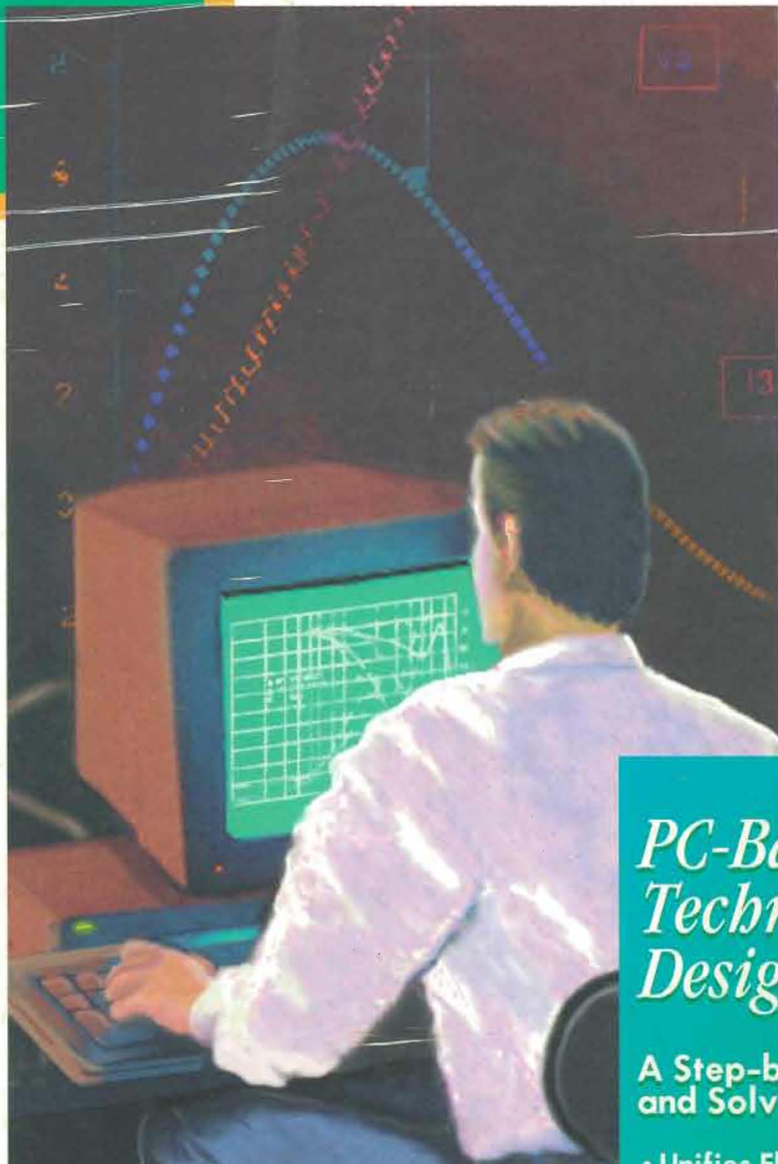
MODELING ENGINEERING SYSTEMS

By Jack W. Lewis



PC-Based Techniques and Design Tools

A Step-by-Step Guide to Building and Solving Math Models:

- **Unifies Electrical, Mechanical, Fluid, and Thermal Systems**
- **Uses Simple Spreadsheet Simulations**
- **Contains Lots of Practical Examples, Graphs, and "Plain English" Explanations**

A Volume in the Engineering Mentor™ Series

Modeling Engineering Systems

PC-Based Techniques and Design Tools

by Jack W. Lewis

HighText publications inc.

Solana Beach, California

This is a volume in the Engineering Mentor™ Series

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

Cover design: Brian McMurdo, Ventana Studio, Valley Center, CA

Developmental editing: Carol Lewis, HighText Publications

Interior design: Sara Patton, Wailuku, HI

Library of Congress Cataloging-in-Publication Data

Lewis, Jack W., 1937–
Modeling engineering systems : PC-based techniques and design tools / by Jack W. Lewis
p. cm. – (Engineering mentor series)
Includes bibliographical references and index.
ISBN 1-878707-08-6 : $19.95
1. Mathematical models–Data processing. 2. Systems engineering–Mathematical models–Data processing. 3. Computer-aided engineering. I. Title. II. Series.
TA342.L48 1994
620'.001'1–dc20 93-40319
CIP

HighText publications inc.

P.O. Box 1489 ▪ Solana Beach, CA 92075

HighText is a trademark of HighText Publications, Inc.

This book is dedicated to Dr. Herbert H. Richardson, a professor of mechanical engineering at MIT during my graduate school years of 1964–66. H^2R, as he was fondly called, probably taught me more about engineering than all the other teachers and professors I had, combined. He was an outstanding teacher who seemed to have a hundred different ways of explaining a complex subject in a simple and humorous manner. I owe much of my success in engineering to this man.

Contents

Preface

If you're like me, when you pick up a book you like to know something about the person you're trying to learn from. Quite simply, I'm a practicing engineer who believes anyone can learn engineering, as long as it is explained by someone who is willing to take the time to present the subject matter in simple enough terms. That's what this book is about.

I graduated from the US Coast Guard Academy in 1960 with a B.S. in Science. I did well there—at least I got good grades and was near the top of my class. After four years at sea, I was selected to go to graduate school at MIT. What a humbling experience that school was for me! During the first semester, I realized that I wasn't nearly as clever as I thought. In fact, I quickly became convinced that I knew nothing about math and science, and that I was going to flunk out!

However, as I gave up the notion that I knew anything, the situation gradually began to change. At MIT I encountered, for the first time it seemed, professors who were not trying to impress me with their knowledge and intelligence. They sincerely wanted me to learn. When I wasn't understanding something, they took it as a failing on their part, not mine. It seemed that most professors had 50 different ways to teach a fundamental principle.

At MIT, I learned engineering in a way I will never forget, but most importantly, I learned that engineering is about *solving practical physical problems by creating mathematical models that can be manipulated.* These models allow you to learn a great deal about the physical problem that you're trying to solve. Engineering is not just mathematics—rather, mathematics is simply a tool used in engineering.

Through my coursework at MIT in the area of automatic control systems, I learned that all engineering systems look alike mathematically—a concept that is at the heart of this book. Incidentally, I did manage to graduate from MIT with a master's degree in mechanical engineering and a naval engineer degree. I started my own consulting engineering business in 1970. I'm still active in the consulting arena, and still love engineering.

Chapter 1

Read Me

1.1 The Engineering Design Process in a Nutshell

I've discovered during the writing of this book that it's important to know who your readers are. As a reader, it's also important for you to know who this book is aimed at. I would like to think that anyone with a strong desire to learn the fundamentals of engineering can do so in the pages of this book. I do believe that, actually, but it will help a lot if you already have a strong foundation in mathematics—at least one or two semesters of calculus.

This book is not a textbook per se. It is primarily intended for those just beginning their engineering careers, or for practicing engineers who are changing fields or who need a brush-up. I also believe that senior technicians who want to press ahead into more advanced engineering design work can also benefit from it. For those whose math is a little rusty, Appendix A provides a "quick and dirty" review of both differential and integral calculus. Other mathematical topics, such as complex number theory, are introduced and explained in the text as they are needed.

Before we get into the nuts and bolts of modeling engineering systems, bear with me for a few more moments while I wax philosophical about the engineering process. My definition of engineering is the application of physics and other branches of science to the creation of products and services that make the world a (hopefully) better place. Your "success" in engineering will likely be closely related to how well you can create products and services that your organization's customers need and want. Unfortunately, I have found that creating new products and services is a lot more difficult than analyzing or criticizing those that already exist. There are far more critics in the world than creators!

Both a creative and a critical person are inside each one of us. To be good in engineering, you have to be able to "turn off" your critical side long enough to allow yourself to create. Once you have created, then you can turn the critical side back on to analyze and pick apart your creation. Watch out for all those other critics in the world. If you constantly listen to them, you will never succeed in engineering (or anything else, for that matter). Learn to encourage yourself. The praise of peers will follow.

This book is about *modeling* and *analyzing* engineering systems. Modeling is the creative side of engineering, and analyzing is the critical side. I use the term "engineering system" in this book to refer to a product or device that may contain mechanical, electrical, fluid, and/or thermal components. An engineering system can therefore be interdisciplinary, and require a designer to have knowledge of many engineering fields.

Creating an engineering design does not have to be a mysterious art. The more you learn about what is available in the way of real-world basic components and services, the more creative you will become. This book contains all of the fundamentals needed to develop mathematical models of engineering systems and to analyze these models. But you must make it a habit to collect and carefully study product and service catalogs of basic components so you know what materials are readily available. Don't be afraid to ask "dumb" questions. Product representatives as well as more experienced engineers and technicians can be the source of a great deal of knowledge. It sounds like a cliche, but asking questions is really the easiest way to learn. Then, when you are trying to create a new design, you will have a lot of information in your head that will (almost subconsciously) help you generate ideas.

Figure 1.1 shows the basic process undergone in developing a new engineering design. The design is created first in your mind, when you become aware of a want or need. You then make a list of specifications that you envision would satisfy that need. Then you will create (in your mind, at first) a product or service that meets these specifications.

In order to "give birth" to this new design idea, you have to get it out of your mind and into reality. Take care at this point. Don't let your (or anyone else's) critical side take over too soon, or your ideas will never be allowed to take root.

You can transfer your "mind model" into a "symbolic" or even a "physical" model. It is usually best to work with symbolic models at first, because they are typically less expensive than physical models (but not always). A symbolic model might be a circuit diagram from which you can derive a number of mathematical expressions describing the behavior of your mind model. You can then solve these mathematical expressions for the answers you need.

Once you have the first solution to your mathematical expressions, then—and only then—allow your critical side to tear apart your design. Don't be discouraged if the first attempt at meeting the specifications seems silly. Use the

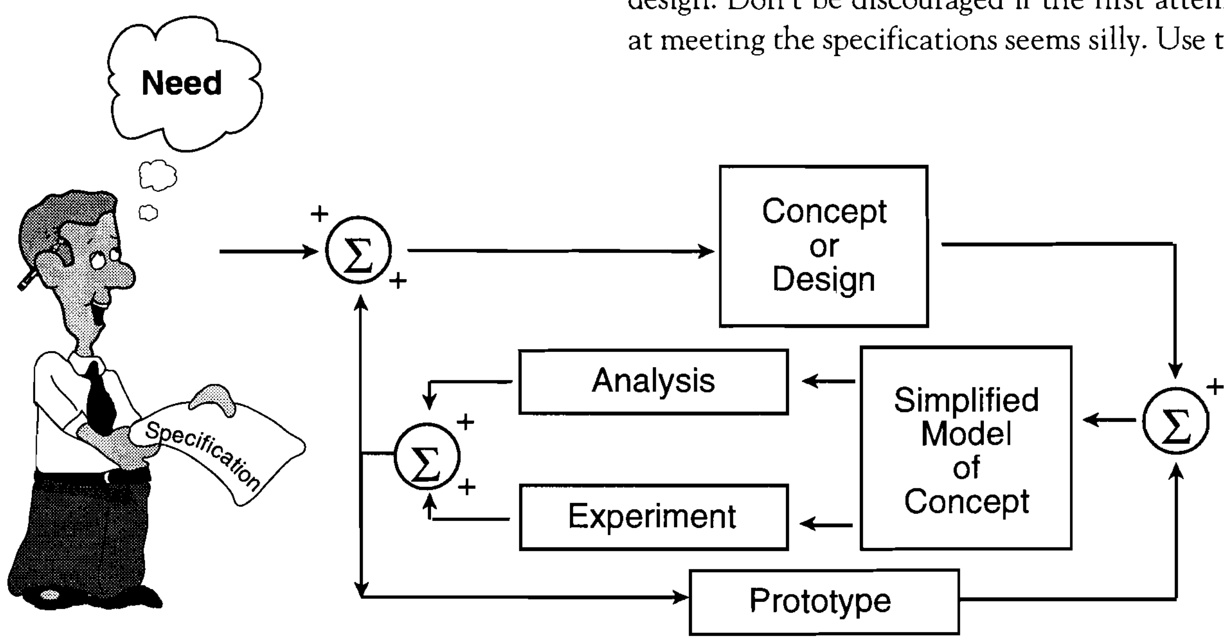


Figure 1.1. The engineering design process.

results of your analysis to see how well your product met the specifications, and alter your specs and modify the design as needed. Then create another model to analyze. Continue to iterate in this manner until you have a product that satisfies the need.

At this point in your design you have to ask yourself, "Am I confident enough in my analytical results to proceed with the construction of a full-scale prototype?" If the answer is no, then consider building and testing a physical model of your product. Physical model experiments can produce empirical solutions to problems that are difficult to solve mathematically or for which too many untested assumptions had to be made in order to derive the analytical model and its solution. Building and testing physical models are usually much less expensive than building and testing a full-scale prototype.

Results of physical model-testing experiments could lead to a revision of your specifications, a new mathematical model, or even another series of physical model experiments. Eventually you will get to the point where you are confident enough with your design to proceed to the construction of a prototype. Seriously consider at this point conducting experiments with this prototype to obtain data to verify your analytical and physical model results. If the prototype does not meet your original specifications, you may have to iterate once again through the design cycle before proceeding further.

This is the engineering design process in a nutshell. It may sound long, complicated, and expensive. It can be, when major, complex engineering systems are involved. But no matter how big or small the system, *never* short circuit this engineering design process! Engineers and technicians are not artists who create their masterpieces on canvas with a brush, or out of marble with a hammer and chisel, with little input from others. Instead, they are people who design, redesign, and then let others check their designs. They convert their designs into engineering drawings and then they allow skilled craftsmen to provide feedback on how to improve the design to reduce production time and costs. I have seen and worked with too many "seat-of-the-pants" engineers who have the engineering design process backwards. They first build prototypes and then they design, often with disastrous results. If you do this, you may be considered a good "artist," but you will never be a good engineer.

1.2 An Engineer's Tool Box

Any skilled craftsman knows that a good set of tools and the knowledge to use them is of fundamental importance in getting a job done properly and safely. An engineer also has "tools." Like the craftsman, some of these tools are physical in nature, but for the most part an engineer's tools consist of mental skills developed through study of mathematics and science.

This book will explain the fundamental mathematical and scientific tools needed to succeed in engineering. It will also show how to use them in practical applications. The tools are mostly mathematical in nature, because mathematics is at the core of engineering. *Please bear with me through the mathematics.* I have attempted to explain each step and to make it as simple and understandable as possible. But significant effort on your part is required.

I have assumed in the book that you have a fairly good grounding in algebra and trigonometry, and have had courses in both differential and integral calculus. If you find yourself getting "lost in the algebra," then brush up using a high school text. Calculus fundamentals are reviewed in Appendix A.

When reading a new engineering text, I usually keep a notebook handy. As I encounter new equations in the text, I jot them down and work through each one until I understand exactly what the author is doing. I highly recommend this method—although it makes for slow reading, it ensures that you understand the material covered.

When I started my engineering career, an engineer's physical tools consisted of a drafting board and supplies, a slide rule, engineering reference books, and textbooks. Many of the engineering reference books contained log and trig tables to help in mathematical calculations. There were no electronic calculators, and digital computers were kept in caged air-conditioned rooms where only the computer folks were allowed.

The electronic calculator and personal computer have completely replaced many of the physical tools that an engineer used to use. The hand calculator made the slide rule and many of the tables in engineering reference books obsolete. It's hard to imagine being without a good calculator.

The personal computer has revolutionized engineering. There are now so many powerful engineering and mathematical programs available for the PC and Macintosh that I think it is fair to say that the capabilities of these packages have surpassed the capabilities of the average user. I see many technicians and engineers using or attempting to use engineering and mathematical software packages who do not understand the fundamentals. For example, dynamic system simulation (at the heart of engineering) and DSP (digital signal processing) software packages are available now that simply astound me with their capabilities. To put these programs to good use, however, you must have the mathematical and engineering fundamentals. That is what this book is all about.

I have therefore assumed that you either have, or have access to, a personal computer. I also have assumed that you know how to use spreadsheets such as Excel, Lotus 1-2-3, or Quattro, and that you know how to use a higher level computer programming language like BASIC. (Spreadsheets are a surprisingly useful engineering tool. I use them frequently to develop very complex simulations.) If you have some of the latest simulation and DSP packages, that's great too. But, let me give you a word of caution. Having a computer and knowing how to run canned engineering programs doesn't make you an engineer any more than carrying around and knowing how to use a slide rule made you an engineer years ago. The latest and fastest computer and the latest version of a software package are the "trappings" and "images" of engineering—they are not engineering. The fundamental knowledge presented in this book is absolutely necessary to make the most of any engineering design software package on the market.

1.3 Analog Computers – An Anachronism?

When I first entered graduate school, I was introduced to an analog computer. What an incredibly powerful machine it was! It could solve differential equations much faster than a digital computer could at that time (and it still can). In fact, an analog computer solves dynamic system problems instantaneously, something no digital computer can do. So don't get the idea that an analog computer is "old fashioned" and that digital computers are the only computer you need in your engineering tool box.

In this book you will study operational calculus and block diagrams. These concepts are particularly useful in building mathematical models that can be solved on analog computers. Once an operational block diagram of the dynamics of an engineering system has been constructed, it is easy to put it on an analog computer and let it solve the problem. You can also build electrical analogs of physical systems and interface these to a computer without the need for expensive sensors. Then you can quickly develop digital control algorithms.

You may be wondering where you are going to find an "old" analog computer to do that. While it is true that you can't readily buy an analog computer these days, it's not because they are old and obsolete. It's simply because they were replaced by the operational amplifier—an integrated circuit that became so cheap and easy to use that anyone could build an analog computer for peanuts. Unfortunately, it seems that many educators have forgotten this, as I have run into young engineers, even electronics engineers, who have no idea how to use operational amplifiers to build analog computers to solve differential equations.

A few years ago while I was studying the problem of controlling a water wavemaker (see Chapter 7), I built an electrical analog of the wave tank and associated hydraulic piston and valve so I could study a digital feedback controller. The software engineer working with me was amazed. How could a breadboard full of op amps behave like a water wavemaker?

You will learn in this book that the dynamics of all engineering systems—whether electrical, mechanical, thermal, or fluid—can be described by the same mathematical equations. That means mechanical, fluid, and thermal systems, which are difficult and expensive to construct and test, can be converted into electrical circuits, which are cheap and easy to test. This is the whole fundamental concept behind the "analog computer." The word "analog" really doesn't have anything to do with electronics, even though it is generally accepted these days that analog is anything that is not digital. The word analog is used to describe a system that behaves like another system yet has a different physical form.

Today you can go to Radio Shack or an electronics mail-order house and buy IC operational amplifiers, resistors, capacitors, and other components for pennies. All you need is a little skill, a power supply, a breadboard, and some wires, and you can build an analog computer that can solve mechanical, fluid, and thermal dynamic system problems as well as electrical circuit problems. Once you have such an analog model of your system, you can also very easily conduct experiments with it using a digital computer equipped with an analog-to-digital converter. Since your analog model will already be in electrical form, ready to interface to your digital computer, you won't have to go out and buy expensive sensors that convert mechanical, fluid, and thermal variables into electrical signals.

1.4 A Few Words About Units

Throughout this book I use the English system of engineering units, with the corresponding SI units in parenthesis. I do this because, in spite of efforts made to convert the United States and some other leading industrial countries to the SI system of units, I find that engineers stick to the units they "think" in. Let me explain with a story. In the early 1970s I made a concerted effort to get everyone in my firm to convert to the SI system. Then we got a job in Japan and I was happy that we had converted, since I knew the Japanese used the metric system and assumed they had made the easy conversion to the SI system. When I made my first presentation, much to my amazement I found that our Japanese clients (who were shipbuilders) did not understand the units I was using and asked me to explain what they meant in terms of units they were familiar with. The units they used turned out to be a rather strange system that I can only call a "Japanese naval architectural metric system." Even today, the ship-building industry throughout the world uses some of the strangest sets of measurement units you will ever encounter, and I doubt they will ever change to the SI system.

Just how important are units in engineering? First of all, never lose sight of the fact that the laws of nature have no inherent system of units. Units are man-made. Indeed, if you run into an equation that cannot be made unitless (that is, dimensionless or without dimensions) by dividing through by some combination of variables, then the equation does not truly describe nature and is probably wrong or valid only over a very small range of the independent variables. Nondimensionalizing an equation is a good way to check your equations and an excellent way to present your results.

Of course, units are important in engineering. Many components you purchase will have weight, volume, or linear dimensions, and will consume or require power, produce a force or a torque, and so forth. These components may come from every part of the world and will make use of every conceivable system of units known to man. What do you do? You simply need to know how to convert from one set of units to another. If you can't think in the supplier's units, convert them to the ones you are familiar with or intend to use in your product. There are no right or wrong units to use. Just make sure your customers readily understand the units *you* use.

1.5 Overview of Book

Now—finally—I'll tell you a little bit about the content of this book and the way it's organized.

Chapter 2 is probably the most important chapter. You will learn that there are only three basic types of engineering building blocks, two that can store energy and one that dissipates energy. The concepts are deceptively simple, yet extremely powerful.

Chapter 3 begins the process of teaching you how to build math models of any engineering system. It covers all of the model-building techniques that have been developed over the years in many branches of engineering. You will discover the mighty first-order linear ordinary differential equation, the very backbone of engineering. From my experience, learning how to build math models is the hardest part of engineering, because it involves creating. I have spent a great deal of time on this chapter and tried to make it as easy as possible to grasp.

Chapter 4 introduces you to the analytical side of engineering. You'll learn how to solve the first-order linear differential equation math models you developed in Chapter 3. This is also a very important chapter. First-order linear differential equations can be used to describe many facets of engineering, from the flow of ground water through porous soils to the flow of electrons through electrical circuits. The so-called *time and frequency domain* solutions to these equations form the basis of all engineering analysis.

Chapter 5 is one of two chapters in the book intended to help you bridge the gap between theory and the solution of practical engineering problems. A problem is selected that involves modeling and analyzing a combined mechanical and electrical engineering system. The example helps emphasize that all engineering systems look alike mathematically. It shows how two first-order linear differential equations lead to a second-order linear differential equation, the subject of the next chapter.

Chapter 6 introduces you to the next important subject, second-order linear differential equations. This chapter addresses both the modeling and analysis of systems that can be described by these equations. You will learn that modeling systems containing two independent ideal energy storage devices always leads to a second-order linear differential equation. The material contained in this chapter, along with that contained in Chapters 3 and 4, is the foundation of any branch of engineering.

Chapter 7 is the second chapter in the book that relates theory to practical engineering. This time a real-world problem is selected that combines fluid, mechanical, and electrical systems modeling and analysis. You will see how everything you learned in the previous chapters is applied to solving complex engineering problems.

Chapter 8 is intended to lead you into the world of more complex engineering systems with the confidence that what you learned in the previous chapters is all that is required to understand such systems. You will learn that no matter how complex an engineering system is, it can be broken down into combinations of first- and second-order linear differential equations.

Appendix A provides an opportunity for you to review engineering calculus, for those who may need it. It covers the high points of both differential and integral calculus. You need to be familiar with at least the amount of calculus presented in this appendix to understand the rest of the material in this book.

Appendix B contains the physics behind the engineering building blocks covered in Chapter 2. At first I was going to put this information in Chapter 2, but it made for a very long chapter and I didn't want readers to get bogged down in all the math at the very beginning of the book. The information in this appendix is very powerful and very useful, however, and I strongly recommend that you peruse and understand it.

I hope you enjoy the book and that it helps you achieve your goals in engineering.

Chapter 2 Basic Building Blocks for Modeling Engineering Systems

Objectives

At the completion of this chapter, you will be able to:

- **Define the basic concepts of voltage, current, work, power, and energy as related to electrical systems.**
- **Identify the fundamental electrical circuit elements and write their describing equations.**
- **Recognize two important tools used in formulating math models: impedance and operational block diagrams.**
- **Define the basic concepts of motion and force as they relate to mechanical components.**
- **Identify the fundamental mechanical components and write their describing equations.**
- **Define the basic concepts of pressure and mass/volume rate of flow in fluid systems.**
- **Identify the fundamental fluid components and write their describing equations.**
- **Define the basic concepts of temperature and heat flow in thermal systems.**
- **Identify the fundamental thermal elements and write their describing equations.**
- **Recognize the analogies that can be drawn between the fundamental elements of all four types of systems: electrical, mechanical, fluid, and thermal.**

2.1 Introduction

We'll now discuss how to build mathematical models of engineering systems. All of the material you will encounter in this chapter should be somewhat familiar, since it is covered in most first courses in physics. However, the subject matter is presented differently. If you don't understand something, I strongly suggest that you get a first-year college physics text and refer to it as required as you read though this chapter.

Throughout this book I deal primarily with *linear* equations because they are far easier to work with than nonlinear equations, and because they give you the quickest insight into the physical behavior of engineering systems. A linear equation is an equation in which a change in the input or independent variable results in a *proportional* change in the output or dependent variable. Take, for example, the equation for the volume of a cylinder V given by

$$V = \pi r^2 h \tag{2.1}$$

where r = radius of the cylinder
h = height of the cylinder.

This equation is linear with respect to h, but it is nonlinear with respect to r. You can see this by holding one variable constant while varying the other. For a cylinder where r is fixed, doubling h from h_0 to $2h_0$ doubles the volume of the cylinder, a proportional increase. Therefore the equation is linear with respect to h. For a cylinder where h is fixed, doubling the radius from r_0 to $2r_0$, does not result in a proportional increase in volume—it quadruples the volume. Therefore, the equation is nonlinear with respect to r.

Quite often you will want to linearize an equation so you can study the behavior of an engineering system model about a certain set of values for the independent variables. This set of values is frequently called the operating or steady-state point of the system. A very powerful equation called Taylor's Theorem allows any function of any number of independent variables to be expanded about an operating point. To fully understand and appreciate the power of this equation you need to understand the concept of partial derivatives. (See the review of these topics in Appendix A if you need a quick brush-up.)

Throughout the book, we'll go from the simplest elements to more and more complex systems. In this chapter, you will learn how to break down components found in electrical, mechanical, fluid, and thermal systems into rudimentary elements that can be described by simple differential and integral calculus. You will discover that there are only three fundamental elements in electrical, mechanical, and fluid systems, and only two in thermal systems. You will also discover that, from a mathematical point-of-view, all of the fundamental elements in each of these diverse fields look and behave exactly alike. You will learn that design and analysis problems in one field of engineering can be easily converted to another field, where other tools for obtaining solutions might be available.

In the interests of space and practicality, I've left out some of the physics in this chapter and placed it in Appendix B, *The Physics of Work, Power, and Energy in Engineering Systems*. In order to make sure that you understand the physics behind all of the elements described here, it would be a good idea to read through this appendix.

In this chapter, I'll begin a convention that I'll use throughout the book. Equations that I feel are important enough to commit to memory will be boxed in. You might wonder why you should commit *anything* to memory, when you could just look it up in a reference book when you need it. I strongly believe that a certain core set of equations should be memorized if you're going to be successful at engineering. These equations could be compared to a minimal vocabulary for someone who is learning to read. If we didn't remember the meanings of all the words we use in English, how would we read or speak? I suppose we could sit with a dictionary and look each word up, but that would be rather painful! If you understand how these equations are derived and how they relate to each other, your engineering practice will be smoother and more successful.

2.2 Electrical Elements

I'll begin this introduction with electrical systems, for the following reasons. First, every engineer and advanced technician, no matter what their field of specialty, should know everything contained in this section about electrical engineering, just on general principles! Second, many modeling and analysis concepts and techniques are easier to explain and easier to grasp using electrical elements. Third, numerous tools and methods for analyzing electrical circuits have been developed over the years and all of these can be applied to modeling and analyzing mechanical, fluid, and thermal circuits as well.

Unfortunately, I've discovered that many civil, mechanical, and other nonelectrical engineers and technicians shy away from electrical circuits, circuit modeling, circuit analysis, and other such nasties. *Please* don't do this. If you want to be a really good engineer or advanced technician, you should know all of the electrical engineering fundamentals discussed in this section.

Concepts of Voltage and Current

The concepts of *voltage* and *current* are used in electrical and electronics engineering to describe the behavior of engineering systems that use electricity. Electrical power supplies, generators, motors, transformers, and computers are examples of such systems.

The term *voltage* is used to describe the work that must be performed to move a unit of electrical *charge* (an electron is a minute electrical charge) from one point to another. Consequently, voltage is a relative term and is often referred to as the *potential difference* between two points. The units for voltage are *volts*, and the units for electrical charge are *coulombs*. One coulomb of electrical charge is equal, but opposite in sign, to 6.22×10^{18} electrons.

Figure 2.1 is a *definition sketch* which shows a symbolic, or circuit, diagram of an electrical element. A voltage V_1 is at one end and a voltage V_2 at the other. If V_2 is not equal to V_1, then electrical charge flows from one side of the element to the other. This flow of electrical charge per unit of time is called *current* and is given the symbol i. Current is measured in units called *amperes* or *amps*.

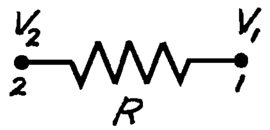


Figure 2.1. Definition sketch of electrical element.

Figure 2.2 shows the same electrical element given in Figure 2.1 with different voltage values at each end. This figure indicates that current has a direction and a magnitude. If the voltage at point 2 is greater than that at point 1, then by convention we say a current flows from point 2 to point 1. Conversely, the current flows from point 1 to point 2 if the voltage at point 1 is greater than it is at point 2. If the voltages at the two points are the same, no current flows.

Since voltage is a relative term, showing a voltage V_2 at point 2 and a voltage V_1 at point 1,

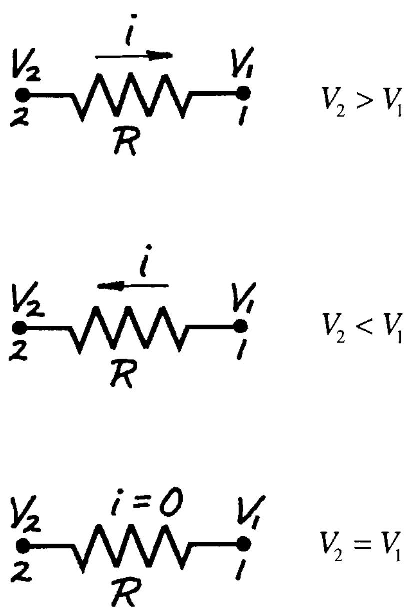


Figure 2.2. Definition sketch of electrical element with different voltage/current values.

as was done in Figures 2.1 and 2.2, implies that there is some reference point associated with these voltages. That point is generally called *ground* or *earth*. Ground has no potential; that is, there is no place surrounding ground where work must be done on an electrical charge to move it from that point to ground. The voltage at point 2 can be referenced to the voltage at point 1. We will use the symbol V_{21}, meaning $V_{21} = V_2 - V_1$, to denote this reference.

Concepts of Work, Power, and Energy in Electrical Elements

Since voltage is defined as the work that must be done to move a unit of electrical charge from one point to another, we can write voltage between two points as

$$V_{21} = \frac{dW_{21}}{dq} \tag{2.2}$$

where dW_{21} is the work that must be done to move the electrical charge dq from point 1 to point 2. A unit of measure for work is the *joule*. Equation (2.2) then defines volts as joules per coulomb. That is, one volt is equal to one joule of work per one coulomb of charge. One joule of work is equivalent to one watt-sec or 0.737 ft-lbs.

Current was defined above as the flow of electrical charge per unit of time. This can be written in the form of a derivative. That is,

$$i = \frac{dq}{dt} \tag{2.3}$$

The units which apply to this equation are coulombs per second, called amperes.

Power is defined as the rate at which work is performed. This too can be written in the form of a derivative. That is,

$$P = \frac{dW}{dt} \tag{2.4}$$

The product of voltage differential across an electrical element and the current flowing through the element is equal to power. This is a very important concept. You can see this by multiplying equation (2.2) and (2.3). That is,

$$V_{21} \times i = \frac{dW_{21}}{dq} \times \frac{dq}{dt} = \frac{dW_{21}}{dt} = P \tag{2.5}$$

The units for power are

$$P = V_{21}i = 1 \text{ volt} \times 1 \text{ ampere}$$

or

$$P = V_{21}i = 1\frac{\text{joule}}{\text{coulomb}} \times 1\frac{\text{coulomb}}{\text{second}}$$

$$= \frac{\text{joules}}{\text{second}} = 1 \text{ watt}$$

Since work is a transitory form of energy, we can rewrite equation (2.4) as

$$\frac{dE}{dt} = P \tag{2.6}$$

Equation (2.6) can be integrated to obtain the energy stored in, or dissipated by, an electrical element over a time interval from $t = t_a$ to $t = t_b$. That is,

$$dE = Pdt$$

$$E = \int Pdt$$

$$E_b - E_a = \int_{t_a}^{t_b} Pdt = \int_{t_a}^{t_b} (V_{21}i)dt \tag{2.7}$$

The Resistor

The most common of all electrical elements is the resistor. It is intentionally or unintentionally present in every real electrical system. Figure 2.3 shows a symbolic (circuit) diagram and a graphical representation of this element. Also shown are the fundamental describing equations for an *ideal resistor*. An idealization of the resistor is given by

$$\boxed{V_{21} = Ri} \tag{2.8}$$

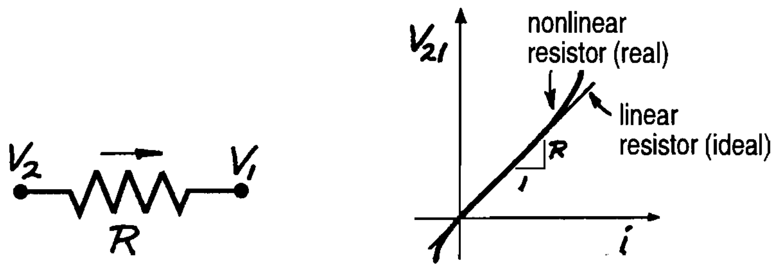
Ri is a linear function in which the voltage is proportional to the current and R is the constant of proportionality. You will also often see the equation for a resistor written as

$$\boxed{i = \frac{V_{21}}{R}} \tag{2.9}$$

Equations (2.8) and (2.9) are often called *Ohm's Law*. The value of R is usually given in *ohms*, which is really volts per amp.

The energy delivered to a resistor in the interval from $t = t_a$ to $t = t_b$ is given by equation (2.7). We can substitute equation (2.9) into (2.7) and eliminate the current as follows:

$$E_b - E_a = \int_{t_a}^{t_b} (V_{21}i)dt \qquad \text{(2.7) repeated}$$



$$i = \frac{V_2 - V_1}{R} = \frac{V_{21}}{R}$$

or

$$V_{21} = Ri$$

(a). *Symbolic or circuit diagram.*
(b). *Graphical representation.*
(c). *Describing equations for ideal damper.*

Figure 2.3. The resistor.

Substitute V_{21} / R for i and obtain

$$E_b - E_a = \int_{t_a}^{t_b} V_{21}\left(\frac{V_{21}}{R}\right)dt \qquad (2.10)$$

Since R is a constant, it can be pulled outside of the integral sign, giving

$$E_b - E_a = \frac{1}{R}\int_{t_a}^{t_b} V_{21}^2 dt \qquad (2.11)$$

We could have substituted equation (2.8) into equation (2.7) and eliminated V_{21}. If we take this route then we get

$$E_b - E_a = \int_{t_a}^{t_b} (Ri)i dt \qquad (2.12)$$

Pulling the constant R outside the integral sign gives

$$E_b - E_a = R\int_{t_a}^{t_b} i^2 dt \qquad (2.13)$$

One very important thing that equations (2.11) and (2.13) reveal is that a resistor dissipates power. Regardless of the direction of the current or the sign of the voltage, both are squared in the equation. Therefore, energy can't be retrieved from a resistor. This element only dissipates energy.

By comparing equation (2.6) with (2.11) and (2.13) you can see that the power dissipated by a resistor at any instant in time is

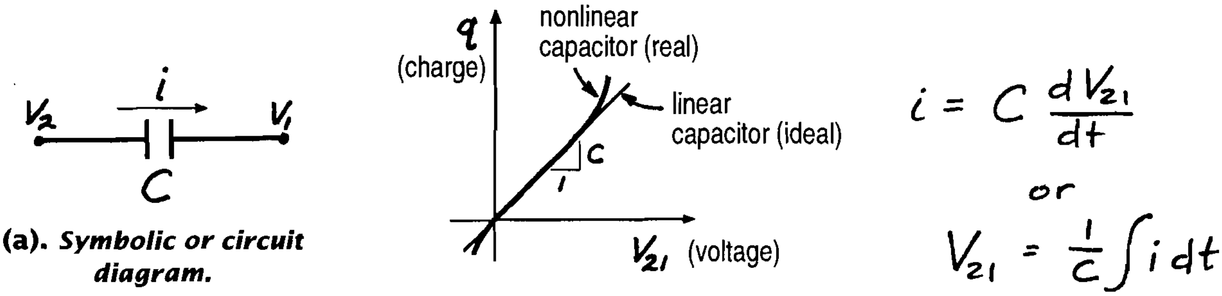
$$\boxed{P = \frac{V_{21}^2}{R} = i^2 R} \qquad (2.14)$$

In real electrical circuits, the energy dissipated by a resistor is converted into heat. Unless this heat is removed the resistor could burn out.

The Capacitor

Another fundamental electrical element is the *capacitor*. Like the resistor, it is intentionally or unintentionally present in every real electrical system. Figure 2.4 shows a symbolic (circuit) diagram and a graphical representation of this element. Also shown are the fundamental describing equations for an *ideal capacitor*.

A capacitor is constructed of two pieces of conducting material separated by another material that allows an electrostatic field to be established without allowing charge to flow between



(a). *Symbolic or circuit diagram.*

(b). *Graphical representation.*

(c). *Describing equations for ideal capacitor.*

Figure 2.4. The capacitor.

the two pieces of conducting material. A capacitor stores electrical energy in this electrostatic field. In an ideal capacitor, all of the energy stored in the device can be retrieved and used.

An idealization of the capacitor is given by the linear relationship

$$q = CV_{21} \tag{2.15}$$

You can see from equation (2.15) that units of capacitance are coulombs per volt. One coulomb per volt is called a *farad*. A farad is a very large number, so most capacitors you will run into will have values given in microfarads (μf) which is one-millionth of a farad.

Differentiating both sides of equation (2.15) with respect to t gives

$$\frac{dq}{dt} = C\frac{dV_{21}}{dt} \tag{2.16}$$

Combining (2.16) and (2.3) gives the following equation for the current-voltage relationship of an ideal capacitor

$$\boxed{i = C\frac{dV_{21}}{dt}} \tag{2.17}$$

The energy delivered to a capacitor in the time interval from $t = t_a$ to $t = t_b$ is from equation (2.7) given by

$$E_b - E_a = \int_{t_a}^{t_b} V_{21} i\,dt \tag{2.18}$$

substituting CdV_{21} for $i\,dt$ from equation (2.17) gives

$$E_b - E_a = C\int_{V_a}^{V_b} V_{21}\,dV_{21} \tag{2.19}$$

Note that the limits of integration have been changed. V_b is the voltage across the capacitor at time $t = t_b$ and V_a is the voltage at time $t = t_a$. The equation can be integrated to give

$$\boxed{E_b - E_a = C\left[\frac{V_{21}^2}{2}\right]_{V_a}^{V_b} = C\frac{V_b^2}{2} - C\frac{V_a^2}{2}} \tag{2.20}$$

The quantity $C(V_a^2/2)$ is the energy that was initially stored in the capacitor at $t = t_a$ and $C(V_b^2/2)$ represents the energy stored at time $t = t_b$. During the time interval $t_b - t_a$, the energy $(E_b - E_a)$ was added to the capacitor. The energy storage feature of a capacitor is extremely important. Numerous electrical circuits, including memory chips in computers make use of the energy storage capability of a capacitor.

The Inductor

The third and last electrical element we will discuss is the inductor. Like the resistor and capacitor, it is intentionally or unintentionally present in every real electrical system. Figure 2.5 shows a symbolic (circuit) diagram and a graphical representation of this element. Also shown are the fundamental describing equations for an *ideal inductor*. It is essentially the opposite of a capacitor. The symbol for an inductor looks like a wire coil because, in its simplest form, that's all it is. Any wire or conductor carrying a current is surrounded by a magnetic field. This magnetic field can be concentrated by winding the wire into a tight coil about a tube. The strength of the magnetic field due to one turn φ can be increased by increasing the number of turns N, so the strength of the total magnetic field is given by $N\varphi$. The magnetic field strength can be increased even further by wrapping coils of the wire around an iron core. However, this generally makes the inductor nonlinear and creates hysteresis, as shown in the graph of Figure 2.5.

The strength of a magnetic field around and the voltage across an inductor are related by the following equation, often referred to as *Faraday's law of induction:*

$$V_{21} = \frac{d(N\varphi)}{dt} \tag{2.21}$$

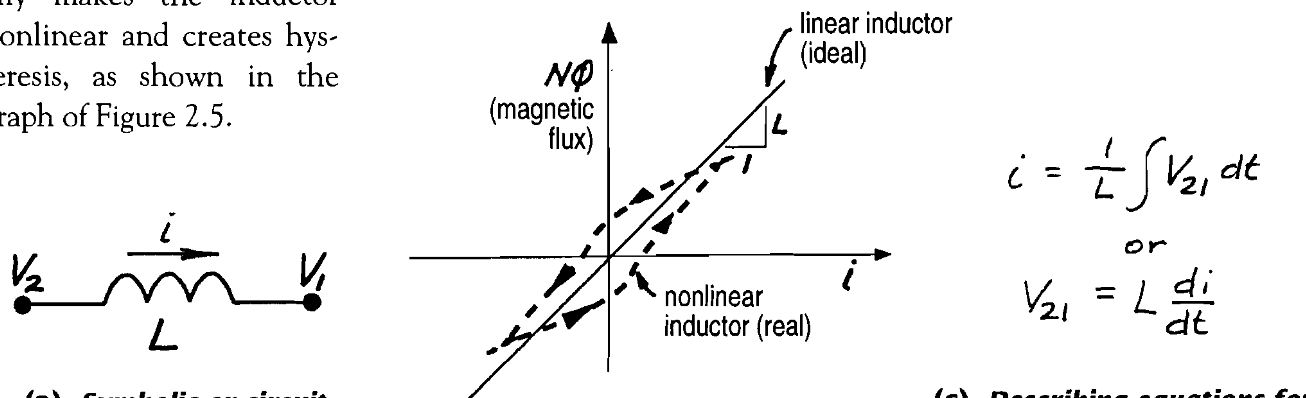
If no magnetic materials are present or used in the core of the inductor, then the strength of the magnetic field is linearly dependent on the current. That is,

$$\frac{d(N\varphi)}{dt} = L\frac{di}{dt} \tag{2.22}$$

Combining equations (2.21) and (2.22) gives

$$\boxed{V_{21} = L\frac{di}{dt}} \tag{2.23}$$

Note that the units of an inductor as determined from equation (2.23) are volts times seconds per ampere. One volt-second per ampere is called a *henry* (H). Like the farad, a henry is a large unit. Consequently, most real-life induc-



(a). ***Symbolic or circuit diagram.*** **(b).** ***Graphical representation.*** **(c).** ***Describing equations for ideal inductor.***

Figure 2.5. The inductor.

tors you will run into will have values in the millihenry (mH) or microhenry (μH) range. A millihenry is one-thousandth of a henry and a microhenry is one-millionth of a henry.

The energy delivered to an inductor in the time interval $t = t_a$ to $t = t_b$ is

$$E_b - E_a = \int_{t_a}^{t_b} V_{21} i dt \qquad \text{(2.7) repeated}$$

Substituting equation (2.23) for V_{21} gives

$$E_b - E_a = \int_{t_a}^{t_b} \left(L\frac{di}{dt} \right) idt = L\int_{i_a}^{i_b} idi$$

Carrying out the integration gives

$$E_b - E_a = L\left[\frac{i^2}{2} \right]_{i_a}^{i_b} = L\frac{i_b^2}{2} - L\frac{i_a^2}{2} \qquad (2.24)$$

The quantity $L(i_a^2/2)$ is the energy that was initially stored in the inductor (magnetic field) at $t = t_a$ and $L(i_b^2/2)$ is the energy stored at $t = t_b$. During the time interval $t_b - t_a$, the energy $(E_b - E_b)$ was added to the magnetic field of the inductor. The energy storage feature of an inductor is extremely important. Numerous electrical circuits including electromagnets, radios, and switching power supplies make use of it.

What to Commit to Memory

Figure 2.6 shows the three fundamental electrical elements and one describing equation for each. You should commit these equations to memory. Also commit to memory the fact that power is the product of voltage and current. If you remember all this, you will be able at any time to derive all of the energy equations discussed here and used later on in the book.

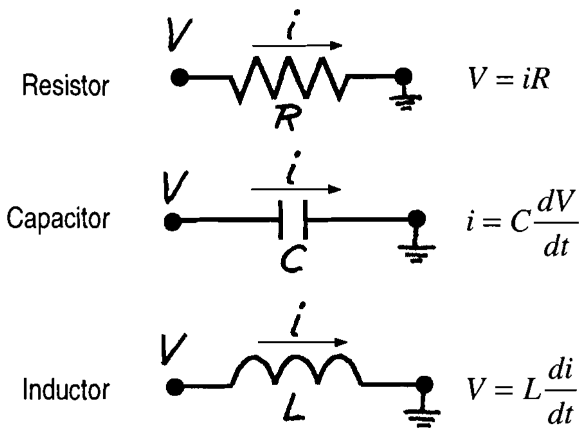


Figure 2.6. The three fundamental electrical elements.

Impedance and Operational Block Diagrams

Now that you know the equations that describe the relations between voltage and current for ideal resistors, capacitors, and inductors, I want to introduce some very important tools that will help you remember these equations and that will make it easier to derive mathematical models of systems containing the basic electrical components. You will discover in later chapters that these tools are applicable not only to electrical systems, but also to mechanical, fluid, and thermal systems.

Tool #1 – Impedance

The concept of impedance is based on observing that a variable which flows through an element is impeded by the element. For example, in electrical elements a current flows *through* a resistor due to a voltage difference *across* the resistor. The current is called the *through* variable and voltage is called the *across* variable. These terms apply to inductors and capacitors also.

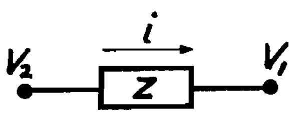


Figure 2.7. Representation of the impedance Z of any electrical element.

The impedance of any electrical component, shown in Figure 2.7, is given by

$$Z = \frac{V_{21}}{i} \quad (2.25)$$

For a resistor, the impedance Z_R is given by

$$\boxed{Z_R = \frac{V_{21}}{i} = R} \quad (2.26)$$

That is, the impedance of a resistor is simply its resistance.

The relationship between voltage and current for a capacitor is given by

$$V_{21} = \frac{1}{C}\int i dt \quad (2.27)$$

I will often use the operator $D = d/dt$ to express derivatives (and $1/D$ to express integrals). This notation helps to reduce the derivation of math models to mere multiplication. (It's discussed in more detail in Appendix A.) Using the operator notation, we can write this equation as

$$V_{21} = \frac{1}{CD} i \quad (2.28)$$

The impedance is then given by

$$\boxed{Z_C = \frac{V_{21}}{i} = \frac{1}{CD}} \quad (2.29)$$

The relationship between voltage and current for an inductor is given by

$$V_{21} = L\frac{di}{dt} \quad (2.30)$$

Again using the operator notation, we can write this equation as

$$V_{21} = LDi \quad (2.31)$$

The impedance is then given by

$$\boxed{Z_L = \frac{V_{21}}{i} = LD} \quad (2.32)$$

Figure 2.8 provides a summary of the impedance of electrical circuit elements.

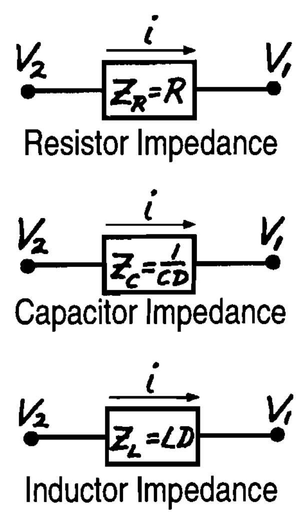


Figure 2.8. Impedance of electrical circuit elements.

Tool #2 – Operational Block Diagrams

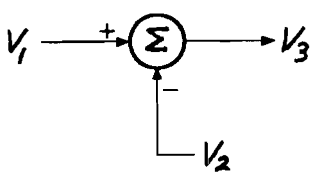
Operational block diagrams are extremely helpful in visualizing an engineering system and in communicating your design ideas to others. The completed diagram can also assist with solving the equations using analog and digital computers.

An operational block represents a mathematical operation. It operates on the input or forcing variable and transforms it into the output or response variable. An operational block is often called a transfer function. That is it transfers (transforms) the input variable into the output variable and clearly establishes cause and effect in a system.

There are four basic operational blocks:

(1) the summer
(2) the constant multiplier
(3) the integrator
(4) the differentiator.

An example of a summer is shown in Figure 2.9. There are two input signals in this example, V_1 and V_2. Input signals are designated as such by showing an arrow pointing into the summation box along with a plus or minus sign. There can only be one output from a summer. In this case it is V_3.



$$V_3 = V_1 - V_2$$

(a). *Block diagram.* **(b). *Equation.***

Figure 2.9. The operational block diagram of a summer.

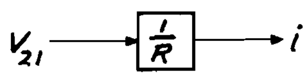
An example of a constant multiplier block is shown in Figure 2.10. I use the notation that the variable on the right side of an equal sign in an equation is the input or forcing variable, and the variable on the left side is the output or response variable. The block diagram makes this very clear.

$$V_{21} = Ri$$

(a). *Block diagram.* **(b). *Equation.***

Figure 2.10. The operational block diagram of a multiplier.

Even though we know we can solve for i in terms of V_{21}, the block diagram does not permit this. You must rewrite the equation and draw a new block diagram as shown in Figure 2.11.

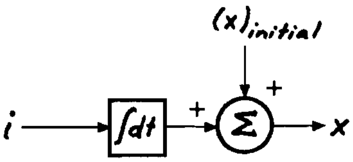


$$i = \frac{1}{R} V_{21}$$

(a). *Block diagram.* **(b). *Equation.***

Figure 2.11. The operational block diagram of a multiplier showing importance of distinguishing between input and output.

The next block diagram we will discuss is the integrator. As you know, integration must account for the *constant of integration*. To handle this in a block diagram, a summer is added *after* the integration block, as shown in Figure 2.12. Be careful here. The summer and the integration



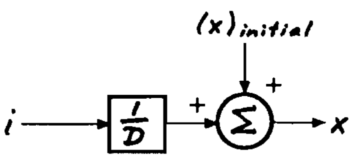
$$x = \int_0^t i dt + (x)_{init}$$

(a). *Block diagram.* **(b). *Equation.***

Figure 2.12. The operational block diagram of an integrator.

block go together, and the summer must be *after* the integrator. If it were in front and had a value, it would produce an incorrect answer.

I used the integral sign in Figure 2.12 to remind you that it is equivalent to the 1/D operator. An equivalent integrator block diagram is shown in Figure 2.13.

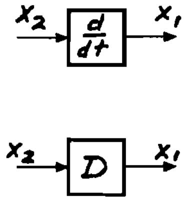


$$x = \frac{1}{D} i + (x)_{initial}$$

(a). *Block diagram.* **(b). *Equation.***

Figure 2.13. Alternative operational block diagram for an integrator.

The final block diagram we'll discuss is the differentiator. An example is shown in Figure 2.14. A differentiator can sometimes be useful, but great care must be taken in their use. For example, I've designed a number of instrumentation systems that measure ship motions and accelerations. It is possible to directly measure only displacement and then differentiate to get velocity and acceleration. However, since differentiators tend to magnify noise and are inaccurate on a digital computer compared to integration, I usually use accelerometers to measure acceleration directly. Integration, on the other hand, is a smoothing operation and can be handled accurately on a digital computer. You should typically avoid using differentiators in your block diagrams for the three basic electrical components.



$$x_1 = \frac{dx_2}{dt}$$

$$x_1 = Dx_2$$

(a). *Block diagrams.* **(b). *Equations.***

Figure 2.14. Operational block diagrams of a differentiator.

We will use these tools extensively throughout this book. Figure 2.15 provides a summary of the three electrical circuit elements. It shows the symbolic circuit diagram, the operational block diagrams, and the impedance circuit.

2.3 Mechanical Components (Translational)

Concepts of Mechanical Motions and Forces

Motion and *force* are concepts used to describe the behavior of engineering systems that employ mechanical components. Heads of disk drives, armatures of motors, gears, ships, and automobiles are just a few examples of engineering systems that employ mechanical components. Indeed, it is hard to imagine any engineering system that does not have mechanical components.

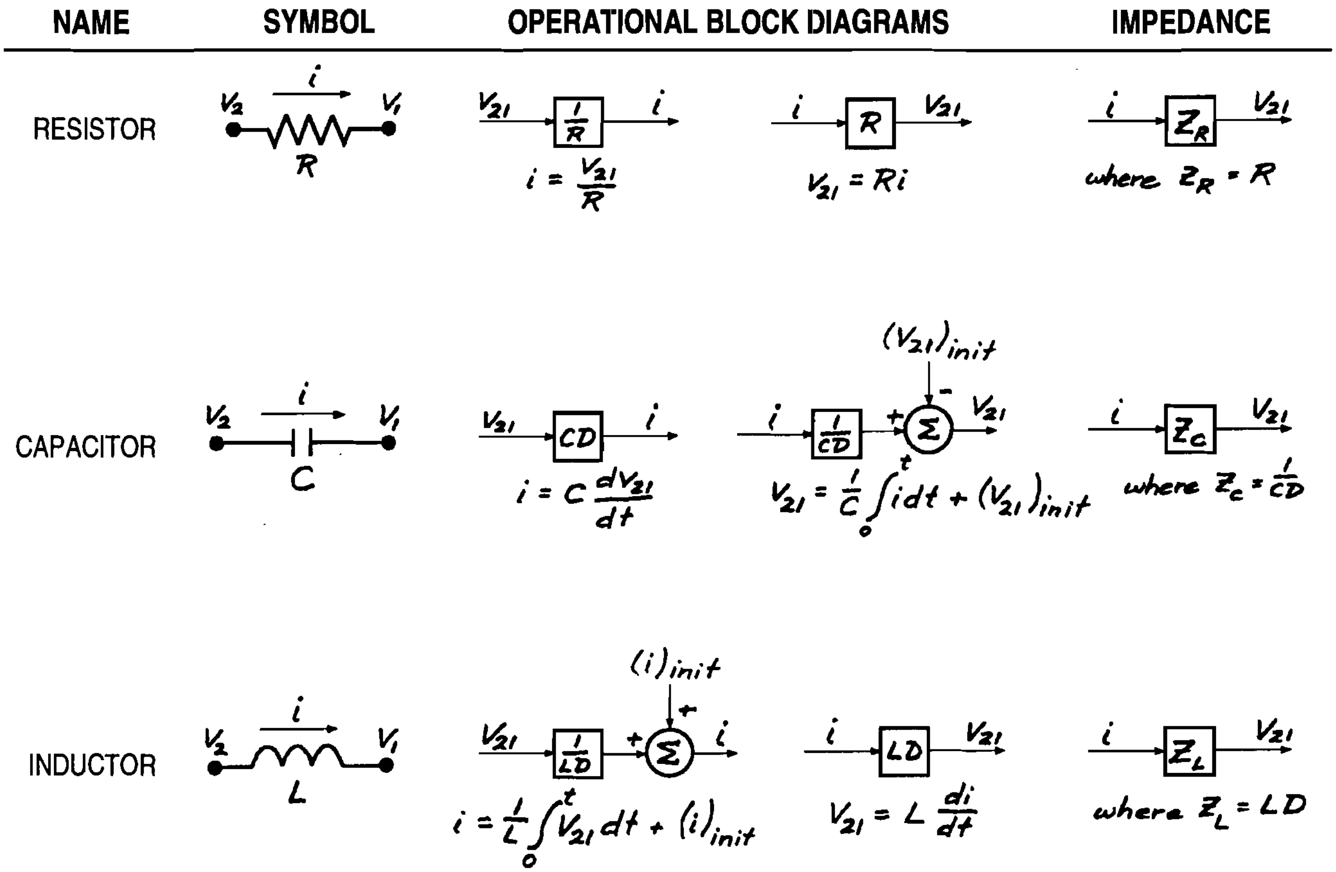


Figure 2.15. Summary of fundamental electrical elements.

Motion is a term used to describe the movement of a point relative to another, and it is described using the terms *distance*, *velocity*, and *acceleration* (see Appendix A if you need a brush-up). The three are related by differentiation or integration. If you know one, you can obtain the other two.

Figure 2.16 is a symbolic or "circuit" diagram of a mechanical component whose ends are undergoing translational movement. One end of this component is moving in a straight line at a velocity v_2 and the other is moving in the same straight line at a velocity v_1. Since velocity is a relative term, this figure implies the existence of a reference that is fixed.

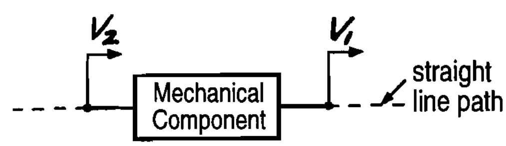


Figure 2.16. Symbolic diagram of a translational mechanical component.

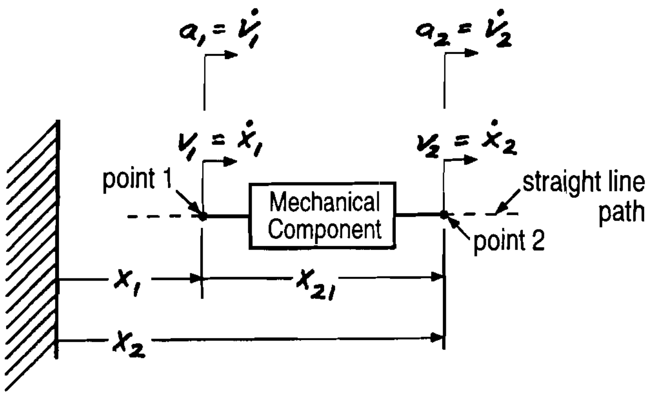


Figure 2.17. Symbolic diagram of a translational mechanical component showing reference point.

Figure 2.17 shows the same mechanical component as in Figure 2.16 but with the fixed reference shown. The component is moving in a straight line and I've shown the distance, velocity, and acceleration of points 1 and 2.

From this figure we can write the following equations describing the motion of point 2 relative to point 1:

$$x_{21} = x_2 - x_1 \tag{2.33}$$

$$v_{21} = v_2 - v_1 = \dot{x}_2 - \dot{x}_1 = \dot{x}_{21} \tag{2.34}$$

$$\begin{aligned} a_{21} &= a_2 - a_1 = \dot{v}_2 - \dot{v}_1 \\ &= \dot{v}_{21} = \ddot{x}_2 - \ddot{x}_1 = \ddot{x}_{21} \end{aligned} \tag{2.35}$$

Relative motion between the ends of a mechanical component can't exist without a force being present. When relative motion exists, the mechanical component is placed in a state of tension or compression. The associated force has both a magnitude and a sign. The conventions used are shown in Figure 2.18.

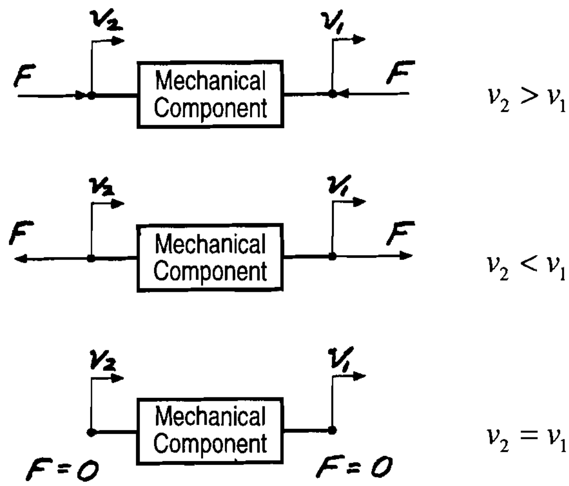


Figure 2.18. Symbolic diagram of a translational mechanical component showing direction and magnitude of force.

Note that power is the product of the force and velocity:

$$Power = Fv$$

Refer to Appendix B.2 for more details on work, energy, and power.

The Damper or Dashpot

A damper (sometimes called a dashpot) is a mechanical component often found in engineering systems. The shock absorbers in your car are an example of a mechanical damper that is intentionally designed into every car. A typical shock absorber is shown in Figure 2.19. The device consists of a piston that moves inside a cylinder filled with hydraulic fluid. Small holes are drilled through the piston so fluid can move from one chamber to the other. As the piston moves relative to the cylinder, the fluid is forced through these small openings, creating resisting fluid shearing forces. If the mass and springiness

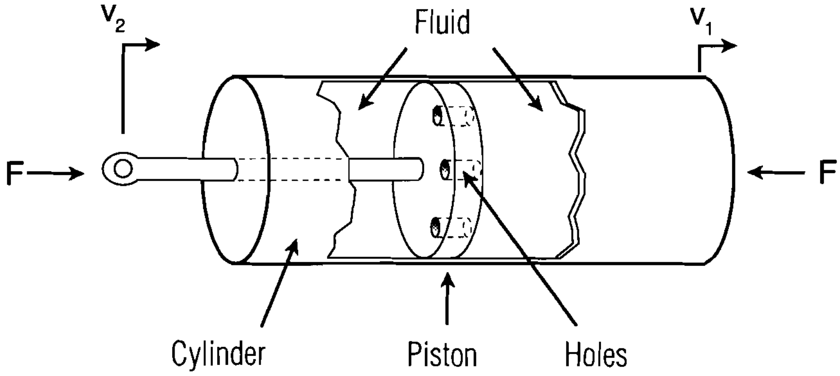
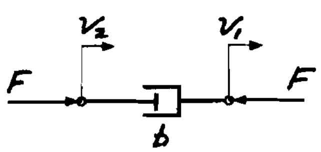


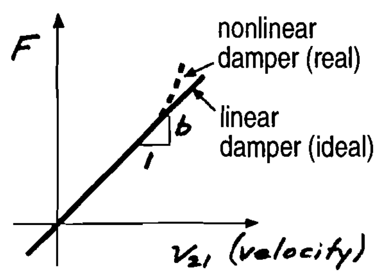
Figure 2.19. A typical mechanical damper.

of the piston and cylinder are small, then the force will be a function of the relative velocity between the piston and the cylinder.

Figure 2.20 shows a symbolic (circuit) diagram of this component along with a graphical representation and the fundamental describing equations for an ideal damper. The symbol implies that this component has no mass and the connecting rods have no springiness. Furthermore, because this is a translational component, the forces act along the same single straight line that characterizes the motion of the two ends.



(a). Symbolic or circuit diagram.



(b). Graphical representation.

$$F = b(v_2 - v_1) = bv_{21} \qquad \text{OR} \qquad v_{21} = \frac{1}{b}F$$

(c). Describing equations for ideal damper.

Figure 2.20. The ideal damper.

An ideal damper is a linear component described by

$$\boxed{F = b(v_2 - v_1) = bv_{21}} \tag{2.36}$$

or

$$\boxed{v_{21} = \frac{1}{b}F} \tag{2.37}$$

Knowing the units of F and v, you can see that the units of the damping force constant b are lb-sec/in.

Like the electrical resistor, the energy delivered to a mechanical damper cannot be retrieved. This energy is dissipated to the surroundings in the form of heat. (Refer to Appendix B for the derivation equations.)

The power dissipated by the damper is

$$P = F \times v_{21} = (bv_{21}) \times v_{21}$$
$$= bv_{21}^2 \tag{2.38}$$

or

$$P = F \times v_{21} = F \times \left(\frac{1}{b}F\right)$$
$$= \frac{1}{b}F^2 \tag{2.39}$$

The Translational Mass

All real mechanical components used in engineering systems have mass. In general, the mass is distributed in three-dimensional space. However, it is frequently possible (and very convenient) to treat a component as if all of its mass were concentrated at a single point called the center of gravity, or c.g. That is, we lump the mass together into one point and end up with an ideal, *lumped parameter* component. If we further consider only a lumped parameter mass moving in a straight line, then we have a *translational* mass.

The symbolic diagram that will be used for a translational mass component is a little peculiar. So, let's build up to it slowly. Figure 2.21 shows a lumped mass moving in a straight line. According to Newton, if the mass m is not changing then

$$F = ma_2 = m\frac{dv_2}{dt} = m\frac{d^2x_2}{dt^2} \qquad (2.40)$$

Now let's write equation (2.40) in terms of relative velocity between points 2 and 1

$$F = m\frac{dv_{21}}{dt} \qquad (2.41)$$

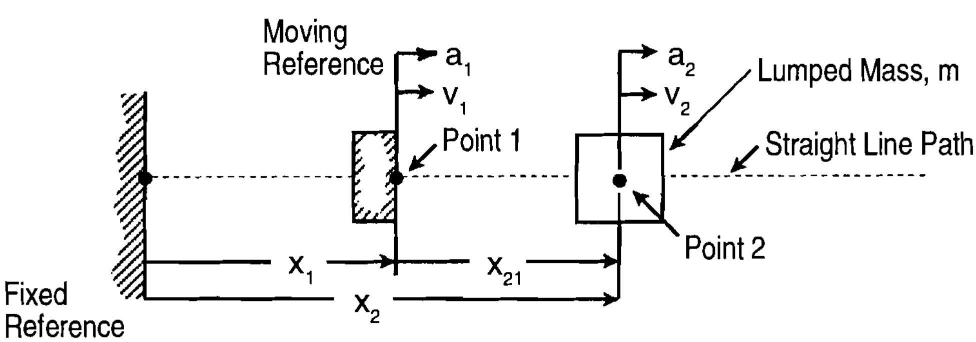


Figure 2.21. Lumped mass moving in a straight line.

and then determine what conditions we have to place on point 1 for equation (2.41) to apply. If we differentiate equation (2.34) we have

$$\frac{dv_{21}}{dt} = \frac{dv_2}{dt} - \frac{dv_1}{dt} \qquad (2.42)$$

Now if we combine equations (2.41) and (2.42) we get

$$F = m\frac{dv_2}{dt} - m\frac{dv_1}{dt} \qquad (2.43)$$

Finally, compare equation (2.43) with equation (2.40). The two equations are equal only when v_1 is constant. In that case dv_1 / dt will equal zero. This means that point 1, the reference, must be stationary or moving at a constant velocity in order for equation (2.41) to be true.

The symbolic diagram we will use to represent the translational mass component is given in Figure 2.22. The dotted line at one end of the element will serve to remind us that

(a) there is no physical connection between point 2 (the mass) and point 1

(b) point 1 must be stationary or moving at a constant velocity.

(c) no force is transmitted to point 1.

Figure 2.23 shows the symbolic (circuit) diagram of the translational mass along with a graphical representation and the fundamental describing equations. Note that the graph shows a plot of *momentum* versus

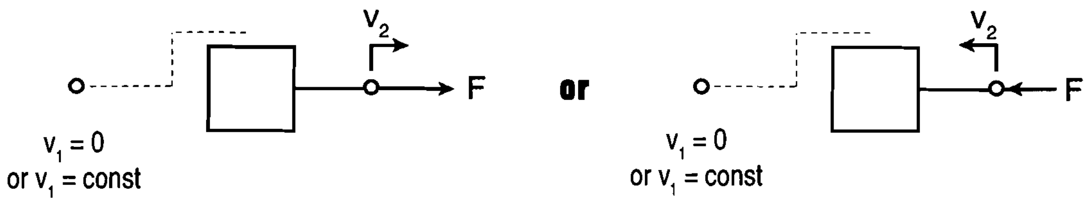
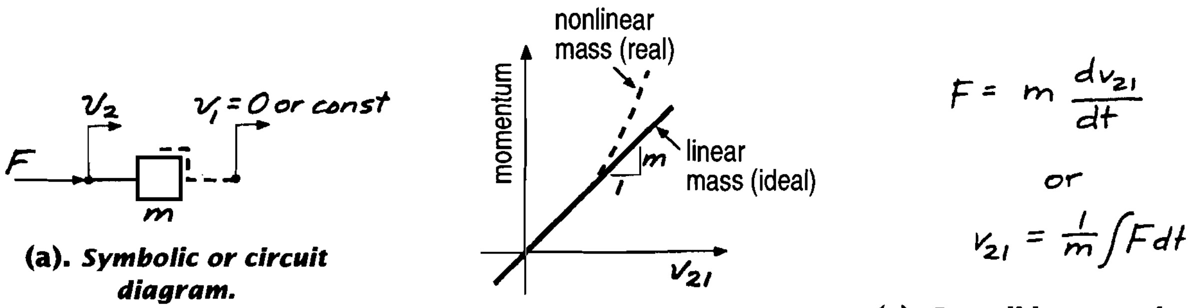


Figure 2.22. Symbolic diagram for translational mass.



(a). Symbolic or circuit diagram.

(b). Graphical representation.

(c). Describing equations for ideal mass.

Figure 2.23. The ideal translational mass.

velocity. Momentum is the product of the mass of a moving object and its velocity. Force is equal to the rate of change of momentum. That is

$$F = \frac{d(mv)}{dt} \tag{2.44}$$

Only when m is constant can we pull the m outside of the differential. If m is constant then the mass is called an ideal (or Newtonian) mass. That is,

$$\boxed{F = m\frac{dv_{21}}{dt}} \tag{2.45}$$

or

$$\boxed{v_{21} = \frac{1}{m}\int F dt} \tag{2.46}$$

Knowing the units of F and v, you can always determine the units of mass as $F/(dv/dt)$ = lb-sec^2/in.

The energy delivered to the mass in the time interval $t = t_a$ to $t = t_b$ by a force $F(t)$ is

$$E_b - E_a = \int_{t_a}^{t_b} F v_{21} dt \tag{2.47}$$

Substituting (2.45) into (2.47) to eliminate F gives

$$E_b - E_a = \int_{t_a}^{t_b}\left(m\frac{dv_{21}}{dt}\right)v_{21}dt$$

$$= m\int_{v_{21a}}^{v_{21b}} v_{21} dv_{21} \tag{2.48}$$

Carrying out the integration gives

$$E_b - E_a = m\left[\frac{v_{21}^2}{2}\right]_{v_{21a}}^{v_{21b}} = \frac{m}{2}\left(v_{21b}^2 - v_{21a}^2\right)$$

$$= \frac{m}{2}v_{21b}^2 - \frac{m}{2}v_{21a}^2 \qquad (2.49)$$

Let's examine equation (2.49). The quantity $(m/2)v^2_{21a}$ represents the energy of the mass at time $t = t_a$ and the quantity $(m/2)v^2_{21b}$ represents the energy at time $t = t_b$. These quantities are identical to those of the capacitor. Like the capacitor, the mass is an energy storage device. The energy stored can be completely retrieved.

The Translational Spring

Nearly all materials used in mechanical systems exhibit an elastic effect that we call a spring. When a force is applied to these materials they deform. When the force is removed, they return to their original shape. A spring is a fundamental mechanical component found intentionally or unintentionally in almost every mechanical engineering system. The springs in your car are a good example. Steel is shaped into a coil so the mass of the spring is minimized and its elasticity maximized.

Figure 2.24 shows a symbolic (circuit) diagram of a spring along with a graphical representation and the fundamental describing equations. The symbol implies that this component has no mass or damping. Furthermore, because this is a translational component, the forces act along the same single straight line that characterizes the motion of the two ends.

An ideal spring is described by

$$F = kx_{21} \qquad (2.50)$$

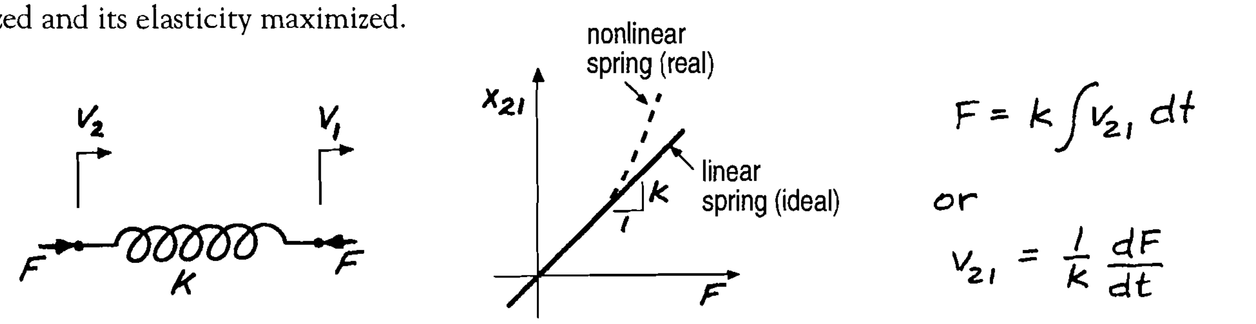
where k is a constant of proportionality relating the force F to the deformation x_{21} of the spring. Knowing the units of F and x, you can easily determine the units of k as F/x = lb/in.

Since we want to describe all of the components in terms of velocity and force, we'll differentiate equation (2.50) as follows

$$\frac{dF}{dt} = k\frac{dx_{21}}{dt} = kv_{21} \qquad (2.51)$$

Rearranging gives

$$\boxed{v_{21} = \frac{1}{k}\frac{dF}{dt}} \qquad (2.52)$$



(a). ***Symbolic or circuit diagram.*** **(b).** ***Graphical representation.*** **(c).** ***Describing equations for ideal spring.***

Figure 2.24. The ideal spring.

Also we can integrate (2.51) as follows

$$F = \int dF = \int k v_{21} dt$$

or

$$\boxed{F = k\int v_{21} dt} \tag{2.53}$$

The energy delivered to the spring in the time interval $t = t_a$ to $t = t_b$ by a force $F(t)$ is

$$E_b - E_a = \int_{t_a}^{t_b} F v_{21} dt \tag{2.54}$$

Substituting (2.52) into (2.54) to eliminate v gives

$$E_b - E_a = \int_{t_a}^{t_b} F\left(\frac{1}{k}\frac{dF}{dt}\right)dt$$

$$= \frac{1}{k}\int_{F_a}^{F_b} F dF \tag{2.55}$$

Carrying out the integration gives

$$E_b - E_a = \frac{1}{k}\left[\frac{F^2}{2}\right]_{F_a}^{F_b}$$

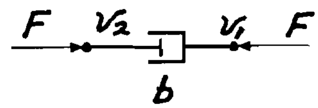
$$= \frac{F_b^2}{2k} - \frac{F_a^2}{2k} \tag{2.56}$$

Let's examine equation (2.56). The quantity $F_a^2/2k$ represents the energy of the spring at time $t = t_a$ and the quantity $F_b^2/2k$ represents the energy at time $t = t_b$. These equations are identical to those of the inductor. Like the inductor, the spring is an energy storage device. The energy stored can be completely retrieved.

What to Commit to Memory

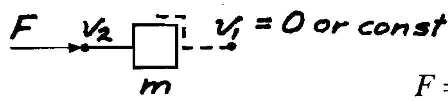
You now have all the building blocks you need to mathematically model engineering systems comprised of translational mechanical components. In the next section we will look at rotational mechanical components, so we can model *any* mechanical system. However, before you move on, the following basic equations should be committed to memory:

DASHPOT



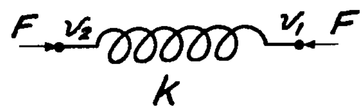
$$F = b v_{21}$$

MASS



$$F = m\frac{dv_{21}}{dt}$$

SPRING



$$F = kx_{21} \quad \text{OR} \quad \frac{dF}{dt} = kv_{21}$$

Figure 2.25. Translational mechanical elements.

Just as I asked you to do with the basic electrical components, make these equations a part of your life and know them as well as you know your name!

The tools associated with operational block diagrams and impedance that you were introduced to when we investigated the basic electrical components also apply to the three basic translational mechanical components. You should now review ***What to Commit to Memory*** in the previous section on electrical elements. The block diagrams and impedances for the basic translational mechanical elements are given in Figure 2.26.

Analogies and Similarities

It has probably occurred to you by now that there are a lot of similarities between the equations describing mechanical and electrical components. Let's look into this in more detail. In Table 2.1 I have arranged the equations in such a way that it is obvious that they have the same form. If we were to agree that:

(1) voltage in an electrical system is analogous to velocity in a mechanical system, and

(2) current in an electrical system is analogous to force in a mechanical system,

then the electrical and mechanical components are *analogs* of one another. Resistors behave like dampers, capacitors like masses, and inductors like springs.

We could, however, just as easily arrange the equations as shown in Table 2.2. Now, if we were to agree that:

(1) voltage is analogous to force, and
(2) current is analogous to velocity

then we have again made electrical and mechanical components analogs of one another.

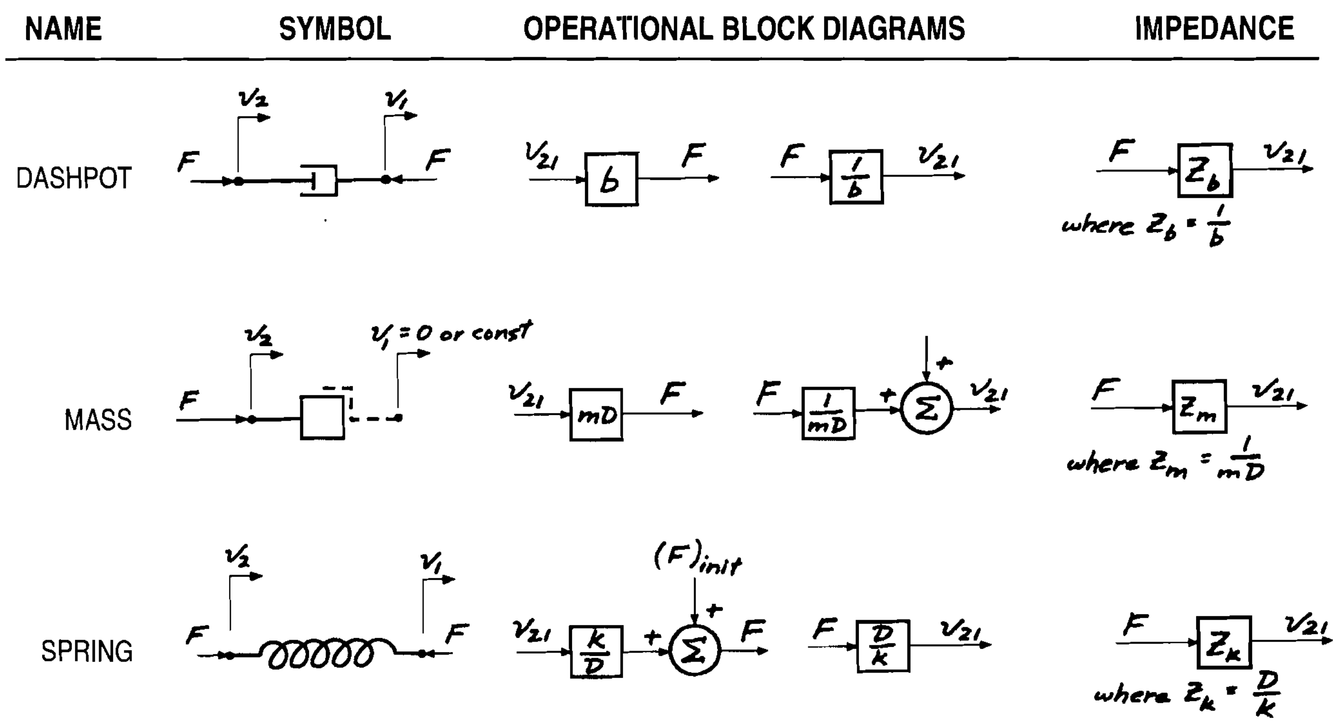


Figure 2.26. Summary of translational mechanical elements.

Table 2.1. Electrical–Mechanical Analogies – Version 1

	Electrical	Mechanical	
Resistor	$V_{21} = Ri$	$v_{21} = \frac{1}{b}F$	Damper
Capacitor	$i = C\frac{dV_{21}}{dt}$	$F = m\frac{dv_{21}}{dt}$	Mass
Inductor	$V_{21} = L\frac{di}{dt}$	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	Spring

Table 2.2. Electrical–Mechanical Analogies – Version 2

	Electrical	Mechanical	
Resistor	$V_{21} = Ri$	$F = bv_{21}$	Damper
Capacitor	$i = C\frac{dV_{21}}{dt}$	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	Spring
Inductor	$V_{21} = L\frac{di}{dt}$	$F = m\frac{dv_{21}}{dt}$	Mass

I cannot overemphasize the importance of these mathematical analogies! In my opinion, they are one of the wonders of the world, and a terrific tool that can make your engineering design life easier. These analogs allow you to "see" and "feel" the behavior of all systems, regardless of your engineering background.

Which analogy is "correct"? Both! It really doesn't matter which one you choose, as long as you are consistent. I prefer the voltage–velocity and current–force analogy (Table 2.1), because it keeps the concepts of *across* and *through* variables the same in both electrical and mechanical engineering fields. That can be important when drawing schematic diagrams of systems and using the impedance tools introduced earlier. I'll be using the analogies that maintain the across and through variable relationships (i.e., those in Table 2.1) throughout this book.

2.4 Mechanical Components (Rotational)

Concepts of Angular Motions and Torques

Angular motion and *torque* are concepts used to describe the behavior of a certain class of mechanical components that undergo rotation. If an axis can be found about which all particles in a mechanical element rotate, then pure rotational motion exists. Motor armatures, ship propellers, and engine drive shafts are just a few examples of mechanical components in pure rotation.

Angular motion is a simple extension of linear motion and is easy to understand. Figure 2.27 shows a rotating shaft. The angular displacement θ, angular velocity ω, and angular acceleration α are all similar to linear displacement, velocity, and acceleration. However, instead of distance as a unit of measurement, we use an arc of a circle, called a radian or degree, to measure angular displacement. There are 2π radians and 360 degrees in a circle, so we can express angular velocity in radians per second or degrees per second. Similarly, we can express angular acceleration in radians per second per second or degrees per second per second.

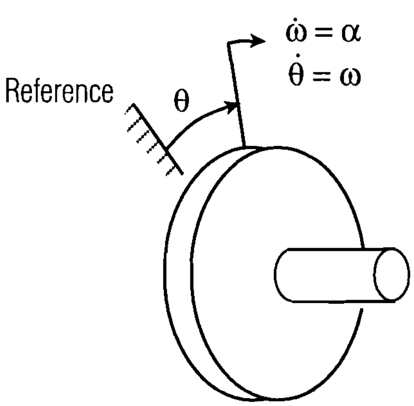


Figure 2.27. Rotating shaft.

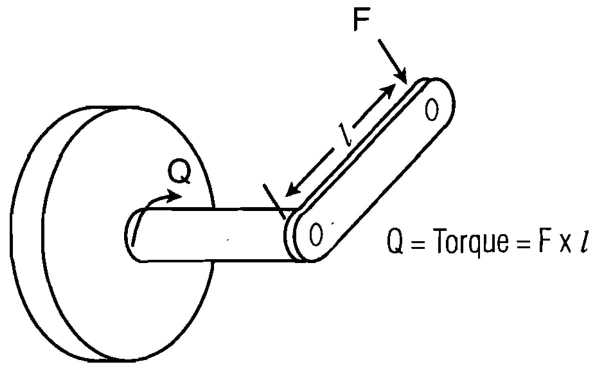


Figure 2.28. Rotating shaft with lever attached.

Torque is to rotational mechanical components as force is to translational components. Torque is best viewed as a force acting on a lever arm. For example, Figure 2.28 shows a lever attached to the end of the shaft shown in Figure 2.27. A force F is applied a distance l away from the center of the shaft. A torque Q, equal to $F \times l$, is applied to the shaft. The units of torque are force times length or in-lbs.

As we did with translational mechanical elements, we can now visualize a generalized rotational mechanical element, as in Figure 2.29, and write the relative motion equations

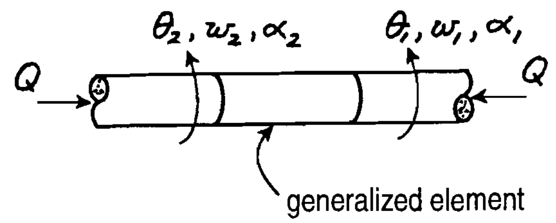


Figure 2.29. Definition sketch for generalized rotational mechanical element.

$$\theta_{21} = \theta_2 - \theta_1 \qquad (2.57)$$

$$\omega_{21} = \dot{\theta}_{21} = \dot{\theta}_2 - \dot{\theta}_1 = \omega_2 - \omega_1 \qquad (2.58)$$

$$\alpha_{21} = \dot{\omega}_{21} = \dot{\omega}_2 - \dot{\omega}_1$$
$$= \ddot{\theta}_2 - \ddot{\theta}_1 = \ddot{\theta}_{21} \qquad (2.59)$$

For rotational mechanical components, power is expressed as

$$Power = Q\omega_{21} \qquad (2.60)$$

Refer to Appendix B.3 for more details.

The Basic Rotational Mechanical Components

As you might expect, the same three fundamental mechanical elements, the damper, mass, and spring that we discussed in the translational mechanical section, also exist as rotational mechanical elements. We will therefore pass through these quickly, pointing out significant differences when appropriate.

The rotational mechanical damper symbolic diagram is shown in Figure 2.30. The torque is given by

$$Q = B(\omega_2 - \omega_1) = B\omega_{21} \qquad (2.61)$$

This equation is essentially identical to the one for the translational damper. The capital B is used to distinguish the rotational damper value symbol from that used with the translational damper. B has units of torque per unit of angular velocity, or in-lb-sec.

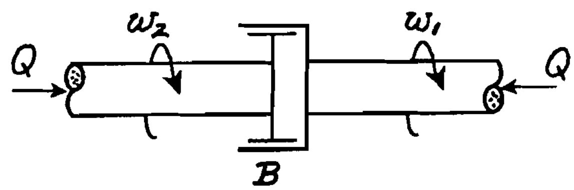


Figure 2.30. Rotational damper/dashpot.

The rotational mechanical mass symbolic diagram is shown in Figure 2.31. Newton's laws of motion applied to a pure rotational mass result in

$$Q = I\frac{d\omega_{21}}{dt} \qquad (2.62)$$

This equation is essentially identical to the one for the translational mass. The same restrictions apply to the reference angular velocity as were noted for the translational mass; that is, the reference point for angular velocity must be either stationary or moving at a constant angular velocity.

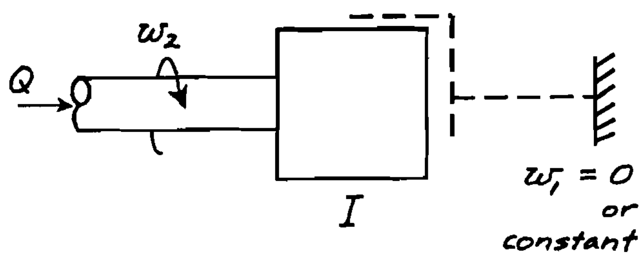


Figure 2.31. Rotational mass.

The rotational mechanical spring symbolic diagram is shown in Figure 2.32. The torque is given by

$$\frac{dQ}{dt} = K(\omega_2 - \omega_1) = K\omega_{21} \qquad (2.63)$$

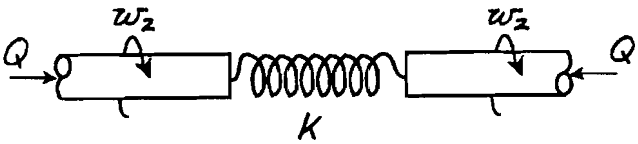


Figure 2.32.
Rotational spring.

or by

$$Q = K\int \omega_{21}dt = K\theta_{21} \tag{2.64}$$

This equation is essentially identical to the equation for the pure translational spring. The capital letter K is used to distinguish rotational springs from translational springs.

What to Commit to Memory

Figure 2.33 shows the three fundamental rotational mechanical components, their symbolic diagrams and the equations you should always remember. Figure 2.34 provides a summary of the symbolic diagrams, operational block diagrams, and impedances for these building blocks. Go over each of these and note how similar they are to those shown in Figure 2.26 for the mechanical translational elements.

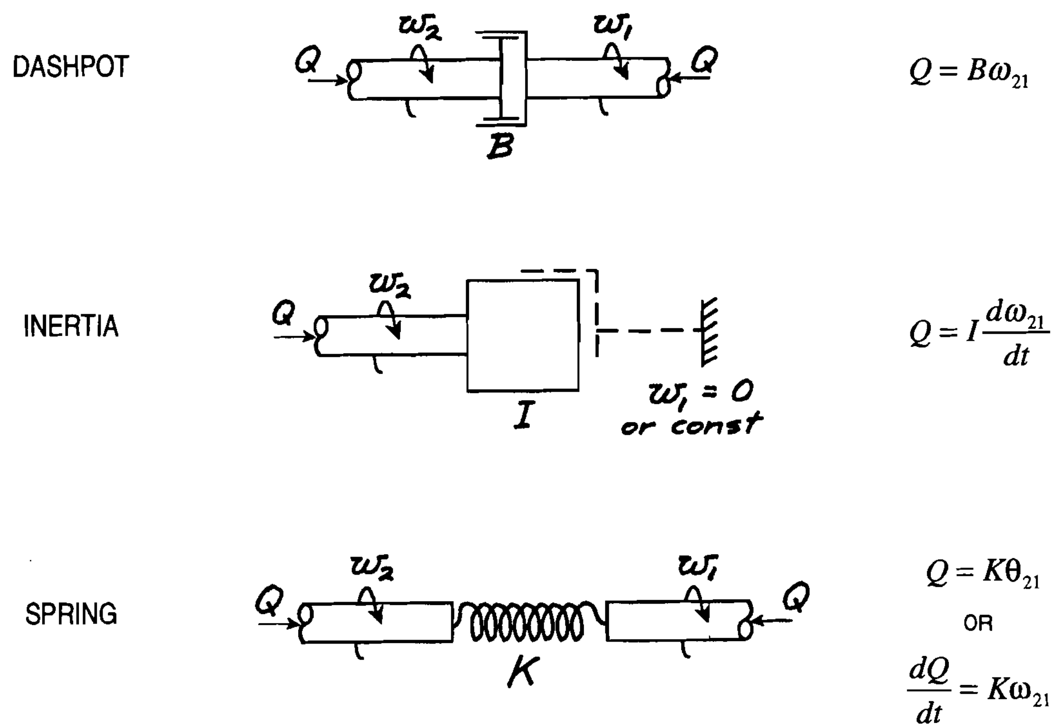


Figure 2.33. Rotational mechanical elements.

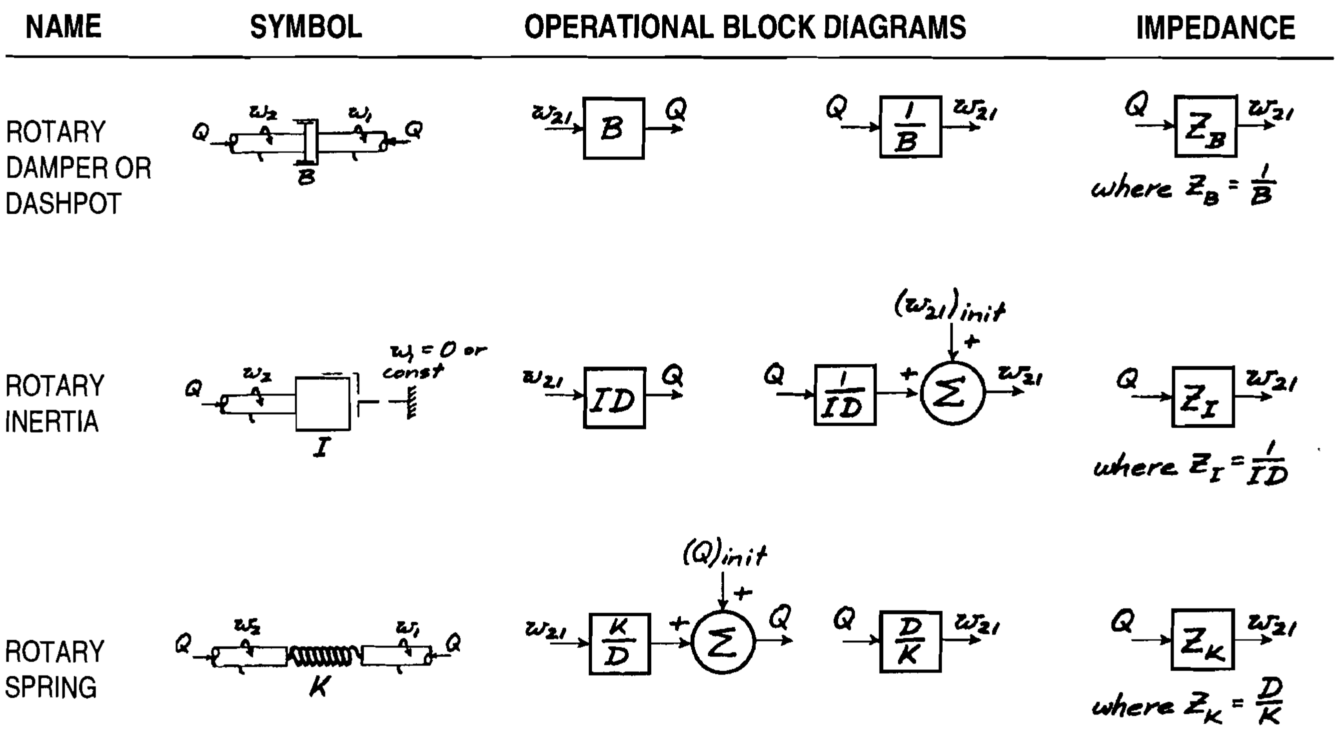


Figure 2.34. Summary of rotational mechanical elements.

2.5 Fluid Elements

Concepts of Pressure and Flow

A fluid is matter in a state which cannot sustain a shear force when it is at rest. For example, Figure 2.35 shows a circular "shear box" filled with a fluid. A minute force F will cause the contents of the upper half to spill. When the box is filled with a solid, a large force is required to shear the material. A fluid can be a liquid or a gas. The only difference between the two is related to the ease with which a gas can be compressed relative to a liquid. A fluid can be looked on as a quantity of small particles of matter that behave in such a manner that properties (pressure, mass per unit volume, etc.) can be measured at every point in the fluid.

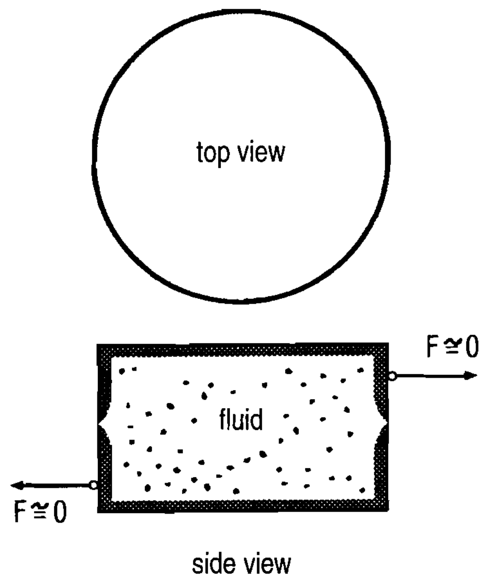


Figure 2.35. Circular box filled with a fluid. Upper half will slide over lower half immediately when a force F is applied.

Pressure in a fluid is defined as force per unit of area exerted in a direction that is perpendicular to the unit area. Pressure has the dimensions of force per length squared, the same as stress. Pressure is similar to the concept of normal stress in a solid.

In actuality, a fluid can sustain a shear force, but only when there is relative motion between fluid particles. For example, the flow of water in a river exerts a shear force on the bed of the river because the velocity of the water u varies as a function of the distance from the river bed y. At the river bed ($y = 0$) the water velocity is zero. Near the surface of the river (y = depth of the water) the water velocity is near its maximum. A so-called *velocity gradient* therefore exists in the water causing relative motion between fluid particles. The velocity gradient can be expressed as a derivative (du/dy) where du is the incremental change in water velocity and dy is the incremental change in depth. The shear stress τ, defined as a force per unit area exerted tangentially to that unit area, is proportional to the velocity gradient. That is:

$$\tau = \mu \frac{du}{dy} \tag{2.65}$$

where μ is the constant of proportionality, called the *absolute viscosity* of the fluid. Fluids which obey equation (2.65) are often called *Newtonian fluids*. (This, of course, is a simplification of fluid behavior; real fluids are more complex.)

Rate of flow of a fluid is defined as either the amount of mass or volume of fluid moving past a boundary in a unit of time. *Mass rate of flow* is generally used when dealing with a gas and *volume rate of flow* when dealing with a liquid.

The law of conservation of mass is often applied to a fluid to determine flow conditions at one point in a fluid, given the conditions at another. For example, Figure 2.36 shows two pipes with a gas flowing in one and a liquid in the other. At one end of each pipe the pressure is p_1 and the area is A_1. At the other the pressure is p_2 and the area is A_2. In the pipe carrying the gas, the mass density (mass per unit volume) varies and is ρ_1 at one end and ρ_2 at the other. In the pipe carrying the liquid, the mass density is a constant, ρ. The mass rate of flow q_{m1} entering the pipe must equal the mass rate of flow q_{m2} exiting the pipe since no mass is being stored inside the pipe. That is

$$q_{m1} = q_{m2}$$

or

$$\rho_1 A_1 U_1 = \rho_2 A_2 U_2 \tag{2.66}$$

(This is commonly called the "mass continuity" equation.)

Note from equation (2.66) that the units of mass rate of flow are slugs/ft^3 × ft^2 × ft/sec = slugs/sec. (Note: a "slug" is the amount of mass which accelerates at the rate of 1 ft/sec^2 under the action of 1 lb of force.)

If the fluid is incompressible, ρ_1 and ρ_2 are equal to ρ which cancels out of both sides of equation (2.66). Equation (2.66) then reduces to

$$A_1 U_1 = A_2 U_2 \tag{2.67}$$

That is, the volume rate of flow q_v must be equal at both ends of the pipe.

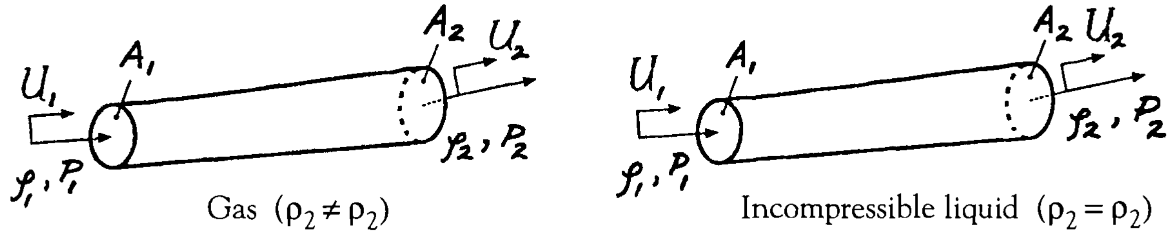


Figure 2.36. Two pipes with a gas flowing in one and a liquid in the other.

Another very useful equation can be found by applying Newton's laws of motion to an *ideal fluid* in *steady flow* conditions. An ideal fluid is one that is incompressible and has no viscosity. Steady flow means that particles of fluid are following a path in space that does not vary with time. Such paths are often called *streamlines* and bundles of streamlines are called *streamtubes*. A streamline or streamtube looks and behaves exactly like the tubes shown in Figure 2.36.

Let's examine the streamtube shown in Figure 2.37. If the pressure p_1, elevation h_1, and velocity U_1 of a point along the tube is known, then the pressure p_2, elevation h_2, and velocity U_2 of another point along the tube are related by

$$(p_2 - p_1) + \rho g(h_2 - h_1) + \frac{\rho}{2}(U_2^2 - U_1^2) = 0 \qquad (2.68)$$

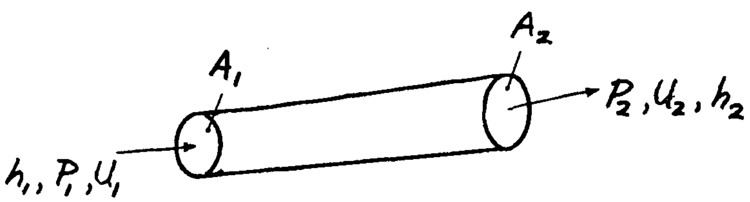


Figure 2.37. A streamtube.

Equation (2.68) is often called *Bernoulli's equation*. It is extremely useful when dealing with fluids and will be now used, together with the mass continuity equations given by equations (2.66) and (2.67), to describe equations for the three fundamental fluid elements.

As we know, power is the rate at which work is done. Referring again to the streamtube in Figure 2.37, we can write an expression for the power required to force the fluid into the entrance of the tube at point 1 as

$$Power_1 = \frac{dW_1}{dt} = p_1 \frac{dvol}{dt} = p_1 q_v \qquad (2.69)$$

(See Appendix B.4 for more details.) Similarly, the power expended by the fluid exiting the streamtube can be found by

$$Power_2 = p_2 q_v \qquad (2.70)$$

Subtracting (2.70) from (2.69) gives the net power as the product of the pressure difference across the tube ends and the volume rate of flow through the tube. That is,

$$\begin{aligned} Power_1 - Power_2 &= p_1 q_v - p_2 q_v \\ &= p_{12} q_v \qquad (2.71) \end{aligned}$$

The Fluid Resistor

Figure 2.38 shows a symbolic (circuit) diagram of a fluid resistor along with the fundamental describing equations for an *ideal fluid resistor*. Due to the form of Bernoulli's Equation, most fluid resistors are nonlinear. For example, sharp-edged orifices are frequently used in fluid systems to measure flow. Figure 2.39 shows such an orifice installed in a section of pipe. If the pipe is level so that gravity has no effect on the flow, then from equation (2.68) we can write

$$(p_2 - p_1) + \frac{\rho}{2}\left(U_2^2 - U_1^2\right) = 0 \qquad (2.72)$$

From equation (2.67) we can write

$$U_2 = \frac{A_1}{A_2} U_1 \qquad (2.73)$$

Substituting (2.73) into (2.72) gives

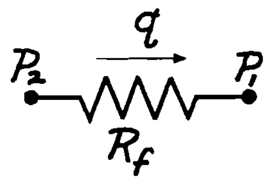
$$(p_2 - p_1) + \frac{\rho}{2}\left(\frac{A_1^2}{A_2^2} U_1^2 - U_1^2\right) = 0$$

or

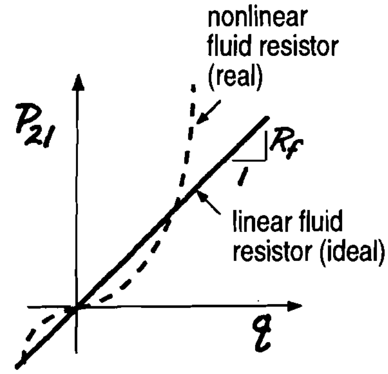
$$(p_2 - p_1) + \frac{\rho}{2} U_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right) = 0 \qquad (2.74)$$

You can see that if A_1 is much smaller than A_2, then the quantity A_1^2 / A_2^2 will be much smaller than 1 and can be neglected. Equation (2.74) then reduces to

$$(p_2 - p_1) = \frac{\rho}{2} U_1^2 \qquad (2.75)$$



(a). *Symbolic or circuit diagram.*



(b). *Graphical representation.*

$$q = \frac{P_2 - P_1}{R_f} = \frac{P_{21}}{R_f}$$

or

$$P_{21} = R_f q$$

(c). *Describing equations for ideal fluid resistor.*

Figure 2.38. The ideal fluid resistor

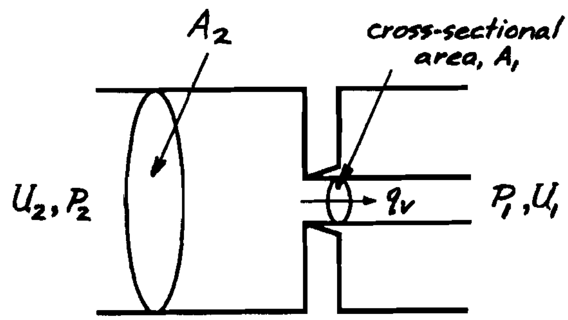


Figure 2.39. A sharp-edged orifice.

Using the relationship

$$q_v = A_1 U_1$$

we can also write (2.75) as

$$p_{21} = \left(\frac{\rho}{2A_1^2}\right) q_v^2$$

or more precisely as

$$p_{21} = \left(\frac{\rho}{2A_1^2}\right) q_v |q_v| \qquad (2.76)$$

where the absolute value allows the sign of p_{21} to change as the flow direction changes. You can see that the pressure difference is a function of the square of the volume rate of flow, causing the orifice to be a nonlinear resistor.

Of course equation (2.76) could be linearized about an operating point defined as q_{vo}, using Taylor's Theorem described in Appendix A, if a linear resistor were required. However, there *are* linear fluid resistors. Incompressible flow through a very small diameter tube (capillary) results in a linear relationship with the pressure differential across the ends of the tube and the volume rate of flow through the tube. Many soils and porous plugs made out of cinder material exhibit a linear relationship between pressure and flow.

A capillary tube is basically an orifice whose length is much greater than its diameter. The flow in such tubes is laminar; that is, it moves in layers, or laminas, and fluid particles do not bounce around from one layer to another. The pressure differential across the ends of such a tube is related to the volume rate of flow through the tube by

$$p_{21} = \left(\frac{128\mu l}{\pi D^4}\right) q_v \qquad (2.77)$$

where D is the tube diameter, l is the tube length, and μ is the absolute viscosity of the liquid.

Compare (2.77) with the equation for an ideal electrical resistor

$$V_{21} = Ri$$

If we let $128\mu l / \pi D^4 = R_f$ in (2.77) then we can rewrite (2.77) as

$$\boxed{p_{21} = R_f q_v} \qquad (2.78)$$

A comparison of (2.78) with the electrical resistor equation shows they are completely analogous when the volume rate of flow is taken as the analog of electrical current and pressure differential is taken as the analog of electrical voltage differential.

The energy delivered to a fluid resistor in the interval from $t = t_a$ to $t = t_b$ is

$$E_b - E_a = \int_{t_a}^{t_b} p_{21} q_v dt \qquad (2.79)$$

Substituting equation (2.78) into (2.79) and eliminating q_v gives

$$E_b - E_a = \int_{t_a}^{t_b} p_{21}\left(\frac{p_{21}}{R_f}\right) dt = \frac{1}{R_f}\int_{t_a}^{t_b} p_{21}^2 dt \qquad (2.80)$$

Alternatively, we can use equation (2.78) to eliminate p_{21}

$$E_b - E_a = \int_{t_a}^{t_b} \left(R_f q_v\right) q_v dt$$
$$= R_f \int_{t_a}^{t_b} q_v^2 dt \qquad (2.81)$$

Just as we saw with the electrical resistor, equations (2.80) and (2.81) reveal that a fixed resistor dissipates power regardless of the direction of the fluid flow. *Mathematically, an ideal fluid resistor behaves exactly like an ideal electrical resistor when the voltage-pressure and current-flow analogy is used.*

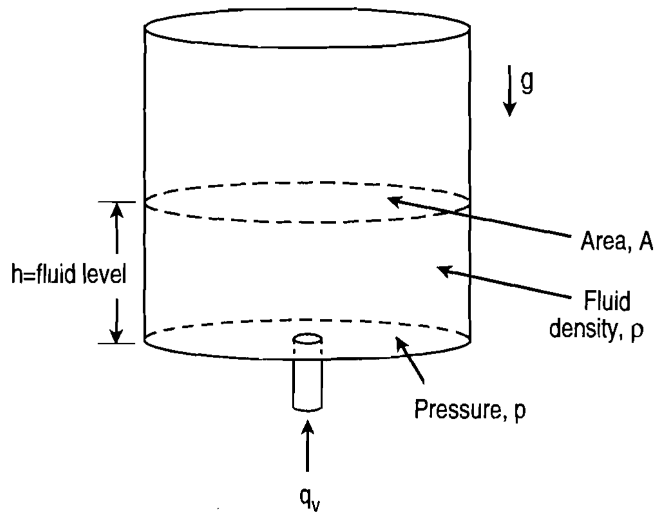


Figure 2.40. Fluid reservoir.

The Fluid Capacitor

Fluid capacitors are found in numerous hydraulic and pneumatic systems. Reservoirs, pressurized tanks, spring-loaded accumulators, and air-charged accumulators are examples of commonly encountered fluid capacitors.

An open reservoir is often used in a hydraulic system as a capacitor. Figure 2.40 shows the general arrangement of such a capacitor. A volume rate of flow q_v enters the bottom of the tank causing the level of the tank h to increase. This increased fluid level also increases the pressure p at the bottom of the tank.

The mass continuity equation applied to the tank gives us

$$q_v = A\frac{dh}{dt} \qquad (2.82)$$

We can also see that the pressure at the bottom of the tank is equal to the weight of the fluid in the tank divided by the area. That is

$$p = \frac{\rho g h A}{A} = \rho g h \qquad (2.83)$$

Equation (2.83) can be rearranged and then differentiated to give

$$\frac{dh}{dt} = \frac{1}{\rho g}\frac{dp}{dt} \qquad (2.84)$$

Substituting (2.84) into (2.82) to eliminate dh/dt gives

$$q_v = \left(\frac{A}{\rho g}\right)\frac{dp}{dt} \qquad (2.85)$$

Compare (2.85) to the equation for an ideal electrical capacitor:

$$i = C\frac{dV_{21}}{dt}$$

Clearly the two equations are mathematically the same if we define $(A/\rho g)$ to be a fluid capacitance C_f and we use the electrical voltage–fluid pressure and electrical current–fluid flow analogy. Equation (2.85) can then be written as

$$q_v = C_f \frac{dp}{dt} \qquad (2.86)$$

where $C_f = A/\rho g$.

An accumulator is another form of fluid capacitor. A spring-loaded accumulator is shown in Figure 2.41. In this type of accumulator a spring rather than gravity provides the pressure increase. A volume rate of flow q_v entering the bottom of the tank causes the spring to compress a distance x. This increases the pressure p in the tank.

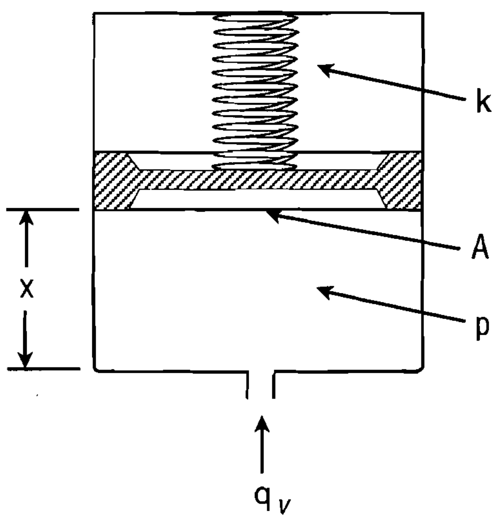


Figure 2.41. Spring-loaded accumulator.

The mass continuity equation applied to the tank gives

$$q_v = A\frac{dx}{dt} \qquad (2.87)$$

The pressure in the tank is also equal to the force exerted by the spring on the fluid divided by the area. That is

$$p = \frac{kx}{A} \qquad (2.88)$$

where k is the spring constant. Equation (2.88) can be rearranged and then differentiated to give

$$\frac{dx}{dt} = \frac{A}{k}\frac{dp}{dt} \qquad (2.89)$$

Substituting (2.89) into (2.87) to eliminate dx/dt gives

$$q_v = \left(\frac{A^2}{k}\right)\frac{dp}{dt} \qquad (2.90)$$

Equation (2.90) can then be written as a fluid capacitor

$$q_v = C_f \frac{dp}{dt} \qquad (2.91)$$

where $C_f = A^2/k$.

The energy delivered to a fluid capacitor in the time interval from $t = t_a$ to $t = t_b$ is given by

$$E_b - E_a = \int_{t_a}^{t_b} p_{12} q_v dt \qquad (2.92)$$

Equation (2.91) can be rewritten using differentials as

$$q_v dt = C_f dp \qquad (2.91) \text{ rewritten}$$

Substituting $C_f dp$ for $q_v dt$ in equation (2.92) gives

$$E_b - E_a = C_f \int_{p_a}^{p_b} p dp \qquad (2.93)$$

Equation (2.93) can be integrated to give

$$E_b - E_a = C_f \left[\frac{p^2}{2} \right]_{p_a}^{p_b}$$

$$= C_f \frac{p_b^2}{2} - C_f \frac{p_a^2}{2} \qquad (2.94)$$

The quantity $C_f(p_a^2/2)$ is the energy that was initially stored in the capacitor at $t = t_a$ and $C_f(p_b^2/2)$ represents the energy stored at time $t = t_b$. During the time interval $t_b - t_a$, the energy $E_b - E_a$ was added to the fluid capacitor. You can see that fluid capacitors, like their electrical cousins, can store energy. This energy can be retrieved and used later when needed.

The Fluid Inductor

A mass of fluid in motion is quite similar to a solid mass in motion. The fluid mass has inertia and a force is required to accelerate or decelerate the fluid. Figure 2.42 shows an *ideal* (no viscosity and hence no friction forces) *incompressible* fluid in *unsteady* (flow velocity is not a constant) flow through a pipe. Let's apply Newton's laws of motion to the mass of fluid in the pipe.

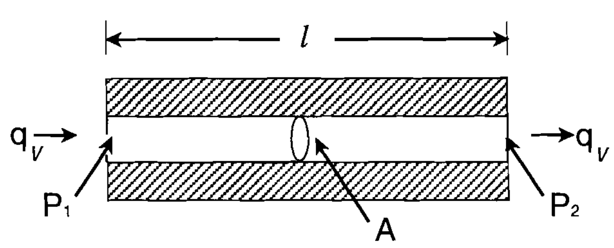


Figure 2.42. Ideal incompressible fluid in unsteady flow through a pipe.

The fluid mass is ρAl, the net force acting on this mass is $A(p_1 - p_2)$, and the acceleration of the fluid mass is dU/dt. From Newton's laws we can write

$$A(p_1 - p_2) = \rho A l \frac{dU}{dt} \qquad (2.95)$$

Since $q_v = AU$ and $p_{12} = p_1 - p_2$, equation (2.95) can also be rewritten as

$$p_{12} = \left(\frac{\rho l}{A} \right) \frac{dq_v}{dt} \qquad (2.96)$$

Compare equation (2.96) with the equation for an electrical inductor:

$$V_{21} = L \frac{di}{dt}$$

Clearly this equation is analogous to (2.96) if we define $(\rho l/A)$ to be a fluid inductance I_f and we use the electrical voltage–fluid pressure and electrical current–fluid flow analogy. Equation (2.96) can therefore be written as

$$\boxed{p_{12} = I_f \frac{dq_v}{dt}} \qquad (2.97)$$

You can see that, mathematically, *an ideal fluid inductor behaves exactly like an ideal electrical inductor.*

The energy delivered to a fluid inductor in the time interval from $t = t_a$ to $t = t_b$ is given by

$$E_b - E_a = \int_{t_a}^{t_b} p_{12} q_v dt$$

Substituting equation (2.97) into this equation to eliminate p_{12} gives

$$E_b - E_a = \int_{t_a}^{t_b} \left(I_f \frac{dq_v}{dt} \right) q_v dt$$
$$= I_f \int_{q_{va}}^{q_{vb}} q_v dq_v \qquad (2.98)$$

Equation (2.98) can be integrated to give

$$E_b - E_a = I_f \left[\frac{q_v^2}{2} \right]_{q_{va}}^{q_{vb}}$$
$$= I_f \frac{q_{vb}^2}{2} - I_f \frac{q_{va}^2}{2} \qquad (2.99)$$

The quantity $I_f(q_{va}^2 / 2)$ is the kinetic energy associated with the initial flow rate at $t = t_a$ and $I_f(q_{vb}^2 / 2)$ is kinetic energy associated with the final flow rate at $t = t_b$. During the time interval $t_b - t_a$, the kinetic energy $E_b - E_a$ was added to the fluid.

What to Commit to Memory

You should always remember that equations associated with fluid mechanics are based on applying the mass continuity (conservation of mass) and Newton's motion laws to a fluid. You should be able to write mass continuity from memory in the form

$$\rho_1 A_1 U_1 = \rho_2 A_2 U_2$$

You should also be able to write Bernoulli's equation from memory and know that it strictly applies to a frictionless fluid in steady flow. Remember the equation in one of these two forms

$$(p_2 - p_1) + \rho g(h_2 - h_1) + \frac{\rho}{2}(U_2^2 - U_1^2) = 0$$

or

$$p_2 + \rho g h_2 + \frac{\rho}{2} U_2^2 = p_1 + \rho g h_1 + \frac{\rho}{2} U_1^2$$

If you remember these equations, you will always be able to derive the elemental equations for a fluid resistor, fluid capacitor, and a fluid inductor. Also keep in mind when working with fluid systems that the three fundamental building blocks are not as easy to spot as are their electrical counterparts. So make it a habit to try to identify the three fluid building blocks whenever you encounter a fluid system. (You could start with the fresh and hot water supplies in your home.)

Figure 2.43 shows the symbolic diagrams, operational block diagrams, and impedances for the three fundamental fluid elements. As you review these, note how similar they are to the three fundamental electrical and mechanical elements in Figures 2.15, 2.26, and 2.34.

2.6 Thermal Elements

Concepts of Temperature and Heat

The concepts of *temperature* and *heat* are encountered in almost all engineering systems. Heat must be dissipated from electrical circuits or the elements will burn out. Heat produced by friction between mechanical elements can cause them to seize if the heat is not removed. Fluids are frequently used to transfer heat from one location to another.

We qualitatively think of temperature as a measure of how "hot" or "cold" an object is. This implies that temperature is a relative term, and indeed it is. A quantitative measure of tempera-

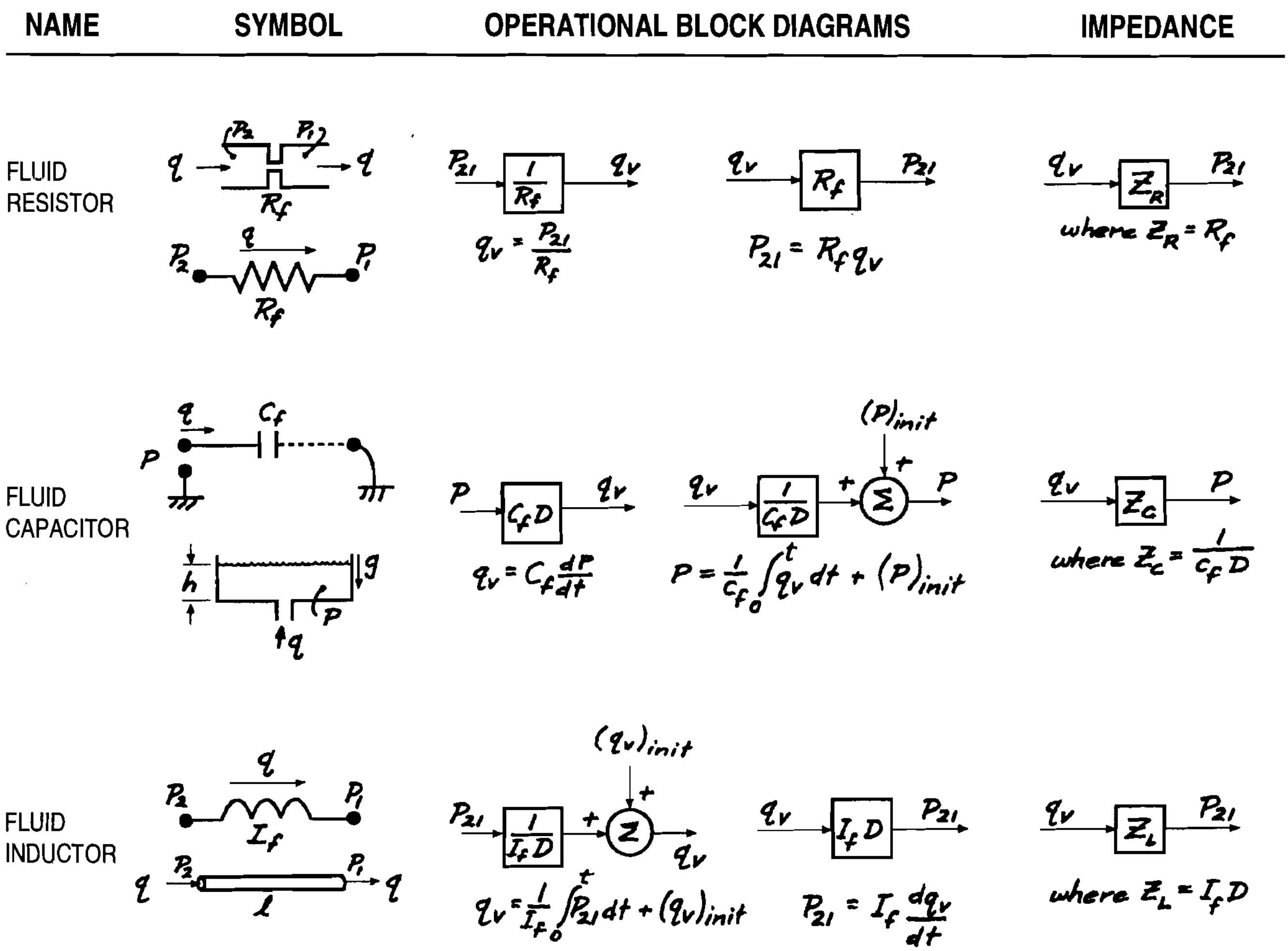


Figure 2.43. Summary of fluid elements.

ture is obtained with a thermometer and a variety of scales are in use today. For example, the Celsius or centigrade scale was developed based on the freezing and boiling points of water. The temperature at which solid water (ice) changes to liquid water is equal to 0 degrees Celsius (0°C) and the temperature at which liquid water changes to gaseous water (steam) is equal to 100 degrees Celsius (100°C).

When two bodies at different temperatures are brought in contact with one another, *heat* flows from the body with the higher temperature to the body with the lower temperature. When this happens, energy is transferred from the hotter to the colder body. Heat is defined as the energy that is transferred from one body or system to another as the result of the temperature difference between the two.

Concepts of Work, Power, and Energy in Thermal Elements

When mechanical work is done on a body, its temperature will rise unless heat is removed from the body. The First Law of Thermodynamics states that work and heat can be converted from one to the other. Consequently, the units of heat and work are equivalent. In the English system, temperature is usually measured in Fahrenheit degrees and heat in British Thermal Units (BTU). One BTU is equivalent to 778.172 ft-lbs. In the SI system, temperature is measured in degrees centigrade and heat in joules. You will recall from our study of electrical elements that one joule is equal to one watt-sec, or 0.737 ft-lbs.

There are three basic ways in which heat can be transferred from a hotter to a colder body: *convection*, *radiation* and *conduction*. Convection-type heat transfer takes place primarily in liquids and gases. Heat is transferred as the result of matter moving from one location to another due to currents set up by the temperature differentials. Radiation-type heat transfer takes place as a result of energy carried by electromagnetic waves. Conduction-type heat transfer usually involves substances in the solid phase. Heat is transferred at the atomic level without any visible motion of matter.

Thermal Resistance

All material offers some resistance to heat flow. When a material offers a large degree of resistance it is often called an insulator. Figure 2.44 shows a section of an insulative material.

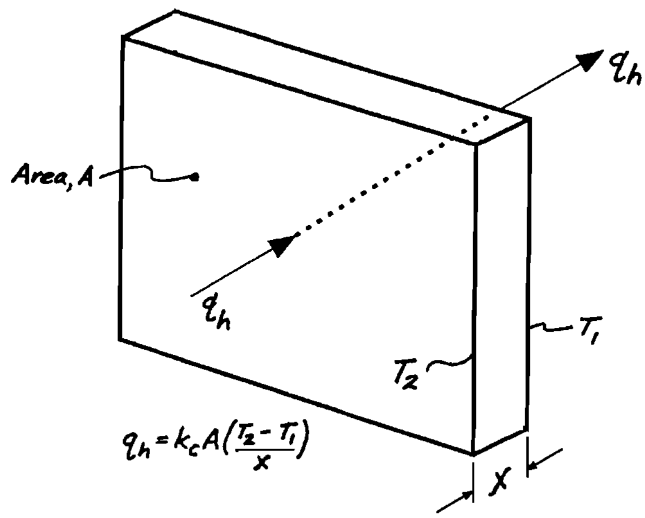


Figure 2.44. Heat flow through a conducting material.

It has been found through experimentation that the rate of heat transfer q_h between two surfaces of area A is proportional to the area and the temperature gradient dT/dx. In equation form,

$$q_h \propto -A\frac{dT}{dx} \qquad (2.100)$$

By using a constant of proportionality k_c we can write equation (2.100) as

$$q_h = -k_c A\frac{dT}{dx} \qquad (2.101)$$

The constant of proportionality k_c is called the thermal conductivity. The negative sign indicates that heat flows from the direction of decreasing temperature.

Applying equation (2.101) to Figure 2.44 gives

$$q_h = k_c A\frac{(T_2 - T_1)}{X} \qquad (2.102)$$

Symbol	Meaning	English units	SI units
q_h	rate of heat flow	BTU/hr	joules/sec
k_c	thermal conductivity	$\frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\cdot(\text{deg F}/\text{ft})}$	$\frac{\text{joules}}{\text{sec}\cdot\text{m}^2\cdot(\text{deg C}/\text{m})}$
A	cross-sectional area	ft^2	m^2
T_2	hotter temperature	deg F	deg C
T_1	colder temperature	deg F	deg C
X	thickness of material	ft	m

Take careful note of the units in this equation. They are given above in both English and SI units.

If we rewrite equation (2.102) in the form

$$T_2 - T_1 = T_{21} = \left(\frac{X}{k_c A}\right) q_h \qquad (2.103)$$

and then compare this with the equation for an electrical resistor

$$V_{21} = Ri$$

we can see that the two equations are identical provided we think of X/k_cA as a resistor (thermal), temperature as a voltage, and rate of heat flow as an electrical current. Drawing this analogy, we can then write (2.103) as

$$\boxed{T_{21} = R_t q_h} \qquad (2.104)$$

where

$$R_t = \frac{X}{k_c A}$$

We must be very careful with this analogy when it comes to looking at power relationships. The rate of heat flow q_h corresponds to power, as a consequence of the equivalency of heat and work stated by the First Law of Thermodynamics. Consequently, power is given by equation (2.102), or in terms of R_t as

$$Power = q_h = \frac{1}{R_t} T_{21} \qquad (2.105)$$

Thermal Capacitance

All materials have some capacity to store heat. The capacity of a material to store heat is called its *specific heat.* It is defined as the amount of heat per unit mass that must be added to the material to raise its temperature one degree. Specific heat c is, in general, a function of temperature and can be written as

$$c = c(T) = \frac{1}{m}\frac{dQ}{dT} \qquad (2.106)$$

where m = mass of the material

dQ = amount of heat required

dT = temperature rise due to addition of dQ

If we restrict attention to small variations of temperature, then c can be taken as a constant. In that case equation (2.106) can be written as

$$dQ = cmdT \qquad (2.107)$$

Dividing through by an incremental time dt, gives

$$\frac{dQ}{dt} = q_h = cm\frac{dT}{dt} \qquad (2.108)$$

Let's compare equation (2.108) with the equation for an electrical capacitor:

$$i = C\frac{dV}{dt}$$

You can see that the two equations are identical if we think of cm as a capacitor (thermal), temperature as a voltage, and rate of heat flow as an electrical current. In this case we rewrite equation (2.108) as

$$q_h = C_t\frac{dT}{dt} \qquad (2.109)$$

where

$$C_t = cm$$

Once again, you must be careful with this analogy when it comes to looking at power relationships. Quite often we are interested in how much heat is stored in a material. This can be obtained by integrating equation (2.109). That is,

$$Q = m\int_{T_1}^{T_2} cdT \qquad (2.110)$$

If c is taken as a constant, then equation (2.110) can be integrated giving

$$Q = mc(T_2 - T_1) \qquad (2.111)$$

Thermal Inductance

There is no known thermal phenomena which stores thermal energy as a function of the rate of change of the heat flow rate. Consequently no analogies with electrical inductance can be drawn.

What to Commit to Memory

The equations for thermal resistance and thermal capacitance given above will likely serve many of your needs when modeling engineering systems. You should commit to memory the analogies with the electrical resistor and capacitor.

Figure 2.45 shows symbolic diagrams, operational diagrams and impedances for the two fundamental thermal elements. Compare these with the electrical, mechanical, and fluid "resistors" and "capacitors." You will see that the equations are identical. However, don't forget that power in thermal systems is *not* equal to the product of the across variable (temperature) and the through variable (heat flow rate), as is the case with the electrical, mechanical, and fluid systems. The through variable for thermal systems, heat flow rate, is itself power.

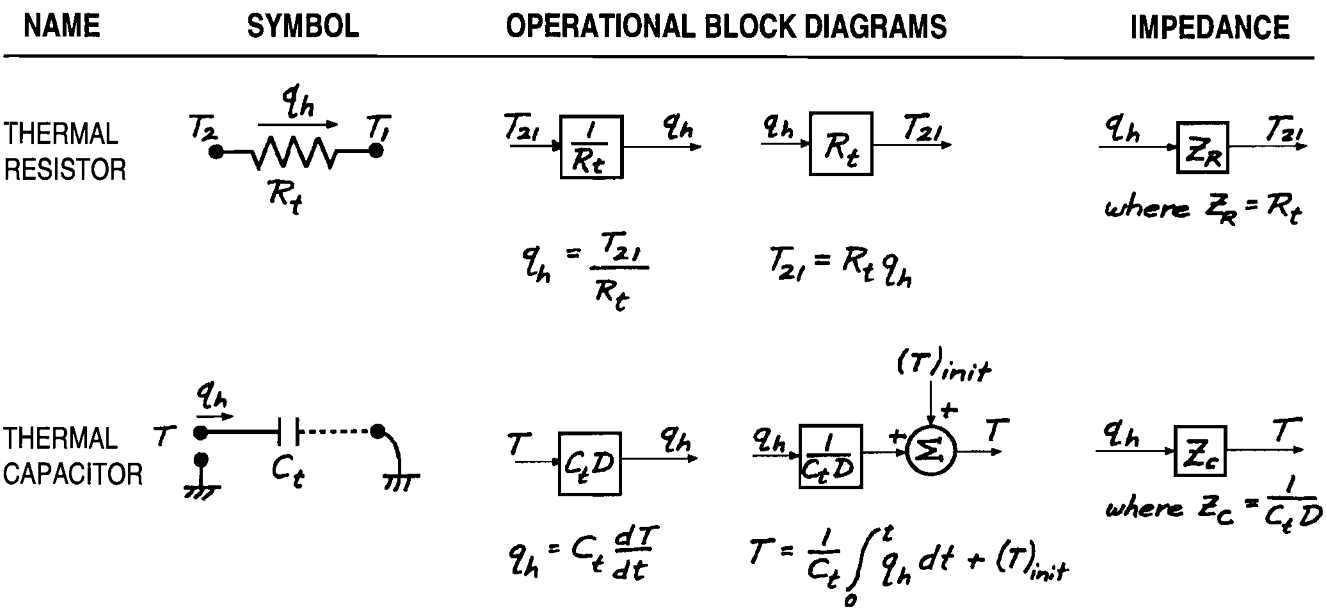


Figure 2.45. Summary of thermal elements.

2.7 The Importance of Analogies

This chapter is probably the most important chapter in this book. It covers what I believe to be the most important fundamentals of engineering. If you understand this material and commit to memory the basic formulas, you will have a solid grounding for any engineering work.

Throughout this chapter I have repeatedly pointed out how the equations describing the three fundamental components in electrical, mechanical, and fluid systems are identical if the proper analogies are drawn between *across* and *through* variables. I have also pointed out that the *product* of the across and through variables for these three vastly different engineering system building blocks is equal to power. Table 2.3 summarizes what we've covered so far.

Note that I have included thermal systems in this summary, but placed them in a box to remind you that the product of temperature (across variable) and rate of heat flow (through variable) is not power.

The real beauty of analogies lies in the way in which mathematics unifies these diverse fields of engineering into one subject. That means tools developed for solving problems in one field can be used to solve problems in another. This is an important concept, since some fields, particularly electrical engineering, have rich sets of problem-solving tools that are fully applicable to other engineering fields. The more you work with engineering problems and these analogies, the more intuitive your engineering skills will become. The result—the good engineering judgment so essential to success in any type of engineering field.

Table 2.3. Analogies summary.

System	Across Variable	Through Variable	Power Equation	Resistor Equation	Capacitor Equation	Inductor Equation
Electrical	Voltage (V)	Current (i)	$V \times i$	$V = Ri$	$i = C\frac{dV}{dt}$	$V = L\frac{di}{dt}$
Mechanical Translation	Velocity (v)	Force (F)	$v \times F$	$v = \frac{1}{b}F$	$F = m\frac{dv}{dt}$	$v = \frac{1}{k}\frac{dF}{dt}$
Mechanical Rotational	Angular Velocity (ω)	Torque (Q)	$\omega \times Q$	$\omega = \frac{1}{B}Q$	$Q = I\frac{d\omega}{dt}$	$\omega = \frac{1}{K}\frac{dQ}{dt}$
Fluid	Pressure (p)	Flow Rate (q_v)	$p \times q_v$	$p = R_f q_v$	$q_v = C_f\frac{dp}{dt}$	$p = I_f\frac{dq_v}{dt}$
Thermal	Temperature (T)	Heat Flow Rate (q_h)	q_h	$T = R_t q_h$	$q_h = C_t\frac{dT}{dt}$	None

Chapter 3 Constructing First-Order Math Models

Objectives

When you have completed this chapter, you will be able to:

- **Develop simple math models of engineering systems.**
- **Define the *path-vertex-elemental equation*, the *impedance*, the *operational block diagram*, and the *free-body diagram* methods for developing math models and know how to use each method.**
- **Recognize when the fundamental elements are connected in *series* or in *parallel* and use the associated simplifying equations in the development of math models.**
- **Define and understand what is meant when a math model is described as a *first-* or *second-order ordinary linear differential equation with constant coefficients.***
- **Recognize and use the power of the *impedance method* for rapidly developing math models.**

3.1 Introduction to Math Models

In the previous chapter you learned how to develop mathematical models of single-element, ideal components found in electrical, mechanical, fluid, and thermal systems. You can model numerous systems found in engineering by breaking the system down into these fundamental elements and then combining the elemental equations to form a mathematical model of the complete system. An electrical system, for example, might be modeled as a combination of an ideal resistor and an ideal capacitor. Or a mechanical system might be modeled as a combination of an ideal spring and an ideal mass.

In this chapter, you will learn that when equations for the ideal elements are combined, a mathematical model referred to as a *differential equation* is obtained. This equation describes the behavior in time of the *output* (or *response*) variable of the system to a time-varying *input* or *forcing* variable. You will also learn that the coefficients of the differential equation are related to the constants associated with the ideal elemental equations (that is, the value of the resistor, capacitor, inductor, mass, spring, etc.). The ideal element constants are combined and form *parameters* which govern the most basic characteristics of the differential equation.

Based on my own experience and observation of other engineers, I've discovered that it is generally much harder to create mathematical models of systems (that is, derive the governing differential equations) than it is to solve the equations once they have been derived. In fact, there are many excellent software programs available now that can be run on an inexpensive PC. These programs can be used to prepare math models of engineering systems and to solve the resultant equations. But these programs can be dangerous! You *must* have the engineering basics before you can model an engineering system, and you need to know how to solve at least basic equations so you can judge whether or not the output you are getting from a computer program makes sense.

I consider the material covered in this chapter to be the true dividing line between a "good" engineer and a mediocre one. A good engineer has internalized this material so that he or she is able to conceptualize an engineering system—to "see" the differential equations that describe its behavior—almost automatically. When you step on the accelerator of your car, for example, you should be able to visualize the graphs of thrust vs. velocity and resistance vs. velocity that determine the acceleration of its mass. Don't expect this to happen overnight, however. It comes with lots of practice!

I spend a great deal of time in this chapter explaining techniques and developing tools that you can use to make the equation derivation process easier. More emphasis will be placed on the methods that I have found produce answers in the shortest period of time and that are easiest to remember. To keep things as simple as possible, only engineering systems that contain two fundamental elements will be discussed in this chapter. While this may at first seem trivial, you will discover that nearly every engineering system contains at least one component that can be described fairly accurately by a combination of two ideal elements.

3.2 Tools for Developing Math Models

Four methods for developing mathematical models of engineering systems are presented in this section. They are usually referred to by the following names:

- *Path-Vertex-Elemental Equation Method*
- *Impedance Method*
- *Operational Block Diagram Method*
- *Free-body Diagram Method*

The path-vertex-elemental equation method has a variety of names depending on the field of engineering in which you're involved. In electrical engineering the method is referred to as Kirchoff's Laws (voltage and current). Other fields often call this method "derivation by first principles."

The impedance method and the operational block diagram method seem to be used mostly by electrical engineers. That's too bad, because both methods are extremely powerful when applied to other fields as well. Learn them and you will save time and gain new insights into the systems you are trying to design or analyze, no matter what type of system it is.

The free-body diagram is most commonly used in mechanical engineering. It is an extremely powerful method and, since nearly all engineering systems have some mechanical components, all engineers should learn how to use it.

Again, I will present the first three methods using simple electrical circuits. This will allow you to better understand the important concepts of series and parallel "circuits" and how these concepts can be used to simplify the development of math models in general.

Path-Vertex-Elemental Equation Method

When several of the fundamental elements described in the previous chapter are connected together, the result looks like a network of highways with junctions and interconnections. For example, Figure 3.1 shows a complex electrical circuit comprised of interconnected resistors, capacitors, and inductors. The point where two or more elemental components are connected is called the *vertex*. The highway leading from one vertex to another is called the *path*.

A vertex is always associated with the elemental *across* variables. At each vertex, the across variables for each interconnected element are identical.

A path is always associated with the elemental *through* variables. The sum of the through variables flowing into or out of a vertex must equal zero. For example, in Figure 3.1, the currents flowing into and out of the vertex labelled V_2 must equal zero. That is, i_1 (flowing into the vertex) must equal the sum of i_2 and i_3 (flowing out of the vertex).

The path-vertex equations are often called "laws" and are frequently named after the person who discovered them. In electrical engineering, the path-vertex equations are called Kirchoff's voltage and current laws. While it's great to honor past engineers for their contributions, don't let names confuse you. You learned in Chapter 2 that you can create a "circuit" for any

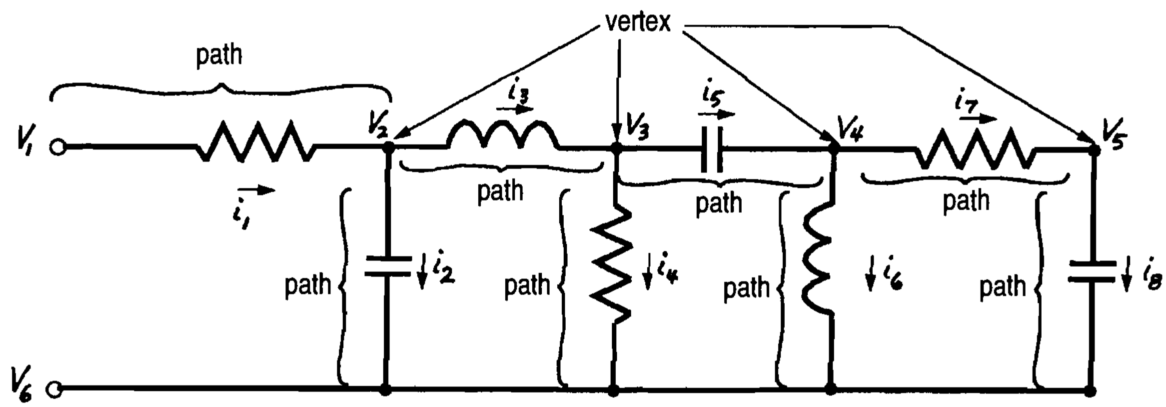


Figure 3.1. A complex electrical circuit showing *vertices* and *paths*. Note that V_1, V_2, ..., V_5 are "across" variables and i_1, i_2, ... i_8 are "through" variables.

electrical, mechanical, fluid, and thermal element. The path-vertex-elemental equation method is simply a procedure that allows you to derive equations for any "circuit," whether it is electrical, mechanical, fluid, or thermal, by writing the elemental equations associated with each element in the "circuit" and using the path and vertex laws associated with the across and through variables at each vertex. Now let's look at this method in more detail.

Series and Parallel Circuits

When two fundamental engineering system elements are connected together, they can only be connected in one of two ways: in series or in parallel. Figure 3.2a shows two electrical resistors (they could just as well be mechanical, fluid, or thermal resistors) connected in series and Figure 3.2b shows them connected in parallel. The voltages (across variable) at each of the vertices are labelled and the electrical current flowing (through variable) in each resistor and into each vertex is shown. The directions of the currents are shown by arrows, but this is arbitrary since the relative values of the voltages are not given. In Figure 3.2a, there can only be one current and it flows through both resistors. In Figure 3.2b, the current flowing in the circuit splits and part flows through each resistor.

You can find the voltage V_1 at the juncture of the two resistors in Figure 3.2a by writing the elemental equation for the current flowing in each resistor. For R_a, the current is

$$i = \frac{V_2 - V_1}{R_a} \qquad (3.1)$$

and for R_b, the current is

$$i = \frac{V_1 - V_0}{R_b} \qquad (3.2)$$

Since the current flowing out of resistor R_a must flow into R_b, we can equate (3.1) and (3.2), giving

$$\frac{V_2 - V_1}{R_a} = \frac{V_1 - V_0}{R_b} \qquad (3.3)$$

Solving this equation for V_l gives

$$V_1 = \frac{R_b}{R_a + R_b} V_2 - \frac{R_a}{R_a + R_b} V_0 \qquad (3.4)$$

If V_0 is taken as ground, or reference, and set to zero, then (3.4) becomes

$$V_1 = \frac{R_b}{R_a + R_b} V_2 \qquad (3.5)$$

Equation (3.5) indicates that the voltage V_l is equal to the voltage V_2 times the ratio of the resistor R_a and the sum of the two resistors. If the two resistors were equal, the ratio would be

$$\frac{R_a}{R_a + R_b} = \frac{R}{R + R} = \frac{R}{2R} = \frac{1}{2}$$

If R_a' were twice the size of R_b, the ratio would be 2/3. In essence, the circuit shown in Figure 3.2a is acting as a voltage divider (or constant multiplier). If V_2 is considered an input to the circuit and V_l the output, then the circuit multiplies the input by the ratio $R_a / (R_a+R_b)$ and provides the product as the output.

The circuit given in Figure 3.2a and the describing equation given in (3.5) are extremely important. You will encounter them often throughout your career and they will be used many times throughout this book.

Now look at the circuit in Figure 3.2b. Note that the voltage across each resistor is the same and is equal to $V_l - V_0$. The current i_a is given by

$$i_a = \frac{V_1 - V_0}{R_a} = \frac{V_1}{R_a} \qquad (3.6)$$

when V_0 is taken as the reference or zero voltage. Similarly the current i_b is given by

$$i_b = \frac{V_1 - V_0}{R_b} = \frac{V_1}{R_b} \qquad (3.7)$$

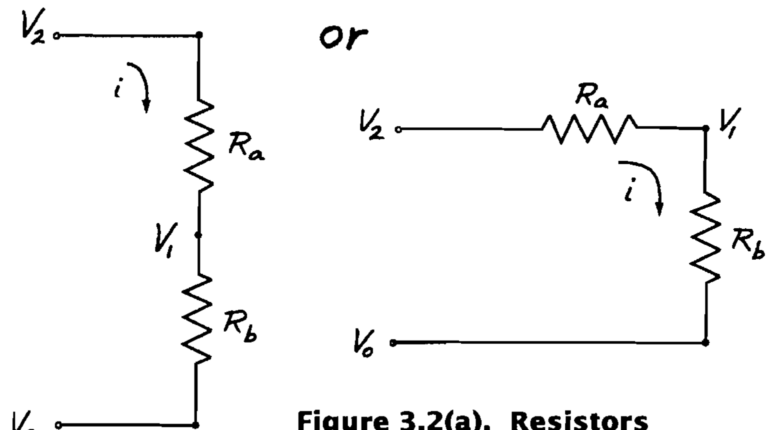


Figure 3.2(a). Resistors connected in series.

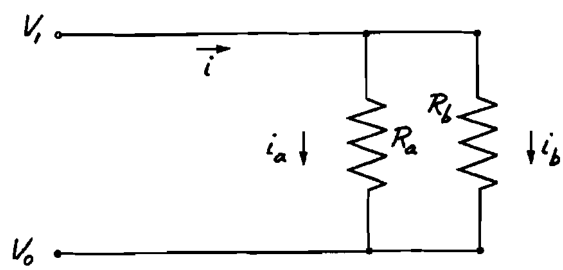


Figure 3.2(b). Resistors connected in parallel.

The current, i, flowing into the junction can be written as

$$i = i_a + i_b \tag{3.8}$$

since the sum of the currents flowing into a vertex must equal the sum of the currents flowing out of the junction.

Substituting (3.6) and (3.7) into (3.8) gives

$$i = \frac{V_1}{R_a} + \frac{V_1}{R_b} = \left(\frac{R_a + R_b}{R_a R_b}\right) V_1 \tag{3.9}$$

or

$$V_1 = \left(\frac{R_a R_b}{R_a + R_b}\right) i \tag{3.10}$$

Equation (3.10) reveals that resistors in parallel can be combined into an equivalent single resistor, R, given by

$$R = \frac{R_a R_b}{R_a + R_b} \tag{3.11}$$

as shown in Figure 3.3. You should note that, in essence, equation (3.11) allows you to reduce circuit complexity by replacing resistors in parallel with a single resistor.

Series and Parallel Circuits Involving One Energy Storage and One Energy Dissipative Element

So far we have only looked at resistors connected in series and in parallel. Figure 3.4a shows an electrical resistor and capacitor in series and Figure 3.4b shows the same elements connected in parallel.

Let's follow the same procedures used above and determine if we get similar equations. We will first write the elemental equations for the series circuit. The current through the resistor is

$$i = \frac{V_2 - V_1}{R} \tag{3.12}$$

The current through the capacitor is

$$i = C\frac{dV_1}{dt} \tag{3.13}$$

As before, let's equate (3.12) and (3.13) to eliminate i. That gives

$$\frac{V_2 - V_1}{R} = C\frac{dV_1}{dt} \tag{3.14}$$

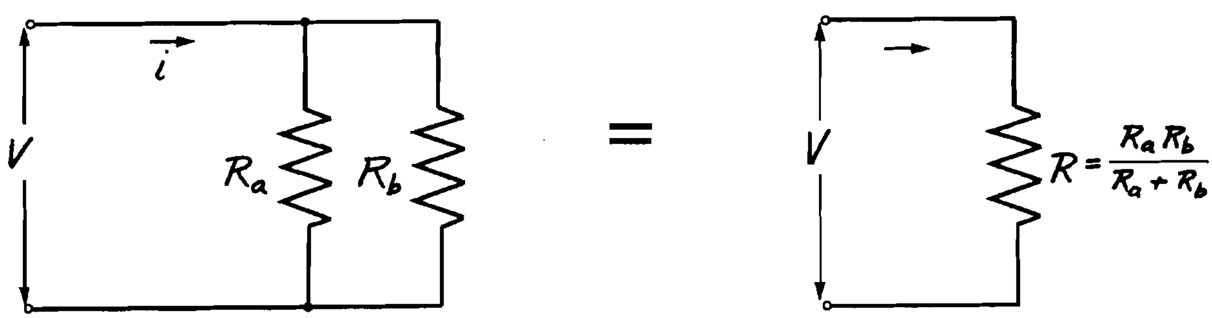


Figure 3.3. Reducing circuit complexity by combining elements in parallel.

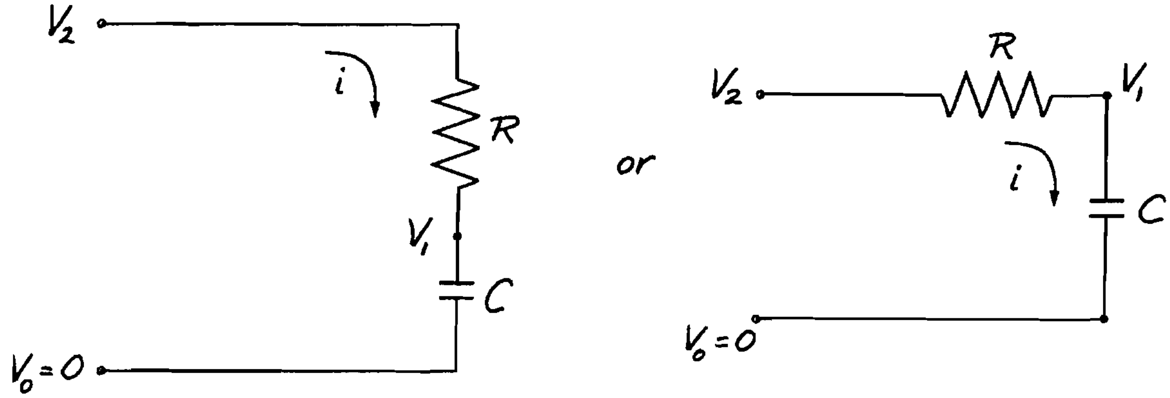


Figure 3.4(a). A resistor and capacitor in series.

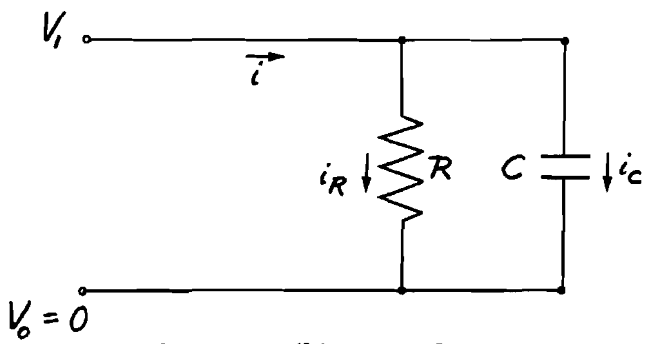


Figure 3.4(b). A resistor and a capacitor in parallel.

Now let's rearrange this equation so all the terms containing V_1 are on the left side of the equal sign and all others are on the right side. That is

$$C\frac{dV_1}{dt}+\frac{1}{R}V_1=\frac{1}{R}V_2 \qquad (3.15a)$$

Multiplying both sides by R gives

$$RC\frac{dV_1}{dt}+V_1=V_2 \qquad (3.15b)$$

While we did not derive an equation which gives the output voltage, V_1, directly as a function of the input voltage V_2, we did obtain a relationship between V_1 and V_2 that contains the time derivative of V_1, V_1 by itself, and V_2. Now let's check the units of each term in the equation to make sure they are correct. Each term on the left side of equation 3.15b must have the units of volts because only volts appear on the right side of the equation. That is,

$$\underbrace{RC\frac{dV_1}{dt}}_{\text{volts}}+\underbrace{V_1}_{\text{volts}}=\underbrace{V_2}_{\text{volts}}$$

Look at the first term. Since dV_1/dt has units of volts per unit of time, RC must have units of time. That is,

$$\left[RC\frac{dV_1}{dt}\right]=\text{sec}\times\frac{\text{volts}}{\text{sec}}$$

The product RC is called the *time constant* of the circuit and is given the Greek symbol, τ.

The relationship given by (3.15) is a *first-order linear ordinary differential equation with constant coefficients*. It is called a *differential equation* because it has both a variable and its derivative in the same equation. It's called *ordinary* because it contains no partial derivative. It's called *first-order* because it involves only the first derivative of the variable. It is qualified as having *constant coefficients* [C and $1/R$ in (3.15a) and RC and 1 in (3.15b)] to distinguish it from similar equations that have coefficients which vary with time. It is called *linear* for reasons which will become clearer later.

You will find out later that (3.15) can be solved for V_1 (often called the *dependent*, *response*, or *output* variable) but the solution will give V_1 as a function of time (often called the *independent* variable) as well as a function of V_2 (often called the *input* or *forcing* variable).

Now let's derive the equation for the circuit in Figure 3.4b. The current flowing through the resistor is

$$i_R = \frac{V_1}{R} \tag{3.16}$$

and through the capacitor is

$$i_C = C\frac{dV_1}{dt} \tag{3.17}$$

The current summation at the junction gives

$$i = i_C + i_R \tag{3.18}$$

Substituting (3.16) and (3.17) into (3.18) gives

$$i = C\frac{dV_1}{dt} + \frac{V_1}{R} \tag{3.19}$$

Once again our final result is a first-order linear differential equation with constant coefficients. The equation is very similar to (3.15) in that it involves the time derivative of V_1 and V_1 by itself.

In equation (3.19) I have purposely placed the time derivative of V_1 and V_1 on the right side of the equal sign to make a point. Ordinarily, you place the output (dependent) variable on the left side of an equation and the input (forcing) variable on the right. But in this case, which is which? Is V_1 the input or is *i*? If V_1 is the input as (3.19) implies, then we have succeeded in arriving at the desired equation and need go no further in solving the equation. That is, if V_1 is the input variable, then it is known as a function of time and therefore its first derivative can be determined. On the other hand, if *i* is the input variable then (3.19) should be rearranged in the form

$$RC\frac{dV_1}{dt} + V_1 = Ri \tag{3.20}$$

Now compare (3.20) and (3.15b). Clearly they are identical except for the right sides. As I indicated with (3.15), you can solve (3.20) for V_1, but the solution will be a function of time as well as the input current, *i*.

Impedance Method

You discovered in the previous section that symbolic (circuit) diagrams can often be simplified for purposes of analysis by recognizing elements that are in series and parallel. You found out that resistors in series act as voltage dividers and input/output relations can be written directly without the need for lengthy equation derivations. You also discovered that resistors in parallel can be combined into an equivalent

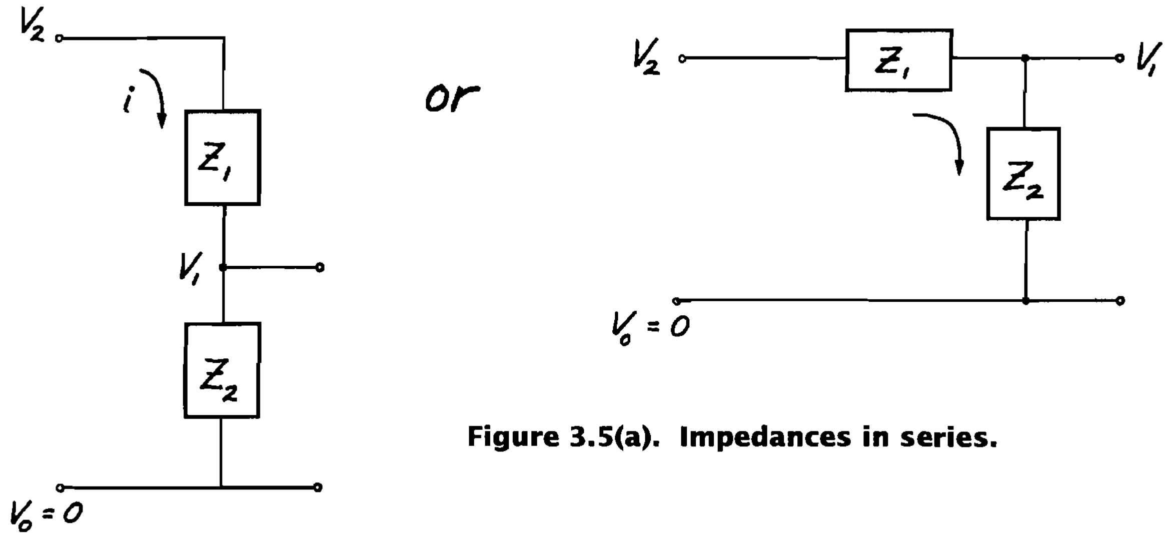


Figure 3.5(a). Impedances in series.

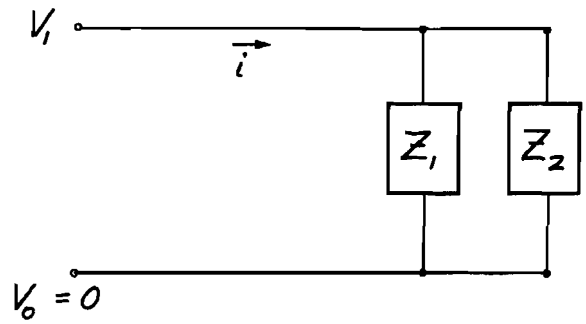


Figure 3.5(b). Impedances in parallel.

resistance. This too provides direct input/output relations without lengthy derivations.

In the last chapter you were introduced to the concept of impedance. You learned that inductors and capacitors can be represented as an electrical impedance just as a resistor can. You will now discover that any circuit element in series or in parallel can be treated just like a resistor when impedances are used. By combining the concepts of impedance and the tools for analyzing elements in series and in parallel, an extremely powerful method of deriving equations for complex circuits evolves.

Figure 3.5 shows the same two circuits shown in Figure 3.4 and studied using the path-vertex-elemental equation approach to deriving equations. All that has been changed between Figures 3.4 and 3.5 is the circuit element representation. The elements are shown in Figure 3.5 as generalized impedances.

Using equation (3.5) as a guide, we can write the output voltage in terms of the input voltage and the ratio of the impedances:

$$V_1 = \frac{Z_2}{Z_1 + Z_2} V_2 \qquad (3.21)$$

where

$$Z_1 = R \qquad (3.22)$$

and

$$Z_2 = \frac{1}{CD} \qquad (3.23)$$

Substituting (3.22) and (3.23) into (3.21) gives

$$V_1 = \frac{\frac{1}{CD}}{R + \frac{1}{CD}} V_2 \qquad (3.24)$$

This equation can be simplified by treating the operator 1/D as if it were just an algebraic variable:

$$V_1 = \frac{\frac{1}{\cancel{CD}}}{\frac{RCD+1}{\cancel{CD}}} V_2 = \frac{1}{RCD+1} V_2 \qquad (3.25)$$

Cross-multiplying by (RCD + 1) gives:

$$(RCD+1)V_1 = RCDV_1 + V_1 = V_2 \qquad (3.26)$$

Now note that the operator D, or $d(\ )/dt$, is operating on V_1. Equation (3.26) can then be written as

$$RC\frac{dV_1}{dt} + V_1 = V_2 \qquad (3.27)$$

Comparing (3.27) to (3.15b) reveals they are identical.

Let's look at what we accomplished. Using simple algebra, we have taken the circuit shown in Figure 3.4a and derived the differential equation relating the output, V_1, to the input, V_2. No lengthy equation derivation was involved as in the path-vertex-elemental equation method. Furthermore, using the operational block diagram notation you learned in Chapter 2, equation (3.25) can be reduced to a single transfer function block as shown in Figure 3.6.

Now let's derive the equation for the circuit shown in Figure 3.4b using the impedance method. From the general impedance diagram shown in Figure 3.5b and equation (3.10), we can write the relationship between the output voltage and input current as

$$V_1 = \left(\frac{Z_1 Z_2}{Z_1 + Z_2}\right) i \qquad (3.28)$$

where from above

$$Z_1 = R \qquad (3.22)\ \text{repeated}$$

and

$$Z_2 = \frac{1}{CD} \qquad (3.23)\ \text{repeated}$$

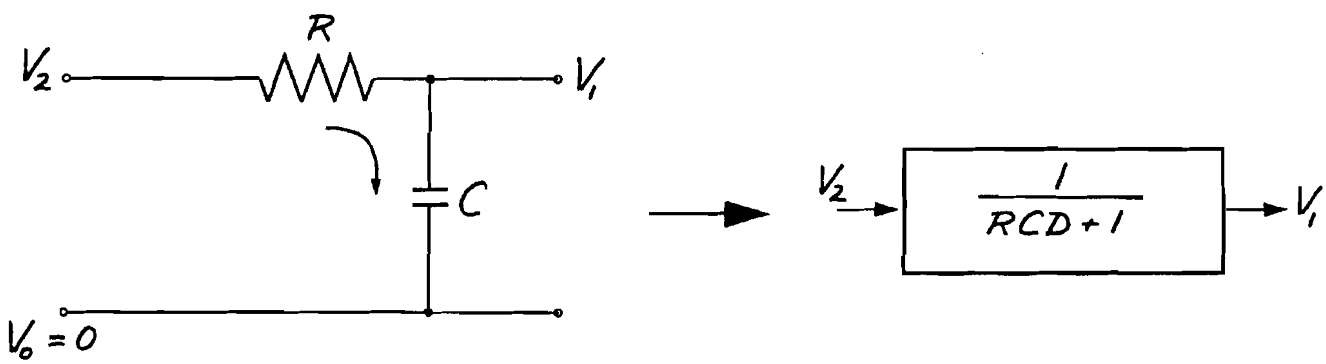


Figure 3.6. Transfer function block diagram equivalent for a resistor and a capacitor in series.

Substituting (3.22) and (3.23) into (3.28) gives

$$V_1 = \left(\frac{R\frac{1}{CD}}{R + \frac{1}{CD}} \right) i \qquad (3.29)$$

which can be simplified to

$$V_1 = \left(\frac{R}{RCD + 1} \right) i \qquad (3.30)$$

or

$$RCDV_1 + V_1 = Ri \qquad (3.31)$$

Here again we note that the operator D, or $d(\ )/dt$ is operating on the output voltage variable. So

$$RC\frac{dV_1}{dt} + V_1 = Ri \qquad (3.32)$$

Comparing (3.32) to (3.20) reveals that they are the same. So again we have been able to write the differential equation relating the output to the input without the need for the lengthy equation derivation we found using the path-vertex-elemental equation method.

Many electrical circuits you will encounter are simply combinations of elements arranged as voltage dividers or as elements in parallel. For example, the circuit shown in Figure 3.7a might look complicated, but it can easily be solved for the relationship between V_{out} and V_{in} in just a few steps as follows:

Step 1. Replace circuit elements with impedances as in Figure 3.7b.

Step 2. Combine parallel elements into a single equivalent element as in Figure 3.7c.

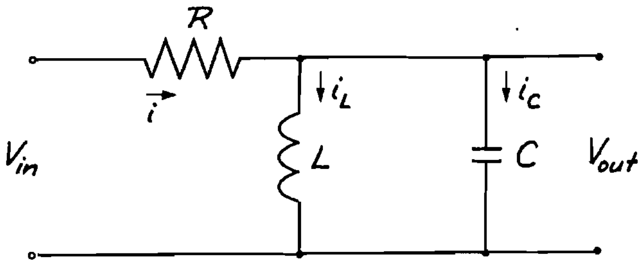


Figure 3.7(a). Electrical circuit containing all three elements.

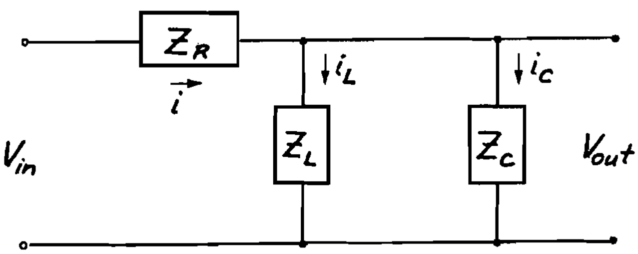


Figure 3.7(b). Replace elements with impedances.

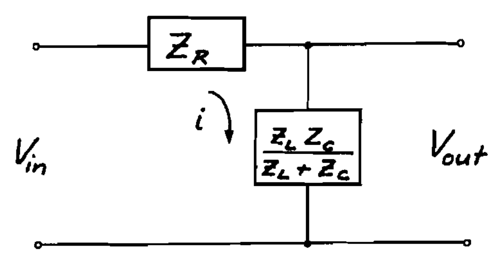


Figure 3.7(c). Reduce parallel elements to equivalent impedance and recognize reduced circuit as a voltage divider.

Step 3. Recognize the simplified circuit as a voltage divider and write the output-input equation by inspection in terms of impedances:

$$\frac{V_{out}}{V_{in}} = \frac{Z_{eq}}{Z_R + Z_{eq}} = \frac{\dfrac{Z_L Z_C}{Z_L + Z_C}}{Z_R + \dfrac{Z_L Z_c}{Z_L + Z_c}}$$

$$= \frac{Z_L Z_c}{Z_R Z_L + Z_R Z_c + Z_L Z_c}$$

Step 4. Substitute elemental impedances and perform algebra:

$$\frac{V_{out}}{V_{in}} = \frac{LD \times \dfrac{1}{CD}}{RLD + R \times \dfrac{1}{CD} + LD \times \dfrac{1}{CD}}$$

$$\frac{V_{out}}{V_{in}} = \frac{\dfrac{LD}{CD}}{\dfrac{RLCD^2 + R + LD}{CD}} = \frac{\dfrac{L}{R}D}{LCD^2 + \dfrac{L}{R}D + 1}$$

Step 5. Write differential equation by substituting $d(\ )/dt$ for the operator D:

$$(LC)\frac{d^2 V_{out}}{dt^2} + \left(\frac{L}{R}\right)\frac{dV_{out}}{dt} + V_{out} = \left(\frac{L}{R}\right)\frac{dV_{in}}{dt}$$

The final equation arrived at in Step 5 is a *second*-order ordinary linear differential equation with constant coefficients. It is called *second-order* because the second derivative of the output (dependent) variable is involved. You will study these types of equations later in Chapter 6. For now, look at what we have accomplished. In five easy steps, we derived the math model for a fairly complex electrical circuit. To prove to yourself just how easy and fast the impedance method is, stop now and try to derive the equation given at the end of step 5 using the path-vertex-elemental equation method.

Operational Block Diagram Method

Another tool that is very useful in visualizing how a system is working is the operational block diagram. In Chapter 2 we developed block diagrams for each of the three circuit elements and I stated then that it was best to develop these diagrams without using differentiator blocks. Now we will prepare a block diagram for the circuit of Figure 3.4.

When you develop a block diagram, first write the elemental equation for the circuit. We did this earlier, so from equation (3.12) we have

$$i = \frac{V_2 - V_1}{R} = \frac{1}{R}(V_2 - V_1)$$

or

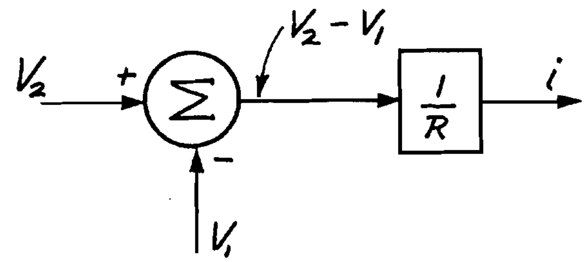


Figure 3.8.

Note how this block diagram was prepared. First you decide which variable is going to be the output. Then you solve for that variable and put it on the left side of the equation. Start drawing the diagram with the input or forcing variable, in this case V_2, at the far left of the diagram. Then just follow the equation drawing summers and multiplication blocks as needed.

The block diagram so far shows current as the output. We ultimately want V_1 as the output, so a block diagram for the capacitor is needed. Equation 3.13 is in the wrong form because it shows current as an output instead of an input. Following the steps, we rearrange the equation and draw the diagram.

$$\frac{dV_1}{dt} = \frac{1}{C} i$$

$$dV_1 = \frac{1}{C} i\, dt$$

$$V_1 = \int_0^t \left(\frac{1}{C}\right) i\, dt + (V_1)_{init}$$

or

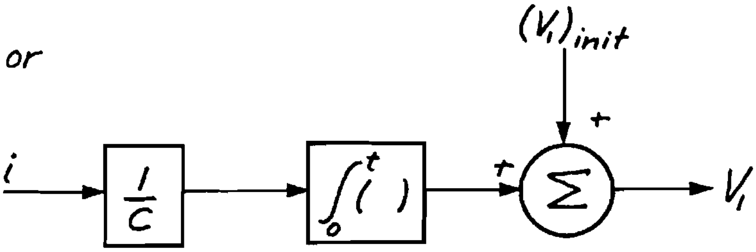


Figure 3.9.

Now we can add this to our previous diagram to give Figure 3.10. Notice how the output voltage V_1 on the right side "feeds back" and is subtracted from V_2. This creates a voltage difference, $V_2 - V_1$, which in turn causes a current to flow across the resistor. This current then flows into the capacitor where it is integrated and produces the output voltage.

Look at how this circuit works as viewed by the block diagram. Start with no initial charge on the capacitor and the input voltage at zero. If a constant voltage V_2^* were suddenly applied to the circuit, the voltage difference out of the summer would equal V_2^* because V_1 is initially zero. A current equal to V_2^* / R will initially flow. The output voltage V_1 will not suddenly increase because the integrator takes time to convert its input into an output. After some time, Δt, a voltage $V_1^{(1)}$ will exist. This voltage is then subtracted from V_2^* to give a new current equal to $(V_2^* - V_1^{(1)}) / R$. Again this new current, now lower than it was before, gets integrated during another time interval, Δt, and creates a higher output voltage $V_1^{(2)}$. Once again, the higher output voltage is fed back, where it reduces the current to the capacitor even more. The process continues until the current is reduced to zero. That occurs when $V_1 = V_2$. So you can see that a sudden change in the input voltage to this circuit passes through to the output, but only after the passage of time.

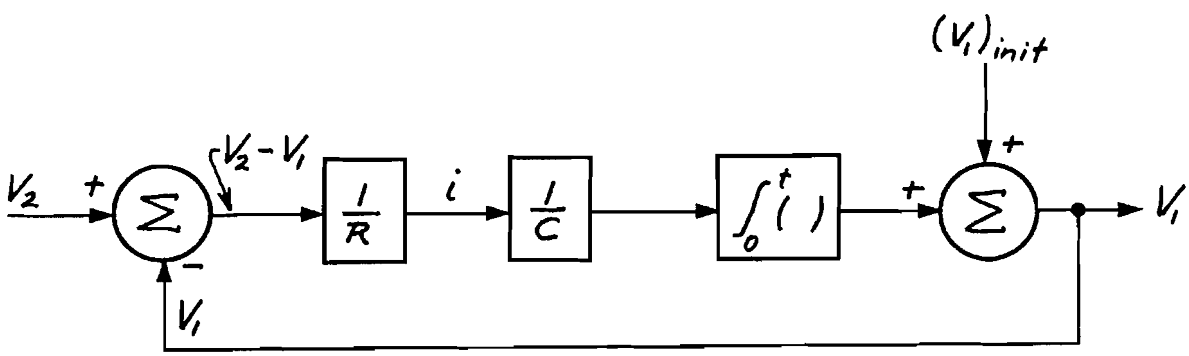


Figure 3.10.

If we let the initial charge on the capacitor $(V_l)_{init}$ be zero, use the operator $1/D$ for the integral, and replace RC with τ, then we can reduce the block diagram into that shown in Figure 3.11.

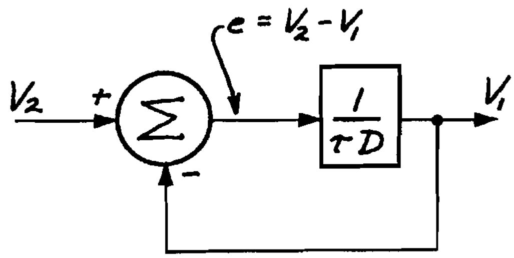


Figure 3.11.

From this block diagram we can write

$$e = V_2 - V_l$$

and

$$V_1 = \frac{1}{\tau D} e$$

(In feedback-control terminology, the symbol e is used to indicate error between the input and output.)

Combining these two equations to eliminate e gives

$$V_1 = \frac{1}{\tau D}(V_2 - V_1)$$

Rearranging gives

$$\tau D V_1 + V_1 = V_2$$

$$V_1(\tau D + 1) = V_2$$

$$V_1 = \frac{V_2}{\tau D + 1}$$

The block diagram for this last equation can be drawn as Figure 3.12.

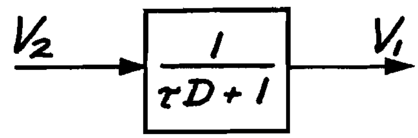


Figure 3.12.

As you can now see, the block diagram in Figure 3.10 can be reduced to the form shown in Figure 3.12. This latter diagram is often called the *transfer function* relating the output V_l to the input V_2. The transfer function is expressed as output divided by input, or

$$\frac{V_1}{V_2} = \frac{1}{\tau D + 1}$$

Note that this equation is identical to equation 3.15b when τ is substituted for RC and D for $d(\ )/dt$ in (3.15b). You can see that if you can draw a block diagram, you can reduce it to the desired math model you're looking for.

Incidentally, the transfer function shown in Figure 3.12 is very common. You will encounter it many times in engineering systems. It is often called a *first-order lag* because it describes a first-order differential equation and because it causes the output to lag the input. We'll discuss the importance of this transfer function in more detail in the next chapter.

Free-body Diagram Method

You learned in Chapter 2 that symbolic (circuit) diagrams can be made for mechanical components. The "circuit" diagrams can be analyzed using the techniques presented in the previous two sections to arrive at math models for mechanical systems. You will now learn another powerful method for deriving equations for mechanical systems. It has its origins with Newton's laws of motion and involves the construction of *free-body diagrams*.

The free-body diagram method involves drawing a sketch of a mechanical component and labeling all of the forces and moments acting on the component. If the body is known to be at rest or is not accelerating, the sum of all the forces and the sum of all the moments acting on the body must equal zero. If the forces and moments acting on the body do not sum to zero, then the body will undergo translational and/or angular acceleration. Equations describing the motion of the body can be obtained by setting the sum of the forces equal to the mass of the body times the acceleration and the sum of the moments equal to the rotational inertia of the body times the angular acceleration.

Figure 3.13a shows a block with a mass m being pulled by a rope along a horizontal surface. Constructing a free-body diagram of the block involves isolating it in free space and showing all forces acting on the block. Next, choose a convenient axis so forces and moments can be summed and set either to zero, if no motion along that reference axis occurs, or to the mass (rotational inertia for moments) times the acceleration (linear for translation, angular for rotation). You must be careful to show *all* forces acting on the body, but *only* the forces acting on the body.

Figure 3.13b shows a correct free-body diagram of the block. The tension in the rope is identified as the force F. Note that the hand has nothing to do with the free-body diagram and that I assumed the tension in the rope acts horizontally to the surface. The force due to earth's gravity is shown as W and is equal to mg, where g is the gravitational constant. Since the block is not moving downward, there is an unknown force N acting at an unknown distance l which supports the block in the vertical direction. Since we know the block is sliding along the surface, we assume there is a resistive force f acting along the sliding surface of the block.

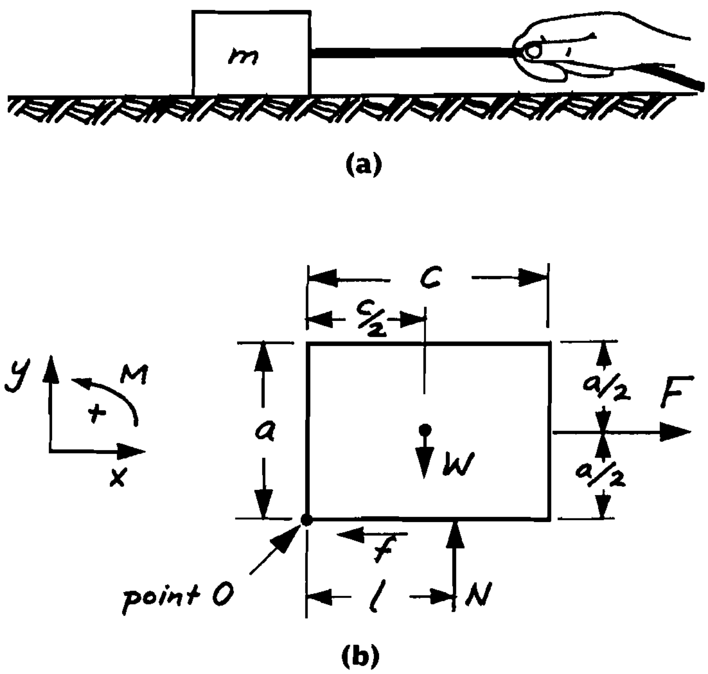


Figure 3.13. Constructing a free-body diagram.

We select a convenient set of axes and denote the positive direction as shown in Figure 3.13b. Then we write the equations in the three axes shown as follows:

Sum of forces in y direction equals zero (no motion in this axis)

$$\sum F_y = N - W = 0 \qquad (3.33)$$

Sum of forces in x direction equals mass times acceleration in x direction

$$\sum F_x = F - f = ma_x \qquad (3.34)$$

Sum of moments about point 0 equals zero (no rotational motion of block)

$$\sum M_0 = Nl - W\frac{c}{2} - F\frac{a}{2} = 0 \qquad (3.35)$$

You can see from (3.33) that the unknown force N is equal to the weight of the block. This information can be substituted into (3.35) and that equation solved for the unknown distance l as follows:

$$Wl = W\frac{c}{2} + F\frac{a}{2}$$

$$l = \frac{c}{2} + \frac{F}{W}\frac{a}{2} \qquad (3.36)$$

Equation (3.36) shows that the point of application of N increases as the force F increases. If l increases to the point where it becomes greater than c, the block will tip over.

Now let's examine (3.34) in more detail. The frictional force f is still unknown and another equation is needed to determine it. One possible assumption is that f is related to N. This assumption is often made when dry surfaces slide against one another. In this case we use a coefficient of friction μ and write

$$f = \mu N \qquad (3.37)$$

This allows us to determine f since we already know N is equal to W.

Another possible assumption is that the surfaces are lubricated causing the force f to be proportional to the velocity of the block v. We can then write

$$f = bv \qquad (3.38)$$

where b is the constant of proportionality. You will immediately recognize (3.38) as the equation for a damper (dashpot) discussed in Chapter 2.

Substituting (3.38) into (3.34) and expressing a_x as dv/dt gives

$$F - bv = m\frac{dv}{dt}$$

$$\frac{m}{b}\frac{dv}{dt} + v = \frac{1}{b}F \qquad (3.39)$$

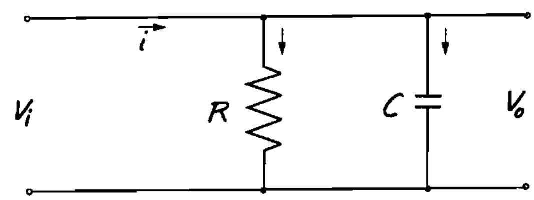
Comparing (3.39) with (3.20) shows them to be identical to each other when the preferred electrical/mechanical analog discussed in Chapter 2 is used. The damper and the mass are therefore in parallel. The through variable F divides into two parts. One part overcomes the damper force and the other part accelerates the mass—you can see this clearly in (3.34).

3.3 First-Order Math Models

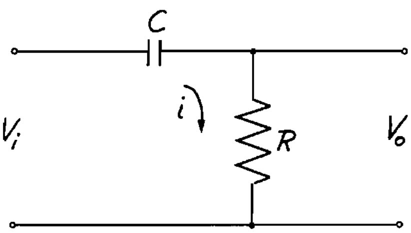
Now that you have tools for developing math models, let's use them to develop models for systems that are comprised of one or more energy dissipative elements and one energy storage element. We'll look at electrical systems first and then move on to translational and rotational mechanical systems, fluid systems, and thermal systems. As you read through this section, take the time to *practice* developing the math models as I suggest. I have found that the more circuits I analyze, the better I get at analyzing circuits and the longer the analysis techniques stay with me.

Electrical Math Models

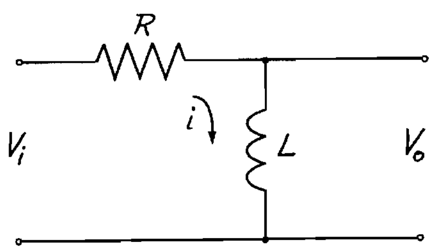
Figure 3.14 shows five electrical circuits. Prepare the math models that describe the output variable as a function of the input variable as indicated by each circuit. Use any method you wish, but I strongly suggest that you use: (1) the path-vertex-elemental equation method; (2) the impedance method; or (3) the block diagram method. (Or, try all three if you feel ambitious!) Try not to look at my solutions until you have at least attempted to derive each of the math models.



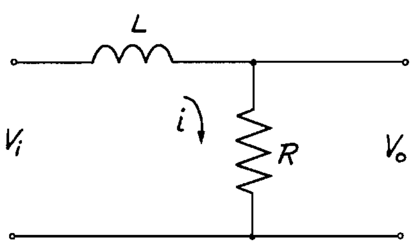
(a). ***Prepare math model describing V_o as a function of i.***



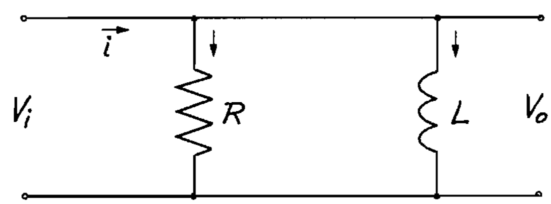
(b). ***Prepare math model describing V_o as a function of V_i.***



(c). ***Prepare math model describing V_o as a function of V_i.***



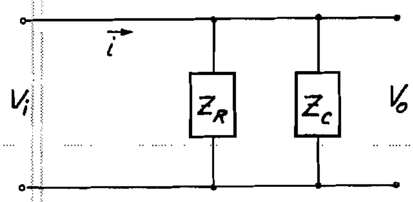
(d). ***Prepare math model describing V_o as a function of V_i.***



(e). ***Prepare math model describing V_o as a function of V_i.***

Figure 3.14. Develop math models for these electrical systems.

(a).



V_o and V_i are equal

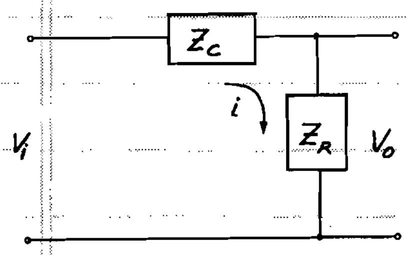
$$V_o = i\,Z_{equiv} = i\left(\frac{Z_R Z_C}{Z_R+Z_C}\right) = i\left(\frac{R\cdot\frac{1}{CD}}{R+\frac{1}{CD}}\right) = i\,\frac{R}{RCD+1}$$

$$(RCD+1)V_o = Ri \Rightarrow RC\frac{dV_o}{dt} + V_o = Ri$$

or

$$\boxed{\tau\frac{dV_o}{dt} + V_o = Ri \quad \text{where } \tau = RC}$$

(b).



$$V_o = \frac{Z_R}{Z_R+Z_C}V_i = \left(\frac{R}{R+\frac{1}{CD}}\right)V_i = \frac{RCD}{RCD+1}V_i$$

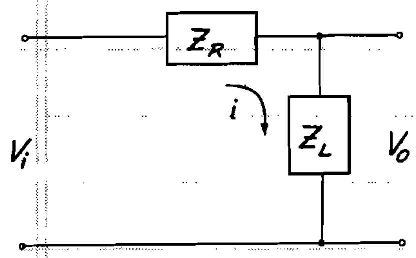
$$(RCD+1)V_o = RCD\,V_i$$

$$\Rightarrow RC\frac{dV_o}{dt} + V_o = RC\frac{dV_i}{dt}$$

or

$$\boxed{\tau\frac{dV_o}{dt} + V_o = \tau\frac{dV_i}{dt} \quad \text{where } \tau = RC}$$

(c).



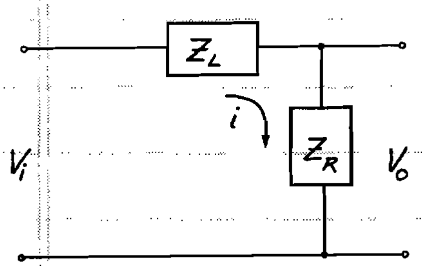
$$V_o = \frac{Z_L}{Z_R+Z_L}V_i = \frac{LD}{R+LD}V_i = \frac{\frac{L}{R}D}{\frac{L}{R}D+1}V_i$$

$$\left(\frac{L}{R}D+1\right)V_o = \frac{L}{R}DV_i \Rightarrow \frac{L}{R}\frac{dV_o}{dt} + V_o = \frac{L}{R}\frac{dV_i}{dt}$$

or

$$\boxed{\tau\frac{dV_o}{dt} + V_o = \tau\frac{dV_i}{dt} \qquad \tau = \frac{L}{R}}$$

(d).



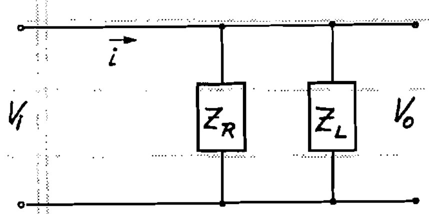
$$V_o = \frac{Z_R}{Z_R+Z_L}V_i = \frac{R}{R+LD}V_i = \frac{1}{\frac{L}{R}D+1}V_i$$

$$\left(\frac{L}{R}D+1\right)V_o = V_i \Rightarrow \frac{L}{R}\frac{dV_o}{dt} + V_o = V_i$$

or

$$\boxed{\tau\frac{dV_o}{dt} + V_o = V_i \qquad \tau = \frac{L}{R}}$$

(e).



V_o and V_i are equal

$$V_o = i\,Z_{equiv} = i\left(\frac{Z_R Z_L}{Z_R+Z_L}\right) = i\left(\frac{RLD}{R+LD}\right) = i\left(\frac{\frac{L}{R}D}{\frac{L}{R}D+1}\right)R$$

$$\left(\frac{L}{R}D+1\right)V_o = R\frac{L}{R}Di \Rightarrow \frac{L}{R}\frac{dV_o}{dt} + V_o = \frac{L}{R}\frac{di}{dt}R$$

or

$$\boxed{\tau\frac{dV_o}{dt} + V_o = R\tau\frac{di}{dt} \qquad \tau = \frac{L}{R}}$$

Figure 3.15. Math models for the electrical systems shown in Figure 3.14.

The method I prefer to use when analyzing electrical circuits is the impedance method. While the circuits shown in Figure 3.14 are relatively easy to analyze by either of the other two methods, those methods become very tedious when the circuits are more complex. The math models I developed and the ways in which they were developed are shown in Figure 3.15.

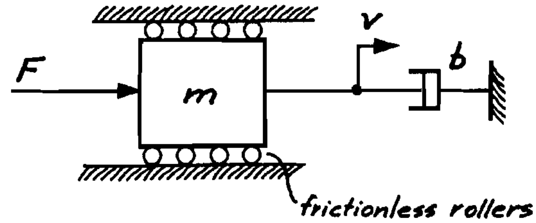
You should note several important common features that each of these models possess. First, they are all first-order ordinary linear differential equations with constant coefficients. Second, the left sides of all of the equations are identical except for the way the time constant is expressed. Finally, only the right, or input, side of each equation is different.

Mechanical Math Models

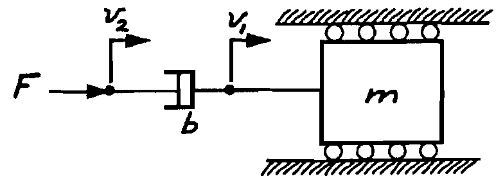
Figure 3.16 shows four mechanical systems to practice with. Prepare the math models that describe the output variable as a function of the input variable as indicated alongside each schematic. Use any method you wish, but I strongly suggest you use all four of the methods we discussed in this chapter. Once again, try not to look at my solutions until you have at least attempted to derive each of the math models.

The method I prefer to use when analyzing mechanical systems is the free-body diagram method. However, if the systems are very complicated, I frequently find myself reverting back to the impedance method. The math models I developed and the ways in which I developed them are shown in Figure 3.17.

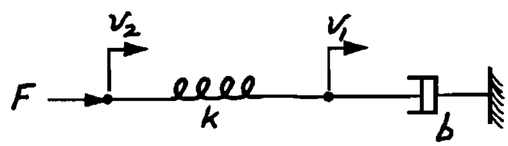
Once again, note the common features that each of these models possess. They are all first-



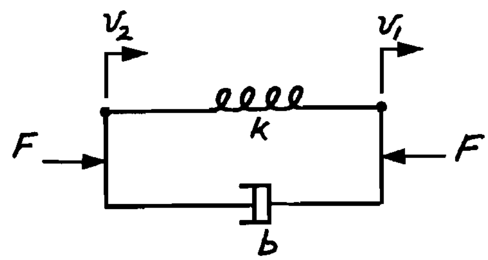
(a). ***Prepare math model describing the velocity v of the mass as a function of the force F.***



(b). ***Prepare math model describing the velocity v_1 as a function of the velocity v_2.***



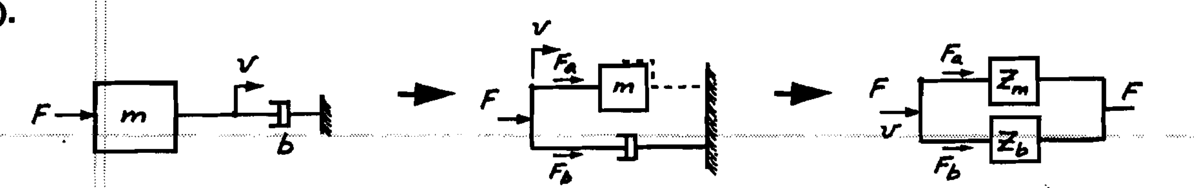
(c). ***Prepare math model describing the velocity v_1 as a function of the velocity v_2.***



(d). ***Prepare math model describing the velocity differential v_{21} as a function of the force F.***

Figure 3.16. Develop math models for these mechanical systems.

(a).



$$F_a + F_b = F$$

$$F_a = m\frac{dv}{dt}$$

$$F_b = bv$$

$$\therefore\; bv + m\frac{dv}{dt} = F$$

$$\frac{m}{b}\frac{dv}{dt} + v = \frac{1}{b}F$$

or

$$\boxed{\tau\frac{dv}{dt} + v = \frac{1}{b}F \qquad \tau = \frac{m}{b}}$$

elements in parallel

$$v = F Z_{equiv} = F\left(\frac{Z_m Z_b}{Z_m + Z_b}\right)$$

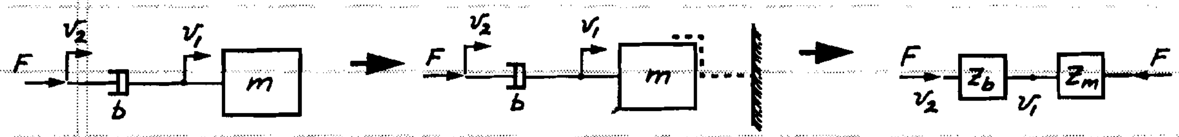
$$= F\left(\frac{\frac{1}{mD}\,\frac{1}{b}}{\frac{1}{mD} + \frac{1}{b}}\right)$$

$$v = F\frac{1}{mD + b}$$

$$(mD + b)v = F$$

$$m\frac{dv}{dt} + bv = F \Rightarrow \frac{m}{b}\frac{dv}{dt} + v = \frac{1}{b}F$$

(b).



$$F = b(v_2 - v_1)$$

$$F = m\frac{dv_1}{dt}$$

$$\therefore\; m\frac{dv_1}{dt} = bv_2 - bv_1$$

$$\frac{m}{b}\frac{dv_1}{dt} + v_1 = v_2$$

or

$$\boxed{\tau\frac{dv_1}{dt} + v_1 = v_2 \qquad \tau = \frac{m}{b}}$$

elements in series

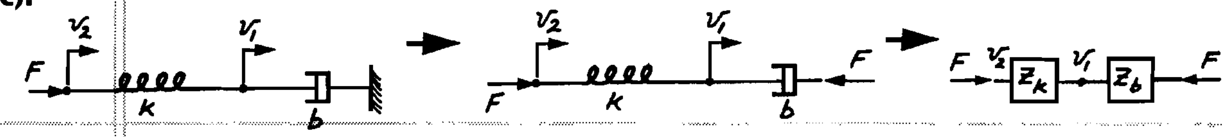
$$v_1 = \frac{Z_m}{Z_m + Z_b}v_2 = \frac{\frac{1}{mD}}{\frac{1}{mD} + \frac{1}{D}}v_2$$

$$v_1 = \frac{b}{mD + b}v_2 \Rightarrow m\frac{dv_1}{dt} + bv_1 = v_2$$

Figure 3.17. Math models for the mechanical systems shown in Figure 3.16.

(continued on next page)

(c).



$v_2 - v_1 = \frac{1}{k}\frac{dF}{dt}$ can see this from $F = k(x_2 - x_1)$

$\frac{dF}{dt} = k(v_2 - v_1)$

elements in series

$v_1 = \frac{Z_b}{Z_k + Z_b} v_2 = \frac{\frac{1}{b}}{\frac{1}{b} + \frac{D}{k}} v_2$

$F = bv_1$ or $\frac{dF}{dt} = b\frac{dv_1}{dt}$

$v_1 = \frac{k}{bD+k} v_2 \Rightarrow$

$\therefore v_2 - v_1 = \frac{b}{k}\frac{dv_1}{dt} \Rightarrow \frac{b}{k}\frac{dv_1}{dt} + v_1 = v_2$

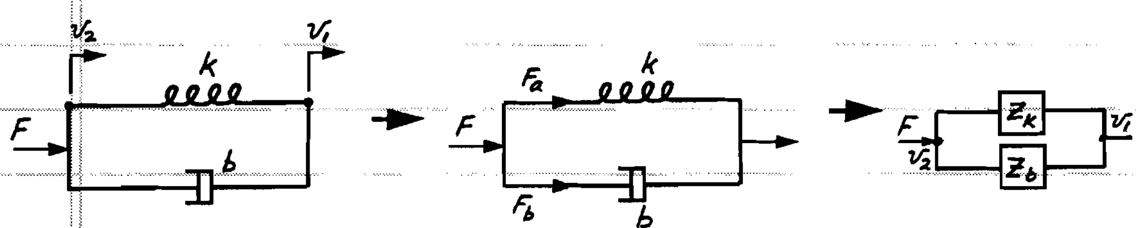
$b\frac{dv_1}{dt} + kv_1 = kv_2$

or

or $\frac{b}{k}\frac{dv_1}{dt} + v_1 = v_2$

$$\boxed{\tau\frac{dv_1}{dt} + v_1 = v_2 \qquad \tau = \frac{b}{k}}$$

(d).



elements in parallel

$F = F_a + F_b$ or $\frac{dF}{dt} = \frac{dF_a}{dt} + \frac{dF_b}{dt}$

$v_{21} = F Z_{equiv} = F\left(\frac{Z_k Z_b}{Z_k + Z_b}\right)$

$\frac{dF_a}{dt} = k(v_2 - v_1) = kv_{21}$

$\therefore \frac{dF}{dt} = kv_{21} + b\frac{dv_{21}}{dt}$

$= \frac{\frac{D}{k}\cdot\frac{1}{b}}{\frac{D}{k} + \frac{1}{b}} F$

$\frac{dF_b}{dt} = \frac{d}{dt}\left[b(v_2 - v_1)\right] = b\frac{dv_{21}}{dt}$

or $\frac{b}{k}\frac{dv_{21}}{dt} + v_{21} = \frac{1}{k}\frac{dF}{dt}$

$v_{21} = \frac{D}{bD+k} F \Rightarrow$

or

$$\boxed{\tau\frac{dv_{21}}{dt} + v_{21} = \frac{1}{k}\frac{dF}{dt} \qquad \tau = \frac{b}{k}}$$

$b\frac{dv_{21}}{dt} + kv_{21} = \frac{dF}{dt}$

Figure 3.17 ***(continued).*** **Math models for the mechanical systems shown in Figure 3.16.**

order ordinary linear differential equations with constant coefficients. The left sides are all identical except for the way the time constant is expressed. Only the right, or the input, side of each equation is different.

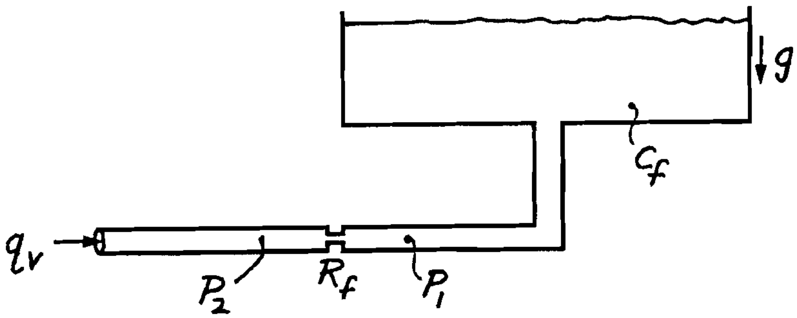
Now compare these equations with those that were developed for the electrical circuits. If you use the force-current and velocity-voltage analogy, you will see that the equations are essentially identical.

Fluid Math Models

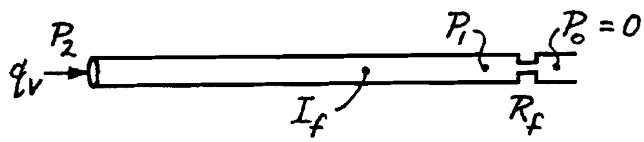
Figure 3.18 shows three fluid systems for you to practice with. Prepare the math models that describe the output variable as a function of the input variable as indicated next to each schematic. Use any method you wish, but again I suggest that you use as many of the methods described as possible. Once again, try not to look at my solutions until you have at least attempted to derive each of the math models.

I really have no preferred method for analyzing fluid systems. If I get stuck, I revert back to a combination of the vertex-path-elemental equation method and the impedance method. The math models I developed and the ways in which I developed them are shown in Figure 3.19. Once again, note the common features that each of these models possess. They are all first-order ordinary linear differential equations with constant coefficients. The left sides are all identical except for the way the time constant is expressed. Only the right, or input, side of each equation is different.

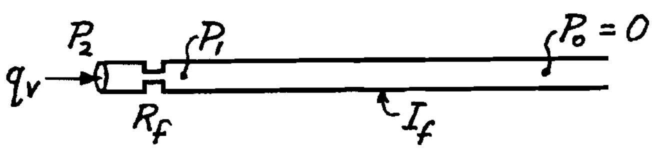
Now compare these equations with those that were developed for the electrical circuits and the mechanical systems. If you use the force-current-flow rate and velocity-voltage-pressure analogy, you will see that the equations are essentially identical.



(a). *Prepare math model describing pressure p_1 as a function of input pressure p_2.*



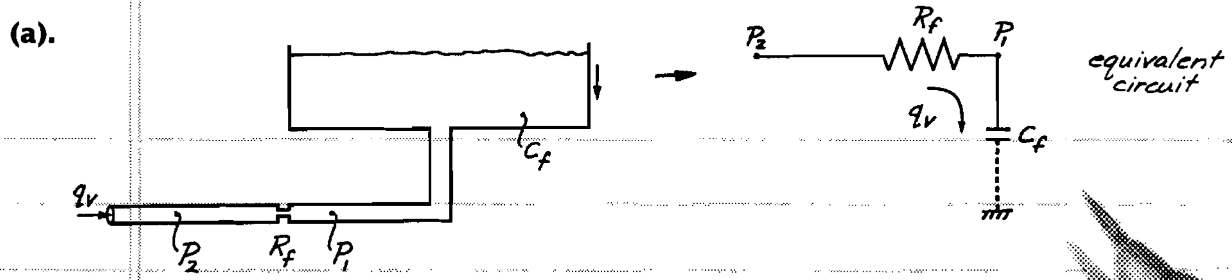
(b). *Prepare math model describing: (1) q_v as a function of p_2; (2) p_1 as a function of p_2.*



(c). *Prepare math model describing: (1) q_v as a function of p_2; (2) p_1 as a function of p_2.*

Figure 3.18. Develop math models for these fluid systems.

(a).



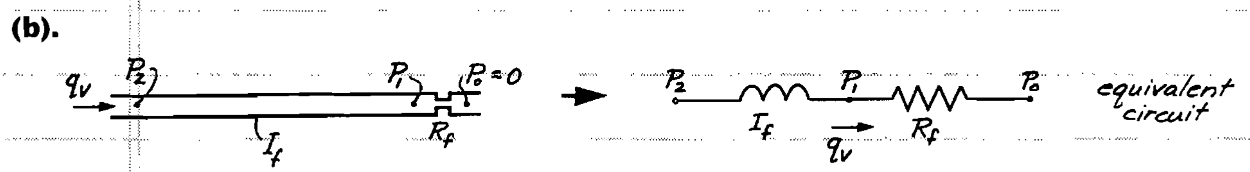
$$\left.\begin{aligned} P_2 - P_1 &= R_f q_v \\ q_v &= C_f \frac{dP_1}{dt} \end{aligned}\right\} \Rightarrow P_2 - P_1 = R_f C_f \frac{dP_1}{dt}$$

$$\Rightarrow R_f C_f \frac{dP_1}{dt} + P_1 = P_2$$

or

$$\boxed{\tau \frac{dP_1}{dt} + P_1 = P_2 \qquad \tau = R_f C_f}$$

(b).



$$\left.\begin{aligned} P_2 - P_1 &= I_f \frac{dq_v}{dt} \\ P_1 - P_0 &= R_f q_v \Rightarrow P_1 = R_f q_v \end{aligned}\right\} P_2 - R_f q_v = I_f \frac{dq_v}{dt} \Rightarrow I_f \frac{dq_v}{dt} + R_f q_v = P_2$$

or

$$\boxed{\tau \frac{dq_v}{dt} + q_v = \frac{1}{R_f} P_2 \qquad \tau = \frac{I_f}{R_f}}$$

$$\frac{dq_v}{dt} = \frac{1}{R_f} \frac{dP_1}{dt}$$

$$P_2 - P_1 = \frac{I_f}{R_f} \frac{dP_1}{dt} \Rightarrow \frac{I_f}{R_f} \frac{dP_1}{dt} + P_1 = P_2$$

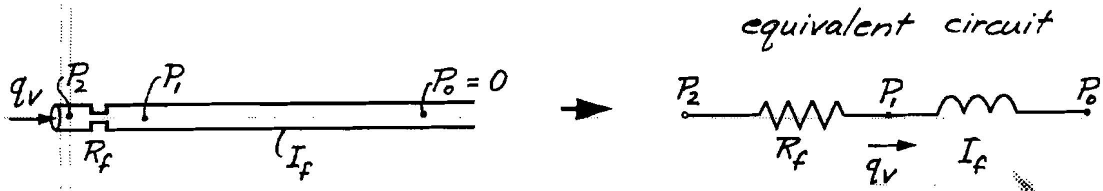
or

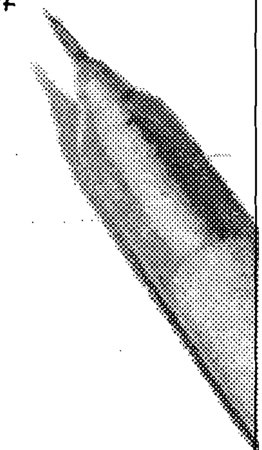
$$\boxed{\tau \frac{dP_1}{dt} + P_1 = P_2 \qquad \tau = \frac{I_f}{R_f}}$$

Figure 3.19. Math models for the fluid systems shown in Figure 3.18.

(continued on next page)

(c).





$$\left.\begin{array}{l} P_2 - P_1 = R_f q_v \\ P_1 - P_0 = I_f \dfrac{dq_v}{dt} \Rightarrow P_1 = I_f \dfrac{dq_v}{dt} \end{array}\right\} P_2 - I_f \frac{dq_v}{dt} = R_f q_v$$

$$\Rightarrow I_f \frac{dq_v}{dt} + R_f q_v = P_2$$

or

$$\boxed{\tau \frac{dq_v}{dt} + q_v = \frac{1}{R_f} P_2 \qquad \tau = \frac{I_f}{R_f}}$$

$$\left.\begin{array}{l} \dfrac{dP_2}{dt} - \dfrac{dP_1}{dt} = R_f \dfrac{dq_v}{dt} \\ \dfrac{dq_v}{dt} = \dfrac{1}{I_f} P_1 \end{array}\right\} \frac{dP_2}{dt} - \frac{dP_1}{dt} = \frac{R_f}{I_f} P_1$$

$$\Rightarrow \frac{I_f}{R_f} \frac{dP_1}{dt} + P_1 = \frac{I_f}{R_f} \frac{dP_2}{dt}$$

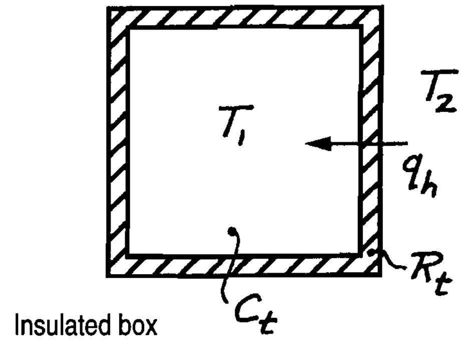
or

$$\boxed{\tau \frac{dP_1}{dt} + P_1 = \tau \frac{dP_2}{dt} \qquad \tau = \frac{I_f}{R_f}}$$

Figure 3.19 ***(continued).*** **Math models for the fluid systems shown in Figure 3.18.**

Thermal Math Models

Figure 3.20 shows one thermal system for you to practice with. Prepare the math model that describes the output variable as a function of the input variable as indicated. Use any method you wish, but again I suggest that you use as many of the methods described as possible. Don't look at my solution until you have at least attempted to derive the math model.



Prepare math model describing the temperature T_1 inside the box as a function of temperature T_2 outside the box.

Figure 3.20. Develop math model for this thermal system.

I have no preferred method for analyzing thermal systems. Should I get stuck, I revert back to a combination of the vertex-path-elemental equation method and the impedance method. The math model I developed and the way in which I developed it is shown in Figure 3.21. Note that this is a first-order ordinary linear differential equations with constant coefficients.

Compare this equation with those that were developed for the electrical circuits, the mechanical systems, and the fluid system. If you use the force-current-flow rate-heat flow rate and velocity-voltage-pressure-temperature analogy, you will see that the equations are essentially identical.

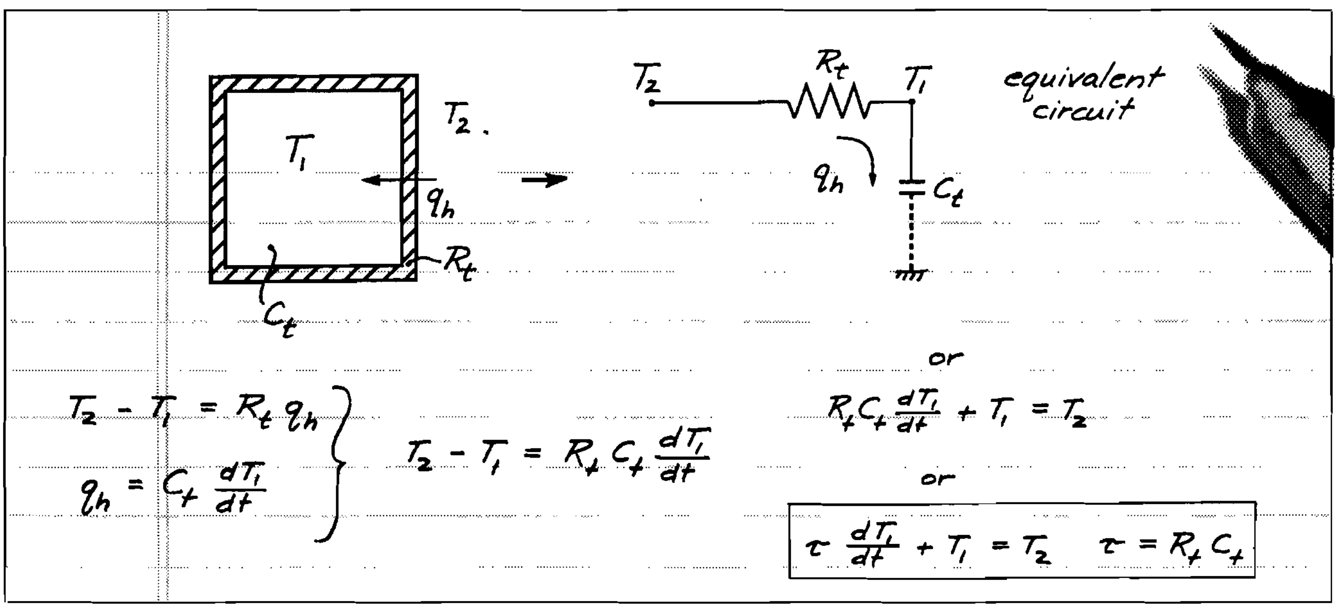


Figure 3.21. Math model for the thermal system shown in Figure 3.20.

Chapter 4 Analyzing First-Order Math Models

Objectives

When you have completed this chapter, you will be able to:

- **Solve first-order ordinary differential equations using both numerical and exact solution methods.**
- **Perform frequency analysis of first-order engineering systems and plot the frequency responses.**
- **Find solutions to first-order differential equations for step, ramp, pulse and arbitrary inputs.**
- **Perform a power analysis for systems responding to a sinusoidal input.**

4.1 Introduction

In the previous chapter you discovered that systems modeled as a combination of an ideal energy storage element and an ideal energy dissipative element lead to first-order linear ordinary differential equations with constant coefficients. You found out that this differential equation usually must be "solved" in order to obtain the time-varying response of the system to a time-varying input forcing function.

You will discover in this chapter that the solution provides a great deal of information about the behavior of the real system being modeled. You will first learn a simple approximation method of solution that can be used to solve any ordinary differential equation. This method is extremely powerful and does not require a great deal of mathematical skill. It will be used to introduce you to the various types of input forcing functions commonly encountered when modeling engineering systems.

Once you understand the basic nature of the solutions to differential equations, you will then learn how to obtain the exact solutions. You will discover that exact solutions provide more insight into the system being modeled than do approximate solutions. However, exact methods do involve more mathematical manipulations.

The first exact solution method you will learn involves separating variables so the differential equation can be directly integrated. Following this you will learn a more methodical approach that takes advantage of the fact that the differential equation is linear and has constant coefficients. You will discover that this method is easy and can be used to obtain exact solutions to higher-order linear differential equations with constant coefficients.

You will also be introduced in this chapter to so-called "frequency response" or "frequency domain" solutions and analysis. You will find these extremely important methods of analysis are nothing more than the solution of the differential equation to a sinusoidal input function.

I can't emphasize enough just how important first-order linear ordinary differential equations with constant coefficients are to engineering and science. Probably 80% of all engineering systems you will ever encounter can be completely described by, or contain at least one component that can be described by, one of these equations. Make friends with them—they will serve you well!

4.2 Response to a Step Input

In this section, we will investigate the behavior of an engineering system that can be described by a first-order linear ordinary differential equation with constant coefficients in response to a sudden change in the input. The sudden change of the input variable from one level to another is called a *step* input. The step input is an approximation of many real-world inputs to engineering systems. For example, when we tramp on a car's accelerator or flip on a switch that applies voltage to a circuit, we're applying a step input. We want to find out how the system responds to such an input—that is, we want to find the output as a function of time. You will first learn how to numerically solve the equation subjected to a step input. Then you will learn how to find the exact solution.

Numerical Solution Method

Let's begin solving first-order math models using a model we developed in Chapter 3 that described the electrical circuit in Figure 3.4. The circuit and math model are repeated in Figure 4.1 and equation (4.1).

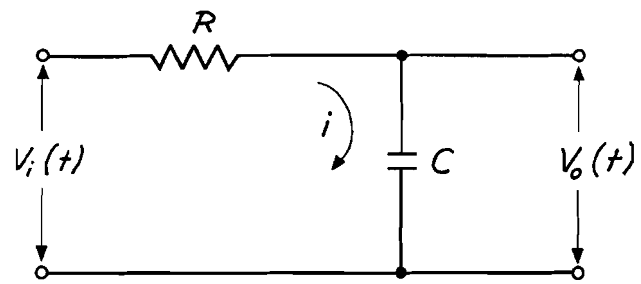


Figure 4.1.

$$RC\frac{dV_o}{dt} + V_o = V_i \qquad (4.1)$$

New labels for the voltages are used to clearly indicate which one is the input voltage, V_i, and which is the output voltage V_O. If we had the solution to this differential equation, $V_O(t)$, we could use it to determine: (1) how various values of R and C affect the output voltage for a given input voltage; and (2) how the output voltage varies with different input voltage functions, given values for R and C.

First I'll show you an easy method for solving differential equations. It is called the "numerical solution" method and it can easily be set up on a spreadsheet or by writing a simple BASIC, FORTRAN, or C computer program. Take the math model as given by equation (4.1) and rewrite it as

$$RC\frac{\Delta V_o}{\Delta t} + V_o = V_i \qquad (4.2)$$

In this equation, $\Delta V_O / \Delta t$ is an approximation for the derivative, dV_O / dt. Now solve for ΔV_O

$$\Delta V_o = \left[\frac{1}{RC}(V_i - V_o)\right]\Delta t \qquad (4.3)$$

Equation (4.3) tells us that if we know V_i at every point in time and if we know the initial value of V_O, then we can use this equation to determine the change in the output voltage ΔV_O that occurs in the time interval Δt. We can therefore solve equation (4.1) using a time stepping process. That is, we start at time $t = 0$ where we know V_O and V_i. Then we take a time step Δt and use equation (4.3) to compute the *change* in the output voltage that occurs during this time interval. We add this change to the previous value of V_O to get the new V_O. The process is continued until we reach some point in time where the variables are no longer changing or where we are no longer interested in the results.

The algorithm for numerically solving a differential equation is very straightforward. One is provided for a first-order differential equation in Table 4.1. Notice in this table that selecting a value for the time step is arbitrary. I've selected a value equal to 1/10th of the constant $\tau = RC$. We'll investigate this selection later but for now keep in mind that the size of the time step affects the speed of computing the solution and its accuracy.

Table 4.1.
Algorithm for numerically solving a first-order differential equation.

Given:

(1) V_o at $t = 0$ (that is, the initial condition of the dependent variable)

(2) V_i as a function of time (that is, the forcing function at any point in time)

(3) Values for R and C

Step 1 Initialize variables:

$\tau = RC$
$t = 0$
$t_{end} = 5\tau$
$\Delta t = \tau/10$
$V_o = (V_o)_{init}$

Step 2 Increment time and check if done

$t = t + \Delta t$
If $t = t_{end}$ then stop.

Step 3 Compute $V_i(t)$ from the given function

Step 4 Solve for ΔV_o

$$\Delta V_o = \left[\frac{1}{RC}(V_i - V_o)\right]\Delta t$$

Step 5 Determine new V_o

$V_o = V_o + \Delta V_o$

Step 6 Go back to **Step 2**

Spreadsheets are very useful in solving differential equations numerically, as math models can be set up very quickly. An Excel® spreadsheet implementing the algorithm given in Table 4.1 is shown in Table 4.2. Lines 1 through 7 accept the given data (R = 1, C = 1, $(V_o)_{init}$ = 0 and $V_i(t)$ = 10) and compute the time step. Note that I have chosen R and C so their product is equal to unity. That is, the time constant of the circuit is equal to 1 second. Line 9 provides a label for the results showing t, V_i, V_o and ΔV_o. Lines 10 through the end implement the iterative solution technique.

The results from this spreadsheet are shown in Table 4.3, and Figure 4.2 shows an Excel graph of the results. You can see that a constant voltage, suddenly applied at t = 0, does not produce an instantaneous output. The output voltage builds rapidly at first, having an initial rate of increase of 1 volt per 0.1 seconds (10 volts per second). Since V_o is initially zero, you can see that the initial rate is from equation (4.3):

$$\frac{\Delta V_o}{\Delta t} = \left[\frac{1}{RC}(V_i - V_o)\right] = \left[\frac{1}{1}(10-0)\right] = 10 \quad (4.4)$$

The rate slows as the output voltage builds. You will recall from the block diagram discussion of this circuit given in Section 3.3 that this is due to the output voltage feeding back and reducing the current flowing into the capacitor. As this current is reduced, the charge on the capacitor asymptotically builds to the value of the input voltage. Note also that in 1 second the output voltage is 6.51 volts, or 65.1% of its final value.

Table 4.2. First-order step response spreadsheet implementation.

	A	B	C	D
1	R	1		
2	C	1		
3	Tau	=B1*B2		
4	del t	=B3/10		
5	(Vo)init	0		
6	Vi*	10		
7	Tend	=5*B3		
8				
9	t	Vi	Vo	del Vo
10	0	=$B$6	=B5	=(1/$B$3*(B10-C10))*$B$4
11	=A10+$B$4	=$B$6	=C10+D10	=(1/$B$3*(B11-C11))*$B$4
12	=A11+$B$4	=$B$6	=C11+D11	=(1/$B$3*(B12-C12))*$B$4
13	=A12+$B$4	=$B$6	=C12+D12	=(1/$B$3*(B13-C13))*$B$4
14	=A13+$B$4	=$B$6	=C13+D13	=(1/$B$3*(B14-C14))*$B$4
15	=A14+$B$4	=$B$6	=C14+D14	=(1/$B$3*(B15-C15))*$B$4
16	=A15+$B$4	=$B$6	=C15+D15	=(1/$B$3*(B16-C16))*$B$4
17	=A16+$B$4	=$B$6	=C16+D16	=(1/$B$3*(B17-C17))*$B$4
18	=A17+$B$4	=$B$6	=C17+D17	=(1/$B$3*(B18-C18))*$B$4
19	=A18+$B$4	=$B$6	=C18+D18	=(1/$B$3*(B19-C19))*$B$4
20	=A19+$B$4	=$B$6	=C19+D19	=(1/$B$3*(B20-C20))*$B$4
21	=A20+$B$4	=$B$6	=C20+D20	=(1/$B$3*(B21-C21))*$B$4
22	=A21+$B$4	=$B$6	=C21+D21	=(1/$B$3*(B22-C22))*$B$4
23	=A22+$B$4	=$B$6	=C22+D22	=(1/$B$3*(B23-C23))*$B$4
24	=A23+$B$4	=$B$6	=C23+D23	=(1/$B$3*(B24-C24))*$B$4
25	=A24+$B$4	=$B$6	=C24+D24	=(1/$B$3*(B25-C25))*$B$4
26	=A25+$B$4	=$B$6	=C25+D25	=(1/$B$3*(B26-C26))*$B$4
27	=A26+$B$4	=$B$6	=C26+D26	=(1/$B$3*(B27-C27))*$B$4
28	=A27+$B$4	=$B$6	=C27+D27	=(1/$B$3*(B28-C28))*$B$4
29	=A28+$B$4	=$B$6	=C28+D28	=(1/$B$3*(B29-C29))*$B$4
30	=A29+$B$4	=$B$6	=C29+D29	=(1/$B$3*(B30-C30))*$B$4
31	=A30+$B$4	=$B$6	=C30+D30	=(1/$B$3*(B31-C31))*$B$4
32	=A31+$B$4	=$B$6	=C31+D31	=(1/$B$3*(B32-C32))*$B$4
33	=A32+$B$4	=$B$6	=C32+D32	=(1/$B$3*(B33-C33))*$B$4
34	=A33+$B$4	=$B$6	=C33+D33	=(1/$B$3*(B34-C34))*$B$4
35	=A34+$B$4	=$B$6	=C34+D34	=(1/$B$3*(B35-C35))*$B$4
36	=A35+$B$4	=$B$6	=C35+D35	=(1/$B$3*(B36-C36))*$B$4
37	=A36+$B$4	=$B$6	=C36+D36	=(1/$B$3*(B37-C37))*$B$4
38	=A37+$B$4	=$B$6	=C37+D37	=(1/$B$3*(B38-C38))*$B$4
39	=A38+$B$4	=$B$6	=C38+D38	=(1/$B$3*(B39-C39))*$B$4
40	=A39+$B$4	=$B$6	=C39+D39	=(1/$B$3*(B40-C40))*$B$4
41	=A40+$B$4	=$B$6	=C40+D40	=(1/$B$3*(B41-C41))*$B$4
42	=A41+$B$4	=$B$6	=C41+D41	=(1/$B$3*(B42-C42))*$B$4
43	=A42+$B$4	=$B$6	=C42+D42	=(1/$B$3*(B43-C43))*$B$4
44	=A43+$B$4	=$B$6	=C43+D43	=(1/$B$3*(B44-C44))*$B$4
45	=A44+$B$4	=$B$6	=C44+D44	=(1/$B$3*(B45-C45))*$B$4
46	=A45+$B$4	=$B$6	=C45+D45	=(1/$B$3*(B46-C46))*$B$4

Table 4.3. First-order step response results.

	A	B	C	D
1	R	1		
2	C	1		
3	Tau	1		
4	del t	0.1		
5	(Vo)init	0		
6	Vi*	10		
7	Tend	5		
8				
9	t	Vi	Vo	del Vo
10	0	10	0.00	1.00
11	0.10	10	1.00	0.90
12	0.20	10	1.90	0.81
13	0.30	10	2.71	0.73
14	0.40	10	3.44	0.66
15	0.50	10	4.10	0.59
16	0.60	10	4.69	0.53
17	0.70	10	5.22	0.48
18	0.80	10	5.70	0.43
19	0.90	10	6.13	0.39
20	1.00	10	6.51	0.35
21	1.10	10	6.86	0.31
22	1.20	10	7.18	0.28
23	1.30	10	7.46	0.25
24	1.40	10	7.71	0.23
25	1.50	10	7.94	0.21
26	1.60	10	8.15	0.19
27	1.70	10	8.33	0.17
28	1.80	10	8.50	0.15
29	1.90	10	8.65	0.14
30	2.00	10	8.78	0.12
31	2.10	10	8.91	0.11
32	2.20	10	9.02	0.10
33	2.30	10	9.11	0.09
34	2.40	10	9.20	0.08
35	2.50	10	9.28	0.07
36	2.60	10	9.35	0.06
37	2.70	10	9.42	0.06
38	2.80	10	9.48	0.05
39	2.90	10	9.53	0.05
40	3.00	10	9.58	0.04
41	3.10	10	9.62	0.04
42	3.20	10	9.66	0.03
43	3.30	10	9.69	0.03
44	3.40	10	9.72	0.03
45	3.50	10	9.75	0.03
46	3.60	10	9.77	0.02

In 2 seconds, it's 8.78 volts or 87.8%, and in 3 seconds it reaches 9.58 volts, or 95.8% of the final value.

The response of the circuit shown in Figure 4.2 is called a step response. The input forcing function takes a step at time $t = 0$ from its value of 0 at $t < 0$ to a value of 10 volts. There are other types of input forcing functions and we will discuss these later. For now, let's keep the input function equal to a step and change its value to 5 volts. The results are shown in Figure 4.3. Note that the output voltage appears to have exactly the same shape as it did in Figure

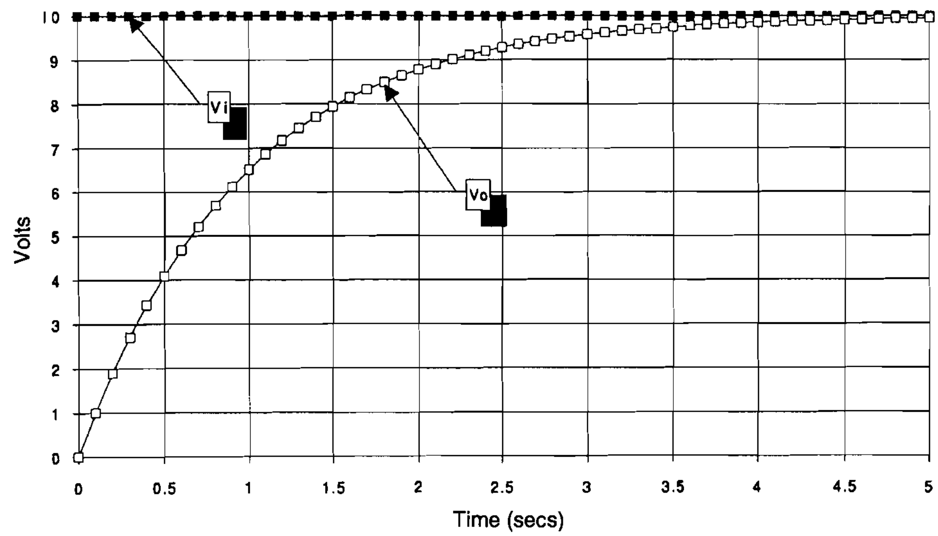


Figure 4.2. Numerical solution of equation (4.1) math model to a step change in input voltage (Vi = 10, R = 1, C = 1, Tau = RC = 1, delt = 0.1).

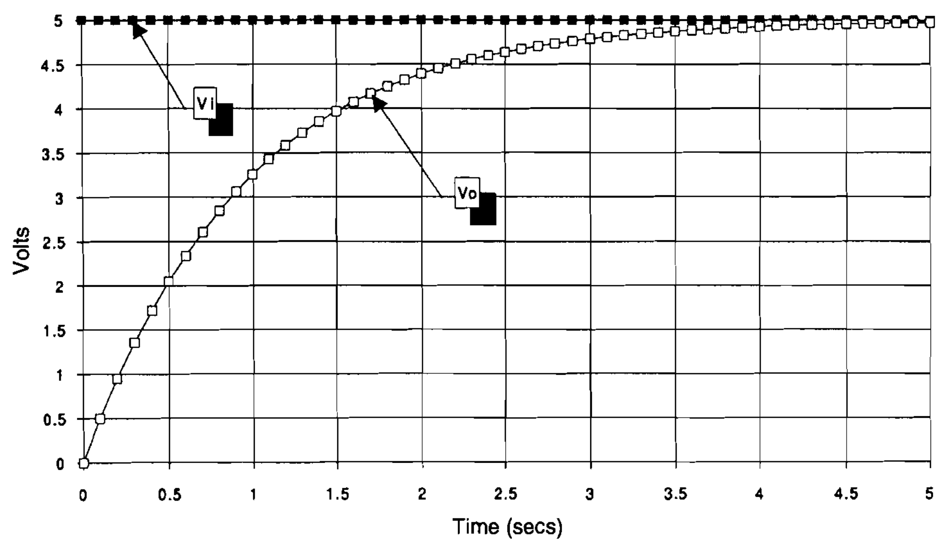


Figure 4.3. Numerical solution of equation (4.1) math model to a step change in input voltage (Vi = 5, R = 1, C = 1, Tau = RC = 1, delt = 0.1) .

4.2. Also note that at the end of 1 second the output voltage is approximately 3.25 volts, or 65.1% of the input voltage. This is the same percentage we found in Figure 4.2.

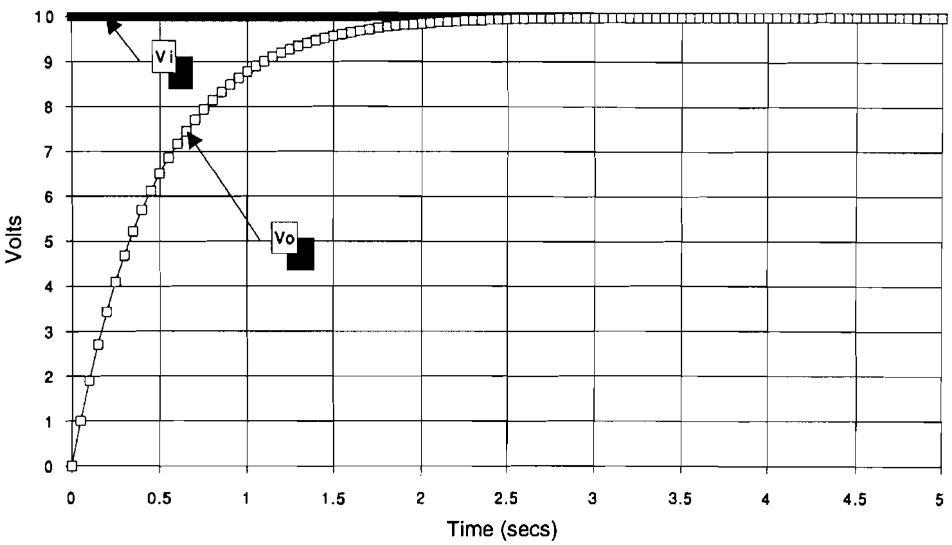


Figure 4.4. Numerical solution of equation (4.1) math model to a step change in input voltage (Vi = 10, R = 0.5, C = 1, Tau = RC = 0.5, delt = 0.05).

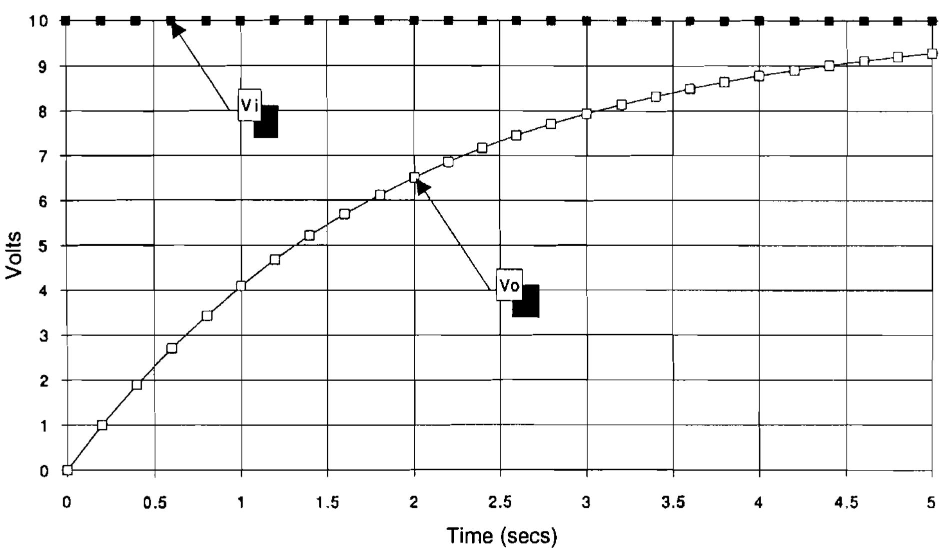


Figure 4.5. Numerical solution of equation (4.1) math model to a step change in input voltage (Vi = 10, R = 2, C = 1, Tau = RC = 2, delt = 0.2).

Now let's put the value of the step input voltage back to 10 volts and vary the values of *R* and C. Before we do this, look at equation (4.4). When the product of *R* and C is small, the initial rate of change of the output voltage will be large. Also, it is the *product* of *R* and C that affects this rate. Thus, if *R* were to decrease to 0.5 and C increase to 2, the product would remain equal to unity. Figures 4.4 and 4.5 show the voltage output when τ = 0.5 and 2.0, respectively. You can see that when τ is small, the output voltage reaches the input voltage level more rapidly. Note in both of these figures that when the output voltage reaches approximately 65% of the input, the time is equal to τ. That's because the time constant is a *characteristic* of first-order linear ordinary differential equations. It controls the form of the relationship between the input and the output. When we investigate exact solutions of first-order differential equations you will find the exact values given in Table 4.4 hold for the step responses.

Note that the results we obtained for these percentages are not quite equal to the exact values listed in Table 4.4. That's because the numerical solution is approximate and not exact. The accuracy of the solution is determined to a large extent by the size of the time step used in the solution. Table 4.5 and Figure 4.6 show the results previously shown in Table 4.3 and Figure 4.2, but with the time step changed to 0.05 seconds ($\tau/20$). You can see from the table of results that the output voltage is now equal to 6.42 volts, or 64.2% of the input voltage when t = 1.0 seconds. As Δt is made smaller, the solution approaches the exact value.

Table 4.4.
Step response times for first-order linear differential equations.

Time	Output as a percentage of input
1 τ	63.2
2 τ	86.5
3 τ	95.0
4 τ	98.2
5 τ	99.3
6 τ	99.8

Table 4.5.
First-order step response results (delt = 0.05 seconds).

	A	B	C	D
1	R	1		
2	C	1		
3	Tau	1		
4	del t	0.05		
5	(Vo)init	0		
6	Vi*	10		
7	Tend	5		
8				
9	t	Vi	Vo	del Vo
10	0	10	0.00	0.50
11	0.05	10	0.50	0.48
12	0.10	10	0.98	0.45
13	0.15	10	1.43	0.43
14	0.20	10	1.85	0.41
15	0.25	10	2.26	0.39
16	0.30	10	2.65	0.37
17	0.35	10	3.02	0.35
18	0.40	10	3.37	0.33
19	0.45	10	3.70	0.32
20	0.50	10	4.01	0.30
21	0.55	10	4.31	0.28
22	0.60	10	4.60	0.27
23	0.65	10	4.87	0.26
24	0.70	10	5.12	0.24
25	0.75	10	5.37	0.23
26	0.80	10	5.60	0.22
27	0.85	10	5.82	0.21
28	0.90	10	6.03	0.20
29	0.95	10	6.23	0.19
30	1.00	10	6.42	0.18
31	1.05	10	6.59	0.17
32	1.10	10	6.76	0.16
33	1.15	10	6.93	0.15
34	1.20	10	7.08	0.15
35	1.25	10	7.23	0.14
36	1.30	10	7.36	0.13
37	1.35	10	7.50	0.13
38	1.40	10	7.62	0.12
39	1.45	10	7.74	0.11
40	1.50	10	7.85	0.11
41	1.55	10	7.96	0.10
42	1.60	10	8.06	0.10
43	1.65	10	8.16	0.09
44	1.70	10	8.25	0.09
45	1.75	10	8.34	0.08
46	1.80	10	8.42	0.08

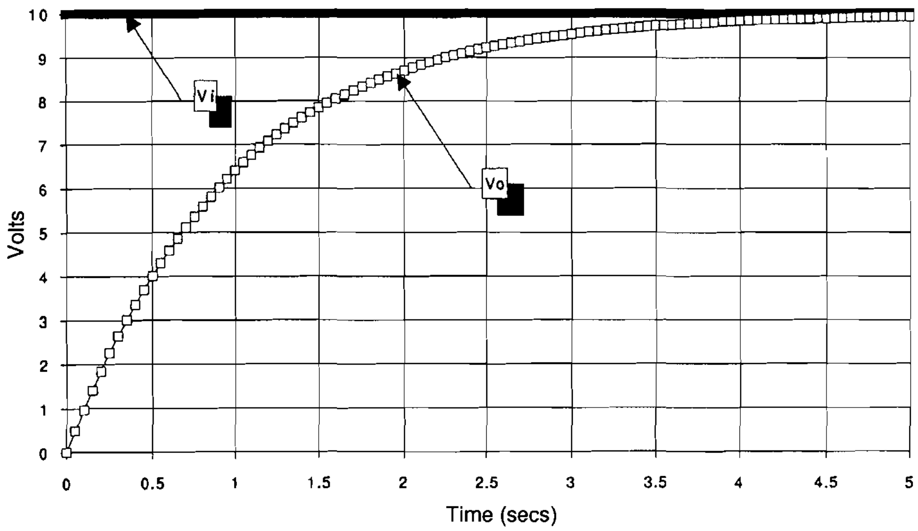


Figure 4.6. Numerical solution of equation (4.1) math model to a step change in input voltage (Vi = 10, R = 1, C = 1, Tau = RC = 1, delt = 0.05).

Exact Solution Method

You've seen how easy it is to solve differential equations approximately. Now I want to show you how to obtain the exact solution.

First write the math model given in equation (4.1) in the form

$$\tau \frac{dV_o}{dt} + V_o = V_{is} \qquad (4.5)$$

where V_{is} equals the value of the step input function as shown in Figure 4.7.

Since V_{is} is a constant, we can divide both sides of equation (4.5) by V_{is} and obtain the following *nondimensional* version of the equation

$$\tau \frac{d\left(\dfrac{V_o}{V_{is}}\right)}{dt} + \left(\frac{V_o}{V_{is}}\right) = 1 \qquad (4.6)$$

I hope this doesn't confuse you. All I have done is divide the output voltage by the *constant* input voltage. Since both have units of volts, (V_O / V_{is}) has units of volts/volt, which is dimensionless.

Note that the new dependent variable is now (V_O / V_{is}) instead of just V_O. This will make the solution of the equation easier. Also note that the step input is now a *unit step* input,

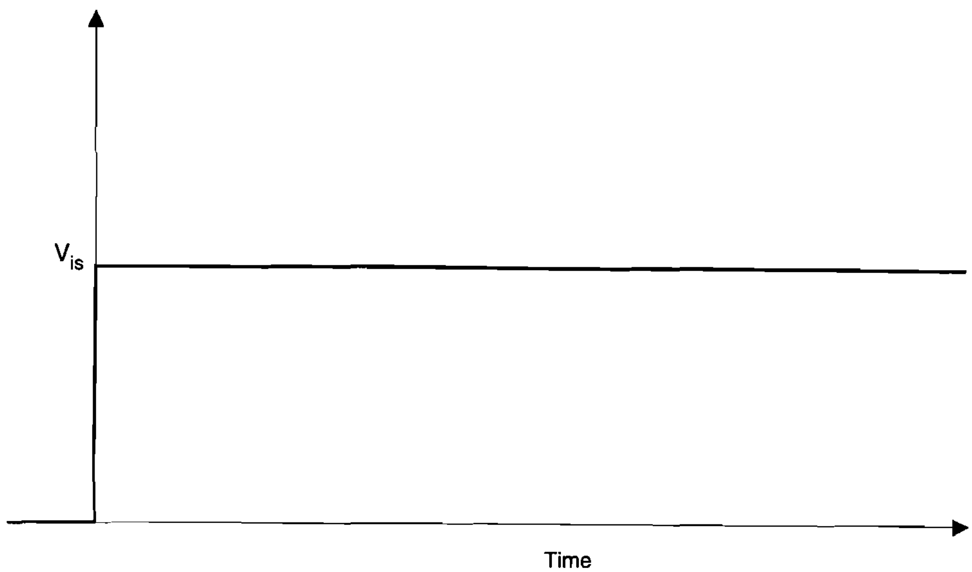


Figure 4.7. The step input function.

which is commonly used in linear system analysis. Let $(V_o / V_{is}) = V$ and rewrite (4.6) as

$$\tau\frac{dV}{dt} + V = 1 \qquad (4.7)$$

This is identical to (4.5) except the output variable is dimensionless and the input is now a unit step function.

It is possible to solve equation (4.7) by direct integration. The method of solution is called *separation of variables*. You arrange the equation so all variables involving V are on one side of the equation and all involving t are on the other. First write (4.7) as

$$\tau\frac{dV}{dt} = 1 - V \qquad (4.8)$$

Now divide both sides by $1-V$ and multiply both sides by dt/τ

$$\frac{dV}{1-V} = \frac{1}{\tau}dt \qquad (4.9)$$

The variables have been successfully separated since only terms containing V are on the left side and only terms containing t are on the right side. You can now integrate both sides of the equation

$$\int_0^V \frac{dV}{1-V} = \frac{1}{\tau}\int_0^t dt \qquad (4.10)$$

The integral on the right side of the equation is easy to solve, but you may have to look up the one on the left side in a table of integrals. The result is

$$-\ln(1-V) = \frac{1}{\tau}t$$

or

$$\ln(1-V) = -\frac{t}{\tau} \qquad (4.11)$$

Now take the antilog of both sides of (4.11)

$$\begin{aligned} e^{\ln(1-V)} &= e^{-t/\tau} \\ 1-V &= e^{-t/\tau} \\ V &= 1-e^{-t/\tau} \qquad (4.12) \end{aligned}$$

Equation (4.12) is the exact solution of equation (4.7). We can get it back to dimensional values if necessary simply by substituting (V_O / V_{is}) for V. That is

$$V = \frac{V_o}{V_{is}} = 1-e^{-t/\tau}$$

or

$$V_o = V_{is}\left(1-e^{-t/\tau}\right) \qquad (4.13)$$

This is the exact solution of equation (4.5).

One of the great advantages of an exact solution is that it provides you with more insight into the behavior of the system than the numerical solution. For example, both equations (4.12) and (4.13) make it very clear that the output voltage is a function only of (t / τ). Since τ has units of time, (t / τ) has units of seconds/second—that is, dimensionless time. Equation (4.12) is a completely general solution that is valid for all values of τ. Equation (4.13) makes it clear that the output voltage is directly proportional to the input voltage. *This is a distinguishing characteristic of all linear differential equations and is why they are called linear.* That is, the output is directly proportional to the input.

Another great advantage an exact solution has over a numerical solution is you can precisely calculate the response of the system for any value of a step input voltage at any point in time. Figure 4.8 shows a plot of equation (4.12) and Figure 4.9 shows a plot of equation (4.13) with τ = 1 and V_{is} = 10 volts. Figure 4.8 shows it all very concisely. *Regardless* of the values selected for τ and V_{is}, this graph shows the solution. (This is another advantage of nondimensionalizing, as we did in equation (4.6).) Figure 4.9, on the other hand, shows a particular solution for τ = 1 second and V_{is} = 10 volts.

Let's go back now and determine how accurate our numerical solution was as a function of the size of the time step. You will recall that we solved the equation with time steps equal to τ / 10 and τ / 20. Table 4.6 shows the exact solution versus the numerical solution using time steps equal to τ / 5, τ / 10, and τ / 20. You can see that acceptable accuracy is achieved using a time step equal to 1/10th of the time constant.

You might be wondering at this time why we bother with numerical solutions if exact solutions to our math models can be obtained. The answer is, if you can obtain an exact solution, do so. It is always easier and more insightful to work with exact solutions. Unfortunately, it's not always as easy as you saw here to find exact solutions to differential equations. Sometimes it takes more time to find a solution than it does to solve the equations numerically. Furthermore, numerically solving equations also works for arbitrary input functions and nonlinear differential equations. It is very difficult, and sometimes impossible, to find exact solutions to nonlinear differential equations. Of course you will have to experiment with the size of the time step when solving nonlinear differential equations to make sure your solution is sufficiently accurate for your purposes. That is the major drawback with using numerical methods. However, there are numerical methods that can automatically adjust the time step and there are better methods of numerically solving differential equations than the one I presented above. Two good references on this subject are provided at the end of this chapter.

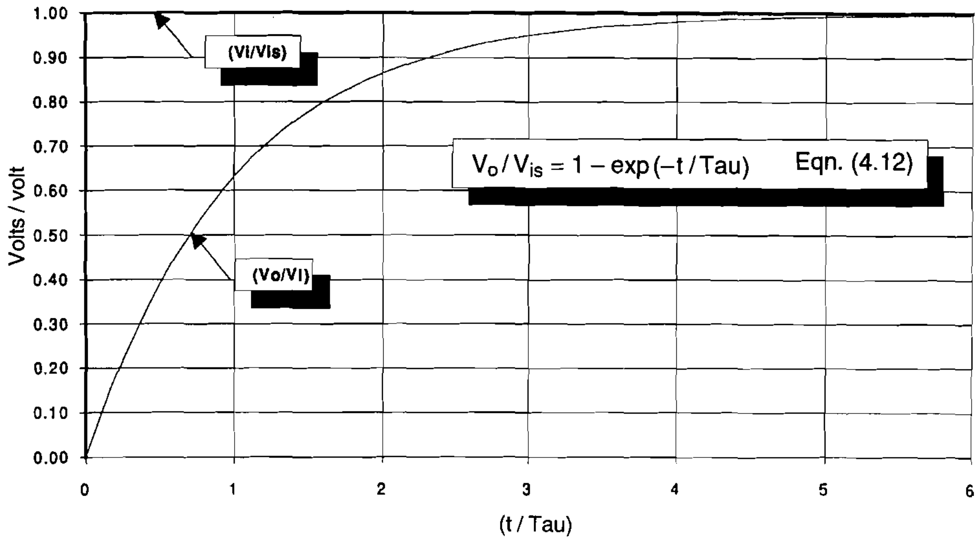


Figure 4.8. Exact dimensionless solution of equation (4.1) math model to a dimensionless step change in input voltage.

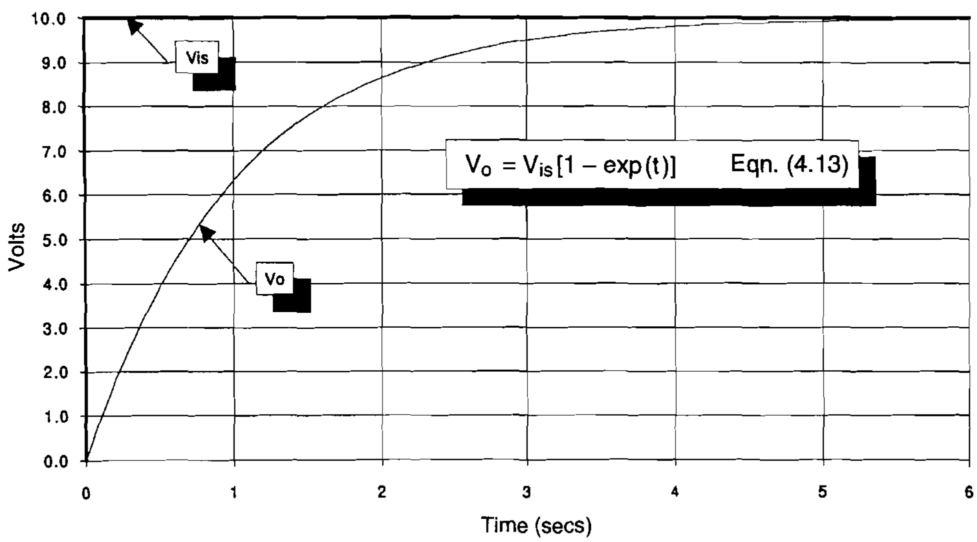


Figure 4.9. Exact solution of equation (4.1) math model to a step change in input voltage (Vi = 10 volts, Tau = 1 sec).

Take some time now and study Figure 4.10. Learn to recog-

Table 4.6.
Accuracy of numerical solution of a differential equation.

Time	Output Voltage (volts)			
	$\Delta t = \tau/5$	$\Delta t = \tau/10$	$\Delta t = \tau/20$	Exact
0.0τ	0.00	0.00	0.00	0.00
0.2τ	2.00	1.90	1.85	1.81
0.4τ	3.60	3.44	3.37	3.30
0.6t	4.88	4.69	4.60	4.51
0.8τ	5.90	5.70	5.60	5.51
1.0τ	6.72	6.51	6.42	6.32
1.2τ	7.38	7.18	7.08	6.99
1.4τ	7.90	7.71	7.62	7.53
1.6τ	8.32	8.15	8.06	7.98
1.8τ	8.66	8.50	8.42	8.35
2.0τ	8.93	8.78	8.71	8.65
2.2τ	9.14	9.02	8.95	8.89
2.4τ	9.31	9.20	9.15	9.09
2.6τ	9.45	9.35	9.31	9.26
2.8τ	9.56	9.48	9.43	9.39
3.0τ	9.65	9.58	9.54	9.50

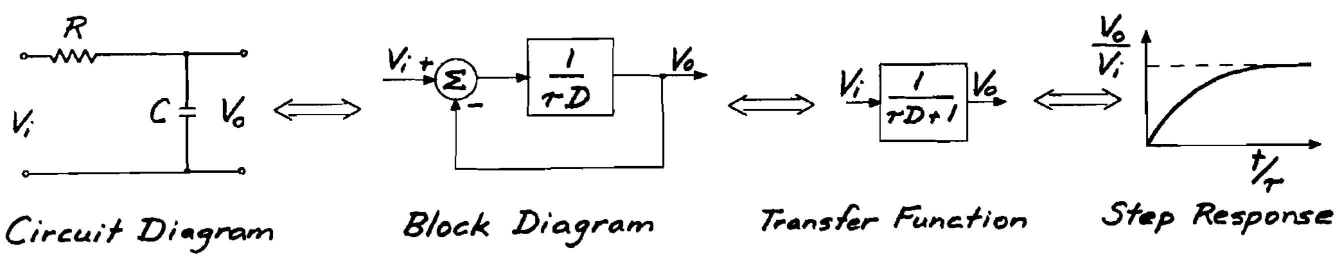


Figure 4.10.

nize first-order differential equations and visualize their symbolic representations and solutions to a step response. Remember that this important class of equations represents about 80% of the dynamics you will encounter in engineering system design and analysis.

4.3 Response to a Sinusoidal Input

You will now learn about so-called *frequency analysis* and *frequency response* of engineering systems. This basically means investigating the response of the system to sinusoidal inputs of different frequencies. This is an extremely important method of analysis, as poor frequency response can be a drawback in many types of engineering systems. For example, a telephone that attenuates high frequencies too much does not deliver intelligible speech. In an instrumentation system, if the sensor and amplifier don't have a good frequency response, false measurements may result. Frequency analysis is actually very easy to understand, but for some reason is often misunderstood.

We just investigated the response of our system to a step input. That is, in equation (4.1) we let $V_i(t)$ equal a constant V_{is} at $t = 0$. The system equation and its step response were found approximately and exactly.

Now you will investigate the response of the same system to a sinusoidal input. That is, the input to equation (4.1) will be of the form

$$V_i = V_{is}\sin(2\pi ft) \qquad (4.14)$$

This is a sinusoid with a maximum amplitude of V_{is} and a frequency of f cycles per second (hertz or Hz). Often you will see equation (4.14) expressed in terms of *circular frequency*, denoted by the symbol ω. That is,

$$V_i = V_{is}\sin(\omega t) \qquad (4.15)$$

where $\omega = 2\pi f$.

You will also sometimes see sinusoids expressed in terms of their *period*, denoted by the symbol T. That is,

$$V_i = V_{is}\sin\left(2\pi\frac{t}{T}\right) \qquad (4.16)$$

where $T = 1/f = 2\pi/\omega$.

Any of these forms are fine to use and Figure 4.11 shows all of these relationships.

Numerical Solution Method

We will first obtain the response of the system to the sinusoidal input by numerically integrating the differential equation. Solve the following equation

$$\tau\frac{\Delta V_o}{\Delta t} + V_o = V_{is}\sin 2\pi ft \qquad (4.17)$$

where $V_{is} = 10$ volts, $(V_o)_{init} = 0$, $\tau = 1$ second, and $f = 1$ Hz,

and use $\Delta t = \tau/20$.

I obtained the results shown in Figure 4.12 using the Excel spreadsheet given in Table 4.7. There are several important things to note in this figure. The response V_o appears to have a frequency identical to the input, but it is shifted in time so it lags behind the input voltage V_i. The maximum amplitude of the output is also less than the maximum amplitude of the input. Notice also that there is an initial start-up transient, during which the output voltage tries to catch up with the input voltage but never quite makes it. After a while, this transient appears to die away.

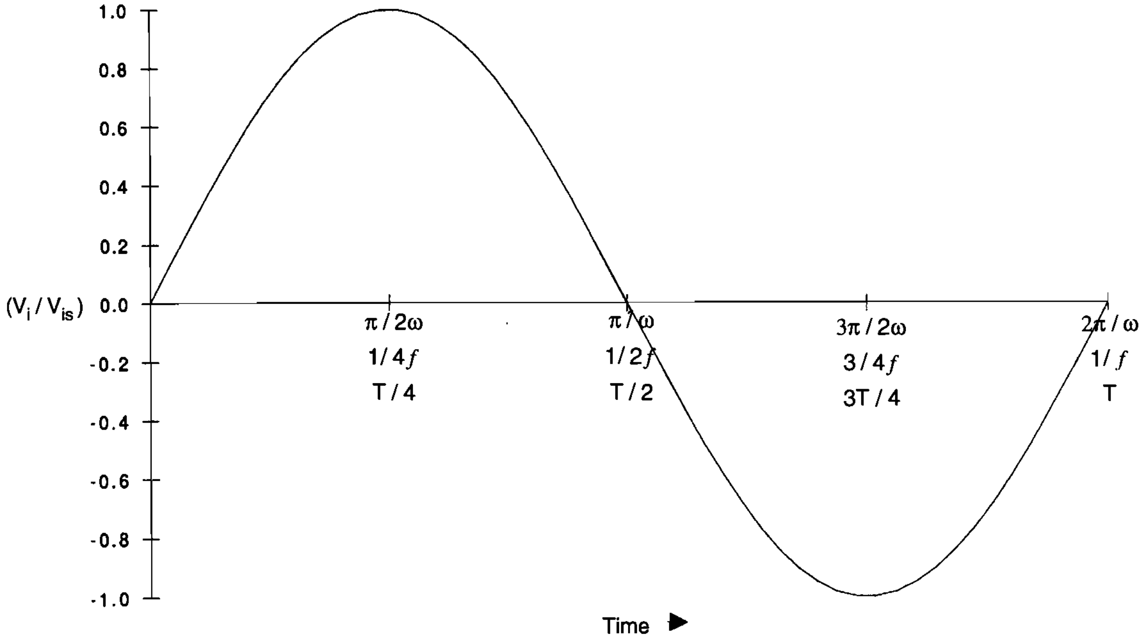


Figure 4.11. Various ways to express a sinusoid.

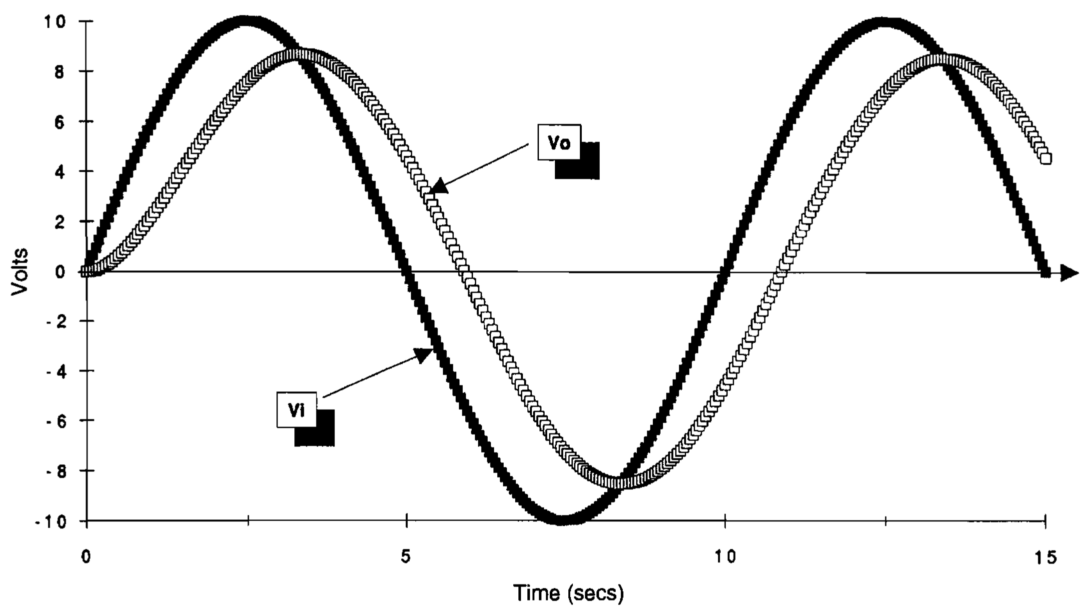


Figure 4.12. Numerical solution of equation (4.1) math model to a sinusoidal input voltage (f = 0.1, Tau = 1, delt = 0.05).

Table 4.7.
First-order sin response spreadsheet implementation.

	A	B	C	D
1	R	1		
2	C	1		
3	Tau	=B1*B2		
4	del t	=B3/20		
5	(Vo)init	0		
6	Vi*	10		
7	f	0.5		
8	Tend	=5*B3		
9				
10	t	Vi	Vo	del Vo
11	0	=$B$6*SIN(2*PI()*$B$7*A11)	=B5	=(1/$B$3*(B11-C11))*$B$4
12	=A11+$B$4	=$B$6*SIN(2*PI()*$B$7*A12)	=C11+D11	=(1/$B$3*(B12-C12))*$B$4
13	=A12+$B$4	=$B$6*SIN(2*PI()*$B$7*A13)	=C12+D12	=(1/$B$3*(B13-C13))*$B$4
14	=A13+$B$4	=$B$6*SIN(2*PI()*$B$7*A14)	=C13+D13	=(1/$B$3*(B14-C14))*$B$4
15	=A14+$B$4	=$B$6*SIN(2*PI()*$B$7*A15)	=C14+D14	=(1/$B$3*(B15-C15))*$B$4
16	=A15+$B$4	=$B$6*SIN(2*PI()*$B$7*A16)	=C15+D15	=(1/$B$3*(B16-C16))*$B$4
17	=A16+$B$4	=$B$6*SIN(2*PI()*$B$7*A17)	=C16+D16	=(1/$B$3*(B17-C17))*$B$4
18	=A17+$B$4	=$B$6*SIN(2*PI()*$B$7*A18)	=C17+D17	=(1/$B$3*(B18-C18))*$B$4
19	=A18+$B$4	=$B$6*SIN(2*PI()*$B$7*A19)	=C18+D18	=(1/$B$3*(B19-C19))*$B$4
20	=A19+$B$4	=$B$6*SIN(2*PI()*$B$7*A20)	=C19+D19	=(1/$B$3*(B20-C20))*$B$4
21	=A20+$B$4	=$B$6*SIN(2*PI()*$B$7*A21)	=C20+D20	=(1/$B$3*(B21-C21))*$B$4
22	=A21+$B$4	=$B$6*SIN(2*PI()*$B$7*A22)	=C21+D21	=(1/$B$3*(B22-C22))*$B$4
23	=A22+$B$4	=$B$6*SIN(2*PI()*$B$7*A23)	=C22+D22	=(1/$B$3*(B23-C23))*$B$4
24	=A23+$B$4	=$B$6*SIN(2*PI()*$B$7*A24)	=C23+D23	=(1/$B$3*(B24-C24))*$B$4
25	=A24+$B$4	=$B$6*SIN(2*PI()*$B$7*A25)	=C24+D24	=(1/$B$3*(B25-C25))*$B$4
26	=A25+$B$4	=$B$6*SIN(2*PI()*$B$7*A26)	=C25+D25	=(1/$B$3*(B26-C26))*$B$4
27	=A26+$B$4	=$B$6*SIN(2*PI()*$B$7*A27)	=C26+D26	=(1/$B$3*(B27-C27))*$B$4
28	=A27+$B$4	=$B$6*SIN(2*PI()*$B$7*A28)	=C27+D27	=(1/$B$3*(B28-C28))*$B$4
29	=A28+$B$4	=$B$6*SIN(2*PI()*$B$7*A29)	=C28+D28	=(1/$B$3*(B29-C29))*$B$4
30	=A29+$B$4	=$B$6*SIN(2*PI()*$B$7*A30)	=C29+D29	=(1/$B$3*(B30-C30))*$B$4
31	=A30+$B$4	=$B$6*SIN(2*PI()*$B$7*A31)	=C30+D30	=(1/$B$3*(B31-C31))*$B$4
32	=A31+$B$4	=$B$6*SIN(2*PI()*$B$7*A32)	=C31+D31	=(1/$B$3*(B32-C32))*$B$4
33	=A32+$B$4	=$B$6*SIN(2*PI()*$B$7*A33)	=C32+D32	=(1/$B$3*(B33-C33))*$B$4
34	=A33+$B$4	=$B$6*SIN(2*PI()*$B$7*A34)	=C33+D33	=(1/$B$3*(B34-C34))*$B$4
35	=A34+$B$4	=$B$6*SIN(2*PI()*$B$7*A35)	=C34+D34	=(1/$B$3*(B35-C35))*$B$4
36	=A35+$B$4	=$B$6*SIN(2*PI()*$B$7*A36)	=C35+D35	=(1/$B$3*(B36-C36))*$B$4
37	=A36+$B$4	=$B$6*SIN(2*PI()*$B$7*A37)	=C36+D36	=(1/$B$3*(B37-C37))*$B$4
38	=A37+$B$4	=$B$6*SIN(2*PI()*$B$7*A38)	=C37+D37	=(1/$B$3*(B38-C38))*$B$4
39	=A38+$B$4	=$B$6*SIN(2*PI()*$B$7*A39)	=C38+D38	=(1/$B$3*(B39-C39))*$B$4
40	=A39+$B$4	=$B$6*SIN(2*PI()*$B$7*A40)	=C39+D39	=(1/$B$3*(B40-C40))*$B$4
41	=A40+$B$4	=$B$6*SIN(2*PI()*$B$7*A41)	=C40+D40	=(1/$B$3*(B41-C41))*$B$4
42	=A41+$B$4	=$B$6*SIN(2*PI()*$B$7*A42)	=C41+D41	=(1/$B$3*(B42-C42))*$B$4
43	=A42+$B$4	=$B$6*SIN(2*PI()*$B$7*A43)	=C42+D42	=(1/$B$3*(B43-C43))*$B$4
44	=A43+$B$4	=$B$6*SIN(2*PI()*$B$7*A44)	=C43+D43	=(1/$B$3*(B44-C44))*$B$4
45	=A44+$B$4	=$B$6*SIN(2*PI()*$B$7*A45)	=C44+D44	=(1/$B$3*(B45-C45))*$B$4
46	=A45+$B$4	=$B$6*SIN(2*PI()*$B$7*A46)	=C45+D45	=(1/$B$3*(B46-C46))*$B$4

I repeated the numerical solution using the same input amplitude, initial condition, and time constant, while increasing the frequency of the input to 0.5, 1, and 2 Hz. The results are shown in Figures 4.13 through 4.15. You can see that the amplitude of the output signal decreased each time the frequency was increased and the input signal lagged further behind the input signal.

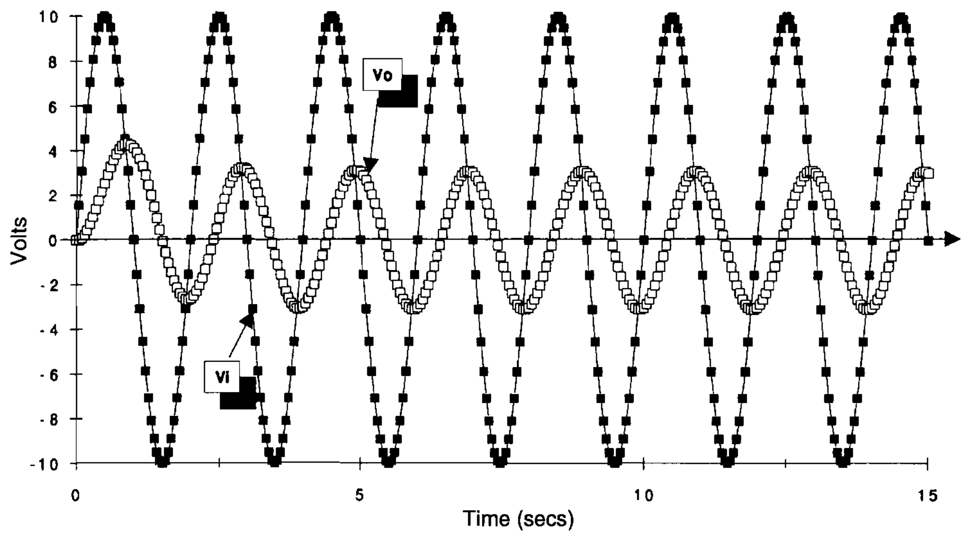


Figure 4.13. Numerical solution of equation (4.1) math model to a sinusoidal input voltage (f = 0.5, Tau = 1, delt = 0.05).

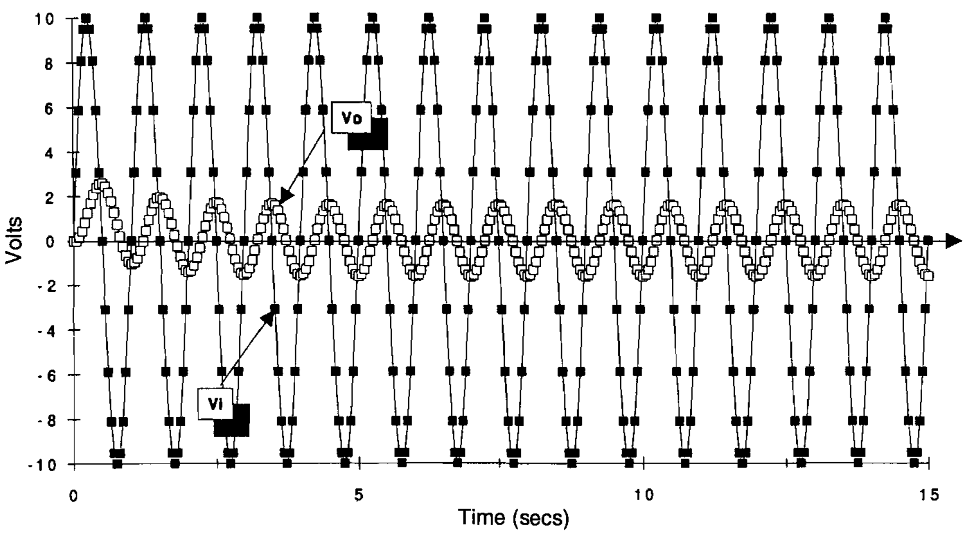


Figure 4.14. Numerical solution of equation (4.1) math model to a sinusoidal input voltage (f = 1, Tau = 1, delt = 0.05).

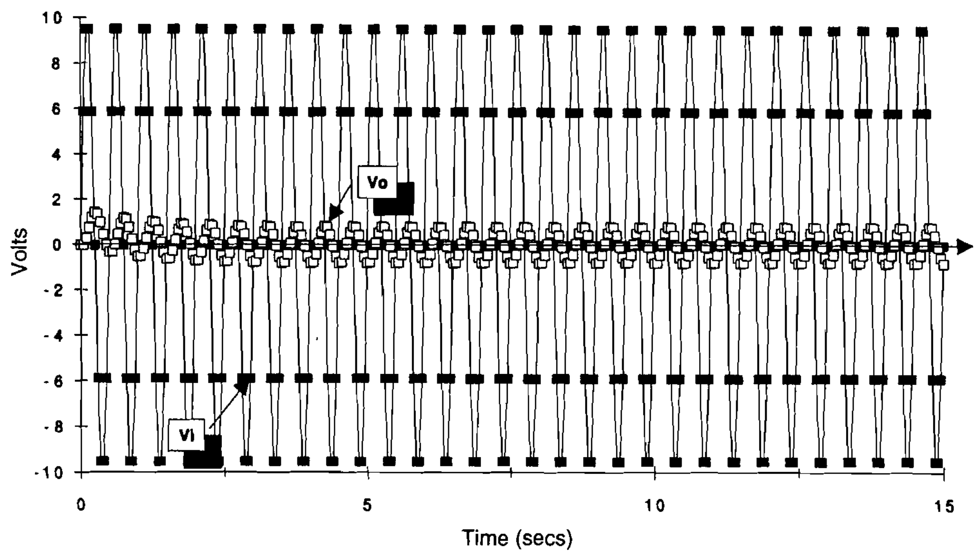


Figure 4.15. Numerical solution of equation (4.1) math model to a sinusoidal input voltage (f = 2, Tau = 1, delt = 0.05).

You can compute a so-called *signal attenuation factor* by dividing the maximum amplitude of the output signal by the maximum amplitude of the input signal. It is given the symbol A_T. If these attenuation factors are then plotted against the corresponding value of the input frequency, a graph similar to that shown in Figure 4.16 results. You can see that this system passes low-frequency input signals without too much attenuation, but it definitely attenuates the high-frequency signals. This system is often called a *first-order low-pass filter*. Many engineering systems other than electrical circuits have exactly the same type of frequency response characteristic shown in this figure. You can see this characteristic in a ship responding to waves, for example. Very high-frequency waves cause very little ship motion, but long-period (low-frequency) waves can cause a lot of motion.

You can also compute the amount the output signal leads or lags behind the input signal and plot this against the corresponding frequency. The amount the output signal leads or lags the input signal is generally shown in degrees and is called the *phase angle*. One complete cycle of a sinusoid is 360 degrees, so a lead is shown as 0 to +180 degrees and a lag as 0 to −180 degrees. In our system the output signal is lagging behind the input signal. This is called *phase lag*, and we can compute the angle by measuring the number of seconds the output signal lags behind the input, dividing this by the number of seconds for half of a cycle and multiplying by 180 degrees. Figure 4.17 shows the approximate results for this system.

The plots shown in Figure 4.16 and 4.17 are called the *frequency response of the system*. In several engineering fields Figure 4.16 is called the *response amplitude operator* (or RAO for short) and Figure 4.17 is called the *phase angle plot*. Regardless of their name, they show in a summary form: (1) the *ratio* of the output and input signals as a function of frequency or period of the input signal; and (2) the *phase shift* between the output and input signals as a function of frequency or period of the input signal. Many times you will see these plots shown using log scales. (I'll explain why in the next section.)

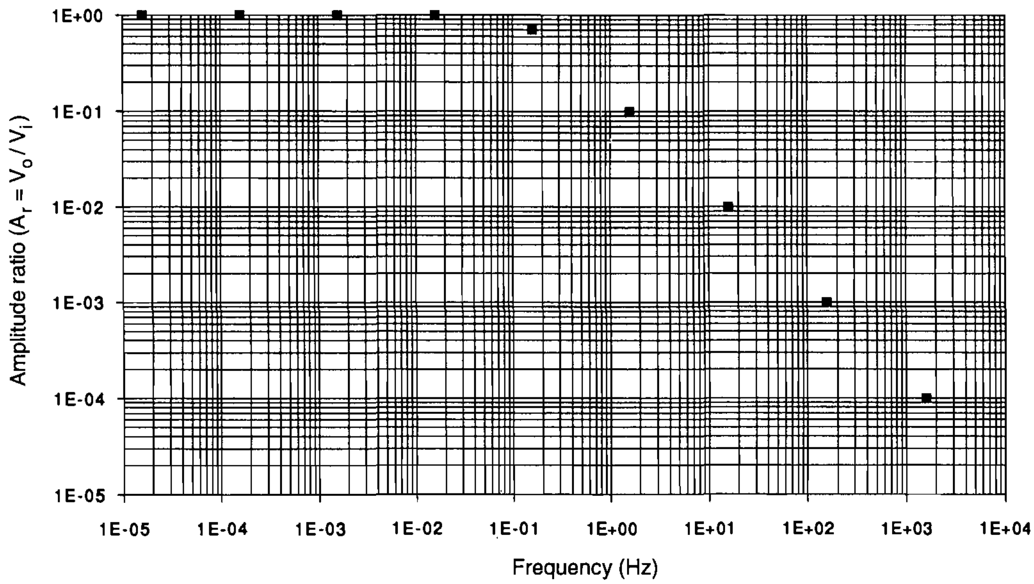


Figure 4.16. Estimated ratio of output and input voltages for equation (4.1) math model (Tau = 1).

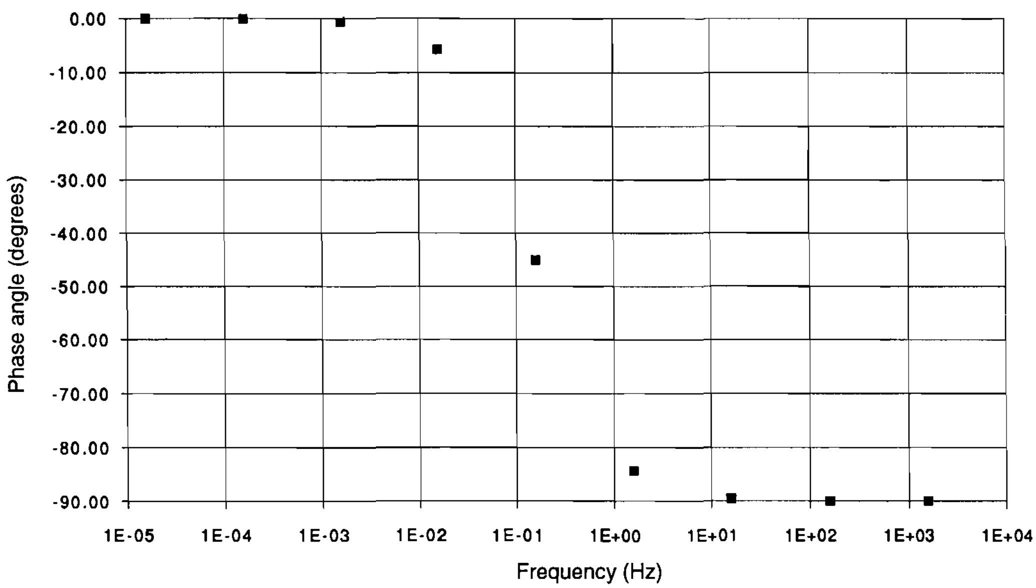


Figure 4.17. Estimated phase angle between output and input voltages for equation (4.1) math model (Tau = 1).

Exact Solution Method

I'll now explain how to determine the exact solution to a differential equation with a sinusoidal input. Before I do this, however, I'm going to digress in order to show you a more general way to solve differential equations. Study this method carefully, as we will be using it for the remainder of this book.

A characteristic of a linear ordinary differential equation of any order is its adherence to the *superposition principle*. That means if $G_1(t)$ and $G_2(t)$ are the outputs of a linear system in response to inputs $F_1(t)$ and $F_2(t)$, respectively, then an output corresponding to a *linear combination* of inputs such as $C_1F_1(t) + C_2F_2(t)$ [C_1 and C_2 are constants] is the linear combination $C_1G_1(t) + C_2G_2(t)$. This is really what is meant when the adjective *linear* is used to describe a first-order *linear* ordinary differential equation with constant coefficients. The superposition principle of linear systems is extremely important.

The solution of any linear differential equation is made up of two additive parts. One part is called the *homogeneous solution* and the other the *particular solution*. When the differential equation is arranged in its proper form with the response or output variable and its derivatives on the left side of the equal sign and the input variable and its derivatives on the right side, then the homogeneous solution is associated with the left side of the equation and the particular solution with the right. The left side of a differential equation is often called the *characteristic equation* and the right side the *forcing function*.

The *homogeneous solution* is found by setting the characteristic equation to zero. That is, we solve the differential equation as if there were no input. This may at first seem odd but, as you will soon discover, it's easy to find this solution and the solution is a fundamental characteristic of the differential equation and the system it represents.

For example, in our system presently under study, the characteristic equation is

$$\tau\frac{dV_o}{dt} + V_o$$

We set this to zero in order to obtain the homogeneous solution. That is

$$\tau\frac{dV_o}{dt} + V_o = 0 \qquad (4.18)$$

The homogeneous solution always has at least one solution of the form

$$V_o = Ae^{rt} \qquad (4.19)$$

where A and r are unknown values. We can differentiate this solution once to get dV_0 / dt. Then we can substitute the assumed solution for V_0 into (4.18) giving

$$\tau Are^{rt} + Ae^{rt} = 0 \qquad (4.20)$$

Dividing both sides by Ae^{rt} results in

$$\tau r + 1 = 0 \qquad (4.21)$$

This equation must hold for all time if (4.19) is to be a solution to (4.18) for all time. That is, (4.21) requires that

$$r = -\frac{1}{\tau} \tag{4.22}$$

and gives the homogeneous solution as

$$V_{oH} = Ae^{-t/\tau} \tag{4.23}$$

The constant A is still unknown and must be determined from the initial conditions after the particular solution is found, but (4.23) is a solution of (4.18).

Several methods are available for finding the *particular solution* to linear ordinary differential equations with constant coefficients. We'll use the so-called *method of undetermined coefficients.* This method involves finding a function that is similar in appearance to the input or forcing function, but which contains undetermined coefficients. The undetermined coefficients are found by substituting the function into the differential equation and determining the coefficients which allow both sides of the differential equation to remain equal for all time.

For example, in the case of a step input function we can use an undetermined coefficient that is a constant *B*. That is,

$$V_{oP} = B \tag{4.24}$$

Then we substitute this particular solution into our system math model and get

$$\tau\frac{dB}{dt} + B = V_{is} \tag{4.25}$$

Since the derivative of a constant is zero, dB / dt is zero and we are left with

$$B = V_{is}$$

which holds for all time. Thus, the particular solution is

$$V_{oP} = V_{is} \tag{4.26}$$

The complete solution is obtained by adding the homogeneous and particular solutions. That is,

$$V_o = V_{oH} + V_{oP}$$

or

$$V_o = Ae^{-t/\tau} + V_{is} \tag{4.27}$$

We can now determine the value for the coefficient A using the initial condition

$$V_o = 0 \text{ at } t = 0 \tag{4.28}$$

Substituting (4.28) into (4.27) gives

$$0 = Ae^{-0/\tau} + V_{is} = A + V_{is}$$

or

$$A = -V_{is}$$

The final solution then is equal to

$$V_o = V_{is} - V_{is}e^{-t/\tau}$$

or

$$V_o = V_{is}\left(1 - e^{-t/\tau}\right) \qquad (4.29)$$

which is the same as we obtained previously using the method of separation of variables.

The method of undetermined coefficients for finding a particular solution can be generalized as follows. The forcing function $F(t)$ will always have the general form

$$F(t) = e^{at}\cos bt\left(p_m t^m + p_{m-1}t^{m-1} + \cdots + p_0\right) + e^{at}\sin bt\left(q_m t^m + q_{m-1}t^{m-1} + \cdots + q_0\right)$$

where some of the constants a, b, p_0, p_m, ... q_0, q_m, ... q_0 may be zero. Then a particular solution to the differential equation of form similar to that of $F(t)$ is

$$y_p(t) = e^{at}\cos bt\left(k_m t^m + k_{m-1}t^{m-1} + \cdots + k_0\right) + e^{at}\sin bt\left(l_m t^m + l_{m-1}t^{m-1} + \cdots + l_0\right)$$

The coefficients k_m, k_{m-1}, ... k_0, l_m, l_{m-1}, ... l_0 are determined by substituting $y_p(t)$ into the differential equation and choosing the coefficients so the equation remains equal on both sides for all time.

For example, let's find the particular solution for the forcing function

$$V_i = V_{is}\sin 2\pi ft$$

We will use as the general solution

$$V_{oP} = k_o\cos 2\pi ft + l_o\sin 2\pi ft \qquad (4.30)$$

where k_o and l_o must be determined. Substituting (4.30) into our system math model

$$\tau\frac{dV_o}{dt} + V_o = V_{is}\sin 2\pi ft$$

gives

$$\tau\frac{d}{dt}\left(k_o\cos 2\pi ft + l_o\sin 2\pi ft\right) + \left(k_o\cos 2\pi f + l_o\sin 2\pi ft\right) = V_{is}\sin 2\pi ft$$

Carrying out the differentiation and collecting like terms gives

$$\left(\tau 2\pi fk_o + l_o\right)\cos 2\pi ft + \left(k_o - \tau 2\pi fl_o\right)\sin 2\pi ft = V_{is}\sin 2\pi ft \qquad (4.31)$$

We determine the coefficients k_o and l_o so (4.31) is satisfied for all time. For this to happen, the coefficient for the cosine term must be zero and the coefficient for the sine term must equal V_{is}. That means the following must be true for all time

$$\tau 2\pi fk_o + l_o = 0 \qquad (4.32)$$

and

$$k_o - \tau 2\pi fl_o = V_{is} \qquad (4.33)$$

Equations (4.32) and (4.33) are two equations in two unknowns, k_o and l_o. The simultaneous solution of these algebraic equations gives

$$k_o = \frac{V_{is}}{1+(\tau 2\pi f)^2}$$

$$l_o = -\frac{\tau 2\pi f V_{is}}{1+(\tau 2\pi f)^2}$$

Substituting these constants into (4.30) gives the particular solution as

$$V_{oP} = \frac{V_{is}}{1+(\tau 2\pi f)^2}\sin 2\pi ft - \frac{\tau 2\pi f V_{is}}{1+(\tau 2\pi f)^2}\cos 2\pi ft \qquad (4.34)$$

Now add the homogeneous solution given by equation (4.23) and the particular solution given by equation (4.34) to get the complete solution

$$V_o = Ae^{-t/\tau} + \frac{V_{is}}{1+(\tau 2\pi f)^2}\sin 2\pi ft - \frac{\tau 2\pi f V_{is}}{1+(\tau 2\pi f)^2}\cos 2\pi ft \qquad (4.35)$$

Obtain A using the initial condition (V_o = 0 at t = 0). That is,

$$V_o = 0 = A - \frac{\tau 2\pi f V_{is}}{1+(\tau 2\pi f)^2}$$

or

$$A = \frac{\tau 2\pi f}{1+(\tau 2\pi f)^2}V_{is}$$

The exact solution then is

$$\frac{V_o}{V_{is}} = \underbrace{\frac{\tau 2\pi f}{1+(\tau 2\pi f)^2}e^{-t/\tau}}_{\text{transient}} + \underbrace{\frac{1}{1+(\tau 2\pi f)^2}\sin 2\pi ft - \frac{\tau 2\pi f}{1+(\tau 2\pi f)^2}\cos 2\pi ft}_{\text{nontransient}} \qquad (4.36)$$

Note the first term in this exact solution is the transient part noticed when we solved the equation numerically. After around t = 3τ seconds, this part of the solution approaches zero because of the exponential term. We are then left with the nontransient part of the solution which lasts for all time.

The *nontransient* part of the solution can also be placed in the following simpler form

$$\frac{V_o}{V_{is}} = C\sin(2\pi ft + \varphi) \qquad (4.37)$$

where C is the amplitude attenuation factor
and φ is the phase lag.

To do this, all we need are a few trigonometric identities and algebra.

Recall the trigonometric identity

$$C\sin(\varphi + \theta) = C\cos\varphi\sin\theta + C\sin\varphi\cos\theta \qquad (4.38)$$

Compare (4.38) with the nontransient part of (4.36). You can see they are identical if we let

$$\theta = 2\pi ft \quad (4.39)$$

$$C\cos\varphi = \frac{1}{1+(\tau 2\pi f)^2} \quad (4.40)$$

$$C\sin\varphi = -\frac{(\tau 2\pi f)}{1+(\tau 2\pi f)^2} \quad (4.41)$$

The angle φ is the phase lag angle. We can obtain an equation for it by dividing (4.41) by (4.40). That is,

$$\frac{C\sin\varphi}{C\cos\varphi} = \tan\varphi = -(\tau 2\pi f)$$

or

$$\varphi = \tan^{-1}(-\tau 2\pi f) \quad (4.42)$$

You can obtain C using another trigonometric identity

$$\sin^2\varphi + \cos^2\varphi = 1 \quad (4.43)$$

Substituting (4.40) and (4.41) into (4.43) and solving for C gives

$$C = \frac{1}{\sqrt{1+(\tau 2\pi f)^2}} \quad (4.44)$$

Now we rewrite (4.37) using (4.42) and (4.44)

$$V_o = \frac{V_{is}}{\sqrt{1+(\tau 2\pi f)^2}}\sin[2\pi ft + \tan^{-1}(-\tau 2\pi f)] \quad (4.45)$$

Equation (4.45) is the exact solution to our system math model given in (4.1) for a sinusoidal input whose maximum amplitude is V_{is} and whose frequency is f, after the initial transient has died out. The equation shows that the maximum amplitude of the output voltage is dependent on the frequency, as well as the amplitude, of the input signal. The equation also shows that the phase lag between the input and output signals is dependent on the frequency of the input signal. Also note that the time constant, τ, plays a key role in determining the frequency response. The time constant appears in the equation as a multiplier of frequency. This makes it behave as a scaling factor for the frequency.

We can use (4.45) to calculate the ratio of the maximum output amplitude to the maximum input amplitude, as well as the phase shift between the two signals, as a function of frequency. Table 4.8 shows the results. The frequency is listed in terms of the time constant and is increased each entry by a multiple of 10 (an *order of magnitude* increase) so that a broad range of frequencies is covered. If t = 1 second as we have been using, the frequency range in the table goes from 0.0000159 Hz to 159,000 Hz. For some unknown reason it is still customary, particularly when presenting frequency response data of electrical components, to show the amplitude ratio using an acoustics unit called decibels. A decibel, abbreviated dB, is simply 20 times the base 10 log of the amplitude ratio; that is,

$$dB = 20\log\left|\left(\frac{V_o}{V_{is}}\right)\right|$$

If you have a gut feel for decibels, then use them. Otherwise, just use the base 10 log of the