1 **(b)** ln 0.82

**21.** Use the change-of-base rule to express $\log_3 19$

**(a)** in terms of common logarithms **(b)** in terms of natural logarithms

**(c)** correct to four decimal places.

**22.** Solve $3^x = 78$, giving the solution to three decimal places.

**23.** Solve $\log_8 (x + 5) + \log_8 (x - 2) = 1$.

**24.** Suppose that \$10,000 is invested at 4.5% annual interest, compounded quarterly. How much will be in the account in 5 yr if no money is withdrawn?

**25.** Suppose that \$15,000 is invested at 5% annual interest, compounded continuously.

**(a)** How much will be in the account in 5 yr if no money is withdrawn?

**(b)** How long will it take for the initial principal to double?

## CHAPTERS 1–12 CUMULATIVE REVIEW EXERCISES

*Let* $S = \left\{-\frac{9}{4}, -2, -\sqrt{2}, 0, 0.6, \sqrt{11}, \sqrt{-8}, 6, \frac{30}{3}\right\}$. *List the elements of S that are members of each set.*

**1.** Integers **2.** Rational numbers **3.** Irrational numbers

*Simplify each expression.*

**4.** $|-8| + 6 - |-2| - (-6 + 2)$

**5.** $2(-5) + (-8)(4) - (-3)$

*Solve each equation or inequality.*

**6.** $7 - (3 + 4x) + 2x = -5(x - 1) - 3$

**7.** $2x + 2 \leq 5x - 1$

**8.** $|2x - 5| = 9$

**9.** $|4x + 2| > 10$

**10.** $\sqrt{2x + 1} - \sqrt{x} = 1$

**11.** $3x^2 - x - 1 = 0$

**12.** $x^2 + 2x - 8 > 0$

**13.** $x^4 - 5x^2 + 4 = 0$

**14.** $5^{x+3} = \left(\frac{1}{25}\right)^{3x+2}$

*Graph.*

**15.** $5x + 2y = 10$

**16.** $-4x + y \leq 5$

**17.** $f(x) = \frac{1}{3}(x - 1)^2 + 2$

**18.** $f(x) = 2^x$

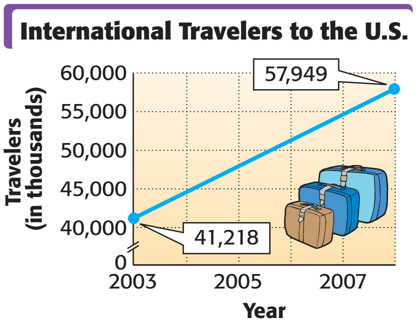
**19.** $f(x) = \log_3 x$

**20.** The graph indicates that the number of international travelers to the United States increased from 41,218 thousand in 2003 to 57,949 thousand in 2008.

**(a)** Is this the graph of a function?

**(b)** What is the slope, to the nearest tenth, of the line in the graph? Interpret the slope in the context of U.S. travelers to foreign countries.

**21.** Find an equation of the line through $(5, -1)$ and parallel to the line with equation $3x - 4y = 12$. Write the equation in slope-intercept form.



*Source:* U.S. Department of Commerce.

*Perform the indicated operations.*

**22.** $(2p + 3)(3p - 1)$

**23.** $(4k - 3)^2$

**24.** $(3m^3 + 2m^2 - 5m) - (8m^3 + 2m - 4)$

**25.** Divide $6t^4 + 17t^3 - 4t^2 + 9t + 4$ by $3t + 1$.

*Factor.*

**26.** $8x + x^3$

**27.** $24y^2 - 7y - 6$

**28.** $5z^3 - 19z^2 - 4z$

**29.** $16a^2 - 25b^4$

**30.** $8c^3 + d^3$

**31.** $16r^2 + 56rq + 49q^2$

*Perform the indicated operations.*

**32.** $\dfrac{(5p^3)^4(-3p^7)}{2p^2(4p^4)}$

**33.** $\dfrac{x^2 - 9}{x^2 + 7x + 12} \div \dfrac{x - 3}{x + 5}$

**34.** $\dfrac{2}{k + 3} - \dfrac{5}{k - 2}$

*Solve each system.*

**35.**
$$\begin{aligned} 5x - 3y &= 14 \\ 2x + 5y &= 18 \end{aligned}$$

**36.**
$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x - y + z &= 8 \\ 2x + 2y - 3z &= -12 \end{aligned}$$

**37.** Candy worth $1.00 per lb is to be mixed with 10 lb of candy worth $1.96 per lb to get a mixture that will be sold for $1.60 per lb. How many pounds of the $1.00 candy should be used?

| *Number of Pounds* | *Price per Pound* | *Value* |
|---|---|---|
| $x$ | $1.00 | $1x$ |
| | | |
| | $1.60 | |

*Simplify.*

**38.** $\sqrt{288}$

**39.** $2\sqrt{32} - 5\sqrt{98}$

**40.** $(5 + 4i)(5 - 4i)$

**41.** Rewrite the following using the product, quotient, and power properties of logarithms.

$$\log \frac{x^3\sqrt{y}}{z}$$

**42.** Let the number of bacteria present in a certain culture be given by

$$B(t) = 25{,}000e^{0.2t},$$

where $t$ is time measured in hours, and $t = 0$ corresponds to noon. Approximate, to the nearest hundred, the number of bacteria present at each time.

**(a)** noon **(b)** 1 P.M. **(c)** 2 P.M.

**(d)** When will the population double?

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CHAPTER 13

# Nonlinear Functions, Conic Sections, and Nonlinear Systems

In this chapter, we study a group of curves known as *conic sections*. One conic section, the *ellipse*, has a special reflecting property responsible for “whispering galleries.” In a whispering gallery, a person whispering at a certain point in the room can be heard clearly at another point across the room.

The Old House Chamber of the U.S. Capitol, now called Statuary Hall, is a whispering gallery. History has it that John Quincy Adams, whose desk was positioned at exactly the right point beneath the ellipsoidal ceiling, often pretended to sleep there as he listened to political opponents whispering strategies across the room. (*Source:* Aikman, Lonnelle, *We, the People, The Story of the United States Capitol*.)

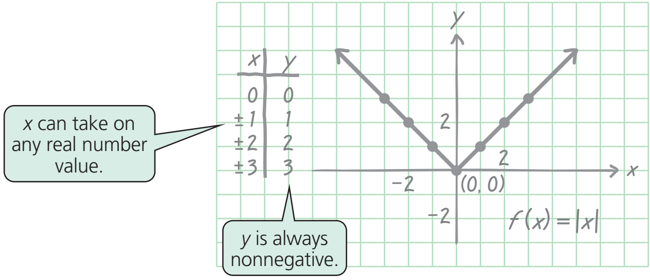
In **Section 13.2,** we investigate ellipses.

# 13.1 Additional Graphs of Functions

**OBJECTIVES**

1. Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations.
2. Recognize and graph step functions.

**OBJECTIVE 1 Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations.** Earlier, we introduced the function defined by $f(x) = x^2$, sometimes called the **squaring function.** Another elementary function is the **absolute value function,** defined by $\boldsymbol{f(x) = |x|}$. This function pairs each real number with its absolute value. Its graph is shown in **FIGURE 1**.



*x* can take on any real number value.

*y* is always nonnegative.

**FIGURE 1**

**Absolute value function**

$f(x) = |x|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

The **reciprocal function,** defined by $\boldsymbol{f(x) = \frac{1}{x}}$, is a *rational function*. Its graph is shown in **FIGURE 2**. Since $x$ can never equal 0, as $x$ gets closer and closer to 0, $\frac{1}{x}$ approaches either $\infty$ or $-\infty$. Also, $\frac{1}{x}$ can never equal 0, and as $x$ approaches $\infty$ or $-\infty$, $\frac{1}{x}$ approaches 0. The axes are called **asymptotes** for the function.

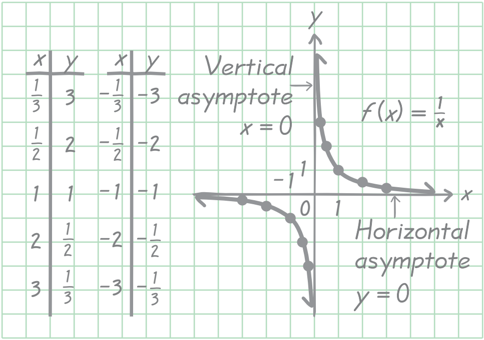


**FIGURE 2**

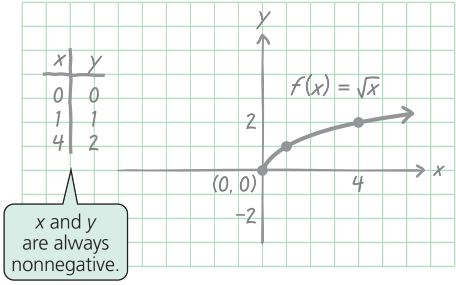
**Reciprocal function**

$f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

The **square root function,** defined by $\boldsymbol{f(x) = \sqrt{x}}$ and introduced in **Section 10.1,** is shown in **FIGURE 3**.



*x* and *y* are always nonnegative.

**FIGURE 3**

**Square root function**

$f(x) = \sqrt{x}$

Domain: $[0, \infty)$

Range: $[0, \infty)$

The graphs of these elementary functions can be shifted, or translated, just as we did with the graph of $f(x) = x^2$ in **Section 11.6.**

**NOW TRY EXERCISE 1**

Graph $f(x) = \frac{1}{x+3}$. Give the domain and range.

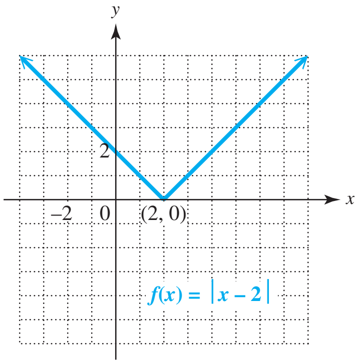
## EXAMPLE 1 Applying a Horizontal Shift

Graph $f(x) = |x - 2|$. Give the domain and range.

The graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ two units to the right. In a similar manner, the graph of $f(x) = |x - 2|$ is found by shifting the graph of $y = |x|$ two units to the right, as shown in **FIGURE 4**.

| x | y |
|---|---|
| 0 | 2 |
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 2 |

Compare this table of values to that with **FIGURE 1**.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

**FIGURE 4**

NOW TRY

**NOW TRY EXERCISE 2**

Graph $f(x) = \sqrt{x} + 2$. Give the domain and range.

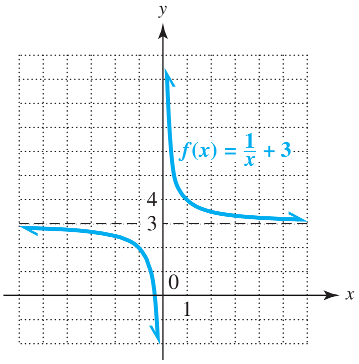
## EXAMPLE 2 Applying a Vertical Shift

Graph $f(x) = \frac{1}{x} + 3$. Give the domain and range.

The graph is found by shifting the graph of $y = \frac{1}{x}$ three units up. See **FIGURE 5**.

| x | y | x | y |
|---|---|---|---|
| $\frac{1}{3}$ | 6 | $-\frac{1}{3}$ | 0 |
| $\frac{1}{2}$ | 5 | $-\frac{1}{2}$ | 1 |
| 1 | 4 | $-1$ | 2 |
| 2 | 3.5 | $-2$ | 2.5 |

Compare this table of values to that with **FIGURE 2**.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 3$

**FIGURE 5**

NOW TRY

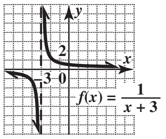
## EXAMPLE 3 Applying Both Horizontal and Vertical Shifts

Graph $f(x) = \sqrt{x + 1} - 4$. Give the domain and range.

The graph of $y = (x + 1)^2 - 4$ is obtained by shifting the graph of $y = x^2$ one unit to the left and four units down. Following this pattern, we shift the graph of $y = \sqrt{x}$ one unit to the left and four units down to get the graph of $f(x) = \sqrt{x + 1} - 4$. See **FIGURE 6** on the next page.

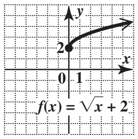
**NOW TRY ANSWERS**

**1.**



domain: $(-\infty, -3) \cup (-3, \infty)$; range: $(-\infty, 0) \cup (0, \infty)$

**2.**



domain: $[0, \infty)$; range: $[2, \infty)$

NOW TRY EXERCISE 3

Graph $f(x) = |x + 1| - 3$. Give the domain and range.

| $x$ | $y$ |
|---|---|
| $-1$ | $-4$ |
| $0$ | $-3$ |
| $3$ | $-2$ |

Compare this table of values to that with **FIGURE 3**.

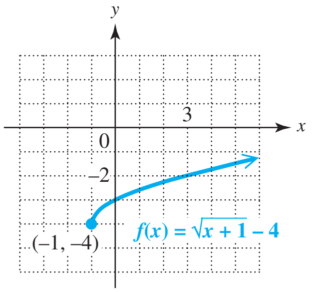


**FIGURE 6**

Domain: $[-1, \infty)$
Range: $[-4, \infty)$

NOW TRY

**OBJECTIVE 2** **Recognize and graph step functions.** The greatest integer function is defined as follows.

### $f(x) = [\![x]\!]$

The **greatest integer function,** written $f(x) = [\![x]\!]$, pairs every real number $x$ with the greatest integer less than or equal to $x$.

NOW TRY EXERCISE 4

Evaluate each expression.

**(a)** $[\![5]\!]$ **(b)** $[\![-6]\!]$
**(c)** $[\![3.5]\!]$ **(d)** $[\![-4.1]\!]$

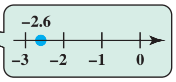
## EXAMPLE 4 Finding the Greatest Integer

Evaluate each expression.

**(a)** $[\![8]\!] = 8$ **(b)** $[\![-1]\!] = -1$ **(c)** $[\![0]\!] = 0$

**(d)** $[\![7.45]\!] = 7$ The greatest integer *less than or equal to* 7.45 is 7.

**(e)** $[\![-2.6]\!] = -3$



Think of a number line with $-2.6$ graphed on it. Since $-3$ is to the *left of* (and is, therefore, *less than*) $-2.6$, the greatest integer less than or equal to $-2.6$ is $-3$, ***not*** $-2$.

NOW TRY

## EXAMPLE 5 Graphing the Greatest Integer Function

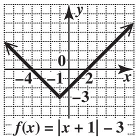
Graph $f(x) = [\![x]\!]$. Give the domain and range.

$$\begin{aligned} \text{For } [\![x]\!], \quad &\text{if } -1 \le x < 0, \quad \text{then } [\![x]\!] = -1; \\ &\text{if } \phantom{-}0 \le x < 1, \quad \text{then } [\![x]\!] = 0; \\ &\text{if } \phantom{-}1 \le x < 2, \quad \text{then } [\![x]\!] = 1; \\ &\text{if } \phantom{-}2 \le x < 3, \quad \text{then } [\![x]\!] = 2; \\ &\text{if } \phantom{-}3 \le x < 4, \quad \text{then } [\![x]\!] = 3, \quad \text{and so on.} \end{aligned}$$

Thus, the graph, as shown in **FIGURE 7** on the next page, consists of a series of horizontal line segments. In each one, the left endpoint is included and the right endpoint is excluded. These segments continue infinitely following this pattern to the left and right. The appearance of the graph is the reason that this function is called a **step function.**

NOW TRY ANSWERS

3.



domain: $(-\infty, \infty)$; range: $[-3, \infty)$

4. **(a)** 5 **(b)** $-6$ **(c)** 3 **(d)** $-5$

**NOW TRY EXERCISE 5**

Graph $f(x) = [\![x - 1]\!]$. Give the domain and range.

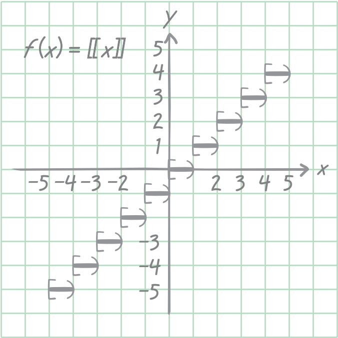


FIGURE 7

**Greatest integer function**

$f(x) = [\![x]\!]$

Domain: $(-\infty, \infty)$

Range: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
(the set of integers)

The graph of a step function also may be shifted. For example, the graph of

$$h(x) = [\![x - 2]\!]$$

is the same as the graph of $f(x) = [\![x]\!]$ shifted two units to the right. Similarly, the graph of

$$g(x) = [\![x]\!] + 2$$

is the graph of $f(x)$ shifted two units up. NOW TRY

**NOW TRY EXERCISE 6**

The cost of parking a car at an airport hourly parking lot is \$4 for the first hour and \$2 for each additional hour or fraction thereof. Let $f(x) =$ the cost of parking a car for $x$ hours. Graph $f(x)$ for $x$ in the interval $(0, 5]$.

## EXAMPLE 6 Applying a Greatest Integer Function

An overnight delivery service charges \$25 for a package weighing up to 2 lb. For each additional pound or fraction of a pound there is an additional charge of \$3. Let $D(x)$, or $y$, represent the cost to send a package weighing $x$ pounds. Graph $D(x)$ for $x$ in the interval $(0, 6]$.

For $x$ in the interval $(0, 2]$, $y = 25$.

For $x$ in the interval $(2, 3]$, $y = 25 + 3 = 28$.

For $x$ in the interval $(3, 4]$, $y = 28 + 3 = 31$.

For $x$ in the interval $(4, 5]$, $y = 31 + 3 = 34$.

For $x$ in the interval $(5, 6]$, $y = 34 + 3 = 37$.

The graph, which is that of a step function, is shown in **FIGURE 8**.

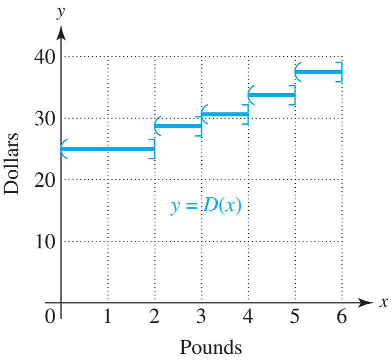
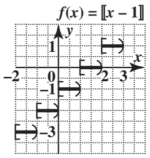


FIGURE 8

NOW TRY

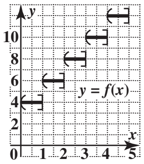
**NOW TRY ANSWERS**

5.



domain: $(-\infty, \infty)$;
range: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

6.



## 13.1 EXERCISES

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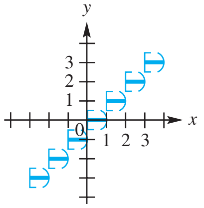
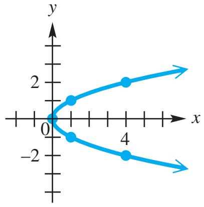
*Complete solution available on the Video Resources on DVD*

**Concept Check** *For Exercises 1–6, refer to the basic graphs in A–F.*

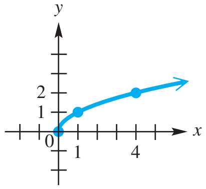
**A.**



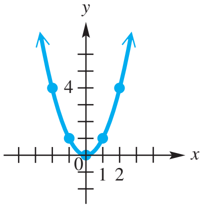
**B.**



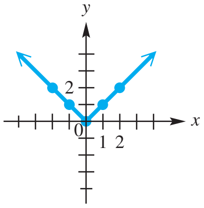
**C.**



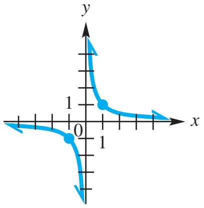
**D.**



**E.**



**F.**



**1.** Which is the graph of $f(x) = |x|$? The lowest point on its graph has coordinates (______, ______).

**2.** Which is the graph of $f(x) = x^2$? Give the domain and range.

**3.** Which is the graph of $f(x) = [\![x]\!]$? Give the domain and range.

**4.** Which is the graph of $f(x) = \sqrt{x}$? Give the domain and range.

**5.** Which is not the graph of a function? Why?

**6.** Which is the graph of $f(x) = \frac{1}{x}$? The lines with equations $x = 0$ and $y = 0$ are called its ____________.

**Concept Check** *Without actually plotting points, match each function defined by the absolute value expression with its graph.*

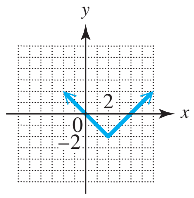
**7.** $f(x) = |x - 2| + 2$

**8.** $f(x) = |x + 2| + 2$

**9.** $f(x) = |x - 2| - 2$

**10.** $f(x) = |x + 2| - 2$

**A.**



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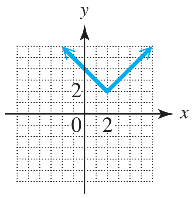
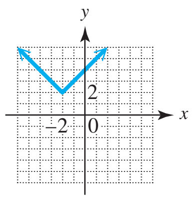
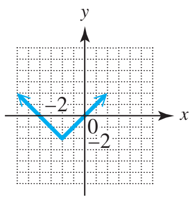


**C.**



**D.**



*Graph each function. Give the domain and range.* ***See Examples 1–3.***

**11.** $f(x) = |x + 1|$

**12.** $f(x) = |x - 1|$

**13.** $f(x) = \dfrac{1}{x} + 1$

**14.** $f(x) = \dfrac{1}{x} - 1$

**15.** $f(x) = \sqrt{x - 2}$

**16.** $f(x) = \sqrt{x + 5}$

**17.** $f(x) = \dfrac{1}{x - 2}$ **18.** $f(x) = \dfrac{1}{x + 2}$ **19.** $f(x) = \sqrt{x + 3} - 3$

**20.** $f(x) = \sqrt{x - 2} + 2$ **21.** $f(x) = |x - 3| + 1$ **22.** $f(x) = |x + 1| - 4$

**23.** ***Concept Check*** How is the graph of $f(x) = \frac{1}{x - 3} + 2$ obtained from the graph of $g(x) = \frac{1}{x}$?

**24.** ***Concept Check*** How is the graph of $f(x) = \frac{1}{x + 5} - 3$ obtained from the graph of $g(x) = \frac{1}{x}$?

*Evaluate each expression. See* ***Example 4.***

**25.** $\llbracket 3 \rrbracket$ **26.** $\llbracket 18 \rrbracket$ **27.** $\llbracket 4.5 \rrbracket$ **28.** $\llbracket 8.7 \rrbracket$ **29.** $\left\llbracket \dfrac{1}{2} \right\rrbracket$

**30.** $\left\llbracket \dfrac{3}{4} \right\rrbracket$ **31.** $\llbracket -14 \rrbracket$ **32.** $\llbracket -5 \rrbracket$ **33.** $\llbracket -10.1 \rrbracket$ **34.** $\llbracket -6.9 \rrbracket$

*Graph each step function. See* ***Examples 5 and 6.***

**35.** $f(x) = \llbracket x \rrbracket - 1$ **36.** $f(x) = \llbracket x \rrbracket + 1$

**37.** $f(x) = \llbracket x - 3 \rrbracket$ **38.** $f(x) = \llbracket x + 2 \rrbracket$

**39.** Assume that postage rates are 44¢ for the first ounce, plus 17¢ for each additional ounce, and that each letter carries one 44¢ stamp and as many 17¢ stamps as necessary. Graph the function defined by

$$y = p(x) = \text{the number of stamps}$$

on a letter weighing $x$ ounces. Use the interval $(0, 5]$.

**40.** The cost of parking a car at an airport hourly parking lot is \$3 for the first half-hour and \$2 for each additional half-hour or fraction thereof. Graph the function defined by $y = f(x) =$ the cost of parking a car for $x$ hours. Use the interval $(0, 2]$.

**41.** A certain long-distance carrier provides service between Podunk and Nowhereville. If $x$ represents the number of minutes for the call, where $x > 0$, then the function $f$ defined by

$$f(x) = 0.40\llbracket x \rrbracket + 0.75$$

gives the total cost of the call in dollars. Find the cost of a 5.5-minute call.

**42. See Exercise 41.** Find the cost of a 20.75-minute call.

## *PREVIEW EXERCISES*

*Find the distance between each pair of points.* ***See Section 10.3.***

**43.** $(2, -1)$ and $(4, 3)$ **44.** $(x, y)$ and $(-2, 5)$ **45.** $(x, y)$ and $(h, k)$

# 13.2

# The Circle and the Ellipse

**OBJECTIVES**

1. Find an equation of a circle given the center and radius.
2. Determine the center and radius of a circle given its equation.
3. Recognize an equation of an ellipse.
4. Graph ellipses.

When an infinite cone is intersected by a plane, the resulting figure is called a **conic section.** The parabola is one example of a conic section. Circles, ellipses, and hyperbolas may also result. See **FIGURE 9**.

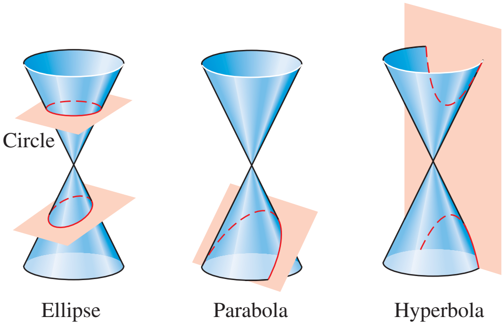


**FIGURE 9**

**OBJECTIVE 1** **Find an equation of a circle given the center and radius.** A **circle** is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed point is called the **center,** and the fixed distance is called the **radius.** We use the distance formula from **Section 10.3** to find an equation of a circle.

*NOW TRY EXERCISE 1*

Find an equation of the circle with radius 6 and center at $(0, 0)$, and graph it.

**EXAMPLE 1** Finding an Equation of a Circle and Graphing It

Find an equation of the circle with radius 3 and center at $(0, 0)$, and graph it.

If the point $(x, y)$ is on the circle, then the distance from $(x, y)$ to the center $(0, 0)$ is 3.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \quad \text{Distance formula}$$

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 3 \quad \text{Let } x_1 = 0,\ y_1 = 0, \text{ and } d = 3.$$

$$x^2 + y^2 = 9 \quad \text{Square each side.}$$

An equation of this circle is $x^2 + y^2 = 9$. The graph is shown in **FIGURE 10**.

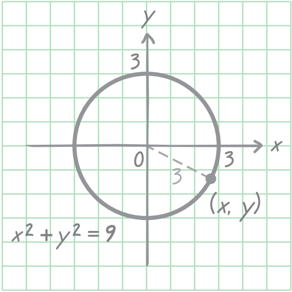


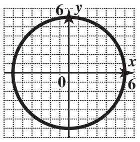
**FIGURE 10**

*NOW TRY*

A circle may not be centered at the origin, as seen in the next example.

*NOW TRY ANSWER*

1. $x^2 + y^2 = 36$



**EXAMPLE 2** Finding an Equation of a Circle and Graphing It

Find an equation of the circle with center at $(4, -3)$ and radius 5, and graph it.

$$\sqrt{(x - 4)^2 + [y - (-3)]^2} = 5 \quad \text{Let } x_1 = 4,\ y_1 = -3, \text{ and } d = 5 \text{ in the distance formula.}$$

$$(x - 4)^2 + (y + 3)^2 = 25 \quad \text{Square each side.}$$

**NOW TRY EXERCISE 2**

Find an equation of the circle with center at $(-2, 2)$ and radius 3, and graph it.

To graph the circle, plot the center $(4, -3)$, then move 5 units right, left, up, and down from the center, plotting the points

$$(9, -3), \quad (-1, -3), \quad (4, 2), \quad \text{and} \quad (4, -8).$$

Draw a smooth curve through these four points, sketching one quarter of the circle at a time. See **FIGURE 11.**

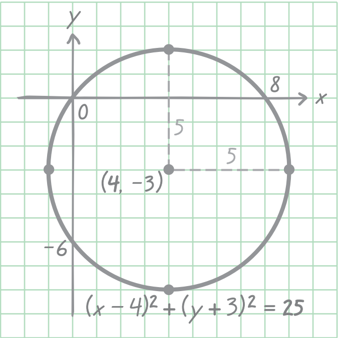


**FIGURE 11**

NOW TRY

**Examples 1 and 2** suggest the form of an equation of a circle with radius $r$ and center at $(h, k)$. If $(x, y)$ is a point on the circle, then the distance from the center $(h, k)$ to the point $(x, y)$ is $r$. By the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2} = r.$$

Squaring both sides gives the **center-radius form** of the equation of a circle.

### Equation of a Circle (Center-Radius Form)

An equation of a circle with radius $r$ and center $(h, k)$ is

$$\boldsymbol{(x-h)^2+(y-k)^2=r^2.}$$

**NOW TRY EXERCISE 3**

Find an equation of the circle with center at $(-5, 4)$ and radius $\sqrt{6}$.

### EXAMPLE 3 Using the Center-Radius Form of the Equation of a Circle

Find an equation of the circle with center at $(-1, 2)$ and radius $\sqrt{7}$.

$$(x-h)^2+(y-k)^2=r^2 \qquad \text{Center-radius form}$$

$$[x-(-1)]^2+(y-2)^2=\left(\sqrt{7}\right)^2 \qquad \text{Let } h=-1,\ k=2, \text{ and } r=\sqrt{7}.$$

Pay attention to signs here.

$$(x+1)^2+(y-2)^2=7 \qquad \text{Simplify; } \left(\sqrt{a}\right)^2=a$$

NOW TRY

**NOTE** If a circle has its center at the origin $(0, 0)$, then its equation becomes

$$(x-0)^2+(y-0)^2=r^2 \qquad \text{Let } h=0,\ k=0 \text{ in the center-radius form.}$$

$$\boldsymbol{x^2+y^2=r^2.} \qquad \text{See } \textbf{Example 1.}$$

**OBJECTIVE 2** Determine the center and radius of a circle given its equation.

In the equation found in **Example 2,** multiplying out $(x-4)^2$ and $(y+3)^2$ gives

$$(x-4)^2+(y+3)^2=25$$

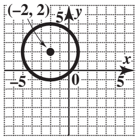
$$x^2-8x+16+y^2+6y+9=25 \qquad \text{Square each binomial.}$$

$$x^2+y^2-8x+6y=0. \qquad \text{Subtract 25.}$$

This general form suggests that an equation with both $x^2$- and $y^2$-terms with equal coefficients may represent a circle.

**NOW TRY ANSWERS**

**2.** $(x+2)^2+(y-2)^2=9$



**3.** $(x+5)^2+(y-4)^2=6$

NOW TRY EXERCISE 4

Find the center and radius of the circle.

$x^2 + y^2 - 8x + 10y - 8 = 0$

### EXAMPLE 4 Completing the Square to Find the Center and Radius

Find the center and radius of the circle $x^2 + y^2 + 2x + 6y - 15 = 0$, and graph it.

Since the equation has $x^2$- and $y^2$-terms with equal coefficients, its graph might be that of a circle. To find the center and radius, complete the squares on $x$ and $y$.

$$x^2 + y^2 + 2x + 6y = 15$$

Transform so that the constant is on the right.

$$(x^2 + 2x \qquad) + (y^2 + 6y \qquad) = 15$$

Write in anticipation of completing the square.

$$\left[\frac{1}{2}(2)\right]^2 = 1 \qquad \left[\frac{1}{2}(6)\right]^2 = 9$$

Square half the coefficient of each middle term.

$$(x^2 + 2x + 1) + (y^2 + 6y + 9) = 15 + 1 + 9$$

Complete the squares on both $x$ and $y$.

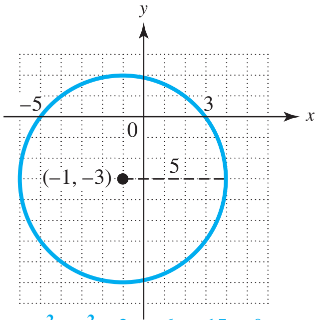
Add 1 and 9 on *both* sides of the equation.

$$(x + 1)^2 + (y + 3)^2 = 25$$

Factor on the left. Add on the right.

$$[x - (-1)]^2 + [y - (-3)]^2 = 5^2$$

Center-radius form



$x^2 + y^2 + 2x + 6y - 15 = 0$

FIGURE 12

The final equation shows that the graph is a circle with center at $(-1, -3)$ and radius 5, as shown in **FIGURE 12**.

NOW TRY

**NOTE** Consider the following.

**1.** If the procedure of **Example 4** leads to an equation of the form

$$(x - h)^2 + (y - k)^2 = 0,$$

then the graph is the single point $(h, k)$.

**2.** If the constant on the right side is *negative,* then the equation has *no graph.*

**OBJECTIVE 3 Recognize an equation of an ellipse.** An **ellipse** is the set of all points in a plane the *sum* of whose distances from two fixed points is constant. These fixed points are called **foci** (singular: *focus*). The ellipse in **FIGURE 13** has foci $(c, 0)$ and $(-c, 0)$, with $x$-intercepts $(a, 0)$ and $(-a, 0)$ and $y$-intercepts $(0, b)$ and $(0, -b)$. It is shown in more advanced courses that $c^2 = a^2 - b^2$ for an ellipse of this type. The origin is the **center** of the ellipse.

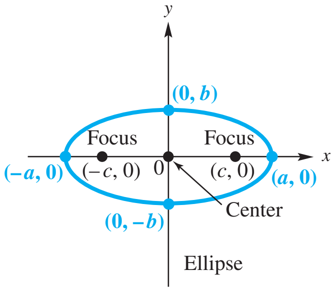


FIGURE 13

An ellipse has the following equation.

### Equation of an Ellipse

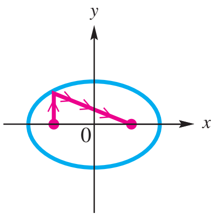
The ellipse whose $x$-intercepts are $(a, 0)$ and $(-a, 0)$ and whose $y$-intercepts are $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

**NOTE** A circle is a special case of an ellipse, where $a^2 = b^2$.

NOW TRY ANSWER

**4.** center: $(4, -5)$; radius: 7



Reflecting property
of an ellipse

**FIGURE 14**

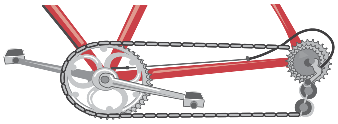


**FIGURE 15**

When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus. See **FIGURE 14**. As mentioned in the chapter introduction, this reflecting property is responsible for whispering galleries. John Quincy Adams was able to listen in on his opponents' conversations because his desk was positioned at one of the foci beneath the ellipsoidal ceiling and his opponents were located across the room at the other focus.

Elliptical bicycle gears are designed to respond to the legs' natural strengths and weaknesses. At the top and bottom of the powerstroke, where the legs have the least leverage, the gear offers little resistance, but as the gear rotates, the resistance increases. This allows the legs to apply more power where it is most naturally available. See **FIGURE 15**.

## OBJECTIVE 4 Graph ellipses.

### EXAMPLE 5 Graphing Ellipses

Graph each ellipse.

**(a)** $\frac{x^2}{49} + \frac{y^2}{36} = 1$

Here, $a^2 = 49$, so $a = 7$, and the $x$-intercepts are $(7, 0)$ and $(-7, 0)$. Similarly, $b^2 = 36$, so $b = 6$, and the $y$-intercepts are $(0, 6)$ and $(0, -6)$. Plotting the intercepts and sketching the ellipse through them gives the graph in **FIGURE 16**.

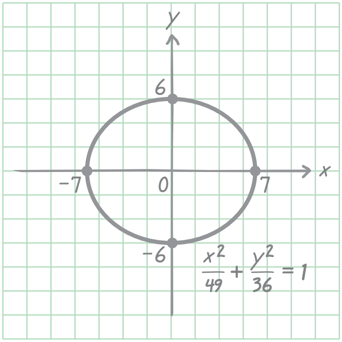


**FIGURE 16**

**(b)** $\frac{x^2}{36} + \frac{y^2}{121} = 1$

The $x$-intercepts are $(6, 0)$ and $(-6, 0)$, and the $y$-intercepts are $(0, 11)$ and $(0, -11)$. Join these with the smooth curve of an ellipse. See **FIGURE 17**.

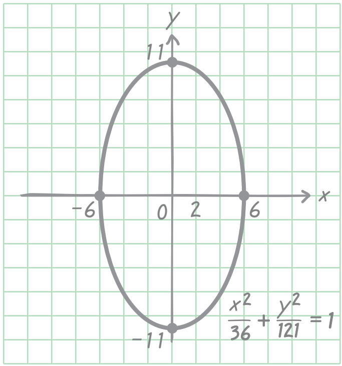


**FIGURE 17**

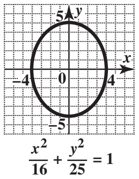
NOW TRY

**NOW TRY EXERCISE 5**

Graph $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

**NOW TRY ANSWER**

**5.**



NOW TRY EXERCISE 6

Graph

$$\frac{(x-3)^2}{36}+\frac{(y-4)^2}{4}=1.$$

## EXAMPLE 6 Graphing an Ellipse Shifted Horizontally and Vertically

Graph $\dfrac{(x-2)^2}{25}+\dfrac{(y+3)^2}{49}=1.$

Just as $(x-2)^2$ and $(y+3)^2$ would indicate that the center of a circle would be $(2, -3)$, so it is with this ellipse. **FIGURE 18** shows that the graph goes through the four points

$$(2, 4), \quad (7, -3), \quad (2, -10), \quad \text{and} \quad (-3, -3).$$

The $x$-values of these points are found by adding $\pm a = \pm 5$ to 2, and the $y$-values come from adding $\pm b = \pm 7$ to $-3$.

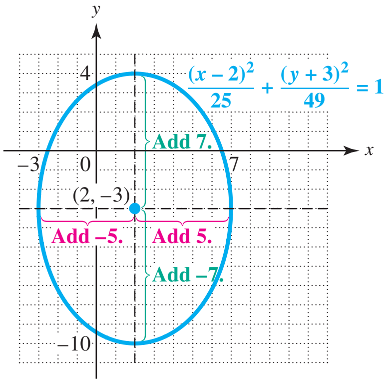


FIGURE 18

NOW TRY

**NOTE** ***Graphs of circles and ellipses are not graphs of functions.*** The only conic section whose graph represents a function is the vertical parabola with equation $f(x) = ax^2 + bx + c$.

## CONNECTIONS

A graphing calculator in function mode cannot directly graph a circle or an ellipse, since they do not represent functions. We must first solve the equation for $y$, getting two functions $y_1$ and $y_2$. The union of these two graphs is the graph of the entire figure.

For example, to graph $(x+3)^2+(y+2)^2=25$, begin by solving for $y$.

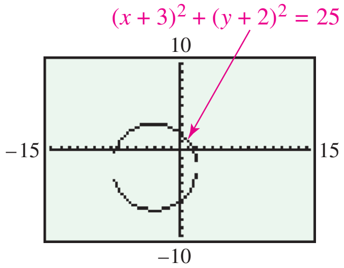
$$(x+3)^2+(y+2)^2=25$$

$$(y+2)^2=25-(x+3)^2 \qquad \text{Subtract } (x+3)^2.$$

$$y+2=\pm\sqrt{25-(x+3)^2} \qquad \text{Take square roots.}$$

Remember both roots.

$$y=-2\pm\sqrt{25-(x+3)^2} \qquad \text{Add } -2.$$



Square Viewing Window

FIGURE 19

The two functions to be graphed are

$$y_1=-2+\sqrt{25-(x+3)^2} \quad \text{and} \quad y_2=-2-\sqrt{25-(x+3)^2}.$$

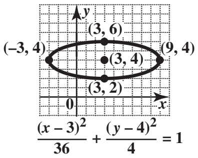
To get an undistorted screen, a **square viewing window** must be used. (Refer to your instruction manual for details.) See **FIGURE 19**. The two semicircles seem to be disconnected. This is because the graphs are nearly vertical at those points, and the calculator cannot show a true picture of the behavior there.

### For Discussion or Writing

Find the two functions $y_1$ and $y_2$ to use to obtain the graph of the circle with equation $(x-3)^2+(y+1)^2=36$. Then graph the circle using a square viewing window.

NOW TRY ANSWER

6.



## 13.2 EXERCISES

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1. **See Example 1.** Consider the circle whose equation is $x^2 + y^2 = 25$.
   **(a)** What are the coordinates of its center? **(b)** What is its radius?
   **(c)** Sketch its graph.

2. Why does a set of points defined by a circle *not* satisfy the definition of a function?

*Concept Check* *Match each equation with the correct graph.*

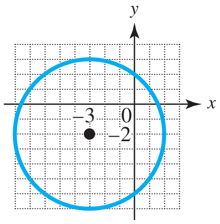
3. $(x - 3)^2 + (y - 2)^2 = 25$

4. $(x - 3)^2 + (y + 2)^2 = 25$

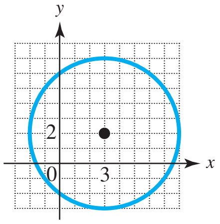
5. $(x + 3)^2 + (y - 2)^2 = 25$

6. $(x + 3)^2 + (y + 2)^2 = 25$

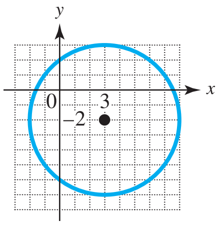
**A.**



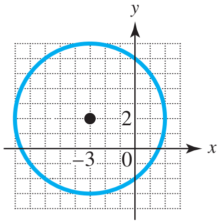
**B.**



**C.**



**D.**



*Find the equation of a circle satisfying the given conditions.* ***See Examples 2 and 3.***

7. Center: $(-4, 3)$; radius: 2

8. Center: $(5, -2)$; radius: 4

9. Center: $(-8, -5)$; radius: $\sqrt{5}$

10. Center: $(-12, 13)$; radius: $\sqrt{7}$

*Find the center and radius of each circle. (Hint: In Exercises 15 and 16, divide each side by a common factor.)* ***See Example 4.***

11. $x^2 + y^2 + 4x + 6y + 9 = 0$

12. $x^2 + y^2 - 8x - 12y + 3 = 0$

13. $x^2 + y^2 + 10x - 14y - 7 = 0$

14. $x^2 + y^2 - 2x + 4y - 4 = 0$

15. $3x^2 + 3y^2 - 12x - 24y + 12 = 0$

16. $2x^2 + 2y^2 + 20x + 16y + 10 = 0$

*Graph each circle. Identify the center if it is not at the origin.* ***See Examples 1, 2, and 4.***

17. $x^2 + y^2 = 9$

18. $x^2 + y^2 = 4$

19. $2y^2 = 10 - 2x^2$

20. $3x^2 = 48 - 3y^2$

21. $(x + 3)^2 + (y - 2)^2 = 9$

22. $(x - 1)^2 + (y + 3)^2 = 16$

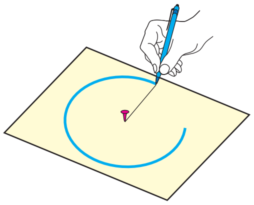
23. $x^2 + y^2 - 4x - 6y + 9 = 0$

24. $x^2 + y^2 + 8x + 2y - 8 = 0$

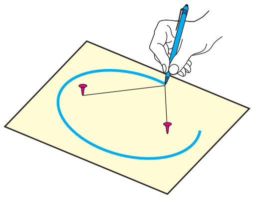
25. $x^2 + y^2 + 6x - 6y + 9 = 0$

26. $x^2 + y^2 - 4x + 10y + 20 = 0$

27. A circle can be drawn on a piece of posterboard by fastening one end of a string with a thumbtack, pulling the string taut with a pencil, and tracing a curve, as shown in the figure. Explain why this method works.



**28.** An ellipse can be drawn on a piece of posterboard by fastening two ends of a length of string with thumbtacks, pulling the string taut with a pencil, and tracing a curve, as shown in the figure. Explain why this method works.



*Graph each ellipse.* ***See Examples 5 and 6.***

**29.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$

**30.** $\frac{x^2}{9} + \frac{y^2}{16} = 1$

**31.** $\frac{x^2}{36} + \frac{y^2}{16} = 1$

**32.** $\frac{x^2}{9} + \frac{y^2}{4} = 1$

**33.** $\frac{x^2}{16} + \frac{y^2}{4} = 1$

**34.** $\frac{x^2}{49} + \frac{y^2}{81} = 1$

**35.** $\frac{y^2}{25} = 1 - \frac{x^2}{49}$

**36.** $\frac{y^2}{9} = 1 - \frac{x^2}{16}$

**37.** $\frac{(x+1)^2}{64} + \frac{(y-2)^2}{49} = 1$

**38.** $\frac{(x-4)^2}{9} + \frac{(y+2)^2}{4} = 1$

**39.** $\frac{(x-2)^2}{16} + \frac{(y-1)^2}{9} = 1$

**40.** $\frac{(x+3)^2}{25} + \frac{(y+2)^2}{36} = 1$

**41.** Explain why a set of ordered pairs whose graph forms an ellipse does not satisfy the definition of a function.

**42.** **(a)** How many points are there on the graph of $(x-4)^2 + (y-1)^2 = 0$? Explain.

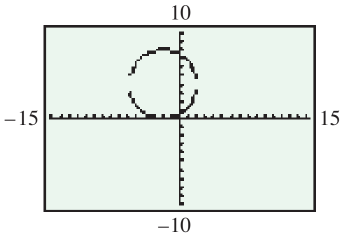
**(b)** How many points are there on the graph of $(x-4)^2 + (y-1)^2 = -1$? Explain.

## *TECHNOLOGY INSIGHTS* EXERCISES 43 AND 44

**43.** The circle shown in the calculator graph was created using function mode, with a square viewing window. It is the graph of

$$(x+2)^2 + (y-4)^2 = 16.$$

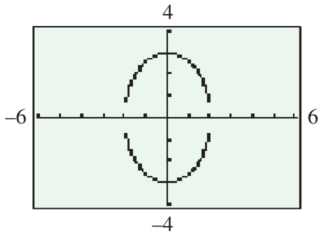
What are the two functions $y_1$ and $y_2$ that were used to obtain this graph?



**44.** The ellipse shown in the calculator graph was graphed using function mode, with a square viewing window. It is the graph of

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

What are the two functions $y_1$ and $y_2$ that were used to obtain this graph?



*Use a graphing calculator in function mode to graph each circle or ellipse. Use a square viewing window.* ***See the Connections box.***

**45.** $x^2 + y^2 = 36$

**46.** $(x-2)^2 + y^2 = 49$

**47.** $\frac{x^2}{16} + \frac{y^2}{4} = 1$

**48.** $\frac{(x-3)^2}{25} + \frac{y^2}{9} = 1$

*A **lithotripter** is a machine used to crush kidney stones using shock waves. The patient is placed in an elliptical tub with the kidney stone at one focus of the ellipse. A beam is projected from the other focus to the tub, so that it reflects to hit the kidney stone. See the figure.*

**49.** Suppose a lithotripter is based on the ellipse with equation

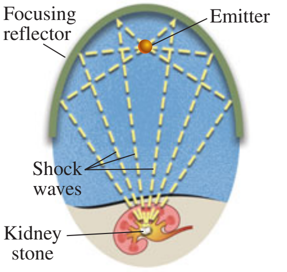
$$\frac{x^2}{36} + \frac{y^2}{9} = 1.$$

How far from the center of the ellipse must the kidney stone and the source of the beam be placed? (*Hint*: Use the fact that $c^2 = a^2 - b^2$, since $a > b$ here.)

**50.** Rework **Exercise 49** if the equation of the ellipse is

$$9x^2 + 4y^2 = 36.$$

(*Hint:* Write the equation in fractional form by dividing each term by 36, and use $c^2 = b^2 - a^2$, since $b > a$ here.)



The top of an ellipse is illustrated in this depiction of how a lithotripter crushes a kidney stone.

*Solve each problem.*

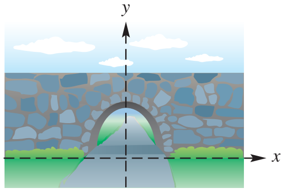
**51.** An arch has the shape of half an ellipse. The equation of the ellipse is

$$100x^2 + 324y^2 = 32{,}400,$$

where $x$ and $y$ are in meters.

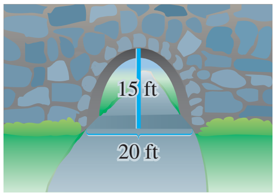
**(a)** How high is the center of the arch?

**(b)** How wide is the arch across the bottom?



**NOT TO SCALE**

**52.** A one-way street passes under an overpass, which is in the form of the top half of an ellipse, as shown in the figure. Suppose that a truck 12 ft wide passes directly under the overpass. What is the maximum possible height of this truck?



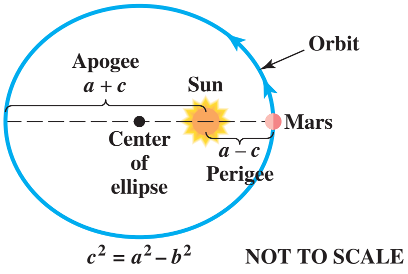
**NOT TO SCALE**

*In Exercises 53 and 54, see* **FIGURE 13** *and use the fact that $c^2 = a^2 - b^2$, where $a^2 > b^2$.*

**53.** The orbit of Mars is an ellipse with the sun at one focus. For $x$ and $y$ in millions of miles, the equation of the orbit is

$$\frac{x^2}{141.7^2} + \frac{y^2}{141.1^2} = 1.$$

(*Source:* Kaler, James B., *Astronomy!*, Addison-Wesley.)



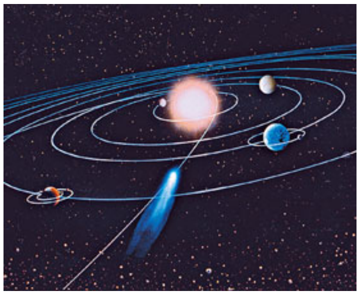
**(a)** Find the greatest distance (the **apogee**) from Mars to the sun.

**(b)** Find the least distance (the **perigee**) from Mars to the sun.

**54.** The orbit of Venus around the sun (one of the foci) is an ellipse with equation

$$\frac{x^2}{5013} + \frac{y^2}{4970} = 1,$$

where $x$ and $y$ are measured in millions of miles. (*Source:* Kaler, James B., *Astronomy!*, Addison-Wesley.)



**(a)** Find the greatest distance between Venus and the sun.

**(b)** Find the least distance between Venus and the sun.

## PREVIEW EXERCISES

*For Exercises 55–57,* ***see Sections 3.1 and 7.1.***

**55.** Plot the points $(3, 4)$, $(-3, 4)$, $(3, -4)$, and $(-3, -4)$.

**56.** Sketch the graphs of $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$ on the same axes.

**57.** Find the $x$- and $y$-intercepts of the graph of $4x + 3y = 12$.

**58.** Solve the equation $x^2 = 121$. **See Section 11.1.**

# 13.3

# The Hyperbola and Functions Defined by Radicals

## OBJECTIVES

- **1** Recognize the equation of a hyperbola.
- **2** Graph hyperbolas by using asymptotes.
- **3** Identify conic sections by their equations.
- **4** Graph certain square root functions.

**OBJECTIVE 1 Recognize the equation of a hyperbola.** A **hyperbola** is the set of all points in a plane such that the absolute value of the *difference* of the distances from two fixed points (the *foci*) is constant. The graph of a hyperbola has two parts, called *branches,* and two intercepts (or *vertices*) that lie on its axis, called the **transverse axis.** The hyperbola in **FIGURE 20** has a horizontal transverse axis, with foci $(c, 0)$ and $(-c, 0)$ and $x$-intercepts $(a, 0)$ and $(-a, 0)$. (A hyperbola with vertical transverse axis would have its intercepts on the $y$-axis.)

A hyperbola centered at the origin has one of the following equations. It is shown in more advanced courses that for a hyperbola, $c^2 = a^2 + b^2$.

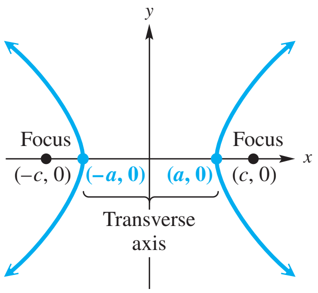


**FIGURE 20**

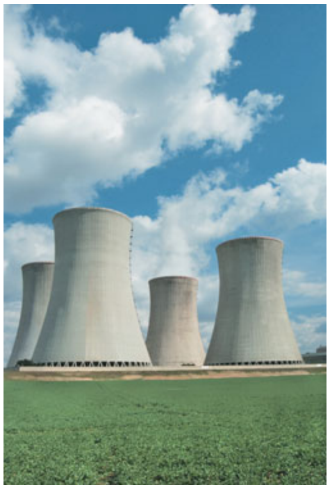
### Equations of Hyperbolas

A hyperbola with $x$-intercepts $(a, 0)$ and $(-a, 0)$ has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \text{Transverse axis on } x\text{-axis}$$

A hyperbola with $y$-intercepts $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1. \qquad \text{Transverse axis on } y\text{-axis}$$



If we were to throw two stones into a pond, the ensuing concentric ripples would be shaped like a hyperbola. A cross-section of the cooling towers for a nuclear power plant is hyperbolic, as shown in the photo.

**OBJECTIVE 2 Graph hyperbolas by using asymptotes.** The two branches of the graph of a hyperbola approach a pair of intersecting straight lines, which are its asymptotes. See **FIGURE 21** on the next page. The asymptotes are useful for sketching the graph of the hyperbola.

### Asymptotes of Hyperbolas

The extended diagonals of the rectangle with vertices (corners) at the points $(\boldsymbol{a}, \boldsymbol{b})$, $(-\boldsymbol{a}, \boldsymbol{b})$, $(-\boldsymbol{a}, -\boldsymbol{b})$, and $(\boldsymbol{a}, -\boldsymbol{b})$ are the **asymptotes** of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This rectangle is called the **fundamental rectangle.** Using the methods of **Chapter 3 or Chapter 7,** we could show that the equations of these asymptotes are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

Equations of the asymptotes of a hyperbola

To graph hyperbolas, follow these steps.

### Graphing a Hyperbola

***Step 1*** **Find the intercepts.** Locate the intercepts at $(a, 0)$ and $(-a, 0)$ if the $x^2$-term has a positive coefficient, or at $(0, b)$ and $(0, -b)$ if the $y^2$-term has a positive coefficient.

***Step 2*** **Find the fundamental rectangle.** Locate the vertices of the fundamental rectangle at $(a, b)$, $(-a, b)$, $(-a, -b)$, and $(a, -b)$.

***Step 3*** **Sketch the asymptotes.** The extended diagonals of the rectangle are the asymptotes of the hyperbola, and they have equations $y = \pm \frac{b}{a}x$.

***Step 4*** **Draw the graph.** Sketch each branch of the hyperbola through an intercept and approaching (but not touching) the asymptotes.

NOW TRY EXERCISE 1

Graph $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

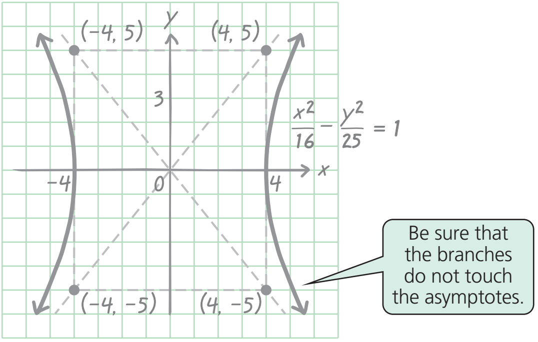
### EXAMPLE 1 Graphing a Horizontal Hyperbola

Graph $\frac{x^2}{16} - \frac{y^2}{25} = 1$.

***Step 1*** Here $a = 4$ and $b = 5$. The $x$-intercepts are $(4, 0)$ and $(-4, 0)$.

***Step 2*** The four points $(4, 5)$, $(-4, 5)$, $(-4, -5)$, and $(4, -5)$ are the vertices of the fundamental rectangle, as shown in **FIGURE 21** below.

***Steps 3 and 4*** The equations of the asymptotes are $y = \pm \frac{5}{4}x$, and the hyperbola approaches these lines as $x$ and $y$ get larger and larger in absolute value. NOW TRY



Be sure that the branches do not touch the asymptotes.

FIGURE 21

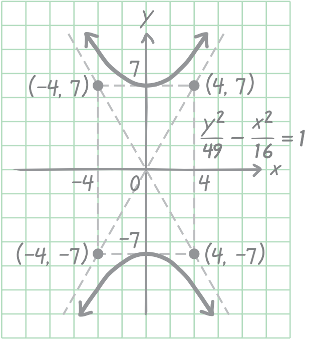


FIGURE 22

NOW TRY EXERCISE 2

Graph $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

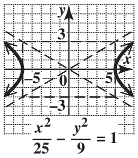
### EXAMPLE 2 Graphing a Vertical Hyperbola

Graph $\frac{y^2}{49} - \frac{x^2}{16} = 1$.

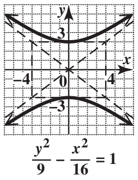
This hyperbola has $y$-intercepts $(0, 7)$ and $(0, -7)$. The asymptotes are the extended diagonals of the rectangle with vertices at $(4, 7)$, $(-4, 7)$, $(-4, -7)$, and $(4, -7)$. Their equations are $y = \pm \frac{7}{4}x$. See **FIGURE 22** above. NOW TRY

NOW TRY ANSWERS

1.

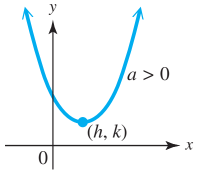
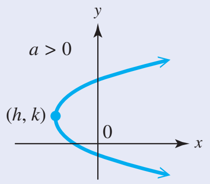
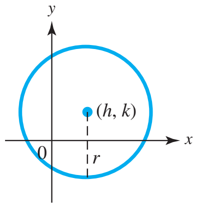
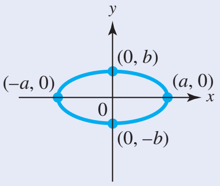
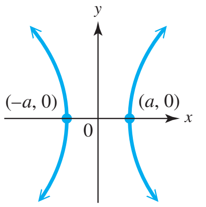
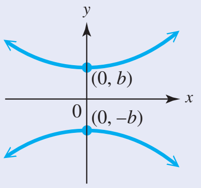


2.



**NOTE** As with circles and ellipses, hyperbolas are graphed with a graphing calculator by first writing the equations of two functions whose union is equivalent to the equation of the hyperbola. A square window gives a truer shape for hyperbolas, too.

## SUMMARY OF CONIC SECTIONS

| *Equation* | *Graph* | *Description* | *Identification* |
|---|---|---|---|
| $y = ax^2 + bx + c$<br>**or**<br>$y = a(x - h)^2 + k$ |  Parabola | It opens up if $a > 0$, down if $a < 0$. The vertex is $(h, k)$. | It has an $x^2$-term. $y$ is not squared. |
| $x = ay^2 + by + c$<br>**or**<br>$x = a(y - k)^2 + h$ |  Parabola | It opens to the right if $a > 0$, to the left if $a < 0$. The vertex is $(h, k)$. | It has a $y^2$-term. $x$ is not squared. |
| $(x - h)^2 + (y - k)^2 = r^2$ |  Circle | The center is $(h, k)$, and the radius is $r$. | $x^2$- and $y^2$-terms have the same positive coefficient. |
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ |  Ellipse | The $x$-intercepts are $(a, 0)$ and $(-a, 0)$. The $y$-intercepts are $(0, b)$ and $(0, -b)$. | $x^2$- and $y^2$-terms have different positive coefficients. |
| $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ |  Hyperbola | The $x$-intercepts are $(a, 0)$ and $(-a, 0)$. The asymptotes are found from $(a, b)$, $(a, -b)$, $(-a, -b)$, and $(-a, b)$. | $x^2$ has a positive coefficient. $y^2$ has a negative coefficient. |
| $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ |  Hyperbola | The $y$-intercepts are $(0, b)$ and $(0, -b)$. The asymptotes are found from $(a, b)$, $(a, -b)$, $(-a, -b)$, and $(-a, b)$. | $y^2$ has a positive coefficient. $x^2$ has a negative coefficient. |

**OBJECTIVE 3** **Identify conic sections by their equations.** Rewriting a second-degree equation in one of the forms given for ellipses, hyperbolas, circles, or parabolas makes it possible to identify the graph of the equation.

*NOW TRY EXERCISE 3*

Identify the graph of each equation.

**(a)** $y^2 - 10 = -x^2$

**(b)** $y - 2x^2 = 8$

**(c)** $3x^2 + y^2 = 4$

## EXAMPLE 3 Identifying the Graphs of Equations

Identify the graph of each equation.

**(a)** $9x^2 = 108 + 12y^2$

Both variables are squared, so the graph is either an ellipse or a hyperbola. (This situation also occurs for a circle, which is a special case of an ellipse.) Rewrite the equation so that the $x^2$- and $y^2$-terms are on one side of the equation and 1 is on the other.

$$9x^2 - 12y^2 = 108 \qquad \text{Subtract } 12y^2.$$

$$\frac{x^2}{12} - \frac{y^2}{9} = 1 \qquad \text{Divide by 108.}$$

The graph of this equation is a hyperbola.

**(b)** $x^2 = y - 3$

Only one of the two variables, $x$, is squared, so this is the vertical parabola $y = x^2 + 3$.

**(c)** $x^2 = 9 - y^2$

Write the variable terms on the same side of the equation.

$$x^2 + y^2 = 9 \qquad \text{Add } y^2.$$

The graph of this equation is a circle with center at the origin and radius 3.

*NOW TRY*

**OBJECTIVE 4** **Graph certain square root functions.** Recall from the vertical line test that no vertical line will intersect the graph of a function in more than one point. Thus, the graphs of horizontal parabolas, all circles and ellipses, and most hyperbolas discussed in this chapter do not satisfy the conditions of a function. However, by considering only a part of each graph, we have the graph of a function, as seen in **FIGURE 23**.

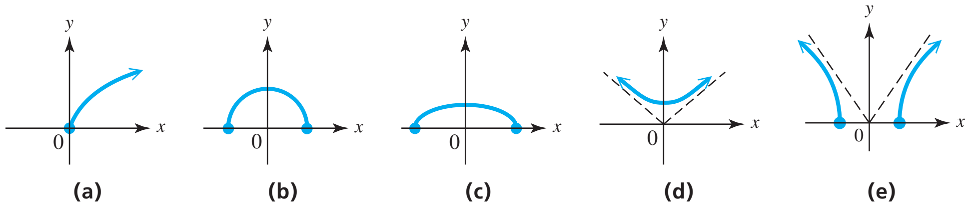


**FIGURE 23**

In parts (a)–(d) of **FIGURE 23**, the top portion of a conic section is shown (parabola, circle, ellipse, and hyperbola, respectively). In part (e), the top two portions of a hyperbola are shown. In each case, the graph is that of a function since the graph satisfies the conditions of the vertical line test.

In **Sections 10.1 and 13.1,** we observed the square root function defined by $f(x) = \sqrt{x}$. To find equations for the types of graphs shown in **FIGURE 23**, we extend its definition.

*NOW TRY ANSWERS*

3\. **(a)** circle **(b)** parabola
**(c)** ellipse

**Generalized Square Root Function**

For an algebraic expression in $x$ defined by $u$, with $u \geq 0$, a function of the form

$$f(x) = \sqrt{u}$$

is a **generalized square root function.**

**NOW TRY EXERCISE 4**

Graph $f(x) = \sqrt{64 - x^2}$. Give the domain and range.

## EXAMPLE 4 Graphing a Semicircle

Graph $f(x) = \sqrt{25 - x^2}$. Give the domain and range.

$f(x) = \sqrt{25 - x^2}$ Given function

$y = \sqrt{25 - x^2}$ Replace $f(x)$ with $y$.

$\left(\sqrt{a}\right)^2 = a$

$y^2 = 25 - x^2$ Square each side.

$x^2 + y^2 = 25$ Add $x^2$.

This is the graph of a circle with center at $(0, 0)$ and radius 5. Since $f(x)$, or $y$, represents a principal square root in the original equation, $f(x)$ must be nonnegative. This restricts the graph to the upper half of the circle, as shown in **FIGURE 24**. The domain is $[-5, 5]$, and the range is $[0, 5]$.

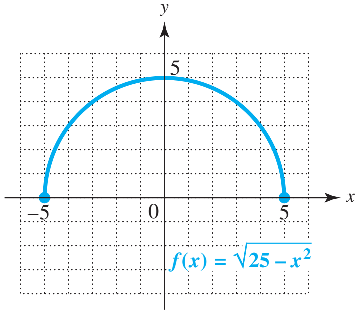


**FIGURE 24**

NOW TRY

**NOW TRY EXERCISE 5**

Graph $\frac{y}{4} = -\sqrt{1 - \frac{x^2}{9}}$.

Give the domain and range.

## EXAMPLE 5 Graphing a Portion of an Ellipse

Graph $\frac{y}{6} = -\sqrt{1 - \frac{x^2}{16}}$. Give the domain and range.

Square each side to get an equation whose form is known.

$$\left(\frac{y}{6}\right)^2 = \left(-\sqrt{1 - \frac{x^2}{16}}\right)^2 \quad \text{Square each side.}$$

$$\frac{y^2}{36} = 1 - \frac{x^2}{16} \quad \text{Apply the exponents.}$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1 \quad \text{Add } \tfrac{x^2}{16}.$$

This is the equation of an ellipse with $x$-intercepts $(4, 0)$ and $(-4, 0)$ and $y$-intercepts $(0, 6)$ and $(0, -6)$. *Since $\frac{y}{6}$ equals a negative square root in the original equation, $y$ must be nonpositive, restricting the graph to the lower half of the ellipse,* as shown in **FIGURE 25**. The domain is $[-4, 4]$, and the range is $[-6, 0]$.

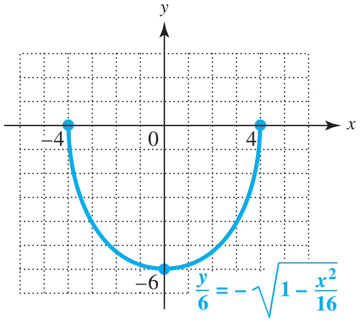


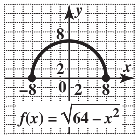
**FIGURE 25**

NOW TRY

**NOTE** Root functions like those graphed in **FIGURES 24 and 25**, can be entered and graphed directly with a graphing calculator.

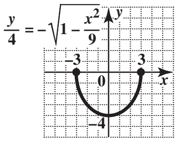
**NOW TRY ANSWERS**

**4.**



domain: $[-8, 8]$; range: $[0, 8]$

**5.**



domain: $[-3, 3]$; range: $[-4, 0]$

# 13.3 EXERCISES

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***Concept Check*** *Based on the discussions of ellipses in the previous section and of hyperbolas in this section, match each equation with its graph.*

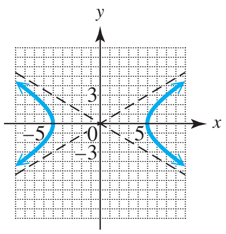
**1.** $\frac{x^2}{25} + \frac{y^2}{9} = 1$

**2.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$

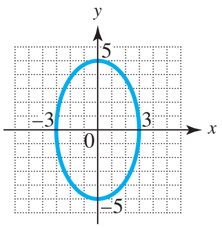
**3.** $\frac{x^2}{9} - \frac{y^2}{25} = 1$

**4.** $\frac{x^2}{25} - \frac{y^2}{9} = 1$

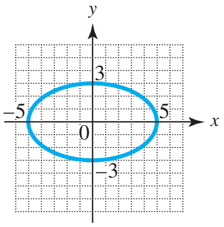
**A.**



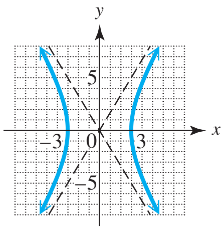
**B.**



**C.**



**D.**



*Graph each hyperbola.* ***See Examples 1 and 2.***

◉ **5.** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

**6.** $\frac{x^2}{25} - \frac{y^2}{9} = 1$

◉ **7.** $\frac{y^2}{4} - \frac{x^2}{25} = 1$

**8.** $\frac{y^2}{9} - \frac{x^2}{4} = 1$

**9.** $\frac{x^2}{25} - \frac{y^2}{36} = 1$

**10.** $\frac{x^2}{49} - \frac{y^2}{16} = 1$

**11.** $\frac{y^2}{16} - \frac{x^2}{16} = 1$

**12.** $\frac{y^2}{9} - \frac{x^2}{9} = 1$

*Identify the graph of each equation as a* parabola, circle, ellipse, *or* hyperbola, *and then sketch the graph.* ***See Example 3.***

**13.** $x^2 - y^2 = 16$

**14.** $x^2 + y^2 = 16$

◉ **15.** $4x^2 + y^2 = 16$

**16.** $9x^2 = 144 + 16y^2$

**17.** $y^2 = 36 - x^2$

**18.** $9x^2 + 25y^2 = 225$

**19.** $x^2 - 2y = 0$

**20.** $x^2 + 9y^2 = 9$

**21.** $y^2 = 4 + x^2$

✎ **22.** State in your own words the major difference between the definitions of *ellipse* and *hyperbola*.

*Graph each generalized square root function. Give the domain and range.* ***See Examples 4 and 5.***

◉ **23.** $f(x) = \sqrt{16 - x^2}$

**24.** $f(x) = \sqrt{9 - x^2}$

◉ **25.** $f(x) = -\sqrt{36 - x^2}$

**26.** $f(x) = -\sqrt{25 - x^2}$

**27.** $y = -2\sqrt{1 - \frac{x^2}{9}}$

**28.** $y = -3\sqrt{1 - \frac{x^2}{25}}$

**29.** $\frac{y}{3} = \sqrt{1 + \frac{x^2}{9}}$

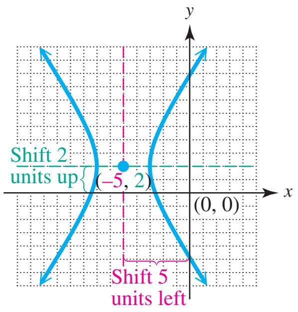
**30.** $\frac{y}{2} = \sqrt{1 + \frac{x^2}{4}}$

*In **Section 13.2, Example 6,** we saw that the center of an ellipse may be shifted away from the origin. The same process applies to hyperbolas. For example, the hyperbola shown at the right,*

$$\frac{(x+5)^2}{4} - \frac{(y-2)^2}{9} = 1,$$

*has the same graph as*

$$\frac{x^2}{4} - \frac{y^2}{9} = 1,$$



*but it is centered at* $(-5, 2)$. *Graph each hyperbola with center shifted away from the origin.*

**31.** $\dfrac{(x-2)^2}{4} - \dfrac{(y+1)^2}{9} = 1$

**32.** $\dfrac{(x+3)^2}{16} - \dfrac{(y-2)^2}{25} = 1$

**33.** $\dfrac{y^2}{36} - \dfrac{(x-2)^2}{49} = 1$

**34.** $\dfrac{(y-5)^2}{9} - \dfrac{x^2}{25} = 1$

*Solve each problem.*

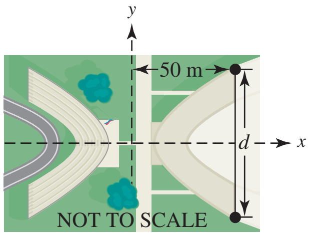
**35.** Two buildings in a sports complex are shaped and positioned like a portion of the branches of the hyperbola with equation

$$400x^2 - 625y^2 = 250{,}000,$$

where $x$ and $y$ are in meters.

**(a)** How far apart are the buildings at their closest point?

**(b)** Find the distance $d$ in the figure.



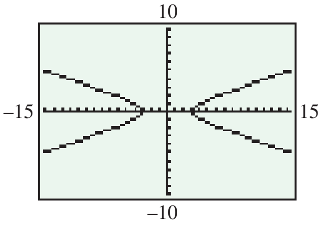
**36.** In rugby, after a *try* (similar to a touchdown in American football) the scoring team attempts a kick for extra points. The ball must be kicked from directly behind the point where the try was scored. The kicker can choose the distance but cannot move the ball sideways. It can be shown that the kicker's best choice is on the hyperbola with equation

$$\frac{x^2}{g^2} - \frac{y^2}{g^2} = 1,$$

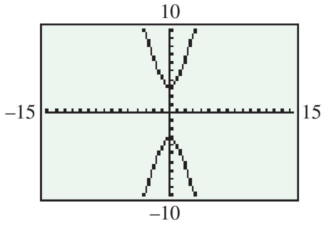
where $2g$ is the distance between the goal posts. Since the hyperbola approaches its asymptotes, it is easier for the kicker to estimate points on the asymptotes instead of on the hyperbola. What are the asymptotes of this hyperbola? Why is it relatively easy to estimate them? (*Source:* Isaksen, Daniel C., "How to Kick a Field Goal," *The College Mathematics Journal*.)

## TECHNOLOGY INSIGHTS EXERCISES 37 AND 38

**37.** The hyperbola shown in the figure was graphed in function mode, with a square viewing window. It is the graph of $\frac{x^2}{9} - y^2 = 1$. What are the two functions $y_1$ and $y_2$ that were used to obtain this graph?



**38.** Repeat **Exercise 37** for the graph of $\frac{y^2}{9} - x^2 = 1$, shown in the figure.



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*Use a graphing calculator in function mode to graph each hyperbola. Use a square viewing window.*

**39.** $\frac{x^2}{25} - \frac{y^2}{49} = 1$ **40.** $\frac{x^2}{4} - \frac{y^2}{16} = 1$ **41.** $y^2 - 9x^2 = 9$ **42.** $y^2 - 9x^2 = 36$

## PREVIEW EXERCISES

*Solve each system.* ***See Sections 8.1–8.3.***

**43.** $2x + y = 13$
$y = 3x + 3$

**44.** $9x + 2y = 10$
$x - y = -5$

**45.** $4x - 3y = -10$
$4x + 6y = 8$

**46.** $5x + 7y = 6$
$10x - 3y = 46$

*Solve each equation.* ***See Section 11.4.***

**47.** $2x^4 - 5x^2 - 3 = 0$ **48.** $x^4 - 7x^2 + 12 = 0$

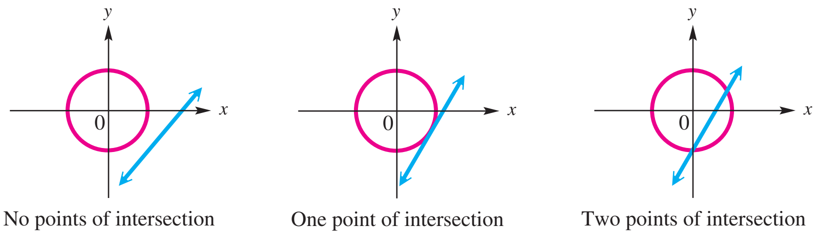
# 13.4 Nonlinear Systems of Equations

**OBJECTIVES**

1. Solve a nonlinear system by substitution.
2. Solve a nonlinear system by elimination.
3. Solve a nonlinear system that requires a combination of methods.

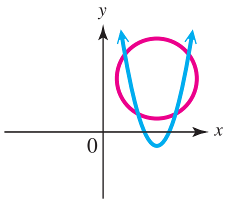
An equation in which some terms have more than one variable or a variable of degree 2 or greater is called a **nonlinear equation.** A **nonlinear system of equations** includes at least one nonlinear equation.

When solving a nonlinear system, it helps to visualize the types of graphs of the equations of the system to determine the possible number of points of intersection. For example, if a system includes two equations where the graph of one is a circle and the graph of the other is a line, then there may be zero, one, or two points of intersection, as illustrated in **FIGURE 26**.



No points of intersection One point of intersection Two points of intersection

**FIGURE 26**



This system has four solutions, since there are four points of intersection.

**FIGURE 27**

If a system consists of two second-degree equations, then there may be zero, one, two, three, or four solutions. **FIGURE 27** shows a case where a system consisting of a circle and a parabola has four solutions, all made up of ordered pairs of real numbers.

**OBJECTIVE 1** **Solve a nonlinear system by substitution.** We can usually solve a nonlinear system by the substitution method (**Section 8.2**) when one equation is linear.

### EXAMPLE 1 Solving a Nonlinear System by Substitution

Solve the system.

$$x^2 + y^2 = 9 \qquad (1)$$
$$2x - y = 3 \qquad (2)$$

The graph of (1) is a circle and the graph of (2) is a line, so the graphs could intersect in zero, one, or two points, as in **FIGURE 26** on the preceding page. We solve the linear equation (2) for one of the two variables and then substitute the resulting expression into the nonlinear equation.

$$\begin{aligned} 2x - y &= 3 && (2) \\ y &= 2x - 3 && \text{Solve for } y. \quad (3) \end{aligned}$$

Substitute $2x - 3$ for $y$ in equation (1).

$$\begin{aligned} x^2 + y^2 &= 9 && (1) \\ x^2 + (2x - 3)^2 &= 9 && \text{Let } y = 2x - 3. \\ x^2 + 4x^2 - 12x + 9 &= 9 && \text{Square } 2x - 3. \\ 5x^2 - 12x &= 0 && \text{Combine like terms. Subtract 9.} \\ x(5x - 12) &= 0 && \text{Factor. The GCF is } x. \\ x = 0 \quad \text{or} \quad 5x - 12 &= 0 && \text{Zero-factor property} \\ x &= \frac{12}{5} \end{aligned}$$

Set *both* factors equal to 0.

Let $x = 0$ in equation (3) to get $y = -3$. If $x = \frac{12}{5}$, then $y = \frac{9}{5}$. The solution set of the system is

$$\left\{(0, -3), \left(\frac{12}{5}, \frac{9}{5}\right)\right\}.$$

See the graph in **FIGURE 28**.

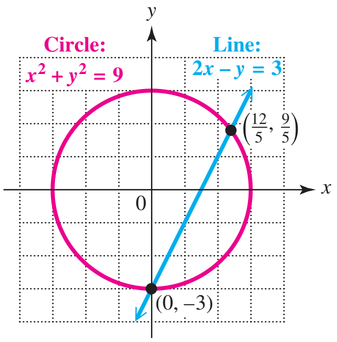


**FIGURE 28**

NOW TRY

NOW TRY EXERCISE 1

Solve the system.

$$4x^2 + y^2 = 36$$
$$x - y = 3$$

### EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system.

$$6x - y = 5 \qquad (1)$$
$$xy = 4 \qquad (2)$$

The graph of (1) is a line. It can be shown by plotting points that the graph of (2) is a hyperbola. Visualizing a line and a hyperbola indicates that there may be zero, one, or two points of intersection.

NOW TRY ANSWER

1. $\left\{(3, 0), \left(-\frac{9}{5}, -\frac{24}{5}\right)\right\}$

**NOW TRY EXERCISE 2**

Solve the system.

$$\begin{aligned} xy &= 2 \\ x - 3y &= 1 \end{aligned}$$

Since neither equation has a squared term, we can solve either equation for one of the variables and then substitute the result into the other equation. Solving $xy = 4$ for $x$ gives $x = \frac{4}{y}$. We substitute $\frac{4}{y}$ for $x$ in equation (1).

$$\begin{aligned} 6x - y &= 5 && \text{(1)} \\ 6\left(\frac{4}{y}\right) - y &= 5 && \text{Let } x = \tfrac{4}{y}. \\ \frac{24}{y} - y &= 5 && \text{Multiply.} \\ 24 - y^2 &= 5y && \text{Multiply by } y,\ y \neq 0. \\ y^2 + 5y - 24 &= 0 && \text{Standard form} \\ (y-3)(y+8) &= 0 && \text{Factor.} \\ y = 3 \quad \text{or} \quad y &= -8 && \text{Zero-factor property} \end{aligned}$$

We substitute these results into $x = \frac{4}{y}$ to obtain the corresponding values of $x$.

If $y = 3$, then $x = \frac{4}{3}$.

If $y = -8$, then $x = -\frac{1}{2}$.

The solution set of the system is

$$\left\{\left(\frac{4}{3}, 3\right), \left(-\frac{1}{2}, -8\right)\right\}.$$

See the graph in **FIGURE 29**.

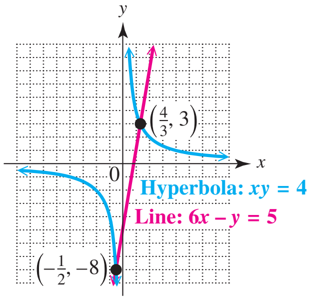


**FIGURE 29**

**NOW TRY**

**OBJECTIVE 2** **Solve a nonlinear system by elimination.** We can often use the elimination method **(Section 8.3)** when both equations of a nonlinear system are second degree.

### EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system.

$$\begin{aligned} x^2 + y^2 &= 9 && \text{(1)} \\ 2x^2 - y^2 &= -6 && \text{(2)} \end{aligned}$$

The graph of (1) is a circle, while the graph of (2) is a hyperbola. By analyzing the possibilities, we conclude that there may be zero, one, two, three, or four points of intersection. Adding the two equations will eliminate $y$.

$$\begin{aligned} x^2 + y^2 &= 9 && \text{(1)} \\ \underline{2x^2 - y^2} &\underline{= -6} && \text{(2)} \\ 3x^2 \qquad &= 3 && \text{Add.} \\ x^2 &= 1 && \text{Divide by 3.} \\ x = 1 \quad \text{or} \quad x &= -1 && \text{Square root property} \end{aligned}$$

**NOW TRY ANSWER**

**2.** $\left\{(-2, -1), \left(3, \frac{2}{3}\right)\right\}$

*NOW TRY EXERCISE 3*

Solve the system.

$$\begin{aligned} x^2 + \quad y^2 &= 16 \\ 4x^2 + 13y^2 &= 100 \end{aligned}$$

Each value of $x$ gives corresponding values for $y$ when substituted into one of the original equations. Using equation (1) gives the following.

$$\begin{aligned} x^2 + y^2 &= 9 && \text{(1)} \\ 1^2 + y^2 &= 9 && \text{Let } x = 1. \\ y^2 &= 8 \\ y = \sqrt{8} \quad &\text{or} \quad y = -\sqrt{8} \\ y = 2\sqrt{2} \quad &\text{or} \quad y = -2\sqrt{2} \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= 9 && \text{(1)} \\ (-1)^2 + y^2 &= 9 && \text{Let } x = -1. \\ y^2 &= 8 \\ y = 2\sqrt{2} \quad &\text{or} \quad y = -2\sqrt{2} \end{aligned}$$

The solution set is

$$\left\{\left(1, 2\sqrt{2}\right), \left(1, -2\sqrt{2}\right), \left(-1, 2\sqrt{2}\right), \left(-1, -2\sqrt{2}\right)\right\}.$$

**FIGURE 30** shows the four points of intersection.

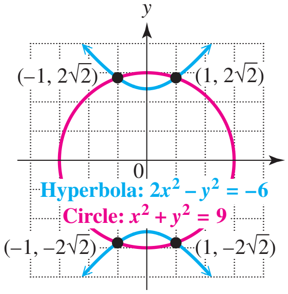


**FIGURE 30**

*NOW TRY*

## OBJECTIVE 3 Solve a nonlinear system that requires a combination of methods.

### EXAMPLE 4 Solving a Nonlinear System by a Combination of Methods

Solve the system.

$$\begin{aligned} x^2 + 2xy - y^2 &= 7 && \text{(1)} \\ x^2 - y^2 &= 3 && \text{(2)} \end{aligned}$$

While we have not graphed equations like (1), its graph is a hyperbola. The graph of (2) is also a hyperbola. Two hyperbolas may have zero, one, two, three, or four points of intersection. We use the elimination method here in combination with the substitution method.

$$\begin{aligned} x^2 + 2xy - y^2 &= \phantom{-}7 && \text{(1)} \\ \underline{-x^2 \qquad\quad + y^2} &= \underline{-3} && \text{Multiply (2) by } -1. \\ 2xy \qquad &= \phantom{-}4 && \text{Add.} \end{aligned}$$

The $x^2$- and $y^2$-terms were eliminated.

Next, we solve $2xy = 4$ for one of the variables. We choose $y$.

$$\begin{aligned} 2xy &= 4 \\ y &= \frac{2}{x} && \text{Divide by } 2x. \quad \text{(3)} \end{aligned}$$

Now, we substitute $y = \frac{2}{x}$ into one of the original equations.

$$\begin{aligned} x^2 - y^2 &= 3 && \text{The substitution is easier in (2).} \\ x^2 - \left(\frac{2}{x}\right)^2 &= 3 && \text{Let } y = \tfrac{2}{x}. \\ x^2 - \frac{4}{x^2} &= 3 && \text{Square } \tfrac{2}{x}. \\ x^4 - 4 &= 3x^2 && \text{Multiply by } x^2,\ x \neq 0. \end{aligned}$$

*NOW TRY ANSWER*

3. $\left\{\left(2\sqrt{3}, 2\right), \left(2\sqrt{3}, -2\right), \left(-2\sqrt{3}, 2\right), \left(-2\sqrt{3}, -2\right)\right\}$

**NOW TRY EXERCISE 4**

Solve the system.

$$x^2 + 3xy - y^2 = 23$$
$$x^2 - y^2 = 5$$

$$\begin{aligned} x^4 - 3x^2 - 4 &= 0 && \text{Subtract } 3x^2. \\ (x^2 - 4)(x^2 + 1) &= 0 && \text{Factor.} \\ x^2 - 4 = 0 \quad \text{or} \quad x^2 + 1 &= 0 && \text{Zero-factor property} \\ x^2 = 4 \quad \text{or} \quad x^2 &= -1 && \text{Solve each equation.} \\ x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = i \quad &\text{or} \quad x = -i \end{aligned}$$

Substituting these four values into $y = \frac{2}{x}$ (equation (3)) gives the corresponding values for $y$.

$$\begin{aligned} &\text{If } x = 2, && \text{then } y = \tfrac{2}{2} = 1. \\ &\text{If } x = -2, && \text{then } y = \tfrac{2}{-2} = -1. \\ &\text{If } x = i, && \text{then } y = \tfrac{2}{i} = \tfrac{2}{i} \cdot \tfrac{-i}{-i} = -2i. \\ &\text{If } x = -i, && \text{then } y = \tfrac{2}{-i} = \tfrac{2}{-i} \cdot \tfrac{i}{i} = 2i. \end{aligned}$$

Multiply by the complex conjugate of the denominator. $i(-i) = 1$

If we substitute the $x$-values we found into equation (1) or (2) instead of into equation (3), we get extraneous solutions. ***It is always wise to check all solutions in both of the given equations.*** There are four ordered pairs in the solution set, two with real values and two with pure imaginary values. The solution set is

$$\{(2, 1), (-2, -1), (i, -2i), (-i, 2i)\}.$$

The graph of the system, shown in **FIGURE 31**, shows only the two real intersection points because the graph is in the real number plane. In general, if solutions contain nonreal complex numbers as components, they do not appear on the graph

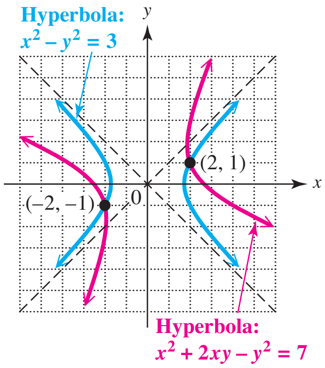


**FIGURE 31**

NOW TRY

**NOTE** It is not essential to visualize the number of points of intersection of the graphs in order to solve a nonlinear system. Sometimes we are unfamiliar with the graphs or, as in **Example 4,** there are nonreal complex solutions that do not appear as points of intersection in the real plane. Visualizing the geometry of the graphs is only an aid to solving these systems.

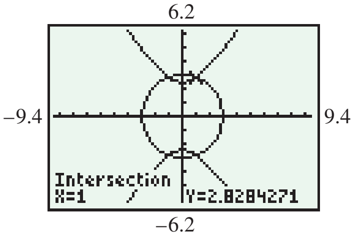
## CONNECTIONS

If the equations in a nonlinear system can be solved for $y$, then we can graph the equations of the system with a graphing calculator and use the capabilities of the calculator to identify all intersection points.

For instance, the two equations in **Example 3** would require graphing four separate functions.

$$Y_1 = \sqrt{9 - X^2}, \quad Y_2 = -\sqrt{9 - X^2}, \quad Y_3 = \sqrt{2X^2 + 6}, \quad \text{and} \quad Y_4 = -\sqrt{2X^2 + 6}$$

**FIGURE 32** indicates the coordinates of one of the points of intersection.



Intersection
X=1 Y=2.8284271

**FIGURE 32**

**NOW TRY ANSWER**

4. $\{(3, 2), (-3, -2), (2i, -3i), (-2i, 3i)\}$

## 13.4 EXERCISES

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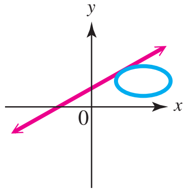
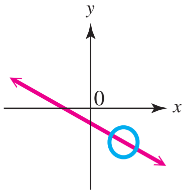
***Concept Check*** *Each sketch represents the graphs of a pair of equations in a system. How many points are in each solution set?*

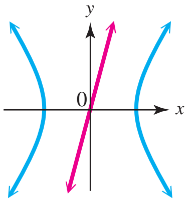
**1.**



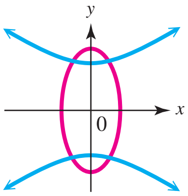
**2.**



**3.**



**4.**



***Concept Check*** *Suppose that a nonlinear system is composed of equations whose graphs are those described, and the number of points of intersection of the two graphs is as given. Make a sketch satisfying these conditions. (There may be more than one way to do this.)*

**5.** A line and a circle; no points

**6.** A line and a circle; one point

**7.** A line and a hyperbola; one point

**8.** A line and an ellipse; no points

**9.** A circle and an ellipse; four points

**10.** A parabola and an ellipse; one point

**11.** A parabola and an ellipse; four points

**12.** A parabola and a hyperbola; two points

*Solve each system by the substitution method.* ***See Examples 1 and 2.***

**13.** $y = 4x^2 - x$
$y = x$

**14.** $y = x^2 + 6x$
$3y = 12x$

**15.** $y = x^2 + 6x + 9$
$x + y = 3$

**16.** $y = x^2 + 8x + 16$
$x - y = -4$

◉ **17.** $x^2 + y^2 = 2$
$2x + y = 1$

**18.** $2x^2 + 4y^2 = 4$
$x = 4y$

◉ **19.** $xy = 4$
$3x + 2y = -10$

**20.** $xy = -5$
$2x + y = 3$

**21.** $xy = -3$
$x + y = -2$

**22.** $xy = 12$
$x + y = 8$

**23.** $y = 3x^2 + 6x$
$y = x^2 - x - 6$

**24.** $y = 2x^2 + 1$
$y = 5x^2 + 2x - 7$

**25.** $2x^2 - y^2 = 6$
$y = x^2 - 3$

**26.** $x^2 + y^2 = 4$
$y = x^2 - 2$

**27.** $x^2 - xy + y^2 = 0$
$x - 2y = 1$

**28.** $x^2 - 3x + y^2 = 4$
$2x - y = 3$

*Solve each system by the elimination method or a combination of the elimination and substitution methods.* ***See Examples 3 and 4.***

**29.** $3x^2 + 2y^2 = 12$
$x^2 + 2y^2 = 4$

**30.** $5x^2 - 2y^2 = -13$
$3x^2 + 4y^2 = 39$

**31.** $2x^2 + 3y^2 = 6$
$x^2 + 3y^2 = 3$

**32.** $6x^2 + y^2 = 9$
$3x^2 + 4y^2 = 36$

**33.** $2x^2 + y^2 = 28$
$4x^2 - 5y^2 = 28$

**34.** $x^2 + 6y^2 = 9$
$4x^2 + 3y^2 = 36$

**35.** $2x^2 = 8 - 2y^2$
$3x^2 = 24 - 4y^2$

**36.** $5x^2 = 20 - 5y^2$
$2y^2 = 2 - x^2$

**37.** $x^2 + xy + y^2 = 15$
$x^2 + y^2 = 10$

**38.** $2x^2 + 3xy + 2y^2 = 21$
$x^2 + y^2 = 6$

**39.** $3x^2 + 2xy - 3y^2 = 5$
$-x^2 - 3xy + y^2 = 3$

**40.** $-2x^2 + 7xy - 3y^2 = 4$
$2x^2 - 3xy + 3y^2 = 4$

*Use a graphing calculator to solve each system. Then confirm your answer algebraically.*

**41.** $xy = -6$
$x + y = -1$

**42.** $y = 2x^2 + 4x$
$y = -x^2 - 1$

*Solve each problem by using a nonlinear system.*

**43.** The area of a rectangular rug is 84 $\text{ft}^2$ and its perimeter is 38 ft. Find the length and width of the rug.

**44.** Find the length and width of a rectangular room whose perimeter is 50 m and whose area is 100 $\text{m}^2$.

**45.** A company has found that the price $p$ (in dollars) of its scientific calculator is related to the supply $x$ (in thousands) by the equation

$$px = 16.$$

The price is related to the demand $x$ (in thousands) for the calculator by the equation

$$p = 10x + 12.$$

The **equilibrium price** is the value of $p$ where demand equals supply. Find the equilibrium price and the supply/demand at that price. (*Hint:* Demand, price, and supply must all be positive.)

**46.** The calculator company in **Exercise 45** has determined that the cost $y$ to make $x$ (thousand) calculators is

$$y = 4x^2 + 36x + 20,$$

while the revenue $y$ from the sale of $x$ (thousand) calculators is

$$36x^2 - 3y = 0.$$

Find the **break-even point,** where cost equals revenue.

## PREVIEW EXERCISES

*Graph each inequality.* ***See Section 9.3.***

**47.** $2x - y \leq 4$

**48.** $-x + 3y > 9$

# 13.5 Second-Degree Inequalities and Systems of Inequalities

**OBJECTIVES**

1. Graph second-degree inequalities.
2. Graph the solution set of a system of inequalities.

**OBJECTIVE 1** **Graph second-degree inequalities.** A **second-degree inequality** is an inequality with at least one variable of degree 2 and no variable with degree greater than 2.

## EXAMPLE 1 Graphing a Second-Degree Inequality

Graph $x^2 + y^2 \le 36$.

The boundary of the inequality $x^2 + y^2 \le 36$ is the graph of the equation $x^2 + y^2 = 36$, a circle with radius 6 and center at the origin, as shown in **FIGURE 33**.

The inequality $x^2 + y^2 \le 36$ will include either the points outside the circle or the points inside the circle, as well as the boundary. To decide which region to shade, we substitute any test point not on the circle into the original inequality.

| | |
|---|---|
| $x^2 + y^2 \le 36$ | Original inequality |
| $0^2 + 0^2 \overset{?}{\le} 36$ | Use $(0, 0)$ as a test point. |
| $0 \le 36$ ✓ | True |

Since a true statement results, the original inequality includes the points *inside* the circle, the shaded region in **FIGURE 33**, and the boundary.

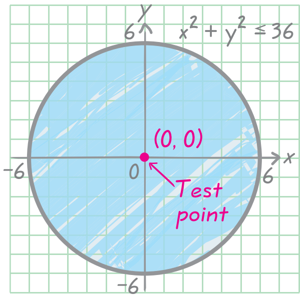


**FIGURE 33**

NOW TRY

NOW TRY EXERCISE 1
Graph $x^2 + y^2 \ge 9$.

**NOTE** Since the substitution is easy, the origin is the test point of choice unless the graph actually passes through $(0, 0)$.

## EXAMPLE 2 Graphing a Second-Degree Inequality

Graph $y < -2(x-4)^2 - 3$.

The boundary, $y = -2(x-4)^2 - 3$, is a parabola that opens down with vertex at $(4, -3)$.

| | |
|---|---|
| $y < -2(x-4)^2 - 3$ | Original inequality |
| $0 \overset{?}{<} -2(0-4)^2 - 3$ | Use $(0, 0)$ as a test point. |
| $0 \overset{?}{<} -32 - 3$ | Simplify. |
| $0 < -35$ | False |

Because the final inequality is a false statement, the points in the region containing $(0, 0)$ do not satisfy the inequality. In **FIGURE 34** the parabola is drawn as a dashed curve since the points of the parabola itself do not satisfy the inequality, and the region inside (or below) the parabola is shaded.

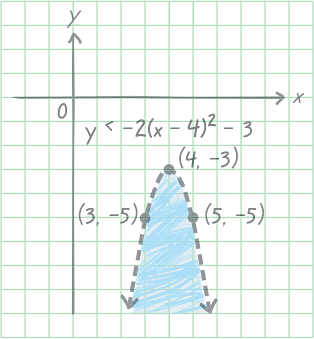


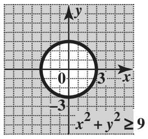
**FIGURE 34**

NOW TRY

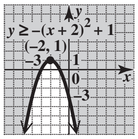
NOW TRY EXERCISE 2
Graph $y \ge -(x+2)^2 + 1$.

NOW TRY ANSWERS

1.



2.



**NOW TRY EXERCISE 3**

Graph $25x^2 - 16y^2 > 400$.

## EXAMPLE 3 Graphing a Second-Degree Inequality

Graph $16y^2 \leq 144 + 9x^2$.

$$16y^2 - 9x^2 \leq 144 \quad \text{Subtract } 9x^2.$$

$$\frac{y^2}{9} - \frac{x^2}{16} \leq 1 \quad \text{Divide by 144.}$$

This form shows that the boundary is the hyperbola given by

$$\frac{y^2}{9} - \frac{x^2}{16} = 1.$$

Since the graph is a vertical hyperbola, the desired region will be either the region between the branches or the regions above the top branch and below the bottom branch. Choose $(0, 0)$ as a test point. Substituting into the original inequality leads to $0 \leq 144$, a true statement, so the region between the branches containing $(0, 0)$ is shaded, as shown in **FIGURE 35**.

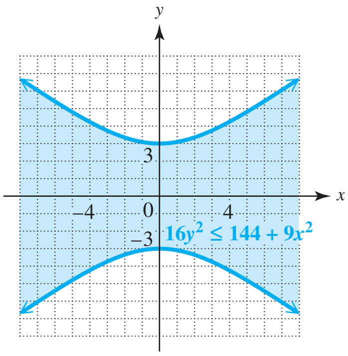


**FIGURE 35**

NOW TRY

**OBJECTIVE 2** **Graph the solution set of a system of inequalities.** If two or more inequalities are considered at the same time, we have a **system of inequalities.** To find the solution set of the system, we find the intersection of the graphs (solution sets) of the inequalities in the system.

**NOW TRY EXERCISE 4**

Graph the solution set of the system.

$$x^2 + y^2 > 9$$
$$y > x^2 - 1$$

## EXAMPLE 4 Graphing a System of Two Inequalities

Graph the solution set of the system.

$$2x + 3y > 6$$
$$x^2 + y^2 < 16$$

Begin by graphing the solution set of $2x + 3y > 6$. The boundary line is the graph of $2x + 3y = 6$ and is a dashed line because of the symbol $>$. The test point $(0, 0)$ leads to a false statement in the inequality $2x + 3y > 6$, so shade the region above the line, as shown in **FIGURE 36**.

The graph of $x^2 + y^2 < 16$ is the interior of a dashed circle centered at the origin with radius 4. This is shown in **FIGURE 37**.

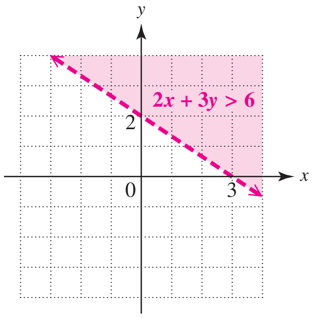


**FIGURE 36**

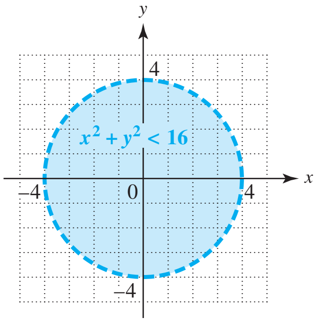


**FIGURE 37**

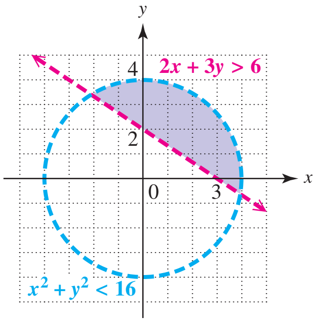
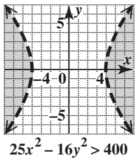


**FIGURE 38**

The graph of the solution set of the system is the intersection of the graphs of the two inequalities. The overlapping region in **FIGURE 38** is the solution set. NOW TRY

**NOW TRY ANSWERS**

3.



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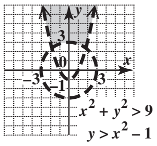


NOW TRY EXERCISE 5

Graph the solution set of the system.

$$3x + 2y > 6$$
$$y \geq \frac{1}{2}x - 2$$
$$x \geq 0$$

### EXAMPLE 5 Graphing a Linear System of Three Inequalities

Graph the solution set of the system.

$$x + y < 1$$
$$y \leq 2x + 3$$
$$y \geq -2$$

Graph each inequality separately, on the same axes. The graph of $x + y < 1$ consists of all points that lie below the dashed line $x + y = 1$. The graph of $y \leq 2x + 3$ is the region that lies below the solid line $y = 2x + 3$. Finally, the graph of $y \geq -2$ is the region above the solid horizontal line $y = -2$.

The graph of the system, the intersection of these three graphs, is the triangular region enclosed by the three boundary lines in **FIGURE 39**, including two of its boundaries.

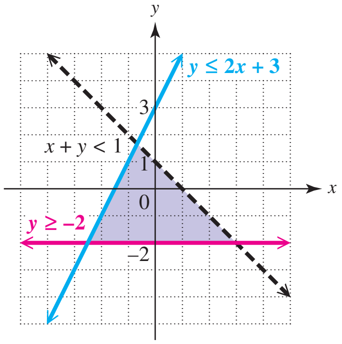


**FIGURE 39**

NOW TRY

NOW TRY EXERCISE 6

Graph the solution set of the system.

$$\frac{x^2}{4} + \frac{y^2}{16} \leq 1$$
$$y \leq x^2 - 2$$
$$y + 3 > 0$$

### EXAMPLE 6 Graphing a System of Three Inequalities

Graph the solution set of the system.

$$y \geq x^2 - 2x + 1$$
$$2x^2 + y^2 > 4$$
$$y < 4$$

The graph of $y = x^2 - 2x + 1$ is a parabola with vertex at $(1, 0)$. Those points above (or in the interior of) the parabola satisfy the condition $y > x^2 - 2x + 1$. Thus, the solution set of $y \geq x^2 - 2x + 1$ includes points on the parabola or in the interior.

The graph of the equation $2x^2 + y^2 = 4$ is an ellipse. We draw it as a dashed curve. To satisfy the inequality $2x^2 + y^2 > 4$, a point must lie outside the ellipse. The graph of $y < 4$ includes all points below the dashed line $y = 4$.

The graph of the system is the shaded region in **FIGURE 40**, which lies outside the ellipse, inside or on the boundary of the parabola, and below the line $y = 4$.

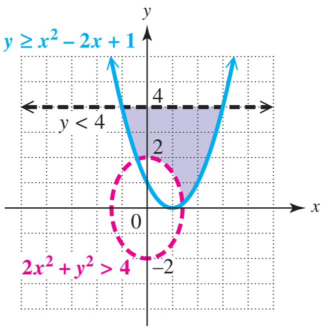
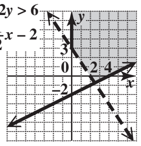


**FIGURE 40**

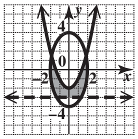
NOW TRY

NOW TRY ANSWERS

5. $3x + 2y > 6$
$y \geq \frac{1}{2}x - 2$
$x \geq 0$



6. $\frac{x^2}{4} + \frac{y^2}{16} \leq 1$
$y \leq x^2 - 2$
$y + 3 > 0$



## 13.5 EXERCISES

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**1.** *Concept Check* Which one of the following is a description of the graph of the solution set of the following system?

$$\begin{aligned} &x^2 + y^2 < 25 \\ &y > -2 \end{aligned}$$

**A.** All points outside the circle $x^2 + y^2 = 25$ and above the line $y = -2$

**B.** All points outside the circle $x^2 + y^2 = 25$ and below the line $y = -2$

**C.** All points inside the circle $x^2 + y^2 = 25$ and above the line $y = -2$

**D.** All points inside the circle $x^2 + y^2 = 25$ and below the line $y = -2$

**2.** *Concept Check* Fill in each blank with the appropriate response. The graph of the system

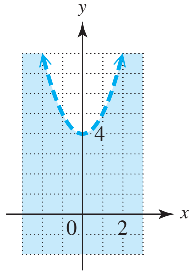
$$\begin{aligned} &y > x^2 + 1 \\ &\frac{x^2}{9} + \frac{y^2}{4} > 1 \\ &y < 5 \end{aligned}$$

consists of all points ______________ (above/below) the parabola $y = x^2 + 1$, ______________ (inside/outside) the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and ______________ (above/below) the line $y = 5$.

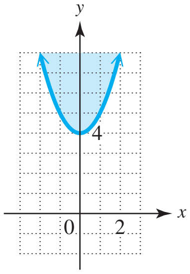
*Concept Check* *Match each nonlinear inequality with its graph.*

**3.** $y \geq x^2 + 4$ **4.** $y \leq x^2 + 4$ **5.** $y < x^2 + 4$ **6.** $y > x^2 + 4$

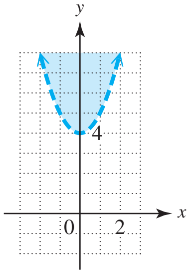
**A.**



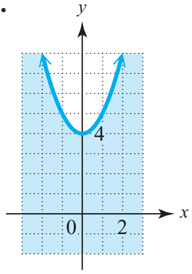
**B.**



**C.**



**D.**



*Graph each nonlinear inequality.* ***See Examples 1–3.***

**7.** $y^2 > 4 + x^2$

**8.** $y^2 \leq 4 - 2x^2$

◉ **9.** $y \geq x^2 - 2$

**10.** $x^2 \leq 16 - y^2$

**11.** $2y^2 \geq 8 - x^2$

**12.** $x^2 \leq 16 + 4y^2$

**13.** $y \leq x^2 + 4x + 2$

**14.** $9x^2 < 16y^2 - 144$

◉ **15.** $9x^2 > 16y^2 + 144$

**16.** $4y^2 \leq 36 - 9x^2$

**17.** $x^2 - 4 \geq -4y^2$

**18.** $x \geq y^2 - 8y + 14$

**19.** $x \leq -y^2 + 6y - 7$

**20.** $y^2 - 16x^2 \leq 16$

*Graph each system of inequalities.* ***See Examples 4–6.***

**21.** $\begin{aligned} 2x + 5y &< 10 \\ x - 2y &< 4 \end{aligned}$

**22.** $\begin{aligned} 3x - y &> -6 \\ 4x + 3y &> 12 \end{aligned}$

**23.** $\begin{aligned} 5x - 3y &\le 15 \\ 4x + y &\ge 4 \end{aligned}$

**24.** $\begin{aligned} 4x - 3y &\le 0 \\ x + y &\le 5 \end{aligned}$

**25.** $\begin{aligned} x &\le 5 \\ y &\le 4 \end{aligned}$

**26.** $\begin{aligned} x &\ge -2 \\ y &\le 4 \end{aligned}$

**27.** $\begin{aligned} y &> x^2 - 4 \\ y &< -x^2 + 3 \end{aligned}$

**28.** $\begin{aligned} x^2 - y^2 &\ge 9 \\ \frac{x^2}{16} + \frac{y^2}{9} &\le 1 \end{aligned}$

**29.** $\begin{aligned} x^2 + y^2 &\ge 4 \\ x + y &\le 5 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

**30.** $\begin{aligned} y^2 - x^2 &\ge 4 \\ -5 \le y &\le 5 \end{aligned}$

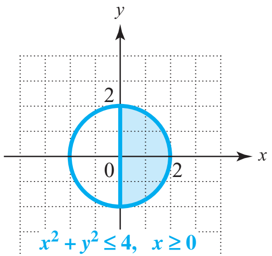
**31.** $\begin{aligned} y &\le -x^2 \\ y &\ge x - 3 \\ y &\le -1 \\ x &< 1 \end{aligned}$

**32.** $\begin{aligned} y &< x^2 \\ y &> -2 \\ x + y &< 3 \\ 3x - 2y &> -6 \end{aligned}$

*For each nonlinear inequality in Exercises 33–40, a restriction is placed on one or both variables. For example, the inequality*

$$x^2 + y^2 \le 4, \quad x \ge 0$$

*is graphed in the figure. Only the right half of the interior of the circle and its boundary is shaded, because of the restriction that x must be nonnegative. Graph each nonlinear inequality with the given restrictions.*



**33.** $x^2 + y^2 > 36, \quad x \ge 0$

**34.** $4x^2 + 25y^2 < 100, \quad y < 0$

**35.** $x < y^2 - 3, \quad x < 0$

**36.** $x^2 - y^2 < 4, \quad x < 0$

**37.** $4x^2 - y^2 > 16, \quad x < 0$

**38.** $x^2 + y^2 > 4, \quad y < 0$

**39.** $x^2 + 4y^2 \ge 1, \quad x \ge 0, y \ge 0$

**40.** $2x^2 - 32y^2 \le 8, \quad x \le 0, y \ge 0$

*Use the shading feature of a graphing calculator to graph each system.*

**41.** $\begin{aligned} y &\ge x - 3 \\ y &\le -x + 4 \end{aligned}$

**42.** $\begin{aligned} y &\ge -x^2 + 5 \\ y &\le x^2 - 3 \end{aligned}$

**43.** $\begin{aligned} y &< x^2 + 4x + 4 \\ y &> -3 \end{aligned}$

**44.** $\begin{aligned} y &> (x - 4)^2 - 3 \\ y &< 5 \end{aligned}$

## PREVIEW EXERCISES

*Evaluate each expression for* ***(a)*** $n = 1$, ***(b)*** $n = 2$, ***(c)*** $n = 3$, *and* ***(d)*** $n = 4$. ***See Sections 1.3 and 1.6.***

**45.** $\frac{n + 5}{n}$

**46.** $\frac{n - 1}{n + 1}$

**47.** $n^2 - n$

**48.** $n(n - 3)$

# CHAPTER 13
# SUMMARY

## *KEY TERMS*

### 13.1
squaring function
absolute value function
reciprocal function
asymptotes
square root function
greatest integer function
step function

### 13.2
conic section
circle
center (of circle)
radius
center-radius form
ellipse
foci (singular: focus)
center (of ellipse)

### 13.3
hyperbola
transverse axis
asymptotes of a hyperbola
fundamental rectangle
generalized square root function

### 13.4
nonlinear equation
nonlinear system of equations

### 13.5
second-degree inequality
system of inequalities

## *NEW SYMBOLS*

$[\![x]\!]$ greatest integer less than or equal to $x$

## *TEST YOUR WORD POWER*

*See how well you have learned the vocabulary in this chapter.*

1. **Conic sections** are
   **A.** graphs of first-degree equations
   **B.** the result of two or more intersecting planes
   **C.** graphs of first-degree inequalities
   **D.** figures that result from the intersection of an infinite cone with a plane.

2. A **circle** is the set of all points in a plane
   **A.** such that the absolute value of the difference of the distances from two fixed points is constant
   **B.** that lie a fixed distance from a fixed point
   **C.** the sum of whose distances from two fixed points is constant
   **D.** that make up the graph of any second-degree equation.

3. An **ellipse** is the set of all points in a plane
   **A.** such that the absolute value of the difference of the distances from two fixed points is constant
   **B.** that lie a fixed distance from a fixed point
   **C.** the sum of whose distances from two fixed points is constant
   **D.** that make up the graph of any second-degree equation.

4. A **hyperbola** is the set of all points in a plane
   **A.** such that the absolute value of the difference of the distances from two fixed points is constant
   **B.** that lie a fixed distance from a fixed point
   **C.** the sum of whose distances from two fixed points is constant
   **D.** that make up the graph of any second-degree equation.

5. A **nonlinear equation** is an equation
   **A.** in which some terms have more than one variable or a variable of degree 2 or greater
   **B.** in which the terms have only one variable
   **C.** of degree 1
   **D.** of a linear function.

6. A **nonlinear system of equations** is a system
   **A.** with at least one linear equation
   **B.** with two or more inequalities
   **C.** with at least one nonlinear equation
   **D.** with at least two linear equations.

### *ANSWERS*

**1.** D; *Example:* Parabolas, circles, ellipses, and hyperbolas are conic sections. **2.** B; *Example:* See the graph of $x^2 + y^2 = 9$ in **FIGURE 10** of **Section 13.2.** **3.** C; *Example:* See the graph of $\frac{x^2}{49} + \frac{y^2}{36} = 1$ in **FIGURE 16** of **Section 13.2.** **4.** A; *Example:* See the graph of $\frac{x^2}{16} - \frac{y^2}{25} = 1$ in **FIGURE 21** of **Section 13.3.** **5.** A; *Examples:* $y = x^2 + 8x + 16$, $xy = 5$, $2x^2 - y^2 = 6$ **6.** C; *Example:* $x^2 + y^2 = 2$
$2x + y = 1$

## QUICK REVIEW

**CONCEPTS**

**EXAMPLES**

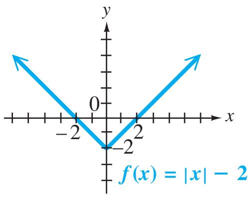
### 13.1 Additional Graphs of Functions

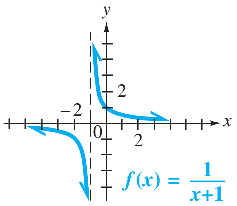
**Other Functions**

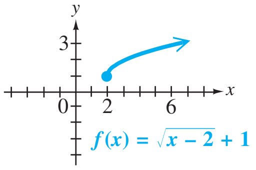
In addition to the squaring function, some other elementary functions include the following:

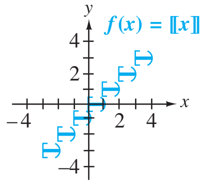
- Absolute value function, defined by $f(x) = |x|$
- Reciprocal function, defined by $f(x) = \frac{1}{x}$
- Square root function, defined by $f(x) = \sqrt{x}$
- Greatest integer function, defined by $f(x) = [\![x]\!]$, which is a step function.

Their graphs can be translated, as shown in the first three examples at the right.









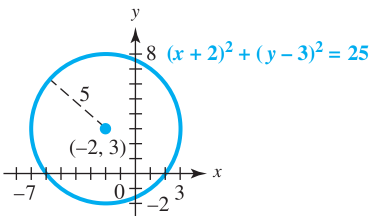
### 13.2 The Circle and the Ellipse

**Circle**

The circle with radius $r$ and center at $(h, k)$ has an equation of the form

$$(x - h)^2 + (y - k)^2 = r^2.$$

The circle with equation $(x + 2)^2 + (y - 3)^2 = 25$, which can be written $[x - (-2)]^2 + (y - 3)^2 = 5^2$, has center $(-2, 3)$ and radius 5.

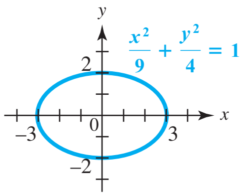


**Ellipse**

The ellipse whose $x$-intercepts are $(a, 0)$ and $(-a, 0)$ and whose $y$-intercepts are $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Graph $\frac{x^2}{9} + \frac{y^2}{4} = 1$.



### 13.3 The Hyperbola and Functions Defined by Radicals

**Hyperbola**

A hyperbola with $x$-intercepts $(a, 0)$ and $(-a, 0)$ has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

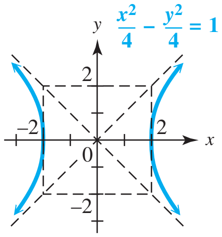
and a hyperbola with $y$-intercepts $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The extended diagonals of the fundamental rectangle with vertices at the points $(a, b)$, $(-a, b)$, $(-a, -b)$, and $(a, -b)$ are the asymptotes of these hyperbolas.

Graph $\frac{x^2}{4} - \frac{y^2}{4} = 1$.

The graph has $x$-intercepts $(2, 0)$ and $(-2, 0)$.



The fundamental rectangle has vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$.

(continued)

***CONCEPTS***

***EXAMPLES***

**Graphing a Generalized Square Root Function**

To graph a generalized square root function defined by

$$f(x) = \sqrt{u}$$

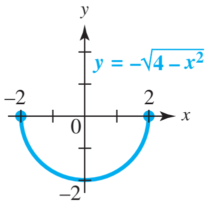
for an algebraic expression $u$, with $u \geq 0$, square each side so that the equation can be easily recognized. Then graph only the part indicated by the original equation.

Graph $y = -\sqrt{4 - x^2}$.

Square each side and rearrange terms to get

$$x^2 + y^2 = 4.$$

This equation has a circle as its graph. However, graph only the lower half of the circle, since the original equation indicates that $y$ cannot be positive.



## 13.4 Nonlinear Systems of Equations

**Solving a Nonlinear System**

A nonlinear system can be solved by the substitution method, the elimination method, or a combination of the two.

Solve the system.

$$\begin{aligned} x^2 + 2xy - y^2 &= 14 \qquad (1) \\ x^2 - y^2 &= -16 \qquad (2) \end{aligned}$$

Multiply equation (2) by $-1$ and use elimination.

$$\begin{array}{rcrcl} x^2 & + 2xy & - y^2 & = & 14 \\ -x^2 & & + y^2 & = & 16 \\ \hline & 2xy & & = & 30 \\ & & xy & = & 15 \end{array}$$

Solve $xy = 15$ for $y$ to obtain $y = \frac{15}{x}$, and substitute into equation (2).

$$\begin{aligned} x^2 - y^2 &= -16 && (2) \\ x^2 - \left(\frac{15}{x}\right)^2 &= -16 && \text{Let } y = \tfrac{15}{x}. \\ x^2 - \frac{225}{x^2} &= -16 && \text{Apply the exponent.} \\ x^4 + 16x^2 - 225 &= 0 && \text{Multiply by } x^2. \text{ Add } 16x^2. \\ (x^2 - 9)(x^2 + 25) &= 0 && \text{Factor.} \\ x = \pm 3 \quad \text{or} \quad x &= \pm 5i && \text{Zero-factor property} \end{aligned}$$

Find corresponding $y$-values to get the solution set

$$\{(3, 5), (-3, -5), (5i, -3i), (-5i, 3i)\}.$$

## 13.5 Second-Degree Inequalities and Systems of Inequalities

**Graphing a Second-Degree Inequality**

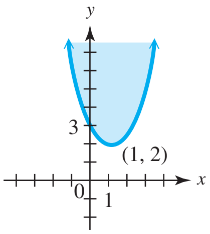
To graph a second-degree inequality, graph the corresponding equation as a boundary and use test points to determine which region(s) form the solution set. Shade the appropriate region(s).

**Graphing a System of Inequalities**

The solution set of a system of inequalities is the intersection of the solution sets of the individual inequalities.

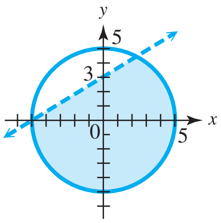
Graph

$$y \geq x^2 - 2x + 3.$$



Graph the solution set of the system

$$\begin{aligned} 3x - 5y &> -15 \\ x^2 + y^2 &\leq 25. \end{aligned}$$



# CHAPTER 13

# REVIEW EXERCISES

**13.1** *Graph each function.*

**1.** $f(x) = |x + 4|$

**2.** $f(x) = \dfrac{1}{x - 4}$

**3.** $f(x) = \sqrt{x} + 3$

**4.** $f(x) = [\![x]\!] - 2$

**13.2** *Write an equation for each circle.*

**5.** Center $(-2, 4)$, $r = 3$

**6.** Center $(-1, -3)$, $r = 5$

**7.** Center $(4, 2)$, $r = 6$

*Find the center and radius of each circle.*

**8.** $x^2 + y^2 + 6x - 4y - 3 = 0$

**9.** $x^2 + y^2 - 8x - 2y + 13 = 0$

**10.** $2x^2 + 2y^2 + 4x + 20y = -34$

**11.** $4x^2 + 4y^2 - 24x + 16y = 48$

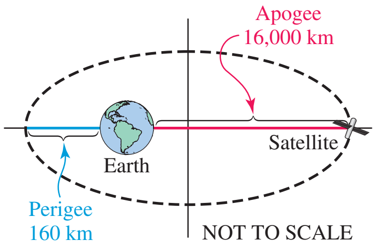
*Graph each equation.*

**12.** $x^2 + y^2 = 16$

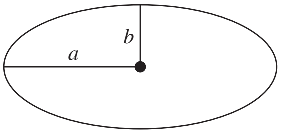
**13.** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

**14.** $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$

**15.** A satellite is in an elliptical orbit around Earth with perigee altitude of 160 km and apogee altitude of 16,000 km. See the figure. (*Source:* Kastner, Bernice, *Space Mathematics,* NASA.) Find the equation of the ellipse. (*Hint:* Use the fact that $c^2 = a^2 - b^2$ here.)



**16.** **(a)** The Roman Colosseum is an ellipse with $a = 310$ ft and $b = \frac{513}{2}$ ft. Find the distance, to the nearest tenth, between the foci of this ellipse.



**(b)** The approximate perimeter of an ellipse is given by

$$P \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}},$$

where $a$ and $b$ are the lengths given in part (a). Use this formula to find the approximate perimeter, to the nearest tenth, of the Roman Colosseum.

**13.3** *Graph each equation.*

**17.** $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$

**18.** $\dfrac{y^2}{25} - \dfrac{x^2}{4} = 1$

**19.** $f(x) = -\sqrt{16 - x^2}$

*Identify the graph of each equation as a* parabola, circle, ellipse, *or* hyperbola.

**20.** $x^2 + y^2 = 64$

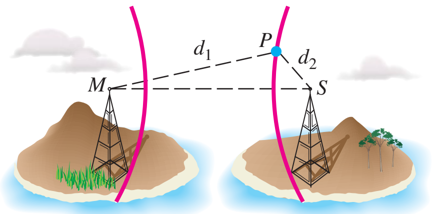
**21.** $y = 2x^2 - 3$

**22.** $y^2 = 2x^2 - 8$

**23.** $y^2 = 8 - 2x^2$

**24.** $x = y^2 + 4$

**25.** $x^2 - y^2 = 64$



**26.** Ships and planes often use a location-finding system called LORAN. With this system, a radio transmitter at $M$ sends out a series of pulses. When each pulse is received at transmitter $S$, it then sends out a pulse. A ship at $P$ receives pulses from both $M$ and $S$. A receiver on the ship measures the difference in the arrival times of the pulses. A special map gives hyperbolas that correspond to the differences in arrival times (which give the distances $d_1$ and $d_2$ in the figure.) The ship can then be located as lying on a branch of a particular hyperbola.

Suppose $d_1 = 80$ mi and $d_2 = 30$ mi, and the distance between transmitters $M$ and $S$ is 100 mi. Use the definition to find an equation of the hyperbola on which the ship is located.

## 13.4 *Solve each system.*

**27.** $2y = 3x - x^2$
$x + 2y = -12$

**28.** $y + 1 = x^2 + 2x$
$y + 2x = 4$

**29.** $x^2 + 3y^2 = 28$
$y - x = -2$

**30.** $xy = 8$
$x - 2y = 6$

**31.** $x^2 + y^2 = 6$
$x^2 - 2y^2 = -6$

**32.** $3x^2 - 2y^2 = 12$
$x^2 + 4y^2 = 18$

**33.** ***Concept Check*** How many solutions are possible for a system of two equations whose graphs are a circle and a line?

**34.** ***Concept Check*** How many solutions are possible for a system of two equations whose graphs are a parabola and a hyperbola?

## 13.5 *Graph each inequality.*

**35.** $9x^2 \geq 16y^2 + 144$

**36.** $4x^2 + y^2 \geq 16$

**37.** $y < -(x + 2)^2 + 1$

*Graph each system of inequalities.*

**38.** $2x + 5y \leq 10$
$3x - y \leq 6$

**39.** $|x| \leq 2$
$|y| > 1$
$4x^2 + 9y^2 \leq 36$

**40.** $9x^2 \leq 4y^2 + 36$
$x^2 + y^2 \leq 16$

## MIXED REVIEW EXERCISES

*Graph.*

**41.** $\dfrac{y^2}{4} - 1 = \dfrac{x^2}{9}$

**42.** $x^2 + y^2 = 25$

**43.** $x^2 + 9y^2 = 9$

**44.** $x^2 - 9y^2 = 9$

**45.** $f(x) = \sqrt{4 - x}$

**46.** $4y > 3x - 12$
$x^2 < 16 - y^2$

# CHAPTER 13 TEST

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

***Concept Check*** *Fill in each blank with the correct response.*

**1.** For the reciprocal function defined by $f(x) = \frac{1}{x}$, ________ is the only real number not in the domain.

**2.** The range of the square root function, given by $f(x) = \sqrt{x}$, is ________.

**3.** The range of $f(x) = [\![x]\!]$, the greatest integer function, is ________.

**4.** ***Concept Check*** Match each function with its graph from choices A–D.

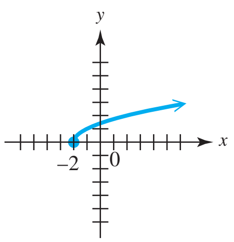
**(a)** $f(x) = \sqrt{x-2}$

**(b)** $f(x) = \sqrt{x+2}$

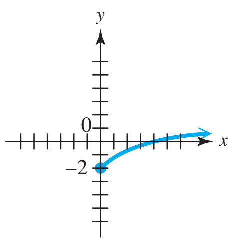
**(c)** $f(x) = \sqrt{x} + 2$

**(d)** $f(x) = \sqrt{x} - 2$

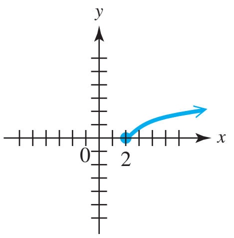
**A.**



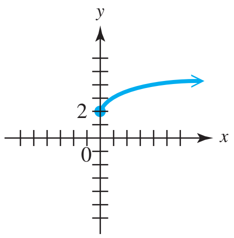
**B.**



**C.**



**D.**



**5.** Sketch the graph of $f(x) = |x - 3| + 4$. Give the domain and range.

**6.** Find the center and radius of the circle whose equation is $(x-2)^2 + (y+3)^2 = 16$. Sketch the graph.

**7.** Find the center and radius of the circle whose equation is $x^2 + y^2 + 8x - 2y = 8$.

*Graph.*

**8.** $f(x) = \sqrt{9 - x^2}$

**9.** $4x^2 + 9y^2 = 36$

**10.** $16y^2 - 4x^2 = 64$

**11.** $\frac{y}{2} = -\sqrt{1 - \frac{x^2}{9}}$

*Identify the graph of each equation as a* parabola, hyperbola, ellipse, *or* circle.

**12.** $6x^2 + 4y^2 = 12$

**13.** $16x^2 = 144 + 9y^2$

**14.** $y^2 = 20 - x^2$

**15.** $4y^2 + 4x = 9$

*Solve each system.*

**16.** $\begin{aligned} 2x - y &= 9 \\ xy &= 5 \end{aligned}$

**17.** $\begin{aligned} x - 4 &= 3y \\ x^2 + y^2 &= 8 \end{aligned}$

**18.** $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - 2y^2 &= 16 \end{aligned}$

**19.** Graph the inequality $y < x^2 - 2$.

**20.** Graph the system $\begin{aligned} x^2 + 25y^2 &\le 25 \\ x^2 + y^2 &\le 9. \end{aligned}$

## CHAPTERS 1–13

# CUMULATIVE REVIEW EXERCISES

**1.** Find the slope of the line through $(2, 5)$ and $(-4, 1)$.

**2.** Find the equation of the line through the point $(-3, -2)$ and perpendicular to the graph of $2x - 3y = 7$.

*Perform the indicated operations.*

**3.** $(5y - 3)^2$

**4.** $\dfrac{8x^4 - 4x^3 + 2x^2 + 13x + 8}{2x + 1}$

*Factor.*

**5.** $12x^2 - 7x - 10$

**6.** $z^4 - 1$

**7.** $a^3 - 27b^3$

*Perform the indicated operations.*

**8.** $\dfrac{y^2 - 4}{y^2 - y - 6} \div \dfrac{y^2 - 2y}{y - 1}$

**9.** $\dfrac{5}{c + 5} - \dfrac{2}{c + 3}$

**10.** $\dfrac{p}{p^2 + p} + \dfrac{1}{p^2 + p}$

**11.** Henry Harris and Lawrence Hawkins want to clean their office. Henry can do the job alone in 3 hr, while Lawrence can do it alone in 2 hr. How long will it take them if they work together?

*Solve each system.*

**12.** $\begin{aligned} 3x - \phantom{3}y &= 12 \\ 2x + 3y &= -3 \end{aligned}$

**13.** $\begin{aligned} x + y - 2z &= 9 \\ 2x + y + z &= 7 \\ 3x - y - z &= 13 \end{aligned}$

**14.** $\begin{aligned} xy &= -5 \\ 2x + y &= 3 \end{aligned}$

**15.** Al and Bev traveled from their apartment to a picnic 20 mi away. Al traveled on his bike while Bev, who left later, took her car. Al's average rate was half of Bev's average rate. The trip took Al $\frac{1}{2}$ hr longer than Bev. What was Bev's average rate?

*Simplify. Assume all variables represent positive real numbers.*

**16.** $\dfrac{(2a)^{-2}a^4}{a^{-3}}$

**17.** $4\sqrt[3]{16} - 2\sqrt[3]{54}$

**18.** $\dfrac{3\sqrt{5x}}{\sqrt{2x}}$

**19.** $\dfrac{5 + 3i}{2 - i}$

*Solve for real solutions.*

**20.** $4 - (2x + 3) + x = 5x - 3$

**21.** $-4x + 7 \geq 6x + 1$

**22.** $|5x| - 6 = 14$

**23.** $|2p - 5| > 15$

**24.** $2\sqrt{x} = \sqrt{5x + 3}$

**25.** $10q^2 + 13q = 3$

**26.** $3x^2 - 3x - 2 = 0$

**27.** $2(x^2 - 3)^2 - 5(x^2 - 3) = 12$

**28.** $\log(x + 2) + \log(x - 1) = 1$

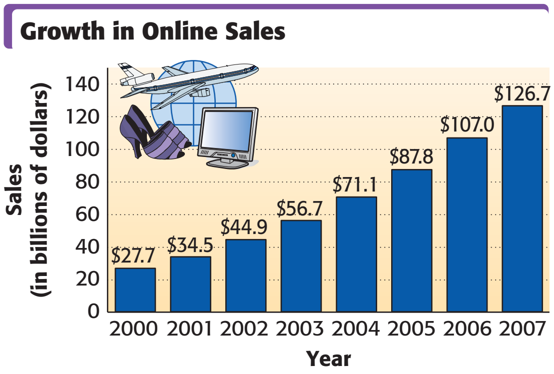
**29.** Solve $F = \dfrac{kwv^2}{r}$ for $v$.

**30.** If $f(x) = x^3 + 4$, find $f^{-1}(x)$.

**31.** Evaluate. **(a)** $3^{\log_3 4}$ **(b)** $e^{\ln 7}$

**32.** Use properties of logarithms to write $2 \log(3x + 7) - \log 4$ as a single logarithm.

**33.** The bar graph shows online U.S. retail sales (in billions of dollars).



***Source:*** U.S. Census Bureau.

A reasonable model for sales $y$ in billions of dollars is the exponential function defined by

$$y = 28.43(1.25)^x,$$

where $x$ is the number of years since 2000.

**(a)** Use the model to estimate sales in 2005. (*Hint:* Let $x = 5$.)

**(b)** Use the model to estimate sales in 2008.

**34.** Give the domain and range of the function defined by $f(x) = |x - 3|$.

*Graph.*

**35.** $f(x) = -3x + 5$

**36.** $f(x) = -2(x - 1)^2 + 3$

**37.** $\frac{x^2}{25} + \frac{y^2}{16} \leq 1$

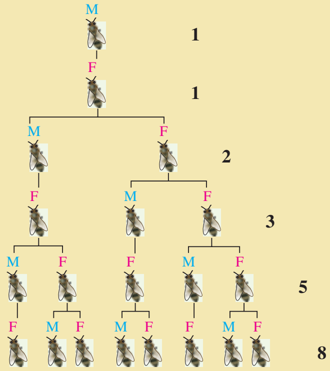
**38.** $f(x) = \sqrt{x - 2}$

**39.** $\frac{x^2}{4} - \frac{y^2}{16} = 1$

**40.** $f(x) = 3^x$

## CHAPTER 14

# Sequences and Series



The male honeybee hatches from an unfertilized egg, while the female hatches from a fertilized one. The “family tree” of a male honeybee is shown at the left, where M represents male and F represents female. Starting with the male honeybee at the top, and counting the number of bees in each generation, we obtain the following numbers in the order shown.

$$1, 1, 2, 3, 5, 8$$

Notice the pattern. After the first two terms (1 and 1), each successive term is obtained by adding the two previous terms. This sequence of numbers is called the *Fibonacci sequence*.

In this chapter, we study *sequences* and sums of terms of sequences, known as *series*.

# 14.1 Sequences and Series

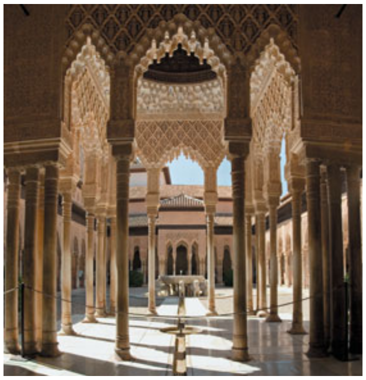
**OBJECTIVES**

1. Find the terms of a sequence, given the general term.
2. Find the general term of a sequence.
3. Use sequences to solve applied problems.
4. Use summation notation to evaluate a series.
5. Write a series with summation notation.
6. Find the arithmetic mean (average) of a group of numbers.

In the Palace of the Alhambra, residence of the Moorish rulers of Granada, Spain, the Sultana's quarters feature an interesting architectural pattern:

> There are 2 matched marble slabs inlaid in the floor, 4 walls, an octagon (8-sided) ceiling, 16 windows, 32 arches, and so on.

If this pattern is continued indefinitely, the set of numbers forms an *infinite sequence* whose *terms* are powers of 2.



### Sequence

An **infinite sequence** is a function with the set of all positive integers as the domain. A **finite sequence** is a function with domain of the form $\{1, 2, 3, \ldots, n\}$, where $n$ is a positive integer.

## OBJECTIVE 1 Find the terms of a sequence, given the general term.

For any positive integer $n$, the function value of a sequence is written as $\boldsymbol{a_n}$ (read **"$a$ sub-$n$"**). The function values $a_1, a_2, a_3, \ldots$, written in order, are the **terms** of the sequence, with $a_1$ the first term, $a_2$ the second term, and so on. The expression $a_n$, which defines the sequence, is called the **general term** of the sequence.

In the Palace of the Alhambra example, the first five terms of the sequence are

$$a_1 = 2, \quad a_2 = 4, \quad a_3 = 8, \quad a_4 = 16, \quad \text{and} \quad a_5 = 32.$$

The general term for this sequence is $a_n = 2^n$.

### EXAMPLE 1 Writing the Terms of Sequences from the General Term

Given an infinite sequence with $a_n = n + \frac{1}{n}$, find the following.

**(a)** The second term of the sequence

$$a_2 = 2 + \frac{1}{2} = \frac{5}{2} \qquad \text{Replace } n \text{ with 2.}$$

**(b)** $a_{10} = 10 + \frac{1}{10} = \frac{101}{10}$ $\quad 10 = \frac{100}{10}$

**(c)** $a_{12} = 12 + \frac{1}{12} = \frac{145}{12}$ $\quad 12 = \frac{144}{12}$

NOW TRY

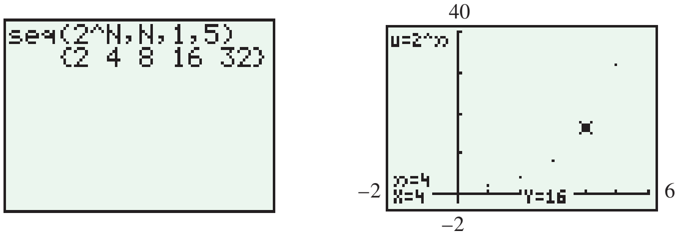
**NOW TRY EXERCISE 1**

Given an infinite sequence with $a_n = 5 - 3n$, find $a_3$.

Graphing calculators can be used to generate and graph sequences, as shown in **FIGURE 1** on the next page. The calculator must be in dot mode, so that the discrete points on the graph are not connected. ***Remember that the domain of a sequence consists only of positive integers.***

**NOW TRY ANSWER**

**1.** $a_3 = -4$



```
seq(2^N,N,1,5)
     {2 4 8 16 32}
```

The first five terms of the sequence $a_n = 2^n$

**(a)**

The first five terms of $a_n = 2^n$ are graphed here. The display indicates that the fourth term is 16; that is, $a_4 = 2^4 = 16$.

**(b)**

**FIGURE 1**

**OBJECTIVE 2** **Find the general term of a sequence.** Sometimes we need to find a general term to fit the first few terms of a given sequence.

*NOW TRY EXERCISE 2*

Find an expression for the general term $a_n$ of the sequence.

$$-3, 9, -27, 81, \dots$$

### EXAMPLE 2 Finding the General Term of a Sequence

Determine an expression for the general term $a_n$ of the sequence.

$$5, 10, 15, 20, 25, \dots$$

Notice that the terms are all multiples of 5. The first term is $5(1)$, the second is $5(2)$, and so on. The general term

$$a_n = 5n$$

will produce the given first five terms. *NOW TRY*

---

**CAUTION** Remember that when determining a general term, as in **Example 2,** there may be more than one way to express it.

---

**OBJECTIVE 3** **Use sequences to solve applied problems.** Practical problems may involve *finite sequences.*

*NOW TRY EXERCISE 3*

Chase borrows $8000 and agrees to pay $400 monthly, plus interest of 2% on the unpaid balance from the beginning of the first month. Find the payments for the first four months and the remaining debt at the end of that period.

### EXAMPLE 3 Using a Sequence in an Application

Saad Alarachi borrows $5000 and agrees to pay $500 monthly, plus interest of 1% on the unpaid balance from the beginning of the first month. Find the payments for the first four months and the remaining debt at the end of that period.

The payments and remaining balances are calculated as follows.

| | | |
|---|---|---|
| *First month* | Payment: | $\$500 + 0.01(\$5000) = \$550$ |
| | Balance: | $\$5000 - \$500 = \$4500$ |
| *Second month* | Payment: | $\$500 + 0.01(\$4500) = \$545$ |
| | Balance: | $\$5000 - 2 \cdot \$500 = \$4000$ |
| *Third month* | Payment: | $\$500 + 0.01(\$4000) = \$540$ |
| | Balance: | $\$5000 - 3 \cdot \$500 = \$3500$ |
| *Fourth month* | Payment: | $\$500 + 0.01(\$3500) = \$535$ |
| | Balance: | $\$5000 - 4 \cdot \$500 = \$3000$ |

The payments for the first four months are

$550, $545, $540, $535

and the remaining debt at the end of the period is $3000. *NOW TRY*

*NOW TRY ANSWERS*

**2.** $a_n = (-3)^n$

**3.** payments: $560, $552, $544, $536; balance: $6400

**OBJECTIVE 4** **Use summation notation to evaluate a series.** By adding the terms of a sequence, we obtain a *series.*

### Series

The indicated sum of the terms of a sequence is called a **series.**

For example, if we consider the sum of the payments listed in **Example 3,** namely,

$$550 + 545 + 540 + 535,$$

we have a series that represents the total payments for the first four months. Since a sequence can be finite or infinite, there are both finite and infinite series.

We use a compact notation, called **summation notation,** to write a series from the general term of the corresponding sequence. In mathematics, the Greek letter **Σ (sigma)** is used to denote summation. For example, the sum of the first six terms of the sequence with general term $a_n = 3n + 2$ is written as

$$\sum_{i=1}^{6} (3i + 2).$$

The letter $i$ is called the **index of summation.** We read this as "the sum from $i = 1$ to 6 of $3i + 2$." To find this sum, we replace the letter $i$ in $3i + 2$ with 1, 2, 3, 4, 5, and 6, and add the resulting terms.

**CAUTION** This use of $i$ as the index of summation has no connection with the complex number $i$.

## EXAMPLE 4 Evaluating Series Written in Summation Notation

Write out the terms and evaluate each series.

**(a)** $\displaystyle\sum_{i=1}^{6} (3i + 2)$

Multiply and then add.

$$\begin{aligned}
&= (3 \cdot 1 + 2) + (3 \cdot 2 + 2) + (3 \cdot 3 + 2) \\
&\quad + (3 \cdot 4 + 2) + (3 \cdot 5 + 2) + (3 \cdot 6 + 2) && \text{Replace } i \text{ with } 1, 2, 3, 4, 5, 6. \\
&= 5 + 8 + 11 + 14 + 17 + 20 && \text{Work inside the parentheses.} \\
&= 75 && \text{Add.}
\end{aligned}$$

**(b)** $\displaystyle\sum_{i=1}^{5} (i - 4)$

$$\begin{aligned}
&= (1 - 4) + (2 - 4) + (3 - 4) + (4 - 4) + (5 - 4) && i = 1, 2, 3, 4, 5 \\
&= -3 - 2 - 1 + 0 + 1 && \text{Subtract.} \\
&= -5 && \text{Simplify.}
\end{aligned}$$

**NOW TRY EXERCISE 4**
Write out the terms and evaluate the series.

$$\sum_{i=1}^{5}(i^2-4)$$

**(c)** $\sum_{i=3}^{7} 3i^2$

$$\begin{aligned} &= 3(3)^2 + 3(4)^2 + 3(5)^2 + 3(6)^2 + 3(7)^2 && i = 3, 4, 5, 6, 7 \\ &= 27 + 48 + 75 + 108 + 147 && \text{Square, and then multiply.} \\ &= 405 && \text{Add.} \end{aligned}$$

NOW TRY

**OBJECTIVE 5** **Write a series with summation notation.** In **Example 4,** we started with summation notation and wrote each series using + signs. Given a series, we can write it with summation notation by observing a pattern in the terms and writing the general term accordingly.

**NOW TRY EXERCISE 5**
Write each sum with summation notation.

**(a)** $3 + 5 + 7 + 9 + 11$

**(b)** $-1 - 4 - 9 - 16 - 25$

### EXAMPLE 5 Writing Series with Summation Notation

Write each sum with summation notation.

**(a)** $2 + 5 + 8 + 11$

First, find a general term $a_n$ that will give these four terms for $a_1$, $a_2$, $a_3$, and $a_4$, respectively. Each term is one less than a multiple of 3, so try $3i - 1$ as the general term.

$$\begin{aligned} 3(1) - 1 &= 2 && i = 1 \\ 3(2) - 1 &= 5 && i = 2 \\ 3(3) - 1 &= 8 && i = 3 \\ 3(4) - 1 &= 11 && i = 4 \end{aligned}$$

(Remember, there may be other expressions that also work.) Since $i$ ranges from 1 to 4,

$$2 + 5 + 8 + 11 = \sum_{i=1}^{4}(3i - 1).$$

**(b)** $8 + 27 + 64 + 125 + 216$

These numbers are the cubes of 2, 3, 4, 5, and 6, so the general term is $i^3$.

$$8 + 27 + 64 + 125 + 216 = \sum_{i=2}^{6} i^3$$

NOW TRY

**OBJECTIVE 6** **Find the arithmetic mean (average) of a group of numbers.**

**Arithmetic Mean or Average**

The **arithmetic mean,** or **average,** of a group of numbers is symbolized $\bar{x}$ and is found by dividing their sum by the number of numbers. That is,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

The values of $x_i$ represent the individual numbers in the group, and $n$ represents the number of numbers.

**NOW TRY ANSWERS**

4. $-3 + 0 + 5 + 12 + 21 = 35$

5. (a) $\sum_{i=1}^{5}(2i+1)$ (b) $\sum_{i=1}^{5} -i^2$

## EXAMPLE 6 Finding the Arithmetic Mean, or Average

The following table shows the number of FDIC-insured financial institutions for each year during the period from 2002 through 2008. What was the average number of institutions per year for this 7-yr period?

| Year | Number of Institutions |
|---|---|
| 2002 | 9369 |
| 2003 | 9194 |
| 2004 | 8988 |
| 2005 | 8845 |
| 2006 | 8691 |
| 2007 | 8544 |
| 2008 | 8314 |

*Source:* U.S. Federal Deposit Insurance Corporation.

$$\bar{x} = \frac{\sum_{i=1}^{7} x_i}{7}$$

Let $x_1 = 9369$, $x_2 = 9194$, and so on. There are 7 numbers in the group, so $n = 7$.

$$= \frac{9369 + 9194 + 8988 + 8845 + 8691 + 8544 + 8314}{7}$$

$$= 8849 \quad \text{(rounded to the nearest unit)}$$

The average number of institutions per year for this 7-yr period was 8849.

NOW TRY

NOW TRY EXERCISE 6

The following table shows the top 5 American Quarter Horse States in 2009 based on the total number of registered Quarter Horses. To the nearest whole number, what is the average number of Quarter Horses registered per state in these top five states?

| *State* | *Number of Registered Quarter Horses* |
|---|---|
| Texas | 461,054 |
| Oklahoma | 188,381 |
| California | 136,583 |
| Missouri | 107,630 |
| Colorado | 93,958 |

*Source:* American Quarter Horse Association.

NOW TRY ANSWER

**6.** 197,521

## 14.1 EXERCISES

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*Write out the first five terms of each sequence.* ***See Example 1.***

◉ **1.** $a_n = n + 1$

**2.** $a_n = n + 4$

**3.** $a_n = \frac{n+3}{n}$

**4.** $a_n = \frac{n+2}{n}$

**5.** $a_n = 3^n$

**6.** $a_n = 2^n$

**7.** $a_n = \frac{1}{n^2}$

**8.** $a_n = \frac{-2}{n^2}$

**9.** $a_n = 5(-1)^{n-1}$

**10.** $a_n = 6(-1)^{n+1}$

**11.** $a_n = n - \frac{1}{n}$

**12.** $a_n = n + \frac{4}{n}$

*Find the indicated term for each sequence.* ***See Example 1.***

**13.** $a_n = -9n + 2$; $a_8$

**14.** $a_n = 3n - 7$; $a_{12}$

**15.** $a_n = \frac{3n+7}{2n-5}$; $a_{14}$

**16.** $a_n = \frac{5n-9}{3n+8}$; $a_{16}$

**17.** $a_n = (n+1)(2n+3)$; $a_8$

**18.** $a_n = (5n-2)(3n+1)$; $a_{10}$

*Find a general term $a_n$ for the given terms of each sequence.* ***See Example 2.***

**19.** $4, 8, 12, 16, \ldots$

**20.** $7, 14, 21, 28, \ldots$

**21.** $-8, -16, -24, -32, \ldots$

**22.** $-10, -20, -30, -40, \ldots$

**23.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

**24.** $\frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \frac{2}{625}, \ldots$

**25.** $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \ldots$

**26.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

*Solve each applied problem by writing the first few terms of a sequence.* ***See Example 3.***

**27.** Horacio Loschak borrows \$1000 and agrees to pay \$100 plus interest of 1% on the unpaid balance each month. Find the payments for the first six months and the remaining debt at the end of that period.

**28.** Leslie Maruri is offered a new modeling job with a salary of $20{,}000 + 2500n$ dollars per year at the end of the $n$th year. Write a sequence showing her salary at the end of each of the first 5 yr. If she continues in this way, what will her salary be at the end of the tenth year?

**29.** Suppose that an automobile loses $\frac{1}{5}$ of its value each year; that is, at the end of any given year, the value is $\frac{4}{5}$ of the value at the beginning of that year. If a car costs \$20,000 new, what is its value at the end of 5 yr, to the nearest dollar?

**30.** A certain car loses $\frac{1}{2}$ of its value each year. If this car cost \$40,000 new, what is its value at the end of 6 yr?

*Write out each series and evaluate it.* ***See Example 4.***

**31.** $\sum_{i=1}^{5} (i + 3)$

**32.** $\sum_{i=1}^{6} (i + 9)$

**33.** $\sum_{i=1}^{3} (i^2 + 2)$

**34.** $\sum_{i=1}^{4} (i^3 + 3)$

**35.** $\sum_{i=1}^{6} (-1)^i$

**36.** $\sum_{i=1}^{5} (-1)^i \cdot i$

**37.** $\sum_{i=3}^{7} (i - 3)(i + 2)$

**38.** $\sum_{i=2}^{6} (i + 3)(i - 4)$

*Write each series with summation notation.* ***See Example 5.***

**39.** $3 + 4 + 5 + 6 + 7$

**40.** $7 + 8 + 9 + 10 + 11$

**41.** $-2 + 4 - 8 + 16 - 32$

**42.** $-1 + 2 - 3 + 4 - 5 + 6$

**43.** $1 + 4 + 9 + 16$

**44.** $1 + 16 + 81 + 256$

**45.** Explain the basic difference between a sequence and a series.

**46.** ***Concept Check*** Consider the following statement. **WHAT WENT WRONG?**

For the sequence defined by $a_n = 2n + 4$, find $a_{1/2}$.

*Find the arithmetic mean for each collection of numbers.* ***See Example 6.***

**47.** 8, 11, 14, 9, 7, 6, 8

**48.** 10, 12, 8, 19, 23, 12

**49.** 5, 9, 8, 2, 4, 7, 3, 2, 0

**50.** 2, 1, 4, 8, 3, 7, 10, 8, 0

*Solve each problem.* ***See Example 6.***

**51.** The number of mutual funds operating in the United States available to investors each year during the period 2004 through 2008 is given in the table.

| *Year* | *Number of Funds Available* |
|---|---|
| 2004 | 8041 |
| 2005 | 7975 |
| 2006 | 8117 |
| 2007 | 8024 |
| 2008 | 8022 |

*Source:* Investment Company Institute.

To the nearest whole number, what was the average number of funds available per year during the given period?

**52.** The total assets of mutual funds operating in the United States, in billions of dollars, for each year during the period 2004 through 2008 are shown in the table. What were the average assets per year during this period?

| *Year* | *Assets (in billions of dollars)* |
|---|---|
| 2004 | 8107 |
| 2005 | 8905 |
| 2006 | 10,397 |
| 2007 | 12,000 |
| 2008 | 9601 |

*Source:* Investment Company Institute.

## *PREVIEW EXERCISES*

*Find the values of a and d by solving each system.* ***See Sections 8.2 and 8.3.***

**53.** $a + 3d = 12$
$a + 8d = 22$

**54.** $a + 7d = 12$
$a + 2d = 7$

**55.** Evaluate $a + (n - 1)d$ for $a = -2$, $n = 5$, and $d = 3$. **See Sections 1.3–1.5.**

# 14.2 Arithmetic Sequences

**OBJECTIVES**

1. Find the common difference of an arithmetic sequence.
2. Find the general term of an arithmetic sequence.
3. Use an arithmetic sequence in an application.
4. Find any specified term or the number of terms of an arithmetic sequence.
5. Find the sum of a specified number of terms of an arithmetic sequence.

**OBJECTIVE 1 Find the common difference of an arithmetic sequence.** In this section, we introduce a special type of sequence that has many applications.

**Arithmetic Sequence**

An **arithmetic sequence,** or **arithmetic progression,** is a sequence in which each term after the first is found by adding a constant number to the preceding term.

For example, the sequence

$$6, 11, 16, 21, 26, \ldots \qquad \text{Arithmetic sequence}$$

is an arithmetic sequence, since the difference between any two adjacent terms is always 5. The number 5 is called the **common difference** of the arithmetic sequence. The common difference, $d$, is found by subtracting $a_n$ from $a_{n+1}$ in any such pair of terms.

$$\boldsymbol{d = a_{n+1} - a_n} \qquad \text{Common difference}$$

**NOW TRY EXERCISE 1**

Determine the common difference $d$ for the arithmetic sequence.

$$-4, -13, -22, -31, -40, \dots$$

### EXAMPLE 1 Finding the Common Difference

Determine the common difference $d$ for the arithmetic sequence.

$$-11, -4, 3, 10, 17, 24, \dots$$

Since the sequence is arithmetic, $d$ is the difference between any two adjacent terms: $a_{n+1} - a_n$. We arbitrarily choose the terms 10 and 17.

$$d = 17 - 10, \quad \text{or} \quad 7$$

Verify that *any* two adjacent terms would give the same result. NOW TRY

**NOW TRY EXERCISE 2**

Write the first five terms of the arithmetic sequence with first term 10 and common difference $-8$.

### EXAMPLE 2 Writing the Terms of a Sequence from the First Term and the Common Difference

Write the first five terms of the arithmetic sequence with first term 3 and common difference $-2$.

The second term is found by adding $-2$ to the first term 3, getting 1. For the next term, add $-2$ to 1, and so on. The first five terms are

$$3, 1, -1, -3, -5.$$

NOW TRY

**OBJECTIVE 2** **Find the general term of an arithmetic sequence.** Generalizing from **Example 2,** if we know the first term $a_1$ and the common difference $d$ of an arithmetic sequence, then the sequence is completely defined as

$$a_1, \quad a_2 = a_1 + d, \quad a_3 = a_1 + 2d, \quad a_4 = a_1 + 3d, \dots .$$

Writing the terms of the sequence in this way suggests the following formula for $a_n$.

#### General Term of an Arithmetic Sequence

The general term of an arithmetic sequence with first term $a_1$ and common difference $d$ is

$$\boldsymbol{a_n = a_1 + (n - 1)d.}$$

Since $a_n = a_1 + (n - 1)d = dn + (a_1 - d)$ is a linear function in $n$, any linear expression of the form $kn + c$, where $k$ and $c$ are real numbers, defines an arithmetic sequence.

### EXAMPLE 3 Finding the General Term of an Arithmetic Sequence

Determine the general term of the arithmetic sequence.

$$-9, -6, -3, 0, 3, 6, \dots$$

Then use the general term to find $a_{20}$.

The first term is $a_1 = -9$.

$$d = -3 - (-6), \quad \text{or} \quad 3. \qquad \text{Let } d = a_3 - a_2.$$

Now find $a_n$.

$$\begin{aligned} a_n &= a_1 + (n - 1)d && \text{Formula for } a_n \\ a_n &= -9 + (n - 1)(3) && \text{Let } a_1 = -9,\ d = 3. \\ a_n &= -9 + 3n - 3 && \text{Distributive property} \\ a_n &= 3n - 12 && \text{Combine like terms.} \end{aligned}$$

**NOW TRY ANSWERS**

**1.** $d = -9$

**2.** 10, 2, $-6$, $-14$, $-22$

The general term is $a_n = 3n - 12$. Now find $a_{20}$.

$$\begin{aligned} a_{20} &= 3(20) - 12 && \text{Let } n = 20. \\ &= 60 - 12 && \text{Multiply.} \\ &= 48 && \text{Subtract.} \end{aligned}$$

NOW TRY

**NOW TRY EXERCISE 3**

Determine the general term of the arithmetic sequence.

$$-5, 0, 5, 10, 15, \dots$$

Then use the general term to find $a_{20}$.

## OBJECTIVE 3 Use an arithmetic sequence in an application.

**NOW TRY EXERCISE 4**

Ginny Tiller is saving money for her son's college education. She makes an initial contribution of \$1000 and deposits an additional \$120 each month for the next 96 months. Disregarding interest, how much money will be in the account after 96 months?

### EXAMPLE 4 Applying an Arithmetic Sequence

Leonid Bekker's uncle decides to start a fund for Leonid's education. He makes an initial contribution of \$3000 and deposits an additional \$500 each month. Thus, after one month the fund will have \$3000 + \$500 = \$3500. How much will it have after 24 months? (Disregard any interest.)

After $n$ months, the fund will contain

$$a_n = 3000 + 500n \text{ dollars.} \qquad \text{Use an arithmetic sequence.}$$

To find the amount in the fund after 24 months, find $a_{24}$.

$$\begin{aligned} a_{24} &= 3000 + 500(24) && \text{Let } n = 24. \\ &= 3000 + 12{,}000 && \text{Multiply.} \\ &= 15{,}000 && \text{Add.} \end{aligned}$$

The account will contain \$15,000 (disregarding interest) after 24 months.

NOW TRY

## OBJECTIVE 4 Find any specified term or the number of terms of an arithmetic sequence.

The formula for the general term of an arithmetic sequence has four variables: $a_n$, $a_1$, $n$, and $d$. If we know any three of these, the formula can be used to find the value of the fourth variable.

### EXAMPLE 5 Finding Specified Terms in Sequences

Evaluate the indicated term for each arithmetic sequence.

**(a)** $a_1 = -6$, $d = 12$; $a_{15}$

$$\begin{aligned} a_n &= a_1 + (n - 1)d && \text{Formula for } a_n \\ a_{15} &= a_1 + (15 - 1)d && \text{Let } n = 15. \\ &= -6 + 14(12) && \text{Let } a_1 = -6,\ d = 12. \\ &= 162 && \text{Multiply, and then add.} \end{aligned}$$

**(b)** $a_5 = 2$ and $a_{11} = -10$; $a_{17}$

Any term can be found if $a_1$ and $d$ are known. Use the formula for $a_n$.

$$\begin{aligned} a_5 &= a_1 + (5 - 1)d \\ a_5 &= a_1 + 4d \\ 2 &= a_1 + 4d && a_5 = 2 \end{aligned}$$

$$\begin{aligned} a_{11} &= a_1 + (11 - 1)d \\ a_{11} &= a_1 + 10d \\ -10 &= a_1 + 10d && a_{11} = -10 \end{aligned}$$

This gives a system of two equations in two variables, $a_1$ and $d$.

$$\begin{aligned} a_1 + 4d &= 2 && (1) \\ a_1 + 10d &= -10 && (2) \end{aligned}$$

**NOW TRY ANSWERS**

3. $a_n = 5n - 10$; $a_{20} = 90$
4. \$12,520

**NOW TRY EXERCISE 5**

Evaluate the indicated term for each arithmetic sequence.

**(a)** $a_1 = 21$ and $d = -3$; $a_{22}$

**(b)** $a_7 = 25$ and $a_{12} = 40$; $a_{19}$

Multiply equation (2) by $-1$ and add to equation (1) to eliminate $a_1$.

$$\begin{aligned} a_1 + 4d &= 2 && \text{(1)} \\ -a_1 - 10d &= 10 && -1 \text{ times (2)} \\ \hline -6d &= 12 && \text{Add.} \\ d &= -2 && \text{Divide by } -6. \end{aligned}$$

Now find $a_1$ by substituting $-2$ for $d$ into either equation.

$$\begin{aligned} a_1 + 10(-2) &= -10 && \text{Let } d = -2 \text{ in (2).} \\ a_1 - 20 &= -10 && \text{Multiply.} \\ a_1 &= 10 && \text{Add 20.} \end{aligned}$$

Use the formula for $a_n$ to find $a_{17}$.

$$\begin{aligned} a_{17} &= a_1 + (17 - 1)d && \text{Let } n = 17. \\ &= a_1 + 16d && \text{Subtract.} \\ &= 10 + 16(-2) && \text{Let } a_1 = 10,\ d = -2. \\ &= -22 && \text{Simplify.} \end{aligned}$$

Multiply and then add.

NOW TRY

**NOW TRY EXERCISE 6**

Evaluate the number of terms in the arithmetic sequence.

$$1, \frac{4}{3}, \frac{5}{3}, 2, \ldots, 11$$

## EXAMPLE 6 Finding the Number of Terms in a Sequence

Evaluate the number of terms in the arithmetic sequence.

$$-8, -2, 4, 10, \ldots, 52$$

Let $n$ represent the number of terms in the sequence. Since $a_n = 52$, $a_1 = -8$, and $d = -2 - (-8) = 6$, use the formula for $a_n$ to find $n$.

$$\begin{aligned} a_n &= a_1 + (n - 1)d && \text{Formula for } a_n \\ 52 &= -8 + (n - 1)(6) && \text{Let } a_n = 52,\ a_1 = -8,\ d = 6. \\ 52 &= -8 + 6n - 6 && \text{Distributive property} \\ 66 &= 6n && \text{Simplify.} \\ n &= 11 && \text{Divide by 6.} \end{aligned}$$

The sequence has 11 terms.

NOW TRY

**OBJECTIVE 5 Find the sum of a specified number of terms of an arithmetic sequence.** To find a formula for the sum $S_n$ of the first $n$ terms of a given arithmetic sequence, we can write out the terms in two ways. We start with the first term, and then with the last term. Then we add the terms in columns.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d] \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d] \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \end{aligned}$$

The right-hand side of this expression contains $n$ terms, each equal to $a_1 + a_n$.

$$2S_n = n(a_1 + a_n)$$

Formula for $S_n$

$$\mathbf{S_n = \frac{n}{2}(a_1 + a_n)} \qquad \text{Divide by 2.}$$

**NOW TRY ANSWERS**

**5.** **(a)** $-42$ **(b)** 61

**6.** 31

**NOW TRY EXERCISE 7**

Evaluate the sum of the first seven terms of the arithmetic sequence in which $a_n = 5n - 7$.

### EXAMPLE 7 Finding the Sum of the First *n* Terms of an Arithmetic Sequence

Evaluate the sum of the first five terms of the arithmetic sequence in which $a_n = 2n - 5$.

Begin by evaluating $a_1$ and $a_5$.

$$\begin{aligned} a_1 &= 2(1) - 5 \\ &= -3 \end{aligned} \qquad \begin{aligned} a_5 &= 2(5) - 5 \\ &= 5 \end{aligned}$$

Now evaluate the sum using $a_1 = -3$, $a_5 = 5$, and $n = 5$.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && \text{Formula for } S_n \\ S_5 &= \frac{5}{2}(-3 + 5) && \text{Substitute.} \\ &= \frac{5}{2}(2) && \text{Add.} \\ &= 5 && \text{Multiply.} \end{aligned}$$

NOW TRY

It is possible to express the sum $S_n$ of an arithmetic sequence in terms of $a_1$ and $d$, the quantities that define the sequence. Since

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{and} \quad a_n = a_1 + (n - 1)d,$$

by substituting the expression for $a_n$ into the expression for $S_n$ we obtain

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + [a_1 + (n - 1)d]) && \text{Substitute for } a_n. \\ \boldsymbol{S_n} &\boldsymbol{= \frac{n}{2}[2a_1 + (n - 1)d].} && \text{Combine like terms.} \end{aligned}$$

The summary box gives both of the alternative forms that may be used to find the sum of the first *n* terms of an arithmetic sequence.

**Sum of the First *n* Terms of an Arithmetic Sequence**

The sum of the first *n* terms of the arithmetic sequence with first term $a_1$, *n*th term $a_n$, and common difference *d* is given by either formula.

$$\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n)} \quad \text{or} \quad \boldsymbol{S_n = \frac{n}{2}[2a_1 + (n - 1)d]}$$

### EXAMPLE 8 Finding the Sum of the First *n* Terms of an Arithmetic Sequence

Evaluate the sum of the first eight terms of the arithmetic sequence having first term 3 and common difference $-2$.

Since the known values, $a_1 = 3$, $d = -2$, and $n = 8$, appear in the second formula for $S_n$, we use it.

**NOW TRY ANSWER**

**7.** 91

**NOW TRY EXERCISE 8**

Evaluate the sum of the first nine terms of the arithmetic sequence having first term $-8$ and common difference $-5$.

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \qquad \text{Second formula for } S_n$$

$$S_8 = \frac{8}{2}[2(3) + (8-1)(-2)] \qquad \text{Let } a_1 = 3,\ d = -2,\ n = 8.$$

$$= 4[6 - 14] \qquad \text{Work inside the brackets.}$$

$$= -32 \qquad \text{Subtract and then multiply.}$$

NOW TRY

As mentioned earlier, linear expressions of the form $kn + c$, where $k$ and $c$ are real numbers, define an arithmetic sequence. For example, the sequences defined by $a_n = 2n + 5$ and $a_n = n - 3$ are arithmetic sequences. For this reason,

$$\sum_{i=1}^{n} (ki + c)$$

represents the sum of the first $n$ terms of an arithmetic sequence having first term $a_1 = k(1) + c = k + c$ and general term $a_n = k(n) + c = kn + c$. We can find this sum with the first formula for $S_n$, as shown in the next example.

**NOW TRY EXERCISE 9**

Evaluate $\sum_{i=1}^{11} (5i - 7)$.

**EXAMPLE 9** Using $S_n$ to Evaluate a Summation

Evaluate $\sum_{i=1}^{12} (2i - 1)$.

This is the sum of the first 12 terms of the arithmetic sequence having $a_n = 2n - 1$. This sum, $S_{12}$, is found with the first formula for $S_n$.

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{First formula for } S_n$$

$$S_{12} = \frac{12}{2}[\overset{a_1}{(2(1) - 1)} + \overset{a_{12}}{(2(12) - 1)}] \qquad \text{Let } n = 12.$$

$$= 6(1 + 23) \qquad \text{Evaluate } a_1 \text{ and } a_{12}.$$

$$= 6(24) \qquad \text{Add.}$$

$$= 144 \qquad \text{Multiply.}$$

NOW TRY

**NOW TRY ANSWERS**

**8.** $-252$ **9.** $253$

## 14.2 EXERCISES

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*If the given sequence is arithmetic, find the common difference d. If the sequence is not arithmetic, say so.* ***See Example 1.***

**1.** 1, 2, 3, 4, 5, . . .

**2.** 2, 5, 8, 11, . . .

**3.** 2, $-4$, 6, $-8$, 10, $-12$, . . .

**4.** 1, 2, 4, 7, 11, 16, . . .

**5.** 10, 5, 0, $-5$, $-10$, . . .

**6.** $-6$, $-10$, $-14$, $-18$, . . .

*Write the first five terms of each arithmetic sequence.* ***See Example 2.***

**7.** $a_1 = 5, d = 4$

**8.** $a_1 = 6, d = 7$

**9.** $a_1 = -2, d = -4$

**10.** $a_1 = -3, d = -5$

*Use the formula for* $a_n$ *to find the general term of each arithmetic sequence.* ***See Example 3.***

**11.** $a_1 = 2, d = 5$

**12.** $a_1 = 5, d = 3$

**13.** $3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}, \ldots$

**14.** $1, \frac{5}{3}, \frac{7}{3}, 3, \ldots$

**15.** $-3, 0, 3, \ldots$

**16.** $-10, -5, 0, \ldots$

*Evaluate the indicated term for each arithmetic sequence.* ***See Examples 3 and 5.***

**17.** $a_1 = 4, d = 3; \quad a_{25}$

**18.** $a_1 = 1, d = -3; \quad a_{12}$

**19.** $2, 4, 6, \ldots; \quad a_{24}$

**20.** $1, 5, 9, \ldots; \quad a_{50}$

**21.** $a_{12} = -45, a_{10} = -37; \quad a_1$

**22.** $a_{10} = -2, a_{15} = -8; \quad a_3$

*Evaluate the number of terms in each arithmetic sequence.* ***See Example 6.***

**23.** $3, 5, 7, \ldots, 33$

**24.** $4, 1, -2, \ldots, -32$

**25.** $\frac{3}{4}, 3, \frac{21}{4}, \ldots, 12$

**26.** $2, \frac{3}{2}, 1, \frac{1}{2}, \ldots, -5$

**27.** ***Concept Check*** In the formula for $S_n$, what does $n$ represent?

**28.** Explain when you would use each of the two formulas for $S_n$.

*Evaluate* $S_6$ *for each arithmetic sequence.* ***See Examples 7 and 8.***

**29.** $a_1 = 6, d = 3$

**30.** $a_1 = 5, d = 4$

**31.** $a_1 = 7, d = -3$

**32.** $a_1 = -5, d = -4$

**33.** $a_n = 4 + 3n$

**34.** $a_n = 9 + 5n$

*Use a formula for* $S_n$ *to evaluate each series.* ***See Example 9.***

**35.** $\sum_{i=1}^{10} (8i - 5)$

**36.** $\sum_{i=1}^{17} (3i - 1)$

**37.** $\sum_{i=1}^{20} \left(\frac{3}{2}i + 4\right)$

**38.** $\sum_{i=1}^{11} \left(\frac{1}{2}i - 1\right)$

**39.** $\sum_{i=1}^{250} i$

**40.** $\sum_{i=1}^{2000} i$

*Solve each problem. (Hint: Immediately after reading the problem, determine whether you need to find a specific term of a sequence or the sum of the terms of a sequence.)* ***See Examples 4, 7, 8, and 9.***

**41.** Nancy Bondy's aunt has promised to deposit \$1 in her account on the first day of her birthday month, \$2 on the second day, \$3 on the third day, and so on for 30 days. How much will this amount to over the entire month?

**42.** Repeat **Exercise 41,** but assume that the deposits are \$2, \$4, \$6, and so on, and that the month is February of a leap year.

**43.** Suppose that Cherian Mathew is offered a job at \$1600 per month with a guaranteed increase of \$50 every six months for 5 yr. What will Cherian's salary be at the end of that time?

**44.** Repeat **Exercise 43,** but assume that the starting salary is \$2000 per month and the guaranteed increase is \$100 every four months for 3 yr.

**45.** A seating section in a theater-in-the-round has 20 seats in the first row, 22 in the second row, 24 in the third row, and so on for 25 rows. How many seats are there in the last row? How many seats are there in the section?

**46.** Constantin Arne has started on a fitness program. He plans to jog 10 min per day for the first week and then add 10 min per day each week until he is jogging an hour each day. In which week will this occur? What is the total number of minutes he will run during the first four weeks?

**47.** A child builds with blocks, placing 35 blocks in the first row, 31 in the second row, 27 in the third row, and so on. Continuing this pattern, can she end with a row containing exactly 1 block? If not, how many blocks will the last row contain? How many rows can she build this way?

**48.** A stack of firewood has 28 pieces on the bottom, 24 on top of those, then 20, and so on. If there are 108 pieces of wood, how many rows are there? (*Hint:* $n \le 7$.)

## *PREVIEW EXERCISES*

*Evaluate* $ar^n$ *for the given values of a, r, and n.* ***See Section 4.1.***

**49.** $a = 2, r = 3, n = 2$

**50.** $a = 3, r = 2, n = 4$

**51.** $a = 4, r = \frac{1}{2}, n = 3$

**52.** $a = 5, r = \frac{1}{4}, n = 2$

# 14.3

# Geometric Sequences

**OBJECTIVES**

1. Find the common ratio of a geometric sequence.
2. Find the general term of a geometric sequence.
3. Find any specified term of a geometric sequence.
4. Find the sum of a specified number of terms of a geometric sequence.
5. Apply the formula for the future value of an ordinary annuity.
6. Find the sum of an infinite number of terms of certain geometric sequences.

In an arithmetic sequence, each term after the first is found by *adding* a fixed number to the previous term. A *geometric sequence* is defined as follows.

### Geometric Sequence

A **geometric sequence,** or **geometric progression,** is a sequence in which each term after the first is found by multiplying the preceding term by a nonzero constant.

**OBJECTIVE 1** **Find the common ratio of a geometric sequence.** We find the constant multiplier, called the **common ratio,** by dividing any term $a_{n+1}$ by the preceding term, $a_n$.

$$\boldsymbol{r = \frac{a_{n+1}}{a_n}} \qquad \text{Common ratio}$$

For example,

$$2, 6, 18, 54, 162, \ldots \qquad \text{Geometric sequence}$$

is a geometric sequence in which the first term, $a_1$, is 2 and the common ratio is

$$r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3. \leftarrow \frac{a_{n+1}}{a_n} = 3 \text{ for all } n.$$

### EXAMPLE 1 Finding the Common Ratio

Determine the common ratio $r$ for the geometric sequence.

$$15, \frac{15}{2}, \frac{15}{4}, \frac{15}{8}, \ldots$$

To find $r$, choose any two successive terms and divide the second one by the first. We choose the second and third terms of the sequence.

$$\begin{aligned} r &= \frac{a_3}{a_2} \\ &= \frac{\frac{15}{4}}{\frac{15}{2}} && \text{Substitute.} \\ &= \frac{15}{4} \div \frac{15}{2} && \text{Write as division.} \\ &= \frac{15}{4} \cdot \frac{2}{15} && \text{Definition of division} \\ &= \frac{1}{2} && \text{Multiply. Write in lowest terms.} \end{aligned}$$

Any other two successive terms could have been used to find $r$. Additional terms of the sequence can be found by multiplying each successive term by $\frac{1}{2}$. NOW TRY

NOW TRY EXERCISE 1

Determine $r$ for the geometric sequence.

$$\frac{1}{4}, -1, 4, -16, 64, \ldots$$

**OBJECTIVE 2** **Find the general term of a geometric sequence.** The general term $a_n$ of a geometric sequence $a_1, a_2, a_3, \ldots$ is expressed in terms of $a_1$ and $r$ by writing the first few terms as

$$a_1, \quad a_2 = a_1 r, \quad a_3 = a_1 r^2, \quad a_4 = a_1 r^3, \ldots,$$

which suggests the next rule.

### General Term of a Geometric Sequence

The general term of the geometric sequence with first term $a_1$ and common ratio $r$ is

$$\boldsymbol{a_n = a_1 r^{n-1}}.$$

**CAUTION** In finding $a_1 r^{n-1}$, be careful to use the correct order of operations. The value of $r^{n-1}$ must be found first. Then multiply the result by $a_1$.

### EXAMPLE 2 Finding the General Term of a Geometric Sequence

Determine the general term of the sequence in **Example 1.**

The first term is $a_1 = 15$ and the common ratio is $r = \frac{1}{2}$.

$$a_n = a_1 r^{n-1} = 15\left(\frac{1}{2}\right)^{n-1} \qquad \text{Substitute into the formula for } a_n.$$

It is not possible to simplify further, because the exponent must be applied before the multiplication can be done. NOW TRY

NOW TRY EXERCISE 2

Determine the general term of the sequence.

$$\frac{1}{4}, -1, 4, -16, 64, \ldots$$

NOW TRY ANSWERS

1. $-4$ 2. $a_n = \frac{1}{4}(-4)^{n-1}$

**OBJECTIVE 3** **Find any specified term of a geometric sequence.** We can use the formula for the general term to find any particular term.

**NOW TRY EXERCISE 3**

Evaluate the indicated term for each geometric sequence.

**(a)** $a_1 = 3, r = 2;\quad a_8$

**(b)** $10, 2, \frac{2}{5}, \frac{2}{25}, \dots;\quad a_7$

## EXAMPLE 3 Finding Specified Terms in Sequences

Evaluate the indicated term for each geometric sequence.

**(a)** $a_1 = 4, r = -3;\quad a_6$

Use the formula for the general term.

$$\begin{aligned} a_n &= a_1 r^{n-1} && \text{Formula for } a_n \\ a_6 &= a_1 \cdot r^{6-1} && \text{Let } n = 6. \\ &= 4 \cdot (-3)^5 && \text{Let } a_1 = 4,\ r = -3. \\ &= -972 && \text{Simplify.} \end{aligned}$$

Evaluate $(-3)^5$ and then multiply.

**(b)** $\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \dots;\quad a_7$

$$\begin{aligned} a_7 &= \frac{3}{4} \cdot \left(\frac{1}{2}\right)^{7-1} && \text{Let } a_1 = \tfrac{3}{4},\ r = \tfrac{1}{2},\ n = 7. \\ &= \frac{3}{4} \cdot \frac{1}{64} && \text{Apply the exponent.} \\ &= \frac{3}{256} && \text{Multiply.} \end{aligned}$$

NOW TRY

**NOW TRY EXERCISE 4**

Write the first five terms of the geometric sequence whose first term is 25 and whose common ratio is $-\frac{1}{5}$.

## EXAMPLE 4 Writing the Terms of a Sequence

Write the first five terms of the geometric sequence whose first term is 5 and whose common ratio is $\frac{1}{2}$.

$$a_1 = 5,\quad a_2 = 5\left(\frac{1}{2}\right) = \frac{5}{2},\quad a_3 = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4},$$

$$a_4 = 5\left(\frac{1}{2}\right)^3 = \frac{5}{8},\quad a_5 = 5\left(\frac{1}{2}\right)^4 = \frac{5}{16}$$

Use $a_n = a_1 r^{n-1}$, with $a_1 = 5$, $r = \frac{1}{2}$, and $n = 1, 2, 3, 4, 5$.

NOW TRY

**OBJECTIVE 4** **Find the sum of a specified number of terms of a geometric sequence.** It is convenient to have a formula for the sum $S_n$ of the first $n$ terms of a geometric sequence. We can develop a formula by first writing out $S_n$.

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

Next, we multiply both sides by $-r$.

$$-rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \cdots - a_1 r^n$$

Now add.

$$\begin{array}{rcl} S_n &=& a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \\ -rS_n &=& \phantom{a_1} - a_1 r - a_1 r^2 - a_1 r^3 - \cdots - a_1 r^{n-1} - a_1 r^n \\ \hline S_n - rS_n &=& a_1 \phantom{+ a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}} - a_1 r^n \end{array}$$

$$\begin{aligned} S_n(1 - r) &= a_1 - a_1 r^n && \text{Factor on the left.} \\ \boldsymbol{S_n} &\boldsymbol{=} \boldsymbol{\frac{a_1(1 - r^n)}{1 - r}} && \text{Factor on the right. Divide each side by } 1 - r. \end{aligned}$$

**NOW TRY ANSWERS**

3. **(a)** $3(2)^7 = 384$

**(b)** $10\left(\frac{1}{5}\right)^6 = \frac{2}{3125}$

4. $a_1 = 25, a_2 = -5, a_3 = 1,$ $a_4 = -\frac{1}{5}, a_5 = \frac{1}{25}$

### Sum of the First *n* Terms of a Geometric Sequence

The sum of the first $n$ terms of the geometric sequence with first term $a_1$ and common ratio $r$ is

$$\boldsymbol{S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1).}$$

If $r = 1$, then $S_n = a_1 + a_1 + a_1 + \cdots + a_1 = na_1$.

Multiplying the formula for $S_n$ by $\frac{-1}{-1}$ gives an alternative form.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \cdot \frac{-1}{-1} = \boldsymbol{\frac{a_1(r^n - 1)}{r - 1}} \qquad \text{Alternative form}$$

**NOW TRY EXERCISE 5**

Evaluate the sum of the first six terms of the geometric sequence with first term 4 and common ratio 2.

### EXAMPLE 5 Finding the Sum of the First *n* Terms of a Geometric Sequence

Evaluate the sum of the first six terms of the geometric sequence with first term $-2$ and common ratio 3.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad \text{Formula for } S_n$$

$$S_6 = \frac{-2(1 - 3^6)}{1 - 3} \qquad \text{Let } n = 6,\ a_1 = -2,\ r = 3.$$

$$= \frac{-2(1 - 729)}{-2} \qquad \text{Evaluate } 3^6. \text{ Subtract in the denominator.}$$

$$= -728 \qquad \text{Simplify.}$$

NOW TRY

A series of the form

$$\boldsymbol{\sum_{i=1}^{n} a \cdot b^i}$$

represents the sum of the first $n$ terms of a geometric sequence having first term $a_1 = a \cdot b^1 = ab$ and common ratio $b$. The next example illustrates this form.

**NOW TRY EXERCISE 6**

Evaluate $\sum_{i=1}^{5} 8\left(\frac{1}{2}\right)^i$.

### EXAMPLE 6 Using the Formula for $S_n$ to Find a Summation

Evaluate $\sum_{i=1}^{4} 3 \cdot 2^i$.

Since the series is in the form $\sum_{i=1}^{n} a \cdot b^i$, it represents the sum of the first $n$ terms of the geometric sequence with $a_1 = a \cdot b^1$ and $r = b$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad \text{Formula for } S_n$$

$$S_4 = \frac{6(1 - 2^4)}{1 - 2} \qquad \text{Let } n = 4,\ a_1 = 6,\ r = 2.$$

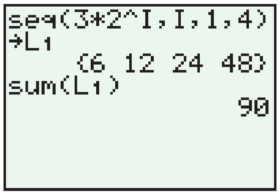
$$= \frac{6(1 - 16)}{-1} \qquad \text{Evaluate } 2^4. \text{ Subtract in the denominator.}$$

$$= 90 \qquad \text{Simplify.}$$

NOW TRY

**NOW TRY ANSWERS**

**5.** 252 **6.** 7.75, or $\frac{31}{4}$



```
seq(3*2^I,I,1,4)
→L₁
   {6 12 24 48}
sum(L₁)
             90
```

FIGURE 2

FIGURE 2 shows how a graphing calculator can store the terms in a list and then find the sum of these terms. The figure supports the result of **Example 6.**

**OBJECTIVE 5 Apply the formula for the future value of an ordinary annuity.** A sequence of equal payments made over equal periods is called an **annuity.** If the payments are made at the end of the period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity.** The time between payments is the **payment period,** and the time from the beginning of the first payment period to the end of the last is called the **term of the annuity.** The **future value of the annuity,** the final sum on deposit, is defined as the sum of the compound amounts of all the payments, compounded to the end of the term.

We state the following formula without proof.

### Future Value of an Ordinary Annuity

The future value of an ordinary annuity is

$$\boldsymbol{S = R\left[\frac{(1+i)^n - 1}{i}\right],}$$

where $S$ is the future value,
$R$ is the payment at the end of each period,
$i$ is the interest rate per period, and
$n$ is the number of periods.

NOW TRY EXERCISE 7

**(a)** Billy Harmon deposits \$600 at the end of each year into an account paying 2.5% per yr, compounded annually. Find the total amount on deposit after 18 yr.

**(b)** How much will be in Billy Harmon's account after 18 yr if he deposits \$100 at the end of each month at 3% interest compounded monthly?

### EXAMPLE 7 Applying the Formula for the Future Value of an Annuity

**(a)** Igor Kalugin is an athlete who believes that his playing career will last 7 yr. He deposits \$22,000 at the end of each year for 7 yr in an account paying 6% compounded annually. How much will he have on deposit after 7 yr?

Igor's payments form an ordinary annuity with $R = 22{,}000$, $n = 7$, and $i = 0.06$. The future value of this annuity (from the formula) is

$$S = 22{,}000\left[\frac{(1.06)^7 - 1}{0.06}\right]$$

$$= 184{,}664.43, \quad \text{or} \quad \$184{,}664.43. \qquad \text{Use a calculator.}$$

**(b)** Amy Loschak has decided to deposit \$200 at the end of each month in an account that pays interest of 4.8% compounded monthly for retirement in 20 yr. How much will be in the account at that time?

Because the interest is compounded monthly, $i = \frac{0.048}{12}$. Also, $R = 200$ and $n = 12(20)$. The future value is

$$S = 200\left[\frac{\left(1 + \frac{0.048}{12}\right)^{12(20)} - 1}{\frac{0.048}{12}}\right] = 80{,}335.01, \quad \text{or} \quad \$80{,}335.01.$$

NOW TRY

**OBJECTIVE 6 Find the sum of an infinite number of terms of certain geometric sequences.** Consider an infinite geometric sequence such as

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots .$$

NOW TRY ANSWERS

7. **(a)** \$13,431.81 **(b)** \$28,594.03

The sum of the first two terms is

$$S_2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} = 0.5.$$

In a similar manner, we can find additional "partial sums."

$$S_3 = S_2 + \frac{1}{12} = \frac{1}{2} + \frac{1}{12} = \frac{7}{12} \approx 0.583, \quad S_4 = S_3 + \frac{1}{24} = \frac{7}{12} + \frac{1}{24} = \frac{15}{24} = 0.625,$$

$$S_5 = \frac{31}{48} \approx 0.64583, \quad S_6 = \frac{21}{32} = 0.65625, \quad S_7 = \frac{127}{192} \approx 0.6614583.$$

Each term of the geometric sequence is less than the preceding one, so each additional term is contributing less and less to the partial sum. In decimal form (to the nearest thousandth), the first 7 terms and the 10th term are given in the table.

| *Term* | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_{10}$ |
|---|---|---|---|---|---|---|---|---|
| *Value* | 0.333 | 0.167 | 0.083 | 0.042 | 0.021 | 0.010 | 0.005 | 0.001 |

As the table suggests, the value of a term gets closer and closer to 0 as the number of the term increases. To express this idea, we say that as $n$ increases without bound (written $n \to \infty$), the limit of the term $a_n$ is 0, written

$$\lim_{n\to\infty} a_n = 0.$$

A number that can be defined as the sum of an infinite number of terms of a geometric sequence is found by starting with the expression for the sum of a finite number of terms.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

If $|r| < 1$, then as $n$ increases without bound, the value of $r^n$ gets closer and closer to 0. As $r^n$ approaches 0, $1 - r^n$ approaches $1 - 0 = 1$, and $S_n$ approaches the quotient $\frac{a_1}{1 - r}$.

$$\lim_{r^n\to 0} S_n = \lim_{r^n\to 0} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}$$

This limit is defined to be the sum of the infinite geometric sequence.

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots = \frac{a_1}{1 - r}, \quad \text{if } |r| < 1$$

### Sum of the Terms of an Infinite Geometric Sequence

The sum $S$ of the terms of an infinite geometric sequence with first term $a_1$ and common ratio $r$, where $|r| < 1$, is

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, then the sum does not exist.

Now consider $|r| > 1$. For example, suppose the sequence is

$$6, 12, 24, \ldots, 3(2)^n, \ldots.$$

In this kind of sequence, as $n$ increases, the value of $r^n$ also increases and so does the sum $S_n$. Since each new term adds a greater and greater amount to the sum, there is no limit to the value of $S_n$. The sum $S$ does not exist. A similar situation exists if $r = 1$.

**NOW TRY EXERCISE 8**

Evaluate the sum of the terms of the infinite geometric sequence with $a_1 = -4$ and $r = \frac{2}{3}$.

### EXAMPLE 8 Finding the Sum of the Terms of an Infinite Geometric Sequence

Evaluate the sum of the terms of the infinite geometric sequence with $a_1 = 3$ and $r = -\frac{1}{3}$.

Substitute into the formula.

$$\begin{aligned} S &= \frac{a_1}{1-r} && \text{Infinite sum formula} \\ &= \frac{3}{1-\left(-\frac{1}{3}\right)} && \text{Let } a_1 = 3,\ r = -\tfrac{1}{3}. \\ &= \frac{3}{\frac{4}{3}} && \text{Simplify the denominator.} \\ &= 3 \div \frac{4}{3} && \text{Write as division.} \\ &= 3 \cdot \frac{3}{4} && \text{Definition of division} \\ &= \frac{9}{4} && \text{Multiply.} \end{aligned}$$

NOW TRY

In summation notation, the sum of an infinite geometric sequence is written as

$$\sum_{i=1}^{\infty} a_i.$$

For instance, the sum in **Example 8** would be written

$$\sum_{i=1}^{\infty} 3\left(-\frac{1}{3}\right)^{i-1}.$$

**NOW TRY EXERCISE 9**

Evaluate $\sum_{i=1}^{\infty} \left(\frac{5}{8}\right)\left(\frac{3}{4}\right)^i$.

### EXAMPLE 9 Finding the Sum of the Terms of an Infinite Geometric Series

Evaluate $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$.

This is the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. Since $|r| < 1$, we find the sum as follows.

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ &= \frac{\frac{1}{2}}{1-\frac{1}{2}} && \text{Let } a_1 = \tfrac{1}{2},\ r = \tfrac{1}{2}. \\ &= \frac{\frac{1}{2}}{\frac{1}{2}} && \text{Simplify the denominator.} \\ &= 1 && \text{Divide.} \end{aligned}$$

NOW TRY

**NOW TRY ANSWERS**

**8.** $-12$ **9.** $\frac{15}{8}$

## 14.3 EXERCISES

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*If the given sequence is geometric, find the common ratio r. If the sequence is not geometric, say so.* **See Example 1.**

**1.** $4, 8, 16, 32, \ldots$

**2.** $5, 15, 45, 135, \ldots$

**3.** $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots$

**4.** $\frac{5}{7}, \frac{8}{7}, \frac{11}{7}, 2, \ldots$

**5.** $1, -3, 9, -27, 81, \ldots$

**6.** $2, -8, 32, -128, \ldots$

**7.** $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

**8.** $\frac{2}{3}, -\frac{2}{15}, \frac{2}{75}, -\frac{2}{375}, \ldots$

*Find a general term for each geometric sequence.* **See Example 2.**

**9.** $-5, -10, -20, \ldots$

**10.** $-2, -6, -18, \ldots$

**11.** $-2, \frac{2}{3}, -\frac{2}{9}, \ldots$

**12.** $-3, \frac{3}{2}, -\frac{3}{4}, \ldots$

**13.** $10, -2, \frac{2}{5}, \ldots$

**14.** $8, -2, \frac{1}{2}, \ldots$

*Evaluate the indicated term for each geometric sequence.* **See Example 3.**

**15.** $a_1 = 2, r = 5;\quad a_{10}$

**16.** $a_1 = 1, r = 3;\quad a_{15}$

**17.** $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots;\quad a_{12}$

**18.** $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \ldots;\quad a_{18}$

**19.** $a_3 = \frac{1}{2}, a_7 = \frac{1}{32};\quad a_{25}$

**20.** $a_5 = 48, a_8 = -384;\quad a_{10}$

*Write the first five terms of each geometric sequence.* **See Example 4.**

**21.** $a_1 = 2, r = 3$

**22.** $a_1 = 4, r = 2$

**23.** $a_1 = 5, r = -\frac{1}{5}$

**24.** $a_1 = 6, r = -\frac{1}{3}$

*Use the formula for $S_n$ to determine the sum of the terms of each geometric sequence.* **See Examples 5 and 6.** *In Exercises 27–32, give the answer to the nearest thousandth.*

**25.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$

**26.** $\frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}$

**27.** $-\frac{4}{3}, -\frac{4}{9}, -\frac{4}{27}, -\frac{4}{81}, -\frac{4}{243}, -\frac{4}{729}$

**28.** $\frac{5}{16}, -\frac{5}{32}, \frac{5}{64}, -\frac{5}{128}, \frac{5}{256}$

**29.** $\sum_{i=1}^{7} 4\left(\frac{2}{5}\right)^i$

**30.** $\sum_{i=1}^{8} 5\left(\frac{2}{3}\right)^i$

**31.** $\sum_{i=1}^{10} (-2)\left(\frac{3}{5}\right)^i$

**32.** $\sum_{i=1}^{6} (-2)\left(-\frac{1}{2}\right)^i$

*Solve each problem involving an ordinary annuity.* **See Example 7.**

**33.** A father opened a savings account for his daughter on her first birthday, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her 21st birthday. If the account pays 4.4% interest compounded annually, how much is in the account at the end of the day on the daughter's 21st birthday?

**34.** B. G. Thompson puts $1000 in a retirement account at the end of each quarter $\left(\frac{1}{4}\text{ of a year}\right)$ for 15 yr. If the account pays 4% annual interest compounded quarterly, how much will be in the account at that time?

**35.** At the end of each quarter, a 50-year-old woman puts $1200 in a retirement account that pays 5% interest compounded quarterly. When she reaches age 60, she withdraws the entire amount and places it in a mutual fund that pays 6% interest compounded monthly. From then on, she deposits $300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65?

**36.** Derrick Ruffin deposits $10,000 at the end of each year for 12 yr in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 yr. Find the final amount on deposit after the entire 21-yr period.

*Find the sum, if it exists, of the terms of each infinite geometric sequence.* ***See Examples 8 and 9.***

**37.** $a_1 = 6, r = \frac{1}{3}$

**38.** $a_1 = 10, r = \frac{1}{5}$

**39.** $a_1 = 1000, r = -\frac{1}{10}$

**40.** $a_1 = 8800, r = -\frac{3}{5}$

**41.** $\sum_{i=1}^{\infty} \frac{9}{8}\left(-\frac{2}{3}\right)^i$

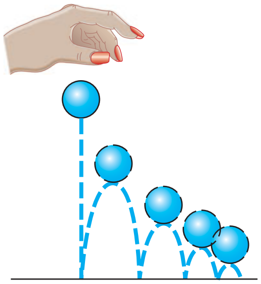
**42.** $\sum_{i=1}^{\infty} \frac{3}{5}\left(\frac{5}{6}\right)^i$

**43.** $\sum_{i=1}^{\infty} \frac{12}{5}\left(\frac{5}{4}\right)^i$

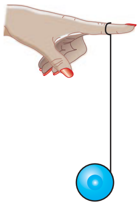
**44.** $\sum_{i=1}^{\infty} \left(-\frac{16}{3}\right)\left(-\frac{9}{8}\right)^i$

*Solve each application.* (*Hint: Immediately after reading the problem, determine whether you need to find a specific term of a sequence or the sum of the terms of a sequence.*)

**45.** When dropped from a certain height, a ball rebounds $\frac{3}{5}$ of the original height. How high will the ball rebound after the fourth bounce if it was dropped from a height of 10 ft?



**46.** A fully wound yo-yo has a string 40 in. long. It is allowed to drop, and on its first rebound it returns to a height 15 in. lower than its original height. Assuming that this "rebound ratio" remains constant until the yo-yo comes to rest, how far does it travel on its third trip up the string?



**47.** A particular substance decays in such a way that it loses half its weight each day. In how many days will 256 g of the substance be reduced to 32 g? How much of the substance is left after 10 days?

**48.** A tracer dye is injected into a system with an ingestion and an excretion. After 1 hr, $\frac{2}{3}$ of the dye is left. At the end of the second hour, $\frac{2}{3}$ of the remaining dye is left, and so on. If one unit of the dye is injected, how much is left after 6 hr?

**49.** In a certain community, the consumption of electricity has increased about 6% per yr.

**(a)** If a community uses 1.1 billion units of electricity now, how much will it use 5 yr from now?

**(b)** Find the number of years it will take for the consumption to double.

**50.** Suppose the community in **Exercise 49** reduces its increase in consumption to 2% per yr.

**(a)** How much will it use 5 yr from now?

**(b)** Find the number of years it will take for the consumption to double.

**51.** A machine depreciates by $\frac{1}{4}$ of its value each year. If it cost \$50,000 new, what is its value after 8 yr?

**52.** Refer to **Exercise 46.** Theoretically, how far does the yo-yo travel before coming to rest?

## *RELATING CONCEPTS* EXERCISES 53–58

### FOR INDIVIDUAL OR GROUP WORK

*In **Chapter 1,** we learned that any repeating decimal is a rational number; that is, it can be expressed as a quotient of integers. Thus, the repeating decimal*

$$0.99999\ldots,$$

*with an endless string of* 9s, *must be a rational number. **Work Exercises 53–58 in order,** to discover the surprising simplest form of this rational number.*

**53.** Use long division to write a repeating decimal representation for $\frac{1}{3}$.

**54.** Use long division to write a repeating decimal representation for $\frac{2}{3}$.

**55.** Because $\frac{1}{3} + \frac{2}{3} = 1$, the sum of the decimal representations in **Exercises 53 and 54** must also equal 1. Line up the decimals in the usual vertical method for addition, and obtain the repeating decimal result. The value of this decimal is exactly 1.

**56.** The repeating decimal 0.99999... can be written as the sum of the terms of a geometric sequence with $a_1 = 0.9$ and $r = 0.1$.

$$0.99999\ldots = 0.9 + 0.9(0.1) + 0.9(0.1)^2 + 0.9(0.1)^3 + 0.9(0.1)^4 + 0.9(0.1)^5 + \cdots$$

Since $|0.1| < 1$, this sum can be found from the formula $S = \frac{a_1}{1 - r}$. Use this formula to support the result you found another way in **Exercises 53–55.**

**57.** Which one of the following is true, based on your results in **Exercises 55 and 56?**

**A.** $0.99999\ldots < 1$ **B.** $0.99999\ldots = 1$ **C.** $0.99999\ldots \approx 1$

**58.** Show that $0.49999\ldots = \frac{1}{2}$.

## *PREVIEW EXERCISES*

*Multiply.* ***See Section 4.6.***

**59.** $(3x + 2y)^2$

**60.** $(4x - 3y)^2$

**61.** $(a - b)^3$

**62.** $(x + y)^4$

# 14.4

# The Binomial Theorem

**OBJECTIVES**

1. Expand a binomial raised to a power.
2. Find any specified term of the expansion of a binomial.

**OBJECTIVE 1** **Expand a binomial raised to a power.** Observe the expansion of the expression $(x+y)^n$ for the first six nonnegative integer values of $n$.

$$\begin{aligned}(x+y)^0 &= 1,\\ (x+y)^1 &= x+y,\\ (x+y)^2 &= x^2+2xy+y^2,\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3,\\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4,\\ (x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\end{aligned}$$

Expansions of $(x+y)^n$

By identifying patterns, we can write a general expansion for $(x+y)^n$.

First, if $n$ is a positive integer, each expansion after $(x+y)^0$ begins with $x$ raised to the same power to which the binomial is raised. That is, the expansion of $(x+y)^1$ has a first term of $x^1$, the expansion of $(x+y)^2$ has a first term of $x^2$, and so on. Also, the last term in each expansion is $y$ to this same power, so the expansion of $(x+y)^n$ should begin with the term $x^n$ and end with the term $y^n$.

The exponents on $x$ decrease by 1 in each term after the first, while the exponents on $y$, beginning with $y$ in the second term, increase by 1 in each succeeding term. Thus, the *variables* in the expansion of $(x+y)^n$ have the following pattern.

$$x^n,\quad x^{n-1}y,\quad x^{n-2}y^2,\quad x^{n-3}y^3,\quad \dots,\quad xy^{n-1},\quad y^n$$

This pattern suggests that the sum of the exponents on $x$ and $y$ in each term is $n$. For example, in the third term shown, the variable part is $x^{n-2}y^2$ and the sum of the exponents, $n-2$ and 2, is $n$.

Now examine the pattern for the *coefficients* of the terms of the preceding expansions. Writing the coefficients alone in a triangular pattern gives **Pascal's triangle,** named in honor of the 17th-century mathematician Blaise Pascal.

**Blaise Pascal (1623–1662)**

**Pascal's Triangle**

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}\quad \text{and so on}$$

The first and last terms of each row are 1. Each number in the interior of the triangle is the sum of the two numbers just above it (one to the right and one to the left). For example, in the fifth row from the top, 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on.

To obtain the coefficients for $(x+y)^6$, we attach the seventh row to the table by starting and ending with 1, and adding pairs of numbers from the sixth row.

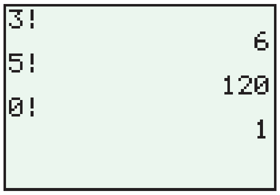
$$1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1$$

Seventh row

We then use these coefficients to expand $(x+y)^6$ as

$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6.$$

Although it is possible to use Pascal's triangle to find the coefficients in $(x + y)^n$ for any positive integer value of $n$, it is impractical for large values of $n$. A more efficient way to determine these coefficients uses the symbol $\boldsymbol{n!}$ (read **"*n* factorial"**), defined as follows.



```
3!
                 6
5!
               120
0!
                 1
```

FIGURE 3

### *n* Factorial ($n!$)

For any positive integer $n$,

$$\boldsymbol{n! = n(n-1)(n-2)(n-3)\cdots(2)(1).}$$

By definition, $\boldsymbol{0! = 1.}$

NOW TRY EXERCISE 1

Evaluate.

$7!$

### EXAMPLE 1 Evaluating Factorials

Evaluate each factorial.

**(a)** $3! = 3 \cdot 2 \cdot 1 = 6$

**(b)** $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

**(c)** $0! = 1$ 0! is defined to be 1.

FIGURE 3 shows how a graphing calculator computes factorials. NOW TRY

NOW TRY EXERCISE 2

Find the value of each expression.

**(a)** $\frac{8!}{6!2!}$ **(b)** $\frac{8!}{5!3!}$

**(c)** $\frac{6!}{6!0!}$ **(d)** $\frac{6!}{5!1!}$

### EXAMPLE 2 Evaluating Expressions Involving Factorials

Find the value of each expression.

**(a)** $$\frac{5!}{4!1!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = 5$$

**(b)** $$\frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

**(c)** $$\frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

**(d)** $$\frac{4!}{4!0!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = 1$$

NOW TRY

Now look again at the coefficients of the expansion

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

The coefficient of the second term is 5, and the exponents on the variables in that term are 4 and 1. From **Example 2(a),** $\frac{5!}{4!1!} = 5$. The coefficient of the third term is 10, and the exponents are 3 and 2. From **Example 2(b),** $\frac{5!}{3!2!} = 10$. Similar results are true for the remaining terms. The first term can be written as $1x^5y^0$, and the last term can be written as $1x^0y^5$. Then the coefficient of the first term should be $\frac{5!}{5!0!} = 1$, and the coefficient of the last term would be $\frac{5!}{0!5!} = 1$.

The coefficient of a term in $(x + y)^n$ in which the variable part is $x^ry^{n-r}$ is

$$\frac{n!}{r!(n-r)!}.$$ This is called a **binomial coefficient.**

The binomial coefficient $\frac{n!}{r!(n-r)!}$ is often represented by the symbol $\boldsymbol{{}_nC_r}$. This notation comes from the fact that if we choose *combinations* of $n$ things taken $r$ at a time, the result is given by that expression. We read ${}_nC_r$ as **"combinations of *n* things taken *r* at a time."** Another common representation is $\boldsymbol{\binom{n}{r}}$.

NOW TRY ANSWERS

**1.** 5040

**2.** **(a)** 28 **(b)** 56 **(c)** 1 **(d)** 6

### Formula for the Binomial Coefficient ${}_nC_r$

For nonnegative integers $n$ and $r$, where $r \leq n$,

$$ {}_nC_r = \frac{n!}{r!(n-r)!}. $$

**NOW TRY EXERCISE 3**

Evaluate ${}_7C_2$.

## EXAMPLE 3 Evaluating Binomial Coefficients

Evaluate each binomial coefficient.

$$
\begin{aligned}
\textbf{(a)}\quad {}_5C_4 &= \frac{5!}{4!(5-4)!} && \text{Let } n = 5,\ r = 4. \\
&= \frac{5!}{4!1!} && \text{Subtract.} \\
&= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} && \text{Definition of } n \text{ factorial} \\
&= 5 && \text{Lowest terms}
\end{aligned}
$$

Binomial coefficients will always be whole numbers.

$$ \textbf{(b)}\quad {}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10 $$

$$ \textbf{(c)}\quad {}_6C_3 = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 20 $$

```
5 nCr 4
                5
5 nCr 3
               10
6 nCr 3
               20
```

**FIGURE 4**

**FIGURE 4** shows how a graphing calculator displays the binomial coefficients computed here.

NOW TRY

We now state the **binomial theorem,** or the **general binomial expansion.**

### Binomial Theorem

For any positive integer $n$,

$$
\begin{aligned}
(x+y)^n = x^n &+ \frac{n!}{1!(n-1)!}x^{n-1}y + \frac{n!}{2!(n-2)!}x^{n-2}y^2 \\
&+ \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \cdots + \frac{n!}{(n-1)!1!}xy^{n-1} + y^n.
\end{aligned}
$$

The binomial theorem can be written in summation notation as

$$ (x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}x^{n-k}y^k. $$

**NOTE** We used the letter $k$ as the summation index letter in the statement just given. This is customary notation in mathematics.

NOW TRY ANSWER

**3.** 21

NOW TRY EXERCISE 4

Expand $(a + 3b)^5$.

## EXAMPLE 4 Using the Binomial Theorem

Expand $(2m + 3)^4$.

$$
\begin{aligned}
&(2m+3)^4\\
&= (2m)^4 + \frac{4!}{1!3!}(2m)^3(3) + \frac{4!}{2!2!}(2m)^2(3)^2 + \frac{4!}{3!1!}(2m)(3)^3 + 3^4\\
&= 16m^4 + 4(8m^3)(3) + 6(4m^2)(9) + 4(2m)(27) + 81\\
&= 16m^4 + 96m^3 + 216m^2 + 216m + 81
\end{aligned}
$$

Remember: $(ab)^m = a^m b^m$.

NOW TRY

NOW TRY EXERCISE 5

Expand $\left(\frac{x}{3} - 2y\right)^4$.

## EXAMPLE 5 Using the Binomial Theorem

Expand $\left(a - \frac{b}{2}\right)^5$.

$$
\begin{aligned}
&\left(a - \frac{b}{2}\right)^5\\
&= a^5 + \frac{5!}{1!4!}a^4\left(-\frac{b}{2}\right) + \frac{5!}{2!3!}a^3\left(-\frac{b}{2}\right)^2 + \frac{5!}{3!2!}a^2\left(-\frac{b}{2}\right)^3\\
&\quad + \frac{5!}{4!1!}a\left(-\frac{b}{2}\right)^4 + \left(-\frac{b}{2}\right)^5\\
&= a^5 + 5a^4\left(-\frac{b}{2}\right) + 10a^3\left(\frac{b^2}{4}\right) + 10a^2\left(-\frac{b^3}{8}\right)\\
&\quad + 5a\left(\frac{b^4}{16}\right) + \left(-\frac{b^5}{32}\right)\\
&= a^5 - \frac{5}{2}a^4b + \frac{5}{2}a^3b^2 - \frac{5}{4}a^2b^3 + \frac{5}{16}ab^4 - \frac{1}{32}b^5
\end{aligned}
$$

Notice that signs alternate positive and negative.

NOW TRY

**CAUTION** When the binomial is the *difference* of two terms, as in **Example 5,** the signs of the terms in the expansion will alternate. Those terms with odd exponents on the second variable expression $\left(-\frac{b}{2}\text{ in } \textbf{Example 5}\right)$ will be negative, while those with even exponents on the second variable expression will be positive.

## OBJECTIVE 2 Find any specified term of the expansion of a binomial.

Any single term of a binomial expansion can be determined without writing out the whole expansion. For example, if $n \geq 10$, then the 10th term of $(x + y)^n$ has $y$ raised to the ninth power (since $y$ has the power of 1 in the second term, the power of 2 in the third term, and so on). Since the exponents on $x$ and $y$ in any term must have a sum of $n$, the exponent on $x$ in the 10th term is $n - 9$. The quantities 9 and $n - 9$ determine the factorials in the denominator of the coefficient. Thus, the 10th term of $(x + y)^n$ is

$$\frac{n!}{9!(n-9)!}x^{n-9}y^9.$$

### *r*th Term of the Binomial Expansion

If $n \geq r - 1$, then the $r$th term of the expansion of $(x + y)^n$ is

$$\frac{\boldsymbol{n!}}{\boldsymbol{(r-1)![n-(r-1)]!}}\boldsymbol{x^{n-(r-1)}y^{r-1}}.$$

NOW TRY ANSWERS

**4.** $a^5 + 15a^4b + 90a^3b^2 + 270a^2b^3 + 405ab^4 + 243b^5$

**5.** $\frac{x^4}{81} - \frac{8x^3y}{27} + \frac{8x^2y^2}{3} - \frac{32xy^3}{3} + 16y^4$

In this general expression, remember to start with the exponent on $y$, which is 1 less than the term number $r$. Then subtract that exponent from $n$ to get the exponent on $x$: $n - (r - 1)$. The two exponents are then used as the factorials in the denominator of the coefficient.

**NOW TRY EXERCISE 6**

Find the sixth term of the expansion of $(2m - n^2)^8$.

## EXAMPLE 6 Finding a Single Term of a Binomial Expansion

Find the fourth term of the expansion of $(a + 2b)^{10}$.

In the fourth term, $2b$ has an exponent of $4 - 1 = 3$ and $a$ has an exponent of $10 - 3 = 7$. The fourth term is determined as follows.

$$\frac{10!}{3!7!}(a^7)(2b)^3$$

Parentheses MUST be used for $2b$.

$$= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}(a^7)(8b^3) \qquad \text{Let } n = 10, x = a, y = 2b, r = 4.$$

$$= 120a^7(8b^3) \qquad \text{Simplify the factorials.}$$

$$= 960a^7b^3 \qquad \text{Multiply.}$$

NOW TRY

**NOW TRY ANSWER**

6. $-448m^3n^{10}$

## 14.4 EXERCISES

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*Evaluate each expression.* ***See Examples 1–3.***

◉ **1.** $6!$
**2.** $4!$
**3.** $8!$
**4.** $9!$
◉ **5.** $\frac{6!}{4!2!}$
**6.** $\frac{7!}{3!4!}$
**7.** $\frac{4!}{0!4!}$
**8.** $\frac{5!}{5!0!}$
**9.** $4! \cdot 5$
**10.** $6! \cdot 7$
◉ **11.** ${}_6C_2$
**12.** ${}_7C_4$
**13.** ${}_{13}C_{11}$
**14.** ${}_{13}C_2$

*Use the binomial theorem to expand each expression.* ***See Examples 4 and 5.***

**15.** $(m + n)^4$
**16.** $(x + r)^5$
**17.** $(a - b)^5$
**18.** $(p - q)^4$
◉ **19.** $(2x + 3)^3$
**20.** $(4x + 2)^3$
◉ **21.** $\left(\frac{x}{2} - y\right)^4$
**22.** $\left(\frac{x}{3} - 2y\right)^5$
**23.** $(x^2 + 1)^4$
**24.** $(y^3 + 2)^4$
**25.** $(3x^2 - y^2)^3$
**26.** $(2p^2 - q^2)^3$

*Write the first four terms of each binomial expansion.* ***See Examples 4 and 5.***

**27.** $(r + 2s)^{12}$
**28.** $(m + 3n)^{20}$
**29.** $(3x - y)^{14}$
**30.** $(2p - 3q)^{11}$
**31.** $(t^2 + u^2)^{10}$
**32.** $(x^2 + y^2)^{15}$

*Find the indicated term of each binomial expansion.* ***See Example 6.***

◉ **33.** $(2m + n)^{10}$; fourth term
**34.** $(a + 3b)^{12}$; fifth term
**35.** $\left(x + \frac{y}{2}\right)^8$; seventh term
**36.** $\left(a + \frac{b}{3}\right)^{15}$; eighth term
**37.** $(k - 1)^9$; third term
**38.** $(r - 4)^{11}$; fourth term
**39.** The middle term of $(x^2 + 2y)^6$
**40.** The middle term of $(m^3 + 2y)^8$
**41.** The term with $x^9y^4$ in $(3x^3 - 4y^2)^5$
**42.** The term with $x^8y^2$ in $(2x^2 + 3y)^6$

# CHAPTER 14 SUMMARY

## KEY TERMS

### 14.1

infinite sequence
finite sequence
terms of a sequence
general term
series
summation notation
index of summation
arithmetic mean (average)

### 14.2

arithmetic sequence (arithmetic progression)
common difference

### 14.3

geometric sequence (geometric progression)
common ratio
annuity
ordinary annuity
payment period
future value of an annuity
term of an annuity

### 14.4

Pascal's triangle
binomial theorem (general binomial expansion)

## NEW SYMBOLS

| | |
|---|---|
| $a_n$ | $n$th term of a sequence |
| $\sum_{i=1}^{n} a_i$ | summation notation |
| $S_n$ | sum of first $n$ terms of a sequence |
| $\lim_{n\to\infty} a_n$ | limit of $a_n$ as $n$ gets larger and larger |
| $\sum_{i=1}^{\infty} a_i$ | sum of an infinite number of terms |
| $n!$ | $n$ factorial |
| ${}_nC_r$ | binomial coefficient (combinations of $n$ things taken $r$ at a time) |

## TEST YOUR WORD POWER

*See how well you have learned the vocabulary in this chapter.*

**1.** An **infinite sequence** is
  **A.** the values of a function
  **B.** a function whose domain is the set of positive integers
  **C.** the sum of the terms of a function
  **D.** the average of a group of numbers.

**2.** A **series** is
  **A.** the sum of the terms of a sequence
  **B.** the product of the terms of a sequence
  **C.** the average of the terms of a sequence
  **D.** the function values of a sequence.

**3.** An **arithmetic sequence** is a sequence in which
  **A.** each term after the first is a constant multiple of the preceding term
  **B.** the numbers are written in a triangular array
  **C.** the terms are added
  **D.** each term after the first differs from the preceding term by a common amount.

**4.** A **geometric sequence** is a sequence in which
  **A.** each term after the first is a constant multiple of the preceding term
  **B.** the numbers are written in a triangular array
  **C.** the terms are multiplied
  **D.** each term after the first differs from the preceding term by a common amount.

**5.** The **common difference** is
  **A.** the average of the terms in a sequence
  **B.** the constant multiplier in a geometric sequence
  **C.** the difference between any two adjacent terms in an arithmetic sequence
  **D.** the sum of the terms of an arithmetic sequence.

**6.** The **common ratio** is
  **A.** the average of the terms in a sequence
  **B.** the constant multiplier in a geometric sequence
  **C.** the difference between any two adjacent terms in an arithmetic sequence
  **D.** the product of the terms of a geometric sequence.

### ANSWERS

**1.** B; *Example:* The ordered list of numbers 3, 6, 9, 12, 15, . . . is an infinite sequence.

**2.** A; *Example:* $3 + 6 + 9 + 12 + 15$, written in summation notation as $\sum_{i=1}^{5} 3i$, is a series.

**3.** D; *Example:* The sequence $-3, 2, 7, 12, 17, \ldots$ is arithmetic.

**4.** A; *Example:* The sequence 1, 4, 16, 64, 256, . . . is geometric.

**5.** C; *Example:* The common difference of the arithmetic sequence in Answer 3 is 5, since $2 - (-3) = 5$, $7 - 2 = 5$, and so on.

**6.** B; *Example:* The common ratio of the geometric sequence in Answer 4 is 4, since $\frac{4}{1} = \frac{16}{4} = \frac{64}{16} = \frac{256}{64} = 4$.

## QUICK REVIEW

| ***CONCEPTS*** | ***EXAMPLES*** |
|---|---|
| **14.1 Sequences and Series** | |
| A finite sequence is a function with domain<br>$\{1, 2, 3, \ldots n\}$,<br>while an infinite sequence has domain<br>$\{1, 2, 3, \ldots\}$.<br>The $n$th term of a sequence is symbolized $a_n$. A series is an indicated sum of the terms of a sequence. | $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}$ has general term $a_n = \frac{1}{n}$.<br>The corresponding series is the *sum*<br>$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$. |
| **14.2 Arithmetic Sequences** | |
| Assume that $a_1$ is the first term, $a_n$ is the $n$th term, and $d$ is the common difference.<br>**Common Difference**<br>$d = a_{n+1} - a_n$ | Consider the arithmetic sequence<br>$2, 5, 8, 11, \ldots$.<br>$a_1 = 2$   $a_1$ is the first term.<br>$d = 5 - 2 = 3$   Use $a_2 - a_1$.<br>(Any two successive terms could have been used.) |
| ***n*th Term**<br>$a_n = a_1 + (n - 1)d$ | The tenth term is<br>$a_{10} = 2 + (10 - 1)(3)$   Let $n = 10$.<br>$= 2 + 9 \cdot 3$, or $29$. |
| **Sum of the First *n* Terms**<br>$S_n = \frac{n}{2}(a_1 + a_n)$<br>or   $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ | The sum of the first ten terms can be found in either way.<br>$S_{10} = \frac{10}{2}(2 + a_{10})$<br>$= 5(2 + 29)$<br>$= 5(31)$<br>$= 155$<br>$S_{10} = \frac{10}{2}[2(2) + (10 - 1)(3)]$<br>$= 5(4 + 9 \cdot 3)$<br>$= 5(4 + 27)$<br>$= 5(31)$<br>$= 155$ |
| **14.3 Geometric Sequences** | |
| Assume that $a_1$ is the first term, $a_n$ is the $n$th term, and $r$ is the common ratio.<br>**Common Ratio**<br>$r = \frac{a_{n+1}}{a_n}$ | Consider the geometric sequence<br>$1, 2, 4, 8, \ldots$.<br>$a_1 = 1$   $a_1$ is the first term.<br>$r = \frac{8}{4} = 2$   Use $\frac{a_4}{a_3}$.<br>(Any two successive terms could have been used.) |
| ***n*th Term**<br>$a_n = a_1 r^{n-1}$ | The sixth term is<br>$a_6 = (1)(2)^{6-1} = 1(2)^5 = 32$.   Let $n = 6$. |
| **Sum of the First *n* Terms**<br>$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)$ | The sum of the first six terms is<br>$S_6 = \frac{1(1 - 2^6)}{1 - 2} = \frac{1 - 64}{-1} = 63$. |

(continued)

| *CONCEPTS* | *EXAMPLES* |
|---|---|
| **Future Value of an Ordinary Annuity**<br>$S = R\left[\frac{(1+i)^n - 1}{i}\right],$<br>where $S$ is the future value, $R$ is the payment at the end of each period, $i$ is the interest rate per period, and $n$ is the number of periods. | If \$5800 is deposited into an ordinary annuity at the end of each quarter for 4 yr and interest is earned at 2.4% compounded quarterly, then<br>$R = \$5800, \quad i = \frac{0.024}{4} = 0.006, \quad n = 4(4) = 16,$<br>and $\quad S = 5800\left[\frac{(1+0.006)^{16} - 1}{0.006}\right] = \$97{,}095.24.$ |
| **Sum of the Terms of an Infinite Geometric Sequence with $\lvert r\rvert < 1$**<br>$S = \frac{a_1}{1-r}$ | The sum $S$ of the terms of an infinite geometric sequence with $a_1 = 1$ and $r = \frac{1}{2}$ is<br>$S = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2.$ |

## 14.4 The Binomial Theorem

| | |
|---|---|
| **Factorials**<br>For any positive integer $n$,<br>$n! = n(n-1)(n-2)\cdots(2)(1).$<br>By definition, $\quad 0! = 1.$ | $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ |
| **Binomial Coefficient**<br>${}_nC_r = \frac{n!}{r!(n-r)!}, \quad r \le n$ | ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10$ |
| **General Binomial Expansion**<br>For any positive integer $n$,<br>$(x+y)^n$<br>$= x^n + \frac{n!}{1!(n-1)!}x^{n-1}y + \frac{n!}{2!(n-2)!}x^{n-2}y^2$<br>$+ \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \cdots + \frac{n!}{(n-1)!1!}xy^{n-1}$<br>$+ y^n.$ | $(2x-3)^4$<br>$= (2x)^4 + \frac{4!}{1!3!}(2x)^3(-3) + \frac{4!}{2!2!}(2x)^2(-3)^2 +$<br>$\frac{4!}{3!1!}(2x)(-3)^3 + (-3)^4$<br>$= 2^4x^4 - 4(2)^3x^3(3) + 6(2)^2x^2(9) - 4(2x)(27) + 81$<br>$= 16x^4 - 12(8)x^3 + 54(4)x^2 - 216x + 81$<br>$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$ |
| ***r*th Term of the Binomial Expansion of $(x+y)^n$**<br>$\frac{n!}{(r-1)![n-(r-1)]!}x^{n-(r-1)}y^{r-1}$ | The eighth term of $(a-2b)^{10}$ is<br>$\frac{10!}{7!3!}a^3(-2b)^7$<br>$= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}a^3(-2)^7b^7 \quad$ $n = 10,\ x = a,$ $y = -2b,\ r = 8$<br>$= 120(-128)a^3b^7 \quad$ Simplify.<br>$= -15{,}360a^3b^7. \quad$ Multiply. |

# CHAPTER 14

# REVIEW EXERCISES

**14.1** *Write out the first four terms of each sequence.*

**1.** $a_n = 2n - 3$

**2.** $a_n = \dfrac{n-1}{n}$

**3.** $a_n = n^2$

**4.** $a_n = \left(\dfrac{1}{2}\right)^n$

**5.** $a_n = (n+1)(n-1)$

**6.** $a_n = n(-1)^{n-1}$

*Write each series as a sum of terms.*

**7.** $\sum_{i=1}^{5} i^2$

**8.** $\sum_{i=1}^{6} (i+1)$

**9.** $\sum_{i=3}^{6} (5i-4)$

*Evaluate each series.*

**10.** $\sum_{i=1}^{4} (i+2)$

**11.** $\sum_{i=1}^{6} 2^i$

**12.** $\sum_{i=4}^{7} \dfrac{i}{i+1}$

**13.** Find the arithmetic mean, or average, of the total retirement assets of Americans for the years 2004 through 2008 shown in the table. Round to the nearest unit (in billions).

| *Year* | *Assets (in billions of dollars)* |
|---|---|
| 2004 | 13,778 |
| 2005 | 14,862 |
| 2006 | 16,680 |
| 2007 | 17,916 |
| 2008 | 13,985 |

*Source:* Investment Company Institute.

**14.2–14.3** *Decide whether each sequence is* arithmetic, geometric, *or* neither. *If the sequence is arithmetic, find the common difference d. If it is geometric, find the common ratio r.*

**14.** 2, 5, 8, 11, . . .

**15.** $-6, -2, 2, 6, 10, \ldots$

**16.** $\dfrac{2}{3}, -\dfrac{1}{3}, \dfrac{1}{6}, -\dfrac{1}{12}, \ldots$

**17.** $-1, 1, -1, 1, -1, \ldots$

**18.** $64, 32, 8, \dfrac{1}{2}, \ldots$

**19.** 64, 32, 16, 8, . . .

**20.** ***Concept Check*** Refer to the Chapter Opener on **page 855.** What is the eleventh term of the Fibonacci sequence?

**14.2** *Determine the indicated term of each arithmetic sequence.*

**21.** $a_1 = -2, d = 5;\quad a_{16}$

**22.** $a_6 = 12, a_8 = 18;\quad a_{25}$

*Determine the general term of each arithmetic sequence.*

**23.** $a_1 = -4, d = -5$

**24.** $6, 3, 0, -3, \ldots$

*Determine the number of terms in each arithmetic sequence.*

**25.** 7, 10, 13, . . . , 49

**26.** $5, 1, -3, \ldots, -79$

*Evaluate $S_8$ for each arithmetic sequence.*

**27.** $a_1 = -2, d = 6$

**28.** $a_n = -2 + 5n$

**14.3** *Determine the general term for each geometric sequence.*

**29.** $-1, -4, -16, \ldots$

**30.** $\dfrac{2}{3}, \dfrac{2}{15}, \dfrac{2}{75}, \ldots$

*Determine the indicated term for each geometric sequence.*

**31.** $2, -6, 18, \dots; \quad a_{11}$

**32.** $a_3 = 20, a_5 = 80; \quad a_{10}$

*Evaluate each sum if it exists.*

**33.** $\sum_{i=1}^{5} \left(\frac{1}{4}\right)^i$

**34.** $\sum_{i=1}^{8} \frac{3}{4}(-1)^i$

**35.** $\sum_{i=1}^{\infty} 4\left(\frac{1}{5}\right)^i$

**36.** $\sum_{i=1}^{\infty} 2(3)^i$

**14.4** *Use the binomial theorem to expand each binomial.*

**37.** $(2p - q)^5$

**38.** $(x^2 + 3y)^4$

**39.** $(3t^3 - s^2)^4$

**40.** Write the fourth term of the expansion of $(3a + 2b)^{19}$.

## MIXED REVIEW EXERCISES

*Determine the indicated term and evaluate $S_{10}$ for each sequence.*

**41.** $a_{10}$: geometric; $-3, 6, -12, \dots$

**42.** $a_{40}$: arithmetic; $1, 7, 13, \dots$

**43.** $a_{15}$: arithmetic; $a_1 = -4, \quad d = 3$

**44.** $a_9$: geometric; $a_1 = 1, \quad r = -3$

*Determine the general term for each arithmetic or geometric sequence.*

**45.** $2, 8, 32, \dots$

**46.** $2, 7, 12, \dots$

**47.** $12, 9, 6, \dots$

**48.** $27, 9, 3, \dots$

*Solve each problem.*

**49.** When Faith's sled goes down the hill near her home, she covers 3 ft in the first second. Then, for each second after that, she goes 4 ft more than in the preceding second. If the distance she covers going down is 210 ft, how long does it take her to reach the bottom?

**50.** An ordinary annuity is set up so that $672 is deposited at the end of each quarter for 7 yr. The money earns 4.5% annual interest compounded quarterly. What is the future value of the annuity?

**51.** The school population in Middleton has been dropping 3% per yr. The current population is 50,000. If this trend continues, what will the population be in 6 yr?

**52.** A pump removes $\frac{1}{2}$ of the liquid in a container with each stroke. What fraction of the liquid is left in the container after seven strokes?

**53.** Consider the repeating decimal number 0.55555 . . . .

**(a)** Write it as the sum of the terms of an infinite geometric sequence.

**(b)** What is $r$ for this sequence?

**(c)** Find this infinite sum if it exists, and write it as a common fraction in lowest terms.

**54.** Can the sum of the terms of the infinite geometric sequence defined by $a_n = 5(2)^n$ be found? Explain.

# CHAPTER 14 Test

CHAPTER Test Prep VIDEOS

Step-by-step test solutions are found on the Chapter Test Prep Videos available via the Video Resources on DVD, in MyMathLab, or on YouTube (search "LialCombinedAlgebra").

*View the complete solutions to all Chapter Test exercises on the Video Resources on DVD.*

*Write the first five terms of each sequence described.*

**1.** $a_n = (-1)^n + 1$

**2.** arithmetic, with $a_1 = 4$ and $d = 2$

**3.** geometric, with $a_4 = 6$ and $r = \frac{1}{2}$

*Determine $a_4$ for each sequence described.*

**4.** arithmetic, with $a_1 = 6$ and $d = -2$

**5.** geometric, with $a_5 = 16$ and $a_7 = 9$

*Evaluate $S_5$ for each sequence described.*

**6.** arithmetic, with $a_2 = 12$ and $a_3 = 15$

**7.** geometric, with $a_5 = 4$ and $a_7 = 1$

**8.** The numbers of commercial bank offices (main offices and branches) in the United States for the years 2004 through 2008 are given in the table. What was the average number of banks per year for that period? Round to the nearest unit.

| Year | Number |
|---|---|
| 2004 | 78,473 |
| 2005 | 80,967 |
| 2006 | 83,860 |
| 2007 | 86,150 |
| 2008 | 97,103 |

*Source:* U.S. Federal Deposit Insurance Corporation.

**9.** If \$4000 is deposited in an ordinary annuity at the end of each quarter for 7 yr and earns 6% interest compounded quarterly, how much will be in the account at the end of this term?

**10.** ***Concept Check*** Under what conditions does an infinite geometric series have a sum?

*Determine each sum that exists.*

**11.** $\sum_{i=1}^{5} (2i + 8)$

**12.** $\sum_{i=1}^{6} (3i - 5)$

**13.** $\sum_{i=1}^{500} i$

**14.** $\sum_{i=1}^{3} \frac{1}{2}(4^i)$

**15.** $\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$

**16.** $\sum_{i=1}^{\infty} 6\left(\frac{3}{2}\right)^i$

*Evaluate.*

**17.** $8!$

**18.** $0!$

**19.** $\frac{6!}{4!2!}$

**20.** ${}_{12}C_{10}$

**21.** Expand $(3k - 5)^4$.

**22.** Write the fifth term of the expansion of $\left(2x - \frac{y}{3}\right)^{12}$.

*Solve each problem.*

**23.** Christian Sabau bought a new dishwasher for \$300. He agreed to pay \$20 per month for 15 months, plus interest of 1% each month, on the unpaid balance. Find the total cost of the machine.

**24.** During the summer months, the population of a certain insect colony triples each week. If there are 20 insects in the colony at the end of the first week in July, how many are present by the end of September? (Assume exactly four weeks in a month.)

# CHAPTERS 1–14 CUMULATIVE REVIEW EXERCISES

*Simplify each expression.*

**1.** $|-7| + 6 - |-10| - (-8 + 3)$

**2.** $4(-6) + (-8)(5) - (-9)$

*Let* $P = \left\{-\frac{8}{3}, 10, 0, \sqrt{13}, -\sqrt{3}, \frac{45}{15}, \sqrt{-7}, 0.82, -3\right\}$. *List the elements of P that are members of each set.*

**3.** Rational numbers

**4.** Irrational numbers

*Solve each equation or inequality.*

**5.** $9 - (5 + 3x) + 5x = -4(x - 3) - 7$

**6.** $7x + 18 \leq 9x - 2$

**7.** $|4x - 3| = 21$

**8.** $\frac{x + 3}{12} - \frac{x - 3}{6} = 0$

**9.** $2x > 8$ or $-3x > 9$

**10.** $|2x - 5| \geq 11$

**11.** $2x^2 + x = 10$

**12.** $x^2 - x - 6 \leq 0$

**13.** $\frac{4}{x - 3} - \frac{6}{x + 3} = \frac{24}{x^2 - 9}$

**14.** $6x^2 + 5x = 8$

**15.** $3^{2x-1} = 81$

**16.** $\log_8 x + \log_8 (x + 2) = 1$

*Perform the indicated operations.*

**17.** $(4p + 2)(5p - 3)$

**18.** $(3k - 7)^2$

**19.** $(2m^3 - 3m^2 + 8m) - (7m^3 + 5m - 8)$

**20.** Divide $6t^4 + 5t^3 - 18t^2 + 14t - 1$ by $3t - 2$.

*Factor.*

**21.** $6z^3 + 5z^2 - 4z$

**22.** $49a^4 - 9b^2$

**23.** $c^3 + 27d^3$

*Simplify.*

**24.** $\left(\frac{2}{3}\right)^{-2}$

**25.** $\frac{(3p^2)^3(-2p^6)}{4p^3(5p^7)}$

**26.** $\frac{x^2 - 16}{x^2 + 2x - 8} \div \frac{x - 4}{x + 7}$

**27.** $\frac{5}{p^2 + 3p} - \frac{2}{p^2 - 4p}$

**28.** $5\sqrt{72} - 4\sqrt{50}$

**29.** $(8 + 3i)(8 - 3i)$

**30.** Find the slope of the line through $(4, -5)$ and $(-12, -17)$.

**31.** Find the standard form of the equation of the line through $(-2, 10)$ and parallel to the line with equation $3x + y = 7$.

**32.** Consider the set of ordered pairs.

$$\{(-3, 2), (-2, 6), (0, 4), (1, 2), (2, 6)\}$$

**(a)** Is this a function? **(b)** What is its domain? **(c)** What is its range?

*Solve each system of equations.*

**33.** $\begin{aligned} y &= 5x + 3 \\ 2x + 3y &= -8 \end{aligned}$

**34.** $\begin{aligned} x + 2y + z &= 8 \\ 2x - y + 3z &= 15 \\ -x + 3y - 3z &= -11 \end{aligned}$

**35.** $\begin{aligned} xy &= -5 \\ 2x + y &= 3 \end{aligned}$

**36.** Nuts worth \$3 per lb are to be mixed with 8 lb of nuts worth \$4.25 per lb to obtain a mixture that will be sold for \$4 per lb. How many pounds of the \$3 nuts should be used?

*Graph.*

**37.** $x - 3y = 6$

**38.** $4x - y < 4$

**39.** $f(x) = 2(x - 2)^2 - 3$

**40.** $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

**41.** $x^2 - y^2 = 9$

**42.** $g(x) = \left(\dfrac{1}{3}\right)^x$

**43.** $y = \log_{1/3} x$

**44.** Find $f^{-1}(x)$ if $f(x) = 9x + 5$.

**45.** Find the equation of a circle with center at $(-5, 12)$ and radius 9.

**46.** Write the first five terms of the sequence defined by $a_n = 5n - 12$.

**47.** Find each sum.

**(a)** The sum of the first six terms of the arithmetic sequence with $a_1 = 8$ and $d = 2$

**(b)** The sum of the geometric series $15 - 6 + \frac{12}{5} - \frac{24}{25} + \cdots$

**48.** Find the sum $\sum_{i=1}^{4} 3i$.

**49.** Use the binomial theorem to expand $(2a - 1)^5$.

**50.** What is the fourth term in the expansion of $\left(3x^4 - \frac{1}{2}y^2\right)^5$?

# APPENDIX A

# Sets

**OBJECTIVES**

1. Learn the vocabulary and symbols used to discuss sets.
2. Decide whether a set is finite or infinite.
3. Decide whether a given set is a subset of another set.
4. Find the complement of a set.
5. Find the union and the intersection of two sets.

**OBJECTIVE 1** **Learn the vocabulary and symbols used to discuss sets.** A **set** is a collection of objects. These objects are called the **elements** of the set. A set is represented by listing its elements between **braces,** $\{\quad\}$.* The order in which the elements of a set are listed is unimportant.

Capital letters are used to name sets. To state that 5 is an element of

$$S = \{1, 2, 3, 4, 5\},$$

write $5 \in S$. The statement $6 \notin S$ means that 6 is not an element of $S$.

The set with no elements is called the **empty set,** or the **null set.** The symbol $\emptyset$ or $\{\quad\}$ is used for the empty set. If we let $A$ be the set of all negative natural numbers, then $A$ is the empty set.

$$A = \emptyset \quad \text{or} \quad A = \{\quad\}$$

**CAUTION** Do not make the common error of writing the empty set as $\{\emptyset\}$.

**EXAMPLE 1** Listing the Elements of Sets

Represent each set by listing its elements.

**(a)** The set of states in the United States that border the Pacific Ocean is

$$\{\text{California, Oregon, Washington, Hawaii, Alaska}\}.$$

**(b)** The set of all counting numbers less than 6 $= \{1, 2, 3, 4, 5\}$.

**(c)** The set of all counting numbers less than 0 $= \emptyset$

*NOW TRY*

*NOW TRY EXERCISE 1*
List the elements of the set of odd natural numbers less than 13.

In any discussion of sets, there is some set that includes all the elements under consideration. This set is called the **universal set** for that situation. For example, if the discussion is about presidents of the United States, then the set of all presidents of the United States is the universal set. The universal set is denoted $U$.

**OBJECTIVE 2** **Decide whether a set is finite or infinite.** In **Example 1,** there are five elements in the set in part (a) and five in part (b). If the number of elements in a set is either 0 or a counting number, then the set is **finite.** By contrast, the set of natural numbers is an **infinite** set, because there is no final natural number. We can list the elements of the set of natural numbers as

$$N = \{1, 2, 3, 4, \ldots\},$$

where the three dots indicate that the set continues indefinitely. Not all infinite sets can be listed in this way. For example, there is no way to list the elements in the set of all real numbers between 1 and 2.

*NOW TRY ANSWER*

**1.** $\{1, 3, 5, 7, 9, 11\}$

---

*Some people refer to this convention as *roster notation*.

**NOW TRY EXERCISE 2**

List the elements of each set if possible. Decide whether each set is finite or infinite.

**(a)** The set of negative integers

**(b)** The set of even natural numbers between 11 and 19

### EXAMPLE 2 Distinguishing between Finite and Infinite Sets

List the elements of each set if possible. Decide whether each set is finite or infinite.

**(a)** The set of all integers
One way to list the elements is $\{\dots, -2, -1, 0, 1, 2, \dots\}$. The set is infinite.

**(b)** The set of all natural numbers between 0 and 5
$\{1, 2, 3, 4\}$ The set is finite.

**(c)** The set of all irrational numbers
This is an infinite set whose elements cannot be listed. NOW TRY

Two sets are equal if they have exactly the same elements. Thus, the set of natural numbers and the set of positive integers are equal sets. Also, the sets

$$\{1, 2, 4, 7\} \quad \text{and} \quad \{4, 2, 7, 1\} \quad \text{are equal.}$$

The order of the elements does not make a difference.

**OBJECTIVE 3** **Decide whether a given set is a subset of another set.** If all elements of a set $A$ are also elements of another set $B$, then we say that $A$ is a **subset** of $B$, written $A \subseteq B$. We use the symbol $A \nsubseteq B$ to mean that $A$ is not a subset of $B$.

**NOW TRY EXERCISE 3**

Let

$A = \{1, 3, 5, 7, 9, 11\}$,
$B = \{1, 5, 7, 9\}$, and
$C = \{1, 9, 11\}$.

Tell whether each statement is *true* or *false*.

**(a)** $B \subseteq A$ **(b)** $C \subseteq B$

**(c)** $C \nsubseteq A$

### EXAMPLE 3 Using Subset Notation

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 4\}$, and $C = \{1\}$. Then

$$B \subseteq A, \quad C \subseteq A, \quad \text{and} \quad C \subseteq B,$$

but

$$A \nsubseteq B, \quad A \nsubseteq C, \quad \text{and} \quad B \nsubseteq C.$$

NOW TRY

The empty set is defined to be a subset of any set. Thus, the set $M = \{a, b\}$ has four subsets:

$$\{a, b\}, \quad \{a\}, \quad \{b\}, \quad \text{and} \quad \emptyset.$$

How many subsets does $N = \{a, b, c\}$ have? There is one subset with three elements: $\{a, b, c\}$. There are three subsets with two elements:

$$\{a, b\}, \quad \{a, c\}, \quad \text{and} \quad \{b, c\}.$$

There are three subsets with one element:

$$\{a\}, \quad \{b\}, \quad \text{and} \quad \{c\}.$$

There is one subset with no elements: $\emptyset$. Thus, set $N$ has eight subsets.

The following generalization can be made and proved in more advanced courses.

### Number of Subsets of a Set

A set with $n$ elements has $2^n$ subsets.

To illustrate the relationships between sets, **Venn diagrams** are often used. A rectangle represents the universal set, $U$. The sets under discussion are represented by regions within the rectangle. The Venn diagram in **FIGURE 1** on the next page shows that $B \subseteq A$.

**NOW TRY ANSWERS**

**2.** **(a)** $\{-1, -2, -3, -4, \dots\}$; infinite
**(b)** $\{12, 14, 16, 18\}$; finite

**3.** **(a)** true **(b)** false **(c)** false

$B \subseteq A$

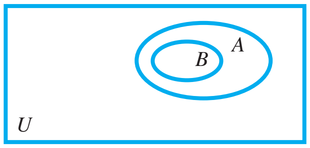

FIGURE 1

**OBJECTIVE 4** **Find the complement of a set.** For every set $A$, there is a set $\boldsymbol{A'}$**,** the **complement** of $A$, that contains all the elements of $U$ that are not in $A$. The shaded region in the Venn diagram in **FIGURE 2** represents $A'$.

$A'$ is shaded.

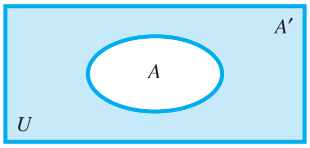

FIGURE 2

**NOW TRY EXERCISE 4**

Let

$$U = \{2, 4, 6, 8, 10, 12, 14\}$$

and $M = \{2, 10, 12, 14\}$.

List the elements in $M'$.

### EXAMPLE 4 Determining Complements of a Set

Given $U = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c\}$, $B = \{a, d, f, g\}$, and $C = \{d, e\}$, list the elements of $A'$, $B'$, and $C'$.

$$A' = \{d, e, f, g\}, \quad B' = \{b, c, e\}, \quad \text{and} \quad C' = \{a, b, c, f, g\}.$$

NOW TRY

**OBJECTIVE 5** **Find the union and the intersection of two sets.** The **union** of two sets $A$ and $B$, written $A \cup B$, is the set of all elements of $A$ together with all elements of $B$. Thus, for the sets in **Example 4,**

$$A \cup B = \{a, b, c, d, f, g\} \quad \text{and} \quad A \cup C = \{a, b, c, d, e\}.$$

In **FIGURE 3**, the shaded region is the union of sets $A$ and $B$.

$A \cup B$ is shaded.

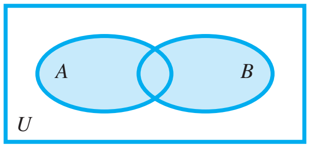

FIGURE 3

**NOW TRY EXERCISE 5**

If $M = \{1, 3, 5, 7, 9\}$ and $N = \{0, 3, 6, 9\}$, find $M \cup N$.

### EXAMPLE 5 Finding the Union of Two Sets

If $M = \{2, 5, 7\}$ and $N = \{1, 2, 3, 4, 5\}$, find $M \cup N$.

$$M \cup N = \{1, 2, 3, 4, 5, 7\}$$

NOW TRY

The **intersection** of two sets $A$ and $B$, written $A \cap B$, is the set of all elements that belong to both $A$ and $B$. For example, if

$$A = \{\text{José}, \text{Ellen}, \text{Marge}, \text{Kevin}\}$$

and

$$B = \{\text{José}, \text{Patrick}, \text{Ellen}, \text{Sue}\},$$

then

$$A \cap B = \{\text{José}, \text{Ellen}\}.$$

**NOW TRY ANSWERS**

**4.** $\{4, 6, 8\}$

**5.** $\{0, 1, 3, 5, 6, 7, 9\}$

The shaded region in **FIGURE 4** represents the intersection of the two sets $A$ and $B$.

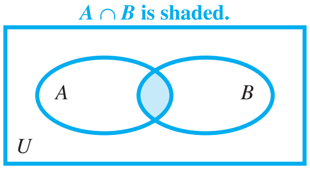

**FIGURE 4**

**NOW TRY EXERCISE 6**

If $M = \{1, 3, 5, 7, 9\}$ and $N = \{0, 3, 6, 9\}$, find $M \cap N$.

## EXAMPLE 6 Finding the Intersection of Two Sets

Suppose that $P = \{3, 9, 27\}$, $Q = \{2, 3, 10, 18, 27, 28\}$, and $R = \{2, 10, 28\}$. Find each of the following.

**(a)** $P \cap Q = \{3, 27\}$ **(b)** $Q \cap R = \{2, 10, 28\} = R$ **(c)** $P \cap R = \emptyset$

NOW TRY

Sets like $P$ and $R$ in **Example 6** that have no elements in common are called **disjoint sets.** The Venn diagram in **FIGURE 5** shows a pair of disjoint sets.

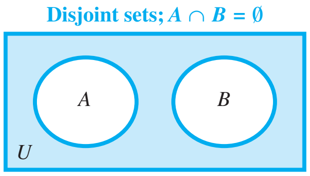

**FIGURE 5**

**NOW TRY EXERCISE 7**

Let

$U = \{1, 2, 4, 5, 7, 8, 9, 10\}$,
$A = \{1, 4, 7, 9, 10\}$,
$B = \{2, 5, 8\}$, and
$C = \{5\}$.

Find each of the following.

**(a)** $B \cup C$ **(b)** $A \cap B$ **(c)** $C'$

## EXAMPLE 7 Using Set Operations

Let $U = \{2, 5, 7, 10, 14, 20\}$, $A = \{2, 10, 14, 20\}$, $B = \{5, 7\}$, and $C = \{2, 5, 7\}$. Find each of the following.

**(a)** $A \cup B = \{2, 5, 7, 10, 14, 20\} = U$ **(b)** $A \cap B = \emptyset$

**(c)** $B \cup C = \{2, 5, 7\} = C$ **(d)** $B \cap C = \{5, 7\} = B$

**(e)** $A' = \{5, 7\} = B$

NOW TRY

**NOW TRY ANSWERS**

**6.** $\{3, 9\}$
**7.** **(a)** $\{2, 5, 8\} = B$ **(b)** $\emptyset$
**(c)** $\{1, 2, 4, 7, 8, 9, 10\}$

## EXERCISES

*List the elements of each set.* ***See Examples 1 and 2.***

1. The set of all natural numbers less than 8
2. The set of all integers between 4 and 10
3. The set of seasons
4. The set of months of the year
5. The set of women presidents of the United States before 2008
6. The set of all living humans who are more than 200 years old
7. The set of letters of the alphabet between K and M
8. The set of letters of the alphabet between D and H

**9.** The set of positive even integers

**10.** The set of all multiples of 5

**11.** Which of the sets described in **Exercises 1–10** are infinite sets?

**12.** Which of the sets described in **Exercises 1–10** are finite sets?

***Concept Check*** *Tell whether each statement is* true *or* false.

**13.** $5 \in \{1, 2, 5, 8\}$

**14.** $6 \in \{1, 2, 3, 4, 5\}$

**15.** $2 \in \{1, 3, 5, 7, 9\}$

**16.** $1 \in \{6, 2, 5, 1\}$

**17.** $7 \notin \{2, 4, 6, 8\}$

**18.** $7 \notin \{1, 3, 5, 7\}$

**19.** $\{2, 4, 9, 12, 13\} = \{13, 12, 9, 4, 2\}$

**20.** $\{7, 11, 4\} = \{7, 11, 4, 0\}$

*Let*

$$A = \{1, 3, 4, 5, 7, 8\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{1, 3, 5, 7\}, \quad D = \{1, 2, 3\},$$
$$E = \{3, 7\}, \quad \text{and} \quad U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

*Tell whether each statement is* true *or* false. ***See Examples 3, 5, 6, and 7.***

**21.** $A \subseteq U$

**22.** $D \subseteq A$

**23.** $\emptyset \subseteq A$

**24.** $\{1, 2\} \subseteq D$

**25.** $C \subseteq A$

**26.** $A \subseteq C$

**27.** $D \subseteq B$

**28.** $E \subseteq C$

**29.** $D \nsubseteq E$

**30.** $E \nsubseteq A$

**31.** There are exactly 4 subsets of $E$.

**32.** There are exactly 8 subsets of $D$.

**33.** There are exactly 12 subsets of $C$.

**34.** There are exactly 16 subsets of $B$.

**35.** $\{4, 6, 8, 12\} \cap \{6, 8, 14, 17\} = \{6, 8\}$

**36.** $\{2, 5, 9\} \cap \{1, 2, 3, 4, 5\} = \{2, 5\}$

**37.** $\{3, 1, 0\} \cap \{0, 2, 4\} = \{0\}$

**38.** $\{4, 2, 1\} \cap \{1, 2, 3, 4\} = \{1, 2, 3\}$

**39.** $\{3, 9, 12\} \cap \emptyset = \{3, 9, 12\}$

**40.** $\{3, 9, 12\} \cup \emptyset = \emptyset$

**41.** $\{3, 5, 7, 9\} \cup \{4, 6, 8\} = \emptyset$

**42.** $\{1, 2, 3\} \cup \{1, 2, 3\} = \{1, 2, 3\}$

**43.** $\{4, 9, 11, 7, 3\} \cup \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5, 7, 9, 11\}$

**44.** $\{5, 10, 15, 20\} \cup \{5, 15, 30\} = \{5, 15\}$

*Let*

$$U = \{\text{a, b, c, d, e, f, g, h}\}, \quad A = \{\text{a, b, c, d, e, f}\},$$
$$B = \{\text{a, c, e}\}, \quad C = \{\text{a, f}\}, \quad \text{and} \quad D = \{\text{d}\}.$$

*List the elements in each set.* ***See Examples 4–7.***

**45.** $A'$

**46.** $B'$

**47.** $C'$

**48.** $D'$

**49.** $A \cap B$

**50.** $B \cap A$

**51.** $A \cap D$

**52.** $B \cap D$

**53.** $B \cap C$

**54.** $A \cup B$

**55.** $B \cup D$

**56.** $B \cup C$

**57.** $C \cup B$

**58.** $C \cup D$

**59.** $A \cap \emptyset$

**60.** $B \cup \emptyset$

**61.** Name every pair of disjoint sets among sets $A$–$D$ in the directions for **Exercises 45–60.**

**62.** Show that for sets $B$ and $D$ in the directions for **Exercises 45–60,**

$$(B \cup D)' = B' \cap D'.$$

## APPENDIX B

# Review of Exponents, Polynomials, and Factoring

## (Transition from Beginning to Intermediate Algebra)

**OBJECTIVES**

1. Review the basic rules for exponents.
2. Review addition, subtraction, and multiplication of polynomials.
3. Review factoring techniques.

**OBJECTIVE 1** **Review the basic rules for exponents.** In **Sections 4.1 and 4.2,** we introduced the following definitions and rules for working with exponents.

### Definitions and Rules for Exponents

If no denominators are 0, the following are true for any integers $m$ and $n$.

| | | **Examples** |
|---|---|---|
| **Product rule** | $a^m \cdot a^n = a^{m+n}$ | $7^4 \cdot 7^5 = 7^9$ |
| **Zero exponent** | $a^0 = 1$ | $(-3)^0 = 1$ |
| **Negative exponent** | $a^{-n} = \frac{1}{a^n}$ | $5^{-3} = \frac{1}{5^3}$ |
| **Quotient rule** | $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{2^2}{2^5} = 2^{-3} = \frac{1}{2^3}$ |
| **Power rules (a)** | $(a^m)^n = a^{mn}$ | $(4^2)^3 = 4^6$ |
| **(b)** | $(ab)^m = a^m b^m$ | $(3k)^4 = 3^4 k^4$ |
| **(c)** | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ | $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$ |
| **Negative-to-positive rules** | $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$ | $\frac{2^{-4}}{5^{-3}} = \frac{5^3}{2^4}$ |
| | $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ | $\left(\frac{4}{7}\right)^{-2} = \left(\frac{7}{4}\right)^2$ |

**EXAMPLE 1** Applying Definitions and Rules for Exponents

Simplify. Write answers using only positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $(x^2y^{-3})(x^{-5}y^7)$

$= (x^{2+(-5)})(y^{-3+7})$ Product rule

$= x^{-3}y^4$ Add exponents.

$= \frac{1}{x^3}y^4$, or $\frac{y^4}{x^3}$ Definition of negative exponent; $\frac{1}{x^3}y^4 = \frac{1}{x^3} \cdot \frac{y^4}{1} = \frac{y^4}{x^3}$

**NOW TRY EXERCISE 1**

Simplify. Write answers using only positive exponents. Assume that all variables represent nonzero real numbers.

**(a)** $(m^{-8}n^4)(m^4n^{-3})$

**(b)** $-8^0 + 8^0$

**(c)** $\dfrac{(p^{-3}q)^4}{(p^2q^5)^2}$

**(d)** $\left(\dfrac{2x^{-2}y}{x^2y^{-4}}\right)^{-4}$

**(b)** $(-5)^0 + (-5^0)$

$= 1 + (-1)$ $\quad (-5^0) = -1 \cdot 5^0 = -1 \cdot 1 = -1$

$= 0$ $\quad$ Add.

**(c)** $\dfrac{(t^5s^{-4})^2}{(t^{-3}s^5)^3}$

$= \dfrac{t^{10}s^{-8}}{t^{-9}s^{15}}$ $\quad$ Power rule (b)

$= \dfrac{t^{10}t^9}{s^{15}s^8}$ $\quad$ Negative-to-positive rule

$= \dfrac{t^{10+9}}{s^{15+8}}$, or $\dfrac{t^{19}}{s^{23}}$ $\quad$ Product rule

**(d)** $\left(\dfrac{-3x^{-4}y}{x^5y^{-4}}\right)^{-2}$

$= \left(\dfrac{x^5y^{-4}}{-3x^{-4}y}\right)^2$ $\quad$ Negative-to-positive rule

$= \dfrac{x^{10}y^{-8}}{9x^{-8}y^2}$ $\quad$ Power rules (b) and (c)

$= \dfrac{x^{18}}{9y^{10}}$ $\quad$ Quotient rule

**(e)** $(2x^2y^3z)^2(x^4y^2)^3$

$= (4x^4y^6z^2)(x^{12}y^6)$ $\quad$ Power rule (b)

$= 4x^{16}y^{12}z^2$ $\quad$ Product rule

NOW TRY

**OBJECTIVE 2** **Review addition, subtraction, and multiplication of polynomials.** These arithmetic operations with polynomials were covered in **Sections 4.4–4.6.**

### Adding and Subtracting Polynomials

To add polynomials, add like terms.

To subtract polynomials, change all signs in the second polynomial and add the result to the first polynomial.

**NOW TRY EXERCISE 2**

Add or subtract as indicated.

**(a)** $(3x^3 + x^2 - 5x - 6) + (-6x^3 + 2x^2 + 4x - 1)$

**(b)** Subtract.

$$\begin{array}{r} 4x^2 + 7x - 5 \\ \underline{-5x^2 - 2x + 3} \end{array}$$

### EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

**(a)** $(-4x^3 + 3x^2 - 8x + 2) + (5x^3 - 8x^2 + 12x - 3)$

$= (-4x^3 + 5x^3) + (3x^2 - 8x^2) + (-8x + 12x) + (2 - 3)$

Commutative and associative properties

$= (-4 + 5)x^3 + (3 - 8)x^2 + (-8 + 12)x + (2 - 3)$

Distributive property

$= x^3 - 5x^2 + 4x - 1$ $\quad$ Simplify.

**(b)** $-4(x^2 + 3x - 6) - (2x^2 - 3x + 7)$

$= -4x^2 - 12x + 24 - 2x^2 + 3x - 7$ $\quad$ Distributive property; definition of subtraction

$= -6x^2 - 9x + 17$ $\quad$ Combine like terms.

**(c)** Subtract.

$$\begin{array}{r} 2t^2 - 3t - 4 \\ \underline{-8t^2 + 4t - 1} \end{array}$$

Change the sign of each term in $-8t^2 + 4t - 1$, and add.

$$\begin{array}{r} 2t^2 - 3t - 4 \\ \underline{8t^2 - 4t + 1} \\ 10t^2 - 7t - 3 \end{array}$$

Change signs.

Add.

NOW TRY

**NOW TRY ANSWERS**

1. (a) $\dfrac{n}{m^4}$ (b) 0 (c) $\dfrac{1}{p^{16}q^6}$ (d) $\dfrac{x^{16}}{16y^{20}}$
2. (a) $-3x^3 + 3x^2 - x - 7$ (b) $9x^2 + 9x - 8$

### Multiplying Polynomials

To multiply two polynomials, multiply each term of the second polynomial by each term of the first polynomial and add the products. In particular, when multiplying two binomials, use the FOIL method. (See **Section 4.5.**)

The special product rules are useful when multiplying binomials.

### Special Product Rules

For $x$ and $y$, the following are true.

$$\left.\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\(x-y)^2 &= x^2-2xy+y^2\end{aligned}\right\}\quad \text{Square of a binomial}$$

$$(x+y)(x-y) = x^2-y^2 \quad \text{Product of the sum and difference of two terms}$$

**NOW TRY EXERCISE 3**

Find each product.

**(a)** $(6x-5)(2x-3)$

**(b)** $(4m-3n)(4m+3n)$

**(c)** $(7z+1)^2$

**(d)** $(r+3)(r^2-3r+9)$

### EXAMPLE 3 Multiplying Polynomials

Find each product.

**(a)** $(4y-1)(3y+2)$

$$\begin{aligned}&= \overbrace{4y(3y)}^{\text{First terms}} + \overbrace{4y(2)}^{\text{Outer terms}} - \overbrace{1(3y)}^{\text{Inner terms}} - \overbrace{1(2)}^{\text{Last terms}} && \text{FOIL method}\\ &= 12y^2 + 8y - 3y - 2 && \text{Multiply.}\\ &= 12y^2 + 5y - 2 && \text{Combine like terms.}\end{aligned}$$

**(b)** $(3x+5y)(3x-5y)$

$(ab)^2 = a^2b^2$, **not** $ab^2$.

$$\begin{aligned}&= (3x)^2 - (5y)^2 && (x+y)(x-y) = x^2-y^2\\ &= 9x^2 - 25y^2 && \text{Power rule (b)}\end{aligned}$$

**(c)** $(2t+3)^2$

$$\begin{aligned}&= (2t)^2 + 2(2t)(3) + 3^2 && (x+y)^2 = x^2+2xy+y^2\\ &= 4t^2 + 12t + 9\end{aligned}$$

Remember the middle term.

**(d)** $(5x-1)^2$

$$\begin{aligned}&= (5x)^2 - 2(5x)(1) + 1^2 && (x-y)^2 = x^2-2xy+y^2\\ &= 25x^2 - 10x + 1 && (5x)^2 = 5^2x^2 = 25x^2\end{aligned}$$

**(e)** $(3x+2)(9x^2-6x+4)$

$$\begin{array}{rrrrl} & & 9x^2 & -\,6x & +\,4 \\ & & & 3x & +\,2 \\ \hline & & 18x^2 & -\,12x & +\,8 \quad \longleftarrow 2(9x^2-6x+4) \\ 27x^3 & -\,18x^2 & +\,12x & & \quad\quad \longleftarrow 3x(9x^2-6x+4) \\ \hline 27x^3 & & & & +\,8 \quad \text{Add.} \end{array}$$

Multiply vertically.

Be sure to write like terms in columns.

The product is the sum of cubes, $27x^3+8$.

NOW TRY

**NOW TRY ANSWERS**

3. (a) $12x^2-28x+15$
   (b) $16m^2-9n^2$
   (c) $49z^2+14z+1$
   (d) $r^3+27$

**OBJECTIVE 3** **Review factoring techniques.** Factoring, which involves writing a polynomial as a product, was covered in **Chapter 5.** Here are some general guidelines to use when factoring.

### Factoring a Polynomial

1. **Is there a common factor?** If so, factor it out.
2. **How many terms are in the polynomial?**

   ***Two terms:*** Check to see whether it is a difference of squares or the sum or difference of cubes. If so, factor as in **Section 5.4.**

   $$x^2 - y^2 = (x + y)(x - y) \quad \text{Difference of squares}$$

   $$x^3 - y^3 = (x - y)(x^2 + xy + y^2) \quad \text{Difference of cubes}$$

   $$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \quad \text{Sum of cubes}$$

   ***Three terms:*** Is it a perfect square trinomial?

   $$\left.\begin{aligned} x^2 + 2xy + y^2 &= (x + y)^2 \\ x^2 - 2xy + y^2 &= (x - y)^2 \end{aligned}\right\} \quad \text{Perfect square trinomials}$$

   If the trinomial is not a perfect square, check to see whether the coefficient of the second-degree term is 1. If so, use the method of **Section 5.2.** If the coefficient of the second-degree term of the trinomial is not 1, use the general factoring methods of **Section 5.3.**

   ***Four terms:*** Try to factor the polynomial by grouping, as in **Section 5.1.**
3. **Can any factors be factored further?** If so, factor them.

### EXAMPLE 4 Factoring Polynomials

Factor each polynomial completely.

**(a)** $6x^2y^3 - 12x^3y^2$

$$= 6x^2y^2 \cdot y - 6x^2y^2 \cdot 2x \quad 6x^2y^2 \text{ is the greatest common factor.}$$

$$= 6x^2y^2(y - 2x) \quad \text{Distributive property}$$

**(b)** $3x^2 - x - 2$

To find the factors, find two terms that multiply to give $3x^2$ (here $3x$ and $x$) and two terms that multiply to give $-2$ (here $+2$ and $-1$). Make sure that the sum of the outer and inner products in the factored form is $-x$.

$$3x^2 - x - 2 \quad \text{factors as} \quad (3x + 2)(x - 1).$$

*CHECK* To check, multiply the factors using the FOIL method.

**(c)** $3x^2 - 27x + 42$

$$= 3(x^2 - 9x + 14) \quad \text{Factor out the common factor.}$$

$$= 3(x - 7)(x - 2) \quad \text{Factor the trinomial.}$$

**(d)** $100t^2 - 81$

$$= (10t)^2 - 9^2 \quad \text{Difference of squares}$$

$$= (10t + 9)(10t - 9) \quad x^2 - y^2 = (x + y)(x - y)$$

**NOW TRY EXERCISE 4**

Factor each polynomial completely.

**(a)** $5x^2 - 20x - 60$
**(b)** $10t^2 + 13t - 3$
**(c)** $49x^2 + 42x + 9$
**(d)** $mn - 2n + 5m - 10$
**(e)** $27x^3 - 1000$

**(e)** $4x^2 + 20xy + 25y^2$

The terms $4x^2$ and $25y^2$ are both perfect squares, so this trinomial might factor as a perfect square trinomial.

$$\text{Try to factor} \quad 4x^2 + 20xy + 25y^2 \quad \text{as} \quad (2x + 5y)^2.$$

**CHECK** Take twice the product of the two terms in the squared binomial.

$$\underset{\text{Twice}}{\underset{\uparrow}{2}} \cdot \underset{\text{First term}}{\underset{\uparrow}{2x}} \cdot \underset{\text{Last term}}{\underset{\uparrow}{5y}} = 20xy \leftarrow \text{Middle term of } 4x^2 + 20xy + 25y^2$$

Since $20xy$ is the middle term of the trinomial, the trinomial is a perfect square.

$$4x^2 + 20xy + 25y^2 \quad \text{factors as} \quad (2x + 5y)^2.$$

**(f)** $1000x^3 - 27$

$$\begin{aligned} &= (10x)^3 - 3^3 && \text{Difference of cubes} \\ &= (10x - 3)[(10x)^2 + 10x(3) + 3^2] && x^3 - y^3 = (x - y)(x^2 + xy + y^2) \\ &= (10x - 3)(100x^2 + 30x + 9) && (10x)^2 = 10^2x^2 \end{aligned}$$

**(g)** $6xy - 3x + 4y - 2$

Since there are four terms, try factoring by grouping.

$$\begin{aligned} &6xy - 3x + 4y - 2 \\ &= (6xy - 3x) + (4y - 2) && \text{Group the terms.} \\ &= 3x(2y - 1) + 2(2y - 1) && \text{Factor each group.} \\ &= (2y - 1)(3x + 2) && \text{Factor out } 2y - 1. \end{aligned}$$

In the final step, factor out the greatest common factor, the binomial $2y - 1$.

NOW TRY

**NOW TRY ANSWERS**

4. **(a)** $5(x - 6)(x + 2)$
   **(b)** $(5t - 1)(2t + 3)$
   **(c)** $(7x + 3)^2$
   **(d)** $(n + 5)(m - 2)$
   **(e)** $(3x - 10)(9x^2 + 30x + 100)$

## EXERCISES

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*Simplify each expression. Write the answers using only positive exponents. Assume that all variables represent positive real numbers.* ***See Example 1.***

**1.** $(a^4b^{-3})(a^{-6}b^2)$
**2.** $(t^{-3}s^{-5})(t^8s^{-2})$
**3.** $(5x^{-2}y)^2(2xy^4)^2$
**4.** $(7x^{-3}y^4)^3(2x^{-1}y^{-4})^2$
**5.** $-6^0 + (-6)^0$
**6.** $(-12)^0 - 12^0$
**7.** $\dfrac{(2w^{-1}x^2y^{-1})^3}{(4w^5x^{-2}y)^2}$
**8.** $\dfrac{(5p^{-3}q^2r^{-4})^2}{(10p^4q^{-1}r^5)^{-1}}$
**9.** $\left(\dfrac{-4a^{-2}b^4}{a^3b^{-1}}\right)^{-3}$
**10.** $\left(\dfrac{r^{-3}s^{-8}}{-6r^2s^{-4}}\right)^{-2}$
**11.** $(7x^{-4}y^2z^{-2})^{-2}(7x^4y^{-1}z^3)^2$
**12.** $(3m^{-5}n^2p^{-4})^3(3m^4n^{-3}p^5)^{-2}$

*Add or subtract as indicated.* ***See Example 2.***

**13.** $(2a^4 + 3a^3 - 6a^2 + 5a - 12) + (-8a^4 + 8a^3 - 14a^2 + 21a - 3)$
**14.** $(-6r^4 - 3r^3 + 12r^2 - 9r + 9) + (8r^4 - 13r^3 - 14r^2 - 10r - 3)$
**15.** $(6x^3 - 12x^2 + 3x - 4) - (-2x^3 + 6x^2 - 3x + 12)$
**16.** $(10y^3 - 4y^2 + 8y + 7) - (7y^3 + 5y^2 - 2y - 13)$

**17.** Add.

$$\begin{array}{r} 5x^2y + 2xy^2 + y^3 \\ \underline{-4x^2y - 3xy^2 + 5y^3} \end{array}$$

**18.** Add.

$$\begin{array}{r} 6ab^3 - 2a^2b^2 + 3b^5 \\ \underline{8ab^3 + 12a^2b^2 - 8b^5} \end{array}$$

**19.** $3(5x^2 - 12x + 4) - 2(9x^2 + 13x - 10)$

**20.** $-4(2t^3 - 3t^2 + 4t - 1) - 3(-8t^3 + 3t^2 - 2t + 9)$

**21.** Subtract.

$$\begin{array}{r} 6x^3 - 2x^2 + 3x - 1 \\ \underline{-4x^3 + 2x^2 - 6x + 3} \end{array}$$

**22.** Subtract.

$$\begin{array}{r} -9y^3 - 2y^2 + 3y - 8 \\ \underline{-8y^3 + 4y^2 + 3y + 1} \end{array}$$

*Find each product.* ***See Example 3.***

**23.** $(3x + 1)(2x - 7)$
**24.** $(5z + 3)(2z - 3)$
**25.** $(4x - 1)(x - 2)$
**26.** $(7t - 3)(t - 4)$
**27.** $(4t + 3)(4t - 3)$
**28.** $(6x + 1)(6x - 1)$
**29.** $(2y^2 + 4)(2y^2 - 4)$
**30.** $(3b^3 + 2t)(3b^3 - 2t)$
**31.** $(4x - 3)^2$
**32.** $(9t + 2)^2$
**33.** $(6r + 5y)^2$
**34.** $(8m - 3n)^2$
**35.** $(c + 2d)(c^2 - 2cd + 4d^2)$
**36.** $(f + 3g)(f^2 - 3fg + 9g^2)$
**37.** $(4x - 1)(16x^2 + 4x + 1)$
**38.** $(5r - 2)(25r^2 + 10r + 4)$
**39.** $(7t + 5s)(2t^2 + 5st - s^2)$
**40.** $(8p + 3q)(2p^2 - 4pq + q^2)$

*Factor each polynomial completely.* ***See Example 4.***

**41.** $8x^3y^4 + 12x^2y^3 + 36xy^4$
**42.** $10m^5n + 4m^2n^3 + 18m^3n^2$
**43.** $x^2 - 2x - 15$
**44.** $x^2 + x - 12$
**45.** $2x^2 - 9x - 18$
**46.** $3x^2 + 2x - 8$
**47.** $36t^2 - 25$
**48.** $49r^2 - 9$
**49.** $16t^2 + 24t + 9$
**50.** $25t^2 + 90t + 81$
**51.** $4m^2p - 12mnp + 9n^2p$
**52.** $16p^2r - 40pqr + 25q^2r$
**53.** $x^3 + 1$
**54.** $x^3 + 27$
**55.** $8t^3 + 125$
**56.** $27s^3 + 64$
**57.** $t^6 - 125$
**58.** $w^6 - 27$
**59.** $5xt + 15xr + 2yt + 6yr$
**60.** $3am + 18mb + 2an + 12nb$
**61.** $6ar + 12br - 5as - 10bs$
**62.** $7mt + 35ms - 2nt - 10ns$
**63.** $t^4 - 1$
**64.** $r^4 - 81$
**65.** $4x^2 + 12xy + 9y^2 - 1$
**66.** $81t^2 + 36ty + 4y^2 - 9$
**67.** $4x^2 - 28x + 40$
**68.** $2x^2 - 18x + 36$

APPENDIX C

# Synthetic Division

**OBJECTIVES**

1. Use synthetic division to divide by a polynomial of the form $x - k$.
2. Use the remainder theorem to evaluate a polynomial.
3. Decide whether a given number is a solution of an equation.

**OBJECTIVE 1 Use synthetic division to divide by a polynomial of the form $x - k$.** If a polynomial in $x$ is divided by a binomial of the form $x - k$, a shortcut method can be used. For an illustration, look at the division on the left below.

$$\begin{array}{rrrrr}
 & & 3x^2 & +\ 9x & +\ 25 \\ \cline{2-5}
x - 3\big) & 3x^3 & +\ 0x^2 & -\ 2x & +\ 5 \\
 & 3x^3 & -\ 9x^2 & & \\ \cline{2-3}
 & & 9x^2 & -\ 2x & \\
 & & 9x^2 & -\ 27x & \\ \cline{3-4}
 & & & 25x & +\ 5 \\
 & & & 25x & -\ 75 \\ \cline{4-5}
 & & & & 80
\end{array}
\qquad\qquad
\begin{array}{rrrrr}
 & & 3 & 9 & 25 \\ \cline{2-5}
1 - 3\big) & 3 & 0 & -2 & 5 \\
 & 3 & -9 & & \\ \cline{2-3}
 & & 9 & -2 & \\
 & & 9 & -27 & \\ \cline{3-4}
 & & & 25 & 5 \\
 & & & 25 & -75 \\ \cline{4-5}
 & & & & 80
\end{array}$$

On the right above, exactly the same division is shown written without the variables. This is why it is *essential* to use 0 as a placeholder in synthetic division. All the numbers in color on the right are repetitions of the numbers directly above them, so we omit them, as shown on the left below.

$$\begin{array}{rrrrr}
 & & 3 & 9 & 25 \\ \cline{2-5}
1 - 3\big) & 3 & 0 & -2 & 5 \\
 & & -9 & & \\ \cline{2-3}
 & & 9 & -2 & \\
 & & & -27 & \\ \cline{3-4}
 & & & 25 & 5 \\
 & & & & -75 \\ \cline{4-5}
 & & & & 80
\end{array}
\qquad\qquad
\begin{array}{rrrrr}
 & & 3 & 9 & 25 \\ \cline{2-5}
1 - 3\big) & 3 & 0 & -2 & 5 \\
 & & -9 & & \\ \cline{2-3}
 & & 9 & & \\
 & & & -27 & \\ \cline{3-4}
 & & & 25 & \\
 & & & & -75 \\ \cline{4-5}
 & & & & 80
\end{array}$$

The numbers in color on the left are again repetitions of the numbers directly above them. They too are omitted, as shown on the right above. If we bring the 3 in the dividend down to the beginning of the bottom row, the top row can be omitted, since it duplicates the bottom row.

$$\begin{array}{rrrrr}
1 - 3\big) & 3 & 0 & -2 & 5 \\
 & & -9 & -27 & -75 \\ \cline{2-5}
 & 3 & 9 & 25 & 80
\end{array}$$

We omit the 1 at the upper left, since it represents $1x$, which will always be the first term in the divisor. Also, to simplify the arithmetic, we replace subtraction in the second row by addition. To compensate for this, we change the $-3$ at the upper left to its additive inverse, 3.

$$\begin{array}{r|rrrl}
\text{Additive inverse} \rightarrow 3 & 3 & 0 & -2 & 5 \\
 & & 9 & 27 & 75 \leftarrow \text{Signs changed} \\
\hline
 & 3 & 9 & 25 & 80 \leftarrow \text{Remainder} \\
 & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}$$

The quotient is read from the bottom row. $\quad 3x^2 + 9x + 25 + \dfrac{80}{x-3}$

The first three numbers in the bottom row are the coefficients of the quotient polynomial with degree 1 less than the degree of the dividend. The last number gives the remainder.

This shortcut procedure is called **synthetic division.** ***It is used only when dividing a polynomial by a binomial of the form $x - k$.***

**NOW TRY EXERCISE 1**

Use synthetic division to divide.

$$\frac{4x^3 + 18x^2 + 19x + 7}{x+3}$$

## EXAMPLE 1 Using Synthetic Division

Use synthetic division to divide $5x^2 + 16x + 15$ by $x + 2$.

We change $x + 2$ into the form $x - k$ by writing it as

$$x + 2 = x - (-2), \quad \text{where } k = -2.$$

Now write the coefficients of $5x^2 + 16x + 15$, placing $-2$ to the left.

$$x + 2 \text{ leads to } -2. \rightarrow \begin{array}{r|rrr} -2 & 5 & 16 & 15 \end{array} \leftarrow \text{Coefficients}$$

$$\begin{array}{r|rrr}
-2 & 5 & 16 & 15 \\
 & \downarrow & -10 & \\
\hline
 & 5 & &
\end{array}$$

Bring down the 5, and multiply: $-2 \cdot 5 = -10$.

$$\begin{array}{r|rrr}
-2 & 5 & 16 & 15 \\
 & & -10 & -12 \\
\hline
 & 5 & 6 &
\end{array}$$

Add 16 and $-10$, getting 6, and multiply 6 and $-2$ to get $-12$.

$$\begin{array}{r|rrl}
-2 & 5 & 16 & 15 \\
 & & -10 & -12 \\
\hline
 & 5 & 6 & 3 \leftarrow \text{Remainder}
\end{array}$$

Add 15 and $-12$, getting 3.

The result is read from the bottom row.

$$\frac{5x^2 + 16x + 15}{x+2} = 5x + 6 + \frac{3}{x+2}$$

NOW TRY

**NOW TRY EXERCISE 2**

Use synthetic division to divide.

$$\frac{-3x^4 + 13x^3 - 6x^2 + 31}{x-4}$$

## EXAMPLE 2 Using Synthetic Division with a Missing Term

Use synthetic division to find $(-4x^5 + x^4 + 6x^3 + 2x^2 + 50) \div (x - 2)$.

$$\begin{array}{r|rrrrrr}
2 & -4 & 1 & 6 & 2 & 0 & 50 \\
 & & -8 & -14 & -16 & -28 & -56 \\
\hline
 & -4 & -7 & -8 & -14 & -28 & -6
\end{array}$$

Use the steps given above, first inserting a 0 for the missing $x$-term.

Read the result from the bottom row.

$$\frac{-4x^5 + x^4 + 6x^3 + 2x^2 + 50}{x-2} = -4x^4 - 7x^3 - 8x^2 - 14x - 28 + \frac{-6}{x-2}$$

NOW TRY

**NOW TRY ANSWERS**

1. $4x^2 + 6x + 1 + \frac{4}{x+3}$
2. $-3x^3 + x^2 - 2x - 8 + \frac{-1}{x-4}$

**OBJECTIVE 2** **Use the remainder theorem to evaluate a polynomial.** We can use synthetic division to evaluate polynomials. For example, in the synthetic division of **Example 2,** where the polynomial was divided by $x - 2$, the remainder was $-6$.

Replacing $x$ in the polynomial with $2$ gives

$$\begin{aligned} &-4x^5 + x^4 + 6x^3 + 2x^2 + 50 \\ &\quad = -4 \cdot 2^5 + 2^4 + 6 \cdot 2^3 + 2 \cdot 2^2 + 50 && \text{Replace } x \text{ with 2.} \\ &\quad = -4 \cdot 32 + 16 + 6 \cdot 8 + 2 \cdot 4 + 50 && \text{Evaluate the powers.} \\ &\quad = -128 + 16 + 48 + 8 + 50 && \text{Multiply.} \\ &\quad = -6, && \text{Add.} \end{aligned}$$

the same number as the remainder. Dividing by $x - 2$ produced a remainder equal to the result when $x$ is replaced with $2$. This always happens, as the following **remainder theorem** states. This result is proved in more advanced courses.

**Remainder Theorem**

If the polynomial $P(x)$ is divided by $x - k$, then the remainder is equal to $P(k)$.

**NOW TRY EXERCISE 3**

Let $P(x) = 3x^3 - 2x^2 + 5x + 30$. Use synthetic division to evaluate $P(-2)$.

### EXAMPLE 3 Using the Remainder Theorem

Let $P(x) = 2x^3 - 5x^2 - 3x + 11$. Use synthetic division to evaluate $P(-2)$.

Use the remainder theorem, and divide $P(x)$ by $x - (-2)$.

$$\begin{array}{r|rrrr} \text{Value of } k \rightarrow -2 & 2 & -5 & -3 & 11 \\ & & -4 & 18 & -30 \\ \cline{2-5} & 2 & -9 & 15 & -19 \leftarrow \text{Remainder} \end{array}$$

Thus, $P(-2) = -19$.

NOW TRY

**OBJECTIVE 3** **Decide whether a given number is a solution of an equation.** We can also use the remainder theorem to do this.

**NOW TRY EXERCISE 4**

Use synthetic division to decide whether $-4$ is a solution of the equation.

$$5x^3 + 19x^2 - 2x + 8 = 0$$

### EXAMPLE 4 Using the Remainder Theorem

Use synthetic division to decide whether $-5$ is a solution of the equation.

$$2x^4 + 12x^3 + 6x^2 - 5x + 75 = 0$$

If synthetic division gives a remainder of 0, then $-5$ is a solution. Otherwise, it is not.

$$\begin{array}{r|rrrrr} \text{Proposed solution} \rightarrow -5 & 2 & 12 & 6 & -5 & 75 \\ & & -10 & -10 & 20 & -75 \\ \cline{2-6} & 2 & 2 & -4 & 15 & 0 \leftarrow \text{Remainder} \end{array}$$

Since the remainder is 0, the polynomial has value 0 when $k = -5$. So $-5$ is a solution of the given equation.

NOW TRY

The synthetic division in **Example 4** shows that $x - (-5)$ divides the polynomial with 0 remainder. Thus $x - (-5) = x + 5$ is a *factor* of the polynomial and

$$2x^4 + 12x^3 + 6x^2 - 5x + 75 \quad \text{factors as} \quad (x + 5)(2x^3 + 2x^2 - 4x + 15).$$

The second factor is the quotient polynomial found in the last row of the synthetic division.

**NOW TRY ANSWERS**

**3.** $-12$ **4.** yes

## EXERCISES

MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

*Use synthetic division to find each quotient.* ***See Examples 1 and 2.***

**1.** $\dfrac{x^2 - 6x + 5}{x - 1}$

**2.** $\dfrac{x^2 - 4x - 21}{x + 3}$

**3.** $\dfrac{4m^2 + 19m - 5}{m + 5}$

**4.** $\dfrac{3x^2 - 5x - 12}{x - 3}$

**5.** $\dfrac{2a^2 + 8a + 13}{a + 2}$

**6.** $\dfrac{4y^2 - 5y - 20}{y - 4}$

**7.** $(p^2 - 3p + 5) \div (p + 1)$

**8.** $(z^2 + 4z - 6) \div (z - 5)$

**9.** $\dfrac{4a^3 - 3a^2 + 2a - 3}{a - 1}$

**10.** $\dfrac{5p^3 - 6p^2 + 3p + 14}{p + 1}$

**11.** $(x^5 - 2x^3 + 3x^2 - 4x - 2) \div (x - 2)$

**12.** $(2y^5 - 5y^4 - 3y^2 - 6y - 23) \div (y - 3)$

**13.** $(-4r^6 - 3r^5 - 3r^4 + 5r^3 - 6r^2 + 3r + 3) \div (r - 1)$

**14.** $(2t^6 - 3t^5 + 2t^4 - 5t^3 + 6t^2 - 3t - 2) \div (t - 2)$

**15.** $(-3y^5 + 2y^4 - 5y^3 - 6y^2 - 1) \div (y + 2)$

**16.** $(m^6 + 2m^4 - 5m + 11) \div (m - 2)$

*Use the remainder theorem to find* $P(k)$. ***See Example 3.***

**17.** $P(x) = 2x^3 - 4x^2 + 5x - 3;\ k = 2$

**18.** $P(x) = x^3 + 3x^2 - x + 5;\ k = -1$

**19.** $P(x) = -x^3 - 5x^2 - 4x - 2;\ k = -4$

**20.** $P(x) = -x^3 + 5x^2 - 3x + 4;\ k = 3$

**21.** $P(x) = 2x^3 - 4x^2 + 5x - 33;\ k = 3$

**22.** $P(x) = x^3 - 3x^2 + 4x - 4;\ k = 2$

**23.** Explain why a 0 remainder in synthetic division of $P(x)$ by $x - k$ indicates that $k$ is a solution of the equation $P(x) = 0$.

**24.** Explain why it is important to insert 0s as placeholders for missing terms before performing synthetic division.

*Use synthetic division to decide whether the given number is a solution of the equation.* ***See Example 4.***

**25.** $x^3 - 2x^2 - 3x + 10 = 0;\ x = -2$

**26.** $x^3 - 3x^2 - x + 10 = 0;\ x = -2$

**27.** $3x^3 + 2x^2 - 2x + 11 = 0;\ x = -2$

**28.** $3x^3 + 10x^2 + 3x - 9 = 0;\ x = -2$

**29.** $2x^3 - x^2 - 13x + 24 = 0;\ x = -3$

**30.** $5x^3 + 22x^2 + x - 28 = 0;\ x = -4$

**31.** $x^4 + 2x^3 - 3x^2 + 8x - 8 = 0;\ x = -2$

**32.** $x^4 - x^3 - 6x^2 + 5x + 10 = 0;\ x = -2$

### RELATING CONCEPTS EXERCISES 33–38

**FOR INDIVIDUAL OR GROUP WORK**

*We can show a connection between dividing one polynomial by another and factoring the first polynomial. Let* $P(x) = 2x^2 + 5x - 12$. ***Work Exercises 33–38 in order.***

**33.** Factor $P(x)$.

**34.** Solve $P(x) = 0$.

**35.** Evaluate $P(-4)$.

**36.** Evaluate $P\left(\frac{3}{2}\right)$.

**37.** Complete the following sentence: If $P(a) = 0$, then $x -$ _____ is a factor of $P(x)$.

**38.** Use the conclusion reached in **Exercise 37** to decide whether $x - 3$ is a factor of $Q(x) = 3x^3 - 4x^2 - 17x + 6$. Factor $Q(x)$ completely.

APPENDIX D

# An Introduction to Calculators

There is little doubt that the appearance of handheld calculators more than three decades ago and the later development of scientific and graphing calculators have changed the methods of learning and studying mathematics forever. For example, computations with tables of logarithms and slide rules made up an important part of mathematics courses prior to 1970. Today, with the widespread availability of calculators, these topics are studied only for their historical significance.

Calculators come in a large array of different types, sizes, and prices. ***For the course for which this textbook is intended, the most appropriate type is the scientific calculator,*** which costs \$10–\$20.

In this introduction, we explain some of the features of scientific and graphing calculators. However, remember that calculators vary among manufacturers and models and that, while the methods explained here apply to many of them, they may not apply to your specific calculator. ***This introduction is only a guide and is not intended to take the place of your owner's manual.*** Always refer to the manual whenever you need an explanation of how to perform a particular operation.

## Scientific Calculators

Scientific calculators are capable of much more than the typical four-function calculator that you might use for balancing your checkbook. Most scientific calculators use *algebraic logic*. (Models sold by Texas Instruments, Sharp, Casio, and Radio Shack, for example, use algebraic logic.) A notable exception is Hewlett-Packard, a company whose calculators use *Reverse Polish Notation* (RPN). In this introduction, we explain the use of calculators with algebraic logic.

**Arithmetic Operations** To perform an operation of arithmetic, simply enter the first number, press the operation key $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, or $\boxed{\div}$, enter the second number, and then press the $\boxed{=}$ key. For example, to add 4 and 3, use the following keystrokes.

$$\boxed{4}\ \boxed{+}\ \boxed{3}\ \boxed{=}\qquad \boxed{\quad\quad 7}$$

**Change Sign Key** The key marked $\boxed{+/-}$ allows you to change the sign of a display. This is particularly useful when you wish to enter a negative number. For example, to enter $-3$, use the following keystrokes.

$$\boxed{3}\ \boxed{+/-}\qquad \boxed{\quad -3}$$

**Memory Key** Scientific calculators can hold a number in memory for later use. The label of the memory key varies among models; two of these are $\boxed{\text{M}}$ and $\boxed{\text{STO}}$. The $\boxed{\text{M+}}$ and $\boxed{\text{M}-}$ keys allow you to add to or subtract from the value currently in memory. The memory recall key, labeled $\boxed{\text{MR}}$, $\boxed{\text{RM}}$, or $\boxed{\text{RCL}}$, allows you to retrieve the value stored in memory.

Suppose that you wish to store the number 5 in memory. Enter 5, and then press the key for memory. You can then perform other calculations. When you need to retrieve the 5, press the key for memory recall.

If a calculator has a constant memory feature, the value in memory will be retained even after the power is turned off. Some advanced calculators have more than one memory. Read the owner's manual for your model to see exactly how memory is activated.

**Clearing/Clear Entry Keys** The key $\boxed{\text{C}}$ or $\boxed{\text{CE}}$ allows you to clear the display or clear the last entry entered into the display. In some models, pressing the $\boxed{\text{C}}$ key once will clear the last entry, while pressing it twice will clear the entire operation in progress.

**Second Function Key** This key, usually marked $\boxed{\text{2nd}}$, is used in conjunction with another key to activate a function that is printed *above* an operation key (and not on the key itself). For example, suppose you wish to find the square of a number, and the squaring function (explained in more detail later) is printed above another key. You would need to press $\boxed{\text{2nd}}$ before the desired squaring function can be activated.

**Square Root Key** Pressing $\boxed{\sqrt{\phantom{x}}}$ or $\boxed{\sqrt{x}}$ will give the square root (or an approximation of the square root) of the number in the display. On some scientific calculators, the square root key is pressed *before* entering the number, while other calculators use the opposite order. Experiment with your calculator to see which method it uses. For example, to find the square root of 36, use the following keystrokes.

$$\boxed{\sqrt{\phantom{x}}}\ \boxed{3}\ \boxed{6}\quad \boxed{\qquad 6}\qquad \text{or}\qquad \boxed{3}\ \boxed{6}\ \boxed{\sqrt{\phantom{x}}}\quad \boxed{\qquad 6}$$

The square root of 2 is an example of an irrational number (**Chapter 10**). The calculator will give an approximation of its value, since the decimal for $\sqrt{2}$ never terminates and never repeats. The number of digits shown will vary among models. To find an approximation for $\sqrt{2}$, use the following keystrokes.

$$\boxed{\sqrt{\phantom{x}}}\ \boxed{2}\quad \boxed{1.4142136}\qquad \text{or}\qquad \boxed{2}\ \boxed{\sqrt{\phantom{x}}}\quad \boxed{1.4142136}$$

An approximation for $\sqrt{2}$

**Squaring Key** The $\boxed{x^2}$ key allows you to square the entry in the display. For example, to square 35.7, use the following keystrokes.

$$\boxed{3}\ \boxed{5}\ \boxed{.}\ \boxed{7}\ \boxed{x^2}\quad \boxed{1274.49}$$

The squaring key and the square root key are often found together, with one of them being a second function (that is, activated by the second function key previously described).

**Reciprocal Key** The key marked $\boxed{1/x}$ is the reciprocal key. (When two numbers have a product of 1, they are called *reciprocals*. See **Chapter 1.**) Suppose that you wish to find the reciprocal of 5. Use the following keystrokes.

$$\boxed{5}\ \boxed{1/x}\quad \boxed{\qquad 0.2}$$

**Inverse Key** Some calculators have an inverse key, marked $\boxed{\text{INV}}$. Inverse operations are operations that "undo" each other. For example, the operations of squaring and taking the square root are inverse operations. The use of the $\boxed{\text{INV}}$ key varies among different models of calculators, so read your owner's manual carefully.

**Exponential Key** The key marked $\boxed{x^y}$ or $\boxed{y^x}$ allows you to raise a number to a power. For example, if you wish to raise 4 to the fifth power (that is, find $4^5$, as explained in **Chapter 1**), use the following keystrokes.

$$\boxed{4}\ \boxed{x^y}\ \boxed{5}\ \boxed{=}\qquad \boxed{\quad 1024}$$

**Root Key** Some calculators have a key specifically marked $\boxed{\sqrt[y]{x}}$ or $\boxed{\sqrt[x]{y}}$; with others, the operation of taking roots is accomplished by using the inverse key in conjunction with the exponential key. Suppose, for example, your calculator is of the latter type and you wish to find the fifth root of 1024. Use the following keystrokes.

$$\boxed{1}\ \boxed{0}\ \boxed{2}\ \boxed{4}\ \boxed{\text{INV}}\ \boxed{x^y}\ \boxed{5}\ \boxed{=}\qquad \boxed{\quad 4}$$

Notice how this "undoes" the operation explained in the discussion of the exponential key.

**Pi Key** The number $\pi$ is an important number in mathematics. It occurs, for example, in the area and circumference formulas for a circle. One popular model gives the following display when the $\boxed{\pi}$ key is pressed. (Because $\pi$ is irrational, the display shows only an approximation.)

$$\boxed{3.1415927}\qquad \text{An approximation for } \pi$$

**Methods of Display** When decimal approximations are shown on scientific calculators, they are either *truncated* or *rounded*. To see how a particular model is programmed, evaluate 1/18 as an example. If the display shows 0.0555555 (last digit 5), the calculator truncates the display. If the display shows 0.0555556 (last digit 6), the calculator rounds the display.

When very large or very small numbers are obtained as answers, scientific calculators often express these numbers in scientific notation (**Chapter 4**). For example, if you multiply 6,265,804 by 8,980,591, the display might look like this:

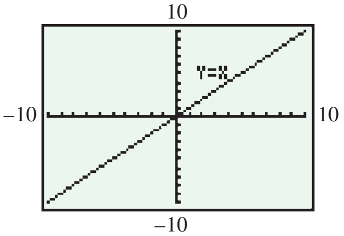
$$\boxed{5.6270623\ \ 13}$$

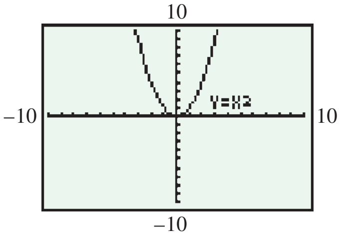
The 13 at the far right means that the number on the left is multiplied by $10^{13}$. This means that the decimal point must be moved 13 places to the right if the answer is to be expressed in its usual form. Even then, the value obtained will only be an approximation: 56,270,623,000,000.

## Graphing Calculators

While you are not expected to have a graphing calculator to study from this book, we include the following as background information and reference should your course or future courses require the use of graphing calculators.

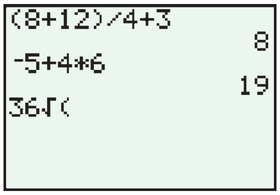
**Basic Features** In addition to possessing the typical keys found on scientific calculators, graphing calculators have keys that can be used to create graphs, make tables, analyze data, and change settings. One of the major differences between graphing and scientific calculators is that a graphing calculator has a larger viewing screen with graphing capabilities. The following screens illustrate the graphs of $Y = X$ and $Y = X^2$. (We use screens from a Texas Instruments calculator in our illustrations.)



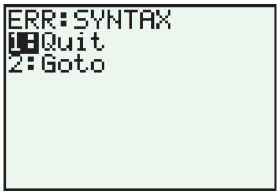


If you look closely at the screens, you will see that the graphs appear to be jagged rather than smooth. The reason for this is that graphing calculators have much lower resolution than computer screens. Because of this, graphs generated by graphing calculators must be interpreted carefully.

**Editing Input** The screen of a graphing calculator can display several lines of text at a time. This feature allows you to view both previous and current expressions. If an incorrect expression is entered, an error message is displayed. The erroneous expression can be viewed and corrected by using various editing keys, much like a word-processing program. You do not need to enter the entire expression again. Many graphing calculators can also recall past expressions for editing or updating. The screen on the left shows how two expressions are evaluated. The final line is entered incorrectly, and the resulting error message is shown in the screen on the right.

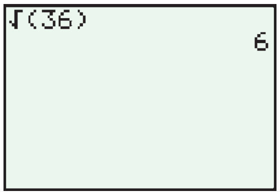


```
(8+12)/4+3
               8
-5+4*6
              19
36√(
```

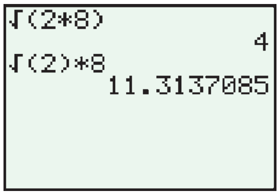


```
ERR:SYNTAX
1:Quit
2:Goto
```

**Order of Operations** Arithmetic operations on graphing calculators are usually entered as they are written in mathematical expressions. For example, to evaluate $\sqrt{36}$ you would first press the square root key and then enter 36. See the left screen below. The order of operations on a graphing calculator is also important, and current models assist the user by inserting parentheses when typical errors might occur. The open parenthesis that follows the square root symbol is automatically entered by the calculator so that an expression such as $\sqrt{2 \times 8}$ will not be calculated incorrectly as $\sqrt{2} \times 8$. Compare the two entries and their results in the screen on the right.



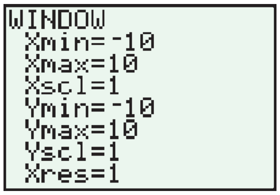
```
√(36)
               6
```



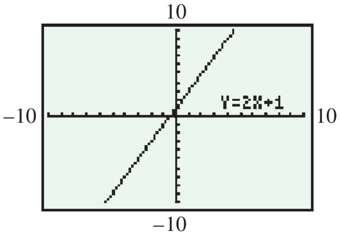
```
√(2*8)
               4
√(2)*8
      11.3137085
```

**Viewing Windows** The viewing window for a graphing calculator is similar to the viewfinder in a camera. A camera usually cannot take a photograph of an entire view of a scene. The camera must be centered on some object and can capture only a portion of the available scenery. A camera with a zoom lens can photograph different views of the same scene by zooming in and out. Graphing calculators have similar capabilities. The *xy*-coordinate plane is infinite. The calculator screen can show only a finite, rectangular region in the plane, and it must be specified before the graph can be drawn. This is done by setting both minimum and maximum values for the *x*- and *y*-axes. The scale (distance between tick marks) is usually specified as well. Determining an appropriate viewing window for a graph is often a challenge, and many times it will take a few attempts before a satisfactory window is found.

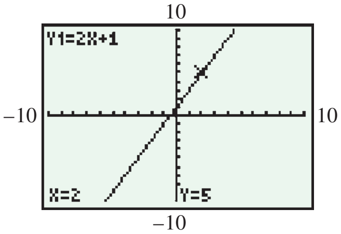
The screen on the left shows a standard viewing window, and the graph of Y = 2X + 1 is shown on the right. Using a different window would give a different view of the line.

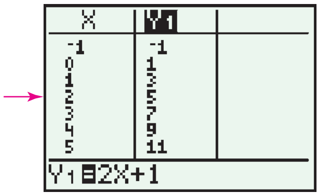


```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



**Locating Points on a Graph: Tracing and Tables** Graphing calculators allow you to trace along the graph of an equation and display the coordinates of points on the graph. For example, the screen on the left below indicates that the point $(2, 5)$ lies on the graph of Y = 2X + 1. Tables for equations can also be displayed. The screen on the right shows a partial table for this same equation. Note the middle of the screen, which indicates that when X = 2, Y = 5.





| X | Y1 |
|---|---|
| -1 | -1 |
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |
| 3 | 7 |
| 4 | 9 |
| 5 | 11 |

Y1=2X+1

**Additional Features** There are many features of graphing calculators that go far beyond the scope of this book. These calculators can be programmed, much like computers. Many of them can solve equations at the stroke of a key, analyze statistical data, and perform symbolic algebraic manipulations. Calculators also provide the opportunity to ask “What if . . . ?” more easily. Values in algebraic expressions can be altered and conjectures tested quickly.

**Final Comments** Despite the power of today’s calculators, they cannot replace human thought. ***In the entire problem-solving process, your brain is the most important component.*** Calculators are only tools, and like any tool, they must be used appropriately in order to enhance our ability to understand mathematics. Mathematical insight may often be the quickest and easiest way to solve a problem; a calculator may be neither needed nor appropriate. By applying mathematical concepts, you can make the decision whether to use a calculator.

# Answers to Selected Exercises

In this section we provide the answers that we think most students will obtain when they work the exercises using the methods explained in the text. If your answer does not look exactly like the one given here, it is not necessarily wrong. In many cases, there are equivalent forms of the answer that are correct. For example, if the answer section shows $\frac{3}{4}$ and your answer is 0.75, you have obtained the right answer, but written it in a different (yet equivalent) form. Unless the directions specify otherwise, 0.75 is just as valid an answer as $\frac{3}{4}$.

In general, if your answer does not agree with the one given in the text, see whether it can be transformed into the other form. If it can, then it is the correct answer. If you still have doubts, talk with your instructor. You might also want to obtain a copy of the *Student's Solutions Manual* that goes with this book. Your college bookstore either has this manual or can order it for you.

## 1 THE REAL NUMBER SYSTEM

### Section 1.1 (pages 10–13)

**1.** true **3.** false; This is an improper fraction. Its value is 1. **5.** false; The fraction $\frac{13}{39}$ is written in lowest terms as $\frac{1}{3}$. **7.** false; *Product* refers to multiplication, so the product of 10 and 2 is 20. **9.** prime **11.** composite; $2 \cdot 3 \cdot 5$ **13.** composite; $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ **15.** neither **17.** composite; $3 \cdot 19$ **19.** prime **21.** composite; $2 \cdot 2 \cdot 31$ **23.** composite; $2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$ **25.** composite; $2 \cdot 7 \cdot 13 \cdot 19$ **27.** $\frac{1}{2}$ **29.** $\frac{5}{6}$ **31.** $\frac{16}{25}$ **33.** $\frac{1}{5}$ **35.** $\frac{6}{5}$ **37.** C **39.** $\frac{24}{35}$ **41.** $\frac{5}{8}$ **43.** $\frac{6}{25}$ **45.** $\frac{6}{5}$, or $1\frac{1}{5}$ **47.** 9 **49.** $\frac{65}{12}$, or $5\frac{5}{12}$ **51.** $\frac{38}{5}$, or $7\frac{3}{5}$ **53.** $\frac{10}{3}$, or $3\frac{1}{3}$ **55.** 12 **57.** $\frac{1}{16}$ **59.** 10 **61.** 18 **63.** $\frac{35}{24}$, or $1\frac{11}{24}$ **65.** $\frac{84}{47}$, or $1\frac{37}{47}$ **67.** A **69.** $\frac{11}{15}$ **71.** $\frac{2}{3}$ **73.** $\frac{8}{9}$ **75.** $\frac{29}{24}$, or $1\frac{5}{24}$ **77.** $\frac{43}{8}$, or $5\frac{3}{8}$ **79.** $\frac{101}{20}$, or $5\frac{1}{20}$ **81.** $\frac{5}{9}$ **83.** $\frac{2}{3}$ **85.** $\frac{1}{4}$ **87.** $\frac{17}{36}$ **89.** $\frac{67}{20}$, or $3\frac{7}{20}$ **91.** $\frac{11}{12}$ **93.** 6 cups **95.** $1\frac{1}{8}$ in. **97.** $\frac{9}{16}$ in. **99.** $618\frac{3}{4}$ ft **101.** $5\frac{5}{24}$ in. **103.** 8 cakes (There will be some sugar left over.) **105.** $16\frac{5}{8}$ yd **107.** $3\frac{3}{8}$ in. **109.** $\frac{1}{20}$ **111.** about $5\frac{8}{25}$ million, or 5,320,000 **113.** **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{6}$

### Section 1.2 (pages 19–21)

**1.** false; $6^2$ means that 6 is used as a factor 2 times; so $6^2 = 6 \cdot 6 = 36$. **3.** false; 1 raised to *any* power is 1. Here, $1^3 = 1 \cdot 1 \cdot 1 = 1$. **5.** false; $4 + 3(8 - 2)$ means $4 + 3 \cdot 6$, which simplifies to $4 + 18$, or 22. The common error leading to 42 is adding 4 to 3 and then multiplying by 6. One must follow the order of operations. **7.** 9 **9.** 49 **11.** 144 **13.** 64 **15.** 1000 **17.** 81 **19.** 1024 **21.** $\frac{1}{36}$ **23.** $\frac{16}{81}$ **25.** 0.064 **27.** 32 **29.** 58 **31.** 22.2 **33.** $\frac{49}{30}$, or $1\frac{19}{30}$ **35.** 12 **37.** 13 **39.** 26 **41.** 4 **43.** 42 **45.** 5 **47.** 41 **49.** 95 **51.** 90 **53.** 14 **55.** 9 **57.** $16 \leq 16$; true **59.** $61 \leq 60$; false **61.** $0 \geq 0$; true **63.** $45 \geq 46$; false **65.** $66 > 72$; false **67.** $2 \geq 3$; false **69.** $3 \geq 3$; true **71.** $3 \cdot (6 + 4) \cdot 2 = 60$ **73.** $10 - (7 - 3) = 6$ **75.** $(8 + 2)^2 = 100$ **77.** Five is less than seventeen; true **79.** Five is not equal to eight; true **81.** Seven is greater than or equal to fourteen; false **83.** Fifteen is less than or equal to 15; true **85.** $15 = 5 + 10$ **87.** $9 > 5 - 4$ **89.** $16 \neq 19$ **91.** $\frac{1}{2} \leq \frac{2}{4}$ **93.** $20 > 5$ **95.** $1.3 \leq 2.5$ **97.** **(a)** $14.7 - 40 \cdot 0.13$ **(b)** 9.5 **(c)** 8.075; walking (5 mph) **99.** Answers will vary.

### Section 1.3 (pages 26–27)

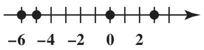
**1.** B **3.** A **5.** $2x^3 = 2 \cdot x \cdot x \cdot x$, while $2x \cdot 2x \cdot 2x = (2x)^3$. **7.** The exponent 2 applies only to its base, which is $x$. **9.** **(a)** 11 **(b)** 13 **11.** **(a)** 16 **(b)** 24 **13.** **(a)** 64 **(b)** 144 **15.** **(a)** $\frac{5}{3}$ **(b)** $\frac{7}{3}$ **17.** **(a)** $\frac{7}{8}$ **(b)** $\frac{13}{12}$ **19.** **(a)** 52 **(b)** 114 **21.** **(a)** 25.836 **(b)** 38.754 **23.** **(a)** 24 **(b)** 28 **25.** **(a)** 12 **(b)** 33 **27.** **(a)** 6 **(b)** $\frac{9}{5}$ **29.** **(a)** $\frac{4}{3}$ **(b)** $\frac{13}{6}$ **31.** **(a)** $\frac{2}{7}$ **(b)** $\frac{16}{27}$ **33.** **(a)** 12 **(b)** 55 **35.** **(a)** 1 **(b)** $\frac{28}{17}$ **37.** **(a)** 3.684 **(b)** 8.841 **39.** $12x$ **41.** $x + 9$ **43.** $x - 4$ **45.** $7 - x$ **47.** $x - 8$ **49.** $\frac{18}{x}$ **51.** $6(x - 4)$ **53.** An expression cannot be solved—it indicates a series of operations to perform. An expression is simplified. An equation is solved. **55.** yes **57.** no **59.** yes **61.** yes **63.** yes **65.** no **67.** $x + 8 = 18$; 10 **69.** $16 - \frac{3}{4}x = 13$; 4 **71.** $2x + 1 = 5$; 2 **73.** $3x = 2x + 8$; 8 **75.** expression **77.** equation **79.** equation **81.** 64.9 yr **83.** 73.8 yr **85.** Life expectancy has increased over 13 yr during this time.

### Section 1.4 (pages 34–36)

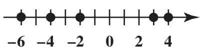
**1.** 2,866,000 **3.** $-52{,}000$ **5.** $-11.2$; 8.6 **7.** 82.60 **9.** 4 **11.** 0 **13.** One example is $\sqrt{13}$. There are others. **15.** true **17.** true **19.** false

*In Exercises 21–25, answers will vary.*

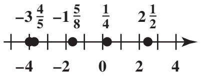
**21.** $\frac{1}{2}, \frac{5}{8}, 1\frac{3}{4}$ **23.** $-3\frac{1}{2}, -\frac{2}{3}, \frac{3}{7}$ **25.** $\sqrt{5}, \pi, -\sqrt{3}$ **27.** **(a)** 3, 7 **(b)** 0, 3, 7 **(c)** $-9, 0, 3, 7$ **(d)** $-9, -1\frac{1}{4}, -\frac{3}{5}, 0, 0.\overline{1}, 3, 5.9, 7$ **(e)** $-\sqrt{7}, \sqrt{5}$ **(f)** All are real numbers. **29.**



**31.**



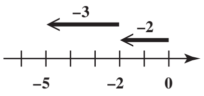
**33.**



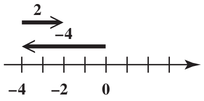
**35.** **(a)** A **(b)** A **(c)** B **(d)** B **37.** **(a)** 7 **(b)** 7 **39.** **(a)** $-8$ **(b)** 8 **41.** **(a)** $\frac{3}{4}$ **(b)** $\frac{3}{4}$ **43.** 6 **45.** $-12$ **47.** $-\frac{2}{3}$ **49.** 3 **51.** This is not true. The absolute value of 0 is 0, and 0 is not positive. A more accurate way of describing absolute value is to say that *absolute value is never negative,* or *absolute value is always nonnegative.* **53.** $-11$ **55.** $-7$ **57.** 4 **59.** $|-3.5|$, or 3.5 **61.** $-|-6|$, or $-6$ **63.** $|5 - 3|$, or 2 **65.** true **67.** true **69.** true **71.** false **73.** true **75.** false **77.** fuel and other utilities, 2004 to 2005 **79.** apparel and upkeep, 2006 to 2007

## Section 1.5 (pages 44–48)

**1.** negative



**3.** negative



**5.** $-8; -6; 2$ **7.** positive **9.** negative **11.** $-8$ **13.** $-12$ **15.** $2$ **17.** $-2$ **19.** $8.9$ **21.** $12$ **23.** $5$ **25.** $2$ **27.** $-9$ **29.** $0$ **31.** $\frac{1}{2}$ **33.** $-\frac{19}{24}$ **35.** $-\frac{3}{4}$ **37.** $-7.7$ **39.** $-8$ **41.** $0$ **43.** $-20$ **45.** $-3$ **47.** $-4$ **49.** $-8$ **51.** $-14$ **53.** $9$ **55.** $-4$ **57.** $4$ **59.** $\frac{3}{4}$ **61.** $-\frac{11}{8}$, or $-1\frac{3}{8}$ **63.** $\frac{15}{8}$, or $1\frac{7}{8}$ **65.** $11.6$ **67.** $-9.9$ **69.** $10$ **71.** $-5$ **73.** $11$ **75.** $-10$ **77.** $22$ **79.** $-2$ **81.** $-\frac{17}{8}$, or $-2\frac{1}{8}$ **83.** $-\frac{1}{4}$, or $-0.25$ **85.** $-6$ **87.** $-12$ **89.** $-5.90617$ **91.** $-5 + 12 + 6; 13$ **93.** $[-19 + (-4)] + 14; -9$ **95.** $[-4 + (-10)] + 12; -2$ **97.** $\left[\frac{5}{7} + \left(-\frac{9}{7}\right)\right] + \frac{2}{7}; -\frac{2}{7}$ **99.** $4 - (-8); 12$ **101.** $-2 - 8; -10$ **103.** $[9 + (-4)] - 7; -2$ **105.** $[8 - (-5)] - 12; 1$ **107.** $-12$ **109.** $-56$°F **111.** $-69$°F **113.** $-184$ m **115.** **(a)** 7.4% **(b)** Americans spent more money than they earned, which means they had to dip into savings or increase borrowing. **117.** \$3173 **119.** 17 **121.** \$1045.55 **123.** \$323.83 **125.** 30.4 billion dollars **127.** 3.1 billion dollars **129.** 50,395 ft **131.** 1345 ft **133.** 136 ft

## Section 1.6 (pages 56–59)

**1.** greater than 0 **3.** less than 0 **5.** greater than 0 **7.** equal to 0 **9.** undefined; 0; Examples include $\frac{1}{0}$, which is undefined, and $\frac{0}{1}$, which equals 0. **11.** $-30$ **13.** $30$ **15.** $120$ **17.** $-33$ **19.** $0$ **21.** $-2.38$ **23.** $\frac{5}{12}$ **25.** $-\frac{1}{6}$ **27.** $6$ **29.** $-32, -16, -8, -4, -2, -1, 1, 2, 4, 8, 16, 32$ **31.** $-40, -20, -10, -8, -5, -4, -2, -1, 1, 2, 4, 5, 8, 10, 20, 40$ **33.** $-31, -1, 1, 31$ **35.** $3$ **37.** $-7$ **39.** $8$ **41.** $-6$ **43.** $\frac{32}{3}$, or $10\frac{2}{3}$ **45.** $-4$ **47.** $0$ **49.** undefined **51.** $-11$ **53.** $-2$ **55.** $35$ **57.** $13$ **59.** $-22$ **61.** $6$ **63.** $-18$ **65.** $67$ **67.** $-8$ **69.** $3$ **71.** $7$ **73.** $4$ **75.** $-1$ **77.** $4$ **79.** $-3$ **81.** $47$ **83.** $72$ **85.** $-\frac{78}{25}$ **87.** $0$ **89.** $-23$ **91.** $2$ **93.** $9 + (-9)(2); -9$ **95.** $-4 - 2(-1)(6); 8$ **97.** $(1.5)(-3.2) - 9; -13.8$ **99.** $12[9 - (-8)]; 204$ **101.** $\frac{-12}{-5 + (-1)}; 2$ **103.** $\frac{15 + (-3)}{4(-3)}; -1$ **105.** $\frac{2}{3}[8 - (-1)]; 6$ **107.** $0.20(-5 \cdot 6); -6$ **109.** $\left(\frac{1}{2} + \frac{5}{8}\right)\left(\frac{3}{5} - \frac{1}{3}\right); \frac{3}{10}$ **111.** $\frac{-\frac{1}{2}\left(\frac{3}{4}\right)}{-\frac{2}{3}}; \frac{9}{16}$ **113.** $\frac{x}{3} = -3; -9$ **115.** $x - 6 = 4; 10$ **117.** $x + 5 = -5; -10$ **119.** $8\frac{2}{5}$ **121.** $4$ **123.** $2$ **125.** **(a)** 6 is divisible by 2. **(b)** 9 is not divisible by 2. **127.** **(a)** 64 is divisible by 4. **(b)** 35 is not divisible by 4. **129.** **(a)** 2 is divisible by 2 and $1 + 5 + 2 + 4 + 8 + 2 + 2 = 24$ is divisible by 3. **(b)** Although 0 is divisible by 2, $2 + 8 + 7 + 3 + 5 + 9 + 0 = 34$ is not divisible by 3. **131.** **(a)** $4 + 1 + 1 + 4 + 1 + 0 + 7 = 18$ is divisible by 9. **(b)** $2 + 2 + 8 + 7 + 3 + 2 + 1 = 25$ is not divisible by 9.

## Summary Exercises on Operations with Real Numbers (pages 59–60)

**1.** $-16$ **2.** $4$ **3.** $0$ **4.** $-24$ **5.** $-17$ **6.** $76$ **7.** $-18$ **8.** $90$ **9.** $38$ **10.** $4$ **11.** $-5$ **12.** $5$ **13.** $-\frac{7}{2}$, or $-3\frac{1}{2}$ **14.** $4$ **15.** $13$ **16.** $\frac{5}{4}$, or $1\frac{1}{4}$ **17.** $9$ **18.** $\frac{37}{10}$, or $3\frac{7}{10}$ **19.** $0$ **20.** $25$ **21.** $14$ **22.** undefined **23.** $-4$ **24.** $\frac{6}{5}$, or $1\frac{1}{5}$ **25.** $-1$ **26.** $\frac{52}{37}$, or $1\frac{15}{37}$ **27.** $\frac{17}{16}$, or $1\frac{1}{16}$ **28.** $-\frac{2}{3}$ **29.** $3.33$ **30.** $1.02$ **31.** $-13$ **32.** $0$ **33.** $24$ **34.** $-7$ **35.** $37$ **36.** $-3$ **37.** $-1$ **38.** $\frac{1}{2}$ **39.** $-\frac{5}{13}$ **40.** $5$ **41.** $-\frac{8}{27}$ **42.** $4$

## Section 1.7 (pages 67–69)

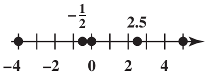
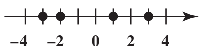
**1.** **(a)** B **(b)** F **(c)** C **(d)** I **(e)** B **(f)** D, F **(g)** B **(h)** A **(i)** G **(j)** H **3.** yes **5.** no **7.** no **9.** (foreign sales) clerk; foreign (sales clerk) **11.** $-15$; commutative property **13.** 3; commutative property **15.** 6; associative property **17.** 7; associative property **19.** Subtraction is not associative. **21.** row 1: $-5, \frac{1}{5}$; row 2: $10, -\frac{1}{10}$; row 3: $\frac{1}{2}, -2$; row 4: $-\frac{3}{8}, \frac{8}{3}$; row 5: $-x, \frac{1}{x}$; row 6: $y, -\frac{1}{y}$; opposite; the same **23.** commutative property **25.** associative property **27.** associative property **29.** inverse property **31.** inverse property **33.** identity property **35.** commutative property **37.** distributive property **39.** identity property **41.** distributive property **43.** $150$ **45.** $2010$ **47.** $400$ **49.** $1400$ **51.** $470$ **53.** $-9300$ **55.** $11$ **57.** $0$ **59.** $-0.38$ **61.** $1$ **63.** The expression following the first equals symbol should be $-3(4) - 3(-6)$. $-3(4 - 6)$ means $-3(4) - 3(-6)$, which simplifies to $-12 + 18$, or 6. **65.** $85$ **67.** $4t + 12$ **69.** $7z - 56$ **71.** $-8r - 24$ **73.** $-2x - \frac{3}{4}$ **75.** $-5y + 20$ **77.** $-16y - 20z$ **79.** $8(z + w)$ **81.** $7(2v + 5r)$ **83.** $24r + 32s - 40y$ **85.** $-24x - 9y - 12z$ **87.** $5(x + 3)$ **89.** $-4t - 3m$ **91.** $5c + 4d$ **93.** $q - 5r + 8s$

## Section 1.8 (pages 72–74)

**1.** B **3.** C **5.** $4r + 11$ **7.** $5 + 2x - 6y$ **9.** $-7 + 3p$ **11.** $2 - 3x$ **13.** $-12$ **15.** $3$ **17.** $1$ **19.** $-1$ **21.** $\frac{1}{2}$ **23.** $\frac{2}{5}$ **25.** $10$ **27.** like **29.** unlike **31.** like **33.** unlike **35.** The student made a sign error when applying the distributive property: $7x - 2(3 - 2x)$ means $7x - 2(3) - 2(-2x)$, which simplifies to $7x - 6 + 4x$, or $11x - 6$. **37.** $13y$ **39.** $-9x$ **41.** $13b$ **43.** $7k + 15$ **45.** $-4y$ **47.** $2x + 6$ **49.** $14 - 7m$ **51.** $-17 + x$ **53.** $23x$ **55.** $-\frac{1}{3}t - \frac{28}{3}$ **57.** $9y^2$ **59.** $-14p^3 + 5p^2$ **61.** $8x + 15$ **63.** $5x + 15$ **65.** $-4y + 22$ **67.** $-\frac{3}{2}y + 16$ **69.** $-16y + 63$ **71.** $4r + 15$ **73.** $12k - 5$ **75.** $-2k - 3$ **77.** $4x - 7$ **79.** $-23.7y - 12.6$ **81.** $(x + 3) + 5x; 6x + 3$ **83.** $(13 + 6x) - (-7x); 13 + 13x$ **85.** $2(3x + 4) - (-4 + 6x); 12$ **87.** $1000 + 5x$ (dollars) **88.** $750 + 3y$ (dollars) **89.** $1000 + 5x + 750 + 3y$ (dollars) **90.** $1750 + 5x + 3y$ (dollars)

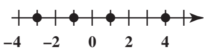
## Chapter 1 Review Exercises (pages 79–83)

**1.** $\frac{3}{4}$ **2.** $\frac{7}{2}$, or $3\frac{1}{2}$ **3.** $\frac{11}{24}$ **4.** $\frac{59}{16}$, or $3\frac{11}{16}$ **5.** about 1270 thousand **6.** about 5079 thousand **7.** $625$ **8.** $\frac{27}{125}$ **9.** $0.0004$ **10.** $0.001$ **11.** $27$ **12.** $17$ **13.** $4$ **14.** $399$ **15.** $39$ **16.** $5$ **17.** true **18.** true **19.** false **20.** $13 < 17$ **21.** $5 + 2 \neq 10$ **22.** $\frac{2}{3} \geq \frac{4}{6}$ **23.** $30$ **24.** $60$ **25.** $14$ **26.** $13$ **27.** $x + 6$ **28.** $8 - x$ **29.** $6x - 9$ **30.** $12 + \frac{3}{5}x$ **31.** yes **32.** no **33.** $2x - 6 = 10; 8$ **34.** $4x = 8; 2$

**35.**  **36.** 

**37.** rational numbers, real numbers **38.** rational numbers, real numbers **39.** natural numbers, whole numbers, integers, rational numbers, real numbers **40.** irrational numbers, real numbers **41.** $-10$ **42.** $-9$ **43.** $-\frac{3}{4}$ **44.** $-|23|$ **45.** true **46.** true **47.** true **48.** true **49.** **(a)** 9 **(b)** 9 **50.** **(a)** 0 **(b)** 0 **51.** **(a)** $-6$ **(b)** 6 **52.** **(a)** $\frac{5}{7}$ **(b)** $\frac{5}{7}$ **53.** 12 **54.** $-3$ **55.** $-19$ **56.** $-7$ **57.** $-6$ **58.** $-4$ **59.** $-17$ **60.** $-\frac{29}{36}$ **61.** $-21.8$ **62.** $-14$ **63.** $-10$ **64.** $-19$ **65.** $-11$ **66.** $-1$ **67.** 7 **68.** $-\frac{43}{35}$, or $-1\frac{8}{35}$ **69.** 10.31 **70.** $-12$ **71.** 2 **72.** $-3$ **73.** $(-31 + 12) + 19; 0$ **74.** $[-4 + (-8)] + 13; 1$ **75.** $-4 - (-6); 2$ **76.** $[4 + (-8)] - 5; -9$ **77.** $-2$ **78.** $-1$ **79.** \$26.25 **80.** −10°F **81.** −\$29 **82.** −10° **83.** 38 **84.** 9544.2 **85.** 36 **86.** $-105$ **87.** $\frac{1}{2}$ **88.** 10.08 **89.** $-20$ **90.** $-10$ **91.** $-24$ **92.** $-35$ **93.** 4 **94.** $-20$ **95.** $-\frac{3}{4}$ **96.** 11.3 **97.** $-1$ **98.** 2 **99.** 1 **100.** 0.5 **101.** $-18$ **102.** $-18$ **103.** 125 **104.** $-423$ **105.** $-4(5) - 9; -29$ **106.** $\frac{5}{6}[12 + (-6)]; 5$ **107.** $\frac{12}{8 + (-4)}; 3$ **108.** $\frac{-20(12)}{15 - (-15)}; -8$ **109.** $8x = -24; -3$ **110.** $\frac{x}{3} = -2; -6$ **111.** 32 **112.** $-3$ **113.** identity property **114.** identity property **115.** inverse property **116.** inverse property **117.** associative property **118.** associative property **119.** distributive property **120.** commutative property **121.** $7(y + 2)$ **122.** $-48 + 12t$ **123.** $3(2s + 5y)$ **124.** $4r - 5s$ **125.** $11m$ **126.** $16p^2$ **127.** $16p^2 + 2p$ **128.** $-4k + 12$ **129.** $-2m + 29$ **130.** $-5k - 1$ **131.** $-2(3x) - 7x; -13x$ **132.** $(5 + 4x) + 8x; 5 + 12x$ **133.** $\frac{8}{3}$, or $2\frac{2}{3}$ **134.** $-\frac{1}{24}$ **135.** 2 **136.** $-\frac{28}{15}$, or $-1\frac{13}{15}$ **137.** $-\frac{3}{2}$, or $-1\frac{1}{2}$ **138.** $\frac{25}{36}$ **139.** 16 **140.** 77.6 **141.** 11 **142.** $16t - 36$ **143.** $8x^2 - 21y^2$ **144.** 24 **145.** Dividing 0 *by* a nonzero number gives a quotient of 0. However, dividing a number *by* 0 is undefined. **146.** −47°F **147.** $-0.84$ million students **148.** $-1.05$ million students **149.** 1.02 million students **150.** 1.39 million students

## Chapter 1 Test (pages 83–84)

[1.1] **1.** $\frac{7}{11}$ **2.** $\frac{241}{120}$, or $2\frac{1}{120}$ **3.** $\frac{19}{18}$, or $1\frac{1}{18}$ [1.2] **4.** true [1.4] **5.**  **6.** rational numbers, real numbers **7.** If $-8$ and $-1$ are both graphed on a number line, we see that the point for $-8$ is to the *left* of the point for $-1$. This indicates $-8 < -1$. [1.6] **8.** $\frac{-6}{2 + (-8)}; 1$ [1.1, 1.4–1.6] **9.** 4 **10.** $-\frac{17}{6}$, or $-2\frac{5}{6}$ **11.** 2 **12.** 6 **13.** 108 **14.** $\frac{30}{7}$, or $4\frac{2}{7}$ [1.3, 1.5, 1.6] **15.** 6 **16.** 4 [1.6] **17.** $-70$ **18.** 3 [1.4–1.6] **19.** 7000 m **20.** 15 **21.** −\$1.42 trillion [1.7] **22.** B **23.** D **24.** E **25.** A **26.** C **27.** distributive property **28.** **(a)** $-18$ **(b)** $-18$ **(c)** The distributive property assures us that the answers must be the same, because $a(b + c) = ab + ac$ for all $a, b, c$. [1.8] **29.** $21x$ **30.** $15x - 3$

## 2 LINEAR EQUATIONS AND INEQUALITIES IN ONE VARIABLE

### Section 2.1 (pages 90–92)

**1.** **(a)** expression; $x + 15$ **(b)** expression; $y + 7$ **(c)** equation; $\{-1\}$ **(d)** equation; $\{-17\}$ **3.** A and B **5.** $\{12\}$ **7.** $\{31\}$ **9.** $\{-3\}$ **11.** $\{4\}$ **13.** $\{-9\}$ **15.** $\left\{-\frac{3}{4}\right\}$ **17.** $\{-10\}$ **19.** $\{-13\}$ **21.** $\{10\}$ **23.** $\left\{\frac{4}{15}\right\}$ **25.** $\{6.3\}$ **27.** $\{-16.9\}$ **29.** $\{7\}$ **31.** $\{-4\}$ **33.** $\{-3\}$ **35.** $\{0\}$ **37.** $\{2\}$ **39.** $\{-6\}$ **41.** $\{-2\}$ **43.** $\{3\}$ **45.** $\{0\}$ **47.** $\{0\}$ **49.** $\{-5\}$ **51.** $\{-7\}$ **53.** $\{13\}$ **55.** $\{-4\}$ **57.** $\{0\}$ **59.** $\left\{\frac{7}{15}\right\}$ **61.** $\{7\}$ **63.** $\{-4\}$ **65.** $\{13\}$ **67.** $\{29\}$ **69.** $\{18\}$ **71.** $\{12\}$ **73.** Answers will vary. One example is $x - 6 = -8$. **75.** $3x = 2x + 17; \{17\}$ **77.** $7x - 6x = -9; \{-9\}$ **79.** 1 **81.** $x$ **83.** $r$

### Section 2.2 (pages 96–97)

**1.** **(a)** multiplication property of equality **(b)** addition property of equality **(c)** multiplication property of equality **(d)** addition property of equality **3.** To find the solution of $-x = 5$, multiply (or divide) each side by $-1$, or use the rule "If $-x = a$, then $x = -a$." **5.** $\frac{5}{4}$ **7.** 10 **9.** $-\frac{2}{9}$ **11.** $-1$ **13.** 6 **15.** $-4$ **17.** 0.12 **19.** $-1$ **21.** $\{6\}$ **23.** $\left\{\frac{15}{2}\right\}$ **25.** $\{-5\}$ **27.** $\{-4\}$ **29.** $\left\{-\frac{18}{5}\right\}$, or $\{-3.6\}$ **31.** $\{12\}$ **33.** $\{0\}$ **35.** $\{-12\}$ **37.** $\left\{\frac{3}{4}\right\}$ **39.** $\{40\}$ **41.** $\{-12.2\}$ **43.** $\{-48\}$ **45.** $\{72\}$ **47.** $\{-35\}$ **49.** $\{14\}$ **51.** $\{18\}$ **53.** $\left\{-\frac{27}{35}\right\}$ **55.** $\{-30\}$ **57.** $\{3\}$ **59.** $\{-5\}$ **61.** $\{20\}$ **63.** $\{7\}$ **65.** $\{0\}$ **67.** $\left\{-\frac{3}{5}\right\}$ **69.** $\{18\}$ **71.** Answers will vary. One example is $\frac{3}{2}x = -6$. **73.** $4x = 6; \left\{\frac{3}{2}\right\}$ **75.** $\frac{x}{-5} = 2; \{-10\}$ **77.** $-3m - 5$ **79.** $-8 + 5p$ **81.** $\{5\}$

### Section 2.3 (pages 104–106)

**1.** Use the addition property of equality to subtract 8 from each side. **3.** Clear parentheses by using the distributive property. **5.** Clear fractions by multiplying by the LCD, 6. **7.** D **9.** $\{4\}$ **11.** $\{-5\}$ **13.** $\left\{\frac{5}{2}\right\}$ **15.** $\{-1\}$ **17.** $\left\{-\frac{1}{2}\right\}$ **19.** $\{-3\}$ **21.** $\{5\}$ **23.** $\{0\}$ **25.** $\left\{\frac{4}{3}\right\}$ **27.** $\left\{-\frac{5}{3}\right\}$ **29.** $\{5\}$ **31.** $\{0\}$ **33.** $\emptyset$ **35.** {all real numbers} **37.** $\emptyset$ **39.** $\{5\}$ **41.** $\{12\}$ **43.** $\{11\}$ **45.** $\{0\}$ **47.** $\{18\}$ **49.** $\{120\}$ **51.** $\{6\}$ **53.** $\{15{,}000\}$ **55.** $\{8\}$ **57.** $\{4\}$ **59.** $\{20\}$ **61.** {all real numbers} **63.** $\emptyset$ **65.** $11 - q$ **67.** $\frac{9}{x}$ **69.** $x + 9$ **71.** $65 - h$ **73.** $x + 15; x - 5$ **75.** $25r$ **77.** $\frac{t}{5}$ **79.** $3x + 2y$ **81.** $-6 + x$ **83.** $-5 - x$ **85.** $12(x - 9)$

### Summary Exercises on Solving Linear Equations (pages 106–107)

**1.** $\{-5\}$ **2.** $\{4\}$ **3.** $\{-5.1\}$ **4.** $\{12\}$ **5.** $\{-25\}$ **6.** $\{-6\}$ **7.** $\{0\}$ **8.** $\{-16\}$ **9.** $\{-6\}$ **10.** $\left\{-\frac{96}{5}\right\}$ **11.** {all real numbers}

**12.** $\left\{\frac{7}{3}\right\}$ **13.** $\{7\}$ **14.** $\{1\}$ **15.** $\{5\}$ **16.** $\{23.7\}$ **17.** $\{6\}$ **18.** $\{3\}$ **19.** $\emptyset$ **20.** $\emptyset$ **21.** $\{25\}$ **22.** $\{-10.8\}$ **23.** $\{3\}$ **24.** $\{7\}$ **25.** $\{2\}$ **26.** {all real numbers} **27.** $\{-2\}$ **28.** $\{70\}$ **29.** $\left\{\frac{14}{17}\right\}$ **30.** $\{0\}$

## Section 2.4 (pages 115–119)

**1.** D; There cannot be a fractional number of cars. **3.** A; Distance cannot be negative. **5.** 7 **7.** 3 **9.** 6 **11.** $-3$ **13.** Pennsylvania: 35 screens; Ohio: 33 screens **15.** Democrats: 58; Republicans: 40 **17.** Bon Jovi: \$210.7 million; Bruce Springsteen: \$204.6 million **19.** wins: 62; losses: 20 **21.** orange: 97 mg; pineapple: 25 mg **23.** 168 DVDs **25.** onions: 81.3 kg; grilled steak: 536.3 kg **27.** 1950 Denver nickel: \$16.00; 1945 Philadelphia nickel: \$8.00 **29.** whole wheat: 25.6 oz; rye: 6.4 oz **31.** American: 18; United: 11; Southwest: 26 **33.** shortest piece: 15 in.; middle piece: 20 in.; longest piece: 24 in. **35.** 36 million mi **37.** $A$ and $B$: 40°; $C$: 100° **39.** 68, 69 **41.** 101, 102 **43.** 10, 12 **45.** 10, 11 **47.** 18 **49.** 15, 17, 19 **51.** 18° **53.** 20° **55.** 39° **57.** 50° **59.** 24 **61.** 20

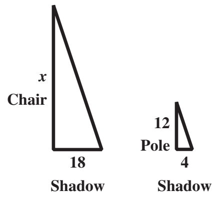
## Section 2.5 (pages 125–129)

**1. (a)** The perimeter of a plane geometric figure is the distance around the figure. **(b)** The area of a plane geometric figure is the measure of the surface covered or enclosed by the figure. **3.** 180°; the same **5.** area **7.** perimeter **9.** area **11.** area **13.** $P = 26$ **15.** $\mathcal{A} = 64$ **17.** $b = 4$ **19.** $t = 5.6$ **21.** $I = 1575$ **23.** $B = 14$ **25.** $r = 2.6$ **27.** $r = 10$ **29.** $\mathcal{A} = 50.24$ **31.** $r = 6$ **33.** $V = 150$ **35.** $V = 52$ **37.** $V = 7234.56$ **39.** length: 18 in.; width: 9 in. **41.** length: 14 m; width: 4 m **43.** shortest: 5 in.; medium: 7 in.; longest: 8 in. **45.** two equal sides: 7 m; third side: 10 m **47.** about 154,000 ft$^2$ **49.** perimeter: 5.4 m; area: 1.8 m$^2$ **51.** 10 ft **53.** 194.48 ft$^2$; 49.42 ft **55.** 23,800.10 ft$^2$ **57.** length: 36 in.; volume: 11,664 in.$^3$ **59.** 48°, 132° **61.** 55°, 35° **63.** 51°, 51° **65.** 105°, 105° **67.** $t = \frac{d}{r}$ **69.** $b = \frac{\mathcal{A}}{h}$ **71.** $d = \frac{C}{\pi}$ **73.** $H = \frac{V}{LW}$ **75.** $r = \frac{I}{pt}$ **77.** $h = \frac{2\mathcal{A}}{b}$ **79.** $h = \frac{3V}{\pi r^2}$ **81.** $b = P - a - c$ **83.** $W = \frac{P - 2L}{2}$ **85.** $m = \frac{y - b}{x}$ **87.** $y = \frac{C - Ax}{B}$ **89.** $r = \frac{M - C}{C}$, or $r = \frac{M}{C} - 1$ **91.** $a = \frac{P - 2b}{2}$, or $a = \frac{P}{2} - b$ **93.** $\{5000\}$ **95.** $\{28\}$ **97.** $\left\{-\frac{1}{12}\right\}$

## Section 2.6 (pages 135–139)

**1. (a)** C **(b)** D **(c)** B **(d)** A **3.** $\frac{4}{3}$ **5.** $\frac{4}{3}$ **7.** $\frac{15}{2}$ **9.** $\frac{1}{5}$ **11.** $\frac{5}{6}$ **13.** 10 lb; \$0.429 **15.** 32 oz; \$0.093 **17.** 128 oz; \$0.051 **19.** 36 oz; \$0.049 **21.** 263 oz; \$0.076 **23.** true **25.** false **27.** true **29.** $\{35\}$ **31.** $\{7\}$ **33.** $\left\{\frac{45}{2}\right\}$ **35.** $\{2\}$ **37.** $\{-1\}$ **39.** $\{5\}$ **41.** $\left\{-\frac{31}{5}\right\}$ **43.** \$30.00 **45.** \$8.75 **47.** \$67.50 **49.** \$48.90 **51.** 4 ft **53.** 2.7 in. **55.** 2.0 in. **57.** $2\frac{5}{8}$ cups **59.** \$428.82 **61.** 50,000 fish **63.** $x = 4$ **65.** $x = 2$ **67.** $x = 1$; $y = 4$

**69. (a)**



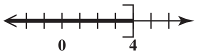
**(b)** 54 ft **71.** \$237 **73.** \$272 **75.** 0.53 **77.** 0.96 **79.** 0.09 **81.** 1.29 **83.** 80% **85.** 2% **87.** 12.5% **89.** 220% **91.** 109.2 **93.** 700 **95.** 425 **97.** 8% **99.** 120% **101.** \$119.25; \$675.75 **103.** 80% **105.** \$3000 **107.** $\{6\}$ **109.** $\{4\}$

## Section 2.7 (pages 145–150)

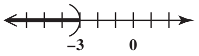
**1.** 45 L **3.** \$750 **5.** \$17.50 **7.** A **9. (a)** 532,000 **(b)** 798,000 **(c)** 494,000 **11.** D **13.** 160 L **15.** $13\frac{1}{3}$ L **17.** 4 L **19.** 20 mL **21.** 4 L **23.** \$2100 at 5%; \$900 at 4% **25.** \$2500 at 6%; \$13,500 at 5% **27.** 10 nickels **29.** 44-cent stamps: 25; 17-cent stamps: 20 **31.** Arabian Mocha: 7 lb; Colombian Decaf: 3.5 lb **33.** A **35.** 530 mi **37.** 3.483 hr **39.** 7.97 m per sec **41.** 8.47 m per sec **43.** 5 hr **45.** $1\frac{3}{4}$ hr **47.** $7\frac{1}{2}$ hr **49.** eastbound: 300 mph; westbound: 450 mph **51.** 40 mph; 60 mph **53.** Bob: 7 yr old; Kevin: 21 yr old **55.** width: 3 ft; length: 9 ft **57.** \$650 **59.** false **61.** true **63.** true

## Section 2.8 (pages 160–163)

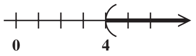
**1.** $>, <$ (or $<, >$); $\geq, \leq$ (or $\leq, \geq$) **3.** $(0, \infty)$ **5.** $x > -4$ **7.** $x \leq 4$ **9.** $(-\infty, 4]$



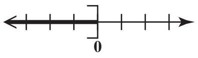
**11.** $(-\infty, -3)$



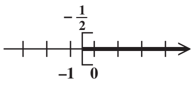
**13.** $(4, \infty)$



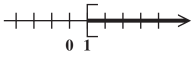
**15.** $(-\infty, 0]$



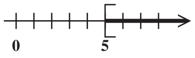
**17.** $\left[-\frac{1}{2}, \infty\right)$



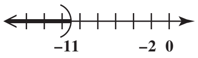
**19.** $[1, \infty)$



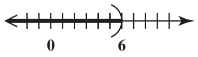
**21.** $[5, \infty)$



**23.** $(-\infty, -11)$

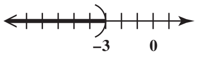


**25.** $(-\infty, 6)$

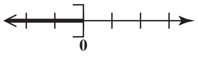


**27.** $[-10, \infty)$

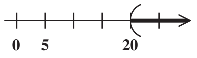
**29.** $(-\infty, -3)$



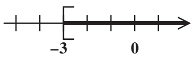
**31.** $(-\infty, 0]$



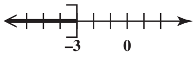
**33.** $(20, \infty)$



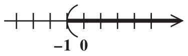
**35.** $[-3, \infty)$



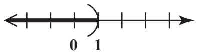
**37.** $(-\infty, -3]$

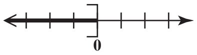


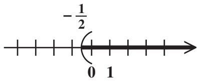
**39.** $(-1, \infty)$

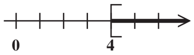


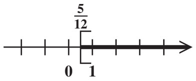
**41.** $[-5, \infty)$


**43.** $(-\infty, 1)$


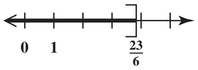
**45.** $(-\infty, 0]$


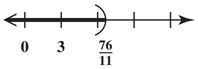
**47.** $\left(-\frac{1}{2}, \infty\right)$


**49.** $[4, \infty)$


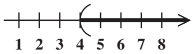
**51.** $\left[\frac{5}{12}, \infty\right)$


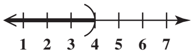
**53.** $(-\infty, 32)$


**55.** $\left(-\infty, \frac{23}{6}\right]$


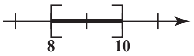
**57.** $\left(-\infty, \frac{76}{11}\right)$


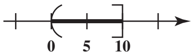
**59.** $\{4\}$


**60.** $(4, \infty)$


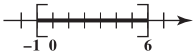
**61.** $(-\infty, 4)$


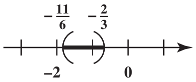
**62.** The graph is the set of all real numbers. **63.** $x \geq 18$ **65.** $x > 5$ **67.** $x \leq 20$ **69.** 83 or greater **71.** all numbers greater than 16 **73.** It is never less than −13°F. **75.** 32 or greater **77.** 15 min **79.** $R = 5x - 100$ **81.** $P = (5x - 100) - (125 + 4x) = x - 225$; $x > 225$ **83.** $-1 < x < 2$ **85.** $-1 < x \leq 2$

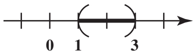
**87.** $[8, 10]$


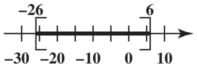
**89.** $(0, 10]$


**91.** $(-3, 4)$

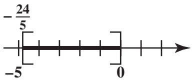

**93.** $[-1, 6]$


**95.** $\left(-\frac{11}{6}, -\frac{2}{3}\right)$


**97.** $(1, 3)$


**99.** $[-26, 6]$


**101.** $[-3, 6]$

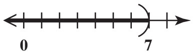

**103.** $\left[-\frac{24}{5}, 0\right]$


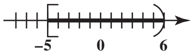
**105.** **(a)** −7 **(b)** 23 **107.** **(a)** −14 **(b)** 22 **109.** **(a)** $-\frac{14}{5}$ **(b)** $-\frac{2}{5}$

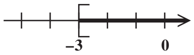
## Chapter 2 Review Exercises (pages 168–172)

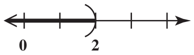
**1.** $\{6\}$ **2.** $\{-12\}$ **3.** $\{7\}$ **4.** $\left\{\frac{2}{3}\right\}$ **5.** $\{11\}$ **6.** $\{17\}$ **7.** $\{5\}$ **8.** $\{-4\}$ **9.** $\{5\}$ **10.** $\{-12\}$ **11.** $\left\{\frac{64}{5}\right\}$ **12.** $\{4\}$ **13.** {all real numbers} **14.** $\{-19\}$ **15.** {all real numbers} **16.** $\{20\}$ **17.** $\emptyset$ **18.** $\{-1\}$ **19.** $-\frac{7}{2}$ **20.** Democrats: 70; Republicans: 48 **21.** Hawaii: 6425 mi²; Rhode Island: 1212 mi² **22.** Seven Falls: 300 ft; Twin Falls: 120 ft **23.** 80° **24.** 11, 13 **25.** $h = 11$ **26.** $\mathcal{A} = 28$ **27.** $r = 4.75$ **28.** $V = 904.32$ **29.** $h = \frac{\mathcal{A}}{b}$ **30.** $h = \frac{2\mathcal{A}}{b + B}$ **31.** 135°; 45° **32.** 100°; 100° **33.** 2 cm **34.** diameter: approximately 19.9 ft; radius: approximately 9.95 ft; area: approximately 311 ft² **35.** 42.2°; 92.8° **36.** $\frac{3}{2}$ **37.** $\frac{5}{14}$ **38.** $\frac{3}{4}$ **39.** $\left\{\frac{7}{2}\right\}$ **40.** $\left\{-\frac{8}{3}\right\}$ **41.** \$3.06 **42.** 375 km **43.** 10 bronze medals **44.** 25.5 oz; \$0.137 **45.** 6 **46.** 175% **47.** 2500 **48.** 3.75 L **49.** \$5000 at 5%; \$5000 at 6% **50.** 8.2 mph **51.** 13 hr **52.** $2\frac{1}{2}$ hr

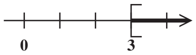
**53.** $[-4, \infty)$

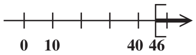

**54.** $(-\infty, 7)$


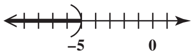
**55.** $[-5, 6)$


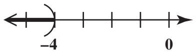
**56.** B **57.** $[-3, \infty)$


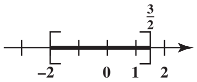
**58.** $(-\infty, 2)$


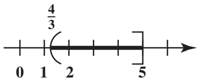
**59.** $[3, \infty)$


**60.** $[46, \infty)$


**61.** $(-\infty, -5)$


**62.** $(-\infty, -4)$


**63.** $\left[-2, \frac{3}{2}\right]$


**64.** $\left(\frac{4}{3}, 5\right]$


**65.** 88 or more **66.** all numbers less than or equal to $-\frac{1}{3}$ **67.** $\{7\}$ **68.** $r = \frac{I}{pt}$ **69.** $(-\infty, 2)$ **70.** $\{-9\}$ **71.** $\{70\}$ **72.** $\left\{\frac{13}{4}\right\}$ **73.** $\emptyset$ **74.** {all real numbers} **75.** \$304 **76.** 4000 calories **77.** Golden Gate Bridge: 4200 ft; Brooklyn Bridge: 1596 ft **78.** 100 oz; \$0.060 **79.** 8 qt **80.** faster train: 80 mph; slower train: 50 mph **81.** 44 m **82.** 50 m or less

## Chapter 2 Test (pages 172–173)

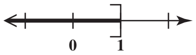
[2.1–2.3] **1.** $\{-6\}$ **2.** $\{21\}$ **3.** $\emptyset$ **4.** $\{30\}$ **5.** {all real numbers} [2.4] **6.** wins: 100; losses: 62 **7.** Hawaii: 4021 mi²; Maui: 728 mi²; Kauai: 551 mi² **8.** 50° [2.5] **9.** **(a)** $W = \frac{P - 2L}{2}$ **(b)** 18 **10.** 75°, 75° [2.6] **11.** $\{6\}$ **12.** $\{-29\}$ **13.** 32 oz; \$0.250 **14.** 2300 mi [2.7] **15.** \$8000 at 3%; \$14,000 at 4.5% **16.** 4 hr [2.8] **17.** $(-\infty, 4]$

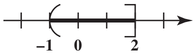

**18.** $(-2, 6]$ 

**19.** 83 or more **20.** When an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed.

## Chapters 1–2 Cumulative Review Exercises (pages 173–174)

[1.1] **1.** $\frac{37}{60}$ **2.** $\frac{48}{5}$ [1.2] **3.** $\frac{1}{2}x - 18$ **4.** $\frac{6}{x + 12} = 2$ [1.4] **5.** true [1.5–1.6] **6.** $-8$ **7.** 28 [1.3] **8.** $-\frac{19}{3}$ [1.7] **9.** distributive property **10.** commutative property [2.1–2.3] **11.** $\{-1\}$ **12.** $\{-1\}$ **13.** $\{-12\}$ [2.6] **14.** $\{26\}$ [2.5] **15.** $y = \frac{24 - 3x}{4}$

[2.8] **16.** $(-\infty, 1]$ 

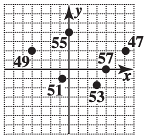
**17.** $(-1, 2]$ 

[2.4] **18.** 4 cm; 9 cm; 27 cm [2.5] **19.** 12.42 cm [2.7] **20.** 40 mph; 60 mph

## 3 LINEAR EQUATIONS IN TWO VARIABLES

### Section 3.1 (pages 183–187)

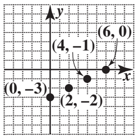
**1.** between 2003 and 2004, 2004 and 2005, and 2005 and 2006 **3.** 2003: 6.0%; 2004: 5.5%; decline: 0.5% **5.** does; do not **7.** II **9.** 3 **11.** yes **13.** yes **15.** no **17.** yes **19.** yes **21.** no **23.** 17 **25.** $-5$ **27.** $-1$ **29.** $-7$ **31.** 8; 6; 3; (0, 8); (6, 0); (3, 4) **33.** $-9$; 4; 9; $(-9, 0)$; (0, 4); (9, 8) **35.** 12; 12; 12; (12, 3); (12, 8); (12, 0) **37.** $-10$; $-10$; $-10$; $(4, -10)$; $(0, -10)$; $(-4, -10)$ **39.** $-2$; $-2$; $-2$; $(9, -2)$; $(2, -2)$; $(0, -2)$ **41.** 4; 4; 4; (4, 4); (4, 0); $(4, -4)$ **43.** No, the ordered pair (3, 4) represents the point 3 units to the right of the origin and 4 units up from the $x$-axis. The ordered pair (4, 3) represents the point 4 units to the right of the origin and 3 units up from the $x$-axis. **45.** $A$: (2, 4), I; $B$: $(-3, 2)$, II; $C$: $(-5, 4)$, II; $D$: $(-5, -2)$, III; $E$: (3, 0), no quadrant; $F$: $(0, -2)$, no quadrant

**47.–57.** 

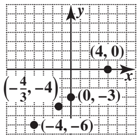
**59.** negative; negative **61.** positive; negative

**63.** If $xy < 0$, then either $x < 0$ and $y > 0$ or $x > 0$ and $y < 0$. If $x < 0$ and $y > 0$, then the point lies in quadrant II. If $x > 0$ and $y < 0$, then the point lies in quadrant IV.

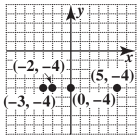
**65.** $-3$; 6; $-2$; 4



**67.** $-3$; 4; $-6$; $-\frac{4}{3}$



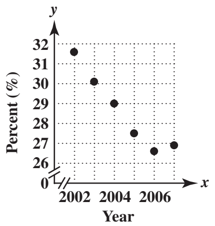
**69.** $-4$; $-4$; $-4$; $-4$



**71.** The points in each graph appear to lie on a straight line.

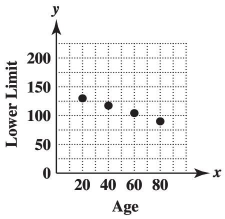
**73.** **(a)** (5, 45) **(b)** (6, 50) **75.** **(a)** (2002, 31.6), (2003, 30.1), (2004, 29.0), (2005, 27.5), (2006, 26.6), (2007, 26.9) **(b)** (2007, 26.9) means that 26.9 percent of 2-year college students in 2007 received a degree within 3 years.

**(c)**



**(d)** With the exception of the point for 2007, the points lie approximately on a straight line. Rates at which 2-year college students complete a degree within 3 years were generally decreasing.

**77.** **(a)** 130; 117; 104; 91 **(b)** (20, 130), (40, 117), (60, 104), (80, 91) **(c)** yes

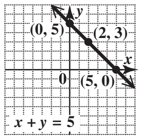


**79.** between 130 and 170 beats per minute; between 117 and 153 beats per minute **81.** $\{-2\}$ **83.** $\{13\}$

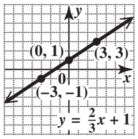
**Connections** (page 195) **1.** $3x + 4 - 2x - 7 - 4x - 3 = 0$ **2.** $5x - 15 - 3(x - 2) = 0$

### Section 3.2 (pages 195–199)

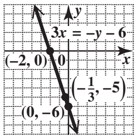
**1.** 5; 5; 3



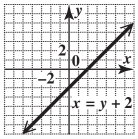
**3.** 1; 3; $-1$

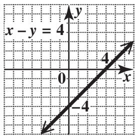


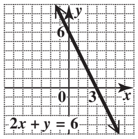
**5.** $-6$; $-2$; $-5$

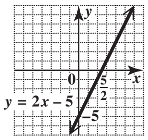


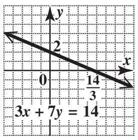
**7.** **(a)** A **(b)** C **(c)** D **(d)** B **9.** $x$-intercept: (4, 0); $y$-intercept: $(0, -4)$ **11.** $x$-intercept: $(-2, 0)$; $y$-intercept: $(0, -3)$ **13.** (8, 0); $(0, -8)$ **15.** (4, 0); $(0, -10)$ **17.** (0, 0); (0, 0) **19.** (2, 0); (0, 4) **21.** (6, 0); $(0, -2)$ **23.** (0, 0); (0, 0) **25.** (4, 0); none **27.** none; (0, 2.5) **29.** **(a)** D **(b)** C **(c)** B **(d)** A

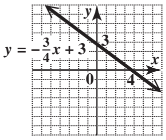
**31.** 

**33.** 

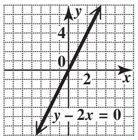
**35.** 

**37.** 

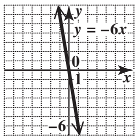
**39.** 

**41.** 

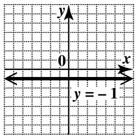
**43.**



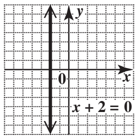
**45.**



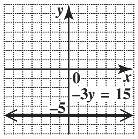
**47.**



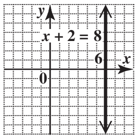
**49.**



**51.**



**53.**



*In Exercises 55–61, descriptions may vary.*

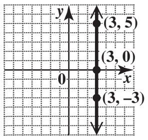
**55.** The graph is a line with $x$-intercept $(-3, 0)$ and $y$-intercept $(0, 9)$.

**57.** The graph is a vertical line with $x$-intercept $(11, 0)$.

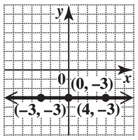
**59.** The graph is a horizontal line with $y$-intercept $(0, -2)$.

**61.** The graph has $x$- and $y$-intercepts $(0, 0)$. It passes through the points $(2, 1)$ and $(4, 2)$.

**63.** $x = 3$

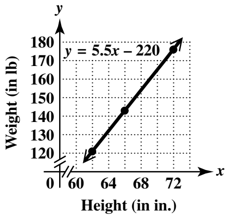


**65.** $y = -3$



**67.** **(a)** 121 lb; 143 lb; 176 lb **(b)** $(62, 121)$, $(66, 143)$, $(72, 176)$

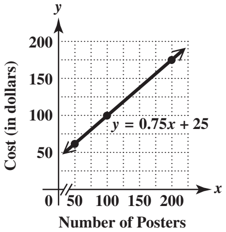
**(c)**



**(d)** 68 in.; 68 in.

**69.** **(a)** \$62.50; \$100 **(b)** 200 **(c)** $(50, 62.50)$, $(100, 100)$, $(200, 175)$

**(d)**



**71.** **(a)** \$30,000 **(b)** \$15,000 **(c)** \$5000 **(d)** After 5 yr, the SUV has a value of \$5000. **73.** **(a)** 2000: \$73 billion; 2004: \$85 billion; 2006: \$91 billion **(b)** 2000: \$74 billion; 2004: \$86 billion; 2006: \$91 billion **(c)** The values are quite close. **75.** $\frac{2}{3}$ **77.** $\frac{1}{2}$

## Section 3.3 (pages 206–210)

**1.** 4 **3.** $-\frac{1}{2}$ **5.** 0 **7.** Rise is the vertical change between two different points on a line. Run is the horizontal change between two different points on a line. **9.** **(a)** C **(b)** A **(c)** D **(d)** B

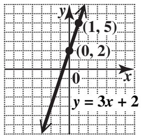
*In Exercises 11 and 13, sketches will vary.*

**11.** The line must fall from left to right. **13.** The line must be vertical. **15.** **(a)** negative **(b)** zero **17.** **(a)** positive **(b)** negative **19.** **(a)** zero **(b)** negative **21.** $\frac{8}{27}$ **23.** $-\frac{2}{3}$ **25.** Because he found the difference $3 - 5 = -2$ in the numerator, he should have subtracted in the same order in the denominator to get $-1 - 2 = -3$. The correct slope is $\frac{-2}{-3} = \frac{2}{3}$. **27.** $\frac{5}{4}$ **29.** $\frac{3}{2}$ **31.** 0 **33.** $-3$ **35.** undefined **37.** $\frac{1}{4}$ **39.** $-\frac{1}{2}$ **41.** 5 **43.** $\frac{1}{4}$ **45.** $\frac{3}{2}$ **47.** $\frac{3}{2}$ **49.** 0 **51.** undefined **53.** $-3$; $\frac{1}{3}$ **55.** A **57.** $-\frac{2}{5}$; $-\frac{2}{5}$; parallel **59.** $\frac{8}{9}$; $-\frac{4}{3}$; neither **61.** $\frac{3}{2}$; $-\frac{2}{3}$; perpendicular **63.** 5; $\frac{1}{5}$; neither **65.** 232 thousand, or 232,000 **66.** positive; increased **67.** 232,000 students **68.** $-0.95$ **69.** negative; decreased **70.** 0.95 students per computer **71.** **(a)** $(2004, 817)$, $(2008, 1513)$ **(b)** 174 **(c)** Music purchases increased by 696 million units in 4 yr, or 174 million units per year. **73.** 0.4 **75.** $(0, 4)$ **77.** $y = -\frac{2}{5}x + 3$ **79.** $y = \frac{10}{3}x - 10$ **81.** $y = 2x - 16$

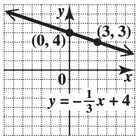
## Section 3.4 (pages 218–221)

**1.** E **3.** B **5.** C **7.** A **9.** slope: $\frac{5}{2}$; $y$-intercept: $(0, -4)$ **11.** slope: $-1$; $y$-intercept: $(0, 9)$ **13.** slope: $\frac{1}{5}$; $y$-intercept: $\left(0, -\frac{3}{10}\right)$ **15.** $y = 3x - 3$ **17.** $y = -x + 3$ **19.** $y = -\frac{1}{2}x + 2$ **21.** $y = 4x - 3$ **23.** $y = -x - 7$ **25.** $y = 3$ **27.** $x = 0$

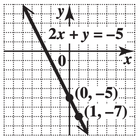
**29.**



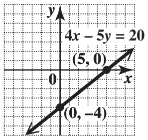
**31.**



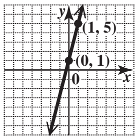
**33.**



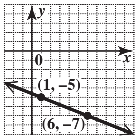
**35.**



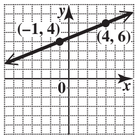
**37.**



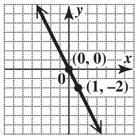
**39.**



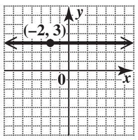
**41.**



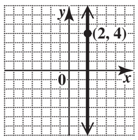
**43.**



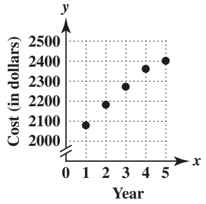
**45.**



**47.**



**49.** the $y$-axis **51.** $y = 2x - 7$ **53.** $y = -4x - 1$ **55.** $y = x - 6$ **57.** $y = \frac{3}{4}x + 4$ **59.** $y = \frac{2}{3}x + \frac{19}{3}$ **61.** $y = -\frac{4}{5}x + \frac{9}{5}$ **63.** A, B, D **65.** **(a)** $y = x + 6$ **(b)** $x - y = -6$ **67.** **(a)** $y = \frac{1}{2}x + 2$ **(b)** $x - 2y = -4$ **69.** **(a)** $y = -\frac{3}{5}x - \frac{11}{5}$ **(b)** $3x + 5y = -11$ **71.** **(a)** $y = -\frac{1}{3}x + \frac{22}{9}$ **(b)** $3x + 9y = 22$ **73.** $y = -2x - 3$ **75.** $y = 4x - 5$ **77.** $y = \frac{3}{4}x - \frac{9}{2}$ **79.** **(a)** \$400 **(b)** \$0.25 **(c)** $y = 0.25x + 400$ **(d)** \$425 **(e)** 1500 **81.** **(a)** $(1, 2079)$, $(2, 2182)$, $(3, 2272)$, $(4, 2361)$, $(5, 2402)$ **(b)** yes

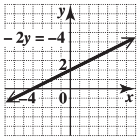


**(c)** $y = 94x + 1985$ **(d)** \$2549

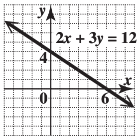
**83.** $y = 15x - 29{,}978$ **85.** 64 **87.** 625 **89.** $\frac{8}{27}$

## Summary Exercises on Linear Equations and Graphs (page 222)

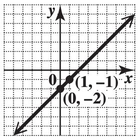
**1.**



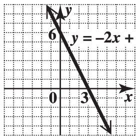
**2.**



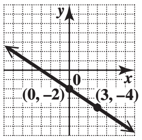
**3.**



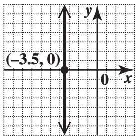
**4.**



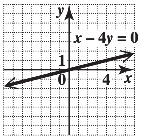
**5.**



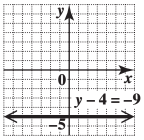
**6.**



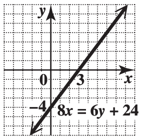
**7.**



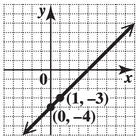
**8.**



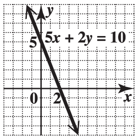
**9.**



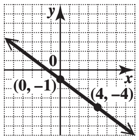
**10.**



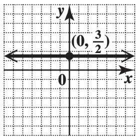
**11.**



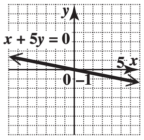
**12.**



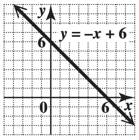
**13.**



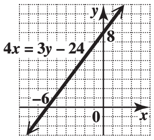
**14.**



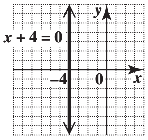
**15.**



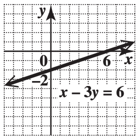
**16.**



**17.**



**18.**

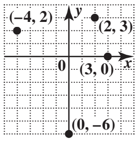


**19.** **(a)** B **(b)** D **(c)** A **(d)** C **20.** A, B **21.** $y = -3x - 6$
**22.** $y = \frac{3}{2}x + 12$ **23.** $y = -4x - 3$ **24.** $y = \frac{3}{5}x$ **25.** $x = 0$
**26.** $y = x - 3$ **27.** $y = \frac{2}{3}x$ **28.** $y = -2x - 4$ **29.** $y = x - 5$
**30.** $y = 0$ **31.** $y = \frac{5}{3}x + 5$ **32.** $y = -5x - 8$

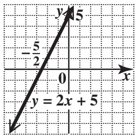
## Chapter 3 Review Exercises (pages 227–228)

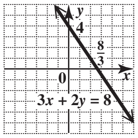
**1.** **(a)** from 2002 to 2005 and 2006 to 2007 **(b)** from 2005 to 2006 **(c)** 2005: about 33 million; 2006: about 29 million **(d)** about 4 million
**2.** $-1$; 2; 1 **3.** 2; $\frac{3}{2}$; $\frac{14}{3}$ **4.** 0; $\frac{8}{3}$; $-9$ **5.** 7; 7; 7 **6.** yes **7.** no
**8.** yes **9.** I **10.** II **11.** none **12.** none **13.** I or III

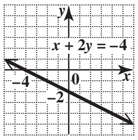
*Graph for* ***Exercises 9–12***



**14.** $\left(-\frac{5}{2}, 0\right)$; $(0, 5)$ **15.** $\left(\frac{8}{3}, 0\right)$; $(0, 4)$ **16.** $(-4, 0)$; $(0, -2)$

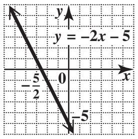


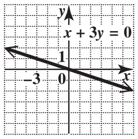




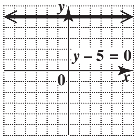
**17.** $-\frac{1}{2}$ **18.** undefined **19.** 3 **20.** 0 **21.** $\frac{3}{2}$ **22.** $-\frac{1}{3}$ **23.** $\frac{3}{2}$
**24.** **(a)** 2 **(b)** $\frac{1}{3}$ **25.** parallel **26.** perpendicular **27.** neither
**28.** $y = -x + \frac{2}{3}$ **29.** $y = -\frac{1}{2}x + 4$ **30.** $y = x - 7$ **31.** $y = \frac{2}{3}x + \frac{14}{3}$
**32.** $y = -\frac{3}{4}x - \frac{1}{4}$ **33.** $y = -\frac{1}{4}x + \frac{3}{2}$ **34.** $y = 1$ **35.** $x = \frac{1}{3}$; It is not possible to express this equation in the form $y = mx + b$.
**36.** A **37.** C, D **38.** A, B, D **39.** D **40.** C **41.** B

**42.** $\left(-\frac{5}{2}, 0\right)$; $(0, -5)$; $-2$ **43.** $(0, 0)$; $(0, 0)$; $-\frac{1}{3}$





**44.** no $x$-intercept; $(0, 5)$; 0 **45.** $y = -\frac{1}{4}x - \frac{5}{4}$ **46.** $y = -3x + 30$
**47.** $y = -\frac{4}{7}x - \frac{23}{7}$ **48.** $y = -5$

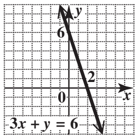


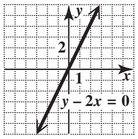
## Chapter 3 Test (page 229)

[3.1] **1.** $-6, -10, -5$ **2.** no [3.2] **3.** To find the $x$-intercept, let $y = 0$, and to find the $y$-intercept, let $x = 0$.

**4.** $x$-intercept: $(2, 0)$; $y$-intercept: $(0, 6)$

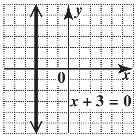
**5.** $x$-intercept: $(0, 0)$; $y$-intercept: $(0, 0)$

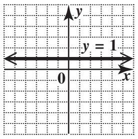




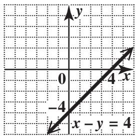
**6.** $x$-intercept: $(-3, 0)$; $y$-intercept: none

**7.** $x$-intercept: none; $y$-intercept: $(0, 1)$





**8.** $x$-intercept: $(4, 0)$; $y$-intercept: $(0, -4)$



[3.3] **9.** $-\frac{8}{3}$ **10.** $-2$ **11.** undefined **12.** 0 **13.** $\frac{5}{2}$
[3.4] **14.** $y = 2x + 6$ **15.** $y = \frac{5}{2}x - 4$
**16.** $y = -9x + 12$

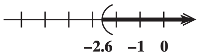
[3.1–3.4] **17.** The slope is negative, since sales are decreasing.
**18.** $(0, 209)$, $(7, 160)$; $y = -7x + 209$
**19.** 174 thousand; The equation gives a good approximation of the actual sales. **20.** In 2007, worldwide snowmobile sales were 160 thousand.

## Chapters 1–3 Cumulative Review Exercises (page 230)

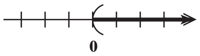
[1.1] **1.** $\frac{301}{40}$, or $7\frac{21}{40}$ **2.** 6 [1.5] **3.** 7 [1.6] **4.** $\frac{73}{18}$, or $4\frac{1}{18}$
[1.2–1.6] **5.** true **6.** $-43$ [1.7] **7.** distributive property

[1.8] **8.** $-p + 2$ [2.5] **9.** $h = \frac{3V}{\pi r^2}$ [2.3] **10.** $\{-1\}$ **11.** $\{2\}$

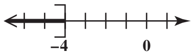
[2.6] **12.** $\{-13\}$ [2.8] **13.** $(-2.6, \infty)$



**14.** $(0, \infty)$



**15.** $(-\infty, -4]$

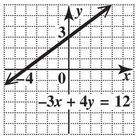


[2.4] **16.** high school diploma: \$30,732; bachelor's degree: \$50,856

[1.1, 2.6] **17.** **(a)** \$7000 **(b)** \$10,000 **18.** about \$30,000

[3.2] **19.** $(-4, 0)$; $(0, 3)$ [3.3] **20.** $\frac{3}{4}$

[3.2] **21.**



[3.3] **22.** perpendicular

[3.4] **23.** $y = 3x - 11$ **24.** $y = 4$

## 4 EXPONENTS AND POLYNOMIALS

### Section 4.1 (pages 237–239)

**1.** false; $3^3 = 3 \cdot 3 \cdot 3 = 27$ **3.** false; $(x^2)^3 = x^{2 \cdot 3} = x^6$ **5.** $w^6$ **7.** $\left(\frac{1}{2}\right)^6$ **9.** $(-4)^4$ **11.** $(-7y)^4$ **13.** In $(-3)^4$, $-3$ is the base; in $-3^4$, 3 is the base. $(-3)^4 = 81$; $-3^4 = -81$. **15.** base: 3; exponent: 5; 243 **17.** base: $-3$; exponent: 5; $-243$ **19.** base: $-6x$; exponent: 4 **21.** base: $x$; exponent: 4 **23.** $5^2 + 5^3$ is a sum, not a product; $5^2 + 5^3 = 25 + 125 = 150$. **25.** $5^8$ **27.** $4^{12}$ **29.** $(-7)^9$ **31.** $t^{24}$ **33.** $-56r^7$ **35.** $42p^{10}$ **37.** $-30x^9$ **39.** The product rule does not apply. **41.** The product rule does not apply. **43.** $4^6$ **45.** $t^{20}$ **47.** $7^3r^3$ **49.** $5^5x^5y^5$ **51.** $5^{12}$ **53.** $-8^{15}$ **55.** $8q^3r^3$ **57.** $\frac{9^8}{5^8}$ **59.** $\frac{1}{2^3}$ **61.** $\frac{a^3}{b^3}$ **63.** $\frac{x^3}{2^3}$ **65.** $\frac{5^5}{2^5}$ **67.** $\frac{9^5}{8^3}$ **69.** $2^{12}x^{12}$ **71.** $-6^5p^5$ **73.** $6^5x^{10}y^{15}$ **75.** $x^{21}$ **77.** $4w^4x^{26}y^7$ **79.** $-r^{18}s^{17}$ **81.** $\frac{5^3a^6b^{15}}{c^{18}}$, or $\frac{125a^6b^{15}}{c^{18}}$ **83.** Using the product rule, it is simplified as follows: $(10^2)^3 = 10^{2 \cdot 3} = 10^6 = 1{,}000{,}000$. **85.** $12x^5$ **87.** $6p^7$ **89.** $125x^6$ **91.** $-a^4, -a^3, -(-a)^3, (-a)^4$; One way is to choose a positive number greater than 1 and substitute it for $a$ in each expression. Then arrange the terms from least to greatest. **93.** \$304.16 **95.** \$1843.88 **97.** $\frac{1}{9}$ **99.** $-8$ **101.** 12 **103.** 3

### Section 4.2 (pages 244–246)

**1.** 1 **3.** 1 **5.** $-1$ **7.** $-1$ **9.** 0 **11.** 0 **13.** 0 **15.** 0 **17.** **(a)** B **(b)** C **(c)** D **(d)** B **(e)** E **(f)** B **19.** 2 **21.** $\frac{1}{64}$ **23.** 16 **25.** $\frac{49}{36}$ **27.** $\frac{1}{81}$ **29.** $\frac{8}{15}$ **31.** $-\frac{7}{18}$ **33.** $\frac{7}{2}$ **35.** $5^3$, or 125 **37.** $\frac{5^3}{3^2}$, or $\frac{125}{9}$ **39.** $5^2$, or 25 **41.** $x^{15}$ **43.** $6^3$, or 216 **45.** $2r^4$ **47.** $\frac{5^2}{4^3}$, or $\frac{25}{64}$ **49.** $\frac{p^5}{q^8}$ **51.** $r^9$ **53.** $\frac{yz^2}{4x^3}$ **55.** $a + b$ **57.** $(x + 2y)^2$ **59.** 1 **60.** $\frac{5^2}{5^2}$ **61.** $5^0$ **62.** $1 = 5^0$; This supports the definition of 0 as an exponent. **63.** $7^3$, or 343 **65.** $\frac{1}{x^2}$ **67.** $\frac{64x}{9}$ **69.** $\frac{x^2z^4}{y^2}$ **71.** $6x$ **73.** $\frac{1}{m^{10}n^5}$ **75.** $\frac{1}{xyz}$ **77.** $x^3y^9$ **79.** $\frac{a^{11}}{2b^5}$ **81.** $\frac{108}{y^5z^3}$ **83.** $\frac{9z^2}{400x^3}$ **85.** The student attempted to use the quotient rule with unequal bases. The correct way to simplify is $\frac{16^3}{2^2} = \frac{(2^4)^3}{2^2} = \frac{2^{12}}{2^2} = 2^{10} = 1024$. **87.** 64,280 **89.** 1530 **91.** 3.8 **93.** 0.277

### Summary Exercises on the Rules for Exponents (page 247)

**1.** $10^5x^7y^{14}$ **2.** $-128a^{10}b^{15}c^4$ **3.** $\frac{729w^3x^9}{y^{12}}$ **4.** $\frac{x^4y^6}{16}$ **5.** $c^{22}$ **6.** $\frac{1}{k^4t^{12}}$ **7.** $\frac{11}{30}$ **8.** $y^{12}z^3$ **9.** $\frac{x^6}{y^5}$ **10.** 0 **11.** $\frac{1}{z^2}$ **12.** $\frac{9}{r^2s^2t^{10}}$ **13.** $\frac{300x^3}{y^3}$ **14.** $\frac{3}{5x^6}$ **15.** $x^8$ **16.** $\frac{y^{11}}{x^{11}}$ **17.** $\frac{a^6}{b^4}$ **18.** $6ab$ **19.** $\frac{61}{900}$ **20.** 1 **21.** $\frac{343a^6b^9}{8}$ **22.** 1 **23.** $-1$ **24.** 0 **25.** $\frac{27y^{18}}{4x^8}$ **26.** $\frac{1}{a^8b^{12}c^{16}}$ **27.** $\frac{x^{15}}{216z^9}$ **28.** $\frac{q}{8p^6r^3}$ **29.** $x^6y^6$ **30.** 0 **31.** $\frac{343}{x^{15}}$ **32.** $\frac{9}{x^6}$ **33.** $5p^{10}q^9$ **34.** $\frac{7}{24}$ **35.** $\frac{r^{14}t}{2s^2}$ **36.** 1 **37.** $8p^{10}q$ **38.** $\frac{1}{mn^3p^3}$ **39.** $-1$ **40.** **(a)** D **(b)** D **(c)** E **(d)** B **(e)** J **(f)** F **(g)** I **(h)** B **(i)** E **(j)** F

**Connections** **(page 252)** **1.** The Indonesia earthquake was 10 times as powerful as the Peru earthquake. **2.** The Afghanistan earthquake had one-hundredth the power of the China earthquake. **3.** The Alaska earthquake was about 19.95 times as powerful as the China earthquake. **4.** "+3.0" corresponds to a factor of 1000 times stronger; "−1.0" corresponds to a factor of one-tenth as strong.

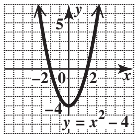
### Section 4.3 (pages 252–255)

**1.** **(a)** C **(b)** A **(c)** B **(d)** D **3.** in scientific notation **5.** not in scientific notation; $5.6 \times 10^6$ **7.** not in scientific notation; $8 \times 10^1$ **9.** not in scientific notation; $4 \times 10^{-3}$ **11.** It is written as the product of a power of 10 and a number whose absolute value is between 1 and 10 (inclusive of 1). Some examples are $2.3 \times 10^{-4}$ and $6.02 \times 10^{23}$. **13.** $5.876 \times 10^9$ **15.** $8.235 \times 10^4$ **17.** $7 \times 10^{-6}$ **19.** $2.03 \times 10^{-3}$ **21.** $-1.3 \times 10^7$ **23.** $-6 \times 10^{-3}$ **25.** 750,000 **27.** 5,677,000,000,000 **29.** 1,000,000,000,000 **31.** 6.21 **33.** 0.00078 **35.** 0.000000005134 **37.** $-0.004$ **39.** $-810{,}000$ **41.** **(a)** $6 \times 10^{11}$ **(b)** 600,000,000,000 **43.** **(a)** $1.5 \times 10^7$ **(b)** 15,000,000 **45.** **(a)** $-6 \times 10^4$ **(b)** $-60{,}000$ **47.** **(a)** $2.4 \times 10^2$ **(b)** 240 **49.** **(a)** $6.3 \times 10^{-2}$ **(b)** 0.063 **51.** **(a)** $3 \times 10^{-4}$ **(b)** 0.0003 **53.** **(a)** $-4 \times 10$ **(b)** $-40$ **55.** **(a)** $1.3 \times 10^{-5}$ **(b)** 0.000013 **57.** **(a)** $5 \times 10^2$ **(b)** 500 **59.** **(a)** $-3 \times 10^6$ **(b)** $-3{,}000{,}000$ **61.** **(a)** $2 \times 10^{-7}$ **(b)** 0.0000002 **63.** 4.7E-7 **65.** 2E7 **67.** 1E1 **69.** $1.04 \times 10^8$ **71.** $9.2 \times 10^{-3}$ **73.** $6 \times 10^9$ **75.** $1 \times 10^{10}$ **77.** 2,000,000,000 **79.** $3.305 \times 10^9$ **81.** \$2.81 × $10^{10}$ **83.** about $2.76 \times 10^{-1}$, or 0.276, lb **85.** $3.59 \times 10^2$, or 359, sec **87.** \$76.26 **89.** \$40,000 **91.** $1.5 \times 10^{17}$ mi **93.** about \$1,220,000,000,000 **95.** \$3252 **97.** $2x - 36$ **99.** 19 **101.** 64

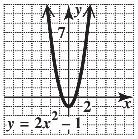
## Section 4.4 (pages 261–265)

**1.** 4; 6 **3.** 9 **5.** 19 **7.** 0 **9.** 1; 6 **11.** 1; 1 **13.** 2; $-19$, $-1$ **15.** 3; 1, 8, 5 **17.** $2m^5$ **19.** $-r^5$ **21.** It cannot be simplified. **23.** $-5x^5$ **25.** $5p^9 + 4p^7$ **27.** $-2xy^2$ **29.** already simplified; 4; binomial **31.** $11m^4 - 7m^3 - 3m^2$; 4; trinomial **33.** $x^4$; 4; monomial **35.** 7; 0; monomial **37.** **(a)** $-3$ **(b)** 0 **39.** **(a)** 14 **(b)** $-19$ **41.** **(a)** 36 **(b)** $-12$ **43.** $5x^2 - 2x$ **45.** $5m^2 + 3m + 2$ **47.** $\frac{7}{6}x^2 - \frac{2}{15}x + \frac{5}{6}$ **49.** $6m^3 + m^2 + 4m - 14$ **51.** $3y^3 - 11y^2$ **53.** $4x^4 - 4x^2 + 4x$ **55.** $15m^3 - 13m^2 + 8m + 11$ **57.** Answers will vary. **59.** $5m^2 - 14m + 6$ **61.** $4x^3 + 2x^2 + 5x$ **63.** $-11y^4 + 8y^2 + y$ **65.** $a^4 - a^2 + 1$ **67.** $5m^2 + 8m - 10$ **69.** $-6x^2 - 12x + 12$ **71.** $-10$ **73.** $4b - 5c$ **75.** $6x - xy - 7$ **77.** $-3x^2y - 15xy - 3xy^2$ **79.** $8x^2 + 8x + 6$ **81.** $2x^2 + 8x$ **83.** $8t^2 + 8t + 13$ **85.** **(a)** $23y + 5t$ **(b)** 25°, 67°, 88° **87.** $-7x - 1$

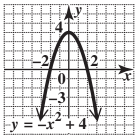
**89.** 0, $-3$, $-4$, $-3$, 0



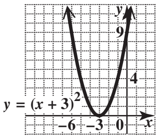
**91.** 7, 1, $-1$, 1, 7



**93.** 0, 3, 4, 3, 0



**95.** 4, 1, 0, 1, 4



**97.** 63; If a dog is 9 in dog years, then it is 63 in human years. **98.** about 26 **99.** 2.5; 130 **100.** 6; $27 **101.** $5x + 20$ **103.** $8a + 24b$ **105.** $-10a^2b$ **107.** $-m^7$

## Section 4.5 (pages 269–270)

**1.** **(a)** B **(b)** D **(c)** A **(d)** C **3.** $15y^{11}$ **5.** $30a^9$ **7.** $15pq^2$ **9.** $-18m^3n^2$ **11.** $9y^{10}$ **13.** $-8x^{10}$ **15.** $6m^2 + 4m$ **17.** $-6p^4 + 12p^3$ **19.** $-16z^2 - 24z^3 - 24z^4$ **21.** $6y^3 + 4y^4 + 10y^7$ **23.** $28r^5 - 32r^4 + 36r^3$ **25.** $6a^4 - 12a^3b + 15a^2b^2$ **27.** $21m^5n^2 + 14m^4n^3 - 7m^3n^5$ **29.** $12x^3 + 26x^2 + 10x + 1$ **31.** $72y^3 - 70y^2 + 21y - 2$ **33.** $20m^4 - m^3 - 8m^2 - 17m - 15$ **35.** $6x^6 - 3x^5 - 4x^4 + 4x^3 - 5x^2 + 8x - 3$ **37.** $5x^4 - 13x^3 + 20x^2 + 7x + 5$ **39.** $3x^5 + 18x^4 - 2x^3 - 8x^2 + 24x$ **41.** $m^2 + 12m + 35$ **43.** $n^2 + 3n - 4$ **45.** $x^2 - 25$ **47.** $12x^2 + 10x - 12$ **49.** $81 - t^2$ **51.** $9x^2 - 12x + 4$ **53.** $10a^2 + 37a + 7$ **55.** $12 + 8m - 15m^2$ **57.** $20 - 7x - 3x^2$ **59.** $3t^2 + 5st - 12s^2$ **61.** $8xy - 4x + 6y - 3$ **63.** $15x^2 + xy - 6y^2$ **65.** $6y^5 - 21y^4 - 45y^3$ **67.** $-200r^7 + 32r^3$ **69.** **(a)** $3y^2 + 10y + 7$ **(b)** $8y + 16$ **71.** $6p^2 - \frac{5}{2}pq - \frac{25}{12}q^2$ **73.** $x^2 + 14x + 49$ **75.** $a^2 - 16$ **77.** $4p^2 - 20p + 25$ **79.** $25k^2 + 30kq + 9q^2$ **81.** $m^3 - 15m^2 + 75m - 125$ **83.** $8a^3 + 12a^2 + 6a + 1$ **85.** $-9a^3 + 33a^2 + 12a$ **87.** $56m^2 - 14m - 21$ **89.** $81r^4 - 216r^3s + 216r^2s^2 - 96rs^3 + 16s^4$ **91.** $6p^8 + 15p^7 + 12p^6 + 36p^5 + 15p^4$ **93.** $-24x^8 - 28x^7 + 32x^6 + 20x^5$ **95.** $14x + 49$ **97.** $\pi x^2 - 9$ **99.** $9m^2$ **101.** $4r^2$ **103.** $16x^4$

## Section 4.6 (pages 274–276)

**1.** **(a)** $16x^2$ **(b)** $24x$ **(c)** 9 **(d)** $16x^2 + 24x + 9$ **3.** $m^2 + 4m + 4$ **5.** $r^2 - 6r + 9$ **7.** $x^2 + 4xy + 4y^2$ **9.** $25p^2 + 20pq + 4q^2$ **11.** $16a^2 + 40ab + 25b^2$ **13.** $36m^2 - \frac{48}{5}mn + \frac{16}{25}n^2$ **15.** $9t^3 - 6t^2 + t$ **17.** $48t^3 + 24t^2 + 3t$ **19.** $-16r^2 + 16r - 4$ **21.** **(a)** $49x^2$ **(b)** 0 **(c)** $-9y^2$ **(d)** $49x^2 - 9y^2$; Because 0 is the identity element for addition, it is not necessary to write " + 0." **23.** $k^2 - 25$ **25.** $16 - 9t^2$ **27.** $25x^2 - 4$ **29.** $25y^2 - 9x^2$ **31.** $100x^2 - 9y^2$ **33.** $4x^4 - 25$ **35.** $\frac{9}{16} - x^2$ **37.** $81y^2 - \frac{4}{9}$ **39.** $25q^3 - q$ **41.** No. In general, $(a + b)^2$ equals $a^2 + 2ab + b^2$, which is not equivalent to $a^2 + b^2$. **43.** $x^3 + 3x^2 + 3x + 1$ **45.** $t^3 - 9t^2 + 27t - 27$ **47.** $r^3 + 15r^2 + 75r + 125$ **49.** $8a^3 + 12a^2 + 6a + 1$ **51.** $256x^4 - 256x^3 + 96x^2 - 16x + 1$ **53.** $81r^4 - 216r^3t + 216r^2t^2 - 96rt^3 + 16t^4$ **55.** $2x^4 + 6x^3 + 6x^2 + 2x$ **57.** $-4t^4 - 36t^3 - 108t^2 - 108t$ **59.** $x^4 - 2x^2y^2 + y^4$ **61.** $(a + b)^2$ **62.** $a^2$ **63.** $2ab$ **64.** $b^2$ **65.** $a^2 + 2ab + b^2$ **66.** They both represent the area of the entire large square. **67.** 1225 **68.** $30^2 + 2(30)(5) + 5^2$ **69.** 1225 **70.** They are equal. **71.** 9999 **73.** 39,999 **75.** $399\frac{3}{4}$ **77.** $\frac{1}{2}m^2 - 2n^2$ **79.** $9a^2 - 4$ **81.** $\pi x^2 + 4\pi x + 4\pi$ **83.** $x^3 + 6x^2 + 12x + 8$ **85.** $2p + 1 + \frac{4}{p}$ **87.** $\frac{m^2}{3} + 3m - 2$ **89.** $-24k^3 + 36k^2 - 6k$ **91.** $-16k^3 - 10k^2 + 3k + 3$ **93.** $5t + 3$

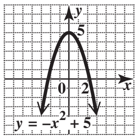
## Section 4.7 (pages 282–284)

**1.** $10x^2 + 8$; 2; $5x^2 + 4$ **3.** $5x^2 + 4$; 2 (These may be reversed.); $10x^2 + 8$ **5.** The first is a polynomial divided by a monomial, covered in Objective 1. This section does not cover dividing a monomial by a polynomial of several terms. **7.** $30x^3 - 10x + 5$ **9.** $4m^3 - 2m^2 + 1$ **11.** $4t^4 - 2t^2 + 2t$ **13.** $a^4 - a + \frac{2}{a}$ **15.** $-3p^2 - 2 + \frac{1}{p}$ **17.** $7r^2 - 6 + \frac{1}{r}$ **19.** $4x^3 - 3x^2 + 2x$ **21.** $-9x^2 + 5x + 1$ **23.** $2x + 8 + \frac{12}{x}$ **25.** $\frac{4x^2}{3} + x + \frac{2}{3x}$ **27.** $-27x^3 + 10x^2 + 4$ **29.** $9r^3 - 12r^2 + 2r + \frac{26}{3} - \frac{2}{3r}$ **31.** $-m^2 + 3m - \frac{4}{m}$ **33.** $-3a + 4 + \frac{5}{a}$ **35.** $\frac{12}{x} - \frac{6}{x^2} + \frac{14}{x^3} - \frac{10}{x^4}$ **37.** $6x^4y^2 - 4xy + 2xy^2 - x^4y$ **39.** 1423 **40.** $(1 \times 10^3) + (4 \times 10^2) + (2 \times 10^1) + (3 \times 10^0)$ **41.** $x^3 + 4x^2 + 2x + 3$ **42.** They are similar in that the coefficients of powers of 10 are equal to the coefficients of the powers of $x$. They are different in that one is a constant while the other is a polynomial. They are equal if $x = 10$ (the base of our decimal system). **43.** $x + 2$ **45.** $2y - 5$ **47.** $p - 4 + \frac{44}{p + 6}$ **49.** $6m - 1$ **51.** $2a - 14 + \frac{74}{2a + 3}$ **53.** $4x^2 - 7x + 3$ **55.** $4k^3 - k + 2$ **57.** $5y^3 + 2y - 3 + \frac{-5}{y + 1}$ **59.** $3k^2 + 2k - 2 + \frac{6}{k - 2}$

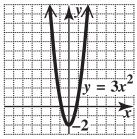
**61.** $2p^3 - 6p^2 + 7p - 4 + \frac{14}{3p+1}$ **63.** $x^2 + 3x + 3$ **65.** $2x^2 - 2x + 3 + \frac{-1}{x+1}$ **67.** $r^2 - 1 + \frac{4}{r^2 - 1}$ **69.** $3x^2 + 3x - 1 + \frac{1}{x-1}$ **71.** $y^2 - y + 1$ **73.** $a^2 + 1$ **75.** $x^2 - 4x + 2 + \frac{9x-4}{x^2+3}$ **77.** $x^3 + 3x^2 - x + 5$ **79.** $\frac{3}{2}a - 10 + \frac{77}{2a+6}$ **81.** $x^2 + \frac{8}{3}x - \frac{1}{3} + \frac{4}{3x-3}$ **83.** $x^2 + x - 3$ **85.** $48m^2 + 96m + 24$ **87.** $5x^2 - 11x + 14$ **89.** 1, 2, 3, 6, 9, 18 **91.** 1, 2, 3, 4, 6, 8, 12, 16, 24, 48

## Chapter 4 Review Exercises (pages 288–291)

**1.** $4^{11}$ **2.** $(-5)^{11}$ **3.** $-72x^7$ **4.** $10x^{14}$ **5.** $19^5x^5$ **6.** $(-4)^7y^7$ **7.** $5p^4t^4$ **8.** $\frac{7^6}{5^6}$ **9.** $3^3x^6y^9$ **10.** $t^{42}$ **11.** $6^2x^{16}y^4z^{16}$ **12.** $\frac{2^3m^9n^3}{p^6}$ **13.** The expression is a *sum* of powers of 7, not a *product*. **14.** 2 **15.** $-1$ **16.** $-1$ **17.** $-\frac{1}{49}$ **18.** $\frac{64}{25}$ **19.** $5^8$ **20.** $\frac{1}{81}$ **21.** $\frac{3}{4}$ **22.** $\frac{1}{36}$ **23.** $x^2$ **24.** $y^7$ **25.** $\frac{r^8}{81}$ **26.** $\frac{3^5}{p^3}$ **27.** $\frac{1}{a^3b^5}$ **28.** $72r^5$ **29.** $4.8 \times 10^7$ **30.** $2.8988 \times 10^{10}$ **31.** $8.24 \times 10^{-8}$ **32.** 24,000 **33.** 78,300,000 **34.** 0.000000897 **35.** 800 **36.** 4,000,000 **37.** 0.025 **38.** 0.000002 **39.** 0.0000000000016 **40.** $4.2 \times 10^{42}$ **41.** $9.7 \times 10^4$; $5 \times 10^3$ **42.** $1 \times 10^{100}$ **43.** $1 \times 10^3$; $2 \times 10^3$; $5 \times 10^4$; $1 \times 10^5$ **44.** $22m^2$; degree 2; monomial **45.** $p^3 - p^2 + 4p + 2$; degree 3; none of these **46.** already in descending powers; degree 5; none of these **47.** $-8y^5 - 7y^4 + 9y$; degree 5; trinomial **48.** $7r^4 - 4r^3 + 1$; degree 4; trinomial **49.** $13x^3y^2 - 5xy^5 + 21x^2$ **50.** $a^3 + 4a^2$ **51.** $y^2 - 10y + 9$ **52.** $-13k^4 - 15k^2 + 18k$ **53.** 1, 4, 5, 4, 1



**54.** 10, 1, −2, 1, 10

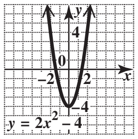


**55.** $a^3 - 2a^2 - 7a + 2$ **56.** $6r^3 + 8r^2 - 17r + 6$ **57.** $5p^5 - 2p^4 - 3p^3 + 25p^2 + 15p$ **58.** $m^2 - 7m - 18$ **59.** $6k^2 - 9k - 6$ **60.** $2a^2 + 5ab - 3b^2$ **61.** $12k^2 - 32kq - 35q^2$ **62.** $s^3 - 3s^2 + 3s - 1$ **63.** $a^2 + 8a + 16$ **64.** $4r^2 + 20rt + 25t^2$ **65.** $36m^2 - 25$ **66.** $25a^2 - 36b^2$ **67.** $r^3 + 6r^2 + 12r + 8$ **68.** $25t^3 - 30t^2 + 9t$ **69.** **(a)** Answers will vary. For example, let $x = 1$ and $y = 2$. $(1 + 2)^2 \neq 1^2 + 2^2$, because $9 \neq 5$. **(b)** Answers will vary. For example, let $x = 1$ and $y = 2$. $(1 + 2)^3 \neq 1^3 + 2^3$, because $27 \neq 9$. **70.** To find the third power of a binomial, such as $(a + b)^3$, first square the binomial and then multiply that result by the binomial. $(a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$ **71.** In both cases, $x = 0$ and $y = 1$ lead to 1 on each side of the inequality. This would not be sufficient to show that, *in general,* the inequality is true. It would be necessary to choose other values of $x$ and $y$. **72.** $x^6 + 6x^4 + 12x^2 + 8$ **73.** $\frac{4}{3}\pi(x + 1)^3$, or $\frac{4}{3}\pi x^3 + 4\pi x^2 + 4\pi x + \frac{4}{3}\pi$ **74.** $\frac{-5y^2}{3}$ **75.** $y^3 - 2y + 3$

**76.** $-2m^2n + mn + \frac{6n^3}{5}$ **77.** $2mn + 3m^4n^2 - 4n$ **78.** The friend wrote the second term of the quotient as $-12x$ rather than $-2x$. Here is the correct method. $\frac{6x^2 - 12x}{6} = \frac{6x^2}{6} - \frac{12x}{6} = x^2 - 2x$ **79.** $2r + 7$ **80.** $2a^2 + 3a - 1 + \frac{6}{5a-3}$ **81.** $x^2 + 3x - 4$ **82.** $m^2 + 4m - 2$ **83.** $4x - 5$ **84.** $5y - 10$ **85.** $y^2 + 2y + 4$ **86.** $100x^4 - 10x^2 + 1$ **87.** $2y^2 - 5y + 4 + \frac{-5}{3y^2+1}$ **88.** $x^3 - 2x^2 + 4 + \frac{-3}{4x^2-3}$ **89.** 2 **90.** $\frac{6^3r^6p^3}{5^3}$ **91.** $144a^2 - 1$ **92.** $\frac{1}{16}$ **93.** $\frac{1}{8^{12}}$ **94.** $p - 3 + \frac{5}{2p}$ **95.** $\frac{2}{3m^3}$ **96.** $6k^3 - 21k - 6$ **97.** $r^{13}$ **98.** $4r^2 + 20rs + 25s^2$ **99.** $-y^2 - 4y + 4$ **100.** $10r^2 + 21r - 10$ **101.** $y^2 + 5y + 1$ **102.** $\frac{5}{2} - \frac{4}{5xy} + \frac{3x}{2y^2}$ **103.** $10p^2 - 3p - 5$ **104.** $3x^2 + 9x + 25 + \frac{80}{x-3}$ **105.** $49 - 28k + 4k^2$ **106.** $\frac{1}{x^4y^{12}}$ **107.** **(a)** $6x - 2$ **(b)** $2x^2 + x - 6$ **108.** **(a)** $20x^4 + 8x^2$ **(b)** $25x^8 + 20x^6 + 4x^4$

## Chapter 4 Test (pages 291–292)

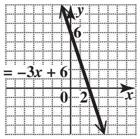
[4.1, 4.2] **1.** $\frac{1}{625}$ **2.** 2 **3.** $\frac{7}{12}$ **4.** $9x^3y^5$ **5.** $8^5$ **6.** $x^2y^6$ **7.** **(a)** positive **(b)** positive **(c)** negative **(d)** positive **(e)** zero **(f)** negative [4.3] **8.** **(a)** $4.5 \times 10^{10}$ **(b)** 0.0000036 **(c)** 0.00019 **9.** **(a)** $1 \times 10^3$; $5.89 \times 10^{12}$ **(b)** $5.89 \times 10^{15}$ mi [4.4] **10.** $-7x^2 + 8x$; 2; binomial **11.** $4n^4 + 13n^3 - 10n^2$; 4; trinomial **12.** 4, −2, −4, −2, 4



**13.** $-2y^2 - 9y + 17$ **14.** $-21a^3b^2 + 7ab^5 - 5a^2b^2$ **15.** $-12t^2 + 5t + 8$ [4.5] **16.** $-27x^5 + 18x^4 - 6x^3 + 3x^2$ **17.** $t^2 - 5t - 24$ **18.** $8x^2 + 2xy - 3y^2$ [4.6] **19.** $25x^2 - 20xy + 4y^2$ **20.** $100v^2 - 9w^2$ [4.5] **21.** $2r^3 + r^2 - 16r + 15$ [4.6] **22.** $12x + 36$; $9x^2 + 54x + 81$ [4.7] **23.** $4y^2 - 3y + 2 + \frac{5}{y}$ **24.** $-3xy^2 + 2x^3y^2 + 4y^2$ **25.** $x - 2$ **26.** $3x^2 + 6x + 11 + \frac{26}{x-2}$

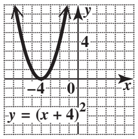
## Chapters 1–4 Cumulative Review Exercises (pages 293–294)

[1.1] **1.** $\frac{7}{4}$ **2.** 5 **3.** $31\frac{1}{4}$ yd³ [1.6] **4.** $1836 **5.** 1, 3, 5, 9, 15, 45 **6.** −8 **7.** $\frac{1}{2}$ [1.5] **8.** −4 [1.7] **9.** associative property **10.** distributive property [1.8] **11.** $-10x^2 + 21x - 29$ [2.1–2.3] **12.** $\left\{\frac{13}{4}\right\}$ **13.** $\emptyset$ [2.5] **14.** $r = \frac{d}{t}$ [2.6] **15.** $\{-5\}$ [2.1–2.3] **16.** $\{-12\}$ **17.** $\{20\}$ **18.** {all real numbers} [2.4] **19.** exertion: 9443 calories; regulating body temperature: 1757 calories [2.8] **20.** 11 ft and 22 ft **21.** $\left(-\infty, -\frac{14}{5}\right)$ **22.** $[-4, 2)$ [3.2] **23.**



[3.3, 3.4] **24.** **(a)** 1 **(b)** $y = x + 6$ [4.1, 4.2] **25.** $\frac{5}{4}$ **26.** 1 **27.** $\frac{2b}{a^{10}}$ [4.3] **28.** about 10,800,000 km

[4.4] **29.**



**30.** $11x^3 - 14x^2 - x + 14$

[4.5] **31.** $63x^2 + 57x + 12$

[4.7] **32.** $y^2 - 2y + 6$

## 5 FACTORING AND APPLICATIONS

### Section 5.1 (pages 301–303)

**1.** 4 **3.** 6 **5.** 1 **7.** 8 **9.** $10x^3$ **11.** $xy^2$ **13.** $6m^3n^2$ **15.** factored **17.** not factored **19.** $3m^2$ **21.** $2z^4$ **23.** $2mn^4$ **25.** $y + 2$ **27.** $a - 2$ **29.** $2 + 3xy$ **31.** First, verify that you have factored completely. Then multiply the factors. The product should be the original polynomial. **33.** $x(x - 4)$ **35.** $3t(2t + 5)$ **37.** $9m(3m^2 - 1)$ **39.** $8z^2(2z^2 + 3)$ **41.** $6x^2(2x + 1)$ **43.** $5y^6(13y^4 + 7)$ **45.** in factored form **47.** $8mn^3(1 + 3m)$ **49.** $13y^2(y^6 + 2y^2 - 3)$ **51.** $9p^3q(4p^3 + 5p^2q^3 + 9q)$ **53.** $a^3(a^2 + 2b^2 - 3a^2b^2 + 4ab^3)$ **55.** $(x + 2)(c - d)$ **57.** $(m + 2n)(m + n)$ **59.** $(p - 4)(q^2 + 1)$ **61.** not in factored form; $(7t + 4)(8 + x)$ **63.** in factored form **65.** not in factored form **67.** The quantities in parentheses are not the same, so there is no common factor of the two terms $18x^2(y + 4)$ and $7(y - 4)$. **69.** $(p + 4)(p + q)$ **71.** $(a - 2)(a + b)$ **73.** $(z + 2)(7z - a)$ **75.** $(3r + 2y)(6r - x)$ **77.** $(a^2 + b^2)(3a + 2b)$ **79.** $(3 - a)(4 - b)$ **81.** $(4m - p^2)(4m^2 - p)$ **83.** $(y + 3)(y + x)$ **85.** $(5 - 2p)(m + 3)$ **87.** $(3r + 2y)(6r - t)$ **89.** $(1 + 2b)(a^5 - 3)$ **91.** commutative property **92.** $2x(y - 4) - 3(y - 4)$ **93.** No, because it is not a product. It is the difference between $2x(y - 4)$ and $3(y - 4)$. **94.** $(2x - 3)(y - 4)$; yes **95.** $x^2 - 3x - 54$ **97.** $x^2 + 9x + 14$ **99.** $2x^4 + 6x^3 + 10x^2$

### Section 5.2 (pages 307–309)

**1.** 1 and 48, $-1$ and $-48$, 2 and 24, $-2$ and $-24$, 3 and 16, $-3$ and $-16$, 4 and 12, $-4$ and $-12$, 6 and 8, $-6$ and $-8$; The pair with a sum of $-19$ is $-3$ and $-16$. **3.** 1 and $-24$, $-1$ and 24, 2 and $-12$, $-2$ and 12, 3 and $-8$, $-3$ and 8, 4 and $-6$, $-4$ and 6; The pair with a sum of $-5$ is 3 and $-8$. **5.** $a$ and $b$ must have different signs, one positive and one negative. **7.** A prime polynomial is a polynomial that cannot be factored by using only integers in the factors. **9.** C **11.** $a^2 + 13a + 36$ **13.** $p + 6$ **15.** $x + 11$ **17.** $x - 8$ **19.** $y - 5$ **21.** $x + 11$ **23.** $y - 9$ **25.** $(y + 8)(y + 1)$ **27.** $(b + 3)(b + 5)$ **29.** $(m + 5)(m - 4)$ **31.** $(y - 5)(y - 3)$ **33.** prime **35.** $(z - 7)(z - 8)$ **37.** $(r - 6)(r + 5)$ **39.** $(a + 4)(a - 12)$ **41.** prime **43.** $(x + 16)(x - 2)$ **45.** $(r + 2a)(r + a)$ **47.** $(t + 2z)(t - 3z)$ **49.** $(x + y)(x + 3y)$ **51.** $(v - 5w)(v - 6w)$ **53.** $4(x + 5)(x - 2)$ **55.** $2t(t + 1)(t + 3)$ **57.** $2x^4(x - 3)(x + 7)$ **59.** $5m^2(m^3 + 5m^2 - 8)$ **61.** $mn(m - 6n)(m - 4n)$ **63.** $a^3(a + 4b)(a - b)$ **65.** $yz(y + 3z)(y - 2z)$ **67.** $z^8(z - 7y)(z + 3y)$ **69.** $(a + b)(x + 4)(x - 3)$ **71.** $(2p + q)(r - 9)(r - 3)$ **73.** $2y^2 + y - 28$ **75.** $15z^2 - 4z - 4$

### Section 5.3 (pages 314–316)

**1.** $(2t + 1)(5t + 2)$ **3.** $(3z - 2)(5z - 3)$ **5.** $(2s - t)(4s + 3t)$ **7.** **(a)** 2, 12, 24, 11 **(b)** 3, 8 (Order is irrelevant.) **(c)** $3m$, $8m$ **(d)** $2m^2 + 3m + 8m + 12$ **(e)** $(2m + 3)(m + 4)$ **(f)** $(2m + 3)(m + 4) = 2m^2 + 11m + 12$ **9.** B **11.** B **13.** A **15.** $2a + 5b$ **17.** $x^2 + 3x - 4$; $x + 4$, $x - 1$, or $x - 1$, $x + 4$ **19.** $2z^2 - 5z - 3$; $2z + 1$, $z - 3$, or $z - 3$, $2z + 1$ **21.** The binomial $2x - 6$ cannot be a factor because its terms have a common factor of 2, which the polynomial terms do not have. **23.** $(3a + 7)(a + 1)$ **25.** $(2y + 3)(y + 2)$ **27.** $(3m - 1)(5m + 2)$ **29.** $(3s - 1)(4s + 5)$ **31.** $(5m - 4)(2m - 3)$ **33.** $(4w - 1)(2w - 3)$ **35.** $(4y + 1)(5y - 11)$ **37.** prime **39.** $2(5x + 3)(2x + 1)$ **41.** $3(4x - 1)(2x - 3)$ **43.** $q(5m + 2)(8m - 3)$ **45.** $3n^2(5n - 3)(n - 2)$ **47.** $y^2(5x - 4)(3x + 1)$ **49.** $(5a + 3b)(a - 2b)$ **51.** $(4s + 5t)(3s - t)$ **53.** $m^4n(3m + 2n)(2m + n)$ **55.** $(x - 5)(x - 1)$ **57.** $(3x + 4)(x + 4)$ **59.** $-5x(2x + 7)(x - 4)$ **61.** $(12x + 1)(x - 4)$ **63.** $(24y + 7x)(y - 2x)$ **65.** $(18x^2 - 5y)(2x^2 - 3y)$ **67.** $2(24a + b)(a - 2b)$ **69.** $x^2y^5(10x - 1)(x + 4)$ **71.** $4ab^2(9a + 1)(a - 3)$ **73.** $(12x - 5)(2x - 3)$ **75.** $(8x^2 - 3)(3x^2 + 8)$ **77.** $(4x + 3y)(6x + 5y)$ **79.** $-1(x + 7)(x - 3)$ **81.** $-1(3x + 4)(x - 1)$ **83.** $-1(a + 2b)(2a + b)$ **85.** $(m + 1)^3(5q - 2)(5q + 1)$ **87.** $(r + 3)^3(3x + 2y)^2$ **89.** $-4$, 4 **91.** $-11$, $-7$, 7, 11 **93.** $49p^2 - 9$ **95.** $x^2 + 12x + 36$

### Section 5.4 (pages 323–325)

**1.** 1; 4; 9; 16; 25; 36; 49; 64; 81; 100; 121; 144; 169; 196; 225; 256; 289; 324; 361; 400 **3.** 1; 8; 27; 64; 125; 216; 343; 512; 729; 1000 **5.** **(a)** both of these **(b)** perfect cube **(c)** perfect square **(d)** perfect square **7.** $(y + 5)(y - 5)$ **9.** $(x + 12)(x - 12)$ **11.** prime **13.** $4(m^2 + 4)$ **15.** $(3r + 2)(3r - 2)$ **17.** $4(3x + 2)(3x - 2)$ **19.** $(14p + 15)(14p - 15)$ **21.** $(4r + 5a)(4r - 5a)$ **23.** prime **25.** $(p^2 + 7)(p^2 - 7)$ **27.** $(x^2 + 1)(x + 1)(x - 1)$ **29.** $(p^2 + 16)(p + 4)(p - 4)$ **31.** $k^2 - 9$ can be factored as $(k + 3)(k - 3)$. The completely factored form is $(k^2 + 9)(k + 3)(k - 3)$. **33.** 10 **35.** 9 **37.** $(w + 1)^2$ **39.** $(x - 4)^2$ **41.** $2(x + 6)^2$ **43.** $(4x - 5)^2$ **45.** $(7x - 2y)^2$ **47.** $(8x + 3y)^2$ **49.** $2(5h - 2y)^2$ **51.** $k(4k^2 - 4k + 9)$ **53.** $z^2(25z^2 + 5z + 1)$ **55.** $(a - 1)(a^2 + a + 1)$ **57.** $(m + 2)(m^2 - 2m + 4)$ **59.** $(k + 10)(k^2 - 10k + 100)$ **61.** $(3x - 4)(9x^2 + 12x + 16)$ **63.** $6(p + 1)(p^2 - p + 1)$ **65.** $5(x + 2)(x^2 - 2x + 4)$ **67.** $(y - 2x)(y^2 + 2yx + 4x^2)$ **69.** $2(x - 2y)(x^2 + 2xy + 4y^2)$ **71.** $(2p + 9q)(4p^2 - 18pq + 81q^2)$ **73.** $(3a + 4b)(9a^2 - 12ab + 16b^2)$ **75.** $(5t + 2s)(25t^2 - 10ts + 4s^2)$ **77.** $(2x - 5y^2)(4x^2 + 10xy^2 + 25y^4)$ **79.** $(3m^2 + 2n)(9m^4 - 6m^2n + 4n^2)$

**81.** $(x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$ **83.** $\left(p + \frac{1}{3}\right)\left(p - \frac{1}{3}\right)$
**85.** $\left(6m + \frac{4}{5}\right)\left(6m - \frac{4}{5}\right)$ **87.** $(x + 0.8)(x - 0.8)$ **89.** $\left(t + \frac{1}{2}\right)^2$
**91.** $(x - 0.5)^2$ **93.** $\left(x + \frac{1}{2}\right)\left(x^2 - \frac{1}{2}x + \frac{1}{4}\right)$ **95.** $4mn$
**97.** $(m - p + 2)(m + p)$ **99.** $\{4\}$ **101.** $\{-5\}$

## Summary Exercises on Factoring (pages 326–327)

**1.** G **2.** H **3.** A **4.** B **5.** E **6.** I **7.** C **8.** F **9.** I **10.** E
**11.** $(a - 6)(a + 2)$ **12.** $(a + 8)(a + 9)$ **13.** $6(y - 2)(y + 1)$
**14.** $7y^4(y + 6)(y - 4)$ **15.** $6(a + 2b + 3c)$
**16.** $(m - 4n)(m + n)$ **17.** $(p - 11)(p - 6)$ **18.** $(z + 7)(z - 6)$
**19.** $(5z - 6)(2z + 1)$ **20.** $2(m - 8)(m + 3)$ **21.** $17xy(x^2y + 3)$
**22.** $5(3y + 1)$ **23.** $8a^3(a - 3)(a + 2)$ **24.** $(4k + 1)(2k - 3)$
**25.** $(z - 5a)(z + 2a)$ **26.** $50(z^2 - 2)$ **27.** $(x - 5)(x - 4)$
**28.** $10nr(10nr + 3r^2 - 5n)$ **29.** $(3n - 2)(2n - 5)$
**30.** $(3y - 1)(3y + 5)$ **31.** $4(4x + 5)$ **32.** $(m + 5)(m - 3)$
**33.** $(3y - 4)(2y + 1)$ **34.** $(m + 9)(m - 9)$ **35.** $(6z + 1)(z + 5)$
**36.** $(12x - 1)(x + 4)$ **37.** $(2k - 3)^2$ **38.** $(8p - 1)(p + 3)$
**39.** $6(3m + 2z)(3m - 2z)$ **40.** $(4m - 3)(2m + 1)$
**41.** $(3k - 2)(k + 2)$ **42.** $15a^3b^2(3b^3 - 4a + 5a^3b^2)$
**43.** $7k(2k + 5)(k - 2)$ **44.** $(5 + r)(1 - s)$
**45.** $(y^2 + 4)(y + 2)(y - 2)$ **46.** $10y^4(2y - 3)$ **47.** $8m(1 - 2m)$
**48.** $(k + 4)(k - 4)$ **49.** $(z - 2)(z^2 + 2z + 4)$
**50.** $(y - 8)(y + 7)$ **51.** prime **52.** $9p^8(3p + 7)(p - 4)$
**53.** $8m^3(4m^6 + 2m^2 + 3)$ **54.** $(2m + 5)(4m^2 - 10m + 25)$
**55.** $(4r + 3m)^2$ **56.** $(z - 6)^2$ **57.** $(5h + 7g)(3h - 2g)$
**58.** $5z(z - 7)(z - 2)$ **59.** $(k - 5)(k - 6)$
**60.** $4(4p - 5m)(4p + 5m)$ **61.** $3k(k - 5)(k + 1)$
**62.** $(y - 6k)(y + 2k)$ **63.** $(10p + 3)(100p^2 - 30p + 9)$
**64.** $(4r - 7)(16r^2 + 28r + 49)$ **65.** $(2 + m)(3 + p)$
**66.** $(2m - 3n)(m + 5n)$ **67.** $(4z - 1)^2$
**68.** $5m^2(5m - 3n)(5m - 13n)$ **69.** $3(6m - 1)^2$
**70.** $(10a + 9y)(10a - 9y)$ **71.** prime **72.** $(2y + 5)(2y - 5)$
**73.** $8z(4z - 1)(z + 2)$ **74.** $5(2m - 3)(m + 4)$
**75.** $(4 + m)(5 + 3n)$ **76.** $(2 - q)(2 - 3p)$
**77.** $2(3a - 1)(a + 2)$ **78.** $6y^4(3y + 4)(2y - 5)$
**79.** $(a - b)(a^2 + ab + b^2 + 2)$ **80.** $4(2k - 3)^2$
**81.** $(8m - 5n)^2$ **82.** $12y^2(6yz^2 + 1 - 2y^2z^2)$
**83.** $(4k - 3h)(2k + h)$ **84.** $(2a + 5)(a - 6)$
**85.** $2(x + 4)(x^2 - 4x + 16)$ **86.** $(2a - 3)(4a^2 + 6a + 9)$
**87.** $(5y - 6z)(2y + z)$ **88.** $(m - 2)^2$ **89.** $(8a - b)(a + 3b)$
**90.** $(a^2 + 25)(a + 5)(a - 5)$ **91.** $(x^3 - 1)(x^3 + 1)$
**92.** $(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$
**93.** $(x^2 - 1)(x^4 + x^2 + 1)$ **94.** $(x - 1)(x + 1)(x^4 + x^2 + 1)$
**95.** The result in **Exercise 92** is factored completely. **96.** Show that $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$. **97.** difference of squares
**98.** $(x - 3)(x^2 + 3x + 9)(x + 3)(x^2 - 3x + 9)$

## Section 5.5 (pages 334–337)

**1.** $ax^2 + bx + c$ **3.** factor **5.** 0; $x$ **7.** To solve $2x(3x - 4) = 0$, set each *variable* factor equal to 0 to get $x = 0$ or $3x - 4 = 0$. The *constant* factor 2 does not introduce solutions into the equation. The solution set is $\left\{0, \frac{4}{3}\right\}$. **9.** The variable $x$ is another factor to set equal to 0, so the solution set is $\left\{0, \frac{1}{7}\right\}$. **11.** $\{-5, 2\}$ **13.** $\left\{3, \frac{7}{2}\right\}$ **15.** $\left\{-\frac{1}{2}, \frac{1}{6}\right\}$
**17.** $\left\{-\frac{5}{6}, 0\right\}$ **19.** $\left\{0, \frac{4}{3}\right\}$ **21.** $\{6\}$ **23.** $\{-2, -1\}$ **25.** $\{1, 2\}$
**27.** $\{-8, 3\}$ **29.** $\{-1, 3\}$ **31.** $\{-2, -1\}$ **33.** $\{-4\}$ **35.** $\left\{-2, \frac{1}{3}\right\}$
**37.** $\left\{-\frac{4}{3}, \frac{1}{2}\right\}$ **39.** $\left\{-\frac{2}{3}\right\}$ **41.** $\{-3, 3\}$ **43.** $\left\{-\frac{7}{4}, \frac{7}{4}\right\}$
**45.** $\{-11, 11\}$ **47.** $\{0, 7\}$ **49.** $\left\{0, \frac{1}{2}\right\}$ **51.** $\{2, 5\}$ **53.** $\left\{-4, \frac{1}{2}\right\}$
**55.** $\{-17, 4\}$ **57.** $\left\{-\frac{5}{2}, \frac{1}{3}, 5\right\}$ **59.** $\left\{-\frac{7}{2}, -3, 1\right\}$ **61.** $\left\{-\frac{7}{3}, 0, \frac{7}{3}\right\}$
**63.** $\{-2, 0, 4\}$ **65.** $\{-5, 0, 4\}$ **67.** $\{-3, 0, 5\}$ **69.** $\{-1, 3\}$
**71.** $\{-1, 3\}$ **73.** $\{3\}$ **75.** $\left\{-\frac{2}{3}, 4\right\}$ **77.** $\left\{-\frac{4}{3}, -1, \frac{1}{2}\right\}$
**79.** **(a)** 64; 144; 4; 6 **(b)** No time has elapsed, so the object hasn't fallen (been released) yet. **81.** $\{-0.5, 0.1\}$ **83.** 1845 **85.** 9, 10

## Section 5.6 (pages 342–347)

**1.** Read; variable; equation; Solve; answer; Check, original
**3.** *Step 3:* $45 = (2x + 1)(x + 1)$; *Step 4:* $x = 4$ or $x = -\frac{11}{2}$; *Step 5:* base: 9 units; height: 5 units; *Step 6:* $9 \cdot 5 = 45$
**5.** *Step 3:* $80 = (x + 8)(x - 8)$; *Step 4:* $x = 12$ or $x = -12$; *Step 5:* length: 20 units; width: 4 units; *Step 6:* $20 \cdot 4 = 80$
**7.** length: 14 cm; width: 12 cm **9.** base: 12 in.; height: 5 in.
**11.** height: 13 in.; width: 10 in. **13.** length: 15 in.; width: 12 in.
**15.** mirror: 7 ft; painting: 9 ft **17.** 20, 21 **19.** 0, 1, 2 or 7, 8, 9
**21.** 7, 9, 11 **23.** −2, 0, 2 or 6, 8, 10 **25.** 12 cm **27.** 12 mi **29.** 8 ft
**31.** 112 ft **33.** 256 ft **35.** **(a)** 1 sec **(b)** $\frac{1}{2}$ sec and $1\frac{1}{2}$ sec **(c)** 3 sec **(d)** The negative solution, −1, does not make sense, since $t$ represents time, which cannot be negative. **37.** **(a)** 104.4 million; The result obtained from the model is less than 109 million, the actual number for 2000. **(b)** 18 **(c)** 272.7 million; The result is more than 263 million. **(d)** 326.6 million **39.** $\frac{25}{36}$ **41.** $\frac{16}{-9}$, or $-\frac{16}{9}$

## Chapter 5 Review Exercises (pages 350–352)

**1.** $7(t + 2)$ **2.** $30z(2z^2 + 1)$ **3.** $(2y + 3)(x - 4)$
**4.** $(3y + 2x)(2y + 3)$ **5.** $(x + 3)(x + 2)$ **6.** $(y - 5)(y - 8)$
**7.** $(q + 9)(q - 3)$ **8.** $(r - 8)(r + 7)$ **9.** $(r + 8s)(r - 12s)$
**10.** $(p + 12q)(p - 10q)$ **11.** $8p(p + 2)(p - 5)$
**12.** $3x^2(x + 2)(x + 8)$ **13.** $p^5(p - 2q)(p + q)$
**14.** $3r^3(r + 3s)(r - 5s)$ **15.** $9x^2y(x + 2)(x - 3)$
**16.** $2x^5(x - 2y)(x + 3y)$ **17.** $r$ and $6r$, $2r$ and $3r$ **18.** Factor out $z$.
**19.** $(2k - 1)(k - 2)$ **20.** $(3r - 1)(r + 4)$ **21.** $(3r + 2)(2r - 3)$
**22.** $(5z + 1)(2z - 1)$ **23.** $(v + 3)(8v - 7)$
**24.** $4x^3(3x - 1)(2x - 1)$ **25.** $-3(x + 2)(2x - 5)$
**26.** $rs(5r + 6s)(2r + s)$ **27.** $4x^2y(3x + y)(4x - y)$ **28.** The student stopped too soon. He needs to factor out the common factor $4x - 1$ to get $(4x - 1)(4x - 5)$ as the correct answer. **29.** B
**30.** D **31.** $(n + 7)(n - 7)$ **32.** $(5b + 11)(5b - 11)$
**33.** $(7y + 5w)(7y - 5w)$ **34.** $36(2p + q)(2p - q)$ **35.** prime
**36.** $(r - 6)^2$ **37.** $(3t - 7)^2$ **38.** $(m + 10)(m^2 - 10m + 100)$
**39.** $(5k + 4x)(25k^2 - 20kx + 16x^2)$
**40.** $(7x - 4)(49x^2 + 28x + 16)$

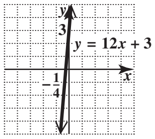
**41.** $(10 - 3x^2)(100 + 30x^2 + 9x^4)$
**42.** $(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
**43.** $\left\{-\frac{3}{4}, 1\right\}$ **44.** $\{-7, -3, 4\}$ **45.** $\left\{0, \frac{5}{2}\right\}$ **46.** $\{-3, -1\}$
**47.** $\{1, 4\}$ **48.** $\{3, 5\}$ **49.** $\left\{-\frac{4}{3}, 5\right\}$ **50.** $\left\{-\frac{8}{9}, \frac{8}{9}\right\}$ **51.** $\{0, 8\}$
**52.** $\{-1, 6\}$ **53.** $\{7\}$ **54.** $\{6\}$ **55.** $\left\{-2, -1, -\frac{2}{5}\right\}$ **56.** $\{-3, 3\}$
**57.** length: 10 ft; width: 4 ft **58.** 5 ft **59.** 6, 7 or −5, −4 **60.** 26 mi **61.** **(a)** 256 ft **(b)** 1024 ft **62.** **(a)** 601,000 vehicles; The result is slightly higher than the actual number for 2005. **(b)** 655,000 vehicles **(c)** The estimate may be unreliable because the conditions that prevailed in the years 2001–2006 may have changed, causing either a greater increase or a decrease predicted by the model for the number of alternative-fueled vehicles. **63.** D **64.** The factor $(2x + 8)$ has a factor of 2. The completely factored form is $2(x + 4)(3x - 4)$.
**65.** $(3k + 5)(k + 2)$ **66.** $(z - x)(z - 10x)$
**67.** $(y^2 + 25)(y + 5)(y - 5)$ **68.** $(3m + 4)(5m - 4p)$
**69.** $8abc(3b^2c - 7ac^2 + 9ab)$ **70.** $3m(2m + 3)(m - 5)$
**71.** $6xyz(2xz^2 + 2y - 5x^2yz^3)$ **72.** prime **73.** $(2r + 3q)(6r - 5)$
**74.** $2a^3(a + 2)(a - 6)$ **75.** $(7t + 4)^2$
**76.** $(10a + 3)(100a^2 - 30a + 9)$ **77.** $\{0, 7\}$ **78.** $\{-5, 2\}$
**79.** $\left\{-\frac{2}{5}\right\}$ **80.** −5, −4, −3 or 5, 6, 7 **81.** length: 6 m; width: 4 m
**82.** 15 m, 36 m, 39 m **83.** 6 m **84.** width: 10 m; length: 17 m

## Chapter 5 Test (page 353)

[5.1–5.4] **1.** D **2.** $6x(2x - 5)$ **3.** $m^2n(2mn + 3m - 5n)$
**4.** $(2x + y)(a - b)$ **5.** $(x + 3)(x - 8)$ **6.** $(2x + 3)(x - 1)$
**7.** $(5z - 1)(2z - 3)$ **8.** prime **9.** prime **10.** $(2 - a)(6 + b)$
**11.** $(3y + 8)(3y - 8)$ **12.** $(2x - 7y)^2$ **13.** $-2(x + 1)^2$
**14.** $3t^2(2t + 9)(t - 4)$ **15.** $(r - 5)(r^2 + 5r + 25)$
**16.** $8(k + 2)(k^2 - 2k + 4)$ **17.** $(x^2 + 9)(x + 3)(x - 3)$
**18.** $(3x + 2y)(3x - 2y)(9x^2 + 4y^2)$ **19.** $(3x^3y^2 + 2)^2$
[5.5] **20.** $\left\{\frac{1}{2}, 6\right\}$ **21.** $\left\{-\frac{2}{5}, \frac{2}{5}\right\}$ **22.** $\{0, 9\}$ **23.** $\{10\}$
**24.** $\left\{-8, -\frac{5}{2}, \frac{1}{3}\right\}$ [5.6] **25.** 6 ft by 9 ft **26.** −2, −1 **27.** 17 ft
**28.** \$8493 billion

## Chapters 1–5 Cumulative Review Exercises (pages 354–355)

[2.1–2.3] **1.** $\{0\}$ **2.** $\{0.05\}$ **3.** $\{6\}$ [2.5] **4.** $P = \dfrac{A}{1 + rt}$
**5.** 110° and 70° [2.4] **6.** gold: 11; silver: 12; bronze: 6
[2.6] **7.** 230; 205; 38%; 12% [3.1] **8.** **(a)** negative, positive **(b)** negative, negative
[3.2, 3.3] **9.** **(a)** $\left(-\frac{1}{4}, 0\right)$, $(0, 3)$ **(b)** 12 **(c)**



[3.3, 3.4] **10.** **(a)** 16; A slope of (approximately) 16 means that retail sales of prescription drugs increased by about \$16 billion per year. **(b)** $(2005, 230)$ [4.1, 4.2] **11.** $\frac{16}{9}$ **12.** 256 **13.** $\dfrac{1}{p^2}$ **14.** $\dfrac{1}{m^6}$

[4.4] **15.** $-4k^2 - 4k + 8$ [4.5] **16.** $45x^2 + 3x - 18$
[4.6] **17.** $9p^2 + 12p + 4$ [4.7] **18.** $4x^3 + 6x^2 - 3x + 10$
[4.3] **19.** $5.5 \times 10^4$; $2.0 \times 10^6$ [5.2, 5.3] **20.** $(2a - 1)(a + 4)$
**21.** $(2m + 3)(5m + 2)$ **22.** $(4t + 3v)(2t + v)$
[5.4] **23.** $(2p - 3)^2$ **24.** $(5r + 9t)(5r - 9t)$
[5.3] **25.** $2pq(3p + 1)(p + 1)$ [5.5] **26.** $\left\{-\frac{2}{3}, \frac{1}{2}\right\}$ **27.** $\{0, 8\}$
[5.6] **28.** 5 m, 12 m, 13 m

# 6 RATIONAL EXPRESSIONS AND APPLICATIONS

**Connections** **(page 363)** **1.** $3x^2 + 11x + 8$ cannot be factored, so this quotient cannot be simplified. By long division, the quotient is $3x + 5 + \dfrac{-2}{x + 2}$. **2.** The numerator factors as $(x - 2)(x^2 + 2x + 4)$, so, after simplification, the quotient is $x - 2$. Long division gives the same quotient.

## Section 6.1 (pages 364–366)

**1.** **(a)** $\frac{7}{10}$ **(b)** $\frac{8}{15}$ **3.** **(a)** 0 **(b)** −1 **5.** **(a)** $-\frac{64}{15}$ **(b)** undefined **7.** **(a)** undefined **(b)** $\frac{8}{25}$ **9.** **(a)** 0 **(b)** 0 **11.** **(a)** 0 **(b)** undefined **13.** A rational expression is a quotient of two polynomials, such as $\dfrac{x^2 + 3x - 6}{x + 4}$. One can think of this as an algebraic fraction. **15.** Division by 0 is undefined. If the denominator of a rational expression equals 0, the expression is undefined. **17.** $y \neq 0$ **19.** $x \neq 6$ **21.** $x \neq -\frac{5}{3}$ **23.** $m \neq -3, m \neq 2$ **25.** It is never undefined. **27.** It is never undefined. **29.** **(a)** numerator: $x^2$, $4x$; denominator: $x$, 4 **(b)** First factor the numerator, getting $x(x + 4)$. Then divide the numerator and denominator by the common factor $x + 4$ to get $\dfrac{x}{1}$, or $x$.
**31.** $3r^2$ **33.** $\frac{2}{5}$ **35.** $\dfrac{x - 1}{x + 1}$ **37.** $\frac{7}{5}$ **39.** $\frac{6}{7}$ **41.** $m - n$ **43.** $\dfrac{2}{t - 3}$
**45.** $\dfrac{3(2m + 1)}{4}$ **47.** $\dfrac{3m}{5}$ **49.** $\dfrac{3r - 2s}{3}$ **51.** $k - 3$ **53.** $\dfrac{x - 3}{x + 1}$
**55.** $\dfrac{x + 1}{x - 1}$ **57.** $\dfrac{x + 2}{x - 4}$ **59.** $-\dfrac{3}{7t}$ **61.** $\dfrac{z - 3}{z + 5}$ **63.** $\dfrac{r + s}{r - s}$ **65.** $\dfrac{a + b}{a - b}$
**67.** $\dfrac{m + n}{2}$ **69.** $\dfrac{x^2 + 1}{x}$ **71.** $1 - p + p^2$ **73.** $x^2 + 3x + 9$
**75.** $-\dfrac{b^2 + ba + a^2}{a + b}$ **77.** $\dfrac{k^2 - 2k + 4}{k - 2}$ **79.** $\dfrac{z + 3}{z}$ **81.** $\dfrac{1 - 2r}{2}$
**83.** B, D **85.** −1 **87.** $-(m + 1)$ **89.** −1 **91.** It is already in lowest terms. **93.** B

*Answers may vary in Exercises 95, 97, and 99.* **95.** $\dfrac{-(x + 4)}{x - 3}$, $\dfrac{-x - 4}{x - 3}$, $\dfrac{x + 4}{-(x - 3)}$, $\dfrac{x + 4}{-x + 3}$ **97.** $\dfrac{-(2x - 3)}{x + 3}$, $\dfrac{-2x + 3}{x + 3}$, $\dfrac{2x - 3}{-(x + 3)}$, $\dfrac{2x - 3}{-x - 3}$
**99.** $\dfrac{-(3x - 1)}{5x - 6}$, $\dfrac{-3x + 1}{5x - 6}$, $\dfrac{3x - 1}{-(5x - 6)}$, $\dfrac{3x - 1}{-5x + 6}$
**101.** $x^2 + 3$ **103.** **(a)** 0 **(b)** 1.6 **(c)** 4.1 **(d)** The waiting time also increases. **105.** $\frac{5}{9}$ **107.** 4

## Section 6.2 (pages 371–372)

**1.** **(a)** B **(b)** D **(c)** C **(d)** A **3.** $\frac{3a}{2}$ **5.** $-\frac{4x^4}{3}$ **7.** $\frac{2}{c+d}$

**9.** $4(x-y)$ **11.** $\frac{t^2}{2}$ **13.** $\frac{x+3}{2x}$ **15.** 5 **17.** $-\frac{3}{2t^4}$ **19.** $\frac{1}{4}$

**21.** $-\frac{35}{8}$ **23.** $\frac{2(x+2)}{x(x-1)}$ **25.** $\frac{x(x-3)}{6}$ **27.** $\frac{10}{9}$ **29.** $-\frac{3}{4}$ **31.** $-\frac{9}{2}$

**33.** $\frac{p+4}{p+2}$ **35.** $-1$ **37.** $\frac{(2x-1)(x+2)}{x-1}$ **39.** $\frac{(k-1)^2}{(k+1)(2k-1)}$

**41.** $\frac{4k-1}{3k-2}$ **43.** $\frac{m+4p}{m+p}$ **45.** $\frac{m+6}{m+3}$ **47.** $\frac{y+3}{y+4}$ **49.** $\frac{m}{m+5}$

**51.** $\frac{r+6s}{r+s}$ **53.** $\frac{(q-3)^2(q+2)^2}{q+1}$ **55.** $\frac{x+10}{10}$ **57.** $\frac{3-a-b}{2a-b}$

**59.** $-\frac{(x+y)^2(x^2-xy+y^2)}{3y(y-x)(x-y)}$, or $\frac{(x+y)^2(x^2-xy+y^2)}{3y(x-y)^2}$ **61.** $\frac{5xy^2}{4q}$

**63.** $2 \cdot 3^2$ **65.** $2^2 \cdot 3^3$ **67.** 6 **69.** $6q^3$

## Section 6.3 (pages 376–378)

**1.** C **3.** C **5.** 60 **7.** 1800 **9.** $x^5$ **11.** $30p$ **13.** $180y^4$ **15.** $84r^5$
**17.** $15a^5b^3$ **19.** $12p(p-2)$ **21.** $28m^2(3m-5)$ **23.** $30(b-2)$
**25.** $18(r-2)$ **27.** $12p(p+5)^2$ **29.** $8(y+2)(y+1)$
**31.** $c-d$ or $d-c$ **33.** $m-3$ or $3-m$ **35.** $p-q$ or $q-p$
**37.** $k(k+5)(k-2)$ **39.** $a(a+6)(a-3)$
**41.** $(p+3)(p+5)(p-6)$ **43.** $(k+3)(k-5)(k+7)(k+8)$
**45.** 7 **46.** 1 **47.** identity property of multiplication
**48.** 7 **49.** 1 **50.** identity property of multiplication

**51.** $\frac{20}{55}$ **53.** $\frac{-45}{9k}$ **55.** $\frac{60m^2k^3}{32k^4}$ **57.** $\frac{57z}{6z-18}$ **59.** $\frac{-4a}{18a-36}$

**61.** $\frac{6(k+1)}{k(k-4)(k+1)}$ **63.** $\frac{36r(r+1)}{(r-3)(r+2)(r+1)}$

**65.** $\frac{ab(a+2b)}{2a^3b+a^2b^2-ab^3}$ **67.** $\frac{(t-r)(4r-t)}{t^3-r^3}$

**69.** $\frac{2y(z-y)(y-z)}{y^4-z^3y}$, or $\frac{-2y(y-z)^2}{y^4-z^3y}$ **71.** $\frac{11}{8}$ **73.** $\frac{13}{20}$

## Section 6.4 (pages 383–386)

**1.** E **3.** C **5.** B **7.** G **9.** $\frac{11}{m}$ **11.** $\frac{4}{y+4}$ **13.** 1 **15.** $\frac{m-1}{m+1}$

**17.** $b$ **19.** $x$ **21.** $y-6$ **23.** $\frac{1}{x-3}$ **25.** $\frac{3z+5}{15}$ **27.** $\frac{10-7r}{14}$

**29.** $\frac{-3x-2}{4x}$ **31.** $\frac{57}{10x}$ **33.** $\frac{x+1}{2}$ **35.** $\frac{5x+9}{6x}$ **37.** $\frac{7-6p}{3p^2}$

**39.** $\frac{-k-8}{k(k+4)}$ **41.** $\frac{x+4}{x+2}$ **43.** $\frac{6m^2+23m-2}{(m+2)(m+1)(m+5)}$

**45.** $\frac{4y^2-y+5}{(y+1)^2(y-1)}$ **47.** $\frac{3}{t}$ **49.** $m-2$ or $2-m$

**51.** $\frac{-2}{x-5}$, or $\frac{2}{5-x}$ **53.** $-4$ **55.** $\frac{-5}{x-y^2}$, or $\frac{5}{y^2-x}$

**57.** $\frac{x+y}{5x-3y}$, or $\frac{-x-y}{3y-5x}$ **59.** $\frac{-6}{4p-5}$, or $\frac{6}{5-4p}$

**61.** $\frac{-m-n}{2(m-n)}$ **63.** $\frac{-x^2+6x+11}{(x+3)(x-3)(x+1)}$

**65.** $\frac{-5q^2-13q+7}{(3q-2)(q+4)(2q-3)}$ **67.** $\frac{9r+2}{r(r+2)(r-1)}$

**69.** $\frac{2(x^2+3xy+4y^2)}{(x+y)(x+y)(x+3y)}$, or $\frac{2(x^2+3xy+4y^2)}{(x+y)^2(x+3y)}$

**71.** $\frac{15r^2+10ry-y^2}{(3r+2y)(6r-y)(6r+y)}$ **73.** **(a)** $\frac{9k^2+6k+26}{5(3k+1)}$ **(b)** $\frac{1}{4}$

**75.** $\frac{10x}{49(101-x)}$ **77.** $\frac{5}{4}$ **79.** $\frac{6}{7}$

## Section 6.5 (pages 392–394)

**1.** **(a)** 6; $\frac{1}{6}$ **(b)** 12; $-\frac{1}{4}$ **(c)** $\frac{1}{6} \div \left(-\frac{1}{4}\right)$ **(d)** $-\frac{2}{3}$ **3.** Choice D is correct, because every sign has been changed in the fraction. This means it was multiplied by $\frac{-1}{-1} = 1$. **5.** $-6$ **7.** $\frac{1}{xy}$ **9.** $\frac{2a^2b}{3}$

**11.** $\frac{m(m+2)}{3(m-4)}$ **13.** $\frac{2}{x}$ **15.** $\frac{8}{x}$ **17.** $\frac{a^2-5}{a^2+1}$ **19.** $\frac{31}{50}$ **21.** $\frac{y^2+x^2}{xy(y-x)}$

**23.** $\frac{40-12p}{85p}$ **25.** $\frac{5y-2x}{3+4xy}$ **27.** $\frac{a-2}{2a}$ **29.** $\frac{z-5}{4}$ **31.** $\frac{-m}{m+2}$

**33.** $\frac{3m(m-3)}{(m-1)(m-8)}$ **35.** $\frac{2x-7}{3x+1}$ **37.** $\frac{y+4}{y-8}$ **39.** $\frac{x^2y^2}{y^2+x^2}$

**41.** $\frac{y^2+x^2}{xy^2+x^2y}$, or $\frac{y^2+x^2}{xy(y+x)}$ **43.** $\frac{1}{2xy}$ **45.** $\frac{x-3}{x-5}$ **47.** division

**49.** $\frac{\frac{3}{8}+\frac{5}{6}}{2}$ **50.** $\frac{29}{48}$ **51.** $\frac{29}{48}$ **52.** Answers will vary. **53.** $\frac{5}{3}$ **55.** $\frac{13}{2}$

**57.** $\frac{19r}{15}$ **59.** $12x+2$ **61.** $-44p^2+27p$ **63.** $\left\{\frac{1}{2}\right\}$ **65.** $\{-5\}$

## Section 6.6 (pages 401–404)

**1.** expression; $\frac{43}{40}x$ **3.** equation; $\left\{\frac{40}{43}\right\}$ **5.** expression; $-\frac{1}{10}x$
**7.** equation; $\{-10\}$ **9.** equation; $\{0\}$ **11.** $x \neq -2, 0$

**13.** $x \neq -3, 4, -\frac{1}{2}$ **15.** $x \neq -9, 1, -2, 2$ **17.** $\frac{2}{3x} + \frac{1}{5x}$ is an expression, not an equation. Only equations and inequalities are "solved."

**19.** $\left\{\frac{1}{4}\right\}$ **21.** $\left\{-\frac{3}{4}\right\}$ **23.** $\{-15\}$ **25.** $\{7\}$ **27.** $\{-15\}$ **29.** $\{-5\}$
**31.** $\{-6\}$ **33.** $\emptyset$ **35.** $\{5\}$ **37.** $\{4\}$ **39.** $\{5\}$ **41.** $\left\{x \mid x \neq \pm\frac{4}{3}\right\}$
**43.** $\{1\}$ **45.** $\{4\}$ **47.** $\{5\}$ **49.** $\{-4\}$ **51.** $\{-2, 12\}$ **53.** $\emptyset$
**55.** $\{3\}$ **57.** $\{3\}$ **59.** $\{-3\}$ **61.** $\left\{-\frac{1}{5}, 3\right\}$ **63.** $\left\{-\frac{1}{2}, 5\right\}$ **65.** $\{3\}$
**67.** $\left\{-\frac{1}{3}, 3\right\}$ **69.** $\{-1\}$ **71.** $\{-6\}$ **73.** $\left\{-6, \frac{1}{2}\right\}$ **75.** $\{6\}$
**77.** Transform so that the terms with $k$ are on one side and the remaining term is on the other. **79.** $F = \frac{ma}{k}$ **81.** $a = \frac{kF}{m}$ **83.** $R = \frac{E-Ir}{I}$, or $R = \frac{E}{I} - r$ **85.** $\mathcal{A} = \frac{h(B+b)}{2}$ **87.** $a = \frac{2S-ndL}{nd}$, or $a = \frac{2S}{nd} - L$

**89.** $y = \frac{xz}{x+z}$ **91.** $t = \frac{rs}{rs-2s-3r}$, or $t = \frac{-rs}{-rs+2s+3r}$

**93.** $z = \frac{3y}{5-9xy}$, or $z = \frac{-3y}{9xy-5}$ **95.** $t = \frac{2x-1}{x+1}$, or $t = \frac{-2x+1}{-x-1}$

**97.** $\frac{288}{t}$ mph **99.** $\frac{289}{z}$ hr

## Summary Exercises on Rational Expressions and Equations (pages 405–406)

**1.** expression; $\frac{10}{p}$ **2.** expression; $\frac{y^3}{x^3}$ **3.** expression; $\frac{1}{2x^2(x+2)}$ **4.** equation; $\{9\}$ **5.** expression; $\frac{y+2}{y-1}$ **6.** expression; $\frac{5k+8}{k(k-4)(k+4)}$ **7.** equation; $\{39\}$ **8.** expression; $\frac{t-5}{3(2t+1)}$ **9.** expression; $\frac{13}{3(p+2)}$ **10.** equation; $\left\{-1, \frac{12}{5}\right\}$ **11.** equation; $\left\{\frac{1}{7}, 2\right\}$ **12.** expression; $\frac{16}{3k}$ **13.** expression; $\frac{7}{12z}$ **14.** equation; $\{13\}$ **15.** expression; $\frac{3m+5}{(m+3)(m+2)(m+1)}$ **16.** expression; $\frac{k+3}{5(k-1)}$ **17.** equation; $\emptyset$ **18.** equation; $\emptyset$ **19.** expression; $\frac{t+2}{2(2t+1)}$ **20.** equation; $\{-7\}$

## Section 6.7 (pages 410–414)

**1.** **(a)** the amount **(b)** $5 + x$ **(c)** $\frac{5+x}{6} = \frac{13}{3}$ **3.** $\frac{12}{18}$ **5.** $\frac{12}{3}$ **7.** 12 **9.** $\frac{1386}{97}$ **11.** 18.809 min **13.** 314.248 m per min **15.** 3.275 hr **17.** $\frac{D}{R} = \frac{d}{r}$ **19.** $\frac{500}{x-10} = \frac{600}{x+10}$ **21.** 8 mph **23.** 32 mph **25.** 165 mph **27.** 3 mph **29.** 18.5 mph **31.** $\frac{1}{10}$ job per hr **33.** $\frac{1}{8}t + \frac{1}{6}t = 1$, or $\frac{1}{8} + \frac{1}{6} = \frac{1}{t}$ **35.** $2\frac{2}{5}$ hr **37.** $5\frac{5}{11}$ hr **39.** 3 hr **41.** $2\frac{7}{10}$ hr **43.** $9\frac{1}{11}$ min **45.** 2 **47.** $\frac{5}{2}$ **49.** 0 **51.** $y = -\frac{3}{2}x + 4$

## Chapter 6 Review Exercises (pages 419–421)

**1.** **(a)** $\frac{11}{8}$ **(b)** $\frac{13}{22}$ **2.** **(a)** undefined **(b)** 1 **3.** $x \neq 3$ **4.** $y \neq 0$ **5.** $k \neq -5, -\frac{2}{3}$ **6.** Set the denominator equal to 0 and solve the equation. Any solutions are values for which the rational expression is undefined. **7.** $\frac{b}{3a}$ **8.** $-1$ **9.** $\frac{-(2x+3)}{2}$ **10.** $\frac{2p+5q}{5p+q}$ *Answers may vary in Exercises 11 and 12.* **11.** $\frac{-(4x-9)}{2x+3}, \frac{-4x+9}{2x+3}, \frac{4x-9}{-(2x+3)}, \frac{4x-9}{-2x-3}$ **12.** $\frac{-(8-3x)}{3-6x}, \frac{-8+3x}{3-6x}, \frac{8-3x}{-(3-6x)}, \frac{8-3x}{-3+6x}$ **13.** $\frac{72}{p}$ **14.** 2 **15.** $\frac{5}{8}$ **16.** $\frac{r+4}{3}$ **17.** $\frac{3a-1}{a+5}$ **18.** $\frac{y-2}{y-3}$ **19.** $\frac{p+5}{p+1}$ **20.** $\frac{3z+1}{z+3}$ **21.** $108y^4$ **22.** $(x+3)(x+1)(x+4)$ **23.** $\frac{15a}{10a^4}$ **24.** $\frac{-54}{18-6x}$ **25.** $\frac{15y}{50-10y}$ **26.** $\frac{4b(b+2)}{(b+3)(b-1)(b+2)}$ **27.** $\frac{15}{x}$ **28.** $-\frac{2}{p}$ **29.** $\frac{4k-45}{k(k-5)}$ **30.** $\frac{28+11y}{y(7+y)}$ **31.** $\frac{-2-3m}{6}$ **32.** $\frac{3(16-x)}{4x^2}$ **33.** $\frac{7a+6b}{(a-2b)(a+2b)}$ **34.** $\frac{-k^2-6k+3}{3(k+3)(k-3)}$ **35.** $\frac{5z-16}{z(z+6)(z-2)}$ **36.** $\frac{-13p+33}{p(p-2)(p-3)}$ **37.** $\frac{4(y-3)}{y+3}$ **38.** $\frac{10}{13}$ **39.** $\frac{xw+1}{xw-1}$ **40.** $\frac{(q-p)^2}{pq}$ **41.** $(x-5)(x-3)$, or $x^2 - 8x + 15$ **42.** $\frac{y+x}{xy}$ **43.** $\emptyset$ **44.** $\{-16\}$ **45.** $\{0\}$ **46.** $\{3\}$ **47.** $t = \frac{Ry}{m}$ **48.** $y = \frac{4x+5}{3}$ **49.** $m = \frac{4+p^2q}{3p^2}$ **50.** $\frac{20}{15}$ **51.** $\frac{3}{18}$ **52.** 10 mph **53.** $3\frac{1}{13}$ hr **54.** 2 hr **55.** $\frac{m+7}{(m-1)(m+1)}$ **56.** $8p^2$ **57.** $\frac{1}{6}$ **58.** $\frac{s^2+t^2}{st(s-t)}$ **59.** 3 **60.** $\frac{z+7}{(z+1)(z-1)^2}$ **61.** $\frac{-t-1}{(t+2)(t-2)}$, or $\frac{t+1}{(2+t)(2-t)}$ **62.** $\{-2, 3\}$ **63.** $v = at + w$ **64.** 150 km per hr **65.** $5\frac{1}{11}$ hr **66.** **(a)** $-3$ **(b)** $-1$ **(c)** $-3, -1$ **67.** $\frac{15}{2x}$ **68.** If $x = 0$, the divisor $R$ is equal to 0, and division by 0 is undefined. **69.** $(x+3)(x+1)$ **70.** $\frac{7}{x+1}$ **71.** $\frac{11x+21}{4x}$ **72.** $\emptyset$ **73.** We know that $-3$ is not allowed, because $P$ and $R$ are undefined for $x = -3$. **74.** Rate is equal to distance divided by time. Here, distance is 6 miles and time is $(x + 3)$ minutes, so rate $= \frac{6}{x+3}$, which is the expression for $P$. **75.** $\frac{6}{5}, \frac{5}{2}$

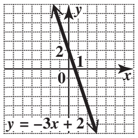
## Chapter 6 Test (page 422)

[6.1] **1.** **(a)** $\frac{11}{6}$ **(b)** undefined **2.** $x \neq -2, 4$ **3.** (Answers may vary.) $\frac{-(6x-5)}{2x+3}, \frac{-6x+5}{2x+3}, \frac{6x-5}{-(2x+3)}, \frac{6x-5}{-2x-3}$ **4.** $-3x^2y^3$ **5.** $\frac{3a+2}{a-1}$ [6.2] **6.** $\frac{25}{27}$ **7.** $\frac{3k-2}{3k+2}$ **8.** $\frac{a-1}{a+4}$ **9.** $\frac{x-5}{3-x}$ [6.3] **10.** $150p^5$ **11.** $(2r+3)(r+2)(r-5)$ **12.** $\frac{240p^2}{64p^3}$ **13.** $\frac{21}{42m-84}$ [6.4] **14.** 2 **15.** $\frac{-14}{5(y+2)}$ **16.** $\frac{-x^2+x+1}{3-x}$, or $\frac{x^2-x-1}{x-3}$ **17.** $\frac{-m^2+7m+2}{(2m+1)(m-5)(m-1)}$ [6.5] **18.** $\frac{2k}{3p}$ **19.** $\frac{-2-x}{4+x}$ **20.** $\frac{2y^2+x^2}{xy(y-x)}$ [6.6] **21.** $\left\{-\frac{1}{2}, 1\right\}$ **22.** $\left\{-\frac{1}{2}\right\}$ **23.** $D = \frac{dF-k}{F}$, or $D = d - \frac{k}{F}$ [6.7] **24.** 3 mph **25.** $2\frac{2}{9}$ hr

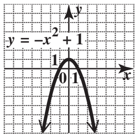
## Chapters 1–6 Cumulative Review Exercises (pages 423–424)

[1.2, 1.5, 1.6] **1.** 2 [2.3] **2.** $\{17\}$ [2.5] **3.** $b = \frac{2\mathscr{A}}{h}$ [2.6] **4.** $\left\{-\frac{2}{7}\right\}$ [2.8] **5.** $[-8, \infty)$ [3.1, 3.2] **6.** **(a)** $(-3, 0)$ **(b)** $(0, -4)$

**7.**



[4.4] **8.**



[4.1, 4.2] **9.** $\frac{1}{2^4x^7}$ **10.** $\frac{1}{m^6}$ [4.4] **11.** $k^2 + 2k + 1$

[4.6] **12.** $4a^2 - 4ab + b^2$ [4.5] **13.** $3y^3 + 8y^2 + 12y - 5$

[4.7] **14.** $6p^2 + 7p + 1 + \frac{3}{p - 1}$ [5.3] **15.** $(4t + 3v)(2t + v)$

**16.** prime [5.4] **17.** $(4x^2 + 1)(2x + 1)(2x - 1)$ [5.5] **18.** $\{-3, 5\}$

**19.** $\left\{5, -\frac{1}{2}, \frac{2}{3}\right\}$ [5.6] **20.** $-2$ or $-1$ **21.** 6 m [6.1] **22.** A **23.** D

[6.4] **24.** $\frac{4}{q}$ **25.** $\frac{3r + 28}{7r}$ **26.** $\frac{7}{15(q - 4)}$ **27.** $\frac{-k - 5}{k(k + 1)(k - 1)}$

[6.2] **28.** $\frac{7(2z + 1)}{24}$ [6.5] **29.** $\frac{195}{29}$ [6.6] **30.** $\left\{\frac{21}{2}\right\}$ **31.** $\{-2, 1\}$

[6.7] **32.** $1\frac{1}{5}$ hr

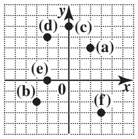
# 7 GRAPHS, LINEAR EQUATIONS, AND FUNCTIONS

Connections **(page 432)** **1.** $x$-intercept: $(-2, 0)$; $y$-intercept: $(0, 3)$ *For Problems 2 and 3, we give each equation solved for y. Graphs are not included.* **2.** $y = 4x + 3$ **3.** $y = -0.5x$

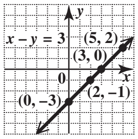
## Section 7.1 (pages 438–443)

**1.** **(a)** I **(b)** III **(c)** II **(d)** IV **(e)** none **(f)** none

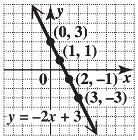
**3.**



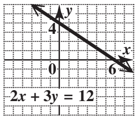
**5.** **(a)** $-3$; 3; 2; $-1$ **(b)**



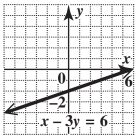
**7.** **(a)** 3; 1; $-1$; 3 **(b)**



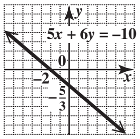
**9.** $(6, 0)$; $(0, 4)$



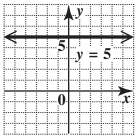
**11.** $(6, 0)$; $(0, -2)$



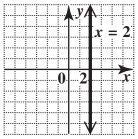
**13.** $(-2, 0)$; $\left(0, -\frac{5}{3}\right)$



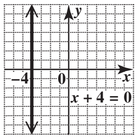
**15.** none; $(0, 5)$



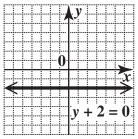
**17.** $(2, 0)$; none



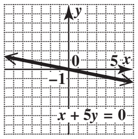
**19.** $(-4, 0)$; none



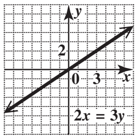
**21.** none; $(0, -2)$



**23.** $(0, 0)$; $(0, 0)$



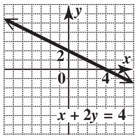
**25.** $(0, 0)$; $(0, 0)$



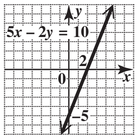
**27.** $(-5, -1)$ **29.** $\left(\frac{9}{2}, -\frac{3}{2}\right)$ **31.** $\left(0, \frac{11}{2}\right)$ **33.** $(2.1, 0.9)$ **35.** $(1, 1)$

**37.** $\left(-\frac{5}{12}, \frac{5}{28}\right)$ **39.** B **41.** A, B, D, G **43.** **(a)** C **(b)** A **(c)** D **(d)** B **45.** **(a)** 2 **(b)** 0 **(c)** undefined **(d)** $-\frac{1}{3}$ **(e)** 1 **(f)** $-4$ **47.** **(a)** 8 **(b)** rises **49.** **(a)** $\frac{5}{6}$ **(b)** rises **51.** **(a)** 0 **(b)** horizontal **53.** **(a)** $-\frac{1}{2}$ **(b)** falls **55.** **(a)** undefined **(b)** vertical **57.** **(a)** $-1$ **(b)** falls **59.** 6 **61.** $-3$ **63.** $-\frac{5}{2}$ **65.** undefined

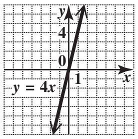
**67.** $-\frac{1}{2}$



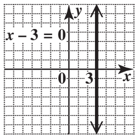
**69.** $\frac{5}{2}$



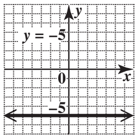
**71.** 4



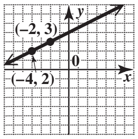
**73.** undefined



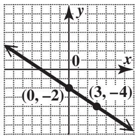
**75.** 0



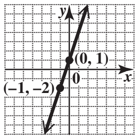
**77.**



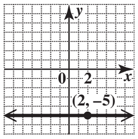
**79.**



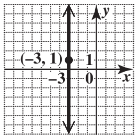
**81.**



**83.**



**85.**

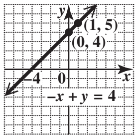


**87.** $-\frac{4}{9}$; $\frac{9}{4}$ **89.** parallel **91.** perpendicular **93.** neither **95.** parallel **97.** neither **99.** perpendicular **101.** $-\$4000$ per yr; The value of the machine is decreasing \$4000 each year during these years. **103.** 0% per yr (or no change); The percent of pay raise is not changing—it is 3% each year during these years. **105.** **(a)** 21.2 **(b)** The number of subscribers increased by an average of 21.2 million each year from 2005 to 2008. **107.** **(a)** $-5$ theaters per yr **(b)** The negative slope means that the number of drive-in theaters decreased by an average of 5 each year from 2000 to 2007. **109.** \$1371.67 million per yr; Sales of plasma TVs increased by an average of \$1371.67 million each year from 2003 to 2006. **111.** $3x - 2y = 19$

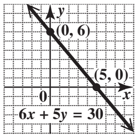
## Section 7.2 (pages 451–455)

**1.** A **3.** A **5.** $3x + y = 10$ **7.** A **9.** C **11.** H **13.** B **15.** $y = 5x + 15$ **17.** $y = -\frac{2}{3}x + \frac{4}{5}$ **19.** $y = x - 1$ **21.** $y = \frac{2}{5}x + 5$ **23.** $y = \frac{2}{3}x + 1$ **25.** $y = -x - 2$

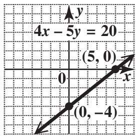
**27.** **(a)** $y = x + 4$ **(b)** 1 **(c)** $(0, 4)$ **(d)**



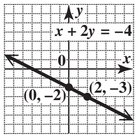
**29.** **(a)** $y = -\frac{6}{5}x + 6$ **(b)** $-\frac{6}{5}$ **(c)** $(0, 6)$ **(d)**



**31.** **(a)** $y = \frac{4}{5}x - 4$ **(b)** $\frac{4}{5}$ **(c)** $(0, -4)$ **(d)**



**33.** **(a)** $y = -\frac{1}{2}x - 2$ **(b)** $-\frac{1}{2}$ **(c)** $(0, -2)$ **(d)**



**35.** **(a)** $2x + y = 18$ **(b)** $y = -2x + 18$ **37.** **(a)** $3x + 4y = 10$ **(b)** $y = -\frac{3}{4}x + \frac{5}{2}$ **39.** **(a)** $x - 2y = -13$ **(b)** $y = \frac{1}{2}x + \frac{13}{2}$ **41.** **(a)** $4x - y = 12$ **(b)** $y = 4x - 12$ **43.** **(a)** $7x - 5y = -20$ **(b)** $y = 1.4x + 4$ **45.** $y = 5$ **47.** $x = 9$ **49.** $y = -\frac{3}{2}$ **51.** $y = 8$ **53.** $x = 0.5$ **55.** **(a)** $2x - y = 2$ **(b)** $y = 2x - 2$ **57.** **(a)** $x + 2y = 8$ **(b)** $y = -\frac{1}{2}x + 4$ **59.** **(a)** $y = 5$ **(b)** $y = 5$ **61.** **(a)** $x = 7$ **(b)** not possible **63.** **(a)** $y = -3$ **(b)** $y = -3$ **65.** **(a)** $2x - 13y = -6$ **(b)** $y = \frac{2}{13}x + \frac{6}{13}$ **67.** **(a)** $y = 3x - 19$ **(b)** $3x - y = 19$ **69.** **(a)** $y = \frac{1}{2}x - 1$ **(b)** $x - 2y = 2$ **71.** **(a)** $y = -\frac{1}{2}x + 9$ **(b)** $x + 2y = 18$ **73.** **(a)** $y = 7$ **(b)** $y = 7$ **75.** $y = 45x$; $(0, 0)$, $(5, 225)$, $(10, 450)$ **77.** $y = 3.10x$; $(0, 0)$, $(5, 15.50)$, $(10, 31.00)$ **79.** $y = 111x$; $(0, 0)$, $(5, 555)$, $(10, 1110)$ **81.** **(a)** $y = 112.50x + 12$ **(b)** $(5, 574.50)$; The cost for 5 tickets and a parking pass is \$574.50. **(c)** \$237 **83.** **(a)** $y = 41x + 99$ **(b)** $(5, 304)$; The cost for a 5-month membership is \$304. **(c)** \$591 **85.** **(a)** $y = 60x + 36$ **(b)** $(5, 336)$; The cost of the plan for 5 months is \$336. **(c)** \$756 **87.** **(a)** $y = 6x + 30$ **(b)** $(5, 60)$; It costs \$60 to rent the saw for 5 days. **(c)** 18 days **89.** **(a)** $y = 1294.7x + 3921$; Sales of digital cameras in the United States increased by \$1294.7 million per yr from 2003 to 2006. **(b)** \$9099.8 million **91.** **(a)** $y = 5.25x + 22.25$ **(b)** \$48.5 billion; It is greater than the actual value. **93.** 32; 212 **94.** $(0, 32)$ and $(100, 212)$ **95.** $\frac{9}{5}$ **96.** $F = \frac{9}{5}C + 32$ **97.** $C = \frac{5}{9}(F - 32)$ **98.** 86° **99.** 10° **100.** −40° **101.** $[0, \infty)$ **103.** $[-4, 4]$

## Summary Exercises on Slopes and Equations of Lines (page 456)

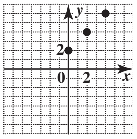
**1.** $-\frac{3}{5}$ **2.** $-\frac{4}{7}$ **3.** 2 **4.** $\frac{5}{2}$ **5.** undefined **6.** 0 **7.** **(a)** $y = -\frac{5}{6}x + \frac{13}{3}$ **(b)** $5x + 6y = 26$ **8.** **(a)** $y = 3x + 11$ **(b)** $3x - y = -11$ **9.** **(a)** $y = -\frac{5}{2}x$ **(b)** $5x + 2y = 0$ **10.** **(a)** $y = -8$ **(b)** $y = -8$ **11.** **(a)** $y = -\frac{7}{9}$ **(b)** $9y = -7$ **12.** **(a)** $y = -3x + 10$ **(b)** $3x + y = 10$ **13.** **(a)** $y = \frac{2}{3}x + \frac{14}{3}$ **(b)** $2x - 3y = -14$ **14.** **(a)** $y = 2x - 10$ **(b)** $2x - y = 10$ **15.** **(a)** $y = -\frac{5}{2}x + 2$ **(b)** $5x + 2y = 4$ **16.** **(a)** $y = \frac{2}{3}x + 8$ **(b)** $2x - 3y = -24$ **17.** **(a)** $y = -7x + 3$ **(b)** $7x + y = 3$ **18.** **(a)** B **(b)** D **(c)** A **(d)** C **(e)** E

## Section 7.3 (pages 462–464)

**1.** Answers will vary. A function is a set of ordered pairs in which each first component corresponds to exactly one second component. For example, $\{(0, 1), (1, 2), (2, 3), (3, 4)\}$ is a function. **3.** independent variable

*In Exercises 5 and 7, answers will vary.*

**5.**



**7.**

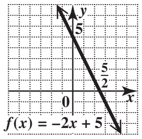
| $x$ | $y$ |
|---|---|
| −3 | −4 |
| −3 | 1 |
| 2 | 0 |

**9.** function; domain: $\{5, 3, 4, 7\}$; range: $\{1, 2, 9, 6\}$ **11.** not a function; domain: $\{2, 0\}$; range: $\{4, 2, 5\}$ **13.** function; domain: $\{-3, 4, -2\}$; range: $\{1, 7\}$ **15.** not a function; domain: $\{1, 0, 2\}$; range: $\{1, -1, 0, 4, -4\}$ **17.** function; domain: $\{2, 5, 11, 17, 3\}$; range: $\{1, 7, 20\}$ **19.** not a function; domain: $\{1\}$; range: $\{5, 2, -1, -4\}$ **21.** function; domain: $\{4, 2, 0, -2\}$; range: $\{-3\}$ **23.** function; domain: $\{-2, 0, 3\}$; range: $\{2, 3\}$ **25.** function; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$ **27.** not a function; domain: $(-\infty, 0]$; range: $(-\infty, \infty)$ **29.** function; domain: $(-\infty, \infty)$; range: $(-\infty, 4]$ **31.** not a function; domain: $[-4, 4]$; range: $[-3, 3]$ **33.** function; $(-\infty, \infty)$ **35.** function; $(-\infty, \infty)$ **37.** function; $(-\infty, \infty)$ **39.** not a function; $[0, \infty)$ **41.** function; $(-\infty, \infty)$ **43.** function; $(-\infty, 0) \cup (0, \infty)$ **45.** function; $(-\infty, 4) \cup (4, \infty)$ **47.** function; $\left(-\infty, -\frac{3}{4}\right) \cup \left(-\frac{3}{4}, \infty\right)$ **49.** not a function; $[1, \infty)$ **51.** function; $(-\infty, 0) \cup (0, \infty)$ **53.** **(a)** yes **(b)** domain: $\{2004, 2005, 2006, 2007, 2008\}$; range: $\{42.3, 42.8, 43.7, 43.8\}$ **(c)** Answers will vary. Two possible answers are $(2005, 42.3)$ and $(2008, 43.8)$. **55.** −9 **57.** 1 **59.** $y = \frac{1}{2}x - \frac{7}{4}$

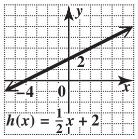
## Section 7.4 (pages 468–472)

**1.** B **3.** 4 **5.** −11 **7.** 3 **9.** 2.75 **11.** $-3p + 4$ **13.** $3x + 4$ **15.** $-3x - 2$ **17.** $-\pi^2 + 4\pi + 1$ **19.** $-3x - 3h + 4$ **21.** −9 **23.** **(a)** −1 **(b)** −1 **25.** **(a)** 2 **(b)** 3 **27.** **(a)** 15 **(b)** 10 **29.** **(a)** 4 **(b)** 1 **31.** **(a)** 3 **(b)** −3 **33.** **(a)** −3 **(b)** 2 **35.** **(a)** 2 **(b)** 0 **(c)** −1 **37.** **(a)** $f(x) = -\frac{1}{3}x + 4$ **(b)** 3 **39.** **(a)** $f(x) = 3 - 2x^2$ **(b)** −15 **41.** **(a)** $f(x) = \frac{4}{3}x - \frac{8}{3}$ **(b)** $\frac{4}{3}$ **43.** line; −2; $-2x + 4$; −2; 3; −2

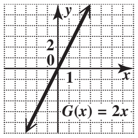
**45.** domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



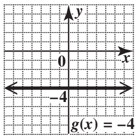
**47.** domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



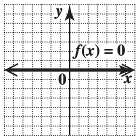
**49.** domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



**51.** domain: $(-\infty, \infty)$; range: $\{-4\}$



**53.** domain: $(-\infty, \infty)$; range: $\{0\}$



**55.** $x$-axis **57. (a)** \$11.25 **(b)** 3 is the value of the independent variable, which represents a package weight of 3 lb; $f(3)$ is the value of the dependent variable, representing the cost to mail a 3-lb package. **(c)** \$18.75; $f(5) = 18.75$

**59. (a)** 194.53 cm **(b)** 177.29 cm **(c)** 177.41 cm **(d)** 163.65 cm **61. (a)** $f(x) = 12x + 100$ **(b)** 1600; The cost to print 125 t-shirts is \$1600. **(c)** 75; $f(75) = 1000$; The cost to print 75 t-shirts is \$1000. **63. (a)** 1.1 **(b)** 5 **(c)** $-1.2$ **(d)** $(0, 3.5)$ **(e)** $f(x) = -1.2x + 3.5$ **65. (a)** $[0, 100]$; $[0, 3000]$ **(b)** 25 hr; 25 hr **(c)** 2000 gal **(d)** $f(0) = 0$; The pool is empty at time 0. **(e)** $f(25) = 3000$; After 25 hr, there are 3000 gal of water in the pool. **67.** $15x^2 - x - 4$ **69.** $12x^2 - 11x - 5$ **71.** $3x^2 - 2x$

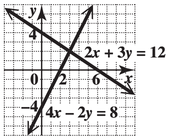
## Section 7.5 (pages 478–480)

**1. (a)** $-10$ **(b)** 8 **3. (a)** 8 **(b)** 2 **5. (a)** 8 **(b)** 74 **7. (a)** $-11$ **(b)** 4 **9. (a)** $8x - 3$ **(b)** $2x - 17$ **11. (a)** $-x^2 + 12x - 12$ **(b)** $9x^2 + 4x + 6$ **13.** $x^2 + 2x - 9$ **15.** 6 **17.** $x^2 - x - 6$ **19.** 6 **21.** $-33$ **23.** 0 **25.** $-\frac{9}{4}$ **27.** $-\frac{9}{2}$ **29.** For example, let $f(x) = 2x^3 + 3x^2 + x + 4$ and $g(x) = 2x^4 + 3x^3 - 9x^2 + 2x - 4$. For these functions, $(f - g)(x) = -2x^4 - x^3 + 12x^2 - x + 8$, and $(g - f)(x) = 2x^4 + x^3 - 12x^2 + x - 8$. Because the two differences are not equal, subtraction of functions is not commutative. **31.** $10x^2 - 2x$ **33.** $2x^2 - x - 3$ **35.** $8x^3 - 27$ **37.** $2x^3 - 18x$ **39.** $-20$ **41.** $2x^2 - 6x$ **43.** 36 **45.** $\frac{35}{4}$ **47.** $\frac{1859}{64}$ **49.** $5x - 1$; 0 **51.** $2x - 3$; $-1$ **53.** $4x^2 + 6x + 9$; $\frac{3}{2}$ **55.** $\frac{x^2 - 9}{2x}$, $x \neq 0$ **57.** $-\frac{5}{4}$ **59.** $\frac{x - 3}{2x}$, $x \neq 0$ **61.** 0 **63.** $-\frac{35}{4}$ **65.** $\frac{7}{2}$ **67.** 6 **69.** 83 **71.** 53 **73.** 13 **75.** $2x^2 + 11$ **77.** $2x - 2$ **79.** $\frac{97}{4}$ **81.** 8 **83.** $(f \circ g)(x) = 63{,}360x$; It computes the number of inches in $x$ miles. **85.** $(\mathscr{A} \circ r)(t) = 4\pi t^2$; This is the area of the circular layer as a function of time. **87.** $\frac{1}{3}$ **89.** 3

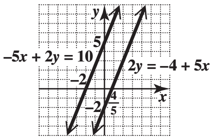
## Section 7.6 (pages 486–489)

**1.** direct **3.** direct **5.** inverse **7.** inverse **9.** inverse **11.** direct **13.** joint **15.** combined **17.** increases; decreases **19.** The perimeter of a square varies directly as the length of its side. **21.** The surface area of a sphere varies directly as the square of its radius. **23.** The area of a triangle varies jointly as the length of its base and height. **25.** 4; 2; $4\pi$; $\frac{4}{3}\pi$; $\frac{1}{2}$; $\frac{1}{3}\pi$ **27.** 36 **29.** $\frac{16}{9}$ **31.** 0.625 **33.** $\frac{16}{5}$ **35.** $222\frac{2}{9}$ **37.** \$2.919, or \$2.91$\frac{9}{10}$ **39.** 8 lb **41.** about 450 cm$^3$ **43.** 256 ft **45.** $106\frac{2}{3}$ mph **47.** 100 cycles per sec **49.** $21\frac{1}{3}$ foot-candles **51.** \$420 **53.** about 11.8 lb **55.** about 448.1 lb **57.** about 68,600 calls **59.** Answers will vary.

**61.**



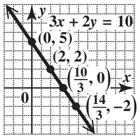
**63.**



## Chapter 7 Review Exercises (pages 493–496)

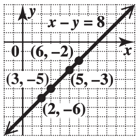
**1.**

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| $\frac{10}{3}$ | 0 |
| 2 | 2 |
| $\frac{14}{3}$ | $-2$ |

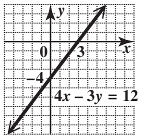


**2.**

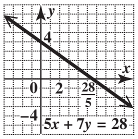
| $x$ | $y$ |
|---|---|
| 2 | $-6$ |
| 5 | $-3$ |
| 3 | $-5$ |
| 6 | $-2$ |



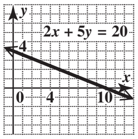
**3.** $(3, 0)$; $(0, -4)$



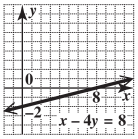
**4.** $\left(\frac{28}{5}, 0\right)$; $(0, 4)$



**5.** $(10, 0)$; $(0, 4)$



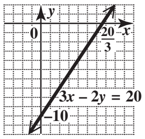
**6.** $(8, 0)$; $(0, -2)$



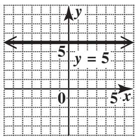
**7.** $(0, 2)$ **8.** $\left(-\frac{9}{2}, \frac{3}{2}\right)$ **9.** $-\frac{7}{5}$ **10.** $-\frac{1}{2}$ **11.** 2 **12.** $\frac{3}{4}$ **13.** undefined **14.** $\frac{2}{3}$ **15.** $-\frac{1}{3}$ **16.** undefined **17.** $-\frac{1}{3}$ **18.** $-1$ **19.** positive **20.** negative **21.** undefined **22.** 0 **23.** 12 ft **24.** \$1496 per yr **25. (a)** $y = -\frac{1}{3}x - 1$ **(b)** $x + 3y = -3$ **26. (a)** $y = -2$ **(b)** $y = -2$ **27. (a)** $y = -\frac{4}{3}x + \frac{29}{3}$ **(b)** $4x + 3y = 29$ **28. (a)** $y = 3x + 7$ **(b)** $3x - y = -7$ **29. (a)** not possible **(b)** $x = 2$ **30. (a)** $y = -9x + 13$ **(b)** $9x + y = 13$ **31. (a)** $y = \frac{7}{5}x + \frac{16}{5}$ **(b)** $7x - 5y = -16$ **32. (a)** $y = -x + 2$ **(b)** $x + y = 2$ **33. (a)** $y = 4x - 29$ **(b)** $4x - y = 29$ **34. (a)** $y = -\frac{5}{2}x + 13$ **(b)** $5x + 2y = 26$ **35. (a)** $y = 57x + 159$; \$843 **(b)** $y = 47x + 159$; \$723 **36. (a)** $y = 143.75x + 1407.75$; The revenue from skiing facilities increased by an average of \$143.75 million each year from 2003 to 2007. **(b)** \$2558 million **37.** domain: $\{-4, 1\}$; range: $\{2, -2, 5, -5\}$; not a function **38.** domain: $\{9, 11, 4, 17, 25\}$; range: $\{32, 47, 69, 14\}$; function **39.** domain: $[-4, 4]$; range: $[0, 2]$; function **40.** domain: $(-\infty, 0]$; range: $(-\infty, \infty)$; not a function **41.** function; domain: $(-\infty, \infty)$; linear function **42.** not a function; domain: $[0, \infty)$ **43.** function; domain: $(-\infty, 6) \cup (6, \infty)$ **44.** $-6$ **45.** $-8.52$ **46.** $-8$ **47.** $-2k^2 + 3k - 6$ **48.** $f(x) = 2x^2$; 18 **49.** C **50. (a)** yes **(b)** domain: $\{1960, 1970, 1980, 1990, 2000, 2009\}$; range: $\{69.7, 70.8, 73.7, 75.4, 76.8, 78.1\}$ **(c)** Answers will vary. Two possible answers are $(1960, 69.7)$ and $(2009, 78.1)$. **(d)** 73.7; In 1980, life expectancy at birth was 73.7 yr. **(e)** 2000 **51. (a)** $-11$ **(b)** 4 **52. (a)** $5x^2 - x + 5$ **(b)** $-5x^2 + 5x + 1$ **(c)** 11 **(d)** $-9$ **53. (a)** $36x^3 - 9x^2$ **(b)** $4x - 1$, $x \neq 0$ **(c)** $-45$ **(d)** 7 **54. (a)** 167 **(b)** 1495 **(c)** 20 **(d)** 42 **(e)** $75x^2 + 220x + 160$ **(f)** $15x^2 + 10x + 2$ **55.** C **56.** 430 mm **57.** 5.59 vibrations per sec **58.** 22.5 ft$^3$

## Chapter 7 Test (pages 496–497)

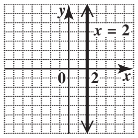
[7.1] **1.** $-\frac{10}{3}$; $-2$; 0 **2.** $\left(\frac{20}{3}, 0\right)$; $(0, -10)$



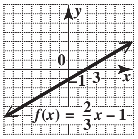
**3.** none; $(0, 5)$



**4.** $(2, 0)$; none



**5.** $\frac{1}{2}$ **6.** It is a vertical line. **7.** perpendicular **8.** neither **9.** $-929$ farms per yr; The number of farms decreased, on the average, by about 929 each year from 1980 to 2008. [7.2] **10.** **(a)** $y = -5x + 19$ **(b)** $5x + y = 19$ **11.** **(a)** $y = 14$ **(b)** $y = 14$ **12.** **(a)** $y = -\frac{1}{2}x + 2$ **(b)** $x + 2y = 4$ **13.** **(a)** not possible **(b)** $x = 5$ **14.** **(a)** $y = -\frac{3}{5}x - \frac{11}{5}$ **(b)** $3x + 5y = -11$ **15.** **(a)** $y = -\frac{1}{2}x - \frac{3}{2}$ **(b)** $x + 2y = -3$ **16.** B **17.** **(a)** $y = 2078x + 51{,}557$ **(b)** \$61,947; It is more than the actual value. [7.3] **18.** D **19.** D **20.** **(a)** domain: $[0, \infty)$; range: $(-\infty, \infty)$ **(b)** domain: $\{0, -2, 4\}$; range: $\{1, 3, 8\}$ [7.4] **21.** **(a)** 0 **(b)** $-a^2 + 2a - 1$ **22.** domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

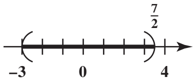


[7.5] **23.** **(a)** $-18$ **(b)** $-2x^2 + 12x - 9$ **(c)** $-2x^2 - 2x - 3$ **(d)** $-7$ **24.** **(a)** $x^3 + 4x^2 + 5x + 2$ **(b)** 0 **25.** **(a)** $x + 2,\quad x \neq -1$ **(b)** 0 **26.** **(a)** 23 **(b)** $3x^2 + 11$ **(c)** $9x^2 + 30x + 27$ [7.6] **27.** 200 amps **28.** 0.8 lb

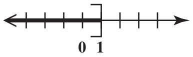
## Chapters 1–7 Cumulative Review Exercises (pages 498–499)

[1.4, 1.5] **1.** always true **2.** never true **3.** sometimes true; For example, $3 + (-3) = 0$, but $3 + (-1) = 2 \neq 0$. **4.** 4 [1.7] **5.** $4m - 3$ [1.6] **6.** $-\frac{19}{2}$ [1.5, 1.6] **7.** $-39$ **8.** undefined [2.3] **9.** $\left\{\frac{7}{6}\right\}$ **10.** $\{-1\}$ [2.5] **11.** 6 in. [2.7] **12.** 2 hr

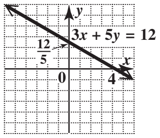
[2.8] **13.** $\left(-3, \frac{7}{2}\right)$



**14.** $(-\infty, 1]$



[3.2, 7.1] **15.** $x$-intercept: $(4, 0)$; $y$-intercept: $\left(0, \frac{12}{5}\right)$



[3.3, 7.1] **16.** **(a)** $-\frac{6}{5}$ **(b)** $\frac{5}{6}$ [3.4, 7.2] **17.** **(a)** $y = -\frac{3}{4}x - 1$ **(b)** $3x + 4y = -4$ **18.** **(a)** $y = -\frac{4}{3}x + \frac{7}{3}$ **(b)** $4x + 3y = 7$ [4.1, 4.2] **19.** $\frac{y}{18x}$ [4.6] **20.** $49x^2 + 42xy + 9y^2$ [4.4] **21.** $x^3 + 12x^2 - 3x - 7$ [4.7] **22.** $m^2 - 2m + 3$ [5.1–5.4] **23.** $(2w + 7z)(8w - 3z)$ **24.** $(2x - 1 + y)(2x - 1 - y)$ **25.** $(2p + 3)(4p^2 - 6p + 9)$ [5.5] **26.** $\left\{\frac{1}{3}\right\}$ [5.6] **27.** 4 ft **28.** longer sides: 18 in.; distance between: 16 in. [6.4] **29.** $\frac{6x + 22}{(x + 1)(x + 3)}$ [6.2] **30.** $\frac{(x + 3)^2}{3x}$ [6.5] **31.** 6 [6.6] **32.** $\{5\}$ [7.3] **33.** domain: $\{14, 91, 75, 23\}$; range: $\{9, 70, 56, 5\}$; not a function; 75 in the domain is paired with two different values, 70 and 56, in the range.

[7.4] **34.** **(a)** domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$ **(b)** 22 **(c)** 1 [7.2] **35.** $-2.02$; The per capita consumption of potatoes in the United States decreased by an average of 2.02 lb per yr from 2003 to 2008.

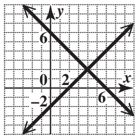
# 8 SYSTEMS OF LINEAR EQUATIONS

**Connections** **(page 507)** **1.** $\{(-1, 5)\}$ **2.** $\{(0.25, -0.5)\}$ **3.** $\{(1.5, -1.5)\}$

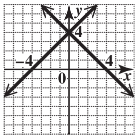
## Section 8.1 (pages 507–511)

**1.** A; The ordered-pair solution must be in quadrant II, and $(-4, -4)$ is in quadrant III. **3.** no **5.** yes **7.** yes **9.** no **11.** yes **13.** **(a)** B **(b)** C **(c)** D **(d)** A

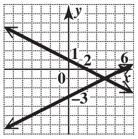
**15.** $\{(4, 2)\}$



**17.** $\{(0, 4)\}$



**19.** $\{(4, -1)\}$



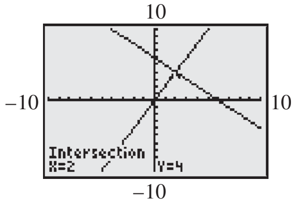
*In Exercises 21–31, we do not show the graphs.*

**21.** $\{(1, 3)\}$ **23.** $\{(0, 2)\}$ **25.** $\emptyset$ (inconsistent system) **27.** $\{(x, y) \mid 3x + y = 5\}$ (dependent equations) **29.** $\{(4, -3)\}$ **31.** $\emptyset$ (inconsistent system) **33.** It is difficult to read the exact coordinates of the solution. Thus, the solution cannot be checked. **35.** **(a)** neither **(b)** intersecting lines **(c)** one solution **37.** **(a)** dependent **(b)** one line **(c)** infinite number of solutions **39.** **(a)** neither **(b)** intersecting lines **(c)** one solution **41.** **(a)** inconsistent **(b)** parallel lines **(c)** no solution **43.** **(a)** 1980–2000 **(b)** 2001; about 750 newspapers **45.** **(a)** 1997–2002 **(b)** 2001 **(c)** 2002 **(d)** $(1998, 30)$ (The $y$-value is approximate.) **47.** 30 **49.** B **51.** A **53.** $y = -3x + 4$ **55.** $y = \frac{9}{2}x - 2$ **57.** $\{2\}$ **59.** $\left\{\frac{4}{5}\right\}$

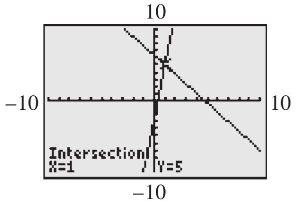
## Section 8.2 (pages 516–518)

**1.** The student must find the value of $y$ and write the solution as an ordered pair. The solution set is $\{(3, 0)\}$. **3.** $\{(3, 9)\}$ **5.** $\{(7, 3)\}$ **7.** $\{(-4, 8)\}$ **9.** $\{(3, -2)\}$ **11.** $\{(0, 5)\}$ **13.** $\{(x, y) \mid 3x - y = 5\}$ **15.** $\left\{\left(\frac{1}{4}, -\frac{1}{2}\right)\right\}$ **17.** $\emptyset$ **19.** $\{(x, y) \mid 2x - y = -12\}$ **21.** $\{(2, 6)\}$ **23.** $\{(2, -4)\}$ **25.** $\{(-2, 1)\}$ **27.** $\{(x, y) \mid x + 2y = 48\}$ **29.** $\{(10, 4)\}$ **31.** $\{(4, -9)\}$ **33.** To find the total cost, multiply the number of bicycles $(x)$ by the cost per bicycle (\$400), and add the fixed cost (\$5000). Thus, $y_1 = 400x + 5000$ gives this total cost (in dollars). **34.** $y_2 = 600x$ **35.** $y_1 = 400x + 5000$, $y_2 = 600x$; solution set: $\{(25, 15{,}000)\}$ **36.** 25; 15,000; 15,000

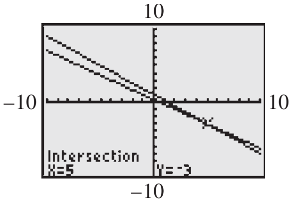
**37.** $\{(2, 4)\}$



**39.** $\{(1, 5)\}$



**41.** $\{(5, -3)\}$; The equations to input are $Y_1 = \frac{5 - 4X}{5}$ and $Y_2 = \frac{1 - 2X}{3}$.



**43.** $16x$ **45.** $10y$ **47.** $4x$ **49.** $-2$

## Section 8.3 (pages 522–524)

**1.** false; The solution set is $\emptyset$. **3.** $\{(4, 6)\}$ **5.** $\{(-1, -3)\}$ **7.** $\{(-2, 3)\}$ **9.** $\{(\frac{1}{2}, 4)\}$ **11.** $\{(3, -6)\}$ **13.** $\{(0, 4)\}$ **15.** $\{(0, 0)\}$ **17.** $\{(7, 4)\}$ **19.** $\{(0, 3)\}$ **21.** $\{(3, 0)\}$ **23.** $\{(x, y) \mid x - 3y = -4\}$ **25.** $\emptyset$ **27.** $\{(-3, 2)\}$ **29.** $\{(11, 15)\}$ **31.** $\{(13, -\frac{7}{5})\}$ **33.** $\{(6, -4)\}$ **35.** $\{(x, y) \mid x + 3y = 6\}$ **37.** $\emptyset$ **39.** $\{(-\frac{5}{7}, -\frac{2}{7})\}$ **41.** $\{(\frac{1}{8}, -\frac{5}{6})\}$ **43.** $5.39 = 2000a + b$ **44.** $7.18 = 2008a + b$ **45.** $2000a + b = 5.39$, $2008a + b = 7.18$; solution set: $\{(0.22375, -442.11)\}$ **46.** **(a)** $y = 0.22375x - 442.11$ **(b)** 6.96 (\$6.96); This is a bit more (\$0.08) than the actual figure. **47.** $8x - 12y + 4z = 20$ **49.** 4 **51.** $-3$

## Summary Exercises on Solving Systems of Linear Equations (page 525)

**1.** **(a)** Use substitution, since the second equation is solved for $y$. **(b)** Use elimination, since the coefficients of the $y$-terms are opposites. **(c)** Use elimination, since the equations are in standard form with no coefficients of 1 or $-1$. Solving by substitution would involve fractions. **2.** System B is easier to solve by substitution because the second equation is already solved for $y$. **3.** **(a)** $\{(1, 4)\}$ **(b)** $\{(1, 4)\}$ **(c)** Answers will vary. **4.** **(a)** $\{(-5, 2)\}$ **(b)** $\{(-5, 2)\}$ **(c)** Answers will vary. **5.** $\{(3, 12)\}$ **6.** $\{(-3, 2)\}$ **7.** $\{(\frac{1}{3}, \frac{1}{2})\}$ **8.** $\emptyset$ **9.** $\{(3, -2)\}$ **10.** $\{(-1, -11)\}$ **11.** $\{(x, y) \mid 2x - 3y = 5\}$ **12.** $\{(9, 4)\}$ **13.** $\{(\frac{45}{31}, \frac{4}{31})\}$ **14.** $\{(4, -5)\}$ **15.** $\emptyset$ **16.** $\{(-4, 6)\}$ **17.** $\{(-3, 2)\}$ **18.** $\{(\frac{22}{13}, -\frac{23}{13})\}$ **19.** $\{(0, 0)\}$ **20.** $\{(2, -3)\}$ **21.** $\{(24, -12)\}$ **22.** $\{(3, 2)\}$ **23.** $\{(10, -12)\}$ **24.** $\{(-4, 2)\}$ **25.** $\{(5, 3)\}$

## Section 8.4 (pages 531–533)

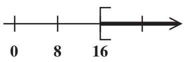
**1.** B **3.** $\{(3, 2, 1)\}$ **5.** $\{(1, 4, -3)\}$ **7.** $\{(0, 2, -5)\}$ **9.** $\{(1, 0, 3)\}$ **11.** $\{(1, \frac{3}{10}, \frac{2}{5})\}$ **13.** $\{(-\frac{7}{3}, \frac{22}{3}, 7)\}$ **15.** $\{(-12, 18, 0)\}$ **17.** $\{(0.8, -1.5, 2.3)\}$ **19.** $\{(4, 5, 3)\}$ **21.** $\{(2, 2, 2)\}$ **23.** $\{(\frac{8}{3}, \frac{2}{3}, 3)\}$ **25.** $\{(-1, 0, 0)\}$ **27.** $\{(-4, 6, 2)\}$ **29.** $\{(-3, 5, -6)\}$ **31.** $\emptyset$; inconsistent system **33.** $\{(x, y, z) \mid x - y + 4z = 8\}$; dependent equations **35.** $\{(3, 0, 2)\}$ **37.** $\{(x, y, z) \mid 2x + y - z = 6\}$; dependent equations **39.** $\{(0, 0, 0)\}$ **41.** $\emptyset$; inconsistent system **43.** $\{(2, 1, 5, 3)\}$ **45.** $\{(-2, 0, 1, 4)\}$ **47.** 100 in., 103 in., 120 in. **49.** $-4, 8, 12$

## Section 8.5 (pages 541–546)

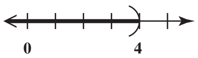
**1.** wins: 95; losses: 67 **3.** length: 78 ft; width: 36 ft **5.** AT&T: \$124.0 billion; Verizon: \$97.4 billion **7.** $x = 40$ and $y = 50$, so the angles measure 40° and 50°. **9.** NHL: \$288.23; NBA: \$291.93 **11.** Junior Roast Beef: \$2.09; Big Montana: \$4.39 **13.** **(a)** 6 oz **(b)** 15 oz **(c)** 24 oz **(d)** 30 oz **15.** \$2.29$x$ **17.** 6 gal of 25%; 14 gal of 35% **19.** pure acid: 6 L; 10% acid: 48 L **21.** nuts: 14 kg; cereal: 16 kg **23.** \$1000 at 2%; \$2000 at 4% **25.** **(a)** $(10 - x)$ mph **(b)** $(10 + x)$ mph **27.** train: 60 km per hr; plane: 160 km per hr **29.** boat: 21 mph; current: 3 mph **31.** \$0.75-per-lb candy: 5.22 lb; \$1.25-per-lb candy: 3.78 lb **33.** general admission: 76; with student ID: 108 **35.** 8 for a citron; 5 for a wood apple **37.** $x + y + z = 180$; angle measures: 70°, 30°, 80° **39.** first: 20°; second: 70°; third: 90° **41.** shortest: 12 cm; middle: 25 cm; longest: 33 cm **43.** gold: 23; silver: 21; bronze: 28 **45.** \$16 tickets: 1170; \$23 tickets: 985; \$40 tickets: 130 **47.** bookstore A: 140; bookstore B: 280; bookstore C: 380 **49.** first chemical: 50 kg; second chemical: 400 kg; third chemical: 300 kg **51.** wins: 53; losses: 19; overtime losses: 10 **53.** **(a)** 6 **(b)** $-\frac{1}{6}$ **55.** **(a)** $-\frac{7}{8}$ **(b)** $\frac{8}{7}$

## Section 8.6 (pages 552–553)

**1.** **(a)** 0, 5, $-3$ **(b)** 1, $-3$, 8 **(c)** yes; The number of rows is the same as the number of columns (three). **(d)** $\begin{bmatrix} 1 & 4 & 8 \\ 0 & 5 & -3 \\ -2 & 3 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 5 & -3 \\ 1 & 4 & 8 \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & 15 & 25 \\ 0 & 5 & -3 \\ 1 & 4 & 8 \end{bmatrix}$ **3.** $3 \times 2$ **5.** $4 \times 2$ **7.** $\{(4, 1)\}$ **9.** $\{(1, 1)\}$ **11.** $\{(-1, 4)\}$ **13.** $\emptyset$ **15.** $\{(x, y) \mid 2x + y = 4\}$ **17.** $\{(0, 0)\}$ **19.** $\{(4, 0, 1)\}$ **21.** $\{(-1, 23, 16)\}$ **23.** $\{(3, 2, -4)\}$ **25.** $\{(x, y, z) \mid x - 2y + z = 4\}$ **27.** $\emptyset$ **29.** $\{(1, 1)\}$ **31.** $\{(-1, 2, 1)\}$ **33.** $\{(1, 7, -4)\}$ **35.** $[16, \infty)$



**37.** $(-\infty, 4)$



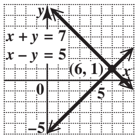
## Chapter 8 Review Exercises (pages 557–559)

**1.** yes **2.** no **3.** $\{(3, 1)\}$ **4.** $\{(0, -2)\}$ **5.** $\emptyset$ **6.** $\{(x, y) \mid x - 2y = 2\}$ **7.** It would be easiest to solve for $x$ in the second equation because its coefficient is $-1$. No fractions would be involved. **8.** The true statement $0 = 0$ is an indication that the system has an infinite number of solutions. Write the solution set using set-builder notation and the equation of the system that is in standard form with integer coefficients having greatest common factor 1. **9.** $\{(2, 1)\}$ **10.** $\{(3, 5)\}$ **11.** $\{(6, 4)\}$ **12.** $\{(x, y) \mid x + 3y = 6\}$ **13.** C **14.** **(a)** 2 **(b)** 9 **15.** $\{(7, 1)\}$ **16.** $\{(-4, 3)\}$ **17.** $\{(x, y) \mid 3x - 4y = 9\}$ **18.** $\emptyset$ **19.** $\{(-4, 1)\}$ **20.** $\{(x, y) \mid 2x - 3y = 0\}$ **21.** $\{(9, 2)\}$ **22.** $\{(8, 9)\}$ **23.** $\{(2, 1)\}$ **24.** $\{(-3, 2)\}$ **25.** $\{(1, -5, 3)\}$ **26.** $\{(1, 2, 3)\}$

**27.** $\emptyset$; inconsistent system **28.** length: 200 ft; width: 85 ft **29.** New York Yankees: \$72.97; Boston Red Sox: \$50.24 **30.** plane: 300 mph; wind: 20 mph **31.** \$2-per-lb nuts: 30 lb; \$1-per-lb candy: 70 lb **32.** 85°, 60°, 35° **33.** \$40,000 at 10%; \$100,000 at 6%; \$140,000 at 5% **34.** 5 L of 8%; 3 L of 20%; none of 10% **35.** Mantle: 54; Maris: 61; Berra: 22 **36.** $\{(3, -2)\}$ **37.** $\{(-1, 5)\}$ **38.** $\{(0, 0, -1)\}$ **39.** $\{(1, 2, -1)\}$ **40.** B; The second equation is already solved for *y*. **41.** $\{(12, 9)\}$ **42.** $\emptyset$ **43.** $\{(3, -1)\}$ **44.** $\{(5, 3)\}$ **45.** $\{(0, 4)\}$ **46.** $\left\{\left(\frac{82}{23}, -\frac{4}{23}\right)\right\}$ **47.** 20 L **48.** U.S.: 37; Germany: 30; Canada: 26

## Chapter 8 Test (pages 560–561)

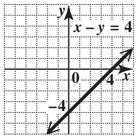
[8.1] **1.** $x = 8$, or 800 items; \$3000 **2.** about \$400 **3.** **(a)** no **(b)** no **(c)** yes **4.** $\{(6, 1)\}$



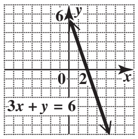
[8.2, 8.3] **5.** $\{(6, -4)\}$ **6.** $\left\{\left(-\frac{9}{4}, \frac{5}{4}\right)\right\}$ **7.** $\{(x, y) \mid 12x - 5y = 8\}$; dependent equations **8.** $\{(3, 3)\}$ **9.** $\{(0, -2)\}$ **10.** $\emptyset$; inconsistent system **11.** $\{(-15, 6)\}$ [8.4] **12.** $\left\{\left(-\frac{2}{3}, \frac{4}{5}, 0\right)\right\}$ **13.** $\{(3, -2, 1)\}$ [8.5] **14.** *Star Wars Episode IV: A New Hope:* \$461.0 million; *Indiana Jones and the Kingdom of the Crystal Skull:* \$317.0 million **15.** 45 mph, 75 mph **16.** 20% solution: 4 L; 50% solution: 8 L **17.** AC adaptor: \$8; rechargeable flashlight: \$15 **18.** Orange Pekoe: 60 oz; Irish Breakfast: 30 oz; Earl Grey: 10 oz [8.6] **19.** $\left\{\left(\frac{2}{5}, \frac{7}{5}\right)\right\}$ **20.** $\{(-1, 2, 3)\}$

## Chapters 1–8 Cumulative Review Exercises (pages 561–563)

[1.6] **1.** −1, 1, −2, 2, −4, 4, −5, 5, −8, 8, −10, 10, −20, 20, −40, 40 **2.** 46 [1.3] **3.** 1 [1.7] **4.** distributive property [2.3] **5.** $\left\{-\frac{13}{11}\right\}$ **6.** $\left\{\frac{9}{11}\right\}$ [2.5] **7.** $T = \frac{PV}{k}$ [2.8] **8.** $(-18, \infty)$ **9.** $\left(-\frac{11}{2}, \infty\right)$ [2.6] **10.** 2010; 1813; 62.8%; 57.2% [2.4] **11.** in favor: 68; against: 31 [2.5] **12.** 46°, 46°, 88° [3.2, 7.1] **13.**



**14.**

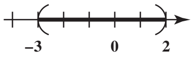


[3.3, 7.1] **15.** $-\frac{4}{3}$ **16.** $-\frac{1}{4}$ [3.4, 7.2] **17.** $y = \frac{1}{2}x + 3$ **18.** $y = 2x + 1$ **19.** **(a)** $x = 9$ **(b)** $y = -1$ [4.1] **20.** $\frac{m}{n}$ [4.4] **21.** $4y^2 - 7y - 6$ [4.5] **22.** $12f^2 + 5f - 3$ [4.6] **23.** $\frac{1}{16}x^2 + \frac{5}{2}x + 25$ [4.7] **24.** $x^2 + 4x - 7$ [5.2, 5.3] **25.** $(2x + 5)(x - 9)$ [5.4] **26.** $25(2t^2 + 1)(2t^2 - 1)$ **27.** $(2p + 5)(4p^2 - 10p + 25)$ [5.5] **28.** $\left\{-\frac{7}{3}, 1\right\}$ [6.1] **29.** $\frac{y + 4}{y - 4}$ [6.2] **30.** $\frac{a(a - b)}{2(a + b)}$ [6.4] **31.** 3 [6.6] **32.** $\{-4\}$ [7.4] **33.** **(a)** $\frac{5x - 8}{3}$, or $\frac{5}{3}x - \frac{8}{3}$ **(b)** −1 [7.5] **34.** **(a)** $2x^3 - 2x^2 + 6x - 4$ **(b)** $2x^3 - 4x^2 + 2x + 2$ **(c)** −14 **(d)** $x^4 + 2x^2 - 3$ [8.1–8.3, 8.6] **35.** $\{(3, -3)\}$ **36.** $\{(x, y) \mid x - 3y = 7\}$ [8.4] **37.** $\{(5, 3, 2)\}$ [8.5] **38.** Tickle Me Elmo: \$27.63; T.M.X.: \$40.00

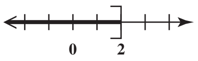
## 9 INEQUALITIES AND ABSOLUTE VALUE

## Section 9.1 (pages 571–573)

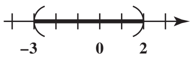
**1.** true **3.** false; The union is $(-\infty, 7) \cup (7, \infty)$. **5.** false; The intersection is $\emptyset$. **7.** $\{1, 3, 5\}$, or *B* **9.** $\{4\}$, or *D* **11.** $\emptyset$ **13.** $\{1, 2, 3, 4, 5, 6\}$, or *A* **15.**



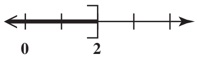
**17.**



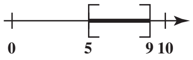
**19.** $(-3, 2)$



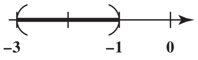
**21.** $(-\infty, 2]$



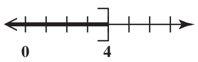
**23.** $\emptyset$ **25.** $[5, 9]$



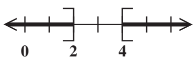
**27.** $(-3, -1)$



**29.** $(-\infty, 4]$

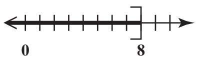


**31.**

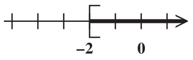


**33.**

**35.** $(-\infty, 8]$

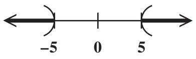


**37.** $[-2, \infty)$

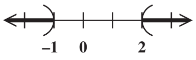


**39.** $(-\infty, \infty)$

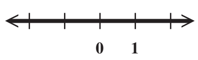
**41.** $(-\infty, -5) \cup (5, \infty)$



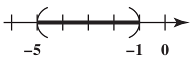
**43.** $(-\infty, -1) \cup (2, \infty)$



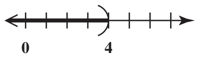
**45.** $(-\infty, \infty)$



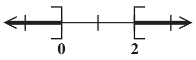
**47.** $[-4, -1]$ **49.** $[-9, -6]$ **51.** $(-\infty, 3)$ **53.** $[3, 9)$ **55.** intersection; $(-5, -1)$



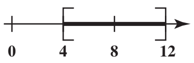
**57.** union; $(-\infty, 4)$



**59.** union; $(-\infty, 0] \cup [2, \infty)$



**61.** intersection; $[4, 12]$



**63.** {Tuition and fees} **65.** {Tuition and fees, Board rates, Dormitory charges} **67.** Maria, Joe **68.** none of them **69.** none of them **70.** Luigi, Than **71.** Maria, Joe **72.** all of them **73.** $(-3, 2)$ **75.** −21 **77.** false

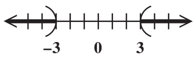
**Connections** **(page 579)** The filled carton may contain between 30.4 and 33.6 oz, inclusive.

## Section 9.2 (pages 580–582)

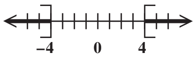
**1.** E; C; D; B; A **3.** **(a)** one **(b)** two **(c)** none **5.** $\{-12, 12\}$ **7.** $\{-5, 5\}$ **9.** $\{-6, 12\}$ **11.** $\{-5, 6\}$ **13.** $\left\{-3, \frac{11}{2}\right\}$

**15.** $\left\{-\frac{19}{2}, \frac{9}{2}\right\}$ **17.** $\{-10, -2\}$ **19.** $\left\{-\frac{32}{3}, 8\right\}$ **21.** $\{-75, 175\}$

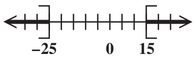
**23.** $(-\infty, -3) \cup (3, \infty)$



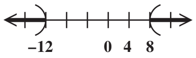
**25.** $(-\infty, -4] \cup [4, \infty)$



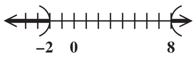
**27.** $(-\infty, -25] \cup [15, \infty)$



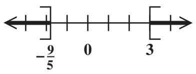
**29.** $(-\infty, -12) \cup (8, \infty)$



**31.** $(-\infty, -2) \cup (8, \infty)$

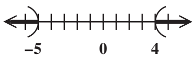


**33.** $\left(-\infty, -\frac{9}{5}\right] \cup [3, \infty)$



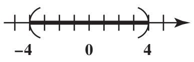
**35. (a)**

**(b)**

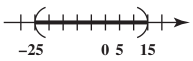


**37.** $[-3, 3]$

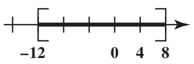
**39.** $(-4, 4)$



**41.** $(-25, 15)$

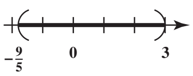


**43.** $[-12, 8]$

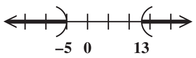


**45.** $[-2, 8]$

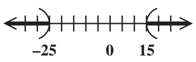
**47.** $\left(-\frac{9}{5}, 3\right)$



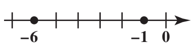
**49.** $(-\infty, -5) \cup (13, \infty)$



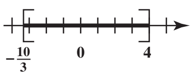
**51.** $(-\infty, -25) \cup (15, \infty)$



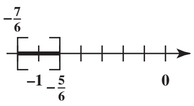
**53.** $\{-6, -1\}$



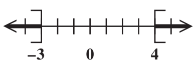
**55.** $\left[-\frac{10}{3}, 4\right]$



**57.** $\left[-\frac{7}{6}, -\frac{5}{6}\right]$



**59.** $(-\infty, -3] \cup [4, \infty)$



**61.** $\{-5, 1\}$

**63.** $\{3, 9\}$ **65.** $\{0, 20\}$ **67.** $\{-5, 5\}$ **69.** $\{-5, -3\}$

**71.** $(-\infty, -3) \cup (2, \infty)$ **73.** $[-10, 0]$ **75.** $\left\{-\frac{5}{3}, \frac{1}{3}\right\}$

**77.** $(-\infty, 20] \cup [30, \infty)$ **79.** $\{-1, 3\}$ **81.** $\left\{-3, \frac{5}{3}\right\}$ **83.** $\left\{-\frac{1}{3}, -\frac{1}{15}\right\}$

**85.** $\left\{-\frac{5}{4}\right\}$ **87.** $(-\infty, \infty)$ **89.** $\emptyset$ **91.** $\left\{-\frac{1}{4}\right\}$ **93.** $\emptyset$ **95.** $(-\infty, \infty)$

**97.** $\left\{-\frac{3}{7}\right\}$ **99.** $\left\{\frac{2}{5}\right\}$ **101.** $(-\infty, \infty)$ **103.** $\emptyset$

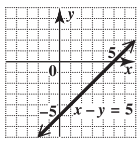
**105.** $|x - 1000| \le 100$; $900 \le x \le 1100$ **107.** 810.5 ft

**108.** Bank of America Center, Texaco Heritage Plaza **109.** Williams Tower, Bank of America Center, Texaco Heritage Plaza, Enterprise Plaza, Centerpoint Energy Plaza, Continental Center I, Fulbright Tower

**110.** $|x - 810.5| \ge 95$; $x \ge 905.5$ or $x \le 715.5$

**(b)** JPMorgan Chase Tower, Wells Fargo Plaza, One Shell Plaza; It makes sense because it includes all buildings *not* listed in the answer to **Exercise 109.**

**111.**



**113.** no

## Summary Exercises on Solving Linear and Absolute Value Equations and Inequalities (page 583)

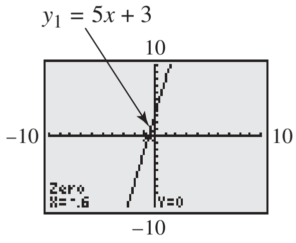
**1.** $\{12\}$ **2.** $\{-5, 7\}$ **3.** $\{7\}$ **4.** $\left\{-\frac{2}{5}\right\}$ **5.** $\emptyset$ **6.** $(-\infty, -1]$ **7.** $\left[-\frac{2}{3}, \infty\right)$ **8.** $\{-1\}$ **9.** $\{-3\}$ **10.** $\left\{1, \frac{11}{3}\right\}$ **11.** $(-\infty, 5]$ **12.** $(-\infty, \infty)$ **13.** $\{2\}$ **14.** $(-\infty, -8] \cup [8, \infty)$ **15.** $\emptyset$ **16.** $(-\infty, \infty)$ **17.** $(-5.5, 5.5)$ **18.** $\left\{\frac{13}{3}\right\}$ **19.** $\left\{-\frac{96}{5}\right\}$ **20.** $(-\infty, 32]$ **21.** $(-\infty, -24)$ **22.** $\left\{\frac{3}{8}\right\}$ **23.** $\left\{\frac{7}{2}\right\}$ **24.** $(-6, 8)$ **25.** {all real numbers} **26.** $(-\infty, 5)$ **27.** $(-\infty, -4) \cup (7, \infty)$ **28.** $\{24\}$ **29.** $\left\{-\frac{1}{5}\right\}$ **30.** $\left(-\infty, -\frac{5}{2}\right]$ **31.** $\left[-\frac{1}{3}, 3\right]$ **32.** $[1, 7]$ **33.** $\left\{-\frac{1}{6}, 2\right\}$ **34.** $\{-3\}$ **35.** $(-\infty, -1] \cup \left[\frac{5}{3}, \infty\right)$ **36.** $\left[\frac{3}{4}, \frac{15}{8}\right]$ **37.** $\left\{-\frac{5}{2}\right\}$ **38.** $\{60\}$ **39.** $\left[-\frac{9}{2}, \frac{15}{2}\right]$ **40.** $(1, 9)$ **41.** $(-\infty, \infty)$ **42.** $\left\{\frac{1}{3}, 9\right\}$ **43.** {all real numbers} **44.** $\left\{-\frac{10}{9}\right\}$ **45.** $\{-2\}$ **46.** $\emptyset$ **47.** $(-\infty, -1) \cup (2, \infty)$ **48.** $[-3, -2]$

Connections **(page 589)** We include a calculator graph and supporting explanation only with the answer to Problem 1.

**1. (a)** $\{-0.6\}$; The graph of $y_1 = 5x + 3$ has $x$-intercept $(-0.6, 0)$.

**(b)** $(-0.6, \infty)$; The graph of $y_1$ lies *above* the $x$-axis for values of *x greater than* $-0.6$.

**(c)** $(-\infty, -0.6)$; The graph of $y_1$ lies *below* the $x$-axis for values of *x less than* $-0.6$.

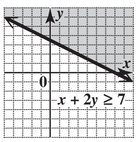


**2. (a)** $\{-0.5\}$ **(b)** $(-0.5, \infty)$ **(c)** $(-\infty, -0.5)$ **3. (a)** $\{-1.2\}$ **(b)** $(-\infty, -1.2]$ **(c)** $[-1.2, \infty)$ **4. (a)** $\{-3\}$ **(b)** $(-\infty, -3]$ **(c)** $[-3, \infty)$

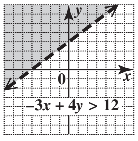
## Section 9.3 (pages 589–592)

**1.** solid; below **3.** dashed; above

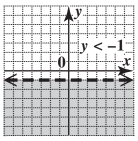
**5.**



**7.**

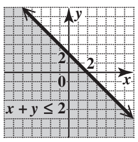


**9.**

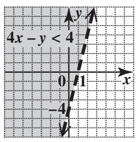


**11.** Use a dashed line if the symbol is < or >. Use a solid line if the symbol is ≤ or ≥.

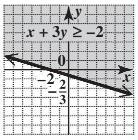
**13.**



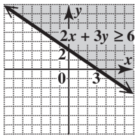
**15.**



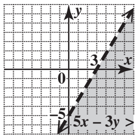
**17.**



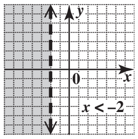
**19.**



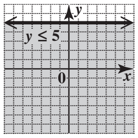
**21.**



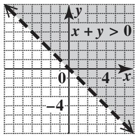
**23.**



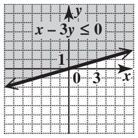
**25.**



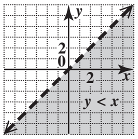
**27.**



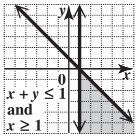
**29.**



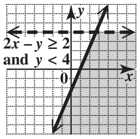
**31.**



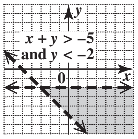
**33.**



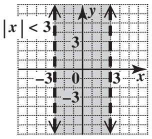
**35.**



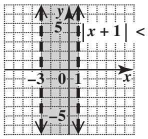
**37.**



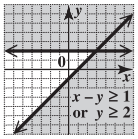
**39.** $-3 < x < 3$



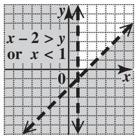
**41.** $-2 < x + 1 < 2$



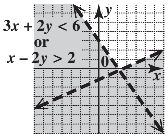
**43.**



**45.**

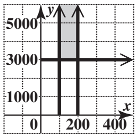


**47.**



**49.** C **51.** A **53.** **(a)** $\{-4\}$ **(b)** $(-\infty, -4)$ **(c)** $(-4, \infty)$
**55.** **(a)** $\{3.5\}$ **(b)** $(3.5, \infty)$ **(c)** $(-\infty, 3.5)$
**57.** $x \le 200$, $x \ge 100$, $y \ge 3000$

**58.**



**59.** $C = 50x + 100y$

**60.** Some examples are $(100, 5000)$, $(150, 3000)$, and $(150, 5000)$. The corner points are $(100, 3000)$ and $(200, 3000)$.

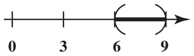
**61.** The least value occurs when $x = 100$ and $y = 3000$.

**62.** The company should use 100 workers and manufacture 3000 units to achieve the least possible cost. **63.** 64 **65.** −144

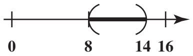
## Chapter 9 Review Exercises (pages 594–595)

**1.** $\{a, c\}$ **2.** $\{a\}$ **3.** $\{a, c, e, f, g\}$ **4.** $\{a, b, c, d, e, f, g\}$

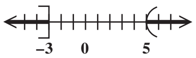
**5.** $(6, 9)$



**6.** $(8, 14)$



**7.** $(-\infty, -3] \cup (5, \infty)$



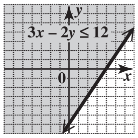
**8.** $(-\infty, \infty)$

**9.** $\emptyset$

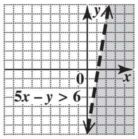
**10.** $(-\infty, -2] \cup [7, \infty)$

**11.** $(-3, 4)$ **12.** $(-\infty, 2)$ **13.** $(4, \infty)$ **14.** $(1, \infty)$ **15.** $\{-7, 7\}$
**16.** $\{-11, 7\}$ **17.** $\left\{-\frac{1}{3}, 5\right\}$ **18.** $\emptyset$ **19.** $\{0, 7\}$ **20.** $\left\{-\frac{3}{2}, \frac{1}{2}\right\}$
**21.** $\left\{-\frac{3}{4}, \frac{1}{2}\right\}$ **22.** $\left\{-\frac{1}{2}\right\}$ **23.** $(-14, 14)$ **24.** $[-1, 13]$
**25.** $[-3, -2]$ **26.** $(-\infty, \infty)$ **27.** $\left(-\infty, -\frac{8}{5}\right) \cup (2, \infty)$ **28.** $\emptyset$
**29.** $(-\infty, \infty)$ **30.** $\left\{\frac{3}{11}\right\}$

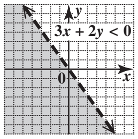
**31.**



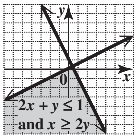
**32.**



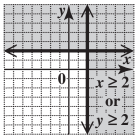
**33.**



**34.**

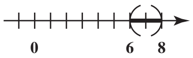


**35.**

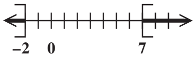


**36.** D **37.** $[-2, 3)$
**38.** $(-\infty, \infty)$ **39.** $\left\{-\frac{7}{3}, 1\right\}$
**40.** $[-16, 10]$
**41.** $\left(-\infty, -\frac{13}{5}\right) \cup (3, \infty)$

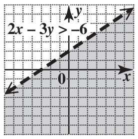
**42.** $(-\infty, \infty)$ **43.** $\left\{-4, -\frac{2}{3}\right\}$ **44.** $[-4, -2]$ **45.** $\left\{1, \frac{11}{3}\right\}$
**46.** **(a)** $\emptyset$ **(b)** $\emptyset$ **(c)** $(-\infty, \infty)$ **47.** $(6, 8)$



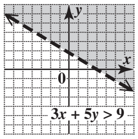
**48.** $(-\infty, -2] \cup [7, \infty)$



**49.**



**50.**



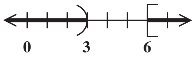
**51.** **(a)** {Illinois} **(b)** {Illinois, Maine, North Carolina, Oregon, Utah}
**(c)** $\emptyset$ **52.** **(a)** $\left(-\infty, -\frac{11}{3}\right] \cup [1, \infty)$ **(b)** $\left[-\frac{11}{3}, 1\right]$

## Chapter 9 Test (page 596)

[9.1] **1.** $\emptyset$ **2.** Managerial/Professional, Technical/Sales/Administrative Support, Service, Operators/Fabricators/Laborers
**3.** $\{1, 5\}$ **4.** $\{1, 2, 5, 7, 9, 12\}$

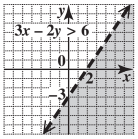
**5.** $[2, 9)$

**6.** $(-\infty, 3) \cup [6, \infty)$

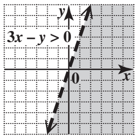


[9.2] **7.** $\left\{-1, \frac{5}{2}\right\}$ **8.** $\left(-\infty, -\frac{7}{6}\right) \cup \left(\frac{17}{6}, \infty\right)$
**9.** $\emptyset$ **10.** $\left\{-\frac{5}{7}, \frac{11}{3}\right\}$ **11.** $\left(\frac{1}{3}, \frac{7}{3}\right)$ **12.** $(-\infty, \infty)$ **13.** **(a)** $\emptyset$
**(b)** $(-\infty, \infty)$ **(c)** $\emptyset$

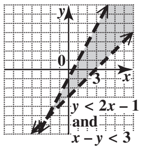
[9.3] **14.**



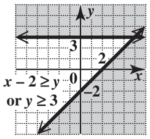
**15.**



**16.**



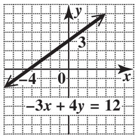
**17.**



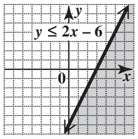
## Chapters 1–9 Cumulative Review Exercises (pages 596–598)

[1.4] **1.** **(a)** A, B, C, D, F **(b)** B, C, D, F **(c)** D, F **(d)** C, D, F
**(e)** E, F **(f)** D, F [1.2] **2.** 32 [1.4, 1.5] **3.** 0 [2.3] **4.** $\{-65\}$
**5.** {all real numbers} [2.5] **6.** $t = \dfrac{A - p}{pr}$ [2.8] **7.** $(-\infty, 6)$
[2.6] **8.** 32%; 390; 270; 10% [2.4] **9.** 15°, 35°, 130°

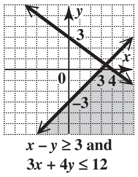
[3.3, 7.1] **10.** $-\frac{4}{3}$ **11.** 0 [3.4, 7.2] **12.** **(a)** $y = -4x + 15$ **(b)** $4x + y = 15$ **13.** **(a)** $y = 4x$ **(b)** $4x - y = 0$ [3.2, 7.1] **14.**



[4.1, 4.2] **15.** $\frac{8m^9n^3}{p^6}$ **16.** $\frac{y^7}{x^{13}z^2}$ [4.4] **17.** $2x^2 - 5x + 10$ [4.5] **18.** $x^3 + 8y^3$ **19.** $15x^2 + 7xy - 2y^2$ [4.7] **20.** $4xy^4 - 2y + \frac{1}{x^2y}$ [5.2, 5.3] **21.** $(m + 8)(m + 4)$ [5.4] **22.** $(5t^2 + 6)(5t^2 - 6)$ **23.** $(9z + 4)^2$ [5.5] **24.** $\{-2, -1\}$ [6.1] **25.** $x \neq -7, 2$ [6.2] **26.** $\frac{x + 1}{x}$ **27.** $(t + 5)(t + 3)$, or $t^2 + 8t + 15$ [6.4] **28.** $\frac{-2x - 14}{(x + 3)(x - 1)}$ [6.5] **29.** $-21$ [6.6] **30.** $\{19\}$ [7.1, 7.2] **31.** **(a)** $-12{,}272.8$ thousand lb per yr; The number of pounds of shrimp caught decreased an average of 12,272.8 thousand lb per yr. **(b)** $y = -12{,}272.8x + 322{,}486$ **(c)** 285,668 thousand lb [7.3] **32.** domain: $\{-4, -1, 2, 5\}$; range: $\{-2, 0, 2\}$; function [7.4] **33.** $-9$ [8.1–8.3, 8.6] **34.** $\{(3, 2)\}$ **35.** $\emptyset$ [8.4, 8.6] **36.** $\{(1, 0, -1)\}$ [8.5] **37.** length: 42 ft; width: 30 ft **38.** 15% solution: 6 L; 30% solution: 3 L [9.1] **39.** $(-4, 4)$ **40.** $(-\infty, 0] \cup (2, \infty)$ [9.2] **41.** $\left\{-\frac{1}{3}, 1\right\}$ **42.** $\left(-\infty, -\frac{8}{3}\right] \cup [2, \infty)$ [9.3] **43.**



**44.**



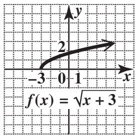
# 10 ROOTS, RADICALS, AND ROOT FUNCTIONS

## Section 10.1 (pages 607–611)

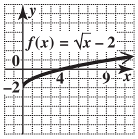
**1.** true **3.** false; Zero has only one square root. **5.** true **7.** $-3, 3$ **9.** $-8, 8$ **11.** $-13, 13$ **13.** $-\frac{5}{14}, \frac{5}{14}$ **15.** $-30, 30$ **17.** 1 **19.** 7 **21.** $-16$ **23.** $-\frac{12}{11}$ **25.** 0.8 **27.** It is not a real number. **29.** It is not a real number. **31.** 19 **33.** 19 **35.** $\frac{2}{3}$ **37.** $3x^2 + 4$ **39.** $a$ must be positive. **41.** $a$ must be negative. **43.** rational; 5 **45.** irrational; 5.385 **47.** rational; $-8$ **49.** irrational; $-17.321$ **51.** It is not a real number. **53.** irrational; 34.641 **55.** 9 and 10 **57.** 7 and 8 **59.** $-7$ and $-6$ **61.** 4 and 5 **63.** 1; 8; 27; 64; 125; 216; 343; 512; 729; 1000 **65.** **(a)** E **(b)** F **(c)** D **(d)** B **(e)** A **(f)** C **67.** $-9$ **69.** 6 **71.** $-4$ **73.** $-8$ **75.** 6 **77.** $-2$ **79.** It is not a real number. **81.** 2 **83.** It is not a real number. **85.** $\frac{8}{9}$ **87.** $\frac{4}{3}$ **89.** $-\frac{1}{2}$ **91.** 3 **93.** 0.5 **95.** $-0.7$ **97.** 0.1

*In Exercises 99–105, we give the domain and then the range.*

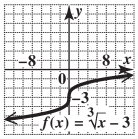
**99.** $[-3, \infty)$; $[0, \infty)$



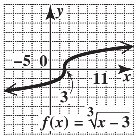
**101.** $[0, \infty)$; $[-2, \infty)$



**103.** $(-\infty, \infty)$; $(-\infty, \infty)$



**105.** $(-\infty, \infty)$; $(-\infty, \infty)$



**107.** 12 **109.** 10 **111.** 2 **113.** $-9$ **115.** $-5$ **117.** $|x|$ **119.** $|z|$ **121.** $x$ **123.** $x^5$ **125.** $|x|^5$ (or $|x^5|$) **127.** C **129.** 97.381 **131.** 16.863 **133.** $-9.055$ **135.** 7.507 **137.** 3.162 **139.** 1.885 **141.** A **143.** 1,183,000 cycles per sec **145.** 10 mi **147.** 392,000 mi$^2$ **149.** **(a)** 1.732 amps **(b)** 2.236 amps **151.** $x^1$, or $x$ **153.** $\frac{3^3}{2^3}$, or $\frac{27}{8}$

## Section 10.2 (pages 617–619)

**1.** C **3.** A **5.** H **7.** B **9.** D **11.** 13 **13.** 9 **15.** 2 **17.** $\frac{8}{9}$ **19.** $-3$ **21.** It is not a real number. **23.** 1000 **25.** 27 **27.** $-1024$ **29.** 16 **31.** $\frac{1}{8}$ **33.** $\frac{1}{512}$ **35.** $\frac{9}{25}$ **37.** $\frac{27}{8}$ **39.** $\sqrt{10}$ **41.** $\left(\sqrt[4]{8}\right)^3$ **43.** $\left(\sqrt[8]{9q}\right)^5 - \left(\sqrt[3]{2x}\right)^2$ **45.** $\frac{1}{\left(\sqrt{2m}\right)^3}$ **47.** $\left(\sqrt[3]{2y + x}\right)^2$ **49.** $\frac{1}{\left(\sqrt[3]{3m^4 + 2k^2}\right)^2}$ **51.** 64 **53.** 64 **55.** $x^{10}$ **57.** $\sqrt[6]{x^5}$ **59.** $\sqrt[15]{t^8}$ **61.** 9 **63.** 4 **65.** $y$ **67.** $x^{5/12}$ **69.** $k^{2/3}$ **71.** $x^3y^8$ **73.** $\frac{1}{x^{10/3}}$ **75.** $\frac{1}{m^{1/4}n^{3/4}}$ **77.** $p^2$ **79.** $\frac{c^{11/3}}{b^{11/4}}$ **81.** $\frac{q^{5/3}}{9p^{7/2}}$ **83.** $p + 2p^2$ **85.** $k^{7/4} - k^{3/4}$ **87.** $6 + 18a$ **89.** $-5x^2 + 5x$ **91.** $x^{17/20}$ **93.** $\frac{1}{x^{3/2}}$ **95.** $y^{5/6}z^{1/3}$ **97.** $m^{1/12}$ **99.** $x^{1/8}$ **101.** $x^{1/24}$ **103.** $\sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5$; $a + b = 3 + 4 = 7$; $5 \neq 7$ **105.** 4.5 hr **107.** 19.0°; The table gives 19°. **109.** 4.2°; The table gives 4°. **111.** 30; 30; They are the same.

## Section 10.3 (pages 626–629)

**1.** $\sqrt{9}$, or 3 **3.** $\sqrt{36}$, or 6 **5.** $\sqrt{30}$ **7.** $\sqrt{14x}$ **9.** $\sqrt{42pqr}$ **11.** $\sqrt[3]{10}$ **13.** $\sqrt[3]{14xy}$ **15.** $\sqrt[4]{33}$ **17.** $\sqrt[4]{6x^3}$ **19.** This expression cannot be simplified by the product rule. **21.** $\frac{8}{11}$ **23.** $\frac{\sqrt{3}}{5}$ **25.** $\frac{\sqrt{x}}{5}$ **27.** $\frac{p^3}{9}$ **29.** $-\frac{3}{4}$ **31.** $\frac{\sqrt[3]{r^2}}{2}$ **33.** $-\frac{3}{x}$ **35.** $\frac{1}{x^3}$ **37.** $2\sqrt{3}$ **39.** $12\sqrt{2}$ **41.** $-4\sqrt{2}$ **43.** $-2\sqrt{7}$ **45.** This radical cannot be simplified further. **47.** $4\sqrt[3]{2}$ **49.** $-2\sqrt[3]{2}$ **51.** $2\sqrt[3]{5}$ **53.** $-4\sqrt[4]{2}$ **55.** $2\sqrt[5]{2}$ **57.** $-3\sqrt[5]{2}$ **59.** $2\sqrt[6]{2}$ **61.** His reasoning was incorrect. Here, 8 is a term, not a factor. **63.** $6k\sqrt{2}$ **65.** $12xy^4\sqrt{xy}$ **67.** $11x^3$ **69.** $-3t^4$ **71.** $-10m^4z^2$ **73.** $5a^2b^3c^4$ **75.** $\frac{1}{2}r^2t^5$ **77.** $5x\sqrt{2x}$ **79.** $-10r^5\sqrt{5r}$ **81.** $x^3y^4\sqrt{13x}$ **83.** $2z^2w^3$ **85.** $-2zt^2\sqrt[3]{2z^2t}$ **87.** $3x^3y^4$ **89.** $-3r^3s^2\sqrt[4]{2r^3s^2}$ **91.** $\frac{y^5\sqrt{y}}{6}$ **93.** $\frac{x^5\sqrt[3]{x}}{3}$ **95.** $4\sqrt{3}$ **97.** $\sqrt{5}$ **99.** $x^2\sqrt{x}$ **101.** $\sqrt[6]{432}$ **103.** $\sqrt[12]{6912}$ **105.** $\sqrt[6]{x^5}$ **107.** 5 **109.** $8\sqrt{2}$ **111.** $2\sqrt{14}$ **113.** 13 **115.** $9\sqrt{2}$ **117.** $\sqrt{17}$ **119.** 5 **121.** $6\sqrt{2}$ **123.** $\sqrt{5y^2 - 2xy + x^2}$

**125.** $2\sqrt{106} + 4\sqrt{2}$ **127.** 15.3 mi **129.** 27.0 in. **131.** 581
**133.** $\sqrt{9} + \sqrt{9} = 3 + 3 = 6$, and $\sqrt{4} = 2$; $6 \neq 2$, so the statement is false. **135.** MSX-77: 18.4 in.; MSX-83: 17.0 in.; MSX-60: 14.1 in.
**137.** $22x^4 - 10x^3$ **139.** $8q^2 - 3q$

## Section 10.4 (pages 632–633)

**1.** $-4$ **3.** $7\sqrt{3}$ **5.** $14\sqrt[3]{2}$ **7.** $5\sqrt[4]{2}$ **9.** $24\sqrt{2}$ **11.** The expression cannot be simplified further. **13.** $20\sqrt{5}$ **15.** $4\sqrt{2x}$
**17.** $-11m\sqrt{2}$ **19.** $7\sqrt[3]{2}$ **21.** $2\sqrt[3]{x}$ **23.** $-7\sqrt[3]{x^2y}$ **25.** $-x\sqrt[3]{xy^2}$
**27.** $19\sqrt[4]{2}$ **29.** $x\sqrt[4]{xy}$ **31.** $9\sqrt[4]{2a^3}$ **33.** $(4 + 3xy)\sqrt[3]{xy^2}$
**35.** $4t\sqrt[3]{3st} - 3s\sqrt{3st}$ **37.** $4x\sqrt[3]{x} + 6x\sqrt[4]{x}$ **39.** $2\sqrt{2} - 2$
**41.** $\frac{5\sqrt{5}}{6}$ **43.** $\frac{7\sqrt{2}}{6}$ **45.** $\frac{5\sqrt{2}}{3}$ **47.** $5\sqrt{2} + 4$ **49.** $\frac{5 + 3x}{x^4}$
**51.** $\frac{m\sqrt[3]{m^2}}{2}$ **53.** $\frac{3x\sqrt[3]{2} - 4\sqrt[3]{5}}{x^3}$ **55.** B
**57.** 15; Each radical expression simplifies to a whole number.
**59.** A; 42 m **61.** $\left(12\sqrt{5} + 5\sqrt{3}\right)$ in. **63.** $\left(24\sqrt{2} + 12\sqrt{3}\right)$ in.
**65.** $10x^3y^4 - 20x^2y$ **67.** $a^4 - b^2$ **69.** $64x^9 + 144x^6 + 108x^3 + 27$
**71.** $\frac{4x - 5}{3x}$

**Connections** **(page 639)** **1.** $\frac{319}{6\left(8\sqrt{5} + 1\right)}$ **2.** $\frac{9a - b}{b\left(3\sqrt{a} - \sqrt{b}\right)}$
**3.** $\frac{9a - b}{\left(\sqrt{b} - \sqrt{a}\right)\left(3\sqrt{a} - \sqrt{b}\right)}$ **4.** $\frac{\left(3\sqrt{a} + \sqrt{b}\right)\left(\sqrt{b} + \sqrt{a}\right)}{b - a}$;
Instead of multiplying by the conjugate of the numerator, we use the conjugate of the denominator.

## Section 10.5 (pages 640–642)

**1.** E **3.** A **5.** D **7.** $3\sqrt{6} + 2\sqrt{3}$ **9.** $20\sqrt{2}$ **11.** $-2$
**13.** $-1$ **15.** 6 **17.** $\sqrt{6} - \sqrt{2} + \sqrt{3} - 1$
**19.** $\sqrt{22} + \sqrt{55} - \sqrt{14} - \sqrt{35}$ **21.** $8 - \sqrt{15}$ **23.** $9 + 4\sqrt{5}$
**25.** $26 - 2\sqrt{105}$ **27.** $4 - \sqrt[3]{36}$ **29.** 10
**31.** $6x + 3\sqrt{x} - 2\sqrt{5x} - \sqrt{5}$ **33.** $9r - s$
**35.** $4\sqrt[3]{4y^2} - 19\sqrt[3]{2y} - 5$ **37.** $3x - 4$ **39.** $4x - y$ **41.** $2\sqrt{6} - 1$
**43.** $\sqrt{7}$ **45.** $5\sqrt{3}$ **47.** $\frac{\sqrt{6}}{2}$ **49.** $\frac{9\sqrt{15}}{5}$ **51.** $-\frac{7\sqrt{3}}{12}$ **53.** $\frac{\sqrt{14}}{2}$
**55.** $-\frac{\sqrt{14}}{10}$ **57.** $\frac{2\sqrt{6x}}{x}$ **59.** $\frac{-8\sqrt{3k}}{k}$ **61.** $\frac{-5m^2\sqrt{6mn}}{n^2}$
**63.** $\frac{12x^3\sqrt{2xy}}{y^5}$ **65.** $\frac{5\sqrt{2my}}{y^2}$ **67.** $-\frac{4k\sqrt{3z}}{z}$ **69.** $\frac{\sqrt[3]{18}}{3}$ **71.** $\frac{\sqrt[3]{12}}{3}$
**73.** $\frac{\sqrt[3]{18}}{4}$ **75.** $-\frac{\sqrt[3]{2pr}}{r}$ **77.** $\frac{x^2\sqrt[3]{y^2}}{y}$ **79.** $\frac{2\sqrt[4]{x^3}}{x}$ **81.** $\frac{\sqrt[4]{2yz^3}}{z}$
**83.** $\frac{3\left(4 - \sqrt{5}\right)}{11}$ **85.** $\frac{6\sqrt{2} + 4}{7}$ **87.** $\frac{2\left(3\sqrt{5} - 2\sqrt{3}\right)}{33}$
**89.** $2\sqrt{3} + \sqrt{10} - 3\sqrt{2} - \sqrt{15}$ **91.** $\sqrt{m} - 2$
**93.** $\frac{4\left(\sqrt{x} + 2\sqrt{y}\right)}{x - 4y}$ **95.** $\frac{x - 2\sqrt{xy} + y}{x - y}$ **97.** $\frac{5\sqrt{k}\left(2\sqrt{k} - \sqrt{q}\right)}{4k - q}$
**99.** $3 - 2\sqrt{6}$ **101.** $1 - \sqrt{5}$ **103.** $\frac{4 - 2\sqrt{2}}{3}$ **105.** $\frac{6 + 2\sqrt{6p}}{3}$
**107.** $\frac{3\sqrt{x + y}}{x + y}$ **109.** $\frac{p\sqrt{p + 2}}{p + 2}$ **111.** Each expression is approximately equal to 0.2588190451. **113.** $\frac{33}{8\left(6 + \sqrt{3}\right)}$ **115.** $\frac{4x - y}{3x\left(2\sqrt{x} + \sqrt{y}\right)}$
**117.** $\left\{\frac{3}{8}\right\}$ **119.** $\left\{-\frac{1}{3}, \frac{3}{2}\right\}$

## Summary Exercises on Operations with Radicals and Rational Exponents (pages 642–643)

**1.** $-6\sqrt{10}$ **2.** $7 - \sqrt{14}$ **3.** $2 + \sqrt{6} - 2\sqrt{3} - 3\sqrt{2}$ **4.** $4\sqrt{2}$
**5.** $73 + 12\sqrt{35}$ **6.** $\frac{-\sqrt{6}}{2}$ **7.** $4\left(\sqrt{7} - \sqrt{5}\right)$ **8.** $-3 + 2\sqrt{2}$
**9.** $-44$ **10.** $\frac{\sqrt{x} + \sqrt{5}}{x - 5}$ **11.** $2abc^3\sqrt[3]{b^2}$ **12.** $5\sqrt[3]{3}$
**13.** $3\left(\sqrt{5} - 2\right)$ **14.** $\frac{\sqrt{15x}}{5x}$ **15.** $\frac{8}{5}$ **16.** $\frac{\sqrt{2}}{8}$ **17.** $-\sqrt[3]{100}$
**18.** $11 + 2\sqrt{30}$ **19.** $-3\sqrt{3x}$ **20.** $52 - 30\sqrt{3}$ **21.** $\frac{\sqrt[3]{117}}{9}$
**22.** $3\sqrt{2} + \sqrt{15} + \sqrt{42} + \sqrt{35}$ **23.** $2\sqrt[4]{27}$ **24.** $\frac{1 + \sqrt[3]{3} + \sqrt[3]{9}}{-2}$
**25.** $\frac{x\sqrt[3]{x^2}}{y}$ **26.** $-4\sqrt{3} - 3$ **27.** $xy^{6/5}$ **28.** $x^{10}y$ **29.** $\frac{1}{25x^2}$
**30.** $7 + 4 \cdot 3^{1/2}$, or $7 + 4\sqrt{3}$ **31.** $3\sqrt[3]{2x^2}$ **32.** $-2$
**33.** **(a)** 8 **(b)** $\{-8, 8\}$ **34.** **(a)** 10 **(b)** $\{-10, 10\}$
**35.** **(a)** $\{-4, 4\}$ **(b)** $-4$ **36.** **(a)** $\{-5, 5\}$ **(b)** $-5$
**37.** **(a)** $-\frac{9}{11}$ **(b)** $\left\{-\frac{9}{11}, \frac{9}{11}\right\}$ **38.** **(a)** $-\frac{7}{10}$ **(b)** $\left\{-\frac{7}{10}, \frac{7}{10}\right\}$
**39.** **(a)** $\{-0.2, 0.2\}$ **(b)** 0.2 **40.** **(a)** $\{-0.3, 0.3\}$ **(b)** 0.3

## Section 10.6 (pages 648–650)

**1.** **(a)** yes **(b)** no **3.** **(a)** yes **(b)** no **5.** No. There is no solution. The radical expression, which is positive, cannot equal a negative number.
**7.** $\{11\}$ **9.** $\left\{\frac{1}{3}\right\}$ **11.** $\emptyset$ **13.** $\{5\}$ **15.** $\{18\}$ **17.** $\{5\}$ **19.** $\{4\}$
**21.** $\{17\}$ **23.** $\{5\}$ **25.** $\emptyset$ **27.** $\{0\}$ **29.** $\{0\}$ **31.** $\emptyset$ **33.** $\{1\}$
**35.** It is incorrect to just square each term. The right side should be $(8 - x)^2 = 64 - 16x + x^2$. The correct first step is $3x + 4 = 64 - 16x + x^2$, and the solution set is $\{4\}$. **37.** $\{1\}$ **39.** $\{-1\}$
**41.** $\{14\}$ **43.** $\{8\}$ **45.** $\{0\}$ **47.** $\emptyset$ **49.** $\{7\}$ **51.** $\{7\}$
**53.** $\{4, 20\}$ **55.** $\emptyset$ **57.** $\left\{\frac{5}{4}\right\}$ **59.** $\{9, 17\}$ **61.** $\left\{\frac{1}{4}, 1\right\}$
**63.** $L = CZ^2$ **65.** $K = \frac{V^2m}{2}$ **67.** $M = \frac{r^2F}{m}$ **69.** $r = \frac{a}{4\pi^2N^2}$
**71.** $1 + x$ **73.** $2x^2 + x - 15$ **75.** $\frac{-7\left(5 + \sqrt{2}\right)}{23}$

## Section 10.7 (pages 655–657)

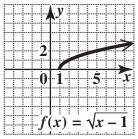
**1.** $i$ **3.** $-1$ **5.** $-i$ **7.** $13i$ **9.** $-12i$ **11.** $i\sqrt{5}$ **13.** $4i\sqrt{3}$
**15.** $-\sqrt{105}$ **17.** $-10$ **19.** $i\sqrt{33}$ **21.** $\sqrt{3}$ **23.** $5i$ **25.** $-2$
**27.** Any real number $a$ can be written as $a + 0i$, a complex number with imaginary part 0. **29.** $-1 + 7i$ **31.** 0 **33.** $7 + 3i$ **35.** $-2$

**37.** $1 + 13i$ **39.** $6 + 6i$ **41.** $4 + 2i$ **43.** $-81$ **45.** $-16$ **47.** $-10 - 30i$ **49.** $10 - 5i$ **51.** $-9 + 40i$ **53.** $-16 + 30i$ **55.** 153 **57.** 97 **59.** 4 **61.** $a - bi$ **63.** $1 + i$ **65.** $2 + 2i$ **67.** $-1 + 2i$ **69.** $-\frac{5}{13} - \frac{12}{13}i$ **71.** $1 - 3i$ **73.** $1 + 3i$ **75.** $-1$ **77.** $i$ **79.** $-1$ **81.** $-i$ **83.** $-i$ **85.** Since $i^{20} = (i^4)^5 = 1^5 = 1$, the student multiplied by 1, which is justified by the identity property for multiplication. **87.** $\frac{1}{2} + \frac{1}{2}i$ **89.** Substitute both $1 + 5i$ and $1 - 5i$ for $x$, and show that the result is $0 = 0$ in each case. **91.** $\frac{37}{10} - \frac{19}{10}i$ **93.** $-\frac{13}{10} + \frac{11}{10}i$ **95.** $\left\{-\frac{13}{6}\right\}$ **97.** $\{-8, 5\}$ **99.** $\left\{-\frac{2}{5}, 1\right\}$

## Chapter 10 Review Exercises (pages 662–665)

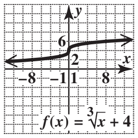
**1.** 42 **2.** $-17$ **3.** 6 **4.** $-5$ **5.** $-3$ **6.** $-2$ **7.** $\sqrt[n]{a}$ is not a real number if $n$ is even and $a$ is negative. **8.** **(a)** $|x|$ **(b)** $-|x|$ **(c)** $x$ **9.** $-6.856$ **10.** $-5.053$ **11.** 4.960 **12.** 0.009 **13.** $-3968.503$ **14.** $-0.189$

**15.**



domain: $[1, \infty)$; range: $[0, \infty)$

**16.**

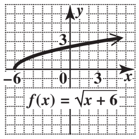


domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

**17.** B **18.** cube (third); 8; 2; second; 4; 4 **19.** A **20.** **(a)** $m$ must be even. **(b)** $m$ must be odd. **21.** no **22.** 7 **23.** $-11$ **24.** 32 **25.** $-4$ **26.** $-\frac{216}{125}$ **27.** $-32$ **28.** $\frac{1000}{27}$ **29.** It is not a real number. **30.** $\left(\sqrt[3]{8}\right)^2$; $\sqrt[3]{8^2}$ **31.** The radical $\sqrt[n]{a^m}$ is equivalent to $a^{m/n}$. For example, $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$, and $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$. **32.** $\sqrt{m + 3n}$ **33.** $\dfrac{1}{\left(\sqrt[3]{3a + b}\right)^5}$, or $\dfrac{1}{\sqrt[3]{(3a + b)^5}}$ **34.** $7^{9/2}$ **35.** $p^{4/5}$ **36.** $5^2$, or 25 **37.** 96 **38.** $a^{2/3}$ **39.** $\dfrac{1}{y^{1/2}}$ **40.** $\dfrac{z^{1/2}x^{8/5}}{4}$ **41.** $r^{1/2} + r$ **42.** $s^{1/2}$ **43.** $r^{3/2}$ **44.** $p^{1/2}$ **45.** $k^{9/4}$ **46.** $m^{13/3}$ **47.** $z^{1/12}$ **48.** $x^{1/8}$ **49.** $x^{1/15}$ **50.** $x^{1/36}$ **51.** The product rule for exponents applies only if the bases are the same. **52.** $\sqrt{66}$ **53.** $\sqrt{5r}$ **54.** $\sqrt[3]{30}$ **55.** $\sqrt[4]{21}$ **56.** $2\sqrt{5}$ **57.** $5\sqrt{3}$ **58.** $-5\sqrt{5}$ **59.** $-3\sqrt[3]{4}$ **60.** $10y^3\sqrt{y}$ **61.** $4pq^2\sqrt[3]{p}$ **62.** $3a^2b\sqrt[3]{4a^2b^2}$ **63.** $2r^2t\sqrt[3]{79r^2t}$ **64.** $\dfrac{y\sqrt{y}}{12}$ **65.** $\dfrac{m^5}{3}$ **66.** $\dfrac{\sqrt[3]{r^2}}{2}$ **67.** $\dfrac{a^2\sqrt[4]{a}}{3}$ **68.** $\sqrt{15}$ **69.** $p\sqrt{p}$ **70.** $\sqrt[12]{2000}$ **71.** $\sqrt[10]{x^7}$ **72.** 10 **73.** $\sqrt{197}$ **74.** $-11\sqrt{2}$ **75.** $23\sqrt{5}$ **76.** $7\sqrt{3y}$ **77.** $26m\sqrt{6m}$ **78.** $19\sqrt[3]{2}$ **79.** $-8\sqrt[4]{2}$ **80.** $\left(16\sqrt{2} + 24\sqrt{3}\right)$ ft **81.** $\left(12\sqrt{3} + 5\sqrt{2}\right)$ ft **82.** $1 - \sqrt{3}$ **83.** 2 **84.** $9 - 7\sqrt{2}$ **85.** $15 - 2\sqrt{26}$ **86.** 29 **87.** $2\sqrt[3]{2y^2} + 2\sqrt[3]{4y} - 3$ **88.** $4.801960973 \neq 66.28725368$ **89.** The denominator would become $\sqrt[3]{6^2} = \sqrt[3]{36}$, which is not rational. **90.** $\dfrac{\sqrt{30}}{5}$ **91.** $-3\sqrt{6}$ **92.** $\dfrac{3\sqrt{7py}}{y}$ **93.** $\dfrac{\sqrt{22}}{4}$ **94.** $-\dfrac{\sqrt[3]{45}}{5}$ **95.** $\dfrac{3m\sqrt[3]{4n}}{n^2}$ **96.** $\dfrac{\sqrt{2} - \sqrt{7}}{-5}$ **97.** $\dfrac{5\left(\sqrt{6} + 3\right)}{3}$ **98.** $\dfrac{1 - \sqrt{5}}{4}$ **99.** $\dfrac{1 - 4\sqrt{2}}{3}$ **100.** $\dfrac{-6 + \sqrt{3}}{2}$ **101.** $\{2\}$ **102.** $\{6\}$ **103.** $\emptyset$ **104.** $\{0, 5\}$ **105.** $\{9\}$ **106.** $\{3\}$ **107.** $\{7\}$ **108.** $\left\{-\frac{1}{2}\right\}$ **109.** $\{-13\}$ **110.** $\{-1\}$ **111.** $\{14\}$ **112.** $\{-4\}$ **113.** $\emptyset$ **114.** $\emptyset$ **115.** $\{7\}$ **116.** $\{4\}$ **117.** **(a)** $H = \sqrt{L^2 - W^2}$ **(b)** 7.9 ft **118.** $5i$ **119.** $10i\sqrt{2}$ **120.** no **121.** $-10 - 2i$ **122.** $14 + 7i$ **123.** $-\sqrt{35}$ **124.** $-45$ **125.** 3 **126.** $5 + i$ **127.** $32 - 24i$ **128.** $1 - i$ **129.** $4 + i$ **130.** $-i$ **131.** 1 **132.** $-1$ **133.** 1 **134.** $-4$ **135.** $\frac{1}{100}$ **136.** $\dfrac{1}{z^{3/5}}$ **137.** $k^6$ **138.** $3z^3t^2\sqrt[3]{2t^2}$ **139.** $57\sqrt{2}$ **140.** $-\dfrac{\sqrt{3}}{6}$ **141.** $\dfrac{\sqrt[3]{60}}{5}$ **142.** 1 **143.** $7i$ **144.** $3 - 7i$ **145.** $-5i$ **146.** $\dfrac{1 + \sqrt{6}}{2}$ **147.** $5 + 12i$ **148.** $6x\sqrt[3]{y^2}$ **149.** The expression cannot be simplified further. **150.** $\sqrt{35} + \sqrt{15} - \sqrt{21} - 3$ **151.** $\{5\}$ **152.** $\{-4\}$ **153.** $\left\{\frac{3}{2}\right\}$ **154.** $\{2\}$ **155.** $\{1\}$ **156.** $\{2\}$ **157.** $\{9\}$ **158.** $\{4\}$ **159.** $\{7\}$ **160.** $\{6\}$

## Chapter 10 Test (pages 665–666)

[10.1] **1.** $-29$ **2.** $-8$ [10.2] **3.** 5 [10.1] **4.** C **5.** 21.863 **6.** $-9.405$ **7.**

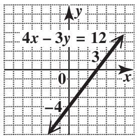


domain: $[-6, \infty)$; range: $[0, \infty)$

[10.2] **8.** $\frac{125}{64}$ **9.** $\frac{1}{256}$ **10.** $\dfrac{9y^{3/10}}{x^2}$ **11.** $x^{4/3}y^6$ **12.** $7^{1/2}$, or $\sqrt{7}$ [10.3] **13.** $a^3\sqrt[3]{a^2}$, or $a^{11/3}$ **14.** $\sqrt{145}$ **15.** 10 **16.** $3x^2y^3\sqrt{6x}$ **17.** $2ab^3\sqrt[4]{2a^3b}$ **18.** $\sqrt[6]{200}$ [10.4] **19.** $26\sqrt{5}$ **20.** $(2ts - 3t^2)\sqrt[3]{2s^2}$ [10.5] **21.** $66 + \sqrt{5}$ **22.** $23 - 4\sqrt{15}$ **23.** $-\dfrac{\sqrt{10}}{4}$ **24.** $\dfrac{2\sqrt[3]{25}}{5}$ **25.** $-2\left(\sqrt{7} - \sqrt{5}\right)$ **26.** $3 + \sqrt{6}$ [10.6] **27.** **(a)** 59.8 **(b)** $T = \dfrac{V_0^2 - V^2}{-V^2k}$, or $T = \dfrac{V^2 - V_0^2}{V^2k}$ **28.** $\{-1\}$ **29.** $\{3\}$ **30.** $\{-3\}$ [10.7] **31.** $-5 - 8i$ **32.** $-2 + 16i$ **33.** $3 + 4i$ **34.** $i$ **35.** **(a)** true **(b)** true **(c)** false **(d)** true

## Chapters 1–10 Cumulative Review Exercises (pages 667–668)

[1.4–1.6] **1.** 1 **2.** $-\frac{14}{9}$ [2.3] **3.** $\{-4\}$ **4.** $\{-12\}$ **5.** $\{6\}$ [2.8] **6.** $(-6, \infty)$ [2.7] **7.** 36 nickels; 64 quarters **8.** $2\dfrac{2}{39}$ L [3.2, 7.1] **9.**



[3.3, 3.4, 7.1, 7.2] **10.** $-\frac{3}{2}$; $y = -\frac{3}{2}x$ [4.4] **11.** $-k^3 - 3k^2 - 8k - 9$ [4.5] **12.** $8x^2 + 17x - 21$ [4.7] **13.** $3y^3 - 3y^2 + 4y + 1 + \dfrac{-10}{2y + 1}$ [5.2, 5.3] **14.** $(2p - 3q)(p - q)$ [5.4] **15.** $(3k^2 + 4)(k - 1)(k + 1)$ **16.** $(x + 8)(x^2 - 8x + 64)$ [5.5] **17.** $\left\{-3, -\frac{5}{2}\right\}$ **18.** $\left\{-\frac{2}{5}, 1\right\}$

[6.1] **19.** $x \neq -3, x \neq 3$ [6.2] **20.** $\dfrac{y}{y+5}$ [6.4] **21.** $\dfrac{4x+2y}{(x+y)(x-y)}$ [6.5] **22.** $-\frac{9}{4}$ **23.** $\dfrac{1}{xy-1}$ [6.6] **24.** $\emptyset$ [6.7] **25.** Danielle: 8 mph; Richard: 4 mph [7.4] **26.** $-37$ [7.6] **27.** \$9.92 [8.1–8.3, 8.6] **28.** $\{(7, -2)\}$ [8.4, 8.6] **29.** $\{(-1, 1, 1)\}$ [8.5] **30.** 2-oz letter: \$0.61; 3-oz letter: \$0.78 [9.1] **31.** $(2, 3)$ **32.** $(-\infty, 2) \cup (3, \infty)$ [9.2] **33.** $\left\{-\frac{10}{3}, 1\right\}$ **34.** $(-\infty, -2] \cup [7, \infty)$ [10.3] **35.** $2x\sqrt[3]{6x^2y^2}$ [10.4] **36.** $7\sqrt{2}$ [10.5] **37.** $\dfrac{\sqrt{10}+2\sqrt{2}}{2}$ [10.3] **38.** $\sqrt{29}$ [10.6] **39.** $\{3, 4\}$ [10.7] **40.** $4 + 2i$

# 11 QUADRATIC EQUATIONS, INEQUALITIES, AND FUNCTIONS

## Section 11.1 (pages 674–675)

**1.** B, C **3.** The zero-factor property requires a product equal to 0. The first step should have been to rewrite the equation with 0 on one side. **5.** $\{-7, 8\}$ **7.** $\{-11, 11\}$, or $\{\pm 11\}$ **9.** $\left\{-\frac{5}{3}, 6\right\}$ **11.** $\{\pm 9\}$ **13.** $\{\pm\sqrt{14}\}$ **15.** $\{\pm 4\sqrt{3}\}$ **17.** $\left\{\pm\frac{5}{2}\right\}$ **19.** $\{\pm 1.5\}$ **21.** $\{\pm\sqrt{3}\}$ **23.** $\{\pm 2\sqrt{5}\}$ **25.** $\left\{\pm\dfrac{2\sqrt{7}}{7}\right\}$ **27.** $\{\pm 2\sqrt{6}\}$ **29.** $\left\{\pm\dfrac{2\sqrt{5}}{5}\right\}$ **31.** $\{\pm 3\sqrt{3}\}$ **33.** $\{\pm 2\sqrt{2}\}$ **35.** $\{-2, 8\}$ **37.** $\{4 \pm \sqrt{3}\}$ **39.** $\{8 \pm 3\sqrt{3}\}$ **41.** $\left\{-3, \frac{5}{3}\right\}$ **43.** $\left\{0, \frac{3}{2}\right\}$ **45.** $\left\{\dfrac{1 \pm \sqrt{7}}{3}\right\}$ **47.** $\left\{\dfrac{-1 \pm 3\sqrt{2}}{3}\right\}$ **49.** $\left\{\dfrac{5 \pm \sqrt{30}}{2}\right\}$ **51.** $\{-10 \pm 4\sqrt{3}\}$ **53.** $\left\{\dfrac{1 \pm 4\sqrt{3}}{4}\right\}$ **55.** $\{-4.48, 0.20\}$ **57.** $\{-3.09, -0.15\}$ **59.** $\{\pm 2i\sqrt{3}\}$ **61.** $\{5 \pm 2i\}$ **63.** $\left\{\dfrac{1}{6} \pm \dfrac{\sqrt{2}}{3}i\right\}$ **65.** 5.6 sec **67.** 9 in. **69.** 2% **71.** $\dfrac{4+4\sqrt{3}}{5}$ **73.** $\dfrac{3+\sqrt{6}}{4}$ **75.** $(x-5)^2$

## Section 11.2 (pages 681–683)

**1.** Solve $(2x+1)^2 = 5$ by the square root property. Solve $x^2 + 4x = 12$ by completing the square. **3.** 9; $(x+3)^2$ **5.** 36; $(p-6)^2$ **7.** $\frac{81}{4}$; $\left(q + \frac{9}{2}\right)^2$ **9.** $\frac{1}{64}$; $\left(x + \frac{1}{8}\right)^2$ **11.** 0.16; $(x - 0.4)^2$ **13.** 4 **15.** 25 **17.** $\frac{1}{36}$ **19.** $\{1, 3\}$ **21.** $\{-1 \pm \sqrt{6}\}$ **23.** $\{-2 \pm \sqrt{6}\}$ **25.** $\{-5 \pm \sqrt{7}\}$ **27.** $\{4 \pm 2\sqrt{3}\}$ **29.** $\left\{\dfrac{-7 \pm \sqrt{53}}{2}\right\}$ **31.** $\left\{-\frac{3}{2}, \frac{1}{2}\right\}$ **33.** $\left\{-\frac{8}{3}, 3\right\}$ **35.** $\left\{\dfrac{-5 \pm \sqrt{41}}{4}\right\}$ **37.** $\left\{\dfrac{5 \pm \sqrt{15}}{5}\right\}$ **39.** $\left\{\dfrac{4 \pm \sqrt{3}}{3}\right\}$ **41.** $\{-4, 2\}$ **43.** $\{4 \pm \sqrt{3}\}$ **45.** $\{1 \pm \sqrt{6}\}$ **47.** $\left\{\dfrac{2 \pm \sqrt{3}}{3}\right\}$ **49.** $\{1 \pm \sqrt{2}\}$ **51. (a)** $\left\{\dfrac{3 \pm 2\sqrt{6}}{3}\right\}$ **(b)** $\{-0.633, 2.633\}$ **53. (a)** $\{-2 \pm \sqrt{3}\}$ **(b)** $\{-3.732, -0.268\}$ **55.** $\{-2 \pm 3i\}$ **57.** $\left\{-\dfrac{2}{3} \pm \dfrac{2\sqrt{2}}{3}i\right\}$ **59.** $\{-3 \pm i\sqrt{3}\}$ **61.** $x^2$ **62.** $x$ **63.** $6x$ **64.** 1 **65.** 9 **66.** $(x+3)^2$, or $x^2 + 6x + 9$ **67.** $\{\pm\sqrt{b}\}$ **69.** $\left\{\pm\dfrac{\sqrt{b^2+16}}{2}\right\}$ **71.** $\left\{\dfrac{2b \pm \sqrt{3a}}{5}\right\}$ **73.** $\sqrt{13}$ **75.** 1

## Section 11.3 (pages 688–690)

**1.** The documentation was incorrect, since the fraction bar should extend under the term $-b$. The correct formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. **3.** The last step is wrong. Because 5 is not a common factor in the numerator, the fraction cannot be simplified. The solution set is $\left\{\dfrac{5 \pm \sqrt{5}}{10}\right\}$. **5.** $\{3, 5\}$ **7.** $\left\{\dfrac{-2 \pm \sqrt{2}}{2}\right\}$ **9.** $\left\{\dfrac{1 \pm \sqrt{3}}{2}\right\}$ **11.** $\{5 \pm \sqrt{7}\}$ **13.** $\left\{\dfrac{-1 \pm \sqrt{2}}{2}\right\}$ **15.** $\left\{\dfrac{-1 \pm \sqrt{7}}{3}\right\}$ **17.** $\{1 \pm \sqrt{5}\}$ **19.** $\left\{\dfrac{-2 \pm \sqrt{10}}{2}\right\}$ **21.** $\{-1 \pm 3\sqrt{2}\}$ **23.** $\left\{\dfrac{1 \pm \sqrt{29}}{2}\right\}$ **25.** $\left\{\dfrac{-4 \pm \sqrt{91}}{3}\right\}$ **27.** $\left\{\dfrac{-3 \pm \sqrt{57}}{8}\right\}$ **29.** $\left\{\dfrac{3}{2} \pm \dfrac{\sqrt{15}}{2}i\right\}$ **31.** $\{3 \pm i\sqrt{5}\}$ **33.** $\left\{\dfrac{1}{2} \pm \dfrac{\sqrt{6}}{2}i\right\}$ **35.** $\left\{-\dfrac{2}{3} \pm \dfrac{\sqrt{2}}{3}i\right\}$ **37.** $\left\{\frac{1}{2} \pm \frac{1}{4}i\right\}$ **39.** B; factoring **41.** C; quadratic formula **43.** A; factoring **45.** D; quadratic formula **47.** $\left\{-\frac{7}{5}\right\}$ **49.** $\left\{-\frac{1}{3}, 2\right\}$ **51. (a)** Discriminant is 25, or $5^2$; solve by factoring; $\left\{-3, -\frac{4}{3}\right\}$ **(b)** Discriminant is 44; use the quadratic formula; $\left\{\dfrac{7 \pm \sqrt{11}}{2}\right\}$ **53.** $-10$ or 10 **55.** 16 **57.** 25 **59.** $b = \frac{44}{5}; \frac{3}{10}$ **61.** $\{-8\}$ **63.** $\{5\}$

## Section 11.4 (pages 697–700)

**1.** Multiply by the LCD, $x$. **3.** Substitute a variable for $x^2 + x$. **5.** The proposed solution $-1$ does not check. The solution set is $\{4\}$. **7.** $\{-2, 7\}$ **9.** $\{-4, 7\}$ **11.** $\left\{-\frac{2}{3}, 1\right\}$ **13.** $\left\{-\frac{14}{17}, 5\right\}$ **15.** $\left\{-\frac{11}{7}, 0\right\}$ **17.** $\left\{\dfrac{-1 \pm \sqrt{13}}{2}\right\}$ **19.** $\left\{-\frac{8}{3}, -1\right\}$ **21.** $\left\{\dfrac{2 \pm \sqrt{22}}{3}\right\}$ **23.** $\left\{\dfrac{-1 \pm \sqrt{5}}{4}\right\}$ **25. (a)** $(20 - t)$ mph **(b)** $(20 + t)$ mph **27.** 25 mph **29.** 50 mph **31.** 3.6 hr **33.** Rusty: 25.0 hr; Nancy: 23.0 hr **35.** 3 hr; 6 hr **37.** $\{2, 5\}$ **39.** $\{3\}$

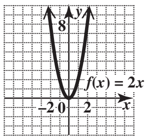
**41.** $\left\{\frac{8}{9}\right\}$ **43.** $\{9\}$ **45.** $\left\{\frac{2}{5}\right\}$ **47.** $\{-2\}$ **49.** $\{\pm 2, \pm 5\}$
**51.** $\left\{\pm 1, \pm \frac{3}{2}\right\}$ **53.** $\left\{\pm 2, \pm 2\sqrt{3}\right\}$ **55.** $\{-6, -5\}$
**57.** $\left\{-\frac{16}{3}, -2\right\}$ **59.** $\{-8, 1\}$ **61.** $\{-64, 27\}$ **63.** $\left\{\pm 1, \pm \frac{27}{8}\right\}$
**65.** $\left\{-\frac{1}{3}, \frac{1}{6}\right\}$ **67.** $\left\{-\frac{1}{2}, 3\right\}$ **69.** $\left\{\pm \frac{\sqrt{6}}{3}, \pm \frac{1}{2}\right\}$ **71.** $\{3, 11\}$
**73.** $\{25\}$ **75.** $\left\{-\sqrt[3]{5}, -\frac{\sqrt[3]{4}}{2}\right\}$ **77.** $\left\{\frac{4}{3}, \frac{9}{4}\right\}$
**79.** $\left\{\pm \frac{\sqrt{9 + \sqrt{65}}}{2}, \pm \frac{\sqrt{9 - \sqrt{65}}}{2}\right\}$ **81.** $\left\{\pm 1, \pm \frac{\sqrt{6}}{2} i\right\}$
**83.** $W = \frac{P - 2L}{2}$, or $W = \frac{P}{2} - L$ **85.** $C = \frac{5}{9}(F - 32)$

## Summary Exercises on Solving Quadratic Equations (page 700)

**1.** square root property **2.** factoring **3.** quadratic formula
**4.** quadratic formula **5.** factoring **6.** square root property
**7.** $\left\{\pm\sqrt{7}\right\}$ **8.** $\left\{-\frac{3}{2}, \frac{5}{3}\right\}$ **9.** $\left\{-3 \pm \sqrt{5}\right\}$ **10.** $\{-2, 8\}$
**11.** $\left\{-\frac{3}{2}, 4\right\}$ **12.** $\left\{-3, \frac{1}{3}\right\}$ **13.** $\left\{\frac{2 \pm \sqrt{2}}{2}\right\}$ **14.** $\left\{\pm 2i\sqrt{3}\right\}$
**15.** $\left\{\frac{1}{2}, 2\right\}$ **16.** $\{\pm 1, \pm 3\}$ **17.** $\left\{\frac{-3 \pm 2\sqrt{2}}{2}\right\}$ **18.** $\left\{\frac{4}{5}, 3\right\}$
**19.** $\left\{\pm\sqrt{2}, \pm\sqrt{7}\right\}$ **20.** $\left\{\frac{1 \pm \sqrt{5}}{4}\right\}$ **21.** $\left\{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$
**22.** $\left\{-\frac{\sqrt[3]{175}}{5}, 1\right\}$ **23.** $\left\{\frac{3}{2}\right\}$ **24.** $\left\{\frac{2}{3}\right\}$ **25.** $\left\{\pm 6\sqrt{2}\right\}$
**26.** $\left\{-\frac{2}{3}, 2\right\}$ **27.** $\{-4, 9\}$ **28.** $\{\pm 13\}$ **29.** $\left\{1 \pm \frac{\sqrt{3}}{3} i\right\}$
**30.** $\{3\}$ **31.** $\left\{\frac{1}{6} \pm \frac{\sqrt{47}}{6} i\right\}$ **32.** $\left\{-\frac{1}{3}, \frac{1}{6}\right\}$

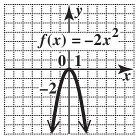
## Section 11.5 (pages 705–709)

**1.** Find a common denominator, and then multiply both sides by the common denominator. **3.** Write it in standard form (with 0 on one side, in decreasing powers of $w$). **5.** $m = \sqrt{p^2 - n^2}$
**7.** $t = \frac{\pm\sqrt{dk}}{k}$ **9.** $d = \frac{\pm\sqrt{skI}}{I}$ **11.** $v = \frac{\pm\sqrt{kAF}}{F}$
**13.** $r = \frac{\pm\sqrt{3\pi Vh}}{\pi h}$ **15.** $t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ **17.** $h = \frac{D^2}{k}$
**19.** $\ell = \frac{p^2 g}{k}$ **21.** $r = \frac{\pm\sqrt{S\pi}}{2\pi}$ **23.** $R = \frac{E^2 - 2pr \pm E\sqrt{E^2 - 4pr}}{2p}$
**25.** $r = \frac{5pc}{4}$ or $r = -\frac{2pc}{3}$ **27.** $I = \frac{-cR \pm \sqrt{c^2R^2 - 4cL}}{2cL}$
**29.** 7.9, 8.9, 11.9 **31.** eastbound ship: 80 mi; southbound ship: 150 mi
**33.** 8 in., 15 in., 17 in. **35.** length: 24 ft; width: 10 ft **37.** 2 ft
**39.** 7 m by 12 m **41.** 20 in. by 12 in. **43.** 1 sec and 8 sec
**45.** 2.4 sec and 5.6 sec **47.** 9.2 sec **49.** It reaches its *maximum* height at 5 sec because this is the only time it reaches 400 ft. **51.** \$0.80
**53.** 0.035, or 3.5% **55.** 5.5 m per sec **57.** 5 or 14
**59.** **(a)** \$420 billion **(b)** \$420 billion; They are the same.
**61.** 2004; The graph indicates that spending first exceeded \$400 billion in 2005. **63.** 9 **65.** domain: $(-\infty, \infty)$; range: $[0, \infty)$

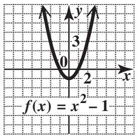


## Section 11.6 (pages 715–718)

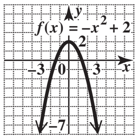
**1.** **(a)** B **(b)** C **(c)** A **(d)** D **3.** $(0, 0)$ **5.** $(0, 4)$ **7.** $(1, 0)$
**9.** $(-3, -4)$ **11.** $(5, 6)$ **13.** down; wider **15.** up; narrower
**17.** down; narrower **19.** **(a)** D **(b)** B **(c)** C **(d)** A
**21.**



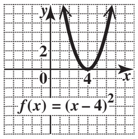
**23.**

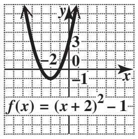


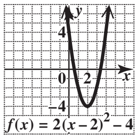
**25.**



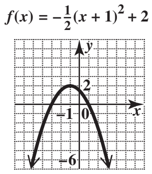
**27.** vertex: $(4, 0)$; axis: $x = 4$; domain: $(-\infty, \infty)$; range: $[0, \infty)$
**29.** vertex: $(-2, -1)$; axis: $x = -2$; domain: $(-\infty, \infty)$; range: $[-1, \infty)$
**31.** vertex: $(2, -4)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $[-4, \infty)$

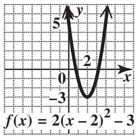




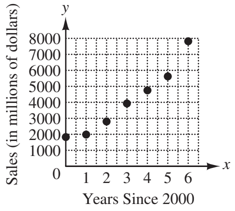


**33.** vertex: $(-1, 2)$; axis: $x = -1$; domain: $(-\infty, \infty)$; range: $(-\infty, 2]$
**35.** vertex: $(2, -3)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $[-3, \infty)$





**37.** linear; positive
**39.** quadratic; positive
**41.** quadratic; negative
**43.** **(a)**



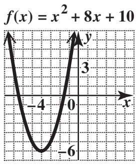
**(b)** quadratic; positive **(c)** $f(x) = 99.3x^2 + 400.7x + 1825$
**(d)** \$9496 million **(e)** No. The number of digital cameras sold in 2007 is far below the number approximated by the model. Rather than continuing to increase, sales of digital cameras fell in 2007. **45.** **(a)** 6105
**(b)** The approximation using the model is low. **47.** $\{-4, 5\}$
**49.** $-2$ **51.** $\{-4, 1\}$ **53.** $\left\{-3 \pm 2\sqrt{3}\right\}$

## Section 11.7 (pages 726–729)

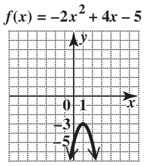
**1.** If $x$ is squared, it has a vertical axis. If $y$ is squared, it has a horizontal axis. **3.** Use the discriminant of the function. If it is positive, there are two $x$-intercepts. If it is 0, there is one $x$-intercept (at the vertex), and if it is negative, there is no $x$-intercept. **5.** $(-4, -6)$ **7.** $(1, -3)$
**9.** $\left(-\frac{1}{2}, -\frac{29}{4}\right)$ **11.** $(-1, 3)$; up; narrower; no $x$-intercepts

**13.** $\left(\frac{5}{2}, \frac{37}{4}\right)$; down; same; two $x$-intercepts **15.** $(-3, -9)$; to the right; wider **17.** F **19.** C **21.** D

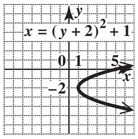
**23.** vertex: $(-4, -6)$; axis: $x = -4$; domain: $(-\infty, \infty)$; range: $[-6, \infty)$



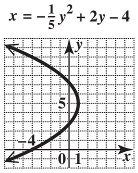
**25.** vertex: $(1, -3)$; axis: $x = 1$; domain: $(-\infty, \infty)$; range: $(-\infty, -3]$



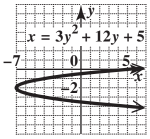
**27.** vertex: $(1, -2)$; axis: $y = -2$; domain: $[1, \infty)$; range: $(-\infty, \infty)$



**29.** vertex: $(1, 5)$; axis: $y = 5$; domain: $(-\infty, 1]$; range: $(-\infty, \infty)$



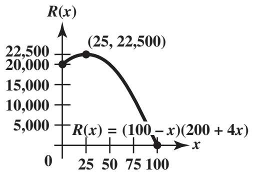
**31.** vertex: $(-7, -2)$; axis: $y = -2$; domain: $[-7, \infty)$; range: $(-\infty, \infty)$



**33.** 20 and 20 **35.** 140 ft by 70 ft; 9800 ft$^2$ **37.** 16 ft; 2 sec **39.** 2 sec; 65 ft **41.** 20 units; \$210 **43. (a)** minimum **(b)** 2003; \$825.8 billion **45. (a)** The coefficient of $x^2$ is negative because a parabola that models the data must open down. **(b)** $(18.45, 3860)$ **(c)** In 2018 Social Security assets will reach their maximum value of \$3860 billion.

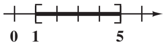
**47. (a)** $R(x) = (100 - x)(200 + 4x) = 20{,}000 + 200x - 4x^2$

**(b)**

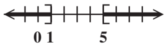


**(c)** 25 **(d)** \$22,500

**49.**



**51.**



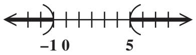
**53.** $\left(-\frac{3}{2}, \infty\right)$

## Section 11.8 (pages 735–736)

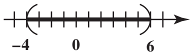
**1. (a)** $\{1, 3\}$ **(b)** $(-\infty, 1) \cup (3, \infty)$ **(c)** $(1, 3)$

**3. (a)** $\{-2, 5\}$ **(b)** $[-2, 5]$ **(c)** $(-\infty, -2] \cup [5, \infty)$

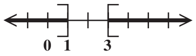
**5.** $(-\infty, -1) \cup (5, \infty)$



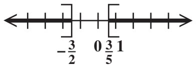
**7.** $(-4, 6)$



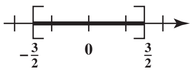
**9.** $(-\infty, 1] \cup [3, \infty)$



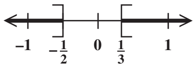
**11.** $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{5}, \infty\right)$



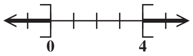
**13.** $\left[-\frac{3}{2}, \frac{3}{2}\right]$



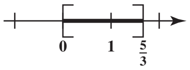
**15.** $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{3}, \infty\right)$



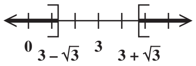
**17.** $(-\infty, 0] \cup [4, \infty)$



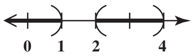
**19.** $\left[0, \frac{5}{3}\right]$



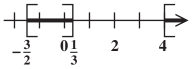
**21.** $\left(-\infty, 3 - \sqrt{3}\right] \cup \left[3 + \sqrt{3}, \infty\right)$



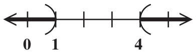
**23.** $(-\infty, \infty)$ **25.** $\emptyset$ **27.** $(-\infty, 1) \cup (2, 4)$



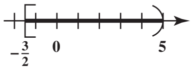
**29.** $\left[-\frac{3}{2}, \frac{1}{3}\right] \cup [4, \infty)$



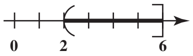
**31.** $(-\infty, 1) \cup (4, \infty)$



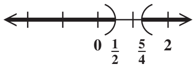
**33.** $\left[-\frac{3}{2}, 5\right)$



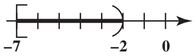
**35.** $(2, 6]$



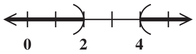
**37.** $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{5}{4}, \infty\right)$



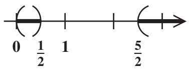
**39.** $[-7, -2)$



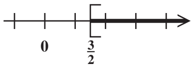
**41.** $(-\infty, 2) \cup (4, \infty)$



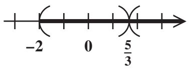
**43.** $\left(0, \frac{1}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$



**45.** $\left[\frac{3}{2}, \infty\right)$



**47.** $\left(-2, \frac{5}{3}\right) \cup \left(\frac{5}{3}, \infty\right)$



**49.** domain: $\{0, 1, 2, 3\}$; range: $\{1, 2, 4, 8\}$ **51.** function

## Chapter 11 Review Exercises (pages 740–744)

**1.** $\{\pm 11\}$ **2.** $\{\pm\sqrt{3}\}$ **3.** $\left\{-\frac{15}{2}, \frac{5}{2}\right\}$ **4.** $\left\{\frac{2}{3} \pm \frac{5}{3}i\right\}$

**5.** By the square root property, the first step should be $x = \sqrt{12}$ or $x = -\sqrt{12}$. The solution set is $\{\pm 2\sqrt{3}\}$. **6.** 5.8 sec

**7.** $\{-2 \pm \sqrt{19}\}$ **8.** $\left\{\frac{1}{2}, 1\right\}$ **9.** $\left\{\frac{-4 \pm \sqrt{22}}{2}\right\}$

**10.** $\left\{\frac{3}{8} \pm \frac{\sqrt{87}}{8}i\right\}$ **11.** $\left\{-\frac{7}{2}, 3\right\}$ **12.** $\left\{\frac{-5 \pm \sqrt{53}}{2}\right\}$

**13.** $\left\{\frac{1 \pm \sqrt{41}}{2}\right\}$ **14.** $\left\{-\frac{3}{4} \pm \frac{\sqrt{23}}{4}i\right\}$ **15.** $\left\{\frac{2}{3} \pm \frac{\sqrt{2}}{3}i\right\}$

**16.** $\left\{\frac{-7 \pm \sqrt{37}}{2}\right\}$ **17. (a)** C **(b)** A **18. (a)** D **(b)** B

**19.** $\left\{-\frac{5}{2}, 3\right\}$ **20.** $\left\{-\frac{1}{2}, 1\right\}$ **21.** $\{-4\}$ **22.** $\left\{-\frac{11}{6}, -\frac{19}{12}\right\}$

**23.** $\left\{-\frac{343}{8}, 64\right\}$ **24.** $\{\pm 1, \pm 3\}$ **25.** 7 mph **26.** 40 mph **27.** 4.6 hr

**28.** Zoran: 2.6 hr; Claude: 3.6 hr **29.** $v = \frac{\pm\sqrt{rFkw}}{kw}$ **30.** $y = \frac{6p^2}{z}$

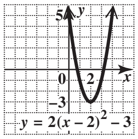
**31.** $t = \frac{3m \pm \sqrt{9m^2 + 24m}}{2m}$ **32.** 9 ft, 12 ft, 15 ft

**33.** 12 cm by 20 cm **34.** 1 in. **35.** 18 in. **36.** 5.2 sec **37.** 3 min

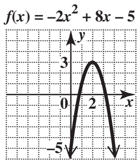
**38.** **(a)** \$15,511 million; It is close to the number suggested by the graph. **(b)** $x = 6$, which represents 2006; Based on the graph, the revenue in 2006 was closer to \$13,000 million than \$14,000 million. **39.** $(1, 0)$

**40.** $(3, 7)$ **41.** $(-4, 3)$ **42.** $\left(\frac{2}{3}, -\frac{2}{3}\right)$

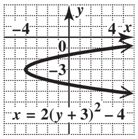
**43.** vertex: $(2, -3)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $[-3, \infty)$



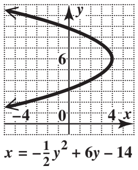
**44.** vertex: $(2, 3)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $(-\infty, 3]$



**45.** vertex: $(-4, -3)$; axis: $y = -3$; domain: $[-4, \infty)$; range: $(-\infty, \infty)$



**46.** vertex: $(4, 6)$; axis: $y = 6$; domain: $(-\infty, 4]$; range: $(-\infty, \infty)$



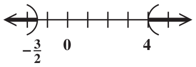
**47.** **(a)**

$$\begin{aligned} c &= 2.9 \\ 100a + 10b + c &= 24.3 \\ 400a + 20b + c &= 56.5 \end{aligned}$$

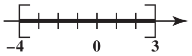
**(b)** $f(x) = 0.054x^2 + 1.6x + 2.9$
**(c)** \$60.3 billion; The result using the model is close, but slightly low.

**48.** 5 sec; 400 ft **49.** length: 50 m; width: 50 m; maximum area: 2500 $\text{m}^2$

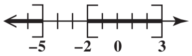
**50.** $\left(-\infty, -\frac{3}{2}\right) \cup (4, \infty)$



**51.** $[-4, 3]$

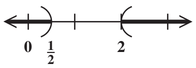


**52.** $(-\infty, -5] \cup [-2, 3]$



**53.** $\emptyset$

**54.** $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$



**55.** $[-3, 2)$

**56.** $R = \frac{\pm\sqrt{Vh - r^2h}}{h}$

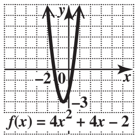
**57.** $\left\{1 \pm \frac{\sqrt{3}}{3}i\right\}$ **58.** $\left\{\frac{-11 \pm \sqrt{7}}{3}\right\}$ **59.** $d = \frac{\pm\sqrt{SkI}}{I}$

**60.** $(-\infty, \infty)$ **61.** $\{4\}$ **62.** $\left\{\pm\sqrt{4 + \sqrt{15}}, \pm\sqrt{4 - \sqrt{15}}\right\}$

**63.** $\left(-5, -\frac{23}{5}\right]$ **64.** $\left\{-\frac{5}{3}, -\frac{3}{2}\right\}$ **65.** $\{-2, -1, 3, 4\}$

**66.** $(-\infty, -6) \cup \left(-\frac{3}{2}, 1\right)$ **67.** **(a)** F **(b)** B **(c)** C **(d)** A **(e)** E **(f)** D

**68.** vertex: $\left(-\frac{1}{2}, -3\right)$; axis: $x = -\frac{1}{2}$; domain: $(-\infty, \infty)$; range: $[-3, \infty)$



**69.** 10 mph **70.** length: 2 cm; width: 1.5 cm

## Chapter 11 Test (pages 744–746)

[11.1] **1.** $\left\{\pm 3\sqrt{6}\right\}$ **2.** $\left\{-\frac{8}{7}, \frac{2}{7}\right\}$ [11.2] **3.** $\left\{-1 \pm \sqrt{5}\right\}$

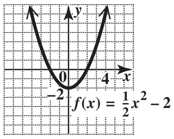
[11.3] **4.** $\left\{\frac{3 \pm \sqrt{17}}{4}\right\}$ **5.** $\left\{\frac{2}{3} \pm \frac{\sqrt{11}}{3}i\right\}$ [11.1] **6.** A

[11.3] **7.** discriminant: 88; There are two irrational solutions.

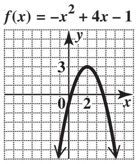
[11.1–11.4] **8.** $\left\{\frac{2}{3}\right\}$ **9.** $\left\{-\frac{2}{3}, 6\right\}$ **10.** $\left\{\frac{-7 \pm \sqrt{97}}{8}\right\}$ **11.** $\left\{\pm\frac{1}{3}, \pm 2\right\}$

**12.** $\left\{-\frac{5}{2}, 1\right\}$ [11.5] **13.** $r = \frac{\pm\sqrt{\pi S}}{2\pi}$ [11.4] **14.** Terry: 11.1 hr; Callie: 9.1 hr **15.** 7 mph [11.5] **16.** 2 ft **17.** 16 m [11.6] **18.** A

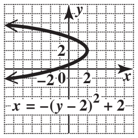
**19.** vertex: $(0, -2)$; axis: $x = 0$; domain: $(-\infty, \infty)$; range: $[-2, \infty)$



[11.7] **20.** vertex: $(2, 3)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $(-\infty, 3]$

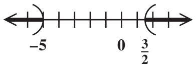


**21.** vertex: $(2, 2)$; axis: $y = 2$; domain: $(-\infty, 2]$; range: $(-\infty, \infty)$

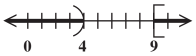


**22.** **(a)** 139 million **(b)** 2007; 145 million **23.** 160 ft by 320 ft

[11.8] **24.** $(-\infty, -5) \cup \left(\frac{3}{2}, \infty\right)$



**25.** $(-\infty, 4) \cup [9, \infty)$



## Chapters 1–11 Cumulative Review Exercises (pages 746–747)

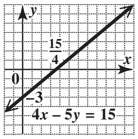
[1.4, 10.7] **1.** **(a)** $-2, 0, 7$ **(b)** $-\frac{7}{3}, -2, 0, 0.7, 7, \frac{32}{3}$ **(c)** All are real except $\sqrt{-8}$. **(c)** All are complex numbers.

[2.3] **2.** $\left\{\frac{4}{5}\right\}$ [9.2] **3.** $\left\{\frac{11}{10}, \frac{7}{2}\right\}$ [10.6] **4.** $\left\{\frac{2}{3}\right\}$ [6.6] **5.** $\emptyset$

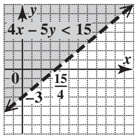
[11.2, 11.3] **6.** $\left\{\frac{7 \pm \sqrt{177}}{4}\right\}$ [11.4] **7.** $\{\pm 1, \pm 2\}$ [2.8] **8.** $[1, \infty)$

[9.2] **9.** $\left[2, \frac{8}{3}\right]$ [11.8] **10.** $(1, 3)$ **11.** $(-2, 1)$

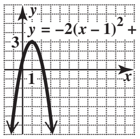
[3.2, 7.1, 7.3, 7.4] **12.** function; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; $f(x) = \frac{4}{5}x - 3$



[7.3, 9.3] **13.** not a function



[11.6] **14.** function; domain: $(-\infty, \infty)$; range: $(-\infty, 3]$; $f(x) = -2(x - 1)^2 + 3$



[3.2, 3.3, 7.1] **15.** $m = \frac{2}{7}$; $x$-intercept: $(-8, 0)$; $y$-intercept: $\left(0, \frac{16}{7}\right)$

[7.2] **16. (a)** $y = -\frac{5}{2}x + 2$ **(b)** $y = \frac{2}{5}x + \frac{13}{5}$ [4.1, 4.2] **17.** $\frac{x^8}{y^4}$

**18.** $\frac{4}{xy^2}$ [4.6] **19.** $\frac{4}{9}t^2 + 12t + 81$ [4.7] **20.** $4x^2 - 6x + 11 + \frac{4}{x + 2}$

[5.1–5.4] **21.** $(4m - 3)(6m + 5)$ **22.** $(2x + 3y)(4x^2 - 6xy + 9y^2)$

**23.** $(3x - 5y)^2$ [6.2] **24.** $-\frac{5}{18}$ [6.4] **25.** $-\frac{8}{x}$ [6.5] **26.** $\frac{r - s}{r}$

[8.1–8.3, 8.6] **27.** $\{(1, -2)\}$ [8.4, 8.6] **28.** $\{(3, -4, 2)\}$

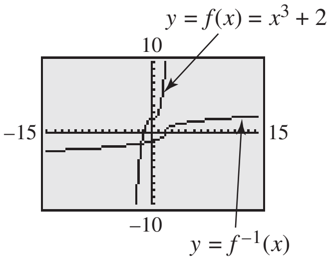
[8.5] **29.** Microsoft: \$60.4 billion; Oracle: \$22.4 billion

[10.3] **30.** $\frac{3\sqrt[3]{4}}{4}$ [10.5] **31.** $\sqrt{7} + \sqrt{5}$

[11.5] **32.** southbound car: 57 mi; eastbound car: 76 mi

# 12 INVERSE, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

**Connections** (page 754)



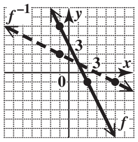
## Section 12.1 (pages 755–758)

**1.** This function is not one-to-one because both France and the United States are paired with the same trans fat percentage, 11. **3.** Yes. By adding 1 to 1058, two distances would be the same, so the function would not be one-to-one. **5.** B **7.** A **9.** $\{(6, 3), (10, 2), (12, 5)\}$ **11.** not one-to-one

**13.** $f^{-1}(x) = \frac{x - 4}{2}$, or $f^{-1}(x) = \frac{1}{2}x - 2$ **15.** $g^{-1}(x) = x^2 + 3$, $x \geq 0$ **17.** not one-to-one **19.** $f^{-1}(x) = \sqrt[3]{x + 4}$

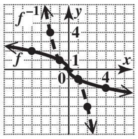
**21. (a)** 8 **(b)** 3 **23. (a)** 1 **(b)** 0

**25. (a)** one-to-one **(b)**

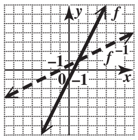


**27. (a)** not one-to-one

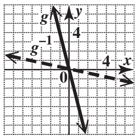
**29. (a)** one-to-one **(b)**



**31.**

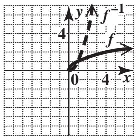


**33.**



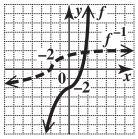
**35.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |



**37.**

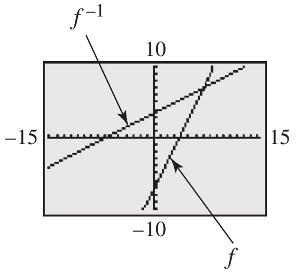
| $x$ | $f(x)$ |
|---|---|
| −1 | −3 |
| 0 | −2 |
| 1 | −1 |
| 2 | 6 |

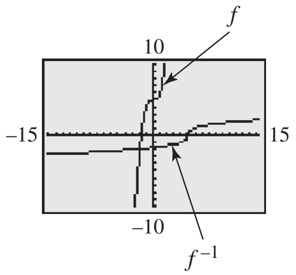


**39.** $f^{-1}(x) = \frac{x + 5}{4}$, or $f^{-1}(x) = \frac{1}{4}x + \frac{5}{4}$

**40.** MY GRAPHING CALCULATOR IS THE GREATEST THING SINCE SLICED BREAD. **41.** If the function were not one-to-one, there would be ambiguity in some of the characters, as they could represent more than one letter. **42.** Answers will vary. For example, Jane Doe is 1004 5 2748 129 68 3379 129.

**43.** $f^{-1}(x) = \frac{x + 7}{2}$, or $f^{-1}(x) = \frac{1}{2}x + \frac{7}{2}$ **45.** $f^{-1}(x) = \sqrt[3]{x - 5}$

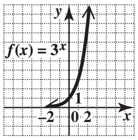




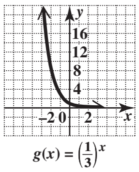
**47.** 64 **49.** $\frac{1}{2}$

## Section 12.2 (pages 763–765)

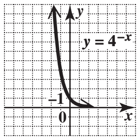
**1.** C **3.** A **5.**



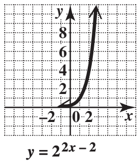
**7.**



**9.**



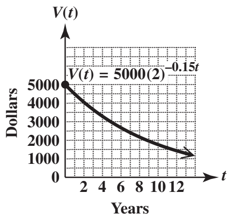
**11.**



**13.** rises; falls **15.** $\{2\}$ **17.** $\left\{\frac{3}{2}\right\}$ **19.** $\{7\}$ **21.** $\{-3\}$ **23.** $\{-1\}$ **25.** $\{-3\}$ **27.** 639.545 **29.** 0.066

**31.** 12.179 **33. (a)** 0.6°C **(b)** 0.3°C **35. (a)** 1.4°C **(b)** 0.5°C **37. (a)** 5028 million tons **(b)** 6512 million tons **(c)** It is less than what the model provides (7099 million tons). **39. (a)** \$5000 **(b)** \$2973 **(c)** \$1768

**(d)**



**41.** 6.67 yr after it was purchased **43.** 4 **45.** 0

## Section 12.3 (pages 770–773)

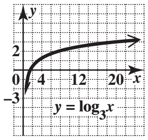
**1. (a)** B **(b)** E **(c)** D **(d)** F **(e)** A **(f)** C **3.** $\log_4 1024 = 5$ **5.** $\log_{1/2} 8 = -3$ **7.** $\log_{10} 0.001 = -3$ **9.** $\log_{625} 5 = \frac{1}{4}$ **11.** $\log_8 \frac{1}{4} = -\frac{2}{3}$ **13.** $\log_5 1 = 0$ **15.** $4^3 = 64$ **17.** $10^{-4} = \frac{1}{10{,}000}$

**19.** $6^0 = 1$ **21.** $9^{1/2} = 3$ **23.** $\left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$ **25.** $5^{-1} = 5^{-1}$

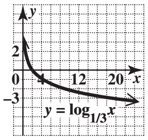
**27.** **(a)** C **(b)** B **(c)** B **(d)** C **29.** $\left\{\frac{1}{3}\right\}$ **31.** $\{81\}$ **33.** $\left\{\frac{1}{5}\right\}$

**35.** $\{1\}$ **37.** $\{x \mid x > 0, x \neq 1\}$ **39.** $\{5\}$ **41.** $\left\{\frac{5}{3}\right\}$ **43.** $\{4\}$

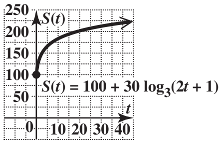
**45.** $\left\{\frac{3}{2}\right\}$ **47.** $\{30\}$ **49.**



**51.**

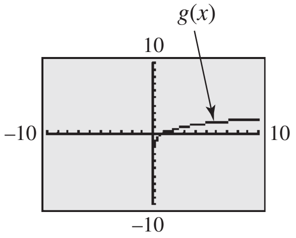


**53.** Every power of 1 is equal to 1, and thus it cannot be used as a base. **55.** $(0, \infty)$; $(-\infty, \infty)$ **57.** 8 **59.** 24 **61.** **(a)** 4385 billion ft³ **(b)** 5555 billion ft³ **(c)** 6140 billion ft³ **63.** **(a)** 130 thousand units **(b)** 190 thousand units **(c)**

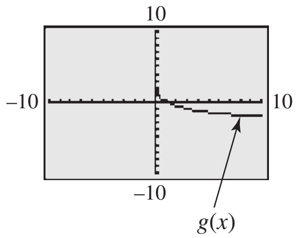


**65.** about 4 times as powerful

**67.**



**69.**



**71.** $4^9$ **73.** $7^{12}$

**Connections** **(page 779)** **1.**

$$\begin{aligned} \log_{10} 458.3 &\approx 2.661149857 \\ +\log_{10} 294.6 &\approx 2.469232743 \\ \hline &\approx 5.130382600 \\ 10^{5.130382600} &\approx 135{,}015.18 \end{aligned}$$

A calculator gives $(458.3)(294.6) = 135{,}015.18$.

**2.** Answers will vary.

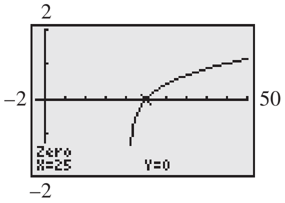
## Section 12.4 (pages 780–781)

**1.** $\log_{10} 7 + \log_{10} 8$ **3.** 4 **5.** 9 **7.** $\log_7 4 + \log_7 5$ **9.** $\log_5 8 - \log_5 3$ **11.** $2 \log_4 6$ **13.** $\frac{1}{3}\log_3 4 - 2\log_3 x - \log_3 y$ **15.** $\frac{1}{2}\log_3 x + \frac{1}{2}\log_3 y - \frac{1}{2}\log_3 5$ **17.** $\frac{1}{3}\log_2 x + \frac{1}{5}\log_2 y - 2\log_2 r$ **19.** In the notation $\log_a(x + y)$, the parentheses do not indicate multiplication. They indicate that $x + y$ is the result of raising $a$ to some power. **21.** $\log_b xy$ **23.** $\log_a \frac{m}{n}$ **25.** $\log_a \frac{rt^3}{s}$ **27.** $\log_a \frac{125}{81}$ **29.** $\log_{10}(x^2 - 9)$ **31.** $\log_p \frac{x^3y^{1/2}}{z^{3/2}a^3}$ **33.** 1.2552 **35.** $-0.6532$ **37.** 1.5562 **39.** 0.2386 **41.** 0.4771 **43.** 4.7710 **45.** false **47.** true **49.** true **51.** false **53.** The exponent of a quotient is the difference between the exponent of the numerator and the exponent of the denominator. **55.** No number allowed as a logarithmic base can be raised to a power with a result of 0. **57.** $\log_{10} 10{,}000 = 4$ **59.** $\log_{10} 0.01 = -2$ **61.** $10^0 = 1$

## Section 12.5 (pages 787–790)

**1.** C **3.** C **5.** 31.6 **7.** 1.6335 **9.** 2.5164 **11.** $-1.4868$ **13.** 9.6776 **15.** 2.0592 **17.** $-2.8896$ **19.** 5.9613 **21.** 4.1506 **23.** 2.3026 **25.** **(a)** 2.552424846 **(b)** 1.552424846 **(c)** 0.552424846 **(d)** The whole number parts will vary, but the decimal parts are the same. **27.** poor fen **29.** bog **31.** rich fen **33.** 11.6 **35.** 4.3 **37.** $4.0 \times 10^{-8}$ **39.** $4.0 \times 10^{-6}$ **41.** **(a)** 107 dB **(b)** 100 dB **(c)** 98 dB **43.** **(a)** 800 yr **(b)** 5200 yr **(c)** 11,500 yr **45.** **(a)** 77% **(b)** 1989 **47.** **(a)** \$54 per ton **(b)** If $p = 0$, then $\ln(1 - p) = \ln 1 = 0$, so $T$ would be negative. If $p = 1$, then $\ln(1 - p) = \ln 0$, but the domain of $\ln x$ is $(0, \infty)$. **49.** 2.2619 **51.** 0.6826 **53.** 0.3155 **55.** 0.8736 **57.** 2.4849 **59.** Answers will vary. Suppose the name is Jeffery Cole, with $m = 7$ and $n = 4$. **(a)** $\log_7 4$ is the exponent to which 7 must be raised to obtain 4. **(b)** 0.7124143742 **(c)** 4 **61.** 6446 billion ft³ **63.** $\left\{-\frac{3}{5}\right\}$ **65.** $\{5\}$ **67.** $\{-3\}$ **69.** $\log(x + 2)(x + 3)$, or $\log(x^2 + 5x + 6)$

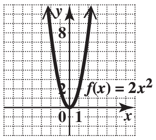
**Connections** **(page 797)**



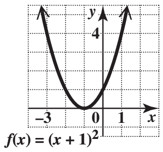
## Section 12.6 (pages 797–800)

**1.** $\{0.827\}$ **3.** $\{0.833\}$ **5.** $\{1.201\}$ **7.** $\{2.269\}$ **9.** $\{15.967\}$ **11.** $\{-6.067\}$ **13.** $\{261.291\}$ **15.** $\{-10.718\}$ **17.** $\{3\}$ **19.** $\{5.879\}$ **21.** $\{-\pi\}$, or $\{-3.142\}$ **23.** $\{1\}$ **25.** Natural logarithms are a better choice because $e$ is the base. **27.** $\left\{\frac{2}{3}\right\}$ **29.** $\left\{\frac{33}{2}\right\}$ **31.** $\left\{-1 + \sqrt[3]{49}\right\}$ **33.** 2 cannot be a solution because $\log(2 - 3) = \log(-1)$, and $-1$ is not in the domain of $\log x$. **35.** $\left\{\frac{1}{3}\right\}$ **37.** $\{2\}$ **39.** $\emptyset$ **41.** $\{8\}$ **43.** $\left\{\frac{4}{3}\right\}$ **45.** $\{8\}$ **47.** **(a)** \$2539.47 **(b)** 10.2 yr **49.** **(a)** \$4934.71 **(b)** 19.8 yr **51.** **(a)** \$11,260.96 **(b)** \$11,416.64 **(c)** \$11,497.99 **(d)** \$11,580.90 **(e)** \$11,581.83 **53.** \$137.41 **55.** **(a)** 15.9 million tons **(b)** 30.7 million tons **(c)** 59.2 million tons **(d)** 93.7 million tons **57.** \$143,598 million **59.** **(a)** 1.62 g **(b)** 1.18 g **(c)** 0.69 g **(d)** 2.00 g **61.** **(a)** 179.73 g **(b)** 21.66 yr **63.** 2012 **65.** It means that after 250 yr, approximately 2.9 g of the original sample remain.

**67.**



**69.**

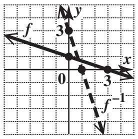


## Chapter 12 Review Exercises (pages 804–808)

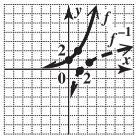
**1.** not one-to-one **2.** one-to-one **3.** $f^{-1}(x) = \frac{x - 7}{-3}$, or $f^{-1}(x) = -\frac{1}{3}x + \frac{7}{3}$ **4.** $f^{-1}(x) = \frac{x^3 + 4}{6}$ **5.** not one-to-one

**6.** This function is not one-to-one because two sodas in the list have 41 mg of caffeine.

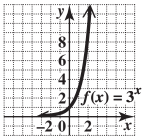
**7.**



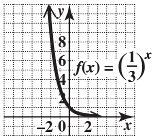
**8.**



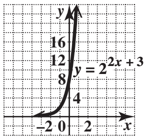
**9.**



**10.**



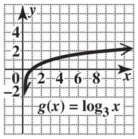
**11.**



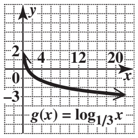
**12.** $\left\{\frac{1}{2}\right\}$ **13.** $\{4\}$ **14.** $\left\{\frac{3}{7}\right\}$

**15.** **(a)** 29.4 million tons **(b)** 18.2 million tons **(c)** 14.4 million tons

**16.**



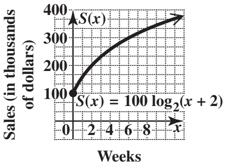
**17.**



**18.** $\{2\}$ **19.** $\left\{\frac{3}{2}\right\}$ **20.** $\{7\}$ **21.** $\{8\}$ **22.** $\{4\}$

**23.** $\{b \mid b > 0, b \neq 1\}$ **24.** $\log_b a$ is the exponent to which $b$ must be raised to obtain $a$. **25.** $a$

**26.** **(a)** \$300,000 **(b)**



**27.** $\log_2 3 + \log_2 x + 2 \log_2 y$ **28.** $\frac{1}{2} \log_4 x + 2 \log_4 w - \log_4 z$

**29.** $\log_b \dfrac{3x}{y^2}$ **30.** $\log_3 \left(\dfrac{x + 7}{4x + 6}\right)$ **31.** 1.4609 **32.** −0.5901

**33.** 3.3638 **34.** −1.3587 **35.** 0.9251 **36.** 1.7925 **37.** 6.4

**38.** 8.4 **39.** $2.5 \times 10^{-5}$ **40.** Magnitude 1 is about 6.3 times as intense as magnitude 3. **41.** **(a)** 18 yr **(b)** 12 yr **(c)** 7 yr **(d)** 6 yr **(e)** Each comparison shows approximately the same number. For example, in part (a) the doubling time is 18 yr (rounded) and $\frac{72}{4} = 18$. Thus, the formula $t = \dfrac{72}{100r}$ (called the *rule of 72*) is an excellent approximation of the doubling time formula. **42.** $\{2.042\}$

**43.** $\{4.907\}$ **44.** $\{18.310\}$ **45.** $\left\{\frac{1}{9}\right\}$ **46.** $\left\{-6 + \sqrt[3]{25}\right\}$

**47.** $\{2\}$ **48.** $\left\{\frac{3}{8}\right\}$ **49.** $\{4\}$ **50.** $\{1\}$ **51.** When the power rule was applied in the second step, the domain was changed from $\{x \mid x \neq 0\}$ to $\{x \mid x > 0\}$. The valid solution −10 was "lost." The solution set is $\{\pm 10\}$. **52.** \$24,403.80 **53.** \$11,190.72 **54.** Plan A is better, since it would pay \$2.92 more. **55.** about 13.9 days **56.** **(a)** about \$4267 **(b)** about 11% **57.** about 67% **58.** D **59.** 7 **60.** 36 **61.** 4 **62.** $e$ **63.** −5 **64.** 5.4 **65.** $\{72\}$ **66.** $\{3\}$ **67.** $\left\{\frac{1}{9}\right\}$ **68.** $\left\{\frac{4}{3}\right\}$

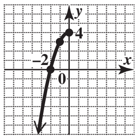
**69.** $\{3\}$ **70.** $\{0\}$ **71.** $\left\{\frac{1}{8}\right\}$ **72.** $\left\{\frac{11}{3}\right\}$ **73.** $\{-2, -1\}$ **74.** **(a)** $\left\{\frac{3}{8}\right\}$ **(b)** The $x$-value of the $x$-intercept is 0.375, the decimal equivalent of $\frac{3}{8}$.

**75.** about 32.28% **76.** **(a)** 0.325 **(c)** 0.673

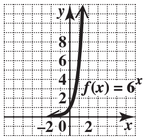
## Chapter 12 Test (pages 808–810)

[12.1] **1.** **(a)** not one-to-one **(b)** one-to-one **2.** $f^{-1}(x) = x^3 - 7$

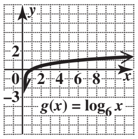
**3.**



[12.2] **4.**



[12.3] **5.**



[12.1–12.3] **6.** Once the graph of $f(x) = 6^x$ is sketched, interchange the $x$- and $y$-values of its ordered pairs. The resulting points will be on the graph of $g(x) = \log_6 x$ since $f$ and $g$ are inverses. [12.2] **7.** $\{-4\}$

**8.** $\left\{-\frac{13}{3}\right\}$ [12.5] **9.** **(a)** 55.8 million **(b)** 80.8 million

[12.3] **10.** $\log_4 0.0625 = -2$ **11.** $7^2 = 49$ **12.** $\{32\}$ **13.** $\left\{\frac{1}{2}\right\}$

**14.** $\{2\}$ **15.** 5; 2; fifth; 32 [12.4] **16.** $2 \log_3 x + \log_3 y$

**17.** $\frac{1}{2} \log_5 x - \log_5 y - \log_5 z$ **18.** $\log_b \dfrac{s^3}{t}$ **19.** $\log_b \dfrac{r^{1/4}s^2}{t^{2/3}}$

[12.5] **20.** **(a)** 1.3636 **(b)** −0.1985 **21.** **(a)** $\dfrac{\log 19}{\log 3}$ **(b)** $\dfrac{\ln 19}{\ln 3}$

**(c)** 2.6801 [12.6] **22.** $\{3.966\}$ **23.** $\{3\}$ **24.** \$12,507.51

**25.** **(a)** \$19,260.38 **(b)** approximately 13.9 yr

## Chapters 1–12 Cumulative Review Exercises (pages 810–811)

[1.4] **1.** $-2, 0, 6, \frac{30}{3}$ (or 10) **2.** $-\frac{9}{4}, -2, 0, 0.6, 6, \frac{30}{3}$ (or 10)

**3.** $-\sqrt{2}, \sqrt{11}$ [1.4–1.6] **4.** 16 **5.** −39 [2.3] **6.** $\left\{-\frac{2}{3}\right\}$

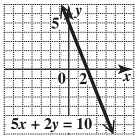
[2.8] **7.** $[1, \infty)$ [9.2] **8.** $\{-2, 7\}$ **9.** $(-\infty, -3) \cup (2, \infty)$

[10.6] **10.** $\{0, 4\}$ [11.1–11.3] **11.** $\left\{\dfrac{1 \pm \sqrt{13}}{6}\right\}$

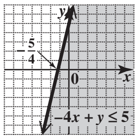
[11.8] **12.** $(-\infty, -4) \cup (2, \infty)$ [11.4] **13.** $\{\pm 1, \pm 2\}$

[12.2] **14.** $\{-1\}$

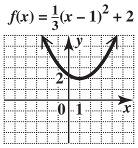
[3.2, 7.1] **15.**



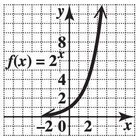
[9.3] **16.**



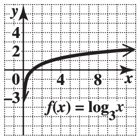
[11.6] **17.**



[12.2] **18.**



[12.3] **19.**



[3.3, 7.1, 7.3] **20.** **(a)** yes **(b)** 3346.2; The number of travelers increased by an average of 3346.2 thousand per year during 2003–2008.

[3.4, 7.2] **21.** $y = \frac{3}{4}x - \frac{19}{4}$ [4.5] **22.** $6p^2 + 7p - 3$

[4.6] **23.** $16k^2 - 24k + 9$ [4.4] **24.** $-5m^3 + 2m^2 - 7m + 4$

[4.7] **25.** $2t^3 + 5t^2 - 3t + 4$ [5.1] **26.** $x(8 + x^2)$

[5.2, 5.3] **27.** $(3y - 2)(8y + 3)$ **28.** $z(5z + 1)(z - 4)$

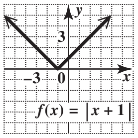
[5.4] **29.** $(4a + 5b^2)(4a - 5b^2)$ **30.** $(2c + d)(4c^2 - 2cd + d^2)$ **31.** $(4r + 7q)^2$ [4.1, 4.2] **32.** $-\dfrac{1875p^{13}}{8}$ [6.2] **33.** $\dfrac{x + 5}{x + 4}$ [6.4] **34.** $\dfrac{-3k - 19}{(k + 3)(k - 2)}$ [8.1–8.3, 8.6] **35.** $\{(4, 2)\}$ [8.4, 8.6] **36.** $\{(1, -1, 4)\}$ [8.5] **37.** 6 lb [10.3] **38.** $12\sqrt{2}$ [10.4] **39.** $-27\sqrt{2}$ [10.7] **40.** 41 [12.4] **41.** $3 \log x + \frac{1}{2} \log y - \log z$ [12.6] **42.** **(a)** 25,000 **(b)** 30,500 **(c)** 37,300 **(d)** in about 3.5 hr, or at about 3:30 P.M.

## 13 NONLINEAR FUNCTIONS, CONIC SECTIONS, AND NONLINEAR SYSTEMS

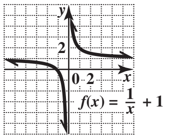
### Section 13.1 (pages 818–819)

**1.** E; 0; 0 **3.** A; $(-\infty, \infty)$; $\{\dots, -2, -1, 0, 1, 2, \dots\}$ **5.** B; It does not satisfy the conditions of the vertical line test. **7.** B **9.** A

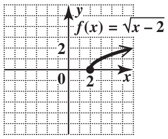
**11.** domain: $(-\infty, \infty)$; range: $[0, \infty)$



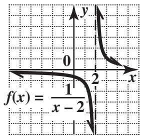
**13.** domain: $(-\infty, 0) \cup (0, \infty)$; range: $(-\infty, 1) \cup (1, \infty)$



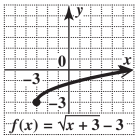
**15.** domain: $[2, \infty)$; range: $[0, \infty)$



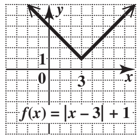
**17.** domain: $(-\infty, 2) \cup (2, \infty)$; range: $(-\infty, 0) \cup (0, \infty)$



**19.** domain: $[-3, \infty)$; range: $[-3, \infty)$



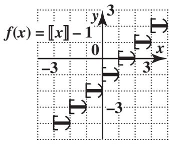
**21.** domain: $(-\infty, \infty)$; range: $[1, \infty)$



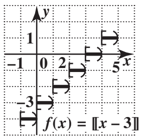
**23.** Shift the graph of $g(x) = \dfrac{1}{x}$ to the right 3 units and up 2 units.

**25.** 3 **27.** 4 **29.** 0 **31.** $-14$ **33.** $-11$

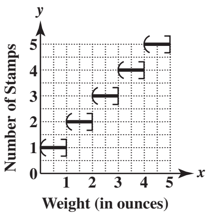
**35.**



**37.**

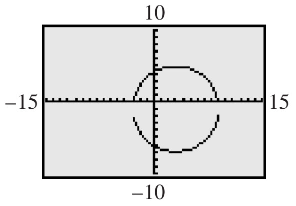


**39.**



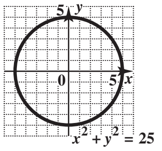
**41.** \$2.75 **43.** $2\sqrt{5}$ **45.** $\sqrt{(x - h)^2 + (y - k)^2}$

**Connections** **(page 824)** $y_1 = -1 + \sqrt{36 - (x - 3)^2}$, $y_2 = -1 - \sqrt{36 - (x - 3)^2}$



### Section 13.2 (pages 825–828)

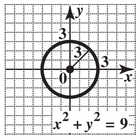
**1.** **(a)** $(0, 0)$ **(b)** 5 **(c)**



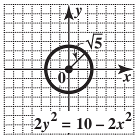
**3.** B **5.** D

**7.** $(x + 4)^2 + (y - 3)^2 = 4$ **9.** $(x + 8)^2 + (y + 5)^2 = 5$ **11.** center: $(-2, -3)$; $r = 2$ **13.** center: $(-5, 7)$; $r = 9$ **15.** center: $(2, 4)$; $r = 4$

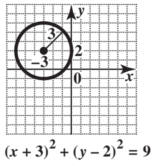
**17.**



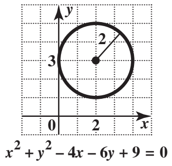
**19.**



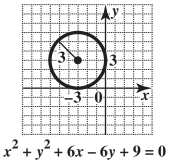
**21.** center: $(-3, 2)$



**23.** center: $(2, 3)$

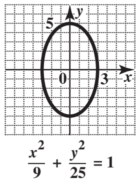


**25.** center: $(-3, 3)$

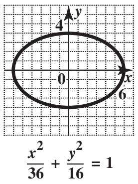


**27.** The thumbtack acts as the center and the length of the string acts as the radius.

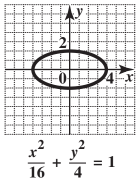
**29.**



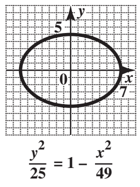
**31.**



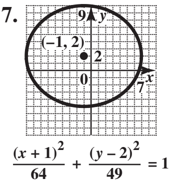
**33.**



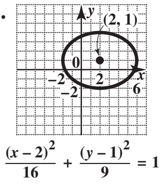
**35.**



**37.**



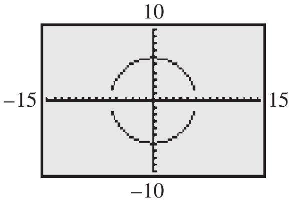
**39.**



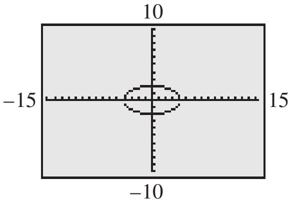
**41.** By the vertical line test the set is not a function, because a vertical line may intersect the graph of an ellipse in two points.

**43.** $y_1 = 4 + \sqrt{16 - (x + 2)^2}$, $y_2 = 4 - \sqrt{16 - (x + 2)^2}$

**45.**



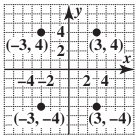
**47.**



**49.** $3\sqrt{3}$ units **51.** **(a)** 10 m **(b)** 36 m

**53.** **(a)** 154.7 million mi **(b)** 128.7 million mi (Answers are rounded.)

**55.**

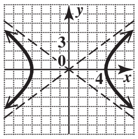


**57.** (3, 0); (0, 4)

## Section 13.3 (pages 833–835)

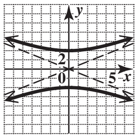
**1.** C **3.** D

**5.**



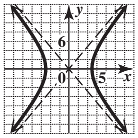
$\frac{x^2}{16}-\frac{y^2}{9}=1$

**7.**



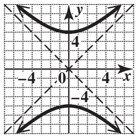
$\frac{y^2}{4}-\frac{x^2}{25}=1$

**9.**



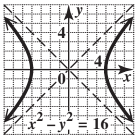
$\frac{x^2}{25}-\frac{y^2}{36}=1$

**11.**

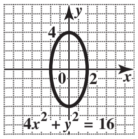


$\frac{y^2}{16}-\frac{x^2}{16}=1$

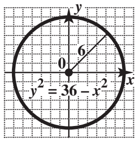
**13.** hyperbola



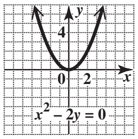
**15.** ellipse



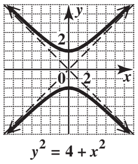
**17.** circle



**19.** parabola

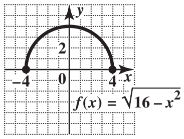


**21.** hyperbola

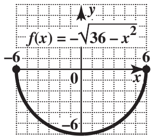


$y^2=4+x^2$

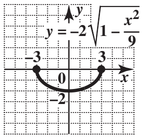
**23.** domain: $[-4, 4]$; range: $[0, 4]$



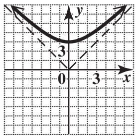
**25.** domain: $[-6, 6]$; range: $[-6, 0]$



**27.** domain: $[-3, 3]$; range: $[-2, 0]$

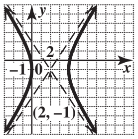


**29.** domain: $(-\infty, \infty)$; range: $[3, \infty)$

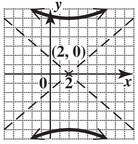


$\frac{y}{3}=\sqrt{1+\frac{x^2}{9}}$

**31.** $\frac{(x-2)^2}{4}-\frac{(y+1)^2}{9}=1$

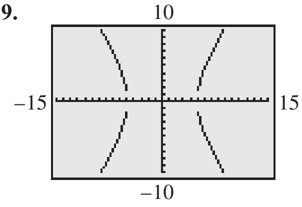


**33.** $\frac{y^2}{36}-\frac{(x-2)^2}{49}=1$

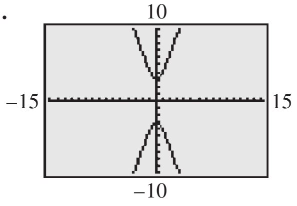


**35.** **(a)** 50 m **(b)** 69.3 m **37.** $y_1=\sqrt{\frac{x^2}{9}-1}$, $y_2=-\sqrt{\frac{x^2}{9}-1}$

**39.**



**41.**

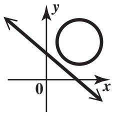


**43.** $\{(2, 9)\}$ **45.** $\{(-1, 2)\}$ **47.** $\left\{\pm\sqrt{3}, \pm\frac{\sqrt{2}}{2}i\right\}$

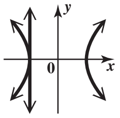
## Section 13.4 (pages 840–841)

**1.** one **3.** none

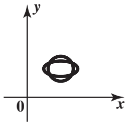
**5.**



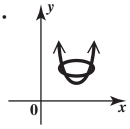
**7.**



**9.**



**11.**



**13.** $\left\{(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ **15.** $\{(-6, 9), (-1, 4)\}$

**17.** $\left\{\left(-\frac{1}{5}, \frac{7}{5}\right), (1, -1)\right\}$ **19.** $\left\{(-2, -2), \left(-\frac{4}{3}, -3\right)\right\}$

**21.** $\{(-3, 1), (1, -3)\}$ **23.** $\left\{\left(-\frac{3}{2}, -\frac{9}{4}\right), (-2, 0)\right\}$

**25.** $\{(-\sqrt{3}, 0), (\sqrt{3}, 0), (-\sqrt{5}, 2), (\sqrt{5}, 2)\}$

**27.** $\left\{\left(\frac{\sqrt{3}}{3}i, -\frac{1}{2}+\frac{\sqrt{3}}{6}i\right), \left(-\frac{\sqrt{3}}{3}i, -\frac{1}{2}-\frac{\sqrt{3}}{6}i\right)\right\}$

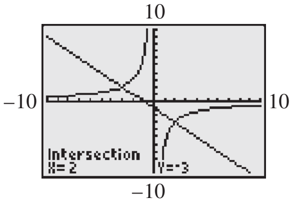
**29.** $\{(-2, 0), (2, 0)\}$ **31.** $\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\}$

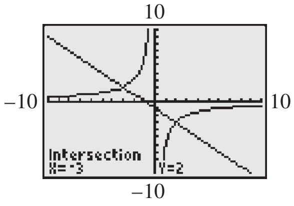
**33.** $\{(-2\sqrt{3}, -2), (-2\sqrt{3}, 2), (2\sqrt{3}, -2), (2\sqrt{3}, 2)\}$

**35.** $\{(-2i\sqrt{2}, -2\sqrt{3}), (-2i\sqrt{2}, 2\sqrt{3}), (2i\sqrt{2}, -2\sqrt{3}),$
$(2i\sqrt{2}, 2\sqrt{3})\}$ **37.** $\{(-\sqrt{5}, -\sqrt{5}), (\sqrt{5}, \sqrt{5})\}$

**39.** $\{(i, 2i), (-i, -2i), (2, -1), (-2, 1)\}$

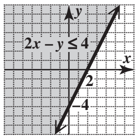
**41.** $\{(2, -3), (-3, 2)\}$





**43.** length: 12 ft; width: 7 ft **45.** \$20; $\frac{4}{5}$ thousand or 800 calculators

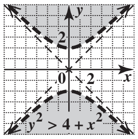
**47.**



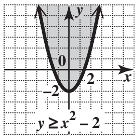
## Section 13.5 (pages 845–846)

**1.** C **3.** B **5.** A

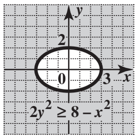
**7.**



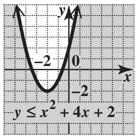
**9.**



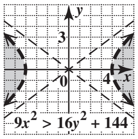
**11.**



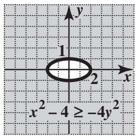
**13.**



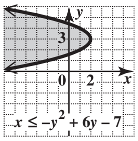
**15.**



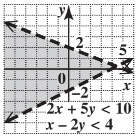
**17.**



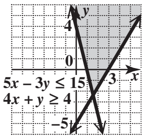
**19.**



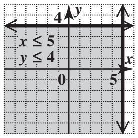
**21.**



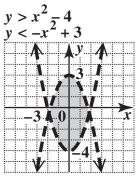
**23.**



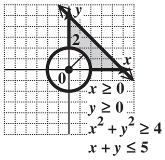
**25.**



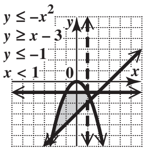
**27.**



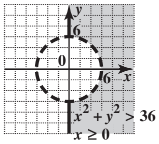
**29.**



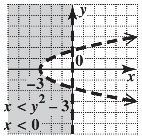
**31.**



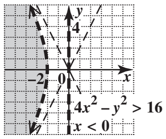
**33.**



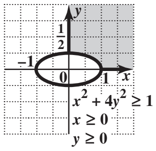
**35.**



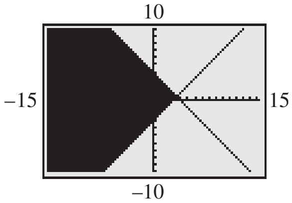
**37.**



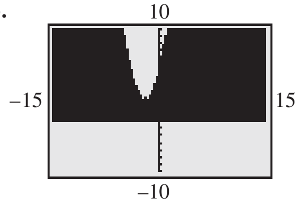
**39.**



**41.**



**43.**

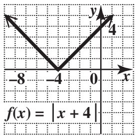


**45.** **(a)** 6 **(b)** $\frac{7}{2}$ **(c)** $\frac{8}{3}$ **(d)** $\frac{9}{4}$

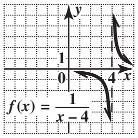
**47.** **(a)** 0 **(b)** 2 **(c)** 6 **(d)** 12

## Chapter 13 Review Exercises (pages 850–851)

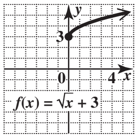
**1.**



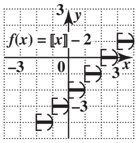
**2.**



**3.**



**4.**



**5.** $(x + 2)^2 + (y - 4)^2 = 9$

**6.** $(x + 1)^2 + (y + 3)^2 = 25$

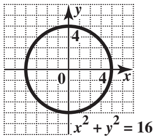
**7.** $(x - 4)^2 + (y - 2)^2 = 36$

**8.** center: $(-3, 2)$; $r = 4$

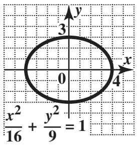
**9.** center: $(4, 1)$; $r = 2$ **10.** center: $(-1, -5)$; $r = 3$

**11.** center: $(3, -2)$; $r = 5$

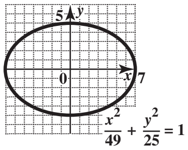
**12.**



**13.**

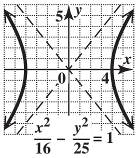


**14.**

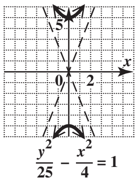


**15.** $\dfrac{x^2}{65{,}286{,}400} + \dfrac{y^2}{2{,}560{,}000} = 1$ **16.** **(a)** 348.2 ft **(b)** 1787.6 ft

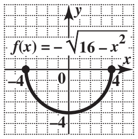
**17.**



**18.**



**19.**



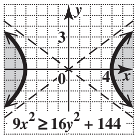
**20.** circle **21.** parabola **22.** hyperbola **23.** ellipse **24.** parabola

**25.** hyperbola **26.** $\dfrac{x^2}{625} - \dfrac{y^2}{1875} = 1$ **27.** $\{(6, -9), (-2, -5)\}$

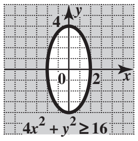
**28.** $\{(1, 2), (-5, 14)\}$ **29.** $\{(4, 2), (-1, -3)\}$

**30.** $\{(-2, -4), (8, 1)\}$ **31.** $\{(-\sqrt{2}, 2), (-\sqrt{2}, -2),$ $(\sqrt{2}, -2), (\sqrt{2}, 2)\}$ **32.** $\{(-\sqrt{6}, -\sqrt{3}), (-\sqrt{6}, \sqrt{3}),$ $(\sqrt{6}, -\sqrt{3}), (\sqrt{6}, \sqrt{3})\}$ **33.** 0, 1, or 2 **34.** 0, 1, 2, 3, or 4

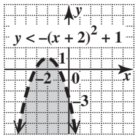
**35.**



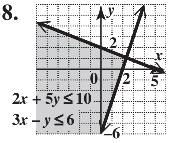
**36.**



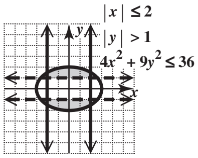
**37.**



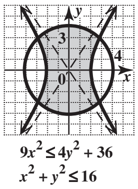
**38.**



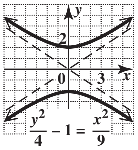
**39.**



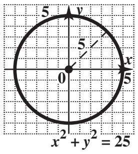
**40.**



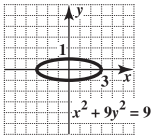
**41.**



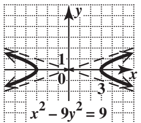
**42.**



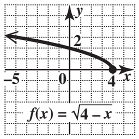
**43.**



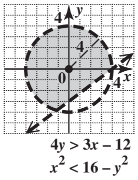
**44.**



**45.**



**46.**

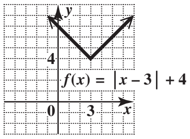


## Chapter 13 Test (page 852)

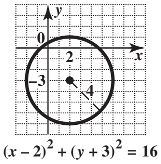
[13.1] **1.** 0 **2.** $[0, \infty)$ **3.** $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

**4.** **(a)** C **(b)** A **(c)** D **(d)** B

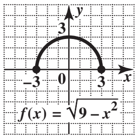
**5.** domain: $(-\infty, \infty)$; range: $[4, \infty)$



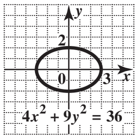
[13.2] **6.** center: $(2, -3)$; radius: 4



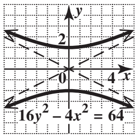
**7.** center: $(-4, 1)$; radius: 5 [13.3] **8.**



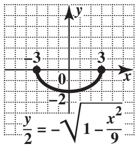
[13.2] **9.**



[13.3] **10.**

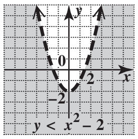


**11.**

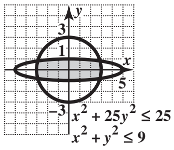


**12.** ellipse **13.** hyperbola **14.** circle **15.** parabola [13.4] **16.** $\left\{\left(-\frac{1}{2}, -10\right), (5, 1)\right\}$ **17.** $\left\{(-2, -2), \left(\frac{14}{5}, -\frac{2}{5}\right)\right\}$ **18.** $\left\{\left(-\sqrt{22}, -\sqrt{3}\right), \left(-\sqrt{22}, \sqrt{3}\right), \left(\sqrt{22}, -\sqrt{3}\right), \left(\sqrt{22}, \sqrt{3}\right)\right\}$

[13.5] **19.**



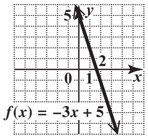
**20.**



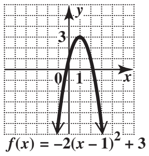
## Chapters 1–13 Cumulative Review Exercises (pages 853–854)

[3.3, 7.1] **1.** $\frac{2}{3}$ [3.4, 7.2] **2.** $3x + 2y = -13$ [4.6] **3.** $25y^2 - 30y + 9$ [4.7] **4.** $4x^3 - 4x^2 + 3x + 5 + \dfrac{3}{2x + 1}$ [5.2, 5.3] **5.** $(3x + 2)(4x - 5)$ [5.4] **6.** $(z^2 + 1)(z + 1)(z - 1)$ **7.** $(a - 3b)(a^2 + 3ab + 9b^2)$ [6.2] **8.** $\dfrac{y - 1}{y(y - 3)}$ [6.4] **9.** $\dfrac{3c + 5}{(c + 5)(c + 3)}$ **10.** $\dfrac{1}{p}$ [6.7] **11.** $1\frac{1}{5}$ hr [8.1–8.3, 8.6] **12.** $\{(3, -3)\}$ [8.4, 8.6] **13.** $\{(4, 1, -2)\}$ [13.4] **14.** $\left\{(-1, 5), \left(\frac{5}{2}, -2\right)\right\}$ [8.5] **15.** 40 mph [4.1, 4.2] **16.** $\dfrac{a^5}{4}$ [10.4] **17.** $2\sqrt[3]{2}$ [10.5] **18.** $\dfrac{3\sqrt{10}}{2}$ [10.7] **19.** $\frac{7}{5} + \frac{11}{5}i$ [2.3] **20.** $\left\{\frac{2}{3}\right\}$ [2.8] **21.** $\left(-\infty, \frac{3}{5}\right]$ [9.2] **22.** $\{-4, 4\}$ **23.** $(-\infty, -5) \cup (10, \infty)$ [10.6] **24.** $\emptyset$ [5.5] **25.** $\left\{\frac{1}{5}, -\frac{3}{2}\right\}$ [11.2, 11.3] **26.** $\left\{\dfrac{3 \pm \sqrt{33}}{6}\right\}$ [11.4] **27.** $\left\{\pm\dfrac{\sqrt{6}}{2}, \pm\sqrt{7}\right\}$ [12.6] **28.** $\{3\}$ [11.5] **29.** $v = \dfrac{\pm\sqrt{rFkw}}{kw}$ [12.1] **30.** $f^{-1}(x) = \sqrt[3]{x - 4}$ [12.4, 12.5] **31.** **(a)** 4 **(b)** 7 [12.4] **32.** $\log \dfrac{(3x + 7)^2}{4}$ [12.2] **33.** **(a)** \$86.8 billion **(b)** \$169.5 billion [13.1] **34.** domain: $(-\infty, \infty)$; range: $[0, \infty)$

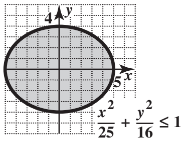
[7.4] **35.**



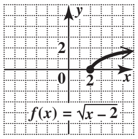
[11.6] **36.**



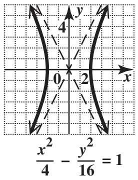
[13.5] **37.**



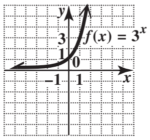
[13.1] **38.**



[13.3] **39.**



[12.2] **40.**



## 14 SEQUENCES AND SERIES

### Section 14.1 (pages 860–862)

**1.** 2, 3, 4, 5, 6 **3.** $4, \frac{5}{2}, 2, \frac{7}{4}, \frac{8}{5}$ **5.** 3, 9, 27, 81, 243 **7.** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$ **9.** $5, -5, 5, -5, 5$ **11.** $0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \frac{24}{5}$ **13.** $-70$ **15.** $\frac{49}{23}$ **17.** 171 **19.** $4n$ **21.** $-8n$ **23.** $\dfrac{1}{3^n}$ **25.** $\dfrac{n + 1}{n + 4}$ **27.** \$110, \$109, \$108, \$107, \$106, \$105; \$400 **29.** \$6554 **31.** $4 + 5 + 6 + 7 + 8 = 30$ **33.** $3 + 6 + 11 = 20$ **35.** $-1 + 1 - 1 + 1 - 1 + 1 = 0$ **37.** $0 + 6 + 14 + 24 + 36 = 80$

*Answers may vary for Exercises 39–43.*

**39.** $\sum_{i=1}^{5}(i + 2)$ **41.** $\sum_{i=1}^{5} 2^i(-1)^i$ **43.** $\sum_{i=1}^{4} i^2$ **45.** A sequence is a list of terms in a specific order, while a series is the indicated sum of the terms of a sequence. **47.** 9 **49.** $\frac{40}{9}$ **51.** 8036 **53.** $a = 6, d = 2$ **55.** 10

### Section 14.2 (pages 867–869)

**1.** $d = 1$ **3.** not arithmetic **5.** $d = -5$ **7.** 5, 9, 13, 17, 21 **9.** $-2, -6, -10, -14, -18$ **11.** $a_n = 5n - 3$ **13.** $a_n = \frac{3}{4}n + \frac{9}{4}$ **15.** $a_n = 3n - 6$ **17.** 76 **19.** 48 **21.** $-1$ **23.** 16 **25.** 6 **27.** $n$ represents the number of terms. **29.** 81 **31.** $-3$ **33.** 87 **35.** 390 **37.** 395 **39.** 31,375 **41.** \$465 **43.** \$2100 per month **45.** 68; 1100 **47.** no; 3; 9 **49.** 18 **51.** $\frac{1}{2}$

### Section 14.3 (pages 876–878)

**1.** $r = 2$ **3.** not geometric **5.** $r = -3$ **7.** $r = -\frac{1}{2}$

*There are alternative forms of the answers in Exercises 9–13.*

**9.** $a_n = -5(2)^{n-1}$ **11.** $a_n = -2\left(-\dfrac{1}{3}\right)^{n-1}$ **13.** $a_n = 10\left(-\dfrac{1}{5}\right)^{n-1}$ **15.** $2(5)^9 = 3{,}906{,}250$ **17.** $\dfrac{1}{2}\left(\dfrac{1}{3}\right)^{11}$, or $\dfrac{1}{354{,}294}$ **19.** $2\left(\dfrac{1}{2}\right)^{24} = \dfrac{1}{2^{23}}$ **21.** 2, 6, 18, 54, 162 **23.** $5, -1, \frac{1}{5}, -\frac{1}{25}, \frac{1}{125}$ **25.** $\frac{121}{243}$ **27.** $-1.997$ **29.** 2.662 **31.** $-2.982$ **33.** \$33,410.84 **35.** \$104,273.05 **37.** 9

**39.** $\frac{10{,}000}{11}$ **41.** $-\frac{9}{20}$ **43.** The sum does not exist. **45.** $10\left(\frac{3}{5}\right)^4 \approx 1.3$ ft
**47.** 3 days; $\frac{1}{4}$ g **49. (a)** $1.1(1.06)^5 \approx 1.5$ billion units
**(b)** approximately 12 yr **51.** $\$50{,}000\left(\frac{3}{4}\right)^8 \approx \$5005.65$
**53.** $0.33333\ldots$ **54.** $0.66666\ldots$ **55.** $0.99999\ldots$
**56.** $\frac{a_1}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$; Therefore, $0.99999\ldots = 1$
**57.** B **58.** $0.49999\ldots = 0.4 + 0.09999\ldots = \frac{4}{10} + \frac{1}{10}(0.9999\ldots) = \frac{4}{10} + \frac{1}{10}(1) = \frac{5}{10} = \frac{1}{2}$ **59.** $9x^2 + 12xy + 4y^2$
**61.** $a^3 - 3a^2b + 3ab^2 - b^3$

## Section 14.4 (page 883)

**1.** 720 **3.** 40,320 **5.** 15 **7.** 1 **9.** 120 **11.** 15 **13.** 78
**15.** $m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$ **17.** $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ **19.** $8x^3 + 36x^2 + 54x + 27$
**21.** $\frac{x^4}{16} - \frac{x^3y}{2} + \frac{3x^2y^2}{2} - 2xy^3 + y^4$ **23.** $x^8 + 4x^6 + 6x^4 + 4x^2 + 1$
**25.** $27x^6 - 27x^4y^2 + 9x^2y^4 - y^6$ **27.** $r^{12} + 24r^{11}s + 264r^{10}s^2 + 1760r^9s^3$ **29.** $3^{14}x^{14} - 14(3^{13})x^{13}y + 91(3^{12})x^{12}y^2 - 364(3^{11})x^{11}y^3$
**31.** $t^{20} + 10t^{18}u^2 + 45t^{16}u^4 + 120t^{14}u^6$ **33.** $120(2^7)m^7n^3$
**35.** $\frac{7x^2y^6}{16}$ **37.** $36k^7$ **39.** $160x^6y^3$ **41.** $4320x^9y^4$

## Chapter 14 Review Exercises (pages 887–888)

**1.** $-1, 1, 3, 5$ **2.** $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ **3.** $1, 4, 9, 16$ **4.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$
**5.** $0, 3, 8, 15$ **6.** $1, -2, 3, -4$ **7.** $1 + 4 + 9 + 16 + 25$
**8.** $2 + 3 + 4 + 5 + 6 + 7$ **9.** $11 + 16 + 21 + 26$ **10.** 18
**11.** 126 **12.** $\frac{2827}{840}$ **13.** \$15,444 billion **14.** arithmetic; $d = 3$
**15.** arithmetic; $d = 4$ **16.** geometric; $r = -\frac{1}{2}$ **17.** geometric; $r = -1$
**18.** neither **19.** geometric; $r = \frac{1}{2}$ **20.** 89 **21.** 73 **22.** 69
**23.** $a_n = -5n + 1$ **24.** $a_n = -3n + 9$ **25.** 15 **26.** 22 **27.** 152
**28.** 164 **29.** $a_n = -1(4)^{n-1}$ **30.** $a_n = \frac{2}{3}\left(\frac{1}{5}\right)^{n-1}$
**31.** $2(-3)^{10} = 118{,}098$ **32.** $5(2)^9 = 2560$ or $5(-2)^9 = -2560$
**33.** $\frac{341}{1024}$ **34.** 0 **35.** 1 **36.** The sum does not exist.
**37.** $32p^5 - 80p^4q + 80p^3q^2 - 40p^2q^3 + 10pq^4 - q^5$
**38.** $x^8 + 12x^6y + 54x^4y^2 + 108x^2y^3 + 81y^4$
**39.** $81t^{12} - 108t^9s^2 + 54t^6s^4 - 12t^3s^6 + s^8$ **40.** $7752(3)^{16}a^{16}b^3$
**41.** $a_{10} = 1536$; $S_{10} = 1023$ **42.** $a_{40} = 235$; $S_{10} = 280$
**43.** $a_{15} = 38$; $S_{10} = 95$ **44.** $a_9 = 6561$; $S_{10} = -14{,}762$
**45.** $a_n = 2(4)^{n-1}$ **46.** $a_n = 5n - 3$ **47.** $a_n = -3n + 15$
**48.** $a_n = 27\left(\frac{1}{3}\right)^{n-1}$ **49.** 10 sec **50.** \$21,973.00
**51.** approximately 42,000 **52.** $\frac{1}{128}$ **53. (a)** $\frac{5}{10} + \frac{5}{10}\left(\frac{1}{10}\right) + \frac{5}{10}\left(\frac{1}{10}\right)^2 + \frac{5}{10}\left(\frac{1}{10}\right)^3 + \cdots$ **(b)** $\frac{1}{10}$ **(c)** $\frac{5}{9}$
**54.** No, the sum cannot be found, because $r = 2$. This value of $r$ does not satisfy $|r| < 1$.

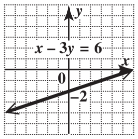
## Chapter 14 Test (page 889)

[14.1] **1.** 0, 2, 0, 2, 0 [14.2] **2.** 4, 6, 8, 10, 12 [14.3] **3.** 48, 24, 12, 6, 3 [14.2] **4.** 0 [14.3] **5.** $\frac{64}{3}$ or $-\frac{64}{3}$ [14.2] **6.** 75
[14.3] **7.** 124 or 44 [14.1] **8.** 85,311 [14.3] **9.** \$137,925.91
**10.** It has a sum if $|r| < 1$. [14.2] **11.** 70 **12.** 33 **13.** 125,250
[14.3] **14.** 42 **15.** $\frac{1}{3}$ **16.** The sum does not exist. [14.4] **17.** 40,320
**18.** 1 **19.** 15 **20.** 66 **21.** $81k^4 - 540k^3 + 1350k^2 - 1500k + 625$
**22.** $\frac{14{,}080x^8y^4}{9}$ [14.1] **23.** \$324 [14.3] **24.** $20(3^{11}) = 3{,}542{,}940$

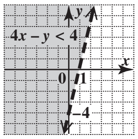
## Chapters 1–14 Cumulative Review Exercises (pages 890–891)

[1.4–1.6] **1.** 8 **2.** $-55$ [1.4] **3.** $-\frac{8}{3}, 10, 0, \frac{45}{15}$ (or 3), $0.82, -3$
**4.** $\sqrt{13}, -\sqrt{3}$ [2.3] **5.** $\left\{\frac{1}{6}\right\}$ [2.8] **6.** $[10, \infty)$ [9.2] **7.** $\left\{-\frac{9}{2}, 6\right\}$
[2.3] **8.** $\{9\}$ [9.1] **9.** $(-\infty, -3) \cup (4, \infty)$
[9.2] **10.** $(-\infty, -3] \cup [8, \infty)$ [5.5] **11.** $\left\{-\frac{5}{2}, 2\right\}$ [11.8] **12.** $[-2, 3]$
[6.6] **13.** $\emptyset$ [11.2, 11.3] **14.** $\left\{\frac{-5 \pm \sqrt{217}}{12}\right\}$ [12.2] **15.** $\left\{\frac{5}{2}\right\}$
[12.6] **16.** $\{2\}$ [4.5] **17.** $20p^2 - 2p - 6$
[4.6] **18.** $9k^2 - 42k + 49$ [4.4] **19.** $-5m^3 - 3m^2 + 3m + 8$
[4.7] **20.** $2t^3 + 3t^2 - 4t + 2 + \frac{3}{3t-2}$
[5.2, 5.3] **21.** $z(3z + 4)(2z - 1)$ [5.4] **22.** $(7a^2 + 3b)(7a^2 - 3b)$
**23.** $(c + 3d)(c^2 - 3cd + 9d^2)$ [4.1, 4.2] **24.** $\frac{9}{4}$ **25.** $-\frac{27p^2}{10}$
[6.2] **26.** $\frac{x+7}{x-2}$ [6.4] **27.** $\frac{3p - 26}{p(p+3)(p-4)}$ [10.4] **28.** $10\sqrt{2}$
[10.7] **29.** 73 [3.3, 7.1] **30.** $\frac{3}{4}$ [3.4, 7.2] **31.** $3x + y = 4$
[7.3] **32. (a)** yes **(b)** $\{-3, -2, 0, 1, 2\}$ **(c)** $\{2, 6, 4\}$
[8.1–8.3, 8.6] **33.** $\{(-1, -2)\}$ [8.4, 8.6] **34.** $\{(2, 1, 4)\}$
[13.4] **35.** $\left\{(-1, 5), \left(\frac{5}{2}, -2\right)\right\}$ [8.5] **36.** 2 lb

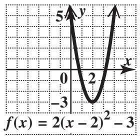
[3.2, 7.1] **37.**



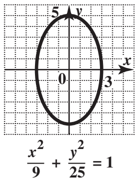
[9.3] **38.**



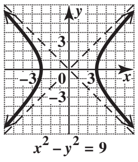
[11.6] **39.**



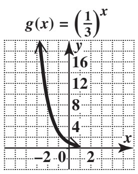
[13.2] **40.**



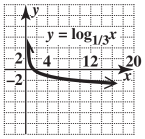
[13.3] **41.**



[12.2] **42.**



[12.3] **43.**



[12.1] **44.** $f^{-1}(x) = \frac{x-5}{9}$, or $f^{-1}(x) = \frac{1}{9}x - \frac{5}{9}$

[13.2] **45.** $(x+5)^2 + (y-12)^2 = 81$ [14.1] **46.** $-7, -2, 3, 8, 13$ [14.2, 14.3] **47.** **(a)** 78 **(b)** $\frac{75}{7}$ [14.2] **48.** 30 [14.4] **49.** $32a^5 - 80a^4 + 80a^3 - 40a^2 + 10a - 1$ **50.** $-\frac{45x^8y^6}{4}$

# APPENDICES

## Appendix A (pages 896–897)

**1.** $\{1, 2, 3, 4, 5, 6, 7\}$ **3.** {winter, spring, summer, fall} **5.** $\emptyset$ **7.** {L} **9.** $\{2, 4, 6, 8, 10, \dots\}$ **11.** The sets in **Exercises 9 and 10** are infinite sets. **13.** true **15.** false **17.** true **19.** true **21.** true **23.** true **25.** true **27.** false **29.** true **31.** true **33.** false **35.** true **37.** true **39.** false **41.** false **43.** true **45.** {g, h} **47.** {b, c, d, e, g, h} **49.** {a, c, e} $= B$ **51.** {d} $= D$ **53.** {a} **55.** {a, c, d, e} **57.** {a, c, e, f} **59.** $\emptyset$ **61.** $B$ and $D$; $C$ and $D$

## Appendix B (pages 903–904)

**1.** $\frac{1}{a^2b}$ **3.** $\frac{100y^{10}}{x^2}$ **5.** 0 **7.** $\frac{x^{10}}{2w^{13}y^5}$ **9.** $\frac{a^{15}}{-64b^{15}}$ **11.** $\frac{x^{16}z^{10}}{y^6}$ **13.** $-6a^4 + 11a^3 - 20a^2 + 26a - 15$ **15.** $8x^3 - 18x^2 + 6x - 16$ **17.** $x^2y - xy^2 + 6y^3$ **19.** $-3x^2 - 62x + 32$ **21.** $10x^3 - 4x^2 + 9x - 4$ **23.** $6x^2 - 19x - 7$ **25.** $4x^2 - 9x + 2$ **27.** $16t^2 - 9$ **29.** $4y^4 - 16$ **31.** $16x^2 - 24x + 9$ **33.** $36r^2 + 60ry + 25y^2$ **35.** $c^3 + 8d^3$ **37.** $64x^3 - 1$ **39.** $14t^3 + 45st^2 + 18s^2t - 5s^3$ **41.** $4xy^3(2x^2y + 3x + 9y)$ **43.** $(x+3)(x-5)$ **45.** $(2x+3)(x-6)$ **47.** $(6t+5)(6t-5)$ **49.** $(4t+3)^2$ **51.** $p(2m-3n)^2$ **53.** $(x+1)(x^2-x+1)$ **55.** $(2t+5)(4t^2-10t+25)$ **57.** $(t^2-5)(t^4+5t^2+25)$ **59.** $(5x+2y)(t+3r)$ **61.** $(6r-5s)(a+2b)$ **63.** $(t^2+1)(t+1)(t-1)$ **65.** $(2x+3y-1)(2x+3y+1)$ **67.** $4(x-5)(x-2)$

## Appendix C (page 908)

**1.** $x - 5$ **3.** $4m - 1$ **5.** $2a + 4 + \frac{5}{a+2}$ **7.** $p - 4 + \frac{9}{p+1}$ **9.** $4a^2 + a + 3$ **11.** $x^4 + 2x^3 + 2x^2 + 7x + 10 + \frac{18}{x-2}$ **13.** $-4r^5 - 7r^4 - 10r^3 - 5r^2 - 11r - 8 + \frac{-5}{r-1}$ **15.** $-3y^4 + 8y^3 - 21y^2 + 36y - 72 + \frac{143}{y+2}$ **17.** 7 **19.** $-2$ **21.** 0 **23.** By the remainder theorem, a 0 remainder means that $P(k) = 0$. That is, $k$ is a number that makes $P(x) = 0$. **25.** yes **27.** no **29.** yes **31.** no **33.** $(2x-3)(x+4)$ **34.** $\left\{-4, \frac{3}{2}\right\}$ **35.** 0 **36.** 0 **37.** $a$ **38.** Yes, $x - 3$ is a factor. $Q(x) = (x-3)(3x-1)(x+2)$

# Glossary

For a more complete discussion, see the section(s) in parentheses.

## A

**absolute value** The absolute value of a number is the distance between 0 and the number on a number line. (Section 1.4)

**absolute value equation** An absolute value equation is an equation that involves the absolute value of a variable expression. (Section 9.2)

**absolute value function** The function defined by $f(x) = |x|$ with a graph that includes portions of two lines is called the absolute value function. (Section 13.1)

**absolute value inequality** An absolute value inequality is an inequality that involves the absolute value of a variable expression. (Section 9.2)

**addition property of equality** The addition property of equality states that the same number can be added to (or subtracted from) both sides of an equation to obtain an equivalent equation. (Section 2.1)

**addition property of inequality** The addition property of inequality states that the same number can be added to (or subtracted from) both sides of an inequality without changing the solution set. (Section 2.8)

**additive inverse (opposite)** The additive inverse of a number $x$, symbolized $-x$, is the number that is the same distance from 0 on the number line as $x$, but on the opposite side of 0. The number 0 is its own additive inverse. For all real numbers $x$, $x + (-x) = (-x) + x = 0$. (Section 1.4)

**algebraic expression** An algebraic expression is a sequence of numbers, variables, operation symbols, and/or grouping symbols (such as parentheses) formed according to the rules of algebra. (Section 1.3)

**annuity** An annuity is a sequence of equal payments made at equal periods of time. (Section 14.3)

**area** Area is a measure of the surface covered by a two-dimensional (flat) figure. (Section 2.5)

**arithmetic mean (average)** The arithmetic mean of a group of numbers is the sum of all the numbers divided by the number of numbers. (Section 14.1)

**arithmetic sequence (arithmetic progression)** An arithmetic sequence is a sequence in which each term after the first differs from the preceding term by a constant difference. (Section 14.2)

**associative property of addition** The associative property of addition states that the grouping of terms in a sum does not affect the sum. (Section 1.7)

**associative property of multiplication** The associative property of multiplication states that the grouping of factors in a product does not affect the product. (Section 1.7)

**asymptote** A line that a graph more and more closely approaches as the graph gets farther away from the origin is called an asymptote of the graph. (Sections 12.2, 13.1)

**asymptotes of a hyperbola** The two intersecting straight lines that the branches of a hyperbola approach are called asymptotes of the hyperbola. (Section 13.3)

**augmented matrix** An augmented matrix is a matrix that has a vertical bar that separates the columns of the matrix into two groups, separating the coefficients from the constants of the corresponding system of equations. (Section 8.6)

**axis (axis of symmetry)** The axis of a parabola is the vertical or horizontal line (depending on the orientation of the graph) through the vertex of the parabola. (Sections 4.4, 11.6, 11.7)

## B

**base** The base in an exponential expression is the expression that is the repeated factor. In $b^x$, $b$ is the base. (Sections 1.2, 4.1)

**binomial** A binomial is a polynomial consisting of exactly two terms. (Section 4.4)

**binomial theorem (general binomial expansion)** The binomial theorem provides a formula used to expand a binomial raised to a power. (Section 14.4)

**boundary line** In the graph of an inequality, the boundary line separates the region that satisfies the inequality from the region that does not satisfy the inequality. (Sections 9.3, 13.5)

## C

**center of a circle** The fixed point that is a fixed distance from all the points that form a circle is the center of the circle. (Section 13.2)

**center of an ellipse** The center of an ellipse is the fixed point located exactly halfway between the two foci. (Section 13.2)

**center-radius form of the equation of a circle** The center-radius form of the equation of a circle with center $(h, k)$ and radius $r$ is $(x - h)^2 + (y - k)^2 = r^2$. (Section 13.2)

**circle** A circle is the set of all points in a plane that lie a fixed distance from a fixed point. (Section 13.2)

**circle graph (pie chart)** A circle graph (or pie chart) is a circle divided into sectors, or wedges, whose sizes show the relative magnitudes of the categories of data being represented. (Section 1.1)

**coefficient** (See **numerical coefficient.**)

**column of a matrix** A column of a matrix is a group of elements that are read vertically. (Section 8.6)

**combined variation** A relationship among variables that involves both direct and inverse variation is called combined variation. (Section 7.6)

**combining like terms** Combining like terms is a method of adding or subtracting terms having exactly the same variable factors by using the properties of real numbers. (Section 1.8)

**common difference** The common difference $d$ is the difference between any two adjacent terms of an arithmetic sequence. (Section 14.2)

**common factor** An integer that is a factor of two or more integers is called a common factor of those integers. (Section 5.1)

**common logarithm** A common logarithm is a logarithm having base 10. (Section 12.5)

**common ratio** The common ratio $r$ is the constant multiplier between adjacent terms in a geometric sequence. (Section 14.3)

**commutative property of addition** The commutative property of addition states that the order of terms in a sum does not affect the sum. (Section 1.7)

**commutative property of multiplication** The commutative property of multiplication states that the order of factors in a product does not affect the product. (Section 1.7)

**complement of a set** The set of elements in the universal set that are not in a set $A$ is the complement of $A$, written $A'$. (Appendix A)

**complementary angles (complements)** Complementary angles are two angles whose measures have a sum of 90°. (Section 2.4)

**completing the square** The process of adding to a binomial the expression that makes it a perfect square trinomial is called completing the square. (Section 11.2)

**complex conjugate** The complex conjugate of $a + bi$ is $a - bi$. (Section 10.7)

**complex fraction** A complex fraction is a quotient with one or more fractions in the numerator, denominator, or both. (Section 6.5)

**complex number** A complex number is any number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is the imaginary unit. (Section 10.7)

**components** In an ordered pair $(x, y)$, $x$ and $y$ are called the components of the ordered pair. (Section 7.1)

**composite function** If $g$ is a function of $x$, and $f$ is a function of $g(x)$, then $f(g(x))$ defines the composite function of $f$ and $g$. It is symbolized $(f \circ g)(x)$. (Section 7.5)

**composite number** A natural number greater than 1 that is not prime is a composite number. It is composed of prime factors represented in one and only one way. (Section 1.1)

**composition of functions** The process of finding a composite function is called composition of functions. (Section 7.5)

**compound inequality** A compound inequality consists of two inequalities linked by a connective word such as *and* or *or*. (Section 9.1)

**conditional equation** A conditional equation is true for some replacements of the variable and false for others. (Section 2.3)

**conic section** When a plane intersects an infinite cone at different angles, the figures formed by the intersections are called conic sections. (Section 13.2)

**conjugate** The conjugate of $a + b$ is $a - b$. (Section 10.5)

**consecutive integers** Two integers that differ by 1 are called consecutive integers. (Sections 2.4, 5.6)

**consistent system** A system of equations with a solution is called a consistent system. (Section 8.1)

**constant function** A linear function of the form $f(x) = b$, where $b$ is a constant, is called a constant function. (Section 7.4)

**constant of variation** In the variation equations $y = kx$, $y = \frac{k}{x}$, or $y = kxz$, the nonzero real number $k$ is called the constant of variation. (Section 7.6)

**contradiction** A contradiction is an equation that is never true. It has no solution. (Section 2.3)

**coordinate on a number line** Every point on a number line is associated with a unique real number, called the coordinate of the point. (Section 1.4)

**coordinates of a point** The numbers in an ordered pair are called the coordinates of the corresponding point in the plane. (Sections 3.1, 7.1)

**cross products** The cross products in the proportion $\frac{a}{b} = \frac{c}{d}$ are $ad$ and $bc$. (Section 2.6)

**cube root** A number $b$ is a cube root of $a$ if $b^3 = a$ is true. (Section 10.1)

**cube root function** The function defined by $f(x) = \sqrt[3]{x}$ is called the cube root function. (Section 10.1)

## D

**degree** A degree is a basic unit of measure for angles in which one degree (1°) is $\frac{1}{360}$ of a complete revolution. (Section 2.4)

**degree of a polynomial** The degree of a polynomial is the greatest degree of any of the terms in the polynomial. (Section 4.4)

**degree of a term** The degree of a term is the sum of the exponents on the variables in the term. (Section 4.4)

**denominator** The number below the fraction bar in a fraction is called the denominator. It indicates the number of equal parts in a whole. (Section 1.1)

**dependent equations** Equations of a system that have the same graph (because they are different forms of the same equation) are called dependent equations. (Section 8.1)

**dependent variable** In an equation relating $x$ and $y$, if the value of the variable $y$ depends on the value of the variable $x$, then $y$ is called the dependent variable. (Section 7.3)

**descending powers** A polynomial in one variable is written in descending powers of the variable if the exponents on the variables of the terms of the polynomial decrease from left to right. (Section 4.4)

**difference** The answer to a subtraction problem is called the difference. (Section 1.1)

**difference of cubes** The difference of cubes, $x^3 - y^3$, can be factored as $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. (Section 5.4)

**difference of squares** The difference of squares, $x^2 - y^2$, can be factored as $x^2 - y^2 = (x + y)(x - y)$. (Section 5.4)

**direct variation** $y$ varies directly as $x$ if there exists a nonzero real number (constant) $k$ such that $y = kx$. (Section 7.6)

**discriminant** The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$ under the radical in the quadratic formula. (Section 11.3)

**disjoint sets** Sets that have no elements in common are disjoint sets. (Appendix A)

**distributive property of multiplication with respect to addition (distributive property)** For any real numbers $a$, $b$, and $c$, the distributive property states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$. (Section 1.7)

**domain** The set of all first components ($x$-values) in the ordered pairs of a relation is called the domain. (Section 7.3)

## E

**element of a matrix** The numbers in a matrix are called the elements of the matrix. (Section 8.6)

**elements (members)** The elements (members) of a set are the objects that belong to the set. (Section 1.3, Appendix A)

**elimination method** The elimination method is an algebraic method used to solve a system of equations in which the equations of the system are combined so that one or more variables is eliminated. (Section 8.3)

**ellipse** An ellipse is the set of all points in a plane such that the sum of the distances from two fixed points is constant. (Section 13.2)

**empty set (null set)** The empty set, denoted by $\{ \}$ or $\emptyset$, is the set containing no elements. (Section 2.3, Appendix A)

**equation** An equation is a statement that two algebraic expressions are equal. (Section 1.3)

**equivalent equations** Equivalent equations are equations that have the same solution set. (Section 2.1)

**equivalent inequalities** Equivalent inequalities are inequalities that have the same solution set. (Section 2.8)

**exponent (power)** An exponent, or power, is a number that indicates how many times its base is used as a factor. In $b^x$, $x$ is the exponent (power). (Sections 1.2, 4.1)

**exponential equation** An exponential equation is an equation that has a variable in at least one exponent. (Section 12.2)

**exponential expression** A number or letter (variable) written with an exponent is an exponential expression. (Sections 1.2, 4.1)

**exponential function with base *a*** An exponential function with base $a$ is a function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$ for all real numbers $x$. (Section 12.2)

**extraneous solution (extraneous value)** A proposed solution to an equation, following any of several procedures in the solution process, that does not satisfy the original equation is called an extraneous solution. (Sections 7.6, 10.6)

**extremes of a proportion** In the proportion $\frac{a}{b} = \frac{c}{d}$, the $a$- and $d$-terms are called the extremes. (Section 2.6)

## F

**factor** If $a$, $b$, and $c$ represent numbers and $a \cdot b = c$, then $a$ and $b$ are factors of $c$. (Sections 1.1, 5.1)

**factored** A number is factored by writing it as the product of two or more numbers. (Section 1.1)

**factored form** An expression is in factored form when it is written as a product. (Section 5.1)

**factoring** Writing a polynomial as the product of two or more simpler polynomials is called factoring. (Section 5.1)

**factoring by grouping** Factoring by grouping is a method for grouping the terms of a polynomial in such a way that the polynomial can be factored. It is used when the greatest common factor of the terms of the polynomial is 1. (Section 5.1)

**factoring out the greatest common factor** Factoring out the greatest common factor is the process of using the distributive property to write a polynomial as a product of the greatest common factor and a simpler polynomial. (Section 5.1)

**finite sequence** A finite sequence has a domain that includes only the first $n$ positive integers. (Section 14.1)

**first-degree equation** A first-degree (linear) equation has no term with the variable to a power other than 1. (Section 7.1)

**foci** (singular, **focus)** Foci are fixed points used to determine the points that form a parabola, an ellipse, or a hyperbola. (Sections 13.2, 13.3)

**FOIL** FOIL is a mnemonic device which represents a method for multiplying two binomials $(a + b)(c + d)$. Multiply **F**irst terms $ac$, **O**uter terms $ad$, **I**nner terms $bc$, and **L**ast terms $bd$. Then combine like terms. (Section 4.5)

**formula** A formula is an equation in which variables are used to describe a relationship among several quantities. (Section 2.5)

**fourth root** A number $b$ is a fourth root of $a$ if $b^4 = a$ is true. (Section 10.1)

**function** A function is a set of ordered pairs $(x, y)$ in which each value of the first component $x$ corresponds to exactly one value of the second component $y$. (Section 7.3)

**function notation** If a function is denoted by $f$, the notation $y = f(x)$ is called function notation. Here $y$, or $f(x)$, represents the value of the function at $x$. (Section 7.4)

**fundamental rectangle** The asymptotes of a hyperbola are the extended diagonals of its fundamental rectangle, with corners at the points $(a, b)$, $(-a, b)$, $(-a, -b)$, and $(a, -b)$. (Section 13.3)

**future value of an annuity** The future value of an annuity is the sum of the compound amounts of all the payments, compounded to the end of the term. (Section 14.3)

## G

**general term of a sequence** The expression $a_n$, which defines a sequence, is called the general term of the sequence. (Section 14.1)

**geometric sequence (geometric progression)** A geometric sequence is a sequence in which each term after the first is a constant multiple of the preceding term. (Section 14.3)

**graph of a number** The point on a number line that corresponds to a number is its graph. (Section 1.4)

**graph of an equation** The graph of an equation in two variables is the set of all points that correspond to all of the ordered pairs that satisfy the equation. (Sections 3.2, 7.1)

**graph of a relation** The graph of a relation is the graph of its ordered pairs. (Section 7.3)

**graphing method** The graphing method for solving a system of equations requires graphing all equations of the system on the same axes and locating the ordered pair(s) of their intersection. (Section 8.1)

**greatest common factor (GCF)** The greatest common factor of a list of integers is the largest factor of all those integers. The greatest common factor of the terms of a polynomial is the largest factor of all the terms in the polynomial. (Sections 1.1, 5.1)

**greatest integer function** The function defined by $f(x) = [\![x]\!]$, where the symbol $[\![x]\!]$ is used to represent the greatest integer less than or equal to $x$, is called the greatest integer function. (Section 13.1)

**grouping symbols** Examples of grouping symbols are parentheses ( ), brackets [ ], and fraction bars. (Section 1.2)

## H

**horizontal line test** The horizontal line test states that a function is one-to-one if every horizontal line intersects the graph of the function at most once. (Section 12.1)

**hyperbola** A hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points is constant. (Section 13.3)

**hypotenuse** The side opposite the right angle in a right triangle is the longest side and is called the hypotenuse. (Sections 5.6, 10.3)

## I

**identity** An identity is an equation that is true for all valid replacements of the variable. It has an infinite number of solutions. (Section 2.3)

**identity element for addition** For all real numbers $a$, $a + 0 = 0 + a = a$. The number 0 is called the identity element for addition. (Section 1.7)

**identity element for multiplication** For all real numbers $a$, $a \cdot 1 = 1 \cdot a = a$. The number 1 is called the identity element for multiplication. (Section 1.7)

**identity property** The identity property for addition states that the sum of 0 and any number equals the number. The identity property for multiplication states that the product of 1 and any number equals the number. (Section 1.7)

**imaginary part** The imaginary part of the complex number $a + bi$ is $b$. (Section 10.7)

**imaginary unit** The symbol $i$, which represents $\sqrt{-1}$, is called the imaginary unit. (Section 10.7)

**inconsistent system** An inconsistent system of equations is a system with no solution. (Section 8.1)

**independent equations** Equations of a system that have different graphs are called independent equations. (Section 8.1)

**independent variable** In an equation relating $x$ and $y$, if the value of the variable $y$ depends on the value of the variable $x$, then $x$ is called the independent variable. (Section 7.3)

**index (order)** In a radical of the form $\sqrt[n]{a}$, $n$ is called the index or order. (Section 10.1)

**index of summation** When using summation notation, $\sum_{i=1}^{n} f(i)$, the letter $i$ is called the index of summation. Other letters can be used. (Section 14.1)

**inequality** An inequality is a statement that two expressions are not equal. (Section 1.2)

**infinite sequence** An infinite sequence is a function with the set of all positive integers as the domain. (Section 14.1)

**inner product** When using the FOIL method to multiply two binomials $(a + b)(c + d)$, the inner product is $bc$. (Section 4.5)

**integers** The set of integers is $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. (Section 1.4)

**intersection** The intersection of two sets $A$ and $B$, written $A \cap B$, is the set of elements that belong to *both A and B*. (Section 9.1, Appendix A)

**interval** An interval is a portion of a number line. (Section 2.8)

**interval notation** Interval notation is a simplified notation that uses parentheses ( ) and/or brackets [ ] and/or the infinity symbol $\infty$ to describe an interval on a number line. (Section 2.8)

**inverse of a function *f*** If $f$ is a one-to-one function, then the inverse of $f$ is the set of all ordered pairs of the form $(y, x)$ where $(x, y)$ belongs to $f$. (Section 12.1)

**inverse property** The inverse property for addition states that a number added to its opposite (additive inverse) is 0. The inverse property for multiplication states that a number multiplied by its reciprocal (multiplicative inverse) is 1. (Section 1.7)

**inverse variation** $y$ varies inversely as $x$ if there exists a nonzero real number (constant) $k$ such that $y = \frac{k}{x}$. (Section 7.6)

**irrational number** An irrational number cannot be written as the quotient of two integers, but can be represented by a point on a number line. (Sections 1.4, 10.1)

## J

**joint variation** $y$ varies jointly as $x$ and $z$ if there exists a nonzero real number (constant) $k$ such that $y = kxz$. (Section 7.6)

## L

**least common denominator (LCD)** Given several denominators, the least multiple that is divisible by all the denominators is called the least common denominator. (Sections 1.1, 6.3)

**legs of a right triangle** The two shorter perpendicular sides of a right triangle are called the legs. (Sections 5.6, 10.3)

**like radicals** Like radicals are multiples of the same root of the same number or expression. (Section 10.3)

**like terms** Terms with exactly the same variables raised to exactly the same powers are called like terms. (Sections 1.8, 4.4)

**linear equation in one variable** A linear equation in one variable can be written in the form $Ax + B = C$, where $A$, $B$, and $C$ are real numbers, with $A \neq 0$. (Section 2.1)

**linear equation in two variables** A linear equation in two variables is an equation that can be written in the form $Ax + By = C$, where $A$, $B$, and $C$ are real numbers, and $A$ and $B$ are not both 0. (Sections 3.1, 7.1)

**linear function** A function defined by an equation of the form $f(x) = ax + b$, for real numbers $a$ and $b$, is a linear function. The value of $a$ is the slope $m$ of the graph of the function. (Section 7.4)

**linear inequality in one variable** A linear inequality in one variable can be written in the form $Ax + B < C$, $Ax + B \leq C$, $Ax + B > C$, or $Ax + B \geq C$, where $A$, $B$, and $C$ are real numbers, with $A \neq 0$. (Section 2.8)

**linear inequality in two variables** A linear inequality in two variables can be written in the form $Ax + By < C$, $Ax + By \leq C$, $Ax + By > C$, or $Ax + By \geq C$, where $A$, $B$, and $C$ are real numbers, and $A$ and $B$ are not both 0. (Section 9.3)

**linear system (system of linear equations)** Two or more linear equations in two or more variables form a linear system. (Section 8.1)

**line graph** A line graph is a series of line segments in two dimensions that connect points representing data. (Section 3.1)

**line of symmetry** The axis of a parabola is a line of symmetry for the graph. It is a line that can be drawn through the vertex of the graph in such a way that the part of the graph on one side of the line is an exact reflection of the part on the opposite side. (Sections 4.4, 11.6)

**logarithm** A logarithm is an exponent. The expression $\log_a x$ represents the exponent to which the base $a$ must be raised to obtain $x$. (Section 12.3)

**logarithmic equation** A logarithmic equation is an equation with a logarithm of a variable expression in at least one term. (Section 12.3)

**logarithmic function with base *a*** If $a$ and $x$ are positive numbers with $a \neq 1$, then $f(x) = \log_a x$ defines the logarithmic function with base $a$. (Section 12.3)

**lowest terms** A fraction is in lowest terms if the greatest common factor of the numerator and denominator is 1. (Sections 1.1, 6.1)

## M

**mathematical model** In a real-world problem, a mathematical model is one or more equations (or inequalities) that describe the situation. (Section 3.1)

**matrix** (plural, **matrices**) A matrix is a rectangular array of numbers consisting of horizontal rows and vertical columns. (Section 8.6)

**means of a proportion** In the proportion $\frac{a}{b} = \frac{c}{d}$, the $b$- and $c$-terms are called the means. (Section 2.6)

**mixed number** A mixed number includes a whole number and a fraction written together and is understood to be the sum of the whole number and the fraction. (Section 1.1)

**monomial** A monomial is a polynomial consisting of exactly one term. (Section 4.4)

**multiplication property of equality** The multiplication property of equality states that the same nonzero number can be multiplied by (or divided into) both sides of an equation to obtain an equivalent equation. (Section 2.2)

**multiplication property of inequality** The multiplication property of inequality states that both sides of an inequality may be multiplied (or divided) by a positive number without changing the direction of the inequality symbol. Multiplying (or dividing) by a negative number reverses the direction of the inequality symbol. (Section 2.8)

**multiplicative inverse (reciprocal)** The multiplicative inverse (reciprocal) of a nonzero number $x$, symbolized $\frac{1}{x}$, is the real number which has the property that the product of the two numbers is 1. For all nonzero real numbers $x$, $\frac{1}{x} \cdot x = x \cdot \frac{1}{x} = 1$. (Section 1.6)

## N

**$n$-factorial ($n!$)** For any positive integer $n$, $n(n-1)(n-2)(n-3)\cdots(2)(1) = n!$. By definition, $0! = 1$. (Section 14.4)

**natural logarithm** A natural logarithm is a logarithm having base $e$. (Section 12.5)

**natural numbers** The set of natural numbers is the set of numbers used for counting: $\{1, 2, 3, 4, \ldots\}$. (Sections 1.1, 1.4)

**negative number** A negative number is located to the left of 0 on a number line. (Section 1.4)

**nonlinear equation** A nonlinear equation is an equation in which some terms have more than one variable or a variable of degree 2 or greater. (Section 13.4)

**nonlinear system of equations** A nonlinear system of equations consists of two or more equations to be considered at the same time, at least one of which is nonlinear. (Section 13.4)

**nonlinear system of inequalities** A nonlinear system of inequalities consists of two or more inequalities to be considered at the same time, at least one of which is nonlinear. (Section 13.5)

**number line** A line that has a point designated to correspond to the real number 0, and a standard unit chosen to represent the distance between 0 and 1, is a number line. All real numbers correspond to one and only one number on such a line. (Section 1.4)

**numerator** The number above the fraction bar in a fraction is called the numerator. It shows how many of the equivalent parts are being considered. (Section 1.1)

**numerical coefficient (coefficient)** The numerical factor in a term is called the numerical coefficient, or simply, the coefficient. (Sections 1.8, 4.4)

## O

**one-to-one function** A one-to-one function is a function in which each $x$-value corresponds to only one $y$-value and each $y$-value corresponds to only one $x$-value. (Section 12.1)

**ordered pair** An ordered pair is a pair of numbers written within parentheses in the form $(x, y)$. (Sections 3.1, 7.1)

**ordered triple** An ordered triple is a triple of numbers written within parentheses in the form $(x, y, z)$. (Section 8.4)

**ordinary annuity** An ordinary annuity is an annuity in which the payments are made at the end of each time period, and the frequency of payments is the same as the frequency of compounding. (Section 14.3)

**origin** The point at which the $x$-axis and $y$-axis of a rectangular coordinate system intersect is called the origin. (Sections 3.1, 7.1)

**outer product** When using the FOIL method to multiply two binomials $(a + b)(c + d)$, the outer product is $ad$. (Section 4.5)

## P

**parabola** The graph of a second-degree (quadratic) equation in two variables is called a parabola. (Sections 4.4, 11.6)

**parallel lines** Parallel lines are two lines in the same plane that never intersect. (Sections 3.3, 7.1)

**Pascal's triangle** Pascal's triangle is a triangular array of numbers that occur as coefficients in the expansion of $(x + y)^n$, using the binomial theorem. (Section 14.4)

**payment period** In an annuity, the time between payments is called the payment period. (Section 14.3)

**percent** Percent, written with the symbol %, means per one hundred. (Section 2.6)

**percentage** A percentage is a part of a whole. (Section 2.6)

**perfect cube** A perfect cube is a number with a rational cube root. (Section 10.1)

**perfect square** A perfect square is a number with a rational square root. (Section 10.1)

**perfect square trinomial** A perfect square trinomial is a trinomial that can be factored as the square of a binomial. (Section 5.4)

**perimeter** The perimeter of a two-dimensional figure is a measure of the distance around the outside edges of the figure—that is, the sum of the lengths of its sides. (Section 2.5)

**perpendicular lines** Perpendicular lines are two lines that intersect to form a right (90°) angle. (Sections 3.3, 7.1)

**plot** To plot an ordered pair is to locate it on a rectangular coordinate system. (Sections 3.1, 7.1)

**point-slope form** A linear equation is written in point-slope form if it is in the form $y - y_1 = m(x - x_1)$, where $m$ is the slope and $(x_1, y_1)$ is a point on the line. (Sections 3.4, 7.2)

**polynomial** A polynomial is a term or a finite sum of terms in which all coefficients are real, all variables have whole number exponents, and no variables appear in denominators. (Section 4.4)

**polynomial function** A function defined by a polynomial in one variable, consisting of one or more terms, is called a polynomial function. (Section 7.5)

**polynomial in $x$** A polynomial whose only variable is $x$ is called a polynomial in $x$. (Section 4.4)

**positive number** A positive number is located to the right of 0 on a number line. (Section 1.4)

**prime factor** A prime factor of a number is a factor greater than 1 whose only factors are 1 and itself. For example, the prime factors of 12 are $2 \cdot 2 \cdot 3$. (Section 1.1)

**prime number** A natural number greater than 1 is prime if it has only 1 and itself as factors. (Section 1.1)

**prime polynomial** A prime polynomial is a polynomial that cannot be factored into factors having only integer coefficients. (Section 5.2)

**principal root (principal $n$th root)** For even indexes, the symbols $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }, \ldots, \sqrt[n]{\ }$ are used for nonnegative roots, which are called principal roots. (Section 10.1)

**product** The answer to a multiplication problem is called the product. (Section 1.1)

**product of the sum and difference of two terms** The product of the sum and difference of two terms is the difference of the squares of the terms, or $(x + y)(x - y) = x^2 - y^2$. (Section 4.6)

**proportion** A proportion is a statement that two ratios are equal. (Section 2.6)

**proportional** If $y$ varies directly as $x$ and there exists some nonzero real number (constant) $k$ such that $y = kx$, then $y$ is said to be proportional to $x$. (Section 7.6)

**proposed solution** A value that appears as an apparent solution after a rational, radical, or logarithmic equation has been solved according to standard methods is called a proposed solution for the original equation. It may or may not be an actual solution and must be checked. (Sections 6.6, 10.6, 12.6)

**pure imaginary number** If $a = 0$ and $b \neq 0$ in the complex number $a + bi$, the complex number is called a pure imaginary number. (Section 10.7)

**Pythagorean theorem** The Pythagorean theorem states that the square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of the two legs. (Sections 5.6, 10.3)

## Q

**quadrant** A quadrant is one of the four regions in the plane determined by the axes in a rectangular coordinate system. (Sections 3.1, 7.1)

**quadratic equation** A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers, with $a \neq 0$. (Sections 5.5, 11.1)

**quadratic formula** The quadratic formula is a general formula used to solve a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. It is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (Section 11.3)

**quadratic function** A function defined by an equation of the form $f(x) = ax^2 + bx + c$, for real numbers $a$, $b$, and $c$, with $a \neq 0$, is a quadratic function. (Section 11.6)

**quadratic inequality** A quadratic inequality is an inequality that can be written in the form $ax^2 + bx + c < 0$ or $ax^2 + bx + c > 0$ (or with $\leq$ or $\geq$), where $a$, $b$, and $c$ are real numbers, with $a \neq 0$. (Section 11.8)

**quadratic in form** An equation is quadratic in form if it can be written in the form $au^2 + bu + c = 0$, for $a \neq 0$ and an algebraic expression $u$. (Section 11.4)

**quotient** The answer to a division problem is called the quotient. (Section 1.1)

## R

**radical** An expression consisting of a radical symbol, root index, and radicand is called a radical. (Section 10.1)

**radical equation** A radical equation is an equation with a variable in at least one radicand. (Section 10.6)

**radical expression** A radical expression is an algebraic expression that contains radicals. (Section 10.1)

**radical symbol** The symbol $\sqrt{\phantom{x}}$ is called a radical symbol. (Section 10.1)

**radicand** The number or expression under a radical symbol is called the radicand. (Section 10.1)

**radius** The radius of a circle is the fixed distance between the center and any point on the circle. (Section 13.2)

**range** The set of all second components ($y$-values) in the ordered pairs of a relation is called the range. (Section 7.3)

**ratio** A ratio is a comparison of two quantities using a quotient. (Section 2.6)

**rational expression** The quotient of two polynomials with denominator not 0 is called a rational expression. (Section 6.1)

**rational function** A function that is defined by a quotient of polynomials is called a rational function. (Section 13.1)

**rational inequality** An inequality that involves rational expressions is called a rational inequality. (Section 11.8)

**rationalizing the denominator** The process of rewriting a radical expression so that the denominator contains no radicals is called rationalizing the denominator. (Section 10.5)

**rational numbers** Rational numbers can be written as the quotient of two integers, with denominator not 0. (Section 1.4)

**real numbers** Real numbers include all numbers that can be represented by points on the number line—that is, all rational and irrational numbers. (Section 1.4)

**real part** The real part of a complex number $a + bi$ is $a$. (Section 10.7)

**reciprocal** (See **multiplicative inverse.**)

**reciprocal function** The reciprocal function is defined by $f(x) = \frac{1}{x}$. (Section 13.1)

**rectangular (Cartesian) coordinate system** The $x$-axis and $y$-axis placed at a right angle at their zero points form a rectangular coordinate system. It is also called the Cartesian coordinate system. (Sections 3.1, 7.1)

**relation** A relation is a set of ordered pairs. (Section 7.3)

**right angle** A right angle measures 90°. (Section 2.4)

**rise** Rise refers to the vertical change between two points on a line—that is, the change in $y$-values. (Sections 3.3, 7.1)

**row echelon form** If a matrix is written with 1s on the diagonal from upper left to lower right and 0s below the 1s, it is said to be in row echelon form. (Section 8.6)

**row of a matrix** A row of a matrix is a group of elements that are read horizontally. (Section 8.6)

**row operations** Row operations are operations on a matrix that produce equivalent matrices, leading to systems that have the same solutions as the original system of equations. (Section 8.6)

**run** Run refers to the horizontal change between two points on a line—that is, the change in $x$-values. (Sections 3.3, 7.1)

## S

**scatter diagram** A scatter diagram is a graph of ordered pairs of data. (Section 3.1)

**scientific notation** A number is written in scientific notation when it is expressed in the form $a \times 10^n$, where $1 \leq |a| < 10$ and $n$ is an integer. (Section 4.3)

**second-degree inequality** A second-degree inequality is an inequality with at least one variable of degree 2 and no variable with degree greater than 2. (Section 13.5)

**sequence** A sequence is a function whose domain is the set of natural numbers or a set of the form $\{1, 2, 3, \ldots, n\}$. (Section 14.1)

**series** The indicated sum of the terms of a sequence is called a series. (Section 14.1)

**set** A set is a collection of objects. (Section 1.3, Appendix A)

**set-builder notation** The special symbolism $\{x | x$ has a certain property$\}$ is called set-builder notation. It is used to describe a set of numbers without actually having to list all of the elements. (Section 1.4)

**signed numbers** Signed numbers are numbers that can be written with a positive or negative sign. (Section 1.4)

**simplified radical** A simplified radical meets four conditions:

1. The radicand has no factor (except 1) that is a perfect square (if the radical is a square root), a perfect cube (if the radical is a cube root), and so on.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.

(Section 10.3)

**slope** The ratio of the change in $y$ to the change in $x$ for any two points on a line is called the slope of the line. (Sections 3.3, 7.1)

**slope-intercept form** A linear equation is written in slope-intercept form if it is in the form $y = mx + b$, where $m$ is the slope and $(0, b)$ is the $y$-intercept. (Sections 3.4, 7.2)

**solution of an equation** A solution of an equation is any replacement for the variable that makes the equation true. (Section 1.3)

**solution of a system** A solution of a system of equations is an ordered pair $(x, y)$ that makes all equations true at the same time. (Section 8.1)

**solution set** The set of all solutions of an equation is called the solution set. (Section 2.1)

**solution set of a linear system** The set of all ordered pairs that satisfy all equations of a system at the same time is called the solution set. (Section 8.1)

**solution set of a system of linear inequalities** The set of all ordered pairs that make all inequalities of a linear system true at the same time is called the solution set of the system of linear inequalities. (Section 13.5)

**square matrix** A square matrix is a matrix that has the same number of rows as columns. (Section 8.6)

**square of a binomial** The square of a binomial is the sum of the square of the first term, twice the product of the two terms, and the square of the last term: $(x + y)^2 = x^2 + 2xy + y^2$ and $(x - y)^2 = x^2 - 2xy + y^2$. (Section 4.6)

**square root** The inverse of squaring a number is called taking its square root. That is, a number $a$ is a square root of $k$ if $a^2 = k$ is true. (Section 10.1)

**square root function** The function defined by $f(x) = \sqrt{x}$, with $x \geq 0$, is called the square root function. (Sections 10.1, 13.3)

**square root property** The square root property (for solving equations) states that if $x^2 = k$, with $k > 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$. (Section 11.1)

**squaring function** The polynomial function defined by $f(x) = x^2$ is called the squaring function. (Section 13.1)

**squaring property** The squaring property (for solving equations) states that if each side of a given equation is squared, then all solutions of the given equation are *among* the solutions of the squared equation. (Section 10.6)

**standard form of a complex number** The standard form of a complex number is $a + bi$. (Section 10.7)

**standard form of a linear equation** A linear equation in two variables written in the form $Ax + By = C$, with $A$ and $B$ not both 0, is in standard form. (Sections 3.4, 7.2)

**standard form of a quadratic equation** A quadratic equation written in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers with $a \neq 0$, is in standard form. (Sections 5.5, 11.1)

**step function** A function that is defined using the greatest integer function and has a graph that resembles a series of steps is called a step function. (Section 13.1)

**straight angle** A straight angle measures 180°. (Section 2.4)

**subscript notation** Subscript notation is a way of indicating nonspecific values. In $x_1$ and $x_2$, 1 and 2 are subscripts on the variable $x$. (Sections 3.3, 7.1)

**subset** If all elements of set $A$ are in set $B$, then $A$ is a subset of $B$, written $A \subseteq B$. (Appendix A)

**substitution method** The substitution method is an algebraic method for solving a system of equations in which one equation is solved for one of the variables, and then the result is substituted into the other equation. (Section 8.2)

**sum** The answer to an addition problem is called the sum. (Section 1.1)

**sum of cubes** The sum of cubes, $x^3 + y^3$, can be factored as $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (Section 5.4)

**summation (sigma) notation** Summation notation is a compact way of writing a series using the general term of the corresponding sequence. It involves the use of the Greek letter sigma, $\Sigma$. (Section 14.1)

**supplementary angles (supplements)** Supplementary angles are two angles whose measures have a sum of 180°. (Section 2.4)

**synthetic division** Synthetic division is a shortcut procedure for dividing a polynomial by a binomial of the form $x - k$. (Appendix C)

**system of inequalities** A system of inequalities consists of two or more inequalities to be solved at the same time. (Section 13.5)

**system of linear equations (linear system)** A system of linear equations consists of two or more linear equations to be solved at the same time. (Section 8.1)

## T

**table of values** A table of values is an organized way of displaying ordered pairs. (Section 3.1)

**term** A term is a number, a variable, or the product or quotient of a number and one or more variables raised to powers. (Section 1.8)

**term of an annuity** The time from the beginning of the first payment period to the end of the last period is called the term of an annuity. (Section 14.3)

**terms of a proportion** The terms of the proportion $\frac{a}{b} = \frac{c}{d}$ are $a$, $b$, $c$, and $d$. (Section 2.6)

**terms of a sequence** The function values in a sequence, written in order, are called terms of the sequence. (Section 14.1)

**three-part inequality** An inequality that says that one number is between two other numbers is called a three-part inequality. (Section 2.8)

**trinomial** A trinomial is a polynomial consisting of exactly three terms. (Section 4.4)

## U

**union** The union of two sets $A$ and $B$, written $A \cup B$, is the set of elements that belong to *either A or B*, or both. (Section 9.1, Appendix A)

**universal constant** The number $e$ is called a universal constant because of its importance in many areas of mathematics. (Section 12.5)

**universal set** The set that includes all elements under consideration is the universal set, symbolized $U$. (Appendix A)

**unlike terms** Unlike terms are terms that do not have the same variable, or terms with the same variables but whose variables are not raised to the same powers. (Section 1.8)

## V

**variable** A variable is a symbol, usually a letter, used to represent an unknown number. (Section 1.3)

**vary directly (is proportional to)** $y$ varies directly as $x$ if there exists a nonzero real number (constant) $k$ such that $y = kx$. (Section 7.6)

**vary inversely** $y$ varies inversely as $x$ if there exists a nonzero real number (constant) $k$ such that $y = \frac{k}{x}$. (Section 7.6)

**vary jointly** If one variable varies as the product of several other variables (possibly raised to powers), then the first variable is said to vary jointly as the others. (Section 7.6)

**Venn diagram** A Venn diagram consists of geometric figures, such as rectangles and circles, that illustrate the relationships among sets. (Appendix A)

**vertex** The point on a parabola that has the least $y$-value (if the parabola opens up) or the greatest $y$-value (if the parabola opens down) is called the vertex of the parabola. (Sections 4.4, 11.6)

**vertical angles** When two intersecting lines are drawn, the angles that lie opposite each other have the same measure and are called vertical angles. (Section 2.5)

**vertical line test** The vertical line test states that any vertical line will intersect the graph of a function in at most one point. (Section 7.3)

**volume** The volume of a three-dimensional figure is a measure of the space occupied by the figure. (Section 2.5)

## W

**whole numbers** The set of whole numbers is $\{0, 1, 2, 3, 4, \ldots\}$. (Sections 1.1, 1.4)

## X

***x*-axis** The horizontal number line in a rectangular coordinate system is called the $x$-axis. (Sections 3.1, 7.1)

***x*-intercept** A point where a graph intersects the $x$-axis is called an $x$-intercept. (Sections 3.2, 7.1)

## Y

***y*-axis** The vertical number line in a rectangular coordinate system is called the $y$-axis. (Sections 3.1, 7.1)

***y*-intercept** A point where a graph intersects the $y$-axis is called a $y$-intercept. (Sections 3.2, 7.2)

## Z

**zero-factor property** The zero-factor property states that if two numbers have a product of 0, then at least one of the numbers is 0. (Sections 5.5, 11.1)

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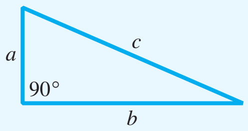
# Triangles and Angles

***Right Triangle***

Triangle has one 90° (right) angle.



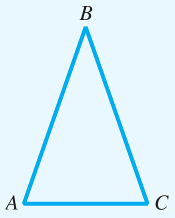
***Pythagorean Theorem (for right triangles)***

$a^2 + b^2 = c^2$

***Isosceles Triangle***

Two sides are equal.

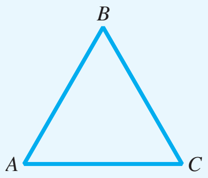
$AB = BC$



***Equilateral Triangle***

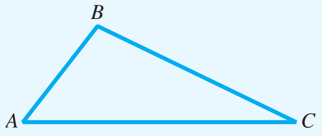
All sides are equal.

$AB = BC = CA$



***Sum of the Angles of Any Triangle***

$A + B + C = 180°$

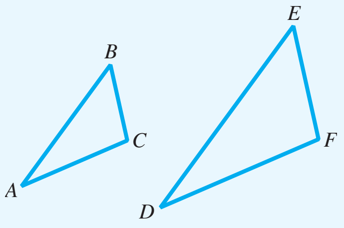


***Similar Triangles***

Corresponding angles are equal. Corresponding sides are proportional.

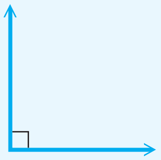
$A = D, B = E, C = F$

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$



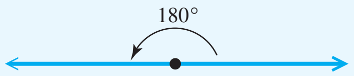
***Right Angle***

Measure is 90°.



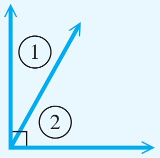
***Straight Angle***

Measure is 180°.



***Complementary Angles***

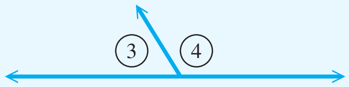
The sum of the measures of two complementary angles is 90°.



Angles ① and ② are complementary.

***Supplementary Angles***

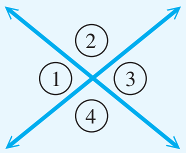
The sum of the measures of two supplementary angles is 180°.



Angles ③ and ④ are supplementary.

***Vertical Angles***

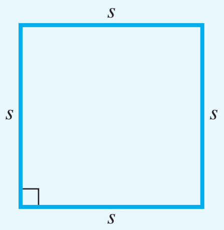
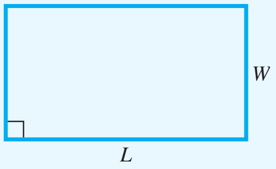
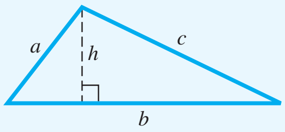
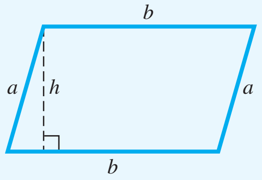
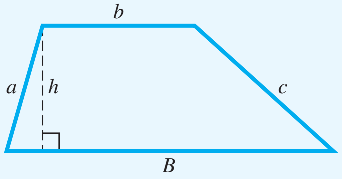
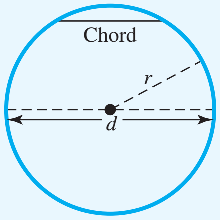
Vertical angles have equal measures.



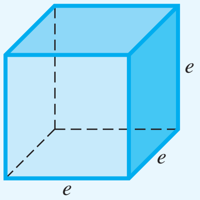
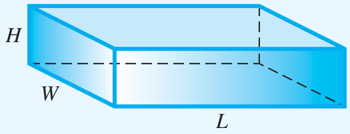
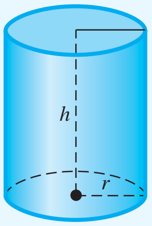
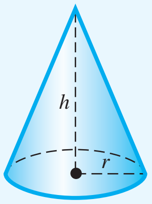
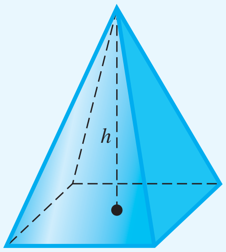
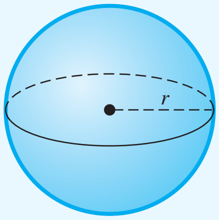
Angle ① = Angle ③

Angle ② = Angle ④

# Formulas

| *Figure* | *Formulas* | *Illustration* |
|---|---|---|
| **Square** | Perimeter: $P = 4s$<br>Area: $\mathcal{A} = s^2$ |  |
| **Rectangle** | Perimeter: $P = 2L + 2W$<br>Area: $\mathcal{A} = LW$ |  |
| **Triangle** | Perimeter: $P = a + b + c$<br>Area: $\mathcal{A} = \frac{1}{2}bh$ |  |
| **Parallelogram** | Perimeter: $P = 2a + 2b$<br>Area: $\mathcal{A} = bh$ |  |
| **Trapezoid** | Perimeter: $P = a + b + c + B$<br>Area: $\mathcal{A} = \frac{1}{2}h(b + B)$ |  |
| **Circle** | Diameter: $d = 2r$<br>Circumference: $C = 2\pi r$<br>$C = \pi d$<br>Area: $\mathcal{A} = \pi r^2$ |  |

# Formulas

| *Figure* | *Formulas* | *Illustration* |
|---|---|---|
| **Cube** | Volume: $V = e^3$<br>Surface area: $S = 6e^2$ |  |
| **Rectangular Solid** | Volume: $V = LWH$<br>Surface area:<br>$\mathscr{A} = 2HW + 2LW + 2LH$ |  |
| **Right Circular Cylinder** | Volume: $V = \pi r^2 h$<br>Surface area: $S = 2\pi rh + 2\pi r^2$<br>(Includes both circular bases) |  |
| **Cone** | Volume: $V = \frac{1}{3}\pi r^2 h$<br>Surface area: $S = \pi r\sqrt{r^2 + h^2} + \pi r^2$<br>(Includes circular base) |  |
| **Right Pyramid** | Volume: $V = \frac{1}{3}Bh$<br>$B =$ area of the base |  |
| **Sphere** | Volume: $V = \frac{4}{3}\pi r^3$<br>Surface area: $S = 4\pi r^2$ |  |

# Other Formulas

**Distance:** $d = rt$ ($r =$ rate or speed, $t =$ time)

**Percent:** $p = br$ ($p =$ percentage, $b =$ base, $r =$ rate)

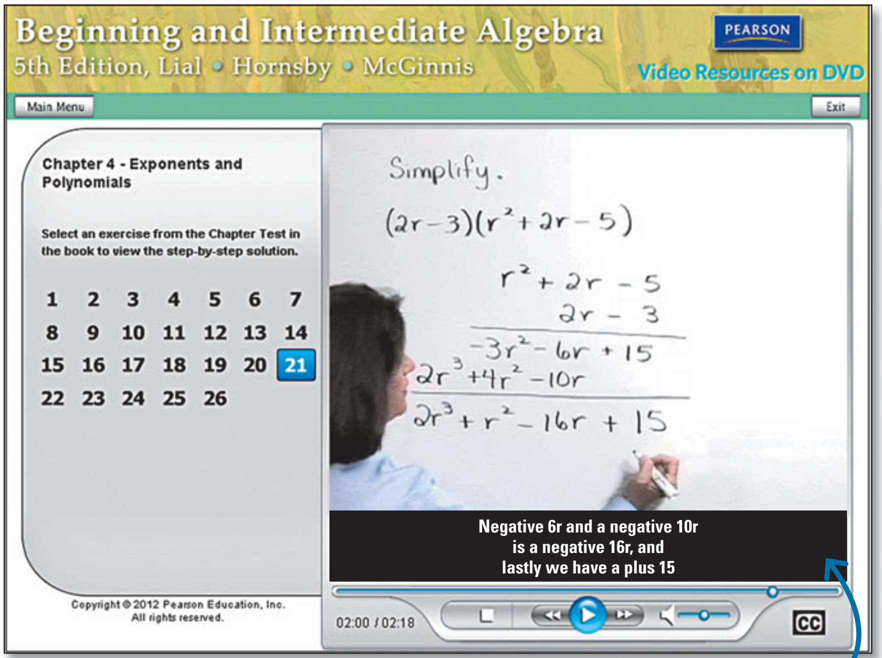
**Temperature:** $F = \frac{9}{5}C + 32$ $\quad C = \frac{5}{9}(F - 32)$

**Simple Interest:** $I = prt$ ($p =$ principal or amount invested, $r =$ rate or percent, $t =$ time in years)

# STUDY SMARTER

CHAPTER
Test Prep
VIDEOS

**Step-by-step solutions** on video for all chapter test exercises from the text


Beginning and Intermediate Algebra
5th Edition, Lial • Hornsby • McGinnis

Chapter 4 - Exponents and Polynomials

Select an exercise from the Chapter Test in the book to view the step-by-step solution.

Negative 6r and a negative 10r
is a negative 16r, and
lastly we have a plus 15

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