PURE AND APPLIED MATHEMATICS
A SERIES OF MONOGRAPHS AND TEXTBOOKS

Frontiers in Interpolation and Approximation

Dedicated to the memory of Ambikeshwar Sharma

N. K. Govil
H. N. Mhaskar
Ram N. Mohapatra
Zuhair Nashed
J. Szabados

Chapman & Hall/CRC
Taylor & Francis Group

Frontiers in Interpolation and Approximation

Dedicated to the memory of Ambikeshwar Sharma

MONOGRAPHS AND TEXTBOOKS IN PURE AND APPLIED MATHEMATICS

Recent Titles

G. S. Ladde and M. Sambandham, Stochastic versus Deterministic Systems of Differential Equations (2004)

B. J. Gardner and R. Wiegandt, Radical Theory of Rings (2004)

J. Haluska, The Mathematical Theory of Tone Systems (2004)

C. Menini and F. Van Oystaeyen, Abstract Algebra: A Comprehensive Treatment (2004)

E. Hansen and G. W. Walster, Global Optimization Using Interval Analysis, Second Edition, Revised and Expanded (2004)

M. M. Rao, Measure Theory and Integration, Second Edition, Revised and Expanded (2004)

W. J. Wickless, A First Graduate Course in Abstract Algebra (2004)

R. P. Agarwal, M. Bohner, and W-T Li, Nonoscillation and Oscillation Theory for Functional Differential Equations (2004)

J. Galambos and I. Simonelli, Products of Random Variables: Applications to Problems of Physics and to Arithmetical Functions (2004)

Walter Ferrer and Alvaro Rittatore, Actions and Invariants of Algebraic Groups (2005)

Christof Eck, Jiri Jarusek, and Miroslav Krbec, Unilateral Contact Problems: Variational Methods and Existence Theorems (2005)

M. M. Rao, Conditional Measures and Applications, Second Edition (2005)

A. B. Kharazishvili, Strange Functions in Real Analysis, Second Edition (2006)

Vincenzo Ancona and Bernard Gaveau, Differential Forms on Singular Varieties: De Rham and Hodge Theory Simplified (2005)

Santiago Alves Tavares, Generation of Multivariate Hermite Interpolating Polynomials (2005)

Sergio Macías, Topics on Continua (2005)

Mircea Sofonea, Weimin Han, and Meir Shillor, Analysis and Approximation of Contact Problems with Adhesion or Damage (2006)

Marwan Moubachir and Jean-Paul Zolésio, Moving Shape Analysis and Control: Applications to Fluid Structure Interactions (2006)

Alfred Geroldinger and Franz Halter-Koch, Non-Unique Factorizations: Algebraic, Combinatorial and Analytic Theory (2006)

Kevin J. Hastings, Introduction to the Mathematics of Operations Research with *Mathematica*®, Second Edition (2006)

Robert Carlson, A Concrete Introduction to Real Analysis (2006)

John Dauns and Yiqiang Zhou, Classes of Modules (2006)

N. K. Govil, H. N. Mhaskar, Ram N. Mohapatra, Zuhair Nashed, and J. Szabados, Frontiers in Interpolation and Approximation (2006)

Frontiers in Interpolation and Approximation

Dedicated to the memory of Ambikeshwar Sharma

N. K. Govil
Auburn Univesity
Alabama, U.S.A.

H. N. Mhaskar
California State University
Los Angeles, U.S.A.

Ram N. Mohapatra
University of Central Florida
Orlando, U.S.A.

Zuhair Nashed
University of Central Florida
Orlando, U.S.A.

J. Szabados
Alfréd Rényi Institute of Mathematics
Budapest, Hungary

Chapman & Hall/CRC
Taylor & Francis Group
Boca Raton London New York

Chapman & Hall/CRC is an imprint of the
Taylor & Francis Group, an informa business

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-10: 1-58488-636-6 (Hardcover)
International Standard Book Number-13: 978-1-58488-636-5 (Hardcover)

Library of Congress Cataloging-in-Publication Data

Frontiers in interpolation and approximation / [edited by] N.K. Govil ... [et al.].
p. cm. -- (Monographs and textbooks in pure and applied mathematics ; 282)
Includes bibliographical references and index.
ISBN-13: 978-1-58488-636-5 (acid-free paper)
ISBN-10: 1-58488-636-6 (acid-free paper)
1. Interpolation. 2. Approximation theory. I. Govil, N. K. (Narendra Kumar) II. Series.

QA281.F76 2007
511'.42--dc22 2006013708

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Contents

Foreword

I am indeed honored to write this foreword for this present volume which is dedicated to Ambikeshwar Sharma, who was well known in the world as a beloved teacher and research mathematician. He was my dear and trusted friend and colleague for many, many years; we first met, some fifty years ago, in the office of Professor Joseph L. Walsh, my adviser at Harvard University. Little did I know how much our lives would intertwine over the years.

Ambikeshwar had two well-recognized strengths which were absolutely infectious to those near him. He loved mathematics and he loved mathematical research with others, both with great vigor. I can vividly recall working with him on some math paper (we wrote 13 papers together) when, at night, I would be exhausted, and he always was ready to continue onward! (I often wondered how much of this came from his excellent and strict vegetarian diet.) His research work was basically in complex function theory and approximation theory, and he is probably best known for his work on splines, interpolation theory, and Walsh over-convergence. On this last topic, Walsh over-convergence, it was his dream to write the first and definitive book on this topic, which would examine in detail all aspects of this theory. Unfortunately, his health gave out before this book was finished, and it was left to his circle of friends in mathematics to complete this task.

This book, which follows, is dedicated to Ambikeshwar Sharma, who will long be remembered for his mathematics, for his enthusiasm for mathematical research, and for the overwhelming kindness and understanding he showered on all who came in contact with him.

Richard S. Varga
Kent State University

Preface

This monograph is a collection of papers in memory of Professor Ambikeshwar Sharma who passed away on December 22, 2003 at his home in Edmonton, Alberta, Canada. Professor Sharma was a leading mathematician whose research has spanned several areas of approximation theory and classical analysis, including interpolation theory and approximation by spline functions. Interpolation was a topic in which Professor Sharma was viewed as a world expert by his collaborators and many other colleagues.

We invited outstanding mathematicians, friends and collaborators of Professor Sharma to submit papers to be included in this volume. This collection contains original research articles and comprehensive survey papers by 30 mathematicians from 11 countries. All the papers were refereed. We hope that the papers will be of interest both to graduate students as well as researchers in analysis and approximation theory.

The paper of Babenko and Kroó deals with Markov inequalities for multivariate polynomials. These inequalities estimate the supremum norm of the derivatives of a polynomial in terms of the norm of the polynomial itself. Babenko and Kroó establish such inequalities for homogeneous polynomials on a nonsymmetric convex body in a Euclidean space, possibly with cusps.

The paper by Brudnyi and Brudnyi studies the analogues of Chebyshev and Bernstein inequalities for multivariate polynomials. These inequalities estimate the norm of a polynomial on a set in a Euclidean space in terms of its norm on a subset of this set.

The paper of Cavaretta and Fontes–Merz gives explicit formulas in some cases for the norm of the operator $L_{n-1}(\,\cdot\,;\zeta) : H^\infty(D) \to C$, where $L_{n-1}(\,\cdot\,;\zeta)$ represents the Lagrange interpolation polynomial of degree $n-1$, evaluated at a complex number ζ, and defined by interpolating functions in $H^\infty(D)$ at the zeros of $z^n - r^n$. Here, $0 < r < 1$ and $|\zeta| > 1$.

The paper of de Bruin is a survey of his joint work with Sharma on interpolation, covering the period 1993–2003.

The paper of Deo and Maitra studies the conditions under which a module of smooth splines on a subdivision of a simplex embedded in a Euclidean space is free.

The paper of Ditzian gives a survey of various measures of smoothness of functions which are defined on the unit sphere $S^{d-1} \subset R^d$.

The paper of Dryanov presents results on existence, uniqueness, and explicit construction of quadrature formulae with maximal trigonometric degree of precision.

The paper of Erdélyi surveys recent results for exponential sums and linear combinations of shifted Gaussians which were obtained via interpolation. In particular, a Chebyshev type inequality and a reverse Markov inequality is obtained in this setting.

The paper of Goodman and Lee investigates the optimality of the uncertainty products for certain approximations to the Gaussian, and the corresponding wavelets, when the refinement masks are polynomials satisfying certain conditions on the locations of their zeros.

The paper of Govil, Qazi, and Rahman deals with some basic facts about interpolation by classes of entire functions like algebraic polynomials, trigonometric polynomials, and non-periodic transcendental entire functions. The authors also explain what Hermite "really did" in his frequently quoted paper.

The paper of Hesse and Sloan describes several known results as well as proves some new ones regarding the degree of approximation by hyperinterpolation operators on the Euclidean sphere. The hyperinterpolation operator is a discretization of the Fourier projection operator onto the space of spherical polynomials, obtained by using a positive quadrature formula, exact for spherical polynomials of an appropriate degree.

The paper of Jakimovski studies the connection between Lagrange and Hermite interpolatory polynomials, interpolating at a set of roots of unity, and the corresponding polynomials interpolating at different subsets of this set.

The paper of Keiner and Prestin presents a fast algorithm for scattered data interpolation and approximation on the Euclidean sphere with spherical radial basis functions of different spatial density.

The paper of Mastroianni and Szabados establishes the analogues of certain classical polynomial inequalities, as well as direct and converse approximation theorems in the context of weighted approximation on the whole line with respect to a generalized Freud weight.

The paper of Mastroianni and Vértesi investigates the truncated Fourier sums and Lagrange interpolation operators in weighted L^p spaces on unbounded intervals $(0, \infty)$ and the whole line.

The paper of Mhaskar proposes alternatives to interpolation for approximation of functions using values of the function at scattered sites on the circle, the real line, the unit interval, and the unit sphere. In particular, it proves the existence of bounded operators, yielding entire functions of finite exponential type, that interpolate a Birkhoff data for a function on a Euclidean space, where a finite number of derivatives, of order not exceeding a fixed number, are prescribed at each point.

The paper of Pai and Indira establishes the equivalence of Hausdorff continuity and pointwise Hausdorff Lipschitz continuity of a restricted center multifunction.

The paper of Schmeisser describes methods to obtain estimates on the zeros of polynomials, in terms of their coefficients in an orthogonal polynomial expansion. In particular, certain L^2 inequalities and lower bounds for Vandermonde type determinants of orthogonal polynomials are proved.

The paper of Shekhtman defines a generalization of Chebyshev spaces, "ideal complements," and demonstrates their uniqueness. Various analogues of Chebyshev spaces (minimal interpolating systems) in several variables are also discussed.

It is a pleasure to express our gratitude to all the authors and referees without whose contributions this volume would not have been possible. We would like to thank Richard Varga for accepting our invitation to write the Foreword, Charles Chui for his encouragement, Darrel Hankerson for his help with TEX issues, Gerhard Schmeisser for modifying our style file, Larry Schumaker for allowing us to use his micros, and Huajun Huang for his help in compiling and formatting some of the papers in this volume. Finally, our thanks are due to the publisher for support and careful handling of the monograph.

N. K. Govil
H. N. Mhaskar
R. N. Mohapatra
Z. Nashed
J. Szabados

EDITORS

N. K. Govil is a professor of mathematics at Auburn University, Auburn, Alabama. He received his Ph.D. degree in mathematics from the University of Montreal, Canada in 1968. Prior to joining Auburn University, he was professor at the Indian Institute of Technology, New Delhi, India.

H. N. Mhaskar is a professor of mathematics at California State University, Los Angeles. He obtained his Ph.D. from Ohio State University, Columbus in 1980.

R. N. Mohapatra is a professor of mathematics at the University of Central Florida, Orlando. He received his Ph.D. degree from University of Jabalpur, India in 1968.

Zuhair Nashed received his S.B. and S.M. degrees in electrical engineering from Massachusetts Institute of Technology, Cambridge, and his Ph.D. in mathematics from the University of Michigan. Prior to joining the University of Central Florida, he was a professor at Georgia Institute of Technology, Atlanta and the University of Delaware, Newark.

J. Szabados is a senior research fellow at the Department of Analysis of the Alfréd Rényi Institute of Mathematics and a professor at the Department of Mathematics and Its Applications of the Central European University, both in Budapest. He received the D.Sc. degree in mathematics from the Hungarian Academy of Sciences, Budapest in 1976.

Contributors

Yuliya Babenko
Department of Mathematics
Vanderbilt University
Nashville, TN 37203
Email address: yuliya.v.babenko@vanderbilt.edu

A. Brudnyi
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4
Email address: albru@math.ucalgary.ca

Yu. Brudnyi
Department of Mathematics
Technion, Haifa, 32000, Israel
Email address: ybrudnyi@math.technion.ac.il

Alfred S. Cavaretta
Department of Mathematics
Kent State University
Kent, Ohio 44242
Email address: cavarett@math.kent.edu

M. G. de Bruin
Delft Institute of Applied Mathematics
Delft University of Technology
Delft, The Netherlands
Email address: m.g.debruin@ewi.tudelft.nl

Satya Deo
Harish Chandra Research Institute
Allahabad 211 019, India
Email address: sdeo@mri.ernet.in, vcsdeo@yahoo.com

Z. Ditzian
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada, T6G 2G1
Email address: zditzian@math.ualberta.ca

Dimiter Dryanov
Department of Mathematics and Statistics
Concordia University
Montréal, Quebec
Canada H3G 1M8
Email address: dryanovd@alcor.concordia.ca

T. Erdélyi
Department of Mathematics
Texas A&M University
College Station, Texas 77843
Email address: terdelyi@math.tamu.edu

Natacha Fontes-Merz
Department of Mathematics and Computer Science
Westminster College
New Wilmington, Pennsylvania 16172
Email address: fontesnc@westminster.edu

T. N. T. Goodman
Department of Mathematics
University of Dundee
Dundee DD1 4HN
Scotland, U.K.
Email address: tgoodman@mcs.dundee.ac.uk

N. K. Govil
Department of Mathematics
Auburn University
Auburn, AL 36849-5310
Email address: govilnk@auburn.edu

Kerstin Hesse
School of Mathematics
The University of New South Wales
Sydney NSW 2052, Australia
Email addresses: k.hesse@unsw.edu.au

A. Jakimovski
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978, Israel
Email address: jakimov@post.tau.ac.il

Indira K
Department of Mathematics
Indian Institute of Technology
Mumbai 400076, India
Email address: dvp@math.iitb.ac.in

J. Keiner
Institute of Mathematics
University of Lübeck
Lübeck, Germany
Email address: keiner@math.uni-luebeck.de

András Kroó
Alfréd Rényi Institute of Mathematics
Budapest, Hungary
Email address: kroo@renyi.hu

S. L. Lee
Department of Mathematics
National University of Singapore
Singapore 117543
Email address: matleesl@nus.edu.sg

J. K. Maitra
Department of Mathematics and Computer Science
R. D. University
Jabalpur 482 001, India
Email address: jkmaitra@yahoo.co.in

G. Mastroianni
Dipartimento di Matematica
Università della Basilicata
Potenza, Italy
Email address: mastroianni@unibas.it

H. N. Mhaskar
Department of Mathematics
California State University
Los Angeles, California 90032
Email address: hmhaska@calstatela.edu

Devidas Pai
Department of Mathematics
Indian Institute of Technology
Mumbai 400076, India
Email address: dvp@math.iitb.ac.in

J. Prestin
Institute of Mathematics
University of Lübeck
Lübeck, Germany
Email address: prestin@math.uni-luebeck.de

M. A. Qazi
Department of Mathematics
Tuskegee University
Tuskegee, AL 36088
Email address: qazima@aol.com

Q. I. Rahman
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, Canada H3G 3J7
Email address: rahmanqi@dms.umontreal.ca

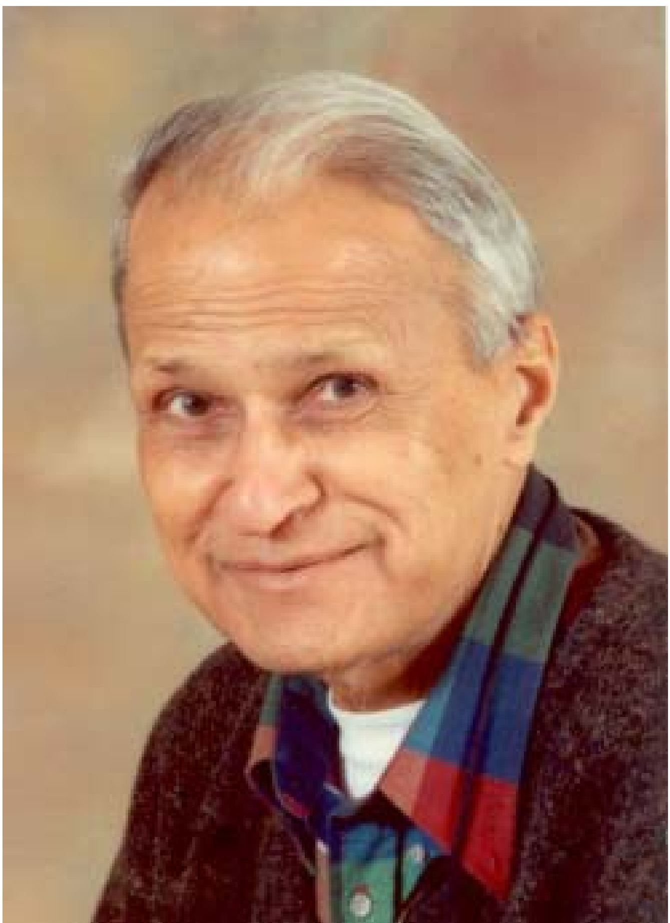
Gerhard Schmeisser
Mathematical Institute
University of Erlangen-Nuremberg
Erlangen, Germany
Email address: schmeisser@mi.uni-erlangen.de

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620
Email address: boris@math.usf.edu

Ian H. Sloan
School of Mathematics
The University of New South Wales
Sydney NSW 2052, Australia
Email address: i.sloan@unsw.edu.au

J. Szabados
Alfréd Rényi Institute of Mathematics
Budapest, Hungary
Email address: szabados@renyi.hu

P. Vértesi
Alfred Renyi Institute of Mathematics
Budapest, Hungary
Email address: vertesi@renyi.hu



Ambikeshwar Sharma (1920-2003)

Ambikeshwar Sharma
(1920-2003)

A. Jakimovski and J. Szabados

Ambikeshwar Sharma passed away on December 22, 2003, after a long period of illness at his home in Edmonton, Alberta, Canada. Sharma is survived by a daughter, Jyotsna Sharma-Srinivasan, and two sons, Someshwar (Raja) Sharma and Yogi Sharma.

Ambikeshwar Sharma was born at Jobner, a small city in the state of Rajasthan, India on July 2, 1920. He received his B.A. (1938) and M.Sc. (1940) from the Maharaja's College, Jaipur, and his Ph.D. (1951) under A. N. Singh from Lucknow University, Lucknow, India.

Sharma held positions at Cornell, Rajasthan, Harvard, and UCLA before joining the University of Alberta in 1962, where he remained until his retirement, in 1985.

He had eight Ph.D. students: A. M. Chak (1956), R. B. Saxena (1964), A. K. Varma (1964), J. Prasad (1968), D. J. Leeming (1969), S. L. Lee (1974), Mario Botto (1975), and M. A. Bokhari (1986). In addition, he co-advised H. M. Srivastava and K. K. Mathur before leaving India.

Sharma worked in classical analysis, concentrating eventually on lacunary polynomial and trigonometric interpolation, and on spline functions, first cubic splines, then cardinal splines, trigonometric splines, and even multivariate splines. In his final years, Sharma focused on various aspects of the Walsh over-convergence theorem.

Sharma's wide-ranging knowledge and intuition, his infectious enthusiasm and engaging personality are reflected in his many publications (more than 200 papers) and in the fact that 56 mathematicians have written papers with him and have become his friends in the process. Among his coauthors are G. Alexits, R. Askey, E. W. Cheney, P. Erdös, G. Freud, C. A. Micchelli, T. S. Motzkin, I. J. Schoenberg, R. S. Varga, J. L. Walsh, and H. Zassenhaus.

Although he was unable to visit the Mathematics Department of the University of Alberta in his final years, his immobility did not prevent him from doing mathematics. He was up-to-date in the literature of his chosen subject, approximation theory. Fortunately, email enabled him to remain in contact with friends and colleagues. He was very eager to stay mobile as long as possible.

The last conference he attended, and even gave a plenary talk, was in the summer of 1999 in Budapest. He made the long trip against the advice of family, doctors and friends, using a wheelchair at airports, and delivered a successful talk. He even attended the conference excursion, a further indication of his unflagging willpower.

He was an expert in the theory of interpolation. His dream for many years was to write a monograph on his favorite subject, the theory of over-convergence of complex polynomials. This theory is based on the classic result of J. Walsh stating that the difference of the partial sums of the Taylor series of an analytic function and the Lagrange interpolation polynomials of the function based on the roots of unity converges to zero in a circle larger than the domain of analyticity, although both diverge there. The project started about ten years ago, but his death prevented him from completing the work. It is our duty now to finish the monograph and thus realize his dream.

He was a person devoted to his profession and did not care much for other worldly pleasures. At the same time, he was very sensitive to his friends' problems and did everything he could to help people.

In particular, he tried to help Ph.D. students and fresh Ph.D.s. He was the most friendly person we have ever met. He was a credit to mathematics and, especially, approximation theory.

Acknowledgment. Reprinted from Journal of Approximation Theory, Vol. 131, No 1, 2004, pp. 1 - 2, with permission from Elsevier.

Scientific Papers of Ambikeshwar Sharma

1. On the minimal interval of ζ in the second mean-value theorem, Proc. Benares Math. Soc. (N.S.) 7(2) (1945) 33–40.

2. On the zeros of a class of functions, Proc. Benares Math. Soc. (N.S.) 8(2) (1946) 1–21.

3. (with S. C. Mitra), On a generalization of Weber's parabolic cylinder functions, Proc. Benares Math. Soc. (N.S.) 9 (1947) 25–31.

4. On a generalization of Legendre polynomials, Bull. Calcutta Math. Soc. 40 (1948) 195- -206.

5. (with S. C. Mitra), On a generalization of Weber's parabolic cylinder functions, Bull. Calcutta Math. Soc. 41 (1949) 87–91.

6. (with S. C. Mitra), On certain self-reciprocal functions, Ganita 1 (1950) 31–38.

7. On the properties of $\theta(x, h)$ in Mazzoni's form of the mean-value theorem, Math. Student 19 (1951) 37–43.

8. On certain relations between ultraspherical polynomials and Bessel functions, Bull. Calcutta Math. Soc. 43 (1951) 61–66.

9. On an application of a method of Shohat to a problem of Lukacs, Ganita 2 (1951) 9–22.

10. On the differentiability of the remainder of Mazzoni's formula, Ganita 2 (1951) 65–67.

11. On the remainder in two theorems of Kloosterman, Nederl. Akad. Wetensch. Proc. Ser.A 54 = Indag. Math. 13 (1951) 418–425.

12. (with S. C. Mitra), On generating functions of polynomials. I. Generalised parabolic cylinder functions of Weber, Bull. Calcutta Math. Soc. 43 (1951) 46–50.

13. On the zeros of a certain polynomial, Proc. Nat. Inst. Sci. India 18 (1952) 491–493.

14. The zeros of a complex polynomial, Math. Student 21 (1953) 52–54.

15. (with A. M. Chak), The basic analogue of a class of polynomials, Riv. Mat. Univ. Parma. 5 (1954) 325–337.

16. (with H. M. Srivastava), On certain functional relations and a generalization of the $M_{k,m}$ function, Ann. Polon. Math. 3 (1957) 76–86.

17. On Gołab's contribution to Simpson's formula, Ann. Polon. Math. 3 (1957) 240–246.

18. q-Bernoulli and Euler numbers of higher order, Duke Math. J. 25 (1958) 343–354.

19. On Newton's method of approximation, Ann. Polon. Math. 6 (1959) 295–300.

20. (with R. B. Saxena), On some interpolatory properties of Legendre polynomials, Acta Math. Acad. Sci. Hungar. 9 (1958) 345–358.

21. Some properties of plane curves, Ann. Polon. Math. 6 (1959) 245–252.

22. (with R. B. Saxena), Convergence properties of interpolatory polynomials, Acta Math. Acad. Sci. Hungar. 10 (1959) 157–175.

23. Remark on a paper of Cinquini, Acta Math. Acad. Sci. Hungar. 11 (1960) 93–96.

24. (with K. K. Mathur), Some interpolatory properties of Hermite polynomials, Acta Math. Acad. Sci. Hungar. 12 (1961) 193–207.

25. (with R. B. Saxena), Some inequalities on polynomials, J. Indian Math. Soc. (N.S.) 25 (1961) 63–102.

26. (with A. K. Varma), Some interpolatory properties of Tchebyche. polynomials (0, 2) case modified, Publ. Math. Debrecen 8 (1961) 336–349.

27. Remarks on quasi-Hermite-Fejér interpolation, Canad. Math. Bull. 7 (1964) 101–119.

28. Interpolation by polynomials in z and z^{-1} in the roots of unity, Canad. J. Math. 19 (1967) 16–23.

29. (with J. L. Walsh), Least square approximation and interpolation in roots of unity, Pacific J. Math. 14 (1964) 727–730.

30. (with T. S. Motzkin), Next-to-interpolatory approximation on sets with multiplicities, Canad. J. Math. 18 (1966) 1196–1211.

31. (with E. W. Cheney), Bernstein Power-Series, Canad. J. Math. 16 (1964) 241–252.

32. (with E. W. Cheney), On a generalization of Bernstein polynomials, Riv. Math. Univ. Parma (2) 5 (1964) 77–84.

33. Some remarks on lacunary interpolation in the roots of unity, Israel J. Math. 2 (1964) 41–49.

34. (with A. K. Varma), Trigonometric interpolation, Duke Math. J. 32 (1965) 341–357.

35. (with E. Frank), Continued fraction expansions and iterations of Newton's formula, J. Reine Angew. Math. 219 (1965) 62–66.

36. (with A. Meir), On the method of Romberg quadrature, J. SIAM Ser. B, Numer. Anal. 2 (1965) 250–258.

37. (with P. Erdös), On Tchebychev. quadrature, Canad. J. Math. 17 (1965) 652–658.

38. (with A. Meir), Degree of approximation of spline interpolation, J. Math. Mech. 15 (1966) 759–767.

39. Lacunary interpolation in the roots of unity (German and Russian summaries), Z. Angew. Math. Mech. 46 (1966) 127–133.

40. (with A. Meir), Span of derivatives of polynomials, Amer. Math. Monthly 74 (1967) 527–531.

41. (with A. Meir), Simultaneous approximation of a function and its derivatives, SIAM J. Numer. Anal. 3 (1966) 553–563.

42. (with A. Meir), Span of linear combinations of derivatives of polynomials, Duke Math. J. 34 (1967) 123–129.

43. (with A. Meir), A variation of the Tchebyschev. quadrature problem, Illinois J. Math. 11 (1967) 535–546.

44. (with A. Meir), Approximation methods by polynomials and power series, Indag. Math. 29 (1967) 395–403.

45. (with A. K. Varma), Trigonometric interpolation (0, 2, 3) case, Ann. Polon. Math. 21 (1968) 51–58.

46. (with J. Prasad), On Abel-Hermite-Birkhoff interpolation, SIAM J. Numer. Anal. 5 (1968) 864–881.

47. (with A. Meir), Symmetric differences and derivatives, Indag. Math. 30 (1968) 353–360.

48. (with A. Meir), A generalization of the S_α-*summation* method, Proc. Cambridge Philos. Soc. 67 (1970) 61–66.

49. (with A. Meir), Convergence of a class of interpolatory splines, J. Approx. Theory 1 (1968) 243–250.

50. (with A. Meir), One-sided spline approximation, Studia Sci. Math. Hungar. 3 (1968) 211–218.

51. (with K. Atkinson), A partial characterization of poised Hermite-Birkhoff interpolation problems, SIAM J. Numer. Anal. 6 (1969) 230–235.

52. (with A. Meir), An extension of Obreshkov's formula, SIAM J. Numer. Anal. 5 (1968) 488–490.

53. (with A. Meir), On zeros of derivatives of polynomials, Canad. Math. Bull. 11 (1968) 443–445.

54. (with D. Leeming), Lacunary interpolation (0, n-1, n) case, Mathematica (Cluj) 11 (34) (1969) 155–162.

55. (with A. Meir), On Ilyeff's conjecture, Pacific J. Math. 31 (1969) 459–467.

56. (with A. Meir), Multipoint expansions of finite differences, in: I. J. Schoenberg (Ed.), Approximation with Special Emphasis on Spline Functions, Academic Press, New York, 1969, pp. 389–404.

57. (with D. Leeming), A generalization of the class of completely convex functions, Inequalities, III (Proc. Third Sympos. Univ. California, Los Angeles, 1969, dedicated to the memory of Theodore S. Motzkin), Academic Press, New York, 1972, pp. 177–199.

58. (with A. Meir), On uniform approximation by cubic splines, J. Approx. Theory 2 (1969) 270–274.

59. (with G. Alexits), On the convergence of multiplicatively orthogonal series, Acta Math. Acad. Sci. Hungar. 22 (1971/2) 257–266.

60. (with S. P. Pethe), Modified Abel expansion and a sub-class of completely convex functions, SIAM J. Math. Anal. 3 (1972) 546–558.

61. Some poised and non-poised problems of interpolation. (Survey Article), SIAM Rev. 14 (1972) 129–151.

62. (with T. S. Motzkin), A sequence of linear polynomial operators and their approximation-theoretic properties, J. Approx. Theory 5 (1972) 176–198.

63. (with I. J. Schoenberg), The interpolatory background of the Euler-Maclaurin quadrature formula, Bull. Amer. Math. Soc. 77 (1971) 1034–1038.

64. (with S. P. Pethe), Functions analogous to completely convex functions, Rocky Mountain J. Math. 3(4) (1973) 591–617.

65. (with A. Meir), Lacunary interpolation by splines, SIAM J. Numer. Anal. 10 (1973) 433–442.

66. (with G. Alexits), The influence of Lebesgue functions on the convergence and summability of function series, Acta Sci. Math. (Szeged) 33 (1972) 1–10.

67. (with I. J. Schoenberg), Cardinal interpolation and spline functions. V. The B-splines for cardinal Hermite interpolation, Linear Algebra Appl. 7 (1973) 1–42.

68. (with T. S. Motzkin and E. G. Straus), Averaging interpolation, in: A. Meir and A. Sharma (Eds.), Spline Functions and Approximation Theory, ISNM 21, Birkhäuser Verlag, Basel, 1973, pp. 191–233.

69. (with E. G. Straus), On the irreducibility of a class of Euler-Frobenius polynomials, Canad. Math. Bull. 17 (1974) 265–273.

70. (with G. Freud), Some good sequences of interpolatory polynomials, Canad. J. Math. 26 (1974) 233–246.

71. Some poised problems of interpolation, in: G. Alexits and S. B. Stechkin (Eds.), Proceedings of Conference on Constructive Theory of Functions held in Budapest, (1969), Akadémiai Kiadó, Budapest, 1972, pp. 435–441.

72. (with R. DeVore and A. Meir), Strongly and weakly non-poised H-B interpolation problems, Canad. J. Math. 25 (1973) 1040–1050.

73. (with R. B. Saxena), Convergence of averaging interpolation operators, Demonstratio Math. 6 (1973) 821–839.

74. (with T. N. E. Greville and I. J. Schoenberg), The spline interpolation of sequences satisfying a linear recurrence relation, J. Approx. Theory 17(3) (1976) 200–221.

75. (with A. Meir and J. Tzimbalario), Hermite-Fejér type interpolation processes, Anal. Math. 1(2) (1975) 121–129.

76. (with M. A. Botto), Averaging interpolation on sets with multiplicities, Aequationes Math. 15(1) (1977) 63–72.

77. (with S. L. Lee), Cardinal lacunary interpolation by g-splines. I. The characteristic polynomials, J. Approx. Theory 16(1) (1976) 85–96.

78. (with J. Tzimbalario), Quasi-Hermite-Fejér type interpolation of higher order, J. Approx. Theory 13 (1975) 431–442.

79. (with S. L. Lee and J. Tzimbalario), A class of cardinal splines with Hermite type interpolation, J. Approx. Theory 18(1) (1976) 30–38.

80. (with J. Tzimbalario), Cardinal interpolation and generalized exponential Euler splines, Canad. J. Math. 28(2) (1976) 291–300.

81. (with J. Tzimbalario), Cardinal t-perfect L-splines, SIAM J. Numer. Anal. 13(6) (1976) 915–922.

82. (with J. Tzimbalario), A class of cardinal trigonometric splines, SIAM J. Math. Anal. 7(6) (1976) 809–819.

83. (with J. Tzimbalario), Landau-type inequalities for some linear differential operators, Illinois J. Math. 20(3) (1976) 443–455.

84. (with J. Tzimbalario), Quadratic splines, J. Approx. Theory 19(2) (1977) 186–193.

85. (with J. Tzimbalario), Classes of functions defined by differential inequalities, J. Math. Anal. Appl. 61(1) (1977) 122–135.

86. (with J. Tzimbalario), Some inequalities for a certain class of C^∞-functions, Period. Math. Hungar. 12(1) (1981) 31–36.

87. (with J. Tzimbalario), Landau-type inequalities for bounded intervals, Period. Math. Hungar. 9(3) (1978) 175–186.

88. (with J. Tzimbalario), Some inequalities between derivatives on bounded intervals, Delta (Waukesha) 6(2) (1976) 78–91.

89. (with J. Tzimbalario), Some linear differential operators, and generalized differences. (Russian), Mat. Zametki 21(2) (1977) 161–172.

90. (with J. Tzimbalario), Some strongly non-poised H-B problems, J. Math. Anal. Appl. 63(2) (1978) 521–524.

91. (with J. Tzimbalario), A generalization of monosplines and perfect splines, in: A. G. Law and B. N. Sahney (Eds.), Theory of Approximation with Applications, Academic Press, New York, 1976, pp. 257–267.

92. (with J. Tzimbalario), A generalization of a result of Subbotin, in: G. G. Lorentz, C. K. Chui, and L. L. Schumaker (Eds.), Approximation Theory, II, Academic Press, New York, 1976, pp. 557–562.

93. (with R. N. Mohapatra), Discrete exponential Abel-Euler splines, J. Indian Math. Soc. (N.S.) 42(1-4) (1978) 367–379.

94. (with R. N. Mohapatra), Convergence of discrete spline interpolants without mesh ratio restrictions, Indian J. of Math. 20(2) (1978) 161–171.

95. (with K. K. Mathur), Discrete polynomials splines on the circle, Acta Math. Acad. Sci. Hungar. 33(1-2) (1979) 143–153.

96. (with Geza Freud), Addendum: "Some good sequences of interpolatory polynomials," Canad. J. Math. 29(6) (1977) 1163–1166.

97. (with A. S. Cavaretta, Jr., and R. S. Varga), Interpolation in the roots of unity: An extension of a theorem of J. L. Walsh, Resultate Math. 3(2) (1980) 155–191.

98. (with C. A. Micchelli), Spline functions on the circle: Cardinal L-splines revisited, Canad. J. Math. 32(6) (1980) 1459–1473.

99. (with A. S. Cavaretta, Jr., and R. S. Varga), Hermite-Birkhoff interpolation in the nth roots of unity, Trans. Amer. Math. Soc. 259(2) (1980) 621–628.

100. (with A. S. Cavaretta, Jr., and R. S. Varga), Lacunary trigonometric interpolation on equidistant nodes, in: R. DeVore and K. Scherer (Eds.), Quantitative Approximation, Academic Press, New York, 1980, pp. 63–80.

101. (with A. S. Cavaretta, Jr. and C. A. Micchelli), Multivariate interpolation and the Radon transform, Math. Z. 174(3) (1980) 263–279.

102. (with P. W. Smith and J. Tzimbalario), Polynomial interpolation in the roots of unity with applications, in: C. Ciesielski (Ed.), Approximation and Function Spaces, North Holland, Amsterdam, 1981, pp. 667–685

103. (with S. Riemenschneider), Birkhoff interpolation at the nth roots of unity: convergence, Canad. J. Math. 33(2) (1981) 362–371.

104. (with P. W. Smith and S. Riemenschneider), Convergence of lacunary trigonometric interpolation on equidistant nodes, Acta Math. Acad. Sci. Hungar. 39(1-3) (1982) 27–37.

105. (with A. S. Cavaretta, Jr., and C. A. Micchelli), Multivariate interpolation and the Radon transform. II. Some further examples, in: R. DeVore and K. Scherer (Eds.), Quantitative Approximation, Academic Press, New York, 1980, pp. 49–62.

106. (with E. B. Saff and R. S. Varga), An extension to rational functions of a theorem of J. L. Walsh on differences of interpolating polynomials, RAIRO Anal. Numér. 15(4) (1981) 371–390.

107. (with R. Askey and I. J. Schoenberg), Hausdorff's moment problem and expansions in Legendre polynomials, J. Math. Anal. Appl. 86(1) (1982) 237–245.

108. (with P. Vértesi), Mean convergence and interpolation in roots of unity, SIAM J. Math. Anal. 14(4) (1983) 800–806.

109. (with A. K. Varma), Lacunary trigonometric interpolation on equidistant nodes (convergence), J. Approx. Theory 35(1) (1982) 45–63.

110. (with A. M. Chak and J. Szabados), On a problem of P. Turán, Studia Scient. Math. Hungar. 15(4) (1980) 441–455.

111. (with A. S. Cavaretta, Jr. T. N. T. Goodman and C. A. Micchelli), Multivariate interpolation and the Radon transform. III. Lagrange representation, Canad. Math. Soc. Conf. Proc. 3 (1983) 37–50.

112. (with T. N. E. Greville and I. J. Schoenberg), The behavior of the exponential Euler spline $S_n(x;t)$ as $n \to \infty$ for negative values of the base t, Canad. Math. Soc. Conf. Proc. 3 (1983) 185–198.

113. (with R. B. Saxena and Z. Ziegler), Hermite-Birkhoff interpolation on roots of unity and Walsh equiconvergence, Linear Algebra Appl. 52/53 (1983) 603–615.

114. (with S. L. Lee, C. A. Micchelli and P. W. Smith), Some properties of periodic B-spline collocation matrices, Proc. Roy. Soc. Edinburgh Sect. A 94 (3-4) (1983) 235–246.

115. (with C. A. Micchelli), On a problem of Turán: multiple node Gaussian quadrature, Rend. Mat. (7) 3(3) (1983) 529–552.

116. (with T. N. T. Goodman), Convergence of multivariate polynomials interpolating on a triangular array, Trans. Amer. Math. Soc. 285(1) (1984) 141–157.

117. (with J. Fabrykowski and H. Zassenhaus), Some Birkhoff interpolation problems on the roots of unity, Linear Algebra Appl. 65 (1985) 1–23.

118. (with A. S. Cavaretta, Jr. and R. S. Varga), A theorem of J. L. Walsh, revisited, Pacific J. Math. 118 (2) (1985) 313–322.

119. (with E. B. Saff), On equiconvergence of certain sequences of rational interpolants, in: Proc. Conf. on Approximation, Tampa, Florida, 1983, Lecture Notes in Math. 1105, Springer-Verlag, Berlin, 1984, pp. 256–271.

120. (with H. P. Dikshit and J. Tzimbalario), Asymptotic error expansions for spline interpolation, Canad. Math. Bull. 27(3) (1984) 337–344.

121. (with C. A. Micchelli), Convergence of complete spline interpolation for holomorphic functions, Ark. Mat. 23(1) (1985) 159–170.

122. (with S. D. Riemenschneider and P. W. Smith), Lacunary trigonometric interpolation: convergence, in: E. W. Cheney (Ed.), Approximation Theory III, Academic Press, New York, 1980, pp. 741–746.

123. Birkhoff interpolation on the roots of unity, in: S. P. Singh, J. H. W. Burry, and B. Watson (Eds.), Approximation Theory and Spline Functions, Reidel, Dordrecht, 1984, pp. 199–205.

124. (with Z. Ziegler), Walsh equiconvergence for best l_2-approximates, Studia Math. 77(5) (1984) 523–528.

125. (with A. S. Cavaretta, Jr. and H. P. Dikshit), An extension of a theorem of Walsh, Resultate Math. 7(2) (1984) 154–163.

126. (with T. N. T. Goodman and S. L. Lee), Approximation by Λ-splines on the circle, Canad. J. Math. 37(6) (1985) 1085–1111.

127. (with A. S. Cavaretta, Jr. and R. S. Varga), Converse results in the Walsh theory of overconvergence, RAIRO Modél. Math. Anal. Numér. 19(4) (1985) 601–609.

128. (with T. N. T. Goodman), A property of Bernstein-Schoenberg spline operators, Proc. Edinburgh Math. Soc. (2) 28(3) (1985) 333–340.

129. (with A. S. Cavaretta, Jr. and J. Tzimbalario), Convergence of a class of interpolatory splines for holomorphic functions, J. Approx. Theory 46(4) (1986) 374–384.

130. (with V. Singh), Some Bernstein type inequalities for polynomials, Analysis 5(4) (1985) 321–341.

131. (with A. Jakimovski), Lacunary trigonometric interpolation on equidistant nodes, Analysis 6(2-3) (1986) 269–284.

132. (with K. G. Ivanov), Some new results on Walsh th eory of equiconvergence, in: J. Szabados (Ed.), Proc. Haar Memo. Conf. Budapest, August 1985, Vol.1, Colloq. Math. Soc. János Bolyai, 49, North-Holland, Amsterdam, 1987, pp. 465–474.

133. (with K. G. Ivanov), More quantitative results on Walsh equiconvergence. I. Lagrange case, Constr. Approx. 3(3) (1987) 265–280.

134. (with K. G. Ivanov), Quantitative results on Walsh equiconvergence II (Hermite interpolation and l_2-approximation), Approx. Theory Appl. 2(1) (1986) 47–64.

135. (with M. A. Bokhari), Equiconvergence of certain sequences of rational interpolants (Hermite case), in: C. A. Micchelli, D. V. Pai and B. V. Limaye (Eds.), Methods of Functional Analysis in Approximation Theory, ISNM 76, Birkhäuser, Basel, 1986, pp. 281–292.

136. Some recent results on Walsh theory of equiconvergence, in: C. Chui, L. Schumaker, and J. Ward (Eds.), Approximation Theory V, Academic Press, New York, 1986, pp. 173–190.

137. Lacunary trigonometric interpolation on equidistant nodes (a survey), Math. Student 55(2-4) (1987) 123–131.

138. (with H. P. Dikshit and A. Ojha), Certain mapping properties of rational complex planar splines, Math. Proc. Cambridge Philos. Soc. 101(1) (1987) 141–149.

139. (with T. N. T. Goodman and I. J. Schoenberg), Piecewise smooth solutions of some difference-differential equations, J. Approx. Theory 48(3) (1986) 262–271.

140. (with T. N. T. Goodman and S. L. Lee), Asymptotic formula for the Bernstein- Schoenberg operator, Approx. Theory Appl. 4(1) (1988) 67–86.

141. (with M. R. Akhlaghi and A. M. Chak), (0, 2, 3) interpolation on zeros of $\pi_n(x)$, Approx. Theory Appl. 4(2) (1988) 55–74.

142. (with M. R. Akhlaghi and A. M. Chak), (0,3) interpolation on zeros of $\pi_n(x)$, Rocky M. J. 19(1) (1989) 9–21.

143. (with J. Szabados and R. S. Varga), 2-periodic lacunary trigonometric interpolation: the (0,M) case, in: B. Sendov, P. Petrushev, K. Ivanov, and R. Maleev (Eds.), Constructive Theory of Functions '87, Bulgarian Academy of Sciences, Sofia, 1987, pp. 420–427.

144. (with H. P. Dikshit, V. Singh and F. Stenger), Rivlin's Theorem on Walsh equiconvergence, J. Approx. Theory 52(3) (1988) 339–349.

145. (with J. Szabados), Convergence rates for some lacunary interpolators on the roots of unity, Approx. Theory Appl. 4(2) (1988) 41–48.

146. (with K. G. Ivanov), Converse results on equiconvergence of interpolating polynomials, Anal. Math. 14(2) (1988) 185–192.

147. (with A. S. Cavaretta, Jr., and H. P. Dikshit), Convergence of certain polynomial interpolants to a function defined on the unit circle, Acta Math. Hungar. 53(1-2) (1989) 143–147.

148. (with T. N. T. Goodman), A modified Bernstein-Schoenberg operator, in: B. Sendov, P. Petrushev, K. Ivanov, and R. Maleev (Eds.), Constructive Theory of Functions '87, Bulgarian Academy of Sciences, Sofia, 1987, pp. 166–173.

149. (with T. N. T. Goodman and S. L. Lee), Approximation and interpolation by complex splines on the torus, Proc. Edinburgh Math. Soc.(2) 32(2) (1989) 197–212.

150. (with M. R. Akhlaghi and A. Jakimovski), Equiconvergence of some complex interpolatory polynomials, Numer. Math. 57 (6-7) (1990) 635–649.

151. (with R. S. Varga), On a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes, Resultate Math. 16(3-4) (1989) 383–404.

152. (with D. P. Dryanov and R. Q. Jia), Quadrature formulae with Birkhoff-type data on equidistant nodes for 2π-periodic functions, in: P. Nevai and A. Pinkus (Eds.), Progress in Approximation Theory, Academic Press, New York, 1991, pp. 243–261.

153. (with A. S. Cavaretta, Jr., and C. A. Micchelli), A multivariate extension of Walsh equiconvergence, in: C. Chui, L. Schumaker, and J. Ward (Eds.), Approximation Theory VI, Vol.I, Academic Press, New York, 1989, pp. 121–124.

154. (with J. Szabados and R. S. Varga), Some 2π-periodic trigonometric interpolation problems on equidistant nodes, Analysis, 11(2-3) (1991) 165–190.

155. (with T. N. T. Goodman), Trigonometric interpolation, Proc. Edinburgh Math. Soc. (2) 35(3) (1992) 457–472.

156. (with Yuan Xu), Mean convergence of trigonometric interpolants on equidistant nodes: Birkhoff data, Bull. Polish Acad. Sci. 39(3.4) (1991) 199–206.

157. (with R. Q. Jia), Solvability of some multivariate interpolation problems, J. Reine Angew. Math. 421 (1991) 73–81.

158. (with J. Szabados and R. S. Varga), Some 2-periodic trigonometric interpolation problems on equidistant nodes. II. Convergence, Studia Sci. Math. Hungar. 29(3-4) (1994) 415–432.

159. (with R. B. Saxena), Almost Hermitian trigonometric interpolation on three equidistant nodes, Aequationes Math. 41(1) (1991) 55–69.

160. (with A. S. Cavaretta, Jr.), Variation diminishing properties and convexity for the tensor product Bernstein operator, Springer-Verlag Lecture Notes in Math. 1511 (1992) 18–32.

161. (with M. A. Bokhari), Equiconvergence of some sequences of rational functions, Internat. J. Math. Math. Sci. 15(2) (1992) 221–228.

162. (with T. N. T. Goodman), A Bernstein type operator on the simplex, Math. Balkanica (N.S) 5(2) (1991) 129–145.

163. (with K. G. Ivanov), Walsh equiconvergence and (l, ρ) distinguished points, Constr. Approx. 7(3) (1991) 315–327.

164. (with T. N. T. Goodman), Factorization of totally positive, symmetric, periodic, banded matrices with applications, Linear Algebra Appl. 178 (1993) 85–107.

165. (with C. A. Micchelli and A. S. Cavaretta),Walsh equiconvergence theorem and optimal recovery, Analysis 12(3-4) (1992) 271–302.

166. (with T. N. T. Goodman), A property of Hermite interpolants in the roots of unity, Ganita, 43(1-2) (1992) 171–180.

167. (with X. Sun), A 2-periodic trigonometric interpolation problem, Approx. Theory Appl. 8(4) (1992) 1–16.

168. (with A. Jakimovski), Walsh equiconvergence and equisummability, J. Anal. 1 (1993) 21–72.

169. (with T. N. T. Goodman and K. G. Ivanov), Hermite interpolation in the roots of unity, J. Approx. Theory 84(1) (1996) 41–60.

170. (with T. N. T. Goodman), A Bernstein-Schoenberg type operator: shape preserving and limiting behaviour, Canad. J. Math. 47(5) (1995) 959–973.

171. (with R. Brück and J. Müller), Equisummability of certain sequences of Hadamard products of Taylor sections and interpolatory polynomials, Methods Appl. Anal. 1(3) (1994) 270–283.

172. (with T. N. T. Goodman), A property of Hermite-Padé interpolation on the roots of unity, Approx. Theory Appl. (N.S.) 12(1) (1996) 31–41.

173. (with R. Brück and R. S. Varga), An extension of a result of Rivilin on Walsh equiconvergence, in: C. Micchelli and H. P. Dikshit (Eds.), Advances in Computational Mathematics (New Delhi, 1993), Ser. Approx. Decompos., 4, World Sci. Publ. River Edge NJ, 1994, pp. 225–234.

174. (with Marcel G. de Bruin), Equiconvergence of some simultaneous Hermite-Padé interpolants, RAIRO Model. Math. Anal. Numér. 29(4) (1995) 477–503.

175. (with R. Brück and R. S. Varga), An extension of a result of Rivlin on Walsh equiconvergence (Faber Nodes), in: R. V. M. Zahar (Ed.), Approximation and Computation: Festschrift in Honor of Walter Gautschi, ISNM 119, Birkhäuser Verlag, Basel-Boston- Berlin, 1994, pp. 40–66.

176. (with J. Szabados, B. Underhill, and A. K. Varma), On some general lacunary interpolation problems, J. Approx. Theory 87(2) (1996) 194–219.

177. (with J. Szabados), Remarks on (0,2) interpolation, Ganita 46(1-2) (1995) 87–110.

178. (with W. Chen), Lacunary interpolation on some non-uniformly distributed nodes on the unit circle, Ann. Univ. Sci. Budapest (Sektio Computeriko) 16 (1996) 69–82.

179. (with W. Chen), Some Bernstein-Durrmeyer type operators, Methods Appl. Anal. 4(3) (1997) 239–249.

180. (with A. Jakimovski), Asymptotic properties of differences of generalized Hermite-Padé interpolants, J. Orissa Math. Soc. 12-15 (1993-96) 339–386.

181. (with M. G. de Bruin), Overconvergence of some simultaneous Hermite-Padé interpolants, Ann. Numer. Math. 4(1-4) (1997) 239–259.

182. (with R. Q. Jia and S. L. Lee), Spectral properties of continuous refinement operators, Proc. Amer. Math. Soc. 126(3) (1998) 729–737.

183. (with A. S. Cavaretta, and C. R. Selvaraj), Lacunary interpolation by cosine polynomials, Ann. Numer. Math. 4(1-4) (1997) 181–192.

184. (with K. G. Ivanov), Quadratic mean radius of a polynomial in C(Z), Serdica Math. J. 22(4) (1996) 497–514.

185. (with A. Jakimovski), Reflections onWalsh equiconvergence and equisummability, Acta Univ. Lodziensis Folia Math. 8 (1996) 9–18.

186. (with S. L. Lee and H. H. Tan), Spline interpolation and wavelet construction, Appl. Comput. Harmon. Anal. 5(3) (1998) 249–276.

187. (with M. G. de Bruin and J. Szabados), Birkhoff type interpolation on perturbed roots of unity, in: N. K. Govil et al (Eds.), Approximation Theory, A. K. Varma Memorial Volume, Monogr. Textbooks Pure Appl. Math. 212, Marcel Dekker, New York, (1998), pp. 167–179.

188. (with A. Jakimovski), Hermite interpolation on Chebyshev nodes and Walsh equiconvergence, in: N. K. Govil et al (Eds.), Approximation Theory, A. K. Varma Memorial Volume, Monogr. Textbooks Pure Appl. Math., 212, Marcel Dekker, New York, 1998, pp. 293–333.

189. (with A. Jakimovski), Hermite interpolation on Chebyshev nodes and Walsh equiconvergence, J. Anal. 7 (1999) 89–102.

190. (with M. G. de Bruin and K. G. Ivanov), A conjecture of Schoenberg, J. Inequal. Appl. 4(3) (1999) 183–213.

191. (with M. G. de Bruin and K. G. Ivanov), A conjecture of Schoenberg, Comm. Anal. Theory Contin. Fractions 6 (1997) 103–105.

192. (with M. R. Akhlaghi), Some Pál type interpolation problems, in: A. G. Law and R. L. Wong (Eds.), Approximation, Optimization and Computing, North-Holland, Amsterdam, 1990, pp. 37–40.

193. (with Z. Ziegler), Hermite interpolation on some perturbed roots of unity, Analysis (Munich) 19(1) (1999) 1–12.

194. Some simple properties of *up(x)*, in: Z. U. Ahmad, N. K. Govil and P. K. Jain (Eds.), Fourier Analysis, Approx. Theory and Applications (Aligarh, 1993), New Age International Publishers, New Delhi, 1996, pp. 263–280.

195. (with M. G. de Bruin), Birkhoff interpolation on perturbed roots of unity on the unit circle, J. Nat. Acad. Math. India 11 (1997) 83–97.

196. (with J. Szabados), Quantitative results in some problems on Walsh equiconvergence, East J. Approx. 4(3) (1998) 291–309.

197. (with A. Jakimovski), Hermite interpolation on Chebyshev nodes and Walsh equiconvergence. II, J. Anal. 6 (1998) 91–119.

198. (with M. G. de Bruin), On a Schoenberg-type conjecture, J. Comput. Appl. Math. 105(1-2) (1999) 221–228.

199. (with M. G. de Bruin), Equiconvergence of some simultaneous Hermite-Padé interpolants: multiple nodes, in: Charles K. Chui and Larry L. Schumaker (Eds.), Approximation Theory VIII, Vol. 1: Approximation and Interpolation, World Scientific Publishing Co., Inc. Singapore, 1995, pp. 103–110.

200. (with M. G. de Bruin and H. P. Dikshit), Birkhoff interpolation on unity and on Möbius transform of the roots of unity, Numer. Algorithms 23(1) (2000) 115–125.

201. (with M. G. de Bruin), Birkhoff interpolation on nonuniformly distributed roots of unity, Numer. Algorithms 25(1-4) (2000) 123–138.

202. (with M. G. de Bruin), Birkhoff interpolation on nonuniformly distributed roots of unity. II, J. Comput. Appl. Math. 133 (1–2) (2001) 295–303.

203. (with M. A. Bokhari and H. P. Dikshit), Birkhoff interpolation on perturbed roots of unity: revisited, Numer. Algorithms 25(1-4) (2000) 47–62.

204. (with J. Szabados), A multivariate extension of Walsh's over-convergence theorem, Rendiconti del Circolo Matematico di Palermo (2) Suppl. 68 part II (2002) 797–803.

205. (with A. Jakimovski), Quantitative results on equiconvergence of certain sequences of rational interpolants, Math. Balkanica (N.S.) 16(2-4) (2002) 249–282.

206. (with Marcel G. de Bruin), Lacunary Pál-type interpolation and over-convergence, Comput. Methods and Function Theory (CMFT) 3 (2003) 305–323.

207. (with A. Cavaretta), Optimal recovery of interpolation operators in Hardy spaces, Adv. Comput. Math. 24 (2006) 131–141.

208. (with A. Jakimovski and J. Szabados), Walsh Equiconvergence of Complex Interpolatory Polynomials, Springer Monographs in Mathematics, (to appear in 2006).

Markov-Type Inequalities for Homogeneous Polynomials on Nonsymmetric Star-Like Domains

Yuliya Babenko
Department of Mathematics
Vanderbilt University
Nashville, TN 37203, U.S.A.
Email address: yuliya.v.babenko@vanderbilt.edu

András Kroó
Alfréd Rényi Institute of Mathematics
P.O.B. 127, H-1364 Budapest, Hungary
Email address: kroo@renyi.hu

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Let H_n^d be the set of homogeneous polynomials of degree n and d variables. Consider a compact set $K \subset \mathbb{R}^d$. Denote by $\|\cdot\|_K$ the usual sup norm on K. It was proved by Harris [4] that if K is a $\mathbf{0}$-symmetric convex body then for every $h \in H_n^d$ with $\|h\|_K \leq 1$ we have $\|D_{\mathbf{u}}h\|_K \leq Cn\log n$ where $D_{\mathbf{u}}h$ is the derivative of h in the direction $\mathbf{u} \in S^{d-1}$. In this paper we extend Harris' result for nonsymmetric star-like domains.

1 Introduction and New Results

Let K be a compact set in $\mathbb{R}^d$, and F be a family of differentiable functions on K. As usual, $S^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ stands for the unit sphere in $\mathbb{R}^d$ in Euclidean norm. (Here and in what follows

$|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^d$.) Denote by $D_{\mathbf{u}}f$ the derivative of f in direction $\mathbf{u} \in S^{d-1}$. Let

$$\|f\|_K = \sup_{\mathbf{x}\in K} |f(\mathbf{x})|$$

be the usual sup norm on K.

Then the Markov Factor of F on K is given by

$$M(F,K) := \sup \left\{ \|D_{\mathbf{u}}f\|_K : f \in F, \|f\|_K \leq 1, \mathbf{u} \in S^{d-1} \right\}. \quad (1.1)$$

This quantity measures the size of the derivatives of functions in F compared to their sup norms on K. The problem of estimating $M(F,K)$ originates from the classical Markov inequality which gives

$$M(P_n^1, [a,b]) = \frac{2n^2}{b-a}, \quad (1.2)$$

where

$$P_n^d := \left\{ \sum_{|\mathbf{k}|_1 \leq n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R} \right\}, \quad \mathbf{x} \in \mathbb{R}^d, \quad d \geq 1, \quad n \geq 1, \quad (1.3)$$

is the space of polynomials of total degree at most n in d variables. (Here $|\cdot|_1$ denotes the l_1-norm.)

Numerous extensions of Markov inequality for various families of univariate and multivariate polynomials are known. For an overview of univariate inequalities see [2] or [3]; a survey of multivariate Markov-type inequalities can be found in [6]. In particular, it is known that for convex bodies $K \subset \mathbb{R}^d$ we have

$$M(P_n^d, K) \asymp n^2, \quad (1.4)$$

while for cuspidal domains in $\mathbb{R}^d$ the Markov Factors of P_n^d are, in general, of higher order. The size of the Markov Factors is essentially different for the set of homogeneous polynomials of degree n defined by

$$H_n^d := \left\{ \sum_{|\mathbf{k}|_1 = n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R} \right\}, \quad \mathbf{x} \in \mathbb{R}^d, \quad d \geq 2, \quad n \geq 1. \quad (1.5)$$

It was first shown by Harris [4] that for **0**-symmetric convex bodies K in $\mathbb{R}^d$ we have

$$M(H_n^d, K) \leq Cn \log n. \tag{1.6}$$

For the sharpness of this upper bound see [5] and [7].

Thus, the rates of Markov Factors for homogeneous polynomials are substantially smaller than for ordinary polynomials. It is also shown in [5] that for smooth **0**-symmetric convex bodies the $\log n$ in (1.6) can be dropped, i.e., $M(H_n^d, K) = O(n)$.

In all of the papers on homogeneous polynomials mentioned above the symmetry of the domain played an essential role. The goal of the present note consists of extending the results on Markov Factors for homogeneous polynomials to *nonsymmetric* domains K for which **0** is on the boundary ∂K of K, rather than inside K. The consideration of nonsymmetric domains K will require a more delicate study of the geometry of K around the origin. Also we shall relax the assumption of convexity of the domain and replace it by the more general star-like property. Since $|h(\mathbf{x})| = |h(-\mathbf{x})|$ for $h \in H_n^d$, it is natural to consider star-like domains contained in the half-space

$$\mathbb{R}_+^d := \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \geq 0 \right\}.$$

Let

$$S_+^{d-1} := \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in S^{d-1} : x_1 \geq 0 \right\}$$

be the upper halfsphere, $r : S_+^{d-1} \to \mathbb{R}_+$ be a continuous nonnegative mapping. Then a star-like domain in the halfspace $\mathbb{R}_+^d$ associated with r is given by

$$K_r := \left\{ t\mathbf{x} : \mathbf{x} \in S_+^{d-1},\ 0 \leq t \leq r(\mathbf{x}) \right\}. \tag{1.7}$$

We shall need to impose some mild smoothness conditions on ∂K_r at the origin.

Assume that there exist $C_1, C_2 > 0$, $\beta > 0$, and $0 < \epsilon < 1$ such that for $\mathbf{x} = (x_1, \ldots, x_d) \in S_+^{d-1}$

$$C_2 e^{-x_1^{\epsilon - 1}} \leq r(\mathbf{x}) \leq C_1 x_1^{\beta}. \tag{1.8}$$

The right inequality in (1.8) insures that $r(\mathbf{x})$ tends to 0 (as $\mathbf{x} \to \mathbf{0}$) with a polynomial rate. On the other hand the left inequality requires that $r(\mathbf{x})$ does not vanish at $\mathbf{0}$ exponentially. Thus if, for example,

$$C_2 x_1^{\alpha} \le r(\mathbf{x}) \le C_1 x_1^{\beta}$$

with some $C_1, C_2 > 0$, $0 < \beta < \alpha$ $(0 < x_1 \le 1)$ then (1.8) will hold. We shall also require that $r \in \mathrm{Lip}_M \alpha$ on S_+^{d-1}, for some $0 < \alpha \le 1$, $M > 0$, i.e. for any $\mathbf{x}_1, \mathbf{x}_2 \in S_+^{d-1}$

$$|r(\mathbf{x}_1) - r(\mathbf{x}_2)| \le M |\mathbf{x}_1 - \mathbf{x}_2|^{\alpha}.$$

Theorem 1.1 *Let K_r be a star-like domain (1.7) with $r \in Lip_M \alpha$, $0 < \alpha \le 1$, satisfying condition (1.8). Then for any $n \ge 2$*

$$M(H_n^d, K_r) \le C_0 \rho_\alpha(n), \tag{1.9}$$

where

$$\rho_\alpha(n) := \begin{cases} n^{1/\alpha}, & \alpha < 1; \\ n \log n, & \alpha = 1, \end{cases} \tag{1.10}$$

and constant $C_0 > 0$ depends only on K_r.

For $\alpha = 1$ the above theorem is a generalization of Harris' result to nonsymmetric star-like domains without cusps. In particular, it yields that $M(H_n^d, K_r) = O(n \log n)$ for a wide family of nonsymmetric convex bodies. The cuspidal case $(0 < \alpha < 1)$ is new.

While we can not prove the necessity of condition (1.8) it can be shown that some kind of smoothness at the origin is needed, in general. Indeed, let

$$h_n(x, y) := x^n \tilde{T}_n\left(\frac{y}{x}\right) \in H_n^2,$$

where $\tilde{T}_n(t) = \cos(n \arccos(2t - 1))$ is the Chebyshev polynomial on $[0, 1]$. Consider the triangle

$$\Delta := \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}.$$

Clearly, $\|h_n\|_\Delta = 1$, and

$$\frac{\partial}{\partial y} h_n(1, 1) = \tilde{T}_n'(1) = 4n^2.$$

Thus, the $O(n \log n)$ bound does not hold for Δ. (Note that condition (1.8) fails for this triangle.)

Now let us address the question of sharpness of estimate (1.9). For $\alpha = 1$, i.e. the $n \log n$ bound, the estimate is known to be sharp even in **0**-symmetric case (see [5], [7]). Consider now the domain

$$D_\alpha := \{(x, y) \in \mathbb{R}^2 : x^2 \le y \le 1 - x^\alpha\}, \quad 0 < \alpha.$$

Clearly, this star-like set satisfies (1.8), and the $\mathrm{Lip}_M \alpha$ condition holds for the function r associated with this set. Consider now $g_n(x, y) := xy^n \in H^2_{n+1}$. An easy calculation yields that

$$\|g_n\|_{D_\alpha} \le \|(1 - |x|^\alpha)^n x\|_{[-1,1]} = \left(\frac{\alpha n}{\alpha n + 1}\right)^n \frac{1}{(1 + \alpha n)^{1/\alpha}} \le Cn^{-1/\alpha}.$$

On the other hand $\|\frac{\partial g_n}{\partial x}\|_{D_\alpha} = 1$. Thus, $M(H^2_n, D_\alpha) \ge \frac{n^{1/\alpha}}{C}$ which is also the rate of the upper bound of (1.9) for $0 < \alpha < 1$. Hence, the estimate (1.9) is sharp, in general.

2 Proof of the Theorem 1.1 in the case $d = 2$

Let the star-like domain $K_r \subset \mathbb{R}^2, \mathbf{0} \in \partial K_r$, be parameterized by polar coordinates:

$$x = \tilde{\rho} \cos \phi, \quad y = \tilde{\rho} \sin \phi, \quad 0 \le \tilde{\rho} \le r(\cos \phi, \sin \phi), \quad |\phi| \le \frac{\pi}{2}. \quad (2.1)$$

Set

$$\tilde{r}(t) := r\left(\frac{1}{\sqrt{1 + t^2}}, \frac{t}{\sqrt{1 + t^2}}\right), \quad t = \tan \phi, \quad t \in \mathbb{R}.$$

Since $r \in \mathrm{Lip}_M \alpha$ on S^1, it immediately follows that $\tilde{r} \in \mathrm{Lip}_M \alpha$ on $\mathbb{R}$. Moreover, by (1.8) (with $x_1 = \cos \phi = 1/\sqrt{1 + t^2}$)

$$e^{-C_3|t|^{1-\epsilon}} \le \tilde{r}(t) \le C_1 |t|^{-\beta}, \quad t \in \mathbb{R}, \quad (2.2)$$

where $C_3 > 0$ depends only on K_r. Suppose we have a homogeneous polynomial $p \in H^2_n$

$$p(x, y) = \sum_{k=0}^{n} a_k x^{n-k} y^k = x^n \sum_{k=0}^{n} a_k \left(\frac{y}{x}\right)^k,$$

satisfying
$$\|p\|_{K_r} \le 1. \tag{2.3}$$

Setting $t = \tan\phi, |\phi| \le \pi/2$, we obtain a weighted univariate polynomial on $\mathbb{R}$

$$p(r(\cos\phi,\sin\phi)\cos\phi, r(\cos\phi,\sin\phi)\sin\phi) = \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^n \sum_{k=0}^{n} a_k t^k. \tag{2.4}$$

Denote $q_n(t) := \sum_{k=0}^{n} a_k t^k$. Then by (2.3) and (2.4)

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^n q_n(t) \right\|_{\mathbb{R}} = \|p\|_{\partial K_r} \le 1. \tag{2.5}$$

Observe that for $(x, y) \in \partial K_r$

$$\frac{\partial p(x,y)}{\partial x} = n \sum_{k=0}^{n} a_k x^{n-k-1} y^k - \sum_{k=1}^{n} k a_k x^{n-k-1} y^k$$

$$= n \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t) - \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} t q_n'(t). \tag{2.6}$$

and

$$\frac{\partial p(x,y)}{\partial y} = x^{n-1} \sum_{k=1}^{n} k a_k \left(\frac{y}{x}\right)^{k-1} = \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t). \tag{2.7}$$

In order to estimate the weighted polynomials appearing in (2.6) and (2.7) we shall need the following two lemmas.

Lemma 2.1 *Let $\tilde{r} \in C(\mathbb{R})$, $\tilde{r} > 0$, satisfy the right inequality in (2.2) for some $\beta > 0$. Then there exists a $t^* > 0$ depending only on $\tilde{r}$ such that for any $n \ge 2 + 2/\beta$ and $q_n \in P_n^1$ we have*

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t) \right\|_{\mathbb{R}} = \left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t) \right\|_{[-t^*,t^*]}.$$

Proof. Note that $\tilde{r}(t) \leq C_1|t|^{-\beta}$ and $n \geq 2 + 2/\beta$ ensure that

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{\mathbb{R}} < \infty.$$

Thus, without loss of generality we can assume

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{\mathbb{R}} = 1.$$

It is known (for example, see [2]) that for any $q_n(t) \in P_n^1$

$$|q_n(t)| \leq \|q_n\|_{[-1,1]} |T_n(t)|, \quad |t| > 1, \tag{2.8}$$

where

$$T_n(t) := \frac{1}{2} \left((t + \sqrt{t^2 - 1})^n + (t - \sqrt{t^2 - 1})^n \right)$$

is the Chebyshev polynomial. Evidently,

$$|T_n(t)| \leq 2^n |t|^n, \quad |t| > 1. \tag{2.9}$$

To estimate the norm $\|q_n\|_{[-1,1]}$, observe that

$$\inf_{|t| \leq 1} \tilde{r}(t) > 0$$

and, hence,

$$|q_n(t)| \leq \left(\frac{\sqrt{1+t^2}}{\tilde{r}(t)} \right)^{n-1} \leq C_4^{n-1}, \quad |t| \leq 1, \tag{2.10}$$

with some positive constant C_4 depending only on $\tilde{r}$. Therefore, combining estimates (2.8), (2.9) and (2.10), we obtain that for $|t| > 1$

$$|q_n(t)| \leq 2^n |t|^n C_4^{n-1} = C_5^n |t|^n.$$

Taking into consideration that

$$\tilde{r}(t) \leq C_1 |t|^{-\beta},$$

we obtain for $|t|>1$

$$\left|\left(\frac{\tilde r(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t)\right| \le \frac{\tilde r^{n-1}(t) C_5^n |t|^n}{|t|^{n-1}} \le \frac{C_6^n}{|t|^{\beta(n-1)-1}},$$

where C_6 depends only on $\tilde r$. Note that assumption $n \ge 2 + 2/\beta$ yields that $\beta(n-1)-1 \ge \beta n/2$. Thus, setting $t^* := C_6^{2/\beta} + 1$ we obtain for $|t| > t^* > 1$

$$\left|\left(\frac{\tilde r(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t)\right| \le \left(\frac{C_6}{|t|^{\beta/2}}\right)^n < 1.$$

Hence, the norm of the weighted polynomial is achieved on the finite interval $[-t^*, t^*]$. $\square$

In what follows $C_k,\ k \in \mathbb{N}$ will stand for constants depending only on $\tilde r$.

Lemma 2.2 *Let $\tilde r \in C(\mathbb{R})$ satisfy (2.2). Then for any $q_n \in P_n^1$ satisfying (2.5)*

$$\left\|\left(\frac{\tilde r(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t)|\right\|_{\mathbb{R}} \le C_7 \rho_\alpha(n), \tag{2.11}$$

where $\rho_\alpha(n)$ is defined in (1.10).

Proof.

First, observe that condition (2.5) can be rewritten in the following form:

$$|e^{-nQ(t)} q_n(t)| \le 1 \tag{2.12}$$

where

$$Q(t) := \log \Omega(t) \ \text{ and } \ \Omega(t) := \frac{\sqrt{1+t^2}}{\tilde r(t)}.$$

By a well-known inequality (see, for example, [1], p. 92) for any $\xi \in [-t^*, t^*]$, where t^* is defined in Lemma 2.1, and any $z = u+iv \in \mathbb{C}$ such that $|z-\xi| \le \rho$, $0<\rho<1/e$,

$$\log|q_n(z)| \le \frac{|v|}{\pi}\int_{\mathbb{R}} \frac{\log|q_n(t)|}{(t-u)^2+v^2}dt \le \frac{n|v|}{\pi}\int_{\mathbb{R}} \frac{Q(t)}{(t-u)^2+v^2}dt$$

$$= \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{Q(t_1 + \xi)}{(t_1 - u_1)^2 + v^2} dt_1, \tag{2.13}$$

where $t_1 := t - \xi$, $u_1 := u - \xi$. Note that $|u_1| = |u - \xi| \le \rho$, $|v| \le \rho$.

Since

$$\int_{\mathbb{R}} \frac{1}{(t - u_1)^2 + v^2} dt = \frac{\pi}{|v|},$$

we have that the last expression in (2.13) is equal to

$$\frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{Q(t + \xi) - Q(\xi)}{(t - u_1)^2 + v^2} dt + nQ(\xi).$$

Hence,

$$\log(e^{-nQ(\xi)}|q_n(z)|) \le \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{\log \Omega(t + \xi) - \log \Omega(\xi)}{(t - u_1)^2 + v^2} dt. \tag{2.14}$$

The numerator in the integrand on the right-hand side can be rewritten as follows

$$\log \Omega(t + \xi) - \log \Omega(\xi) = \log \frac{\Omega(t + \xi)}{\Omega(\xi)} = \log(\frac{\Omega(t + \xi) - \Omega(\xi)}{\Omega(\xi)} + 1). \tag{2.15}$$

Let us introduce the following modulus of continuity of function Ω:

$$\tilde{\omega}(\Omega, t) := \max\{|\Omega(x_1) - \Omega(x_2)| : x_1 \in [-t^*, t^*], |x_1 - x_2| \le t\}, \quad t > 0.$$

It has the following properties:

(i) $\tilde{\omega}(\Omega, t)$ is an increasing function of t.

(ii) For $|t| \ge 1$, by property (2.2), we have

$$\tilde{\omega}(\Omega, t) \le 2 \max_{x \in [-t^* - t, t^* + t]} |\Omega(x)| \le e^{C_8 (t^* + t)^{1-\varepsilon}}. \tag{2.16}$$

This yields

$$\log \tilde{\omega}(\Omega, t) = O(t^{1-\varepsilon}), \quad \text{for} \quad |t| \ge 1. \tag{2.17}$$

(iii) $\tilde{\omega}(\Omega, t) \le C_9 t^\alpha$ for $|t| \le 1$, because $\Omega(t) := \frac{\sqrt{1+t^2}}{\tilde{r}(t)}$, where $\tilde{r}(t) \in \mathrm{Lip}_M \alpha$ on $\mathbb{R}$ and $\tilde{r}(t) \ge C_{10}$ for $|t| \le 1 + t^*$ by property (2.2).

Thus, using (2.14) and substituting $t = u_1 + |v|y,\ t_2 = \rho(1+y)$, we have

$$\log(e^{-nQ(\xi)}|q_n(z)|)$$

$$\begin{aligned}
&\leq \frac{n|v|}{\pi}\int_{\mathbb{R}} \frac{\log\left(\frac{\Omega(t+\xi)-\Omega(\xi)}{\Omega(\xi)}+1\right)}{(t-u_1)^2+v^2}dt \leq \frac{n|v|}{\pi}\int_{\mathbb{R}} \frac{\log\left(\frac{\tilde{\omega}(\Omega,|t|)}{C_{11}}+1\right)}{(t-u_1)^2+v^2}dt \\
&= \frac{n}{\pi}\int_{\mathbb{R}} \frac{\log\left(\frac{\tilde{\omega}(\Omega,|u_1+|v|y|)}{C_{11}}+1\right)}{y^2+1}dy \leq \frac{2n}{\pi}\int_0^\infty \frac{\log\left(\frac{\tilde{\omega}(\Omega,\rho(1+y))}{C_{11}}+1\right)}{y^2+1}dy \\
&\leq \frac{4n}{\pi}\int_0^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,\rho(1+y))}{C_{11}}+1)}{(y+1)^2}dy = \frac{4n\rho}{\pi}\int_\rho^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}}+1)}{t_2^2}dt_2 \\
&= \frac{4n\rho}{\pi}\int_\rho^1 \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}}+1)}{t_2^2}dt_2 + \frac{4n\rho}{\pi}\int_1^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}}+1)}{t_2^2}dt_2. \quad (2.18)
\end{aligned}$$

We can estimate each integral as follows. Using property (iii) of $\tilde{\omega}(\Omega, t)$ we obtain for the first term in (2.18):

$$\frac{4n\rho}{\pi}\int_\rho^1 \frac{\log(\frac{\tilde{\omega}(\Omega,t)}{C_{11}}+1)}{t^2}dt \leq C_{12}n\rho\int_\rho^1 t^{\alpha-2}dt \leq C_{13}n\gamma_\alpha(\rho), \quad (2.19)$$

where

$$\gamma_\alpha(\rho) := \begin{cases} \rho^\alpha, & \alpha < 1; \\ \rho\log\frac{1}{\rho}, & \alpha = 1. \end{cases} \quad (2.20)$$

To estimate the second term in (2.18) we use (ii) which yields

$$\frac{4n\rho}{\pi}\int_1^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t)}{C_{11}}+1)}{t^2}dt \leq C_{14}n\rho\int_1^\infty \frac{t^{1-\varepsilon}}{t^2}dt = \frac{C_{14}}{\varepsilon}n\rho. \quad (2.21)$$

Therefore, by (2.19) and (2.21) for the whole sum in (2.18) we obtain:

$$\log(e^{-nQ(\xi)}|q_n(z)|) \leq C_{15}n(\gamma_\alpha(\rho)+\rho) \leq 2C_{15}n\gamma_\alpha(\rho). \quad (2.22)$$

Setting in (2.22) $\rho := 1/\rho_\alpha(n)$ where $\rho_\alpha(n)$ is defined in (1.10), we obtain

$$\log(e^{-nQ(\xi)}|q_n(z)|) \leq C_{16}. \quad (2.23)$$

Thus, by (2.23) for all z such that $|z-\xi| \le \rho = 1/\rho_\alpha(n)$

$$|q_n(z)| \le C_{17} e^{nQ(\xi)}. \tag{2.24}$$

By the Cauchy Integral Formula

$$q_n'(\xi) = \frac{1}{2\pi i} \int_{|z-\xi|=\rho} \frac{q_n(z)}{(z-\xi)^2} dz.$$

Estimating this integral using (2.24), we have

$$|q_n'(\xi)| \le \max_{z:|z-\xi|\le\rho} |q_n(z)| \frac{1}{\rho} \le C_{17} e^{nQ(\xi)} \frac{1}{\rho} = C_{17} e^{nQ(\xi)} \rho_\alpha(n).$$

Hence,

$$\left(\frac{\tilde{r}(\xi)}{\sqrt{1+\xi^2}}\right)^n |q_n'(\xi)| = e^{-nQ(\xi)} |q_n'(\xi)| \le C_{17} \rho_\alpha(n). \tag{2.25}$$

Recall that this estimate holds for $\xi \in [-t^*, t^*]$, where t^* is defined in Lemma 2.1. Using Lemma 2.1

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t) \right\|_{\mathbb{R}} = \left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t) \right\|_{[-t^*,t^*]}.$$

Furthermore, by (2.2)

$$\left\| \frac{\sqrt{1+t^2}}{\tilde{r}(t)} \right\|_{[-t^*,t^*]} \le C_{18}. \tag{2.26}$$

Finally, by last two estimates and (2.25)

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t) \right\|_{\mathbb{R}} \le C_{17} \rho_\alpha(n) \left\| \frac{\sqrt{1+t^2}}{\tilde{r}(t)} \right\|_{[-t^*,t^*]} \le C_{19} \rho_\alpha(n).$$

This completes the proof of Lemma 2.2.

□

Now in order to prove the statement of the Theorem for $d = 2$ we need to estimate partial derivatives in (2.6) and (2.7) under the assumption (2.5). Note that for $n < 2 + 2/\beta$ the statement of the Theorem is obviously true by the equivalence of norms in finite-dimensional spaces; i.e., we may assume that $n \geq 2+2/\beta$ and, hence, Lemma 2.1 is applicable. Then using (2.6), (2.5) and (2.26) together with Lemmas 2.1 and 2.2, we obtain

$$\left\|\frac{\partial p}{\partial x}\right\|_{K_r} = \left\|\frac{\partial p}{\partial x}\right\|_{\partial K_r}$$

$$\leq n \left\|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t)\right\|_{[-t^*,t^*]} + t^* \left\|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t)\right\|_{[-t^*,t^*]}$$

$$\leq n \left\|\frac{\sqrt{1+t^2}}{\tilde{r}(t)}\right\|_{[-t^*,t^*]} + C_{19}\rho_\alpha(n) \leq C_{18}n + C_{19}\rho_\alpha(n) \leq C_{20}\rho_\alpha(n).$$

Similarly, by (2.7), (2.5) and Lemma 2.2

$$\left\|\frac{\partial p}{\partial y}\right\|_{K_r} = \left\|\frac{\partial p}{\partial y}\right\|_{\partial K_r} \leq C_7\rho_\alpha(n).$$

The last two estimates complete the proof of the Theorem 1.1 for $d = 2$.

3 Proof of the Theorem 1.1 for $d > 2$

First let us observe that for any $h \in H_n^d$ we have $h(t\mathbf{x}) = t^n h(\mathbf{x})$, $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, i.e. $D_{\mathbf{u}}h(\mathbf{0}) = 0$ for any $\mathbf{u} \in S^{d-1}$. Thus, it suffices to estimate $D_{\mathbf{u}}h(\mathbf{x})$ for $\mathbf{x} \in \partial K_r \setminus \{\mathbf{0}\}$. Note that property (2.2) of the star-like domain K_r yields that $\mathrm{Int} K_r \neq \emptyset$. Thus, there exists a d-dimensional ball $B \subset K_r$, $\mathbf{0} \notin B$. Consider the quantity

$$\eta(K) := \inf_{\mathbf{u}\in S^{d-1}} \sup_{\mathbf{w}\in B} | < \mathbf{u}, \mathbf{w} > |.$$

We claim that $\eta(K) > 0$. Indeed, for any $\mathbf{u} \in S^{d-1}$ we clearly have

$$\tau(\mathbf{u}) := \sup_{\mathbf{w}\in B} | < \mathbf{u}, \mathbf{w} > | > 0.$$

Since $\tau(\mathbf{u})$ is continuous on compact set S^{d-1} the claim is obvious. Thus, for every $\mathbf{u} \in S^{d-1}$ there exists $\mathbf{b} \in B$ such that $| < \mathbf{u}, \mathbf{b} > | \geq \eta(K)$.

Let now $h \in H_n^d$ satisfy $\|h\|_{K_r} \leq 1$, and consider an arbitrary $\mathbf{x} \in \partial K_r \setminus \{\mathbf{0}\}$. Denote

$$\mathbf{u} := \frac{\nabla h(\mathbf{x})}{|\nabla h(\mathbf{x})|} \in S^{d-1},$$

where ∇h is the gradient of h. By the above observation we can choose $\mathbf{b} \in B$ so that $| < \mathbf{u}, \mathbf{b} > | \geq \eta(K)$. Set $\mathbf{w} := \mathbf{b}/|\mathbf{b}|$. Then, using that $\nabla h(\mathbf{x}) = |\nabla h(\mathbf{x})|\mathbf{u}$, we have

$$|D_{\mathbf{w}} h(\mathbf{x})| = | < \nabla h(\mathbf{x}), \mathbf{w} > | = \frac{|\nabla h(\mathbf{x})|}{|\mathbf{b}|} | < \mathbf{u}, \mathbf{b} > | \geq \frac{\eta(K)}{|\mathbf{b}|} |\nabla h(\mathbf{x})|.$$

In other words, with some constant $C_{21} > 0$ depending only on K_r

$$|\nabla h(\mathbf{x})| \leq \frac{|\mathbf{b}|}{\eta(K)} |D_{\mathbf{w}} h(\mathbf{x})| \leq C_{21} |D_{\mathbf{w}} h(\mathbf{x})|. \tag{3.1}$$

Thus, it suffices to estimate $|D_{\mathbf{w}} h(\mathbf{x})|$, where $\mathbf{w} = \mathbf{b}/|\mathbf{b}|$ and $\mathbf{b} \in B$. Consider now the 2-dimensional plane $L_{\mathbf{b}} := \operatorname{span}\{\mathbf{x}, \mathbf{b}\}$. Then $\tilde{K}_r := K_r \cap L_{\mathbf{b}}$ is a 2-dimensional star-like domain with $\mathbf{x} \in \tilde{K}_r$. Moreover, by (2.26), if we estimate all directional derivatives of $p|_{L_{\mathbf{b}}} \in H_n^2$ at $\mathbf{x}$ then this will yield an upper bound for $|\nabla h(\mathbf{x})|$. Hence, our considerations can be reduced to the 2-dimensional plane $L_{\mathbf{b}}$ and the star-like domain $\tilde{K}_r$. Let $\tilde{r} := r|_{L_{\mathbf{b}} \cap S^{d-1}}$ be the corresponding radial function associated with $\tilde{K}_r$. In order to complete the proof we need to show that $\tilde{r}$ satisfies condition (2.2) with some constants independent of $\mathbf{x}$. Let

$$l := L_{\mathbf{b}} \cap \left\{ (0, x_2, \ldots, x_d) : (x_2, \ldots, x_d) \in \mathbb{R}^{d-1} \right\}$$

be the line in $L_{\mathbf{b}}$ supporting $\tilde{K}_r$ at $\mathbf{0}$, and denote by $\mathbf{y}^{\perp}$ the orthogonal projection of $\mathbf{y} = (y_1, \ldots, y_d) \in L_{\mathbf{b}}$ to the line l. Clearly,

$$|\mathbf{y} - \mathbf{y}^{\perp}| = \operatorname{dist}(\mathbf{y}, l) \geq y_1. \tag{3.2}$$

Furthermore, set $\tilde{\mathbf{y}} = (0, y_2, \ldots, y_d)$, where $\mathbf{y} = (y_1, y_2, \ldots, y_d)$. Obviously, $\mathbf{y} - \mathbf{y}^{\perp} \in L_{\mathbf{b}}$ is a normal direction to l in $L_{\mathbf{b}}$, $\mathbf{y} - \tilde{\mathbf{y}} = t\mathbf{e}_1$, where $t \in \mathbb{R}$, $\mathbf{e}_1 = (1, 0, \dots, 0)$. This means that the angle between $\mathbf{y} - \mathbf{y}^\perp$ and $\mathbf{y} - \tilde{\mathbf{y}}$ is invariant of the choice of $\mathbf{y} \in L_{\mathbf{b}}$, and so is the angle between $\tilde{\mathbf{y}} - \mathbf{y}^\perp$ and $\mathbf{y} - \mathbf{y}^\perp$. Denote by $\alpha(\mathbf{u}, \mathbf{v})$ the angle between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Then by the above observation using that $\mathbf{b} \in L_{\mathbf{b}}$ we have

$$\sin \alpha(\tilde{\mathbf{y}} - \mathbf{y}^\perp, \mathbf{y} - \mathbf{y}^\perp) = \sin \alpha(\tilde{\mathbf{b}} - \mathbf{b}^\perp, \mathbf{b} - \mathbf{b}^\perp)$$

$$= \frac{|\mathbf{b} - \tilde{\mathbf{b}}|}{|\mathbf{b} - \mathbf{b}^\perp|} \geq \frac{b_1}{|\mathbf{b}|} \geq \frac{\min\limits_{\mathbf{x} \in B} x_1}{\max\limits_{\mathbf{x} \in B} |\mathbf{x}|} = C_{22} > 0.$$

(It was used above that $B \subset K_r \subset \mathbb{R}^d_+$ and $\mathbf{0} \notin B$, i.e. $\min\limits_{\mathbf{x} \in B} x_1 > 0$.)

Thus, for every $y \in L_{\mathbf{b}}$

$$|\mathbf{y} - \mathbf{y}^\perp| = \frac{|\mathbf{y} - \tilde{\mathbf{y}}|}{\sin \alpha(\tilde{\mathbf{y}} - \mathbf{y}^\perp, \mathbf{y} - \mathbf{y}^\perp)} \leq \frac{y_1}{C_{22}}.$$

This together with (3.2) implies that $|\mathbf{y} - \mathbf{y}^\perp| \asymp |y_1|$ whenever $\mathbf{y} \in L_{\mathbf{b}}$ with constants involved being independent of $\mathbf{x}$. Thus, in conditions (1.8) we can replace $|y_1|$ by $|\mathbf{y} - \mathbf{y}^\perp| = \operatorname{dist}(\mathbf{y}, l)$, where l is the supporting line of $\tilde{K}_r$ at $\mathbf{0}$. Without loss of generality we can assume that l is a coordinate axis (this can be achieved by a rotation); i.e., it remains to refer to the already verified case when $d = 2$. This completes the proof of the Theorem 1.1.

References

[1] R. P. Boas, "Entire functions," Academic Press, New York, 1954.

[2] P. Borwein, T. Erdélyi, "Polynomials and polynomial inequalities," Springer, New York, 1995.

[3] B. Boyanov, Markov-type inequalities for polynomials and splines, in "Approximation Theory, X" (St. Louis, MO, 2001), pp. 31–90, Innov. Appl. Math., Vanderbilt University Press.

[4] L. A. Harris, A Bernstein-Markov theorem for normed spaces, J. Math. Anal. Appl., 208 (1997), 476–486.

[5] A. Kroó, Markov-type inequalities for homogeneous polynomials on smooth convex bodies, East J. Approx., 9 (2003), no.4, 487–500.

[6] A. Kroó, On Markov inequality for multivariate polynomials, in “Approximation Theory XI: Gatlinburg 2004” (C.K. Chui, M. Neamtu, L. Schumaker, eds.), pp. 211–229, Nashboro Press, Brentwood.

[7] Sz. Revesz, Y. Sarantopoulos, On Markov constants of homogeneous polynomials over real normed spaces, East J. Approx., 9 (2003), no.3, 277–304.

Local Inequalities for Multivariate Polynomials and Plurisubharmonic Functions

A. Brudnyi
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada T2N 1N4
Email address: albru@math.ucalgary.ca

Yu. Brudnyi
Department of Mathematics
Technion, Haifa, 32000, Israel
Email address: ybrudnyi@math.technion.ac.il

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

In the present paper we discuss some generalizations of the classical Chebyshev and Bernstein inequalities.

1 Introduction

In the present paper we discuss several multivariate generalizations of the classical Chebyshev and Bernstein inequalities. The first of these (in an equivalent form adapted to our exposition) states that for every polynomial $p(x)$ in $x \in \mathbb{R}$ of degree n and a pair of intervals $[c, d] \subset [a, b]$ the inequality

$$\sup_{[a,b]} |p| \leq T_n \left(\frac{2 + \lambda}{\lambda} \right) \sup_{[c,d]} |p| \tag{1.1}$$

holds with $\lambda := \frac{d-c}{b-a}$; here $T_n(x) := \cos(n \arccos x)$ is the Chebyshev polynomial of degree n.

Bernstein's inequality concerns polynomials of a complex variable $z \in \mathbb{C}$ of degree n. It states that for such a polynomial p and every $R > 1$

$$\sup_{|z|\le R} |p(z)| \le R^n \sup_{|z|\le 1} |p(z)| \ . \tag{1.2}$$

Both of these inequalities are sharp. In his 1936 paper Remez [35] generalizes the Chebyshev inequality replacing the subinterval $[c, d]$ by an arbitrary measurable subset $S \subset [a, b]$. The Remez inequality has the same form as (1.1) with $\lambda := \frac{|S|}{b-a}$ and is clearly sharp, as well.

There is no Remez type generalization of Bernstein's inequality: Yu. Brudnyi-Ganzburg inequality which we describe below gives, for this case, the inequality

$$\sup_{|z|\le R} |p(z)| \le (cR)^n \sup_S |p|$$

provided S is a measurable subset of the disk $\{z \in \mathbb{C} \ : \ |z| \le R\}$ of measure π and c is a constant independent of R and p.

Question 1.1 *What is the optimal value of c?*

There are also analogs of Chebyshev's and Remez' inequalities for trigonometric polynomials proved by V. Videnski [38] and T. Erdelyi, respectively; see the references in the book [3]. It is convenient for us to think of these inequalities as results on the traces of polynomials in $x, y \in \mathbb{R}^2$ to the circle $x^2 + y^2 = 1$.

2 Related Problems

2.1. The results discussed pose several problems for multivariate polynomials. Of course, it would be unlikely to establish such inequalities in a sharp form. So we are mostly looking for a sharp order of growth as the *relative measure* tends to zero. Here the relative measure of a d-measurable subset S of a convex body $K \subset \mathbb{R}^d$ is defined as $\frac{|S|}{|K|}$. (Here and below $|U|$ denotes the Lebesgue d-measure of $U \subset \mathbb{R}^d$.) According to the one-dimensional results exhibited above we have the following groups of problems for multivariate polynomials. Throughout this paper $\mathcal{P}_{n,d}$ denotes the linear space of polynomials in $x \in \mathbb{R}^d$ of degree n, and K stands for a bounded convex body in $\mathbb{R}^d$. We begin with a Chebyshev type problem. Let S be a subset of K with $|S| > 0$.

Problem 2.1 *Given K, find the optimal constant in the inequality*

$$\sup_K |p| \le C \sup_S |p| \ , \quad p \in \mathcal{P}_{n,d} \ , \tag{2.3}$$

where S runs over all convex subsets of K of relative measure $\lambda > 0$.

The required optimal constant is denoted by $C_K(n, \lambda)$. The next is a Remez type problem formulated as follows.

Problem 2.2 *(a) Find the optimal constant in (2.3) where S now runs over all measurable subsets of K of relative measure $\lambda > 0$.*
(b) The same in the case of an arbitrary K, i.e., (2.3) should hold for all measurable $S \subset K$ of relative measure λ and all bounded convex bodies.

Let us denote the optimal constant in (a) by $R_K(n, \lambda)$ and that in (b) by $R(n, \lambda)$. It is clear that

$$C_K(n, \lambda) \le R_K(n, \lambda) \le R(n, \lambda) \ .$$

Conjecture 2.3 ([4, 1973]).

$$C_K(n, \lambda) = R_K(n, \lambda).$$

Comparing the Chebyshev and Remez inequalities shows that this is the case for $d = 1$; the conjecture is also true for K a finite convex cone; see [4]. Otherwise, the conjecture remains an open problem. Let $V \subset \mathbb{R}^d$ be a real algebraic variety of pure dimension $\widetilde{d} < d$. Equip V with the metric induced by the Euclidean metric of $\mathbb{R}^d$ and denote by μ_V the measure induced on V by the Lebesgue measure of $\mathbb{R}^d$. Let now $B \subset V$ be a (metric) ball and $S \subset B$ be a measurable subset of the relative measure $\frac{\mu_V(S)}{\mu_V(B)}$ equals $\lambda > 0$.

Problem 2.4 *Find the optimal constant (or its asymptotic as* $\lambda \to 0$*) in the inequality*

$$\sup_B |p| \leq C \sup_S |p| \ , \quad p \in \mathcal{P}_{n,d}|_V \ . \tag{2.4}$$

Let us denote the required optimal constant by $C_V(B, \lambda)$. Note that the behaviour of this constant depends strongly on the structure of V as can be seen in the case of the curve $y^3 = x^2$ in $\mathbb{R}^2$ with the cusp at $(0,0)$. Here for a ball centered at $(0,0)$ the asymptotics of $C_V(B, \lambda)$ as $\lambda \to 0$ are distinct in order from those for a ball centered at a smooth point of the curve and not containing $(0,0)$. But for such varieties as the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ or the upper sheet of the hyperboloid $\mathbb{H}^{d-1} = \{x \in \mathbb{R}^d \ : \ -x_d^2 + \sum_{i=1}^{d-1} x_i^2 = 1\}$ this effect does not exist, since all their points are smooth. Observe that $\mathcal{P}_{n,d}|_{\mathbb{S}^{d-1}}$ is the space of spherical harmonics of degree n (trigonometric polynomials if $d = 1$).

2.2. The problems presented have had a lot of generalizations and variants. We mention only several of them. (a) In all the above problems the uniform norms can be replaced by the corresponding L_p-norms or some other rearrangement invariant norms. (b) Polynomials can be replaced by entire functions on $\mathbb{C}^d$ of exponential type whose L_p-norms are bounded on a subset $S \subset \mathbb{R}^d$ of density $\lambda > 0$. The latter means that the relative d-measure of $S \cap B$ for every closed ball centered at S is at least λ. One-dimensional prototypes of this problem are those of Cartwright and of Plancherel-Polya. (c) Polynomials can be replaced by real analytic or plurisubharmonic functions. (d) Finally, one can replace the Lebesgue d-measure by the Hausdorff α-measure. Of course, we should have

$$\alpha > d - 1 \ , \tag{2.5}$$

since the zero set of a polynomial $p \in \mathcal{P}_{n,d}$ has Hausdorff $(d-1)$-dimension zero and therefore inequality (2.3) cannot be true for a subset S of Hausdorff measure $d - 1$.

3 Formulation of Main Results

3.1. In the sequel we briefly discuss some results related to the problems posed above and their generalizations and prove a new polynomial inequality for the case of Hausdorff α-measures satisfying (2.5). Unfortunately, it is absolutely unrealistic to present in this short paper all the important results of this vast field. Therefore we will restrict our consideration to some results studied in the authors papers and results related to them.

3.2. Based on a simple geometric fact proved in [36] it is established in [4] that the optimal constant of Problem 2.1 can be found by the formula given below in which $\Pi \subset \mathbb{R}^d$ denotes a layer between two parallel hyperplanes such that the relative d-measure of $\Pi \cap K$ is λ. Moreover, we set

$$\rho_K(\Pi) := \sup_{x\in K} \inf_{y\in\Pi} |x-y|$$

and denote by $\nu(\Pi)$ the width of Π (the metric of $\mathbb{R}^d$ is chosen to be Euclidean). Under these notations we have

$$C_K(n,\lambda) = \sup_{\Pi} \, T_n\left(\frac{2\rho_K(\Pi)}{\nu(\Pi)} + 1\right) \tag{3.6}$$

where, as above, T_n is the n-th Chebyshev polynomial. In some cases equality (3.6) allows us to derive the sharp asymptotics of $C_K(n,\lambda)$. For example, for K a d-dimensional ball

$$C_K(n,\lambda) = \frac{1}{2}\left(\frac{c_d}{\lambda}\right)^n + o(\lambda^{-n}) \ , \quad \lambda \to 0 \ ; \tag{3.7}$$

here $c_1 = 8$ and

$$c_d = 4\,\omega_{d-1}\left(1+\frac{1}{d}\right)^{\frac{d+1}{2}}\left(1-\frac{1}{d}\right)^{\frac{d-1}{2}} \ , \quad d > 1 \ ,$$

where ω_{d-1} is the volume of the $(d-1)$-dimensional unit ball. Formula (3.6) allows to derive sharp asymptotics for several other convex bodies, e.g., cubes or ellipsoids, but a general result is unknown. Interesting new results on the multivariate Chebyshev's constant C_K can be found in [28].

3.3. Strikingly, the optimal constant of Problem 2.2 (b) is known [4]. Namely,

$$R(n,\lambda) = T_n\left(\frac{1+\sqrt[d]{1-\lambda}}{1-\sqrt[d]{1-\lambda}}\right). \tag{3.8}$$

Moreover, for this case

$$C_K(n,\lambda) = R_K(n,\lambda) = R(n,\lambda)$$

provided K is a convex cone with a flat base. Estimating the right-hand side we, in particular, get

$$R(n,\lambda) \leq \frac{1}{2}\left(\frac{4d}{\lambda}\right)^n . \tag{3.9}$$

Comparing this with the asymptotic in (3.7) one can see that λ^{-n} is the sharp order of growth for $R(n,\lambda)$ as $\lambda \to 0$. Estimate (3.9) is applied in [4] to derive an inequality for the nondecreasing rearrangement of a polynomial from $\mathcal{P}_{n,d}$ and in this way to obtain Remez type inequalities for integral (or, more generally, rearrangement invariant) metrics.

In particular, we have for every $p \in \mathcal{P}_{n,d}$ and $S \subset K \subset \mathbb{R}^d$ chosen as above:

$$\left(\frac{1}{|K|}\int_K |p|^a\,dx\right)^{1/a} \leq C\left(\frac{1}{|S|}\int_S |p|^b\,dx\right)^{1/b} \tag{3.10}$$

where $0 < a, b \leq \infty$ and C depends only on n, d and $\min(1,a,b)$. Recently there has been a considerable interest in the behaviour of the optimal constant C in (3.10), see [7], [27], [10], [12], [15], [19], [31], [32]. In particular, the dimension d.

3.4. Before discussing the third problem on traces of polynomials to algebraic varieties, we turn to a seemingly distant area of research related to purisubharmonic functions; see [26] for the basic definitions and facts. An inequality for such functions has a surprisingly wide range of applications including for the problems presented in Section 2.2. To describe this inequality, proved in [9], we denote by $B(x,\rho)$ and $B_c(x,\rho)$ the Euclidean balls with center x and radius ρ in $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively. Set also $\Phi(x) := x + \sqrt{x^2-1}$, $|x| \geq 1$.

Theorem 3.1 *Assume that f is a plurisubharmonic function on $B_c(0,\rho)$, $\rho > 1$, such that*

$$\sup_{B_c(0,\rho)} f = 0 \quad \text{and} \quad \sup_{B_c(0,1)} f \geq -1 \ .$$

Then there is a constant $c = c(\rho)$ such that

$$\sup_{B(x,t)} f \leq c \log \Phi \left(\frac{1 + \sqrt[d]{1-\lambda}}{1 - \sqrt[d]{1-\lambda}} \right) + \sup_S f \tag{3.11}$$

for every measurable $S \subset B(x,t)$ with $\lambda = \frac{|S|}{|B(x,t)|}$ and every ball $B(x,t) \subset B_c(0,1)$.

(In fact, in [9] a weaker version of (3.11) is obtained. The very same proof gives inequality (3.11) if one replaces inequalities (2.31) and (2.26) of [9] by their sharp forms presented in [4], see there Lemmas 3 and 1.) To explain the relation of (3.11) to the polynomial inequalities under consideration it suffices to note that $\log |p(z)|$ is plurisubharmonic if p is a (complex) polynomial in $z \in \mathbb{C}^d$. Since the same is true for a holomorphic function defined on a domain of $\mathbb{C}^d$, inequality (3.11) leads also to Remez type results for holomorphic functions. We present here only two such results referring the interested reader to the papers [9]-[13], [15], [17]. The first of these results gives a solution to Problem 2.4. So, suppose that $V \subset \mathbb{R}^d$ is a real algebraic variety of pure dimension $\widetilde{d} < d$.

Theorem 3.2 ([9]) *For every smooth point $x \in V$ there is an open neighbourhood N of x such that*

$$\sup_B |p| \leq \left(\frac{c_0}{\lambda} \right)^{c_1 n \, deg(V)} \sup_S |p| \ , \quad p \in \mathcal{P}_{n,d}|_V \ .$$

Here we use the notation of (2.4), and $c_0 = c_0(\widetilde{d})$, while c_1 is a numerical constant.

Note that $deg(V)$, the degree of an algebraic variety, is defined, e.g., by the Bezout theorem. In particular, for the sphere $\mathbb{S}^{d-1}$ this equals 2. Moreover, the proof in [9] gives some information on the size of N. In the special case of spherical harmonics we have

$$\sup_{\mathbb{S}^{d-1}} |p| \leq \left(\frac{c_0}{\lambda}\right)^{c_2 n} \sup_S |p|, \quad p \in \mathcal{P}_{n,d}|_V,$$

where c_2 is a numerical constant. The next result is a Remez type inequality for a function f holomorphic in the ball $B_c(0,\rho)$ with $\rho > 1$. In this case we consider a convex body $K \subset B(0,1) \subset \mathbb{R}^d$ and a measurable subset $S \subset K$ of relative measure $\lambda > 0$. Then the following is true:

Theorem 3.3 ([10]) *There is a constant $n = n(f,r)$ such that the inequality*

$$\sup_K |f| \leq \left(\frac{4d}{\lambda}\right)^n \sup_S |f|$$

holds.

In some cases the constant n can be effectively estimated, see [10], [15], [17]. As in the case of inequality (3.11) the proof in [10] with the above mentioned modification leads to a sharper inequality with $\Phi\left(\frac{1+\sqrt[d]{1-\lambda}}{1-\sqrt[d]{1-\lambda}}\right)$ instead of $\frac{4d}{\lambda}$.

3.5. Finally, we present a result giving an answer to the question on subsets of positive Hausdorff α-measure with $\alpha > d-1$ (see Section 2.2.(d); its proof will be postponed to the final section). To its formulation we require the notion of (upper) *Ahlfors regular subsets of* $\mathbb{R}^d$. In the next definition $\mu_\alpha(S)$ stands for the Hausdorff α-measure of a Borel subset $S \subset \mathbb{R}^d$.

Definition 3.4 *S is said to be α-regular, if there exist constants $r_0 > 0$ and $\gamma_+ > 0$ such that for every ball B centered at a point of S and of radius less than r_0*

$$\mu_\alpha(S \cap B) \leq \gamma_+ |B|^{\alpha/d} . \tag{3.12}$$

(Recall that $|B|$ stands for the Lebesgue d-measure of B, so $|B| = \mu_d(B)$.)

Theorem 3.5 *Assume that $K \subset \mathbb{R}^d$ is a bounded open set and $S \subset K$ is α-regular with*

$$\alpha > d - 1 \tag{3.13}$$

and with parameters γ_+ and r_0. Assume also that

$$\gamma_- := \frac{\mu_\alpha(S)}{|K|^{\alpha/d}} > 0 \ . \tag{3.14}$$

Then there is a constant $C > 1$ such that for every $p \in \mathcal{P}_{n,d}$

$$\left(\frac{1}{|K|}\int_K |p|^r \, dx\right)^{1/r} \leq C \left(\frac{1}{\mu_\alpha(S)}\int_S |p|^q \, d\mu_\alpha\right)^{1/q} . \tag{3.15}$$

Here $0 < q, r \leq \infty$ and C depends on K, n, q, r, α, d, $\gamma_\pm$ and is increasing in $1/\gamma_-$.

The first result of this type was proved in [16] for univariate polynomials (and more generally for subharmonic functions) by a method which gives a constructive constant C in (3.15). The proof of the present paper is unconstructive (an existence theorem). In a forthcoming paper we will give a constructive proof of this and a more general result concerning plurisubharmonic and real analytic functions.

4 Comments

Univariate inequalities for polynomials have appeared in Approximation Theory and for a long time have been considered as technical tools for proofs of Bernstein's type inverse theorems. At the present time polynomial inequalities have been found a lot of important applications in areas which are well apart from Approximation Theory. We will only briefly mention several of these areas. The papers [23], [7] and [27] apply polynomial inequalities with different integral norms to study some problems of Convex Geometry (in particular, the famous Slice Problem). In the papers [1], [2], [33] and [34] and books [20] and [24]. Chebyshev-Bernstein and related Markov type inequalities are used to explore a wide range of properties of the classical spaces of smooth functions including Sobolev type embeddings and trace theorems, extensions and differentiability. The papers [21] and [22] on Bernstein's type inequalities for traces of polynomials to algebraic varieties were inspired by and would have important applications to some basic problems of the theory of subelliptic differential equations. The paper [6] discovers a profound relation between the exponents in the tangential Markov inequalities for restrictions of polynomials to a smooth manifold $M \subset \mathbb{R}^d$ and the property of M to be an algebraic manifold. An application of polynomial inequalities to Cartwright type theorems for entire functions (see Section 2.2 (b)) is presented in [11] and [13], see also [29], [30], [25]. In the papers [14] and [18] Chebyshev-Bernstein type inequalities are used to prove some distributional inequalities for the number of zeros of families of holomorphic functions depending holomorphically on a multidimensional complex parameter. In the papers [31] and [32] these inequalities are used to estimate the distribution of zeros of certain families of random analytic functions. Finally, we mention the application of polynomial inequalities to the second part of Hilbert's sixteenth problem concerning the number of limit cycles of planar polynomial vector fields, see [37], [14] and [18]. In particular, it was proved in the last paper that locally, the expected number of limit cycles of a random polynomial vector field of degree d is $O(\log d)$, a surprisingly small estimate in comparison with known examples where the number of cycles is of polynomial growth.

5 Proof of Theorem 3.5

Proof. We set for brevity

$$||p; S||_q = \left(\frac{1}{\mu_\alpha(S)}\int_S |p|^q \, d\mu_\alpha\right)^{1/q} \qquad \text{and}$$

$$||p; K||_r = \left(\frac{1}{|K|}\int_K |p|^r \, dx\right)^{1/r}.$$

Let $\Sigma(\gamma_+, \gamma_-)$, $\gamma_\pm > 0$, be the class of μ_α-measurable subsets S of K satisfying condition (3.12) and also

$$\mu_\alpha(S) \geq \gamma_- \ . \tag{5.1}$$

We require to show that there exists a positive constant $C = C(K, n, q, r, \alpha, d, \gamma_+, \gamma_-)$ such that for every polynomial $p \in \mathcal{P}_{n,d}$

$$||p; K||_r \leq C||p; S||_q \ . \tag{5.2}$$

Remark 5.1 Let C_0 be the optimal constant in (5.2). Since the class $\Sigma(\gamma_+, \gamma_-)$ increases as γ_- decreases, C_0 increases in $1/\gamma_-$, as is required in the theorem.

If, on the contrary, the constant in (5.2) does not exist, one can find sequences of polynomials $\{p_j\} \subset \mathcal{P}_{n,d}$ and sets $\{S_j\} \subset \Sigma(\gamma_+, \gamma_-)$ so that

$$||p_j; K||_r = 1 \quad \text{for all} \quad j \in \mathbb{N} \ , \tag{5.3}$$

$$\lim_{j\to\infty} ||p_j; S_j||_q = 0 \ . \tag{5.4}$$

Since all (quasi-) norms on $\mathcal{P}_{n,d}$ are equivalent, (5.3) implies the existence of a subsequence of $\{p_j\}$ that converges uniformly on K to a polynomial $p \in \mathcal{P}_{n,d}$. Assume without loss of generality that $\{p_j\}$ itself converges uniformly to p. Then (5.3), (5.4) imply for this p that

$$||p; K||_r = 1 \ , \tag{5.5}$$

$$\lim_{j\to\infty} ||p; S_j||_q = 0 \ . \tag{5.6}$$

From this we derive the next result, given below.

Lemma 5.2 *There is a sequence of closed subsets $\{\sigma_j\} \subset \overline{K}$ such that for every j bigger than a fixed j_0 the following is true*

$$\mu_\alpha(\sigma_j) \geq \frac{1}{2} \ \gamma_- \ . \tag{5.7}$$

Moreover,

$$\max_{\sigma_j} |p| \to 0 \quad \text{as} \quad j \to \infty \ . \tag{5.8}$$

PROOF. Let first $q < \infty$. By the (probabilistic) Chebyshev inequality

$$\mu_\alpha\{x \in S_j \ : \ |p(x)| \leq t\} \geq \mu_\alpha(S_j) - \frac{\mu_\alpha(S_j)}{t^q} \ ||p; S_j||_q^q \ .$$

Pick here $t = t_j := ||p; S_j||_q^{1/2}$. Then by (5.6) the left-hand side is at least $\frac{1}{2}\mu_\alpha(S_j)$, for j sufficiently large. Denoting the closure of the set in the braces by σ_j we also have

$$\max_{\sigma_j} |p| = t_j \to 0 \quad \text{as} \quad j \to \infty \ .$$

If $q = \infty$, simply set $\sigma_j := S_j$ to produce (5.7) and (5.8). $\square$

Apply now the Hausdorff compactness theorem to find a subsequence of $\{\sigma_j\}$ converging to a closed subset $\sigma \subset \overline{K}$ in the Hausdorff metric. We assume without loss of generality that $\{\sigma_j\} \to \sigma$. By (5.8) this limit set is a subset of the zero set for p. Since p is nontrivial by (5.5), the dimension of its zero set is at most $d-1$; hence $\mu_\alpha(\sigma) = 0$ by (3.13). Then for every $\epsilon > 0$ one can find a finite open cover of σ by open balls B_i of radius at most $r(\epsilon)$ so that

$$\sum_i |B_i|^{\alpha/d} < \epsilon \ . \tag{5.9}$$

Let σ_δ be a δ-neighbourhood of σ such that

$$\sigma_\delta \subset \bigcup_i B_i \quad \text{and} \quad \delta < r(\epsilon) \ .$$

Pick j so large that $\sigma_j \subset \sigma_\delta$. For every B_i intersecting σ_j choose a point $x_i \in B_i \cap \sigma_j$. Consider an open ball $\widetilde{B}_i$ centered at x_i of radius twice that of B_i. Then $B_i \subset \widetilde{B}_i$ and $|\widetilde{B}_i| = 2^d|B_i|$. Moreover, $\{\widetilde{B}_i\}$ is an open cover of σ_j. Hence

$$\mu_\alpha(\sigma_j) \le \sum_i \mu_\alpha(\sigma_j \cap \widetilde{B}_i) \le \gamma_+ \sum_i |\widetilde{B}_i|^{\alpha/d} = 2^d\gamma_+ \sum_i |B_i|^{\alpha/d} \ ,$$

using assumption (3.12). Together with (5.7) and (5.9) this implies that

$$\frac{1}{2}\,\gamma_- \le \mu_\alpha(\sigma_j) \le 2^d\gamma_+ \sum |B_i|^{\alpha/d} \le 2^d\gamma_+\epsilon \ .$$

Letting $\epsilon \to 0$ one gets a contradiction. $\square$

References

[1] Yu. Brudnyi, Spaces that are defined by means of local approximation. Trudy Moskov. Mat. Obsch. **24** (1971), 69-132; English transl. in Trans. Moscow Math. Soc. **24** (1974).

[2] Yu. Brudnyi, Adaptive approximation of functions with singularities. Trudy Moskov. Mat. Obsch. **55** (1994), 123-186; English transl. in Trans. Moscov. Math. Soc. **1994** (1995).

[3] P. Borwein, and T. Erdelyi, Polynomials and polynomial inequalities. GTM, v. 161, Springer, 1995.

[4] Yu. Brudnyi and M. Ganzburg, On an extremal problem for polynomials of n-variables. Math. USSR Izv. **37** (1973), 344-355.

[5] L. P. Bos, A. Brudnyi, N. Levenberg, and V. Totik, Tangential Markov inequalities on transcendental curves. Constr. Approx. **19** (2003), no. 3, 339-354.

[6] L. P. Bos, N. Levenberg, P. Milman, and B. A. Taylor, Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^N$. Indiana Univ. Math. J. **44** (1995), 115-137.

[7] J. Bourgain, On the distribution of polynomials on high dimensional convex sets. Lecture Notes in Math., 1469, 127-131, Springer, Berlin, 1991.

[8] A. Brudnyi, Bernstein-type inequality for algebraic functions. Indiana Univ. Math. J. **46** (1997), no. 1, 93-115.

[9] A. Brudnyi, Local inequalities for plurisubharmonic functions. Ann. Math. **149** (1999), 511-533.

[10] A. Brudnyi, On local behavior of analytic functions. J. Funct. Anal. **169** (1999), no. 2, 481-493.

[11] A. Brudnyi, Cartwright's type theorems for a class of plurisubharmonic functions. Indiana Univ. Math. J. **48** (1999), no. 2, 365-384.

[12] A. Brudnyi, The distribution of values of analytic functions on convex sets. e-print: http://xxx.arxiv/math.CV/0104271 (2000).

[13] A. Brudnyi, Inequalities for entire functions. Israel Math. Conf. Proc. **15** (2001), 47-66.

[14] A. Brudnyi, Small amplitude limit cycles and the distribution of zeros of families of analytic functions. Ann. of Math. (2) **154** (2001), no. 2, 227-243.

[15] A. Brudnyi, Bernstein type inequalities for quasipolynomials. J. Approx. Theory **112** (2001), no. 1, 28-43.

[16] A. Brudnyi, On a BMO-property for subharmonic functions. J. Fourier Anal. Appl. **8** (2002), no. 6, 603-612.

[17] A. Brudnyi, The distribution of values of generalized exponential polynomials on convex sets. New Zealand J. Math. **31** (2002), no. 2, 131-136.

[18] A. Brudnyi, A Jensen inequality for a family of analytic functions and an estimate for the average number of limit cycles. Bull. London Math. Soc. **35** (2003), no. 2, 229-235.

[19] A. Carbery, and J. Wright, Distributional and L^q norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$. Math. Res. Lett. **8** (2001), no. 3, 233-248.

[20] R. DeVore, and R. Sharpley, Maximal functions measuring smoothness. Mem. Amer. Math. Soc. **47** (1984), no. 293.

[21] C. Fefferman, and R. Narasimhan, On the polynomial-like behavior of certain algebraic functions. Ann. Inst. Fourier **44** (1994), no. 4, 1091-1179.

[22] C. Fefferman, and R. Narasimhan, A local Bernstein inequality on real algebraic varieties. Math. Z. **223** (1996), no. 4, 673-692.

[23] M. Gromov, and V. Milman, Brunn theorem and a concentration of volume for convex bodies. GAFA seminar (1983/1984). Tel Aviv University, Tel Aviv. 1984.

[24] A. Jonsson, and H. Wallin, Function spaces on subsets of $\mathbb{R}^n$. Harwood Acad. Publ., 1984.

[25] V. Katznel'son, Equivalent norms in spaces of entire functions. Mat. Sb. **92** (**134**) (1973), 34-54; English transl. in Math. USSR Sb. **21** (1973).

[26] M. Klimek, Pluripotential Theory. Oxford Univ. Press, London, 1991.

[27] R. Kannan, L. Lovász, and M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma. Discrete Comput. Geom. **13** (1995), no. 3-4, 541-599.

[28] A. Kroó, and D. Schmidt, Some extremal problems for multivariate polynomials on convex bodies. J. Approx. Theory **90** (1997), 415-434.

[29] B. Ya. Levin, and V. N. Logvinenko, Classes of functions that are subharmonic in $\mathbb{R}^m$, and bounded on certain sets. Investigations in Linear Operators and Function Theory **17**, 157-175. Zapiski Nauchnyh Seminarov LOMI. Nauka, Leningrad, 1989.

[30] V. Logvinenko, On a multidimensional generalization of a theorem of M. L. Cartwright. Dokl. Akad. Nauk SSSR **219** (1974), 545-548; English transl. in Soviet Math. Dokl. **15** (1974).

[31] F. Nazarov, M. Sodin, and A. Volberg, The geometric Kannan-Lovász-Simonovits lemma, dimension-free estimates for the distribution of the values of polynomials,and the distribution of zeros of random analytic functions. (Russian) Algebra i Analiz **14** (2002), no. 2, 214-234.

[32] F. Nazarov, M. Sodin, and A. Volberg, Local dimension-free estimates for volumes of sublevel sets of analytic functions. Israel J. Math. **133** (2003), 269-283.

[33] W. Pleśniak, Markov's inequality and the existence of an extension operator for C^∞-functions. J. Approx. Theory **61** (1990), 106-117.

[34] W. Pawlucki, and W. Pleśniak, Extension of C^∞-functions from sets with polynomial cusps. Studia Math. LXXX VIII (1988), 279-288.

[35] E. Remez, Sur une propriété extrémale des polynômes des Tchebychef. Zap. Nauk-Doslid. Inst. Math. Meh. Harkiv Mat. Tovar. **13**, (1936), no. 4, 93-95.

[36] Th. Rivlin, and H. Shapiro, A unified approach to certain problems of approximation and minimization. J. Soc. Indust. Appl. Math. **9** (1961), no. 4, 670-699.

[37] N. Roytwarf, and Y. Yomdin, Bernstein classes. Ann. Inst. Fourier **47** (1997), no. 3, 825-858.

[38] V. S. Videnski, Extremal estimates of derivatives of trigonometric polynomials on the interval less than the period. DAN SSSR, **130** (1960), 13-16.

The Norm of an Interpolation Operator on $H^\infty(D)$

Alfred S. Cavaretta
Department of Mathematics
Kent State University
Kent, Ohio 44242, U.S.A.
Email address: cavarett@math.kent.edu

Natacha Fontes-Merz
Department of Mathematics and Computer Science
Westminster College
New Wilmington, Pennsylvania 16172, U.S.A.
Email address: fontesnc@westminster.edu

In memory of A. Sharma, a faithful friend to many

Abstract

In this paper we describe our attempts at calculating the norm of the operator

$$L_{n-1}(\,\cdot\,;\zeta) : H^\infty(D) \to C$$

where $L_{n-1}(\,\cdot\,;\zeta)$ represents the Lagrange interpolation polynomial of degree $n-1$, evaluated at a complex number ζ, and defined by interpolating functions in $H^\infty(D)$ at the zeros of $z^n - r^n$. We assume that $0 < r < 1$ and that $|\zeta| > 1$.

Although our goal is to calculate the norm of the operator for all values $n \geq 2$ and all values of ζ satisfying $|\zeta| > 1$, we will find an explicit formula for the norm of the operator which we can show to hold for $n \geq 3$ and $|\zeta| > 1.36$, for $n = 2$ and $|\zeta| > 1$, and for $n = 3$ and $\zeta = Re^{\frac{i\pi}{3}k}$, where $k = 1, 3$, and 5 and $R > 1$.

1 Introduction

Let $\mathcal{B}$ represent the unit ball of $H^\infty(D)$ where $H^\infty(D)$ is the Hardy space consisting of functions which are bounded and analytic in the open unit disk D. That is, let $\mathcal{B} := \{f \in H^\infty(D) : \|f\|_{L^\infty(D)} \leq 1\}$ where $\|f\|_{L^\infty(D)} = \sup_{z\in D} |f(z)|$. Many mathematicians have studied the norms of operators acting on this space.

In 1913, for example, Landau calculated the norm of $S_{n-1}(\,\cdot\,;1) : H^\infty(D) \to \mathbb{C}$ where $S_{n-1}(f;1) := \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}$ is the $n-1^{st}$ partial sum of the Taylor series of f, evaluated at 1 [5]. In 1918 Szász considered a generalization of Landau's problem. In particular, if the Taylor series of f is given by $f(z) = \sum_{k=0}^\infty a_k z^k$, then Szász investigated the maximization of the linear functional

$$|c_0 a_0 + \cdots + c_{n-1} a_{n-1}|$$

over all $f \in \mathcal{B}$ where $c_0, \ldots, c_n$ are given complex numbers [8]. When $c_j = 1$ we have the Landau case. When $c_j = R^j$ we have the $n-1^{st}$ partial sum of the Taylor series of f evaluated at R.

In a related problem, Sharma and Cavaretta calculated the norm of the operator $L_{n-1}(\,\cdot\,;R) : H^\infty(D) \to \mathbb{C}$ where $L_{n-1}(f;R)$ represents the Lagrange interpolation polynomial of degree $n-1$, evaluated at R, and defined by interpolating f at the zeros of $z^n - r^n$ [1]. Here, we assume that $0 < r < 1$ and that $R > 1$.

It is interesting to note that while the norm of the operator $S_{n-1}(\,\cdot\,;R)$ is invariant under rotations in that $\|S_{n-1}(\,\cdot\,;R)\| = \|S_{n-1}(\,\cdot\,;\zeta)\|$ where ζ is any complex number of modulus R, the norm of the operator $L_{n-1}(\,\cdot\,;R)$ is only invariant under rotations $\zeta \to \omega^j\zeta$, $j = 0, 1, \ldots, n-1$ where ω^j, $j = 0, 1, \ldots, n-1$ are the n roots of unity, i.e., the n solutions of the equation $z^n = 1$. In this paper, we would like to investigate the problem of computing the operator norm $\max_{f\in\mathcal{B}} |L_{n-1}(f;\zeta)|$ where ζ is any complex number of modulus $R > 1$ and where $n \geq 2$. Although we will not solve the problem completely, we will obtain results in the following cases:

CASE I: $\quad |\,\zeta\,| > 1.36$ and $n \geq 3$

CASE II: $|\,\zeta\,| > 1$ and $n = 2$
CASE III: $\zeta = Re^{\frac{i\pi}{3}k}$ for $k = 1, 3, 5$ and $n = 3$

2 A Reduction of the Problem

In this section we wish to provide a formula for the norm of $L_{n-1}(\,\cdot\,;\zeta)$ which will hold for various values of ζ and n under the condition that a certain polynomial (which is dependent on ζ and n) has no zeros in the unit disk. Throughout this section we may assume that ζ is any complex number of modulus R greater than 1, and that n is any integer greater than or equal to 2.

Define $g(z) := \left(\frac{\zeta^n - r^n}{\zeta - z}\right)^{\frac{1}{2}}$ where we consider the square root of $\frac{\zeta^n - r^n}{\zeta - z}$ to be the principle square root. Let $p(z) := L_{n-1}(g; z)$ where $L_{n-1}(g; z)$ represents the polynomial of degree $n-1$, evaluated at z, which interpolates g at the zeros of $z^n - r^n$ for $0 < r < 1$.

Following Cavaretta and Sharma's calculation of $L_{n-1}(f; R)$ [1], we see that for any $f \in \mathcal{B}$,

$$\begin{aligned} |L_{n-1}(f;\zeta)| &= \Big|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^n - \zeta^n}{(z-\zeta)(z^n - r^n)} f(z)\; dz\Big| \\ &= \Big|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(1 - r^{2n})p^2(z)}{(z^n - r^n)(1 - z^n r^n)} f(z)\; dz\Big| \\ &\leq \frac{1 - r^{2n}}{2\pi}\int_0^{2\pi} \frac{|p^2(e^{i\theta})|}{|e^{in\theta} - r^n|^2} d\theta, \end{aligned}$$

where the second equality holds due to the fact that $p^2(r\omega^j) = \frac{\zeta^n - r^n}{\zeta - r\omega^j}$, which implies that the residues of each integrand are equal.

By expanding the factor $(\zeta - z)^{-\frac{1}{2}}$ of g in a binomial series, and by using the linearity of the operator L_{n-1}, we have that (see (3.1))

$$p(z) = \Big(\frac{\zeta^n - r^n}{\zeta}\Big)^{\frac{1}{2}} \sum_{k=0}^{n-1} \lambda_k \Big(\frac{z}{\zeta}\Big)^k$$

where

$$\lambda_k = \sum_{l=0}^{\infty} (-1)^{k+ln} \binom{-\frac{1}{2}}{k + ln} \Big(\frac{r}{\zeta}\Big)^{ln}$$

for $k = 0, \ldots, n-1$. Therefore, Parseval's relation gives us

$$\frac{1-r^{2n}}{2\pi}\int_0^{2\pi}\frac{|p^2(e^{i\theta})|}{|e^{in\theta}-r^n|^2}d\theta = \frac{|\zeta^n - r^n|}{R}\sum_{k=0}^{n-1}\frac{|\lambda_k|^2}{R^{2k}},$$

so that for any $f \in \mathcal{B}$,

$$|L_{n-1}(f;\zeta)| \leq \frac{|\zeta^n - r^n|}{R}\sum_{k=0}^{n-1}\frac{|\lambda_k|^2}{R^{2k}}. \tag{2.1}$$

If p has no zeros in the unit disk, then the function $F(z) := \dfrac{z^{n-1}\overline{p(1/\bar{z})}}{p(z)}$ is in $\mathcal{B}$. Furthermore, if we replace f by F in inequality (2.1) we will get equality, thus obtaining a formula for the norm of $L_{n-1}(\,\cdot\,;\zeta)$. As the general theory states [3], the extremal function F is a Blaschke product and $|F(e^{i\theta})| = 1$ for all θ. We have the following:

Theorem 2.1 *Let $0 < r < 1$ and let ζ be any complex number of modulus $R > 1$. Let the operator $L_{n-1}(\,\cdot\,;\zeta) : H^\infty(D) \to \mathbb{C}$ be the Lagrange interpolation polynomial of degree $n-1$, evaluated at ζ, and defined by interpolating functions in $H^\infty(D)$ at the zeros of $z^n - r^n$. Define $g(z) := \left(\frac{\zeta^n - r^n}{\zeta - z}\right)^{\frac{1}{2}}$ and let $p(z) := L_{n-1}(g;z)$. If p has no zeros in D, then*

$$\|L_{n-1}(\,\cdot\,;\zeta)\| = \frac{|\zeta^n - r^n|}{R}\sum_{k=0}^{n-1}\frac{|\lambda_k|^2}{R^{2k}} \tag{2.2}$$

where $\lambda_k = \displaystyle\sum_{l=0}^{\infty}(-1)^{k+ln}\binom{-\frac{1}{2}}{k+ln}\left(\frac{r}{\zeta}\right)^{ln}$. In other words, the norm of this operator is equal to the square of the ℓ_2 norm of p, as long as p has no zeros in D.

Our problem of calculating the norm of $L_{n-1}(\,\cdot\,;\zeta)$ thus reduces to proving that p has no zeros in the unit disk. In the following sections, we will prove that p has no zeros in D for various values of ζ and n.

3 $n \geq 3$ and $|\zeta| > 1.36$

In this section we wish to calculate the norm of $L_{n-1}(\,\cdot\,;\zeta)$ in the case that $n \geq 3$ and that $|\zeta| > 1.36$. Again, let $g(z) := \left(\frac{\zeta^n - r^n}{\zeta - z}\right)^{\frac{1}{2}}$ and let $p(z) := L_{n-1}(g;z)$. In order to apply Theorem 2.1 we will use Rouché's theorem to show that p has no zeros in D.

Notice that g has no zeros in D and is analytic in D. Also, by expanding the factor $(\zeta - z)^{-\frac{1}{2}}$ of g in a binomial series, we have that

$$g(z) = \left(\frac{\zeta^n - r^n}{\zeta}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \left(\frac{z}{\zeta}\right)^k$$

and, therefore, that

$$\begin{aligned} p(z) &= \left(\frac{\zeta^n - r^n}{\zeta}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \left(\frac{1}{\zeta}\right)^k L_{n-1}(z^k) \\ &= \left(\frac{\zeta^n - r^n}{\zeta}\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{r}{z}\right)^{jn} \sum_{k=jn}^{(j+1)n-1} (-1)^k \binom{-\frac{1}{2}}{k} \left(\frac{1}{\zeta}\right)^k z^k. \end{aligned} \tag{3.1}$$

Let $\mathbb{T}$ denote the unit circle. Then for any $z \in \mathbb{T}$ we obtain the following inequality:

$$\begin{aligned} |g(z) - p(z)| &= \Bigg| \left(\frac{\zeta^n - r^n}{\zeta}\right)^{\frac{1}{2}} \Bigg(\sum_{k=n}^{2n-1} (-1)^k \binom{-\frac{1}{2}}{k} \left(\frac{1}{\zeta}\right)^k \left(1 - \frac{r^n}{z^n}\right) z^k + \\ &\qquad \sum_{k=2n}^{3n-1} (-1)^k \binom{-\frac{1}{2}}{k} \left(\frac{1}{\zeta}\right)^k \left(1 - \frac{r^{2n}}{z^{2n}}\right) z^k + \cdots \Bigg) \Bigg| \\ &\leq \frac{|\zeta^n - r^n|^{\frac{1}{2}}}{R^{\frac{1}{2}}} \left(2 \cdot \sum_{k=n}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{1}{R^k}\right) \\ &\leq 2\frac{|\zeta^n - r^n|^{\frac{1}{2}}}{R^{\frac{1}{2}}} \left(\left(1 - \frac{1}{R}\right)^{\frac{-1}{2}} - 1 - \frac{1}{2R} - \frac{3}{8R^2}\right). \end{aligned} \tag{3.2}$$

In order to apply Rouché's theorem, it would suffice to prove that

$$2\frac{|\zeta^n - r^n|^{\frac{1}{2}}}{R^{\frac{1}{2}}} \left(\left(1 - \frac{1}{R}\right)^{-\frac{1}{2}} - 1 - \frac{1}{2R} - \frac{3}{8R^2}\right)$$

$$< \min_{z\in\mathbb{T}} |g(z)| = \frac{|\zeta^n - r^n|^{\frac{1}{2}}}{(R+1)^{\frac{1}{2}}},$$

so that $|g(z) - p(z)| < |g(z)|$ for all $z \in \mathbb{T}$. However, the inequality holds if and only if

$$\frac{1}{2}\sqrt{\frac{R}{R+1}} + 1 + \frac{1}{2R} + \frac{3}{8R^2} - \sqrt{\frac{R}{R-1}} > 0. \tag{3.3}$$

Inequality (3.3) can be proven to hold for values of $R > 1.36$ by noticing that $\frac{1}{2}\sqrt{\frac{R}{R+1}}+1+\frac{1}{2R}+\frac{3}{8R^2}-\sqrt{\frac{R}{R-1}}$ has a zero approximately at $R = 1.355$ and is increasing for values of R greater than 1.355.

Therefore, by Rouché's theorem, p has no zeros in D. In the case that $|\zeta| > 1.36$ and $n \geq 3$ we then have that the norm of $L_{n-1}(\,\cdot\,;\zeta)$ is given by formula (2.2).

It should be noted that as n increases, we can prove that for any $z \in \mathbb{T}$ the inequality $|g(z) - p(z)| < \min_{z\in\mathbb{T}} |g(z)|$ holds for even smaller values of R. Notice that

$$2 \cdot (-1)^k \binom{-\frac{1}{2}}{k} \leq 1$$

for any value of k. Therefore, by inequality (3.2) we have that

$$|g(z) - p(z)| \leq \frac{|\zeta^n - r^n|^{\frac{1}{2}}}{R^{\frac{1}{2}}} \frac{1}{R^{n-1}} \frac{1}{R-1},$$

which is smaller than

$$\min_{z\in\mathbb{T}} |g(z)| = \frac{|\zeta^n - r^n|^{\frac{1}{2}}}{(R+1)^{\frac{1}{2}}}$$

when

$$n > \frac{\ln\left(\sqrt{\frac{R+1}{R}} \cdot \frac{1}{R-1}\right)}{\ln R} + 1.$$

Therefore, for larger values of n, we are able to show that formula (2.2) remains valid for even smaller values of R. For example, one can verify that when $n = 30$ the estimate holds for $R > 1.1$.

Our motivation for choosing to compare p to the function g when applying Rouché's Theorem is that $p(z) = L_{n-1}(g; z)$. It seems that the quantity $|g(z) - L_{n-1}(g; z)|$ should not be too large for $z \in \mathbb{T}$ since $L_{n-1}(g; z)$ is a polynomial approximation to $g(z)$ in the unit disk. Therefore, it is plausible to expect that $|g(z) - L_{n-1}(g; z)| < |g(z)|$ for $z \in \mathbb{T}$.

However, our choice of g was in some sense arbitrary in that there are many analytic functions, h, which are free of zeros in the unit disk and for which $p(z) = L_{n-1}(h; z)$. If $|h|$ has a larger minimum value on the boundary of D, perhaps we could prove that $|h(z) - p(z)| < \min_{z\in\mathbb{T}} |h(z)|$ for all values of ζ with modulus greater than one.

4 $\mathbf{n = 2}$ and $|\zeta| > 1$

In this section we wish to prove that the calculation of the norm of $L_{n-1}(\,\cdot\,; \zeta)$ given by formula (2.2) holds in the case that $n = 2$ and $|\zeta| > 1$. Proving this case will show that our formula does not need to be restricted by the condition that $|\zeta| > 1.36$.

When $n = 2$ the polynomial defined in Theorem 2.1 is $p(z) = L_1(g; z)$ where $g(z) = \left(\dfrac{\zeta^2 - r^2}{\zeta - z}\right)^{\frac{1}{2}}$. Since $p(z)$ is defined by the interpolation conditions $p(r) = g(r)$ and $p(-r) = g(-r)$ we have that

$$p(z) = \Big(\frac{\sqrt{\zeta + r} - \sqrt{\zeta - r}}{2r}\Big)z + \Big(\frac{\sqrt{\zeta + r} + \sqrt{\zeta - r}}{2}\Big).$$

In order to apply Theorem 2.1, we need to prove that the modulus of the only zero of p, namely

$$|z_0| := r \cdot \left|\frac{\sqrt{\zeta + r} + \sqrt{\zeta - r}}{\sqrt{\zeta + r} - \sqrt{\zeta - r}}\right|,$$

is larger than one when $|\zeta| = R > 1$.

Let

$$\sqrt{\zeta + r} := \alpha e^{i\frac{\theta_1}{2}}$$
$$\sqrt{\zeta - r} := \beta e^{i\frac{\theta_2}{2}}.$$

Then, by the half angle formula for cosine, we have that

$$|z_0|^2 = \frac{r^2\left(\alpha^2+\beta^2+2\alpha\beta\sqrt{\frac{1+\cos(\theta_2-\theta_1)}{2}}\right)}{\alpha^2+\beta^2-2\alpha\beta\sqrt{\frac{1+\cos(\theta_2-\theta_1)}{2}}}. \tag{4.1}$$

In order to calculate $\cos(\theta_2-\theta_1)$ we notice that $\theta_2-\theta_1$ is the angle between the vector connecting the origin to $\zeta-r$ and the vector connecting the origin to $\zeta+r$. These vectors have lengths β^2 and α^2 respectively. Also, the vector connecting $\zeta-r$ to $\zeta+r$ has length $2r$. Therefore, by the law of cosines, we have that

$$\cos(\theta_2-\theta_1) = \frac{\alpha^4+\beta^4-4r^2}{2\alpha^2\beta^2}.$$

Using this fact in conjunction with (4.1) gives us that $|z_0|^2 \geq 1$ if and only if

$$r^2[\alpha^2+\beta^2+\sqrt{(\alpha^2+\beta^2)^2-4r^2}] \geq \alpha^2+\beta^2-\sqrt{(\alpha^2+\beta^2)^2-4r^2},$$

which is equivalent to showing that

$$\sqrt{(\alpha^2+\beta^2)^2-4r^2}(r^2+1) \geq (\alpha^2+\beta^2)(1-r^2).$$

By squaring both sides of the inequality, it can be seen that our problem reduces to proving

$$(\alpha^2+\beta^2)^2 \geq (r^2+1)^2,$$

i.e. that

$$|\zeta-r|+|\zeta+r| \geq r^2+1. \tag{4.2}$$

However, by setting $\zeta = Re^{i\theta}$ and $f(\theta) := |\zeta-r|+|\zeta+r|$, we have

$$f^{'}(\theta) = Rr\sin\theta\frac{|\zeta+r|-|\zeta-r|}{|\zeta-r||\zeta+r|}.$$

so that the critical values of f are $\theta = 0, \pi$, and $\frac{\pi}{2}$. Since $f(0) = f(\pi)$ and $f(\frac{\pi}{2}) > f(0)$, f attains a minimum value of $f(0) = 2R$, which implies that

$$|\zeta-r|+|\zeta+r| \geq 2R.$$

Since $R > 1$ and $0 < r < 1$ we have that $2R \geq r^2 + 1$. Thus, we have proven inequality (4.2) and, therefore, that p has no zeros in the unit disk.

By using Parseval's relation to evaluate the quantity on the right-hand side of equation (2.2) in the specific case that $n = 2$, we obtain the following theorem:

Theorem 4.1 *Let $0 < r < 1$ and let ζ be any complex number of modulus $R > 1$. Let the operator $L_1(\cdot\,;\zeta) : H^\infty(D) \to \mathbb{C}$ be the linear polynomial, evaluated at ζ, which is defined by interpolating functions in $H^\infty(D)$ at r and $-r$. Then*

$$\|L_1(\cdot\,;\zeta)\| = \frac{|\zeta^2 - r^2|}{R}\left(|\lambda_0|^2 + \frac{|\lambda_1|^2}{R^2}\right)$$

where

$$\lambda_0 = \frac{\sqrt{\zeta + r} + \sqrt{\zeta - r}}{2},$$
$$\lambda_1 = \frac{\sqrt{\zeta + r} - \sqrt{\zeta - r}}{2r}.$$

5 $\mathbf{n = 3}$ and $\zeta = \mathbf{Re^{i\frac{k\pi}{3}}}$ for $\mathbf{k = 1, 3, 5}$ and $\mathbf{R > 1}$

In this section we will sketch a proof of why calculation (2.2) holds in the case that $n = 3$ and $\zeta = Re^{i\frac{k\pi}{3}}$ for $k = 1, 3$, and 5, and for $R > 1$.

As Cavaretta and Sharma explain in their paper [1], the calculation of the norm of $L_{n-1}(\cdot\,;\zeta)$ is invariant under rotations $\zeta \to \omega^j\zeta$, $j = 0, 1, \ldots, n-1$ where ω^j, $j = 0, 1, \ldots, n-1$ are the n roots of unity, i.e., the n solutions of the equation $z^n = 1$. Therefore, their calculation of the norm of $L_2(\cdot\,;R)$ is proven to be the same as the norm of $L_2(\cdot\,;\zeta)$ in the case that $\zeta = Re^{i\frac{k\pi}{3}}$ for $k = 2, 4$, and 6. The values of ζ that we wish to consider in this section are in between these values in that their arguments are halfway between the arguments of $\zeta = Re^{i\frac{k\pi}{3}}$ for $k = 2, 4$, and 6.

Since the norm of $L_2(\,\cdot\,;\zeta)$ is invariant under rotations $\zeta \to \zeta\omega^j$, $j = 0, 1, 2$, we only need to calculate the norm of $L_2(\,\cdot\,;-R)$. The norms of the operators $L_2(\,\cdot\,;Re^{i\frac{\pi}{3}})$ and $L_2(\,\cdot\,;Re^{i\frac{5\pi}{3}})$ will then be the same as the norm of $L_2(\,\cdot\,;-R)$.

When $n = 3$ and $\zeta = -R$, the polynomial defined in Theorem 2.1 is $p(z) = L_2(g;z)$ where $g(z) = \left(\dfrac{R^3 + r^3}{R + z}\right)^{\frac{1}{2}}$. Since $p(z)$ is defined by the interpolation conditions $p(z) = g(z)$ for $z = r\omega^k$, $k = 0, 1, 2$, we have that

$$p(z) = \left(\frac{R^3 + r^3}{R}\right)^{\frac{1}{2}} \sum_{k=0}^{2} \lambda_k z^k$$
$$\lambda_k := \sum_{l=0}^{\infty} (-1)^{3l+k} \binom{-\frac{1}{2}}{3l+k} \frac{r^{3l}}{(-R)^{3l+k}}.$$

In order to apply Theorem 2.1, we need to show that p has no zeros in D. This can be proven by using Cauchy's coefficient estimate [7], which states that p has no zeros in D if its coefficients satisfy the following condition:

$$|\lambda_0| - |\lambda_1| - |\lambda_2| > 0.$$

By proving that $\lambda_0 > 0$, $\lambda_1 < 0$, and $\lambda_2 > 0$, Cauchy's coefficient estimate can be reduced to

$$\lambda_0 + \lambda_1 - \lambda_2 > 0.$$

After much calculation, this inequality can be proven to be true by rearranging the terms of the infinite sum $\lambda_0 + \lambda_1 - \lambda_2 > 0$ and by relying on the fact that $(-1)^{k+1}\binom{-\frac{1}{2}}{k+1} = (-1)^k \binom{-\frac{1}{2}}{k} \left(\dfrac{2k+1}{2k+2}\right)$.

References

[1] A.S. Cavaretta and A. Sharma, *Optimal recovery of interpolation operators in Hardy spaces*, to appear in Advances in Comp. Math. (2004).

[2] A.S. Cavaretta, A. Sharma, R.S. Varga, *Interpolation in the roots of unity: An extension of a theorem of J.L. Walsh*, Resultate Math. **3** (1980), no. 2, pp. 155-191.

[3] P.L. Duren, *Theory of H^p Spaces*, New York, Academic Press, 1970.

[4] S. Kakeya, *On the limits of the roots of an algebraic equation with positive coefficients*, Tôhoku Mathematics Journal **2** (1912), pp. 140-142.

[5] E. Landau, *Abschätzung der Doeffizientensumme einer Potenzreihe*, I, II., Arch. Math. und Phys. (3) **21**, (1913), pp. 42-50, 250-55.

[6] A.J. Macintyre and W.W. Rogosinski, *Extremum problems in the theory of analytic functions*, Acta Mathematica **81-82** (1949-1950), pp. 275-325.

[7] M. Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*, American Mathematical Society, 1949.

[8] C.A. Micchelli and T.J. Rivlin, *An optimal recovery view of Walsh's equiconvergence theorem*, Journal of Complexity **3** (1987), pp. 312-330.

Sharma and Interpolation, 1993–2003: The Dutch Connection

M. G. de Bruin
Delft Institute of Applied Mathematics
Delft University of Technology
P.O. Box 5031, 2600 GA Delft, The Netherlands
Email address: m.g.debruin@ewi.tudelft.nl

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

This paper gives a survey of Sharma's work on interpolation during the years 1993–2003 in cooperation with the author.

1 Introduction

This summary paper covers a time period of approximately 10 years and started when A. Sharma invited the author to Edmonton (after the conference in his honor) to work on interpolation linked to simultaneous rational approximation with common denominator; the work included so called *equiconvergence*, *overconvergence* and the concept of *exceptional points*; cf. §4.

After that the focus of interest turned to Birkhoff and Pál-type interpolation problems, where Sharma was very interested in determining the thin line that seperated the regular problems from the non-regular ones; cf.§2, §3.

The cooperation came to an end with his (for everybody, untimely) death on December 22, 2003, around one month before our last joint paper [9] was accepted for publication; it appeared early Spring 2004.

Interpolation has been studied for quite a long time and a rather general definition of the problem can be given as follows

given m points $\{z_i\}_{i=1}^m \in \mathbb{C}$ (*nodes*),
given m *order-sets* $D_i \in \mathbb{N}$ with number of elements $|D_i|$,
given *data* $c_{i,j} \in \mathbb{C}$ $(1 \leq j \leq |D_i|,\ 1 \leq i \leq m)$,
find $P_n \in \Pi_n,\ n = (\sum_{i=1}^m |D_i|) - 1$ with

$$P_n^{(k)}(z_i) = c_{i,k},\ k \in D_i,\ 1 \leq i \leq m.$$

Here Π_n is the set of polynomials of degree at most n with complex coefficients.

When $D_i = \{0\}$ for all i, we have the well-known *Lagrange interpolation* problem, the case $D_i = \{0, 1, \ldots, k_i\}$ is referred to as *Hermite interpolation* and if at least one set D_i is a sequence of non-negative integers containing a gap, the term *Birkhoff interpolation* is used. This case also is sometimes referred to as *lacunary interpolation* (for a survey on $(0,2)$-interpolation, see [20]). The main source for the theory on these problems is [13].

If the interpolation problem has a unique solution for all sets of data, it is called *regular* (or *well poised*). Sometimes authors use the concepts of *real poised*, *circle poised* or *complex poised* for a regular problem where all nodes are *real*, all on the *unit circle* or not all located in either of the previous locations respectively.

In regularity proofs (when the unique solution polynomial, as function of the given data, is not needed) one often uses the following equivalent formulation with $n = (\sum_{i=1}^m |D_i|) - 1$ as before:

$$P_n \in \Pi_n,\ P_n^{(k)}(z_i) = 0\ (k \in D_i,\ 1 \leq i \leq m) \Rightarrow P_n \equiv 0.$$

Up to now, all problems mentioned used the same set of nodes for each of the given order-sets. Another type of problem, referred to as *Pál-type interpolation*, uses two sets of nodes Z_1, Z_2 and as orders of derivative $k = 0$ and $k = m$ (originally $k = 0$ and $k = 1$)):

$$P(z_i) = c_i \text{ for } z_i \in Z_1,\ P'(w_j) = d_j \text{ for } w_j \in Z_2.$$

This type of problem was introduced by L.G. Pál ([10]) in 1975 and has been studied in a.o. [18], [23]. Extensions arise when one uses $q+1,\ q \geq 2$ orders of derivatives $k_j,\ 0 = k_0 < k_1 < \cdots < k_q$ (consecutive or 'lacunary') and sets of nodes which might be different for each order of derivative k_j.

Finally, there has to be distinguished between the cases that the nodes are uniformly distributed or not. Interpolation on the roots of unity (the zeros of $z^n - 1$) for instance, clearly falls in the class of uniformly distributed nodes, while interpolation on the zeros of the Legendre polynomials obviously belongs to the class of nonuniformly distributed nodes.

The layout of the paper is now as follows.

In section 2 the results on Birkhoff interpolation are covered, in section 3 those on Pál-type interpolation and section 4 turns to the *two stage interpolation* in the context of Hermite-Padé approximation.

Finally a (rather short) bibliography is given, containing a limited set of the prolific literature on the subject of interpolation; the list is not intended to give a survey of the entire field, but just displays the papers by Sharma and the author, along with a few extra references.

2 Birkhoff-Type Interpolation

Let Π_N be the set of polynomials of degree at most N with complex coefficients and consider for an arbitrary integer $q \geq 1$ the following problem:

> given n different points $\{z_i\}_{i=1}^n$, (*nodes*),
> given $q+1$ integers $0 = m_0 < m_1 \cdots < m_q$ (*orders*),
> given numbers $\{c_i^{[j]}\}$ $(1 \leq i \leq n,\ 0 \leq j \leq q)$ in $\mathbb{C}$ (*data*),
> find $P_N \in \Pi_N,\ N = (q+1)n - 1$ with
>
> $$P_N^{(m_j)}(z_i) = c_i^{[j]}\ (1 \leq i \leq n,\ 0 \leq j \leq q).$$

As stated in the introduction, the problem is called *regular* when the solution $P_N(z)$ is unique for any set of data; or equivalently:

all data are zero $\Rightarrow P_N(z) \equiv 0$ *is the unique solution.*

After it has been established that a set of nodes leads to a regular problem, it is (sometimes) possible to solve it explicitly, and the unique polynomials $P_N(i,j;z)$ satisfying

$$P_N^{(m_s)}(i,j;z_r) = \begin{cases} 1 & (r,s)=(i,j) \\ 0 & (r,s)\neq(i,j) \end{cases}$$

are referred to as *fundamental polynomials.* They generate the solution for arbitrary data by

$$P_N(z) = \sum_{i=1}^{n}\sum_{j=0}^{q} c^{[m_j]}(z_i)P(i,j;z).$$

This type of problem is called *Hermite-Birkhoff interpolation*, a well-known subject (cf. the excellent book [13]). The problem is called *lacunary* when the *orders* of the derivatives are *not consecutive.*

There are few examples of regular lacunary problems (see [11], [22]) and in the papers [4]–[8] the attention has been focused on the cases of $(0,m)$, $m \geq 2$ and $(0,1,\ldots,r-2,r)$, $r \geq 2$ interpolation problems.

2.1 Interpolation of type $(0,m)$, $m \geq 2$

Results by Chen and Sharma (cf. [21]) were generalised in the direction of:

Theorem 2.1 ([5], Theorem 1) *The problem of* $(0,m)$ *interpolation on the zeros of* $(z^{3n}+1)(z-1)$ *is regular for* $m \leq 3n+1$.

Theorem 2.2 ([5], Theorem 3) *The problem of* $(0,m)$ *interpolation on the zeros of* $(z^{3n}+1)(z^2+z+1)$ *is regular for* $2 \leq m \leq 3n+2$.

Remark. The polynomial z^2+z+1 can be replaced by $(z-1)(z-\omega)$ or $(z-1)(z-\omega^2)$ with $\omega^3=1$, $\omega\neq 1$. In order to replace it by z^2-1, we have to require that n is even.

Theorem 2.3 ([5], Theorem 4) *The problem of* $(0,m)$ *interpolation on the zeros of* $(z^{3n}+1)(z^3-1)$ *is regular for* $m \leq 3n+3$.

The 'missing' cases $(z^k+1)(z^3-1)$ with $k \not\equiv 0 \bmod 3$ were treated later. In those cases it is not possible to reduce the 18×18 systems for $n \geq 3$ to smaller and more manageable systems: another method of proof was necessary, using MAPLE® to handle the cases $2 \leq m \leq 5$.

Theorem 2.4 ([7], Theorem 1) *The problem of* $(0,m)$ *interpolation on the zeros of* $(z^{3n+1}+1)(z^3-1)$ *is regular for* $m \leq 3n+4$, $n \geq 0$.

Theorem 2.5 ([7], Theorem 2) *The problem of* $(0,m)$ *interpolation on the zeros of* $(z^{3n+2}+1)(z^3-1)$ *is regular for* $m \leq 3n+5$, $n \geq 0$.

Remark. Taking Theorems 2.3–2.5 together, we see

$(0,m)$ interpolation on the zeros of $(z^k+1)(z^3-1)$ is regular for $m \leq k+3$.

Finally, an 'isolated result' was found as spin-off in the study of $(0,1,\ldots,r-2,r)$ interpolation.

Theorem 2.6 ([4], section 4) *The problem of* $(0,3)$ *interpolation on the zeros of* $(z+\alpha)^{n+1} - (1+\alpha z)^{n+1}$, $0 < \alpha < 1$, *is regular for* $\alpha < 1/\sqrt{66}$.

Remark. A fundamental determinant for the problem is explicitly calculated for $n = 3, 4$

$$\Delta_n(\alpha) = \begin{cases} c_1(17 + 62\alpha^4 + 17\alpha^8),\ c_1 > 0 \text{ for } n = 3, \\ c_2(11 - 47\alpha^5 + 11\alpha^{10}),\ c_2 > 0 \text{ for } n = 4. \end{cases}$$

This leads to the conjecture: $\Delta_n(\alpha) > 0$ for all $0 < \alpha < 1$ when n is odd, but has only one zero for n even. This would imply that the interpolation problem is regular for all α from $(0,1)$, with the possible exception of one value in the case that n is even.

2.2 Interpolation of type $(0,1,\ldots,r-2,r)$, $r \geq 2$

In the beginning of the 1990s some few papers on $(0,1,\ldots,r-2,r)$ interpolation started to appear (compare [19], [17]). New contributions were given by Sharma et al. in the following direction.

Theorem 2.7 ([5], Theorem 1) *The problem of* $(0,1,\ldots,r-2,r)$ *interpolation on the zeros of* $(z^{3n}+1)(z-1)$ *is regular for* $r \geq 2$.

Theorem 2.8 ([6], Theorem 1) *The problem of* $(0, 1, \ldots, r-2, r)$ *interpolation on the zeros of* $((z+\alpha)^n + (1+\alpha z)^n)(z-1)$ *is regular for* $0 < \alpha < 1$ *and* $r \geq 2$.

Theorem 2.9 ([4], Theorem 2.1) *The problem of* $(0, 1, \ldots, r-2, r)$ *interpolation on the zeros of* $(z+\alpha)^{n+1} - (1+\alpha z)^{n+1}$ *is regular for* $0 < \alpha < 1$ *and* $r \geq 2$. *Moreover, the fundamental polynomials can be given explicitly.*

Theorem 2.10 ([8], Theorem 1) *The problem of* $(0, 1, \ldots, r-2, r)$ *interpolation on the zeros of* $(z^{3n}+1)(z^3-1)$ *is regular for* $r \geq 2$.

3 Pál-Type Interpolation

As stated before, finding the location of nodes that lead to regular interpolation problems is a problem that has not yet been solved in full generality for (lacunary) problems that use the same set of nodes for each order of derivative. Things get even more intricate when the sets of nodes depend on the order of derivative used. Sharma studied several situations of which the results from the references [4], [6] and [9] will be given explicitly.

Let natural numbers $0 = m_0 < m_1 < \cdots < m_q$ be given and $q+1$ polynomials $A_j(z)$ of degrees n_j $(0 \leq j \leq q)$ with simple zeros (node generating polynomials), along with complex numbers c_k^j $(1 \leq k \leq n_j,\, 0 \leq j \leq q)$.

Then the $(m_0, m_1, \ldots, m_q)$ *Pál-type interpolation problem on* the polynomials $\{A_0, A_1, \ldots, A_q\}$ consists of finding a polynomial $P_N(z)$ of degree at most $N = \left(\sum_{j=0}^{q} n_j\right) - 1$ with

$$P_N^{(m_j)}(z_k) = c_k^j,\ 1 \leq k \leq n_j,\, 0 \leq j \leq q,$$

where the $\{z_k\}$ are the zeros of the polynomial $A_j(z)$.

Theorem 3.1 ([4], §5) *The* $(0,1)$ *Pál-type interpolation problem on* $\{(z+\alpha)^{n+1} - (1+\alpha z)^{n+1}, (z+\alpha)^{n+1} + (1+\alpha z)^{n+1}\}$ *is regular.*

Theorem 3.2 ([6], Theorem 2) *The* $(0,1)$ *Pál-type interpolation problem on* $\{(z+\alpha)^{n+1} - (1+\alpha z)^{n+1}, \left[(z+\alpha)^{n+1} + (1+\alpha z)^{n+1}\right] \times (z-1)\}$ *is regular.*

Theorem 3.3 ([9], Theorem 2.1) *There exists a natural number* n_0*, such that the* $(m_0, m_1, \ldots, m_q)$ *Pál-type interpolation problem on* $\{z^n - \alpha_0^n, z^n - \alpha_1^n, \ldots, z^n - \alpha_q^n\}$ *with* $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_q$ *is regular for* $n \geq n_0$*.*

Moreover, it is possible to give the asymptotic behavior of the solution of the problem in the case that the data are derived from a function $f(z)$, analytic on the disk $\mathbf{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ but not on its closure, where $\rho > \alpha_q$. Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \ \limsup_{k \to \infty} \sqrt[k]{|f_k|} = \frac{1}{\rho}. \tag{3.1}$$

The data are given by

$$c_k^j = f^{(m_j)}(\alpha_j \omega^k) \text{ with } \omega = e^{2\pi i/n} \text{ (a primitive root of unity)}. \tag{3.2}$$

The first result concerns the convergence of this interpolation procedure.

Theorem 3.4 ([6], Theorem 2.2) *Let* $P_N(f;z)$ *be the unique solution to the Pál-type interpolation problem stated in Theorem 3.3 using the data (3.2), then*

$$\lim_{n \to \infty} P_N(f;z) = f(z)$$

uniformly on compact subsets of $\mathbf{D}_\rho$*.*

Just as in the case of Birkhoff interpolation on the roots of unity ($\alpha_j = 1,\ 0 \leq j \leq q$, in Theorem 3.3; compare [11]) it is possible to prove so-called *overconvergence*. First introduce the sections of the series for f by

$$S_r(z) = \sum_{k=0}^{r} f_k z^k, \ r \geq 0. \tag{3.3}$$

Theorem 3.5 ([6], Theorem 2.3) *Let $\ell \geq 1$ be a fixed natural number. The interpolation process using the data (3.2) from the function (3.1) exhibits overconvergence on the disk*

$$|z| < \rho \left(\frac{\rho}{\alpha_q}\right)^{\ell-1+\frac{1}{q+1}}. \tag{3.4}$$

Thus, for z in the disk (3.4)

$$\lim_{n\to\infty} P_N(f;z) - P_N(S_{(q+1)n\ell-1};z) = 0,$$

uniformly on compact subsets.

4 Simultaneous Hermite-Padé Interpolation

The study of this subject was started to extend results published in [12], [15], [16] and on quantitative results by [14].

The interpolation process is now a two-stage method.

Let $d, \nu_0, \nu_1, \nu_2, \ldots, \nu_d$ be natural numbers and let $F_1, F_2, \ldots, F_d$ be d functions, meromorphic in the disc $\mathbf{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$, $\rho > 1$, given by

$$F_i(z) = \frac{f_i(z)}{B_i(z)},\ f_i(z) = \sum_{k=0}^{\infty} a_{i,k}\, z^k,\ \limsup_{k\to\infty} |a_{i,k}|^{1/k} = \frac{1}{\rho}, \tag{4.1}$$

where

$$B_i(z) = \prod_{j=1}^{\mu_i} (z - z_{i,j})^{\lambda_{i,j}} = \sum_{k=0}^{\nu_i} \alpha_{i,k} z^k,\ \sum_{j=1}^{\mu_i} \lambda_{i,j} = \nu_i. \tag{4.2}$$

Here it is assumed that the poles given all lie in $\mathbf{D}_\rho$ and the poles of F_i are disjoint from those of $F_k,\ k \neq i$. Let $\ell \geq 1$ be an integer and put $n = \sigma + 1$, where $\sigma = \nu_0 + \nu_1 + \nu_2 + \cdots + \nu_d$. It is important for the sequel to remember that the ν_i with $i = 1, \ldots, d$ are fixed (thus also $\sigma - \nu_0 = n - \nu_0 - 1$ fixed) and that ν_0 will go to infinity (or, equivalently, n or σ). Let now $\alpha \in \mathbb{C} \setminus \{0\}$ satisfy $|\alpha| < \rho$, such that the zeros of $z^n - \alpha^n$ are different from those of the $B_i(z)$ for all i. Finally $r \geq 1$ is a natural number.

Then the Hermite-interpolant to the Taylor sections $\sum_{k=0}^{n\ell r-1} a_{i,k}\, z^k$ on the zeros of $(z^n - \alpha^n)^r$ will be denoted by

$$\widetilde{f_{i,\ell}}(z) = \sum_{j=0}^{n-1} A_{i,j}^{\ell}\, z^j.$$

This is the first stage of the problem; explicit formulae for the $A_{i,j}^{\ell}$ are easily derived (cf. [1]). When $\ell = \infty$, $\widetilde{f_{i,0}}(z)$ is the full Taylor series of $\widetilde{f_i}(z)$.

For $r = 1$ we are actually considering Lagrange-interpolation.

The second, *simultaneous*, stage of the problem can be stated as: find d rational functions $U_i^{\ell}(z)/B^{\ell}(z)$ with *a common denominator* and

1. with $U_i^{\ell}(z) = \sum_{s=0}^{\sigma-\nu_i} p_{i,s}^{\ell} z^s$ $(1 \leq i \leq d)$, $B^{\ell}(z) = \sum_{k=0}^{n-\nu_0-1} \gamma_k^{\ell} z^k$,
2. that interpolate the d rationals $\widetilde{f_{i,\ell}}(z)/B_i(z)$ on the zeros of $z^n - \alpha^n$,
3. and $B^{\ell}(z)$ is monic: $\gamma_{n-\nu_0-1}^{\ell} = 1$.

For $r = 1$ we have ordinary Hermite-Padé interpolation (see [1]) and for $r > 1$ the interpolation is using *multiple nodes* ([2])

The main result is now the following.

Theorem 4.1 ([1], Theorem 1; [2], §2) *Let* $r,\ d,\ \nu_i,\ f_i,\ \rho_i,\ B_i,\ z_{i,j}$ *and* ℓ *be given as before. Then we have:*
A. *For n sufficiently large, the interpolation problem stated above, has a unique solution that moreover satisfies*

$$\lim_{n\to\infty} \gamma_k^{\ell} = \zeta_k, \ \textit{with} \ \sum_{k=0}^{n-\nu_0-1} \zeta_k z^k = \prod_{i=1}^{d} B_i(z);\ 1 \leq \ell \leq \infty.$$

B. *Let $\mathcal{H}$ be a compact subset of $|z| < \tau$, $\tau > 0$, that omits the singularities of the functions F_i $(1 \le i \le d)$, then*

$$\limsup_{n\to\infty}\left(\max_{z\in\mathcal{H}}\left|\frac{U_i^\infty(z)}{B^\infty(z)}-\frac{U_i^\ell(z)}{B^\ell(z)}\right|\right)^{1/n} \le \begin{cases} R^{(\ell-1)r+1}(\tau/\rho_i)^r & \text{for } \tau \ge \rho_i, \\ R^{(\ell-1)r+1} & \text{for } \tau < \rho_i, \end{cases}$$

with $R = \max_{1\le i\le d} 1/\rho_i$.
C. *Specifically we have for $|z| < \rho_i R^{-(\ell-1+1/r)}$:*

$$\lim_{n\to\infty}\frac{U_i^\infty(z)}{B^\infty(z)}-\frac{U_i^\ell(z)}{B^\ell(z)} = 0,$$

uniformly and geometrically in compact subsets omitting the singularities.

The final result to be stated uses the case $r = 1$; i.e. in the first stage of the process Lagrange interpolation is used.

Introducing

$$K(z) := \left(\frac{|\alpha|}{\rho}\right)^\ell \frac{|z|}{\rho},\ |z| \ge \rho;\ K(z) := \left(\frac{|\alpha|}{\rho}\right)^\ell,\ |z| < \rho, \tag{4.3}$$

the results of Theorem 4.1 can be formulated as

$$\lim_{n\to\infty}\gamma_k^\ell = \widetilde{\alpha}_k \text{ with } \sum_{k=0}^{n-\nu_0-1}\widetilde{\alpha}_k z^k = \prod_{i=1}^d B_i(z),\ 1 \le \ell \le \infty, \tag{4.4}$$

and

$$\limsup_{n\to\infty}\left(\max_{z\in\mathcal{H}}\left|\frac{U_i^\infty(z)}{B^\infty(z)}-\frac{U_i^\ell(z)}{B^\ell(z)}\right|\right)^{1/n} \le K(\tau), \tag{4.5}$$

for each compact subset $\mathcal{H}$ of $|z| < \tau$ $(\tau > 0)$, that omits the singularities of the functions $F_i(z)$, $1 \le i \le d$.

The result (4.4) shows that the denominators of the simultaneous interpolants converge to the product of the pole parts of the functions and (4.5) shows the speed of convergence of the interpolating rational functions: it follows that the difference converges to zero on compact subsets (omitting singularities) of the disk $|z| < \rho \cdot \rho^\ell/\alpha^\ell$, a larger disk than that where the given functions were meromorphic! So even when not working with full information, but rather only with the $n\ell - 1$ Taylor section of each f_i, the ℓ-interpolant leads to good results.

For further study of the upper bound it is necessary to reformulate (4.5); as it is more convenient to study the difference of polynomials than that of rational functions, we can multiply by the denominators – not changing the upper bound because of (4.5) – and for reasons that will become clear later on, an extra factor $B_i(z)$ is thrown in.

Define for z not one of the singularities of the function F_i

$$\Delta^{\ell}_{i,\nu_0}(z) := B_i(z)B^{\ell}(z)B^{\infty}(z)\left(\frac{U_i^{\infty}(z)}{B^{\infty}(z)} - \frac{U_i^{\ell}(z)}{B^{\ell}(z)}\right), \tag{4.6}$$

and the quantities

$$S^{\ell}(z, F_i) := \limsup_{n\to\infty} |\Delta^{\ell}_{i,\nu_0}(z)|^{1/n}. \tag{4.7}$$

Then (4.5) takes the form

$$S^{\ell}(z, F_i) \le K(z) \text{ for } z \text{ with } B_i(z) \ne 0. \tag{4.8}$$

The question now arises, whether the upper bound $K(z)$ is attained for any z and, if so, whether this is a natural phenomenon or not. For polynomial interpolation the problem has completely been solved (cf. [14]).

Let us call a point z, $|z| \ne \rho$, *exceptional for the function* F_i when

$$S^{\ell}(z, F_i) < K(z). \tag{4.9}$$

The results then are (cf. [14], [16]):

Theorem 4.2 ([3], Theorem 1) *For each $i \in \{1, \ldots, d\}$ there are at most $\ell - 1$ exceptional points for F_i in $|z| < \rho$.*

Theorem 4.3 ([3], Theorem 2) *For any set of $d(\ell - 1)$ points ω_j in $0 < |z| < \rho$, and any subdivision into d sets of $\ell - 1$ points—say $\omega_{i,j}$ $(1 \le j \le \ell-1,\ 1 \le i \le d)$—there exists a d-tuple of meromorphic functions of the type (4.1),(4.2), such that for each $i \in \{1, \ldots, d\}$ the points $\omega_{i,j}$ $(1 \le j \le \ell - 1)$ are exceptional for F_i.*

Theorem 4.4 ([3], Theorem 3) *For each $i \in \{1, \ldots, d\}$ there are at most $\sigma - \nu_0 + \ell$ exceptional points for F_i in $|z| > \rho$.*

Theorem 4.5 ([3], Theorem 4) *For any set of $d(\ell + 1 - (\sigma - \nu_0))$ points ω_j in $|z| > \rho$, and any subdivision into d sets of $\ell + 1 - (\sigma - \nu_0)$ points—say $\omega_{i,j}$ $(1 \leq j \leq \ell + 1 - (\sigma - \nu_0)$, $1 \leq i \leq d)$—there exists a d-tuple of meromorphic functions of the type (4.1),(4.2), such that for each $i \in \{1, \ldots, d\}$ the points $\omega_{i,j}$ $(1 \leq j \leq \ell + 1 - (\sigma - \nu_0))$ are exceptional for F_i.*

The first 9 references given below have been written by A. Sharma with co-authors M.G. de Bruin ([1]–[9]), H.P. Dikshit ([6]) and J. Szabados ([4]).

References

[1] M. G. de Bruin and A. Sharma, Equiconvergence of some simultaneous Hermite-Padé interpolants, RAIRO Modél. Math. Anal. Numér. 29 (1995), 477–503.

[2] M. G. de Bruin and A. Sharma, Equiconvergence of some simultaneous Hermite-Padé interpolants: multiple nodes, in “Approximation Theory VIII,” Vol. 1 (College Station, TX, 1995), Ser. Approx. Decompos. 6 (Ch. K. Chui, L. L. Schumaker Eds.) , pp. 103–110, World Sci. Publishing, 1995.

[3] M. G. de Bruin and A. Sharma, Overconvergence of some simultaneous Hermite-Padé interpolants, Ann. Numer. Math. 4 (1997), 239–259.

[4] M. G. de Bruin, A. Sharma and J. Szabados, Birkhoff type interpolation on perturbed roots of unity, in “Approximation Theory in memory of A. K. Varma” (N. K. Govil, R. N. Mohapatra, Z. Nashed, A. Sharma, J. Szabados Eds.), pp. 167–179, Marcel Dekker Inc., Pure and Applied Mathematics Vol. 212, 1998.

[5] M. G. de Bruin and A. Sharma, Birkhoff Interpolation on Perturbed Roots of Unity on the Unit Circle, J. Nat. Acad. Math. India 11 (1997), 83–97 (1999).

[6] M. G. de Bruin, H. P. Dikshit and A. Sharma, Birkhoff interpolation on unity and on Möbius transform of the roots of unity, Numer. Algorithms 23 (2000), 115–125.

[7] M. G. de Bruin and A. Sharma, Birkhoff interpolation on non-uniformly distributed roots of unity, Numer. Algorithms 25 (2000), 123–138.

[8] M. G. de Bruin and A. Sharma, Birkhoff interpolation on non-uniformly distributed roots of unity II, J. Comput. Appl. Math. 133 (2001), 295–303.

[9] M. G. de Bruin and A. Sharma, Lacunary Pál-type interpolation and over-convergence, Comput. Methods Funct. Theory 3 (2003), 305–323.

[10] L. G. Pál, A new modification of the Hermite-Fejér interpolation, Anal. Math. 1 (1975), 197–205.

[11] A. S. Cavaretta, Jr., A. Sharma and R. S. Varga, Interpolation in the roots of unity: an extension of a theorem of J. L. Walsh, Resultate Math. 3 (1980), 155–191.

[12] E. B. Saff, A. Sharma and R. S. Varga, An extension to rational functions of a theorem of J. L. Walsh on the difference of interpolating polynomials, RAIRO Anal. Numér. 15 (1981) 371–390.

[13] G. G. Lorentz, K. Jetter and S. Riemenschneider, Birkhoff interpolation, Addison Wesley Publ. Co., Reading, Massachusets, 1983.

[14] V. Totik, Quantitative results in overconvergence of complex interpolating polynomials, J. Approx. Theory 47 (1986) 173–183.

[15] M. P. Stojanova, Equiconvergence in rational approximation of functions, Constr. Approx. 4 (1988) 435–445.

[16] M. P. Stojanova, Overconvergence in rational approximation of meromorphic functions, Math. Balkanica (N. S.) 3 (1989), 12–28.

[17] Shen, Xie-Chang, Introduction of a new class of interpolants, Birkhoff interpolants in the complex plane (*(Chinese, English Summary)*, Adv. in Math. (Beijing) 18 (1989), 412-432.

[18] Y. Bao, On simultaneous approximation to a differentiable function and its derivative by inverse Pál-type interpolation polynomials, Approx. Theory Appl. (N.S.) 11 (1995), 15–23.

[19] Xie Siqing, Regularity of $(0, 1, \ldots, r-2, r)$ and $(0, 1, \ldots, r-2, r)^*$ interpolations on some sets of the unit circle, J. Approx. Theory 82 (1995), 54–59.

[20] L. Szili, A survey on $(0, 2)$ interpolation, Ann. Univ. Sci. Budapest. Sect. Comput. 16 (1996), 377–390.

[21] W. Chen and A. Sharma, Lacunary interpolation on some non-uniformly distributed nodes on the unit circle, Ann. Univ. Sci. Budapest. Sect. Comput. 16 (1996), 69–82.

[22] A. Sharma and J. Szabados, Quantitative results in some problems on Walsh equiconvergence, East J. Approx. 4 (1998), 291–309.

[23] Marcel G. de Bruin, Regularity of some “incomplete” Pál-type interpolation problems, J. Comput. Appl. Math. 145 (2002), 407–415.

Freeness of Spline Modules from a Divided to a Subdivided Domain

Satya Deo
Harish Chandra Research Institute
Chhatnag Road, Jhusi
Allahabad 211 019, India
Email address: sdeo@mri.ernet.in, vcsdeo@yahoo.com

J. K. Maitra
Department of Mathematics and Computer Science
R.D. University
Jabalpur 482 001, India
Email address: jkmaitra@yahoo.co.in

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Let Δ be a simplicial d-complex, $d \geq 1$. Let $S = \mathbb{R}[x_1, x_2, \ldots, x_d]$ be the polynomial ring in d-variables and $C^r(\Delta)$ denote the S-module of all C^r-splines defined on Δ. Let $\hat{\Delta} \subset \mathbb{R}^{d+1}$ be the homogenization of $\Delta \subset \mathbb{R}^d$ (embedded in $x_{d+1} = 1$), and Δ' be any subdivision of Δ. In this paper we study the following question: Suppose $C^r(\hat{\Delta})$ is free over R, where $R = \mathbb{R}[x_1, x_2, \ldots, x_{d+1}]$ is the polynomial ring in $(d+1)$-variables. Under what conditions can we say that the R-module $C^r(\hat{(\Delta')})$ is also free? We show that when $r = 0$, $C^r(\hat{\Delta})$ is free imples $C^r(\hat{(\Delta')})$ is also free for any d-complex Δ. For $d = 1$, it is easy to see that $C^r(\hat{\Delta})$ is free implies $C^r(\hat{(\Delta')})$ is also free for every $r \geq 0$, but for $d \geq 2$, we give examples to show that $C^r(\hat{\Delta})$ is free need not imply that $C^r(\hat{(\Delta')})$ is free, for any $r \geq 1$.

1 Introduction

Let Δ be a simplicial d-complex; i.e., Δ is a pure, d-dimensional, strongly connected simplicial complex embedded in $\mathbb{R}^d$. Let $S = \mathbb{R}[x_1, x_2, \ldots, x_d]$ be the polynomial ring in variables $x_1, x_2, \ldots, x_d$. For every integer $k \geq 0$, let us define

$$\begin{aligned} C_k^r(\Delta) \quad = \quad & \{F : |\Delta| \to \mathbb{R} \text{ s.t. } F|_\sigma \in S \text{ is a polynomial of degree} \leq k \\ & \text{for every } d\text{-simplex } \sigma \in \Delta \text{ and } F \text{ is globally smooth} \\ & \text{ of order } r \text{ on } |\Delta|\} \end{aligned}$$

Then $C_k^r(\Delta)$ is a finite dimensional vector space over $\mathbb{R}$. To determine the $\mathbb{R}$-dimension of the vector space $C_k^r(\Delta)$ for a given $r \geq 0$ and a given $k \geq 0$ is fundamental unresolved problem of multivarate spline theory. Alfeld and Schumaker ([1],[2]) are the first approximation theorists to obtain basic results on this problem using, of course, the classical methods of a system of linear equations and determining the number of independent real constants. In the planer case they obtain a formula which essentially shows that for $d = 2$, the Hilbert polynomial and the Hilbert function are equal if $k = 3r + 1$. The fact that this dimension depends upon the geometry of the triangulation of the region makes the problem more difficult. L. Billera [3] introduced the method of homological algebra to tackle this problem. He considers the following set for any $r \geq 0$;

$$\begin{aligned} C^r(\Delta) \quad = \quad & \{F : |\Delta| \to \mathbb{R} \text{ s.t. } F|_\sigma \in S \text{ for every } \sigma \in \Delta \text{ and } F \text{ is} \\ & \text{globally smooth of order } r \text{ on } |\Delta|\} \end{aligned}$$

It is straightforward to see that $C^r(\Delta)$ is a ring with pointwise operations and S is a subring of this ring. Thus $C^r(\Delta)$ becomes a S-module and is called the **spline module** of Δ. Billera-Rose [5] embedded Δ into the hyperplane $x_{d+1} = 1$ of the space $\mathbb{R}^{d+1}$ and considered the simplicial complex $\hat{\Delta} = v * \Delta$ where v is the origin of $\mathbb{R}^{d+1}$. Let $R = \mathbb{R}[x_1, \cdots x_{d+1}]$ be the polynomial ring in $(d+1)$ variables and note that $C^r(\hat{\Delta})$ is a graded R-module. A useful observation is that, as a vector space over $\mathbb{R}$,

$$C_k^r(\Delta) \cong (C^r(\hat{\Delta}))_k$$

i.e., the spline vector space $C_k^r(\Delta)$ is isomorphic to the vector space of homogenous elements of degree k of the graded R-module $C^r(\hat{\Delta})$. This result has proved very important for solving the dimension problem. They considered the Hilbert series of the graded module $C^r(\hat{\Delta})$ which is always of the form $P(C^r(\hat{\Delta}), t)/(1-t)^{d+1}$ where $P(C^r(\hat{\Delta}), t)$ is a polynomial in $\mathbb{Z}[t]$. This is indeed the generating function of the dimension series $\sum_k (\dim_{\mathbb{R}} C_k^r(\Delta))\lambda^k$ of the spline modules [5].

In view of the preceding discussion, it is important to study the R-module $C^r(\hat{\Delta})$. Billera-Rose studied its freeness and the existence of an R-basis in [6]. Schenck has obtained [13] important results on Hilbert series of the spline module $C^r(\hat{\Delta})$ and has shown that the coefficients of the three largest terms of the polynomial $P(C^r(\hat{\Delta}), t)$ are determined by the combinatorics and local geometry of Δ (Corollary 3.2 of [13]). The most important result on freeness in the form of necessary and sufficient conditions has been given by Schenck (Theorem 4.10 of [13]). In [9], Deo and Mazumdar gave an algorithm to write down a basis for the free spline module $C^r(\square)$ for some polyhedral complexes $\square$. When it is not free, the projective dimension of this module was studied in [8], [17].

Suppose Δ' is a subdivision of Δ. Ruth Haas [11] and the authors [10] studied the question as to how the S-modules $C^r(\Delta)$ and $C^r(\Delta')$ are related. In fact it was shown in the above-mentioned papers that a generating set for the S-module $C^r(\Delta)$ can always be extended to a generating set of the S-module $C^r(\Delta')$ for certain subdivisions of Δ. Because finding an S-basis for the S-module $C^r(\Delta)$ is a problem of practical importance, it is natural to ask the following: If $C^r(\hat{\Delta})$ is free over R, then what can we say about the freeness of $C^r(\hat{(\Delta')})$ for any subdivision Δ' of Δ? It is a consequence of results proved in [4] that $C^0(\hat{\Delta})$ is free over R implies that $C^0(\hat{(\Delta')})$ is also free over R for any d-complex Δ, but here we provide an explicit proof of the result. For $d = 1$, it is trivial to see that $C^r(\hat{\Delta})$ is free implies $C^r(\hat{(\Delta')})$ is free for any $r \geq 0$. For $d \geq 2$, we give examples to show that for any $r \geq 1$, $C^r(\hat{\Delta})$ is free need not imply that $C^r((\hat{\Delta'}))$ is free.

Results of [5], [6], [14], [15] are basic ingredients used in our proofs of the above-mentioned results. We follow the notations of above papers freely, as most of them are now in standard use.

2 Preliminaries

Let Δ be a simplicial d-complex embedded in $\mathbb{R}^d$ and $S = \mathbb{R}[x_1, x_2, \ldots, x_d]$ be the polynomial ring over the field of reals in d-variables $x_1, x_2, \ldots, x_d$. For integers $k \geq 1$ and $r \geq 0$, let $C_k^r(\Delta)$ denote the $\mathbb{R}$-vector space of all splines over the region Δ which are of degree $\leq k$ and smooth up to order r. When we consider the set $C^r(\Delta)$ of all splines over Δ of smoothness r, regardless of their degrees, $C^r(\Delta)$ becomes a module over the graded ring S. Determining the vector space dimension of the vector spaces $C_k^r(\Delta)$ for a given k, r and Δ has been a basic problem of multivariate spline theory and is known as the Dimension Problem. It is difficult because the dimension of $C_k^r(\Delta)$ does not depend only on k, r and Δ but also on the geometry of Δ (see [3], [5] for details).

In this section we give some preliminary definitions and results which are used in section 3. A simplicial complex Δ is called **pure** if all maximal faces are of the same dimension. A **d-complex** is a pure d-dimensional simplicial complex embedded in $\mathbb{R}^d$. For a d-complex Δ and $i \leq d$, we denote the set of i-dimensional faces of Δ by Δ_i and the set of i-dimensional interior faces of Δ by Δ_i^0. The cardinality of these sets is denoted by $f_i(\Delta)$ and $f_i^0(\Delta)$ respectively. A **link** of a simplex σ in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta | \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \phi\}.$$

The **star** of σ is the simplicial complex

$$\mathrm{st}_\Delta(\sigma) = \left\{\tau \cup \tau' | \tau \in \mathrm{lk}_\Delta(\sigma) \text{ and } \tau' \subset \sigma\right\}.$$

In other words, $\mathrm{st}_\Delta(\sigma)$ is the smallest subcomplex of Δ containing all faces which contain σ.

For a d-complex Δ, the **graph** $G(\Delta)$ of Δ is the graph whose vertices correspond to the elements of Δ_d and $< v, v' >$, $v, v' \in \Delta_d$ is an edge of $G(\Delta)$ iff $\sigma \cap \sigma' \in \Delta_{d-1}$, where v corresponds to σ and v' corresponds to σ' respectively. A d-complex Δ is said to be **strongly connected** if the graph $G(\Delta)$ is connected. In other words for any two simplexes $\sigma, \sigma' \in \Delta_d$ there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_t \in \Delta_d$ such that $\sigma = \sigma_1, \sigma' = \sigma_t$ and $\sigma_i \cap \sigma_{i+1} \in \Delta_{d-1}$ for $i = 1, 2 \ldots, (t-1)$. A connected complex Δ is said to be **hereditary** if for all non-empty $\sigma \in \Delta$, $\mathrm{st}_\Delta(\sigma)$ is strongly connected (equivalently $\mathrm{lk}_\Delta(\sigma)$ is strongly connected).

Billera-Rose initiated the study of freeness of the module $C^r(\Delta)$ in [6]. They proved the following results:

Proposition 2.1 ([6], Theorem 2.3) *Let Δ be a d-complex. Then $C^r(\Delta)$ is free if and only if $C^r(st_\Delta(v))$ is free for all verices v of Δ.*

Proposition 2.2 ([6], Theorem 3.1) *Let Δ be a d-complex. If $C^r(\Delta)$ is free then Δ is a hereditary complex.*

When Δ is a 2-complex in $\mathbb{R}^2$, they characterized freeness of $C^r(\Delta)$ totally in terms of the topology of the complex Δ viz.

Theorem 2.3 ([6], Theorem 3.5) *Let Δ be a 2-complex. Then $C^r(\Delta)$ is free if and only if Δ is a manifold with boundary.*

Let Δ be a d-complex with n-vertices $v_1, v_2, \ldots, v_n$. Then Δ is said to be **Cohen-Macaulay** over a field k if the face ring $A_\Delta = A/I_\Delta$, where $A = k[y_1, y_2, \ldots, y_n]$ and I_Δ is the ideal in A generated by square-free monomials corresponding to the vertex sets which are not faces of Δ, is a Cohen-Macaulay ring. A homological characterization of Cohen-Macaulay complexes is given in Reisner [12]. Billera and Rose have given a complete characterization for freeness of $C^0(\Delta)$ in terms of combinatorial property of Δ which is independent of the embedding of Δ in $\mathbb{R}^d$ as follows:

Theorem 2.4 (Theorem 4.5 of [6]) *Let Δ be a d-complex. Then $C^0(\Delta)$ is free if and only if Δ has Cohen-Macaulay links of vertices.*

Schenck-Stillman [14] considered the freeness of $C^r(\hat{\Delta})$ when Δ is a 2-complex by a somewhat different approach. We note that the homogenization $\hat{\Delta}$ of a 2-complex Δ is a 3-complex.

Let Δ be a d-complex embedded in $\mathbb{R}^d$ and R be the polynomial ring in $(d+1)$-variables. Schenck-Stillman have defined a **chain complex** $\mathcal{F}$ of R-modules on Δ^0, the set of interior faces of Δ, as follows:

(1) For each $\sigma \in \Delta^0$, there is an R-module $\mathcal{F}(\sigma)$, and
(2) For each $i = 0, 1, 2, \ldots, d$, there is an R-module homomorphism

$$\partial_i : \oplus_{\sigma_i \in \Delta_i^0} \mathcal{F}(\sigma_i) \rightarrow \oplus_{\sigma_{i-1} \in \Delta_{i-1}^0} \mathcal{F}(\sigma_{i-1})$$

such that $\partial_{i-1} \circ \partial_i = 0$. Thus the chain complex $\mathcal{F}$ is given by:

$$0 \longrightarrow \oplus_{\sigma_d \in \Delta_d} \mathcal{F}(\sigma_d) \xrightarrow{\partial_d} \oplus_{\sigma_{d-1} \in \Delta_{d-1}^0} \mathcal{F}(\sigma_{d-1}) \xrightarrow{\partial_{d-1}}$$

$$\cdots \xrightarrow{\partial_1} \oplus_{\sigma_0 \in \Delta_0^0} \mathcal{F}(\sigma_0) \longrightarrow 0$$

For example if we associate the ring R corresponding to every $\sigma \in \Delta^0$ and take ∂_i to be the usual simplicial boundary map relative to $\partial\Delta$, then it is a chain complex, called the **constant complex**, and is denoted by $\mathcal{R}$.

For another example, we fix an integer $r \geq 0$, and corresponding to every $\sigma \in \Delta^0$ we associate the homogenous ideal of $\hat{\sigma} \subset \mathbb{R}^{d+1}$ which we denote by I_σ. Let $\mathcal{J}$ be the complex on Δ^0 defined as follows:

$$\begin{aligned}
\mathcal{J}(\sigma) &= 0 && \text{for } \sigma \in \Delta_d, \\
\mathcal{J}(\tau) &= I_\tau^{r+1} && \text{for } \tau \in \Delta_{d-1}^0, \\
\mathcal{J}(\xi) &= \sum_{\xi \in \tau} I_\tau^{r+1} && \text{for } \xi \in \Delta_{d-2}^0, \tau \in \Delta_{d-1}^0 \\
&\vdots && \vdots \\
\mathcal{J}(v) &= \sum_{v \in \tau} I_\tau^{r+1} && \text{for } v \in \Delta_0^0, \tau \in \Delta_{d-1}^0.
\end{aligned}$$

Then the complex $\mathcal{J}$ is given by:

$$0 \longrightarrow \oplus_{\tau\in\Delta^0_{d-1}}\mathcal{J}(\tau) \xrightarrow{\partial_{d-1}} \oplus_{\xi\in\Delta^0_{d-2}}\mathcal{J}(\xi) \longrightarrow \cdots \xrightarrow{\partial_1} \oplus_{v\in\Delta^0_0}\mathcal{J}(v) \longrightarrow 0.$$

Hence we can consider the quotient complex $\mathcal{R}/\mathcal{J}$ viz.:

$$0 \to \oplus_{\sigma\in\Delta_d}R \xrightarrow{\partial_d} \oplus_{\tau\in\Delta^0_{d-1}}R/\mathcal{J}(\tau) \xrightarrow{\partial_{d-1}} \oplus_{\xi\in\Delta^0_{d-2}}R/\mathcal{J}(\xi)$$

$$\cdots \xrightarrow{\partial_1} \oplus_{v\in\Delta^0_0}R/\mathcal{J}(v) \to 0$$

It turns out that the top homology module of the above quotient complex $\mathcal{R}/\mathcal{J}$ is the graded spline module $C^r(\hat{\Delta})$. Schenck-Stillman [14] have obtained following results for a 2-complex Δ giving relation between homology of the complex $\mathcal{R}$, $\mathcal{R}/\mathcal{J}$ and the freeness of the spline module $C^(\hat{\Delta})$ over $R = \mathbb{R}[x_1, x_2, x_3]$ in terms of one dimensional homology of the quotient complex $\mathcal{R}/J$.

Lemma 2.5 ([14], Lemma 3.7) *Let Δ be a 2-complex. If $H_1(\mathcal{R}) \neq 0$ then $H_1(\mathcal{R}/\mathcal{J}) \neq 0$.*

Theorem 2.6 ([14], Theorem 4.1) *Let Δ be a 2-complex. Then $C^r(\hat{\Delta})$ is free if and only if $H_1(\mathcal{R}/\mathcal{J}) = 0$.*

From the above results we find that if Δ has genus greater than zero, i.e., $H_1(\mathcal{R}) \neq 0$, then $C^r(\hat{\Delta})$ cannot be free. Hence for the freeness of the spline module $C^r(\hat{\Delta})$ we must have Δ of genus zero; i.e., Δ is a 2-disk. But taking Δ to be a disk in $\mathbb{R}^2$ does not guarentee that spline module $C^r(\hat{\Delta})$ is always free for all $r \geq 0$. We explain their nonfreeness result below.

Let Δ be a triangulation of a 2-disk in the plane $\mathbb{R}^2$. An edge τ of Δ is said to be a **totally interior** if both of its end vertices are interior. Let $\sim$ be a relation on the set of interior edges of Δ defined by $\tau \sim \tau'$ if τ and τ' share a vertex and have the same slope. Taking the transitive closure of this relation gives us an equivalence relation on interior edges. An edge τ is called a **pseudoboundary edge** if

$\tau \sim \tau'$ for some not totally interior edge τ'. In other words τ can be extended to the boundary of Δ.

For each edge τ of Δ, let us define

$$s_\tau = \max\{\text{number of slopes at } v,\ v \text{ a vertex of an edge } \tau' \text{ and } \tau' \sim \tau\}$$

and an integer $s(\Delta)$ by

$$s(\Delta) = \min\{s_\tau \mid \tau \text{ an edge which is not a pseudoboundary edge of } \Delta\}.$$

Schenck-Stillman have obtained the following interesting geometric result. We will use this result besides the result of Billera-Rose stated earlier.

Theorem 2.7 ([14], Theorem 5.3) *If Δ is a triangulation of a disk in the plane, then*

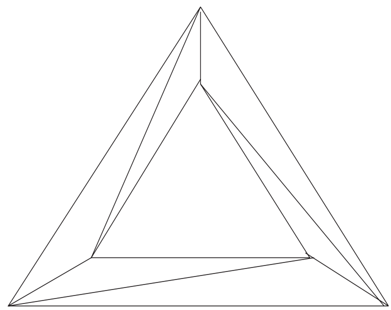
(a) If every edge of Δ is a pseudoboundary edge, then $C^r(\hat{\Delta})$ is free for all $r \geq 0$.

(b) If Δ has at least one edge which is not a pseudoboundary, then for each $r \geq s(\Delta) - 2$, $C^r(\hat{\Delta})$ is not free.

3 Homogenization, Subdivision, and Freeness

We note that if $|\Delta|$ is a k-manifold with boundary then $|\hat{\Delta}|$ need not be a $(k+1)$ manifold with boundary. For example, let Δ be the simplicial complex with a hole (interior of the center triangle is not included) as in Figure 1. Then $|\hat{\Delta}| = |v * \Delta|$ is not a manifold with boundary. However, if Δ is a pseudomanifold with boundary, then $\hat{\Delta}$ is indeed a pseudomanifold with boundary.

Schenck and Stillman have given several examples (see section 6 of [14]) including Morgan-Scott 2-complex of 2-dimensional simplicial complexes Δ, all of genus zero, such that $C^r(\hat{\Delta})$ is not free for $r \geq 2$. Since all these 2-complexes are manifolds with boundary, $C^r(\Delta)$ is free by Theorem 2.3. Thus we conclude that $C^r(\Delta)$ is free need not imply that $C^r(\hat{\Delta})$ is free.



Δ

Figure 1

What about the converse? The answer is yes. We have the following lemma.

Lemma 3.1 *For a 2-complex Δ, $C^r(\hat{\Delta})$ is free implies that $C^r(\Delta)$ is free.*

PROOF. Suppose $C^r(\hat{\Delta})$ is free. Then by Proposition 2.2, it follows that $\hat{\Delta}$ is hereditary; i.e., links of vertices are strongly connected. This means Δ is a 2-manifold with boundary. Hence by Theorem 2.3, $C^r(\Delta)$ is free. □

Now if we take a d-complex Δ and take its subdivision Δ' then what can we say about the freeness of $C^r(\Delta)$ and $C^r(\Delta')$? The following interesting results give an answer of this question using Billera-Rose criterion for $d = 2$.

Proposition 3.2 *If Δ is a 1-complex, then $C^r(\Delta)$ is free if and only if $C^r(\Delta')$ is free for all $r \geq 0$.*

PROOF. See [6], page 491. □

Proposition 3.3 *Let Δ be a 2-complex. Then $C^r(\Delta)$ is free if and only if $C^r(\Delta')$ is free for all $r \geq 0$.*

Proof. $C^r(\Delta)$ is free. $\iff$ Δ is a 2-manifold with boundary. $\iff$ Δ' is a 2-manifold with boundary. $\iff$ $C^r(\Delta')$ is free. $\square$

Let us now consider homogenization of subdivided domains. In view of our comments preceding Lemma 3.1 we observe that for a 2-complex Δ, the following implications break down only at the last level.

(1) $C^r(\hat{\Delta})$ is free. $\Rightarrow C^r(\Delta)$ is free. $\Rightarrow C^r(\Delta')$ is free. $\Rightarrow C^r(\widehat{(\Delta')})$ is free.

(2) $C^r(\widehat{(\Delta')})$ is free. $\Rightarrow C^r(\Delta')$ is free. $\Rightarrow C^r(\Delta)$ is free. $\Rightarrow C^r(\hat{\Delta})$ is free.

Let us start with the following easy result which is true for all $r \geq 0$. This shows that for $d = 1$ the above implications are true everywhere.

Proposition 3.4 *Let Δ be a 1-complex embedded in $\mathbb{R}$. Then for every $r \geq 0$, $C^r(\hat{\Delta})$ is free over R if and only if $C^r(\widehat{(\Delta')})$ is free over R.*

Proof. Since Δ is a 1-complex, Δ is an interval $[a, b]$ with finite number of interior vertices, say

$$a = v_0 < v_1 < \cdots < v_n = b.$$

Hence any subdivision Δ' of Δ will again be of the same type with more interior vertices. Now, clearly $\hat{\Delta}$ and $\widehat{(\Delta')}$ both are 2-complexes which are disks; i.e., $|\hat{\Delta}|$ and $|\widehat{(\Delta')}|$ both are 2-manifolds with boundary. Hence by Theorem 2.3 both $C^r(\hat{\Delta})$ and $C^r(\widehat{(\Delta')})$ are free for every $r \geq 0$, proving the result. $\square$

Proposition 3.5 *Let Δ be a d-complex. If Δ is Cohen-Macaulay then $\hat{\Delta}$ is Cohen-Macaulay.*

Proof. We prove the proposition by induction on the number of simplexes in Δ. If Δ has only one simplex, i.e., Δ is having only one vertex, then $\hat{\Delta}$ is an interval and so by the Reisner criterion (Theorem 1 of [12]), $\hat{\Delta}$ is Cohen-Macaulay.

Let us assume that number of simplexes in Δ is r and $r > 1$. Now Δ is Cohen-Macaulay implies that $\mathrm{lk}_{\Delta}(v)$ is Cohen-Macaulay for all vertices v of Δ, by Reisner criterion (Theorem 1 of [12]). Suppose $\hat{\Delta} = w * \Delta$, then $\mathrm{lk}_{\hat{\Delta}}(w) = \Delta$ which is Cohen-Macaulay. If v is any vertex of $\hat{\Delta}$ other than w (i.e., v is an inside vertex), then $\mathrm{lk}_{\hat{\Delta}}(v) = w * \mathrm{lk}_{\Delta}(v)$ and $w * \mathrm{lk}_{\Delta}(v)$ is Cohen-Macaulay by induction hypothesis. Since $\hat{\Delta}$ is acyclic, $\tilde{H}_i(\hat{\Delta}) = 0$ for all $i \geq 0$, and so it follows from Reisner criterion (Theorem 1 of [12]) that $\hat{\Delta}$ is Cohen-Macaulay. $\square$

Proposition 3.6 *Let Δ be d-complex. If Δ is Cohen-Macaulay then Δ' is a Cohen-Macaulay.*

PROOF.This follows from the fact that Cohen-Macaulayness of Δ is a topolgical property. Since $|\Delta| = |\Delta'|$, the result follows from Corollary 5.4.6 of [7].

Alternatively, we can prove this proposition as follows: If Δ is Cohen-Macaulay, then by Reisner criterion (Theorem 1 of [12]) $\tilde{H}_i(\Delta) = 0$ for all $i < \dim(\Delta)$, and $\mathrm{lk}_{\Delta}(v)$ are Cohen-Macaulay for all vertices v of Δ. This implies $\tilde{H}_i(\Delta') = 0$ for all $i < \dim(\Delta') = \dim(\Delta)$. Hence it is sufficient to show that for every vertex w of Δ', $\mathrm{lk}_{\Delta'}(w)$ is Cohen-Macaulay. If w is a new vertex of Δ' (arising in the subdivision), then $\mathrm{lk}_{\Delta'}(w) \approx S^{d-1}$, and S^{d-1} is Cohen-Macaulay. If v is a vertex of Δ' which is also a vertex of Δ then $\mathrm{lk}_{\Delta'}(v) \approx \mathrm{lk}_{\Delta}(v)$. Since $\mathrm{lk}_{\Delta}(v)$ is Cohen-Macaulay and Cohen-Macaulayness is a topolgical property, it follows that $\mathrm{lk}_{\Delta'}(v)$ is Cohen-Macaulay. Hence by Reisner criterion (Theorem 1 of [12]) Δ' is Cohen-Macaulay. $\square$

For $r = 0$, we have the following general result concerning freeness of the homogenized spline module over a subdivided domain.

Theorem 3.7 *Let Δ be a d-complex and let Δ' be a subdivision of Δ. If $C^0(\hat{\Delta})$ is free over $R = \mathbb{R}[x_1, x_2, \ldots, x_{d+1}]$, then $C^0(\widehat{(\Delta')})$ is free over R.*

Proof. It is clear that $\hat{\Delta}$ and $(\Delta')\hat{}$ both are $(d+1)$-complexes. If $C^0(\hat{\Delta})$ is free over R then by Theorem 2.4, $\mathrm{lk}_{\hat{\Delta}}(v)$ is Cohen-Macaulay for all vertices v of $\hat{\Delta}$. Suppose $\hat{\Delta} = v_0 * \Delta$ and call v_0 as the outside vertex of $\hat{\Delta}$ and other vertices as the inside vertices of $\hat{\Delta}$. Since $\mathrm{lk}_{\hat{\Delta}}(v_0) = \Delta$, we find that Δ is a Cohen-Macaulay complex. Hence by Proposition 3.6, we find that Δ' is Cohen-Macaulay.

Now we show that links of vertices of $(\Delta')\hat{}$ are Cohen-Macaulay. If v is outside vertex of $(\Delta')\hat{}$, then $\mathrm{lk}_{(\Delta')\hat{}}(v) = \Delta'$ which is Cohen-Macaulay. On the other hand if v is an inside vertex of $(\Delta')\hat{}$ then $\mathrm{lk}_{(\Delta')\hat{}}(v) = (\mathrm{lk}_{\Delta'}(v))\hat{}$. Since v is a vertex of Δ' and Δ' is Cohen-Macaulay, it follows by Reisner criterion (Theorem 1 of [12]) that $\mathrm{lk}_{\Delta'}(v)$ is Cohen-Macaulay. Therefore by Proposition 3.5 we deduce that $(\mathrm{lk}_{\Delta'}(v))\hat{}$ is Cohen-Macaulay; i.e., $\mathrm{lk}_{(\Delta')\hat{}}(v)$ is Cohen-Macaulay. Hence, again by Theorem 2.4, $C^0((\Delta')\hat{})$ is free. □

In the above theorem we have shown that for any d-complex Δ if $C^0(\hat{\Delta})$ is free then $C^0((\Delta')\hat{})$ is free. This is the continuous spline case. Next we want to consider the smooth case, i.e., when $r \geq 1$. For $d = 1$, we have proved in Proposition 3.4 that $C^r(\hat{\Delta})$ is free iff $C^r((\Delta')\hat{})$ is free, for all $r \geq 1$. However, as pointed out in the beginning of this section, when $d \geq 2$, and $r \geq 2$ it is not necessarily true that $C^r(\hat{\Delta})$ is free implies $C^r((\Delta')\hat{})$ is free. We now give an example for the case of a 2-complex Δ which has the property that for all $r \geq 1$, $C^r(\hat{\Delta})$ is free but $C^r((\Delta')\hat{})$ is not free. This example is also of interest because it is used in [16] to show that $C^r(\hat{\Delta}')_{2r}$ is not given by Alfeld-Schumaker formula, for all r. To verify this example we will apply Theorem 2.7 for nonfreeness of $C^r(\hat{\Delta})$ when Δ is a 2-complex.

Example 3.8 *Let Δ be 2-complex as in Figure 2. We take Δ' to be the subdivision of Δ as in Figure 2. Then Δ and Δ' are triangulations of a disk in $\mathbb{R}^2$. Since all the edges in Δ are pseudoboundary by Theorem 2.7, $C^r(\hat{\Delta})$ is free over $R = \mathbb{R}[x_1, x_2, x_3]$ for all $r \geq 0$. Now in Δ' we see that there is one edge, say τ, which is not a pseudoboundary. For this edge τ, we have*

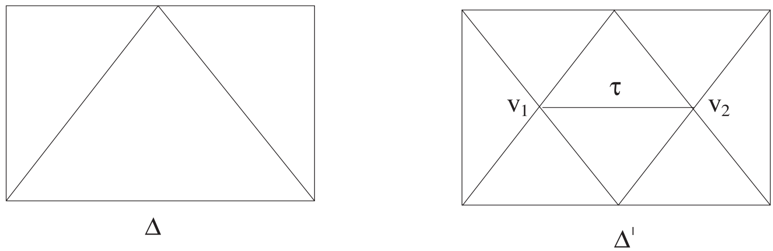


Figure 2

$$\begin{aligned} s_\tau &= \max\{\text{number of slopes at } v_1, \text{ number of slopes at } v_2\} \\ &= \max\{3,\ 3\} = 3 \end{aligned}$$

Hence,

$$\begin{aligned} s(\Delta') &= \min\{s_\tau : \tau \text{ is an edge which is not a pseudoboundary} \\ &\qquad \text{edge of } \Delta'\} \\ &= \quad 3 \end{aligned}$$

Thus $s(\Delta') = 3$. Therefore, by Theorem 2.7, $C^r(\widehat{(\Delta')})$ is not free for $r \geq s(\Delta') - 2 = 3 - 2 = 1$, i.e., for $r \geq 1$.

Acknowledgments

The authors are thankful to the referee for his valuable suggestions. The first author was supported by DST Project Grant No. DST/MS/080/98 while carrying out this work.

References

[1] Alfeld, P and Schumaker, L., *The dimension of bivariate spline spaces of smooothness r for degree $d \geq 4r + 1$,* Constr. Approximation, **3** (1987), 189-197. .

[2] Alfeld, P and Schumaker, L., *On the dimension of bivariate spline spaces of smooothness r and degree $d = 3r + 1$,* Numer. Math., **57** (1990), 651-661.

[3] Billera, L. J., *Homology of smooth splines: Generic triangulations and a conjecture of Strang,* Trans. Amer. Math. Soc. **310** (1988), 325-340 .

[4] Billera, L. J. *The algebra of continuous piecewise polynomials,* Adv. Math. **76** (1989), 170-183.

[5] Billera, L. J. and Rose, L. L. *A dimension series for multivariate splines,* Discrete Comput. Geom. **6** (1991), 107-128.

[6] Billera, L. J. and Rose, L. L. *Modules of piecewise polynomials and their freeness*, Math. Zeitschrift, **209** (1992), 485-497.

[7] Bruns, Winfried and Herzog, Jürgen, *Cohen-Macaulay Rings,* Cambridge University Press, 1993.

[8] Deo, Satya, *On projective dimension of spline modules,* Jour. of Approximation Theory **84** (1996), 12-30.

[9] Deo, Satya and Mazumdar, L., *An algorithm for constructing a basis for C^r-spline modules over polynomial rings,* Math. of Computation **67** (1998), 1107-1120.

[10] Deo, Satya and Maitra, J. K., *Spline modules from a divided domain to a subdivided domain,* Indian J. Pure Appl. Math. **35(8)** (2004) 1033-1041.

[11] Haas, Ruth, *Bases for splines on a subdivided domain,* Acta Appl. Math. **51** (1998), 113-122.

[12] Reisner, G. A. *Cohen-Macaulay quotients of polynomial rings,* Adv. Math. **21** (1976), 30-49.

[13] Schenck, H., *A spectral sequence for splines,* Advances in Applied Mathematics, **19** (1997), 183-199.

[14] Schenck, H. and Stillman, M., *Local cohomology of bivariate splines,* Jour. of Pure and Applied Algebra, **117-118** (1997), 535-548.

[15] Schenck, H. and Stillman, M. *A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$,* Adv. in Applied Math. **19** (1997), 169-182.

[16] Tohaneanu, S., *Smooth planer r-splines of degree $2r$,* Journal of Approximation Theory, **132** (2005), 72-76.

[17] Yuzvinsky, S., *Modules of splines on polyhedral complexes,* Math. Zeitschrift, **210** (1992), 245-254.

Measures of Smoothness on the Sphere

Z. Ditzian
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta, Canada T6G 2G1
Email address: zditzian@math.ualberta.ca

Dedicated to Ambikeshwar Sharma in tribute to our long and meaningful friendship

Abstract

We survey various measures of smoothness of functions which are defined on the unit sphere $S^{d-1} \subset R^d$. Relations among different measures will be given. Some open problems will be indicated.

1 Introduction

Measures of smoothness of functions on a given domain and relations among them play a significant role in approximation theory.

In this article we discuss measures of smoothness of functions on the unit sphere S^{d-1} given by

$$S^{d-1} = \{(x_1, \dots, x_d) : x_1^2 + \cdots + x_d^2 = 1\}. \tag{1.1}$$

We assume $d \geq 3$, but most of the concepts and results (not all) are valid for $d = 2$, that is, the circle (where they are much better known).

In Section 2 we discuss measures using differences. In Sections 3 and 4 we discuss the most investigated measures of smoothness on S^{d-1}, that is, using approximation of the function by its average on the rim of a small cap or on the cap itself about the point. In

Section 5 we describe appropriate $K-$functionals. In Section 6 we relate the above to best harmonic polynomial approximation. In Section 7 we describe various realization results. In Section 8 we survey measuring smoothness by strong converse inequalities. Some additional relations are given in Section 9.

We will define the various concepts and give many of the known relations among them. We will mention advantages and disadvantages of the concepts described. Some open problems will be indicated.

2 Measures of Smoothness Using Differences

For functions on $R = (-\infty, \infty)$ or T (the circle) the most common moduli of smoothness are given by

$$\omega^r(f,t)_X = \sup_{|h|\le t} \|\Delta_h^r f\|_X \tag{2.1}$$

where

$$\Delta_h^r f(x) = \Delta_h\big(\Delta_h^{r-1} f(x)\big), \quad \Delta_h f(x) = f(x+h) - f(x) \tag{2.2}$$

and $\|\cdot\|_X$ is a norm or quasi-norm usually (but not always) satisfying

$$\|f(\cdot + h)\|_X = \|f(\cdot)\|_X. \tag{2.3}$$

For $x \in S^{d-1}$, $x+h$ is not in S^{d-1}, and this is perhaps the reason for the many investigations on and different approaches to the subject.

For a space X of functions on the sphere $S^{d-1} = \{x \in R^d : |x|^2 = x_1^2 + \cdots + x_d^2 = 1\}$ I introduced in [11] the moduli $\omega^r(f,t)_X$ given by

$$\omega^r(f,t)_X = \sup\{\|\Delta_\rho^r f\|_X : \rho \in O_t\} \tag{2.4}$$

where

$$O_t = \{\rho \in SO(d) : \rho x \cdot x \ge \cos t \quad \text{for all} \quad x \in S^{d-1}\} \tag{2.5}$$

(recall that $SO(d)$ is the collection of orthonormal matrices on R^d whose determinants are equal to 1) and

$$\Delta_\rho^r f(x) = \Delta_\rho\big(\Delta_\rho^{r-1} f(x)\big), \quad \Delta_\rho f(x) = f(\rho x) - f(x). \tag{2.6}$$

This definition is applicable to any space of functions for which

$$\|f(\rho\,\cdot)\|_X = \|f(\cdot)\|_X \tag{2.7}$$

including $L_p(S^{d-1})$ and $H_p(S^{d-1})$, $0 < p \le \infty$ and many others for which $\|\quad\|_X$ is a norm or a quasinorm. We note that (2.7) is not a necessary condition for (2.4) to be defined, just that work on (2.4) was done under that assumption. The inequality $\rho x \cdot x \ge \cos t$ for all $x \in S^{d-1}$ is equivalent to $\|I - \rho\| \le 2|\sin\ \frac{t}{2}|$ where I is the identity matrix and the norm is the operator norm $\ell_2 \to \ell_2$ for ℓ_2 of $(x_1, \dots, x_d)$. It should be noted that for $r = 1$ and $X = L_1(S^{d-1})$ (1.1) was already used in [4]. (I believe that in writing $|\rho| < t$ for the "rotation" ρ in definition (1.6) of [4], the authors meant $\|I - \rho\| < t$.)

For $f \in L_\infty(S^{d-1})$, Ragozin [24] introduced the measure of smoothness

$$\omega_*^r(f,t)_\infty = \sup_{\substack{h \le t \\ x \in S^{d-1}}} |\Delta_h^r f(x)| \tag{2.8}$$

with

$$\Delta_h^r f(x) = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} f(x_\nu)$$

where $x_0 = x$ and x_ν are equidistant points on a big circle containing x with $\|x_\nu - x_{\nu-1}\| = h$. (Clearly $\omega_*^r(f,t)_\infty \le \omega^r(f,t)_\infty$ and in fact $\omega_*^r(f,t)_\infty \approx \omega^r(f,t)_\infty$.)

For $1 \le p < \infty$ and $f \in L_p(S^{d-1})$ Fedorov [18] introduced the following measure of smoothness:

$$\omega_*^r(f,t)_p = \\ \sup_{|h| \le t} \Big\{ \int_{S^{d-1}} \int_{S^{d-2}(x)} \Big| \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu}\ f(x \cos \nu t + \xi \sin \nu t) \Big|^p d\xi\, dx \Big\}^{1/p} \tag{2.9}$$

where $S^{d-2}(x) = \{\xi \in S^{d-1} : \xi \perp x\}$. The concept (2.9) generalizes (2.8) and is perhaps a hybrid between measures described in this and the next section. (It is different from either.)

3 Measures of Smoothness Using Average $\mathbf{S_\theta f}$

In most articles on the subject it is not the transformation $T_\rho : f(x) = f(\rho x)$ given in the last section but the transformation $S_\theta f(x)$ that is used. $S_\theta f(x)$ is given by

$$S_\theta f(x) = \frac{1}{m(\theta)} \int_{x\cdot y=\cos\theta} f(y)d\gamma(y), \quad S_\theta 1 = 1 \tag{3.1}$$

where $d\gamma(y)$ is the measure on the set $\{y : x\cdot y = \cos\theta\}$ induced by the Lebesgue measure . The smoothness is described by $S_\theta f(x) - f(x)$ or combinations or iterations of it. It should be noted that because of its symmetry $S_\theta f - f$ corresponds to the second modulus (not the first). It is not possible to mention all papers dealing with expressions using $S_\theta f$ for describing smoothness and proving its relation to other concepts. (I believe there are at least three dozen.) I will highlight several of the articles and some of the related concepts. The description originated from works of Pawelke and Weherens students of Butzer (see [23] and [29]) and continued in many works of Lizorkin, Nikolskii, Teherin, Rustamov (see [20], [22], [25], [26] and [27]) and other Russian mathematicians. In Chapter 5 of [28] a flaw in Rustamov's treatment is fixed. Some recent advances were made in [2], [7] and [13]. I will try to itemize the main different possibilities.

For $f \in L_p(S^{d-1})$, $1 \le p \le \infty$, one has

$$\widetilde{\omega}^{2r}(f,t)_p = \sup_{|\theta|\le t} \|(I - S_\theta)^r f\|_{L_p(S^{d-1})} \tag{3.2}$$

(see for $r = 1$ [29] and [23], and for integer r [22] and [20]). The modulus (3.2) was also treated for non-integer r (see [25] and [28, 183-184]) using

$$(I - S_\theta)^r f = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} S_\theta^k f.$$

As this latter concept requires an infinite sum of higher and higher iterates of the transformation S_θ, I will not dwell on (3.2) with non-integer r. The proofs using this concept (for any $r > 0$) usually treat

$(I - S_\theta)^r$ as a multiplier operator. A measure similar to $\widetilde{\omega}^{2r}(f,t)_p$ is given by

$$\widetilde{\omega}_{\mathrm{I}}^{2r}(f,t)_p = \sup_{|\theta_i| \le t} \|(S_{\theta_1} - I) \ldots (S_{\theta_r} - I) f\|_{L_p(S^{d-1})} \tag{3.3}$$

(see [29]). One can use combinations of $S_{j\theta}$ and obtain

$$\widetilde{\omega}_{\mathrm{II}}^{r}(f,t)_p = \sup_{|\theta| \le t} \left\| \sum_{k=0}^{r} \binom{r}{k} (-1)^k S_{k\theta}(f) \right\|_p \tag{3.4}$$

(see [26, p. 236, (4)] where the first t on the right should be τ) or

$$\widetilde{\omega}_{\mathrm{III}}^{2r}(f,t)_p = \sup_{|\theta| \le t} \left\| \binom{2r}{r} f + 2 \sum_{j=1}^{r} (-1)^j \binom{2r}{r-j} S_{j\theta} f \right\|_p \tag{3.5}$$

(see [26, p. 236 (5)] where the first t on the right should be τ). The advantage of (3.5) over (3.4) is that for the same number of terms one may investigate a higher level of smoothness.

It was further proved in [2] that for $1 \le p \le \infty$ and $d \ge 3$

$$\|(S_t - I)^r f\|_p \approx \sup_{|\theta| \le t} \|(S_\theta - I)^r f\|_{L_p(S^{d-1})} = \widetilde{\omega}^{2r}(f,t)_p, \tag{3.6}$$

and therefore the supremum in (3.2) is redundant, which is an amusing fact since this is not true for $d = 2$ (where the sphere is the circle). Later it was shown in [7] that for $1 \le p \le \infty$ and $d \ge 3$

$$\left\| \binom{2r}{r} f + 2 \sum_{j=1}^{r} (-1)^j \binom{2r}{r-j} S_{jt} f \right\|_p \approx \omega_{\mathrm{III}}^{2r}(f,t)_p, \tag{3.7}$$

and hence the supremum on θ is not necessary in (3.5) as well. One may note that these facts, that is (3.6) and (3.7), would have simplified the proofs of earlier results if they had been known earlier. The equivalences (3.6) and (3.7) are extended from $L_p(S^{d-1})$ to a general class of spaces in [13].

We observe that for L_p, $\omega^r(f,t)_p$ of (2.4) is the closest analogue to the classical modulus of smoothness on R^d or T^d. Moreover, many properties are much easier to prove (see [11]), and it is also applicable to $L_p(S^{d-1})$ when $0 < p$ and to $H_p(S^{d-1})$ when $0 < p$, not just when $1 \le p$. (So is $\omega_*^r(f,t)_p$, for which the case $p < 1$ was not investigated.) On the other hand, the measures of smoothness $\widetilde{\omega}^r$, $\widetilde{\omega}_{\mathrm{I}}^r$ etc. are induced by multiplier operators, and while this prohibits $p < 1$ for $L_p(S^{d-1})$, it makes results such as the Jackson-type inequality easier to prove. For $\omega^r(f,t)_p$, $1 \le p \le \infty$ the Jackson-type inequality was proved in [12], but it was more difficult to prove than with $\widetilde{\omega}^{2r}(f,t)_p$.

4 Measures of Smoothness Using the Averages $\mathbf{B_\theta f}$

In the last section we surveyed measures of smoothness involving the averages of $f(y)$ on the rim of a small cap about x given in (3.1) and denoted by $S_\theta f(x)$. In this section we describe the use of $B_\theta f(x)$ given by

$$B_\theta f(x) = \frac{1}{m_\theta} \int_{x\cdot y \ge \cos\theta} f(y)d\sigma, \quad B_\theta 1 = 1 \tag{4.1}$$

which is an average of $f(y)$ on a small cap about x. The measure of smoothness is given by

$$\widetilde{\widetilde{\omega}}^{2r}(f,t)_p = \sup_{|\theta|\le t} \|(B_\theta - I)^r f\|_{L_p(S^{d-1})} \tag{4.2}$$

where B_θ is given in (4.1). Somewhat simpler (for $r > 1$) is the use of combinations of $B_{j\theta} f$ rather than of $B_\theta^j f$ (see [16]) given by

$$\widetilde{\widetilde{\omega}}_I^{2r}(f,t)_p = \sup_{|h|\le t} \left\| \binom{2r}{r} f + 2\sum_{j=1}^{r} (-1)^j \binom{2r}{r-j} B_{jh} f \right\|_{L_p(S^{d-1})}. \tag{4.3}$$

In [16] a strong converse inequality of type B (in the sense of [15]) was proved for (4.2) and (4.3), that is, the $\sup_{|h|\le t}$ in both was replaced by two terms using only B_t^j and B_{Nt}^j for some fixed N in (4.2) and B_{tj} and B_{Ntj} for some fixed N in (4.3). This was superceded by [9] where it was proved that

$$\|(B_t - I)^r f\|_{L_p(S^{d-1})} \approx \widetilde{\widetilde{\omega}}^{2r}(f,t)_p \tag{4.4}$$

and

$$\Big\| \binom{2r}{r} f + 2\sum_{j=1}^{r}(-1)^j \binom{2r}{r-j} B_{jt} f \Big\|_{L_p(S^{d-1})} \approx \widetilde{\widetilde{\omega}}_I^{2r}(f,t)_p \quad (4.5)$$

which constitute a strong converse inequality of type A (in the sense of [15]).

In fact, both the concepts in the last and the present section can be defined on a class of Banach spaces of functions on S^{d-1} (see [13]). For $L_p(S^{d-1})$, $0 < p < 1$ the moduli in Sections 3 and 4 were not and cannot be defined.

5 K-Functionals

Another common way of describing smoothness is using Peetre K-functionals . A K-functional commonly used for investigation of smoothness of functions on the sphere is

$$\widetilde{K}_{2r}(f,t^{2r})_B = \inf\left(\|f-g\|_B + t^{2r}\|(-\widetilde{\Delta})^r g\|_B\right), \quad r > 0 \quad (5.1)$$

where $\widetilde{\Delta}$ is the Laplace–Beltrami operator (the tangential component of the Laplacian Δ) given by

$$\widetilde{\Delta} f(x) = \Delta f\Big(\frac{x}{|x|}\Big), \quad x \in S^{d-1} \quad (5.2)$$

and for integer r $(-\widetilde{\Delta})^r$ is its $r-$th iterate. For non-integer r $(-\widetilde{\Delta})^r g$ is defined later in (6.4). Another possibility is

$$K_r(f,t^r)_B = \inf\left(\|f-g\|_B + t^r \Big\| \max_{\xi\perp x} \Big|\frac{\partial^r g}{\partial \xi^r}(x)\Big| \Big\|_B\right), \quad r = 1,2,\dots. \quad (5.3)$$

We note that $\max\limits_{\xi\perp x} |\frac{\partial^r g}{\partial \xi^r}(x)|$ is a generalization of the tangential gradient.

For $B = L_p(S^{d-1})$ it was proved that for $1 \le p \le \infty$ (see [20], [22] and [26])

$$\begin{aligned}\widetilde{K}_{2r}(f,t^{2r})_p &\approx \widetilde{\omega}^{2r}(f,t)_p \approx \widetilde{\omega}_{\mathrm{I}}^{2r}(f,t)_p \\ &\approx \widetilde{\omega}_{\mathrm{II}}^{2r}(f,t)_p \approx \widetilde{\omega}_{\mathrm{III}}^{2r}(f,t)_p.\end{aligned} \quad (5.4)$$

In fact it was shown in [2] that

$$\widetilde{K}_{2r}(f,t^{2r})_p \approx \|(S_t - I)^r f\|_p \tag{5.5}$$

and in [7] that

$$\widetilde{K}_{2r}(f,t^{2r})_p \approx \left\| \binom{2r}{r} f + 2\sum_{j=1}^{r}(-1)^j \binom{2r}{r-j} S_{jt}f \right\|_{L_p(S^{d-1})}. \tag{5.6}$$

See also [13] for (5.5) and (5.6) proved for some other Banach spaces. It was shown in [12] that for $1 \le p \le \infty$

$$\omega^r(f,t)_p \approx K_r(f,t^r)_p. \tag{5.7}$$

Clearly,

$$K_{2r}(f,t^{2r})_p \approx \widetilde{K}_{2r}(f,t^{2r})_p \quad \text{for} \quad 1 < p < \infty; \tag{5.8}$$

however, for $p = 1$ and $p = \infty$ (5.8) does not hold. The situation on the sphere, that is, (5.1), and (5.3) – (5.8) are a complete match to corresponding theorems on R^d and T^d. For non-integer r the K-functional $\widetilde{K}_{2r}(f,t^{2r})_B$ was defined and discussed in [10] using the fractional power of $(-\widetilde{\Delta})$, that is $(-\widetilde{\Delta})^r$, which will be defined in the next section.

6 Best Harmonic Polynomial Approximation

The eigenspaces H_k of the Laplace-Beltrami operator $\widetilde{\Delta}$ given in (5.2) on S^{d-1} can be described by

$$\widetilde{\Delta}\varphi = -k(k+d-2)\varphi \quad \text{for} \quad \varphi \in H_k. \tag{6.1}$$

H_k are the harmonic polynomials of degree k [1, Definition 9.1.1]. As the harmonic polynomials are dense in $L_p(S^{d-1})$ for $0 < p < \infty$, $C(S^{d-1})$ and many other spaces and as H_k are finite-dimensional, we may consider the best approximation to f by span $\bigcup_{k<n} H_k$ given by

$$E_n(f)_X = \inf\Big(\|f-\varphi\|_X : \varphi \in \text{ span } \bigcup_{k<n} H_k\Big), \quad n = 1,2,\ldots . \tag{6.2}$$

It clearly follows that this entity measures smoothness as well. $E_n(f)_X$ in general and $E_n(f)_p$ in particular are non-linear entities and hence $E_n(f)_p$ is not and cannot be equivalent to the K-functionals $K_r(f,t^r)_p$, $\widetilde{K}_r(f,t^{2r})_p$ or one of the various concepts $\widetilde{\omega}^{2r}(f,t)_p$, $\widetilde{\widetilde{\omega}}^{2r}(f,t)_p$, etc., which, as we will explain later (and following (3.6), (3.7), (4.3) and (5.4)), are equivalent to approximation by linear operators (for $1 \le p \le \infty$).

Moreover, $\widetilde{K}_{2r}(f,t^{2r})_p$, $K_{2r}(f,t^{2r})_p$, $\omega^{2r}(f,t)_p$, $\widetilde{\omega}^{2r}(f,t)_p$, $\widetilde{\widetilde{\omega}}^{2r}(f,t)_p$ etc. are saturated with the order $O(t^{2r})$ (for $1 \le p \le \infty$), and $E_n(f)_p$ does not have a saturation order. ($\omega^{2r}(f,t)_p$ having t^{2r} as its saturation order means that $\omega^{2r}(f,t)_p = O(t^{2r})$ as $t \to 0+$ for a dense class of functions (for $1 \le p < \infty$) and $\omega^{2r}(f,t)_p = o(t^{2r})$ as $t \to 0+$ implies that f is a constant.) The concept $E_n(f)_{L_p(S^{d-1})}$ like $\omega^{2r}(f,t)_p$ is defined for $0 < p < 1$ as well, but while the modulus $\omega^{2r}(f,t)_p$, $0 < p < 1$ has the saturation order $t^{2r+\frac{1}{p}-1}$ (see [11]), $E_n(f)_p$ does not have a saturation order.

The Jackson inequality, that is the relation between $E_n(f)_B$ and the $K-$functionals $\widetilde{K}_{2r}(f,t^{2r})_B$, is given by

$$E_n(f)_B \le C\widetilde{K}_{2r}(f,n^{-2r})_B \tag{6.3}$$

for any Banach space of functions on S^{d-1} for which some order of the Cesàro summability of the expansion by H_k is bounded (see [5]). Such B clearly include $L_p(S^{d-1})$ (see [3]). In [13] a class of such B is given. The Jackson inequality (6.3) was extended to include a non-integer r in [10] where $g = (-\widetilde{\Delta})^r f$ if $g \in B$ with $f \in B$, $f \sim \sum\limits_{k=0}^{\infty} P_k f$ and

$$g \sim \sum_{k=1}^{\infty} \big(k(k+d-2)\big)^r P_k f. \tag{6.4}$$

For $L_p(S^{d-1})$ results like (6.3) were known much earlier (see [23], [29], [22], [26] and [28, Chapter 5]). As

$$\widetilde{K}_{2r}(f,t^{2r})_p \le CK_{2r}(f,t^{2r})_p \quad \text{for} \quad r \quad \text{integer and, } 1 \le p \le \infty, \tag{6.5}$$

$K_{2r}(f,\ n^{-2r})_p$ can replace $\widetilde{K}_{2r}(f, t^{2r})_p$ in (6.3). In fact, we have (see [12]) for $1 \le p \le \infty$

$$E_n(f)_p \le C\omega^r\Big(f, \frac{1}{n}\Big)_p. \tag{6.6}$$

For $0 < p < 1$, I believe that (6.6) is still valid, but this apparently difficult result was not proved.

7 Realizations

Realizations or realization functionals are a family of concepts that were introduced (not in the context of the sphere) by Hristov and Ivanov in [19] in order to give a useful entity equivalent to K- functionals and for convenience in proving relations between K- functionals of different orders. As it turned out (see [14]), this concept is in some cases useful when K-functionals do not yield any information. (For instance, in the case $0 < p < 1$ for which the K-functionals with the differential operator $Q = \frac{d}{dx}$ was proved in [14] to be zero for all elements of $L_p(T)$.)

A function $\varphi_n \in \text{ span } \bigcup\limits_{k<n} H_k$ is best approximant in the function space X if

$$E_n(f)_X = \|f - \varphi_n\|_X, \quad n = 1, 2, \dots . \tag{7.1}$$

For φ_n given in (7.1) realization functionals are given by

$$\widetilde{R}_{2r}(f, n^{-2r})_X \equiv \|f - \varphi_n\|_X + n^{-2r}\|(-\widetilde{\Delta})^r\varphi_n\|_X \tag{7.2}$$

for $r > 0$ and by

$$R_r(f, n^{-r})_X \equiv \|f - \varphi_n\|_X + n^{-r}\Big\|\sup_{\xi\perp x}\Big(\frac{\partial}{\partial\xi}\Big)^r\varphi_n(x)\Big\|_X \tag{7.3}$$

for integer r.

We note that $\widetilde{R}_{2r}(f, n^{-2r})_X$ and $R_r(f, n^{-r})_X$ are meaningful for $X = L_p(S^{d-1})$ $0 < p$ not just when $p \ge 1$. For a Banach space X for which the Cesàro summability with respect to the expansion by projections on H_k is bounded (for example for $L_p(S^{d-1})$ $1 \le p \le \infty$) it was shown in [5] for integer r and in [10] for $r > 0$ that

$$\widetilde{R}_{2r}(f, n^{-2r})_X \approx \widetilde{K}_{2r}(f, n^{-2r})_X. \tag{7.4}$$

For $X = L_p(S^{d-1})$, $1 \le p \le \infty$ it was shown [12] that for integer r

$$R_r(f, n^{-r})_{L_p(S^{d-1})} \approx K_r(f, n^{-r})_{L_p(S^{d-1})} \approx \omega^r(f, n^{-r})_{L_p(S^{d-1})}. \tag{7.5}$$

I conjecture that for $L_p(S^{d-1})$, $0 < p < 1$, we also have

$$R_r(f, n^{-r})_{L_p(S^{d-1})} \approx \omega^r(f, n^{-r})_{L_p(S^{d-1})}. \tag{7.5$'$}$$

For a Banach space X for which (7.4) and (7.5) were given one can have equivalent functionals to $\widetilde{R}_{2r}(f, n^{-2r})_X$ and $R_r(f, n^{-r})_{L_p(S^{d-1})}$ using $V_n f$ instead of φ_n of (7.1) where $V_n f$ is a delayed means or a de la Vallée-Poussin type linear operator (see [5], [10] and [12]). We remind the reader that such linear operators V_n satisfy

(I) $V_n : X \to \text{ span } \bigcup\limits_{k<Ln} H_k$,

(II) $V_n \varphi = \varphi$ for $\varphi \in \bigcup\limits_{k<n} H_k$

and

(III) $\|V_n f\|_X \le M \|f\|_X$.

Perhaps the simplest example of such a linear operator on $f \sim \sum P_k f$ is

$$V_n f = \sum_{k=0}^{\infty} \eta\Big(\frac{k}{n}\Big) P_k f \tag{7.6}$$

where $\eta \in C^\infty(0, \infty)$, $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ for $x \ge 2$ (in which case L of (I) is 2).

Such $V_n f$ (given in (7.6)) is a delayed mean whenever the Cesàro summability of some order ℓ is bounded on X.

8 Strong Converse Inequalities

For $L_p(S^{d-1})$ $1 \le p \le \infty$ we have already dealt with a description of smoothness that is given by strong converse inequalities in (3.6), (3.7), (4.4) and (4.5), but those appeared as modifications and simplifications (the drop of supremum) of other measures of smoothness.

The Cesàro summability of order ℓ for $f \sim \sum P_k(f)$ is given by

$$C_n^\ell(f) = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) \cdots \Big(1 - \frac{k}{n+\ell}\Big) P_k f. \tag{8.1}$$

As mentioned earlier, a key condition is that for a given ℓ and for the space B

$$\|C_n^\ell(f)\|_B \le A\|f\|_B \tag{8.2}$$

with $A = A(\ell, B)$ independent of n. For $B = L_p(S^{d-1})$ (8.2) is satisfied for $\ell > (d-2)\,|\,\frac{1}{2} - \frac{1}{p}\,|$. Under this condition the Riesz means given by

$$R_{\lambda,\alpha,\ell}(f) = \sum_{\lambda(k)<\lambda} \Big(1 - \big(\frac{\lambda(k)}{\lambda}\big)^\alpha\Big)^\ell P_k f, \tag{8.3}$$

where $\alpha > 0$ and $\lambda(k) = k(k+d-2)$, are bounded (see [10]), that is

$$\|R_{\lambda,\alpha,\ell}(f)\|_B \le A_1\|f\|_B. \tag{8.4}$$

It was shown in [10] that under these conditions for some $M > 1$ we have

$$\|R_{\lambda,\alpha,\ell}(f) - f\|_B + \|R_{M\lambda,\alpha,\ell}(f) - f\|_B \approx \widetilde{K}_{2\alpha}(f, t^{2\alpha})_B \tag{8.5}$$

with $\widetilde{K}_{2\alpha}(f, t^{2\alpha})_B$ given by (5.1) with $(-\widetilde{\Delta})^\alpha$ of (6.4). F. Dai [6] showed that for $L_p(S^{d-1})$ and $\ell > \frac{d-2}{2}$ the second term on the left of (8.5) can be dropped, and his proof can be adapted to Banach spaces and integers ℓ satisfying (8.2). F. Dai described in [6] the smoothness given by $\|C_n^\ell(f) - f\|_{L_p(S^{d-1})}$ as equivalent to the K-functional $\widetilde{K}_1(f, 1/n)_p$ (that is with $\alpha = 1/2$).

9 Some Recent Relations

The relation

$$\widetilde{\omega}^{2r}(f,t)_B \le 2^{r-k}\widetilde{\omega}^{2k}(f,t)_B, \quad 1 \le p \le \infty, \quad r > k \tag{9.1}$$

follows whenever $S_\theta f$ is a contraction on the Banach space B. (This was one of the first properties to be proved for $B = L_p$, $1 \le p \le \infty$.) For $\omega^r(f,t)_X$ with $X = B$ a Banach space satisfying (2.7) one has

$$\omega^r(f,t)_B \le 2^{r-k}\omega^k(f,t)_B, \quad r > k \tag{9.2}$$

and when $X = L_p(S^{d-1}), \quad 0 < p < 1$

$$\omega^r(f,t)_p \le 2^{(r-k)/p}\omega^k(f,t)_p, \quad r > k \tag{9.3}$$

(see [11] for (9.2) and (9.3)). The converse of (9.2) and (9.3) was given in the sharp-Marchaud inequality

$$\omega^r(f,t)_p \le Ct^r\Big(\int_t^{1/2} \frac{\omega^{r+1}(f,u)_p^q}{u^{rq+1}}\,du + \|f\|_p^q\Big)^{1/q} \tag{9.4}$$

where $p > 0$ and $q = \begin{cases} \min(p,2), & p < \infty \\ 1 & p = \infty. \end{cases}$ (It is known now that the second term on the right of (9.4) is redundant.) Using $E_n(f)_p$, $1 \le p \le \infty$, we have (see [11])

$$\omega^r\big(f,\frac{1}{n}\big)_p \le Cn^{-r}\Big(\sum_{k=1}^{n} k^{qr-1}E_k(f)_p^q\Big)^{1/q}, \quad q = \begin{cases} \min(p,2), & p < \infty \\ 1, & p = \infty. \end{cases} \tag{9.5}$$

For other moduli such inequalities were known earlier for $1 \le p \le \infty$ and $q = 1$ (for all such p). For K-functionals we have (see [8, Section 5])

$$\widetilde{K}_\alpha(f,t^{2\alpha})_p \le Ct^{2\alpha}\Big\{\int_t^1 \frac{\widetilde{K}_\beta(f,u^{2\beta})_p^q}{u^{2q\alpha+1}}\,du\Big\}^{1/q}, \quad \alpha < \beta \tag{9.6}$$

and

$$\widetilde{K}_\alpha(f,t^{2\alpha})_p \le Ct^{2\alpha}\Big(\sum_{n\le 1/t} n^{2q\alpha-1}E_n(f)_p^q\Big)^{1/q} \tag{9.7}$$

where $1 < p < \infty$, $q = \min(p, 2)$ and where $K_\alpha(f, t^{2\alpha})_p$ is given by (5.1) for α and β with $(-\widetilde{\Delta})^\alpha g$ given by (6.4) and $E_n(f)_p$ given by (7.1). (Note that $E_n(f)$ in (7.1) is only given for $n \geq 1$.)

For a Banach space B on S^{d-1} which satisfies $\|C_n^\ell f\|_B \leq C\|f\|_B$ for some ℓ one has (see [10, Theorems 6.4, 6.5 and Section 9A])

$$\widetilde{K}_\alpha(f, t^{2\alpha})_B \leq Ct^{2\alpha} \sum_{n \leq 1/t} n^{2\alpha - 1} E_n(f)_B \tag{9.8}$$

and

$$\widetilde{K}_\alpha(f, t^{2\alpha})_B \leq Ct^{2\alpha} \int_t^1 \frac{\widetilde{K}_\beta(f, u^{2\beta})_B}{u^{2\alpha+1}}\, du, \quad \alpha < \beta. \tag{9.9}$$

We observe that special cases of the above were known much earlier.

For relations between $\widetilde{K}_r(f, t^{2r})_p$ with different p we have (see [17, Section 10])

$$\widetilde{K}_\alpha(f, t^{2r})_q \leq C\Big(\int_0^t u^{-(d-1)(\frac{1}{p} - \frac{1}{q})q_1} \widetilde{K}_\alpha(f, t^{2\alpha})_p^{q_1} \frac{du}{u}\Big)^{1/q_1} \tag{9.10}$$

where $1 \leq p < q \leq \infty$ and $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty. \end{cases}$

For $E_n(f)_p$ we have (see [17, Section 10])

$$E_n(f)_q \leq C\Big(\sum_{k=n}^{\infty} k^{(d-1)(\frac{1}{p} - \frac{1}{q})q_1 - 1} E_k(f)_p^{q_1}\Big)^{1/q_1} \tag{9.11}$$

for $0 < p < q \leq \infty$ and $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty. \end{cases}$

We note that in [17, Section 10] somewhat more restrictive versions of (9.10) and (9.11) are given, but (9.10) and (9.11) follow from [17, Section 4].

10 Epilogue

I have attempted to show the main directions in describing smoothness on the sphere and relations among them. I am sure that others would like to see different measures emphasized or different properties displayed. Because of lack of space I have emphasized the latest most general results over important special cases proved earlier.

I would like to thank F. Dai for his helpful comments.

References

[1] G. E. Andrews, R. Askey and R. Roy, "Special Functions," Cambridge Univ. Press, 1999.

[2] E. Belinski, F. Dai and Z. Ditzian, Multivariate approximating averages, J. Approx. Theory 125 (2003), 85–105.

[3] A. Bonami and J. L. Clerc, Sommes de Cesàro et multiplicateur de developpement en harmoniques sphériques, Trans. Amer. Math. Soc. 183 (1973), 223–262.

[4] A. P. Calderon, M. Weiss and A. Zygmund, On the existence of singular integrals, Proc. Symp. Pure Math. 10 (1967), 56–73.

[5] W. Chen and Z. Ditzian, Best approximation and K-functionals, Acta Math. Hungar. 75 (1997), 165–208.

[6] F. Dai, Some equivalence theorems with K-functionals, J. Approx. Theory 121 (1) (2003), 143–157.

[7] F. Dai and Z. Ditzian, Combinations of multivariate averages, J. Approx. Theory 131 (2004), 268–283.

[8] F. Dai and Z. Ditzian, Littlewood Paley theory and sharp Marchaud inequality, Acta Math. Szëged 71 (2005), 65–90.

[9] F. Dai, K. Wang and C. Yu, On a conjecture of Ditzian and Runovskii, J. Approx. Theory 118 (2002), 202–224.

[10] Z. Ditzian, Fractional derivatives and best approximation, Acta Math. Hungar. 81 (4) (1998), 323–348.

[11] Z. Ditzian, A modulus of smoothness on the unit sphere, J. d'Analyse Math. 79 (1999), 189–200.

[12] Z. Ditzian, Jackson-type inequality on the sphere, Acta Math. Hungar. 102 (1-2) (2004), 1–35.

[13] Z. Ditzian, Approximation on Banach spaces of functions on the sphere, J. Approx. Theory, (to appear).

[14] Z. Ditzian, V. H. Hristov and K. G. Ivanov, Moduli of smoothness and K-functionals in L_p, $0 < p < 1$, Constr. Approximation 11 (1995), 67–83.

[15] Z. Ditzian and K. Ivanov, Strong converse inequalities, Jour. d'Analyse Math. 61(1993), 61–111.

[16] Z. Ditzian and K. Runovskii, Averages on caps of S^{d-1}, J. Math. Anal. and Appl. 248 (2000), 260–274.

[17] Z. Ditzian and S. Tikhonov, Ul'yanov and Nikolskii-type inequalities, J. Approx. Theory 123 (2005), 100–133.

[18] M. V. Fedorov, Approximation of functions on a sphere, Moscow Univ. Math. Bull. 45 (1) (1990), 17–23.

[19] V. H. Hristov and K. G. Ivanov, Realization of K-functionals on subsets and constrained approximation, Math. Balknica (New Series) 4 (3) (1990), 236–259.

[20] P. I. Lizorkin and S. M. Nikolski, Approximation by spherical functions, Proc. Steklov Inst. 173 (1987), 195–203.

[21] P. I. Lizorkin and Kh. P. Rustamov, Nikolskii-Besov spaces on the sphere in connection with approximation theory, Trudi Mat. Inst. Steklov 204 (1993), (Russian), Proc. Steklov Inst. 204 (3) (1994), 149–172 (English translatation).

[22] M. Nikolskii and P. I. Lizorkin, Function spaces on the sphere that are connected with approximation theory, Mat. Zametki 41 (1987), 509–515. English translation Math. Notes 41 (1987), 286–291.

[23] S. Pawelke, Über die Approximationsordnung bei Kugel funktionen und algebräischen Polynomen, Tôhoku Math. Jour. 24 (1972), 473–486.

[24] D. L. Ragozin, Constructive polynomial approximation on spheres and projective spaces, Trans. Amer. Math. Soc. 162 (1971), 157–170.

[25] Kh. P. Rustamov, On the equivalence of some moduli of smoothness on the sphere, Soviet Math. Dokl. 44 (3) (1992), 660–664.

[26] Kh. P. Rustamov, On equivalence of difference of different moduli of smoothness on the sphere, Trudy Math. Inst. Steklov 204 (1993), (Russian). Proc. Steklov Inst. Math. 3 (1994), 235–260 (English translation).

[27] A. P. Teherin, Approximation in L_p by spherical polynomials and differentiability classes of functions on the sphere, Proc. Steklov Inst. Math. 187 (1990), 249–261.

[28] K. Wang and L. Li, "Harmonic Analysis and Approximation on the Unit Sphere," Science Press Beijing, 2000.

[29] M. Wehrens, Best approximation on the unit sphere in R^k in "Functional Analysis and Approximation," pp. 233–245, Proceedings of Conference of Oberwolfach, Birkhauser, 1981.

Quadrature Formulae of Maximal Trigonometric Degree of Precision

Dimiter Dryanov
Department of Mathematics and Statistics
Concordia University
Montréal, Quebec, Canada H3G 1M8
Email address: ddryanov@alcor.concordia.ca

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

We present results on existence, uniqueness, and explicit construction of quadrature formulae with maximal trigonometric degree of precision. In Section 1 we discuss Turán type quadrature formulae of maximal trigonometric degree of precision. Birkhoff type quadratures of maximal trigonometric degree of precision are considered in Section 2.

1 Quadrature Formulae with Free Nodes of Maximal Trigonometric Degree of Precision

We discuss the problem of existence, uniqueness, and explicit construction of formulas for approximate (numerical) integration of periodic functions based on fixed number of free nodes, fixed multiplicities at each node, and having maximal trigonometric degree of precision. Formulas for approximate integration are usually called quadrature formulae (quadratures). Why such *extremal quadrature formulae* are useful in approximate integration? According to Weierstrass-Jackson approximation result if f is 2π-periodic continuous function then for each n natural, there exists trigonometric polynomial t of degree at most n such that $\max_{x\in R} |f(x)-t(x)| = O\left(\omega(f,1/n)\right)$, where

$\omega(f, 1/n) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| \le 1/n\}$ is the modulus of continuity of f with parameter $1/n$. As n increases, the approximation will improve and the best approximation will tend to zero, when n tends to infinity. Having a QF (quadrature formula) that is exact for trigonometric polynomial of degree as higher as possible; i.e., $\int_0^{2\pi} t(x)dx = QF(t)$ holds for trigonometric polynomials of maximum degree, then the approximation to $\int_0^{2\pi} f(x)dx$ by $QF(f)$ will be in some sense the best possible. This approach to numerical integration was initiated by Gauss [6] who studied quadrature formulae of maximal algebraic degree of precision based on functional values (the case of multiplicity 1 at each node). Turán [12] extended the approach of Gauss by considering quadrature formulae of maximal degree of precision based on node-multiplicities greater than one (case of equal multiplicities at each node). Turán's extension of Gaussian quadratures attracted considerable interest and still remain an area of active research. Existence, uniqueness, and explicit construction of Gauss-Turán quadratures lead to non-linear extremal problems which require non-standard methods from analysis, functional analysis, optimization theory etc.

We study quadrature formulae of the form

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}\, f^{(\lambda)}(x_j), \tag{1.1}$$

where f is 2π-periodic function, $x_0 \le x_1 \le \cdots \le x_l$, $(x_l - x_0 \le 2\pi)$ are $l+1$ real free nodes. The number $l+1$ of the nodes is fixed and $k_0, k_1, \ldots, k_l$ are fixed non-negative integers to determine the multiplicities $2k_j + 1$ of the nodes x_j for $j = 0, 1, \ldots, l$.

Let $T_{(l-1)/2}$ denote the linear space of all trigonometric polynomials of integer or half-integer degree $\le (l-1)/2$ $(l = 1, 2, \ldots)$.

Definition. A quadrature formula (QF) of a type (1.1) has trigonometric degree of precision (TDP) equal to n if it is exact for each $f \in T_n$ and there exists $g \in T_{n+1}$ for which the QF is not exact.

Problem formulation. Find a QF of type (1.1) with fixed number of nodes and fixed multiplicities at each node, such that the trigonometric degree of precision of this formula is maximum. This maximum is called *maximal trigonometric degree of precision.*

Solution of this problem is given by the following theorem [4].

Theorem 1.1 *There exists a quadrature formula of type (1.1) with trigonometric degree of precision*

$$p(k_0, \ldots, k_l) := l + \sum_{j=0}^{l} k_j$$

and this is the maximal trigonometric degree of precision. The quadrature formula of maximal trigonometric degree of precision is uniquely determined within a translation of the nodes with a real parameter.

Remark 1.1 There are $2l + 2 + 2\sum_{j=0}^{l} k_j$ free parameters in (1.1); the dimension of $T_{p(k_0,\ldots,k_l)}$ is $2l + 1 + 2\sum_{j=0}^{l} k_j$ so, the algebraic expectation is that the nodes and the coefficients of all quadratures type (1.1) of maximal trigonometric degree of precision is a one-parametric set. This is confirmed by Theorem 1.1. Moreover, if we fix one of the nodes $x_j (j = 0, \ldots, l)$ then the quadrature formula with maximal trigonometric degree of precision is unique,i.e., each of the nodes could serve as a parameter to obtain one-parametric representation of the quadrature data (nodes and coefficients).
Remark 1.2 The trigonometric polynomial

$$t(x) := \prod_{j=0}^{l} \left[\sin \frac{x - x_j}{2} \right]^{2k_j+2} \in T_{p(k_0,\ldots,k_l)+1}$$

shows that the QF obtained from (1.1) by adding the next $(2k_j+1)$-th (odd) derivative at the node x_j

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j+1} a_{j,\lambda}\, f^{(\lambda)}(x_j)$$

has degree of precision at most $p(k_0, \ldots, k_l)$. Hence, the degree of precision of QF type (1.1) will not increase by adding the next $(2k_j+1)$-th derivative of f at some of the nodes x_j $(j = 0, 1, \ldots, l)$ to the data of a QF type (1.1).

Remark 1.3 There are many QF of the form

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \sum_{\lambda=0}^{2k+1} a_\lambda \sum_{j=0}^{l} f^{(\lambda)}(-\pi + 2j\pi/(l+1)),$$

with trigonometric degree of precision equal to $p(k,\dots,k) = l + (l+1)k$. If $a_{2k+1} = 0$ then the other coefficients a_λ, $\lambda = 0,\dots,2k$ are uniquely determined and the corresponding QF is unique. This example helps to understand the limit of the theory presented here. In the case l odd, it is a corollary from [2, Theorem 1]. For l even the proof is similar.

Some basic facts on trigonometric polynomials of half-integer degree. Let us start with the well-known fact that each algebraic polynomial of degree n has exactly n zeros in the complex plane counting their multiplicities. The example $\cos(nz) + \mathbf{i}\sin(nz)$ ($\mathbf{i}^2 = -1$) indicates that a trigonometric polynomial of arbitrary degree can be free of zeros (no zeros) in the complex plane. Trigonometric polynomial with complex coefficients $t_{(l+1)/2}(z)$ of integer or half-integer degree $(l+1)/2$

$$\begin{aligned} t_{(l+1)/2}(z) &:= \sum_{k=0}^{l+1} c_{(l+1-2k)/2} \exp\left[\mathbf{i}\frac{l+1-2k}{2}z\right] \\ &= c_{(l+1)/2} \exp\left[\mathbf{i}\frac{l+1}{2}z\right] + c_{-(l+1)/2} \exp\left[-\mathbf{i}\frac{l+1}{2}z\right] + \cdots \end{aligned}$$

has in each vertical stripe $\beta \le Re(z) < \beta + 2\pi$ exactly as many zeros as the non-zero roots of the algebraic polynomial equation

$$p_{l+1}(\zeta) := \sum_{k=0}^{l+1} c_{(l+1-2k)/2}\, \zeta^{l+1-k} = 0,\ \zeta = \exp(\mathbf{i}z).$$

Denote the linear space of trigonometric polynomials of degree $(l+1)/2$ by $T_{(l+1)/2}$. Then $T_{(l+1)/2}$ has dimension $l+2$ and each member of $T_{(l+1)/2}$ has at most $l+1$ zeros in $\beta \le Re(z) < \beta + 2\pi$ or it is identically zero. The trigonometric polynomial

$$t_{(l+1)/2}(z) := \sum_{k=0}^{[(l+1)/2]} \Big(a_{(l+1-2k)/2} \cos\frac{l+1-2k}{2} z$$

$$+b_{(l+1-2k)/2} \sin \frac{l+1-2k}{2} z \Big)$$

has exactly $l+1$ zeros in $\beta \leq Re(z) < \beta + 2\pi$ if $a^2_{(l+1-2k)/2} + b^2_{(l+1-2k)/2} \neq 0$, counting their multiplicities. If

$$a^2_{(l+1-2k)/2} + b^2_{(l+1-2k)/2} = 0,$$

then $t_{(l+1)/2}$ has less than $l+1$ zeros in $\beta \leq Re(z) < \beta + 2\pi$. The above properties can be proved by using the representation $p_{l+1}(\zeta) = \exp(\mathbf{i}[(l+1)/2]z)\, t_{(l+1)/2}(z),\ \zeta = \exp(\mathbf{i}z),$ where $\exp(\mathbf{i}z)$ maps the strip $\beta \leq Re(z) < \beta + 2\pi$ one to one onto the ζ-complex plane without the origin ($\zeta \in C \backslash \{0\}$).

The proof of Theorem 1.1 uses two auxiliary results (Theorem 1.2 and Theorem 1.3), each of them of independent value. The proof of the first one, given by the next theorem, is based on properties of topological degree of a map with respect to an open bounded set and a given point.

Some basic facts on topological degree theory [8], [9]. Let D be an open bounded set in R^n with a closure $\bar{D}$ and boundary ∂D. Let the map $\mathbf{\Phi} : \bar{D} \mapsto R^n$ be continuous. For $\mathbf{c} \in R^n$ and $\mathbf{c} \notin \mathbf{\Phi}(\partial D)$, the topological degree of $\mathbf{\Phi}$ with respect to the open set D and the point $\mathbf{c}$ is denoted by $deg\,(\mathbf{\Phi}, D, \mathbf{c})$ and satisfies the following properties.

Lemma A Let D be a bounded open subset of R^n and $\mathbf{\Phi}$ be a continuous map from $\mathbf{t} \in D$ to R^n. If $\mathbf{c} \notin \mathbf{\Phi}(\partial D)$ and $deg\,(\mathbf{\Phi}, D, \mathbf{c}) \neq 0$ then the vector equation $\mathbf{\Phi}(\mathbf{t}) = \mathbf{0}$ has a solution in D.

Lemma B Let $\mathbf{\Phi}(\mathbf{t}, \alpha)$ be continuous map defined on $\bar{D} \times [0,1]$ with $\mathbf{\Phi}(\mathbf{t}, \alpha) \neq \mathbf{c}$ for any $\mathbf{t} \in \partial D$ and $0 \leq \alpha \leq 1$. Then the topological degree $deg\,(\mathbf{\Phi}, D, \mathbf{c})$ does not depend on α (it is a constant independent of α).

Lemma C Suppose $\mathbf{\Phi} \in C^1(D)$, $\mathbf{c} \notin \mathbf{\Phi}(\partial D)$ and the Jacobian $det\,(\mathbf{\Phi}'(\mathbf{x})) \neq 0$ for any $\mathbf{x} \in D$ such that $\mathbf{\Phi}(\mathbf{x}) = \mathbf{c}$ (in other words for any solution $\mathbf{t} = \mathbf{x}$ of *the system* $\mathbf{\Phi}(\mathbf{t}) = \mathbf{c}$). Then there exist only a finite set of points $\{\mathbf{x}_s\}$ in D for which $\mathbf{\Phi}(\mathbf{x}_s) = \mathbf{c}$ (in other words only finite number of solutions of the system $\mathbf{\Phi}(\mathbf{t}) = \mathbf{c}$) and

$$deg\left(\mathbf{\Phi}(.,\alpha), D_{\varepsilon/2}, \mathbf{0}\right) = \sum_{\{x_s\}} sign\left[det\left(\mathbf{\Phi}'(x_s)\right)\right] .$$

Let $\alpha \in [0,1]$, l positive integer, and $k_0, k_1, \ldots, k_l$ non-negative integers be given. Let us denote

$$\Phi\left(\mathbf{t}, \alpha, r_{\frac{l-1}{2}}\right) = \int_{-\pi}^{\pi} \left|\sin\frac{x+\pi}{2}\right|^{2\alpha k_0+1} \prod_{j=1}^{l} \left|\sin\frac{x-t_j}{2}\right|^{2\alpha k_j+1}$$
$$\times\, sign\left(\sin\frac{x+\pi}{2}\prod_{j=1}^{l}\sin\frac{x-t_j}{2}\right) \times r_{\frac{l-1}{2}}(x)\, dx\,,$$

where $\mathbf{t} := (t_1, t_2, \ldots, t_l)$ belongs to the R^l simplex

$$\bar{D} := \{\mathbf{t} : -\pi \le t_1 \le t_2 \le \cdots \le t_l \le \pi\}$$

and $r_{\frac{l-1}{2}} \in T_{(l-1)/2}$ is a trigonometric polynomial of degree (integer or half-integer) $(l-1)/2$.

Theorem 1.2 *The problem to find all* $\mathbf{t} \in \bar{D}$ *such that*

$$\Phi\left(\mathbf{t}, \alpha, r_{\frac{l-1}{2}}\right) = 0 \tag{1.2}$$

for each trigonometric polynomial $r_{\frac{l-1}{2}} \in T_{(l-1)/2}$ *has a unique solution* $\mathbf{x}_\alpha := (x_1^\alpha, x_2^\alpha, \ldots, x_l^\alpha)$ *satisfying* $-\pi < x_1^\alpha < x_2^\alpha < \cdots < x_l^\alpha < \pi$.

The proof of Theorem 1.2 is based on the following lemmas. Details can be found in [4].

Lemma 1.1 *Let* α *be fixed. Then each solution* $\mathbf{x} := (x_1, x_2, \ldots, x_l)$ *of the problem (1.2) belongs to the interior*

$$D := \{\mathbf{t} : -\pi < t_1 < t_2 < \cdots < t_l < \pi\}\ .$$

of $\bar{D}$.

Lemma 1.2 *There exists $\varepsilon > 0$ such that for every $\alpha \in [0,1]$ all solutions of (1.2) belong to the simplex*

$$\bar{D}_\varepsilon := \{\varepsilon \leq t_1 + \pi,\, \varepsilon \leq t_2 - t_1, \ldots, \varepsilon \leq t_l - t_{l-1},\, \varepsilon \leq \pi - t_l\},$$

i.e., all solutions of (1.2) are ε-inside the simplex $\bar{D}$, uniformly with respect to α.

Let us denote

$$r_q(t) := \prod_{j=1, j\neq q}^{l} \sin \frac{t - t_j}{2}$$

for $q = 1, 2, \ldots, l$. Evidently, $r_q \in T_{(l-1)/2}$. The next lemma is based on simple interpolation formula in $T_{(l-1)/2}$ with interpolation nodes, the distinct $t_1 < \cdots < t_l$. Let

$$D_{\varepsilon/2} := \{\varepsilon/2 < t_1 + \pi,\, \varepsilon/2 < t_2 - t_1, \ldots, \varepsilon/2 < t_l - t_{l-1},\, \varepsilon/2 < \pi - t_l\}.$$

Lemma 1.3 *In the simplex $D_{\varepsilon/2}$, the problems*

$$a) \qquad \Phi\left(\mathbf{t}, \alpha, r_{\frac{l-1}{2}}\right) = 0 \quad \left(r_{\frac{l-1}{2}} \in T_{\frac{l-1}{2}}\right)$$

and

$$b) \qquad \Phi\left(\mathbf{t}, \alpha, r_q\right) = 0 \quad (q = 1, 2, \ldots, l)$$

are equivalent; i.e., their solutions coincide.

Remark 1.4 Note that the two problems are not equivalent in $\bar{D}$; i.e., their solutions do not coincide. For example $t_1 = t_2 = \cdots = t_l = 0$ belongs to $\bar{D}$ and it is a solution of the problem b), Lemma 1.3 but is not a solution of the problem a), Lemma 1.3 as Lemma 1.2 shows.

The next Lemma describes properties of each solution of the non-linear system b), Lemma 1.3 in terms of its Jacobian.

Lemma 1.4 *Let $\Phi_q(\mathbf{t}, \alpha) := \Phi(\mathbf{t}, \alpha, r_q)$. Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ is a solution of the problem*

$$\Phi_q(\mathbf{t}, \alpha) = 0 \quad (q = 1, 2, \ldots, l)$$

in the simplex $D_{\varepsilon/2}$. Then

$$i. \quad \frac{\partial \Phi_q}{\partial t_m}\Big|_{\mathbf{t}=\mathbf{x}} = 0 \quad (q \neq m, \;\; q, m = 1, 2, \ldots, l\,;$$

$$ii. \quad sign\left(\frac{\partial \Phi_q}{\partial t_q}|_{\mathbf{t}=\mathbf{x}}\right) \;=\; -1 \quad (q=1,2,\ldots,l\,;$$

$$iii. \quad sign\left(det\left(\frac{\partial \Phi_q}{\partial t_m}|_{\mathbf{t}=\mathbf{x}}\right)_{q=1,m=1}^{l,l}\right) \;=\; (-1)^l\,.$$

The idea of the proof of Theorem 1.2. We shall make a sort of *extrapolation* based on the next lemma and *a topological degree of a map with respect to an open bounded set and a given point.* By the above lemmas it is clear that instead of studding the problem of Theorem 1.2 in $\bar{D}$, we shall study its equivalent problem b), Lemma 1.3 in $D_{\varepsilon/2}$. Note that the two problems are not equivalent in $\bar{D}$, but they are equivalent in $D_{\varepsilon/2}$. On the other hand Lemma 1.2 states that all solutions in $\bar{D}$ of the problem (1.2) formulated in Theorem 1.2 must be uniformly (with respect to $\alpha \in [0,1]$) in $\bar{D}_\varepsilon \subset D_{\varepsilon/2}$. In other words all solutions of b), Lemma 1.3 in $D_{\varepsilon/2}$ are in fact all solutions of the problem (1.2) in $\bar{D}$.
Lemma 1.5 *The problem*

$$\Phi_q\,(\mathbf{t},0) \;=\; 0 \quad (q=1,2,\ldots,l)$$

has a unique solution

$$\mathbf{x}=(x_1,x_2,\ldots,x_l),\;\; x_j=\pi\left(\frac{2j}{l+1}-1\right)(j=1,2,\ldots,l)$$

in the simplex $D_{\varepsilon/2}$. According to the Lemma 1.3 it must be in $\bar{D}_\varepsilon$, also.
Proof. We present **the proof of Theorem 1.2 (Topological Degree Approach).** Define a map

$$\mathbf{\Phi}(\mathbf{t},\alpha)\;:\;D_{\varepsilon/2}\times[0,1]\mapsto R^l, \quad \mathbf{t}=(t_1,t_2,\ldots,t_l)\,,$$

where

$$\mathbf{\Phi}(\mathbf{t},\alpha) := (\Phi_1(\mathbf{t},\alpha),\Phi_2(\mathbf{t},\alpha),\ldots,\Phi_l(\mathbf{t},\alpha))\,,$$

$\mathbf{t}=(t_1,t_2,\ldots,t_l)\in D_{\varepsilon/2}$, $\alpha\in[0,1]$, and $\Phi_q(\mathbf{t},\alpha)=\Phi(\mathbf{t},\alpha,r_q)$. We study the *topological degree of the map* $\mathbf{\Phi}(\mathbf{t},\alpha)$ *with respect to the open bounded set* $D_{\varepsilon/2}$ *and the point* $\mathbf{0}:=(0,0,\ldots,0)$ *(the zero vector in l-dimensional space).*

By Lemma 1.2 it follows that all solutions of (1.2) in $\bar{D}$ must belong to $\bar{D}_\varepsilon$. Hence, all solutions of (1.2) are in $D_{\varepsilon/2}$. Now, by Lemma 1.3, finding all solutions of (1.2) in $\bar{D}$ is equivalent to obtaining all solutions of

$$\mathbf{\Phi}(\mathbf{t},\alpha)=\mathbf{0} \quad (\mathbf{t}\in D_{\varepsilon/2}),$$

where α is a fixed number in $[0,1]$. In other words to find all points $\mathbf{t}\in D_{\varepsilon/2}$ which are mapped to $\mathbf{0}$ by $\mathbf{\Phi}(\mathbf{t},\alpha)$. All solutions must be in $\bar{D}_\varepsilon$, uniformly with respect to α. Hence,

$$\mathbf{\Phi}(\partial D_{\varepsilon/2},\alpha)\neq\mathbf{0}$$

for each α in $[0,1]$. In other words, the solutions are uniformly with respect to α far from the boundary of $D_{\varepsilon/2}$. By Lemma B it follows that *the topological degree*

$$deg\left(\mathbf{\Phi}(.,\alpha),D_{\varepsilon/2},\mathbf{0}\right)$$

is a constant not depending on parameter α. According to Lemma 1.4, Lemma A, and Lemma C, for each fixed $\alpha\in[0,1]$, there is a finite number of solutions (points in R^l) $\mathbf{t}=\mathbf{x}_s:=(x_{1,s},x_{2,s},\ldots,x_{l,s})$ in $D_{\varepsilon/2}$ to $\mathbf{\Phi}(\mathbf{t},\alpha)=\mathbf{0}$, i.e., a finite number of solutions to the problem (1.2) in $\bar{D}$. By using the finite solution set $\{\mathbf{x}_s\}$ for a fixed α and by Lemma C, we have the following formula for *the topological degree of the map* $\mathbf{\Phi}(\mathbf{t},\alpha)$ *with respect to the open bounded set* $D_{\varepsilon/2}$ *and the point* $\mathbf{0}:=(0,0,\ldots,0)$*:*

$$deg\left(\mathbf{\Phi}(.,\alpha),D_{\varepsilon/2},\mathbf{0}\right)=\sum_{\{\mathbf{x}_s\}} sign\left[det\left(\mathbf{\Phi}'(\mathbf{x}_s)\right)\right].$$

By Lemma 1.4 and Lemma 1.5 we conclude that for $(\alpha=0)$, *the topological degree* $deg\left(\mathbf{\Phi}(.,0),D_{\varepsilon/2},\mathbf{0}\right)$ *of the map* $\mathbf{\Phi}(\mathbf{t},0)$ *with respect to the open bounded set* $D_{\varepsilon/2}$ *and the point* $\mathbf{0}:=(0,0,\ldots,0)$ is $(-1)^l$; i.e.,

$$deg\left(\mathbf{\Phi}(.,0),D_{\varepsilon/2},\mathbf{0}\right)=(-1)^l$$

and this is *the initial step of the extrapolation procedure to the existence and the uniqueness of the solution* from $\alpha=0$ to $\alpha=1$. Taking into account that the topological degree does not depend on α we have

$$deg\left(\mathbf{\Phi}(.,\alpha), D_{\varepsilon/2}, \mathbf{0}\right) = \sum_{\{\mathbf{x}_s\}} sign\left[det\left(\mathbf{\Phi}'(x_s)\right)\right] = (-1)^l,$$

where the solution set $\{\mathbf{x}_s\}$ for a fixed α is a finite solution set, according to Lemma C. By Lemma 1.4 there is one and only one solution $\mathbf{x}_s$ and from here, for a fixed $\alpha \in [0,1]$, the problem b), Lemma 1.3 has a unique solution in $D_{\varepsilon/2}$. Hence, the problem (1.2) has a unique solution in $\bar{D}$. The proof of Theorem 1.2 is completed. □

In order to prove our main result we shall need two more auxiliary results. The first one is on **Hermite Interpolation by Trigonometric Polynomials.** Details are given in [4]. Let $x_0 < x_1 < \cdots < x_l$ be $l+1$ real interpolation nodes, $x_l - x_0 < 2\pi$ and $\lambda_0, \lambda_1, \ldots, \lambda_l$ be positive integer numbers. We consider Hermite interpolation problem $(\mathbf{x}, \lambda)$ with interpolation nodes $\mathbf{x} := (x_0, x_1, \ldots, x_l)$ and corresponding multiplicities $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_l)$. Let $s(u) := u/2$ and $\tilde{\lambda} = \lambda_0 + \lambda_1 + \ldots + \lambda_l$. In terms of these notation

$$s(\tilde{\lambda}) = \frac{1}{2}\sum_{j=0}^{l} \lambda_j \qquad \text{and} \qquad s(\lambda \cdot \mathbf{x}) = \frac{1}{2}\sum_{j=0}^{l} \lambda_j\, x_j$$

and the result on Hermite trigonometric interpolation states the following:

Theorem 1.3 *(a) If $2s(\tilde{\lambda})$ is odd, then the interpolation problem $(\mathbf{x}, \lambda)$ has a unique solution in the linear space $T_{s(\tilde{\lambda})-1/2}$.*

(b) If $2s(\tilde{\lambda})$ is even and $s(\lambda \cdot \mathbf{x}) = 0$ then the interpolation problem $(\mathbf{x}, \lambda)$ has a unique solution in the linear space $\left\{T_{s(\tilde{\lambda})-1}, \sin[s(\tilde{\lambda})x]\right\}$.

(c) If $2s(\tilde{\lambda})$ is even then the interpolation problem $(\mathbf{x}, \lambda)$ has a unique solution in the linear space $\left\{T_{s(\tilde{\lambda})-1}, \sin[s(\tilde{\lambda})x - s(\lambda \cdot \mathbf{x})]\right\}$.

Remark 1.5 Obviously, the definite integral over a segment with length 2π of the trigonometric interpolation solution in Theorem 1.3 (a), (c) is invariant with respect to an arbitrary real translation of the interpolating nodes (each interpolation node shifted by the same real number).

Now we shall formulate an auxiliary result on *long division* by trigonometric polynomials. With $\lambda = (\lambda_0, \ldots, \lambda_l)$ and $\mathbf{x} = (x_0, \ldots, x_l)$ we define the trigonometric polynomial

$$\omega_{\lambda,\mathbf{x}}(x) := \prod_{j=0}^{l} \left(\sin \frac{x - x_j}{2} \right)^{\lambda_j} \in T_{s(\tilde{\lambda})},$$

of degree $s(\tilde{\lambda})$ subject to the condition $s(\lambda \cdot \mathbf{x}) = 0$.

Lemma 1.6 *Let $t_n(x) \in T_n$ and $n \geq s(\tilde{\lambda})$. Then*

$$t_n(x) = \omega_{\lambda,\mathbf{x}}(x)\, p(x) \, + \, r(x), p(x) \in T_{n-s(\tilde{\lambda})}.$$

In addition $r(x) = \beta \sin(s(\tilde{\lambda})x) + q(x)$, $q(x) \in T_{s(\tilde{\lambda})-1}$ if $2s(\tilde{\lambda})$ is even; and $r(x) \in T_{s(\tilde{\lambda})-1/2}$ if $2s(\tilde{\lambda})$ is odd.

Proof. We present the **proof of Theorem 1.1.** Evidently, the trigonometric degree of precision of (1.1) does not change if we translate the nodes with a real parameter. On the other hand the example

$$t(x) := \prod_{j=0}^{l} \left[\sin \frac{x - x_j}{2} \right]^{2k_j+2} \in T_{p(k_0,\ldots,k_l)+1}$$

shows that (1.1) has TDP less than or equal to $p(k_0, \ldots, k_l)$.

A. Uniqueness of a quadrature formula of type (1.1) having TDP equal to $p(k_0, \ldots, k_l)$. Assume to the contrary: There are at least two QF of type (1.1)

$$\int_{-\pi}^{\pi} f(x)\, dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a'_{j,\lambda}\, f^{(\lambda)}(x'_j); \tag{1.3}$$

$$\int_{-\pi}^{\pi} f(x)\, dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a''_{j,\lambda}\, f^{(\lambda)}(x''_j) \tag{1.4}$$

both with TDP$\,=p(k_0, \ldots, k_l)$. Construct by shifting the two node sets $(x'_0, x'_1, \ldots, x'_l)$, $(x''_0, x''_1, \ldots, x''_l)$ another two node sets

$$(y'_0, y'_1, \ldots, y'_l), \;\; (y''_0, y''_1, \ldots, y''_l)$$

satisfying: $y_0^{'} = y_0^{''} = -\pi$; $y_j^{'} = x_j^{'} + \alpha^{'}$, $y_j^{''} = x_j^{''} + \alpha^{''}$ for $j = 0, \dots, l$. By construction $-\pi = y_0^{'} \le y_1^{'} \le \cdots \le y_l^{'} \le \pi$, $-\pi = y_0^{''} \le y_1^{''} \le \cdots \le y_l^{''} \le \pi$, and from (1.3) and (1.4) we obtain the quadrature formulae

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}^{'} f^{(\lambda)}(y_j^{'});$$

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}^{''} f^{(\lambda)}(y_j^{''})$$

having TDP equal to the maximal trigonometric degree of precision $p(k_0, \dots, k_l)$. Hence,

$$\int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2k_0+1} \prod_{j=1}^{l} \left(\sin\frac{x - y_j^{'}}{2}\right)^{2k_j+1} \times r_{\frac{l-1}{2}}(x)\,dx = 0$$

and

$$\int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2k_0+1} \prod_{j=1}^{l} \left(\sin\frac{x - y_j^{''}}{2}\right)^{2k_j+1} \times r_{\frac{l-1}{2}}(x)\,dx = 0$$

for each $r_{\frac{l-1}{2}} \in T_{(l-1)/2}$. Now, by Theorem 1.2 $y_j^{'} = y_j^{''}$, $j = 1, 2, \dots, l$ and from here $x_j^{'} = x_j^{''} + \alpha$, $j = 0, 1, 2, \dots, l$, i.e., **the nodes are uniquely determined within a translation of a real parameter.**

B. Existence of a quadrature formula of type (1.1) having TDP equal to $p(k_0, \dots, k_l)$. First of all, if we show existence, from here will follow that $p(k_0, \dots, k_l)$ is the maximal trigonometric degree of precision for the class of all QFe of type (1.1).

Let $(-\pi, y_1, \dots, y_l)$ be the unique solution of the problem stated by Theorem 1.2. We shift $(-\pi, y_1, \dots, y_l)$ to obtain $(x_0, x_1, \dots, x_l)$ satisfying $\sum_{j=0}^{l}(2k_j + 1)x_j = 0$. Let the trigonometric polynomial $t(x) \in T_{l/2+\sum_{j=0}^{l} k_j}$ for l even and let

$$t(x) \in \left\{ T_{(l-1)/2+\sum_{j=0}^{l} k_j}, \sin\left[\left((l+1)/2 + \sum_{j=0}^{l} k_j\right) x\right] \right\}$$

for l odd. By using Theorem 1.3 and by integrating the corresponding Hermite interpolation representation for $t(x)$ in terms of the basic interpolating polynomials $t_{j,\lambda}(x)$

$$t(x) = \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} t^{(\lambda)}(x_j)\, t_{j,\lambda}(x)$$

we obtain a QF

$$\int_{-\pi}^{\pi} f(x)\, dx \approx \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}^{*}\, f^{(\lambda)}(x_j) \tag{1.5}$$

which is exact in the corresponding interpolation trigonometric spaces, given above.

Now, take an arbitrary trigonometric polynomial $t_*(x)$ of degree $p(k_0, \ldots, k_l)$. By Lemma 1.6, using a long division, the trigonometric polynomial $t_*(x)$ can be represented in the form

$$t_*(x) = \omega_{\mathbf{k},\mathbf{x}}(x)\, P_{(l-1)/2}(x) \;+\; r(x), \quad \mathbf{k} = (2k_0 + 1, \ldots, 2k_l + 1)$$

and the QF (1.5) must be exact for $r(x)$ because $s(\tilde{\mathbf{k}}) = (l+1)/2 + \sum_{j=0}^{l} k_j$.

On the other hand, by Theorem 1.2

$$\int_{-\pi}^{\pi} \omega_{\mathbf{k},\mathbf{x}}(x)\, P_{(l-1)/2}(x)\, dx = 0$$

and obviously

$$\sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}^{*} \left[\omega_{\mathbf{k},\mathbf{x}}\, P_{(l-1)/2}\right]^{(\lambda)}(x_j) = 0\,.$$

In view of this

$$\int_{-\pi}^{\pi} t_*(x)\, dx = \sum_{j=0}^{l} \sum_{\lambda=0}^{2k_j} a_{j,\lambda}^{*}\, t_*^{(\lambda)}(x_j)$$

and from here the QF (1.5) has TDP$=p(k_0,\dots,k_l)$; i.e., *the interpolation QF (1.5) obtained by using as interpolation nodes an appropriate shift of the unique solution of the problem in Theorem 1.1 is the unique within a real translation of the nodes QF of trigonometric degree precision $p(k_0,\dots,k_l)$ which is the maximal trigonometric degree of precision.* The proof is completed. □

Remark 1.6 The affirmative answer to the question of existence and uniqueness of a quadrature formulae with maximal degree of precision is not so useful by itself from a practical point of view if there is no **complete numerical characterization for the nodes and the coefficients of the extremal quadratures** . The next two examples present explicit quadrature formulae of maximal trigonometric degree of precision. Details are given in [4].

Example 1.1 Let $k_0=k_1=\cdots=k_l=k$ in (1.1); i.e., we consider QF type (1.1) with equal multiplicities $2k+1$ at each node. Then the quadrature formula of maximal trigonometric degree of precision equal to $k(l+1)+l$ is uniquely determined within a real translation θ^* of the nodes. *The quadrature formula of maximal trigonometric degree of precision has equally spaced nodes and it uses only even order derivatives.* In other words the extremal coefficients before the odd order derivatives are zero; i.e., *the extremal QFe possesses Birkhoff-type effect and the odd order derivatives do not contribute to the extremal formula for numerical integration.* Its explicit form is the following.

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \frac{2\pi}{l+1}\sum_{j=0}^{l} f\left(\frac{2j\pi}{l+1}+\theta^*\right)$$

$$+\frac{2\pi}{l+1}\sum_{\lambda=1}^{k}\frac{c_\lambda}{(l+1)^{2\lambda}}\sum_{j=0}^{l} f^{(2\lambda)}\left(\frac{2j\pi}{l+1}+\theta^*\right),$$

where

$$c_\lambda=\sum_{1\le\nu_1<\cdots<\nu_\lambda\le k}(\nu_1\cdots\nu_\lambda)^{-2},\quad \lambda=1,\dots,k.$$

Remark 1.7 *To the best of our knowledge the next Example gives one of the few known cases of explicit Gaussian formula in the case* *of non-equal multiplicities at each node.* It is an explicit formula for the unique (up to a real shifting of the nodes) quadrature formula having maximum degree of precision, in the case of *2-periodic* multiplicities,i.e., multiplicities satisfying $k_j = k_{j+2}, \quad 0 \le j \le l.$

Example 1.2 If $l+1 = 2n$ and $\{k_j,\ j = 0, \ldots, 2n-1\}$ are two-periodic sequences of non-equal multiplicities at each node, then the unique quadrature formulae within a real node-shifting with maximal trigonometric degree of precision equal to $n(k_0 + k_1 + 2) - 1$ is given by

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \frac{2\pi}{n} \sum_{\lambda=0}^{k_0} \frac{c_\lambda}{n^{2\lambda}} \sum_{j=0}^{n-1} f^{(2\lambda)}(2j\pi/n + \theta^*)$$

$$+ \frac{2\pi}{n} \sum_{\lambda=0}^{k_1} \frac{d_\lambda}{n^{2\lambda}} \sum_{j=0}^{n-1} f^{(2\lambda)}((2j+1)\pi/n + \theta^*),$$

where the coefficients $c_0, \ldots, c_{k_0}$ and $d_0, \ldots, d_{k_1}$ are uniquely determined by *the linear system of equations*

$$\begin{aligned} &c_0 + d_0 = 1 \\ &\sum_{\lambda=0}^{k_0} c_\lambda (-1)^\lambda \nu^{2\lambda} + (-1)^\nu \sum_{\lambda=0}^{k_1} d_\lambda (-1)^\lambda \nu^{2\lambda} = 0 \\ &\nu = 1, 2, \ldots, k_0 + k_1 + 1\,. \end{aligned}$$

Example 1.3 In order to estimate the non-linear structure of the complete constructive characterization of a QF type (1.1) of maximal trigonometric degree of precision the reader may consult [3], where a particular case of 4-periodic case of multiplicities is completely characterized. For example the reader will see that in this case the extremal nodes are not equally spaced.

2 Birkhoff Type Quadrature Formulae of Maximal Trigonometric Degree of Precision

We present results on existence, uniqueness, and explicit characterization of Birkhoff type quadrature formulae on equidistant nodes having maximal trigonometric degree of precision. Here the data means lacunary data; i.e., the derivatives taken at a given node are not necessarily consecutive [7]. The corresponding trigonometric interpolation problem in the case of k-periodic data was first proposed and solved in [11]. However, the necessary and sufficient conditions for the existence and uniqueness of the trigonometric interpolant were found in a simple form only for $k = 1$ [1] (equal multiplicities). In the case of trigonometric interpolation with two-periodic multiplicities data some special cases have been explicitly solved. Details can be found in [10]. *In view of this we found interesting to find explicitly quadrature formulae of maximal trigonometric degree of precision based on two-periodic, Birkhoff type data, to study their existence and uniqueness by making use of a direct method, without using any prior knowledge of the corresponding interpolants that even may not exist [2].*

Let us remind that with T_n we denote the linear space of all trigonometric polynomials $t(x) := \sum_{j=-n}^{n} a_j\, e^{\mathbf{i}jx},\ a_j \in C$ of degree at most n. Let $\mathbf{k} = (k_0, k_1, \ldots, k_{m-1}),\ 0 = k_0 < k_1 < \ldots < k_{m-1}$, and let $\mathbf{k}^{'} = (k_0^{'}, k_1^{'}, \ldots, k_{m_1-1}^{'}),\ 0 \le k_0^{'} < k_1^{'} < \ldots < k_{m_1-1}^{'}$ be two vectors whose components are non-negative distinct integers. Suppose that for a certain 2π-periodic function f we are given the following Birkhoff type information: $f^{(k_s)}(-\pi + 2\nu\pi/n),\ \nu = 0, 1, \ldots, n-1,\ s = 0, 1, \ldots, m-1$ and $f^{(k_j^{'})}(-\pi + (2\nu+1)\pi/n),\ \nu = 0, 1, \ldots, n-1,\ j = 0, 1, \ldots, m_1-1$. We construct from the above data a sort of *discrete differential levels* supposing that the coefficients of the quadrature depend only on the parity (2ν even or $(2\nu+1)$ odd) of the nodes:

$$f_{n,e}^{(k_s)} := \sum_{\nu=0}^{n-1} f^{(k_s)}(-\pi + 2\nu\pi/n) \quad (s = 0, 1, \ldots, m-1)$$

and

$$f_{n,o}^{(k_j^{'})} := \sum_{\nu=0}^{n-1} f^{(k_j^{'})}(-\pi + (2\nu+1)\pi/n) \quad (j = 0, 1, \dots, m_1 - 1)\,.$$

Problem formulation. Find quadrature formulae of the type

$$\int_{-\pi}^{\pi} f(x)\,dx \approx \frac{2\pi}{n} \sum_{s=0}^{m-1} \frac{c_s}{n^{k_s}}\, f_{n,e}^{(k_s)} + \frac{2\pi}{n} \sum_{j=0}^{m_1-1} \frac{d_j}{n^{k_j{'}}}\, f_{n,o}^{(k_j^{'})} \tag{2.1}$$

with maximal trigonometric degree of precision. *The solution of the problem depends on the number of even integers in the vectors* $\mathbf{k}$ *and* $\mathbf{k}^{'}$. *In fact the odd-order derivatives participating in the quadrature formula (2.1) do not influence the maximal degree of precision.* Thus, let ω_e and $\omega_e^{'}$ denote the number of even integers in the sets $\{0 = k_0 < k_1 < \cdots < k_{m-1}\}$ and $\{0 \le k_0^{'} < k_1^{'} < \cdots < k_{m_1-1}^{'})$, respectively. Let ω_o and $\omega_o^{'}$ denote the cardinality of the odd integers in the above sets so that $\omega_e + \omega_o = m$ and $\omega_o^{'} + \omega_e^{'} = m_1$. In terms of the notation, the solution of the problem is given by the next theorem (see [2] for details).

Theorem 2.1 *A quadrature of the form (2.1) has maximal trigonometric degree of precision equal to* $n(\omega_e + \omega_e^{'}) - 1$. *Concerning the uniqueness and the explicit characterization of the quadrature with maximal trigonometric degree of precision we have:*

(a) If $\omega_o + \omega_o' \le \omega_e + \omega_e' - 1$, *then the quadrature of type (2.1) with maximal trigonometric degree of precision exists and it is unique. Moreover,* $c_s = 0$ *for* k_s *odd* $(s = 1, \dots, m-1)$ *and* $d_j = 0$ *for* k_j' *odd* $(j = 0, \dots, m_1 - 1)$. *The values of* $\{c_s,\ (k_s \text{ even})\}_{s=0}^{m-1}$ *and* $\{d_j,\ (k_j^{'} \text{ even})\}_{j=0}^{m_1-1}$ *are uniquely determined by the linear system of equations:*

$$c_0 + d_0 = 1$$

$$\sum_{s=0,k_s\,even}^{m-1} c_s(-1)^{k_s/2}\nu^{k_s} + (-1)^{\nu} \sum_{j=0,k_j'\,even}^{m_1-1} d_j(-1)^{k_j'/2}\nu^{k_j'} = 0$$

$$\nu = 1, \dots, \omega_e + \omega_e' - 1;$$

(b) If $\omega_o+\omega_o' > \omega_e+\omega_e'-1$, *then there are many quadrature formulae of type (2.1) with maximal trigonometric degree of precision.*

(c) If in the quadrature formula (2.1) we are free to choose the $m-1$ *derivatives* $\{k_1 < k_2 < \cdots < k_{m-1}\}$ *and the* m_1 *derivatives* $\{k_0' < k_1' < k_2' < \cdots < k_{m_1-1}'\}$, *then the quadrature formula of maximal trigonometric degree of precision is obtained when* $\omega_e = m$, *and* $\omega_e' = m_1$ *and in this case the maximal trigonometric degree of precision is* $(m+m_1)n-1$.

Remark 2.1 If $\mathbf{k} = \mathbf{k}'$ we get a particular case of Birkoff (2.1) type quadrature with maximal trigonometric degree of precision with equidistant nodes and equal multiplicities at each node. In this form the problem was proposed and solved in [5].

The proof of Theorem 2.1 can be seen as an application of **Poisson summation formula** . Let g be 2π-periodic function of bounded variation and let g be regular; i.e., the value of g at any jump x_0 is $(g(x_0+0)+g(x_0-0))/2$. Then

$$g(x) \sim \sum_{m=-\infty}^{\infty} \hat{g}(m)\, e^{\mathbf{i}mx}, \quad \hat{g}(m) := \frac{1}{2\pi}\int_0^{2\pi} g(x)\, e^{-\mathbf{i}mx}\, dx,$$

where $\hat{g}(m)$ is the mth Fourier coefficient of g. Applying this formula to the function

$$g(x) = \sum_{j=0}^{n-1} f(x+2j\pi/n)$$

that is $2\pi/n$-periodic we obtain a Poisson type summation formula

$$\frac{1}{n}\sum_{j=0}^{n-1} f(a+2j\pi/n) = \sum_{\nu=-\infty}^{\infty} \hat{f}(n\nu)\, e^{\mathbf{i}n\nu\, a}\,. \tag{2.2}$$

By using the formula (2.2) with $a=-\pi$ and $f^{(k_s)}$ for f, we obtain

$$\begin{aligned} \frac{1}{n} f_{n,e}^{(k_s)} &= \sum_{\nu=-\infty}^{\infty} \widehat{f^{(k_s)}}(n\nu)(-1)^{n\nu} \qquad (2.3) \\ &= \sum_{\nu=-\omega}^{\omega} (-1)^{n\nu}(n\nu\mathbf{i})^{k_s}\, \hat{f}(n\nu) + \sum_{|\nu|\ge\omega+1} (-1)^{n\nu}\widehat{f^{k_s}}(n\nu) \end{aligned}$$

and analogously, substituting $a = -\pi + \pi/n$ in (2.2) we obtain

$$\begin{aligned}
\frac{1}{n}\, f_{n,o}^{(k_j^{'})} &= \sum_{\nu=-\infty}^{\infty} \widehat{f^{(k_j^{'})}}(n\nu)(-1)^{n\nu-\nu} \qquad (2.4)\\
&= \sum_{\nu=-\omega}^{\omega} (-1)^{n\nu-\nu}(n\nu\mathbf{i})^{k_j^{'}}\, \hat{f}(n\nu) + \sum_{|\nu|\geq\omega+1} (-1)^{n\nu-\nu}\widehat{f^{k_j^{'}}}(n\nu).
\end{aligned}$$

Multiplying (2.3) by a complex number c_s/n^{k_s} and (2.4) by a complex number $d_j/n^{k_j^{'}}$ and summing them in s from 0 to $m-1$ and in j from 0 to $m_1 - 1$, we get the following identity

$$\begin{aligned}
\frac{2\pi}{n}\sum_{s=0}^{m-1} \frac{c_s}{n^{k_s}}\, f_{n,e}^{(k_s)} &+ \frac{2\pi}{n}\sum_{j=0}^{m_1-1} \frac{d_j}{n^{k_j^{'}}}\, f_{n,o}^{(k_j^{'})} \qquad (2.5)\\
&= 2\pi \sum_{s=0}^{m-1} c_s \sum_{\nu=-\omega}^{\omega} (-1)^{n\nu}(\mathbf{i}\nu)^{k_s}\, \hat{f}(n\nu)\\
&+ 2\pi \sum_{j=0}^{m_1-1} d_j \sum_{\nu=-\omega}^{\omega} (-1)^{n\nu-\nu}(\mathbf{i}\nu)^{k_j^{'}}\, \hat{f}(n\nu)\\
&+ 2\pi \sum_{s=0}^{m-1} \frac{c_s}{n^{k_s}} \sum_{|\nu|\geq\omega+1} (-1)^{n\nu}\widehat{f^{k_s}}(n\nu)\\
&+ 2\pi \sum_{j=0}^{m_1-1} \frac{d_j}{n^{k_j^{'}}} \sum_{|\nu|\geq\omega+1} (-1)^{n\nu-\nu}\widehat{f^{k_j^{'}}}(n\nu)\,.
\end{aligned}$$

The method of the proof. The above formula holds for any $\mathbf{c} = (c_0, c_1, \ldots, c_{m-1})$ and $\mathbf{d} = (d_0, d_1, \ldots, d_{m_1-1})$ and each non-negative integer ω. *Our goal now is to find a formula of type (2.1) that holds for all trigonometric polynomials f in $T_{(\omega+1)n-1}$ with maximal ω.*

Lemma 2.1 *Suppose that a quadrature formula of type (2.1) is exact for each trigonometric polynomial $f \in T_{(\omega+1)n-1}$, where ω is non-negative integer. There are three possible cases:*

(1) If $\omega < \omega_e + \omega_e' - 1$, then such a quadrature type (2.1) is not unique.

(2) If $\omega > \omega_e + \omega_e' - 1$, then such a quadrature of type (2.1) does not exist.

(3) If $\omega=\omega_e+\omega'_e-1$, then such a quadrature type (2.1) is uniquely determined if $\omega_o+\omega'_o\le\omega_e+\omega'_e-1$, and in this case the uniquely determined $\mathbf{c}$ *and* $\mathbf{d}$ *are real. Moreover, if $\omega_o+\omega'_o>\omega_e+\omega'_e-1$, then there are many quadrature formulae type (2.1) which are exact in $T_{(\omega_e+\omega'_e)n-1}$.*

Conclusion. The maximum ω such that a QF type (2.1) is exact in $T_{(\omega+1)n-1}$ is $\omega=\omega_e+\omega'_e-1$. In other words the maximal trigonometric degree of precision of a QF type (2.1) is $(\omega_e+\omega'_e)n-1$ and it is determined by the cardinality of only the even integers in the data $\mathbf{k},\mathbf{k}^{'}$.

PROOF. We give the proof of Lemma 2.1. The proof of Lemma 2.1 requires a known result on determinants: Let $m_1<m_2<\cdots<m_q$ be distinct real numbers and let $t_1<t_2<\cdots<t_q$ be positive numbers. Then the determinant $det[t_k^{m_j}]_{k=1}^{q}{}_{j=1}^{q}>0$. By the identity (2.5) it follows that a QF type (2.1) holds for $f\in T_{(\omega+1)n-1}$ if and only if the following equality holds for each $f\in T_{n(\omega+1)-1}$:

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)\,dt &= \sum_{s=0}^{m-1} c_s \sum_{\nu=-\omega}^{\omega} (\mathbf{i}\nu)^{k_s}(-1)^{n\nu}\hat{f}(n\nu)\\ &+ \sum_{j=0}^{m_1-1} d_j \sum_{\nu=-\omega}^{\omega} (\mathbf{i}\nu)^{k'_j}(-1)^{n\nu-\nu}\hat{f}(n\nu)\,.\end{aligned}$$

The above is a linear equation with respect to f; hence, it will hold in $T_{n(\omega+1)-1}$ if it holds for $f(t)=e^{\mathbf{i}qnt}$ $(q=0,\pm1,\pm2,\ldots,\pm\omega)$. This leads to the following system of equations to determine $\mathbf{c}$ and $\mathbf{d}$. In case $k'_0=0$ we have

$$\begin{aligned}&c_0+d_0=1\\ &\sum_{s=0}^{m-1}(\mathbf{i}\nu)^{k_s}\,c_s+\,(-1)^{\nu}\sum_{j=0}^{m_1-1}(\mathbf{i}\nu)^{k_j^{'}}\,d_j\;=\;0\;\;(\nu=\pm1,\ldots,\pm\omega).\end{aligned} \tag{2.6}$$

In the case $k_0^{'}>0$ the first equation of the system (2.6) becomes $c_0=1$ and the other equations remain the same. First we consider the case $k_0^{'}=0$. Set $c_s=c_s^{'}+\mathbf{i}c''_s$ $s=0,1,\ldots,m-1$ and $d_j=d'_j+\mathbf{i}d''_j$ $j=0,1,\ldots,m_1-1$ in (2.6) and separate the equations into real and imaginary parts. This leads to the following two systems of equations that are *coupled* (mixed) with respect to the even and odd order integers:

$$
\begin{aligned}
& c_0' + d_0' = 1 \\
& \sum_{s=0, k_s\, even}^{m-1} c_s'(-1)^{k_s/2}\nu^{k_s} + \sum_{s=0, k_s\, odd}^{m-1} c_s''(-1)^{(k_s+1)/2}\nu^{k_s} \\
& +(-1)^{\nu} \sum_{j=0, k_j'\, even}^{m_1-1} d_j'(-1)^{k_j'/2}\nu^{k_j'} \\
& +(-1)^{\nu} \sum_{j=0, k_j'\, odd}^{m_1-1} d_j''(-1)^{(k_j'+1)/2}\nu^{k_j'} = 0, \\
& \nu = \pm 1, \ldots, \pm\omega.
\end{aligned}
$$

and

$$
\begin{aligned}
& c_0'' + d_0'' = 1 \\
& \sum_{s=0, k_s\, even}^{m-1} c_s''(-1)^{k_s/2}\nu^{k_s} + \sum_{s=0, k_s\, odd}^{m-1} c_s'(-1)^{(k_s-1)/2}\nu^{k_s} \\
& +(-1)^{\nu} \sum_{j=0, k_j'\, even}^{m_1-1} d_j''(-1)^{k_j'/2}\nu^{k_j'} \\
& +(-1)^{\nu} \sum_{j=0, k_j'\, odd}^{m_1-1} d_j'(-1)^{(k_j'-1)/2}\nu^{k_j'} = 0 \\
& \nu = \pm 1, \ldots, \pm\omega.
\end{aligned}
$$

The above two systems are *coupled* with respect to the even and the odd components of $\mathbf{k}$ and $\mathbf{k}'$. By simply adding and subtracting the equations for ν and $-\nu$, we *uncoupled* the above two systems in four systems of linear equations for c_s', d_j', c_s'', d_j'':

$$\begin{aligned}
&c'_0 + d'_0 = 1 \\
&\sum_{s=0, k_s\, even}^{m-1} c'_s (-1)^{k_s/2} \nu^{k_s} \\
&+(-1)^{\nu} \sum_{j=0, k'_j\, even}^{m_1-1} d'_j (-1)^{k'_j/2} \nu^{k'_j} = 0 \\
&\nu = 1, \ldots, \omega\,;
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&c''_0 + d''_0 = 0 \\
&\sum_{s=0, k_s\, even}^{m-1} c''_s (-1)^{k_s/2} \nu^{k_s} \\
&+(-1)^{\nu} \sum_{j=0, k'_j\, even}^{m_1-1} d''_j (-1)^{k'_j/2} \nu^{k'_j} = 0 \\
&\nu = 1, \ldots, \omega\,;
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\sum_{s=1, k_s\, odd}^{m-1} c'_s (-1)^{(k_s-1)/2} \nu^{k_s} \\
&+(-1)^{\nu} \sum_{j=1, k'_j odd}^{m_1-1} d'_j (-1)^{(k'_j-1)/2} \nu^{k'_j} = 0 \\
&\nu = 1, \ldots, \omega\,;
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\sum_{s=1, k_s\, odd}^{m-1} c''_s (-1)^{(k_s+1)/2} \nu^{k_s} \\
&+(-1)^{\nu} \sum_{j=1, k'_j odd}^{m_1-1} d''_j (-1)^{(k'_j+1)/2} \nu^{k'_j} = 0 \\
&\nu = 1, \ldots, \omega\,;
\end{aligned} \tag{2.10}$$

The system (2.7) is non-homogeneous while the other three (2.8)–(2.10) are homogeneous. The linear systems (2.7) and (2.8) have the same coefficient matrix. If ω_e and ω_e' denote the number of even integers in $\mathbf{k}$ and $\mathbf{k}'$, respectively, it is clear that we have infinitely many solutions in the case $\omega+1 < \omega_e+\omega_e'$. However, if $\omega+1 = \omega_e+\omega_e'$ then by using Laplace expansion with respect to the first ω_e columns of the square determinant Δ of the coefficient matrix we obtain (see [2] for details) $sign(\Delta) = (-1)(l_1+1)(l+1) - l_1(l_1+1)/2 \neq 0$. Thus, when $\omega = \omega_e + \omega_e' - 1$ the system (2.7) has a unique solution. Since the determinant of the coefficient matrix in (2.8) is also Δ, and since (2.8) is homogeneous, we have $c_s'' = 0, s = 0, 1, \ldots, m-1$ (k_s even) and $d_j'' = 0, j = 0, 1, \ldots, m_1 - 1$ (k_j' even). Thus, c_s and d_j are real for k_s and k_j' even. On the other hand (2.9) and (2.10) have the same coefficient matrix and if $\omega_o + \omega_o' < \omega_e + \omega_e' - 1 = \omega$ then $c_s' = c_s'' = 0,\ s = 1, \ldots, m-1$ (k_s odd) and $d_j' = d_j'' = 0,\ j = 1, \ldots, m_1 - 1$ (k_j' odd). However, if $\omega_o + \omega_o' > \omega_e + \omega_e' - 1$ there are infinitely many solutions. It remains to show that if $\omega > \omega_e + \omega_e' - 1$ then a QF type (2.1) having degree of precision $(\omega+1)n-1$ does not exist. This follows easily, because if $\omega = \omega_e + \omega_e'$, then the system of equations (2.7) with $\nu = 1, \ldots, \omega_e + \omega_e'$ is a homogeneous system with $\omega_e + \omega_e^{'}$ unknowns and the determinant of its coefficient matrix is nonzero. So, $c_0' = d_0' = 0$ which contradicts the first equation $c_0' + d_0' = 1$. This completes the proof of Lemma 2.1 in the case $k_0' = 0$. In the case $k_0' > 0$, the system of equations (2.7) becomes

$$
\begin{aligned}
&c_0' = 1 \\
&\sum_{s=0, k_s\ even}^{m-1} c_s'(-1)^{k_s/2}\nu^{k_s} + (-1)^{\nu} \sum_{j=0, k_j'\ even}^{m_1-1} d_j'(-1)^{k_j'/2}\nu^{k_j'} = 0 \\
&\nu = 1, \ldots, \omega
\end{aligned}
$$

and the proofs of (1), (2), and (3) remains unchanged. This completes the proof of Lemma 2.1. □

Proof. The **proof of Theorem 2.1** follows directly from Lemma 2.1. □

Example 2.1 Let $m = 1$, $m_1 = 1$, $k_0 = 0$, and $k_0^{'} > 0$ even. The corresponding Birkhoff $(0, k_0^{'})$ trigonometric interpolation problem was studied in [10]. In this case, the unique QF of maximal trigonometric degree of precision is:

$$\int_{-\pi}^{\pi} f(t)dt = \frac{2\pi}{n}\left(f_{n,e}^{(0)} + \frac{(-1)^{k_0'/2}}{n^{k_0'}} f_{n,o}^{(k_0')}\right)$$

and the maximal trigonometric degree of precision is equal to $2n-1$. This QF can be verified by integration of the $(0, k_0')$ interpolation formula in [10].

Example 2.2 In the particular case $\mathbf{k} = \mathbf{k}'$ the system of equations to determine the coefficients can be simplified and the maximal trigonometric degree of precision is $2\omega_e - 1$ (details can be found in [5]).

Example 2.3 Let $\mathbf{k}' = (k_1, k_2, \ldots, k_{m-1})$ and $\mathbf{k} = (0, k_1, k_2, \ldots, k_{m-1})$. In this case the systems of equations analogous to (2.7) and (2.8) are as follows, provided $k_1, k_2, \ldots, k_l$ are even and the rest are odd:

$$\begin{aligned}
&c_0' = 1,\ c_0' + \sum_{s=1}^{l}(c_s' + d_s')(-1)^{k_s/2}(2\nu)^{k_s} = 0 \\
&\nu = 1, 2, \ldots, \omega_e - 1 = l \quad (k_s \text{ even}) \\
&c_0' + \sum_{s=1}^{l}(c_s' - d_s')(-1)^{k_s/2}(2\nu+1)^{k_s} = 0 \\
&\nu = 0, 1, 2, \ldots, l-1 \quad (k_s \text{ even}).
\end{aligned}$$

References

[1] A. S. Cavaretta, A. Sharma and R. S. Varga, Lacunary trigonometric interpolation on equidistant nodes, in “Quantitative Approximation” (R. DeVore and K. Scherer, Eds.), pp.68–80, Academic Press, New York, 1980.

[2] D. Dryanov, R. Q. Jia and A. Sharma, Quadrature formulae with Birkhoff-type data on equidistant nodes for 2π-periodic functions, in “Progress in Approximation Theory” (P. Nevai and A. Pinkus, Eds.), pp. 243–261, Academic Press, Boston, 1991.

[3] D. Dryanov and V. Vatchev, On quadrature formulae with maximal trigonometric degree of precision, Numer. Math. 77 (1997), 59–82.

[4] D. Dryanov, Quadrature formulae with free nodes for periodic functions, Numer. Math. 67 (1994), 441–464.

[5] D. Dryanov, Birkhoff type quadrature formulae for 2π-periodic functions, C. R. Acad. Bulgare Sci. 42 (1989), 9–12.

[6] C. F. Gauss, Methodus nova integralium valores per approximationem inveniendi, Commentationes Societatis Regiae Scientarium Gottingensis Recentiores 3 (1814), 163-196.

[7] G. G. Lorentz, K. Jetter and S. Riemenschneider, "Birkhoff Interpolation," Chap. Addison-Wesley, Reading, MA, 1983.

[8] J. M. Ortega and W. C. Rheinboldt, "Iterative Solution of Nonlinear Equations in Several Variables," Academic Press, New York, 1970.

[9] J. T. Schwartz, "Nonlinear Functional Analysis," Gordon and Breach, New York, 1969.

[10] A. Sharma, J. Szabados and R. S. Varga, 2-Periodic lacunary trigonometric interpolation: the $(0; m)$ case, in "Constructive Theory of Functions" (B. l. Sendov, Ed.), pp. 420-427, Publ. House of the Bulg. Acad. of Sciences, Sofia, 1988.

[11] A. Sharma, P. W. Smith and J. Tzimbalario, Polynomial interpolation in the roots of unity with applications, in "Approximation and Function spaces," (Z. Cieselski, Ed.), pp. 667–685, Gdansk, Poland, 1979.

[12] P. Turán, On the theory of mechanical quadrature, Acta Sci. Math. Szeged 2 (1950), 30-37.

Inequalities for Exponential Sums via Interpolation and Turán-Type Reverse Markov Inequalities

T. Erdélyi
Department of Mathematics
Texas A&M University
College Station, Texas 77843, U.S.A.
Email address: terdelyi@math.tamu.edu

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Interpolation was a topic in which Sharma was viewed as an almost uncontested world expert by his collaborators and many other colleagues. We survey recent results for exponential sums and linear combinations of shifted Gaussians which were obtained via interpolation. To illustrate the method exploiting the Pinkus-Smith Improvement Theorem for spans of Descartes systems, we present the proof of a Chebyshev-type inequality. Finally, in Section 6 we present three simply formulated new results concerning Turán-type reverse Markov inequalities.

1 Introduction and Notation

In his book [2] Braess writes "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form $\sum_{j=1}^{n} a_j e^{\lambda_j t}$, where the parameters a_j and λ_j are to be determined, while n is fixed." Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j\,, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [21] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$. Here, and in what follows, $\|\cdot\|_{[a,b]}$ denotes the uniform norm on $[a,b]$. The main result, Theorem 3.2, of [5] shows that Schmidt's inequality holds with $c(n) = 2n-1$. That is,

$$\sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a, b-y\}}\,, \qquad y \in (a,b)\,. \tag{1.1}$$

In this Bernstein-type inequality even the pointwise factor is sharp up to a multiplicative absolute constant; the inequality

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \le \sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}\,, \qquad y \in (a,b)\,,$$

is established by Theorem 3.3 in [5].

Bernstein-type inequalities play a central role in approximation theory via a method developed by Bernstein himself, which turns Bernstein-type inequalities into what are called inverse theorems of approximation; see, for example, the books by Lorentz [16] and by DeVore and Lorentz [8]. From (1.1) one can deduce in a standard fashion that if there is a sequence $(f_n)_{n=1}^{\infty}$ of exponential sums with $f_n \in E_n$ and

$$\|f - f_n\|_{[a,b]} = O(n^{-m}(\log n)^{-2})\,, \qquad n = 2, 3, \ldots\,,$$

where $m \in \mathbb{N}$ is a fixed integer, then f is m times continuously differentiable on (a,b). Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. Inequality (1.1) can be extended to E_n replaced by $\widetilde{E}_n$, where $\widetilde{E}_n$ is the collection of all functions f of the form

$$f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t}\,,$$

$$a_0\,, \lambda_j \in \mathbb{R}\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,.$$

In fact, it is well-known that $\widetilde{E}_n$ is the uniform closure of E_n on any finite subinterval of the real number line. For a complex-valued function f defined on a set A let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup_{x \in A} \{|f(x)|\}\,,$$

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. We focus on the class

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j\,, \lambda_j \in \mathbb{R} \right\},$$

the class $\widetilde{G}_n$, the collection of all functions f of the form

$$f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,,$$

$$\lambda_j \in \mathbb{R}\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,,$$

and the class $\widetilde{G}_n^*$, the collection of all functions f of the form

$$f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,,$$

$$\lambda_j \in [-n^{1/2}, n^{1/2}], \quad P_{m_j} \in \mathcal{P}_{m_j}, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,.$$

In other words, G_n is the collection of n term linear combinations (over $\mathbb{R}$) of shifted Gaussians. Note that $\widetilde{G}_n$ is the uniform closure of G_n on any finite subinterval of the real line. Let $W(t) := \exp(-t^2)$. Combining Corollaries 1.5 and 1.8 in [9] and recalling that for the weight W the Mhaskar-Rachmanov-Saff number a_n defined by (1.4) in [9] satisfies $a_n \le c_1 n^{1/2}$ with a constant c_1 independent of n, we obtain that

$$\inf_{P \in \mathcal{P}_n} \|(P-g)W\|_{L_q(\mathbb{R})} \le c_2 n^{-m/2} \|g^{(m)} W\|_{L_q(\mathbb{R})}$$

with a constant c_2 independent of n, whenever the norm on the right-hand side is finite for some $m \in \mathbb{N}$ and $q \in [1,\infty]$. As a consequence

$$\inf_{f \in \widetilde{G}_n^*} \|f - gW\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} (gW)^{(k)}(t)\|_{L_q(\mathbb{R})}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \ldots, m$ with some $q \in [1,\infty]$. Replacing gW by g, we conclude that

$$\inf_{f \in \widetilde{G}_n^*} \|f - g\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} g^{(k)}(t)\|_{L_q(\mathbb{R})} \tag{1.2}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \ldots, m$ with some $q \in [1,\infty]$.

2 A Survey of Recent Results

Theorems 2.1–2.5 were proved in [12].

Theorem 2.1 *There is an absolute constant c_4 such that*

$$|U_n'(0)| \le c_4 n^{1/2} \, \|U_n\|_{\mathbb{R}}$$

for all U_n of the form $U_n = P_n Q_n$ with $P_n \in \widetilde{G}_n$ and an even $Q_n \in \mathcal{P}_n$. As a consequence

$$\|P_n'\|_{\mathbb{R}} \le c_4 n^{1/2} \, \|P_n\|_{\mathbb{R}}$$

for all $P_n \in \widetilde{G}_n$.

We remark that a closer look at the proof shows that $c_4 = 5$ in the above theorem is an appropriate choice in the theorem above.

Theorem 2.2 *There is an absolute constant c_5 such that*

$$\|U_n'\|_{L_q(\mathbb{R})} \le c_5^{1+1/q} n^{1/2} \, \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$ and $q \in (0, \infty)$.

Theorem 2.3 *There is an absolute constant c_6 such that*

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \le (c_6^{1+1/q} nm)^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, and $m = 1, 2, \ldots$.

We remark that a closer look at the proofs shows that $c_5 = 180\pi$ in Theorem 2.2 and $c_6 = 180\pi$ in Theorem 2.3 are suitable choices.

Our next theorem may be viewed as a slightly weak version of the right inverse theorem of approximation that can be coupled with the direct theorem of approximation formulated in (1.2).

Theorem 2.4 *Suppose $q \in [1, \infty]$, m is a positive integer, $\varepsilon > 0$, and f is a function defined on $\mathbb{R}$. Suppose also that*

$$\inf_{f_n \in \widetilde{G}_n} \|f_n - f\|_{L_q(\mathbb{R})} \le c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \ldots,$$

with a constant c_7 independent of n. Then f is m times differentiable almost everywhere in $\mathbb{R}$. Also, if

$$\inf_{f_n \in \widetilde{G}_n^*} \|f_n - f\|_{L_q(\mathbb{R})} = c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \ldots,$$

with a constant c_7 independent of n, then, in addition to the fact that f is m times differentiable almost everywhere in $\mathbb{R}$, we also have

$$\|(1 + |t|)^{m-j} f^{(j)}(t)\|_{L_q(\mathbb{R})} < \infty, \qquad k = 0, 1, \ldots, m.$$

Theorem 2.5 *There is an absolute constant c_8 such that*

$$\|U_n'\|_{L_q[y-\delta/2,y+\delta/2]} \leq c_8^{1+1/q}\left(\frac{n}{\delta}\right)\|U_n\|_{L_q[y-\delta,y+\delta]}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0,\infty]$, $y \in \mathbb{R}$, and $\delta \in (0,n^{1/2}]$.

In [18] H. Mhaskar writes "Professor Ward at Texas A&M University has pointed out that our results implicitly contain an inequality, known as Bernstein inequality, in terms of the number of neurons, under some conditions on the minimal separation. Professor Erdélyi at Texas A&M University has kindly sent us a manuscript in preparation, where he proves this inequality purely in terms of the number of neurons, with no further conditions. This inequality leads to the converse theorems in terms of the number of neurons, matching our direct theorem in this theory. Our direct theorem in [17] is sharp in the sense of n-widths. However, the converse theorem applies to individual functions rather than a class of functions. In particular, it appears that even if the cost of approximation is measured in terms of the number of neurons, if the degrees of approximation of a particular function by Gaussian networks decay polynomially, then a linear operator will yield the same order of magnitude in the error in approximating this function. We find this astonishing, since many people have told us based on numerical experiments that one can achieve a better degree of approximation by non-linear procedures by stacking the centers near the bad points of the target functions,"

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. Newman's inequality (see [3] and [19]) is an essentially sharp Markov-type inequality for $E(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is non-negative.

Theorem 2.6 (Newman's Inequality) *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of nonnegative real numbers. Then

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \le \sup_{0\neq P\in E(\Lambda_n)} \frac{\|P'\|_{(-\infty,0]}}{\|P\|_{(-\infty,0]}} \le 9\sum_{j=0}^{n}\lambda_j\,.$$

An L_p version of this may be found in [3], [6], and [10].

Theorem 2.7 *Let* $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ *be a set of nonnegative real numbers. Let* $1 \le p \le \infty$*. Then*

$$\|Q'\|_{L_p(-\infty,0]} \le 9\left(\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{L_p(-\infty,0]}$$

for every $Q \in E(\Lambda_n)$*.*

The following $L_p[a,b]$ $(1 \le p \le \infty)$ analogue of Theorem 2.7 has been established in [1].

Theorem 2.8 *Let* $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ *be a set of real numbers,* $a, b \in \mathbb{R}$*,* $a < b$*, and* $1 \le p \le \infty$*. There is a positive constant* $c_9 = c_9(a,b)$ *depending only on* a *and* b *such that*

$$\sup_{0\neq P\in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_p[a,b]}} \le c_9\left(n^2 + \sum_{j=0}^{n}|\lambda_j|\right).$$

Theorem 2.8 was proved earlier in [4] and [10] under the additional assumptions that $\lambda_j \ge \delta j$ for each j with a constant $\delta > 0$ and with $c_9 = c_9(a,b)$ replaced by $c_9 = c_9(a,b,\delta)$ depending only on a, b, and δ. The novelty of Theorem 2.8 was the fact that $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is an arbitrary set of real numbers; not even the non-negativity of the exponents λ_j is needed.

In [11] the following Nikolskii-Markov type inequality has been proved for $E(\Lambda_n)$ on $(-\infty,0]$.

Theorem 2.9 *Let* $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ *be a set of non-negative real numbers. Suppose* $0 < q \le p \le \infty$*. Let* μ *be a non-negative integer. There are constants* $c_{10} = c_{10}(p,q,\mu) > 0$

and $c_{11} = c_{11}(p,q,\mu)$ *depending only on* p*,* q*, and* μ *such that for* $A := (-\infty, 0]$ *we have*

$$c_{10}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} \leq \sup_{P\in E(\Lambda_n)}\frac{\|P^{(\mu)}\|_{L_pA}}{\|P\|_{L_qA}} \leq c_{11}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}},$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ *and for all* $\mu \geq 0$*, while the upper bound holds when* $\mu = 0$ *and* $0 < q \leq p \leq \infty$*, and when* $\mu \geq 1$*,* $p \geq 1$*, and* $0 < q \leq p \leq \infty$*. Also, there are constants* $c_{10} = c_{10}(q,\mu) > 0$ *and* $c_{11} = c_{11}(q,\mu)$ *depending only on* q *and* μ *such that*

$$c_{10}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}} \leq \sup_{P\in E(\Lambda_n)}\frac{|P^{(\mu)}(y)|}{\|P\|_{L_q(-\infty,y]}} \leq c_{11}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}}$$

for every $y \in \mathbb{R}$*.*

Motivated by a question of Michel Weber (Strasbourg) we proved the following two theorems in [13].

Theorem 2.10 *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Let $a, b \in \mathbb{R}$*,* $a < b$*,* $0 < q \leq p \leq \infty$*, and*

$$M(\Lambda_n, p, q) := \left(n^2 + \sum_{j=1}^{n}|\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}}.$$

There are constants $c_{12} = c_{12}(p,q,a,b) > 0$ *and* $c_{13} = c_{13}(p,q,a,b)$ *depending only on* p*,* q*,* a*, and* b *such that*

$$c_{12}M(\Lambda_n,p,q) \leq \sup_{P\in E(\Lambda_n)}\frac{\|P\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \leq c_{13}M(\Lambda_n,p,q)\,.$$

Theorem 2.11 *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Let $a, b \in \mathbb{R}$, $a < b$, $0 < q \le p \le \infty$, *and*

$$M(\Lambda_n, p, q) := \left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}}.$$

There are constants $c_{14} = c_{14}(p,q,a,b) > 0$ *and* $c_{15} = c_{15}(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_{14} M(\Lambda_n, p, q) \le \sup_{P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_{15} M(\Lambda_n, p, q)\,,$$

where the lower bound holds for all $0 < q \le p \le \infty$, *while the upper bound holds when* $p \ge 1$ *and* $0 < q \le p \le \infty$.

The lower bounds in these inequalities were shown by a method in which the Pinkus-Smith Improvement Theorem plays a central role. We formulate the useful lemmas applied in the proofs of these lower bounds. To emphasize the power of the technique of interpolation, we present the short proofs of these lemmas. Then these lemmas are used to establish the Chebyshev-type inequality below for exponential sums.

Theorem 2.12 *We have*

$$|f(y)| \le \exp(\gamma(|y| + \delta)) \left(\frac{2|y|}{\delta}\right)^n \|f\|_{[-\delta,\delta]}\,, \qquad y \in \mathbb{R} \setminus [-\delta, \delta]\,,$$

for all $f \in \widetilde{E}_n$ *of the form*

$$f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t}\,,$$

$$a_0 \in \mathbb{R}\,, \quad \lambda_j \in [-\gamma, \gamma]\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,,$$

and for all $\gamma > 0$.

3 Lemmas

Our first lemma, which can be proved by a simple compactness argument, may be viewed as a simple exercise.

Lemma 3.1 *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$, $a < b$. Let $w \neq 0$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. Then there exists a $0 \neq T \in E(\Delta_n)$ such that*

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}},$$

and there exists a $0 \neq S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Our next result is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2 *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$, $a < b < c$. Let $q \in (0, \infty]$. Let T and S be the same as in Lemma 3.1. Then T has exactly n zeros in $[a, b]$ by counting multiplicities. If $\delta_n \geq 0$, then S also has exactly n zeros in $[a, b]$ by counting multiplicities.*

The heart of the proof of our theorems is the following pair of comparison lemmas. The proofs of these are based on basic properties of Descartes systems, in particular on Descartes' Rule of Signs, and on a technique used earlier by P.W. Smith and Pinkus. Lorentz ascribes this result to Pinkus, although it was Smith [22] who published it. I learned about the method of proofs of these lemmas from Peter Borwein, who also ascribes it to Pinkus. This is the proof we present here. Section 3.2 of [3], for instance, gives an introduction to Descartes systems. Descartes' Rule of Signs is stated and proved on page 102 of [3].

Lemma 3.3 *Let*

$$\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$, $a < b \leq c$. Let $0 \neq w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. Then

$$\sup_{P \in E(\Delta_n)} \frac{|(P(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Under the additional assumption $\delta_n \geq 0$, we also have

$$\sup_{P \in E(\Delta_n)} \frac{|(P'(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{P \in E(\Gamma_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Lemma 3.4 *Let*

$$\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$, $c \leq a < b$. Let $0 \neq w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. Then

$$\sup_{P \in E(\Delta_n)} \frac{|(P(c)|}{\|Pw\|_{L_q[a,b]}} \geq \sup_{P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Under the additional assumption $\gamma_0 \leq 0$, we also have

$$\sup_{P \in E(\Delta_n)} \frac{|(Q'(c)|}{\|Qw\|_{L_q[a,b]}} \geq \sup_{P \in E(\Gamma_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}}.$$

4 Proofs of the Lemmas

Proof of Lemma 3.1 Since Δ_n is fixed, the proof is a standard compactness argument. We omit the details. $\square$

To prove Lemma 3.2 we need the following two facts: (a) Every $f \in E(\Delta_n)$ has at most n real zeros by counting multiplicities. (b) If $t_1 < t_2 < \cdots < t_m$ are real numbers and $k_1, k_2, \ldots, k_m$ are positive integers such that $\sum_{j=1}^{m} k_j = n$, then there is a $f \in E(\Delta_n)$, $f \neq 0$ having a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$.
Proof of Lemma 3.2 We prove the statement for T first. Suppose to the contrary that

$$t_1 < t_2 < \cdots < t_m$$

are real numbers in $[a,b]$ such that t_j is a zero of T with multiplicity k_j for each $j = 1,2,\ldots,m$, $k := \sum_{j=1}^m k_j < n$, and T has no other zeros in $[a,b]$ different from $t_1, t_2, \ldots, t_m$. Let $t_{m+1} := c$ and $k_{m+1} := n - k \geq 1$. Choose an $0 \neq R \in E(\Delta_n)$ such that R has a zero at t_j with multiplicity k_j for each $j = 1,2,\ldots,m+1$, and normalize so that $T(t)$ and $R(t)$ have the same sign at every $t \in [a,b]$. Let $T_\varepsilon := T - \varepsilon R$. Note that T and R are of the form

$$T(t) = \widetilde{T}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{T}$ and $\widetilde{R}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|T_\varepsilon(t)| < |T(t)|$ at every $t \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}$, so

$$\|T_\varepsilon w\|_{L_q[a,b]} < \|Tw\|_{L_q[a,b]}\,.$$

This, together with $T_\varepsilon(c) = T(c)$, contradicts the maximality of T.

Now we prove the statement for S. Without loss of generality we may assume that $S'(c) > 0$. Suppose to the contrary that

$$t_1 < t_2 < \cdots < t_m$$

are real numbers in $[a,b]$ such that t_j is a zero of S with multiplicity k_j for each $j = 1,2,\ldots,m$, $k := \sum_{j=1}^m k_j < n$, and S has no other zeros in $[a,b]$ different from $t_1, t_2, \ldots, t_m$. Choose a

$$0 \neq Q \in \operatorname{span}\{e^{\delta_{n-k} t}, e^{\delta_{n-k+1} t}, \ldots, e^{\delta_n t}\} \subset E(\Delta_n)\,,$$

such that Q has a zero at t_j with multiplicity k_j for each $j = 1,2,\ldots,m$, and normalize so that $S(t)$ and $Q(t)$ have the same sign at every $t \in [a,b]$. Note that S and Q are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad Q(t) = \widetilde{Q}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{Q}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Let $t_{m+1} := c$ and $k_{m+1} := 1$. Choose an

$$0 \neq R \in \operatorname{span}\{e^{\delta_{n-k-1} t}, e^{\delta_{n-k} t}, \ldots, e^{\delta_n t}\} \subset E(\Delta_n)$$

such that R has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m+1$, and normalize so that $S(t)$ and $R(t)$ have the same sign at every $t \in [a,b]$. Note that S and R are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^{m} (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^{m} (t - t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{R}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Since $\delta_n \geq 0$, it is easy to see that $Q'(c)R'(c) < 0$, so the sign of $Q'(c)$ is different from the sign of $R'(c)$. Let $U := Q$ if $Q'(c) < 0$, and let $U := R$ if $R'(c) < 0$. Let $S_\varepsilon := S - \varepsilon U$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|S_\varepsilon(t)| < |T(t)|$ at every $t \in [a,b] \setminus \{t_1, t_2, \dots, t_m\}$, so

$$\|S_\varepsilon w\|_{L_q[a,b]} < \|Sw\|_{L_q[a,b]}\,.$$

This, together with the inequalities $S'_\varepsilon(c) > S'(c) > 0$, contradicts the maximality of S. □

Proof of Lemma 3.3 We begin with the first inequality. We may assume that $a < b < c$. The general case when $a < b \leq c$ follows by a standard continuity argument. Let $k \in \{0, 1, \dots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n\,, \qquad \gamma_j = \delta_j\,, \quad j \neq k\,, \quad \text{and} \quad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is a $0 \neq T \in E(\Delta_n)$ such that

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

where T has exactly n zeros in $[a,b]$ by counting multiplicities. Denote the distinct zeros of T in $[a,b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of T with multiplicity k_j for each $j = 1, 2, \dots, m$, and $\sum_{j=1}^{m} k_j = n$. Then T has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$. Let

$$T(t) =: \sum_{j=0}^{n} a_j e^{\delta_j t}\,, \qquad a_j \in \mathbb{R}\,.$$

Without loss of generality we may assume that $T(c) > 0$. We have $T(t) > 0$ for every $t > c$; otherwise, in addition to its n zeros in $[a, b]$ (by counting multiplicities), T would have at least one more zero in (c, ∞), which is impossible. Hence

$$a_n := \lim_{t \to \infty} T(t) e^{-\delta_n t} \geq 0\,.$$

Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty, \infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} a_j > 0\,, \qquad j = 0, 1, \ldots, n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$, and normalize so that $R(c) = T(c) (> 0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs yields

$$(-1)^{n-j} b_j > 0\,, \qquad j = 0, 1, \ldots, n\,.$$

We have

$$(T - R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0 \\ j \neq k}}^{n} (a_j - b_j) e^{\delta_j t}\,.$$

Since $T - R$ has altogether at least $n + 1$ zeros at $t_1, t_2, \ldots, t_m$, and c (by counting multiplicities), it does not have any zero in $\mathbb{R}$ different from $t_1, t_2, \ldots, t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \ldots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \ldots, e^{\delta_n t})$$

is a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0, a_1 - b_1, \ldots, a_{k-1} - b_{k-1}, a_k, -b_k, a_{k+1} - b_{k+1}, \ldots, a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k}a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$, and $-b_n < 0$ if $k = n$, so

$$(T - R)(t) < 0\,, \qquad t > c\,.$$

Since each of T, R, and $T-R$ has a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$; $\sum_{j=1}^m k_j = n$, and $T - R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of T, R, and $T - R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0, 1, \ldots, m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \leq |T(t)|$ holds for all $t \in [a, b] \subset [a, c]$ with strict inequality at every t different from $t_1, t_2, \ldots t_m$. Combining this with $R(c) = T(c)$, we obtain

$$\frac{|R(c)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Since $R \in E(\Gamma_n)$, the first conclusion of the lemma follows from this.

Now we start the proof of the second inequality of the lemma. Although it is quite similar to that of the first inequality, we present the details. We may assume that $a < b < c$ and $\delta_n > 0$. The general case when $a < b \leq c$ and $\delta_n \geq 0$ follows by a standard continuity argument. Let $k \in \{0, 1, \ldots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n\,, \quad \gamma_j = \delta_j\,, \quad j \neq k\,, \quad \text{and} \quad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is an $0 \neq S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

where S has exactly n zeros in $[a, b]$ by counting multiplicities. Denote the distinct zeros of S in $[a, b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of S with multiplicity k_j for each $j = 1, 2, \ldots, m$, and $\sum_{j=1}^m k_j = n$. Then S has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \ldots, t_m$. Let

$$S(t) =: \sum_{j=0}^{n} a_j e^{\delta_j t}\,, \qquad a_j \in \mathbb{R}\,.$$

Without loss of generality we may assume that $S(c) > 0$. Since $\delta_n > 0$, we have $\lim_{t\to\infty} S(t) = \infty$; otherwise, in addition to its n zeros in (a,b), S would have at least one more zero in (c,∞), which is impossible.

Because of the extremal property of S, we have $S'(c) \neq 0$. We show that $S'(c) > 0$. To see this observe that Rolle's Theorem implies that $S' \in E(\Delta_n)$ has at least $n-1$ zeros in $[t_1, t_m]$. If $S'(c) < 0$, then $S(t_m) = 0$ and $\lim_{t\to\infty} S(t) = \infty$ imply that S' has at least 2 more zeros in (t_m, ∞) (by counting multiplicities). Thus $S'(c) < 0$ would imply that S' has at least $n+1$ zeros in $[a,\infty)$, which is impossible. Hence $S'(c) > 0$, indeed. Also $a_n := \lim_{t\to\infty} S(t)e^{-\delta_n t} \geq 0$. Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty.\infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} a_j > 0\,, \qquad j = 0, 1, \dots, n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1, 2, \dots, m$, and normalize so that $R(c) = S(c) (> 0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs implies that

$$(-1)^{n-j} b_j > 0\,, \qquad j = 0, 1, \dots, n\,.$$

We have

$$(S-R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0\\ j\neq k}}^{n} (a_j - b_j) e^{\delta_j t}\,.$$

Since $S-R$ has altogether at least $n+1$ zeros at $t_1, t_2, \dots, t_m$, and c (by counting multiplicities), it does not have any zero in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \dots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \dots, e^{\delta_n t})$$

is a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0, a_1 - b_1, \dots, a_{k-1} - b_{k-1}, a_k, -b_k, a_{k+1} - b_{k+1}, \dots, a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k} a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$ and $-b_n < 0$ if $k = n$, so

$$(S - R)(t) < 0 \,, \qquad t > c \,.$$

Since each of S, R, and $S - R$ has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m$; $\sum_{j=1}^{m} k_j = n$, and $S - R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of S, R, and $S - R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0, 1, \dots, m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \le |S(t)|$ holds for all $t \in [a, b] \subset [a, c]$ with strict inequality at every t different from $t_1, t_2, \dots t_m$. Combining this with $0 < S'(c) < R'(c)$ (recall that $R(c) = S(c) > 0$), we obtain

$$\frac{|R'(c)|}{\|Rw\|_{L_q[a,b]}} \ge \frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} \,.$$

Since $R \in E(\Gamma_n)$, the second conclusion of the lemma follows from this. □

PROOF OF LEMMA 3.4 The lemma follows from Lemma 3.3 via the substitution $u = -t$. □

5 Proof of the Theorem 2.12

PROOF OF THEOREM 2.12 By a well-known and simple limiting argument we may assume that

$$f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t}, \qquad -\gamma \le \lambda_0 < \lambda_1 < \cdots < \lambda_n \le \gamma \,.$$

By reasons of symmetry it is sufficient to examine only the case $y > \delta$. By Lemmas 3.1 – 3.4 we may assume that

$$\lambda_j = \gamma - (n - j)\varepsilon \,, \qquad j = 0, 1, \dots, n \,,$$

for sufficiently small values of $\varepsilon > 0$, that is,

$$f(t) = e^{\gamma t} P_n(e^{-\varepsilon t}) \,, \qquad P_n \in \mathcal{P}_n \,.$$

Now Chebyshev's inequality [8, Proposition 2.3, p. 101] implies that

$$\begin{aligned} |f(y)| &= e^{\gamma y} |P_n(e^{-\varepsilon y})| \le e^{\gamma y} \left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon \delta} - e^{-\varepsilon \delta}} \right)^n \|P_n(e^{-\varepsilon t})\|_{[-\delta,\delta]} \\ &\le e^{\gamma y} \left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon \delta} - e^{-\varepsilon \delta}} \right)^n e^{\delta y} \|f\|_{[-\delta,\delta]} \\ &\le e^{\gamma(y+\delta)} \left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon \delta} - e^{-\varepsilon \delta}} \right)^n \|f\|_{[-\delta,\delta]} \,, \end{aligned}$$

and by taking the limit when $\varepsilon > 0$ tends to 0, the theorem follows. $\square$

6 Turán-Type Reverse Markov Inequalities on Diamonds

Let $\varepsilon \in [0,1]$ and let D_ε be the ellipse in the complex plane with axes $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(D_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in D_ε. Let

$$\|f\|_A := \sup_{z \in A} |f(z)|$$

for complex-valued functions defined on A. Extending a result of Turán [23], Erőd [14, III. tétel] claimed that there are absolute constants $c_1 > 0$ and c_2 such that

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_{p \in \mathcal{P}_n^c(D_\varepsilon)} \frac{\|p'\|_{D_\varepsilon}}{\|p\|_{D_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n}) \,.$$

However, Erőd [14] presented a proof with only $c_1 n\varepsilon$ in the lower bound. It was Levenberg and Poletcky [15] who first published a correct proof of a result implying the lower bound claimed by Erőd.

Let $\varepsilon \in [0,1]$ and let S_ε be the diamond in the complex plane with diagonals $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(S_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in S_ε.

Theorem 6.1 *There are absolute constants $c_1 > 0$ and c_2 such that*

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_p \frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n}),$$

where the infimum is taken over all $p \in \mathcal{P}_n^c(S_\varepsilon)$ with the property

$$|p(z)| = |p(-z)|, \qquad z \in \mathbb{C}, \tag{6.1}$$

or where the infimum is taken over all real $p \in \mathcal{P}_n^c(S_\varepsilon)$.

It is an interesting question whether or not the lower bound in Theorem 6.1 holds when the infimum is taken for all $p \in \mathcal{P}_n^c(\varepsilon)$. As our next result shows this is the case at least when $\varepsilon = 1$.

Theorem 6.2 *There are absolute constants $c_1 > 0$ and c_2 such that*

$$c_1 n \le \inf_{p \in \mathcal{P}_n^c(S_1)} \frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \le c_2 n.$$

The following lemma is the main tool we need for the proofs of the theorems above.

Lemma 6.3 *Let $\Gamma(a, r)$ be the circle in the complex plane centered at a with radius r. Let $z_0 \in \Gamma(a, r)$. Suppose $p \in \mathcal{P}_n^c$ has at least m zeros in the disk $D(a, r)$ bounded by $\Gamma(a, r)$ and it has all its zeros in the half-plane $H(a, r, z_0)$ containing a and bounded by the line tangent to $\Gamma(a, r)$ at z_0. Then*

$$\left|\frac{p'(z_0)}{p(z_0)}\right| \ge \frac{m}{2r}.$$

Proof. Let $p \in \mathcal{P}_n^c$ be of the form

$$p(z) = c \prod_{k=1}^{n} (z - z_k), \qquad c, z_k \in \mathbb{C}.$$

Then

$$r\left|\frac{p'(z_0)}{p(z_0)}\right| = \left|\frac{p'(z_0)(z_0 - a)}{p(z_0)}\right| = \left|\sum_{k=1}^{n} \frac{z_0 - a}{z_0 - z_k}\right|$$

$$\begin{aligned} &= \left| \sum_{k=1}^{n} \left(1 - \frac{z_k - a}{z_0 - a}\right)^{-1} \right| \\ &\geq \left| \mathrm{Re}\left(\sum_{k=1}^{n} \left(1 - \frac{z_k - a}{z_0 - a}\right)^{-1} \right) \right| \\ &\geq \sum_{k=1}^{n} \mathrm{Re}\left(\left(1 - \frac{z_k - a}{z_0 - a}\right)^{-1} \right) \\ &\geq \frac{m}{2}\,, \end{aligned}$$

since

$$\mathrm{Re}\left(\left(1 - \frac{z_k - a}{z_0 - a}\right)^{-1} \right) \geq \frac{1}{2}\,, \qquad z_k \in D(a, r)\,,$$

and

$$\mathrm{Re}\left(\left(1 - \frac{z_k - a}{z_0 - a}\right)^{-1} \right) = \mathrm{Re}\left(\frac{z_0 - z_k}{z_0 - a} \right) \geq 0\,, \qquad z_k \in H(a, r, z_0)\,.$$

□

Proof of Theorem 6.1 The upper bound can be obtained by considering

$$p_n(z) := (z^2 - 1)^{\lfloor n/2 \rfloor}(z - 1)^{n - 2\lfloor n/2 \rfloor}\,.$$

We omit the simple calculation. To prove the lower bound we consider three cases.
Case 1: Property (6.1) holds and $\varepsilon \in [n^{-1/2}, 1]$. Choose a point z_0 on the boundary of S_ε such that

$$|p(z_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.2}$$

Property (6.1) implies that

$$|p(-z_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.3}$$

Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. A simple calculation shows that there are disks $D_1 := D_1(\varepsilon, c, z_0)$ and $D_2 := D_2(\varepsilon, c, -z_0)$ in the complex plane such that D_1 has radius $r = c\varepsilon^{-1}$

and is tangent to $[i\varepsilon, 1]$ at z_0, D_2 has radius $r = c\varepsilon^{-1}$ and is tangent to $[-1, -i\varepsilon]$ at $-z_0$, and $S_\varepsilon \subset D_1 \cup D_2$ for every sufficiently large absolute constant $c > 0$. Since $p \in \mathcal{P}_n^c$ has each of its zeros in S_ε, either p has at least $n/2$ zeros in D_1 or p has at least $n/2$ zeros in D_2. In the first case Lemma 6.3 and (6.2) imply

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{n}{4r} = \frac{1}{4c} n\varepsilon \,.$$

In the other case Lemma 6.3 and (6.3) imply

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(-z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(-z_0)}{p(-z_0)}\right| \geq \frac{n}{4r} = \frac{1}{4c} n\varepsilon \,.$$

Case 2: $p \in \mathcal{P}_n^c(\varepsilon)$ is real and $\varepsilon \in [n^{-1/2}, 1]$. Choose a point z_0 on the boundary of S_ε such that

$$|p(z_0)| = \|p\|_{S_\varepsilon} \,. \tag{6.4}$$

Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. Since $p \in \mathcal{P}_n^c(\varepsilon)$ is real, we have

$$|p(\overline{z}_0)| = \|p\|_{S_\varepsilon} \,. \tag{6.5}$$

Let $D_1 := D_1(\varepsilon, c, z_0)$ and $D_2 := D_2(\varepsilon, c, \overline{z}_0)$ be disks of the complex plane such that D_1 has radius $r = c\varepsilon^{-1}$ and is tangent to $[i\varepsilon, 1]$ at z_0 from below, D_2 has radius $r = c\varepsilon^{-1}$ and is tangent to $[-1, -i\varepsilon]$ at $\overline{z}_0$ from above. Denote the boundary of D_1 by Γ_1 and the boundary of D_2 by Γ_2. A simple calculation shows that if the absolute constant $c > 0$ is sufficiently large, then Γ_1 intersects the boundary of S_ε only at $a_1 \in [-1, i\varepsilon]$ and $b_1 \in [-i\varepsilon, 1]$, while Γ_2 intersects the boundary of S_ε only at $a_2 \in [-1, -i\varepsilon]$ and $b_2 \in [i\varepsilon, 1]$. Also, if the absolute constant $c > 0$ is sufficiently large, then

$$|a_1 - i\varepsilon| \leq \frac{1}{64}, \quad |a_2 + i\varepsilon| \leq \frac{1}{64}, \quad |b_1 - 1| \leq \frac{1}{64}, \quad |b_2 - 1| \leq \frac{1}{64} \,. \tag{6.6}$$

In the sequel let the absolute constant $c > 0$ be so large that inequalities (6.6) hold. If $p \in \mathcal{P}_n^c(\varepsilon)$ has at least αn zeros in D_1, then by using Lemma 6.3 and (6.4), we deduce

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{\alpha n}{2r} = \frac{\alpha}{2c} n\varepsilon \,.$$

If $p \in \mathcal{P}_n^c(\varepsilon)$ has at least αn zeros in D_2, then by using Lemma 6.3 and (6.5), we deduce

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(\overline{z}_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(\overline{z}_0)}{p(\overline{z}_0)}\right| \geq \frac{\alpha n}{2r} = \frac{\alpha}{2c}\, n\varepsilon\,.$$

Hence we may assume that $p \in \mathcal{P}_n^c(\varepsilon)$ has at least $(1-\alpha)n$ zeros in $S_\varepsilon \setminus D_1$ and it has at least $(1-\alpha)n$ zeros in $S_\varepsilon \setminus D_2$. Combining this with (6.6), we obtain that $p \in \mathcal{P}_n^c(\varepsilon)$ has at least $(1-2\alpha)n$ zeros in the disk centered at 1 with radius $1/32$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n^c(\varepsilon)$ be of the form $p = fg$ with

$$f(z) = \prod_{j=1}^{n_1} (z - u_j) \qquad \text{and} \qquad g(z) = \prod_{j=1}^{n_2} (z - v_j)\,,$$

where

$$u_j \in \mathbb{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \leq 2\alpha n\,, \tag{6.7}$$

and

$$|v_j - 1| \leq \frac{1}{32}\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \geq (1-2\alpha)n\,. \tag{6.8}$$

Let I be the subinterval of $[-1, i\varepsilon]$ with endpoint -1 and length $1/32$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality [8, Theorem 6.1, p. 75] and (6.7) we have

$$|f(y_0)| \geq \left(\frac{1}{128}\right)^{n_1} \geq \left(\frac{1}{128}\right)^{2\alpha n}\,,$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \geq \left(\frac{1}{256}\right)^{2\alpha n}\,. \tag{6.9}$$

Also, (6.8) implies

$$\left|\frac{g(y_0)}{g(z_0)}\right| \geq \frac{\left(\frac{31}{16}\right)^{n_2}}{\left(\sqrt{2} + \frac{1}{32}\right)^{n_2}} \geq \left(\frac{31}{24}\right)^{(1-2\alpha)n}\,. \tag{6.10}$$

By (6.9) and (6.10),

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{1}{256}\right)^{2\alpha} \left(\frac{31}{24}\right)^{(1-2\alpha)}\right)^n > 1\,,$$

if $\alpha > 0$ is a sufficiently small absolute constant. This finishes the proof in this case.
Case 3: $\varepsilon \in [0, n^{-1/2}]$. The lower bound of the theorem follows now from a result of Erőd [14, III. tétel] proved by Levenberg and Poletcky [15]. □

Proof of Theorem 6.2 Choose a point $z_0 \in S_1$ such that $|p(z_0)| = \|p\|_{S_1}$. Without loss of generality we may assume that $z_0 \in \left[1, \frac{1}{2}(1+i)\right]$. A simple calculation shows that there is an absolute constant $r > 0$ such that the circle $\Gamma := \Gamma(r, z_0)$ with radius r that is tangent to $[1, i]$ at z_0 and intersects the boundary of S_1 only at $a \in [-1, i]$ and $b \in [-i, 1]$. Moreover, if the $r > 0$ is sufficiently large, then

$$|a - i| \leq \frac{\sqrt{2}}{64} \qquad \text{and} \qquad |b - 1| \leq \frac{\sqrt{2}}{64}\,. \tag{6.11}$$

We denote the disk with boundary Γ by $D := D(r, z_0)$. If $p \in \mathcal{P}_n^c(1)$ has at least αn zeros in D, then by Lemma 6.3 we deduce

$$\frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \geq \frac{|p'(z_0)|}{\|p\|_{S_1}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{\alpha n}{2r}\,.$$

Hence we may assume that $p \in \mathcal{P}_n^c(1)$ has at most αn zeros in D, and hence that $p \in \mathcal{P}_n^c(1)$ has at least $(1-\alpha)n$ zeros in $S_1 \setminus D$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n^c(1)$ be of the form $p = fg$ with

$$f(z) = \prod_{j=1}^{n_1} (z - u_j) \qquad \text{and} \qquad g(z) = \prod_{j=1}^{n_2} (z - v_j)\,,$$

where

$$u_j \in \mathbb{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \leq \alpha n\,, \tag{6.12}$$

and

$$v_j \in S_1 \setminus D\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \geq (1-\alpha)n\,. \tag{6.13}$$

Let I be the subinterval of $[-1,-i]$ with endpoint -1 and length $\sqrt{2}/4$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality [8, Theorem 6.1, p. 75] and (6.12) we have

$$|f(y_0)| \geq \left(\frac{\sqrt{2}}{16}\right)^{n_1} \geq \left(\frac{\sqrt{2}}{16}\right)^{\alpha n},$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \geq \left(\frac{\sqrt{2}}{32}\right)^{\alpha n}. \tag{6.14}$$

Also, (6.11) and (6.13) imply

$$\begin{aligned} \left|\frac{g(y_0)}{g(z_0)}\right| &\geq \frac{\left(\sqrt{2}\left(\left(1-\frac{1}{64}\right)^2+\left(\frac{1}{4}\right)^2\right)^{1/2}\right)^{n_2}}{\left(\sqrt{2}\left(1+\left(\frac{1}{64}\right)2\right)^{1/2}\right)^{n_2}} \\ &\geq \left(\frac{66}{65}\right)^{n_2/2} \geq \left(\frac{66}{65}\right)^{(1/2-\alpha)n}. \end{aligned} \tag{6.15}$$

By (6.14) and (6.15)

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{\sqrt{2}}{32}\right)^{\alpha}\left(\frac{66}{65}\right)^{(1/2-\alpha)}\right)^{n} > 1$$

if $\alpha > 0$ is a sufficiently small absolute constant. □

Motivated by the initial results in this section, Sz. Révész [20] established the right order Turán type converse Markov inequalities on convex domains of the complex plane. His main theorem contains the results in this section as special cases. Révész's proof is also elementary, but rather subtle. It is expected to appear in the Journal of Approximation Theory soon.

References

[1] D. Benko, T. Erdélyi, and J. Szabados, The full Markov-Newman inequality for Müntz polynomials on positive intervals, Proc. Amer. Math. Soc. 131 (2003), 2385-2391.

[2] D. Braess, "Nonlinear Approximation Theory," Springer-Verlag, Berlin, 1986.

[3] P. Borwein and T. Erdélyi, "Polynomials and Polynomial Inequalities," Springer-Verlag, Graduate Texts in Mathematics, Volume 161, 486 p., New York, NY 1995.

[4] P. Borwein and T. Erdélyi, Newman's inequality for Müntz polynomials on positive intervals, J. Approx. Theory, 85 (1996), 132-139.

[5] P. Borwein and T. Erdélyi, A sharp Bernstein-type inequality for exponential sums, J. Reine Angew. Math. 476 (1996), 127-141.

[6] P. Borwein and T. Erdélyi, The L_p version of Newman's inequality for lacunary polynomials, Proc. Amer. Math. Soc. 124 (1996), 101-109.

[7] P. Borwein and T. Erdélyi, Pointwise Remez- and Nikolskii-type inequalities for exponential sums, Math. Annalen, 316 (2000), 39-60.

[8] R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, Berlin, 1993.

[9] Z. Ditzian and D. Lubinsky, Jackson and smoothness theorems for Freud weights, Constr. Approx. 13 (1997), 99-152.

[10] T. Erdélyi, Markov- and Bernstein-type inequalities for Müntz polynomials and exponential sums in L_p, J. Approx. Theory, 104 (2000), 142-152.

[11] T. Erdélyi, Extremal properties of the derivatives of the Newman polynomials, Proc. Amer. Math. Soc. 131 (2003), 3129-3134.

[12] T. Erdélyi, Bernstein-type inequalities for linear combinations of shifted Gaussians, Bull. London Math. Soc. to appear.

[13] T. Erdélyi, Markov-Nikolskii type inequalities for exponential sums on finite intervals, Adv. in Math. to appear.

[14] J. Erőd, Bizonyos polinomok maximumának alsó korlátjáról, Matematikai és Fizikai Lapok, XLVI (1939).

[15] N. Levenberg and J. Poletcky, A reverse Markov inequality, Ann. Acad. Sci. Fenn. Math. 37 (2002), 173-182.

[16] G. G. Lorentz, "Approximation of Functions," 2nd ed. Chelsea, New York, NY 1986.

[17] H. N. Mhaskar, "Introduction to the Theory of Weighted Polynomial Approximation," World Scientific, Singapore, 1996.

[18] H. N. Mhaskar, When is approximation by Gaussian networks necessarily a linear process?, manuscript (2003).

[19] D. J. Newman, Derivative bounds for Müntz polynomials, J. Approx. Theory, 18 (1976), 360-362.

[20] Sz. Révész, Right order Turán-type converse Markov inequalities for convex domains on the plane, J. Approx. Theory, to appear.

[21] E. Schmidt, Zur Kompaktheit der Exponentialsummen, J. Approx. Theory, 3 (1970), 445-459.

[22] P. W. Smith, An improvement theorem for Descartes systems, Proc. Amer. Math. Soc. 70 (1978), 26-30.

[23] P. Turán, Über die Ableitung von Polynomen, Comp. Math. (1939), 89-95.

Asymptotic Optimality in Time-Frequency Localization of Scaling Functions and Wavelets

T. N. T. Goodman
Department of Mathematics
University of Dundee
Dundee DD1 4HN
Scotland, U.K.
Email address: tgoodman@mcs.dundee.ac.uk

S. L. Lee
Department of Mathematics
National University of Singapore
2 Science Drive 2, Singapore 117543
Email address: matleesl@nus.edu.sg

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

For a triangular array of numbers $a_{n,j}$, $n = 2, 3, \dots$, $j = 0, 1, \dots, n$, the refinement equation with mask $P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j$, has for each n, a unique solution ϕ_n, which when suitably normalized, converges to the Gaussian if $P_n(z)$ has all roots in a sector inside the left half-plane and satisfying $P_n(1) = 1$, $P_n(-1) = 0$. The Gaussian, being the extremal function that attains the optimal constant in the uncertainty product, passes this characteristic to ϕ_n, whose uncertainty product converges to the optimal constant. The object is to analyse this phenomena for ϕ_n as well as for the corresponding wavelets.

1 Introduction

For $n \geq 2$, let

$$P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j, \tag{1.1}$$

be a polynomial with all its roots in the left half-plane $\{z : \Re z \leq 0\}$ and satisfying

$$P_n(1) = 1, \quad P_n(-1) = 0. \tag{1.2}$$

It is known that the *refinement equation*

$$\phi_n(x) = \sum_{j=0}^{n} 2a_{n,j}\phi_n(2x - j), \quad x \in \mathbb{R}, \tag{1.3}$$

has a unique solution satisfying $\int_{-\infty}^{\infty} \phi_n = 1$. Moreover it is shown in [5] that ϕ_n is continuous, non-negative and has support in $[0, n]$. In the special case $P_n(z) = 2^{-n}(z+1)^n$, ϕ_n is the uniform B-spline of degree $n-1$ with knots at $0, 1, \ldots, n$. We shall refer to ϕ_n as a *refinable function* with *symbol* P_n.

Now suppose that the roots of P_n are $-r_{n,j}$, $j = 1, \ldots, n$. It is shown in [2] that if we assume the stronger condition that for some β in $[0, \frac{\pi}{2})$,

$$|\arg r_{n,j}| \leq \beta, \text{ for } j = 1, \ldots, n, \ n = 2, 3, \ldots,$$

and furthermore

$$\sigma_n^2 := \frac{1}{3}\sum_{j=1}^{n} \frac{r_{n,j}}{(1 + r_{n,j})^2} \to \infty \quad \text{as } n \to \infty, \tag{1.4}$$

then a suitable shift and scaling of ϕ_n converges to the Gaussian function $G(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. To be precise, let

$$\mu_n := \sum_{j=1}^{n} \frac{1}{1 + r_{n,j}} \tag{1.5}$$

and

$$\widetilde{\phi}_n(x) := \sigma_n \phi_n(\sigma_n x + \mu_n), \quad x \in \mathbb{R}. \tag{1.6}$$

Then $\widetilde{\phi}_n$ has mean zero and standard deviation 1 and $\lim_{n\to\infty}\widetilde{\phi}_n = G$ in $L^p(\mathbb{R})$, $1 \le p \le \infty$. Moreover we have convergence in the frequency domain, i.e., $\lim_{n\to\infty}\widehat{\widetilde{\phi}}_n(u) = \widehat{G}(u) = e^{-u^2/2}$, where the convergence is uniform on $\mathbb{R}$.

The above properties are useful for many applications, e.g., signal processing, because the Gaussian function gives optimal time-frequency localisation. Thus we may approximate the Gaussian function by a refinable function ϕ_n with compact support which has fast algorithms for practical implementation. A natural choice is the uniform B-spline of degree $n-1$, but it was shown in [2] that other choices of ϕ_n give, in a sense, faster rates of convergence to the Gaussian.

To make precise the optimal time-frequency localisation of the Gaussian, we recall *Heisenberg's Uncertainty Principle.* For an L^2 function ϕ for which $\int_{-\infty}^{\infty} x^j|\phi(x)|^2 dx$ exists, $j = 1, 2$, we write

$$\mu_\phi := \frac{\int_{-\infty}^{\infty} x|\phi(x)|^2 dx}{\|\phi\|_2^2} \tag{1.7}$$

and

$$\Delta_\phi := \frac{\left\{\int_{-\infty}^{\infty}(x-\mu_\phi)^2|\phi(x)|^2 dx\right\}^{\frac{1}{2}}}{\|\phi\|_2}. \tag{1.8}$$

Similarly if $\int_{-\infty}^{\infty} u^j|\widehat{\phi}(u)|^2 du$ exists, $j = 1, 2$, we may define $\mu_{\widehat{\phi}}$ and $\Delta_{\widehat{\phi}}$. Thus Δ_ϕ is the standard deviation of the density function $|\phi|^2/\|\phi\|_2^2$ and gives a measure of the localisation of ϕ in the time domain. Similarly $\Delta_{\widehat{\phi}}$ is the standard deviation of $\left|\widehat{\phi}\right|^2/\|\widehat{\phi}\|_2^2$ and measures the localisation of ϕ in the frequency domain. The *uncertainty product* $\Delta_\phi\Delta_{\widehat{\phi}}$ gives an overall measure of the time-frequency localisation of ϕ. Clearly for any constants c, $\mu \in \mathbb{R}$, and $\sigma > 0$, the function $c\phi(\sigma\cdot-\mu)$ has the same uncertainty product as ϕ. *Heisenberg's Uncertainty Principle* states that for any ϕ as above,

$$\Delta_\phi\Delta_{\widehat{\phi}} \ge \frac{1}{2}, \tag{1.9}$$

and equality holds if and only if $\phi = cG(\sigma\cdot-\mu)$ for some c, $\mu \in \mathbb{R}$, and $\sigma > 0$ (see [4] for a general discussion). So for the sequence (ϕ_n)

of refinable functions, as above, to approach optimal time-frequency localisation, we would desire

$$\lim_{n\to\infty} \Delta_{\phi_n}\Delta_{\widehat{\phi}_n} = \frac{1}{2}.$$

This is proved in Section 3 under the further assumption $P_n'(-1) = 0$.

In applications such as signal processing it is also important to have good time-frequency localisation for the wavelet corresponding to the refinable function ϕ_n, as above, which is defined as follows. Let

$$\Phi_n(x) := \int_{-\infty}^{\infty} \phi_n(x+y)\phi_n(y)dy, \quad x \in \mathbb{R}, \tag{1.10}$$

$$Q_n(z) := \sum_{j=-n+1}^{n-1} \Phi_n(j)z^j, \quad z \in \mathbb{C}\setminus\{0\}. \tag{1.11}$$

Then the wavelet corresponding to ϕ_n is defined, up to multiplication by a constant, as $\psi_n(2\cdot -1)$, where ψ_n is defined by

$$\widehat{\psi}_n(u) = e^{inu}Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})\widehat{\phi}_n(u), \quad u \in \mathbb{R}. \tag{1.12}$$

We note that since $P_n(-1) = 0$, we have $\widehat{\psi}_n(0) = 0$. For a function ψ with $\widehat{\psi}(0) = 0$ (associated with a bandpass filter), the definition of the uncertainty product is modified to reflect the fact that $\widehat{\psi}$ treats positive and negative frequency bands separately. Let

$$\mu^+_{\widehat{\psi}} := \int_0^{\infty} u\left|\widehat{\psi}(u)\right|^2 du \Big/ \int_0^{\infty}\left|\widehat{\psi}(u)\right|^2 du, \tag{1.13}$$

$$\Delta^+_{\widehat{\psi}} := \left\{\int_0^{\infty}(u-\mu^+_{\widehat{\psi}})^2\left|\widehat{\psi}(u)\right|^2 du \Big/ \int_0^{\infty}\left|\widehat{\psi}(u)\right|^2 du\right\}^{\frac{1}{2}}, \tag{1.14}$$

where we assume these are well-defined. (Note that for a real-valued function ψ, $|\widehat{\psi}|$ is even and so $\int_0^\infty |\widehat{\psi}(u)|^2 du = \frac{1}{2}\|\widehat{\psi}\|_2^2$ and the definition of $\Delta^+_{\widehat{\psi}}$ is unaltered by replacing $\int_0^\infty$ by $\int_{-\infty}^0$ in (1.13) and (1.14).) Then a measure of the time-frequency localisation of ψ is given by $\Delta_\psi\Delta^+_{\widehat{\psi}}$.

It is shown in ([3], Theorem 1) that if $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function that satisfies $t\psi(t) \in L^2$, $\psi' \in L^2$ and $\widehat{\psi}(0) = 0$, then

$$\Delta_\psi \Delta^+_{\widehat{\psi}} > \frac{1}{2},$$

and the lower bound cannot be improved or attained.

A sequence of functions which are asymptotically optimal are the *modulated Gaussians*

$$G_n(x) := \sin(\lambda_n x)G(x), \quad n = 1, 2, \ldots,$$

where $\lambda_n \to \infty$ as $n \to \infty$. To see this we note that

$$2i\widehat{G}_n(u) = e^{-(u-\lambda_n)^2/2} - e^{-(u+\lambda_n)^2/2}.$$

Thus $\mu^+_{\widehat{G}_n} - \lambda_n \to 0$ as $n \to \infty$ and so

$$\begin{aligned}\lim_{n\to\infty} \left(\Delta^+_{\widehat{G}_n}\right)^2 &= \lim_{n\to\infty} \frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u-\lambda_n)^2} du}{\int_0^\infty e^{-(u-\lambda_n)^2} du} \\ &= \frac{\int_{-\infty}^\infty u^2 e^{-u^2} du}{\int_{-\infty}^\infty e^{-u^2} du} = \frac{1}{2}.\end{aligned}$$

Also

$$\Delta^2_{G_n} = \frac{\int_0^\infty \left|\widehat{G}'_n(u)\right|^2 du}{\int_0^\infty \left|\widehat{G}_n(u)\right|^2 du}$$

and so

$$\lim_{n\to\infty} \Delta^2_{G_n} = \lim_{n\to\infty} \frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u-\lambda_n)^2} du}{\int_0^\infty e^{-(u-\lambda_n)^2} du} = \frac{1}{2}.$$

Thus

$$\lim_{n\to\infty} \Delta_{G_n} \Delta^+_{\widehat{G}_n} = \frac{1}{2}.$$

In Section 4, we consider the wavelets ψ_n, as above. We assume further that P_n is reciprocal (which ensures ψ_n is symmetric or anti-symmetric), the roots of P_n are real (and hence negative), and $P_n'(-1) = 0$. It is then proved that for certain k_n, σ_n, α_n with $\sigma_n \to \infty$ as $n \to \infty$,

$$k_n \widehat{\psi}_n(u/\sigma_n) - e^{-\frac{1}{2}(u-\sigma_n\alpha_n)^2} \to 0$$

in $L^p(0,\infty)$, $1 \leq p \leq \infty$. Taking inverse Fourier transforms shows that the difference between $|k_n|\psi_n(\sigma_n x)$ and either $2\cos(\sigma_n\alpha_n x)G(x)$ or $2\sin(\sigma_n\alpha_n x)G(x)$ converges to 0 in $L^q(\mathbb{R})$, $2 \leq q \leq \infty$.

Finally we deduce that if we also have $P_n''(-1) = 0$, then

$$\lim_{n\to\infty} \Delta_{\psi_n} \Delta^+_{\widehat{\psi}_n} = \frac{1}{2},$$

and so the wavelets ψ_n have asymptotically optimal time-frequency location.

In [6] it was shown that the uniform B-spline converged to the Gaussian and the difference between the B-spline wavelets and modulated Gaussians converged to zero, both in time and frequency domains. In [3] this was extended to more general scaling functions ϕ_n and corresponding wavelets ψ_n and it was shown that the appropriate uncertainty products converged to the optimal value as $n \to \infty$ for both ϕ_n and ψ_n. It was assumed that the symbol P_n for ϕ_n was reciprocal and had real roots. Moreover it was assumed that $P_n(z)$ had a factor $(z+1)^{m_n}$ where $m_n \geq Cn$ for a constant C. These assumptions allowed the use of similar techniques to those used for the case of B-splines. In this paper we use the different techniques in [2] to allow us to relax all of these conditions for the case of ϕ_n, and for the case of ψ_n to relax the final condition to require $P_n(z)$ to have merely a factor of $(z+1)^3$. These together with the results in [2] will provide a more complete undertsanding of the asymptotic behaviour of scaling functions that approximate the Gaussian and the asymptotic properties of the corresponding wavelets.

2 A Preliminary Result

In this section we state a result that will be needed later. Taking Fourier transforms of (1.3) gives

$$\widehat{\phi}_n(u) = P_n(e^{-iu/2})\widehat{\phi}_n(u/2), \quad u \in \mathbb{R}, \tag{2.1}$$

and it follows that

$$\widehat{\phi}_n(u) = \prod_{k=1}^{\infty} P_n(e^{-iu/2^k}), \quad u \in \mathbb{R}, \tag{2.2}$$

where the infinite product converges uniformly. Recalling (1.2) and that the roots of P_n are $-r_{n,j}$, $j = 1, \ldots, n$ gives

$$\widehat{\phi}_n(u) = \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{e^{-iu/2^k} + r_{n,j}}{1 + r_{n,j}}, \quad u \in \mathbb{R}. \tag{2.3}$$

In order to consider the convergence of $\widehat{\phi}_n(u)$ to $e^{-u^2/2}$, we need to consider the covergence of products as in (2.3). The following result actually covers more general products which will be encountered in Section 4. It extends a corresponding result in [2] on asymptotic normality and also gives convergence of derivatives. Although its proof is of the same form as that of Theorem 1.2 of [2], we sketch it here for completeness.

Theorem 2.1 *For $\gamma \in [0, \frac{\pi}{2})$, let D_γ comprise all $z \in \mathbb{C}$ satisfying the inequality*

$$\left| \Im \frac{z}{(1+z)^2} \right| \leq \tan\gamma \, \Re \frac{z}{(1+z)^2} .$$

Suppose that for $n = 1, 2, \ldots$, f_n is defined by

$$f_n(u) = \prod_{j=1}^{r_n} \frac{e^{-iu} + \mu_{n,j}}{1 + \mu_{n,j}} \prod_{j=1}^{s_n} \prod_{k=1}^{\infty} \frac{e^{-iu/2^k} + \lambda_{n,j}}{1 + \lambda_{n,j}}, \quad u \in \mathbb{R}, \tag{2.4}$$

where $\mu_{n,j}$, $j = 1, \ldots, r_n$, and $\lambda_{n,j}$, $j = 1, \ldots, s_n$, $n = 1, 2, \ldots$, lie in D_γ for some $\gamma \in [0, \frac{\pi}{2})$ and are bounded away from -1.

Suppose that for $\mu_n \in \mathbb{R}$ and $\sigma_n > 0$, with $\sigma_n \to \infty$ as $n \to \infty$,

$$\widetilde{f}_n(u) := e^{iu\mu_n/\sigma_n} f_n(u/\sigma_n) \tag{2.5}$$

satisfies $\widetilde{f}_n'(0) = 0$ and $\widetilde{f}_n''(0) = -1$. Then as $n \to \infty$,

$$\widetilde{f}_n(u) \to e^{-u^2/2} \qquad \widetilde{f}_n'(u) \to -ue^{-u^2/2},$$

locally uniformly on $\mathbb{R}$.

PROOF. A simple calculation shows that

$$\mu_n = \sum_{j=1}^{r_n} \frac{1}{1+\mu_{n,j}} + \sum_{j=1}^{s_n} \frac{1}{1+\lambda_{n,j}}, \tag{2.6}$$

$$\sigma_n^2 = \sum_{j=1}^{r_n} \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{3}\sum_{j=1}^{s_n} \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}. \tag{2.7}$$

Thus the condition $\lim_{n\to\infty}\sigma_n = \infty$ implies $\lim_{n\to\infty}(r_n + s_n) = \infty$.

As in the proof of Theorem 1.2 of [2] we write

$$\log \widetilde{f}_n(u) = \frac{iu\mu_n}{\sigma_n} + \sum_{j=1}^{r_n} F(\mu_{n,j}, \frac{-iu}{\sigma_n}) + \sum_{j=1}^{s_n}\sum_{k=1}^{\infty} F(\lambda_{n,j}, \frac{-iu}{2^k\sigma_n}),$$

where

$$F(\mu, t) = \log\left(\frac{e^t+\mu}{1+\mu}\right),$$

and expanding $F(\mu, t)$ in a Taylor series about $t = 0$ gives

$$\log \widetilde{f}_n(u) = -\frac{u^2}{2} + \sum_{\nu=3}^{\infty} \sigma_n^{-\nu}(-iu)^{\nu}\left\{\sum_{j=1}^{r_n} a_\nu(\mu_{n,j}) + \sum_{j=1}^{s_n} \frac{a_\nu(\lambda_{n,j})}{2^\nu - 1}\right\}, \tag{2.8}$$

where, under the conditions on $\mu_{n,j}$, $\lambda_{n,j}$, there is a constant $A > 0$ with

$$|a_\nu(\mu_{n,j})| \leq \frac{A^{\nu-2}}{\nu}\sec\gamma\, \Re\, \frac{\mu_{n,j}}{(1+\mu_{n,j})^2},$$

$$|a_\nu(\lambda_{n,j})| \leq \frac{A^{\nu-2}}{\nu}\sec\gamma\, \Re\, \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}.$$

Thus, from (2.8), (2.6) and (2.7),

$$\left|\log \widetilde{f}_n(u) + \frac{u^2}{2}\right| \leq \sec\gamma \sum_{\nu=3}^{\infty} \frac{\sigma_n^{-\nu}|u|^\nu}{\nu} A^{\nu-2} \times \left\{\sum_{j=1}^{r_n} \Re\, \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{2^\nu-1}\sum_{j=1}^{s_n} \Re\, \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}\right\}$$

$$\begin{aligned} &\leq \quad \sec\gamma \sum_{\nu=3}^{\infty} \frac{|u|^\nu}{\nu} \left(\frac{A}{\sigma_n}\right)^{\nu-2} \\ &\leq \quad \sec\gamma\, \frac{A|u|^3}{\sigma_n} \left(1 - \frac{A|u|}{\sigma_n}\right)^{-1} \end{aligned}$$

whenever $A|u| < \sigma_n$. Since $\sigma_n \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} \widetilde{f}_n(u) = e^{-u^2/2}$ locally uniformly on $\mathbb{R}$.

Differentiating (2.8) gives

$$\frac{\widetilde{f}_n'(u)}{\widetilde{f}_n(u)} = -u - i \sum_{\nu=3}^{\infty} \sigma_n^{-\nu} (-iu)^{\nu-1} \left\{ \sum_{j=1}^{r_n} a_\nu(\mu_{n,j}) + \sum_{j=1}^{s_n} \frac{a_\nu(\lambda_{n,j})}{2^\nu - 1} \right\}$$

and, as before,

$$\begin{aligned} \left| \frac{\widetilde{f}_n'(u)}{\widetilde{f}_n(u)} + u \right| \quad &\leq \quad \sec\gamma \sum_{\nu=3}^{\infty} \sigma_n^{-\nu} |u|^{\nu-1} A^{\nu-2} \\ &\times \quad \left\{ \sum_{j=1}^{r_n} \Re\, \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{2^\nu - 1} \sum_{j=1}^{s_n} \Re\, \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2} \right\} \\ &\leq \quad \sec\gamma \sum_{\nu=3}^{\infty} |u|^{\nu-1} \left(\frac{A}{\sigma_n}\right)^{\nu-2} \\ &= \quad \sec\gamma\, \frac{A|u|^2}{\sigma_n} \left(1 - \frac{A|u|}{\sigma_n}\right)^{-1} \end{aligned}$$

whenever $A|u| < \sigma_n$. Thus $\lim_{n\to\infty} \widetilde{f}_n'(u) = -ue^{-u^2/2}$ locally uniformly in $\mathbb{R}$. □

As noted in [2], the set D_γ contains the sector $|\arg z| \leq \gamma$ and for $z = \pm re^{i\theta}$, $r > 0$, $\gamma \leq \theta \leq \pi$, z lies in D_γ if and only if

$$\frac{\sin(\frac{\theta-\gamma}{2})}{\sin(\frac{\theta+\gamma}{2})} \leq r \leq \frac{\sin(\frac{\theta+\gamma}{2})}{\sin(\frac{\theta-\gamma}{2})}.$$

In particular, D_γ contains the unit circle $r = 1$.

3 Asymptotic Optimality of Refinable Functions

We suppose that (ϕ_n) is a sequence of refinable functions as in Section 1.

Theorem 3.1 *If $P_n'(-1) = 0$, then*

$$\lim_{n\to\infty} \Delta_{\phi_n} \Delta_{\widehat{\phi}_n} = \frac{1}{2}. \tag{3.1}$$

Proof. We recall that $\lim_{n\to\infty} \widetilde{\phi}_n = G$ in $L^p(\mathbb{R})$, $1 \le p \le \infty$. From (1.6) we see that (3.1) is equivalent to

$$\lim_{n\to\infty} \Delta_{\widetilde{\phi}_n} \Delta_{\widehat{\widetilde{\phi}}_n} = \frac{1}{2}. \tag{3.2}$$

We also recall that $\widetilde{\phi}_n$ is defined so that

$$\int_{-\infty}^{\infty} x^2 \widetilde{\phi}_n(x)dx = \int_{-\infty}^{\infty} x^2 G(x)dx = 1. \tag{3.3}$$

Firstly we shall show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^2 \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} x^2 G(x)^2 dx. \tag{3.4}$$

Take $\epsilon > 0$ and choose $A > 1$ so that

$$\int_{|x|>A} x^2 G(x)dx < \epsilon. \tag{3.5}$$

Choose N so that for all $n > N$ and $|x| \le A$,

$$|x^2 \widetilde{\phi}_n(x)^k - x^2 G(x)^k| < \frac{\epsilon}{2A}, \quad k = 1, 2. \tag{3.6}$$

Take any $n > N$. Then

$$\left| \int_{-A}^{A} x^2 \widetilde{\phi}_n(x)dx - \int_{-A}^{A} x^2 G(x)dx \right| < \epsilon$$

and so by (3.3),

$$\left| \int_{|x|>A} x^2 \widetilde{\phi}_n(x) dx - \int_{|x|>A} x^2 G(x) dx \right| < \epsilon.$$

So by (3.5),

$$\int_{|x|>A} x^2 \tilde{\phi}_n(x) dx < 2\varepsilon.$$

Thus for large enough n,

$$\int_{|x|>A} x^2 \widetilde{\phi}_n(x)^2 dx < \frac{1}{\sqrt{2\pi}} \int_{|x|>A} x^2 \widetilde{\phi}_n(x) dx < \varepsilon,$$

and so

$$\left| \int_{|x|>A} x^2 \tilde{\phi}_n(x)^2 dx - \int_{|x|>A} x^2 G(x)^2 dx \right| < 2\epsilon.$$

By (3.6), for all large enough n,

$$\left| \int_{-A}^{A} x^2 \widetilde{\phi}_n(x)^2 dx - \int_{-A}^{A} x^2 G(x)^2 dx \right| < \epsilon$$

and (3.4) follows.

Also for large enough n,

$$\left| \int_{-A}^{A} x \widetilde{\phi}_n(x)^2 dx - \int_{-A}^{A} x G(x)^2 dx \right| < \epsilon,$$

$$\int_{|x|>A} |x| \widetilde{\phi}_n(x)^2 dx < \int_{|x|>A} x^2 \widetilde{\phi}_n(x)^2 dx < \epsilon,$$

and as before,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} x G(x)^2 dx = 0. \tag{3.7}$$

Since

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} G(x)^2 dx,$$

(3.4) and (3.7) give

$$\lim_{n\to\infty} \Delta_{\widetilde{\phi}_n} = \Delta_G = \frac{1}{\sqrt{2}}.$$

To complete the proof we need to show that $\lim_{n\to\infty} \Delta_{\widehat{\widetilde{\phi}}_n} = \Delta_{\widehat{G}}$ and since $|\widehat{\widetilde{\phi}}_n(u)|^2$ is even and $\lim_{n\to\infty} \|\widehat{\widetilde{\phi}}_n\|_2 = \|\widehat{G}\|_2$, it remains to show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} u^2 |\widehat{\widetilde{\phi}}_n(u)|^2 du = \int_{-\infty}^{\infty} u^2 e^{-u^2} du. \tag{3.8}$$

Let $\widehat{\Phi}_n(u) = |\widehat{\widetilde{\phi}}_n(u)|^2$ and $f(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$, so that $\widehat{f}(u) = e^{-u^2}$. Then (3.8) is equivalent to

$$\lim_{n\to\infty} \Phi_n''(0) = f''(0). \tag{3.9}$$

Recall that ϕ_n is the refinable function with symbol $P_n(z)$, which we may write as $(z+1)^2 p_n(z)$. Then Φ_n is, up to scaling, the refinable function with symbol $P_n(z)P_n(z^{-1}) = (z+1)^2(z^{-1}+1)^2 p_n(z)p_n(z^{-1})$. It follows that

$$\Phi_n''(x) = \eta_n(x+1) - 2\eta_n(x) + \eta_n(x-1),$$

where η_n mentioned above is the refinable function with symbol $(z+1)(z^{-1}+1)p_n(z)p_n(z^{-1})$, up to scaling. Since this polynomial has all roots in the left half-plane, it follows from [5], and [1], that η_n satisfies the following property: for any sequence (λ_j) in $\mathbb{R}$, the number of inflections of the function $\sum_{j=-\infty}^{\infty} \lambda_j \eta_n(\cdot - j)$ is bounded by the number of inflections in the polygonal arc with vertices (j, λ_j), $j \in \mathbb{Z}$. Thus the function Φ_n'' has at most four inflections.

Now $\lim_{n\to\infty} \widehat{\widetilde{\phi}}_n(u) = e^{-u^2/2}$ locally uniformly on $\mathbb{R}$ and so $\lim_{n\to\infty} (\Phi_n'')\widehat{\ }(u) = (f'')\widehat{\ }(u)$ locally uniformly on $\mathbb{R}$. Now f'' has exactly four inflections and it follows as in the proof of Lemma 5.1 of [2] that Φ_n'' converges to f'' uniformly on a neighborhood of 0. So (3.9) holds. $\square$

4 Asymptotic Optimality of Wavelets

Throughout this section we shall assume that the roots of P_n in (1.1) are real and negative, and that P_n is reciprocal; i.e., the roots are invariant under the transformation $z \to z^{-1}$. It follows that ϕ_n is symmetric; i.e. $\phi_n(x) = \phi_n(n-x)$, $x \in \mathbb{R}$.

We define Φ_n and Q_n as in (1.10) and (1.11). It is easily seen that Φ_n is a refinable function with mask $P_n(z)P_n(z^{-1})$. It follows that Φ_n is continuous with support $[-n, n]$ and satisfies $\Phi_n(-x) = \Phi_n(x)$, $x \in \mathbb{R}$. The Laurent polynomial Q_n is called the Euler-Frobenius polynomial corresponding to ϕ_n. From the work of [5] we know that Q_n has real negative roots. As in (1.12) we define the wavelet, up to multiplication by a constant as $\psi_n(2\ \cdot\ -1)$, where ψ_n is defined by

$$\widehat{\psi}_n(u) = e^{inu} Q_n(e^{-i(u+\pi)}) P_n(e^{-i(u+\pi)}) \widehat{\phi}_n(u), \quad u \in \mathbb{R}. \tag{4.1}$$

Suppose that the roots of P_n are $-\lambda_{n,1}, \dots, -\lambda_{n,n}$. By (1.2) we may suppose $\lambda_{n,1} = 1$. We write

$$\tilde{\lambda}_{n,j} := \frac{1}{2}(\lambda_{n,j} + \lambda_{n,j}^{-1}), \quad j = 1, \dots, n,$$

and for simplicity drop the first subscript n when it is unambiguous to do so. Note that $\tilde{\lambda}_j \geq 1$, $j = 1, \dots, n$, and

$$(e^{-iu} + \lambda_j)(e^{-iu} + \lambda_j^{-1}) = 2e^{-iu}(\tilde{\lambda}_j + \cos u)\ .$$

Also

$$\begin{aligned} e^{-iu} + 1 &= 2e^{-iu/2} \cos\frac{u}{2} \\ &= \sqrt{2} e^{-iu/2} (1 + \cos u)^{1/2}, \quad -\pi \leq u \leq \pi, \end{aligned}$$

and therefore

$$P_n(e^{-iu}) = e^{-inu/2} \prod_{j=1}^{n} \left(\frac{\tilde{\lambda}_j + \cos u}{\tilde{\lambda}_j + 1} \right)^{1/2}, \quad -\pi \leq u \leq \pi.$$

Similarly, if the roots of Q_n are $-\mu_1, \dots, -\mu_{n-1}, -\mu_1^{-1}, \dots, -\mu_{n-1}^{-1}$, and setting $\tilde{\mu}_j := \frac{1}{2}(\mu_j + \mu_j^{-1})$, $j = 1, \dots, n-1$, then for a constant

$A_n > 0$,

$$Q_n(e^{-iu}) = A_n \prod_{j=1}^{n-1} (\tilde{\mu}_j + \cos u).$$

So from (4.1) and (2.2),

$$\widehat{\psi}_n(u) = B_n e^{-in\pi/2} f_n(u), \quad u \in \mathbb{R}, \tag{4.2}$$

where $B_n > 0$, $f_n(u)$ is real,

$$\begin{aligned} |f_n(u)| &= \prod_{j=1}^{n-1} (\tilde{\mu}_j - \cos u) \prod_{j=1}^{n} (\tilde{\lambda}_j - \cos u)^{1/2} \\ &\quad \times \prod_{k=1}^{\infty} \prod_{j=1}^{n} \left(\frac{\tilde{\lambda}_j + \cos(2^{-k}u)}{\tilde{\lambda}_j + 1} \right)^{1/2}, \quad u \in \mathbb{R}, \end{aligned} \tag{4.3}$$

$f_n(u) > 0$, $0 < u < 2\pi$, $f_n(0) = f_n(2\pi) = 0$, $f_n(-u) = (-1)^n f_n(u)$.

We shall make the following mild assumption on the growth rate of $\tilde{\lambda}_{n,j}$ with n. There are constants $K > 0$ and $0 < \sigma \le 1$ such that

$$\left| \{ j : \tilde{\lambda}_{n,j} \le K \} \right| \ge \sigma n, \quad n = 2, 3, \ldots. \tag{4.4}$$

This can be reformulated as follows.

Lemma 4.1 *Condition (4.4) holds if and only if there is a constant $C > 0$ with*

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j} + 1)^2} \ge Cn, \quad n = 2, 3, \ldots. \tag{4.5}$$

Proof. If (4.4) holds, then

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j} + 1)^2} \ge \frac{\sigma n}{(K+1)^2}, \quad n = 2, 3, \ldots,$$

which gives (4.5).

Conversely, suppose that (4.4) does not hold. Take any $\epsilon > 0$. Then there exists n such that

$$|\{ j : \tilde{\lambda}_{n,j} \le \epsilon^{-1/2} \}| < \epsilon n.$$

If $\tilde{\lambda}_{n,j} > \epsilon^{-1/2}$, then $(\tilde{\lambda}_{n,j}+1)^{-2} < \epsilon$ and so

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j}+1)^2} < 2n\epsilon.$$

Thus there is no constant C for which (4.5) holds. $\square$

We can now study the shape of the graph of f_n in $[0, 2\pi]$, which will lead to the asymptotic behaviour of $\widehat{\psi}_n$.

Lemma 4.2 *There is a number $\alpha > 0$ such that f_n in (4.2) has a unique local maximum in $[0, 2\pi]$ at α_n with*

$$\frac{2\pi}{3} < \alpha_n < \pi - \alpha, \quad n = 2, 3, \ldots.$$

PROOF. For $n = 2, 3, \ldots$, putting $F_n(u) := \log f_n(u)$, $0 < u < 2\pi$, gives by (4.3),

$$F_n'(u) = \sum_{j=1}^{n-1} \frac{\sin u}{\tilde{\mu}_j - \cos u} + \frac{1}{2}\sum_{j=1}^{n} \frac{\sin u}{\tilde{\lambda}_j - \cos u} - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{2^{-k}\sin(2^{-k}u)}{\tilde{\lambda}_j + \cos(2^{-k}u)}. \tag{4.6}$$

For $\pi \le u < 2\pi$, all these terms are negative and so

$$F_n'(u) < 0, \quad \pi \le u < 2\pi.$$

For $0 < u \le \frac{2\pi}{3}$,

$$\begin{aligned} F_n'(u) \quad &> \quad \frac{1}{2}\sum_{j=1}^{n} \frac{\sin u}{\tilde{\lambda}_j - \cos u} - \frac{1}{2}\sum_{j=1}^{n} \frac{\sin(u/2)}{\tilde{\lambda}_j + \cos(u/2)} \\ &= \quad \frac{1}{2}\sum_{j=1}^{n} \frac{\tilde{\lambda}_j(\sin u - \sin(u/2)) + \sin(3u/2)}{(\tilde{\lambda}_j - \cos u)(\tilde{\lambda}_j + \cos(u/2))} \ge 0, \end{aligned}$$

since $\sin u \ge \sin(u/2)$.

Now by condition (4.4), for $0 < u \le \pi$,

$$F_n'(u) \le \frac{3n\sin u}{2(1-\cos u)} - \frac{1}{4}\sigma n \frac{\sin(u/2)}{K + \cos(u/2)} = nG(u),$$

say. Since $G(\pi) < 0$, we can choose $\alpha > 0$ with $G(u) < 0$ for $\pi - \alpha \le u \le \pi$. Thus for $n = 2, 3, \ldots,$

$$F_n'(u) < 0, \quad \pi - \alpha \le u \le \pi.$$

Finally we note that for $\frac{\pi}{2} \le u \le \pi$,

$$\begin{aligned} F_n''(u) &= \sum_{j=1}^{n-1} \frac{\tilde{\mu}_j \cos u - 1}{(\tilde{\mu}_j - \cos u)^2} + \frac{1}{2} \sum_{j=1}^{n} \frac{\tilde{\lambda}_j \cos u - 1}{(\tilde{\lambda}_j - \cos u)^2} \\ &\quad - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}_j \cos(2^{-k}u) + 1}{2^{2k}(\tilde{\lambda}_j + \cos(2^{-k}u))^2} < 0, \end{aligned} \tag{4.7}$$

since all terms are negative.

Thus we have shown that for $n = 2, 3, \ldots$, $F_n'(u) > 0$, $0 < u \le \frac{2\pi}{3}$, $F_n'(u) < 0$, $\pi - \alpha \le u < 2\pi$, and $F_n''(u) < 0$, $\frac{\pi}{2} \le u \le \pi$. Hence F_n has a unique local maximum in $(0, 2\pi)$ at α_n with $\frac{2\pi}{3} < \alpha_n < \pi - \alpha$. The result follows. □

It will be convenient to renormalise the wavelet ψ_n by defining

$$\Psi_n(x) = \frac{(-1)^{[n/2]}}{B_n f_n(\alpha_n)} \psi_n(x), \quad x \in \mathbb{R},$$

where B_n is as in (4.2), from which it follows that for even n,

$$\widehat{\Psi}_n(u) = \frac{f_n(u)}{f_n(\alpha_n)}, \tag{4.8}$$

while for odd n,

$$\widehat{\Psi}_n(u) = \frac{f_n(u)}{i f_n(\alpha_n)}.$$

We next define $\sigma_n > 0$ by

$$\sigma_n^2 = -F_n''(\alpha_n). \tag{4.9}$$

By (4.7) we see that

$$\sigma_n^2 > \sum_{j=1}^{n-1} \frac{1}{(\tilde{\mu}_j + 1)^2} + \frac{2}{3} \sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_j + 1)^2}.$$

So by condition (4.4) and Lemma 4.1,

$$\sigma_n \geq a\sqrt{n}, \quad n = 2, 3, \ldots, \tag{4.10}$$

for some $a > 0$. Also from (4.7),

$$\sigma_n < \sqrt{2n}, \quad n = 2, 3, \ldots. \tag{4.11}$$

Theorem 4.3 *If j_n denotes the number 1 or i as n is even or odd, then*

$$\begin{aligned} \lim_{n\to\infty} j_n \widehat{\Psi}_n(u/\sigma_n + \alpha_n) &= e^{-u^2/2}, \\ \lim_{n\to\infty} \frac{j_n}{\sigma_n} \widehat{\Psi}'_n(u/\sigma_n + \alpha_n) &= -ue^{-u^2/2}, \end{aligned}$$

where the convergence is locally uniform on $\mathbb{R}$.

Proof. We consider even n, the case for odd n differing only by a factor of i. Putting $\widetilde{f}_n(u) := \widehat{\Psi}_n(u/\sigma_n + \alpha_n)$, we see from (4.8), Lemma 4.2 and (4.9) that

$$\widetilde{f}_n(0) = 1, \; \widetilde{f}'_n(0) = 0, \; \widetilde{f}''_n(0) = -1.$$

We also have from (4.1),

$$\begin{aligned} \widetilde{f}_n(u) &= e^{i(2n-1)u/\sigma_n} \prod_{j=1}^{n-1} \frac{(e^{-iu/\sigma_n} - \mu_j e^{i\alpha_n})(e^{-iu/\sigma_n} - \mu_j^{-1} e^{i\alpha_n})}{(1 - \mu_j e^{i\alpha_n})(1 - \mu_j^{-1} e^{i\alpha_n})} \\ &\quad \times \prod_{j=1}^{n} \frac{e^{-iu/\sigma_n} - \lambda_j e^{i\alpha_n}}{1 - \lambda_j e^{i\alpha_n}} \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{e^{iu/2^k\sigma_n} + \lambda_j e^{i\alpha_n/2^k}}{1 + \lambda_j e^{i\alpha_n/2^k}}, \end{aligned}$$

which is of form (2.5), (2.4) in Theorem 2.1. In order to apply Theorem 2.1 we need that for $n = 2, 3, \ldots$, $-\mu_j e^{i\alpha_n}, -\mu_j^{-1} e^{i\alpha_n}$, $j = 1, \ldots, n-1$, $-\lambda_j e^{i\alpha_n}$, $j = 1, \ldots, n$, $\lambda_j e^{i\alpha n/2^k}$, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, all lie in D_γ for some γ in $[0, \frac{\pi}{2})$ and are bounded away from -1. From Lemma 4.2, $\pi - \alpha_n < \frac{\pi}{3}$ and $\frac{\alpha_n}{2^k} < \frac{\pi}{2} - \frac{\alpha}{2}$, $k = 1, 2, \ldots$, and hence all the above numbers lie in the sector $|\arg z| \leq \gamma$ for $\gamma = \max\{\frac{\pi}{3}, \frac{\pi}{2} - \frac{\alpha}{2}\}$. Since D_γ contains this sector, the condition is satisfied.

We see from (4.10) that $\sigma_n \to \infty$ as $n \to \infty$. Then Theorem 2.1 gives the desired result. $\square$

In Theorem 4.3 we consider convergence of $\widehat{\Psi}'_n$ because this will be needed later for convergence of the uncertainty products. In order to extend Theorem 4.3 to convergence in $L^P(\mathbb{R})$ we shall need to apply the Dominated Convergence Theorem. For this we shall need the further condition that P_n has a double root at -1, i.e. $P'_n(-1) = 0$.

Lemma 4.4 *Let*

$$\mu_\lambda(u) = \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k}u)}{\lambda + 1}.$$

Then for any $K > 1$, there is a constant $\rho < 1$ such that $\mu_\lambda(u) \le \rho\mu_\lambda(\pi)$, $\quad |\mu'_\lambda(u)| \le \rho\mu_\lambda(\pi)$, for $u \ge \frac{3\pi}{2}$ and $1 \le \lambda \le K$.

Proof. Fix $\lambda \ge 1$ and let $R(u) = \frac{\lambda + \cos u}{\lambda + 1}$. For $\pi \le u \le 2\pi$, $R(u/2^k) \le R(\pi/2^k)$, $k = 1, 2, \ldots,$ and so

$$\frac{\mu_\lambda(u)}{\mu_\lambda(\pi)} = \prod_{k=1}^{\infty} \frac{R(2^{-k}u)}{R(2^{-k}\pi)} \le \frac{R(u/2)}{R(\pi/2)} = \frac{\lambda + \cos(u/2)}{\lambda}.$$

Thus

$$\mu_\lambda(u) \le \mu_\lambda(\pi), \quad \pi \le u \le 2\pi, \tag{4.12}$$

and

$$\mu_\lambda(u) \le \left(1 - \frac{1}{\lambda\sqrt{2}}\right)\mu_\lambda(\pi), \quad \frac{3\pi}{2} \le u \le 2\pi.$$

For $2\pi \le u \le 3\pi$,

$$\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \le \frac{\lambda}{\lambda + 1}\mu_\lambda(\pi),$$

by (4.12). So for any $K > 1$ there is a constant $\rho < 1$ such that for $1 \le \lambda \le K$, $\frac{3\pi}{2} \le u \le 3\pi$,

$$\mu_\lambda(u) \le \rho\mu_\lambda(\pi). \tag{4.13}$$

Now for all $u \ge 3\pi$,

$$\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \le \mu_\lambda(u/2)$$

and successive application of this gives (4.13) for all $u \geq \frac{3\pi}{2}$.

Now $\mu'_\lambda(u) = -\sum_{k=1}^{\infty} 2^{-k}\nu_k(u)$, where

$$\nu_k(u) := \prod_{\substack{j=1 \\ j\neq k}}^{\infty} \frac{\lambda + \cos(2^{-j}u)}{\lambda+1} \cdot \frac{\sin(2^{-k}u)}{\lambda+1} \ .$$

For $k \geq 1$, $\pi \leq u \leq 2\pi$,

$$\frac{|\nu_k(u)|}{\mu_\lambda(\pi)} = \prod_{\substack{j=1 \\ j\neq k}}^{\infty} \frac{R(2^{-j}u)}{R(2^{-j}\pi)} \cdot \frac{\sin(2^{-k}u)}{\lambda + \cos(2^{-k}u)} \ \leq \ \frac{1}{\lambda} \ \leq \ 1.$$

For $\frac{3\pi}{2} \leq u \leq 2\pi$,

$$\frac{|\nu_1(u)|}{\mu_\lambda(\pi)} \leq \frac{\sin(3\pi/4)}{\lambda} \leq \frac{1}{\sqrt{2}} \ .$$

So

$$|\mu'_\lambda(u)| \leq \mu_\lambda(\pi), \quad \pi \leq u \leq \frac{3\pi}{2},$$
$$|\mu'_\lambda(u)| \leq \rho\mu_\lambda(\pi), \quad \frac{3\pi}{2} \leq u \leq 2\pi,$$

for a constant $\rho < 1$. Now since $\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2)$,

$$\begin{aligned} |\mu'_\lambda(u)| &= \left|\frac{1}{2}R'(u/2)\mu_\lambda(u/2) + \frac{1}{2}R(u/2)\mu'_\lambda(u/2)\right| \\ &\leq \frac{1}{4}\left|\mu_\lambda(u/2)\right| + \frac{1}{2}\left|\mu'_\lambda(u/2)\right|. \end{aligned}$$

So recalling (4.12),

$$|\mu'_\lambda(u)| \leq \frac{3}{4}\mu_\lambda(\pi), \quad u \geq 2\pi.$$

$\square$

Lemma 4.5 *If $P'_n(-1) = 0$, then for constants $A > 0$, $0 < \rho < 1$, $0 < \sigma \leq 1$, $u \geq \frac{3\pi}{2}$, $n = 2, 3, \ldots$,*

$$\begin{aligned} |\widehat{\phi}_n(u)| &\leq A\rho^{\sigma n}u^{-2}|\widehat{\phi}_n(\pi)|, \\ |\widehat{\phi}'_n(u)| &\leq An\rho^{\sigma n}u^{-2}|\widehat{\phi}_n(\pi)|. \end{aligned}$$

PROOF. We may write

$$P_n(e^{-iu}) = \prod_{j=1}^{(n-m)/2} \left(\frac{\tilde{\lambda}_j + \cos u}{\tilde{\lambda}_j + 1} \right) \cos^m \left(\frac{u}{2} \right),$$

for $m = 2$ or 3. We note that

$$\prod_{k=1}^{\infty} \cos\left(\frac{u}{2^{k+1}}\right) = \prod_{k=1}^{\infty} \frac{\sin(2^{-k}u)}{2\sin(2^{-k-1}u)} = \lim_{n\to\infty} \frac{\sin(u/2)}{2^n \sin(2^{-n-1}u)} = \frac{\sin(u/2)}{u/2}.$$

Recalling (2.2), Lemma 4.4 and condition (4.4) give the result. □

Lemma 4.6 *If $P_n'(-1) = 0$, then there is a constant $C > 0$ such that for $u \geq \sigma_n(3\pi/2 - \alpha_n)$, $n = 2, 3, \ldots$, $\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|$ and $\left|\widehat{\Psi}_n'\left(u/\sigma_n + \alpha_n\right)\right|$ are bounded by $C(1+u)^{-2}$.*

PROOF. For all u,

$$|Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})| \leq Q_n(1)P_n(1),$$

and so by Lemma 4.5, on recalling (4.1), for $u \geq \frac{3\pi}{2}$,

$$|\widehat{\psi}_n(u)| \leq A\rho^{\sigma n}u^{-2}|\widehat{\psi}_n(\pi)|.$$

Since $|\widehat{\Psi}_n(\pi)| \leq |\widehat{\Psi}_n(\alpha_n)| = 1$, for $u \geq \sigma_n(\frac{3\pi}{2} - \alpha_n)$,

$$\begin{aligned} \left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right| &\leq A\rho^{\sigma n}\left(u/\sigma_n + \alpha_n\right)^{-2} \\ &\leq C(1+u)^{-2}, \end{aligned}$$

for some $C > 0$, by (4.11).

Similarly we see that for a constant $B > 0$,

$$\left|\widehat{\Psi}_n'\left(u/\sigma_n + \alpha_n\right)\right| \leq Bn\rho^{\sigma n}\left(u/\sigma_n + \alpha_n\right)^{-2}$$

and the result follows. □

Remark

The above method of proof shows that if P_n has a root at $z = -1$ of multiplicity m, then Lemma 4.6 holds with $(1+u)^{-2}$ replaced by $(1+u)^{-m}$.

Lemma 4.7 *There are constants a, $b > 0$ such that for $-\sigma_n\alpha_n \le u \le \sigma_n(\frac{3\pi}{2} - \alpha_n)$, $n = 2, 3, \ldots$, $\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|$ and $\left|\frac{1}{\sigma_n}\Psi'_n\left(u/\sigma_n + \alpha_n\right)\right| \le ae^{-bu^2}$.*

PROOF. As in the proof of Lemma 4.2 we put $F_n(u) = \log f_n(u)$, $0 < u < 2\pi$. Recall from (4.7) that for $\frac{\pi}{2} \le u \le \pi$,

$$|F''_n(u)| = \sum_{j=1}^{n-1} \frac{1 - \tilde{\mu}_j \cos u}{(\tilde{\mu}_j - \cos u)^2} + \frac{1}{2}\sum_{j=1}^{n} \frac{1 - \tilde{\lambda}_j \cos u}{(\tilde{\lambda}_j - \cos u)^2} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1 + \tilde{\lambda}_j \cos(2^{-k}u)}{2^{2k}(\tilde{\lambda}_j + \cos(2^{-k}u))^2}. \tag{4.14}$$

We also recall condition (4.4) and note that for $\mu > K$, $\frac{\pi}{2} \le u < \pi$,

$$\frac{1 - \mu\cos u}{(\mu - \cos u)^2} < \frac{1 - K\cos u}{(K - \cos u)^2} \le \frac{1}{\sigma n}\sum_{\substack{j=1 \\ \tilde{\lambda}_j \le K}}^{n} \frac{1 - \tilde{\lambda}_j \cos u}{(\tilde{\lambda}_j - \cos u)^2}.$$

If $G_n(u)$ denotes the right-hand side of (4.14) with the summations taken only for $\tilde{\mu}_j \le K$, $\tilde{\lambda}_j \le K$, then for $\frac{\pi}{2} \le u \le \pi$,

$$|F''_n(u)| \le \left(1 + \frac{2}{\sigma}\right) G_n(u).$$

Since $|G_n(u)| \le |F''_n(u)|$, $\frac{\pi}{2} \le u \le \pi$, there is a constant $C > 0$ with

$$\frac{|F''_n(u)|}{|F''_n(v)|} \le \left(1 + \frac{2}{\sigma}\right)\frac{G_n(u)}{G_n(v)} \le C, \quad \frac{\pi}{2} \le u,\ v \le \pi,$$

since

$$(K+1)^{-2} \le \frac{1 + \lambda x}{(\lambda + x)^2} \le K + 1, \quad 0 \le x \le 1,\ 1 \le \lambda \le K.$$

As $F'_n(\alpha_n) = 0$,

$$F_n(u) \le F_n(\alpha_n) + \frac{1}{2}C^{-1}(u - \alpha_n)^2 F''_n(\alpha_n), \quad \frac{\pi}{2} \le u \le \pi.$$

Recalling that $\sigma_n^2 = -F_n''(\alpha_n)$ we have

$$f_n(u) \le f_n(\alpha_n) \exp\left(-\frac{1}{2}C^{-1}(u-\alpha_n)^2\sigma_n^2\right), \quad \frac{\pi}{2} \le u \le \pi,$$

and hence

$$\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right| \le e^{-\frac{1}{2}C^{-1}u^2}, \quad \sigma_n\left(\pi/2 - \alpha_n\right) \le u \le \sigma_n(\pi - \alpha_n). \tag{4.15}$$

Also for $\frac{\pi}{2} \le u \le \pi$,

$$|F_n'(u)| \le C|u-\alpha_n||F_n''(\alpha_n)|$$

and so

$$|f_n'(u)| = |f_n(u)||F_n'(u)| \le C\sigma_n^2|u-\alpha_n||f_n(u)|,$$

which gives

$$\left|\widehat{\Psi}_n'\left(u/\sigma_n + \alpha_n\right)\right| \le C|u|\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|, \quad \sigma_n\left(\frac{\pi}{2} - \alpha_n\right) \le u \le \sigma_n(\pi-\alpha_n) \tag{4.16}$$

By Lemma 4.2 we may choose α, $0 < \alpha < \frac{\pi}{6}$, with $\pi - \alpha_n > \alpha$, $n = 2, 3, \ldots$. Let $E(u) = \exp(-\alpha^2 u^2/8C\pi^2)$. For $\pi \le u \le \frac{3\pi}{2}$, $f_n(u) \le f_n(\pi)$ and so for $\sigma_n(\pi - \alpha_n) \le u \le \sigma_n(\frac{3\pi}{2} - \alpha_n)$, (4.15) gives

$$\begin{aligned} \widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right) &\le \exp\left(-\frac{1}{2}C^{-1}(\pi-\alpha_n)^2\sigma_n^2\right) \\ &= E(2\pi\sigma_n\alpha^{-1}(\pi-\alpha_n)) \\ &\le E(2\pi\sigma_n) \le E(u), \end{aligned} \tag{4.17}$$

since $\pi - \alpha_n \ge \alpha$ and $u \le \sigma_n(\frac{3\pi}{2} - \alpha_n) < 2\pi\sigma_n$.

Now from (4.6) and (4.10), there is a constant $A > 0$ such that for $\pi \le u \le \frac{3\pi}{2}$,

$$|F_n'(u)| \le a^2 A_n \le A\sigma_n^2.$$

Thus

$$|f_n'(u)| \le A\sigma_n^2 f_n(u), \quad \pi \le u \le \frac{3\pi}{2}$$

and so for $\sigma_n(\pi - \alpha_n) \le u \le \sigma_n(\frac{3\pi}{2} - \alpha_n)$,

$$\begin{aligned}\left|\widehat{\Psi}'_n\left(u/\sigma_n + \alpha_n\right)\right| &\le A\sigma_n^2\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right| \\ &\le A\alpha^{-2}u^2\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|, \end{aligned} \tag{4.18}$$

since $u \ge \sigma_n(\pi - \alpha_n) > \sigma_n\alpha$.

For $0 \le u \le \frac{\pi}{2}$, $f_n(u) \le f_n(\frac{\pi}{2})$ and so for $-\sigma_n\alpha_n \le u \le \sigma_n(\frac{\pi}{2} - \alpha_n)$, (4.15) gives

$$\begin{aligned}\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right| &\le \exp\left(-\frac{1}{2}C^{-1}\left(\frac{\pi}{2} - \alpha_n\right)^2\sigma_n^2\right) \\ &= E\left(2\pi\sigma_n\alpha^{-1}\left(\frac{\pi}{2} - \alpha_n\right)\right) \\ &\le E(2\pi\sigma_n) \le E(u), \end{aligned} \tag{4.19}$$

since $\alpha_n - \frac{\pi}{2} \ge \frac{\pi}{6} \ge \alpha$ and $|u| \le \sigma_n\alpha_n < 2\pi\sigma_n$.

Now we see from (4.6) that for $0 \le u \le \frac{\pi}{2}$,

$$|f'_n(u)| \le \left\{\sum_{j=1}^{n-1}\frac{\sin u}{\tilde{\mu}_j - \cos u} + \frac{1}{2}\sum_{j=1}^{n}\frac{\sin u}{\tilde{\lambda}_j - \cos u}\right\} f_n(u).$$

Noting that

$$\frac{\sin u}{\lambda}\prod_{k=1}^{\infty}\frac{\lambda + \cos(2^{-k}u)}{\lambda + 1}\left(\prod_{k=1}^{\infty}\frac{\lambda + \cos(2^{-k-1}\pi)}{\lambda + 1}\right)^{-1}$$

is bounded for $\lambda \ge 1$, $0 \le u \le \frac{\pi}{2}$, and recalling from Lemma 4.2 that f_n is increasing in $[0, \frac{\pi}{2}]$, we see that for a constant K,

$$|f'_n(u)| \le Knf_n\left(\pi/2\right), \quad 0 \le u \le \pi/2.$$

So by (4.10) there is a constant B such that for $-\sigma_n\alpha_n \le u \le \sigma_n(\frac{\pi}{2} - \alpha_n)$,

$$\left|\widehat{\Psi}'_n\left(u/\sigma_n + \alpha_n\right)\right| \le B\sigma_n^2\widehat{\Psi}_n\left(\pi/2\sigma_n + \alpha_n\right) \le B\frac{36}{\pi^2}u^2E(u), \tag{4.20}$$

as in (4.19) and since $|u| \ge \sigma_n(\alpha_n - \frac{\pi}{2}) \ge \sigma_n\pi/6$.

The result now follows from (4.15)–(4.20). □

Lemmas 4.6 and 4.7 allow us to apply the Dominated Convergence Theorem to prove the following:

Theorem 4.8 *Suppose $P_n'(-1) = 0$. If j_n denotes the number 1 or i as n is even or odd, then as $n \to \infty$,*

$$j_n \widehat{\Psi}_n\left(u/\sigma_n\right) - e^{-\frac{1}{2}(u-\sigma_n\alpha_n)^2} \to 0,$$

$$\frac{j_n}{\sigma_n}\widehat{\Psi}_n'(u/\sigma_n) + (u - \sigma_n\alpha_n)e^{-\frac{1}{2}(u-\sigma_n\alpha_n)^2} \to 0,$$

in $L^p(0, \infty)$, $1 \le p \le \infty$.

Proof. We need consider only even n. Let

$$h_n(u) = \begin{cases} \widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right), & u \ge -\sigma_n\alpha_n, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 4.3, $\lim_{n\to\infty} h_n(u) = e^{-u^2/2}$ and $\lim_{n\to\infty} h_n'(u) = -ue^{-u^2/2}$ locally uniformly. By Lemmas 4.5 and 4.7, h_n and h_n' are dominated by

$$h(u) = \max\{C(1 + |u|)^{-2}, ae^{-bu^2}\}, \quad u \in \mathbb{R}.$$

So by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} h_n(u) = e^{-u^2/2}, \quad \lim_{n\to\infty} h_n'(u) = -ue^{-u^2/2}, \tag{4.21}$$

in $L^p(\mathbb{R})$, $1 \le p < \infty$. Also since h, $e^{-u^2/2}$ and $ue^{-u^2/2}$ all tend to 0 as $u \to \pm\infty$, it follows that the convergence in (4.21) is also uniform. The result follows. $\square$

Since $\widehat{\Psi}_n(-u) = (-1)^n\widehat{\Psi}_n(u)$, $u \in \mathbb{R}$, we have a corresponding result on $(-\infty, 0)$. Adding these together and taking inverse Fourier transforms gives the following:

Theorem 4.9 *Suppose $P_n'(-1) = 0$. As $n \to \infty$, for even n,*

$$\sigma_n\Psi_n(\sigma_n x) - 2\cos(\sigma_n\alpha_n x)G(x) \to 0,$$

and for odd n,

$$\sigma_n\Psi_n(\sigma_n x) - 2\sin(\sigma_n\alpha_n x)G(x) \to 0,$$

where the convergence is in $L^q(\mathbb{R})$, $2 \le q \le \infty$.

Finally we show that the wavelets Ψ_n have asymptotically optimal time-frequency localization, as deseribed in Section 1 (see (1.13), (1.14)).

Theorem 4.10 *If* $P_n'(-1) = 0$, *then*

$$\lim_{n\to\infty} \Delta_{\Psi_n} \Delta^+_{\widehat{\Psi}_n} = \frac{1}{2}.$$

PROOF. Since $\Psi_n(-x) = (-1)^n \Psi_n(x)$, $x \in \mathbb{R}$, we see that μ_Ψ, as in (1.7), is 0 and so from (1.8),

$$\begin{aligned}
\Delta^2_{\Psi_n} &= \frac{\int_{-\infty}^{\infty} x^2 |\Psi_n(x)|^2 dx}{\int_{-\infty}^{\infty} |\Psi_n(x)|^2 dx} \\
&= \frac{\int_{-\infty}^{\infty} |\widehat{\Psi}_n'(u)|^2 du}{\int_{-\infty}^{\infty} |\widehat{\Psi}_n(u)|^2 du} \\
&= \frac{\int_0^{\infty} |\widehat{\Psi}_n'(u)|^2 du}{\int_0^{\infty} |\widehat{\Psi}_n(u)|^2 du} \ .
\end{aligned}$$

By Theorem 4.8, as $n \to \infty$,

$$\int_0^\infty \left|\widehat{\Psi}_n\left(u/\sigma_n\right)\right|^2 du - \int_0^\infty e^{-(u-\sigma_n\alpha_n)^2} du \to 0$$

and so

$$\sigma_n \int_0^\infty |\widehat{\Psi}_n(u)|^2 du \to \int_{-\infty}^\infty e^{-u^2} du, \tag{4.22}$$

since $\sigma_n \to \infty$. Also by Theorem 4.8, as $n \to \infty$,

$$\frac{1}{\sigma_n^2} \int_0^\infty \left|\widehat{\Psi}_n'\left(u/\sigma_n\right)\right|^2 du - \int_0^\infty (u - \sigma_n\alpha_n)^2 e^{-(u-\sigma_n\alpha_n)^2} du \to 0$$

and so

$$\frac{1}{\sigma_n} \int_0^\infty |\hat{\Psi}_n'(u)|^2 du \to \int_{-\infty}^\infty u^2 e^{-u^2} du.$$

Thus

$$\lim_{n\to\infty} \sigma_n^{-2} \Delta^2_{\Psi_n} = \frac{\int_{-\infty}^\infty u^2 e^{-u^2} du}{\int_{-\infty}^\infty e^{-u^2} du} = \frac{1}{2}. \tag{4.23}$$

By Theorem 4.3 and Lemmas 4.6 and 4.7, for $j = 1, 2$,

$$(u - \sigma_n \alpha_n)^j \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 - (u - \sigma_n \alpha_n)^j e^{-(u - \sigma_n \alpha_n)^2} \to 0 \text{ as } n \to \infty$$

in $L^1(0, \infty)$. Thus as $n \to \infty$,

$$\int_0^\infty (u - \sigma_n \alpha_n) \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 du \to 0, \tag{4.24}$$

$$\int_0^\infty (u - \sigma_n \alpha_n)^2 \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 du \to \int_{-\infty}^\infty u^2 e^{-u^2} du. \tag{4.25}$$

Now

$$\begin{aligned}
\int_0^\infty u \left| \widehat{\Psi}_n(u) \right|^2 du &= \frac{1}{\sigma_n^2} \int_0^\infty u \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 du \\
&= \frac{1}{\sigma_n^2} \int_0^\infty (u - \sigma_n \alpha_n) \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 du + \frac{\alpha_n}{\sigma_n} \int_0^\infty \left| \widehat{\Psi}_n\left(u/\sigma_n\right) \right|^2 du.
\end{aligned}$$

Thus from (1.13),

$$\mu^+_{\widehat{\Psi}_n} = \frac{\int_0^\infty (u - \sigma_n \alpha_n) |\widehat{\Psi}_n(u/\sigma_n)|^2 du}{\sigma_n \int_0^\infty |\widehat{\Psi}_n(u/\sigma_n)|^2 du} + \alpha_n$$

and hence from (4.22) and (4.24),

$$\lim_{n \to \infty} \sigma_n (\mu^+_{\widehat{\Psi}_n} - \alpha_n) = 0. \tag{4.26}$$

By (1.14),

$$\begin{aligned}
\sigma_n^2 \left(\Delta^+_{\widehat{\Psi}_n} \right)^2 &= \frac{\sigma_n^2 \int_0^\infty (u - \mu^+_{\widehat{\Psi}_n})^2 |\widehat{\Psi}_n(u)|^2 du}{\int_0^\infty |\widehat{\Psi}_n(u)|^2 du} \\
&= \frac{\int_0^\infty (u - \sigma_n \mu^+_{\widehat{\Psi}_n})^2 |\widehat{\Psi}_n(u/\sigma_n)|^2 du}{\int_0^\infty |\widehat{\Psi}_n(u/\sigma_n)|^2 du} \\
&\to \frac{\int_{-\infty}^\infty u^2 e^{-u^2} du}{\int_{-\infty}^\infty e^{-u^2} du} = \frac{1}{2},
\end{aligned} \tag{4.27}$$

as $n \to \infty$, by (4.26), (4.25) and (4.22). The result then follows from (4.23) and (4.27). $\square$

References

[1] J. M. Carnicer, T. N. T. Goodman and J. M. Peña, *A generalisation of the variation diminishing property,* Adv. Comp. Math. 3 (1995), 375–394.

[2] Louis H. Y. Chen, T. N. T. Goodman and S. L. Lee, *Asymptotic normality of refinable functions,* SIAM J. Math. Anal. 36 (2004), 323–346.

[3] C. K. Chui and J. Wang, *A study of asymptotically optimal time-frequency localization by scaling functions and wavelets,* Ann. Numer. Math. 4 (1997), 193–216.

[4] S. S. Goh and C. A. Micchelli, *Uncertainty principles in Hilbert spaces,* J. Fourier Anal. Appl. 8 (2002), 335–373.

[5] T. N. T. Goodman and C. A. Micchelli, *On refinement equations determined by Pólya frequency sequences,* SIAM J. Math. Anal. 23 (1992), 766–784.

[6] M. Unser, A. Aldroubi and M. Eden, *On the asymptotic convergence of B-spline wavelets to Gabor functions,* IEEE Trans. Inform. Theory 38 (1992), 864–871.

Interpolation by Polynomials and Transcendental Entire Functions

N. K. Govil
Department of Mathematics
Auburn University
Auburn, AL 36849, U.S.A.
Email address: govilnk@auburn.edu

M. A. Qazi
Department of Mathematics
Tuskegee University
Tuskegee, AL 36088, U.S.A.
Email address: qazima@aol.com

Q. I. Rahman
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, Canada H3G 3J7
Email address: rahmanqi@dms.umontreal.ca

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

This paper deals with some basic facts about interpolation by classes of entire functions like polynomials, trigonometric polynomials (= 2π-periodic entire functions of exponential type), and non-periodic transcendental entire functions. The results discussed here are closely related to the work of the late Professor Ambikeshwar Sharma on interpolation. Some,like the uniqueness theorems for $(0\,,\,m)$ interpolation by entire functions of exponential type, were inspired by his work. Certain details presented in our discussion of Hermite interpolation may be new and so of some special interest. We also explain what Hermite really did in his often quoted paper.

1 Preliminaries about Entire Functions

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be entire if it is analytic throughout the complex plane. Polynomials $\sum_{\nu=0}^{n} a_\nu z^\nu$ are entire functions. Entire functions which are not polynomials are called *transcendental*. The exponential function

$$e^z = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} z^\nu$$

is a transcendental entire function and so are $\cos z$ and $\sin z$. In fact, $t(z) := \sum_{\nu=-n}^{n} c_\nu e^{i\nu z}$, $c_\nu \in \mathbb{C}$ is an entire function. Since any trigonometric polynomial t can be written in the form $t(x) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu x}$ for some n, we see that a trigonometric polynomial is the "restriction of an entire function to the real axis."

It is useful to have a notation for the class of all polynomials of degree at most n. We shall use $\mathcal{P}_n$ to denote it, that is

$$\mathcal{P}_n := \left\{ p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu : a_\nu \in \mathbb{C} \text{ for } \nu = 0, 1, \ldots, n \right\}.$$

Thus $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for $n = 0, 1, 2, \ldots$.

For any entire function f set $M(r) = M_f(r) := \max_{|z| \le r} |f(z)|$. Clearly, $M(r)$ is a non-decreasing function of r. An entire function f is a polynomial if and only if there exist $C \ge 0$ and a non-negative integer m, depending on f, such that

$$\log M_f(r_k) \le m \log r_k + C$$

for an increasing sequence $r_1, r_2, r_3, \cdots \to \infty$. This inequality with $m = n$ characterizes the class $\mathcal{P}_n$.

An entire function f is transcendental if and only if

$$\lim_{r\to\infty} \frac{\log M_f(r)}{\log r} \to \infty\,.$$

The order of an entire function f is defined to be

$$\rho := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}\,.$$

All polynomials are of order 0, but not all entire functions of order 0 are polynomials. The order of an entire function can be ∞. As an example we mention $f(z) := \mathrm{e}^{\mathrm{e}^z}$. The order of an entire function can be calculated from the coefficients in its Maclaurin series development. If $f(z) := \sum_{n=0}^{\infty} a_n z^n$, then

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} \qquad (0 \le \rho \le \infty)\,.$$

This formula for ρ shows that $f(z) := 1 + \sum_{n=1}^{\infty} n^{-n/\rho}\, z^n$ is an entire function of order ρ $(0 < \rho < \infty)$.

The type of an entire function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ of finite positive order ρ is defined to be

$$T := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho} = \limsup_{n\to\infty} \frac{1}{\rho \mathrm{e}} n |a_n|^{\rho/n}\,.$$

For $0 < \rho < \infty$ and $0 \le \tau \le \infty$ an entire function f is said to be of growth $(\rho\,,\,\tau)$ if it is of order not exceeding ρ and of type not exceeding τ if of order ρ. Let $C_{\rho,\tau}$ denote the class of all entire functions of growth $(\rho\,,\,\tau)$. Then

$$C_{\rho_1,\tau_1} \subset C_{\rho_2,\tau_2}$$

if $\rho_1 < \rho_2$ whatever τ_1 and τ_2 may be, and also if $\rho_1 = \rho_2$ provided that $\tau_1 < \tau_2$. Functions of order 0 belong to $C_{\rho,\tau}$ for any $\rho \in (0,\infty)$ and any $\tau \ge 0$.

Functions of growth $(1,\tau)$, $\tau < \infty$, are said to be of exponential type. This is because f is of growth $(1,\tau)$, $\tau < \infty$ if and only if for *any given* $\varepsilon > 0$ there exists a constant $k = k_\varepsilon$, depending on ε, such that

$$|f(z)| \le k\, \mathrm{e}^{(\tau+\varepsilon)|z|} \qquad (z \in \mathbb{C})\,.$$

Note that any entire function of order less than 1 is of exponential type τ, and so is any entire function of order 1 type T not exceeding τ. A trigonometric polynomial $t(x) := \sum_{\nu=-n}^{n} c_\nu \mathrm{e}^{\mathrm{i}\nu x}$ is an entire function of exponential type n when x is allowed to vary in the complex plane. Trigonometric polynomials are, of course, 2π-periodic. It is known [9, p. 109 (Theorem 6.10.1)] that any entire function f of exponential type τ that is 2π-periodic is necessarily a trigonometric polynomial of degree at most $n := \lfloor \tau \rfloor$. However, not every entire function of exponential type n is a trigonometric polynomial. For example, the function $f(z) := 1 + \sum_{k=1}^{\infty} k^{-k/\rho} z^k$, $0 < \rho < 1$, is of exponential type n for any $n \geq 0$ but is not 2π-periodic. Another example is $f(z) := (\sin nz)/z$, $n \geq 1$.

2 Lagrange Interpolation by Polynomials and Trigonometric Polynomials

For any given set of n distinct points $z_1, \ldots, z_n$ in the complex plane and n values $w_1, \ldots, w_n$ in $\mathbb{C}$, distinct or not, there exists a polynomial $p \in \mathcal{P}_{n-1}$ such that

$$p(z_\nu) = w_\nu \qquad (\nu = 1, \ldots, n). \tag{2.1}$$

This well-known result on polynomial interpolation is covered by Cramer's rule which may be stated as follows.

Lemma 2.1 *Let the $n \times n$ matrix $A = (a_{ij})$ be non-singular, that is its determinant $|A|$ is different from zero. Then the system of linear equations*

$$\left.\begin{array}{ccccccccc} a_{11}\,\zeta_1 & + & a_{12}\,\zeta_2 & + & \cdots & + & a_{1n}\,\zeta_n & = & w_1 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ a_{n1}\,\zeta_1 & + & a_{n2}\,\zeta_2 & + & \cdots & + & a_{nn}\,\zeta_n & = & w_n \end{array}\right\}, \tag{2.2}$$

in the unknowns $\zeta_1, \ldots, \zeta_n$, possesses a unique solution. The solution is given by

$$\zeta_\nu = \frac{|A^{(\nu)}|}{|A|} \qquad (\nu = 1, \ldots, n), \tag{2.3}$$

where $A^{(\nu)}$ is the matrix obtained when the νth column of A, is replaced by the vector $\overrightarrow{w} = (w_1, \ldots, w_n)^T$. □

Existence of a polynomial $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ of degree at most $n-1$ satisfying (2.1) is the same as the system of n equations

$$\left.\begin{array}{ccccccccc} 1 \cdot a_0 & + & z_1 \cdot a_1 & + & \cdots & + & z_1^{n-1} \cdot a_{n-1} & = & w_1 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 1 \cdot a_0 & + & z_n \cdot a_1 & + & \cdots & + & z_n^{n-1} \cdot a_{n-1} & = & w_n \end{array}\right\}, \qquad (2.1')$$

in the unknowns $a_0, a_1, \ldots, a_{n-1}$, having a solution. By Lemma 2.1, applied with $a_{jk} = z_j^{k-1}$ and $\zeta_\nu = a_{\nu-1}$, this system has a unique solution if the determinant

$$V(z_1, \ldots, z_{n-1}) := \begin{vmatrix} 1 & z_1 & \cdots & z_1^\nu & \cdots & z_1^{n-1} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & z_n & \cdots & z_n^\nu & \cdots & z_n^{n-1} \end{vmatrix}$$

is different from zero provided that $z_1, \ldots, z_n$ are all distinct. This is indeed the case since

$$V(z_1, \ldots, z_n) = \prod_{1 \le j < k \le n} (z_k - z_j) \, .$$

In order to determine the polynomial p of degree at most $n-1$ having the interpolation property (2.1) we could always use (2.3) but it is shorter to recognize that p must be of the form

$$p(z) = \sum_{\nu=1}^{n} w_\nu \, l_\nu(z),$$

where l_ν is the polynomial of degree $n-1$ which satisfies

$$l_\nu(z_\mu) = \begin{cases} 1 & \text{if} \quad \mu = \nu, \\ 0 & \text{if} \quad \mu \ne \nu. \end{cases} \qquad (2.4)$$

Setting $\psi(z) := \prod_{\mu=1}^{n} (z - z_\mu)$, we readily see that

$$l_\nu(z) = \frac{1}{\psi'(z_\nu)} \, \frac{\psi(z)}{z - z_\nu} \qquad (\nu = 1, \ldots, n).$$

Thus
$$p(z) = \sum_{\nu=1}^{n} \frac{w_\nu}{\psi'(z_\nu)} \frac{\psi(z)}{z-z_\nu} = \sum_{\nu=1}^{n} \frac{p(z_\nu)}{\psi'(z_\nu)} \frac{\psi(z)}{z-z_\nu}\,. \tag{2.5}$$

For an application of Lemma 2.1 it is important to decide whether the determinant $|A|$ of the matrix $(a_{j\,k})$ is different from zero or not. This could be hard if we always had to calculate $|A|$. The following lemma can often be used to avoid this.

Lemma 2.2 ([30, p. 27]). *A necessary and sufficient condition for the existence of numbers $\zeta_1, \ldots, \zeta_n$, not all zero, satisfying the system of equations*

$$\left.\begin{array}{ccccccccc} a_{11}\,\zeta_1 & + & a_{12}\,\zeta_2 & + & \cdots & + & a_{1n}\,\zeta_n & = & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ a_{n1}\,\zeta_1 & + & a_{n2}\,\zeta_2 & + & \cdots & + & a_{nn}\,\zeta_n & = & 0 \end{array}\right\}, \tag{2.6}$$

is that the determinant $|A| = |a_{j\,k}|_{n\times n}$ be zero. In other words, $|A| \neq 0$ if and only if the homogeneous system (2.6) *is satisfied only for $\zeta_1 = \cdots = \zeta_n = 0$.*

Since a polynomial $p(z) := \sum_{\nu=0}^{n-1} a_\nu\, z^\nu$ of degree at most $n-1$ cannot vanish at n different points $z_1, \ldots, z_n$ without being identically zero, we may apply Lemma 2.2 with $\zeta_\nu = a_{\nu-1}$ for $\nu = 1, \ldots, n$ to conclude that the system $(2.1')$ in the unknowns $a_0, a_1, \ldots, a_{n-1}$ has a solution, without knowing the value of $V(z_1, \ldots, z_n)$.

For any $\zeta \neq 0$ and any $\alpha \in \mathbb{R}$, the equation $e^{\mathrm{i}z} = \zeta$ has one and only one solution in the strip

$$S_\alpha := \{z \in \mathbb{C} : \alpha \leq \Re\, z < \alpha + 2\pi\}\,.$$

Hence, a trigonometric polynomial $t(z) := \sum_{\nu=-n+1}^{n-1} c_\nu\, e^{\mathrm{i}\nu z}$ of degree not exceeding $n-1$ cannot vanish at more than $2n-2$ points in S_α unless $c_{-n+1}, \ldots, c_0, \ldots, c_{n-1}$ are all zero. Applying Lemma 2.2 with $2n-1$ instead of n and $c_{-n+1}, \ldots, c_0, \ldots, c_{n-1}$ as the unknowns we conclude that *for any set of $2n-1$ distinct points $z_1, \ldots, z_{2n-1}$ in the strip S_α and $2n-1$ values $w_1, \ldots, w_{2n-1}$ in $\mathbb{C}$, distinct or* *not, there exists a trigonometric polynomial t of degree at most $n-1$ such that*

$$t(z_\nu) = w_\nu \qquad (\nu = 1, \ldots, 2n-1)\,. \tag{2.7}$$

Here we could have also calculated the underlying determinant and seen that it was not zero.

As usual, let

$$T_k(x) := \frac{k}{2}\sum_{j=0}^{\lfloor k/2 \rfloor}(-1)^j\,\frac{(k-j-1)!}{j!\,(-2j)!}(2x)^{k-2j} = \cos\left(k \arccos x\right)$$

be the Chebyshev polynomial of the first kind of degree k. Then, $\cos k\theta = T_k(\cos\theta)$. Hence, if C is an *even* trigonometric polynomial of degree at most n, that is if $C(\theta) := \sum_{k=0}^n c_k \cos k\theta$, then

$$C(\theta) = \sum_{k=0}^{n} c_k T_k(\cos\theta) = p(\cos\theta)\,,$$

where $p(x) := \sum_{\nu=0}^n a_\nu x^\nu = \sum_{k=0}^n c_k T_k(x)$ is a polynomial of degree at most n. Now note that $\cos\theta$ decreases from 1 to -1 monotonically as θ increases from 0 to π. Hence, to any point $x_\nu \in [-1,1]$ there corresponds one and only one point θ_ν in $[-\pi,\pi]$ such that $x_\nu = \cos\theta_\nu$. Since p cannot have more than n zeros in $[-1,1]$ without being identically zero, we conclude that C cannot have more than n distinct zeros in $[0,\pi]$ unless it is identically zero. Hence, Lemma 2.2 implies that *for any set of $n+1$ distinct points $\theta_0, \theta_1, \ldots, \theta_n$* in $[0,\pi]$, *and any numbers $w_0, w_1, \ldots, w_n$, there exists a unique cosine polynomial of degree at most n satisfying $C(\theta_\nu) = w_\nu$ for $\nu = 0, 1, \ldots, n$.*

Lemma 2.2 also implies that *given any distinct points $\theta_1, \ldots, \theta_n$ in the open interval $(0, 2\pi)$ and any n numbers $w_1, \ldots, w_n$, there is a unique sine polynomial $S(\theta)$ of degree at most n satisfying $S(\theta_\nu) = w_\nu$ for $\nu = 1, \ldots, n$.*

Lemma 2.2 becomes almost indispensable as we move on to another kind of interpolation.

3 Hermite Interpolation by Polynomials and Trigonometric Polynomials

Let $\alpha_1, \ldots, \alpha_n$ be any set of n positive integers. The problem is to find the smallest N such that, for any set of n distinct points $z_1, \ldots, z_n$ in the complex plane and $\alpha_1 + \cdots + \alpha_n$ arbitrarily prescribed values

$$w_{0,\nu}, \ldots, w_{\alpha_\nu - 1,\nu} \qquad (\nu = 1, \ldots, n),$$

there exists a polynomial $p(z) := \sum_{\nu=0}^{N} a_\nu z^\nu$ of degree not exceeding N satisfying

$$\left.\begin{array}{ll} p^{(j)}(z_1) = w_{j,1} & (j = 0, \ldots, \alpha_1 - 1) \\ \quad\cdot & \\ \quad\cdot & \\ \quad\cdot & \\ p^{(j)}(z_n) = w_{j,n} & (j = 0, \ldots, \alpha_n - 1) \end{array}\right\} . \tag{3.1}$$

This is a system of $\alpha_1 + \cdots + \alpha_n$ equations in $N + 1$ unknowns $a_0, \ldots, a_N$. In order to apply Lemma 2.1 we should take $N = \alpha_1 + \cdots + \alpha_n - 1$, and then see if the determinant $|A|$ of the corresponding matrix is different from zero. It would be a formidable job to calculate the value of the determinant in this case, but here Lemma 2.2 comes in handy. We only need to show that if $p \in \mathcal{P}_N$, $N = \alpha_1 + \cdots + \alpha_n - 1$, and

$$p^{(j)}(z_\nu) = 0 \qquad (j = 0, \ldots, \alpha_\nu - 1;\ \nu = 1, \ldots, n), \tag{3.2}$$

then p must be identically zero. This is trivial since (3.2) implies that p has zeros of multiplicities $\alpha_1, \ldots, \alpha_n$ at $z_1, \ldots, z_n$, respectively. Thus, counting each zero as many times as its multiplicity, we see that p has at least $\alpha_1 + \cdots + \alpha_n = N + 1$ zeros, and so must be identically zero. Hence, by Lemma 2.2, the *determinant* of the matrix corresponding to the system (3.1) of $N = \alpha_1 + \cdots + \alpha_n$ equations in $N + 1$ unknowns (the coefficients of the interpolating polynomial p of degree at most N) is different from zero, and so by Lemma 2.1 the system has a unique solution.

In order to find a formula for the polynomial p of degree at most $N := \alpha_1 + \cdots + \alpha_n - 1$ satisfying (3.1), let us find for any given $\mu \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, \alpha_\mu - 1\}$ the "fundamental polynomial" $l_{k,\mu}$ of degree N such that

$$l_{k,\mu}^{(j)}(z_\nu) = \begin{cases} 1 & \text{if} \quad \nu = \mu \ \text{ and } \ j = k, \\ 0 & \text{if} \quad \nu = \mu \ \text{ and } \ j \in \{0, \ldots, \alpha_\nu - 1\} \setminus \{k\}, \\ 0 & \text{if} \quad \nu \neq \mu \ \text{ and } \ j = 0, \ldots, \alpha_\nu - 1. \end{cases}$$

For this set $\psi(z) := \prod_{\nu=1}^{n} (z - z_\nu)^{\alpha_\nu}$. Then clearly

$$l_{k,\mu}(z) = \frac{\psi(z)}{(z - z_\mu)^{\alpha_\mu}} \, \varphi(z),$$

where φ is a polynomial of degree $\alpha_\mu - 1$. Developing φ in powers of $z - z_\mu$ and using the fact that $l_{k,\mu}^{(j)}(z_\mu) = 0$ for $j = 0, \ldots, k - 1$, we may write

$$\varphi(z) = \lambda_k \, (z - z_\mu)^k + \cdots + \lambda_{\alpha_\mu - 1} \, (z - z_\mu)^{\alpha_\mu - 1},$$

so that

$$\begin{aligned} l_{k,\mu}(z) &= \frac{\psi(z)}{(z - z_\mu)^{\alpha_\mu}} \left\{ \lambda_k \, (z - z_\mu)^k + \cdots + \lambda_{\alpha_\mu - 1} \, (z - z_\mu)^{\alpha_\mu - 1} \right\} \\ &= b_k (z - z_\mu)^k + b_{k+1} (z - z_\mu)^{k+1} + \cdots . \end{aligned}$$

Since

$$l_{k,\mu}^{(j)}(z_\mu) = \begin{cases} 1 & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \in \{0, \ldots, \alpha_\mu - 1\} \setminus \{k\}, \end{cases}$$

we must have

$$b_k = \frac{1}{k!} \quad \text{and} \quad b_\ell = 0 \quad \text{for} \quad \ell \in \{k, \ldots, \alpha_\mu - 1\} \setminus \{k\}.$$

(Note that the set of values of ℓ to which the statement "$b_\ell = 0$" applies is empty in the case where $k = \alpha_\mu - 1$). Thus

$$\begin{aligned} &\lambda_k + \cdots + \lambda_{\alpha_\mu - 1} (z - z_\mu)^{\alpha_\mu - k - 1} = \\ &\qquad\qquad \left\{ \frac{1}{k!} + b_{\alpha_\mu} (z - z_\mu)^{\alpha_\mu - k} + \cdots \right\} \frac{(z - z_\mu)^{\alpha_\mu}}{\psi(z)} . \end{aligned} \tag{3.3}$$

The function $(z-z_\mu)^{\alpha_\mu}/\psi(z)$ is holomorphic in the largest disk $|z-z_\mu|<\rho$ that does not contain any of the other z_ν's and so is defined therein by a power series, say $c_0+c_1(z-z_\mu)+c_2(z-z_\mu)^2+\cdots$, where

$$c_\ell := \frac{1}{\ell!}\left[\frac{d^\ell}{dz^\ell}\left\{\frac{(z-z_\mu)^{\alpha_\mu}}{\psi(z)}\right\}\right]_{z=z_\mu} \qquad (\ell=0,1,2,\ldots)\,.$$

Since the left-hand side of (3.3) is a polynomial of degree $\alpha_\mu-k-1$, we must have

$$\begin{aligned}\lambda_k + \cdots + \lambda_{\alpha_\mu-1}&(z-z_\mu)^{\alpha_\mu-k-1}\\ &= \frac{1}{k!}\left\{c_0+c_1(z-z_\mu)+\cdots+c_{\alpha_\mu-k-1}(z-z_\mu)^{\alpha_\mu-k-1}\right\}\\ &= \frac{1}{k!}\sum_{\ell=0}^{\alpha_\mu-k-1}\frac{1}{\ell!}\left[\frac{d^\ell}{dz^\ell}\left\{\frac{(z-z_\mu)^{\alpha_\mu}}{\psi(z)}\right\}\right]_{z=z_\mu}(z-z_\mu)^\ell\,.\end{aligned}$$

Hence, for any $\mu\in\{1,\ldots,n\}$ and $k=0,\ldots,\alpha_\mu-1$, the fundamental polynomial $l_{k,\mu}$ is represented by

$$\begin{aligned}l_{k,\mu}(z) \;=\; &\psi(z)(z-z_\mu)^{k-\alpha_\mu}\times\\ &\frac{1}{k!}\sum_{\ell=0}^{\alpha_\mu-k-1}\frac{1}{\ell!}\left[\frac{d^\ell}{dz^\ell}\left\{\frac{(z-z_\mu)^{\alpha_\mu}}{\psi(z)}\right\}\right]_{z=z_\mu}(z-z_\mu)^\ell\,.\end{aligned}\tag{3.4}$$

Hence, the following result holds.

Theorem 3.1 *Let $z_1,\ldots,z_n$ be n distinct points, and $\alpha_1,\ldots,\alpha_n$ positive integers. Set $\psi(z):=\prod_{\mu=1}^n(z-z_\mu)^{\alpha_\mu}$ and for $\mu\in\{1,\ldots,n\}$ and $k=0,\ldots,\alpha_\mu-1$ define $l_{k,\mu}(z)$ by* (3.4)*. Then the unique polynomial p of degree at most $\alpha_1+\cdots+\alpha_n-1$ satisfying the interpolation conditions* (3.1) *is given by*

$$p(z)=\sum_{k=0}^{\alpha_1-1}w_{k,1}\,l_{k,1}(z)+\sum_{k=0}^{\alpha_2-1}w_{k,2}\,l_{k,2}(z)+\cdots+\sum_{k=0}^{\alpha_n-1}w_{k,n}\,l_{k,n}(z)\,. \tag{3.5}$$

$\square$

NOTE. A reader familiar with the well-known book of P. J. Davis [11] might wonder why our formula for $l_{k,\mu}$ does not agree with the one appearing in that book on page 37 (see (2.5.25)). We believe that the latter contains a misprint and is not correct as stated.

The study of "Hermite interpolation" was initiated by Ch. Hermite in a paper [23] entitled "Sur la formule d'interpolation de Lagrange." The term "Hermite interpolation" was of course not coined by him. By the way, he was not interested in a formula like (3.5). Then, what exactly was his motivation? He considered a function f holomorphic in a nonempty simply-connected open set D and took n distinct points $z_1, \ldots, z_n$ in it; taking in addition n positive integers $\alpha_1, \ldots, \alpha_n$ he proposed to himself (in his own words: "Je me suis proposé ") to find a polynomial p of degree not exceeding $\alpha_1+\cdots+\alpha_n-1$ satisfying the conditions:

$$\left.\begin{array}{ll} p^{(j)}(z_1) = f^{(j)}(z_1) & (j=0,\ldots,\alpha_1-1) \\ \quad\vdots & \\ p^{(j)}(z_n) = f^{(j)}(z_n) & (j=0,\ldots,\alpha_n-1) \end{array}\right\} . \tag{3.6}$$

He goes on to say: "En supposant $\alpha_1 + \cdots + \alpha_n = n$ *la question comme on voit est déterminée.*" Thus, it was known to him when he wrote the paper that a polynomial p satisfying (3.1), where the values $w_{0,\nu}, \ldots, w_{\alpha_\nu-1,\nu}$ $(\nu = 1, \ldots, n)$ are completely arbitrary, does exist if its degree is allowed to be as large as $\alpha_1 + \cdots + \alpha_n - 1$. He was simply looking for an integral representation for $p(z)$. His result may be stated as follows.

Theorem 3.2 *Let f be holomorphic in a region (open simply–connected set D and let $z_1, \cdots, z_n$ be any set of n distinct points all lying in D. In addition, let $\alpha_1, \ldots, \alpha_n$ be arbitrary integers, all positive, and set $\psi(\zeta) := \prod_{\nu=1}^{n}(\zeta - z_\nu)^{\alpha_\nu}$. Take any rectifiable Jordan curve Γ which together with its "inside," the bounded component of $\mathbb{C}\setminus\Gamma$, is contained in D and contains the points $z_1, \ldots, z_n$. Then, the polynomial p satisfying* (3.6) *is given by the formula*

$$p(z)=f(z)+\frac{1}{2\pi \mathrm{i}}\,\psi(z)\int_\Gamma \frac{f(\zeta)}{(z-\zeta)\psi(\zeta)}\,\mathrm{d}\zeta \quad \text{for all } z \text{ inside } \Gamma\,. \tag{3.7}$$

We shall discuss the motivation behind this theorem. The proof as explained by Hermite in his paper is not particularly easy to grasp. Davis [11, pp. 67–68] proves the result in the case where $\alpha_1 = \cdots = \alpha_n = 1$, and gives a "brief indication" (see the proof of Corollary 3.6.3 on page 68) for the general case. There is no easily available book, if there exists any, that contains a comprehensive proof of the result in its full generality. So, we find it desirable to make the necessary details available to the reader.

Proof of Theorem 3.2. It is well known (see for example [1, p. 121]) that if $\varphi(z)$ is continuous on an arc γ then the function $\displaystyle\int_\gamma \frac{\varphi(\zeta)}{z-\zeta}\,\mathrm{d}\zeta$ is analytic in each of the regions "determined" by γ. We may apply this with $\varphi(\zeta) := \dfrac{f(\zeta)}{\psi(\zeta)}$ to conclude that $\displaystyle\int_\Gamma \frac{f(\zeta)}{(z-\zeta)\psi(\zeta)}\,\mathrm{d}\zeta$ defines a function of z that is analytic in D. Thus

$$I(z) := \frac{1}{2\pi\mathrm{i}}\;\psi(z)\int_\Gamma \frac{f(\zeta)}{(z-\zeta)\psi(\zeta)}\,\mathrm{d}\zeta$$

is analytic inside Γ and for any $\nu \in \{1, \ldots, n\}$,

$$I^{(j)}(z_\nu) = 0 \quad \text{for} \quad j = 0, \ldots, \alpha_\nu - 1. \tag{3.8}$$

Now, we only need to show that the expression on the right-hand side of (3.7) is a polynomial whose degree does not exceed $\alpha_1 + \cdots + \alpha_n - 1$. Applying the residue theorem we see that

$$I(z) := \frac{\psi(z)}{2\pi\mathrm{i}}\int_\Gamma \frac{f(\zeta)}{(z-\zeta)\psi(\zeta)}\,\mathrm{d}\zeta \;=\; -f(z) + \psi(z)\sum_{\nu=1}^{n} R_\nu(z)\,, \tag{3.9}$$

where $R_\nu(z)$ is the residue of the meromorphic function $F(z) := \dfrac{f(\zeta)}{(z-\zeta)\,\psi(\zeta)}$ at its pole z_ν. Let $z \neq z_\nu$. Since the multiplicity of the pole at z_ν does not exceed α_ν, the Laurent development of F around z_ν has the form

$$F(\zeta) = \sum_{k=0}^{\alpha_\nu} \frac{a_{-\alpha_\nu+k,\nu}}{(\zeta - z_\nu)^{\alpha_\nu - k}} + a_{1,\nu}(\zeta - z_\nu) + a_{2,\nu}(\zeta - z_\nu)^2 + \cdots\,,$$

and so

$$R_\nu(z) = a_{-1,\nu} = a_{-1,\nu}(z) = \frac{1}{(\alpha_\nu - 1)!} \left[D^{\alpha_\nu - 1} \left\{ (\zeta - z_\nu)^{\alpha_\nu} F(\zeta) \right\} \right]_{\zeta = z_\nu} .$$

Writing $(\zeta - z_\nu)^{\alpha_\nu} F(\zeta)$ as the product of $\dfrac{1}{z-\zeta}$ and $\dfrac{f(\zeta)\,(\zeta - z_\nu)^{\alpha_\nu}}{\psi(\zeta)}$, both analytic in the immediate neighbourhood of the point z_ν, we see that

$$\begin{aligned}
&\left[D^{\alpha_\nu - 1} \left\{ (\zeta - z_\nu)^{\alpha_\nu} F(\zeta) \right\} \right]_{\zeta = z_\nu} \\
&\qquad = \left[\sum_{j=0}^{\alpha_\nu - 1} \binom{\alpha_\nu - 1}{j} \frac{(\alpha_\nu - j - 1)!}{(z-\zeta)^{\alpha_\nu - j}} \, D^j \left\{ \frac{f(\zeta)(\zeta - z_\nu)^{\alpha_\nu}}{\psi(\zeta)} \right\} \right]_{\zeta = z_\nu} \\
&\qquad = \sum_{j=0}^{\alpha_\nu - 1} \frac{(\alpha_\nu - 1)!}{j!} \, b_{j,\nu} \, \frac{1}{(z - z_\nu)^{\alpha_\nu - j}} ,
\end{aligned}$$

with $b_{j,\nu} := \left[D^j \left\{ \dfrac{f(\zeta)(\zeta - z_\nu)^{\alpha_\nu}}{\psi(\zeta)} \right\} \right]_{\zeta = z_\nu} \in \mathbb{C}$. Thus

$$\begin{aligned}
\psi(z) R_\nu(z) &= \sum_{j=0}^{\alpha_\nu - 1} \frac{1}{j!} \, b_{j,\nu} \frac{\psi(z)}{(z - z_\nu)^{\alpha_\nu - j}} \\
&= \left\{ \sum_{j=0}^{\alpha_\nu - 1} \frac{1}{j!} \, b_{j,\nu} \, (z - z_\nu)^j \right\} \frac{\psi(z)}{(z - z_\nu)^{\alpha_\nu}} ,
\end{aligned}$$

which shows that $\psi(z) R_\nu(z)$ is a polynomial whose degree cannot be larger than $(\alpha_\nu - 1) + (\alpha_1 + \cdots + \alpha_n) - \alpha_\nu = \alpha_1 + \cdots + \alpha_n - 1$. Since this is true for $\nu = 1, \ldots, n$ we conclude that $\psi(z) \sum_{\nu=1}^{n} R_\nu(z)$ is a polynomial of degree at most $\alpha_1 + \cdots + \alpha_n - 1$. In view of (3.8) and (3.9) the polynomial

$$\begin{aligned}
p(z) := \psi(z) \sum_{\nu=1}^{n} R_\nu(z) &= f(z) + I(z) \\
&= f(z) + \frac{1}{2\pi \mathrm{i}} \, \psi(z) \int_\Gamma \frac{f(\zeta)}{(z-\zeta)\,\psi(\zeta)} \, \mathrm{d}\zeta
\end{aligned}$$

has the desired interpolating property (3.6). □

Hermite's motivation behind Theorem 3.2

Let f, D, z_1, α_1 and Γ be as in Theorem 3.2. Furthermore, let ρ be such that the closed disk $\{\zeta \in \mathbb{C} : |\zeta - z_1| \le \rho\}$ lies inside γ and denote by C_ρ the positively oriented circle $|\zeta - z_1| = \rho$. It is standard (see, for example [1 , p. 125]) that the familiar Taylor polynomial p of degree at most $\alpha_1 - 1$ satisfying

$$p(z_1) = f(z_1), \ldots, p^{(\alpha_1-1)}(z_1) = f^{(\alpha_1-1)}(z_1) \tag{3.10}$$

is given by

$$f(z) = p(z) + \frac{1}{2\pi \mathrm{i}}(z - z_1)^{\alpha_1} \int_{C_\rho} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_1)^{\alpha_1}} \, \mathrm{d}\zeta$$

for $|z - z_1| < \rho$. Since the integrand has no singularities in $D \setminus \{|\zeta - z_1| < \rho\}$, the circle C_ρ in this representation may be replaced by Γ. Thus

$$f(z) = p(z) + \frac{1}{2\pi \mathrm{i}}(z - z_1)^{\alpha_1} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_1)^{\alpha_1}} \, \mathrm{d}\zeta \tag{3.11}$$

for $|z - z_1| < \rho$. The function $f(\zeta)/(\zeta - z_1)^{\alpha_1}$ is continuous on Γ, and so the integral on the right-hand side of (3.11) defines [1, p. 121] a function of z that is analytic throughout the bounded component $\Gamma^{(i)}$ of $\hat{\mathbb{C}} \backslash \Gamma$. Since the function $f(z)$ appearing on the left-hand side of (3.11) is also holomorphic in $\Gamma^{(i)}$, by analytic continuation, the equality must hold not only for $|z - z_1| < \rho$ but throughout $\Gamma^{(i)}$. Thus

$$\begin{aligned} p(z) \;=\; & f(z) - \frac{1}{2\pi \mathrm{i}}(z - z_1)^{\alpha_1} \times \\ & \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_1)^{\alpha_1}} \, \mathrm{d}\zeta \quad \text{for all } z \text{ inside } \Gamma, \end{aligned} \tag{3.12}$$

which is what Theorem 3.2 says in the case where $n = 1$.

REMARK 1. If $\Delta_\rho(z_1) := \{z \in \mathbb{C} : |z - z_1| \le \rho\} \subset D$, then the Taylor polynomial $p(z)$ satisfying (3.10) converges uniformly to $f(z)$ for $z \in \Delta_\rho(z_1)$ as α_1 tends to infinity. Hermite pointed out that the polynomial p satisfying the more general interpolation condition (3.6) tends to $f(z)$ as $\alpha_1, \ldots, \alpha_n$ tend to infinity, if the circles centered at z_ν, $\nu = 1, \ldots, n$ and passing through z lie inside Γ.

4 Lagrange Interpolation versus Hermite Interpolation

Let $f : [-1, 1] \to \mathbb{R}$ be an arbitrary continuous function, let $p_n(f; \cdot) \in \mathcal{P}_n$ be such that

$$p_n\left(-1 + \frac{2\nu}{n}\right) = f\left(-1 + \frac{2\nu}{n}\right) \qquad (\nu = 0, 1, \ldots, n).$$

It was observed by Runge [34] that the sequence $\{p_n(f; x)\}$ may not converge uniformly to $f(x)$ as $n \to \infty$. Subsequently, Bernstein (see [31, pp. 30–35] for details and pertinent remarks) noted that the sequence $\{p_n(|x|; \cdot)\}$ converges to $|x|$ at no point of $[-1, 1]$ other than $-1, 0, 1$. Does there exist a universal infinite triangular matrix, whose n-th row consists of points $x_{1,n}, \ldots, x_{n,n}$ belonging to $[-1, 1]$, such that for any continuous function $f : [-1, 1] \to \mathbb{R}$ the sequence $\{p_{n-1}(f\,;\,\cdot)\}$ of polynomials $p_{n-1} \in \mathcal{P}_{n-1}$, satisfying $p_{n-1}(x_{\nu,n}) = f(x_{\nu,n})$ for $\nu = 1, \cdots, n$, converges uniformly to f? The famous theorem of Faber (see [13] ; also see [6]) says that the answer to this question is "no." Similarly, there does not exist a universal infinite triangular matrix, whose n-th row consists of points $\theta_{0,n}, \theta_{1,n}, \ldots, \theta_{n,2n}$ belonging to $[-1, 1]$, such that for any continuous 2π-periodic function $f : [0, 2\pi] \to \mathbb{R}$ the sequence $\{T_n(f\,;\,\theta)\}$ of trigonometric polynomials $T_n(f; .)$ of degree at most n, satisfying $T_n(\theta_{\nu,2n}) = f(\theta_{\nu,2n})$ for $\nu = 0, 1, \cdots, 2n$, converges uniformly to $f(\theta)$ (see [38 , Chapter VIII]).

Fejér discovered that the situation changes if instead of Lagrange interpolation we consider Hermite interpolation in the points $x_{\nu,n} := \cos\dfrac{2\nu - 1}{2n}\pi\,,\ \nu = 1, \ldots, n$. These points are the zeros of the n-th Chebyshev polynomial of the first kind. The result of Fejér (see [14 , 15]) may be stated as follows.

Theorem 4.1 *Let* $x_{\nu,n} := \cos\dfrac{2\nu - 1}{2n}\pi\,,\ \nu = 1, \ldots, n$. *For any continuous function* $f : [-1, 1] \to \mathbb{R}$, *the sequence* $\{H_{2n-1}(f; \cdot)\}$ *of polynomials belonging to* $\mathcal{P}_{2n-1}$ *satisfying the conditions*

$$H_{2n-1}(f; x_{\nu,n}) = f(x_{\nu,n}) \quad \text{and} \quad H'_{2n-1}(f; x_{\nu,n}) = y'_{\nu,n}$$

converges uniformly to f *on* $[-1,1]$ *provided that*

$$\lim_{n\to\infty}\max_{1\le\nu\le n}\frac{|y'_{\nu,n}|\log n}{n}=0\,. \qquad \square$$

There is an analogous result about the convergence of Hermite interpolating trigonometric polynomials. It has some instructional value in connection with Hermite interpolation by non-periodic entire functions of exponential type , and so we shall discuss it.

Note that the Fejér kernel

$$K_n(u):=\frac{2}{n+1}\left\{\frac{\sin{(n+1)u/2}}{2\sin u/2}\right\}^2$$

is a trigonometric polynomial of degree n that vanishes at the points $t_\nu:=2\pi\nu/(n+1)$ for $\nu=1,2,\ldots,n$ and equals $(n+1)/2$ at $t_0=0$. Thus

$$J_n(x)=J_n(f;x):=\frac{2}{n+1}\sum_{\nu=0}^{n}f(t_\nu)K_n(x-t_\nu) \tag{4.1}$$

is a trigonometric polynomial (J for D. Jackson) of degree at most n coinciding with f at the points $t_0,\ldots,t_n$. It is interesting that $K_n'(2\pi\nu/(n+1))=0$ for $\nu=0,1,\ldots,n$ and so $J_n(f;x)$ is a trigonometric polynomial of degree at most n coinciding with f at the points $t_0,\ldots,t_n$ and having a vanishing derivative there, a fact first observed by Bernstein [5]. *A priori* it might seem that J_n satisfies $2n+2(>2n+1)$ conditions, but the conditions governing the derivative J_n' are not independent. In fact, for any trigonometric polynomial S of degree at most n,

$$S'(x)=\sum_{\nu=-n}^{n}d_\nu\,\mathrm{e}^{\mathrm{i}\nu x}\quad\text{where}\quad d_0=0\,.$$

Hence, using a well-known property of the primitive $(n+1)$-st roots of unity, namely

$$\sum_{j=0}^{n}\mathrm{e}^{2\nu\pi\mathrm{i}j/(n+1)}=0\qquad(\nu=\pm1,\ldots,\pm n)\,,$$

we conclude that

$$0 = (n+1)\, d_0 = \sum_{j=0}^{n} S'\left(\frac{2j\pi}{n+1}\right), \tag{4.2}$$

i.e., the sum of the values, the derivative of a trigonometric polynomial of degree at most n takes at the points $t_0, t_1, \ldots, t_n$, is always zero. It is not obvious but *given any set of $n+1$ numbers $y_0, y_1, \ldots, y_n$ and another set of $n+1$ numbers $y'_0, y'_1, \ldots, y'_n$ satisfying the condition $\sum_{\nu=0}^{n} y'_\nu = 0$ there exists a unique trigonometric polynomial S_n of degree at most n such that*

$$S_n(t_\nu) = y_\nu,\ S'_n(t_\nu) = y'_\nu \qquad (\nu = 0, 1, \ldots, n). \tag{4.3}$$

To see this, note that

$$\begin{aligned}\tau_{n+1}(x) \quad &:= \quad \sum_{\mu=1}^{n} \sin \mu x + \frac{1}{2} \sin\,(n+1)x \\ &= \quad \{1 - \cos\,(n+1)x\}\frac{1}{2} \cot \frac{x}{2}\end{aligned} \tag{4.4}$$

is a trigonometric polynomial of degree $n+1$ which vanishes at the points $t_\nu = 2\pi\nu/(n+1)$, $\nu = 0, 1, \ldots, n$. Its derivative also vanishes at each of these points except at t_0; in fact $\tau'_{n+1}(0) = (n+1)^2/2$. Hence

$$\begin{aligned}\hbar_n(x) \ &= \ 2\frac{1}{(n+1)^2} \sum_{\nu=0}^{n} y'_\nu\, \tau_{n+1}(x - t_\nu) \\ &= \ 2\frac{1}{(n+1)^2} \sum_{\nu=0}^{n} y'_\nu \sum_{\mu=0}^{n} \sin \mu(x - t_\nu) + \\ &\qquad\qquad \frac{1}{(n+1)^2} \sin\{(n+1)(x - t_0)\} \sum_{\nu=0}^{n} y'_\nu \\ &= \ 2\frac{1}{(n+1)^2} \sum_{\nu=0}^{n} y'_\nu \sum_{\mu=0}^{n} \sin \mu(x - t_\nu)\end{aligned} \tag{4.5}$$

is a trigonometric polynomial of degree at most n vanishing at the points t_ν and having derivatives y'_ν there. Consequently, with $K_n(u)$

as above,

$$S_n(x) := \frac{2}{n+1}\sum_{\nu=0}^{n} y_\nu K_n(x-t_\nu) + \hbar_n(x) \tag{4.6}$$

is a trigonometric polynomial of degree at most n that satisfies (4.3). If there were more than one, the difference of two such $S(x)$ would have a double zero at each of the $n+1$ points $t_0, t_1, \ldots, t_n$, and so would vanish identically, giving us a contradiction.

Having discussed the relevant properties of the trigonometric polynomials J_n and $\hbar_n$ defined in (4.1) and (4.5), respectively, we are ready to state the analogue of Theorem 4.1 for Hermite interpolating trigonometric polynomials, we had alluded to above. Here we must quote [42, p. 24] and also [42, p. 331] where Zygmund attributes the result to Fejér [15].

Theorem 4.2 *Let $f(x)$ be any bounded periodic function with period 2π. Furthermore, for any given natural number n, define*

$$t_{\nu,n} := \frac{2\pi\nu}{n+1} \qquad (\nu = 0, 1, \ldots, n)\,.$$

Also, let $y'_{0,n}$, $y'_{1,n}$, $\ldots$, $y'_{n,n}$ be an arbitrary set of $n+1$ numbers satisfying

$$\sum_{\nu=0}^{n} y'_{\nu,n} = 0 \quad \text{and} \quad \max\{|y'_{0,n}|,\, |y'_{1,n}|,\, \ldots,\, |y'_{n,n}|\} = o\left(\frac{n}{\log n}\right).$$

Now, let $S_n(f;x)$ be the trigonometric polynomial of degree at most n such that

$$S_n(f;t_{\nu,n}) = f(t_{\nu,n})\,, \quad S'_n(f;t_{\nu,n}) = y'_{\nu,n} \qquad (\nu = 0, 1, \ldots, n)\,.$$

Then $\lim_{n\to\infty} S_n(f;x) = f(x)$ at every point of continuity of f, and the convergence is uniform over each closed interval of continuity. □

The negative results of Faber and Bernstein about the uniform convergence of Lagrange interpolating polynomials and of trigonometric polynomials notwithstanding, when it comes to convergence in the mean, Lagrange interpolating trigonometric polynomials which interpolate a 2π-periodic function in uniformly distributed points behave very well, as the following result of Marcinkiewicz [28] illustrates.

Theorem 4.3 *Let $f : \mathbb{R} \to \mathbb{C}$ be any continuous 2π-periodic function, and for any $n \in \mathbb{N}$, let*

$$\theta_{\nu,n} := \frac{2\nu\pi}{2n+1} \qquad (\nu = 0, \pm 1, \ldots, \pm n)\,.$$

In addition, let $L_n(.; f)$ be the trigonometric polynomial of degree at most n with $L_n(\theta_{\nu,n}; f) = f(\theta_{\nu,n})$ for $\nu = 0, \pm 1, \ldots, \pm n$. Then

$$\lim_{n\to\infty} \int_0^{2\pi} |f(\theta) - L_n(\theta_{\nu,n}; f)|^p \,\mathrm{d}\theta = 0 \qquad (p > 0)\,.$$

$\square$

This is interesting since $\sup_{n\to\infty} |L_n(\theta_{\nu,n}; f)| = \infty$ for every θ if the continuous 2π-periodic function f is chosen appropriately ([22], [29]). Theorem 4.3 and another analogous result due to Erdős and Feldheim [12] have stimulated a lot of research on mean convergence. Later in the paper, we shall mention an analogue of this theorem for Lagrange interpolation of non-periodic functions $f : \mathbb{R} \to \mathbb{C}$ in uniformly distributed points $\nu\pi/\tau$, $\nu = 0, \pm 1, \pm 2, \ldots$.

5 Lacunary Interpolation by Polynomials and Trigonometric Polynomials

The theory of lacunary interpolation was greatly enriched by contributions made by the late Professor Ambikeshwar Sharma and his associates. His work with A. K. Varma on (lacunary) trigonometric interpolation motivated the third named author of this paper to consider extending the notion of "(0, m) interpolation by trigonometric polynomials" to "(0, m) interpolation by entire functions of exponential type" which will be discussed in a later section.

Lacunary interpolation by polynomials

Given n points $\{x_\nu\}_{\nu=1}^{n}$, and corresponding to each x_ν a set of non-negative integers $m_{1,\nu}, \ldots, m_{\alpha_\nu,\nu}$ and arbitrary numbers

$w_{1,\nu}, \ldots, w_{\alpha_\nu,\nu}$, we may ask if there always exists a polynomial p of degree $\alpha_1 + \cdots + \alpha_n - 1$ or less satisfying the $\alpha_1 + \cdots + \alpha_n$ conditions

$$p^{(m_{k,\nu})}(x_\nu) = w_{k,\nu} \quad \text{for} \;\; k = 1, \ldots, \alpha_\nu \;\; \text{and} \;\; \nu = 1, \ldots, n.$$

This is the central problem of *lacunary interpolation*, called "lacunary" because there may be lacunae (gaps) in the sequence $m_{1,\nu}, \ldots, m_{\alpha_\nu,\nu}$. It is also referred to as Birkhoff interpolation since G. D. Birkhoff [8] was the first to consider this kind of interpolation. It reduces to Hermite interpolation when $m_{k,\nu} = k - 1$ for $k = 1, \ldots, \alpha_\nu$.

In order to get some insight into the problem of lacunary interpolation let us suppose that $\alpha_\nu = 2$ for $\nu = 1, \ldots, n$ and that $m_{1,\nu} = 0$, $m_{2,\nu} = 2$ for each ν. Then the question is: does there always exist a polynomial p of degree $2n - 1$ or less such that

$$\left.\begin{aligned} p(x_\nu) &= w_{1,\nu}, \\ p''(x_\nu) &= w_{2,\nu} \end{aligned}\right\} \qquad (\nu = 1, \ldots, n)\,? \tag{5.1}$$

This kind of interpolation, also known as $(0, 2)$ interpolation, is considerably more complicated than the case $(0, 1)$ of Hermite interpolation. For example, there may be no polynomial p of degree $2n - 1$ or less such that $p(x_\nu) = w_{1,\nu}$ and $p''(x_\nu) = w_{2,\nu}$ for $\nu = 1, \ldots, n$; it is also possible that there may be an infinity of polynomials of degree not exceeding $2n - 1$ satisfying (5.1); and then there are situations of considerable significance and interest where the interpolation problem has *one and only one solution*. These claims need to be substantiated. The following example serves to justify the first two.

Example. Opting for simplicity which adds to the clarity without missing the point we take only three nodes $x_1 = 1, x_2 = 0, x_3 = -1$, that is $n = 3$, and see if there does or does not exist a polynomial p of degree at most $5\,(= 2n - 1)$ that takes arbitrarily prescribed values $w_{1,1}, w_{1,2}, w_{1,3}$ at x_1, x_2, x_3, respectively, and whose second derivative p'' can be assigned any three values $w_{2,1}, w_{2,2}, w_{2,3}$ at these points. Let us take $w_{1,1} = w_{1,2} = w_{1,3} = 0$. Then, writing $p(x) := \sum_{\nu=0}^{5} a_\nu x^\nu$ we see that the coefficients of p must satisfy the conditions

$$a_0 = 0, \quad a_1 + a_3 + a_5 = 0, \quad a_2 + a_4 = 0\,.$$

Let us also choose $w_{2,2}$ to be 0. This forces a_2 and a_4 to be 0. Hence, in view of the condition $a_1 + a_3 + a_5 = 0$, we are led to conclude that the polynomial p has to be of the form

$$p(x) = a_1(x - x^3) + a_5(x^5 - x^3)\,,$$

and so necessarily $p''(x_3) = -p''(x_1)$. Indeed

$$\left.\begin{aligned} p''(x_1) &= p''(1) = -2(3\,a_1 - 7\,a_5), \\ p''(x_3) &= p''(-1) = 2(3\,a_1 - 7\,a_5). \end{aligned}\right\} \tag{5.2}$$

Thus, we can only choose $p''(1)$ or $p''(-1)$ freely but not both. In particular we see that there is no polynomial p of degree 5 or less satisfying the conditions

$$p(1) = p(0) = p(-1) = 0, \quad p''(1) = 1, \quad p''(0) = 0, \quad p''(-1) = 1\,.$$

On the other hand, if we respect the restriction imposed by (5.2) on the choice of $w_{2,1}$ and $w_{2,3}$, that is require $p''(-1)$ to be equal to $-w_{2,1}$, then we can certainly find polynomials p of degree not exceeding 5 for which $p(1) = p(0) = p(-1) = 0$, $p''(1) = w_{2,1}$, $p''(0) = 0$, $p''(-1) = -w_{2,1}$. Any polynomial of the form

$$p(x) = a_1(x - x^3) - \frac{1}{14}\,(w_{2,1} + 6a_1)(x^3 - x^5) \qquad (a_1 \in \mathbb{R})$$

has this property. In fact, any of the polynomials

$$p(x) = a_1(x - x^3) - \frac{1}{14}\,(w_{2,1} + 6a_1)(x^3 - x^5) + c\,x(1 - x^2)(3x^2 - 7),$$

where a_1 and c are arbitrary real numbers, satisfies the specified interpolation conditions.

The preceding aspect of $(0, 2)$ interpolation was discussed from a more general point of view by Surányi and Turán [36].

Let

$$P_m(x) := \frac{1}{2^m} \sum_{\mu=0}^{\lfloor m/2 \rfloor} (-1)^\mu \binom{m}{\mu} \binom{2m - 2\mu}{m} x^{m-2\mu}$$

be the Legendre polynomial of degree m, normalized so that $P_m(1) = 1$. It was observed by Surányi and Turán [36] that the zeros of the polynomial $\pi_n(x) := (1-x^2)P'_{n-1}(x)$ are of special significance in connection with the problem of $(0,2)$ interpolation by polynomials. This is illustrated by the following result.

Theorem 5.1 *Let $\xi_1 = 1 > \cdots > \xi_n = -1$ be the zeros of the polynomial $\pi_n(x) := (1-x^2)P'_{n-1}(x)$. Furthermore, let $w_{1,n}, \ldots, w_{n,n}$ and $y_{1,n}, \ldots, y_{n,n}$ be two sets of n real numbers each. Then, if n is even, there exists a polynomial f of degree not exceeding $2n-1$ such that*

$$f(\xi_{\nu,n}) = w_{\nu,n} \ , \quad f''(\xi_{\nu,n}) = y_{\nu,n} \qquad (\nu = 1, \ldots, n)\,.$$

The same cannot be said in the case where n is odd.

Proof. Let us recall that P_{n-1} satisfies the differential equation

$$\begin{aligned}\left[(1-x^2)P'_{n-1}(x)\right]' + n(n-1)P_{n-1}(x) = {}&\\ (1-x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) &= 0\,,\end{aligned}$$

and so

$$(1-x^2)\left[(1-x^2)P'_{n-1}(x)\right]'' + n(n-1)(1-x^2)P'_{n-1}(x) = 0\,.$$

Thus π_n satisfies the differential equation

$$(1-x^2)\pi''_n(x) + n(n-1)\pi_n(x) = 0\,. \tag{5.3}$$

Differentiating the two sides of (5.3) and putting $x = 1$, $x = -1$ we find that

$$\pi''_n(1) = \frac{n(n-1)}{2}\pi'_n(1) \ \text{ and } \ \pi''_n(-1) = -\frac{n(n-1)}{2}\pi'_n(-1)\,. \tag{5.4}$$

Let

$$-1 = \xi_{n,n} < \xi_{n-1,n} < \cdots < \xi_{2,n} < \xi_{1,n} = 1$$

be the zeros of π_n. Then from (5.3) it follows that

$$\pi''_n(\xi_{\nu,n}) = 0 \qquad (\nu = 2, \ldots, n-1)\,. \tag{5.5}$$

We shall use this property of π_n to determine all those polynomials p in $\mathcal{P}_{2n-1}$ that satisfy

$$\left.\begin{array}{rcl} p(\xi_{\nu,n}) & = & 0 \\ p''(\xi_{\nu,n}) & = & 0 \end{array}\right\} \qquad (\nu = 1, \ldots, n)\,. \tag{5.6}$$

The identically zero polynomial certainly satisfies (5.6). So the class C_n of all polynomials p in $\mathcal{P}_{2n-1}$ that satisfy (5.6) is not empty. Now note that if $p \in C_n$, then p must be of the form

$$p(x) = \pi_n(x)g(x)\,,$$

where $g \in \mathcal{P}_{n-1}$. Using (5.5) in

$$p''(x) = \pi_n''(x)g(x) + 2\pi_n'(x)g'(x) + \pi_n(x)g''(x)$$

we see that for $\nu = 2, \ldots, n-1$, we have

$$0 = p''(\xi_{\nu,n}) = 2\pi_n'(\xi_{\nu,n})g'(\xi_{\nu,n})\,.$$

Since the zeros of π_n are all simple, $\pi_n'(\xi_{\nu,n}) \neq 0$ and so g' and P_{n-1}' have the same zeros. Hence, there exists a constant c such that $g'(x) \equiv cP_{n-1}'(x)$. This, in turn, implies that for some constant d,

$$g(x) = cP_{n-1}(x) + d\,.$$

Thus

$$p(x) = \pi_n(x)\{cP_{n-1}(x) + d\}\,. \tag{5.7}$$

We have not as yet used the fact that $p''(x)$ vanishes at 1 and -1. Since also $\pi_n(x)$ vanishes at 1 and -1, we have

$$0 = p''(1) = \pi_n''(1)(c+d) + 2c\pi_n'(1)P_{n-1}'(1)\,,$$

$$0 = p''(-1) = \pi_n''(-1)\{c\,(-1)^{n-1} + d\} + 2c\pi_n'(-1)P_{n-1}'(-1)\,.$$

Using (5.4) we see that c and d satisfy the two equations

$$\frac{n(n-1)}{2}(c+d) + 2cP_{n-1}'(1) = 0$$

and

$$-\frac{n(n-1)}{2}\{(-1)^{n-1}c + d\} + 2cP_{n-1}'(-1) = 0\,.$$

From the differential equation for P_{n-1} we deduce that

$$P'_{n-1}(1) = \frac{n(n-1)}{2}\ ,\ P'_{n-1}(-1) = (-1)^n \frac{n(n-1)}{2}\ ,$$

and consequently

$$3c + d = 0 \quad \text{and} \quad (-1)^n \cdot 3c - d = 0\,. \tag{5.8}$$

At this stage we should distinguish the case where n is even from the case where n is odd. Let n be even. Then (5.8) reduces to

$$3c + d = 0 \quad \text{and} \quad 3c - d = 0\,.$$

This is possible only if c and d both vanish. Returning to (5.7) we see that $p(x)$ is identically zero. Thus $p = 0$ is the only polynomial $p \in \mathcal{P}_{2n-1}$ that satisfies (5.6).

In the case where n is odd, (5.7) in conjunction with (5.8) implies that a polynomial $p \in \mathcal{P}_{2n-1}$ satisfies (5.6) if and only if $p(x) = c\,\pi_n(x)\{P_{n-1}(x) - 3\}$. $\square$

For a variety of further results about (0, 2) interpolation by polynomials we refer the reader to [2], [3], [4], [32], [40] and to [37].

Lacunary interpolation by trigonometric polynomials

For any integer n let $\vartheta_{\nu,N} := 2\pi\nu/(N+1)$ for $\nu = 0, \ldots, N$, and denote by $\mathcal{T}_N$ the class of all real trigonometric polynomials of degree at most N. We start with the following result [37, Theorem 7.5].

Theorem 5.2 *Let n and m be two positive integers. Also, in the notation just introduced, let $\vartheta_{\nu,n-1} := 2\pi\nu/n$ for $\nu = 0, \ldots, n-1$. Furthermore, let $w_0, \ldots, w_{n-1}$ and $y_0, \ldots, y_{n-1}$ be two sets of numbers subject only to the restriction that $y_0 + \cdots + y_{n-1} = 0$. Then, there exists a unique trigonometric polynomial $T \in \mathcal{T}_{n-1}$ such that*

$$T(\vartheta_{\nu,n-1}) = w_\nu\ ,\ \ T^{(m)}(\vartheta_{\nu,n-1}) = y_\nu \qquad (\nu = 0, \ldots, n-1)\,, \tag{5.9}$$

if either m is odd and n is arbitrary or m is even and n is odd.

NOTE. Analogously to (4.2) it can be shown that for any trigonometric polynomial S of degree at most $n-1$,

$$\sum_{\nu=0}^{n-1} S^{(m)}\left(\frac{2\pi\nu}{n}\right) = 0\,, \tag{4.2$'$}$$

i.e., the sum of the values the m-th derivative of a trigonometric polynomial in $\mathcal{T}_{n-1}$ takes at the points $\vartheta_{0,n-1}, \vartheta_{1,n-1}, \ldots, \vartheta_{n-1,n-1}$ is always zero.

We shall deduce Theorem 5.2 from the following uniqueness theorem for entire functions of exponential type (see [16, Theorem 4], also see [17, Lemma 2]).

Lemma 5.3 *Let m be any positive integer, and λ an arbitrary number in $[0,1)$. Furthermore, let $f(z)$ be an entire function of exponential type 2π such that*

(i) $|f(x)| \le A + B|x|^{\lambda}$ for all real x and certain constants A, B,

(ii) $f(k) = f^{(m)}(k) = 0$ for $k = 0, \pm 1, \pm 2, \ldots$.

Then

$$f(z) = \begin{cases} c_1 \sin(\pi z) + c_2 \sin(2\pi z) & \text{if } m \text{ is even,} \\ c \sin^2(\pi z) & \text{if } m \text{ is odd,} \end{cases}$$

where c_1, c_2 and c are constants. Here λ cannot be allowed to be equal to 1.

$\square$

Proof of Theorem 5.2. In view of Lemma 2.2, it is enough to show that if $T \in \mathcal{T}_{n-1}$ and

$$T(\vartheta_{\nu,n-1}) = T^{(m)}(\vartheta_{\nu,n-1}) = 0 \qquad (\nu = 0, \ldots, n-1)\,, \tag{5.10}$$

then $T(\theta) \equiv 0$. For this note that $f(z) := T\left(\dfrac{2\pi}{n} z\right)$ is an entire function of exponential type $\dfrac{(n-1)\,2\pi}{n}$ and so of exponential type *less than* 2π. Besides,

$$f(k) = f^{(m)}(k) = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots)\,.$$

Hence, by Lemma 5.3, $f(z)$ is identically zero if m is odd. In the case where m is even, there exists a constant c_1 such that $f(z) := c_1 \sin \pi z$. This implies that T is identically zero if m is odd, whatever n may be. If m is even, then T must be of the form $T(\theta) := c_1 \sin\left(\frac{n}{2}\,\theta\right)$ for some constant c_1. However, c_1 must be zero if n happens to be odd, since otherwise $T(\theta)$ would not be 2π-periodic and so not a proper trigonometric polynomial. $\square$

It was shown by Sharma and Varma (see [35, formula (4)]) that if m is odd and n is arbitrary or if m is even and n is odd, then the trigonometric polynomial

$$\mathcal{A}_{0,m}(\theta) := \frac{1}{n}\left\{1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^m \cos j\theta}{(n-j)^m - (-j)^m}\right\},$$

which is clearly of degree $n-1$, satisfies

$$\mathcal{A}_{0,m}\left(\frac{2\pi\nu}{n}\right) = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu = 1, \ldots, n-1, \end{cases}$$

and $\mathcal{A}_{0,m}^{(m)}\left(\frac{2\pi\nu}{n}\right) = 0$ for $\nu = 0, 1, \ldots, n-1$.

They also observed that if m is odd and n is arbitrary, then the trigonometric polynomial

$$\mathcal{B}_{0,m}(\theta) := (-1)^{(m-1)/2}\left\{\frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin j\theta}{(n-j)^m + j^m} + \frac{\sin n\theta}{n^{m+1}}\right\}$$

satisfies

$$\mathcal{B}_{0,m}^{(m)}\left(\frac{2\pi\nu}{n}\right) = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu = 1, \ldots, n-1, \end{cases}$$

and $\mathcal{B}_{0,m}\left(\frac{2\pi\nu}{n}\right) = 0$ for $\nu = 0, 1, \ldots, n-1$. The degree of the trigonometric polynomial $\mathcal{B}_{0,m}(\theta)$ is n, but never mind. This is a situation analogous to (4.4). For any set of n real numbers $\beta_0, \ldots, \beta_{n-1}$, where $\sum_{\nu=0}^{n-1} \beta_\nu = 0$, and any continuous 2π-periodic function f,

$$\begin{aligned} R_{n-1}(\theta) \ &:= \ \sum_{\nu=0}^{n-1} f\left(\frac{2\pi\nu}{n}\right) \mathcal{A}_{0,m}\left(\theta - \frac{2\pi\nu}{n}\right) + \\ &\qquad \sum_{\nu=0}^{n-1} \beta_\nu \mathcal{B}_{0,m}\left(\theta - \frac{2\nu\pi}{n}\right) \end{aligned} \tag{5.11}$$

is the unique trigonometric polynomial of degree not exceeding $n-1$ such that

$$R_{n-1}\left(\frac{2\pi\nu}{n}\right) = f\left(\frac{2\pi\nu}{n}\right) \text{ and } R_{n-1}^{(m)}\left(\frac{2\pi\nu}{n}\right) = \beta_\nu \quad (\nu = 0, 1, \ldots, n-1)\,.$$

In analogy with Theorem 4.2 they [35, Theorem 2] proved the following result.

Theorem 5.4 *Let m be odd and n arbitrary. Also, let f be a continuous 2π-periodic function, and $\beta_0, \ldots, \beta_{n-1}$ real numbers such that*

$$\sum_{\nu=0}^{n-1} \beta_\nu = 0 \text{ and } \max\{|\beta_0|, \ldots, |\beta_{n-1}|\} = o\left(\frac{n^m}{\log n}\right)\,.$$

In addition, let $R_{n-1}(\theta)$ be as in (5.11)*. Then $R_{n-1}(\theta)$ tends uniformly to $f(\theta)$ as n tends to ∞.*

Sharma and Varma [35] also considered the case where m is even and n is odd and obtained a slightly different result.

Due to limitation of time and space we shall not discuss this topic any further and simply refer to [37] for relevant references. However, we must quote [27] which does not appear in their list of references but is quite interesting in our opinion.

6 Interpolation by Transcendental Entire Functions

Looking back at Theorem 3.1, we ask if an analogous result holds for an infinite set of interpolation points. Obviously, a polynomial of degree n cannot assume prescribed values at more than $n+1$ distinct points unless the prescribed values are the same. So, if we are looking for functions which are "just as smooth" as polynomials, we have to move on to transcendental entire functions. However, we need to observe that an entire function f cannot assume the value 0 at an infinite set of points $z_1, z_2, \ldots, z_n, \ldots$ having a finite limit point without being identically zero. For this reason, we shall require the interpolation points to have no finite limit point. With this in mind, the following result may be seen as "an analogue" of Theorem 3.1.

Theorem 6.1 *Let $z_1, \ldots, z_n, \ldots$ be an infinite sequence of points in the complex plane $\mathbb{C}$ such that $z_n \neq 0$ and $|z_n| \to \infty$ as $n \to \infty$. Furthermore, for each $n \in \mathbb{N}$, prescribe α_n values*

$$w_{0,n}, \ldots, w_{\alpha_n - 1,n}\,,$$

not necessarily distinct. Then, there exists an entire function f such that

$$\left.\begin{array}{ll} f^{(j)}(z_1) = w_{j,1} & (j = 0, \ldots, \alpha_1 - 1) \\ \quad\vdots & \\ f^{(j)}(z_n) = w_{j,n} & (j = 0, \ldots, \alpha_n - 1) \\ \quad\vdots & \end{array}\right\}. \tag{6.1}$$

Proof. The idea of the proof is simply ingenious. Let $z_0 = 0$ and $\alpha_0 \in \mathbb{N}$. Construct an entire function g with zeros of multiplicity α_ν at z_ν for $\nu = 0, 1, 2, \ldots$. This is possible by a slightly modified version of a classical theorem of Weierstrass (see [1, p. 194] or [39, pp. 246–247]), stated and proved below as Lemma 6.2. Let

$$g(z) = a_{\alpha_\nu,\nu}\,(z - z_\nu)^{\alpha_\nu} + \cdots + a_{2\alpha_\nu - 1,\nu}\,(z - z_\nu)^{2\alpha_\nu - 1} + \cdots \qquad (a_{\alpha_\nu,\nu} \neq 0)$$

be the Taylor series development of g at the point z_ν. Now construct a meromorphic function $\chi(z)$ having a pole of multiplicity α_ν at z_ν for each ν, and no other poles. This is not only possible but by a theorem of Mittag-Leffler [1, p. 185], stated below as Lemma 6.3, we can also arrange for $\chi(z)$ to have at each z_ν a "principal part" of our choice. Clearly, $f(z) := \chi(z)\,g(z)$ is an entire function, and the freedom in the choice of the principal part of $\chi(z)$ at each z_ν allows us to arrange for $f(z)$ to have the property (6.1), which means that the Taylor series development of $f(z)$ at z_ν should be of the form

$$f(z) = \sum_{k=0}^{\alpha_\nu - 1} \frac{1}{k!}\, w_{k,\alpha_\nu} (z - z_\nu)^k + \sum_{k=\alpha_\nu}^{\infty} c_k\,(z - z_\nu)^k\,.$$

Thus, considering the Laurent development

$$\chi(z) = \frac{b_{-\alpha_\nu,\nu}}{(z-z_\nu)^{\alpha_\nu}} + \cdots + \frac{b_{-1,\nu}}{z-z_\nu} + \sum_{k=0}^{\infty} b_{k,\nu}(z-z_\nu)^k$$

of $\chi(z)$ at z_ν we see that the coefficients $b_{-\alpha_\nu,\nu}, \ldots, b_{-1,\nu}$ of the principal part

$$\frac{b_{-\alpha_\nu,\nu}}{(z-z_\nu)^{\alpha_\nu}} + \cdots + \frac{b_{-1,\nu}}{z-z_\nu}$$

need to be chosen in such a manner that

$$\chi(z)\, g(z) = \sum_{k=0}^{\alpha_\nu - 1} \frac{1}{k!}\, w_{k,\alpha_\nu}(z-z_\nu)^k + \sum_{k=\alpha_\nu}^{\infty} c_k\,(z-z_\nu)^k \,.$$

This means that the the first α_ν terms of the product of

$$\left\{ \frac{b_{-\alpha_\nu,\nu}}{(z-z_\nu)^{\alpha_\nu}} + \cdots + \frac{b_{-1,\nu}}{z-z_\nu} \right\}$$

and

$$\left\{ a_{\alpha_\nu,\nu}\,(z-z_\nu)^{\alpha_\nu} + \cdots + a_{2\alpha_\nu-1,\nu}\,(z-z_\nu)^{2\alpha_\nu-1} \right\}$$

should be

$$w_{0,\alpha_\nu} + \cdots + \frac{1}{k!} w_{k,\alpha_\nu}(z-z_\nu)^k + \cdots + \frac{1}{(\alpha_\nu-1)!} w_{\alpha_\nu-1,\alpha_\nu}(z-z_\nu)^{\alpha_\nu-1} \,.$$

This observation helps us determine the coefficients $b_{-\alpha_\nu,\nu}, \ldots, b_{-1,\nu}$ successively, as desired. Clearly,

$$b_{-\alpha_\nu,\nu} = \frac{w_{0,\alpha_\nu}}{a_{\alpha_\nu,\nu}}\,,\ b_{-\alpha_\nu+1,\nu} = \frac{1}{a_{\alpha_\nu,\nu}} \left(\frac{1}{1!} w_{1,\alpha_\nu} - b_{\alpha_\nu,\nu}\, a_{\alpha_\nu+1,\nu} \right),\ \text{et cetera}\,.$$

□

Lemma 6.2 (K. Weierstrass). *Let $z_1, z_2, \ldots, z_n, \ldots$ be an infinite sequence of points in the complex plane $\mathbb{C}$ such that $0 < |z_n| \le |z_{n+1}|$ for $n = 1, 2, \ldots,$ and $|z_n| \to \infty$ as $n \to \infty$. Furthermore, for each $n \in \mathbb{N}$, let m_n be a positive integer. Then, for any integer $m_0 \ge 0$, there exists an entire function f, which has a zero of multiplicity m_n at z_n for $n = 1, 2, \ldots,$ and a zero of multiplicity m_0 at 0.*

Proof. For $|u| \le \frac{1}{2}$ and $p_n := \max\{n, m_n\}$, let

$$E(u; p_n) := (1-u)^{m_n} \exp\left(m_n \sum_{\nu=1}^{p_n} \frac{1}{\nu}\, u^\nu \right).$$

Then

$$\begin{aligned} |\log E(u;p_n)| &= \left| m_n \left\{ \sum_{\nu=1}^{\infty} \frac{1}{\nu} u^\nu - \sum_{\nu=1}^{p_n} \frac{1}{\nu} u^\nu \right\} \right| \\ &\le \frac{m_n}{p_n+1} \left(u^{p_n+1} + u^{p_n+2} + \cdots \right) \\ &\le |u|^{p_n+1} + |u|^{p_n+2} + \cdots \\ &\le |u|^{p_n+1} \left\{ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots \right\} = 2\,|u|^{p_n+1}. \end{aligned}$$

Now, let ρ be any given positive number. Then, for $|z_n| = r_n > 2\rho$, we have

$$\left| \log E\left(\frac{z}{z_n}; p_n \right) \right| \le 2 \left(\frac{\rho}{r_n} \right)^{p_n+1} \le 2 \left(\frac{1}{2} \right)^{n+1},$$

that is $\displaystyle\sum_{|z_n|>2\rho} \left| \log E\left(\frac{z}{z_n}; p_n \right) \right| \le 2$. Hence, the series

$$\sum_{|z_n|>2R} \log E\left(\frac{z}{z_n}; p_n \right)$$

is uniformly convergent for $|z| \le R$, and so is the product

$$\prod_{|z_n|>2R} E\left(\frac{z}{z_n}, p_n \right) = \exp\left\{ \sum_{|z_n|>2R} \log E\left(\frac{z}{z_n}, p_n \right) \right\}.$$

It follows that the function

$$f(z) := z^{m_0} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}; p_n \right)$$

is regular in $|z| \le R$, and by a well-known theorem of Hurwitz [39, p. 119] its only zeros in this closed disc are those of

$$z^{m_0} \prod_{|z_n| \le 2R} E\left(\frac{z}{z_n}, p_n\right),$$

i.e., the points $0, z_1, z_2, \ldots$ with multiplicities $m_0, m_1, m_2, \ldots$, respectively. Since R may be as large as we please, Lemma 6.2 is proved. □

Lemma 6.3 (G. Mittag-Leffler). *Let $z_0 = 0, z_1, \ldots, z_n, \ldots$ be an infinite sequence of points in the complex plane $\mathbb{C}$ such that $0 < |z_n| \le |z_{n+1}|$ for $n = 1, 2, \ldots$, and $|z_n| \to \infty$ as $n \to \infty$. Associate with each $n \in \mathbb{N}$ a positive integer m_n and with 0 a non-negative integer m_0. In addition, let*

$$G_n(z) := \sum_{\nu=1}^{m_n} \frac{a_{\nu,n}}{(z - z_n)^\nu} \qquad (n = 1, 2, \ldots),$$

and

$$G_0(z) := \sum_{\nu=0}^{m_0} \frac{a_{\nu,0}}{(z - z_0)^\nu}.$$

Then there exists a meromorphic function $f(z)$ whose poles coincide with the points $z_0 = 0, z_1, \ldots, z_n, \ldots$, and whose principal part at the point z_n equals G_n for $n = 0, 1, 2, \ldots$. □

Although there is a clear analogy between Theorem 5.2 and Theorem 3.1, there is an important difference. In order to explain it, we need to recall the notion of "growth" (ρ, τ) of an entire function that was discussed in §1. We remind the reader that this notion has the same significance for a transcendental entire function as the degree has for a polynomial, and the class $C_{\rho,\tau}$ of all entire functions of growth (ρ, τ) is analogous to the class $\mathcal{P}_n$ of all polynomials of degree at most n. When we say that a polynomial f is of degree at most n we imply that $|f(z)| \le C|z|^n$ for some $C > 0$ and all z of sufficiently large modulus. Similarly, when we say that an entire function f is of growth (ρ, τ) we mean to indicate that $|f(z)| < C\mathrm{e}^{(\tau+\varepsilon)|z|^\rho}$

for any $\varepsilon > 0$, some C depending on ε, and all z of sufficiently large modulus. Theorem 3.1 says that not only there is a polynomial that satisfies (3.5) but that we can find one in $\mathcal{P}_N$, $N := \alpha_1 + \cdots + \alpha_n - 1$, that is one which does not grow faster than $|z|^N$. Although we have proved the existence of an entire function f satisfying (6.1) and have also indicated how such a function can be constructed the result is too general for us to be able to say anything about its order. It is quite possible that there may be no entire function f of finite order that satisfies (6.1). Even if we knew that the function f that has been constructed is of growth (ρ', τ'), we can rarely determine the smallest class $C_{\rho,\tau}$ that necessarily contains a function f with this property. We need to know more about the distribution of the points $z_1, \ldots, z_n, \ldots$ and also about the values $w_{j,n}$. For example, let

$$z_1 := p_1,\ z_2 := p_2\ \ldots, z_n := p_n, \ldots\ ,$$

where $p_1 < p_2 < \cdots < p_n < \cdots$ are the primes arranged in increasing order, and let $w_n := p_n^{p_n}$ for $n = 1, 2, \ldots$. Then there does not exist an entire function f of exponential type such that $f(z_n) = w_n := \mathrm{e}^{p_n \log p_n}$ for $n = 1, 2, \ldots$ since for an entire function f exponential type one can always find constants k and c such that $|f(z)| < c\,\mathrm{e}^{k|z|}$ for any $z \in \mathbb{C}$. Then there is the question of uniqueness. It arises when we know a function f to satisfy (6.1) and we know in addition the smallest class $C_{\rho,\tau}$ to which it belongs. We would then like to specify some characteristic property of f so as to set it apart from possibly other functions of the same class that satisfy (6.1). The results that help us in this respect are called uniqueness theorems . In order to illustrate this we shall give some examples.

The following two uniqueness theorems are relevant for Lagrange interpolation in uniformly distributed points on the real line. Both these results are due to Valiron [41]. We refer the reader to [9, Chapter 9] for numerous other results of this nature.

Theorem 6.4 (Valiron) *Let $f(z)$ be an entire function of exponential type π such that $f(z) = O(|z|^p)\,\mathrm{e}^{\pi|z|}$ as $|z| \to \infty$. Furthermore, let $f(z) = 0$ for $z = 0, \pm 1, \pm 2, \ldots$. Then $f(z) = P(z)\sin \pi z$, where $P(z)$ is a polynomial of degree not exceeding p.*

□

Clearly, $(\sin \pi z)/(z-n)$, $n \in \mathbb{Z}$ is an entire function of exponential type π that takes the value 1 at the point n and vanishes at any other positive or negative integer. By the preceding theorem it is *the only such function that tends to zero as $z \to \infty$ along the positive real axis.* So, this latter property sets this function apart from the others of exponential type 2π which also take the value 1 at the point n and vanish at all the other positive or negative integers, namely the functions $(\sin \pi z)/(z-n) + c \sin \pi z$, $c \neq 0$. These others are bounded on the real line but do not tend to zero as $z \to \infty$ along the positive real axis.

Theorem 6.5 (Valiron) *Let $f(z)$ be an entire function of exponential type π such that $z f(z)\, \mathrm{e}^{-\pi|z|}$ tends to zero uniformly as $|z| \to \infty$ and vanishes at least once in each of the intervals $[n, n+1)$ for all $n \in \mathbb{Z}$. Then $f(z) \equiv 0$.*

□

The case $m = 1$ of Lemma 5.3 is a uniqueness theorem useful for Hermite interpolation in uniformly distributed points on the real line. It may be noted that, $((\sin \pi z)/(z-n))^2$ is an entire function of exponential type 2π which takes the value 1 at the point n and vanishes at any other positive or negative integer. Besides, its derivative vanishes at all the integers. By the case $m = 1$ of Lemma 5.3, it is *the only such function that tends to zero as $z \to \infty$ along the positive real axis.* Similarly, $(\sin \pi z)^2/(z-n)$ is an entire function of exponential type 2π which vanishes at all the integers, positive or negative, so does its derivative except at $z = n$ where the derivative takes the value 1. It is *the only such function that tends to zero as $z \to \infty$ along the positive real axis.*

The reader may consult [24], [25] for some other uniqueness theorems relevant to Hermite interpolation by entire functions of exponential type.

In addition to the above mentioned Lemma 5.3 the following three uniqueness theorems also appear in [16]. They are all relevant to lacunary interpolation in uniformly distributed points on the real line.

Theorem 6.6 *Let $f(z)$ be an entire function of exponential type $\tau < 2\pi$ such that*

$$f(n) = f''(n) = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

then $f(z) \equiv c \sin \pi z$, where c is a constant. Here $\tau = 2\pi$ is inadmissible.

□

Let $\tau < 2\pi$. By Theorem 6.6 if there exists an entire function f of exponential type τ that takes *prescribed* values at the points $z = 0, \pm 1, \pm 2, \ldots$ and whose second derivative also takes *prescribed* values at the same set of points, then it has to be unique. However, the existence of such a function is not guaranteed. It depends on the prescribed values.

Theorem 6.7 *Let k be an even integer ≥ 4. In addition let $f(z)$ be an entire function of exponential type $\tau < \pi \sec(\pi/k)$ such that*

$$f(n) = f^{(k)}(n) = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then $f(z) \equiv c \sin \pi z$, where c is a constant. Here τ cannot be allowed to be $\pi \sec(\pi/k)$.

□

Theorem 6.8 *Let k be an odd integer ≥ 3. In addition let $f(z)$ be an entire function of exponential type $\tau < \pi \sec(\pi/2k)$ such that*

$$f(n) = f^{(k)}(n) = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then $f(z) \equiv 0$. Here τ cannot be allowed to be $\pi \sec(\pi/2k)$. □

In his doctoral dissertation, R. Brück [10] obtained various extensions of the uniqueness theorems presented in [16]. They all have direct bearing on lacunary interpolation in uniformly distributed points on the real line.

Approximation via interpolation

Any polynomial other than a constant does not remain bounded on any ray. This can also be said about entire functions of growth $(1/2, 0)$. No non-constant entire function of growth $(1, 0)$ is bounded on a line. Hence functions that are continuous and bounded on the real axis cannot be approximated uniformly closely by functions in the class $C_{1,0}$. However, the class $C_{1,\tau}$ does contain non-constant entire functions that remain bounded on the real axis, provided that τ is positive. This makes it reasonable to explore the "possibility" if any bounded continuous function $g : \mathbb{R} \to \mathbb{R}$ can be approximated arbitrarily closely by an entire function of exponential type. S. Bernstein [7] proved that this was possible *if and only if g was not only bounded but also "uniformly continuous."* Could this be done via Lagrange interpolation? In view of the negative results of Faber and Bernstein about the convergence properties of Lagrange interpolating polynomials and Lagrange interpolating trigonometric polynomials one cannot be optimistic about the possibility. For a discussion of this question we refer the reader to [18]. In that paper it is shown that Hermite interpolation in uniformly distributed points on the real line does lead to a proof of the fact that *a bounded uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$ can be approximated arbitrarily closely by entire functions entire functions of exponential type.*

For results on uniform approximation on the whole real line via lacunary interpolation by entire functions of exponential type we refer the reader to [17], [18], [19], [20], [26] and [27]. We would have liked to discuss some of those results here but due to certain constraints we cannot.

Results on mean convergence of Lagrange interpolating entire functions of exponential type, analogous to Theorem 4.3, appear in [33] and [21].

References

[1] L. Ahlfors, "Complex analysis," 2nd ed., McGraw-Hill, New York, 1966.

[2] J. Balász and P. Turán, Notes on interpolation. II. (Explicit formulae), Acta Math. Acad. Sci. Hung., **8** (1957), 201–215.

[3] J. Balász and P. Turán, Notes on interpolation. III. (Convergence), Acta Math. Acad. Sci. Hung., **9** (1958), 195–214.

[4] J. Balász and P. Turán, Notes on interpolation. IV. (Inequalities), Acta Math. Acad. Sci. Hung., **9** (1958), 243–258.

[5] S. Bernstein, Sur la convergence absolue des séries trigonométriques, Comm. de la Soc. Math. Kharkov, 2ième série, **14** (1914), 139–144.

[6] S. Bernstein, Quelques remarques sur l'interpolation, (1916) (see: Collected Works, Vol. I, 253–263).

[7] S. Bernstein, Sur la meilleure approximation sur tout l'axe réel des fonctions continues par des fonctions entières de degré fini. I, C.R. (Doklady) Acad. Sci. URSS (N.S.), **51** (1946), 331–334.

[8] G. D. Birkhoff, General mean value and remainder theorems with applications to mechanical differentiation and quadrature, Trans. Amer. Math. Soc., **7** (1906), 107–137.

[9] R.P. Boas, Jr., "Entire functions," Academic Press, New York, 1954.

[10] R. Brück, Identitätssätze für ganze Funktionen vom Exponentialtyp, Miiieilungen aus dem Mathem. Seminar Giessen, **168** (1984), 1–57.

[11] P.J. Davis, "Interpolation and approximation," 2nd ed., Blaisdell, New York, 1965.

[12] P. Erdős and E. Feldheim, Sur la mode de convergence pour l'interpolation de Lagrange, C. R. Acad. Sci. Paris **203** (1936), 913–915.

[13] G. Faber, Über die interpolatorische Darstellung stetiger Funktionen, Deutsche Mathematiker–Vereinigung Jahresbericht, **23** (1914), 192–210.

[14] L. Fejér, Über Interpolation, Nachrichten der Akademie der Wissenschaften in Göttingen (1916), 66–91.

[15] L. Fejér, Die Abschätzung eines Polynoms in einem Intervalle, wen Schranken für seine Werte und ersten Ableitungswerte in

einzelnen Punkten des Intervalles gegeben sind, und ihre Anwendung auf die Konvergenzfrage Hermitescher Interpolationsreihen, Math. Zeitschrift, **32** (1930), 426–457.

[16] R. Gervais and Q.I. Rahman, An extension of Carlson's theorem for entire functions of exponential type, Trans. Amer. Math. Soc. **235** (1978), 387–394.

[17] R. Gervais, Q.I. Rahman and G. Schmeisser, Simultaneous interpolation and approximation by entire functions of exponential type, in Numerische Methoden der Approximationstheorie, Band **4** (L. Collatz, G. Meinardus, and H. Werner, eds.), pp. 145–153, Birkhäuser Verlag, Basel, 1978. ISNM **42**, 1978.

[18] R. Gervais, Q.I. Rahman and G. Schmeisser, Simultaneous interpolation and approximation, in Polynomial and Spline Approximation, Proc. NATO Adv. Study Inst. Univ. Calgary, Calgary, Alta., pp. 203–223, NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci. **49**, Reidel, Dordrecht.

[19] R. Gervais, Q.I. Rahman and G. Schmeisser, Approximation by $(0, 2)$ interpolating entire functions of exponential type, J. Math. Anal. Appl. **82** (1981), 184–199.

[20] R. Gervais, Q.I. Rahman and G. Schmeisser, Representation and approximation of functions via $(0, 2)$ interpolation, J. Approx. Theory **50** (1987), 89–110.

[21] G.R. Grozev and Q.I. Rahman, Lagrange interpolation in the zeros of Bessel functions by entire functions of exponential type and mean convergence, Methods Appl. Anal. **3** (1996), 46–79.

[22] G. Grünwald, Über Divergenzersceinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, Ann. of Math. **37** (1936), 908-918.

[23] Ch. Hermite, Sur la formule d'interpolation de Lagrange, Journal für die reine und angewandte Mathematik **84** (1878), 70–79.

[24] J. Korevaar, "Approximation and interpolation applied to entire functions," Dissertation, Leiden, 1949.

[25] J. Korevaar, Functions of exponential type bounded on sequences of points, Ann. Soc. Polon. Math. **22** (1949), 207–234.

[26] Liu Yongping, On the trigonometric interpolation and the entire interpolatin, Approx. Theory Appl. **6** (1990), 85–106.

[27] Liu Yongping, On entire $(0, m_1, \ldots, m_q)$- interpolation on equidistant nodes, Approx. Theory Appl. **7** (1991), 27–34.

[28] J. Marcinkiewicz, Sur l'interpolation (1), Studia Math. **6** (1936), 1–17.

[29] J. Marcinkiewicz, Sur la divergence des polyñomes d'interpolation, Acta Litter. Sci. (Szeged) **8** (1937), 131–135.

[30] L. Mirsky, "An introduction to linear algebra," Dover Publications Inc., New York, 1990.

[31] I.P. Natanson, "Constructive function theory" Volume III (Interpolation and approximate quadratures), Frederick Ungar Publishing Co., Inc., New York, 1965.

[32] Q.I. Rahman, G. Schmeisser and P. Turán, On a phenomenon concerning $(0, 2)$- interpolation, Periodica Mathematica Hungarica **9** (1978), 163–171.

[33] Q.I. Rahman and P. Vértesi, On the L^p convergence of Lagrange interpolating entire functions of exponential type, J. Approx. Theory **69** (1992), 302–317.

[34] C. Runge, Über empirische Funktionen und die Interpolation zweischen äquidistanten Ordinaten, Zeitschrift für Mathematik und Physik, **46** (1901), 224–243.

[35] A. Sharma and A.K. Varma, Trigonometric interpolation, Duke Math. J., **32** (1965), 341–357.

[36] J. Surányi and P. Turán, Notes on interpolation. I. (On some interpolatorical properties of ultraspherical polynomials, Acta Math. Acad. Sci. Hung., **6** (1955), 67–79.

[37] J. Szabados and P. Vértesi, "Interpolation of functions," World Scientific, Singapore, 1990.

[38] A.F. Timan, "Theory of approximation of functions of a real variable," Pergamon Press, Oxford, 1963.

[39] E.C. Titchmarsh, "The theory of functions," 2nd ed., Oxford University Press, 1939.

[40] P. Turán, On some open problems of approximation theory, J. Approx. Theory, **29** (1980), 23–85.

[41] G. Valiron, Sur la formule d'interpolation de Lagrange, Bull. Sci. Math. (2) **49** (1925), 181–192, 203–224.

[42] A. Zygmund, "Trigonometric series," Volume II, 2nd ed., Cambridge University Press, New York, 1959.

Hyperinterpolation on the Sphere

Kerstin Hesse and Ian H. Sloan
School of Mathematics
The University of New South Wales
Sydney NSW 2052, Australia
Email addresses: k.hesse@unsw.edu.au, i.sloan@unsw.edu.au

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

In this paper we survey hyperinterpolation on the sphere $\mathbb{S}^d$, $d \geq 2$. The hyperinterpolation operator L_n is a linear projection onto the space $\mathbb{P}_n(\mathbb{S}^d)$ of spherical polynomials of degree $\leq n$, which is obtained from $L_2(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ by discretizing the integrals in the $L_2(\mathbb{S}^d)$ inner products by a positive-weight numerical integration rule of polynomial degree of exactness $2n$. Thus hyperinterpolation is a kind of "discretized orthogonal projection" onto $\mathbb{P}_n(\mathbb{S}^d)$, which is relatively easy and inexpensive to compute. In contrast, the $L_2(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ cannot generally be computed without some discretization of the integrals in the inner products; hyperinterpolation is a realization of such a discretization. We compare hyperinterpolation with $L_2(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ and with polynomial interpolation onto $\mathbb{P}_n(\mathbb{S}^d)$: we discuss the properties, estimates of the operator norms in terms of n, and estimates of the approximation error. We also present a new estimate of the approximation error of hyperinterpolation in the Sobolev space setting, that is, $L_n : H^t(\mathbb{S}^d) \to H^s(\mathbb{S}^d)$, with $t \geq s \geq 0$ and $t > \frac{d}{2}$, where $H^s(\mathbb{S}^d)$ is for integer s roughly the Sobolev space of those functions whose generalized derivatives up to the order s are square-integrable.

1 Introduction: Orthogonal Projection, Polynomial Interpolation, and Hyperinterpolation

Let $\mathbb{S}^d$, with $d \geq 2$, denote the unit sphere in $\mathbb{R}^{d+1}$,

$$\mathbb{S}^d := \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \,\middle|\, |\mathbf{x}| = 1 \right\},$$

where $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$ denotes the Euclidean norm in $\mathbb{R}^{d+1}$, and $\mathbf{x} \cdot \mathbf{y}$ is the Euclidean inner product for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d+1}$. Let us further denote by $\mathbb{P}_n(\mathbb{S}^d)$ the space of all spherical polynomials of degree $\leq n$ on $\mathbb{S}^d$, that is, the space of the restrictions to $\mathbb{S}^d$ of all polynomials on $\mathbb{R}^{d+1}$ of degree $\leq n$. Then $L_2(\mathbb{S}^d)$-orthogonal projection, interpolation, and hyperinterpolation are all linear projections onto $\mathbb{P}_n(\mathbb{S}^d)$. Hyperinterpolation is, roughly speaking, obtained from $L_2(\mathbb{S}^d)$-orthogonal projection by discretizing the inner products in the $L_2(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ with a positive-weight numerical integration rule that has polynomial degree of exactness $2n$.

This paper summarizes the known results on hyperinterpolation and compares them with the corresponding results for orthogonal projection and polynomial interpolation. Hyperinterpolation was introduced by Sloan in [21]. We start by briefly summarizing all three methods in this introductory section. Then in the next section we show that while hyperinterpolation and interpolation onto $\mathbb{P}_n(\mathbb{S}^d)$ are essentially the same on $\mathbb{S}^1$, they can never be the same on $\mathbb{S}^d$ for $d \geq 2$ if $n \geq 3$. In the third section we give estimates for the norm of the hyperinterpolation operator, and also for the approximation error, in terms of powers of n. In the last part of that section we present a new estimate of the approximation error of hyperinterpolation in a Sobolev space setting with certain assumptions on the indices. We compare the estimates with known results for $L_2(\mathbb{S}^d)$-orthogonal projection and polynomial interpolation. In the last section we make some concluding remarks about the advantages and disadvantages of hyperinterpolation compared to orthogonal projection and polynomial interpolation.

1.1 Orthogonal projection onto $\mathbb{P}_{\mathbf{n}}(\mathbb{S}^{\mathbf{d}})$

The space of continuous functions on $\mathbb{S}^d$ is denoted by $C(\mathbb{S}^d)$, and as usual it is endowed with the supremum norm

$$\|f\|_{C(\mathbb{S}^d)} := \sup_{\mathbf{x}\in\mathbb{S}^d} |f(\mathbf{x})|.$$

Let us denote by $L_2(\mathbb{S}^d)$ the Hilbert space of square-integrable functions on $\mathbb{S}^d$ with the inner product

$$(f,g)_{L_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} f(\mathbf{x})\, g(\mathbf{x})\, d\omega_d(\mathbf{x})$$

and the induced norm $\|f\|_{L_2(\mathbb{S}^d)} := (f,f)^{\frac{1}{2}}_{L_2(\mathbb{S}^d)}$, where $d\omega_d$ is the Lebesgue surface measure on $\mathbb{S}^d$. The surface area of $\mathbb{S}^d$ is denoted by ω_d,

$$\omega_d = |\mathbb{S}^d| = \frac{2\,\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}.$$

The $L_2(\mathbb{S}^d)$-*orthogonal projection* $T_n : L_2(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ is then uniquely defined by

$$T_n f \in \mathbb{P}_n(\mathbb{S}^d) \qquad \forall f \in L_2(\mathbb{S}^d)$$

and

$$(T_n f, p)_{L_2(\mathbb{S}^d)} = (f,p)_{L_2(\mathbb{S}^d)} \qquad \forall f \in L_2(\mathbb{S}^d),\ \forall p \in \mathbb{P}_n(\mathbb{S}^d).$$

Clearly we have $T_n^2 = T_n$.

From now on let $d \geq 2$, unless specified otherwise.

The restriction of any real harmonic homogeneous polynomial on $\mathbb{R}^{d+1}$ of exact degree ℓ to $\mathbb{S}^d$ is called a (real) spherical harmonic of degree ℓ. The space $\mathbb{H}_\ell(\mathbb{S}^d)$ of all (real) spherical harmonics of degree ℓ has the dimension

$$N(d,0) := 1, \qquad N(d,\ell) := \frac{(2\ell+d-1)\,(\ell+d-2)!}{(d-1)!\,\ell!}, \quad \ell \in \mathbb{N}.$$

We denote by

$$\left\{ Y_{\ell k}^{(d)} \,\middle|\, k = 1,\ldots,N(d,\ell) \right\} \tag{1.1}$$

a fixed $L_2(\mathbb{S}^d)$-orthonormal system of real spherical harmonics of degree ℓ. Clearly (1.1) is an $L_2(\mathbb{S}^d)$-orthonormal basis for $\mathbb{H}_\ell(\mathbb{S}^d)$. Moreover

$$\mathbb{P}_n(\mathbb{S}^d) = \bigoplus_{\ell=0}^{n} \mathbb{H}_\ell(\mathbb{S}^d),$$

and

$$\begin{aligned} d_n := \dim(\mathbb{P}_n(\mathbb{S}^d)) &= \sum_{\ell=0}^{n} N(d,\ell) = N(d+1,n) \\ &= \frac{(2n+d)\,(n+d-1)!}{d!\,n!}. \end{aligned}$$

Furthermore

$$L_2(\mathbb{S}^d) = \overline{\bigoplus_{\ell=0}^{\infty} \mathbb{H}_\ell(\mathbb{S}^d)}^{\|\cdot\|_{L_2(\mathbb{S}^d)}} = \overline{\bigcup_{n=0}^{\infty} \mathbb{P}_n(\mathbb{S}^d)}^{\|\cdot\|_{L_2(\mathbb{S}^d)}}.$$

Any two spherical harmonics of different degree are orthogonal, and hence the union of the sets (1.1) over all $\ell \in \mathbb{N}_0$ is a complete orthonormal system in $L_2(\mathbb{S}^d)$. Thus any function $f \in L_2(\mathbb{S}^d)$ can be represented in the $L_2(\mathbb{S}^d)$ sense by its Fourier series (or Laplace series) with respect to this complete orthonormal system of spherical harmonics:

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)}\, Y_{\ell k}^{(d)},$$

with the Fourier coefficients

$$\hat{f}_{\ell k}^{(d)} := \left(f, Y_{\ell k}^{(d)}\right)_{L_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{x})\, Y_{\ell k}^{(d)}(\mathbf{x})\, d\omega_d(\mathbf{x}).$$

The orthogonal projection operator $T_n : L_2(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ onto $\mathbb{P}_n(\mathbb{S}^d)$ can now be represented by

$$T_n f := \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)}\, Y_{\ell k}^{(d)} = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \left(f, Y_{\ell k}^{(d)}\right)_{L_2(\mathbb{S}^d)} Y_{\ell k}^{(d)}. \quad (1.2)$$

An alternative representation of T_n can be given with the help of the reproducing kernel of $\mathbb{P}_n(\mathbb{S}^d)$. The reproducing kernel of $\mathbb{P}_n(\mathbb{S}^d)$

is the uniquely determined kernel $G_n : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ with the following properties: (i) $G_n(\mathbf{x}, \cdot) \in \mathbb{P}_n(\mathbb{S}^d)$ for every fixed $\mathbf{x} \in \mathbb{S}^d$, (ii) $G_n(\mathbf{x}, \mathbf{y}) = G_n(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$, and (iii) the reproducing property

$$(f, G_n(\mathbf{x}, \cdot))_{L_2(\mathbb{S}^d)} = f(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{S}^d,\ \forall f \in \mathbb{P}_n(\mathbb{S}^d).$$

It can be easily seen that the reproducing kernel is given by

$$\begin{aligned} G_n(\mathbf{x}, \mathbf{y}) &:= \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} Y_{\ell k}^{(d)}(\mathbf{x})\, Y_{\ell k}^{(d)}(\mathbf{y}) \\ &= \frac{1}{\omega_d} \sum_{\ell=0}^{n} N(d, \ell)\, \frac{C_\ell^{\frac{d-1}{2}}(\mathbf{x} \cdot \mathbf{y})}{C_\ell^{\frac{d-1}{2}}(1)}, \end{aligned} \tag{1.3}$$

where in the last step we have applied the addition theorem for the spherical harmonics of degree $\ell \in \mathbb{N}_0$: for any $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$ we have

$$\sum_{k=1}^{N(d,\ell)} Y_{\ell k}^{(d)}(\mathbf{x})\, Y_{\ell k}^{(d)}(\mathbf{y}) = \frac{N(d, \ell)}{\omega_d}\, \frac{C_\ell^{\frac{d-1}{2}}(\mathbf{x} \cdot \mathbf{y})}{C_\ell^{\frac{d-1}{2}}(1)}. \tag{1.4}$$

In (1.4) and (1.3) the function $C_\ell^{\frac{d-1}{2}}$ is the ultraspherical (or Gegenbauer) polynomial C_ℓ^λ of degree ℓ with index $\lambda = \frac{d-1}{2}$, where

$$C_\ell^\lambda(t) = \frac{(2\lambda)_\ell}{\left(\lambda + \frac{1}{2}\right)_\ell}\, P_\ell^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(t), \qquad t \in [-1, 1], \tag{1.5}$$

with

$$(a)_0 := 1, \qquad (a)_\ell := a(a+1) \cdots (a + \ell - 1), \quad a \in \mathbb{R},\ \ell \in \mathbb{N},$$

and $P_\ell^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})} : [-1, 1] \to \mathbb{R}$ is the Jacobi polynomial $P_\ell^{(\alpha,\beta)}$ of degree ℓ with indices $\alpha = \beta = \lambda - \frac{1}{2}$. From (1.3), (1.5), and [25, (4.5.3)] we obtain

$$G_n(\mathbf{x}, \mathbf{y}) = \frac{1}{\omega_d}\, \frac{(d)_n}{\left(\frac{d}{2}\right)_n}\, P_n^{(\frac{d}{2}, \frac{d-2}{2})}(\mathbf{x} \cdot \mathbf{y}), \tag{1.6}$$

where $P_n^{(\frac{d}{2},\frac{d-2}{2})}$ is the Jacobi polynomial of degree n with indices $\alpha = \frac{d}{2}$ and $\beta = \frac{d-2}{2}$.

With the reproducing kernel G_n of $\mathbb{P}_n(\mathbb{S}^d)$ we can write the $L_2(\mathbb{S}^d)$-orthogonal projection T_n onto $\mathbb{P}_n(\mathbb{S}^d)$ as

$$T_n f(\mathbf{x}) = \int_{\mathbb{S}^d} f(\mathbf{y})\, G_n(\mathbf{x},\mathbf{y})\, d\omega_d(\mathbf{y}) = (f, G_n(\mathbf{x},\cdot))_{L_2(\mathbb{S}^d)}\,, \qquad (1.7)$$

$\mathbf{x} \in \mathbb{S}^d$. The formulas (1.2) and (1.7) are our starting point for the discussion of hyperinterpolation.

Clearly the $L_2(\mathbb{S}^d)$-orthogonal projection T_n onto $\mathbb{P}_n(\mathbb{S}^d)$ is the "optimal" projection onto $\mathbb{P}_n(\mathbb{S}^d)$ in the $L_2(\mathbb{S}^d)$ sense. However, $T_n f$ cannot usually be computed exactly since the Fourier coefficients $\hat{f}_{\ell k}^{(d)} = (f, Y_{\ell k}^{(d)})_{L_2(\mathbb{S}^d)}$ are not known. Instead we might only know f at a discrete set of points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and want to compute a projection of f onto $\mathbb{P}_n(\mathbb{S}^d)$ which is a suitable approximation of f. Two possible strategies come directly to mind: One option is to discretize the Fourier coefficients of f in (1.2) in a suitable way with the help of a numerical integration rule; this is the idea behind hyperinterpolation. Another possible strategy, if $m = d_n$, is to compute the polynomial interpolant of f onto $\mathbb{P}_n(\mathbb{S}^d)$ from the given data of f. Both methods assume of course that the points $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ satisfy certain assumptions.

1.2 Polynomial interpolation onto $\mathbb{P}_\mathbf{n}(\mathbb{S}^\mathbf{d})$

Let $\{\Phi_1, \ldots, \Phi_{d_n}\}$ be an arbitrary basis of $\mathbb{P}_n(\mathbb{S}^d)$. Any point set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{d_n}\}$ for which the matrix $(\Phi_i(\mathbf{x}_j))_{i,j=1,\ldots,d_n}$ is invertible is called a *fundamental system*. The definition of a fundamental system is independent of the choice of the basis $\{\Phi_1, \ldots, \Phi_{d_n}\}$.

For a fundamental system $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{d_n}\}$ the interpolation problem in $\mathbb{P}_n(\mathbb{S}^d)$ has a unique solution: for a given continuous function f, there exists exactly one polynomial $\Lambda_n f \in \mathbb{P}_n(\mathbb{S}^d)$ which satisfies the interpolation conditions

$$\Lambda_n f(\mathbf{x}_j) = f(\mathbf{x}_j), \qquad j = 1, \ldots, d_n. \qquad (1.8)$$

The *interpolation operator* $\Lambda_n : C(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ is a linear projection operator onto $\mathbb{P}_n(\mathbb{S}^d)$ because $\Lambda_n p = p$ for all $p \in \mathbb{P}_n(\mathbb{S}^d)$, and hence $\Lambda_n^2 = \Lambda_n$.

With respect to the basis $\{\Phi_1, \ldots, \Phi_{d_n}\}$, the polynomial interpolant $\Lambda_n f$ of a given continuous function f has the form

$$\Lambda_n f = \sum_{i=1}^{d_n} a_i \, \Phi_i$$

where the coefficients are the solutions of the linear system of equations

$$\sum_{i=1}^{d_n} a_i \, \Phi_i(\mathbf{x}_j) = f(\mathbf{x}_j), \qquad j = 1, \ldots, d_n.$$

The condition number of the matrix $(\Phi_i(\mathbf{x}_j))_{i,\,j=1,\ldots,d_n}$ depends strongly on both the choice of basis and the quality of the fundamental system $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{d_n}\}$. There are many "bad" fundamental systems for which the matrix $(\Phi_i(\mathbf{x}_j))_{i,j=1,\ldots,d_n}$ is so close to singular that the interpolation problem cannot be solved in practice.

Another representation of (1.8) can be obtained with the help of the Lagrangians (or Lagrange polynomials) $\ell_1, \ldots, \ell_{d_n}$ of the fundamental system $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{d_n}\}$. The Lagrangian ℓ_i, $i \in \{1, \ldots, d_n\}$, is defined as the uniquely determined polynomial $\ell_i \in \mathbb{P}_n(\mathbb{S}^d)$ with the property that

$$\ell_i(\mathbf{x}_j) = \delta_{i,j}, \qquad j = 1, \ldots, d_n,$$

where $\delta_{i,j}$ is the Kronecker symbol, defined by $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. In terms of the Lagrangians the polynomial interpolant $\Lambda_n f$ of a continuous function f can be written as

$$\Lambda_n f(\mathbf{x}) = \sum_{i=1}^{d_n} f(\mathbf{x}_i) \, \ell_i(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d. \tag{1.9}$$

The formulation (1.9) may appear to avoid the necessity to solve a linear system, but this is misleading, because the computation of even one individual Lagrangian ℓ_i demands the solution of a linear system.

We observe that interpolation allows much freedom on the points in which the data is given, as the only assumption is that the points form a fundamental system. A number of explicit constructions of fundamental systems on $\mathbb{S}^2$ are known (see [4, 7, 28, 29]). However, for computational purposes it is also important that the interpolation matrix should have a reasonable condition number, a requirement which imposes a severe restriction.

Of particular interest among fundamental systems are the *extremal* (fundamental) systems, obtained by maximizing the determinant of the matrix $(\Phi_i(\mathbf{x}_j))_{i,j=1,\ldots,d_n}$ with respect to an arbitrary basis $\{\Phi_1,\ldots,\Phi_{d_n}\}$. Properties of extremal systems are summarized in [24].

1.3 Hyperinterpolation onto $\mathbb{P}_\mathbf{n}(\mathbb{S}^\mathbf{d})$

Hyperinterpolation is a numerical discretization of the $L_2(\mathbb{S}^d)$-orthogonal projection $T_n : L_2(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ for continuous functions. It was first introduced in [21] by Sloan for general regions, with the sphere as one example.

Let $Q_{m(n)}$ be a positive-weight $m(n)$-point numerical integration rule

$$Q_{m(n)}f := \sum_{j=1}^{m(n)} w_j\, f(\mathbf{x}_j), \qquad f \in C(\mathbb{S}^d), \tag{1.10}$$

with points $\mathbf{x}_1,\ldots,\mathbf{x}_{m(n)} \in \mathbb{S}^d$ and corresponding positive weights $w_1,\ldots,w_{m(n)}$, which integrates all polynomials in $\mathbb{P}_{2n}(\mathbb{S}^d)$ exactly, that is,

$$Q_{m(n)}p = Ip := \int_{\mathbb{S}^d} p(\mathbf{x})\, d\omega_d(\mathbf{x}) \qquad \forall p \in \mathbb{P}_{2n}(\mathbb{S}^d).$$

With the help of the numerical integration rule $Q_{m(n)}$ we can now define a *discrete (semi) inner product* on $C(\mathbb{S}^d)$ by

$$(f,g)_{m(n)} := Q_{m(n)}(fg) = \sum_{j=1}^{m(n)} w_j\, f(\mathbf{x}_j)\, g(\mathbf{x}_j), \qquad f,g \in C(\mathbb{S}^d). \tag{1.11}$$

For any two spherical polynomials $p,q \in \mathbb{P}_n(\mathbb{S}^d)$ the product $p\,q$ is a spherical polynomial in $\mathbb{P}_{2n}(\mathbb{S}^d)$. Thus it follows from the exactness of $Q_{m(n)}$ for polynomials of degree $\leq 2n$ that

$$(p,q)_{m(n)} = (p,q)_{L_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} p(\mathbf{x})\, q(\mathbf{x})\, d\omega(\mathbf{x}) \qquad \forall p, q \in \mathbb{P}_n(\mathbb{S}^d). \tag{1.12}$$

We remark that (1.11) does not have all the properties of an inner product on $C(\mathbb{S}^d)$. In particular, for f a continuous function on $\mathbb{S}^d$, $(f,f,)_{m(n)} = 0$ does not imply that $f \equiv 0$, since it is possible to construct a function f which is not identically zero but vanishes at all the nodes $\mathbf{x}_1, \ldots, \mathbf{x}_{m(n)}$ of the numerical integration rule $Q_{m(n)}$.

After these preparations we can define the *hyperinterpolation operator* $L_n : C(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$. For $f \in C(\mathbb{S}^d)$ the hyperinterpolant $L_n f$ is defined to be the projection of f onto $\mathbb{P}_n(\mathbb{S}^d)$ obtained by replacing the $L_2(\mathbb{S}^d)$-inner products in the $L_2(\mathbb{S}^d)$-orthogonal projection $T_n f$ by the discrete inner products (1.11). Thus from (1.2)

$$L_n f(\mathbf{x}) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \left(f, Y_{\ell k}^{(d)}\right)_{m(n)} Y_{\ell k}^{(d)}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d, \tag{1.13}$$

or with respect to an arbitrary orthonormal basis $\{\Phi_1, \ldots, \Phi_{d_n}\}$ of $\mathbb{P}_n(\mathbb{S}^d)$

$$L_n f(\mathbf{x}) = \sum_{i=1}^{d_n} (f, \Phi_i)_{m(n)}\, \Phi_i(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d. \tag{1.14}$$

Corresponding to (1.7) we can write

$$L_n f(\mathbf{x}) = (f, G_n(\mathbf{x}, \cdot))_{m(n)} = \sum_{j=1}^{m(n)} w_j\, f(\mathbf{x}_j)\, G_n(\mathbf{x}, \mathbf{x}_j), \tag{1.15}$$

$\mathbf{x} \in \mathbb{S}^d$. The last representation (1.15) of the hyperinterpolant is the easiest and usually least expensive to evaluate, since we have the nice representation (1.6) of the reproducing kernel G_n of $\mathbb{P}_n(\mathbb{S}^d)$.

From (1.12) we have for $p \in \mathbb{P}_n(\mathbb{S}^d)$ that $(p, Y_{\ell k}^{(d)})_{m(n)} = \hat{p}_{\ell k}^{(d)}$, and hence

$$L_n p = T_n p = p \qquad \forall p \in \mathbb{P}_n(\mathbb{S}^d). \tag{1.16}$$

In particular (1.16) implies that $L_n^2 = L_n$, that is, the hyperinterpolation operator is a linear projector onto $\mathbb{P}_n(\mathbb{S}^d)$.

We mention that the hyperinterpolation operator and the discrete inner product have also the following properties: for f in $C(\mathbb{S}^d)$

(i) $(f - L_n f, p)_{m(n)} = 0$ for all $p \in \mathbb{P}_n(\mathbb{S}^d)$,

(ii) $(L_n f, L_n f)_{m(n)} + (f - L_n f, f - L_n f)_{m(n)} = (f, f)_{m(n)}$,

(iii) $(L_n f, L_n f)_{m(n)} \leq (f, f)_{m(n)}$, and

(iv) $(f - L_n f, f - L_n f)_{m(n)} = \min_{p \in \mathbb{P}_n(\mathbb{S}^d)} (f - p, f - p)_{m(n)}$.

The property (i) shows that L_n is the orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ with respect to the discrete inner product $(\cdot,\cdot)_{m(n)}$. The property (ii) is the corresponding Pythagoras theorem. The property (iii) is a trivial consequence of (ii), and (iv) states that $L_n f$ is the best discrete least-squares approximation (weighted by the numerical integration weights) of f at the nodes of the numerical integration rule. The proofs of the properties (i) to (iv) can be found in [21, Lemma 5].

Compared to polynomial interpolation, the hyperinterpolation approximation $L_n f$ can be much more easily and less expensively computed than the polynomial interpolant $\Lambda_n f$, provided that we know the function f in the points $\mathbf{x}_1, \ldots, \mathbf{x}_{m(n)}$ of the numerical integration rule $Q_{m(n)}$.

2 Why Is Hyperinterpolation Different from Polynomial Interpolation?

It is useful to consider first the case of the unit circle

$$\mathbb{S}^1 := \{ (\cos\theta, \sin\theta) \mid 0 \leq \theta \leq 2\pi \} .$$

In this case $\mathbb{P}_n(\mathbb{S}^1)$ is the space of trigonometric polynomials of degree $\leq n$,

$$\mathbb{P}_n(\mathbb{S}^1) = \mathrm{span}\,\{1, \cos(\theta), \ldots, \cos(n\theta), \sin(\theta), \ldots, \sin(n\theta)\} ,$$

a space of dimension $2n+1$. Interpolation onto the space $\mathbb{P}_n(\mathbb{S}^1)$ at the $d_n := 2n+1$ uniformly distributed points

$$\mathbf{x}_j := \left(\cos\left(j\,\frac{2\pi}{2n+1}\right), \sin\left(j\,\frac{2\pi}{2n+1}\right)\right), \qquad j = 1, \ldots, 2n+1, \tag{2.1}$$

has been well understood since at least the 1930s, see [30, Chapter 10]. For example, it is known classically that the interpolant $\Lambda_n f$ with respect to the set of equally spaced points (2.1) and the $L_2(\mathbb{S}^1)$-orthogonal projection $T_n f$ of a continuous function f onto $\mathbb{P}_n(\mathbb{S}^1)$ satisfy

$$\begin{aligned} \|\Lambda_n f\|_{L_2(\mathbb{S}^1)} &\leq \sqrt{2\pi}\,\|f\|_{C(\mathbb{S}^1)}, \\ \|T_n f\|_{L_2(\mathbb{S}^1)} &\leq \sqrt{2\pi}\,\|f\|_{C(\mathbb{S}^1)}. \end{aligned} \tag{2.2}$$

Neither result can be improved, since the inequalities are sharp for $f(x) \equiv 1$.

The proof of (2.2) is simple but interesting, in that it uses quadrature as a tool in the argument, despite the fact that the statement itself has no direct relation to quadrature. The key is the easily verified fact that the $2n+1$ equally spaced interpolation points $\mathbf{x}_j$, $j = 1, \ldots, 2n+1$, given by (2.1) form also the nodes of an equal-weight quadrature rule

$$Q_{2n+1} f := \frac{2\pi}{2n+1} \sum_{j=1}^{2n+1} f(\mathbf{x}_j), \tag{2.3}$$

which can easily be seen to have polynomial degree of exactness $2n$. The steps of the proof of (2.2) are

$$\begin{aligned} \|\Lambda_n f\|^2_{L_2(\mathbb{S}^1)} &:= \int_{\mathbb{S}^1} |\Lambda_n f(\mathbf{x})|^2 \, d\omega(x) \\ &= \frac{2\pi}{2n+1} \sum_{j=1}^{2n+1} |\Lambda_n f(\mathbf{x}_j)|^2 \\ &= \frac{2\pi}{2n+1} \sum_{j=1}^{2n+1} |f(\mathbf{x}_j)|^2 \\ &\leq 2\pi\,\|f\|^2_{C(\mathbb{S}^1)}, \end{aligned}$$

where in the second line we used the fact that $|\Lambda_n f|^2$ is a trigonometric polynomial of degree $\leq 2n$ and is hence integrated exactly by the equal-weight quadrature rule (2.3), and in the next that $\Lambda_n f(\mathbf{x}_j) = f(\mathbf{x}_j)$ for $j = 1, \ldots, 2n+1$ from the interpolation property.

Now consider hyperinterpolation for the case of the circle $\mathbb{S}^1$, using the positive-weight quadrature rule (2.3) which, because it has polynomial degree of exactness $2n$, is a valid rule for generating the hyperinterpolant $L_n f$ by way of (1.13) or (1.15). But by replacing, from the interpolation property, $f(\mathbf{x}_j)$ by $\Lambda_n f(\mathbf{x}_j)$ in (1.15) we find, making use of the exactness of Q_{2n+1} on $\mathbb{P}_{2n}(\mathbb{S}^1)$,

$$\begin{aligned} L_n f(\mathbf{x}) &= \frac{2\pi}{2n+1} \sum_{j=1}^{2n+1} \Lambda_n f(\mathbf{x}_j)\, G_n(\mathbf{x}, \mathbf{x}_j) \\ &= \int_{\mathbb{S}^1} \Lambda_n f(\mathbf{y})\, G_n(\mathbf{x}, \mathbf{y})\, d\omega(\mathbf{y}) \\ &= \Lambda_n f(\mathbf{x}). \end{aligned}$$

Thus hyperinterpolation with the equal-weight quadrature rule (2.3) on $\mathbb{S}^1$ is just the same as interpolation with respect to the set of uniformly distributed points (2.1), and there is no need for another definition. This explains why hyperinterpolation does not arise in the classical literature.

But for $d \geq 2$ the situation is quite different, with hyperinterpolation in general being distinct from interpolation. Specifically, in 1995 Sloan [21] showed, in the case of $\mathbb{S}^d$ with $d \geq 2$, that for $n \geq 3$ the hyperinterpolation operator L_n and the interpolation operator Λ_n cannot be the same. More precisely the following theorem holds.

Theorem 2.1 *Let $d \geq 2$, and let $Q_{m(n)}$ be a positive-weight numerical integration rule (1.10), with points $\mathbf{x}_1, \ldots, \mathbf{x}_{m(n)} \in \mathbb{S}^d$ and corresponding positive weights $w_1, \ldots, w_{m(n)}$, which has polynomial degree of exactness $2n$, that is, $Q_{m(n)} p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$. Let $L_n : C(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ be the hyperinterpolation operator defined with the numerical integration rule $Q_{m(n)}$. Then*

(i) $m(n) \geq d_n$.

(ii) If $m(n) = d_n$ then the nodes of $Q_{m(n)}$ form a fundamental system and we have $L_n f = \Lambda_n f$ for all $f \in C(\mathbb{S}^d)$, where Λ_n is the interpolation operator onto $\mathbb{P}_n(\mathbb{S}^d)$ with respect to the nodes of $Q_{m(n)}$.

(iii) If $n \geq 3$ then $m(n) > d_n$.

We show the proof from [21] below, because the proof gives rather interesting insights, resting as it does on deep properties of spherical designs. In the proof we will need some information about minimal numerical integration rules and spherical t-designs.

First we observe that any positive-weight $m(n)$-point numerical integration rule $Q_{m(n)}$ that satisfies $Q_{m(n)}p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$ uses $m(n) \geq d_n$ points. This can easily be seen as follows (see for example [12, 13]): Let $\{\Phi_1, \ldots, \Phi_{d_n}\}$ be an arbitrary orthonormal basis of $\mathbb{P}_n(\mathbb{S}^d)$. Then from the exactness of $Q_{m(n)}$ on $\mathbb{P}_{2n}(\mathbb{S}^d)$ we have

$$\sum_{j=1}^{m(n)} w_j \, \Phi_i(\mathbf{x}_j) \, \Phi_k(\mathbf{x}_j) = (\Phi_i, \Phi_k)_{L_2(\mathbb{S}^d)} = \delta_{i,k}, \qquad i, k = 1, \ldots, d_n.$$

This implies that the $m(n) \times d_n$-matrix

$$\mathbf{A} := (\sqrt{w_j} \, \Phi_i(\mathbf{x}_j))_{j=1,\ldots,m(n);\, i=1,\ldots,d_n}$$

satisfies $\mathbf{A}^T \mathbf{A} = \mathbf{I}_{\mathbb{R}^{d_n}}$, where $\mathbf{I}_{\mathbb{R}^{d_n}}$ is the identity matrix of $\mathbb{R}^{d_n}$. Thus the columns of $\mathbf{A}$ are orthogonal, from which we deduce $m(n) \geq d_n$.

We speak of a *minimal positive-weight numerical integration rule* $Q_{m(n)}$ on $\mathbb{S}^d$ if $Q_{m(n)}$ has polynomial degree of exactness $2n$ and uses $m(n) = d_n$ points.

The notion of a *spherical t-design* was first introduced in [6] by Delsarte, Goethals, and Seidel. A spherical t-design on $\mathbb{S}^d$ is a point set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ which gives rise to an equal-weight numerical integration rule that is exact on $\mathbb{P}_t(\mathbb{S}^d)$, that is, $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is a spherical t-design if

$$Q_m p := \frac{\omega_d}{m} \sum_{j=1}^{m} p(\mathbf{x}_j) = \int_{\mathbb{S}^d} p(\mathbf{x}) \, d\omega_d(\mathbf{x}) \qquad \forall p \in \mathbb{P}_t(\mathbb{S}^d).$$

Seymour and Zaslavsky showed in [20] that spherical t-designs exist on $\mathbb{S}^d$, for every $d \geq 1$ and every value of t.

A spherical $2n$-design $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is called a *tight spherical $2n$-design* if the number of points m satisfies $m = d_n$, that is, if it is a minimal numerical integration rule.

From the discussion of minimal positive-weight numerical integration rules we know that there are no spherical $2n$-designs with less than d_n points. The key to the proof of the last part of Theorem 2.1 is a deep result of Bannai and Damerell [1], that there are *no* tight spherical $2n$-designs on $\mathbb{S}^d$ with $d \geq 2$ and $n \geq 3$.

Now we can give the proof of Theorem 2.1.

PROOF. In the discussion above we have seen that there are no numerical integration rules $Q_{m(n)}$ that satisfy $Q_{m(n)}p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$ with less than d_n points. This verifies (i).

To prove (ii) and (iii), assume that the hyperinterpolation operator L_n is defined with a positive-weight numerical integration rule $Q_{m(n)}$ that uses $m(n) = d_n$ points. Then the matrix $\mathbf{A}$ defined above is now a square matrix, $\mathbf{A} = (\sqrt{w_j}\,\Phi_i(\mathbf{x}_j))_{j,i=1,\ldots,d_n}$, with (as noted already) orthogonal columns. Therefore $\mathbf{A}$ is an orthogonal matrix. In particular, $\mathbf{A}$ is invertible, implying that $(\Phi_i(\mathbf{x}_j))_{i,j=1,\ldots,d_n}$ is also invertible, and hence $\{\Phi_1, \ldots, \Phi_{d_n}\}$ is a fundamental system. This proves the first part of the statement (ii).

Now we prove the second part of (ii). Because $\mathbf{A}$ is orthogonal, we deduce $\mathbf{A}\,\mathbf{A}^T = \mathbf{I}_{\mathbb{R}^{d_n}}$, that is,

$$\sum_{i=1}^{d_n} \sqrt{w_j}\,\Phi_i(\mathbf{x}_j)\,\sqrt{w_k}\,\Phi_i(\mathbf{x}_k) = \delta_{j,k}, \qquad j,k = 1,\ldots,d_n,$$

or

$$\sum_{i=1}^{d_n} \Phi_i(\mathbf{x}_j)\,\Phi_i(\mathbf{x}_k) = w_j^{-1}\,\delta_{j,k}, \qquad j,k = 1,\ldots,d_n. \tag{2.4}$$

Using this in the definition of the hyperinterpolation operator (1.14), we obtain for every $f \in C(\mathbb{S}^d)$

$$L_n f(\mathbf{x}_k) \;=\; \sum_{i=1}^{d_n} \left(\sum_{j=1}^{d_n} w_j\, f(\mathbf{x}_j)\,\Phi_i(\mathbf{x}_j) \right) \Phi_i(\mathbf{x}_k)$$

$$
\begin{aligned}
&= \sum_{j=1}^{d_n} w_j\, f(\mathbf{x}_j) \sum_{i=1}^{d_n} \Phi_i(\mathbf{x}_j)\, \Phi_i(\mathbf{x}_k) \\
&= \sum_{j=1}^{d_n} w_j\, f(\mathbf{x}_j)\, w_j^{-1}\, \delta_{j,k} \\
&= f(\mathbf{x}_k), \qquad k = 1, \ldots, d_n.
\end{aligned}
$$

This establishes the interpolation condition, and as the interpolant in $\mathbb{P}_n(\mathbb{S}^d)$ is uniquely determined, we conclude that $L_n f = \Lambda_n f$ for all $f \in C(\mathbb{S}^d)$. This verifies (ii).

Now we prove (iii), by showing a contradiction. We choose the $L_2(\mathbb{S}^d)$-orthonormal basis $\{\Phi_1, \ldots, \Phi_{d_n}\}$ in the definition of $\mathbf{A}$ above to be the $L_2(\mathbb{S}^d)$-orthonormal basis

$$
\left\{ Y_{\ell k}^{(d)} \,\middle|\, \ell = 0, \ldots, n;\ k = 1, \ldots, N(d, \ell) \right\}
$$

of spherical harmonics of degree $\leq n$. For this choice we have from (2.4) together with the addition theorem (1.4)

$$
\begin{aligned}
\sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} Y_{\ell k}^{(d)}(\mathbf{x}_i)\, Y_{\ell k}^{(d)}(\mathbf{x}_j) &= \frac{1}{\omega_d} \sum_{\ell=0}^{n} N(d, \ell)\, \frac{C_\ell^{\frac{d-1}{2}}(\mathbf{x}_i \cdot \mathbf{x}_j)}{C_\ell^{\frac{d-1}{2}}(1)} \\
&= \frac{1}{w_j}\, \delta_{i,j}, \qquad i, j = 1, \ldots, d_n.
\end{aligned}
$$

For $i = j$ we obtain in particular

$$
\frac{w_j}{\omega_d} \sum_{\ell=0}^{n} N(d, \ell) = w_j\, \frac{d_n}{\omega_d} = 1, \qquad j = 1, \ldots, d_n,
$$

or

$$
w_j = \frac{\omega_d}{d_n}, \qquad j = 1, \ldots, d_n.
$$

That is, the numerical integration rule $Q_{m(n)}$ has to have equal weights. As it uses $m(n) = d_n$ points and has polynomial degree of exactness $2n$, the numerical integration rule $Q_{m(n)}$ is a tight spherical $2n$-design. As we have noted already, from [1] tight spherical $2n$-designs do not exist for $n \geq 3$, giving a contradiction, and proving (iii). $\square$

Remark 2.2 *Note that the last part of the proof establishes that any* minimal positive-weight *numerical integration rule on* $\mathbb{S}^d$ *(with* $d \geq 2$*) has* equal *weights. As tight spherical* $2n$*-designs for* $\mathbb{S}^d$ *do not exist for* $n \geq 3$*, there are no minimal positive-weight numerical integration rules on* $\mathbb{S}^d$ *with polynomial degree of exactness* $2n \geq 6$*.*

3 Analysis of the Hyperinterpolation Error

In this section we discuss error estimates and estimates of the operator norm of the hyperinterpolation operator in the following settings: $L_n : C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)$; $L_n : C(\mathbb{S}^d) \to C(\mathbb{S}^d)$; and finally $L_n : H^t(\mathbb{S}^d) \to H^s(\mathbb{S}^d)$, where in the last case $H^s(\mathbb{S}^d)$ and $H^t(\mathbb{S}^d)$ are Sobolev spaces with $t \geq s \geq 0$ and $t > \frac{d}{2}$. In most cases we will compare the estimates of the operator norms with estimates of the norm of the $L_2(\mathbb{S}^d)$-orthogonal projection operator T_n and the polynomial interpolation operator Λ_n.

3.1 Estimates in the $\mathbf{L_2}(\mathbb{S}^{\mathbf{d}})$-norm

For the hyperinterpolation operator $L_n : C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)$, Sloan proved in [21] the following result.

Theorem 3.1 *Let* $Q_{m(n)}$ *be a positive-weight* $m(n)$*-point numerical integration rule with* $Q_{m(n)}p = Ip$ *for all* $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$*, and let* $L_n\colon C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)$ *be the hyperinterpolation projection onto* $\mathbb{P}_n(\mathbb{S}^d)$ *defined with the rule* $Q_{m(n)}$*. Then*

(i)

$$\|L_n\|_{C(\mathbb{S}^d)\to L_2(\mathbb{S}^d)} = \sqrt{\omega_d}. \tag{3.1}$$

(ii) For all $f \in C(\mathbb{S}^d)$

$$\|L_n f - f\|_{L_2(\mathbb{S}^d)} \leq 2\sqrt{\omega_d}\, E_n\Big(f; C(\mathbb{S}^d)\Big), \tag{3.2}$$

where $E_n(f; C(\mathbb{S}^d)) := \min_{p\in\mathbb{P}_n(\mathbb{S}^d)} \|f - p\|_{C(\mathbb{S}^d)}$ *is the error of the best uniform approximation of* f *in* $\mathbb{P}_n(\mathbb{S}^d)$*.*

The theorem and the proof in [21] are given for general regions, with $\mathbb{S}^d$ as a special case. As the proof is rather short we show it for the case of the region $\mathbb{S}^d$.

PROOF. Let us again denote by $(f,g)_{m(n)} := Q_{m(n)}(fg)$ the discrete inner product induced by the numerical integration rule $Q_{m(n)}$. Let $f \in C(\mathbb{S}^d)$ be arbitrary. Then from $L_n f \in \mathbb{P}_n(\mathbb{S}^d)$ together with the exactness of $Q_{m(n)}$ on $\mathbb{P}_{2n}(\mathbb{S}^d)$ and the property (iii) of the discrete inner product $(\cdot,\cdot)_{m(n)}$, we find

$$\begin{aligned} \|L_n f\|^2_{L_2(\mathbb{S}^d)} &= (L_n f, L_n f)_{m(n)} \\ &\leq (f,f)_{m(n)} \\ &= \sum_{j=1}^{m(n)} w_j\, |f(\mathbf{x}_j)|^2 \\ &\leq \sum_{j=1}^{m(n)} w_j\, \|f\|^2_{C(\mathbb{S}^d)} \\ &= \omega_d\, \|f\|^2_{C(\mathbb{S}^d)}. \end{aligned}$$

This implies

$$\|L_n\|_{C(\mathbb{S}^d)\to L_2(\mathbb{S}^d)} \leq \sqrt{\omega_d}.$$

It remains to show that we obtain equality. This is achieved by choosing $f \equiv 1$, since then $L_n f = 1$, and

$$\|L_n f\|_{L_2(\mathbb{S}^d)} = \left(\int_{\mathbb{S}^d} 1\, d\omega_d(\mathbf{x})\right)^{\frac{1}{2}} = \sqrt{\omega_d} = \sqrt{\omega_d}\, \|f\|_{C(\mathbb{S}^d)},$$

implying $\|L_n\|_{C(\mathbb{S}^d)\to L_2(\mathbb{S}^d)} \geq \sqrt{\omega_d}$.

The statement (ii) follows by a standard trick: for any $p \in \mathbb{P}_n(\mathbb{S}^d)$, we have $L_n p = p$, and hence from the estimate (i)

$$\begin{aligned} \|L_n f - f\|_{L_2(\mathbb{S}^d)} &= \|L_n(f-p) - (f-p)\|_{L_2(\mathbb{S}^d)} \\ &\leq \|L_n(f-p)\|_{L_2(\mathbb{S}^d)} + \|f-p\|_{L_2(\mathbb{S}^d)} \\ &\leq \sqrt{\omega_d}\, \|f-p\|_{C(\mathbb{S}^d)} + \sqrt{\omega_d}\, \|f-p\|_{C(\mathbb{S}^d)} \\ &= 2\sqrt{\omega_d}\, \|f-p\|_{C(\mathbb{S}^d)}. \end{aligned} \tag{3.3}$$

As the estimate (3.3) is true for any $p \in \mathbb{P}_n(\mathbb{S}^d)$, we can replace $\|f - p\|_{C(\mathbb{S}^d)}$ by $E_n(f; C(\mathbb{S}^d))$, obtaining (ii). $\square$

Finally we compare the results in Theorem 3.1 with corresponding results for T_n and Λ_n. First, for $T_n : C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)$ we have, for any $f \in C(\mathbb{S}^d)$,

$$\|T_n f\|_{L_2(\mathbb{S}^d)} \leq \|f\|_{L_2(\mathbb{S}^d)} \leq \sqrt{\omega_d}\, \|f\|_{C(\mathbb{S}^d)},$$

and again all inequalities are replaced by equalities for the case $f \equiv 1$. Thus

$$\|T_n\|_{C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)} = \sqrt{\omega_d},$$

which is the same value as for $\|L_n\|_{C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)}$. The same argument as in (3.3) then yields

$$\|T_n f - f\|_{L_2(\mathbb{S}^d)} \leq 2\sqrt{\omega_d}\, E_n\Big(f; C(\mathbb{S}^d)\Big), \qquad f \in C(\mathbb{S}^d). \tag{3.4}$$

Thus for L_n and T_n we have identical estimates for the approximation error, from (3.2) and (3.4).

In contrast, Sloan in [22] showed that the interpolation operator onto $\mathbb{P}_n(\mathbb{S}^d)$ in the setting $\Lambda_n : C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)$ satisfies

$$\|\Lambda_n\|_{C(\mathbb{S}^d) \to L_2(\mathbb{S}^d)} \geq \sqrt{\omega_d},$$

with strict inequality for $d \geq 2$ and $n \geq 3$. The strictness of the inequality was proved by showing that equality for $d \geq 2$ and $n \geq 3$ would contradict the previously mentioned non-existence of tight spherical designs. As far as we know, there are no useful estimates, either theoretical or empirical, of the minimal values of $\|\Lambda_n\|_{C(\mathbb{S}^2) \to L_2(\mathbb{S}^d)}$. It is not even known whether for a particular d, say $d = 2$, the minimal norm approaches ∞ as $n \to \infty$, or alternatively is bounded as $n \to \infty$.

3.2 Estimates in the uniform norm

It is convenient to begin the discussion of the $C(\mathbb{S}^d)$ to $C(\mathbb{S}^d)$ setting with the case of the orthogonal projection operator T_n. The $L_2(\mathbb{S}^d)$-orthogonal projection $T_n : C(\mathbb{S}^d) \to C(\mathbb{S}^d)$ is the minimal norm projection among all linear projection operators $\mathcal{P}_n : C(\mathbb{S}^d) \to C(\mathbb{S}^d)$

onto $\mathbb{P}_n(\mathbb{S}^d)$. That is, if $\mathcal{P}_n : C(\mathbb{S}^d) \to C(\mathbb{S}^d)$ is an arbitrary linear projection onto $\mathbb{P}_n(\mathbb{S}^d)$, then

$$\|T_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} \leq \|\mathcal{P}_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}. \tag{3.5}$$

This result was originally proved by Berman in [2] for the case $d = 1$ and then generalized by Daugavet in [5] to general d. For a proof for $d \geq 2$ see also [17, Section 12]. The norm $\|T_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$ satisfies

$$\tilde{c}_1\, n^{\frac{d-1}{2}} \leq \|T_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} \leq \tilde{c}_2\, n^{\frac{d-1}{2}}, \tag{3.6}$$

where the positive constants $\tilde{c}_1$ and $\tilde{c}_2$ are independent of n. The estimate (3.6) was proved by Gronwall in [9] for $d = 2$, and a proof of the generalization to arbitrary $d \geq 2$ can be found in [17, Section 11].

Turning now to the hyperinterpolation operator L_n, we see already that the best result we can hope to obtain for the norm of L_n in the $C(\mathbb{S}^d)$ to $C(\mathbb{S}^d)$ setting is growth of the order $n^{\frac{d-1}{2}}$. The following theorem shows that this best possible result is in fact achieved. Sloan and Womersley in [23, Theorem 5.5.4] proved a modified version of the following result, for the case $d = 2$, and that result was extended to general $d \geq 2$ by Le Gia and Sloan in [10, Theorem 6.2] and by Reimer in [18]. The theorem was proved in the form stated here by Reimer in [18, Theorem 1].

Theorem 3.2 *There exist positive constants $\tilde{c}$ and c such that for any hyperinterpolation operator $L_n : C(\mathbb{S}^2) \to C(\mathbb{S}^d)$ (defined with a positive-weight numerical integration rule $Q_{m(n)}$ with $Q_{m(n)}p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$)*

$$\tilde{c}\, n^{\frac{d-1}{2}} \leq \|L_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} \leq c\, n^{\frac{d-1}{2}}. \tag{3.7}$$

The positive constants $\tilde{c}$ and c are independent of n and of the particular positive-weight numerical integration rule $Q_{m(n)}$ which is used in the definition of L_n.

In this paper c, $\tilde{c}$, and $\hat{c}$ denote generic constants that may have different values at different places, whereas $c_0, c_1, \ldots$, $\tilde{c}_1, \tilde{c}_2, \ldots$, and $\hat{c}_1, \hat{c}_2, \ldots$ denote constants with fixed values.

We sketch below the proof of the upper bound. The lower bound follows trivially from (3.5) and (3.6).

Before we sketch the proof we need the following regularity property of positive-weight numerical integration rules which is due to Reimer [18, Lemma 1]. In the following $S(\mathbf{x};\phi)$ denotes the spherical cap on $\mathbb{S}^d$ with center $\mathbf{x}$ and angular radius ϕ,

$$S(\mathbf{x};\phi) = \left\{\mathbf{y} \in \mathbb{S}^d \mid \mathbf{y}\cdot\mathbf{x} \geq \cos\phi\right\}.$$

The surface area of $S(\mathbf{x};\phi)$ is given by

$$|S(\mathbf{x};\phi)| = \omega_{d-1}\int_0^\phi (\sin\theta)^{d-1}\,d\theta,$$

and has for $\phi \in [0,\frac{\pi}{2}]$ the upper and lower bound

$$\left(\frac{2}{\pi}\right)^{d-1}\frac{\omega_{d-1}}{d}\,\phi^d \leq |S(\mathbf{x};\phi)| \leq \frac{\omega_{d-1}}{d}\,\phi^d. \tag{3.8}$$

Lemma 3.3 *There exist positive constants c_0 and $c_1 \leq \frac{\pi}{2}$, such that for any $n \in \mathbb{N}_0$ and for any positive-weight numerical integration rule $Q_{m(n)}$, which satisfies $Q_{m(n)}p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$, the points $\mathbf{x}_j$ and corresponding weights w_j, $j = 1,\ldots,m(n)$, satisfy*

$$\sum_{j=1}^{m(n)} w_j\;\chi\left(S\left(\mathbf{x};\frac{c_1}{n}\right)\right)(\mathbf{x}_j) \leq c_0\left|S\left(\mathbf{x};\frac{c_1}{n}\right)\right| \qquad \forall \mathbf{x}\in\mathbb{S}^d. \tag{3.9}$$

Here for a set $\mathcal{U} \subset \mathbb{S}^d$, $\chi(\mathcal{U})$, denotes the characteristic function of the set $\mathcal{U}$.

The regularity property (3.9) implies an analogous estimate for larger spherical caps (see [10, Lemma 5.1], [18], [23, Assumption 1 and Lemma 5.5.3] and also [3] for a particularly simple proof): under the same assumptions as in Lemma 3.3 there exists a constant c_d (which depends only on the sphere dimension d) such that the weights and points of the positive-weight numerical integration rule $Q_{m(n)}$ satisfy

$$\sum_{j=1}^{m(n)} w_j\;\chi(S(\mathbf{x};\theta))(\mathbf{x}_j) \leq c_0\,c_d\,|S(\mathbf{x},\theta)| \quad \forall \mathbf{x}\in\mathbb{S}^d,\ \forall\theta \text{ with } \frac{c_1}{n} \leq \theta \leq \pi. \tag{3.10}$$

After these preparations we can sketch the proof of Theorem 3.2.
PROOF. From (1.15) we have for any $f \in C(\mathbb{S}^d)$ and any $\mathbf{x} \in \mathbb{S}^d$ that

$$|L_n f(\mathbf{x})| \le \sum_{j=1}^{m(n)} w_j \, |f(\mathbf{x}_j)| \, |G_n(\mathbf{x}, \mathbf{x}_j)| \le \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}, \mathbf{x}_j)| \, \|f\|_{C(\mathbb{S}^d)}.$$

This implies that

$$\|L_n f\|_{C(\mathbb{S}^d)} \le \|f\|_{C(\mathbb{S}^d)} \max_{\mathbf{x} \in \mathbb{S}^d} \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}, \mathbf{x}_j)|.$$

Hence

$$\|L_n\|_{C(\mathbb{S}^d) \to C(\mathbb{S}^d)} \le \max_{\mathbf{x} \in \mathbb{S}^d} \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}, \mathbf{x}_j)|. \tag{3.11}$$

To show that this is an equality, note first that there exists an $\mathbf{x}_0 \in \mathbb{S}^d$ such that

$$\sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}_0, \mathbf{x}_j)| = \max_{\mathbf{x} \in \mathbb{S}^d} \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}, \mathbf{x}_j)|.$$

It is possible to construct a continuous function $f^* \in C(\mathbb{S}^d)$, with $\|f^*\|_{C(\mathbb{S}^d)} = 1$, such that

$$f^*(\mathbf{x}_j) = \operatorname{sign}(G_n(\mathbf{x}_0, \mathbf{x}_j)), \qquad j = 1, \ldots, m(n),$$

and for this f^*

$$\|L_n f^*\|_{C(\mathbb{S}^d)} \ge \left| \sum_{j=1}^{m(n)} w_j \, f^*(\mathbf{x}_j) \, G_n(\mathbf{x}_0, \mathbf{x}_j) \right| = \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}_0, \mathbf{x}_j)|.$$

Thus (3.11) is an equality, and

$$\begin{aligned} \|L_n\|_{C(\mathbb{S}^d) \to C(\mathbb{S}^d)} &= \max_{\mathbf{x} \in \mathbb{S}^d} \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}, \mathbf{x}_j)| \\ &= \sum_{j=1}^{m(n)} w_j \, |G_n(\mathbf{x}_0, \mathbf{x}_j)|. \end{aligned} \tag{3.12}$$

To estimate $\|L_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$ we estimate the right-most expression in (3.12). The reproducing kernel $G_n(\mathbf{x},\mathbf{y})$ of $\mathbb{P}_n(\mathbb{S}^d)$, given by (1.6), satisfies from [25, (7.32.2)]

$$\begin{aligned}
\max_{(\mathbf{x},\mathbf{y})\in\mathbb{S}^d\times\mathbb{S}^d} |G_n(\mathbf{x},\mathbf{y})| &= \frac{1}{\omega_d}\,\frac{(d)_n}{\left(\frac{d}{2}\right)_n}\, P_n^{(\frac{d}{2},\frac{d-2}{2})}(1) \\
&= \frac{1}{\omega_d}\,\frac{(d)_n}{\left(\frac{d}{2}\right)_n}\,\frac{\left(\frac{d+2}{2}\right)_n}{n!} \\
&\leq c_3\, n^d
\end{aligned} \tag{3.13}$$

and from [25, Theorem 7.32.2 and (4.1.3)] and the elementary estimate $\sin\theta \leq \theta$ for all $\theta \in [0,\pi]$, for $(\mathbf{x},\mathbf{y}) \in \mathbb{S}^d\times\mathbb{S}^d$ with $\mathbf{x}\cdot\mathbf{y} = \cos\theta$ and $c_1\, n^{-1} \leq \theta \leq \pi - c_1\, n^{-1}$

$$|G_n(\mathbf{x},\mathbf{y})| \leq c_4\, n^{\frac{d-1}{2}}\,(\sin\theta)^{-\frac{d+1}{2}}, \tag{3.14}$$

with positive constants c_3 and c_4 that depend only on d. Here the constant c_1 is the constant from the regularity property (3.9).

Now we split the sum in (3.12) into one sum over the weights corresponding to those points in the northern hemisphere with respect to $\mathbf{x}_0$ as the north pole and one sum over the weights corresponding to those points in the corresponding southern hemisphere, where we count the equator arbitrarily to the northern hemisphere. Because the estimate (3.14) "away" from $\mathbf{x} = \pm\mathbf{y}$ is clearly much better than the global estimate (3.13), it is useful to split the points in each hemisphere (with respect to $\mathbf{x}_0$ as north pole) further into those points in the spherical cap $S(\mathbf{x}_0;\frac{c_1}{n})$ and $S(-\mathbf{x}_0;\frac{c_1}{n})$, respectively, and the remaining points. Thus with the notation

$$H^+ := \left\{\mathbf{y}\in\mathbb{S}^d \,\middle|\, \mathbf{y}\cdot\mathbf{x}_0 \geq 0\right\}, \qquad H^- := \mathbb{S}^d \setminus H^+,$$

we can rewrite (3.12) as

$$\|L_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} = D^+ + D^- + R^+ + R^-, \tag{3.15}$$

where

$$D^\pm \;:=\; \sum_{j=1}^{m(n)} w_j\,\chi\left(S\left(\pm\mathbf{x}_0;\frac{c_1}{n}\right)\cap H^\pm\right)(\mathbf{x}_j)\;|G_n(\mathbf{x}_0,\mathbf{x}_j)|,$$

$$R^{\pm} \;:=\; \sum_{j=1}^{m(n)} w_j \, \chi\left(H^{\pm} \setminus S\left(\pm\mathbf{x}_0; \frac{c_1}{n}\right)\right)(\mathbf{x}_j)\; |G_n(\mathbf{x}_0, \mathbf{x}_j)|.$$

The "diagonal" contributions $D^{\pm}$ can easily be estimated with the help of (3.13), (3.9), and (3.8), and we obtain

$$D^{\pm} \leq c, \tag{3.16}$$

with a constant c independent of n.

In order to estimate $R^{\pm}$ we define the angles $\theta_j^{\pm} \in [\frac{c_1}{n}, \pi - \frac{c_1}{n}]$ by $\pm\mathbf{x}_0 \cdot \mathbf{x}_j = \cos\theta_j^{\pm}$ for $\mathbf{x}_j \in H^{\pm} \setminus S(\pm\mathbf{x}_0; \frac{c_1}{n})$, and use (3.14), thus obtaining

$$R^{\pm} \leq c_4\, n^{\frac{d-1}{2}} \sum_{j=1}^{m(n)} w_j \, \chi\left(H^{\pm} \setminus S\left(\pm\mathbf{x}_0; \frac{c_1}{n}\right)\right)(\mathbf{x}_j)\; (\sin\theta_j^{\pm})^{-\frac{d+1}{2}}. \tag{3.17}$$

It should be noted that all the angles $\theta_j^{\pm}$ that actually occur in the sum are in $[\frac{c_1}{n}, \frac{\pi}{2}]$. Defining the piecewise-constant functions $g^{\pm}$, given by $g^{\pm} : [\frac{c_1}{n}, \frac{\pi}{2}] \to \mathbb{R}$,

$$g^{\pm}(\theta) := \sum_{j=1}^{m(n)} w_j \, \chi\left(S(\pm\mathbf{x}_0; \theta) \cap \left(H^{\pm} \setminus S\left(\pm\mathbf{x}_0; \frac{c_1}{n}\right)\right)\right)(\mathbf{x}_j),$$

and the strictly monotonically declining function

$$h(\theta) := (\sin\theta)^{-\frac{d+1}{2}}, \qquad \theta \in \left[\frac{c_1}{n}, \frac{\pi}{2}\right],$$

we can write the sum in (3.17) as a Riemann-Stieltjes integral,

$$R^{\pm} \leq c_4\, n^{\frac{d-1}{2}} \int_{\frac{c_1}{n}}^{\frac{\pi}{2}} h(\theta)\, dg^{\pm}(\theta).$$

With integration by parts and the estimate

$$g^{\pm}(\theta) \leq c_0\, c_d\, \frac{\omega_{d-1}}{d} \left(\frac{\pi}{2}\right)^d (\sin\theta)^d = c_5\, (\sin\theta)^d, \qquad \theta \in \left[\frac{c_1}{n}, \frac{\pi}{2}\right],$$

(which follows from (3.10) and (3.8)), we find that

$$R^{\pm} \leq c\, n^{\frac{d-1}{2}}, \tag{3.18}$$

with a positive constant c independent of n. The formulas (3.15), (3.16), and (3.18) imply now the upper bound in (3.7) in Theorem 3.2. As previously mentioned, the lower bound in (3.7) follows trivially from (3.5) and (3.6). □

Summarizing the results for hyperinterpolation and orthogonal projection in the $C(\mathbb{S}^d)$ to $C(\mathbb{S}^d)$ setting, we find that the operator norm $\|L_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$ is of the same order as the operator norm $\|T_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}$ of the minimal norm projection operator, the best result possible with respect to order.

With an analogous argument to that in (3.3) we obtain for the hyperinterpolation approximation error from (3.7) the estimate

$$\begin{aligned} \|L_n f - f\|_{C(\mathbb{S}^d)} &\leq \Big(1 + \|L_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)}\Big)\, E_n\Big(f; C(\mathbb{S}^d)\Big) \\ &\leq c\, n^{\frac{d-1}{2}}\, E_n\Big(f; C(\mathbb{S}^d)\Big), \end{aligned}$$

where $f \in C(\mathbb{S}^d)$ and as before

$$E_n(f; C(\mathbb{S}^d)) := \min_{p\in\mathbb{P}_n(\mathbb{S}^d)} \|f - p\|_{C(\mathbb{S}^d)}.$$

The same estimate holds for the error of the $L_2(\mathbb{S}^d)$-orthogonal projection $T_n f$.

Now we want to compare hyperinterpolation with polynomial interpolation. From the formula (1.9) for the interpolant in terms of the Lagrange polynomials, it can be relatively easily seen that

$$\|\Lambda_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} = \max_{x\in\mathbb{S}^d} \sum_{j=1}^{d_n} |\ell_j(\mathbf{x})|,$$

which is often called the *Lebesgue constant* for interpolation. For a badly chosen fundamental system of interpolation points, the Lebesgue constant can be arbitrarily large, but how small can it become for a nicely chosen fundamental system of interpolation points?

For extremal (fundamental) systems (see Section 1.2 and [17, 24]) we obtain

$$\|\Lambda_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} \le d_n \tag{3.19}$$

because the properties of an extremal fundamental system imply that $\|\ell_j\|_{C(\mathbb{S}^d)} = 1$ for $j = 1, \ldots, d_n$.

Reimer showed in [16, Corollary 2] (see also [8, (7.2.21), (7.2.22)]) that

$$\|\Lambda_n\|_{C(\mathbb{S}^d)\to C(\mathbb{S}^d)} \le d_n^{\frac{1}{2}} \left(\frac{\lambda_{\mathrm{avg}}}{\lambda_{\mathrm{min}}}\right)^{\frac{1}{2}}, \tag{3.20}$$

where λ_{min} is the smallest eigenvalue and λ_{avg} the average of all eigenvalues (counted with multiplicity) of the matrix $(G_n(\mathbf{x}_i, \mathbf{x}_j))_{i,j=1,\ldots,d_n}$. (This is the matrix of the interpolation problem if the interpolant is represented as a linear combination of the linearly independent functions $G_n(\cdot, \mathbf{x}_1), \ldots, G_n(\cdot, \mathbf{x}_{d_n})$.)

Whether either of the estimates (3.19) or (3.20) is optimal is not known theoretically, although numerical evidence suggests (see [27]) that for $\mathbb{S}^2$

$$\|\Lambda_n\|_{C(\mathbb{S}^2)\to C(\mathbb{S}^2)} \le c\,n \le c\,d_n^{\frac{1}{2}}.$$

In the other direction, that same numerical evidence also suggests that it is most unlikely that interpolation can achieve the same order $O(\sqrt{n})$ as hyperinterpolation or orthogonal projection in the setting $C(\mathbb{S}^2)$ to $C(\mathbb{S}^2)$.

3.3 Estimates in Sobolev spaces

In this section we consider hyperinterpolation as a map from one Sobolev space to another. Specifically, we consider $L_n : H^t(\mathbb{S}^d) \to H^s(\mathbb{S}^d)$, where $H^s(\mathbb{S}^d)$ and $H^t(\mathbb{S}^d)$ are Sobolev spaces with the following assumptions on the indices: $t \ge s \ge 0$ and $t > \frac{d}{2}$. The second assumption $t > \frac{d}{2}$ guarantees that $H^t(\mathbb{S}^d)$ is embedded in $C(\mathbb{S}^d)$ so that hyperinterpolation can be defined on $H^t(\mathbb{S}^d)$. Intuitively, $H^s(\mathbb{S}^d)$ can for integer s be thought of as the space of those functions whose generalized (distributional) derivatives up to (and including) the order s are square-integrable. Before we can formulate the theorem we need to introduce the spaces and state some of their properties.

The Sobolev space $H^s(\mathbb{S}^d)$, with $s \geq 0$, is defined as the closure of $\bigoplus_{\ell=0}^{\infty} \mathbb{H}_\ell(\mathbb{S}^d)$ with respect to the norm

$$\|f\|_{H^s(\mathbb{S}^d)} := \left(\sum_{\ell=0}^{\infty} \left(\ell + \frac{d-1}{2}\right)^{2s} \sum_{k=1}^{N(d,\ell)} \left| \hat{f}_{\ell k}^{(d)} \right|^2 \right)^{\frac{1}{2}} .$$

The space $H^s(\mathbb{S}^d)$ is a Hilbert space with the inner product

$$(f,g)_{H^s(\mathbb{S}^d)} := \sum_{\ell=0}^{\infty} \left(\ell + \frac{d-1}{2}\right)^{2s} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)} \, \hat{g}_{\ell k}^{(d)}, \qquad f,g \in H^s(\mathbb{S}^d),$$

which induces the norm $\|\cdot\|_{H^s(\mathbb{S}^d)}$. For $s > \frac{d}{2}$ there exists a constant c_s such that

$$\|f\|_{C(\mathbb{S}^d)} \leq c_s \, \|f\|_{H^s(\mathbb{S}^d)} \qquad \forall f \in H^s(\mathbb{S}^d),$$

that is, $H^s(\mathbb{S}^d)$ is embedded in $C(\mathbb{S}^d)$. We mention also that for $s > \frac{d}{2}$ the space $H^s(\mathbb{S}^d)$ is a reproducing kernel Hilbert space, although this fact is of no consequence for the result in this subsection. The spaces $H^s(\mathbb{S}^d)$ are nested, that is, $H^t(\mathbb{S}^d) \subset H^s(\mathbb{S}^d)$ whenever $t \geq s$.

After these preparations we can formulate the result.

Theorem 3.4 *Let $d \geq 2$, and let s and t be fixed real numbers with $t \geq s \geq 0$ and $t > \frac{d}{2}$. There exists a positive constant c such that for any hyperinterpolation operator $L_n : H^t(\mathbb{S}^d) \to H^s(\mathbb{S}^d)$ (defined with a positive-weight numerical integration rule $Q_{m(n)}$ with $Q_{m(n)}p = Ip$ for all $p \in \mathbb{P}_{2n}(\mathbb{S}^d)$) the following holds true:*

(i) for any $f \in H^t(\mathbb{S}^d)$

$$\|L_n f - f\|_{H^s(\mathbb{S}^d)} \leq c \left(n + \frac{d-1}{2}\right)^{\frac{d}{2}+s-t} E_n\Big(f; H^t(\mathbb{S}^d)\Big), \tag{3.21}$$

where $E_n(f; H^t(\mathbb{S}^d)) := \min_{p \in \mathbb{P}_n(\mathbb{S}^d)} \|f - p\|_{H^t(\mathbb{S}^d)}$, and

(ii) $\|L_n\|_{H^t(\mathbb{S}^d) \to H^s(\mathbb{S}^d)} \leq c \left(n + \frac{d-1}{2}\right)^{\frac{d}{2}+s-t} + \left(\frac{d-1}{2}\right)^{s-t}.$

The positive constant c is independent of n and of the particular positive-weight numerical integration rule $Q_{m(n)}$ in the definition of L_n.

Theorem 3.4 is new to the best knowledge of the authors. A much weaker result than (3.21) was proved in [26] for $\mathbb{S}^2$ and for the special case of hyperinterpolation defined with a particular type of numerical integration rule. We do not know if the estimates in Theorem 3.4 are the best possible.

As preparation for the proof of Theorem 3.4 we derive some estimates that are independently of interest.

First we observe that, because $\{Y_{\ell k}^{(d)} \mid \ell \in \mathbb{N}_0;\ k = 1, \ldots, N(d, \ell)\}$ is a complete orthogonal system with respect to both $L_2(\mathbb{S}^d)$ and $H^t(\mathbb{S}^d)$, the restriction $T_n|_{H^t(\mathbb{S}^d)}$ of the $L_2(\mathbb{S}^d)$-orthogonal projection $T_n : L_2(\mathbb{S}^d) \to \mathbb{P}_n(\mathbb{S}^d)$ to $H^t(\mathbb{S}^d)$ is just the $H^t(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$. This implies that

$$E_n\Big(f; H^t(\mathbb{S}^d)\Big) = \|f - T_n f\|_{H^t(\mathbb{S}^d)}, \tag{3.22}$$

that is, the $L_2(\mathbb{S}^d)$-orthogonal projection $T_n f$ of $f \in H^t(\mathbb{S}^d)$ is the best approximation of f in $\mathbb{P}_n(\mathbb{S}^d)$ in the $H^t(\mathbb{S}^d)$ sense. We denote by

$$\mathbb{P}_n^{\perp}(\mathbb{S}^d) := \left\{ f \in L_2(\mathbb{S}^d) \;\middle|\; (f, p)_{L_2(\mathbb{S}^d)} = 0 \ \forall p \in \mathbb{P}_n(\mathbb{S}^d) \right\}$$

the orthogonal complement of $\mathbb{P}_n(\mathbb{S}^d)$ in $L_2(\mathbb{S}^d)$, that is, the space of all those functions in $L_2(\mathbb{S}^d)$ which are $L_2(\mathbb{S}^d)$-orthogonal to $\mathbb{P}_n(\mathbb{S}^d)$. From the definition of the inner product $(\cdot, \cdot)_{H^t(\mathbb{S}^d)}$, the orthogonal complement of $\mathbb{P}_n(\mathbb{S}^d)$ in $H^t(\mathbb{S}^d)$ is simply $\mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$.

The elementary estimates that follow are either for functions in $\mathbb{P}_n(\mathbb{S}^d)$ or functions in $\mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$.

Lemma 3.5 *The following estimates are valid in the Sobolev spaces $H^t(\mathbb{S}^d)$:*

(i) Let $t > \frac{d}{2}$. There exists a positive constant $\tilde{c}$ such that for all $n \in \mathbb{N}_0$ and for any function $f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$

$$\|f\|_{C(\mathbb{S}^d)} \le \tilde{c} \left(n + \frac{d-1}{2} \right)^{\frac{d}{2} - t} \|f\|_{H^t(\mathbb{S}^d)}. \tag{3.23}$$

(ii) Let $t \geq 0$. For any $f \in \mathbb{P}_n(\mathbb{S}^d)$

$$\|f\|_{H^t(\mathbb{S}^d)} \leq \left(n + \frac{d-1}{2}\right)^t \|f\|_{L_2(\mathbb{S}^d)}. \tag{3.24}$$

(iii) Let $t \geq s \geq 0$. Then for any $f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$

$$\|f\|_{H^s(\mathbb{S}^d)} \leq \left(n + \frac{d-1}{2}\right)^{s-t} \|f\|_{H^t(\mathbb{S}^d)}. \tag{3.25}$$

The estimates in Lemma 3.5 can certainly be found in several places. Because they are rather useful in many contexts we show quickly the proofs.

PROOF. For any $f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$ we have

$$f(\mathbf{x}) = \sum_{\ell=n+1}^{\infty} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)} \, Y_{\ell k}^{(d)}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d, \tag{3.26}$$

and it can be easily shown that for $t > \frac{d}{2}$ the Fourier series converges uniformly, so that (3.26) is true in the pointwise sense. Hence, from the Cauchy-Schwarz inequality and the addition theorem (1.4)

$$\begin{aligned}
|f(\mathbf{x})| &= \left| \sum_{\ell=n+1}^{\infty} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)} \, Y_{\ell k}^{(d)}(\mathbf{x}) \right| \\
&\leq \sum_{\ell=n+1}^{\infty} \sum_{k=1}^{N(d,\ell)} \left[\left(\ell + \frac{d-1}{2}\right)^t \left|\hat{f}_{\ell k}^{(d)}\right| \right] \left[\left(\ell + \frac{d-1}{2}\right)^{-t} \left|Y_{\ell k}^{(d)}(\mathbf{x})\right| \right] \\
&\leq \|f\|_{H^t(\mathbb{S}^d)} \left(\sum_{\ell=n+1}^{\infty} \sum_{k=1}^{N(d,\ell)} \left(\ell + \frac{d-1}{2}\right)^{-2t} \left|Y_{\ell k}^{(d)}(\mathbf{x})\right|^2 \right)^{\frac{1}{2}} \\
&= \|f\|_{H^t(\mathbb{S}^d)} \left(\frac{1}{\omega_d} \sum_{\ell=n+1}^{\infty} \frac{N(d,\ell)}{\left(\ell + \frac{d-1}{2}\right)^{2t}} \right)^{\frac{1}{2}}.
\end{aligned}$$

As $\hat{c}_1 \, (\ell + \frac{d-1}{2})^{d-1} \leq N(d,\ell) \leq \hat{c}_2 \, (\ell + \frac{d-1}{2})^{d-1}$ with positive constants $\hat{c}_1$ and $\hat{c}_2$ independent of ℓ, we see that the sum in the second term is finite if and only if $d - 2t < 0$, that is, $t > \frac{d}{2}$. Furthermore, the sum satisfies for $t > \frac{d}{2}$

$$\hat{c}_3 \left(n + \frac{d-1}{2} \right)^{d-2t} \leq \frac{1}{\omega_d} \sum_{\ell=n+1}^{\infty} \frac{N(d,\ell)}{\left(\ell + \frac{d-1}{2}\right)^{2t}} \leq \hat{c}_4 \left(n + \frac{d-1}{2} \right)^{d-2t},$$

with suitable positive constants $\hat{c}_3$ and $\hat{c}_4$ independent of n. Thus for $t > \frac{d}{2}$ with the positive constant $\tilde{c} = \sqrt{\hat{c}_4}$

$$\sup_{\mathbf{x} \in \mathbb{S}^d} |f(\mathbf{x})| \leq \tilde{c} \left(n + \frac{d-1}{2} \right)^{\frac{d}{2}-t} \|f\|_{H^t(\mathbb{S}^d)},$$

which proves (3.23).

For $f \in \mathbb{P}_n(\mathbb{S}^d)$ we have, pointwise,

$$f(\mathbf{x}) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell k}^{(d)} \, Y_{\ell k}^{(d)}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d,$$

and thus

$$\begin{aligned} \|f\|_{H^t(\mathbb{S}^d)}^2 &= \sum_{\ell=0}^{n} \left(\ell + \frac{d-1}{2} \right)^{2t} \sum_{k=1}^{N(d,\ell)} \left| \hat{f}_{\ell k}^{(d)} \right|^2 \\ &\leq \left(n + \frac{d-1}{2} \right)^{2t} \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \left| \hat{f}_{\ell k}^{(d)} \right|^2 \\ &= \left(n + \frac{d-1}{2} \right)^{2t} \|f\|_{L_2(\mathbb{S}^d)}^2, \end{aligned}$$

which proves (3.24).

For any $f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$ we have for $t \geq s \geq 0$

$$\begin{aligned} \|f\|_{H^s(\mathbb{S}^d)}^2 &= \sum_{\ell=n+1}^{\infty} \left(\ell + \frac{d-1}{2} \right)^{2s} \sum_{k=1}^{N(d,\ell)} \left| \hat{f}_{\ell k}^{(d)} \right|^2 \\ &\leq \left(n + 1 + \frac{d-1}{2} \right)^{2(s-t)} \sum_{\ell=n+1}^{\infty} \left(\ell + \frac{d-1}{2} \right)^{2t} \sum_{k=1}^{N(d,\ell)} \left| \hat{f}_{\ell k}^{(d)} \right|^2 \\ &\leq \left(n + \frac{d-1}{2} \right)^{2(s-t)} \|f\|_{H^t(\mathbb{S}^d)}^2, \end{aligned}$$

which implies (3.25). □

After these preparations we can relatively easily prove Theorem 3.4.

PROOF. Let $f \in H^t(\mathbb{S}^d)$, where $t \geq s \geq 0$ and $t > \frac{d}{2}$. Then $L_n(T_n f) = T_n f$ because L_n is a projection onto $\mathbb{P}_n(\mathbb{S}^d)$ and $T_n f$ is in $\mathbb{P}_n(\mathbb{S}^d)$. Hence

$$\begin{aligned}\|L_n f - f\|_{H^s(\mathbb{S}^d)} &= \|L_n(f - T_n f) - (f - T_n f)\|_{H^s(\mathbb{S}^d)} \\ &= \left(\|L_n(f - T_n f)\|^2_{H^s(\mathbb{S}^d)} + \|f - T_n f\|^2_{H^s(\mathbb{S}^d)}\right)^{1/2},\end{aligned} \tag{3.27}$$

where we have made use of the fact that the functions $f - T_n f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$ and $L_n(f - T_n f) \in \mathbb{P}_n(\mathbb{S}^d)$ are $H^s(\mathbb{S}^d)$-orthogonal. Now we will estimate both quantities in (3.27) separately, making use of the three estimates in Lemma 3.5 and also of (3.1) in Theorem 3.1.

First we consider the second term. As $f - T_n f \in \mathbb{P}_n^{\perp}(\mathbb{S}^d) \cap H^t(\mathbb{S}^d)$, we have from (3.25) in Lemma 3.5 that

$$\|f - T_n f\|^2_{H^s(\mathbb{S}^d)} \leq \left(n + \frac{d-1}{2}\right)^{2(s-t)} \|f - T_n f\|^2_{H^t(\mathbb{S}^d)}. \tag{3.28}$$

Now we estimate the first term on the right-hand side of (3.27). Since $L_n(f - T_n f)$ is in $\mathbb{P}_n(\mathbb{S}^d)$, the estimate (3.24) in Lemma 3.5 yields

$$\|L_n(f - T_n f)\|^2_{H^s(\mathbb{S}^d)} \leq \left(n + \frac{d-1}{2}\right)^{2s} \|L_n(f - T_n f)\|^2_{L_2(\mathbb{S}^d)}. \tag{3.29}$$

From (3.1) in Theorem 3.1 we get

$$\|L_n(f - T_n f)\|^2_{L_2(\mathbb{S}^d)} \leq \omega_d \, \|f - T_n f\|^2_{C(\mathbb{S}^d)}, \tag{3.30}$$

and finally from (3.23) in Lemma 3.5

$$\|f - T_n f\|^2_{C(\mathbb{S}^d)} \leq \tilde{c} \left(n + \frac{d-1}{2}\right)^{d-2t} \|f - T_n f\|^2_{H^t(\mathbb{S}^d)}. \tag{3.31}$$

The combination of (3.29), (3.30), and (3.31) yields

$$\|L_n(f - T_n f)\|^2_{H^s(\mathbb{S}^d)} \leq \tilde{c}\, \omega_d \left(n + \frac{d-1}{2}\right)^{d+2(s-t)} \|f - T_n f\|^2_{H^t(\mathbb{S}^d)}. \tag{3.32}$$

Now from (3.27), (3.28), and (3.32)

$$\begin{aligned}&\|L_n f - f\|_{H^s(\mathbb{S}^d)}\\ &\quad\le \left(\left(\frac{d-1}{2}\right)^{-d} + \tilde{c}\,\omega_d\right)^{\frac{1}{2}} \left(n + \frac{d-1}{2}\right)^{\frac{d}{2}+s-t} \|f - T_n f\|_{H^t(\mathbb{S}^d)}.\end{aligned}$$

Finally we use, from (3.22), $\|f - T_n f\|_{H^t(\mathbb{S}^d)} = E_n(f; H^t(\mathbb{S}^d))$ to obtain

$$\|L_n f - f\|_{H^s(\mathbb{S}^d)} \le c \left(n + \frac{d-1}{2}\right)^{\frac{d}{2}+s-t} E_n\Big(f; H^t(\mathbb{S}^d)\Big),$$

with

$$c = \left(\left(\frac{d-1}{2}\right)^{-d} + \tilde{c}\,\omega_d\right)^{\frac{1}{2}}.$$

This proves (3.21), and we have proved (i) in Theorem 3.4.

To prove the estimate of the norm in (ii), let $f \in H^t(\mathbb{S}^d)$ with $t \ge s \ge 0$ and $t > \frac{d}{2}$. From the triangle inequality

$$\|L_n f\|_{H^s(\mathbb{S}^d)} \le \|L_n f - f\|_{H^s(\mathbb{S}^d)} + \|f\|_{H^s(\mathbb{S}^d)}. \tag{3.33}$$

The first part satisfies (3.21). From the identity (3.22) and the estimate $\|f - T_n f\|_{H^t(\mathbb{S}^d)} \le \|f\|_{H^t(\mathbb{S}^d)}$

$$E_n\Big(f; H^t(\mathbb{S}^d)\Big) = \|f - T_n f\|_{H^t(\mathbb{S}^d)} \le \|f\|_{H^t(\mathbb{S}^d)},$$

and it follows from (3.21) that

$$\|L_n f - f\|_{H^s(\mathbb{S}^d)} \le c \left(n + \frac{d-1}{2}\right)^{\frac{d}{2}+s-t} \|f\|_{H^t(\mathbb{S}^d)}. \tag{3.34}$$

We only have to find a suitable estimate for the second part. As by assumption $t \ge s$, we find

$$\|f\|_{H^s(\mathbb{S}^d)} = \left(\sum_{\ell=0}^{\infty} \left(\ell + \frac{d-1}{2}\right)^{2s} \sum_{k=1}^{N(d,\ell)} \left|\hat{f}_{\ell k}^{(d)}\right|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{d-1}{2}\right)^{s-t} \left(\sum_{\ell=0}^{\infty}\left(\ell+\frac{d-1}{2}\right)^{2t}\sum_{k=1}^{N(d,\ell)}\left|\hat{f}_{\ell k}^{(d)}\right|^2\right)^{\frac{1}{2}} \tag{3.35}$$
$$= \left(\frac{d-1}{2}\right)^{s-t} \|f\|_{H^t(\mathbb{S}^d)}.$$

The combination of (3.33), (3.34), and (3.35) yields now the estimate in (ii). □

Finally, in this Sobolev space setting we want to compare the hyperinterpolation result with that for the $L_2(\mathbb{S}^d)$-orthogonal projection. (To our knowledge no Sobolev space estimates are known for polynomial interpolation.) For the $L_2(\mathbb{S}^d)$-orthogonal projection we have from (3.25) in Lemma 3.5 and (3.22) that for any $f \in H^t(\mathbb{S}^d)$, where $t \geq s \geq 0$ and $t > \frac{d}{2}$,

$$\|T_n f - f\|_{H^s(\mathbb{S}^d)} \leq \tilde{c}\left(n+\frac{d-1}{2}\right)^{s-t} E_n\Big(f; H^t(\mathbb{S}^d)\Big). \tag{3.36}$$

For $\|T_n f\|_{H^s(\mathbb{S}^d)}$ we obtain from (3.35)

$$\|T_n f\|_{H^s(\mathbb{S}^d)} \leq \|f\|_{H^s(\mathbb{S}^d)} \leq \left(\frac{d-1}{2}\right)^{s-t} \|f\|_{H^t(\mathbb{S}^d)},$$

and hence

$$\|T_n\|_{H^t(\mathbb{S}^d)\to H^s(\mathbb{S}^d)} \leq \left(\frac{d-1}{2}\right)^{s-t}. \tag{3.37}$$

As $T_n f$ is the best approximation of f in $\mathbb{P}_n(\mathbb{S}^d)$ in the $H^s(\mathbb{S}^d)$ sense (see (3.22)), it is clear that $\|T_n f - f\|_{H^s(\mathbb{S}^d)}$ should satisfy the best estimate. The estimate (3.21) for the approximation error of $L_n f$ is worse by the order $(n+\frac{d-1}{2})^{\frac{d}{2}}$, while the estimate of the norm $\|L_n\|_{H^t(\mathbb{S}^d)\to H^s(\mathbb{S}^d)}$ in (ii) in Theorem 3.4 has compared with (3.37) an additional additive term $c\,(n+\frac{d-1}{2})^{\frac{d}{2}+s-t}$ which is only bounded as $n \to \infty$ if $t \geq s + \frac{d}{2}$. We do not know whether the estimates in Theorem 3.4 are optimal.

4 Some Concluding Remarks

From the results summarized in this paper it is clear that the hyperinterpolation projection onto $\mathbb{P}_n(\mathbb{S}^d)$ has good properties. The $L_2(\mathbb{S}^d)$-orthogonal projection onto $\mathbb{P}_n(\mathbb{S}^d)$ has better properties in theory, but as it is not feasible for numerical computation, hyperinterpolation provides a reasonable practical alternative.

Hyperinterpolation is easy to compute. As a disadvantage, hyperinterpolation needs function values at the given points of the positive-weight numerical integration rule with degree of polynomial exactness $2n$ in the definition of the hyperinterpolation operator. However, starting from scattered data, Mhaskar, Narcowich, and Ward [11] (see also [14]) have proved that positive-weight numerical integration rules with polynomial exactness can be constructed which use a subset of the points of the given set of scattered data as nodes.

Polynomial interpolation has, compared with hyperinterpolation, the apparent advantage that it can work with scattered data, as long as the data forms a fundamental system, but in practice the advantage is illusory, since for scattered data the condition number is typically very large. The computation of the polynomial interpolant is rather expensive, and we have seen in Subsections 3.1 and 3.2 that even with the best choice of points the norms of Λ_n seem to grow faster than those of L_n and T_n.

The known results for hyperinterpolation (with the exception of the new ones in the Sobolev space setting) have also been summarized in much more detail in the thesis [15].

Finally, we note that all three of the polynomial approximations considered here are linear projections onto $\mathbb{P}_n(\mathbb{S}^d)$; thus we do not consider polynomial approximations obtained by Cesaro summation and similar methods. In particular, the generalized hyperinterpolation method of Reimer (see [19]), which has this character, is outside the scope of this review.

5 Acknowledgment

The support of the Australian Research Council under its Centres of Excellence program is gratefully acknowledged.

References

[1] E. Bannai and R. M. Damerell, Tight spherical designs, I, J. Math. Soc. Japan, 31 (1979), 199–207.

[2] D. L. Berman, On a class of linear operators, Dokl. Akad. Nauk SSSR, 85 (1952), 13–16 (Russian), Math. Reviews, 14, 57.

[3] J. S. Brauchart and K. Hesse, Numerical integration over spheres of arbitrary dimension, Constr. Approx., to appear.

[4] W. zu Castell, N. Laín Fernández and Y. Xu, Polynomial interpolation on the unit sphere II, Preprint, arXiv:math.NA/0407448 v1, 2004.

[5] I. K. Daugavet, Some applications of the Marcinkiewicz-Berman identity, Vestnik Leningrad Univ. Math., 1 (1974), 321–327.

[6] P. Delsarte, J. M. Goethals and J. J. Seidel, Spherical codes and designs, Geom. Dedicata, 6 (1977), 363–388.

[7] N. Laín Fernández, Localized polynomial bases on the sphere, Preprint, Universität zu Lübeck, 2003.

[8] W. Freeden, T. Gervens and M. Schreiner, "Constructive Approximation on the Sphere," Clarendon Press, Oxford, 1998.

[9] T. H. Gronwall, On the degree of convergence of Laplace's series, Trans. Amer. Math. Soc., 15 (1914), 1–30.

[10] T. Le Gia and I. H. Sloan, The uniform norm of hyperinterpolation on the unit sphere in an arbitrary number of dimensions, Constr. Approx., 17 (2001), 249–265.

[11] H. N Mhaskar, F. J. Narcowich and J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comp., 70 (2001), 1113-1130. (Corrigendum: Math. Comp., 71 (2002), 453–454.)

[12] I. P. Mysovskikh, On the construction of cubature formulas with fewest nodes, Soviet Math. Dokl., 9 (1968), 277–280.

[13] I. P. Mysovskikh, On the evaluation of integrals over the surface of a sphere, Soviet Math. Dokl., 18 (1977), 925–929.

[14] F. J. Narcowich, P. Petrushev and J. D. Ward, Localized tight frames on spheres, Preprint, 2004.

[15] A. C. Piantella, Interpolação e hiperinterpolação em esferas, thesis, Universidade de São Paulo, 2003.

[16] M. Reimer, Interpolation on the sphere and bounds for the Lagrangian square sums, Results Math., 11 (1987), 144–164.

[17] M. Reimer, "Constructive Theory of Multivariate Functions," BI Wissenschaftsverlag, Mannheim, Wien, Zürich, 1990.

[18] M. Reimer, Hyperinterpolation on the sphere at the minimal projection order, J. Approx. Theory, 104 (2000), 272–286.

[19] M. Reimer, On the implementation of a generalized hyperinterpolation process, Ergebnisbericht Angewandte Mathematik No. 243, Universität Dortmund, Nov. 2003. Electronic publication (revised and completed) Jan. 2004.

[20] P. D. Seymour and T. Zaslavsky, Averaging sets: A generalization of mean values and spherical designs, Adv. in Math., 52 (1984), 213–240.

[21] I. H. Sloan, Polynomial interpolation and hyperinterpolation over general regions. J. Approx. Theory, 83 (1995), 238–254.

[22] I. H. Sloan, Interpolation and hyperinterpolation on the sphere, in "Multivariate Approximation: Recent Trends and Results" (W. Haussmann, K. Jetter and M. Reimer, Eds.), pp. 255–268, Mathematical Research, 101, Akademie Verlag GmbH, Berlin (Wiley-VCH), 1997.

[23] I. H. Sloan and R. S. Womersley, Constructive polynomial approximation on the sphere, J. Approx. Theory, 103 (2000), 91–118.

[24] I. H. Sloan and R. S. Womersley, Extremal systems of points and numerical integration on the sphere, Adv. Comput. Math., 21 (2004), 107–125.

[25] G. Szegö, "Orthogonal Polynomials," in American Mathematical Society Colloquium Publications, Volume 23, American Mathematical Society, Providence, Rhode Island, 1975, 4th edition.

[26] V. Turunen, Sampling at equiangular grids on the 2-sphere and estimates for Sobolev space interpolation, Helsinki University of Technology, Institute of Mathematics, Research Reports, A464, 2003.

[27] R. S. Womersley and I. H. Sloan, How good can polynomial interpolation on the sphere be?, Adv. Comput. Math., 14 (2001), 195–226.

[28] Y. Xu, Polynomial interpolation on the unit sphere, SIAM J. Numer. Anal., 41 (2003), 751–766.

[29] Y. Xu, Polynomial interpolation on the unit sphere and on the unit ball, Adv. Comput. Math., 20 (2004), 247–260.

[30] A. Zygmund, "Trigonometric Series. Volume II," The Syndics of the Cambridge University Press, University Press, Cambridge, 1959, 2nd edition.

Lagrange Interpolation at Lacunary Roots of Unity

A. Jakimovski
School of Mathematical Sciences
Tel-Aviv University
Ramat-Aviv Tel-Aviv 69978, Israel
Email address: jakimov@post.tau.ac.il

Dedicated to the memory of Professor Ambikeshwar Sharma

1 Introduction

Assume that the finite complex function $f(z)$ is defined on a set G that includes the pairwise different points $(w_k)_{1\le k\le n}$. The (unique) polynomial of degree at most $n-1$, that takes the value $f(z_k)$ at the points z_k, i.e., $L_{n-1}(f;w_k)=f(w_k)$ for $1\le k\le n$ is called the Lagrange interpolant and is denoted by

$$L_{n-1}(f;z) \ \equiv L_{n-1}(f;z;(z_k)_{k=1}^{n})$$

Applying the notation $\omega_n(z):=\prod_{k=1}^{n}(z-z_k)$, this Lagrange interpolant can be represented in the form

$$L_{n-1}(f;z)=\sum_{k=1}^{n}\frac{f(w_k)}{z-w_k}\cdot\frac{\omega_n(z)}{\omega_n'(w_k)}. \tag{1.1}$$

Suppose the set of points $(z_k)_{1\le k\le n}$ is split into two disjoint subsets $(z_k^{(1)})_{1\le k\le n_1}$ and $(z_k^{(2)})_{1\le k\le n_2}$ such that $n_1+n_2=n$. Then the Lagrange polynomial $L_{n-1}(f;z;(z_k)_{k=1}^{n})$ is the sum of the two Lagrange polynomials, where the first polynomial $L_{n-1,*}(f;z)$ interpolates $f(z)$ for $z=z_k^{(1)}$ $,1\le k\le n_1$ and takes the values 0 at all the points $(z_k^{(2)})_{1\le k\le n_2}$ and the second Lagrange polynomial

$L_{n-1,**}(f;z)$ takes the value 0 at all the points $(z_k)_{1\le k\le n}$ and interpolates the function $f(z)$ at the points $(z_k^{(2)})_{1\le k\le n_2}$. It will be of interest to see the connections between properties of the Lagrange polynomial $L_{n-1}(f;z)$ and the two polynomials $L_{n-1,*}(z)$ and $L_{n-1,**}(f;z)$.

Properties of the Lagrange interpolation polynomials interpolating at the roots of unity were investigated in many papers. Here we consider some properties of the Lagrange and Hermite interpolating polynomials that interpolate a function at a real subset of the sets of roots of unity.

The subsets of the sets of the roots of unity considered in this paper will be chosen in the following way. For a given non-negative integer n denote the sequence of zeros of the polynomial $z^{3(n+1)}-1$ by $(w_{3(n+1),s})_{0\le s\le 3(n+1)-1}$. These zeros are given by $w_{3(n+1),s}=e^{\frac{2\pi i s}{3(n+1)}}$, where $0\le s\le 3(n+1)-1$. From the factorization

$$z^{3(n+1)}-1=(z^{n+1}-1)(z^{n+1}-e^{\frac{2\pi i}{3}})(z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2})$$

it follows that the sequence of zeros of $z^{3(n+1)}-1$ is the union of three disjoint subsequences $(w_{3(n+1),3k+j})_{0\le k\le n}$ for $j=0,1,2$, which are, respectively, the zeros of the polynomials $z^{n+1}-1$, $z^{n+1}-e^{\frac{2\pi i}{3}}$ and $z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2}$.

Write

$$\omega_{0,3(n+1)}(z):=\tfrac{z^{3(n+1)}-1}{z^{n+1}-1}=(z^{n+1}-e^{\frac{2\pi i}{3}})(z^{n}-e^{\frac{2\pi i}{3}\cdot 2})\ ,$$

$$\omega_{1,3(n+1)}(z):=\tfrac{z^{3(n+1)}-1}{z^{n+1}-e^{\frac{2\pi i}{3}}}=(z^{n+1}-1)(z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2})\ ,$$

$$\omega_{2,3(n+1)}(z):=\tfrac{z^{3(n+1)}-1}{z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2}}=(z^{n+1}-1)(z^{n+1}-e^{\frac{2\pi i}{3}})\ ,$$

For each $v=0,1,2$, the set of the roots of the polynomial $\omega_{v,2(n+1)}(z)$ can be looked upon in two ways. First as the union of the zeros of two out of the three polynomials $z^{n+1}-1$, $z^{n+1}-e^{\frac{2\pi i}{3}}$ and $z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2}$ which is the union of two different rotations of the zeros of $z^{n+1}-1$, or, as being obtained by deleting a third of the set of zeros of $z^{3(n+1)}-1$, i.e., the zeros of one of the three polynomials $z^{n+1}-1$, $z^{n+1}-e^{\frac{2\pi i}{3}}$ and $z^{n+1}-e^{\frac{2\pi i}{3}\cdot 2}$.

Instead of investigating the Lagrange and Hermite interpolation polynomials of a function associated at the zeros of the each of the polynomials $\omega_{v,3(n+1)}(z)$, $v = 0, 1, 2$ we consider a slightly more general case.

Assume $0 \le \alpha < \beta < 3$. Let $L_{2(n+1)-1;\alpha;\beta}(f;z)$ denote the Lagrange interpolating polynomial taking the values of a function $f(z)$ at the zeros of the polynomial

$$\begin{aligned}\omega_{2(n+1);\alpha;\beta}(z) \quad &:= \quad \left(z^{n+1} - e^{2\pi i\alpha/3}\right)\left(z^{n+1} - e^{2\pi i\beta/3}\right) \\ &= \quad z^{2(n+1)} - \left(e^{2\pi i\alpha/3} + e^{2\pi i\beta/3}\right) z^{n+1} + e^{2\pi i(\alpha+\beta)/3} ,\end{aligned} \tag{1.2}$$

where the zeros of the polynomial $z^{n+1} - e^{\frac{2\pi i\alpha}{3}}$ are $w_{2(n+1);\alpha;k} = e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}$, $0 \le k \le n$ and the zeros of the polynomial $z^{n+1} - e^{2\pi i\beta/3}$ are $w_{2(n+1);\beta;k} = e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\beta)}$, $0 \le k \le n$. Write also $\omega_{n+1;\alpha}(z) := z^{n+1} - e^{2\pi i\alpha/3}$. Let $L_{n+1;\alpha}(f;z)$ denote the Lagrange polynomial of degree n at most interpolating the function $f(z)$ at the zeros of $\omega_{n+1;\alpha}(z)$. Let $L_{2(n+1)-1;\alpha;*}(f;z)$ denote the Lagrange polynomial interpolating the function $f(z)$ at the zeros of $z^{n+1} - e^{2\pi i\alpha/3}$ and taking the value zero at the zeros of $z^{n+1} - e^{2\pi i\beta/3}$ and, similarly, let $L_{2(n+1)-1;*;\beta}(f;z)$ denote the Lagrange polynomial interpolating $f(z)$ at the zeros of $z^{n+1} - e^{2\pi i\beta/3}$ and taking the value zero at the zeros of $z^{n+1} - e^{2\pi i\alpha/3}$.

2 The Main Results

For functions that are not necessarily analytic, we have the following theorem.

Theorem 2.1 *Let $f(z)$ be Riemann integrable on the unit circle $\gamma := \{z : |z| = 1\}$. Then we have for each non-negative integer r*

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;\beta}(f;z) = \frac{r!}{2\pi i}\oint_{|t|=1} \frac{f(t)}{(t-z)^{r+1}}dt \tag{2.1}$$

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;*}(f;z) = \frac{e^{2\pi i\beta/3}}{e^{2\pi i\alpha/3} - e^{2\pi i\beta/3}}\frac{r!}{2\pi i}\oint_{|t|=1} \frac{f(t)}{(t-z)^{r+1}}dt , \tag{2.2}$$

and

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;*;\beta}(f;z) = \frac{e^{2\pi i\alpha/3}}{e^{2\pi i\beta/3}-e^{2\pi i\alpha/3}}\frac{r!}{2\pi i}\oint_{|t|=1}\frac{f(t)}{(t-z)^{r+1}}dt \tag{2.3}$$

uniformly for $|z|\le\delta<1\ ,\ 0<\delta<1$.

Definition 2.2 *A function* $f(z)$ *is in* A_ρ , $\rho>1$, *if it is a regular analytic function in the disk* $D_\rho:=\{z\ :|z|<\rho\}$ *and not regular on* $\overline{D}_\rho$.

If it is assumed in Theorem 2.1 that $f(z)\in A_\rho$, $\rho>1$, then by applying the the Cauchy formula

$$\tfrac{r!}{2\pi i}\oint_{|t|=1}\tfrac{f(t)}{(t-z)^{r+1}}dt = f^{(r)}(z) \quad \text{for each } z\ , |z|<1,$$

the conclusions of Theorem 2.1 take the form

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;\beta}(f;z) = f^{(r)}(z)\ ,$$

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;*}(f;z) = \tfrac{e^{2\pi i\beta/3}}{e^{2\pi i\alpha/3}-e^{2\pi i\beta/3}}f^{(r)}(z)\ ,$$

and

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;*;\beta}(f;z) = f^{(r)}(z)\tfrac{e^{2\pi i\alpha/3}}{e^{2\pi i\beta/3}-e^{2\pi i\alpha/3}}\ ,$$

uniformly for $|z|\le\delta<1\ ,\ 0<\delta<1$.

When $f(z)\in A_\rho,\ \ \rho>1,$ we shall see by Corollary 2.4, stated later in this paper, that we have the stronger result

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;\beta}(f;z) = f^{(r)}(z),\ \text{uniformly for}\ \ |z|\le\delta \quad \text{and}\quad 0<\delta<\rho\ .$$

We do not know what happens for $1<|z|<\rho$ with the other two conclusions of Theorem 2.1,

$$\lim_{n\to\infty} L^{(r)}_{2(n+1)-1;\alpha;*}(f;z) \quad \text{and} \quad \lim_{n\to\infty} L^{(r)}_{2(n+1)-1;*;\beta}(f;z)\ .$$

When $f(z)\in A_\rho$ then the Taylor series $\sum_{k=0}^{\infty}a_kz^k$ (where for each R with $0<R<\rho$ the coefficients are given by $a_k=\frac{1}{2\pi i}\oint_{|t|=R}\frac{f(t)}{t^{k+1}}dt$) is convergent for $|z| < \rho$ and its sum is $f(z)$, and the series is divergent for each $|z| > \rho$. For the partial sums of the Taylor series $s_n(z) := \sum_{k=0}^{n} a_k z^k$ we have for each R , $0 < R < \rho$,

$$\begin{aligned} s_n(z) &:= \sum_{k=0}^{n} a_k z^k = \frac{1}{2\pi i} \oint_{|t|=R} \sum_{k=0}^{n} \frac{z^k}{t^{k+1}} f(t)dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R} \frac{t^{n+1} - z^{n+1}}{t^{n+1}} \frac{f(t)}{t-z} dt. \end{aligned} \tag{2.4}$$

Assume $f(z) \in A_\rho$, $\rho > 1$, further let n and r be positive integers, and let $z_1, \ldots, z_n$ be pairwise different points in the disk $D_\rho := \{z : |z| < \rho\}$. Write $\omega_n(z) := \prod_{k=1}^{n}(z - z_k)$. Again, denote by $h_{r,rn-1}(f;z)$ the (unique) Hermite polynomial of degree $rn-1$ at most and of order r that satisfies

$$h_{r,rn-1}^{(j)}(f; z_k) = f^{(j)}(z_k), \qquad k = 0, \ldots, n-1;\ j = 0, \ldots, r-1.$$

This polynomial can be represented in the form

$$h_{r,rn-1}(f;z) = \tfrac{1}{2\pi i} \oint_{|t|=R} \tfrac{\omega_n(t)^r - \omega_n(z)^r}{\omega_n(t)^r} \tfrac{f(t)}{t-z}\, dt,$$

where R is any number such that $1 < R < \rho$ and $|z_k| < R$ for $1 \le k \le n$.

Theorem 2.3 *Assume $f(z) \in A_\rho$, $\rho > 1$. Let r be a positive integer. Then for the Hermite interpolating polynomial $h_{r,r\cdot 2(n+1)-1;\alpha;\beta}(f;z)$ interpolating $f(z)$ at the zeros of $\omega_{2(n+1);\alpha;\beta}(z)$ we have*

$$\begin{aligned} &\lim_{n\to\infty} \left| s_{r\cdot 2(n+1)-1}(f;z) - h_{r,r\cdot 2(n+1)-1}(f;z) \right|^{1/(n+1)} \le \\ &\qquad\qquad \le \begin{cases} \frac{|z|^{2r-1}}{\rho^{2r}} & \text{when } 1 < |z| < \rho \\ \frac{|z|^{2r}}{\rho^{2r+1}} & \text{when } |z| > \rho\ . \end{cases} \end{aligned} \tag{2.5}$$

and for each non-negative integer p

$$\lim_{n\to\infty} \left(s^{(p)}_{r\cdot 2(n+1)-1}(f;z) - h^{(p)}_{r,r\cdot 2(n+1)-1;\alpha;\beta}(z) \right) = 0 \tag{2.6}$$

uniformly on each compact subset of the disk $\{z : |z| < \rho^{\frac{2r+1}{2r}}\}$.

Since the Taylor series of a function $f(z) \in A_\rho$ is uniformly convergent to $f(z)$ on each compact subset of the disk $\{z : |z| < \rho\}$, we obtain from Theorem 2.3 the following result.

Corollary 2.4 *If the assumptions of Theorem 2.3 are satisfied then*

$$\lim_{n\to\infty} h^{(p)}_{r\cdot 2(n+1)-1;\alpha;\beta}(z) = f^{(p)}(z),$$

uniformly on each compact subset of $\{z : |z| < \rho\}$.

Results similar to those given in this section for $L_{n+1;\alpha}(z)$ with $\alpha = 0$ are given in Walsh [4, Ch. 7, Theorems 10 and 11], Walsh [4, Ch. 7, Theorem 1]). Additional results for other operators are given in [2, chapters 1 and 2].

3 Proof of Theorem 2.1

Proof of Theorem 2.1 (i) First we prove the theorem for $r = 0$. We have

$$L_{2(n+1)-1;\alpha;*}(f;z) = \sum_{k=0}^{n} \frac{f(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)})}{z-e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}} \times$$
$$\times \frac{\omega_{2(n+1);\alpha;\beta}(z)}{(n+1)e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)n}\left(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)(n+1)} - e^{2\pi i\beta/3}\right)}$$

$$= \omega_{2(n+1);\alpha;\beta}(z) \sum_{k=0}^{n} \frac{f(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)})}{z-e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}} \times$$
$$\times \frac{e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}}{(n+1)e^{2\pi i\alpha/3}\left(e^{2\pi i\alpha/3} - e^{2\pi i\beta/3}\right)}$$

$$= \frac{\omega_{2(n+1);\alpha;\beta}(z)}{(n+1)e^{2\pi i\alpha/3}\left(e^{2\pi i\alpha/3} - e^{2\pi i\beta/3}\right)} \times$$
$$\times \sum_{k=0}^{n} \frac{f(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)})}{z-e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}} \times$$
$$\times \frac{e^{\frac{2\pi i}{3(n+1)}(3k+\alpha)}}{e^{\frac{2\pi i}{3(n+1)}\cdot 3(k+1)} - e^{\frac{2\pi i}{3(n+1)}\cdot 3k)}} \times$$

$$\times\left(e^{\frac{2\pi i}{3(n+1)}\cdot 3(k+1)}-e^{\frac{2\pi i}{3(n+1)}\cdot 3k}\right)$$

$$=\frac{\omega_{2(n+1);\alpha;\beta}(z)e^{\frac{2\pi i}{3(n+1)}\cdot\alpha}}{(n+1)\left(e^{\frac{2\pi i}{3(n+1)}\cdot 3}-1\right)e^{2\pi i\alpha/3}\left(e^{2\pi i\alpha/3}-e^{2\pi i\beta/3}\right)}\times$$

$$\times\sum_{k=0}^{n}\frac{f(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)})}{z-e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}}\left(e^{\frac{2\pi i}{3(n+1)}\cdot 3(k+1)}-e^{\frac{2\pi i}{3(n+1)}\cdot 3k}\right)$$

From the definition of the Riemann integral we have

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{f(e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)})}{z-e^{\frac{2\pi i}{3(n+1)}\cdot(3k+\alpha)}}\quad\left(e^{\frac{2\pi i}{3(n+1)}\cdot 3(k+1)}-e^{\frac{2\pi i}{3(n+1)}\cdot 3k}\right)=$$
$$=\oint_{|t|=1}\frac{f(t)}{z-t}dt\ .$$

Hence we have uniformly in $\{z\ |z|\le\delta\ \}$, $0<\delta<1$,

$$\lim_{n\to\infty}L_{2(n+1);\alpha;*}(f;z)=\frac{e^{2\pi i\beta/3}}{e^{2\pi i\alpha/3}-e^{2\pi i\beta/3}}\frac{1}{2\pi i}\oint_{|t|=1}\frac{f(t)}{z-t}\,dt\ .$$

Similarly we get that we have uniformly in $\{z\ |z|\le\delta\ \}$, $0<\delta<1$,

$$\lim_{n\to\infty}L_{2(n+1);*;\beta}(f;z)=\frac{e^{2\pi i\alpha/3}}{e^{2\pi i\beta/3}-e^{2\pi i\alpha/3}}\frac{1}{2\pi i}\oint_{|t|=1}\frac{f(t)}{z-t}\,dt\ .$$

By adding the last two results we see that we have uniformly in $\{z\ :\ |z|\le\delta\ \}$, $0<\delta<1$,

$$\lim_{n\to\infty}\ L_{2(n+1)-1;\alpha;\beta}(f;z)=\frac{1}{2\pi i}\oint_{|t|=1}\frac{f(t)}{t-z}\,dt\ .$$

This completes the proof of Theorem 2.1 for $r=0$.

(ii) We prove now the theorem for $r\ge 1$. The functions

$$L_{2(n+1)-1;\alpha;\beta}(f;z),\ L_{2(n+1)-1;\alpha;*}(f;z)\ \text{ and } L_{2(n+1)-1;*;\beta}(f;z)\ ,$$

are entire functions. The proof of Theorem 2.1 for $r\ge 1$, follows by applying the Weierstrass double-series theorem ([3, p.95, Sec.2.8]) and differentiating r times the formulas (2.1) , (2.2) and (2.3) for $r=0$. □

4 Proof of Theorem 2.3

Proof of Theorem 2.3 Assume $|z| > 1$. Choose R , $1 < R < \rho$. We have by (2.4)

$$s_{r\cdot 2(n+1)-1} = \frac{1}{2\pi i}\oint_{|t|=R}\frac{f(t)}{t-z}\left(1-\left(\frac{z}{t}\right)^{r\cdot 2(n+1)}\right)dt \tag{4.1}$$

and

$$\begin{aligned} h_{r.\cdot 2(n+1)-1;\alpha;\beta}(f;z) &= \tfrac{1}{2\pi i}\oint_{|t|=R}\tfrac{f(t)}{t-z}\left(1-\frac{\left(\omega_{2(n+1);\alpha;\beta}(z)\right)^r}{\left(\omega_{2(n+1);\alpha;\beta}(t)\right)^r}\right)dt \\ &= \tfrac{1}{2\pi i}\oint_{|t|=R}\tfrac{f(t)}{t-z}\left(1-\frac{(z^{n+1}-e^{2\pi i\alpha/3})^r(z^{n+1}-e^{2\pi i\beta/3})^r}{(t^{n+1}-e^{2\pi i\alpha/3})^r(t^{n+1}-e^{2\pi i\beta/3})^r}\right)dt \\ &= \tfrac{1}{2\pi i}\oint_{|t|=R}\tfrac{f(t)}{t-z}\left(1-\left(\tfrac{z}{t}\right)^{r\cdot 2(n+1)}\frac{(1-\frac{e^{2\pi i\alpha/3}}{z^{n+1}})^r(1-\frac{e^{2\pi i\beta/3}}{z^{n+1}})^r}{(1-\frac{e^{2\pi i\alpha/3}}{t^{n+1}})^r(1-\frac{e^{2\pi i\beta/3}}{t^{n+1}})^r}\right)dt \end{aligned} \tag{4.2}$$

We have for $|z| > 1$ and $|t| > 1$

$$\begin{aligned} 1-\left(\tfrac{z}{t}\right)^{r\cdot 2(n+1)}&\frac{(1-\frac{e^{2\pi i\alpha/3}}{z^{n+1}})^r(1-\frac{e^{2\pi i\beta/3}}{z^{n+1}})^r}{(1-\frac{e^{2\pi i\alpha/3}}{t^{n+1}})^r(1-\frac{e^{2\pi i\beta/3}}{t^{n+1}})^r} = \\ &= 1-\left(\tfrac{z}{t}\right)^{r\cdot 2(n+1)}\left(1-\tfrac{re^{2\pi i\alpha/3}}{z^{n+1}}+\mathcal{O}\Big(\tfrac{1}{z^{2(n+1)}}\Big)\right)\times \\ &\qquad\times\left(1-\tfrac{re^{2\pi i\beta/3}}{z^{n+1}}+\mathcal{O}\Big(\tfrac{1}{z^{2(n+1)}}\Big)\right)\times \\ &\qquad\times\left(1+\tfrac{re^{2\pi i\alpha/3}}{t^{n+1}}+\mathcal{O}\Big(\tfrac{1}{t^{2(n+1)}}\Big)\right)\times \\ &\qquad\times\left(1+\tfrac{re^{2\pi i\beta/3}}{t^{n+1}}+\mathcal{O}\Big(\tfrac{1}{t^{2(n+1)}}\Big)\right) \\ &= \left(\tfrac{z}{t}\right)^{r\cdot(2n+1)}\left(1-\tfrac{r(e^{2\pi i\alpha/3}+e^{2\pi i\beta/3})}{z^{n+1}}+\tfrac{r(e^{2\pi i\alpha/3}+e^{2\pi i\beta/3})}{t^{n+1}}\right. \\ &\qquad\left.+\mathcal{O}\Big(\tfrac{1}{z^{2(n+1)}}\Big)+\mathcal{O}\Big(\tfrac{1}{t^{2(n+1)}}\Big)+\mathcal{O}\Big(\tfrac{1}{(zt)^{n+1}}\Big)\right) \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we get

$$s_{r\cdot 2(n+1)-1} \quad -h_{r.\cdot 2(n+1)-1;\alpha;\beta}(f;z) = \frac{1}{2\pi i}\oint_{|t|=R}\frac{f(t)}{t-z}\times$$
$$\times\left(\frac{z}{t}\right)^{r\cdot 2(n+1)}\times\Bigg(-\frac{r(e^{2\pi i\alpha/3}+e^{2\pi i\beta/3})}{z^{n+1}}+\frac{r(e^{2\pi i\alpha/3}+e^{2\pi i\beta/3})}{t^{n+1}}+$$
$$+\mathcal{O}(\frac{1}{z^{2(n+1)}}\Big)+\mathcal{O}\Big(\frac{1}{t^{2(n+1)}}\Big)+\mathcal{O}\Big(\frac{1}{(zt)^{n+1}}\Big)\Bigg)$$

Considering for $1 < R < \rho$ the cases $1 < |z| < R$ and $|z| > R$ we find

$$\limsup_{n\to\infty} \quad |s_{r\cdot 2(n+1)-1}(f;z) - h_{r.\cdot 2(n+1)-1;\alpha;\beta}(f;z)|^{\frac{1}{n+1}} =$$
$$= \left(\frac{|z|}{R}\right)^{2r}\cdot\max\left(\frac{1}{|z|}\,,\frac{1}{R}\right)$$
$$= \begin{cases} \frac{|z|^{2r-1}}{R^{2r}} & \text{when } \ 1 < |z| < R \quad \text{and } 1 < R < \rho \\ \frac{|z|^{2r}}{R^{2r+1}} & \text{when } \ |z| > R \ \ \text{and } 1 < R < \rho. \end{cases}$$

Letting $R \nearrow \rho$ we get

$$\limsup_{n\to\infty} \quad |s_{r\cdot 2(n+1)-1}(f;z) - h_{r.\cdot 2(n+1)-1;\alpha;\beta}(f;z)|^{\frac{1}{n+1}} =$$
$$= \begin{cases} \frac{|z|^{2r-1}}{\rho^{2r}} & \text{when } \ 1 < |z| < \rho\ , \\ \frac{|z|^{2r}}{\rho^{2r+1}} & \text{when } \ |z| > \rho\ . \end{cases}$$

Now (2.6) follows from (2.5) and the maximum principle for analytic functions. □

References

[1] A. S. Cavaretta, Jr., A. Sharma and R. S. Varga, Interpolation in the roots of unity: An extension of a theorem of J. L. Walsh, Resultate Math. 3(1981), 155-191

[2] A. Jakimovski, A. Sharma and J. Szabados, Walsh equiconvergence of complex interpolating polynomials, Springer, Dordrect, The Netherlands, 2006.

[3] E. C. Titchmarsh, The theory of functions, 2nd ed., Oxford University Press, London 1939.

[4] J. L. Walsh, Interpolation and approximation by rational functions in the complex domain, A.M.S. Colloq. Publications X, Providence, R.I., 5th ed., 1969.

A Fast Algorithm for Spherical Basis Approximation

J. Keiner and J. Prestin
Institute of Mathematics
University of Lübeck
Wallstraße 40, 23560, Lübeck, Germany
Email address: {keiner, prestin}@math.uni-luebeck.de

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Radial basis functions appear in a wide field of applications in numerical mathematics and computer science. We present a fast algorithm for scattered data interpolation and approximation on the sphere with spherical radial basis functions of different spatial density. We discuss three settings, each leading to a special structure of the interpolation matrix allowing for an efficient implementation using discrete Fourier transforms. A numerical example is given to show the advantages of spherical radial basis functions with different spatial densities.

1 Introduction

Radial basis functions have spread into a wide field of topics in numerical mathematics and computer science. Applications can be found in approximation of high dimensional and/or scattered data and the modelling of partial differential equations, as well as in neuroinformatics where so-called *radial basis function networks* are prominent. Not only for the mentioned applications, these functions are of special interest, since they show several features which make them well suited for a wide range of problems and, at the same time, computationally attractive (cf. e.g. [3] and the references therein).

The characteristic property of radial basis functions is that their value depends only on the "distance" of the argument to a fixed element of the function's domain. To be more exact, a radial basis function f is given by

$$f : V^2 \to \mathbb{C},\ f(x,y) := \tilde{f}\,(\|x-y\|)\,,\ \tilde{f} : [0,\infty) \to \mathbb{C},\ x, y \in V,$$

where $(V, \|\cdot\|)$ is a metric space. The definition shows clearly that calculations are simplified since radial basis functions behave like univariate functions, although their domain, in general, is multi-dimensional. Especially for higher dimensions, this fact contributes at a considerable amount to the effectiveness of algorithms which utilise these functions.

Moreover, many radial basis functions have physical interpretations making them a reasonable choice for the modelling of many physically motivated problems. The Poisson kernel for example, a function we will study in this paper, can be viewed as a solution of a potential problem, which makes it particularly useful for a range of problems on spherical geometries.

From another point of view, the use of radial basis functions for approximation problems becomes more clear: They can be used to interpolate functions from a given set of scattered data points without requiring a certain structure with respect to their distribution in the domain. We can think of a set of these functions, where each of them is associated with exactly one of the data points and models the influence of it on a probabilistic model of the function to be approximated. A model function can be derived as a linear combination of these functions. For reasonable choices, they are often unimodal and show exactly one global maximum at a certain point, often referred to as their *centre*. Therefore, each function's influence on the model function decreases as one moves away from it's centre, which, besides from being somewhat reasonable in many cases, leads to a stability property of the interpolating function. Deviations of a single data point become visible only in its neighborhood.

In this paper, we deal with a setting where we like to describe real-valued functions, defined on a two-dimensional sphere embedded in the Euclidean space $\mathbb{R}^3$. Section 2 gives a quick introduction to notational conventions used and special properties of spherical geometry.

In Section 3 we review *Legendre polynomials*, *associated Legendre functions* and *spherical harmonics*. These functions and the underlying concepts form the classical basement for spherical approximation.

The following Section 4 introduces the concept of *spherical basis functions* and in particular *positive definite functions*. They represent an entirely different approach compared to spherical harmonics, which is well suited in cases where spherical harmonics tend to exhibit unwanted ripple structures in the approximating function. This can be often observed when somewhat *smooth* data with only a few protruding peaks serves as input. As we will see, spherical basis functions can master this situation quite well. For error estimates and further applications we refer to [7, 6]. Moreover, we exploit the idea of using different kinds of functions to represent regions of different smoothness in the data. The geometry of the sphere itself renders this uniform approach often useless by causing the same type of problems. In conclusion, we show the linear independence of Poisson kernels of pairwise different parametrisation. For a more general approach to multiscale kernels see [11].

Section 5 formulates some algorithmic aspects which arise in spherical approximation with radial basis functions. We refer the reader also to [8]. Some symmetry properties are derived and it is shown how they can be exploited to reduce computational costs. But as a warning, the amount of reduction that can be achieved, strongly depends on the concrete distribution of the given data.

Finally, Section 6 shows applications of the concepts introduced in the previous sections to real-life data from texture analysis in crystallography (see [2]). We demonstrate that for these data sets, a multiscale approach incorporating Poisson kernels or other radial basis functions of different "shape" is obligate for a good approximation result.

2 Basics

Every point $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ given in Cartesian coordinates by the vector $(x_1, x_2, x_3)^{\mathrm{T}}$ can be described in spherical coordinates by a vector $(r, \vartheta, \varphi)^{\mathrm{T}}$ with $r > 0$, $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ (see Figure 1). We have

$$\begin{aligned} (x_1, x_2, x_3)^{\mathrm{T}} &= (r \sin\vartheta \cos\varphi, r \sin\vartheta \sin\varphi, r \cos\vartheta)^{\mathrm{T}}, \\ r &= \sqrt{x_1^2 + x_2^2 + x_3^2} = \|\mathbf{x}\|_2. \end{aligned}$$

We denote by $\mathbb{S}^2$ the unit sphere embedded into $\mathbb{R}^3$; i.e.,

$$\mathbb{S}^2 := \left\{\mathbf{x} \in \mathbb{R}^3 \,:\, \|\mathbf{x}\|_2 = 1\right\}$$

and identify $\boldsymbol{\xi} \in \mathbb{S}^2$ with the vector $(\vartheta, \varphi)^{\mathrm{T}}$. Let $\boldsymbol{\xi} = (\vartheta, \varphi)^{\mathrm{T}}$, $\boldsymbol{\eta} = (\vartheta', \varphi')^{\mathrm{T}} \in \mathbb{S}^2$ and α be the angle spanned by the origin, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Then the standard inner product $\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \cos\alpha$ is given by

$$\cos\alpha = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi').$$

The space of *homogeneous polynomials* of degree $k \in \mathbb{N}_0$ in $\mathbb{R}^3$ is denoted by $\mathrm{Hom}_k\left(\mathbb{R}^3\right)$, comprising all polynomials $Q_k \in \Pi_k(\mathbb{R})$ fulfilling $Q_k(\alpha\,\mathbf{x}) = \alpha^k Q_k(\mathbf{x})$ for arbitrary $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$. The proper subspace of *harmonic homogeneous polynomials* of degree k is defined by

$$\mathrm{Harm}_k\left(\mathbb{R}^3\right) := \left\{Q_k \in \mathrm{Hom}_k\left(\mathbb{R}^3\right) \,:\, \Delta_{\mathbf{x}} Q = 0\right\}, \tag{2.1}$$

where $\Delta_{\mathbf{x}}$ is the Laplacian

$$\Delta_{\mathbf{x}} := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{2.2}$$

Furthermore, we have

$$\dim\left(\mathrm{Hom}_k\left(\mathbb{R}^3\right)\right) = \frac{(k+1)(k+2)}{2}, \quad \dim\left(\mathrm{Harm}_k\left(\mathbb{R}^3\right)\right) = 2k+1.$$

To keep it short, we let $\mathcal{H}_k := \mathrm{Harm}_k\left(\mathbb{R}^3\right)\big|_{\mathbb{S}^2}$. For further details see [5] or [10].

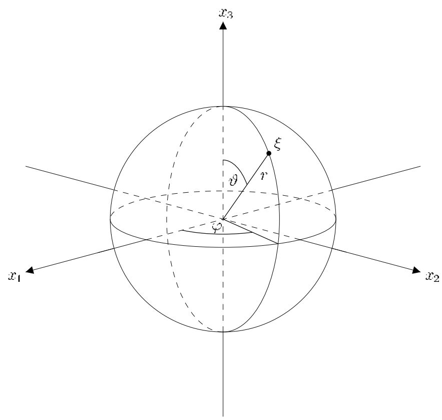


Figure 1: The spherical coordinate system in $\mathbb{R}^3$. Every point $\boldsymbol{\xi}$ on a sphere with radius r around the origin can be described by angles ϑ, φ and the radius r. For $\vartheta = 0$ or $\vartheta = \pi$ the point $\boldsymbol{\xi}$ coincides with the North or the South pole, respectively.

The description of the spherical approximation problem starts with a given finite dimensional space $\mathcal{V}$ with dimension $K \in \mathbb{N}$ of square integrable functions $\psi : \mathbb{S}^2 \to \mathbb{R}$; hence

$$\int_{\mathbb{S}^2} |\psi(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} := \int_0^{2\pi} \int_0^{\pi} |\psi(\vartheta, \varphi)|^2 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi < \infty.$$

With respect to a basis $\{\psi_k\}_{k=1}^K$ of $\mathcal{V}$, every function $f \in \mathcal{V}$ has a unique representation

$$f = \sum_{k=1}^{K} a_k(f)\psi_k \quad (a_k \in \mathbb{R}) \,.$$

In our setting, we are given data points $(\boldsymbol{\xi}_l, f_l)_{l=1}^L$, $L \in \mathbb{N}$, with $\boldsymbol{\xi}_l \in \mathbb{S}^2$ and $f_l \in \mathbb{R}$. We let

$$\mathbf{f} := (f_1, \ldots, f_L)^{\mathrm{T}} \in \mathbb{R}^L, \quad \mathbf{a} := (a_1, \ldots, a_K)^{\mathrm{T}} \in \mathbb{R}^K$$

and

$$\boldsymbol{\Psi} := \begin{pmatrix} \psi_1(\boldsymbol{\xi}_1) & \ldots & \psi_K(\boldsymbol{\xi}_1) \\ \vdots & \ddots & \vdots \\ \psi_1(\boldsymbol{\xi}_L) & \ldots & \psi_K(\boldsymbol{\xi}_L) \end{pmatrix} \in \mathbb{R}^{L \times K}. \tag{2.3}$$

The approximation problem on the sphere reads as follows:

$$\text{Find } \tilde{\mathbf{a}} \in \mathbb{R}^K \text{ satisfying } \tilde{\mathbf{a}} \in \arg\min_{\mathbf{a}\in\mathbb{R}^K} \|\mathbf{f} - \mathbf{\Psi}\,\mathbf{a}\|_2. \tag{2.4}$$

Depending on which case holds, $K \leq L$ or $K > L$, the problem can be viewed as a *least-squares-problem* or as a so-called *special optimisation problem*. For the topics treated in this text, $K = L$ holds and $\mathbf{\Psi}$ will be assumed to be non-singular, so that the solution can be explicitly written as

$$\tilde{\mathbf{a}} = \mathbf{\Psi}^{-1}\mathbf{f}. \tag{2.5}$$

For further information we refer the interested reader to [1].

3 Legendre Functions and Spherical Harmonics

We briefly mention some facts on *Legendre polynomials* and the closely related *associated Legendre functions*. Based on this foundation, we describe the function space of spherical harmonics and how it is related to spherical approximation.

The Legendre polynomials $P_k : [-1, 1] \to \mathbb{R}$, $k \in \mathbb{N}_0$, as classical orthogonal polynomials are given by their corresponding *Rodrigues formula*

$$P_k(t) := \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(t^2 - 1\right)^k. \tag{3.1}$$

The *Formula of Laplace-Heine* ([13, p. 194]) provides a classical asymptotic approximation formula for Legendre polynomials: it says that for $k \in \mathbb{N}$ and $\vartheta \in [\epsilon, \pi - \epsilon]$ with $\epsilon > 0$ we have

$$P_k(\cos\vartheta) = \sqrt{\frac{2}{\pi k \sin\vartheta}}\cos\left(\left(k + \frac{1}{2}\right)\vartheta - \frac{\pi}{4}\right) + \mathcal{O}\left(k^{-3/2}\right). \tag{3.2}$$

Concerning the generating series of the Legendre polynomials

$$\phi(h, t) := \sum_{k=0}^{\infty} P_k(t)h^k \tag{3.3}$$

for arbitrary but fixed $t \in [-1, 1]$, which is absolutely and uniformly convergent for $h \in (-1, 1)$, we have

$$\sum_{k=0}^{\infty} P_k(t) h^k = \frac{1}{\sqrt{1 - 2ht + h^2}}. \tag{3.4}$$

This representation follows from the ordinary differential equation

$$\left(1 + h^2 - 2ht\right) \left(\frac{\partial}{\partial h} \phi\right) (h, t) = (t - h) \phi(h, t) \tag{3.5}$$

obtained by differentiation with respect to h and comparing coefficients in line with (3.3). Using the initial condition $\phi(0, t) = 1$ yields the unique solution (3.4). From this result, the identity

$$\sum_{k=0}^{\infty} (2k + 1) P_k(t) h^k = \frac{1 - h^2}{\left(1 - 2ht + h^2\right)^{3/2}} \tag{3.6}$$

follows easily.

When h is restricted to $(0, 1)$, the function $Q_h : [-1, 1] \to \mathbb{R}$ with

$$Q_h(t) := \frac{1 - h^2}{\left(1 - 2ht + h^2\right)^{3/2}} \tag{3.7}$$

is called *Poisson kernel*. We refer to Figure 2 and notice that the parameter h allows for controlling the concentration of the function's energy around $t = 1$.

The Legendre polynomials can be viewed as a special case of a more general set of orthogonal functions. Let $k, n \in \mathbb{N}_0$ with $n \leq k$. The functions $P_k^n : [-1, 1] \to \mathbb{R}$, given by

$$P_k^n(t) := \left(\frac{(k - n)!}{(k + n)!}\right)^{1/2} \left(1 - t^2\right)^{n/2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} P_k(t),$$

are called associated Legendre functions.

Notice that the associated Legendre function P_k^0 coincides with the Legendre polynomial P_k. The associated Legendre functions fulfill the orthogonality condition

$$\int_{-1}^{1} P_k^n(t) P_l^n(t) \, \mathrm{d}t = \frac{2}{2k + 1} \delta_{k,l} \quad (n \leq \min\{k, l\}). \tag{3.8}$$

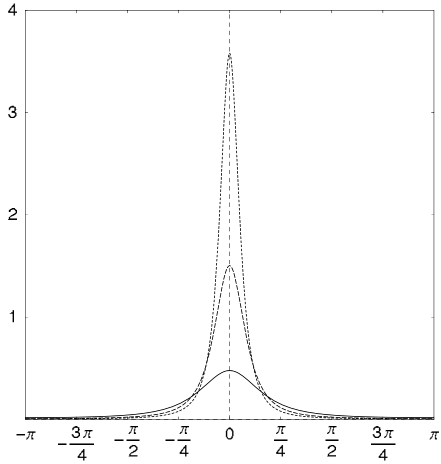


Figure 2: The Poisson kernel $Q_h(\cos\vartheta)$ for $h = 0.5, 0.7, 0.8$. The energy concentrates more and more around $\vartheta = 0$ as h increases.

We now introduce the function space of *spherical harmonics*, a key to the treatment of spherical approximation problems. From Laplace's differential equation $\Delta f = \mathbf{0}$ in $\mathbb{R}^3$, one obtains in spherical coordinates

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \cdot \frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial^2 f}{\partial\varphi^2} = 0. \tag{3.9}$$

Using an ansatz based on separation of variables and taking into account that $r = 1$ when restricting (3.9) to $\mathbb{S}^2$, one obtains the solutions

$$\begin{aligned} Y_k^n &: \mathbb{S}^2 \to \mathbb{C} \quad (k \in \mathbb{N}_0;\ n = -k, -k+1, \ldots, k)\,, \\ Y_k^n(\vartheta,\varphi) &:= \sqrt{\tfrac{2k+1}{4\pi}}\, P_k^{|n|}(\cos\vartheta)\mathrm{e}^{\mathrm{i}n\varphi}. \end{aligned} \tag{3.10}$$

An important result is that these functions Y_k^n are contained in $\mathcal{H}_k$. Owing to the separability, one proves easily that they also fulfill the orthogonality condition

$$\langle Y_k^n, Y_l^m \rangle_{\mathbb{S}^2} = \delta_{k,l}\,\delta_{n,m} \tag{3.11}$$

with respect to the $\mathrm{L}^2\left(\mathbb{S}^2\right)$-inner product

$$\langle Y_k^n, Y_l^m \rangle_{\mathbb{S}^2} := \int_0^{2\pi} \int_0^{\pi} Y_k^n(\vartheta, \varphi) \overline{Y_l^m(\vartheta, \varphi)} \sin \vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi. \tag{3.12}$$

Since $\dim \mathcal{H}_k = 2k+1$, the set $\{Y_k^n : n = -k, -k+1, \ldots, k\}$ forms an orthonormal basis of $\mathcal{H}_k$ for every $k \in \mathbb{N}_0$. Moreover, the spaces $\mathcal{H}_k$ are orthogonal to each other and the set

$$\{Y_k^n : k = 0, 1, \ldots, K;\ n = -k, -k+1, \ldots, k\} \quad (K \in \mathbb{N}_0)$$

provides an orthonormal basis for the direct sum of spaces $\bigoplus_{k=0}^{K} \mathcal{H}_k$ called the space of *spherical harmonics* of degree K.

At first glance, the restriction to homogeneous and harmonic polynomials might exclude various functions from $\Pi_K\left(\mathbb{S}^2\right)$. But as a matter of fact, the spaces are identical (see [5, p. 29]), i.e.

$$\Pi_K\left(\mathbb{S}^2\right) = \bigoplus_{k=0}^{K} \mathcal{H}_k.$$

Finally, we mention the well known *Addition Theorem* that relates any set of functions $\{H_k^n\}_{n=-k}^{k}$ forming an orthonormal basis of the space $\mathcal{H}_k$ to the Legendre polynomials P_k. It particularly holds for the basis given in (3.10).

Proposition 3.1 (Addition Theorem) *For every* $\mathrm{L}^2\left(\mathbb{S}^2\right)$*-orthonormal basis* $\{H_k^n\}_{n=-k}^{k}$ *of* $\mathcal{H}_k$*, we have*

$$\sum_{n=-k}^{k} H_k^n(\boldsymbol{\xi}) \overline{H_k^n(\boldsymbol{\eta})} = \frac{2k+1}{4\pi} P_k(\boldsymbol{\xi} \cdot \boldsymbol{\eta}).$$

For a proof see [10] or [5, p. 37].

4 Spherical Basis Functions

In this section we introduce n suitable alternative class of functions for approximation on the sphere, namely *spherical basis functions*. Instead of providing a basis for a certain function space on the sphere directly, the space of spherical basis functions covers a wide range of functions, where each of them generates a basis for an approximation space that is suited for a given set of data points. This allows to adjust the used space to the properties of the given data in order to achieve optimal results, making spherical basis functions more flexible and useful especially for scattered data. Strongly correlated with spherical basis functions is the class of *positive definite functions.*

Definition 4.1 *A continuous function* $G : [-1, 1] \to \mathbb{R}$ *is called* positive definite, *if and only if for every set of points* $\{\boldsymbol{\xi}_l\}_{l=1}^{L}$ *on* $\mathbb{S}^2$, $L \in \mathbb{N}$, *the corresponding* Gramian matrix $\mathbf{A} := (a_{i,j})_{i,j=1}^{L}$ *with* $a_{i,j} := G\left(\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j\right)$ *is positive semi-definite. If* $\mathbf{A}$ *is even positive definite,* G *is called a* strictly positive definite function.

In general, it is hard to prove directly that a function is positive definite according to Definition 4.1. A fundamental characterisation is given in the following theorem due to Schoenberg (see [12]).

Proposition 4.2 *Let* $G : [-1, 1] \to \mathbb{R}$ *be a function of the form* $G = \sum_{k=0}^{\infty} a_k P_k$, *with* $\sum_{k=0}^{\infty} |a_k| < \infty$. *Then the following statements are equivalent:*

1. *The function* G *is positive definite on* $\mathbb{S}^2$.
2. *The coefficients* a_k *fulfill* $a_k \geq 0$ *for all* $k \in \mathbb{N}_0$.

For strictly positive definite functions a similar necessary and sufficient condition was proved recently by Chen, Menegatto and Sun in [4]: A function G is strictly positive definite if and only if all coefficients a_k are greater than or equal to zero and infinitely many coefficients a_k with odd k and infinitely many coefficients a_k with even k are greater than zero. Now, for our purpose spherical basis functions are defined as follows:

Definition 4.3 *Every function* $G : [-1, 1] \to \mathbb{R}$ *with*

$$G(t) = \sum_{k=0}^{\infty} a_k P_k(t), \tag{4.13}$$

satisfying $a_k > 0$ *for all* $k \in \mathbb{N}_0$ *and* $\sum_{k=0}^{\infty} a_k < \infty$ *is called a* spherical basis function.

The Poisson kernel Q_h defined in (3.7), for example, is a spherical basis function. This follows immediately taking into account that $(2k+1)\,h^k > 0$ for all $k \in \mathbb{N}_0$ and $\sum_{k=0}^{\infty}(2k+1)h^k < \infty$ for $h \in (0,1)$.

Using spherical basis functions, we define the approximation space and a basis in order to use the scheme from Section 2. Given data points $(\boldsymbol{\xi}_l, f_l)_{l=1}^{L}$, $L \in \mathbb{N}$ and having chosen a spherical basis function G, we obtain the functions $G_l : \mathbb{S}^2 \mapsto \mathbb{R}$ by

$$G_l(\boldsymbol{\xi}) := G\left(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}\right).$$

The values of the generated functions G_l solely depend on the geodesic distance of the argument $\boldsymbol{\xi}$ to the fixed point $\boldsymbol{\xi}_l$. For many reasonable choices, the function value reaches a maximum at $\boldsymbol{\xi} = \boldsymbol{\xi}_l$ and decreases towards a minimum for $\boldsymbol{\xi} = -\boldsymbol{\xi}_l$. Figure 3 shows the Poisson-kernel $Q_h\left((0,0)^{\mathrm{T}} \cdot \boldsymbol{\xi}\right)$ centred at the North pole for different values of h.

Now, the matrix $\boldsymbol{\Psi}$ introduced in (2.3) is in this setting identical to the Gramian matrix from Definition 4.1, which is known to be positive definite, hence regular. Notice that this property immediately implies the linear independence of the functions G_l.

From a more practical point of view, a drawback of the described method becomes clear. Independent of the distribution of the points $\boldsymbol{\xi}_l$ on the sphere, all data samples are represented by rotated versions of the same function with same spatial density generated from a single spherical basis function G. As it is often the case in practical situations, the data do not need to be distributed uniformly. Data points can be clustered in certain regions, while in others their density might be low. This can cause problems in the quality of the computed approximation result and leads to numerical instability. Another aspect is that these functions can also be used for multiscale representations, where, depending on the required accuracy, a subset of the basis functions is used to represent either a fine or a coarse approximation. Here, the need for functions of different spatial density is also essential.

As well as the use of this more flexible approach seems to be working (see for example [9]), theoretic results ensuring the solvability of the problem for sets of possibly different basis functions are

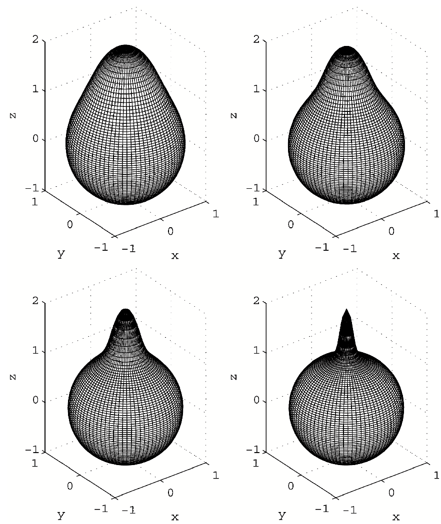


Figure 3: The Poisson kernel $Q_h\left((0,0)^{\mathrm{T}} \cdot \boldsymbol{\xi}\right)$ centred at the North pole as a function of $\boldsymbol{\xi}$ and evaluated on the sphere $\mathbb{S}^2$ for different values of h. Starting with $h = 0.6$ in the upper left picture, the value increases in steps of 0.1 to $h = 0.9$ in the lower right picture.

not easy to obtain. In the next section, we will show the linear independence of Poisson kernels for pairwise different parameter h. But it will remain open whether the matrix $\boldsymbol{\Psi}$ still remains non-singular.

4.1 Extension to the Multiscale Case

In order to investigate the linear independence of Poisson kernels at different scales; i.e., for different values of h, we first need some basic results.

Lemma 4.4 *Let $\vartheta \in [0, \pi]$ be fixed. There exists a constant $c(\vartheta) > 0$ such that for arbitrary $k \in \mathbb{N}_0$ there exists an index $k^* \in \mathbb{N}_0$ with $k^* \geq k$ and*

$$\sqrt{\frac{2k^*+1}{4\pi}}\,|P_{k^*}(\cos\vartheta)| > c(\vartheta). \tag{4.14}$$

PROOF. The case $k = 0$ is trivial since $P_0 = 1$. So let $k > 0$ and assume at first $\vartheta = 0$ or $\vartheta = \pi$. By observing $|P_k(\cos\vartheta)| =$

$|P_k(\pm 1)| = 1$, we obtain the estimate

$$\sqrt{\frac{2k+1}{4\pi}}\,|P_k(\cos\vartheta)| = \sqrt{\frac{2k+1}{4\pi}} \geq \frac{1}{2\sqrt{\pi}} > 0. \tag{4.15}$$

By choosing $k^* = k$ and $c(\vartheta) \in (0, \frac{1}{2\sqrt{\pi}})$ arbitrary, assertion (4.14) is fulfilled. Now fix $\vartheta \in (0,\pi)$. Employing the approximation formula from (3.2) we conclude

$$\sqrt{\frac{2k+1}{4\pi}}\,|P_k(\cos\vartheta)| = \frac{1}{\pi\sqrt{\sin\vartheta}}\left|\cos\left(\left(k+\frac{1}{2}\right)\vartheta - \frac{\pi}{4}\right)\right| + \mathcal{O}\left(k^{-1}\right).$$

The asymptotic part $\mathcal{O}\left(k^{-1}\right)$ vanishes for $k \to \infty$. The constant $\frac{1}{\pi\sqrt{\sin\vartheta}}$ is strictly positive. So let us assume

$$\cos\left(\left(k+\frac{1}{2}\right)\vartheta - \frac{\pi}{4}\right) \xrightarrow{k\to\infty} 0.$$

We now immediately get a contradiction, since this would require ϑ to be of the form $\vartheta = j\pi$ for certain $j \in \mathbb{Z}\setminus\{0\}$ and would therefore violate the assumption $0 < \vartheta < \pi$. $\square$

Corollary 4.5 *Let $\boldsymbol{\xi} = (\vartheta,\varphi) \in \mathbb{S}^2$ be fixed. There exists a constant $c(\vartheta) > 0$ such that for arbitrary $k \in \mathbb{N}_0$ there exists an index $k^* \in \mathbb{N}_0$ with $k^* \geq k$ and*

$$\left|Y_{k^*}^0(\boldsymbol{\xi})\right| > c(\vartheta). \tag{4.16}$$

Proof. We utilise the definition of the functions Y_k^n in (3.10) with

$$Y_k^n(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}}P_k^{|n|}(\cos\vartheta)e^{in\varphi}, \tag{4.17}$$

and close with the remark that for $n = 0$, the proof reduces to an application of Lemma 4.4. $\square$

Corollary 4.5 now allows for an investigation of the linear independence of Poisson kernels Q_h for pairwise different parameters h.

Theorem 4.6 *Let $L \in \mathbb{N}$, $0 < h_1 < h_2 < \ldots < h_L < 1$ and*

$$\boldsymbol{\xi}_l := (\vartheta_l, \varphi_l) \in \mathbb{S}^2 \quad (l = 1, \ldots, L) \tag{4.18}$$

be L pairwise different points on the sphere. Then the functions $G_l : \mathbb{S}^2 \to \mathbb{R}$ with

$$G_l(\boldsymbol{\xi}) := Q_{h_l}(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}) = \sum_{k=0}^{\infty} (2k+1) P_k(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}) h_l^k \tag{4.19}$$

are linearly independent.

Proof. We assume

$$\sum_{l=1}^{L} \lambda_l G_l(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}) = 0 \quad \left(\boldsymbol{\xi} \in \mathbb{S}^2\right) \tag{4.20}$$

for certain coefficients $\lambda_i \in \mathbb{R}$ and prove that $\lambda_l = 0$ holds for $l = 1, \ldots, L$. Applying the definition of the Poisson kernel from (3.7) and using the Addition Theorem from Proposition 3.1, we obtain

$$\begin{aligned}
0 &= \sum_{l=1}^{L} \lambda_l G_l(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}) \\
&= \sum_{l=1}^{L} \lambda_l \sum_{k=0}^{\infty} (2k+1) P_k(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}) h_l^k \\
&= \sum_{l=1}^{L} \lambda_l \sum_{k=0}^{\infty} 4\pi\, h_l^k \sum_{n=-k}^{k} Y_k^n\left(\boldsymbol{\xi}_l\right) \overline{Y_k^n\left(\boldsymbol{\xi}\right)} \\
&= \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \left(\sum_{l=1}^{L} \lambda_l\, 4\pi\, h_l^k\, Y_k^n\left(\boldsymbol{\xi}_l\right) \right) \overline{Y_k^n\left(\boldsymbol{\xi}\right)}.
\end{aligned}$$

In view of the fact that the set $\left\{\overline{Y_k^n}\right\}_{k \in \mathbb{N}_0, n=-k,\ldots,k}$ forms a basis of $\mathrm{L}^2\left(\mathbb{S}^2\right)$, the following infinite set of equations

$$\sum_{l=1}^{L} \lambda_l\, h_l^k\, Y_k^n\left(\boldsymbol{\xi}_l\right) = 0 \quad \left(k \in \mathbb{N}_0,\ n = -k, \ldots, k\right) \tag{4.21}$$

must be fulfilled. So assume now that there exists at least one index l such that $\lambda_l \neq 0$ and define

$$l_{\max} := \max_{l=1,\ldots,L} \| l \,:\, \lambda_l \neq 0\} \,. \tag{4.22}$$

Owing to the estimates

$$\left|\frac{(L-1)\lambda_l}{c_1\,\lambda_{l_{\max}}}\right| = \mathcal{O}(1), \quad \left(\frac{h_l}{h_{l_{\max}}}\right)^k = \mathcal{O}\left(c_2^{-k}\right), \quad \left|Y_k^0\left(\boldsymbol{\xi}_l\right)\right| = \mathcal{O}\left(\sqrt{k}\right) \tag{4.23}$$

for $l = 1, \ldots, L$, where c_1 is an arbitrary and c_2 a fixed positive constant, we get

$$\lim_{k\to\infty} \frac{\left|\lambda_l\, h_l^k\, Y_k^0\left(\boldsymbol{\xi}_l\right)\right|}{\left|\frac{c}{L-1}\lambda_{l_{\max}}\, h_{l_{\max}}^k\right|} = \lim_{k\to\infty}\left|\frac{(L-1)\lambda_l}{c\,\lambda_{l_{\max}}}\right|\left(\frac{h_l}{h_{l_{\max}}}\right)^k \left|Y_k^0\left(\boldsymbol{\xi}_l\right)\right| = 0. \tag{4.24}$$

Now, using Corollary 4.5, let $k^* \in \mathbb{N}_0$ be large enough such that

$$\left|Y_{k^*}^0\left(\boldsymbol{\xi}_{l_{\max}}\right)\right| > c\left(\vartheta_{l_{\max}}\right) > 0 \tag{4.25}$$

and

$$\left|\lambda_l\, h_l^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_l\right)\right| < \left|\frac{c\left(\vartheta_{l_{\max}}\right)}{L-1}\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\right| \quad (l = 1, \ldots, L; l \neq l_{\max}) \tag{4.26}$$

are simultaneously satisfied. We finally obtain

$$\begin{aligned}
&\left|\sum_{l=1}^{L}\lambda_l\, h_l^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_i\right)\right| \\
\geq\;& \left|\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_{l_{\max}}\right)\right| - \sum_{l=1,\, l\neq l_{\max}}^{L}\left|\lambda_l\, h_l^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_l\right)\right| \\
\geq\;& \left|\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_{l_{\max}}\right)\right| \\
&- \sum_{l=1,\, l\neq l_{\max}}^{L}\left|\frac{c\left(\vartheta_{l_{\max}}\right)}{L-1}\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\right| \\
=\;& \left|\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\, Y_{k^*}^0\left(\boldsymbol{\xi}_{l_{\max}}\right)\right| - \left|c\left(\vartheta_{l_{\max}}\right)\,\lambda_{l_{\max}}\, h_{l_{\max}}^{k^*}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left|Y_{k^*}^0\left(\boldsymbol{\xi}_{l_{\max}}\right)\right| - c\left(\vartheta_{l_{\max}}\right)\right)\left|\lambda_{l_{\max}} h_{l_{\max}}^{k^*}\right| \\
&> 0,
\end{aligned}
$$

which contradicts (4.21). □

5 Algorithmic Aspects

This section concentrates on facets in the numerical treatment of the spherical approximation problem with radial basis functions. We mention properties of the interpolation matrix $\boldsymbol{\Psi}$, explain under which conditions with respect to the data points they are present and briefly discuss how they can be exploited to reduce computational costs. We finally give an algorithm for a certain class of grids that achieves better asymptotic complexity than a naive approach.

When we talk about a spherical grid, we refer to a set of points $(\boldsymbol{\xi}_L)_{l=1}^L$, $L \in \mathbb{N}$, on the sphere, which, when viewed in the ϑ-φ-plane, exhibits the structure of a rectangular grid consisting of rows and columns aligned to the two principal axes ϑ and φ. In the general case, we allow rows and columns to have different distances as illustrated in Figure 4. In the most common case in practical settings, the rows and columns are distributed uniformly in a certain domain $I \subseteq [0, \pi] \times [0, 2\pi)$. We call a grid *regular*, if the distance between adjacent columns is constant. Furthermore we call it *complete* if its columns are even distributed uniformly on the torus $2\pi\,\mathbb{T}^1 := 2\pi(\mathbb{R}/\mathbb{Z})$ identified with the interval $[0, 2\pi)$. We number the nodes row by row in increasing order where ϑ determines the row while φ determines the column. Furthermore, we let m be the number of rows and n the number of columns in a grid and we have therefore $\boldsymbol{\Psi} \in \mathbb{R}^{mn \times mn}$.

We now discuss three different setups of nodes and basis functions and derive the properties of $\boldsymbol{\Psi}$ they imply.

- **All basis functions are identical up to rotation.**

  This is the classical case for which the non-singularity of the resulting interpolation matrix $\boldsymbol{\Psi}$ is assured. The nodes might

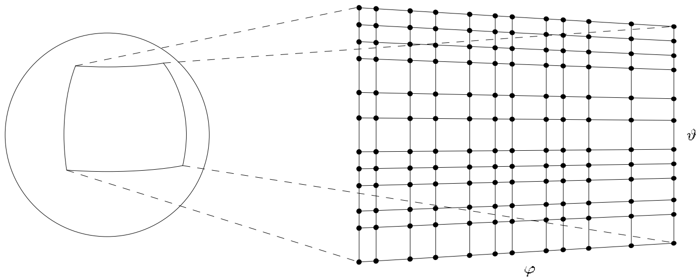


Figure 4: Mapping of a two-dimensional grid to a sphere. Every dot represents a node of the grid. Note that the distance between adjacent rows and columns varies and that the radius of the arcs, representing the rows of the grid, depends on the longitudinal angle ϑ, while the arcs corresponding to grid-columns all have equal radius. This is a direct consequence of the spherical coordinate system, we use.

be distributed arbitrarily. Using the definition of the matrix $\mathbf{\Psi}$ from (2.3) it follows

$$\psi_i\left(\boldsymbol{\xi}_j\right) = G\left(\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j\right) = \psi_j\left(\boldsymbol{\xi}_i\right) \quad (1 \leq i, j \leq L)\,. \tag{5.27}$$

Hence, $\mathbf{\Psi}$ is symmetric and allows one to roughly halve the memory space needed to store the matrix.

- **The grid is regular and the basis function is the same for points in the same row.**

  This restriction allows for a decomposition of $\mathbf{\Psi}$ into quadratic blocks. A sub-matrix $\mathbf{\Psi}^{(k,l)}$ of $\mathbf{\Psi}$, defined by

$$\mathbf{\Psi}^{(k,l)} := \left(\psi_{i,j}^{(k,l)}\right)_{i,j=1}^{n} \in \mathbb{R}^{n\times n} \quad (1 \leq k, l \leq m) \tag{5.28}$$

  and

$$\psi_{i,j}^{(k,l)} := \psi\left(\boldsymbol{\xi}_{(k-1)n+i} \cdot \boldsymbol{\xi}_{(l-1)n+j}\right) \tag{5.29}$$

  contains all values that depend only on the inner products between all points in rows k and l. Therefore, the matrix $\mathbf{\Psi}$ has the representation

$$\mathbf{\Psi} = \begin{pmatrix} \mathbf{\Psi}^{(1,1)} & \mathbf{\Psi}^{(1,2)} & \dots & \mathbf{\Psi}^{(1,m)} \\ \mathbf{\Psi}^{(2,1)} & \mathbf{\Psi}^{(2,2)} & \dots & \mathbf{\Psi}^{(2,m)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Psi}^{(m,1)} & \mathbf{\Psi}^{(m,2)} & \dots & \mathbf{\Psi}^{(m,m)} \end{pmatrix} \in \mathbb{R}^{mn\times mn}. \quad (5.30)$$

Note that generally the matrix $\mathbf{\Psi}$ is no longer symmetric. But as a consequence of the data layout, the blocks $\mathbf{\Psi}^{(k,l)}$ have a very simple structure. Let ϑ_k, ϑ_l be the angles corresponding to rows k, l and φ_i, φ_j be the angles corresponding to columns i, j. Furthermore, we let $\delta > 0$ be the fixed latitudinal angle separating adjacent points in the same row. With G_k as the basis function used for row k, we get for a component

$$\psi_{(i,j)}^{(k,l)} = G_k\left((\vartheta_k,\varphi_i)^{\mathrm{T}} \cdot (\vartheta_l,\varphi_j)^{\mathrm{T}}\right)$$

of a block $\mathbf{\Psi}^{(k,l)}$ and analogously $\psi_{i-1,j-1}^{(k,l)}$ with $i,j > 1$, the identity

$$\begin{aligned} & \psi_{i-1,j-1}^{(k,l)} \\ = \; & G_k\left((\vartheta_k,\varphi_i-\delta)^{\mathrm{T}} \cdot (\vartheta_l,\varphi_j-\delta)^{\mathrm{T}}\right) \\ = \; & G_k\left(\cos\vartheta_k\cos\vartheta_l + \sin\vartheta_k\sin\vartheta_l\cos\left(\varphi_i-\delta-(\varphi_j-\delta)\right)\right) \\ = \; & G_k\left(\cos\vartheta_k\cos\vartheta_l + \sin\vartheta_k\sin\vartheta_l\cos\left(\varphi_i-\varphi_j\right)\right) \\ = \; & G_k\left((\vartheta_k,\varphi_i)^{\mathrm{T}} \cdot (\vartheta_l,\varphi_j)^{\mathrm{T}}\right) \\ = \; & \psi_{i,j}^{(k,l)}. \end{aligned}$$

Therefore, every block $\mathbf{\Psi}^{(k,l)}$ has the form of a *Toeplitz matrix*

$$\mathbf{\Psi}^{(k,l)} = \begin{pmatrix} \psi_{1,1}^{(k,l)} & \psi_{1,2}^{(k,l)} & \dots & \psi_{1,n-1}^{(k,l)} & \psi_{1,n}^{(k,l)} \\ \psi_{2,1}^{(k,l)} & \psi_{1,1}^{(k,l)} & \dots & \psi_{1,n-2}^{(k,l)} & \psi_{1,n-1}^{(k,l)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_{n-1,1}^{(k,l)} & \psi_{n-2,1}^{(k,l)} & \dots & \psi_{1,1}^{(k,l)} & \psi_{1,2}^{(k,l)} \\ \psi_{n,1}^{(k,l)} & \psi_{n-1,1}^{(k,l)} & \dots & \psi_{2,1}^{(k,l)} & \psi_{1,1}^{(k,l)} \end{pmatrix}. \quad (5.31)$$

The first row and column completely determine all its entries. Every Toeplitz matrix can be embedded into a circulant matrix of almost twice the size for every dimension. Since circulant matrices can be diagonalised by means of a multiplication with two Fourier matrices, this allows for the computation of a matrix-vector product with $\mathcal{O}(n \log n)$ arithmetic operations. Similarly, the solution of a linear system of equations, whose matrix is circulant, can also be calculated with $\mathcal{O}(n \log n)$ operations.

- **The grid is complete and the basis function is the same for points in the same row.**

  This case is very similar to the last one, except that further symmetries appear. We now require that also the angle that separates the first and the last point in each row equals δ. First, owing to the fact that the grid is now invariant to rotations along the axis through North and South pole by angles which are multiples of δ, each row of a block $\mathbf{\Psi}^{(k,l)}$ is a circularly shifted version of its successor:

$$\mathbf{\Psi}^{(k,l)} = \begin{pmatrix} \psi_{1,1}^{(k,l)} & \psi_{1,2}^{(k,l)} & \cdots & \psi_{1,n-1}^{(k,l)} & \psi_{1,n}^{(k,l)} \\ \psi_{1,n}^{(k,l)} & \psi_{1,1}^{(k,l)} & \cdots & \psi_{1,n-2}^{(k,l)} & \psi_{1,n-1}^{(k,l)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_{1;3}^{(k,l)} & \psi_{1,4}^{(k,l)} & \cdots & \psi_{1,1}^{(k,l)} & \psi_{1,2}^{(k,l)} \\ \psi_{1;2}^{(k,l)} & \psi_{1,3}^{(k,l)} & \cdots & \psi_{1,n}^{(k,l)} & \psi_{1,1}^{(k,l)} \end{pmatrix}. \tag{5.32}$$

  Circulant matrices are completely determined by their first row or column. Second, another symmetry can be exploited. Since for every point the grid is also symmetric to the plane that contains the point and the two poles, the first row of a matrix-block $\mathbf{\Psi}^{(k,l)}$ is also symmetric to its centre.

We have described three cases, where symmetry properties can be used to reduce the storage space needed for $\mathbf{\Psi}$. Especially the third case is interesting, since there exist algorithms to treat the occurring types of matrices efficiently. How these algorithms work is presented in the following section.

5.1 Circulant Matrices and the Discrete Fourier Transform

Circulant matrices can be multiplied with vectors efficiently. Let $\mathbf{A} := (a_{i,j}) \in \mathbb{R}^{n\times n}$ be a circulant matrix and $\mathbf{x} := (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be a column-vector of length $n \in \mathbb{N}$. We have

$$(\mathbf{Ax})_i = \sum_{j=1}^{n} a_{i,j}\, x_j \quad (1 \le i \le n)\,. \tag{5.33}$$

The circulant structure of $\mathbf{A}$ now gives $a_{i,j} = a_{(i-j \bmod n)+1,1}$ which leads to

$$(\mathbf{Ax})_i = \sum_{j=1}^{n} a_{(i-j \bmod n)+1,1}\, x_j. \tag{5.34}$$

If we now define the n-periodic sequences $\tilde{a} := (\tilde{a}_i)_{i\in\mathbb{Z}}$, $\tilde{x} := (\tilde{x}_i)_{i\in\mathbb{Z}}$, containing the elements of the first column of $\mathbf{A}$ and the elements of the vector $\mathbf{x}$, respectively, by

$$\tilde{a}_i := a_{(i \bmod n)+1,1},\ \tilde{x}_i := x_{(i \bmod n)+1}, \tag{5.35}$$

we can write the result of their discrete periodic convolution $\tilde{a} * \tilde{x}$ as

$$\begin{aligned}
(\tilde{a} * \tilde{x})_i &= \sum_{j=0}^{n-1} \tilde{a}_{i-j}\, \tilde{x}_j \\
&= \sum_{j=0}^{n-1} a_{(i-j \bmod n)+1,1}\, x_{(j \bmod n)+1} \\
&= \sum_{j=0}^{n-1} a_{(i-j \bmod n)+1,1}\, x_{j+1} \\
&= \sum_{j=1}^{n-1} a_{(i+1-j \bmod n)+1,1}\, x_j = (\mathbf{Ax})_{i+1} \quad (0 \le i \le n-1)\,.
\end{aligned}$$

We can calculate the result of the matrix-vector product $\mathbf{A}\,\mathbf{x}$ by a discrete periodic convolution of the vectors $\mathbf{a}$ and $\mathbf{x}$, where $\mathbf{a}$ is the column-vector containing the first column of $\mathbf{A}$. The *Discrete Convolution Theorem* tells that a discrete periodic convolution corresponds to a component-wise multiplication in the frequency domain. If we now denote an application of the discrete Fourier transform (DFT) to a vector $\mathbf{x}$ by $\mathrm{DFT}(\mathbf{x})$ and analogously an application of the inverse transform (IDFT) by $\mathrm{IDFT}(\mathbf{x})$, we therefore get

$$\mathrm{DFT}(\mathbf{A}\,\mathbf{x}) = \mathrm{DFT}(\mathbf{a}) \odot \mathrm{DFT}(\mathbf{x}) \tag{5.36}$$

and

$$\mathbf{A}\,\mathbf{x} = \mathrm{IDFT}(\mathrm{DFT}(\mathbf{a})) \odot \mathrm{DFT}(\mathbf{x}), \tag{5.37}$$

where $\odot$ means the component-wise multiplication or *Hadamard product*. But there is more we can learn from (5.36): If we rewrite the application of a DFT as a multiplication with a Fourier matrix $\mathbf{F_n}$, then the component-wise multiplication of $\mathrm{DFT}(\mathbf{a}) = \mathbf{F}_n\,\mathbf{a}$ and $\mathrm{DFT}(\mathbf{x}) = \mathbf{F}_n\,\mathbf{x}$ can be written as $\mathrm{diag}(\mathbf{F}_n\,\mathbf{a})\;\mathbf{F}_n\,\mathbf{x}$. This yields

$$\mathbf{F}_n\,\mathbf{A}\,\mathbf{x} = \mathrm{diag}(\mathbf{F}_n\,\mathbf{a})\;\mathbf{F}_n\,\mathbf{x}. \tag{5.38}$$

Since Fourier matrices are orthogonal, i.e., $\mathbf{I} = \mathbf{F}_n^{-1}\,\mathbf{F}_n = \mathbf{F}_n^{\mathrm{H}}\,\mathbf{F}_n$ with $\mathbf{I}$ being the identity matrix, we obtain

$$\mathbf{F}_n\,\mathbf{A}\,\mathbf{F}_n^{\mathrm{H}}\,\mathbf{F}_n\,\mathbf{x} = \mathrm{diag}(\mathbf{F}_n\,\mathbf{a})\;\mathbf{F}_n\,\mathbf{x}. \tag{5.39}$$

We conclude $\mathbf{F}_n\,\mathbf{A}\,\mathbf{F}_n^{\mathrm{H}} = \mathrm{diag}(\mathbf{F}_n\,\mathbf{a})$, i.e., the matrix $\mathbf{A}$ can be diagonalised by two Fourier matrices $\mathbf{F}_n$.

The component-wise multiplication of $\mathrm{DFT}(\mathbf{a})$ and $\mathrm{DFT}(\mathbf{x})$ can be computed with $\mathcal{O}(n)$ arithmetic operations. The computational complexity of the multiplication algorithm is therefore determined by the application of DFT and IDFT. These steps require $\mathcal{O}(n\log n)$ *floating point operations (flops)*, if a fast Fourier transform algorithm (FFT) is used.

We now turn to systems of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ whose system matrix is circulant. We multiply with the Fourier matrix $\mathbf{F}_n$ from the left and by recalling $\mathbf{F}_n^{\mathrm{H}}\,\mathbf{F}_n = \mathbf{I}$ we get

$$\mathbf{F}_n\,\mathbf{A}\,\mathbf{F}_n^{\mathrm{H}}\,\mathbf{F}_n\,\mathbf{x} = \mathbf{F}_n\,\mathbf{b}. \tag{5.40}$$

Using that $\mathbf{F}_n\mathbf{A}\mathbf{F}_n^{\mathrm{H}}$ is a diagonal matrix whose main diagonal entries can be computed with $\mathcal{O}(n\log n)$ flops using an FFT, the system has been transformed into a very simple form and we can solve it with

$\mathcal{O}(n)$ arithmetic operations and obtain the vector $\mathbf{F}_n\,\mathbf{x}$. A final application of the inverse FFT gives the sought solution $\mathbf{x}$. In total, we need $\mathcal{O}(n\log n)$ flops.

Block matrices $\boldsymbol{\Psi}$ with circulant blocks can be handled by a reduction to the algorithm for circulant matrices. For example, we write the system $\boldsymbol{\Psi}\,\mathbf{x} = \mathbf{b}$ as

$$\begin{pmatrix} \boldsymbol{\Psi}^{(1,1)} & \boldsymbol{\Psi}^{(1,2)} & \dots & \boldsymbol{\Psi}^{(1,m)} \\ \boldsymbol{\Psi}^{(2,1)} & \boldsymbol{\Psi}^{(2,2)} & \dots & \boldsymbol{\Psi}^{(2,m)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Psi}^{(m,1)} & \boldsymbol{\Psi}^{(m,2)} & \dots & \boldsymbol{\Psi}^{(m,m)} \end{pmatrix} \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \mathbf{x}^{(m)} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^{(1)} \\ \mathbf{b}^{(2)} \\ \vdots \\ \mathbf{b}^{(m)} \end{pmatrix}, \tag{5.41}$$

with circulant blocks $\boldsymbol{\Psi}^{(i,j)}$, where $\boldsymbol{\Psi}^{(i,j)} \in \mathbb{R}^{n\times n}$, $\mathbf{x}^{(i)}, \mathbf{b}^{(i)} \in \mathbb{R}^n$. Using the results for circulant matrices and defining

$$\mathbf{F}_{m,n} := \mathbf{I}_m \otimes \mathbf{F}_n,$$

with the usual *Kronecker product* $\otimes$, one can write

$$\mathbf{F}_{m,n}\,\boldsymbol{\Psi}\,\mathbf{F}_{m,n}^{\mathrm{H}}\,\mathbf{F}_{m,n}\,\mathbf{x} = \mathbf{F}_{m,n}\,\mathbf{b} \tag{5.42}$$

where $\mathbf{F}_{m,n}\,\boldsymbol{\Psi}\,\mathbf{F}_{m,n}^{\mathrm{H}}$ is a matrix consisting of m^2 diagonal blocks. By reordering rows and columns properly, which would correspond to further multiplications with permutation matrices, we obtain a system of linear equations with a block-diagonal matrix. Each of the m blocks of dimension $n \times n$ represents an independent system of linear equations.

For an asymptotic complexity analysis, we now require m to be of comparable size as n; hence $m = cn$ for some $c \in \mathbb{R}$. In the algorithm, one must first calculate $m^2 + m$ discrete Fourier transforms. Each of these transforms has length n and therefore requires $\mathcal{O}(n\log n)$ flops. In total, this step accumulates to $\mathcal{O}\left(m^2 n\log n\right)$ flops. Next, we must calculate m matrix-vector products or must solve m linear systems of equations, in each case of dimension $n \times n$. Depending on the concrete algorithms used for these steps, we can count the complexity for each of them by $C(n)$. We know only that one can't get better than $C(n) = \mathcal{O}\left(n^2\right)$ and that $C(n) = \mathcal{O}\left(n^3\right)$ is always possible. Together, this gives a complexity of $\mathcal{O}\left(mC(n)\right)$ for this

Algorithm 1 Solving systems of linear equations with a block-matrix with circulant blocks

Input: $n, m \in \mathbb{N}$, for $j, k = 1, \ldots, m$ the block-matrix $\mathbf{\Psi} \in \mathbb{R}^{nm \times nm}$ with circulant blocks, represented by vectors $\mathbf{a}^{(j,k)} \in \mathbb{R}^n$ containing the first column of each block, the right-hand side $\mathbf{b} \in \mathbb{R}^{nm}$ given by sub-vectors $\mathbf{b}^{(j)} \in \mathbb{R}^n$ for $j, 1, \ldots, m$.

for $j = 1, \ldots, m$ **do**
  Compute $\hat{\mathbf{b}}^{(j)} := \mathrm{DFT}\left(\mathbf{b}_j\right)$ by an FFT of length n.
  for $k = 1, \ldots, m$ **do**
    Compute $\hat{\mathbf{a}}^{(j,k)} := \mathrm{DFT}\left(\mathbf{a}^{(j,k)}\right)$ by an FFT of length n.
  end for
end for

for $j = 1, \ldots, m$ **do**
  Compute $\hat{\mathbf{x}}^{(j)}$ as the solution of the m-th system of linear equations.
  Compute $\mathbf{x}^{(j)} := \mathrm{IDFT}\left(\hat{\mathbf{x}}^{(j)}\right)$ by an inverse FFT.
end for

Output: $\mathbf{x} \in \mathbb{R}^{nm}$ as solution of the system $\mathbf{\Psi}\,\mathbf{x} = \mathbf{b}$.
Asymptotic complexity: $\mathcal{O}\left(m^2 n \log n + mC(n) + mn \log n\right)$ flops.

step. The last step, the computation of IDFTs for each of the m vector-blocks of the intermediate result, needs $\mathcal{O}\left(mn \log n\right)$ flops. So the described method requires in total

$$\mathcal{O}\left(m^2 n \log n + mC(n) + mn \log n\right) = \mathcal{O}\left(m^3 \log n + mC(m)\right)$$

flops. In general, methods not exploiting the matrix structure have a complexity of $\mathcal{O}\left(C(nm)\right)$ flops which is at least $\mathcal{O}\left(m^4\right)$ flops. Algorithm 1 summarises the described method for systems of linear equations whose system matrix consists of circulant blocks.

6 An Application

In this final section, we present an application of the previously described approximation schemes to real-life data.

For texture analysis in crystallography, spherical data on a regular grid are processed and reviewed. However, these measurements today are still very time consuming, therefore limiting the affordable resolution of the result.

On this basis, representations with Poisson kernels were calculated for a given data set. These representations can be viewed as a stochastic model. Since we solve an interpolation problem, the resulting function as a linear combination of rotated kernel functions coincides with the measured data on the grid, regardless of the parametrisation of the kernels. But from a numerical point of view, a good adjustment of the parameters is the key. If a small value h is used for all basis functions, the interpolation matrix $\boldsymbol{\Psi}$ becomes nearly singular, since the kernels are close to constant functions. On the other hand, the use of a value close to 1 pushes $\boldsymbol{\Psi}$ towards the unity matrix, making the calculation more stable, but the model represented might not be very reasonable. Therefore, the calculated representation was evaluated at a refined resolution.

At first, only rotated versions of a single kernel with a fixed h were used. Results for different values of h are shown in Figures 5 and 6. One observes that, depending on h, the quality of the result strongly differs comparing the equatorial and the polar regions. Clearly, due to the topology of the sphere, a single basis function near the poles has a noticeably higher impact on its neighbour points than one located near the equator, a fact that causes numerical instabilities. On the other hand, increasing h makes the effective support of the basis functions in the equatorial region very small.

This leads to the idea of using kernels with different spatial densities. According to [9, p. 933], a properly normalised Poisson kernel $Q_h\left(\boldsymbol{\eta} \cdot \boldsymbol{\xi}\right)$ as a function of $\boldsymbol{\xi} \in \mathbb{S}^2$ for fixed $\boldsymbol{\eta} \in \mathbb{S}^2$ can be regarded as a probability density function and the variance σ is given by

$$\sigma^2 = \left(\frac{1-h^2}{1+h^2}\right). \tag{6.43}$$

This formula can be used to derive an automatic procedure that adjusts the parameter h for each row of a spherical grid, thereby controlling the overlap of the spherical basis functions $G_l(\boldsymbol{\xi}) = Q_{h_l}(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi})$ for $l = 1, \ldots, L$. Figure 7 shows the result after application of this procedure.

This closes our investigation. For further improvements, the representation of sharp peaks in the input data must be taken into account. But therefore, one must leave the regular structure of the interpolation matrix $\boldsymbol{\Psi}$. Here it might be useful to compute a coarse approximation on a regular grid with the proposed method and incorporate sharp peaks afterwards. For example, one could solve the interpolation problem by using an iterative method for systems of linear equations. The solution calculated first could here be used as the initial solution, in order to reduce the number of iterations needed.

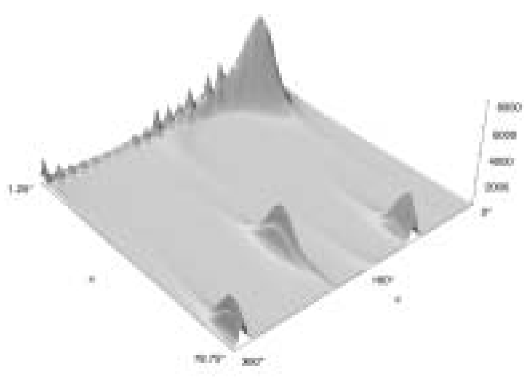


Figure 5: The interpolated data evaluated on a grid with four-fold resolution. The kernels were parametrised with $h = 0.96$. The result shows ripple artifacts near the North pole and also negative values – in this case not admissible from the application.

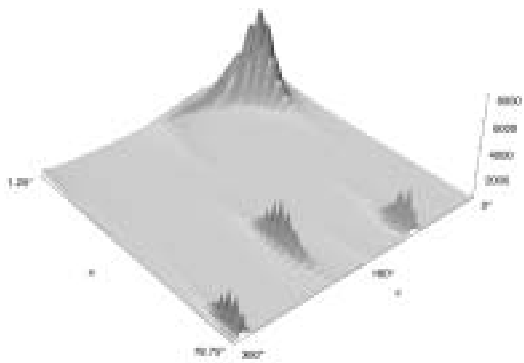


Figure 6: The same data as in Figure 5 but now for $h = 0.98$. Artifacts near the pole are reduced but the spatial density of the spherical basis functions is too low to provide a reasonable approximation far from the pole.

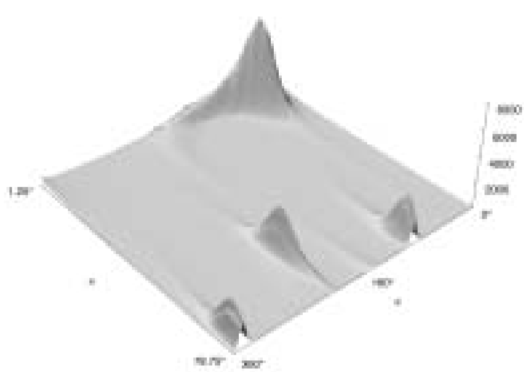


Figure 7: The same data as in Figure 5 now using automatic spatial density adjustment for the spherical basis functions. Some artifacts remain near the pole but the overall approximation result is smooth.

References

[1] Å. Björk, Numerical Methods for Least Squares Problems, SIAM, Philadelphia, 1996.

[2] K. G. v.d. Boogaart, R. Hielscher, J. Prestin, and H. Schaeben, Application of the radial basis function method to texture analysis, J. Comput. Appl. Math. (to appear).

[3] M. D. Buhmann, Radial Basis Functions, Cambridge University Press, Cambridge, 2003.

[4] D. Chen, D., V. A. Menegatto, and Sun, X., A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2733–2740.

[5] W. Freeden, T. Gervens, and M. Schreiner, Constructive Approximation on the Sphere, Oxford University Press, Oxford, 1998.

[6] Q. T. Le Gia, Galerkin approximation for elliptic PDEs on spheres, J. Approx. Theory **130** (2004), 125– 149.

[7] F. J. Narcowich and J. D. Ward, Scattered data interpolation on spheres: error estimates and locally supported basis functions, SIAM J. Math. Anal. **33** (2002), 1393–1410.

[8] S. Kunis. and D. Potts, Fast spherical Fourier algorithms, J. Comput. Appl. Math. **161** (2003), 75–98

[9] Ta-Hsin Li, Multiscale Representation and Analysis of Spherical Data by Spherical Wavelets, SIAM J. Sci. Comput. **21** (1999), no. 3, 924–953

[10] C. Müller, Spherical Harmonics, Lecture Notes in Mathematics, Vol. 17, Springer Verlag, Berlin, 1966

[11] R. Opfer, Multiscale Kernels, Shaker Verlag, Aachen, 2005

[12] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J. **9** (1942), 98–108

[13] Gábor Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc. Providence, 1975

Direct and Converse Polynomial Approximation Theorems on the Real Line with Weights Having Zeros

G. Mastroianni
Dipartimento di Matematica
Università della Basilicata
Contrada Macchia Romana
I-85100 Potenza
Italy
Email address: mastroianni@unibas.it

J. Szabados
Alfréd Rényi Institute of Mathematics
P.O.B. 127, H-1364 Budapest
Hungary
Email address: szabados@renyi.hu

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

We consider polynomial approximation problems on the real line with generalized Freud weights. The generalization means multiplying these weights by so-called generalized polynomials which have finitely many roots on the corresponding intervals. Analogues of classical polynomial inequalities, as well as direct and converse approximation theorems, will be proved.

1 Introduction

We consider polynomial approximation problems on the real line with generalized Freud weights . The generalization means multiplying these weights by so-called generalized polynomials which have finitely many roots on the corresponding intervals. Analogues of classical polynomial inequalities, as well as direct and converse approximation theorems will be proved.

In order to formulate our results, we introduce some general notations. For an arbitrary weight $u \geq 0$ in $\mathbf{R}$ let $L^p_u(\mathbf{R})$, $1 \leq p < \infty$ be the set of all measurable functions f such that $\|uf\|_p^p = \int_{\mathbf{R}} |fu|^p(x)\,dx < \infty$. When $p = \infty$ and $u(x) > 0$ $(x \neq 0), u(0) = 0$, let $L^\infty_u(\mathbf{R})$ denote the set of all continuous functions in $(-\infty, 0) \cup (0, \infty)$ $(f \in C(\mathbf{R}\backslash\{0\}))$ with the further condition $\lim\limits_{x\to 0 \text{ or } \pm\infty} (fu)(x) = 0$. For smoother functions we define

$$W^p_{r,u}(\mathbf{R}) = \{f \in L^p_u(\mathbf{R}) : f^{(r-1)} \in \mathrm{AC}(\mathbf{R}) \text{ and } \|uf^{(r)}\|_p < \infty\}$$

with $1 \leq p \leq \infty$, $r \geq 1$ where $\mathrm{AC}(\mathbf{R})$ is the collection of all absolutely continuous functions in $\mathbf{R}$. Let $\mathbf{P}_n$ be the set of polynomials of degree at most n, and let

$$E_n(f)_{u,p} = \inf_{P\in\mathbf{P}_n} \|u(f-P)\|_p, \qquad 1 \leq p \leq \infty$$

be the error of best polynomial approximation of $f \in L^p_u(\mathbf{R})$.

2 Results

Let

$$u_\alpha(x) = e^{-|x|^\alpha},\ \alpha > 1,\ x \in \mathbf{R} \tag{2.1}$$

be the Freud weight, and a_n be the corresponding Mhaskar-Rahmanov-Saff number, i.e., a quantity for which $\|u_\alpha p\|_\infty = \|u_\alpha p\|_{L^\infty[-a_n,a_n]}$ for all $p \in P_n$. It is well known that

$$a_n \sim n^{1/\alpha}.$$

Denote by $\mathcal{V}$ the set of continuous functions v which are even in $\mathbf{R}$, $v(0) = 0$, $v'(x) > 0$ in $\mathbf{R}^+$ and

$$A(v) := \lim_{x\to 0} \frac{xv'(x)}{v(x)} < \infty, \quad B(v) := \sup_{x\in\mathbf{R}^+} \frac{xv'(x)}{v(x)} < \infty. \tag{2.2}$$

We will be concerned with the so-called *generalized Freud weight*

$$u(x) = v(x)u_\alpha(x) \in \mathrm{GF}, \qquad x \in \mathbf{R}.$$

It has an algebraic type zero at 0 which presents new phenomena concerning polynomial inequalities and weighted polynomial approximation. We could have considered more generally several (finitely many) singularities instead of 0, but this would create only technical difficulties but no theoretical novelties. Such weights were introduced in [5], where its properties were investigated (cf. Lemma 2 there).

Theorem 2.1 *Let* $u \in \mathrm{GF}$ *and* $1 \le p \le \infty$. *Then, for any* $f \in W^p_{r,u}(\mathbf{R})$, $r \ge 1$, *we have*

$$E_n(f)_{u,p} \le C\left(\frac{a_n}{n}\right)^r \|u_\alpha v_n f^{(r)}\|_p, \tag{2.3}$$

where $v_n(x) = v\left(|x| + \frac{a_n}{n}\right)$.

In addition, if $v \in A(p)$, *then*

$$E_n(f)_{u,p} \le C\left(\frac{a_n}{n}\right)^r \|uf^{(r)}\|_p, \qquad 1 < p < \infty. \tag{2.4)}$$

Remarks. 1. For $v(x) = 1$, the estimates (2.3)–(2.4) are equivalent, and this case was proved in [3] (see also DT).

2. If $u \in \mathrm{GF}$ and $1 \le p \le \infty$ then, in general, v_n cannot be replaced by v in (2.3). This can be seen similarly as in [6, Example 3.5 on p. 87]. Therefore the condition $v \in A(p)$, $1 < p < \infty$ is necessary for (2.4).

To define a suitable modulus of smoothness, for an $f \in L^p_{u_\alpha}$, $1 \le p \le \infty$ let

$$\tilde{\Delta}^r_h f(x) = \sum_{j=0}^{r} (-1)^{j+1} \binom{r}{j} f(x - jh\,\mathrm{sgn}\,x)$$

be the so-called *backward-forward difference* . While this may be a discontinuous function at $x = 0$ even if f is continuous, it still keeps the important property of rth order differences that $\tilde{\Delta}^r_h f(x)$ is identically 0 if and only if f is a polynomial of degree at most $r - 1$.

In order to establish Jackson type results, for the weight u_α we define the modulus of smoothness as

$$\overline{\omega}^r(f,t)_{u_\alpha,p} := \sup_{0<h\le t} \|u_\alpha(x)\tilde{\Delta}_h^r f(x)\|_p. \tag{2.5}$$

Compared to other moduli of smoothness, (2.5) involves only the smoothness properties of the function f and, in virtue of (3.3) of Lemma 3.2 below, it is equivalent to the one defined in [2, p. 182]. Consequently, for $f \in L^p_{u_\alpha}$, the Jackson and Stechkin inequalities are true with the modulus (2.5).

For the weight u the modulus (2.5) is useless. Namely, what we expect would be a Jackson type theorem

$$E_n(f)_{u,p} \le \overline{\omega}\left(f, \frac{a_n}{n}\right)_{u,p}.$$

But this cannot hold. Namely, it would imply

$$E_n(f)_{u,p} \le C\left(\frac{a_n}{n}\right)^r \|uf^{(r)}\|_p,$$

which contradicts Remark 2 made after Theorem 2.1.

Therefore first we define the so-called main part modulus

$$\Omega^r(f,t)_{u,p} := \sup_{0<h\le t} \|u(x)\tilde{\Delta}_h^r f(x)\|_{L^p(\mathbf{R}\setminus I_h)} \tag{2.6}$$

where $I_h = [-4rh, 4rh]$, and then the full modulus will be

$$\omega^r(f,t)_{u,p} = \Omega^r(f,t)_{u,p} + \inf_{q\in \mathbf{P}_{r-1}} \|u(f-q)\|_{L^p(I_t)}. \tag{2.7}$$

The inconvenience of this modulus is that we must have some knowledge about a fixed degree of polynomial approximation of the function on a small interval. Concerning this problem, see Lemma 3.3 below.

We can establish the following:

Theorem 2.2 *Let $\alpha > 1$ and $f \in L^p_u(\mathbf{R})$, $1 \le p \le \infty$. Then*

$$E_n(f)_{u,p} \le C\omega^r\left(f, \frac{a_n}{n}\right)_{u,p}, \quad n > r \ge 1 \tag{2.8}$$

and

$$\omega^r(f,t)_{u,p} \le Ct^r \sum_{k=0}^{\left[t^{\frac{\alpha}{1-\alpha}}\right]} (k+1)^{r(1-1/\alpha)-1} E_k(f)_{u,p}. \qquad (2.9)$$

Remarks. 1. In particular, if $f^{(r-1)} \in \mathrm{AC}(\mathbf{R}\setminus\{0\})$ and $\|uf^{(r)}\|_p < \infty$ then by (2.7) it follows that

$$E_n(f)_{u,p} \le C\left(\frac{a_n}{n}\right)^r \|uf^{(r)}\|_p + C \inf_{q\in\mathbf{P}_{r-1}} \|u(f-q)\|_{L^p(I_{Ca_n/n})}$$

with some constant $C>0$ depending on α.

2. We also observe that in the previous statements of this section we assumed only $v \in \mathcal{V}$.

In what follows we will say that p_n is a "near best approximant" polynomial of $f \in L^p_u(\mathbf{R})$ if $\|u(f-p_n)\|_p \le CE_n(f)_{u,p}$. The following theorem estimates the derivatives of such polynomials. We recall the definition of A_p weights; $v \in A_p$ if

$$\sup\left(\frac{1}{b-a}\int_a^b v^p\right)^{1/p} \cdot \left(\frac{1}{b-a}\int_a^b v^{-q}\right)^{1/q} < \infty, \qquad \frac{1}{p}+\frac{1}{q}=1$$

where the sup is taken for all finite intervals $[a,b]\subset\mathbf{R}$.

Theorem 2.3 *Let p_n be a near best polynomial approximant of degree n of $f \in L^p_u(\mathbf{R})$. Then we have*

$$\|up_n^{(r)}\|_p \le \begin{cases} C\left(\frac{n}{a_n}\right)^r \omega^r\left(f,\frac{a_n}{n}\right)_{u,p}, \\ \qquad\qquad \textit{if } v\in A_p,\ 1<p<\infty, \\[2ex] C\left(\left(\frac{n}{a_n}\right)^r \omega^r\left(f,\frac{a_n}{n}\right)_{u,p} + \|uf\|_p\right), \\ \qquad\qquad \textit{if } A(v)+1/p \ne 0,\dots,r;\ 1\le p\le\infty. \end{cases}$$

Remark. Note that Theorems 2.1–2.2 hold for $u=u_\alpha$ as well if we replace ω in (2.7) by $\bar\omega$ in (2.5).

3 Auxiliary Statements

First we establish some polynomial inequalities useful in several contexts. To this end let

$$J_n = [-a_n, a_n] \setminus \left[-C\frac{a_n}{n}, C\frac{a_n}{n}\right]$$

where C is a fixed positive number. In what follows, C will always denote a positive constant that may assume different values in different formulae, but it is always independent of n, p and f.

Lemma 3.1 *Let u be as in* (2.1), $0 < p \le \infty$ *and* $p_n \in \mathbf{P}_n$. *Then, for n sufficiently large (say $n > n_0$), we have*

$$\|up_n\|_p \le C\|up_n\|_{L^p(J_n)}$$

and

$$\|up_n'\|_p \le C\frac{n}{a_n}\|up_n\|_p.$$

Proceeding as in the proofs of Lemmas 3–4 in [5], this lemma follows by Theorem 5.1 in [8]. When the factor v of the weight u is a generalized Jacobi weight, Lemma 3.1 was proved in [4].

Now we introduce the K-functional

$$K_r(f, t^r)_{u,p} = \inf_{g \in W^p_{r,u}(\mathbf{R})} \{\|u(f-g)\|_p + t^r\|ug^{(r)}\|_p\}.$$

Lemma 3.2 *For any function $f \in L^p_u(\mathbf{R})$, with $1 \le p \le \infty$, we have*

$$K_r(f, t^r)_{u,p} \le C\omega^r(f, t)_{u,p} \tag{3.1}$$

and

$$\Omega^r(f, t)_{u,p} \le CK_r(f, t^r)_{u,p}. \tag{3.2}$$

Moreover

$$\bar{\omega}^r(f, t)_{u_\alpha,p} \sim K_r(f, t^r)_{u_\alpha,p}. \tag{3.3}$$

Remark. (3.3) shows that if the weight has no zero then the modulus is completely equivalent to the K-functional (in contrast to (3.1)-(3.2)). Moreover, it will be clear from the proof that (3.3) holds also in the limiting case $\alpha = 1$.

PROOF OF LEMMA 3.2 Let

$$g_t(x) = \\ r^r \int_0^{1/r} \cdots \int_0^{1/r} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} f\left(x - jt(\sum_{j=1}^{r} u_j) \operatorname{sgn} x\right) du_1 \dots du_r, \\ |x| \geq 2rt,$$

and let $q_{r-1}(x) \in \mathbf{P}_{r-1}$ be the polynomial representing the infimum in (2.7). Further it is easy to see that there exists an infinitely differentiable function $\psi_t(x) \geq 0$ such that

$$\psi_t(x) \begin{cases} = 1, & \text{if } |x| \geq 4rt, \\ = 0, & \text{if } |x| \leq 2rt, \\ \leq 1, & \text{if } 2rt \leq |x| \leq 4rt \end{cases}$$

such that

$$|\psi_t^{(j)}(x)| \leq \frac{C}{t^j}, \qquad x \in \mathbf{R},\ j = 0, \dots, r. \tag{3.4}$$

Now consider the function

$$G_t(x) = (1 - \psi_t(x)) q_{r-1}(x) + \psi_t(x) g_t(x), \qquad x \in \mathbf{R}.$$

Evidently, $G_t \in W^p_{r,u}(\mathbf{R})$, and

$$G_t(x) = \begin{cases} q_{r-1}(x), & \text{if } |x| \leq 2rt, \\ g_t(x), & \text{if } |x| \geq 4rt. \end{cases}$$

Thus, since

$$\|u(f - q_{r-1})\|_{L^p(|x| \leq 4rt)} \leq \omega^r(f, t)_{u,p}$$

and

$$\begin{aligned}\|u(f-g_t)\|_{L^p(|x|\geq 4rt)} &\leq \sup_{0<h\leq t}\|u\tilde{\Delta}_h^r f\|_{L^p(|x|\geq 4rt)}\\ &\leq \sup_{0<h\leq t}\|u\tilde{\Delta}_h^r f\|_{L^p(|x|\geq 4rh)} \leq \omega^r(f,t)_{u,p},\end{aligned}$$

these inequalities show that

$$\|u(f-G_t)\|_{L^p(|x|\geq 4rt \,\mathrm{or}\, |x|\leq 2rt)} \leq \omega^r(f,t)_{u,p}.$$

The remaining intervals can be easily settled by using the above estimates:

$$\begin{gathered}\|u(f-G_t)\|_{L^p(2rt\leq|x|\leq 4rt)} \leq \|u(f-q_{r-1})\|_{L^p(2rt\leq|x|\leq 4rt)}\\ +\|u(f-g_t)\|_{L^p(2rt\leq|x|\leq 4rt)} \leq 2\omega^r(f,t)_{u,p}.\end{gathered}$$

Summarizing

$$\|u(f-G_t)\|_p \leq 4\omega^r(f,t)_{u,p}.$$

Now we estimate the rth derivative of G_t. Since q_{r-1} is a polynomial of degree at most $r-1$, we may assume that $|x|\geq 2rt$. When $|x|\geq 4rt$, then by the definition of g_t,

$$g_t^{(r)}(x) = t^{-r}\tilde{\Delta}_t^r f(x),$$

and thus

$$\begin{aligned}\|uG_t^{(r)}\|_{L^p(\mathbf{R}\setminus I_t)} &= \|ug_t^{(r)}\|_{L^p(\mathbf{R}\setminus I_t)}\\ &\leq Ct^{-r}\|u\sup_{0<h\leq t}\tilde{\Delta}_h^r f\| \leq Ct^{-r}\omega^r(f,t)_{u,p}.\end{aligned}$$

Finally, for $2rt\leq|x|\leq 4rt$ we get

$$|G_t^{(r)}| = |\{\psi_t[q_{r-1}-g_t]\}^{(r)}| \leq C\sum_{j=0}^{r}|\psi_t^{(r-j)}|\cdot|[q_{r-1}-g_t]^{(j)}| \tag{3.5}$$

Here, using

$$\begin{aligned}|f(x)-g_t(x)| &= r^r\left|\int_0^{1/r}\cdots\int_0^{1/r}\tilde{\Delta}^r_{t(u_1+\cdots+u_r)}f(x)\,du_1\ldots du_r\right|\\ &\leq \sup_{0\leq h\leq t}\left|\tilde{\Delta}_h^r f(x)\right|\end{aligned}$$

and (2.5),

$$\|u(q_{r-1}-g_t)\|_{L^p(2rt\le|x|\le 4rt)} \le \|u(q_{r-1}-f)\|_{L^p(2rt\le|x|\le 4rt)}$$
$$+\|u(g_t-f)_{L^p(2rt\le|x|\le 4rt)} \le 2\omega^r(f,t)_{u,p}$$

and

$$\begin{aligned}\|u[q_{r-1}-g_t]^{(r)}\|_{L^p(2rt\le|x|\le 4rt)} &= \|ug_t^{(r)}\|_{L^p(2rt\le|x|\le 4rt)}\\ &\le Ct^{-r}\omega^r(f,t)_{u,p}.\end{aligned}$$

Using $u(x)\sim u(t)$ hence we get

$$\|q_{r-1}(x)-g_t(x)\|_{L^p(2rt\le|x|\le 4rt)} \le C\frac{\omega^r(f,t)_{u,p}}{u(t)},$$

$$\|[q_{r-1}(x)-g_t(x)]^{(r)}\|_{L^p(2rt\le|x|\le 4rt)} \le C\frac{\omega^r(f,t)_{u,p}}{t^r u(t)}.$$

To estimate the intermediate derivatives we use an inequality of Ditzian (cf. [2, formula (2.2.14)]) :

$$\begin{aligned}&\|u(x)[q_{r-1}(x)-g_t(x)]^{(j)}\|_{L^p(2rt\le|x|\le 4rt)}\\ &\le Cu(t)\|[q_{r-1}(x)-g_t(x)]^{(j)}\|_{L^p(2rt\le|x|\le 4rt)}\\ &\le C\frac{\omega^r(f,t)_{u,p}}{t^j}, \qquad j=0,\dots,r,\end{aligned}$$

whence and from (3.4)-(3.5) we get

$$\|uG_t^{(r)}\|_{L^p(2rt\le|x|\le 4rt)} \le Ct^{-r}\omega^r(f,t)_{u,p}.$$

This completes the proof of (3.1).

As for (3.2), we take a function $g\in W_r^p(u_\alpha)_{\mathbf{R}}$ such that

$$\|u(f-g)\|_p + t^r\|ug^{(r)}\|_p\} \le 2K_r(f,t^r)_{u,p}.$$

Choosing an arbitrary $0<h\le t$, for $|x|\ge 4rh$ we obtain (since now $u(x)\sim u(x\pm jh)$, $j=0,\dots,r$ using properties of the weight and the presence of backward-forward differences)

$$\|u\tilde{\Delta}_h^r f\|_{L^p(|x|\ge 4rh)} \le \|u\tilde{\Delta}_h^r(f-g)\|_{L^p(|x|\ge 4rh)} + \|u\tilde{\Delta}_h^r g\|_{L^p(|x|\ge 4rh)} \tag{3.6}$$

$\le \|u(f-g)\|_p+\|u\tilde{\Delta}_h^r g\|_{L^p(|x|\ge 4rh)} \le 2K_r(f,t^r)_{u,p}+\|u\tilde{\Delta}_h^r g\|_{L^p(|x|\ge 4rh)}$.
Here, because of symmetry, we may assume $x \ge 4rh$ and get (using the generalized Minkowski inequality)

$$\|u\tilde{\Delta}_h^r g\|_{L^p(4rh,\infty)} \le C\left(\int_{4rh}^{\infty}\left(\int_0^{rh} u(x)g^{(r)}(x-y)(rh-y)^{r-1}\,dy\right)^p dx\right)^{1/p} \tag{3.7}$$

$$\le Ch^{r-1}\int_0^{rh}\left(\int_{3rh}^{\infty} u(x)^p|g^{(r)}(x)|^p\,dx\right)^{1/p} dy \le Ct^r\|ug^{(r)}\|_p,$$

where we used the fact that

$$\frac{u(x)}{u(x-y)} \le C\frac{u(x)}{u(x-rh)} \le C\frac{u(x)}{u(\frac{3}{4}x)} \le C, \qquad 0 \le y \le rh \le \frac{x}{4}$$

(cf. Lemma 1 from [5]). This proves (3.2).

The proof of (3.3) is simpler, so we just outline the changes in the above argument. Introduce the functions

$$g_{t,1}(x) = r^r\int_0^{1/r}\cdots\int_0^{1/r}\sum_{j=1}^r(-1)^{j+1}\binom{r}{j}f\left(x-jt(\sum_{j=1}^r u_j)\operatorname{sgn}x\right)du_1\ldots du_r,$$

$$g_{t,2}(x) = r^r\int_0^{1/r}\cdots\int_0^{1/r}\sum_{j=1}^r(-1)^{j+1}\binom{r}{j}f\left(x+jt(\sum_{j=1}^r u_j)\operatorname{sgn}x\right)du_1\ldots du_r,$$

and

$$G_t(x) := \psi(x)g_{t,1}(x) + (1-\psi(x))g_{t,2}(x)$$

where now $\psi(x) \ge 0$ is an infinitely differentiable function such that

$$\psi(x)\begin{cases} = 0, & \text{if } x \le -1,\\ \le 1, & \text{if } -1 \le x \le 1,\\ = 1, & \text{if } x \ge 1.\end{cases}$$

Then, using that a forward rth order difference with step h at x is the same as the backward rth order difference with step h at $x+rh$, we can see that

$$\|u_\alpha(f-G_t)\|_p \le C\bar{\omega}^r(f,t)_{u_\alpha,p}$$

and

$$\|u_\alpha G_t^{(r)}\|_{L^p(\mathrm{R})} \le Ct^{-r}\bar{\omega}^r(f,t)_{u_\alpha,p},$$

which shows that the K-functional is majorized by the modulus. The opposite inequality in (3.3) can be seen like (3.6)-(3.7), the interval $[4rh,\infty)$ and the limit of integration $4rh$ replaced by $\mathbf{R}^+$ and 0, respectively. The lemma is proved. In order to estimate $\inf_{q\in\mathbf{P}_{r-1}}\|(f-q)u\|_p$ we can use the following lemma that is a slight reformulation of Lemma 9 in [5], so we omit the proof.

Lemma 3.3 (a) *Let $a>0$. If $f\in W^p_{r,u_\alpha}(\mathbf{R})$ and $v\in\mathcal{V}$ is an A_p-weight, $1<p<\infty$, then there exists a polynomial $p\in\mathbf{P}_{r-1}$ such that*

$$\|v(f-p)\|_{L^p(-t,t)} \le Ct^r\|uf^{(r)}\|_{L^p(-t,t)}, \qquad t\le a.$$

(b) *Let $f^{(r-1)}\in\mathrm{AC}(\mathbf{R}\setminus\{0\})$, $1\le p\le\infty$, $r\ge 1$. If $A(v)+\frac{1}{p}>r$ then*

$$\|fv\|_{L^p(-t,t)} \le Ct^r(\|vf^{(r)}\|_{L^p(-a,a)}+\|fv\|_{L^p(-a,a)}), \qquad 0<t\le a,$$

while if $A(v)+\frac{1}{p}\le r$ with $A(v)+\frac{1}{p}\ne 1,\dots,r$ or $A(v)=0$ for $p=\infty$, and $f^{(r-\tau-1)}(0)$ with $\tau=\left[A(v)+\frac{1}{p}\right]$ exists, then there exist polynomials $p\in\mathbf{P}_{r-1-\tau}$ such that

$$\|v(f-p)\|_{L^p(-t,t)} \le Ct^r[\|vf^{(r)}\|_{L^p(-a,a)}+\|vf\|_{L^p(-a,a)}], \qquad 0<t\le a.$$

Here the constants C depend on a.

Lemma 3.4 *Let $1\le p\le\infty$. Then, for any polynomial $P\in\mathbf{P}_n$ we have*

$$\inf_{q\in\mathbf{P}_r}\|u(P-q)\|_{\left[-4r\frac{a_n}{n},4r\frac{a_n}{n}\right]} \le C\left(\frac{a_n}{n}\right)^r\|uP^{(r)}\|_p.$$

PROOF Let $L_r(P)$ be a Lagrange polynomial that interpolates P at r arbitrary points of $[-\alpha_n, \alpha_n]$, $\alpha_n := \frac{a_n}{n}4r \sim n^{1/\alpha-1}$. Using Peano Theorem we have

$$|P(x) - L_r(P,x)| \le C_r \alpha_n^{r-1} \int_{-\alpha_n}^{\alpha_n} |P^{(r)}(t)|\, dt, \qquad |x| \le \alpha_n,$$

whence for $1 \le p < \infty$, $v_n(t) = v\left(t + \frac{a_n}{n}\right)$, using Cauchy–Schwarz inequality

$$\begin{aligned} &\left(\int_{-\alpha_n}^{\alpha_n} u^p(x)|P(x) - L_r(P,x)|^p\, dx\right)^{1/p} \\ \le\ & C_r \alpha_n^{r-1} \left(\int_{-\alpha_n}^{\alpha_n} u^p(x)\, dx\right)^{1/p} \int_{-\alpha_n}^{\alpha_n} |P^{(r)}(t)|\, dt \\ \le\ & C_r \alpha_n^{r-1+\frac{1}{p}} \int_{-\alpha_n}^{\alpha_n} v_n(t)|P^{(r)}(t)| u_\alpha(t)\, dt \\ \le\ & C_r \alpha_n^r \|u_\alpha v_n P^{(r)}\|_{L^p(-\alpha_n,\alpha_n)}. \end{aligned}$$

In the second inequality we used the doubling property of v_n. Now we extend the last norm to all $\mathbf{R}$ and then use the first inequality of Lemma 2.1. Then the last norm is majorized by

$$C\|u_\alpha v_n P^{(r)}\|_{L^p(\mathbf{R}\setminus(-\alpha_n,\alpha_n))} \sim \|uP^{(r)}\|_p$$

whence the assertion easily follows for $1 \le p < \infty$. The case $p = \infty$ is simpler and we omit the details.

Let

$$\tilde{u}_\alpha(x) = e^{-\frac{|x|^\alpha}{2A}}, \tag{3.8}$$

with $2A \ge 1$ integer. Then, for the MRS number, $a_n = a_n(\tilde{u}_\alpha)$ related to $\tilde{u}_\alpha$ we have $a_n(\tilde{u}_\alpha) = a_{2An}(u_\alpha) = C(\alpha)(2An)^{1/\alpha}$.

Let $\{p_n(\tilde{u}_\alpha)\}$ be the system of orthonormal polynomials with positive leading coefficients and denote by $x_1 < \ldots < x_n$ the zeros of $p_n(\tilde{u}_\alpha)$. Then

$$\Delta x_k = x_{k+1} - x_k \sim n^{1/\alpha-1} \frac{1}{(\sqrt{\psi_n(x_k)})^{1/4}}, \;\; \psi_n(x) = \left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}.$$

Now let x_0 be such that $|\sqrt{\tilde{u}_\alpha(x_0)}p_n(\tilde{u}_\alpha, x_0)| = \|\sqrt{\tilde{u}_\alpha}p_n(\tilde{u}_\alpha)\|_\infty$ and denote by $l_k(x) = l_{n+1,k}(\tilde{u}_\alpha, x)$, $k = 0, 1, \ldots, n+1$ the k-th fundamental Lagrange polynomial based on the zeros $-x_0, x_1, \ldots, x_n, x_0$. For n sufficiently large define $j = j(n)$ by

$$x_j = \max\{x_k : x_k \le \theta a_n(\tilde{u}_\alpha)\} \tag{3.9}$$

where $\theta \in (0,1)$ is chosen such that $\pm x_0 \notin [-x_j, x_j]$. This is possible since x_0 is near x_n.

The following lemma was proved in [7].

Lemma 3.5 *Let $|x_k| \le x_j$, $x \in [-b_n, b_n]$ with $b_n = \max(x_0, a_n(\tilde{u}_\alpha))$ and $x \sim x_i \in \{-x_0, x_1, \ldots, x_n, x_0\}$. Then*

$$u_\alpha(x) l_k^{2A}(x) \le C \frac{u_\alpha(x_k)}{(|i-k|+1)^{2A}}.$$

Lemma 3.6 *For $f \in L^p_u(\mathbf{R})$ and $t_2 > t_1$, we have*

$$\omega^r(f, t_2)_{u,p} \le \begin{cases} C\left(\frac{t_2}{t_1}\right)^r \omega^r(f, t_1)_{u,p} \\ \qquad\qquad \textit{if } \ v \in A_p \ \textit{and} \ \ 1 < p < \infty, \\ C\left(\frac{t_2}{t_1}\right)^r \left[\omega^r(f, t_1)_{u,p} + t_1^r \|uf\|_p\right], \\ \qquad\qquad \textit{if } \ A(v) + 1/p \ \textit{is not an integer} \\ \qquad\qquad\qquad \textit{and } 1 \le p \le \infty, \end{cases}$$

PROOF By the definition (2.7) of the modulus and (3.2),

$$\omega^r(f,t)_{u,p} = \Omega^r(f,t)_{u,p} + \inf_{q \in \mathbf{P}_{r-1}} ||u(f-q)||_{L^p(I_t)} \le$$

$$\le CK_r(f, t^r)_{u,p} + \inf_{q \in \mathbf{P}_{r-1}} ||u(g-q)||_{L^p(I_t)},$$

where $g \in W^p_{r,u}(\mathbf{R})$ is chosen such that

$$||u(f-g)||_p + t^r ||ug^{(r)}||_p \le 2K_r(f, t^r)_{u,p}.$$

Using Lemma 3.3 we get

$$\inf_{q\in\mathbf{P}_{r-1}} ||u(g-q)||_{L^p(I_t)}$$
$$\leq \begin{cases} Ct^r||ug^{(r)}||_p, & \text{if } v \in A_p \text{ and } 1 < p < \infty. \\ Ct^r(||ug^{(r)}||_p + ||ug||_p), & \text{if } A(v)+1/p \text{ is not an integer} \\ & \text{and } 1 \leq p \leq \infty, \end{cases}$$

Now using $||ug||_p \leq 2K_r(f,t^r)_{u,p} + ||uf||_p$ and (3.1) we obtain the statements of the lemma.

4 Proof of the Theorems

Proof of Theorem 2.1 We follow the same procedure used in the proof of Theorem 1 in [5]. Nevertheless we have to point out some necessary changes and steps.

Proceeding as in Lemma 6 in [5], it is possible to prove the following relations:

$$v_n(x) := v\left(|x| + \frac{a_n}{n}\right) \sim \frac{n}{a_n(\tilde{u}_\alpha)} \int_{x-\frac{a_n}{2n}}^{x+\frac{a_n}{2n}} v(t)\,dt, \quad x \in \mathbf{R} \tag{4.1}$$

where $\tilde{u}_\alpha$ is defined in (3.8) and

$$\int_{x_{i-1}}^{x_{i+1}} v(x)\,dx \leq C(|k-i|+1)^\gamma \int_{x_{k-1}}^{x_{k+1}} v(x)\,dx \tag{4.2}$$

where $-x_0 \leq x_i, x_k \leq x_0$ and γ depends only on the weight v.

Finally, using the definition (3.9) of the index j,

$$v(x) \leq C(|k-i|+1)^{\gamma+1/4} v_n(t) \tag{4.3}$$

where $x \in [x_{i-1}, x_{i+1}]$, $x_i \in \{-x_0, \ldots, x_0\}$ and $t \in [x_{k-1}, x_k]$, $|x_k| \leq x_j$. (Here the constant C is independent of x, t, n, k, i.) Observe that, for $x_k \in [-x_{j+1}, x_{j+1}]$,

$$\Delta x_k \sim \frac{a_n}{n} \quad \text{and} \quad \tilde{u}_\alpha(x) \sim \tilde{u}_\alpha(t), \quad |x-t| \leq C\frac{a_n}{n}. \tag{4.4}$$

These are the principal facts that appear in the proof of Theorem 2.1.

For any $f \in W^p_{1,u_\alpha}(\mathbf{R})$, consider the function

$$f_j = f_{j(n)} = \begin{cases} f(-x_j) & \text{in}\,(-\infty, -x_j], \\ f & \text{in}\ [-x_j, x_j], \\ f(x_j) & \text{in}\ [x_j, \infty) \end{cases}$$

($j(n)$ is defined in (3.9)). Obviously $f_j \in W^p_{1,u_\alpha}(\mathbf{R})$ and

$$E_M(f)_{u,p} \le \|u(f - f_j)\|_p + E_M(f_j)_{u,p}$$

with $M = 2A(n+1)$, where $2A > \gamma + 2$. The first term, using Lemma 4.1 in [2], is majorized by $C\frac{a_n}{n}\|f'u\|_p$, $1 \le p \le \infty$. Then we have to estimate $E_{2An}(f_j)_{u,p}$. To this end the following two steps are necessary:

First step. We introduce the functions

$$(S^+ f_j)(x) = f(-x_j) + \sum_{k=-j}^{j} (x - x_k)^0_+ \Delta M_k$$

and

$$(S^- f_j)(x) = f(-x_j) + \sum_{k=-j}^{j} (x - x_k)^0_+ \Delta m_k$$

where $M_k = \max_{[x_{k-1},x_k]} f_j(x)$, $m_k = \min_{[x_{k-1},x_k]} f_j(x)$. $\Delta b_k = b_{k+1} - b_k$ and

$$(x - x_k)^0_+ = \begin{cases} 1 & \text{if}\ x_k < x, \\ 0 & \text{if}\ x_k \ge x. \end{cases}$$

Then it is simple to verify that $S^- f_j \le f_j \le S^+ f_j$ and, since $\Delta M_k = \Delta m_k = 0$, $|k| > j$, also

$$S^- f_j = f_j = S^+ f_j \quad \text{in} \quad (-\infty, -x_j] \cup [x_j, \infty).$$

Moreover, proceeding as in the proof of Lemma 7 in [5] and using (4.1)–(4.4), we prove that the norm

$$\|u(S^+ f_j - S^- f_j)\|_p = \|u(S^+ f_j - S^- f_j)\|_{L^p(-x_j,x_j)}$$

is majorized by $C\frac{a_n}{n}\|u_\alpha v_n f'\|_p$, $1 \le p \le \infty$ and by $C\frac{a_n}{n}\|uf'\|_p$ if $v \in A_p$ and $1 < p < \infty$.

Second step. Again proceeding as in Lemma 7 of [5] we construct the polynomials $p_k^\pm \in \mathbf{P}_{2A(n+1)}$, $|k| < j$ such that

$$p_k^-(x) \le (x - x_k)_+^0 \le p_k^+(x), \quad x \in \mathbf{R}$$

and

$$p_k^+(x) - p_k^-(x) = l_k^{2A}(x)$$

with l_k the k-th fundamental Lagrange polynomial based on the knots $-x_0, x_1, \dots, x_m, x_0$. With the previous polynomials we define

$$Q_n^\pm(x) = \sum_{\Delta M_k > 0} p_k^\pm(x)\Delta M_k + \sum_{\Delta M_k < 0} p_k^\mp(x)\Delta M_k + f(-x_j),$$

$$q_n^\pm(x) = \sum_{\Delta m_k > 0} p_k^\pm \Delta m_k + \sum_{\Delta m_k < 0} p_k^\mp(x)\Delta m_k + f(-x_j).$$

From these definitions it follows that in $\mathbf{R}$ we have

$$q_n^- \le S^- f_j \le q_n^+, \quad Q_n^- \le S^+ f_j \le Q_n^+$$

and then

$$q_n^- \le S^- f_j \le f_j \le S^+ f_j \le Q_n^+.$$

Consequently

$$E_{2An}(f_j)_{u,p} \le \|u(f_j - Q_n^+)\|_p \le \|u(Q_n^+ - q_n^-)\|_p \le$$

$$\le \|u()Q_n^+ - Q_n^-)\|_p + \|u(q_n^+ - q_n^-)\|_p + \|u(S^+ f_j - S^- f_j)\|.$$

Here the third term was estimated above while the first two terms can be estimated as in Lemma 8 of [5], using (4.1)–(4.4) and Lemma 3.5. The theorem easily follows.

Proof of Theorem 2.2 We consider the function G_t defined in the proof of Lemma 2 with $t = \frac{a_n}{n}$ and set $G_{\frac{a_n}{n}} = G_n$. Then for each $f \in L^p_u$ we have

$$E_n(f)_{u,p} \leq \|u(f - G_n)\|_p + E_n(G_n)_{u,p}.$$

The first term is majorized by $\omega^r\left(f, \frac{a_n}{n}\right)_{u,p}$. For the second term we use Theorem 2.1 and get

$$E_n(G_n)_{u,p} \leq C\left(\frac{a_n}{n}\right)^r \|u_\alpha v_n G_n^{(r)}\|_p.$$

Proceeding as in the previous proof and observing that, for $x \notin \left[-4r\frac{a_n}{n}, 4r\frac{a_n}{n}\right]$, we have $v_n \sim v$, it immediately follows that

$$\left(\frac{a_n}{n}\right)^r \|u_\alpha v_n G_n^{(r)}\|_p \leq C\omega^r\left(f, \frac{a_n}{n}\right)_{u,p}.$$

This proves (2.8).

In order to prove (2.9) we use (3.2) and, with p_n the best approximation of $f \in L^p_u$, we have

$$\begin{aligned}
\omega^r\left(f, \frac{a_n}{n}\right)_{u,p} &\leq C\Bigg[\|u(f-p_n)\|_p + \left(\frac{a_n}{n}\right)^r \|up_n^{(r)}\|_p + \\
&\qquad + \inf_{q \in \mathbf{P}_{r-1}} \|u(f-q)\|_{L^p\left(\left[-4r\frac{a_n}{n}, 4r\frac{a_n}{n}\right]\right)}\Bigg] \\
&\leq C\Bigg[\|u(f-p_n)\|_p + \left(\frac{a_n}{n}\right)^r \|up_n^{(r)}\|_p + \\
&\qquad \inf_{q \in \mathbf{P}_{r-1}} \|u(p_n-q)\|_{L^p\left(\left[-4r\frac{a_n}{n}, 4r\frac{a_n}{n}\right]\right)}\Bigg].
\end{aligned}$$

If we prove that the third term does not exceed a constant multiple of the second term, then (2.9) follows with a well-known procedure using Bernstein inequality. To this end we can use Lemma 3.4 and the theorem is proved.

Proof of Theorem 2.3 Following an argument in [2], we first prove that

$$\|up_n^{(r+1)}\|_p \leq C\left(\frac{n}{a_n}\right)^{r+1}\left[\omega^r\left(f, \frac{a_n}{n}\right)_{u,p} + \left(\frac{a_n}{n}\right)^r \|uf\|_p\right].$$

From the relation

$$p_n - p_0 = p_n - p_{2^l} + \sum_{k=0}^{l-1}(p_{2^{k+1}} - p_{2^k}),$$

where $2^l \le m \le 2^{l+1}$, and with $1 \le p \le \infty$ follows that

$$\begin{aligned}
\|up_n^{(r+1)}\|_p &\le \|u(p_n - p_{2^l})^{(r+1)}\|_p + \sum_{k=1}^{l-1} \|u(p_{2^{k+1}} - p_{2^k})^{(r+1)}\|_p \\
&\le C\left[\left(\frac{n}{a_n}\right)^{r+1} \|u(p_n - p_{2^l})\|_p \right. \\
&\qquad \left. + \sum_{k=1}^{l-1}\left(\frac{2^{k+1}}{a_{2^{k+1}}}\right)^{r+1} \|u(p_{2^{k+1}} - p_{2^k})\|_p\right] \\
&\le C\sum_{k=1}^{l+1}\left(\frac{2^k}{a_{2^k}}\right)^{r+1} E_{2^k}(f)_{u,p} \le C\sum_{k=1}^{l+1}\left(\frac{2^k}{a_{2^k}}\right)^{r+1} \omega^r\left(f, \frac{a_{2^k}}{2^k}\right)_{u,p}.
\end{aligned}$$

Now suppose that $A(v) + \frac{1}{p}$ is not an integer. Then, using Lemma 3.6 we have

$$\omega^r\left(f, \frac{a_{2^k}}{2^k}\right)_{u,p} \le C\left(\frac{a_{2^k}}{2^k}\frac{2^{l+1}}{a_{2^{l+1}}}\right)^{r+1}\left(\omega^r\left(f, \frac{a_n}{n}\right)_{u,p} + \left(\frac{a_n}{n}\right)^r \|fu\|_p\right).$$

Thus

$$\|up_n^{(r+1)}\|_p \le C\left(\frac{n}{a_n}\right)^{r+1}\left[\omega^r\left(f, \frac{a_n}{n}\right)_{u,p} + \left(\frac{a_n}{n}\right)^r \|fu\|_p\right].$$

In case $v \in A_p$, $1 < p < \infty$, the second term in the brackets is omitted by Lemma 3.6.

Let $B_n = \left[-a_n, -4r\frac{a_n}{n}\right] \cup \left[4r\frac{a_n}{n}, a_n\right]$. Then

$$\begin{aligned}
&\left(\frac{a_n}{n}\right)^r \|up^{(r)}\|_{L^p(B_n)} \\
&\quad \le \left\|u\left[\left(\frac{a_n}{n}\right)^r p_n^{(r)} - \tilde{\Delta}^r_{\frac{a_n}{n}} p_n\right]\right\|_{L^p(B_n)} + \|u\tilde{\Delta}^r_{\frac{a_n}{n}} p_n\|_{L^p(B_n)}.
\end{aligned}$$

Now we easily have

$$\|u\tilde{\Delta}^r_{\frac{a_n}{n}} p_n\|_{L^p(B_n)} \le \left\|u\tilde{\Delta}^r_{\frac{a_n}{n}}(f - p_n)\right\|_{L^p(B_n)} + \left\|u\tilde{\Delta}^r_{\frac{a_n}{n}} f\right\|_{L^p(B_n)} \le$$

$$\le 2^r E_n(f)_{u,p} + \Omega^r\left(f, \frac{a_n}{n}\right)_{u,p} \le C\omega^r\left(f, \frac{a_n}{n}\right)_{u,p}.$$

Then,

$$\begin{aligned}
& u(x)\left|\left(\frac{a_n}{n}\right)^r p_n(x)^r - \tilde{\Delta}^r_{\frac{a_n}{n}} p_n(x)\right| \\
= \quad & u(x)\left|\int_0^{a_n/n} \dots \int_0^{a_n/n} [p_n^{(r)}(x) - \right. \\
& \qquad\qquad \left. p_n^{(r)}\left(x + (u_1 + \ldots + u_r)\right)] du_1 \ldots du_r\right| \\
= \quad & u(x)\left|\int_0^{a_n/n} \dots \int_0^{a_n/n} \int_x^{x+u_1+\ldots+u_r} p_n^{(r+1)}(z)\, dz\, du_1 \ldots du_r\right| \\
\le \quad & \left(\frac{a_n}{n}\right)^{r+1} u(x) \frac{n}{a_n} \left|\int_x^{x+r\frac{a_n}{n}} |p_n^{(r+1)}(z)|\, dz\right| \\
\le \quad & \left(\frac{a_n}{n}\right)^{r+1} \frac{Cn}{a_n} \int_x^{x+r\frac{a_n}{n}} u(z)|p_n^{(r+1)}(z)|\, dz.
\end{aligned}$$

Consequently

$$\left\|u\left[\left(\frac{a_n}{n}\right)^r p_n^{(r)} - \tilde{\Delta}^r_{\frac{a_n}{n}} p_n\right]\right\|_{L^p(B_n)} \le C\left(\frac{a_n}{n}\right)^{r+1} \|up_n^{(r+1)}\|_{L^p(B_n)}$$

using the boundedness of the maximal function for $p \in (1, \infty]$ and Fubini theorem for $p = 1$. Now the theorem easily follows by observing that, by Lemma 3.1,

$$\|up_n^{(r)}\|_p \le C\|up_n^{(r)}\|_{L^p(B_n)}.$$

5 Acknowledgments

The first author acknowledges partial support by GNIM-INDAM, progetto speciale and by the Paul Erdős Summer Research Center of the János Bolyai Mathematical Society in Budapest, while he was holding a Paul Erdős visiting professorship position. Research of the second author was supported by OTKA No. T049196. The authors thank the referee for the careful reading of the manuscript and for useful suggestions.

References

[1] M. C. De Bonis, G. Mastroianni and M. Viggiano, K-functionals, moduli of smoothness and weighted best approximation on the semiaxis, in: *Functions, Series, Operators,* Proccedings of the Alexits Memorial Conference (Budapest, 2002), pp. 181-211.

[2] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Verlag (1987).

[3] G. Freud and H. N. Mhaskar, Weighted polynomial approximation in rearrangement invariant Banach function spaces on the whole real line, *Indian J. Math.,* **22** (1980), 209-224.

[4] H. Joung, Generalized polynomial inequalities, Dissertation (1995).

[5] G. Mastroianni and J. Szabados, Polynomial approximation on infinite intervals with weights having inner zeros, *Acta Math. Hungar.,* **96** (2002), 221-258.

[6] G. Mastroianni and V. Totik, Best approximation and moduli of smoothness for doubling weights, *J. Approx. Theory,* **110** (2001), 180-199.

[7] J. Szabados, Weighted Lagrange and Hermite–Fejér interpolation on the real line, *J. Inequalities and Applications,* **1** (1997), 99-123.

[8] E. Saff and V. Totik, *Logarithmic Potential with Extremal Fields,* Springer (1997).

Fourier Sums and Lagrange Interpolation on $(0, +\infty)$ and $(-\infty, +\infty)$

G. Mastroianni
Dipartimento di Matematica
Università della Basilicata
v.le dell'Ateneo Lucano 10, 85100 Potenza ITALY
Email address: mastroianni@unibas.it

P. Vértesi
Alfréd Rényi Institute of Mathematics
Budapest, P.O. Box 127
Hungary 1364
Email address: vertesi@renyi.hu

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

In order to approximate functions on unbounded intervals, the authors show the convergence of truncated Fourier Sums and truncated Lagrange Polynomials.

1 Introduction

This paper concerns the approximation of functions on unbounded intervals by means of orthogonal polynomials: Laguerre polynomials on $(0, \infty)$ and Freud polynomials on $(-\infty, +\infty)$. Unfortunately, the behavior of the Fourier Sum, the interpolating polynomials or the Gaussian quadrature rules related to the previous systems, is poor in the sense that it can be applied only to a restricted class of functions (see for instance [1, 16, 3, 12, 9]).

On the other hand, the polynomial approximation of functions defined on infinite intervals appears in several contexts.In order to overcome this problem, we will modify the Fourier Sum and the interpolating polynomials. Our results have many applications and are analogous to results on finite intervals.

More precisely, starting from well known polynomial inequalities, we propose to approximate a "finite section" of the function on a special finite interval (see Proposition 2.1). This procedure is simple and is convenient in many situations, especially in the numerical treatment of integral equations.

2 Fourier Sums and Lagrange Interpolation in $(0,+\infty)$

Some preliminary results

Let $L^p(a,b)$, $0 \le a < b \le +\infty$ the set of all measurable functions f such that $\|f\|_{L^p(a,b)} = \left(\int_a^b |f|^p(x)dx\right)^{\frac{1}{p}} < +\infty$, $1 \le p < +\infty$. If $a = 0$ and $b = +\infty$ we write $\|f\|_p$ instead of $\|f\|_{L^p(0,+\infty)}$, and $L^p = L^p(0,+\infty)$. With $u(x) = x^\gamma e^{-\frac{x}{2}}$, $\gamma > -\frac{1}{p}$, $x > 0$ a Laguerre weight, we set $f \in L^p_u$ if and only if $fu \in L^p$, $1 \le p < +\infty$. When $p = +\infty$ and $\gamma > 0$ we define

$$L^\infty_u = \left\{ f \in C^0(\mathbb{R}^+) : \lim_{x\to 0}(fu)(x) = 0 = \lim_{x\to+\infty}(fu)(x) \right\},$$

where $\mathbb{R}^+ = (0,+\infty)$ and $C^0(\mathbb{R}^+)$ is the set of all continuous functions in $\mathbb{R}^+$. In the case $p = +\infty$ and $\gamma = 0$, L^p_u is the set of all continuous function in $[0,+\infty)$ with the condition $\lim_{x\to+\infty}(fu)(x) = 0$. In the sequel $\mathcal{C}$ denotes a positive constant. We write $\mathcal{C} \ne \mathcal{C}(a,b,\ldots)$ if $\mathcal{C}$ is independent of the parameters $a,b,\ldots$ Finally $A \sim B$ means that there exists a constant $\mathcal{C} > 0$, independent of the parameters of A and B such that $\mathcal{C}^{-1}A \le B \le \mathcal{C}A$.

We recall the following polynomial inequality (see [2] or [11]):

$$\left(\int_{4m(1+\delta)}^{\infty} |P_m u|^p(x)dx\right)^{\frac{1}{p}} \le \mathcal{C}e^{-Am}\left(\int_0^{+\infty} |P_m u|^p(x)dx\right)^{\frac{1}{p}}, \quad (2.1)$$

holding for $1 \le p \le +\infty$, where $\delta > 0$ is fixed, P_m is an arbitrary polynomial of degree m ($P_m \in \mathbb{P}_m$), the positive constants $\mathcal{C}$ and A depend on δ and are independent on P_m.

Now we establish a proposition to be used later. To this end let $\theta \in (0,1)$ fixed and m a positive integer sufficiently large (say $m > m_0$). We denote by $\Delta_\theta = \Delta_{\theta,m}$ the characteristic function of the interval $[0, 4\theta m]$ and let $M = \left[\frac{\theta}{1+\theta}m\right]$. Obviously, $M \sim m$ if $m \ge m_0(\theta)$.

Proposition 2.1 *Let $f \in L^p_u$ and $1 \le p \le +\infty$. Then, for $m > m_0$, we have*

$$\|f(1-\Delta_\theta)u\|_p \le E_M(f)_{u,p} + \mathcal{C}e^{-Am}\|fu\|_p \quad \textit{and} \quad (2.2)$$

$$\|fu\|_p \le \mathcal{C}(\|f\Delta_\theta u\|_p + E_M(f)_{u,p}), \quad (2.3)$$

where $E_M(f)_{u,p} = \inf_{P\in\mathbb{P}_M} \|(f-P)u\|_p$ and the constants $\mathcal{C}$ and A are independent on m and f.

PROOF. For all $P_M \in \mathbb{P}_M$ and $1 \le p \le +\infty$ we have

$$\|f(1-\Delta_\theta)u\|_p \le \|(f-P_M)u\|_p + \|P_M(1-\Delta_\theta)u\|_p.$$

By (2.1),

$$\|P_M(1-\Delta_\theta)u\|_p \le \mathcal{C}e^{-Am}\|P_M u\|_p$$

and making the infimum on P_M (2.2) easily follows. Now (2.3) follows from the identity $f = \Delta_\theta f + (1-\Delta_\theta)f$ and (2.2). □

From Proposition 2.1 follows that, if $\{\Gamma_m : L^p_u \to \mathbb{P}_m\}_m$ is a sequence of linear operators, we can approximate a function $f \in L^p_u$ by the sequence $\{\Delta_{\theta,m}\Gamma_m(\Delta_{\theta,m}f)\}_m$. Indeed, since

$$\begin{aligned}\|[f-\Delta_{\theta,m}\Gamma_m(\Delta_{\theta,m}f)]u\|_p &\le \|[f-\Gamma_m(\Delta_{\theta,m}f)]\Delta_{\theta,m}u\|_p \\ &+ \|(1-\Delta_{\theta,m})fu\|_p,\end{aligned}$$

the second term on the right, by (2.2), is small. Consequently it is sufficient to consider only the first term or, in other words, approximate f by the sequence of polynomials $\{\Gamma_m(\Delta_{\theta,m} f)\}_m$ on the interval $[0, 4\theta m]$. We will show that such procedure is much simpler and more convenient than the ordinary polynomial approximation.

Fourier sums.

Let $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x > 0$, be a Laguerre weight and $\{p_m(w_\alpha)\}_m$ the correspondent sequence of orthonormal polynomials with positive leading coefficient γ_m. With $f \in L^p_u$ we consider the Fourier sum

$$S_m(w_\alpha, f) = \sum_{k=0}^{m-1} c_k p_k(w_\alpha), \quad c_k = \int_0^{+\infty} f p_k(w_\alpha) w_\alpha.$$

By using Darboux kernel

$$K_m(w_\alpha, x, y) = \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(w_\alpha, x) p_{m-1}(w_\alpha, y) - p_m(w_\alpha, y) p_{m-1}(w_\alpha, x)}{x - t}$$

we can write

$$S_m(w_\alpha, f, x) = \int_0^{+\infty} K_m(w_\alpha, x, y) f(y) w_\alpha(y) dy. \tag{2.4}$$

In [1] (see also [16]) the following expression of $K_m(w_\alpha, x, y)$ is suggested

$$\begin{aligned} K_m(w_\alpha, x, y) &= p_m(w_\alpha, x) p_m(w_\alpha, y) \qquad (2.5) \\ &+ b_m m \left(F_m(x) \frac{p_m(w_\alpha, y)}{y - x} - \frac{p_m(w_\alpha, x) F_m(y)}{y - x} \right) \end{aligned}$$

where $F_m(z) = p_{m+1}(w_\alpha, z) - p_{m-1}(w_\alpha, z)$ and $b_m \sim 1$. Then we can write the Fourier sum as follows

$$\begin{aligned} & S_m(w_\alpha, f, x) = c_m p_m(w_\alpha, x) \qquad (2.6) \\ + \quad & m b_m \left[F_m(x) H(f w_\alpha p_m(w_\alpha), x) - p_m(w_\alpha, x) H(F_m f w_\alpha, x) \right] \end{aligned}$$

where $H(g,z) = \int_0^{+\infty} \frac{g(y)}{y-z} dy$, $z > 0$, is the Hilbert transform. According with Proposition 2.1, we approximate in L^p the function fu by means the sequence $\{S_m(w_\alpha, \Delta_{\theta,m} f) u \Delta_{\theta,m}\}_m$.

If $v^\beta(x) = x^\beta$, the following theorem holds.

Theorem 2.2 *For any $f \in L^p_u$ and $1 < p < +\infty$, we have*

$$\|S_m(w_\alpha, \Delta_{\theta,m} f)\Delta_{\theta,m} u\|_p \le \mathcal{C}\|f\Delta_{\theta,m} u\|_p, \quad \mathcal{C} \ne \mathcal{C}(m,f) \tag{2.7}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p(0,1) \text{ and } \frac{v^\alpha}{v^\gamma}, \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L^q(0,1), \tag{2.8}$$

where $\varphi(x) = \sqrt{x}$ and $q = \frac{p}{p-1}$.

The previous theorem is the best possible in the following sense.

Proposition 2.3 *The following equivalences are true*

$$\|S_m(w_\alpha, f)u\|_p \le \mathcal{C}\|fu\|_p \Leftrightarrow (2.8) \text{ holds with } \frac{4}{3} < p < 4 \tag{2.9}$$

$$\begin{aligned} \|S_m(w_\alpha, f)u\|_p \quad &\le \quad \mathcal{C}\begin{cases} m^{\frac{1}{3}}\|fu\|_p \\ or \\ \|fu(1+\cdot)^{\frac{1}{3}}\|_p \end{cases} \\ &\Leftrightarrow \quad (2.8) \text{ holds with } p \in (1,\infty) \setminus \left(\frac{4}{3}, 4\right) \end{aligned} \tag{2.10}$$

$$\|S_m(w_\alpha, \Delta_{\theta,m} f)u\|_p \le \mathcal{C}\|fu\|_p \Leftrightarrow (2.8) \text{ holds with } 1 < p < 4 \tag{2.11}$$

$$\|S_m(w_\alpha, f)\Delta_{\theta,m} u\|_p \le \mathcal{C}\|fu\|_p \Leftrightarrow (2.8) \text{ holds with } p > \frac{4}{3}. \tag{2.12}$$

The constants $\mathcal{C} > 0$ are independent on m and f.

We will prove Theorem 2.2 and Proposition 2.3 in Section 4. Now we establish

Corollary 2.4 *Under the assumptions of Theorem* 2.2, *for any* $f \in L^p_u$ *and* $1 < p < +\infty$, *we have*

$$\|[f - \Delta_{\theta,m} S_m(w_\alpha, \Delta_{\theta,m} f)]u\|_p \le \mathcal{C}[E_M(f)_{u,p} + e^{-Am}\|fu\|_p] \quad (2.13)$$

where $\mathcal{C}$ *and* A *are positive constants independent on* m *and* f.

From Proposition 2.1 it is possible to deduce estimates similar to (2.13). For example, if the parameters α and γ of the weights satisfy (2.8), then

$$\|[f - S_m(w_\alpha, \Delta_{\theta,m} f)]u\|_p \le \mathcal{C}[E_M(f)_{u,p} + e^{-Am}\|fu\|_p]$$

with $1 < p < 4$ and

$$\|[f - S_m(w_\alpha, f)]u\|_p \le \mathcal{C} E_m(f)_{u,p}$$

with $\frac{4}{3} < p < 4$. If, in the last estimate, we put $u = \sqrt{w_\alpha}$, we obtain the result of Askey and Wanger in [1]. Finally the equivalences (2.11) and (2.12) are "essentially" Theorems 9-10 in [16], in which the weight $u(x) = \sqrt{w_\alpha(x)}\left(\frac{x}{1+x}\right)^a (1+x)^b$ is considered.

Lagrange interpolation.

We denote by $L_m(w_\alpha, f)$, $f \in C^0(\mathbb{R}^+)$, the Lagrange polynomial that interpolates the function f on the zeros $x_1 < \ldots < x_m$, $(x_k = x_{km})$, of $p_m(w_\alpha)$. Recalling (2.4), (2.5) and using the Gaussian rule, we can write

$$L_m(w_\alpha, f, x) = m b_m \sum_{k=1}^{m} p_m(w_\alpha, x)\lambda_k(w_\alpha)\frac{F_m(x_k)}{x - x_k} f(x_k) \qquad (2.14)$$

where $\lambda_k(w_\alpha) = \lambda_{km}(w_\alpha) = \frac{1}{K_m(w_\alpha, x_{km})}$ is the k–th Christoffel number, $F_m(x) = p_{m+1}(w_\alpha, x) - p_{m-1}(w_\alpha, x)$ and $b_m \sim 1$. In order to study the behavior of $L_m(w_\alpha, f)$ is useful to introduce the following notation. Recalling that (see [19]): $\frac{\mathcal{C}}{m} \le x_1 < x_m < 4m - \mathcal{C}\sqrt[3]{m}$, we define the integer $j = j(m)$ by

$$x_j = \min_k (x_k \ge 4\theta m), \quad \theta \in (0,1) \text{ fixed.}$$

Let $\psi \in C^\infty(\mathbb{R})$, nondecreasing; moreover $\psi(x) = \begin{cases} 0 \text{ if } x \le 0 \\ 1 \text{ if } x \ge 1 \end{cases}$ and $\psi_j(x) = \psi\left(\frac{x-x_j}{x_{j+1}-x_j}\right)$. For any $f \in C^0(\mathbb{R}^+)$, we define $f_j = (1-\psi_j)f$. Obviously $f_j = f$ in $(0, x_j]$ and $f_j = 0$ in $[x_{j+1}, +\infty)$. Then we have

$$L_m(w_\alpha, f_j, x) = \sum_{k=1}^{j} \lambda_k(w_\alpha) \frac{p_m(w_\alpha, x)}{x - x_k} F_m(x_k) f(x_k) \tag{2.15}$$

by removing in (2.14) $[cm]$ $(0 < c < 1)$ terms. Finally with $\theta_1 \in (\theta, 1)$, fixed and with $\Delta_{\theta_1} = \Delta_{\theta_1,m}$ (the characteristic function of $[0, 4\theta_1 m]$), we introduce the sequence $\{\Delta_{\theta_1,m} L_m(w_\alpha, f_j)\}_m$ to approximate the function $f \in C^0(\mathbb{R}^+)$ in L^p_u norm with $u(x) = x^\gamma e^{-x/2}$. In order to study the behaviour of the above mentioned sequence, we state a lemma that will be useful in the sequel.

Lemma 2.5 *Let $0 < \theta < \theta_1 < 1$ and let $1 \le p < +\infty$. Then, for an arbitrary polynomial $P \in \mathbb{P}_{lm}$ (with l fixed integer), we have*

$$\left(\sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k)\right)^{\frac{1}{p}} \le \mathcal{C} \left(\int_{x_1}^{4\theta_1 m} |Pu|^p(x) dx\right)^{\frac{1}{p}} \tag{2.16}$$

where $\Delta x_k = x_{k+1} - x_k$, $\mathcal{C}$ is a positive constant independent on m, P and p.

We can now state the following theorem, which is the main result of this section. Letting $v^\beta(x) = x^\beta$, we have:

Theorem 2.6 *Let $1 < p < +\infty$. Then, for all function $f \in C^0(\mathbb{R}^+)$, we have*

$$\|L_m(w_\alpha, f_j) u \Delta_{\theta_1}\|_p \le \mathcal{C} \left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k)\right)^{\frac{1}{p}}, \quad \mathcal{C} \ne \mathcal{C}(m, f) \tag{2.17}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p(0,1) \text{ and } \frac{\sqrt{v^\alpha \varphi}}{v^\gamma} \in L^q(0,1), \quad \varphi(x) = \sqrt{x}, \quad q = \frac{p}{p-1}. \tag{2.18}$$

Moreover, if we fix w_α, u and p then the inequality

$$\|L_m(w_\alpha,f)u\|_p \le \mathcal{C}\left(\sum_{i=1}^{m}\Delta x_i|fu|^p(x_i)\right)^{\frac{1}{p}} \quad (x_{m+1}=4m) \tag{2.19}$$

is not true for a proper $f\in C^0(\mathbb{R}^+)$ where $\mathcal{C}\ne\mathcal{C}(m,f)$.

The next lemma estimates the term on the right-hand side of (2.17). To this end we recall (see [2])

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\le t}\|u\Delta^r_{h\varphi}f\|_{I_{rh}}, \quad r\ge 1, \quad \varphi(x)=\sqrt{x}$$

where $I_{rh}=[\mathcal{C}(2rh)^2,\mathcal{C}/h^2]$, $\mathcal{C}$ arbitrary fixed constant and

$$\Delta^r_{h\varphi}f(x)=\sum_{i=0}^{r}\left(\begin{array}{c} r\\ i\end{array}\right)(-1)^i f\left(x+\frac{h}{2}\sqrt{x}(r-2i)\right),$$

t sufficiently small; i.e., $t<t_0$.

Lemma 2.7 *For all continuous function in $\mathbb{R}^+$ we have*

$$\begin{aligned}\left(\sum_{k=1}^{j}\Delta x_k|fu|^p(x_k)\right)^{\frac{1}{p}} &\le \mathcal{C}\Big[\|fu\|_{L^p(0,x_j)} \\ &+ \frac{1}{(\sqrt{m})^{\frac{1}{p}}}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}}dt\Big]\end{aligned} \tag{2.20}$$

with $\mathcal{C}\ne\mathcal{C}(m,f)$.

Now we can establish the following theorem:

Theorem 2.8 *For all continuous function in $\mathbb{R}^+$ and $p\in(1,+\infty)$ we have:*

$$\begin{aligned}\|[f-\Delta_{\theta_1}L_m(w_\alpha,f_j)]u\|_p &\le \mathcal{C}\Big(\frac{1}{(\sqrt{m})^{\frac{1}{p}}}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}dt \\ &+ e^{-Am}\|fu\|_p\Big)\end{aligned} \tag{2.21}$$

with $\mathcal{C}\ne\mathcal{C}(m,f)$ if and only if relations (2.18) are true.

In particular if we define the Zygmund space

$$Z_s^p = Z_s^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < +\infty, \quad r > s \right\},$$

where $1 < p < +\infty$, $s \in \mathbb{R}^+$ and we introduce the usual norm

$$\|f\|_{Z_s^p} = \|fu\|_p + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s},$$

then (2.21) becomes

$$\|[f - \Delta_{\theta_1} L_m(w_\alpha, f_j)]u\|_p \le \frac{\mathcal{C}}{m^s} \|f\|_{Z_s^p}, \quad s > \frac{1}{p}, \quad 1 < p < +\infty,$$

that has the same order of the error of best approximation of functions in Z_s^p, $s > \frac{1}{p}$. Similar estimates about the interpolation in $(-1,1)$ can be found in [15] and the special case $p = 2$ was considered in [12]. For completeness we note that theorems 2.6 and 2.8 hold true for $1 < p < 4$ if the norms at the left-hand side of (2.17) and (2.21) are replaced by $\|L_m(w_\alpha, f_j)u\|_p$ and $\|[f - L_m(w_\alpha, f_j)]u\|_p$ respectively.

3 Fourier Sums and Lagrange Interpolation in $\mathbb{R}$

Some preliminary results

In this section we consider Freud weight of the form $w(x) = e^{-Q(x)}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous, $Q' > 0$ in $(0, +\infty)$, Q'' is continuous in $(0, +\infty)$ and, for some $\hat{A}, \hat{B} > 1$ it results

$$\hat{A} \le \frac{(xQ'(x))'}{Q'(x)} \le \hat{B}, \quad x \in (0, +\infty). \tag{3.1}$$

Denote by $\{p_m(w)\}_m$ the sequence of the polynomials that are orthonormal in $\mathbb{R}$ and whose leading coefficient $\gamma_m := \gamma_m(w)$ is positive. Since w is even, the related recurrence relation has the following form

$$xp_k(w,x) = \alpha_{k+1} p_{k+1}(w,x) + \alpha_k p_k(w,x), \quad k \ge 1.$$

For a wide class of the weights w, the coefficients $\{\alpha_m\}_m$ satisfy the conditions

$$\frac{\alpha_{m+1}}{\alpha_m} = 1 + \mathcal{O}\left(\frac{1}{m}\right), \quad \frac{\alpha_m}{a_m} = \frac{1}{2}(1 + \mathcal{O}(m^{-2/3})), \tag{3.2}$$

where $m \to 0$ and a_m is the M-R-S number related to the weight $\sqrt{w}$ and it is uniquely defined by

$$u = \frac{1}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}.$$

This class of weight has been considered by S.W. Jha and S.D. Lubinsky in [8]. Now we recall the well-known polynomial inequality [4]

$$\int_{\Gamma_m} |P_m\sqrt{w}|^p(x)dx \le \mathcal{C}e^{-Am} \int_{\mathbb{R}} |P_m\sqrt{w}|^p(x)dx, \quad 1 \le p \le +\infty, \tag{3.3}$$

where $\Gamma_m = \{x \in \mathbb{R} \ : \ |x| > (1+\delta)a_m\}$, $\delta \in (0,1)$, $\mathcal{C}$ and A are positive constant depending on δ and independent of m. As in the Laguerre case, with $\theta \in (0,1)$, we define $\Delta_{\theta,m}$ the characteristic function of the interval $[-\theta a_m, \theta a_m]$ and $M = \left[\frac{\theta}{1+\theta}m\right]$. As before $M \sim m$. For all functions $f \in L^p_{\sqrt{w}}$,

$$\|(1-\Delta_{\theta,m})f\sqrt{w}\|_p \le \mathcal{C}\left[E_M(f)_{\sqrt{w},p} + e^{-Am}\|f\sqrt{w}\|_p\right], \tag{3.4}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$, $E_M(f)_{\sqrt{w},p} = \inf_{P \in \mathbb{P}_M} \|(f-P)\sqrt{w}\|_p$ and $1 \le p \le +\infty$ (see (2.2)).

Fourier Sums

For all $f \in L^p_{\sqrt{w}}$, let

$$S_m(w,f) = \sum_{k=0}^{m-1} c_k p_k(w), \quad c_k = c_k(f) = \int_{\mathbb{R}} f p_k(w) w,$$

be its m–th Fourier Sum with the respect to the system $\{p_m(w)\}_m$. In the above mentioned paper [8] the authors, assuming (3.2), assigned necessary and sufficient conditions for the boundedness of the operator $S_m(w) : L^p_{\sigma_1} \to L^p_{\sigma_2}$, where $\sigma_1(x) = \sqrt{w(x)}(1+|x|)^b, \sigma_2(x) = \sqrt{w(x)}(1+|x|)^\beta$, $b \le \beta, 1 < p < +\infty$. As a consequence of these results the following equivalences hold

$$\|S_m(w,f)\sqrt{w}\|_p \le \mathcal{C}\|f\sqrt{w}\|_p \Leftrightarrow \frac{4}{3} < p < 4 \tag{3.5}$$

and

$$\|S_m(w,f)\Delta_{\theta,m}\sqrt{w}\|_p \le \mathcal{C}\|f\Delta_{\theta,m}\sqrt{w}\|_p \Leftrightarrow p > \frac{4}{3}, \tag{3.6}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$. Proceeding as in the Laguerre case, we consider the sequence $\{\Delta_{\theta,m}S_m(w,\Delta_{\theta,m}f)\}_m$ and we state the following

Theorem 3.1 *With the previous notation and without the assumption* (3.2), *for all* $f \in L^p_{\sqrt{w}}, 1 < p < +\infty$, *we have*

$$\|S_m(w,\Delta_{\theta,m}f)\Delta_{\theta,m}\sqrt{w}\|_p \le \mathcal{C}\|f\Delta_{\theta,m}\sqrt{w}\|_p \tag{3.7}$$

and

$$\begin{aligned}\|[f-\Delta_{\theta,m}S_m(w,\Delta_{\theta,m}f)]\sqrt{w}\|_p &\le \mathcal{C}\Big[E_M(f)_{\sqrt{w},p} + \\ &+ e^{-Am}\|f\sqrt{w}\|_\infty\Big]\end{aligned} \tag{3.8}$$

where $\mathcal{C}$ *and* A *are independent of* m *and* f.

In order to complete Theorem 3.1, (3.5) and (3.6), we can state the following proposition

Proposition 3.2 *For all* $f \in L^p_{\sqrt{w}}$, *we have*

$$\|S_m(w,\Delta_{\theta,m}f)\sqrt{w}\|_p \le \mathcal{C}\|f\sqrt{w}\Delta_{\theta,m}\|_p, \quad \mathcal{C} \ne \mathcal{C}(m,f), \tag{3.9}$$

if and only if $1 < p < 4$.

Of course an inequality similar to (3.8) can be easily stated for

$$\|[f - S_m(w,\Delta_{\theta,m}f)]\sqrt{w}\|_p, \quad 1 < p < 4.$$

Finally we note that Theorem 3.1 and Proposition 3.2 hold true for more general Freud weights $w \in F(Lip\frac{1}{2})$ (see for example [10] p. 12).

Lagrange Interpolation

For convenience we introduce the following notations. We denote by $x_k(=x_{km})$ the nonnegative zeros of $p_m(w)$ ($x_0=0$ if m is odd) and we set $x_{-k}=-x_k, 0\le k\le \left[\frac{m}{2}\right]$. With $f\in C^0(\mathbb{R})$, let

$$L_m(w,f,x)=\sum_{k=-\left[\frac{m}{2}\right]}^{\left[\frac{m}{2}\right]} l_k(x)f(x_k), \quad l_k(x)=\frac{p_m(w,x)}{p'_m(w,x_k)(x-x_k)}$$

be the Lagrange polynomial interpolating the function f on the knots x_k.

With $\theta\in(0,1)$, let $j=j(m)$ be defined as $x_j=\min\{x_k\ge\theta a_m\}$, let Ψ_j be the characteristic function of the interval $[-x_j,x_j]$ and set $f_j=\Psi_j f$. Then

$$L_m(w,f_j,x)=\sum_{k=-j}^{j} l_k(x)f(x_k)$$

we want to study the behaviour of the sequences

$$\{\Psi_j L_m(w,f_j)\}_m \quad \text{and} \quad \{L_m(w,f_j)\}_m$$

in $L^p_{\sqrt{w}}$ assuming $f\in C^0(\mathbb{R})$. To this end we state the following lemmas.

Lemma 3.3 *For all polynomials $P\in\mathbb{P}_{lm}$ (l fixed integer) and for all $p\in[1,+\infty)$ there exists δ, $0<\theta<\delta<1$, such that*

$$\left(\sum_{i=-j}^{j}\Delta x_i|P\sqrt{w}|^p(x_i)\right)^{\frac{1}{p}}\le\mathcal{C}\left(\int_{-\delta a_m}^{\delta a_m}|P\sqrt{w}|^p(x)dx\right)^{\frac{1}{p}},$$

with $\mathcal{C}\ne\mathcal{C}(m,f)$ and $\Delta x_i=x_{i+1}-x_i$.

Lemma 3.4 *Let $f\in C^0(\mathbb{R})$. Then for $p\in(1,+\infty)$, we have*

$$\|L_m(w,f_j)\Psi_j\sqrt{w}\|_p\le\mathcal{C}\left(\sum_{k=-j}^{j}\Delta x_k|f\sqrt{w}|^p(x_k)\right)^{\frac{1}{p}}. \tag{3.10}$$

Moreover, for $1 < p < 4$, *it results*

$$\|L_m(w, f_j)\sqrt{w}\|_p \le \mathcal{C} \left(\sum_{k=-j}^{j} \Delta x_k |f\sqrt{w}|^p(x_k) \right)^{\frac{1}{p}}. \tag{3.11}$$

The constants $\mathcal{C}$ *are independent of* m *and* f. *Finally, for every fixed* $p \in (1, +\infty)$ *there exists a function* $f \in C^0(\mathbb{R})$ *such that*

$$\|L_m(w, f)\sqrt{w}\|_p \le \mathcal{C} \left(\sum_{k=-\left[\frac{m}{2}\right]}^{\left[\frac{m}{2}\right]} \Delta x_k |f\sqrt{w}|^p(x_k) \right)^{\frac{1}{p}},$$

(with $\mathcal{C} \ne \mathcal{C}(m, f)$*) does not hold.*

Now following [5], p. 182, let $t^* = t^*(t)$ be uniquely defined by $tQ'(t^*) = 1$ and define the following modulus of continuity

$$\Omega^k(f, t)_{\sqrt{w},p} = \sup_{0<h\le t} \|(\Delta_h^k f)\sqrt{w}\|_{L^p(-\mathcal{C}h^*, \mathcal{C}h^*)}.$$

Lemma 3.5 *Let* $f \in C^0(\mathbb{R})$. *Then, for* $1 < p < +\infty$ *we have*

$$\begin{aligned} \left(\sum_{k=-j}^{j} \Delta x_k |f\sqrt{w}|^p(x_k) \right)^{\frac{1}{p}} &\le \mathcal{C} \Bigg(\|f\sqrt{w}\|_{L^p(-x_j, x_j)} + \\ &+ \left(\frac{a_m}{m}\right)^{\frac{1}{p}} \int_0^{\frac{a_m}{m}} \frac{\Omega^k(f, t)_{\sqrt{w},p}}{t^{1+1/p}} dt \Bigg), \end{aligned} \tag{3.12}$$

where $\mathcal{C}$ *is independent of* m *and* f.

The following theorem comes from the previous lemmas.

Theorem 3.6 *For all continuous function in* $\mathbb{R}$, *we have*

$$\begin{aligned} \|[f - L_m(w, f_j)\Psi_j]\sqrt{w}\|_p &\le \mathcal{C} \left\{ \left(\frac{a_m}{m}\right)^{\frac{1}{p}} \int_0^{\frac{a_m}{m}} \frac{\Omega^k(f, t)_{\sqrt{w},p}}{t^{1+1/p}} dt + \right. \\ &+ \left. e^{-Am} \|f\sqrt{w}\|_p \right\}, \end{aligned} \tag{3.13}$$

with $1 < p < +\infty$. *Moreover, if* $1 < p < 4$ *then it results*

$$\begin{aligned} \|[f - L_m(w, f_j)]\sqrt{w}\|_p &\le \mathcal{C}\left\{\left(\frac{a_m}{m}\right)^{\frac{1}{p}} \int_0^{\frac{a_m}{m}} \frac{\Omega^k(f,t)_{\sqrt{w},p}}{t^{1+1/p}} dt + \right. \\ &+ \left. e^{-Am}\|f\sqrt{w}\|_p\right\}. \end{aligned} \tag{3.14}$$

Here the constants $\mathcal{C}$ *are independent of* m *and* f.

Especially, if $f^{(r-1)}$ absolutely continuous on $\mathbb{R}$ and $\|f^{(r)}\sqrt{w}\|_p < +\infty$, then

$$\Omega_\varphi^k(f,t)_{\sqrt{w},p} \le \mathcal{C}\|f^{(r)}\sqrt{w}\|_p\, t^r, \quad k > r,$$

and the right-hand sides of (3.13) and (3.14) with the corresponding value of p, are dominated by $\left(\frac{a_m}{m}\right)^r \left(\|f\sqrt{w}\|_p + \|f^{(r)}\sqrt{w}\|_p\right)$.

4 Proofs

Later we use the following:

Proposition 4.1 *Let* $\sigma(x) = v^\beta(x)e^{-x/2}$, $v^\beta(x) = x^\beta$, $\theta \in (0,1)$ *and* $1 \le p < +\infty$. *Then*

$$\sqrt[4]{m}\|p_m(w_\alpha)\sigma\|_{L^p(0,4\theta m)} \ge \mathcal{C}\left\|\frac{v^\beta}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(0,1)} \tag{4.1}$$

where $\varphi(x) = \sqrt{x}$ *and* $\mathcal{C}$ *is a positive constant independent of* m.

Proof. With $\delta > 0$ "small" we define $\delta_k = \frac{\delta}{8}\Delta x_k$, $k = 1, 2, \dots, m$, $I_m = [0, 4\theta m] \cap \left(\cup_{k=1}^m \left[x_k - \frac{\delta_k}{2}, x_k + \frac{\delta_k}{2}\right]\right)$ and $\mathcal{C}I_m = [0, 4\theta m] \setminus I_m$. Since (see [13]) for $x \in [0, 4m]$ if $x_d = x_{d_k}$ denotes the closest node to x, it results

$$p_m^2(w_\alpha, x)e^{-x}\left(x + \frac{1}{m}\right)^{\alpha+\frac{1}{2}} \sqrt{4m - x + (4m)^{\frac{1}{3}}} \sim \left(\frac{x - x_d}{\Delta x_d}\right)^2, \tag{4.2}$$

we have

$$|p_m(w_\alpha, x)\sigma(x)| \ge \mathcal{C}\frac{x^{\beta-\alpha/2}}{\sqrt[4]{xm}}, \quad x \in CI_m.$$

Consequently

$$\sqrt[4]{m}\|p_m(w_\alpha)\sigma\|_{L^p(0,4\theta m)} \geq \mathcal{C}\left\|\frac{v^\beta}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(CI_m)}.$$

Moreover

$$\begin{aligned}\left\|\frac{v^\beta}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(CI_m)} &= (4m\theta)^{\beta-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p}}\left(\int_0^1\left|\frac{v^\beta}{\sqrt{v^\alpha\varphi}}\right|^p(x)dx-\right.\\ &- \left.\int_{I_m^*}\left|\frac{v^\beta}{\sqrt{v^\alpha\varphi}}\right|^p(x)dx\right)^{\frac{1}{p}} \qquad (4.3)\end{aligned}$$

with $I_m^*=\cup_{k=1}^m\left(\frac{x_k}{4m\theta}-\frac{\delta_k}{4m\theta},\frac{x_k}{4m\theta}+\frac{\delta_k}{4m\theta}\right)\cap[0,1]$. By definition, the measure of I_m^* is bounded by $\mathcal{C}\delta$. Then, for a suitable δ, the second integral in the brackets is the half part of the first one. □

PROOF OF THEOREM 2.2. First we prove that (2.7) implies (2.8). Set $f^*=\Delta_{\theta,m-1}f$ where $\Delta_{\theta,m-1}$ denotes the characteristic function of $[0,4\theta(m-1)]$. By (2.7)

$$\|S_m(w_\alpha,\Delta_{\theta,m}f^*)u\Delta_{\theta,m}\|_p \leq \mathcal{C}\|f^*\Delta_{\theta,m}u\|_p=\mathcal{C}\|f^*u\|_p.$$

Then

$$\begin{aligned}&\|S_{m-1}(w_\alpha,\Delta_{\theta,m}f^*)u\Delta_{\theta,m}\|_p=\|S_{m-1}(w_\alpha,\Delta_{\theta,m-1}f)u\Delta_{\theta,m}\|_p\\ &\leq\mathcal{C}\|S_{m-1}(w_\alpha,\Delta_{\theta,m-1}f)u\Delta_{\theta,m-1}\|_p\leq\mathcal{C}\|\Delta_{\theta,m-1}fu\|=\mathcal{C}\|f^*u\|_p.\end{aligned}$$

In the first inequality we used a Remez–type inequality applied to the interval $[0,4\theta m]$ and we recall that, in the same interval, $e^{-\frac{x}{2}}\sim Q\in\mathbb{P}_{lm}$ (l fixed) (see [2]). In the second inequality we used (2.7). Getting together the previous inequalities we have

$$\|[S_m(w_\alpha,\Delta_{\theta,m}f^*)-S_{m-1}(w_\alpha,\Delta_{\theta,m}f^*)]u\Delta_{\theta,m}\|_p\leq 2\mathcal{C}\|f^*u\|_p.$$

Then

$$\|p_m(w_\alpha)u\Delta_{\theta,m}\|_p\left|\int_0^{4\theta m}p_m(w_\alpha,x)w_\alpha(x)\Delta_{\theta,m}f^*(x)dx\right|\leq 2\mathcal{C}\|f^*u\|_p$$

and

$$\|p_m(w_\alpha)u\Delta_{\theta,m}\|_p \sup_{\|g\|_{L^p(0,4\theta m)}=1} \left| \int_0^{4\theta m} p_m(w_\alpha,x)\frac{w_\alpha}{u}(x)g(x)dx \right| \le 2\mathcal{C}$$

or equivalently

$$\|p_m(w_\alpha)u\Delta_{\theta,m}\|_p \|p_m(w_\alpha)\frac{w_\alpha}{u}\Delta_{\theta,m}\|_q \le \mathcal{C}.$$

From boundedness of the second factor follows that $\frac{v^\alpha}{v^\gamma} \in L^q(0,1)$ and, by Proposition 4.1, follows the remaining part of (2.8). Now we suppose that (2.8) is true. In order to obtain (2.7), we use (2.6) with $b_m \sim 1$. We estimate

$$\begin{aligned} A_1 &:= \|p_m(w_\alpha)u\Delta_\theta\|_p \left| \int_0^\infty p_m(w_\alpha,x)f(x)\Delta_\theta(x)w_\alpha(x)dx \right| \\ A_2 &:= m\|F_m uH(p_m(w_\alpha)f\Delta_\theta w_\alpha)\Delta_\theta\|_{L^p\left(\frac{\mathcal{C}}{m},\infty\right)} \\ A_3 &:= m\|p_m(w_\alpha)u\Delta_\theta H(F_m f\Delta_\theta w_\alpha)\|_{L^p\left(\frac{\mathcal{C}}{m},\infty\right)}. \end{aligned}$$

Since $|(\sqrt{w_\alpha}p_m(w_\alpha))(x)| \le \mathcal{C}x^{-\frac{1}{4}}m^{-\frac{1}{4}}$, $x \in \left(\frac{\mathcal{C}}{m}, 4\theta m\right)$, we have

$$\begin{aligned} A_1 &\le \frac{\mathcal{C}}{m^{\frac{1}{4}}} \left\| \frac{v^\gamma}{\sqrt{v^\alpha \varphi}}\Delta_\theta \right\|_p \left\| p_m(w_\alpha)\frac{w_\alpha}{u}\Delta_F\theta \right\|_q \|\Delta_\theta f u\|_p \le \\ &\le \frac{\mathcal{C}}{\sqrt{m}} \left\| \frac{v^\gamma\Delta_\theta}{\sqrt{v^\alpha\varphi}} \right\|_p \left\| \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{u}\Delta_\theta \right\|_q \|\Delta_\theta f u\|_p = \\ &= \mathcal{C}\left(\int_0^\theta t^{\left(\gamma-\frac{\alpha}{2}-\frac{1}{4}\right)p}dt \right)^{\frac{1}{p}} \left(\int_0^\theta t^{\left(\frac{\alpha}{2}-\frac{1}{4}-\gamma\right)q}dt \right)^{\frac{1}{q}} \|fu\Delta_\theta\|_p \le \\ &\le \mathcal{C}\|f\Delta_\theta u\|_p. \end{aligned}$$

The last estimate follows by (2.8). Moreover, since $|(\sqrt{w_\alpha}F_m)(x)| \le \mathcal{C}x^{\frac{1}{4}}m^{-3/4}$ (see [16] p. 435), we have

$$A_2 \le \mathcal{C}m^{\frac{1}{4}} \left(\int_{\frac{\mathcal{C}}{m}}^{4\theta m} \left| x^{\frac{1}{4}+\gamma-\frac{\alpha}{2}} \int_0^{4\theta m} \frac{(p_m(w_\alpha)fw_\alpha)(y)}{y-x}dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\le \mathcal{C}m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_0^\theta \left| t^{\left(\frac{1}{4}+\gamma-\frac{\alpha}{2}\right)} \int_0^\theta \frac{(p_m(w_\alpha)fw_\alpha)(4mz)}{z-t} \right|^p dt \right)^{\frac{1}{p}},$$

where, in the first integral, we use $y \to 4mz$. But relation (2.8) implies that $t^{\frac{1}{4}+\gamma-\frac{\alpha}{2}}$ is an $\mathcal{A}_p$ weight in $[0,1]$ (see [14] p. 314) and

$$A_2 \leq \mathcal{C} m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_0^{\theta} \left| t^{\gamma-\frac{\alpha}{2}+\frac{1}{4}} (p_m(w_\alpha) f w_\alpha)(4mz) \right|^p dt \right)^{\frac{1}{p}}$$

$$= \mathcal{C} m^{\frac{1}{4}} \left(\int_0^{4m\theta} \left| x^{\gamma-\frac{\alpha}{2}+\frac{1}{4}} p_m(w_\alpha, x) \frac{w_\alpha(x)}{u(x)} (f u \Delta_\theta)(x) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq \mathcal{C} \left(\int_0^{4m\theta} |f u \Delta_\theta|^p(x) dx \right)^{\frac{1}{p}}$$

since

$$\left| p_m(w_\alpha, x) \frac{w_\alpha(x)}{u(x)} \right| \leq \frac{\mathcal{C}}{\sqrt{1-\theta} m^{\frac{1}{4}}} x^{\frac{\alpha}{2}-\frac{1}{4}-\gamma}, \quad x \in (0, 4\theta m).$$

I.e., we get

$$\left| x^{\gamma-\frac{\alpha}{2}+\frac{1}{4}} p_m(w_\alpha, x) \frac{w_\alpha(x)}{u(x)} \right| \leq \frac{\mathcal{C}}{m^{\frac{1}{4}}}.$$

The estimate of A_3 is similar to A_2. □

Proof of Proposition 2.3. The proofs of the equivalences are similar to the corresponding part of Theorem 2.2.

For the sake of brevity we prove only that (2.8) implies

$$\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C} m^{\frac{1}{3}} \|fu\|_p,$$

being the " $\Rightarrow$ " very easy. For this end we recall the inequality [2]

$$\|Qu\|_p \leq \mathcal{C} \|Qu\|_{L^p(\frac{a}{m}, 4m)} \tag{4.4}$$

which holds true for any $Q \in \mathbb{P}_m$ and $u(x) = e^{-x/2} x^\gamma$, $\gamma > -\frac{1}{p}$. Then, since $\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C} \|S_m(w_\alpha, f)u\|_{L^p(\frac{a}{m}, 4m)}$, by (2.6), it is sufficiently to estimate the norms

$$B_1 := |c_m| \|p_m(w_\alpha)u\|_{L^p(\frac{a}{m}, 4m)}$$

$$B_2 := m \|F_m H(f p_m(w_\alpha) w_\alpha)u\|_{L^p(\frac{a}{m}, 4m)}.$$

Now, for $x \in \left[\frac{a}{m}, 4m\right]$, by [13]

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \le \frac{\mathcal{C}}{\sqrt[4]{x(4m - x + m^{\frac{1}{3}})}} \le \mathcal{C}x^{-\frac{1}{4}}m^{-\frac{1}{12}}. \qquad (4.5)$$

Therefore, since $\frac{v^\alpha}{v^\gamma} \in L^q(0,1)$, we have

$$B_1 \le \|fu\|_p \left\|p_m(w_\alpha)\frac{w_\alpha}{u}\right\|_{L^q(0,4m)} \|p_m(w_\alpha)u\|_{L^p(0,4m)}.$$

Then we can write

$$\begin{aligned}
\|p_m(w_\alpha)u\|_{L^p(0,4m)} &\le \mathcal{C}m^{-\frac{1}{12}} \left\|\frac{v^\gamma}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(0,4m)} \\
&= \mathcal{C}m^{-\frac{1}{12}+\gamma-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p}} \left\|\frac{v^\gamma}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(0,1)} \\
\|p_m(w_\alpha)\frac{w_\alpha}{u}\|_{L^p(0,4m)} &\le \mathcal{C}m^{-\frac{1}{12}} \left\|\sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma}\right\|_{L^q(0,4m)} \\
&= \mathcal{C}m^{-\frac{1}{12}-\gamma+\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{q}} \left\|\sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{u}\right\|_{L^q(0,1)}
\end{aligned}$$

and, consequently, recalling (2.8), $B_1 \le \mathcal{C}m^{\frac{1}{3}}\|fu\|_p$.

In order to estimate B_2, with $0 < \delta \le \frac{1}{2}$, we can write

$$\begin{aligned}
B_2 \le m &\left(\int_0^{4m} \left|F_m(t)u(t)\int_0^{4m(1+\delta)} p_m(w_\alpha, x)w_\alpha(x)\frac{f(x)}{x-t}dx\right|^p dt\right)^{\frac{1}{p}} \\
&+ m\left(\int_{\frac{a}{m}}^{4m} \left|F_m(t)u(t)\int_{4m(1+\delta)}^\infty p_m(w_\alpha, x)w_\alpha(x)\frac{f(x)}{x-t}dx\right|^p dt\right)^{\frac{1}{p}} \\
=: &\quad I_1 + I_2.
\end{aligned}$$

In order to estimate I_1, we observe that ([16, eq.(2.6)])

$$|F_m(t)u(t)| \le \mathcal{C}m^{-3/4}t^{\frac{1}{4}}m^{\frac{1}{12}}, \qquad t \in \left(\frac{a}{m}, 4m\right).$$

By $G := p_m(w_\alpha)w_\alpha f$, we have

$$I_1 \le \mathcal{C}m^{\frac{1}{4}+\frac{1}{12}}\left(\int_0^{4m(1+\delta)}\left|t^{\gamma-\frac{\alpha}{2}+\frac{1}{4}}\int_0^{4m(1+\delta)}\frac{G(x)}{x-t}dx\right|^p dt\right)^{\frac{1}{p}}$$

$$= \mathcal{C}m^{\frac{1}{4}+\frac{1}{12}}m^{\gamma-\frac{\alpha}{2}+\frac{1}{4}+\frac{1}{p}}\left(\int_0^{1+\delta}\left|z^{\gamma-\frac{\alpha}{2}+\frac{1}{4}}\int_0^{1+\delta}\frac{G(4mz)}{y-z}\right|dz\right)^{\frac{1}{p}}.$$

(2.8) results that $z^{\gamma-\frac{\alpha}{2}+\frac{1}{4}}$ is an $\mathcal{A}_p$–weight whence

$$\begin{aligned} I_1 &\le \mathcal{C}m^{\frac{1}{4}+\frac{1}{12}}\cdot m^{\gamma-\frac{\alpha}{2}+\frac{1}{4}+\frac{1}{p}}\left(\int_0^{1+\delta}|z^{\gamma-\frac{\alpha}{2}+\frac{1}{4}}G(4mz)|^p dz\right)^{\frac{1}{p}} \\ &= \mathcal{C}m^{\frac{1}{4}+\frac{1}{12}}\left(\int_0^{4m(1+\delta)}\left|t^{\gamma-\frac{\alpha}{2}+\frac{1}{4}}p_m(w_\alpha)\frac{w_\alpha}{u}(fu)(t)\right|^p dt\right)^{\frac{1}{p}}. \end{aligned}$$

In $\left(\frac{a}{m}, 4m\right)$ (see [13])

$$|p_m(w_\alpha x)\sqrt{w_\alpha(x)}| \le \mathcal{C}x^{-\frac{1}{4}}m^{-\frac{1}{12}},$$

moreover, in $[4m, 4m(1+\delta)]$, $0<\delta\le\frac{1}{2}$, we have

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \le m^{-\frac{1}{4}-\frac{1}{12}} \quad \text{(see [16] eq. (2.5)]}$$

whence in both cases,

$$I_1 \le \mathcal{C}\sqrt[4]{m}\|fu\|_p.$$

Finally

$$\begin{aligned} I_2 &\le \frac{1}{\delta}\|F_m u\|_{L^p\left(\frac{a}{m},4m\right)}\left\|p_m(w_\alpha)\frac{w_\alpha}{u}(fu)\right\|_{L^q(4m(1+\delta),\infty)} \\ &\le \frac{1}{\delta}\|fu\|_p\|F_m u\|_{L^p\left(\frac{a}{m},4m\right)}\left\|p_m(w_\alpha)\frac{w_\alpha}{u}\right\|_{L^q(4m(1+\delta),\infty)}, \end{aligned}$$

from which, using (2.1), follows that $I_2 = o(1)\|fu\|_p$. The proof is complete. □

Proof of Corollary 2.4. If $P\in\mathbb{P}_M$, $M = \left[\frac{\theta}{1+\theta}m\right]$,

$$\begin{aligned} R_m(f) &:= f-\Delta_\theta S_m(w_\alpha,\Delta_\theta f) = (f-P)+(1-\Delta_\theta)P+ \\ &+ \Delta_\theta S_m(w_\alpha,(1-\Delta_\theta)P)+\Delta_\theta S_m(w_\alpha,(P-f)\Delta_\theta). \end{aligned}$$

Then

$$\begin{aligned} &\|R_m(f)u\|_p \le \|(f-P)u\|_p + \|(1-\Delta_\theta)P\| + \\ + \quad &\|\Delta_\theta S_m(w_\alpha,(1-\Delta_\theta)P)u\|_p + \|S_m(w_\alpha,(P-f)\Delta_\theta)u\Delta_\theta\|_p. \end{aligned}$$

We may estimate the third term by (2.10) and the last one by Theorem 2.2. Then, taking the infimum on $P \in P_M$, we get

$$\|R_m(f)u\|_p \le E_M(f)_{u,p} + \mathcal{C}(1+m^{\frac{1}{3}})\|(1-\Delta_\theta)Pu\|_p + CE_M(f)_{u,p}.$$

By (2.1) the corollary follows. □

Proof of Lemma 2.5. Let $k > 1$ and $P \in \mathbb{P}_{lm}$. We start from the identity

$$\Delta x_k P(x_k) = \int_{x_{k-1}}^{x_k} P(x)dx + \int_{x_{k-1}}^{x_k} (x-x_{k-1})P'(x)dx.$$

Using Hölder inequality and $u(x) \sim u(x_k)$ for $x \in [x_{k-1}, x_k]$, we get for $k \ge 2$

$$\Delta x_k |Pu|^p(x_k) \le 2^{p-1}\left[\int_{x_{k-1}}^{x_k} |Pu|^p(t)dt + (\Delta x_k)^p \int_{x_{k-1}}^{x_k} |P'u|^p(t)dt\right]. \tag{4.6}$$

If $k = 1$,

$$\Delta x_1 P(x_1) = \int_{x_1}^{x_2} P(x)dx - \int_{x_1}^{x_2} (x_2 - x)P'(x)dx,$$

from which

$$\begin{aligned} &\Delta x_1 |P(x_1)u(x_1)|^p \le \\ \le \quad &2^{p-1}\left[\int_{x_1}^{x_2} |Pu|^p(t)dt + (\Delta x_1)^p \int_{x_1}^{x_2} |P'u|^p(t)dt\right]. \end{aligned} \tag{4.7}$$

Recalling that for $k \le j$, $\Delta x_k \sim \sqrt{\frac{x_k}{m}} \sim \sqrt{\frac{t}{m}}$, we have

$$\begin{aligned} \sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k) \quad &\le \quad \mathcal{C}\left[\int_{x_1}^{x_j} |Pu|^p(t)dt + \right. \\ &+ \quad \left. \frac{1}{(\sqrt{m})^p}\int_{x_1}^{x_j} |P'(t)\sqrt{t}u(t)|^p dt\right] \end{aligned}$$

with $\mathcal{C} \neq \mathcal{C}(m, P)$. Since $x_j < 4\theta_1 m$, we have to prove only that the second term on the right can be estimated by $\mathcal{C}\|Pu\|_{L^p(0,4\theta_1 m)}$. To this end we recall that, for some fixed integer r, there exists a polynomial $Q \in \mathbb{P}_{rm}$ such that, in $[0, 4m]$ (see [2]),

$$|Q(x)| \sim e^{-\frac{x}{2}} \quad \text{and} \quad \sqrt{\frac{x}{m}}|Q'(x)| \leq \mathcal{C}e^{-\frac{x}{2}}.$$

Therefore

$$\begin{aligned} \int_0^{x_j} |P'(t)\sqrt{t}u(t)|^p dt &\sim \int_{x_1}^{x_j} |P'(t)\sqrt{t}t^\gamma Q(t)|^p dt \\ \leq & \int_{x_1}^{x_j} |(PQ)'(t)\sqrt{t}t^\gamma|^p dt + \int_{x_1}^{x_j} |P(t)\sqrt{t}Q'(t)t^\gamma|^p dt. \end{aligned}$$

The last integral is dominated by $\mathcal{C}(\sqrt{m})^p \int_0^{x_j} |P(t)u(t)|^p dt$. Moreover,

$$\begin{aligned} &\int_{x_1}^{x_j} |(PQ)'\sqrt{t}t^\gamma|^p dt \leq \\ \leq & \frac{1}{(4\theta_1 m - x_j)^{\frac{p}{2}}} \int_{x_1}^{4m\theta_1} |(PQ)'(t)\sqrt{t(4m\theta_1 - t)}t^\gamma|^p dt \\ \sim & \frac{1}{(\sqrt{m})^p} \int_{x_1}^{4\theta_1 m} |(PQ)'(t)\sqrt{t(4m\theta_1 - t)}t^\gamma|^p dt. \end{aligned}$$

Using Bernstein inequality in $[x_1, 4\theta_1 m]$, the last integral is smaller than

$$\mathcal{C}(\sqrt{m})^p \int_{x_1}^{4\theta_1 m} |P(t)Q(t)t^\gamma|^p dt \sim (\sqrt{m})^p \int_0^{4\theta_1 m} |Pu|^p(t) dt.$$

Lemma 2.5 easily follows. $\square$

Proof of Theorem 2.6 We first prove that (2.18) implies (2.17). We have

$$\begin{aligned} &\|L_m(w_\alpha, f_j)u\Delta_{\theta_1}\|_p \leq \mathcal{C}\|L_m(w_\alpha, f_j)u\|_{L^p(x_1, 4\theta_1 m)} = \\ = & \mathcal{C} \sup_{\|g\|_q=1} \int_{x_1}^{4\theta_1 m} L_m(w_\alpha, f_j, x)u(x)g(x)dx =: \mathcal{C} \sup_{\|g\|_q=1} A_m(g) \end{aligned}$$

where

$$A_m(g) = mb_m \sum_{k=1}^{j} \lambda_k(w_\alpha) F_m(x_k) f(x_k) \int_{x_1}^{4\theta_1 m} \frac{p_m(w_\alpha, x)}{x - x_k} g(x) u(x) dx.$$

If, for all $Q \in \mathbb{P}_m$, $Q > 0$, we set

$$\pi(t) = \int_{x_1}^{4\theta_1 m} \frac{(Qp_m(w_\alpha))(x)F_m(t) - (Qp_m(w_\alpha))(t)F_m(x)}{x - t} \frac{u(x)}{Q(x)} g(x) dx,$$

then $\pi \in \mathbb{P}_{3m}$. Using π we have

$$A_m(g) = mb_m \sum_{k=1}^{j} \lambda_k(w_\alpha) f(x_k) \pi(x_k).$$

Since $\lambda_k(w_\alpha) \sim w_\alpha(x_k) \Delta x_k$ [13]

$$A_m(g) \leq \mathcal{C} \left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k) \right)^{\frac{1}{p}} m \left(\sum_{k=1}^{j} \Delta x_k \left| x_k^{\alpha-\gamma} e^{-\frac{x_k}{2}} \pi(x_k) \right|^q \right)^{\frac{1}{q}}.$$

By Lemma 2.5

$$\begin{aligned} & m \left(\sum_{k=1}^{j} \Delta x_k \left| x_k^{\alpha-\gamma} e^{-\frac{x_k}{2}} \pi(x_k) \right|^q \right)^{\frac{1}{q}} \leq \\ \leq \ & \mathcal{C} m \left(\int_{x_1}^{4\theta_1 m} \left| t^{\frac{\alpha}{2}-\gamma} \sqrt{w_\alpha(t)} \pi(t) \right|^q \right)^{\frac{1}{q}} =: \mathcal{C} m B. \end{aligned}$$

In order to prove that mB is bounded, we set

$$H(g, t) = \int_{x_1}^{4\theta_1 m} \frac{g(x)}{x - t} dx.$$

Then $\pi(t) = F_m(t) H(p_m(w_\alpha) ug, t) - Q(t) p_m(w_\alpha, t) H\left(F_m \frac{u}{Q} g, t\right)$. Now we choose Q such that $Q(t) \sim \sqrt{t}$, $t \in (x_1, 4\theta_1 m)$. Then we have

$$\begin{aligned}
mB &\leq m\left(\int_{x_1}^{4\theta_1 m}\left|t^{\frac{\alpha}{2}-\gamma}\sqrt{w_\alpha(t)}F_m(t)H(p_m(w_\alpha)ug,t)\right|^q dt\right)^{\frac{1}{q}}\\
&+ \quad m\left(\int_{x_1}^{4\theta_1 m}\left|t^{\frac{\alpha}{2}-\gamma}\sqrt{w_\alpha(t)}Q(t)p_m(w_\alpha,t)H\left(F_m\frac{u}{Q}g,t\right)\right|^q dt\right)^{\frac{1}{q}}\\
&:= \quad I_1+I_2.
\end{aligned}$$

To estimate I_1, we observe that [16] $|(\sqrt{w_\alpha}F_m)(t)|\leq \mathcal{C}t^{\frac{1}{4}}m^{-\frac{3}{4}}$, $x_1\leq t\leq 4\theta_1 m$. Therefore, if $p_m(w_\alpha)ug =: G$, we have

$$\begin{aligned}
I_1 &\leq \mathcal{C}\sqrt[4]{m}\left(\int_{x_1}^{4\theta_1 m}\left|\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}(t)H(G,t)\right|^q dt\right)^{\frac{1}{q}}\\
&= \mathcal{C}\sqrt[4]{m}m^{\frac{\alpha}{2}+\frac{1}{2}-\gamma+\frac{1}{q}}\left(\int_{\frac{x_1}{4m}}^{\theta_1}\left|\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}(\tau)\int_{\frac{x_1}{4m}}^{\theta_1}\frac{G(4my)}{y-\tau}dy\right|^q d\tau\right)^{\frac{1}{q}}\\
&\leq \mathcal{C}\sqrt[4]{m}m^{\frac{\alpha}{2}+\frac{1}{2}-\gamma+\frac{1}{q}}\left(\int_{\frac{x_1}{4m}}^{\theta_1}\left|\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}(\tau)G(4m\tau)\right|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

By (2.18), $\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}$ is an $\mathcal{A}_q$–weight in $(0,1)$. That means

$$\begin{aligned}
I_1 &\leq \mathcal{C}\sqrt[4]{m}\left(\int_{x_1}^{4m\theta_1}\left|\frac{\sqrt{v^\alpha\varphi}}{v^\gamma}(t)p_m(w_\alpha,t)u(t)g(t)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \mathcal{C}\left(\int_0^{4m\theta_1}|g(t)|^q dt\right)^{\frac{1}{q}} = \mathcal{C}
\end{aligned}$$

having used that $u(t)=\sqrt{w_\alpha(t)}t^{\gamma-\frac{\alpha}{2}}$ and $|p_m(w_\alpha,t)\sqrt{w_\alpha(t)}|\leq \frac{\mathcal{C}t^{-\frac{1}{4}}}{\sqrt[4]{m}}$, $t\in[x_1,4m\theta_1]$.

The estimate of I_2 is similar. In fact, recalling that $Q(t)\sim t^{\frac{1}{2}}$ and the estimate of $p_m(w_\alpha)\sqrt{w_\alpha}$ in $[x_1,4\theta_1 m]$, we have

$$\left|t^{\frac{\alpha}{2}-\gamma}Q(t)\sqrt{w_\alpha(t)}p_m(w_\alpha,t)\right|\leq \frac{\mathcal{C}\sqrt{v^\alpha\varphi}}{\sqrt[4]{m}v^\gamma}(t)$$

that, for hypothesis, is an $\mathcal{A}_q$–weight in $(0,1)$. Then

$$\begin{aligned} I_2 &\le \mathcal{C}m^{3/4}\left(\int_{x_1}^{4\theta_1 m}\left|t^{\frac{\alpha}{2}+\frac14-\gamma}\frac{u(t)}{Q(t)}F_m(t)g(t)\right|^q dx\right)^{\frac1q}\\ &\le \mathcal{C}m^{3/4}\left(\int_{x_1}^{4\theta_1 m}\left|t^{\frac{\alpha}{2}+\frac14-\gamma}\frac{t^{\gamma-\frac{\alpha}{2}}t^{\frac14}}{t^{\frac12}m^{3/4}}g(t)\right|^q dx\right)^{\frac1q}\\ &= \mathcal{C}\left(\int_{x_1}^{4\theta_1 m}|g|^q(t)dt\right)^{\frac1q}=\mathcal{C} \end{aligned}$$

and (2.18) implies (2.17).

Now we prove that (2.17) implies (2.18). Let g be a function such that $g(x_k)=\text{sgn}\,[p_m'(w_\alpha,x_k)(x-x_k)]$, $x_k\le 1, 0<x<1$, $g(x_k)=0$, $x_k\ge 1$ and $|g(x)|\le 1$. By (2.17)

$$\|L_m(w_\alpha,g_j)\Delta\theta_1 u\|_p\le\mathcal{C}\int_0^\infty u^p(x)dx=:M$$

whence

$$\begin{aligned} u(x)|L_m(w_\alpha,g_j,x)| &= \sum_{x_1\le x_k\le 1}\frac{|p_m(w_\alpha,x)u(x)|}{|x-x_k||p_m'(w_\alpha,x_k)|}\\ &\ge |u(x)p_m(w_\alpha,x)|\sum_{x_1\le x_k\le 1}\frac{1}{|p_m'(w_\alpha,x_k)|}. \end{aligned}$$

Since [17]

$$\begin{aligned} \frac{1}{|p_m'(w_\alpha,x_k)|} &= \sqrt{x_k\lambda_k(w_\alpha)}\sim\sqrt[4]{m}\,x_k^{\frac{\alpha}{2}+\frac14}e^{-\frac{x_k}{2}}\Delta x_k\\ &\ge \mathcal{C}\sqrt[4]{m}\int_{x_{k-1}}^{x_k}t^{\frac{\alpha}{2}+\frac14}e^{-\frac{t}{2}}dt, \quad (x_0=0), \end{aligned}$$

we have

$$\sum_{x_1\le x_k\le 1}\frac{1}{|p_m'(w_\alpha,x_k)|}\ge\mathcal{C}\sqrt[4]{m}\int_0^{\frac12}t^{\frac{\alpha}{2}+\frac14}e^{-\frac{t}{2}}dt\ge\mathcal{C}\sqrt[4]{m}$$

and, by Proposition 4.1, we get:

$$\frac{1}{\mathcal{C}}\left\|\frac{v^\gamma}{\sqrt{v^\alpha\varphi}}\right\|_{L^p(0,1)}\le\|p_m(w_\alpha)u\Delta_{\theta_1}\|_p\sqrt[4]{m}\le\mathcal{C}. \tag{4.8}$$

In order to prove that (2.17) implies the L^q–condition in (2.18), let g be the continuous function defined as follows $g(x_1) = 1$, $g(0) = 0 = g(x_2)$, g is linear in $[0, x_1] \cup [x_1, x_2]$ and $g(x) = 0$ elsewhere. Now by (2.17) we have

$$\|l_1(w_\alpha)u\Delta_{\theta_1}\|_p \leq \mathcal{C}u(x_1)(\Delta x_1)^{\frac{1}{p}}, \text{ or}$$

$$\left\|\frac{p_m(w_\alpha)u\Delta\theta_1}{\cdot - x_1}\right\| \sqrt{x_1\lambda_1(w_\alpha)} \leq \mathcal{C}x_1^\gamma e^{-\frac{x_1}{2}}(\Delta x_1)^{\frac{1}{p}}$$

and, since $\sqrt{x_1\lambda_1(w_\alpha)} \sim \sqrt[4]{m}x_1^{\frac{\alpha}{2}+\frac{1}{4}}e^{-\frac{x_1}{2}}(\Delta x_1)$ and $x_1 \sim \Delta x_1 \sim \frac{1}{m}$, we can also write

$$\sqrt[4]{m}\left\|\frac{p_m(w_\alpha)u\Delta\theta_1}{\cdot - x_1}\right\|_p \left(\frac{1}{m}\right)^{\frac{\alpha}{2}+\frac{1}{4}-\gamma+\frac{1}{q}} \leq \mathcal{C}, \tag{4.9}$$

from which

$$\left[\sqrt[4]{m}\|p_m(w_\alpha)u\|_{L^p(0,1)}\right]\left(\frac{1}{m}\right)^{\frac{\alpha}{2}+\frac{1}{4}-\gamma+\frac{1}{q}} \leq \mathcal{C},$$

using that $|x - x_1| < 1$ in $(0,1)$. Since we proved that the first factor on the left of the inequality is bounded, we get that $\frac{\alpha}{2}+\frac{1}{4}-\gamma+\frac{1}{q} \geq 0$. If $\frac{\alpha}{2}+\frac{1}{4}-\gamma+\frac{1}{q} = 0$, by (4.9) we have that

$$\sqrt[4]{m}\left(\int_{x_2}^1 \left|\frac{p_m(w_\alpha, x)u(x)}{x}\right|^p dx\right)^{\frac{1}{p}} \leq \mathcal{C},$$

whence, by (4.1),

$$\left(\int_{x_2}^1 x^{(\gamma-1-\frac{\alpha}{2}-\frac{1}{4})p}dx\right)^{\frac{1}{p}} = \left(\int_{x_2}^1 \frac{dx}{x}\right)^{\frac{1}{p}} \leq \mathcal{C},$$

a contradiction. So $\frac{\alpha}{2}+\frac{1}{4}-\gamma+\frac{1}{q} > 0$ i.e. $\frac{\sqrt{v^\alpha\varphi}}{u} \in L^q$. In order to prove that (2.19) is not true for all continuous functions, we proceed by contradiction. Assume that (2.19) is true for every $f \in C^0(\mathbb{R}^+)$. Define g as follows:

$$g(x) = 0 \text{ for } x \in [0, x_{m-1}] \cup [4m, \infty), \quad g(x_m) = 1$$
$$g \text{ is linear for } x \in [x_{m-1}, x_m] \cup [x_m, 4m].$$

Then (2.19) becomes

$$\|l_m(w_\alpha)u\|_p \le \mathcal{C}u(x_m)(\Delta x_m)^{\frac{1}{p}}, \quad x_{m+1} = 4m$$

from which

$$\left\|\frac{p_m(w_\alpha)u}{\cdot - x_m}\right\|_p \sqrt{x_m\lambda_m(w_\alpha)} \le \mathcal{C}u(x_m)(\Delta x_m)^{\frac{1}{p}}.$$

Since

$$\sqrt{x_m\lambda_m(w_\alpha)} \sim \sqrt[4]{m}x_m^{\frac{\alpha}{2}+\frac{1}{4}}e^{-\frac{x_m}{2}}\Delta x_m,$$

we get

$$\left\|\frac{p_m(w_\alpha)u}{\cdot - x_m}\right\|_p \sqrt[4]{m}x_m^{\frac{\alpha}{2}+\frac{1}{4}-\gamma}(\Delta x_m)^{1-\frac{1}{p}} \le \mathcal{C} \ne \mathcal{C}(m). \tag{4.10}$$

Now, with $A = \left[x_m + \frac{\Delta x_m}{3}, 4m - \frac{\Delta x_m}{3}\right]$, we have

$$\left\|\frac{p_m(w_\alpha)u}{\cdot - x_m}\right\|_p > \left\|\frac{p_m(w_\alpha)u}{\cdot - x_m}\right\|_{L^p(A)} \ge \frac{3}{\Delta x_m}\|p_m(w_\alpha)u\|_{L^p(A)}.$$

Using (4.2), we have

$$\left\|\frac{p_m(w_\alpha)u}{\cdot - x_m}\right\|_p \ge \frac{\mathcal{C}}{m^{\frac{1}{12}}(\Delta x_m)}\left(\int_A x^{(\gamma-\frac{\alpha}{2}-\frac{1}{4})p}dx\right)^{\frac{1}{p}} \sim \frac{\mathcal{C}x_m^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}}{m^{\frac{1}{12}}(\Delta x_m)^{1-\frac{1}{p}}}.$$

Recalling (4.10), we get the contradiction $m^{\frac{1}{6}} \le \mathcal{C}$. The proof is complete. □

Proof of Lemma 2.7. We recall the following inequality (see [7])

$$\delta^{\frac{1}{p}}\max_A|f(x)| \le \mathcal{C}\left(\|f\|_{L^p(A)} + \delta^{\frac{1}{p}}\int_0^\delta \frac{\omega^r(f,t)_{L^p(A)}}{t^{1+\frac{1}{p}}}dt\right),$$

that holds for any continuous function in $A = [a, a+\delta]$. Above, ω^k the k–th ordinary modulus of continuity, $1 < p < \infty$ and $\mathcal{C} \ne \mathcal{C}(f,\delta)$. Then, with $A = I_k = [x_k, x_{k+1}]$, $\delta = \Delta x_k \sim \sqrt{\frac{x_k}{m}}$, $k = 1, 2, \dots, j$, we can write (we use $\|\cdot\|_{I_k} = \|\cdot\|_{L^p(I_k)}$)

$$(\Delta x_k)^{\frac{1}{p}}|f(x_k)| \le \mathcal{C}\left(\|f\|_{I_k} + (\Delta x_k)^{\frac{1}{p}}\int_0^{\Delta x_k}\frac{\omega^r(f,t)_{I_k}}{t^{1+\frac{1}{p}}}dt\right).$$

Since $u(t) \sim u(x_k)$ for $t \in I_k$, (see [2]), we have

$$u(x_k)(\Delta x_k)^{\frac{1}{p}}|f(x_k)| \le \mathcal{C}\left(\|fu\|_{I_k} + (\Delta x_k)^{\frac{1}{p}} \int_0^{\Delta x_k} \frac{\omega^r(f,t)_{u,I_k}}{t^{1+\frac{1}{p}}} dt\right)$$

where $\omega^r(f,t)_{u,I_k} = \sup_{h\le t}\left(\int_0^{\Delta x_k - rh} |\Delta_h^r f(x)|^p u^p(x)dx\right)^{\frac{1}{p}}$, with Δ_h^r r–th finite forward difference. Making a change of variables $t \to \sqrt{x_k}\tau$, we have

$$(\Delta x_k)^{\frac{1}{p}}|fu|(x_k) \le \mathcal{C}\left(\|fu\|_{I_k} + \frac{1}{(\sqrt{m})^{\frac{1}{p}}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\omega^r(f,t\sqrt{x_k})_{u,I_k}}{t^{1+\frac{1}{p}}} dt\right).$$

Now let g be a function such that $g^{(r-1)}$ is absolutely continuous in $\mathbb{R}^+$ ($f^{(r-1)} \in AC(\mathbb{R}^+)$) and $\|g^{(r)}\varphi^r u\|_p < \infty$, $\varphi(x) = \sqrt{x}$. Then, since $\sqrt{x_k} \sim \sqrt{t} = \varphi(t)$, $t \in I_k$,

$$\begin{aligned} \omega^r(f,t\sqrt{x_k})_{u,I_k} &\le \|(f-g)u\|_{I_k} + (t\sqrt{x_k})^r\|f^{(r)}u\|_{I_k} \sim \\ &\sim \|(f-g)u\|_{I_k} + t^r\|f^{(r)}\varphi^r u\|_{I_k} =: A_k(t) \end{aligned}$$

and

$$\Delta x_k|fu|^p(x_k) \le 2^{p-1}\mathcal{C}\|fu\|_{I_k}^p + \frac{1}{(\sqrt{m})}\left(\int_0^{\frac{1}{\sqrt{m}}} \frac{A_k(t)}{t^{1+\frac{1}{p}}} dt\right)^p.$$

Summing on k and using the Minkovski inequality ([6] p. 148) it follows:

$$\begin{aligned} \left(\sum_{k=1}^{j} \Delta x_k|fu|^p(x_k)\right)^{\frac{1}{p}} &\le \mathcal{C}\Bigg(\|fu\|_{L^p(x_1,x_{j+1})} + \\ &+ \frac{1}{(\sqrt{m})^{\frac{1}{p}}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\left(\sum_{k=1}^{j} A_k^p(t)\right)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}} dt\Bigg). \end{aligned}$$

Now

$$\left(\sum_{k=1}^{j} A_k^p(t)\right)^{\frac{1}{p}} \le \|(f-g)u\|_{L^p(x_1,x_{j+1})} + t^r\|g^{(r)}\varphi^r u\|_{L^p(x_1,x_{j+1})}.$$

Now let $0 < h \le t$. Since $t \le \frac{1}{\sqrt{m}}$, we get $\frac{A}{h^2} \ge 4m > x_{j+1}$ for some constant A. Moreover, since $x_1 \sim \frac{1}{m}$, for some $\mathcal{C} > 0$ it results $\mathcal{C}(2rh)^2 \le \mathcal{C}t^2 \le \frac{\mathcal{C}}{m}$. Then, setting $I_h = ((2rh)^2\mathcal{C}, A/h^2)$ we also have

$$\left(\sum_{k=1}^{j} A_k^p(t)\right)^{\frac{1}{p}} \le \sup_{0<h\le t} \left\{\|(f-g)u\|_{L^p(I_h)} + h^r \|g^{(r)}\varphi^r u\|_{L^p(I_h)}\right\}.$$

Finally taking the infimum on $g \in AC(\mathbb{R}^+)$ we have [2]

$$\begin{aligned}\left(\sum_{k=1}^{j} A_k^p(t)\right)^{\frac{1}{p}} &\le \sup_{0<h\le t} \inf_{g} \left\{\|(f-g)u\|_{L^p(I_h)} + h^r \|g^{(r)}\varphi^r u\|_{L^p(I_h)}\right\} \\ &\sim \Omega_\varphi^r(f,t)_{u,p}.\end{aligned}$$

The lemma follows. □

Proof of Theorem 2.8. We use the following decompositions:

$$[f - \Delta_{\theta_1} L_m(w_\alpha, f_j)]u = [f - L_m(w_\alpha, f_j)]u\Delta_{\theta_1} + [f(1-\Delta_{\theta_1})]u\Delta_{\theta_1}$$

$$\begin{aligned} &= (f-P)\Delta_{\theta_1} u + L_m(w_\alpha, P - f_j)u\Delta_{\theta_1} + [f(1-\Delta_{\theta_1})]u = \\ &= (f-P)\Delta_{\theta_1} u + L_m(w_\alpha, \psi_j P)u\Delta_{\theta_1} + L_m(w_\alpha, (f-P)_j)u\Delta_{\theta_1} + \\ &+ f(1-\Delta_{\theta_1})u\Delta_{\theta_1} =: A_1 + A_2 + A_3 + A_4\end{aligned}$$

where $P = P_M$, $M = \left[\frac{\theta}{1+\theta} m\right] \sim m$ such that $\|(f-P)u\|_p \le \mathcal{C}E_M(f)_{u,p}$, $0 < \theta < \theta_1 < 1$, fixed, $\psi_j(x) = \psi\left(\frac{x - x_j}{x_{j+1} - x_j}\right)$. Above, $\|A_1\| \le \mathcal{C}E_M(f)_{u,p}$ and by (2.2)

$$\|A_4\| \le E_M(f)_{u,p} + \mathcal{C}e^{-Am}\|fu\|_p.$$

Moreover, by Theorem 2.6 and Lemma 2.7, we have

$$\begin{aligned}\|A_3\| &\le E_M(f)_{u,p} + \frac{\mathcal{C}}{(\sqrt{m})^{\frac{1}{p}}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f-P,t)_{u,p}}{t^{1+\frac{1}{p}}} dt \\ &\le \mathcal{C}\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t} dt + \frac{\mathcal{C}}{(\sqrt{m})^{\frac{1}{p}}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt \\ &+ \frac{\mathcal{C}}{(\sqrt{m})^r} \|P^{(r)}\varphi^r u\|_p,\end{aligned}$$

here the first integral is less than the second one. Moreover, following, with some small changes, the proof in [5, Th. 8.3.1, pp. 98-100], we can also get that

$$\frac{\mathcal{C}}{(\sqrt{m})^{\frac{1}{p}}}\|P^{(r)}\varphi^r u\|_p \le \frac{\mathcal{C}}{(\sqrt{m})^{\frac{1}{p}}}\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}dt.$$

In order to estimate $\|A_2\|$, we note that

$$\max_{0\le x\le 4\theta m}|L_m(w_\alpha,\psi_j P,x)u(x)| \le \mathcal{C}\log m\max_{k\ge j}|Pu|(x_k) := |Pu|(x_\nu).$$

Consequently

$$\|A_2\| \le \mathcal{C}m^{\frac{1}{p}}(\log m)|Pu|(x_\nu).$$

Moreover, by the identity

$$\int_{x_\nu}^{x_\nu+1}(Pu)(x)dx = (Pu)(x_\nu) + \int_{x_\nu}^{x_\nu+1}(x_\nu+1-x)(Pu)'(x)dx,$$

the Hölder inequality and with some simple computation

$$\|A_2\| \le \mathcal{C}m^{\frac{1}{p}}(\log m)\left[\|Pu\|_{L^p(4\theta m,\infty)} + \frac{1}{\sqrt{m}}\|P'\varphi u\|_{L^p(4\theta m,\infty)}\right].$$

Recalling that $P \in \mathbb{P}_M$, we apply first (2.1) to both terms in brackets then the Bernstein inequality to the second one to obtain

$$\|A_2\| \le \mathcal{C}(m\log m)e^{-Am}\|Pu\|_p \le \mathcal{C}e^{-Am}\|fu\|_p.$$

The proof is complete. □

Now we consider the Freud weights. First we recall the following polynomial inequality [10]:

$$\|p_m\sqrt{w}\|_p \le \mathcal{C}\|p_m\sqrt{w}\|_{L^p\left(-a_m+\frac{\mathcal{C}}{m^{2/3}},\, a_m-\frac{\mathcal{C}}{m^{2/3}}\right)}. \tag{4.11}$$

Moreover if

$$K_m(w,x,t) = \frac{\gamma_{m-1}}{\gamma_m}\frac{p_m(w,x)p_{m-1}(w,t) - p_m(w,t)p_{m-1}(w,x)}{x-t}$$

with $\gamma_m = \gamma_m(w)$ is the Darboux kernel, then for $f \in L^1_w$ we have

$$\begin{aligned} S_m(w,f) &= \sum_{k=0}^{m-1} c_k(f)p_k(w) = \int_{\mathbb{R}} K_m(w,x,t)f(t)w(t)dt \\ &= \frac{\gamma_{m-1}}{\gamma_m} p_m(w,x)H(p_{m-1}(w)fw,x) + \\ &+ \frac{\gamma_{m-1}}{\gamma_m} p_{m-1}(w,x)H(p_m(w)fw,x) \end{aligned} \tag{4.12}$$

where

$$H(g,x) = \int_{\mathbb{R}} \frac{g(t)}{x-t}dt \quad \text{and} \quad c_k(f,w) = \int_{\mathbb{R}} p_k(w,x)f(x)w(x)dx.$$

Proofs of Theorem 3.1 and Proposition 3.2. First we proof (3.7). By using (4.11), (4.12) and recalling that $\frac{\gamma_{m-1}}{\gamma_m} \sim a_m$, we can write:

$$\begin{aligned} &\|S_m(w,f\Delta_\theta)\sqrt{w}\|_p \le \mathcal{C}a_m\|S_m(w,f\Delta_\theta)\sqrt{w}\|_{L^p(-a_m,a_m)} \quad (4.13) \\ \le\ & \mathcal{C}a_m\|p_m(w)\sqrt{w}H(p_{m-1}(w)wf\Delta_\theta)\|_{L^p(-a_m,a_m)} \\ +\ & \mathcal{C}a_m\|p_{m-1}(w)\sqrt{w}H(p_{m-1}(w)wf\Delta_\theta)\|_{L^p(-a_m,a_m)} =: I_1 + I_2. \end{aligned}$$

We estimate only the first term because the second one is similar. Since (see for instance [8] p. 351)

$$\sqrt{w(x)}|p_m(w,x)|\sqrt[4]{a_m^2 - x^2} \le \mathcal{C}, \qquad x \in \mathbb{R}, \quad m \ge 1, \tag{4.14}$$

we have

$$I_1 \le \mathcal{C}a_m \left(\int_{-a_m}^{a_m} \left| (a_m^2 - x^2)^{-\frac14} \int_{-a_m}^{a_m} \frac{[p_{m-1}(w)wf\Delta_\theta](t)}{t-x} dt \right|^p \right)^{\frac1p}$$

$$= \mathcal{C}a_m^{\frac34+\frac1p} \left(\int_{-1}^{1} \left| (1 - x^2)^{-\frac14} \int_{-1}^{1} \frac{[p_{m-1}(w)wf\Delta_\theta](a_m t)}{t-x} dt \right|^p dx \right)^{\frac1p}$$

Since $1 < p < 4$ (and only in this case), the function $(1-x^2)^{-\frac14}$ is an $\mathcal{A}_p-$weight in $(-1,1)$, whence the Hilbert transform, is bounded. We continue as follows.

$$I_1 \le \mathcal{C}a_m^{\frac34+\frac1p} \left(\int_{-1}^{1} \left| (1-x^2)^{-\frac14}(p_{m-1}(w)wf\Delta_\theta)(a_m x) \right|^p dx \right)^{\frac1p}$$

$$
\begin{aligned}
&= \mathcal{C}a_m \left(\int_{-a_m}^{a_m} \left| (a_m^2 - x^2)^{-\frac{1}{4}} (p_m(w)\sqrt{w})(x)(\sqrt{w} f \Delta_\theta)(x) \right|^p dx \right)^{\frac{1}{p}} \\
&\le \mathcal{C}a_m \left(\int_{-a_m}^{a_m} \left(\frac{|\sqrt{w} f \Delta_\theta|(x)}{\sqrt{a_m^2 - x^2}} \right)^p dx \right)^{\frac{1}{p}} \\
&= \mathcal{C}a_m \left(\int_{-\theta a_m}^{\theta a_m} \frac{|f\sqrt{w}|^p(x)}{(\sqrt{a_m^2 - x^2})^p} dx \right)^{\frac{1}{p}} \le \frac{\mathcal{C}}{\sqrt{1-\theta}} \|f \Delta_\theta \sqrt{w}\|_p.
\end{aligned}
$$

□

Let us see the cases $p = 1$ and $p \ge 4$. If (3.9) were true with $\mathcal{C} \ne \mathcal{C}(m, f)$ then

$$
\begin{aligned}
& |c_m(f\Delta_\theta, w)| \; \|p_m(w)\sqrt{w}\|_p = \\
= \; & \|[S_{m+1}(w, f\Delta_\theta) - S_m(w, f\Delta_\theta)]\sqrt{w}\|_p \le 2\mathcal{C}\|f\Delta_\theta\sqrt{w}\|_p.
\end{aligned}
$$

Therefore

$$
\|p_m(w)\sqrt{w}\|_p \sup_{\|f\Delta_\theta\sqrt{w}\|_p = 1} \left| \int_{\mathbb{R}} (\sqrt{w} f \Delta_\theta)(x)(p_m(w)\sqrt{w})(x) dx \right| \le 2\mathcal{C}.
$$

Since the second factor on the left is equal to $\|p_m(w)\sqrt{w}\|_q$, with $p^{-1} + q^{-1} = 1$, it follows that

$$
\sup_m (\|p_m(w)\sqrt{w}\|_p \|p_m(w)\sqrt{w}\|_q) \le 2\mathcal{C},
$$

a contradiction, if $p \ge 4$ and $p = 1$ (see [8, Lemma 4.3,4.5]).

The proof of (3.7) is simpler. The previous proof can be repeated replacing the interval $(-a_m, a_m)$ with $(-\theta a_m, \theta a_m)$, $\theta \in (0,1)$. But, by (4.14), it results

$$
|\sqrt{w} p_m(w)|(x) \le \frac{\mathcal{C}}{\sqrt{a_m}}
$$

and therefore the restriction on p disappears. To prove (3.7) we use the following lemma.

Lemma 4.2 *Let $1 < p < \infty$ and $P \in \mathbb{P}_M$ arbitrary. Then*

$$
\|S_m(w, (1 - \Delta_\theta)P)\sqrt{w}\|_p \le \mathcal{C}e^{-Am}\|P\sqrt{w}\|_p
$$

where $\mathcal{C}$ and A dependent of θ and are independent of m and P.

PROOF. As in the proof of (3.9), we can write

$$\begin{aligned} & \|S_m(w,(1-\Delta_\theta)P)\sqrt{w}\|_p \le \\ \le \;& \mathcal{C} a_m [\|p_{m-1}(w)\sqrt{w} H(p_m(w)(1-\Delta_\theta)Pw)\|_{L^p(-a_m,a_m)} \\ + \;& \|p_m(w)\sqrt{w}H(p_{m-1}(w)(1-\Delta_\theta)Pw)\|_{L^p(-a_m,a_m)}] =: A_1 + A_2. \end{aligned}$$

Here we estimate only the first term A_1. To this end we use ([8, p. 351]) the relation

$$|\sqrt{w}p_m(w)|(x) \le \mathcal{C}\frac{m^{\frac{1}{6}}}{\sqrt{a_m}}, \quad x \in \mathbb{R},\, m \ge 1$$

and the boundedness of the Hilbert transform in L^p, $1 < p < \infty$. We get

$$\begin{aligned} A_1 & \le \mathcal{C}\sqrt{a_m} m^{\frac{1}{6}} \|H(p_m(w)(1-\Delta_\theta)Pw)\|_{L^p(A_m)} \\ & \le \mathcal{C}\sqrt{a_m} m^{\frac{1}{6}} \|\sqrt{w}p_m(w)(1-\Delta_\theta)P\sqrt{w}\|_p \\ & \le \mathcal{C} m^{\frac{1}{3}} \|(1-\Delta_\theta)P\sqrt{w}\|_p \le \mathcal{C} m^{\frac{1}{3}} e^{-Am} \|P\sqrt{w}\|_p \\ & \le \mathcal{C} e^{-Am} \|P\sqrt{w}\|_p. \end{aligned}$$

Now, the proof of (3.8) easily follows using the same argument as in the proof of the Corollary 2.4. □

PROOF OF LEMMA 3.3. By the inequality

$$(b-a)\left\{ \begin{array}{l} |f(a)|^p \\ |f(b)|^p \end{array} \right. \le 2^{p-1}\int_a^b |P(t)|^p dt + (b-a)^p \int_a^b |P'(t)|^p dt$$

with $a < b$ and $1 < p < \infty$, setting $\Delta x_k = x_{k+1} - x_k$ it follows

$$\Delta x_k |P(x_k)|^p \le 2^{p-1} \int_{x_k}^{x_{k+1}} |P(t)|^p dt + (\Delta x_k)^p \int_{x_k}^{x_{k+1}} |P'(t)|^p dt.$$

By monotonic property of the weight w, recalling that $\Delta x_k \sim \frac{a_m}{m}$ for $|x_k| \le x_j$

$$\begin{aligned} \sum_{k=-j}^{j} \Delta x_k |P\sqrt{w}|^p(x_k) \; & \le \; 2^{p-1}\left(\int_{-x_j}^{x_j} |P\sqrt{w}|^p(t)dt + \right. \\ & + \; \left. \left(\frac{a_m}{m}\right)^p \int_{-x_j}^{x_j} |P'\sqrt{w}|^p(t)dt \right). \end{aligned}$$

We have to estimate only the last integral. To this end we recall that in $[-a_m, a_m]$ there is a polynomial $Q \in \mathbb{P}_{lm}$ (l fixed integer) such that

$$Q(t) \sim \sqrt{w(t)} \qquad \text{and} \qquad \frac{a_m}{m}|Q'(t)| \le \mathcal{C}\sqrt{w(t)} \qquad \text{(see [10, 18])}.$$

Then

$$|P'\sqrt{w}|(t) \sim |P'Q|(t) \le |PQ'|(t) + |(QP)'|(t)$$

and

$$\frac{a_m}{m}\|P'\sqrt{w}\|_{L^p(-x_j,x_j)} \le \mathcal{C}\|P\sqrt{w}\|_{L^p(-x_j,x_j)} + \frac{a_m}{m}\|(QP)'\|_{L^p(-x_j,x_j)}.$$

Setting $x_j = \bar{\delta}a_m$, $0 < \bar{\delta} < 1$ and $T = QP$, we have:

$$\left(\frac{a_m}{m}\right)^p \int_{-\bar{\delta}a_m}^{\bar{\delta}a_m} |T'(x)|^p dx = \left(\frac{a_m}{m}\right)^p \frac{a_m}{a_m^p} \int_{-\bar{\delta}}^{\bar{\delta}} \left|\frac{d}{dt}T(ta_m)\right|^p dt.$$

Then with δ, $\bar{\delta} < \delta < 1$, the last integral is dominated by

$$\frac{1}{(\sqrt{\delta^2 - \bar{\delta}^2})^p} \frac{a_m}{m^p} \int_{-\delta}^{\delta} \left|\frac{d}{dt}T(ta_m)\sqrt{\delta^2 - t^2}\right|^p dt \le$$

$$\le \frac{\mathcal{C}}{(\sqrt{\delta^2 - \bar{\delta}^2})^p} \int_{-\delta a_m}^{\delta a_m} |PQ|^p(t)dt \le \frac{\mathcal{C}}{(\sqrt{\delta^2 - \bar{\delta}^2})^p} \int_{-\delta a_m}^{\delta a_m} |P\sqrt{w}|^p(t)dt.$$

In the first inequality we used the ordinary Bernstein inequality and then we changed the variables. Finally

$$\frac{a_m}{m}\|P'\sqrt{w}\|_{L^p(-x_j,x_j)} \le \mathcal{C}\|P\sqrt{w}\|_{[-\delta a_m, \delta a_m]}$$

with $(-\delta a_m, \delta a_m) \supset [-x_j, x_j]$. Lemma 3.3 easily follows. □

In order to prove Lemma 3.4 we have to introduce some preliminary relations. Let $|x| \le \delta a_m$, $0 < \delta < 1$. We have

$$\frac{\sqrt[4]{a_m^2 - x^2}}{\sqrt{w(x)}} \sim \frac{\sqrt{a_m}}{\frac{m}{a_m}\lambda_m(\sqrt{w}, x)} =: R(x) \qquad \text{and} \qquad R \in \mathbb{P}_{2m-2}. \tag{4.15}$$

If $|x_k| \leq x_j$, using [10], we have

$$\begin{aligned}
\frac{1}{|p_m'(w,x_k)|} &\sim \sqrt{w(x_k)}\,\frac{a_m^{\frac{3}{2}}}{m} = \sqrt{w(x_k)}\,\lambda_m(\sqrt{w},x_k)\frac{\sqrt{a_m}}{\frac{m}{a_m}\lambda_m(\sqrt{w},x_k)} \\
&\sim w(x_k)\Delta x_k R(x_k). \qquad (4.16)
\end{aligned}$$

On the other hand, it is easy to prove that there exists a polynomial $Q \in \mathbb{P}_{lm}$ (l fixed integer) such that

$$Q(t) \sim \sqrt[4]{a_m^2 - t^2}, \quad |t| \leq a_m - \frac{a_m}{m^2} \qquad (4.17)$$

pointing out that $\frac{a_m}{m^2} \leq \frac{\mathcal{C}}{m^{\frac{2}{3}}}$. □

Now we can prove Lemma 3.4.

Proof of Lemma 3.4. We only prove (3.11). By $\epsilon_m = \frac{a_m}{m^2} \left(\leq \frac{\mathcal{C}}{m^{\frac{2}{3}}}\right)$, we can write

$$\begin{aligned}
\|L_m(w,f_j)\sqrt{w}\|_p &\leq \sup_g \int_{-a_m+\epsilon_m}^{a_m-\epsilon_m} L_m(w,f_j,x)\sqrt{w(x)}g(x)dx \\
&=: \sup_g A_m(g)
\end{aligned}$$

where $\|g\|_{L^q(-a_m+\epsilon_m, a_m-\epsilon_m)} = 1$. Recalling (4.15) we have

$$\begin{aligned}
A_m(g) &\leq \mathcal{C}\sum_{k=-j}^{j} |f\sqrt{w}|(x_k)\Delta x_k R(x_k)\sqrt{w(x_k)} \\
&\cdot \left|\int_{-a_m+\epsilon_m}^{a_m-\epsilon_m} \frac{(p_m(w)\sqrt{w}g)(x)}{x-x_k}dx\right|.
\end{aligned}$$

Set

$$\pi(t) = \int_{-a_m+\epsilon_m}^{a_m-\epsilon_m} \frac{(p_m(w)Q)(x)R(t) - (p_m(w)Q)(t)R(x)}{Q(x)(x-t)}(g\sqrt{w})(x)dx$$

i.e.,

$$\pi(t) = R(t)H(p_m(w)g\sqrt{w},t) - p_m(w,t)Q(t)H\left(\frac{R}{Qg}\sqrt{w},t\right)$$

where Q and R are above defined and H is the Hilbert transform on $[-a_m+\epsilon_m, a_m-\epsilon_m]$. $\pi(t)$ is a polynomial of degree lm, for some fixed integer l. By Hölder inequality, we deduce that

$$\begin{aligned} A_m(g) &\le \mathcal{C}\sum_{k=-j}^{j}|f\sqrt{w}|(x_k)\Delta x_k|\sqrt{w}\pi|(x_k) \\ &\le \mathcal{C}\left(\sum_{k=-j}^{j}\Delta x_k|f\sqrt{w}|^p(x_k)\right)^{\frac{1}{p}}\left(\sum_{k=-j}^{j}\Delta x_k|\sqrt{w}\pi|^q(x_k)\right)^{\frac{1}{q}}. \end{aligned}$$

Now we prove that the second sum is bounded by $\mathcal{C}\|g\|_q$. Denote by B the second sum. By using Lemma 3.5, for some $\delta<1$, we have:

$$\begin{aligned} B &\le \mathcal{C}\|\sqrt{w}\pi\|_{L^q(-a_m\delta,a_m\delta)} \le \mathcal{C}\|\sqrt{w}RH(p_m(w)\sqrt{w}g)\|_{L^q(-a_m\delta,a_m\delta)} \\ &+ \mathcal{C}\|p_m(w)Q\sqrt{w}H(R/Qg\sqrt{w})\|_{L^q(-a_m\delta,a_m\delta)} =: B_1+B_2. \end{aligned}$$

Now let Δ the characteristic function of $[-a_m+\epsilon_m, a_m-\epsilon_m]$. Recalling that in $[-a_m\delta, a_m\delta]$ is $R(t)\sim\frac{\sqrt[4]{a_m^2-t^2}}{\sqrt{w(t)}}$ we have

$$\begin{aligned} B_1 &\le \mathcal{C}\left(\int_{-a_m\delta}^{a_m\delta}\left|\sqrt[4]{a_m^2-t^2}\int_{-a_m}^{a_m}[p_m(w)\sqrt{w}g\Delta](x)\frac{dx}{x-t}\right|^q dt\right)^{\frac{1}{q}} \\ &\le \mathcal{C}\left(\int_{-a_m}^{a_m}\left|\sqrt[4]{a_m^2-t^2}\int_{-a_m}^{a_m}[p_m(w)\sqrt{w}g\Delta](x)\frac{dx}{x-t}\right|^q dt\right)^{\frac{1}{q}} \\ &= \mathcal{C}a_m^{\frac{1}{2}+\frac{1}{q}}\left(\int_{-1}^{1}\left|\sqrt[4]{1-t^2}\int_{-1}^{1}[p_m(w)\sqrt{w}g\Delta]\frac{(a_m x)}{x-t}dx\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

Since $1<p<4$, the function $\sqrt[4]{1-t^2}$ is an $\mathcal{A}_q$−weight in $(-1,1)$ so the Hilbert transform is bounded. Therefore

$$\begin{aligned} B_1 &\le \mathcal{C}a_m^{\frac{1}{2}+\frac{1}{q}}\left(\int_{-1}^{1}|\sqrt[4]{1-t^2}[p_m(w)\sqrt{w}g\Delta](a_m t)|^q dt\right)^{\frac{1}{q}} \\ &= \mathcal{C}\left(\int_{-a_m}^{a_m}\left[\sqrt[4]{a_m^2-t^2}|\sqrt{w(t)}p_m(w,t)|\ |(g\Delta)(t)|\right]^q\right)^{\frac{1}{q}} \\ &\le \mathcal{C}\|g\Delta\|_{L^q(-a_m,a_m)} = \mathcal{C}, \end{aligned}$$

since $|\sqrt[4]{a_m^2 - t^2}\sqrt{w(t)}p_m(w,t)| \le \mathcal{C}$, $t \in \mathbb{R}$, $m > 1$. In order to estimate B_2 we recall (4.17). Using the same notations we obtain

$$\begin{aligned} B_2 &\le \mathcal{C}\left(\int_{-a_m}^{a_m}\left|\int_{-a_m}^{a_m}(R/Q)(x)\sqrt{w(x)}(\Delta g)(x)\frac{dx}{x-t}\right|^q dt\right)^{\frac{1}{q}} \\ &\le \mathcal{C}\left(\int_{-a_m+\epsilon_m}^{a_m-\epsilon_m}\left|\frac{R(t)}{Q(t)}\sqrt{w(t)}g(x)\right|^q dt\right)^{\frac{1}{q}} \\ &\le \mathcal{C}\left(\int_{-a_m+\epsilon_m}^{a_m-\epsilon_m}|g(t)|^q dt\right)^{\frac{1}{q}} = \mathcal{C}, \end{aligned}$$

by the boundedness of the Hilbert transform and the fact that, in $[-a_m + \epsilon_m, a_m - \epsilon_m]$, $R(t) \sim \frac{\sqrt[4]{a_m^2 - t^2}}{\sqrt{w(t)}}$ and $Q(t) \sim \sqrt[4]{a_m^2 - t^2}$. The last part of Lemma 3.4 can be proved by the same argument used to prove (2.19) in Theorem 2.6. The proof is complete. □

We may omit the proof of Lemma 3.5 (in fact it is simpler than Lemma 2.7).

Finally, in order to prove Theorem 3.6, we have to use only the following inequality

$$\|L_m(w, \psi_j P)\psi_j\sqrt{w}\|_p \le \mathcal{C}e^{-Am}\|P\sqrt{w}\|_p,$$

where $P \in \mathbb{P}_M$, $M = \left[\frac{\theta}{1+\theta}m\right]$. It can be proved with the same procedure used to estimate $\|A_2\|$ in the proof of Theorem 2.8.

References

[1] R. Askey and S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, Amer. J. Math., 87 (1965), 695–708.

[2] MĊ. De Bonis, G. Mastroianni and M. Viggiano, K–functionals, moduli of smoothness and weighted best approximation of the semiaxis, in “Functions, series, operators” (Budapest, 1999), pp. 181–211, János Bolyai Math. Soc., Budapest, 2002.

[3] B. Della Vecchia and G. Mastroianni, Gaussian Rules on unbounded intervals, in "Numerical integration and its complexity" (Oberwolfach, 2001), J. Complexity 19 (2003), no.3, 247–258.

[4] Z. Ditzian and D. S. Lubinsky, Jackson and smoothness theorems for Freud weight in L^p, $0 < p \leq \infty$, Constr. Approx., 13 (1997), 99–152.

[5] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer Series in Computational Mathematics, 9. Springer–Verlag, New York, 1987.

[6] G. H. Hardy, J.E. Littlewood and G. Polya,"Inequality," 2nd ed., Cambridge University Press, Cambridge, UK, 1952.

[7] K. G. Ivanov, On the behaviour of two moduli of functions. II, Serdica, 12 (1986), no. 2, 196–203.

[8] S. W. Jha and D. S. Lubinsky, Necessary and Sufficient Conditions for Mean Convergence of Orthogonal Expansions for Freuds Weights, Constr. Approx. , 11 (1995), no. 3, 331–363.

[9] H. König, Vector-Valued Lagrange Interpolation and Mean Convergence of Hermite Series, "Functional Analysis" (Essen, 1991), 227–247, Lecture Notes in Pure and Appl. Math. 150, Dekker, New York, 1994

[10] A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials and Nevai's conjecture for Freud weights, Constr. Approx., 8 (1992), no. 4, 463–535.

[11] G. Mastroianni, Polynomial inequalities, functional spaces and best approximation on the real semiaxis with Laguerre weights. "Orthogonal polynomials, approximation theory and harmonic analysis" (Inzel, 2000), Electron. Trans. Numer. Anal. 14 (2002), 142–151.

[12] G. Mastroianni and G. Monegato, Truncated quadrature rules over $(0, \infty)$ and Nyström type methods, SIAM J. Numer. Anal., 41 (2003), no. 5, 1870–1892 (electronic).

[13] G. Mastroianni and D. Occorsio, Lagrange interpolation at Laguerre zeros in some weighted uniform spaces, Acta Math. Hungar. 91 (2001), no. 1-2, 27–52.

[14] S. G. Mikhlin and S. Prössdorf, "Singular Integral Operators," Springer–Verlag, Berlin, 1986.

[15] G. Mastroianni and M. G. Russo, Lagrange Interpolation in Weighted Besov Spaces, Constr. Approx. 15 (1999), no. 1, 101–116.

[16] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series II., Trans. Amer. Math. Soc. 147 (1970), 433–460.

[17] P. Nevai, A Laguerre-polinomok gyōkein alapuló Lagrange-féle interpolációról, Mat. Lapok, 22 (1971), 149–164.

[18] E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Grundlehren der Mathematischen Wissenschaften,316, Springer–Verlag, Berlin, 1997.

[19] G. Szegö, "Orthogonal Polynomials," 4th ed., American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

On Bounded Interpolatory and Quasi-Interpolatory Polynomial Operators

H. N. Mhaskar
Department of Mathematics
California State University
Los Angeles, California 90032, U.S.A.
Email address: hmhaska@calstatela.edu

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

We discuss our recent work in the theory of approximation of functions using values of the function at scattered sites on the circle, the real line, the unit interval, and the unit sphere. As an alternative to interpolation, we present quasi-interpolatory operators for this purpose. We also prove the existence of bounded operators, yielding entire functions of finite exponential type, that interpolate a Birkhoff data for a function on a Euclidean space, where a finite number of derivatives, of order not exceeding a fixed number, are prescribed at each point.

1 Introduction

In most applications of approximation theory, one wishes to approximate a function given its values at finitely many points. Typically, the approximation is desired in the space $C(\Omega)$ comprising of uniformly continuous, bounded, real functions on a subset Ω of a Euclidean space; the space being endowed with the supremum norm: $\|f\|_\Omega := \sup_{\mathbf{x}\in\Omega} |f(\mathbf{x})|$, $f \in C(\Omega)$. In the theoretical setup, one has a nested sequence of subspaces $V_n \subset V_{n+1} \subset C(\Omega)$, with the dimension of V_n being n. Given a data of the form $\{(\mathbf{x}_j, f(\mathbf{x}_j)\}_{j=1}^N$, $\mathbf{x}_j \in \Omega$,

$j = 1, \cdots, N$, the problem of interpolation consists of finding a function $\mathcal{I}_N(f) \in V_N$, such that $\mathcal{I}_N(f)(\mathbf{x}_j) = f(\mathbf{x}_j)$, $j = 1, \cdots, N$. The subject is studied in great detail in a variety of situations (cf. for example, [24, 25, 3, 6]).

In the case of multivariate approximation, it is often not guaranteed that the interpolation problem from a given space will have a solution. Even if a solution exists, its numerical computation involves a matrix inversion and is, therefore, costly. Moreover, the sequence of interpolants, $\{\mathcal{I}_N(f)\}$ does not converge for *every* continuous function f. There are two ways to remedy the last problem. One approach is to seek bounded operators taking values in V_M for some $M > N$ which interpolate at the given data. This approach has been explored in great detail in the univariate context in the book [24, Chapter II] of Szabados and Vértesi. The other approach is to abandon the requirement that the operators should interpolate the data, and seek bounded operators taking values in V_M for some $M < N$, and constructed from the data in some other way.

The purpose of this paper is to survey some of our recent work in both directions.

2 Bounded Interpolatory Operators

The motivation behind the work in this section is provided by the following theorem (cf. [24, Chapter II, Theorem 2.7]), where we denote the class of all univariate algebraic polynomials of degree at most n by Π_n. Throughout this paper, the symbols $c, c_1, \cdots$ will denote generic constants, depending only on the fixed parameters in the discussion, and any other explicitly indicated parameters.

Theorem 2.1 *Let $x_{k,n} = \cos\theta_{k,n} \in [-1,1]$ be an arbitrary system of nodes ($0 \le \theta_{1,n} < \cdots < \theta_{n,n} \le \pi$) and let*

$$d_n := \min_{1\le k\le n-1} \theta_{k+1,n} - \theta_{k,n}.$$

Then for any $\epsilon > 0$, there exist linear polynomial operators P_n on $C([-1,1])$ with the following properties: (a) *If $m = \lfloor \pi(1+\epsilon)/d_n \rfloor$*

then $P_n(P) = P$ *for all* $P \in \Pi_m$, (b) *for* $f \in C([-1,1])$, $P_n(f) \in \Pi_N$ *where* $N = (\pi/d_n + 1)(1+3\epsilon)$, (c) $P_n(f, x_{k,n}) = f(x_{k,n})$ *for* $k = 1, \cdots, n$, *and* (d) $\|P_n(f)\|_{[-1,1]} \le c(\epsilon)\|f\|_{[-1,1]}$.

We note that the conditions (a) and (d) imply that

$$\|f - P_n(f)\|_{[-1,1]} \le c(\epsilon) \min_{P \in \Pi_m} \|f - P\|_{[-1,1]}.$$

Giving up the requirement that the operator should be linear, we obtained a far reaching generalization of this result in [15]. The following theorem of Narcowich and Ward [20, Proposition 3.1] is a further generalization of the result in [15]. In the sequel, if Y is a Banach space, $V \subset Y$, we will write $\|\cdot\|_Y$ to denote the norm on Y, and write

$$\mathsf{dist}\ (Y; f, V) := \mathsf{dist}\ (f, V) := \inf_{g \in V} \|f - g\|_Y, \qquad f \in Y. \tag{2.1}$$

We note that our notation $\|f\|_\Omega$ for the supremum norm on $C(\Omega)$ may be thought of as an abbreviation for $\|f\|_{C(\Omega)}$.

Theorem 2.2 *Let* $\mathcal{Y}$ *be a (possibly complex) Banach space,* $\mathcal{V}$ *be a subspace of* $\mathcal{Y}$*, and* Z^* *be a finite dimensional subspace of* $\mathcal{Y}^*$*, the dual of* $\mathcal{Y}$*. If there exists* $\beta > 0$ *such that*

$$\|z^*\|_{\mathcal{Y}^*} \le \beta \|z^*|_{\mathcal{V}}\|_{\mathcal{V}^*}, \qquad z^* \in Z^*, \tag{2.2}$$

then for every $y \in \mathcal{Y}$*, there exists* $v \in \mathcal{V}$*, such that* $z^*(v) = z^*(y)$ *for every* $z^* \in Z^*$*, and in addition,* $\|y - v\|_{\mathcal{Y}} \le (1 + 2\beta)\ \mathsf{dist}\ (y, \mathcal{V})$.

The main difference between this theorem and the corresponding theorem in [15] is that the space $\mathcal{V}$ is not required to be finite dimensional. This is accomplished by an appeal to the notion of local reflexivity principle, rather than the fact that finite dimensional spaces are reflexive. This principle is formulated as follows (cf., for example, [7]), where the space Y is identified with a subspace of the dual Y^{**} of the space Y^* in the standard way.

Proposition 2.3 *Let* Y *be a Banach space,* $V \subset Y^{**}$ *and* $W \subset Y^*$ *be finite dimensional spaces. Given* $\epsilon > 0$*, there exists a linear* *operator* $T : V \to Y$ *such that* $T(y) = y$ *if* $y \in V \cap Y$, $y^*(T(v)) = v(y^*)$ *for all* $v \in V$, $y^* \in W$, *and* $(1-\epsilon)\|v\|_{Y^{**}} \le \|T(v)\|_Y \le (1+\epsilon)\|v\|_{Y^{**}}$, *for all* $v \in V$.

In [15], we showed that an analogue of Theorem 2.1 always holds if a Jackson–type theorem holds. In particular, we obtained the analogues in the case when the approximating spaces consist of spherical polynomials and zonal function networks. Theorem 2.2 was used in [20] to obtain the analogue of Theorem 2.1 in the case of approximation by entire functions of finite exponential type.

One restriction of all of the above theorems is that they require interpolation at distinct points. It seems reasonable to expect that a similar theorem will hold in the case of Birkhoff data; i.e., where the values of certain derivatives of the function are prescribed as well, provided there are only finitely many such conditions at each point, the number being independent of the number of points. In this section, we explore this question further. The following theorem gives a recipe for applying Theorem 2.2.

Theorem 2.4 *Let* $\mathcal{Y}$ *be a (possibly complex) Banach space,* $\mathcal{V}$ *be a subspace of* $\mathcal{Y}$*, and* Z^* *be a finite dimensional subspace of* $\mathcal{Y}^*$*. Suppose there exists a compact set* $K \subset \mathcal{Y}$ *such that*

$$\kappa := \sup_{y\in K} \textit{dist}\,(y, \mathcal{V}) < \inf_{y^*\in Z^*, \|y^*\|_{\mathcal{Y}^*}=1} \sup_{y\in K} |y^*(y)| =: m. \tag{2.3}$$

Let $B := \max_{y\in K} \|y\|_{\mathcal{Y}}$*, and* $\beta := 2(2B+m+\kappa)/(m-\kappa)$*. Then for every* $y \in \mathcal{Y}$*, there exists* $v \in \mathcal{V}$*, such that* $z^*(v) = z^*(y)$ *for every* $z^* \in Z^*$*, and in addition,* $\|y - v\|_{\mathcal{Y}} \le (1+2\beta)$ *dist* $(y, \mathcal{V})$.

PROOF. We observe that (2.3) implies that

$$m = \inf_{y^*\in Z^*, \|y^*\|_{\mathcal{Y}^*}=1} \sup_{y\in K} |y^*(y)| > 0. \tag{2.4}$$

Now, let $y^* \in Z^*$, $\|y^*\|_{\mathcal{Y}^*} = 1$. In view of the fact that $\kappa < m$, we find $y \in K$ such that $|y^*(y)| \ge (3/4)m + (1/4)\kappa$ (cf. (2.4)). The estimate (2.3) implies that there exists $v \in \mathcal{V}$ such that $\|y - v\|_{\mathcal{Y}} \le (m+\kappa)/2$. Then

$$\begin{aligned} |y^*(v)| &\ge |y^*(y)| - |y^*(y-v)| \ge (3/4)m + (1/4)\kappa - (m+\kappa)/2 \\ &= (m-\kappa)/4. \end{aligned}$$

Also, $\|v\|_{\mathcal{Y}} \le \|y\|_{\mathcal{Y}} + (m+\kappa)/2 \le (2B+m+\kappa)/2$. Thus,

$$\|y^*\|_{\mathcal{Y}^*} = 1 \le \frac{2(2B+m+\kappa)}{m-\kappa}\frac{|y^*(v)|}{\|v\|_{\mathcal{Y}}} \le \frac{2(2B+m+\kappa)}{m-\kappa}\|y^*|_{\mathcal{V}}\|_{\mathcal{V}^*}.$$

We may now apply Theorem 2.2 to complete the proof. □

One way to construct a compact set as in Theorem 2.4 is the following. Let $\{y_1^*,\cdots,y_N^*\}$ be a basis for Z^*, with each $\|y_\ell^*\|_{\mathcal{Y}^*} = 1$. The Hahn–Banach theorem yields a set $\{x_1^{**},\cdots,x_N^{**}\}$ in the dual $\mathcal{Y}^{**}$ of $\mathcal{Y}^*$ such that each $\|x_\ell^{**}\|_{\mathcal{Y}^{**}} = 1$ and $\mathbf{x}_\ell^{**}(y_j^*) = \delta_{\ell,j}$, $\ell, j = 1,\cdots,N$. Let $\epsilon > 0$. The principle of local reflexivity then implies the existence of $\{x_1,\cdots,x_N\} \in \mathcal{Y}$ such that $1-\epsilon \le \|x_\ell\|_{\mathcal{Y}} \le 1+\epsilon$, $1 \le \ell \le N$ and $y_j^*(x_\ell) = \delta_{\ell,j}$, $\ell, j = 1,\cdots,N$. We may choose K to be the set $\{\sum_{\ell=1}^N a_\ell x_\ell : \max_{1\le\ell\le N}|a_j| = 1\}$.

If $y^* = \sum_{j=1}^N b_j y_j^*$, and $\|y^*\|_{\mathcal{Y}^*} = 1$, then

$$1 = \|y^*\|_{\mathcal{Y}^*} \le \sum_{j=1}^N |b_j| = y^*\left(\sum_{j=1}^N (\text{sgn } b_j)x_j\right) \le \sup_{y\in K}|y^*(y)|.$$

Therefore, the conclusion of Theorem 2.4 holds if $\sup_{y\in K} \mathsf{dist}\,(y,V) < 1$.

We illustrate this process by giving a qualitative generalization of [20, Theorem 3.5]. Thus, we will not have explicit constants as in [20, Theorem 3.5], but our theorem will be valid when Birkhoff interpolatory conditions are required on the approximation.

Let $s \ge 1$ be a fixed integer. We will write $\mathbf{x} = (x_1,\cdots,x_s)$, $|\mathbf{x}| = \sum_{j=1}^s |x_j|$, $|\mathbf{x}|_\infty = \max_{1\le j\le s}|x_j|$. The symbols $\mathbf{k},\mathbf{m},\mathbf{n}$ will denote multiintegers with nonnegative components. We write D_j for the derivative with respect to x_j, $f^{(\mathbf{k})} := (\prod_{j=1}^s D_j^{k_j})f$. For an integer $r \ge 0$, $C^r(\mathbb{R}^s)$ is defined to be the space of functions f such that $f^{(\mathbf{k})} \in C(\mathbb{R}^s)$ for all $\mathbf{k}$ with $|\mathbf{k}| \le r$. For $\tau \ge 0$, we denote by $\mathcal{M}_\tau = \mathcal{M}_{\tau,s}$ the class of all entire functions g of s complex variables, such that the restriction of g to $\mathbb{R}^s$ is real valued, and for some $A = A(g) > 0$ and every $(x_1+iy_1,\cdots,x_s+iy_s) \in \mathbb{C}^s$,

$$|g(x_1+iy_1,\cdots,x_s+iy_s)| \le A\exp\left(\tau\sum_{j=1}^s |y_j|\right).$$

We note that the class $\mathcal{M}_\tau$ consists of entire functions of finite exponential type at most τ, that are bounded and real valued on $\mathbb{R}^s$. The Bernstein inequality [22, Section 3.2.2, eqn. (8)],

$$\|g^{(\mathbf{k})}\|_{\mathbb{R}^s} \le \tau^{|\mathbf{k}|}\|g\|_{\mathbb{R}^s}, \qquad g \in \mathcal{M}_\tau, \tag{2.5}$$

shows that $\mathcal{M}_\tau \subset C^r(\mathbb{R}^s)$ for all $\tau \ge 0$ and integer $r \ge 0$.

Theorem 2.5 *Let $R \ge 0$ be an integer, $\mathbf{x}_j$, $j = 1, \cdots, N$, be a set of distinct points in $\mathbb{R}^s$, and*

$$\eta := \min_{j \ne \ell} |\mathbf{x}_j - \mathbf{x}_\ell|_\infty, \tag{2.6}$$

and $S_j \subseteq \{\mathbf{m} : |\mathbf{m}| \le R\}$, $j = 1, \cdots, N$. Then there exists a constant α with the following property. For any $f \in C^R(\mathbb{R}^s)$, there exists $g \in \mathcal{M}_{\alpha/\eta}$ such that

$$g^{(\mathbf{k})}(\mathbf{x}_j) = f^{(\mathbf{k})}(\mathbf{x}_j), \qquad \mathbf{k} \in S_j,\ j = 1, \cdots, N, \tag{2.7}$$

and

$$\max_{|\mathbf{k}| \le R} \eta^{|\mathbf{k}|}\|f^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \le c(R) \max_{|\mathbf{m}| \le R} \eta^{|\mathbf{m}|} \inf_{P \in \mathcal{M}_{\alpha/(4\eta)}} \|f^{(\mathbf{m})} - P\|_{\mathbb{R}^s}. \tag{2.8}$$

In the rest of this section, R will be treated as a fixed parameter, and the dependence of the various constants on R will not be indicated explicitly.

In order to prove the theorem, we recall from [22, Section 8.6] some facts regarding approximation by entire functions of finite exponential type. For $\tau > 0$, $\mathbf{x} \in \mathbb{R}^s$, and $f \in C(\mathbb{R}^s)$, let

$$V_\tau(\mathbf{x}) := \frac{1}{\tau^s}\prod_{j=1}^{s} \frac{\cos \tau x_j - \cos 2\tau x_j}{x_j^2}, \tag{2.9}$$

and

$$\mathcal{F}_\tau(f, \mathbf{x}) := \frac{1}{\pi^s}\int_{\mathbb{R}^s} V_\tau(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}. \tag{2.10}$$

Lemma 2.6 *If $g \in \mathcal{M}_\tau$ then $\mathcal{F}_\tau(g) = g$. For any $f \in C(\mathbb{R}^s)$, $\mathcal{F}_\tau(f) \in \mathcal{M}_{2\tau}$, and*

$$\|\mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \le c\|f\|_{\mathbb{R}^s}, \qquad \|f - \mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \le c \ \mathsf{dist}\ (f, \mathcal{M}_\tau). \quad (2.11)$$

Moreover, if $r \ge 1$ is an integer, $f \in C^r(\mathbb{R}^s)$, $g \in \mathcal{M}_{2\tau}$, and $\|f - g\|_{\mathbb{R}^s} \le c \ \mathsf{dist}\ (f, \mathcal{M}_\tau)$, then for any multiinteger $\mathbf{k}$ with $|\mathbf{k}| \le r$,

$$\tau^{-|\mathbf{k}|}\|f^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \le c(r)\tau^{-r} \max_{|\mathbf{m}|=r} \ \mathsf{dist}\ (f^{(\mathbf{m})}, \mathcal{M}_{\tau/2}). \quad (2.12)$$

In particular,

$$\max_{|\mathbf{k}|\le r} \tau^{-|\mathbf{k}|}\|f^{(\mathbf{k})} - \mathcal{F}_\tau(f)^{(\mathbf{k})}\|_{\mathbb{R}^s} \le c(r)\tau^{-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s}, \quad (2.13)$$

and

$$\begin{aligned} &\inf_{P\in\mathcal{M}_{2\tau}} \max_{|\mathbf{m}|\le R} \tau^{-|\mathbf{m}|}\|f^{(\mathbf{m})} - P^{(\mathbf{m})}\|_{\mathbb{R}^s} \\ &\quad \le \ c \max_{|\mathbf{m}|\le R} \tau^{-|\mathbf{m}|} \inf_{P\in\mathcal{M}_{\tau/2}} \|f^{(\mathbf{m})} - P\|_{\mathbb{R}^s}. \end{aligned} \quad (2.14)$$

PROOF. The first two statements in the lemma and the estimates (2.11) are proved in [22, Section 8.6]. Let $f \in C^r(\mathbb{R}^s)$, $g \in \mathcal{M}_{2\tau}$, $\|f-g\|_{\mathbb{R}^s} \le c \ \mathsf{dist}\ (f, \mathcal{M}_\tau)$, and $\mathbf{k}$ be a multiinteger with $|\mathbf{k}| \le r$. The second estimate in (2.11) implies that $\|g-\mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \le c \ \mathsf{dist}\ (f, \mathcal{M}_\tau)$. Since $\mathcal{F}_\tau(f) - g \in \mathcal{M}_{2\tau}$, the Bernstein inequality (2.5) shows that

$$\begin{aligned} \|\mathcal{F}_\tau(f^{(\mathbf{k})}) - g^{(\mathbf{k})}\|_{\mathbb{R}^s} &= \|\mathcal{F}_\tau(f)^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \le c(r)\tau^{|\mathbf{k}|}\|\mathcal{F}_\tau(f) - g\|_{\mathbb{R}^s} \\ &\le c(r)\tau^{|\mathbf{k}|} \ \mathsf{dist}\ (f, \mathcal{M}_\tau). \end{aligned} \quad (2.15)$$

In view of the direct theorem for approximation from $\mathcal{M}_\tau$ (cf. [22, Section 5.2.2, eqn. (4), Section 4.2, eqn. (15)]), it follows that

$$\begin{aligned} \tau^{|\mathbf{k}|} \ \mathsf{dist}\ (f, \mathcal{M}_\tau) &\le c(r)\tau^{|\mathbf{k}|-r} \sum_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s} \\ &\le c(r)\tau^{|\mathbf{k}|-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s}. \end{aligned}$$

Therefore, (2.11) leads to

$$\begin{aligned}
\mathsf{dist}\,(f,\mathcal{M}_\tau) &= \mathsf{dist}\,(f-\mathcal{F}_{\tau/2}(f),\mathcal{M}_\tau)\\
&\le c(r)\tau^{-r}\max_{|\mathbf{m}|=r}\|f^{(\mathbf{m})}-\mathcal{F}_{\tau/2}(f)^{(\mathbf{m})}\|_{\mathbb{R}^s}\\
&= c(r)\tau^{-r}\max_{|\mathbf{m}|=r}\|f^{(\mathbf{m})}-\mathcal{F}_{\tau/2}(f^{(\mathbf{m})})\|_{\mathbb{R}^s}\\
&\le c(r)\tau^{-r}\max_{|\mathbf{m}|=r}\ \mathsf{dist}\,(f^{(\mathbf{m})},\mathcal{M}_{\tau/2}). \qquad (2.16)
\end{aligned}$$

Using this estimate with $f^{(\mathbf{k})}$ in place of f, and $r-|\mathbf{k}|$ in place of r, we obtain in view of (2.11) that

$$\begin{aligned}
\|f^{(\mathbf{k})}-\mathcal{F}_\tau(f^{(\mathbf{k})})\|_{\mathbb{R}^s} &\le c(r)\ \mathsf{dist}\,(f^{(\mathbf{k})},\mathcal{M}_\tau)\\
&\le c(r)\tau^{|\mathbf{k}|-r}\max_{|\mathbf{m}|=r}\ \mathsf{dist}\,(f^{(\mathbf{m})},\mathcal{M}_{\tau/2}). \qquad (2.17)
\end{aligned}$$

The estimates (2.17), (2.15), and (2.16) lead to

$$\begin{aligned}
\|f^{(\mathbf{k})}-g^{(\mathbf{k})}\|_{\mathbb{R}^s} &\le \|f^{(\mathbf{k})}-\mathcal{F}_\tau(f^{(\mathbf{k})})\|_{\mathbb{R}^s}+\|\mathcal{F}_\tau(f^{(\mathbf{k})})-g^{(\mathbf{k})}\|_{\mathbb{R}^s}\\
&\le c(r)\tau^{|\mathbf{k}|-r}\max_{|\mathbf{m}|=r}\ \mathsf{dist}\,(f^{(\mathbf{m})},\mathcal{M}_{\tau/2}).
\end{aligned}$$

This completes the proof of (2.12). The estimates (2.13) and (2.14) follow from (2.11) and (2.12). □

PROOF OF THEOREM 2.5. In this proof, for any integer $r\ge 0$ and a function $f\in C^r(\mathbb{R}^s)$, we write

$$\|f\|_r := \max_{|\mathbf{k}|=r}\|f^{(\mathbf{k})}\|_{\mathbb{R}^s}. \qquad (2.18)$$

We will apply Theorem 2.4 with $Y=C^R(\mathbb{R}^s)$, and for $f\in Y$, define

$$\|f\|_Y := \max_{0\le r\le R}\eta^r\|f\|_r. \qquad (2.19)$$

We will write $\tau=\alpha/\eta$ for a constant α to be chosen later, and take $\mathcal{M}_\tau$ for the subspace V of Y. In the remainder of this proof, $\mathsf{dist}\,(f,V)$ is defined as in (2.1) with respect to the norm $\|\cdot\|_Y$. For multiinteger $\mathbf{m}$ and $\ell=1,\cdots,N$, let $y^*_{\mathbf{m},\ell}$ denote the functional on Y defined by $y^*_{\mathbf{m},\ell}(f)=f^{(\mathbf{m})}(\mathbf{x}_\ell)$. Let Z^* be the subspace of

Y^* spanned by $\{y^*_{\mathbf{m},\ell} : \mathbf{m} \in S_\ell,\ \ell = 1, \cdots, N\}$. We now construct the compact set K as required in Theorem 2.4, following the ideas outlined after the proof of that theorem. In this proof only, we find it useful to retain the values of the constants $c, c_1, \cdots$.

Let $\psi : \mathbb{R} \to [0, \infty)$ be an infinitely often differentible function such that $\psi(t) = 1$ if $|t| \le 1/6$ and $\psi(t) = 0$ if $|t| \ge 1/3$. For any multiinteger $\mathbf{k} \ge 0$, let

$$\phi_{\mathbf{k}}(\mathbf{x}) := \prod_{\ell=1}^{s} \psi(x_\ell/\eta) \frac{x_\ell^{k_\ell}}{k_\ell!},$$

and

$$\Phi_{\mathbf{k},j}(\mathbf{x}) = \phi_{\mathbf{k}}(\mathbf{x} - \mathbf{x}_j), \qquad j = 1, \cdots, N. \tag{2.20}$$

We note that if $|\mathbf{x} - \mathbf{x}_j|_\infty \ge \eta/3$, then $\Phi_{\mathbf{k},j}(\mathbf{x}) = 0$ for all $\mathbf{k} \ge 0$. In particular, for any multiinteger $\mathbf{m} \ge 0$,

$$\|{\Phi_{\mathbf{k},j}}^{(\mathbf{m})}\|_{\mathbb{R}^s} \le c_1 \eta^{|\mathbf{k}|-|\mathbf{m}|}. \tag{2.21}$$

Since $\Phi_{\mathbf{k},j}(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_j)^{\mathbf{k}}/\mathbf{k}!$ for all $\mathbf{x}$ with $|\mathbf{x}-\mathbf{x}_j|_\infty \le \eta/6$, it follows that

$$y^*_{\mathbf{m},\ell}(\Phi_{\mathbf{k},j}) = {\Phi_{\mathbf{k},j}}^{(\mathbf{m})}(\mathbf{x}_\ell) = \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{m},\ j = \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{2.22}$$

We define

$$K := \{\sum_{j=1}^{N} \sum_{\mathbf{k} \in S_j} b_{\mathbf{k},j} \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k},j} \ : \ \max_{1 \le j \le N,\ \mathbf{k} \in S_j} |b_{\mathbf{k},j}| \le 1\}.$$

We now estimate the quantities B, m, κ appearing in (2.3).

Let $g = \sum_{j=1}^{N} \sum_{\mathbf{k} \in S_j} b_{\mathbf{k},j} \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k},j} \in K$. Let r be any integer, $0 \le r \le \mathcal{R} + 1$, $\mathbf{n}$ be any multiinteger with $|\mathbf{n}| = r$, and $\mathbf{x} \in \mathbb{R}^s$. If $|\mathbf{x} - \mathbf{x}_j|_\infty > \eta/3$ for every j, then $g^{(\mathbf{n})}(\mathbf{x}) = 0$. Otherwise, there is a unique j such that $|\mathbf{x} - \mathbf{x}_j|_\infty \le \eta/3$, and

$$g^{(\mathbf{n})}(\mathbf{x}) = \sum_{\mathbf{k} \in S_j} b_{\mathbf{k},j} \eta^{-|\mathbf{k}|} {\Phi_{\mathbf{k},j}}^{(\mathbf{n})}(\mathbf{x}).$$

Since the number of elements in S_j is bounded independently of N, and $\max_{\mathbf{k}\in S_j}|b_{\mathbf{k},j}|\le 1$, (2.21) implies that

$$|g^{(\mathbf{n})}(\mathbf{x})| \le \sum_{\mathbf{k}\in S_j} \eta^{-|\mathbf{k}|}|\Phi_{\mathbf{k},j}{}^{(\mathbf{n})}(\mathbf{x})| \le c_2\eta^{-|\mathbf{n}|}.$$

Thus, $\|g^{(\mathbf{n})}\|_{\mathbb{R}^s} \le c_2\eta^{-r}$ for all $\mathbf{n}$ with $|\mathbf{n}| = r$; i.e.,

$$\|g\|_r \le c_2\eta^{-r}, \qquad g\in K,\ r\ge 0. \tag{2.23}$$

It follows that

$$B = \sup_{g\in K}\|g\|_Y \le c_2. \tag{2.24}$$

Let $y^* = \sum_{\ell=1}^N \sum_{\mathbf{m}\in S_\ell} a_{\mathbf{m},\ell}y^*_{\mathbf{m},\ell} \in Z^*$, and $g\in Y$. Then

$$|y^*(g)| = \left|\sum_{\ell=1}^N \sum_{\mathbf{m}\in S_\ell} a_{\mathbf{m},\ell}g^{(\mathbf{m})}(\mathbf{x}_\ell)\right| \le \|g\|_Y \sum_{\ell=1}^N \sum_{\mathbf{m}\in S_\ell} |a_{\mathbf{m},\ell}|\eta^{-|\mathbf{m}|}.$$

Therefore,

$$\|y^*\|_{Y^*} \le \sum_{\ell=1}^N \sum_{\mathbf{m}\in S_\ell} |a_{\mathbf{m},\ell}|\eta^{-|\mathbf{m}|}. \tag{2.25}$$

Let $g_{y^*} = \sum_{j=1}^N \sum_{\mathbf{k}\in S_j}(\operatorname{sgn} a_{\mathbf{k},j})\eta^{-|\mathbf{k}|}\Phi_{\mathbf{k},j} \in K$. It is easy to verify using (2.22) that

$$y^*(g_{y^*}) = \sum_{\ell=1}^N \sum_{\mathbf{m}\in S_\ell} |a_{\mathbf{m},\ell}|\eta^{-|\mathbf{m}|} \ge \|y^*\|_{Y^*}.$$

Thus,

$$m = \inf_{y^*\in Z^*,\ \|y^*\|_{Y^*}=1}\ \sup_{g\in K}|y^*(g)| \ge 1. \tag{2.26}$$

Next, let $g\in K$. For any multiinteger $\mathbf{m}\ge 0$, $|\mathbf{m}|\le R$, (2.13) implies that

$$\|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \le \frac{c_3}{\tau^{R+1-|\mathbf{m}|}}\|g\|_{R+1}. \tag{2.27}$$

Now, (2.23) with $R+1$ in place of r implies that

$$\eta^{|\mathbf{m}|}\|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \le \frac{c_4}{(\tau\eta)^{R+1-|\mathbf{m}|}}.$$

Since this estimate holds for all $\mathbf{m}$ with $0 \le |\mathbf{m}| \le R$, we have

$$\begin{aligned} \mathsf{dist}\ (g, \mathcal{M}_\tau) &\le \|g - \mathcal{F}_{\tau/2}(g)\|_Y \\ &= \max_{0\le|\mathbf{m}|\le R} \eta^{|\mathbf{m}|}\|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \le \frac{c_4}{\tau\eta}. \end{aligned} \tag{2.28}$$

With the choice $\alpha = 2c_4$, we conclude from (2.28) and (2.26) that (2.3) is satisfied for all $\tau \ge \alpha/\eta$. In view of Theorem 2.4 and (2.14), this completes the proof. □

Similar theorems can be obtained in a variety of other situations, where a sequence of simultaneously approximating operators is known; for example, in approximation by trigonometric polynomials [2], approximation by spherical polynomials [5], approximation by periodic basis functions [12], and approximation by Gaussian networks [8, Chapter 11.2].

3 Quasi-Interpolatory Operators

In many practical applications, although one needs to construct an approximation using point evaluations, interpolation is not necessarily desirable. For example, the data may be noisy, or too large to allow for an efficient computation of the interpolant. While the theorems in the previous section assert the existence of a bounded interpolatory operator, it is desirable to obtain explicit, computationally efficient constructions for approximations, whether they interpolate or not.

Given a sequence of subspaces $\{\mathcal{V}_n\}$ with $\mathcal{V}_n \subset \mathcal{V}_{n+1} \subset C(\Omega)$, $n = 0, 1, \cdots$, a quasi-interpolatory operator $\mathcal{T}_{n,N}$ is a linear operator that is constructed from the N data points but has properties similar to the operator $\mathcal{F}_\tau$ as in Lemma 2.6; i.e., we require that $\mathcal{T}_{n,N} : C(\Omega) \to \mathcal{V}_n$, $\|\mathcal{T}_{n,N}(f)\|_\Omega \le c\|f\|_\Omega$ for some $c > 0$ independent of n and N, and for some $\alpha > 0$, $\mathcal{V}_{\alpha n}$ is invariant under $\mathcal{T}_{n,N}$. Here, and in the sequel, the symbol $\mathcal{V}_x$ denotes the space $\mathcal{V}_{\lfloor x \rfloor}$. These assumptions imply that for any $P \in \mathcal{V}_{\alpha n}$,

$$\|f - \mathcal{T}_{n,N}(f)\|_\Omega = \|(f-P) - \mathcal{T}_{n,N}(f-P)\|_\Omega \le c\|f-P\|_\Omega,$$

and hence,

$$\|f - \mathcal{T}_{n,N}(f)\|_\Omega \le c \ \mathsf{dist}\ (f, \mathcal{V}_{\alpha n}). \tag{3.1}$$

Thus, if $\bigcup_{n=0}^\infty \mathcal{V}_n$ is dense in $C(\Omega)$, the sequence $\mathcal{T}_{n,N}(f)$ always converges to f for every $f \in C(\Omega)$, and at a near optimal rate in the sense of (3.1).

In [18], we have described a very general construction for quasi–interpolatory operators. Let (Ω, Σ) be *any* measure space. We will assume that all measures on Ω to be discussed below will be defined for all the subsets $A \in \Sigma$. Let μ^* be a sigma–finite measure on Ω, ν be a signed measure (necessarily, with bounded variation) or a positive, sigma–finite measure on Ω, $|\nu|$ denote ν if ν is a positive measure, and its total variation measure if it is a signed measure. If $A \subseteq \Omega$ is $|\nu|$-measurable, and $f : A \to \mathbb{R}$ is $|\nu|$-measurable, we write

$$\|f\|_{\nu;p,A} := \left\{ \begin{array}{ll} \left\{ \int_A |f(t)|^p d|\nu|(t) \right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ |\nu| - \text{ess sup}_{t \in A} |f(t)|, & \text{if } p = \infty. \end{array} \right.$$

The class of measurable functions f for which $\|f\|_{\nu;p,A} < \infty$ is denoted by $L^p(\nu; A)$, with the standard convention that two functions are considered equal if they are equal $|\nu|$-almost everywhere on A. We will work with a fixed orthonormal set $\{\phi_k\}_{k=0}^\infty \subset L^2(\mu^*; \Omega)$; i.e.,

$$\int_\Omega \phi_k(t)\phi_j(t) d\mu^*(t) = \left\{ \begin{array}{ll} 0, & \text{if } k \ne j, \\ 1, & \text{if } k = j, \end{array} \right. \tag{3.2}$$

and assume that it is complete in the space $L^2(\mu^*; \Omega)$. We assume also that each $\phi_k \in L^1(\mu^*; \Omega) \cap L^\infty(\mu^*; \Omega)$. We will write $\mathbb{P}_n := \ \mathsf{span}\ \{\phi_0, \cdots, \phi_n\}$, and the symbol $X^p(\Omega)$ will denote the $L^p(\mu^*; \Omega)$ – closure of $\bigcup_{n=0}^\infty \mathbb{P}_n$.

We pause in the general discussion to give a few examples.

1. $\Omega = [-1, 1]$, μ^* is the Jacobi measure $(1-x)^\alpha (1+x)^\beta dx$, $\alpha, \beta \ge -1/2$, ϕ_k is the orthonormalized Jacobi polynomial $p_k^{(\alpha,\beta)}$ of degree k with respect to μ^* .

2. $\Omega = \mathbb{R}$, μ^* is the Lebesgue measure on Ω, ϕ_k is the orthonormalized weighted Freud polynomial $\phi_k^F = w_Q p_k$, where w_Q is a Freud weight (cf. [8, Definition 3.1.1]), and p_k is the degree k orthonormal polynomial such that for all integers $k, j = 0, 1, \cdots$,

$$\int_{\mathbb{R}} p_k(x)p_j(x)w_Q^2(x)dx = \begin{cases} 0, & \text{if } k \neq j, \\ 1, & \text{if } k = j. \end{cases}$$

3. $\Omega = [-\pi, \pi]$, μ^* is the measure dx/π on Ω,

$$\{\phi_k\} = \{1/\sqrt{2}, \cos x, \sin x, \cdots, \cos kx, \sin kx, \cdots\}.$$

4. Ω is the unit sphere $\mathbb{S}^q$ embedded in the Euclidean space $\mathbb{R}^{q+1}$, μ^* is the area measure on Ω, $\{\phi_k\}$ is the set of all orthonormal spherical harmonics [19, 23].

A convenient way to define a quasi-interpolatory operator is the following. Let $h : [0, \infty) \to [0, \infty)$, $h(x) = 1$ if $x \leq 1/2$, and $h(x) = 0$ if $x > 1$. We define the kernels

$$\Phi_n(h; x, t) := \sum_{k=0}^{n} h(k/n)\phi_k(x)\phi_k(t), \tag{3.3}$$

and the operators

$$\sigma_n(\nu; h, f, x) := \int_\Omega f(t)\Phi_n(h; x, t)d\nu(t). \tag{3.4}$$

We note that in the case when ν is a discrete measure, the operator $\sigma_n(\nu; h, f)$ is defined in terms of the values of f at the points in the support of ν. In this case, we may view $\sigma_n(\nu; h, f)$ as a discretization of the "continuous" operator $\sigma_n(\mu^*; h, f)$. It is clear that

$$\|\sigma_n(\nu; h, f)\|_{\mu^*;\infty,\Omega} \leq \|f\|_{\nu;\infty,\Omega} \sup_{x\in\Omega} \int_\Omega |\Phi_n(h, x, t)|d|\nu|(t). \tag{3.5}$$

The quantity on the right hand side above is often uniformly bounded for suitable choices of h for the case when $\nu = \mu^*$. The bounds in the discrete case are then obtained using the Marcinkiewicz-Zygmund inequalities (3.6) . Formally, we say that ν is an M–Z measure of order n if

$$\|P\|_{\nu;p,\Omega} \le c\|P\|_{\mu^*;p,\Omega}, \qquad P \in \mathbb{P}_n,\ 1 \le p \le \infty. \tag{3.6}$$

The property that $\sigma_n(\nu; h, P) = P$ for $P \in \mathbb{P}_{n/2}$ is ensured if

$$\int_\Omega P_1 P_2 d\nu = \int_\Omega P_1 P_2 d\mu^*, \qquad P_1, P_2 \in \mathbb{P}_n. \tag{3.7}$$

We say that ν is a quadrature measure of order n if (3.7) holds, and that ν is an M–Z quadrature measure of order n if it is both an M–Z measure of order n and a quadrature measure of order n.

M–Z quadrature measures based on equidistant data for the case of trigonometric polynomials are given in [26, Chapter X, Theorems 7.5, 7.28]. Nevai [21, Theorem 25, p. 168] has given an example of M–Z quadrature measures for the Jacobi weights. For $m \ge 1$, let $\{x_{k,m}\}_{k=1}^m$ be the zeros of $p_m^{(\alpha,\beta)}$, and $\lambda_{k,m}$ be the corresponding Cotes' numbers. Nevai has proved that for $m \ge cn$, the measure ν_m^* that associates the mass $\lambda_{k,m}$ with each $x_{k,m}$ is an M–Z quadrature measure of order n. Similar results in the context of Freud polynomials are given in [16]. The analogues in the case of the sphere are obtained in [13] based on an arbitrary set of points. In [11], we have given an intrinsic characterization of M–Z measures on the sphere $\mathbb{S}^q$. Given a sequence of measures $\{\nu_n\}$ on $\mathbb{S}^q$, we have shown that each ν_n is an M–Z measure of order at least n if and only if for each spherical cap C,

$$|\nu_n|(C) \le c(\mu_q^*(C) + 1/n^q),$$

where μ_q^* is the surface area measure for $\mathbb{S}^q$.

The following theorem explains the construction of quasi-interpolatory operators from their "continuous" analogues using M–Z quadrature measures.

Theorem 3.1 *Let $\{\nu_n\}$ be a sequence of signed or positive sigma-finite measures, such that each ν_n is an M–Z quadrature measure of order at least n. Then $\sigma_n(\nu_n; h, P) = P$ for all $P \in \mathbb{P}_{n/2}$. For*

$f \in L^1(\mu^*;\Omega)$, $\sigma_n(\nu_n; h, f) \in \mathbb{P}_n$. Let $1 \le p \le \infty$. If

$$\sup_{n\ge 0,\ x\in\Omega} \int_\Omega |\Phi_n(h,x,t)|d\mu^*(t) =: B < \infty, \tag{3.8}$$

then

$$\|\sigma_n(\nu_n; h, f)\|_{\mu^*;p,\Omega} \le B\|f\|_{\nu_n;p,\Omega}. \tag{3.9}$$

In particular, if $\|f\|_{\nu_n;p,\Omega} \le c\|f\|_{\mu^*;p,\Omega}$ for all $f \in L^p(\mu^*;\Omega)$, then

$$\|\sigma_n(\nu_n; h, f)\|_{\mu^*;p,\Omega} \le B\|f\|_{\mu^*;p,\Omega}, \qquad f \in L^p(\mu^*;\Omega). \tag{3.10}$$

In [18], we have surveyed various conditions on the function h to ensure that (3.8) is satisfied in each of the four examples listed above. In each of these cases, a sufficient condition is that h should have sufficiently many derivatives having bounded variation. In general, the greater the smoothness of h, the more localized is the kernel. To illustrate this phenomenon, we recall that for $q \ge 1$, the cardinal B-spline of order q is the function defined by (cf. [1, p. 131])

$$\begin{aligned} M_1(x) &:= \begin{cases} 1, & \text{if } 0 < x \le 1, \\ 0, & \text{otherwise,} \end{cases} \\ M_q(x) &:= \frac{1}{q-1}\{xM_{q-1}(x) + (q-x)M_{q-1}(x-1)\}, \quad q \ge 2. \end{aligned} \tag{3.11}$$

We consider the function h defined by

$$h_q(x) = \sum_{j=-q}^{q} M_q(2qx - j), \qquad x \ge 0.$$

The function h_q is $q-2$ times continuously differentiable on $\mathbb{R}$, $h_q^{(q)}$ is a piecewise constant function, $h_q(x) = 1$ if $|x| < 1/2$, and $h(x) = 0$ if $|x| > 1$. The kernel

$$F_{q,n}(x) := 1 + \sum_{k=1}^{n} h_q(k/n)\cos kx \tag{3.12}$$

is the well-known de la Vallée Poussin kernel when $q = 2$. In Figure 1, we demonstrate the graphs of the kernels $F_{q,64}$ for the whole interval $[-\pi,\pi]$ for $q=2$ and $q=5$. The graphs of the same kernels on $[\pi/2,\pi]$ in Figure 2 clearly illustrate the effect of the smoothness of h on the localization. The mathematical details for the localization properties depend upon the special function properties of the orthogonal systems involved; we refer to [18] for further references and details.

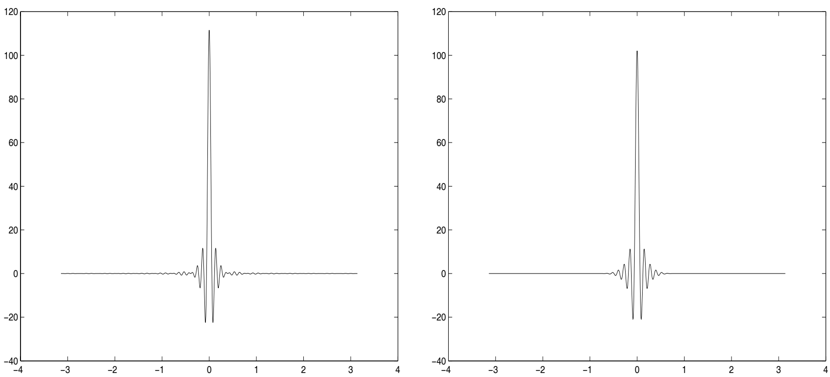


Figure 1: The graph of the de la Vallée Poussin kernel $F_{2,64}$ on the left, the graph of $F_{5,64}$ on the right.

We note that the most difficult part of these constructions is the construction of quadrature measures of high orders. It is much simpler to construct M–Z measures. For example, let $N\geq 2$ be an integer, and $-\pi\leq\theta_1<\cdots<\theta_N\leq\pi$ be points such that each subinterval of $[-\pi,\pi]$ of length $2\pi/N$ contains exactly one point θ_k. It is not difficult to check using the Bernstein inequality (cf. [17, Lemma 3.1]) that the measure ν_N that associates the mass $2\pi/N$ with each of the points θ_k satisfies for all trigonometric polynomials T of order at most n:

$$\big|\|T\|_{\nu_N;1,[-\pi,\pi]}-\|T\|_{\mu^*;1,[-\pi,\pi]}\big|\leq\frac{2\pi n}{N}\|T\|_{\mu^*;1,[-\pi,\pi]}\ . \tag{3.13}$$

In [4], we have studied the construction of quasi-interpolatory operators based on M–Z measures satisfying an inequality analogous to (3.13). The measures are constructed using scattered data on $[0,\pi]$, extended symmetrically to $[-\pi,\pi]$, but are not necessarily quadrature measures. Instead, we use orthogonal polynomials with respect

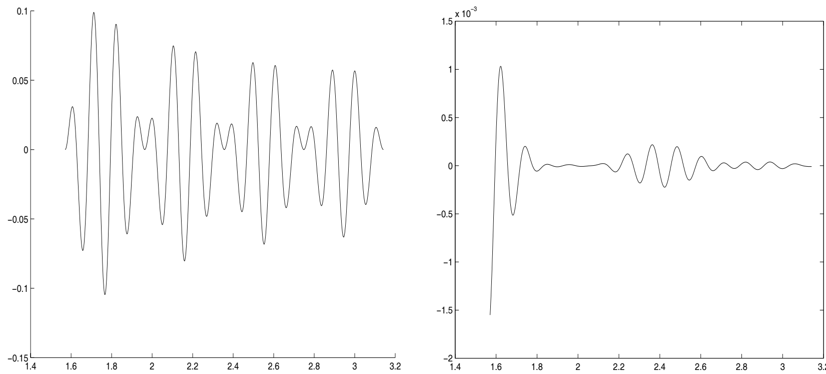


Figure 2: The graph of the de la Vallée Poussin kernel $F_{2,64}$ on $[\pi/2, \pi]$ on the left, the graph of $F_{5,64}$ on $[\pi/2, \pi]$ on the right. We note that the maximum absolute value of the graph on the right is nearly 10^{-3} times that for the graph on the left.

to the measures projected to $[-1, 1]$ in the standard way, and prove a perturbation theorem to estimate the norms of the resulting quasi–interpolatory operators. The perturbation theorem is proved in a very general setting. We apply our operators for approximation of functions on the sphere using scattered, tensor-product data.

Finally, we note that quasi–interpolatory operators have been used to prove theorems about approximation by neural networks [9], zonal function networks [14], detection of singularities ([18] and references therein), solution of pseudo–differential equations on the sphere [5], and representation of smooth functions on the sphere using finitely many bits [10].

4 Acknowledgments

The research of this author was supported, in part, by grant DMS-0204704 from the National Science Foundation and grant W911NF-04-1-0339 from the U.S. Army Research Office.

References

[1] C. de Boor, "A practical guide to splines," Springer Verlag, New York, 1978.

[2] J. Czipszer and G. Freud, Sur l'approximation d'uns fonction periodique et ses derivees successives par un polynomie trigonometrique et par ses derivees successives, Acta Math., **5** (1957), 285–290.

[3] G. Freud, "Orthogonal polynomials," Académiai Kiado, Budapest, 1971.

[4] M. Ganesh and H. N. Mhaskar, Quadrature-free quasi-interpolation on the sphere, Submitted for publication.

[5] Q. T. Le Gia and H. N. Mhaskar, Polynomial operators and local approximation of solutions of pseudo-differential equations on the sphere, Accepted for publication in Numer. Math.

[6] A. Jakimovski, A. Sharma, and J. Szabados, "Walsh equiconvergence of complex interpolating polynomials," Springer Verlag, Berlin, 2005.

[7] A. Martínez-Abejón, An elementary proof of the principle of local reflexivity, Proc. Amer. Math. Soc. **127** (1999), 1397–1398.

[8] H. N. Mhaskar, "Introduction to the theory of weighted polynomial approximation," World Scientific, Singapore, 1996.

[9] H. N. Mhaskar, Approximation theory and neural networks, in "Wavelet Analysis and Applications, Proceedings of the international workshop in Delhi, 1999" (P. K. Jain, M. Krishnan, H. N. Mhaskar, J. Prestin, and D. Singh, Eds.), Narosa Publishing, New Delhi, India, 2001, 247–289.

[10] H. N. Mhaskar, On the representation of smooth functions on the sphere using finitely many bits, Applied and Computational Harmonic Analysis **18**, Issue 3 (May 2005), pp. 215-233.

[11] H. N. Mhaskar, Weighted quadrature formulas and approximation by zonal function networks on the sphere, Accepted for publication in J. Complexity Theory.

[12] H. N. Mhaskar and C. A. Micchelli, Degree of approximation by neural and translation networks with a single hidden layer, Advances in Applied Mathematics, **16** (1995), 151-183.

[13] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comp. **70** (2001), 1113–1130 (Corrigendum: Math. Comp. **71** (2001), 453–454).

[14] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward, Approximation properties of zonal function networks using scattered data on the sphere, Advances in Computational Mathematics, **11** (1999), 121–137.

[15] H. N. Mhaskar, F. J. Narcowich, N. Sivakumar, and J. D. Ward, Approximation with interpolatory constraints, Proc. Amer. Math. Soc. **130** (2002), no. 5, 1355–1364.

[16] H. N. Mhaskar and J. Prestin, On Marcinkiewicz-Zygmund-type inequalities, in "Approximation theory: in memory of A. K. Varma" (N. K. Govil, R. N. Mohapatra, Z. Nashed, A. Sharma, and J. Szabados, Eds.), Marcel Dekker, 1998, pp. 389–404.

[17] H. N. Mhaskar and J. Prestin, On the detection of singularities of a periodic function, Adv. Comput. Math. **12** (2000), 95–131.

[18] H. N. Mhaskar and J. Prestin, Polynomial frames: a fast tour, in "Approximation Theory XI, Gatlinburg, 2004" (C. K. Chui, M. Neamtu, and L. Schumaker, Eds.), Nashboro Press, Brentwood, 2005, pp. 287–318.

[19] C. Müller, "Spherical harmonics," Lecture Notes in Mathematics, Vol. 17, Springer Verlag, Berlin, 1966.

[20] F. J. Narcowich and J. D. Ward, Scattered-data interpolation on $\mathbb{R}^n$: error estimates for radial basis and band-limited functions, SIAM J. Math. Anal. **36** (2004), no. 1, 284–300.

[21] P. Nevai, "Orthogonal polynomials," Mem. Amer. Math. Soc. 203. Providence, Amer. Math. Soc. 1979.

[22] S. M. Nikolskii, "Approximation of functions of several variables and imbedding theorems," Springer Verlag, New York, 1975.

[23] E. M. Stein and G. Weiss, "Fourier analysis on Euclidean spaces," Princeton University Press, Princeton, New Jersey, 1971.

[24] J. Szabados and P. Vértesi, "Interpolation of functions," World Scientific Publishing Co., Singapore, 1990.

[25] J. L. Walsh, "Interpolation and approximation by rational functions in the complex domain," Amer. Math. Soc. Colloq. publ. **20**, American Mathematical Society, Providence, 1960.

[26] A. Zygmund, "Trigonometric Series," Cambridge University Press, Cambridge, 1977.

Hausdorff Strong Uniqueness in Simultaneous Approximation

Devidas Pai and Indira K
Department of Mathematics
Indian Institute of Technology
Powai, Mumbai 400076, India.
Email address: dvp@math.iitb.ac.in

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Given a finite dimensional subspace V and a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of $\mathcal{C}_0(T)$, in an earlier paper, which is Part I of the paper with the same title, lower semicontinuity of the restricted center multifunction C_V from $\mathcal{F}$ into V and an intrinsic characterization of the subspace V yielding both lower semicontinuity of C_V as well as Hausdorff strongly unique best simultaneous approximation property of the triplet $(X, V, \mathcal{F})$ were investigated. In the present paper, we complement our earlier study and are mainly concerned in establishing the equivalence of Hausdorff continuity and pointwise Hausdorff Lipschitz continuity of the multifunction C_V.

1 Introduction

Given a nonempty subset V of a metric space X and a function $I : X \to (-\infty, \infty]$ which is a proper extended real-valued function, consider well-posedness of the following abstract minimization problem:

$$\min I(v),\ v \in V,$$

which we denote by (V, I). Let $v_V(I) := \inf\{I(v) : v \in V\}$ denote the *optimal value function*. We assume I to be lower bounded on V, i.e., $v_V(I) > -\infty$, and let $\arg\min_V(I)$ denote the (possibly void) set $\{v \in V : I(v) = v_V(I)\}$ of optimal solutions of problem (V, I). For $\epsilon \geq 0$, let us also denote by ϵ- $\arg\min_V(I)$ the nonempty set $\{v \in V : I(v) \leq v_V(I) + \epsilon\}$ of ϵ-approximate minimizers of I. Recall (cf., e.g., [4], p.1) that problem (V, I) is said to be **Tykhonov well-posed** if I has a unique global minimizer on V towards which every **minimizing sequence** (i.e., a sequence $\{v_n\} \subset V$, such that $I(v_n) \to v_V(I)$) converges. Put differently, there exists a point $v_0 \in V$ such that $\arg\min_V(I) = \{v_0\}$, and whenever a sequence $\{v_n\} \subset V$ is such that $Iv_n \to Iv_0$, one has $v_n \to v_0$. The concept of Tykhonov well-posedness has been extended to minimization problems admitting non-unique optimal solutions. For our purpose in this paper, the most appropriate well-posedness notion for such problems is the one introduced in Bednarczuk and Penot [2] (cf. also [4], p.26):

Problem (V, I) is called **metrically well-set** (or **M-well set**) if $\arg\min_V(I) \neq \emptyset$ and for every minimizing sequence $\{v_n\}$, one has

$$\operatorname{dist}(v_n, \arg\min_V(I)) \to 0 \text{ as } n \to \infty.$$

(Here $\operatorname{dist}(v, S)$ denotes the distance of v from the set S.) Equivalently, it is easily seen that problem (V, I) is M-well set if and only if $\arg\min_V(I) \neq \emptyset$ and the multifunction

$$\epsilon \rightrightarrows \epsilon\text{-} \arg\min_V(I)$$

is upper Hausdorff semicontinuous (uHsc) at $\epsilon = 0$. (For the relevant definition, see Section 2.) In [4], p.46, problem (V, I) is also called **stable** in this case.

Tykhonov well-posedness as well as M-well setness of problem (V, I) are conveniently characterized in terms of the notion of a firm function (or a forcing function). A function $c : T \to [0, \infty)$ is called a **firm function** or a **forcing function** if

$$0 \in T \subset [0, \infty), c(0) = 0 \text{ and } t_n \in T, c(t_n) \to 0 \Rightarrow t_n \to 0.$$

It is well known (cf., e.g., [4], p.6) that *problem (V, I) is Tykhonov well-posed if and only if there exists a firm function c and a point*

$v_0 \in V$ *such that*

$$I(v) \geq I(v_0) + c[d(v, v_0)], \quad \text{for all } v \in V.$$

Likewise, it is well known (cf. [2]) that *if I is a proper lower semicontinuous function then problem (V, I) is M-well set if and only if* $\arg\min_V(I) \neq \emptyset$ *and I is firmly conditioned, i.e., there exists a firm function c on* $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ *such that*

$$I(v) \geq v_V(I) + c(\operatorname{dist}(v, \arg\min_V(I)), \text{ for all } v \in V.$$

In the classical Chebyshev theory (cf.[13, 3, 12]) as well as in the more recent theory of best approximants in normed linear spaces, there has been a lot of interest in studying "strong unicity" of best approximants: An element $v_0 \in V$, a finite dimensional linear subspace of a normed linear space X, is called a **strongly unique best approximant** (SUBA) to x in V if there exists a constant $\lambda = \lambda(x), 0 < \lambda < 1$, such that

$$\|x - v\| \geq \|x - v_0\| + \lambda\|v - v_0\|, \text{ for all } v \in V.$$

Put differently, the strong uniqueness of a best approximant $v_0 \in V$ to x is precisely the Tykhonov well-posedness of problem (V, I_x) where $I_x(v) := \|x - v\|, v \in V$, with the associated firm function being linear: $c(t) = \lambda t,\ t \in T$. The problem (V, I_x) is also said to be **linearly conditioned** in this case.

Given a finite dimensional subspace V of a normed linear space X and $x \in X$, let us denote by $P_V(x)$ the (nonempty) set $\{v_0 \in V : \|x - v_0\| = \operatorname{dist}(x, V)\}$ of **best approximants** to x in V. In this case the multifunction $X : x \rightrightarrows P_V(x)$ of X into V is called the **metric projection multifunction.** Recall (cf., e.g., [12], p.372) that V is said to be **Chebyshev** if $P_V(x) \neq \emptyset$, for each $x \in X$. In case V is non-Chebyshev, Li [11] introduced the following definition: The metric projection multifunction $P_V : X \rightrightarrows V$ is said to be **Hausdorff strongly uniquely** at $x \in X$ if there exists a constant $\lambda(x) > 0$, such that $\|x - v\| \geq \operatorname{dist}(x, V) + \lambda(x) \operatorname{dist}(v, P_V(x))$, for all $v \in V$. Note that Hausdorff strong uniqueness of the multifunction P_V at x is precisely M-well setness of the problem (V, I_x) with the associated firm function c_x being linear: $c_x(t) = \lambda(x)t$. In this case problem (V, I_x) is also said to be **linearly conditioned**.

The problem of approximating simultaneously a *data set* in a given space by a single element of an approximating family arises naturally in many practical situations (cf., e.g., [6, 9]). One way to do this is to cover the given data set (assumed to be bounded) by a ball of minimal radius among those centered at the points of the approximating family. The problem of best simultaneous approximation in this sense coincides with problem (V, I_F), where V, a finite dimensional subspace of a normed linear space X, is the approximating family, and F, a nonempty bounded subset of X, is the data set. The objective function in this problem is $I_F : V \to \mathbb{R}$, which measures "worstness" of an element $v \in V$ as a representer of F, defined by

$$I_F(v) = r(F; v), \text{ where } r(F; v) := \sup\{\|f - v\| : f \in F\}.$$

The optimal value function $v_V(I_F)$ in this case is denoted $r_V(F)$. Thus the "intrinsic error" in the problem of approximating simultaneously all the elements $f \in F$ by the elements of V is the number $r_V(F) := \inf\{r(F; v) : v \in V\}$, called the **Chebyshev radius** of F in V. It is the minimal radius of a ball (if one such exists) centered at a point in V and covering F. The centers of all such balls are precisely the elements of the set $\arg\min_V(I_F)$ which in this case will be denoted by $C_V(F)$. A typical element of the set

$$C_V(F) := \{v_0 \in V : r(F; v_0) = r_V(F)\}$$

is called a **best simultaneous approximant** or a **restricted center** of F in V. When the bounded sets F are allowed to range over a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of X, the multifunctions $C_V : \mathcal{F} \rightrightarrows V$, with values $C_V(F), F \in \mathcal{F}$, is called the **restricted center multifunction.** Note that in case F is a singleton $\{x\}, x \in X, r_V(F)$ is the distance of x from V, denoted by $\operatorname{dist}(x, V)$, and $C_V(F)$ is precisely the set $P_V(x)$ of all best approximants to x in V.

Let $F \in \mathcal{F}$. Analogously, as in the case of a SUBA, an element $v_0 \in V$ is called a **strongly unique best simultaneous approximant (SUBSA)** to F in V if there exists a constant $\lambda = \lambda_V(F) > 0$

such that

$$r(F;v) \geq r(F;v_0) + \lambda\|v - v_0\|, \text{ for all } v \in V.$$

Likewise, in case $C_V(F)$ is not a singleton, the set F is said to admit **Hausdorff strongly unique best simultaneous approximant** (**H-SUBSA**) in V if there exists a constant $\lambda = \lambda_V(F)) > 0$ such that for all $v \in V$,

$$r(F;v) \geq r_V(F) + \lambda \operatorname{dist}(v, C_V(F)).$$

Clearly, F admits a SUBSA (resp. a H-SUBSA) in V if and only if problem (V, I_F) is Tykhonov well-posed (resp. M-well set) and linearly conditioned. The triplet $(X, V, \mathcal{F})$ is said to satisfy **property SUBSA** (resp. **property H-SUBSA**) if F admits SUBSA (resp. H-SUBSA) in V for every $F \in \mathcal{F}$.

Although uniqueness of best simultaneous approximants was studied previously in many articles (cf., e.g., [5, 8, 1, 16]), it is surprising that strong uniqueness was not treated in these articles. Apparently, in a general framework, strong uniqueness of best simultaneous approximation was explored for the first time in [10]. Triplets $(X, V, \mathcal{F})$ satisfying SUBSA and other related properties were investigated in [14], and in [15] certain well-posedness aspects of these notions were studied. More recently in [7], Hausdorff strong uniqueness of best simultaneous approximation was studied. Given a finite dimensional subspace V and a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of $\mathcal{C}_0(T)$ we mainly studied there the lower semicontinuity of the restricted center multifunction $\mathcal{C}_V : \mathcal{F} \rightrightarrows V$. We also explored in [7] an intrinsic characterization of the subspace V (called property (Li) which was introduced earlier in [11] for characterizing lower semicontinuity of the metric projection multifunction) which yields lower semicontinuity of C_V as well as Hausdorff strong uniqueness of best simultaneous approximants in V.

In the present paper which complements for the most part our study in [7], we are mainly concerned in establishing the equivalence of Hausdorff continuity and Hausdorff Lipschitz continuity of the restricted center multifunction $C_V : \mathcal{F} \rightrightarrows V$, where V is a finite dimensional subspace of $\mathcal{C}_0(T)$ and $\mathcal{F}$ is a certain family of nonempty closed and bounded subsets of $\mathcal{C}_0(T)$. This is presented in Section 3. A similar investigation in [11] for the metric projection multifunction is extended here to the restricted center multifunction. Section 2 mainly describes the preliminaries required for this purpose.

2 Preliminaries

Throughout the following, X will be a (real) normed linear space which for the most part will be the Banach space $\mathcal{C}_0(T)$, where T is a locally compact Hausdorff space, and V will be a finite dimensional subspace of X.

Let us recall that $\mathcal{C}_0(T)$ consists of all continuous function $f : T \to \mathbb{R}$ **vanishing at infinity**; i.e., a continuous function f is in $\mathcal{C}_0(T)$ if and only if for each $\epsilon > 0$, the set $\{t \in T : \|f(t)\| \geq \epsilon\}$ is compact. We endow $\mathcal{C}_0(T)$ with the norm:

$$\|f\| := \max\{|f(t)| : t \in T\}, \;\; f \in \mathcal{C}_0(T).$$

With X as a normed linear space, we denote by $CL(X)$ (resp. $CLB(X)$, resp. $K(X)$) the class of all nonempty closed (resp. nonempty closed and bounded, resp. nonempty compact) subsets of X. The **lower** (resp. **upper**) **Vietoris topology** τ_V^- (resp. τ_V^+) on $CL(X)$ is the one generated by all sets of the form $V^- := \{A \in CL(X) : A \cap V \neq \emptyset\}$ (resp. $V^+ := \{A \in CL(X) : A \subset V\}$) as V runs over all open subsets of X. If T is a topological space, by a **multifunction** $\Gamma : T \rightrightarrows X$, we mean a set-valued function from T to $CL(X)$. A multifunction $\Gamma : T \rightrightarrows X$ is said to be **lower semi-continuous** (resp. **upper semicontinuous**) abbreviated lsc (resp. usc) if it is continuous as a function from T to $CL(X)$ equipped with τ_V^- (resp. τ_V^+). The **lower** (resp. **upper**) **Hausdorff topology** τ_H^- (resp. τ_H^+) on $CL(X)$ is the one for which a neighbourhood base at $A_0 \in CL(X)$ consists of classes of the type $\{A \in CL(X) : A_0 \subset B_\epsilon(A)\}$ (resp. $\{A \in CL(X) : A \subset B_\epsilon(A_0)\}$). Here $B_\epsilon(A_0)$ denotes the set $\{x \in X : \operatorname{dist}(x, A_0) < \epsilon\}$. A multifunction $\Gamma : T \rightrightarrows X$ is said to be **upper Hausdorff semicontinuous** (resp. **lower Hausdorff semicontinuous**), abbreviated uHsc (resp. lHsc), if it is continuous as a function from T to $CL(X)$ equipped with τ_H^+ (resp. τ_H^-). It is said to be **Hausdorff continuous** if it is both uHsc as well as lHsc. For the most part, we are concerned here with $CLB(X)$ which is equipped with the Hausdorff metric H defined by

$$H(A,B) := \max\{e(A,B), e(B,A)\}, \quad A, B \in CLB(X).$$

Here $e(A,B) := \sup\{\operatorname{dist}(a,B) : a \in A\}$ denotes the *excess* of A over B. Whenever $\mathcal{F} \subset CLB(X)$, we shall regard $\mathcal{F}$ as a metric space endowed with the induced Hausdorff metric topology. We need to recall the following result (cf. [15], Theorem 4.4.8) which is easy to prove.

Theorem 2.1 *Let X be a normed linear space, V be a finite dimensional subspace of X and $\mathcal{F} \subset CLB(X)$. If the triplet $(X, V, \mathcal{F})$ satisfies property H-SUBSA, then the multifunction $C_V : \mathcal{F} \rightrightarrows V$ is pointwise Lipschitz uHsc. More precisely at each $F_0 \in \mathcal{F}$, we have*

$$e(C_V(F), C_V(F_0)) \leq 2(\lambda_V(F_0))^{-1} H(F, F_0), \text{ for all } F \in \mathcal{F}.$$

Now let $X = \mathcal{C}_0(T)$. Recall that a finite dimensional subspace V of $\mathcal{C}_0(T)$ is called a **Haar subspace** if for each $v \in V \setminus \{0\}$, card $Z(v) \leq \dim V - 1$. Here we use the notation *card*(A) to denote the cardinality of A and $Z(v)$ to denote the set of all zeros of v. Let

$$\Omega_V(X) := \{F \in CLB(X) : r_X(F) < r_V(F)\}.$$

It is convenient to restate here the following theorem from [14] which summarizes the main characteristizations of Haar subspaces of $\mathcal{C}_0(T)$ in terms of best simultaneous approximants of sets.

Theorem 2.2 *For a finite dimensional subspace V of $\mathcal{C}_0(T)$, the following statements are equivalent.*

(i) V is Haar.

(ii) The triplet $(X, V, \Omega_V(X))$ satisfies property SUBSA.

(iii) *For each $F \in \Omega_V(X), C_V(F)$ is a singleton and the multifunction $C_V : \Omega_V(X) \rightrightarrows V$ is point-wise Hausdorff Lipschitz continuous, i.e., for each $F \in \Omega_V(X)$, there exists $\beta = \beta(F) \geq 2$ such that*

$$\|C_V(F) - C_V(G)\| \leq \beta H(F, G)$$

for every $G \in \Omega_V(X)$. Here if $C_V(F)$ is a singleton and $C_V(F) = \{v_0\}$, we simply write $C_V(F)$ for the element v_0.

Furthermore, if T is a connected metric space, then all the above statements are equivalent to:

(iv) *$U_V = SU_V$, where*

$U_V := \{F \in \Omega_V(X) : F$ has a unique best simultaneous approximant in $V\}$ and

$$SU_V := \{F \in \Omega_V(X) : F \text{ has a SUBSA in } V\}.$$

Let us now recall the following extension of Haar property due to W. Li [11].

Definition 2.3 *[11] A finite dimensional subspace V of $\mathcal{C}_0(T)$ is said to satisfy* **property (Li)** *if for every $v \in V \setminus \{0\}$,*

$$card\,(bd\ Z(v)) \leq \dim\{p \in V : p|_{int\ Z(v)} = 0\} - 1.$$

(Here bd (A) (resp. int (A) denotes the boundary (resp. the interior) of set A.)

Clearly, if T is connected, property (Li) coincides with the Haar property. Li [11] has shown that the above property of V is equivalent to Hausdorff Lipschitz continuity of the metric projection multifunction $P_V : X \rightrightarrows V$. This result was extended in [7] to the restricted center multifunction as follows. Denoting by $K_V(X)$ the class of sets

$$\{F \in K(X) : r_X(F) < r_V(F)\},$$

We have:

Theorem 2.4 *[7] For a finite dimensional subspace V of $\mathcal{C}_0(T)$ the following statements are equivalent.*

(i) The multifunction $C_V : K_V(X) \rightrightarrows V$ is lsc.

(ii) V satisfies property (Li).

Let us remark that Theorem 2.3 was established in [7] for a slightly more general class of sets denoted by s-$K_V(X)$ there, which consists of sets F in $CLB(X)$ that are *sup-compact* w.r.t. V and satisfy $r_V(F) > r_X(F)$. However, throughout the sequel in this paper, we shall confine ourselves to the class $\mathcal{F} = K_V(X)$ equipped with the Hausdorff metric H.

We also need to recall here the following theorem which was established in [7]. This theorem extends to restricted center multifunction a similar result due to W. Li [11] for metric projection multifunction.

Theorem 2.5 *[7] Let V be a finite dimensional subspace of $\mathcal{C}_0(T)$. If V satisfies property (Li) then the triplet $(\mathcal{C}_0(T), V, K_V(X))$ satisfies property H-SUBSA.*

3 Equivalence of Hausdorff Continuity and Hausdorff Lipschitz Continuity of Restricted Center Multifunction in $\mathcal{C}_0$(T)

We have seen in Theorem 2.1 that if V is a finite dimensional subspace of a normed linear space X and $\mathcal{F} \subset CLB(X)$, then H-SUBSA property of the triplet $(X, V, \mathcal{F})$ entails pointwise Lipschitz upper Hausdorff semicontinuity of the restricted center multifunction C_V. Here we show that for a finite dimensional subspace V of $\mathcal{C}_0(T)$, property (Li), in fact, ensures Lipschitz continuity of the multifunction C_V. Since by Theorem 2.4, property (Li) of V yields property H-SUBSA of the triplet $(\mathcal{C}_0(T), V, K_V(X))$, which in turn, gives pointwise Lipschitz upper Hausdorff semicontinuity of C_V, it is enough to establish pointwise Lipschitz lower Hausdorff semicontinuity of the multifunction C_V in this case.

Lemma 3.1 *The restricted center multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Lipschitz continuous at $F \in K_V(X)$ if and only if there exist constants $\lambda > 0$ and $\epsilon > 0$ such that $H(C_V(F), C_V(G)) \leq \lambda H(F,G)$ for all $G \in K_V(X)$ with $H(F,G) \leq \epsilon$.*

Proof. Suppose that there exist $\lambda > 0$ and $\epsilon > 0$ such that for all $G \in K_V(X)$ with $H(F,G) \leq \epsilon$, we have

$$H(C_V(F), C_V(G)) \leq \lambda H(F,G). \tag{3.1}$$

Consider $G \in K_V(X)$ with $H(F,G) > \epsilon$. Let $f_0 \in F$ be arbitrary and pick $g_0 \in G$ such that $\sup_{g\in G} \|f_0 - g\| = \|f_0 - g_0\|$. Let $f_1 \in F$ be such that $\|f_1 - g_0\| < d(g_0, F) + \delta H(F,G)$ for a small $\delta > 0$. Then,

$$\begin{aligned} \|f_0 - g_0\| &\leq \|f_0\| + \|f_1 - g_0\| + \|f_1\| \\ &< 2\sup_{f\in F} \|f\| + H(F,G) + \delta H(F,G). \end{aligned}$$

Now for any $u \in C_V(F)$ and $v \in C_V(G)$, we have

$$\begin{aligned} \|u - v\| \leq \|u\| + \|v\| &\leq 2(\sup_{f\in F} \|f\| + \sup_{g\in G} \|g\|) \\ &\leq 2(\sup_{f\in F} \|f\| + \sup_{g\in G} \|f_0 - g\| + \|f_0\|) \\ &\leq 4\sup_{f\in F} \|f\| + 2\sup_{g\in G} \|f_0 - g\| \\ &= 4\sup_{f\in F} \|f\| + 2\|f_0 - g_0\| \\ &< 8\sup_{f\in F} \|f\| + 2(1+\delta)H(F,G). \end{aligned}$$

Therefore,

$$\frac{\|u - v\|}{H(F,G)} \leq \frac{8\sup_{f\in F} \|f\| + 2(1+\delta)H(F,G)}{H(F,G)}.$$

If $H(F,G) \geq \sup_{f\in F} \|f\|$, then $\|u - v\| \leq (10 + 2\delta)H(F,G)$.

If $H(F,G) < \sup_{f\in F}\|f\|$ and $H(F,G) > \epsilon$, then

$$\|u-v\| \le \epsilon^{-1}(10+2\delta)\left(\sup_{f\in F}\|f\|\right) H(F,G).$$

Hence,

$$\begin{aligned} &H(C_V(F), C_V(G)) \\ &\quad\le \quad \max\left\{(10+2\delta), \epsilon^{-1}(10+2\delta)\sup_{f\in F}\|f\|\right\} H(F,G). \end{aligned}$$

Since this holds for $\delta > 0$ arbitrary, we conclude that

$$H(C_V(F), C_V(G)) \le \max\{10, 10\epsilon^{-1}\sup_{f\in F}\|f\|\} H(F,G).$$

This together with (3.1) gives the required Lipschitz continuity. The converse part is obvious. □

Theorem 3.2 *Let V be a finite dimensional subspace of $\mathcal{C}_0(T)$. If V satisfies property (Li) then the restricted center multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Hausdorff Lipschitz continuous on $K_V(X)$.*

Proof. Since V satisfies property (Li) using Theorems 2.4 and 2.1, we get pointwise Lipschitz upper Hausdorff semicontinuity of the multifunction C_V on $K_V(X)$; i.e., there exists $\lambda(F) > 0$ such that for every $G \in K_V(X)$,

$$e(C_V(G), C_V(F)) \le \lambda(F) H(F,G).$$

Denoting by $\mathcal{G}_F$ the subspace span $\{v_2 - v_1 : v_1, v_2 \in C_V(F)\}$, we have for any $p \in C_V(G)$ and $v \in C_V(F)$.

$$d(p-v, \mathcal{G}_F) \le d(p, C_V(F)) \le \lambda(F) H(F,G).$$

Let $q \in \mathcal{G}_F$ be such that $\|p-v-q\| \le \lambda(F)H(F,G)$.

Let $Z(\mathcal{G}_F) := \bigcap\{Z(v_1 - v_2) : v_2, v_2 \in C_V(F)\}$ and $M = intZ(\mathcal{G}_F)$.

For any $v \in C_V(F)$, let

$$\delta(v) = r_V(F) - \max_{f\in F}\max_{t\in T\setminus M}|f(t) - v(t)|.$$

If $\delta(v) \geq (\lambda(F)+2)H(F,G)$ then we claim that $p+q \in C_V(G)$. Indeed, for any $t \in T \setminus M$ and $g \in G$,

$$\begin{aligned}
|g(t)-p(t)-q(t)| &\leq |g(t)-f(t)|+|f(t)-v(t)|+ \\
&\qquad |v(t)-p(t)-q(t)| \\
&\leq \|g-f\|+r_V(F)-(\lambda(F)+2)H(F,G) \\
&\qquad +\lambda(F)H(F,G) \\
&\leq r_V(F)-H(F,G) \leq r_V(G).
\end{aligned}$$

For $t \in M, q(t)=0$. Hence for all $g \in G$ and $t \in M$,

$$|g(t)-p(t)-q(t)| = |g(t)-p(t)| \leq r_V(G).$$

Hence, $\sup_{g \in G} \|g-p-q\| \leq r_V(G)$, i.e., $p+q \in C_V(G)$. Therefore,

$$d(v, C_V(G)) \leq \|v-p-q\| \leq \lambda(F)H(F,G).$$

Now for $v \in C_V(F)$ with $\delta(v) \leq (\lambda(F)+2)H(F,G)$ we will prove that there exists $k(F) > 0$ such that $d(v, C_V(G)) \leq k(F)H(F,G)$ whenever $H(F,G) \leq \epsilon$, for some $\epsilon > 0$.

For any fixed relative interior point v_0 of $C_V(F)$, denote by δ the value of $\delta(v_0)$. Hence $\delta > 0$. Let us take

$$\epsilon = \min\left\{\frac{\delta/4}{\alpha\nu}, \frac{\delta/4}{\alpha}\right\},$$

where $\nu = \dfrac{\sup\limits_{v \in C_V(F)} \|v-v_0\|}{\delta/2}$ and $\alpha = \lambda(F)+2$.

Now let $v \in C_V(F)$ be such that

$$\delta(v) \leq (\lambda(F)+2)H(F,G) \quad \text{for } H(F,G) \leq \epsilon.$$

Then

$$\begin{aligned}
\max_{f \in F} \max_{t \in T \setminus M} |f(t)-v(t)| &= r_V(F)-\delta(v) \\
&\geq r_V(F)-\alpha H(F,G) \\
&\geq r_V(F)-\frac{\delta}{2}.
\end{aligned}$$

Hence the set $W = \{(f,t) \in F \times (T \setminus M) : |f(t) - v(t)| \geq r_V(F) - \frac{\delta}{2}\}$ is nonempty. Also for $(f,t) \in W$, if $f(t) - v(t) \geq 0$, then

$$f(t) - v(t) \geq r_V(F) - \frac{\delta}{2} \geq f(t) - v_0(t),$$

i.e., $v_0(t) - v(t) \geq 0$. If $f(t) - v(t) \leq 0$, then

$$f(t) - v(t) \leq \frac{\delta}{2} - r_V(F) \leq f(t) - v_0(t),$$

i.e., $v_0(t) - v(t) \leq 0$. Hence

$$(f(t) - v(t))(v_0(t) - v(t)) \geq 0 \tag{3.2}$$

and

$$\begin{aligned} \|v - v_0\| &\geq |f(t) - v(t)| - |f(t) - v_0(t)| \\ &\geq r_V(F) - \frac{\delta}{2} - r_V(F) + \delta \\ &= \frac{\delta}{2}. \end{aligned} \tag{3.3}$$

Thus $\nu \geq 1$, and consequently $\epsilon = \frac{\delta/4}{\alpha\nu}$. We now define $\mu = \frac{\alpha H(F,G)}{\delta/2}$ so that $0 < \mu < 1$ and prove that $v + \mu(v_0 - v)$ is an element of $C_V(F)$ satisfying $\delta(v + \mu(v_0 - v)) \geq \alpha H(F,G)$. For $(f,t) \in W$, using (3.2) and (3.3), we have

$$\begin{aligned} |f(t) - v(t) - \mu(v_0(t) - v(t))| &= |f(t) - v(t)| - \mu|v_0(t) - v(t)| \\ &\leq r_V(F) - \frac{\mu\delta}{2} \\ &= r_V(F) - \alpha H(F,G). \end{aligned}$$

For $(f,t) \in F \times (T \setminus M)$ and $(f,t) \notin W$,

$$\begin{aligned} |f(t) - v(t) - \mu(v_0(t) - v(t))| &\leq |f(t) - v(t)| + \mu|v_0(t) - v(t)| \\ &\leq r_V(F) - \frac{\delta}{2} + \frac{\alpha H(F,G)}{\delta/2}\|v_0 - v\| \\ &\leq r_V(F) - 2\alpha\nu H(F,G) + \\ &\qquad \alpha\nu H(F,G) \\ &\leq r_V(F) - \alpha H(F,G). \end{aligned}$$

Hence, $\sup_{f\in F}\|f - v - \mu(v_0 - v)\|_{T\setminus M} \leq r_V(F) - \alpha H(F,G)$, i.e.,

$$\delta(v + \mu(v_0 - v)) \geq \alpha H(F,G).$$

Thus, $d(v + \mu(v_0 - v), C_V(G)) \leq \lambda(F)H(F,G)$. Therefore,

$$\begin{aligned} d(v, C_V(G)) &\leq \lambda(F)H(F,G) + \mu\|v_0 - v\| \\ &\leq \lambda(F)H(F,G) + \alpha\nu H(F,G). \end{aligned}$$

Hence, for $H(F,G) \leq \epsilon$,

$$d(v, C_V(G)) \leq \min\{\alpha, (\lambda(F) + \alpha\nu)\}H(F,G).$$

Now using Lemma 3.1, there exists $k(F) > 0$ such that for every $v \in C_V(F)$,

$$d(v, C_V(G)) \leq k(F)H(F,G).$$

This proves the pointwise Lipschitz lower Hausdorff semicontinuity of the multifunction C_V.

□

We note that since for a finite dimensional subspace V of $\mathcal{C}_0(T)$, the multifunction C_V is compact-valued, lower semicontinuity is equivalent to lower Hausdorff semicontinuity for C_V. Hence we can summarize Theorems 2.1, 2.3, 2.4 and 3.1 into the next theorem.

Theorem 3.3 *Let V be an n-dimensional subspace of $\mathcal{C}_0(T)$. Then the following statements are equivalent.*

(i) V satisfies property (Li).

(ii) The multifunction $C_V : K_V(X) \rightrightarrows V$ is Hausdorff continuous.

(iii) The triplet $(X, V, K_V(X))$ satisfies property H-SUBSA.

(iv) The multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Hausdorff Lipchitz continuous.

References

[1] D. Amir, Uniqueness of best simultaneous approximation and strictly interpolating subspaces, *J. Approx. Theory*, **40** (1984), 196–201.

[2] E. Bednarczuk and J. P. Penot, Metrically well-set minimization problems, *Appl. Math. Optim.* **26** (1992), 273–285.

[3] E. W. Cheney, *Introduction to Approximation Theory*, Chelsea, New York, 1982.

[4] A. L. Dontchev and T. Zolezzi, *Well-posed Optimization Problems*, Lecture Notes in Mathematics, No. 1543, Springer-Verlag, Berlin, 1993.

[5] M. Golomb, On the uniformly best approximation of functions given by incomplete data, M.R.C. Technical Summary Report #121, University of Wisconsin, Madison, 1959.

[6] R. B. Holmes, *A Course on Optimization and Best Approximation*, Lecture Notes in Mathematics 257, Springer-Verlag, Berlin, 1972.

[7] Indira K and D. Pai, Hausdorff strong uniqueness in simultaneous approximation. Part I, in *Approximation Theory XI : Gatlinburg 2004*, C.K. Chui, M. Neamtu, and L.L. Schumaker (eds.), Nashboro Press, Brentwood, TN, USA, 101–118.

[8] J. M. Lambert and D. D. Milman, Restricted Chebyshev centers of bounded sets in arbitrary Banach spaces, *J. Approx. Theory* **26** (1979), 71–78.

[9] P. J. Laurent and P. D. Tuan, Global approximation of a compact set by elements of a convex set in a normed space, *Numer. Math* **15** (1970), 137–150.

[10] P. Laurent and D. Pai, On simultaneous approximation, *Numer. Funct. Anal. and Optimiz.* **19** (1998), 1045–1064.

[11] W. Li, Strong uniqueness and Lipschitz continuity of metric projections: A generalization of the classical Haar theory, *J. Approx. Theory* **56** (1989), 164–189.

[12] H. N. Mhaskar and D. V. Pai, *Fundamentals of Approximation Theory*, CRC Press, Boca Raton, Florida, 2000.

[13] D. J. Newman and H. S. Shapiro, Some theorems on Chebyshev approximation, *Duke Math. J.* **30** (1963), 673-684.

[14] D. V. Pai, Strong uniqueness of best simultaneous approximation, *J. Indian Math. Soc.* **67** (2000), no. 1-4, 201–215.

[15] D. V. Pai and Indira K, On well-posedness of some problems in approximation theory, in *Advances in Constructive Approximation*, M. Neamtu and E. B. Saff (eds.), Nashboro Press, Brentwood, TN, 2004, 371–392.

[16] E. R. Rozema and P. W. Smith, Global approximation with bounded coefficients, *J. Approx. Theory* **16** (1976), 162–174.

Zeros of Polynomials Given as an Orthogonal Expansion

Gerhard Schmeisser
Mathematical Institute
University of Erlangen-Nuremberg
91054 Erlangen, Germany
Email address: schmeisser@mi.uni-erlangen.de

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Let $f(z) = a_0P_0(z) + \cdots + a_nP_n(z)$ be a polynomial of degree n, given as an orthogonal expansion. There exist numerous estimates for the imaginary parts of the zeros of f in terms of the coefficients $a_0, \ldots, a_n$. We describe two methods which give access to most of these results. The first method is based on a simple *invariance principle of a norm* and applies to polynomials with arbitrary complex coefficients. The second method, which we call *division by inserting a weight*, applies to polynomials with real coefficients. As applications, we establish certain L^2 inequalities and lower bounds for Vandermonde type determinants of orthogonal polynomials. The paper contains some new results as well as new proofs of known results.

1 Introduction

Polynomials were one of the favorite subjects of the late Professor Ambikeshwar Sharma. He particularly admired the work of P. Turán who became one of his close friends.

This paper deals with a topic that was initiated by P. Turán [19]. In 1950, he asked for bounds for the imaginary parts of the zeros of a polynomial f. He pointed out that the moduli of the coefficients of the standard representation (Maclaurin expansion)

$$f(z) = b_0 + b_1 z + \cdots + b_n z^n \qquad (b_n \neq 0) \tag{1.1}$$

are not appropriate quantities for obtaining good results. His discussion led him to the conclusion that the coefficients of an orthogonal expansion

$$f(z) = a_0 P_0(z) + a_1 P_1(z) + \cdots + a_n P_n(z) \qquad (a_n \neq 0) \tag{1.2}$$

will be more suitable.

Utilizing the special properties of the Hermite polynomials

$$H_n(z) := (-1)^n \mathrm{e}^{z^2} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\mathrm{e}^{-z^2}\right) \qquad (n \in \mathbb{N}_0),$$

Turán himself [19, 20, 21] obtained various results for the Hermite expansion of a polynomial. Later Specht [14, 15, 16, 17] proved several theorems for an arbitrary orthogonal expansion (1.2) with an orthogonality relation supported on the real line. Further contributions to this topic were given by Giroux [4], Gol'berg–Malozemov [5], Lajos [7], and later by the author [10, 11, 13]. In this article, we describe two general methods which allow us a systematic approach to most of these results. We also include some applications.

The paper is organized as follows. In Section 2, we recall some basic facts about orthogonal polynomials and fix our notation. In Section 3, we show that a number of known results on zeros of orthogonal expansions with arbitrary complex coefficients can be seen as consequences of a simple invariance principle of the norm induced by the orthogonality relation. In Section 4, we present a method that applies to orthogonal expansions with *real* coefficients. The idea of this method is a polynomial *division by inserting a weight* into the orthogonality relation.

The results of Sections 3 and 4 lead to upper bounds that can be expressed in terms of an L^2 norm. Furthermore, the method of inserting a weight leads to Vandermonde type determinants of orthogonal polynomials. Therefore, in Section 5, we deduce new lower bounds for L^2 norms and for Vandermonde type determinants. Some of these results are sharp.

Finally, in Section 6 we discuss variants and refinements where the role of the moduli of the imaginary parts, or equivalently, of the distances from the real line, is taken by the distances from a finite interval.

2 Notation

Now we introduce some notation and recall some facts about orthogonal polynomials referring to [1, 3, 18] for details and proofs.

Let σ be an m-distribution, that is, a non-decreasing bounded function $\sigma : \mathbb{R} \to \mathbb{R}$ which attains infinitely many distinct values and is such that the moments

$$\mu_n := \int_{-\infty}^{\infty} x^n \mathrm{d}\sigma(x) \qquad (n \in \mathbb{N}_0)$$

exist (see [3], where the term m-distribution seems to have been used for the first time). Then there exists a uniquely determined sequence of polynomials

$$P_0(z),\, P_1(z),\, \ldots\, ,\, P_n(z),\, \ldots,$$

called *the system of monic orthogonal polynomials with respect to* σ, with the following properties:

(i) each P_n is a monic polynomial of degree n;

(ii) $\int_{-\infty}^{\infty} P_m(x)P_n(x)\,\mathrm{d}\sigma(x) = 0$ for $m \neq n$.

The zeros of P_n, where $n \in \mathbb{N}$, are real and those of any two consecutive polynomials P_n and P_{n+1} interlace. Furthermore, there exists a recurrence formula

$$P_\nu(z) = (z - \alpha_\nu)P_{\nu-1}(z) - \beta_{\nu-1}P_{\nu-2}(z) \qquad (\nu = 1, 2, \ldots), \quad (2.1)$$

where $P_{-1}(z) \equiv 0$, $P_0(z) \equiv 1$, $\beta_0 := 1$, and $(\alpha_\nu)_{\nu\in\mathbb{N}}$ is a sequence of real numbers and $(\beta_\nu)_{\nu\in\mathbb{N}}$ is a sequence of positive numbers. Conversely, if a sequence of polynomials $(P_\nu)_{\nu\in\mathbb{N}_0}$ is defined by such a recurrence formula, then it is necessarily a monic orthogonal system with respect to some m-distribution σ. This result is known as

Favard's theorem [2], [1, pp. 21–22] although equivalent forms were stated earlier; see [9, p. 64] for a historical comment.

In some of our considerations, the coefficient of z^{n-1} in the polynomial P_n will be needed. We denote it by

$$q_{n-1} := \frac{P_n^{(n-1)}(0)}{(n-1)!}\,. \tag{2.2}$$

Clearly, $q_{n-1} \in \mathbb{R}$.

For arbitrary polynomials f and g, we introduce the inner product

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}\sigma(x) \tag{2.3}$$

and consider the norm

$$\|f\| := \langle f, f\rangle^{1/2} = \left(\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}\sigma(x)\right)^{1/2}. \tag{2.4}$$

In dealing with orthogonal expansions, important quantities are the numbers

$$\gamma_\nu := \|P_\nu\|^2 \qquad (\nu \in \mathbb{N}_0). \tag{2.5}$$

Let us call them the *metric constants* of the orthogonal system. If f is given by (1.2), then

$$\|f\| = \left(\sum_{\nu=0}^{n} \gamma_\nu\,|a_\nu|^2\right)^{1/2}. \tag{2.6}$$

The numbers β_ν appearing in (2.1) and the numbers (2.5) are related by the equations

$$\beta_\nu = \frac{\gamma_\nu}{\gamma_{\nu-1}} \qquad (\nu \in \mathbb{N}). \tag{2.7}$$

Further significant constants of an orthogonal system are the numbers

$$C_{m,k} := \max\left\{\frac{\|Q_k\|^2}{\gamma_k} \,:\, Q_k \text{ a monic divisor of } P_m,\ \deg Q_k = k\right\}, \tag{2.8}$$

where $k < m$. They are always greater than 1 since amongst all monic polynomials of degree k, the orthogonal polynomial P_k has smallest norm.

The following statement will be of interest. For its proof, we may refer to the considerations in [13, pp. 206–207].

Proposition 2.1 *Let $(P_\nu)_{\nu\in\mathbb{N}_0}$ be a system of monic orthogonal polynomials. Then, for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, there exists a sequence of monic orthogonal polynomials $(\widetilde{P}_\nu)_{\nu\in\mathbb{N}_0}$ with respect to some m-distribution $\widetilde{\sigma}$ such that $\widetilde{P}_\nu = P_\nu$ for $\nu = 0, \ldots, n-1$ and $\widetilde{P}_n = P_n + \alpha P_{n-1}$. Moreover, $\widetilde{\sigma}$ can be chosen such that the metric constants of the two systems are the same.*

Thus, if we take $\alpha = \Re(a_{n-1}/a_n)$, then

$$f(z) = \sum_{\nu=0}^{n-2} a_\nu P_\nu(z) + \mathrm{i}a_n \left(\Im \frac{a_{n-1}}{a_n}\right) P_{n-1}(z) + a_n \widetilde{P}_n(z)$$

is an orthogonal expansion of (1.2) for which the corresponding norm of f has decreased if $\alpha \neq 0$.

We also note that for $k < m < n$, the constants $C_{m,k}$ of the two orthogonal systems in Proposition 2.1 are the same.

3 Polynomials with Complex Coefficients

In [13], we established a unified approach to zeros of an orthogonal expansion with arbitrary complex coefficients. Using the concept of majorization [8] in conjunction with matrix theory [6], we proved the following general theorem which includes most of the former results as special cases.

Theorem 3.1 *Let f be a polynomial given as an orthogonal expansion (1.2) with complex coefficients and denote by $z_1, \ldots, z_n$ the zeros of f arranged in any order. Let ϕ be a non-decreasing convex function on $[0,\infty)$ such that $\phi(0) = 0$. Define*

$$\kappa := \Im \frac{a_{n-1}}{a_n}, \qquad I(f) := \sqrt{\kappa^2 + \sum_{\nu=0}^{n-2} \frac{\gamma_\nu}{\gamma_{n-1}} \left|\frac{a_\nu}{a_n}\right|^2} \tag{3.1}$$

and let $c = -t\kappa$ *for some* $t \in [0, 1/2]$. *Then,*

$$|\Im z_1 - c| \leq \frac{I(f) + |\kappa + 2c|}{2} \tag{3.2}$$

and

$$\begin{aligned}
&\sum_{j=1}^{k} \phi\left(|\Im z_j - c|\right) \\
&\quad \leq \phi\left(\frac{I(f) + |\kappa + 2c|}{2}\right) + \phi\left(\frac{I(f) - |\kappa + 2c|}{2}\right) \\
&\qquad + (k-2)\phi(|c|) \qquad\qquad (k = 2, \ldots, n).
\end{aligned} \tag{3.3}$$

Choosing $\phi(x) \equiv x$ and $c = -\kappa/2$, we obtain from (3.2) the following result by Specht [17, Satz 1] and, independently, by Gol'berg–Malozemov [5, Theorem 1]. It also improves upon an earlier result by Specht [14, Satz 3].

Corollary 3.2 *In the notation of Theorem 3.1, we have*

$$\left|\Im z_\nu + \frac{\kappa}{2}\right| \leq \frac{I(f)}{2} \qquad (\nu = 1, \ldots, n). \tag{3.4}$$

Next, choosing $\phi(x) \equiv x$, $c = 0$, and $k = n$, we obtain from (3.3) the following slight improvement of a result by Giroux [4].

Corollary 3.3 *In the notation of Theorem 3.1, we have*

$$\sum_{j=1}^{n} |\Im z_j| \leq I(f). \tag{3.5}$$

There is a relatively large family of polynomials for which equality is attained in (3.5).

Proposition 3.4 *In (3.5), equality occurs for all polynomials of the form*

$$f(z) = a_{n-1}P_{n-1}(z) + a_n P_n(z) \qquad (a_{n-1}, a_n \in \mathbb{C},\ a_n \neq 0) \tag{3.6}$$

and for all polynomials that can be deduced from (3.6) by replacing some of the zeros of f *by their conjugates.*

Finally, choosing $\phi(x) \equiv x^2$ and $k = n$, and noting that

$$\sum_{j=1}^{n} \Im z_j \;=\; -\kappa,$$

we obtain from (3.3) the following result by Lajos [7].

Corollary 3.5 *In the notation of Theorem 3.1, we have*

$$\sum_{j=1}^{n} \left(\Im z_j\right)^2 \;\leq\; \frac{\kappa^2 + I(f)^2}{2}\,.$$

Amongst all of these results, Corollary 3.3 appears to be the strongest as Proposition 3.4 suggests. We shall show that Corollaries 3.2 and 3.5 are very easy consequences and even Theorem 3.1 in its full generality can be deduced from Corollary 3.3 without involving any special tools apart from convexity properties. Moreover, following Giroux [4], the proof of Corollary 3.3 can be based on a simple and geometrically evident invariance principle of the norm (2.4).

Proof of Corollary 3.3. By (2.6), we have

$$\|f\|^2 \;\geq\; |a_n|^2 \left(\gamma_n + \gamma_{n-1}\left|\frac{a_{n-1}}{a_n}\right|^2\right). \tag{3.7}$$

Using the notation (2.2), we conclude from the orthogonal expansion of f that

$$f(z) \;=\; a_n z^n + \left(a_n q_{n-1} + a_{n-1}\right) z^{n-1} \;+\; \mathcal{O}(z^{n-2}) \qquad \text{as } z \to \infty.$$

Hence, by the first formula of Viete,

$$q_{n-1} + \frac{a_{n-1}}{a_n} \;=\; -\sum_{j=1}^{n} z_j, \tag{3.8}$$

and so (3.7) may be rewritten as

$$\|f\|^2 \;\geq\; |a_n|^2 \left\{\gamma_n + \gamma_{n-1}\left[\left(\Re\frac{a_{n-1}}{a_n}\right)^2 + \left(\sum_{j=1}^{n}\Im z_j\right)^2\right]\right\}. \tag{3.9}$$

Now, as is geometrically evident, for $x \in \mathbb{R}$, the value of $|f(x)|$ does not change if we replace one of the zeros of f by its conjugate. Hence $\|f\|$ is invariant under a reflection of some of the zeros on the real line and the same is true for $\Re(a_{n-1}/a_n)$. Therefore (3.9) can be strengthened to

$$\|f\|^2 \geq |a_n|^2 \left\{ \gamma_n + \gamma_{n-1} \left[\left(\Re \frac{a_{n-1}}{a_n} \right)^2 + \left(\sum_{j=1}^{n} |\Im z_j| \right)^2 \right] \right\}, \quad (3.10)$$

which is an equivalent form of (3.5). □

Proof of Proposition 3.4. Clearly, for the polynomial (3.6) equality holds in (3.7) and in (3.9). Consequently, it must also hold in (3.10). Finally, we note that equality in (3.10) is preserved under a reflection of some of the zeros of f on the real line although the orthogonal expansion of the resulting new polynomial will have more than two non-trivial terms. □

Proof of Corollary 3.2. By (3.8), we have

$$\left(2\Im z_\nu + \kappa\right)^2 = \left(2\Im z_\nu - \sum_{j=1}^{n} \Im z_j\right)^2 \leq \left(\sum_{j=1}^{n} |\Im z_j|\right)^2,$$

which shows that Corollary 3.3 implies Corollary 3.2. □

Proof of Corollary 3.5. Using (3.8), we easily verify that

$$\begin{aligned} 2\sum_{j=1}^{n} \left(\Im z_j\right)^2 - \left(\Im \frac{a_{n-1}}{a_n}\right)^2 &= 2\sum_{j=1}^{n} \left(\Im z_j\right)^2 - \left(\sum_{j=1}^{n} \Im z_j\right)^2 \\ &\leq \left(\sum_{j=1}^{n} |\Im z_j|\right)^2, \end{aligned}$$

which shows that Corollary 3.3 implies Corollary 3.5. □

For the proof of Theorem 3.1, we need an auxiliary result which may be seen as an exercise for students.

Lemma 3.6 *Let* $y_1, \ldots, y_n \in \mathbb{R}$ *and let* $Y := \frac{1}{2}(y_1 + \cdots + y_n)$. *Then*

$$\sum_{j=1}^{k} |y_j - Y| \le \sum_{j=1}^{n} |y_j| + (k-2)\,|Y| \qquad (k = 2, \ldots, n). \tag{3.11}$$

Proof. Take any $k \in \{2, \ldots, n\}$. Let N_1 be the set comprising the indices of the non-negative differences amongst $y_1 - Y, \ldots, y_k - Y$ and let $N_2 := \{1, \ldots, k\} \setminus N_1$.

If $N_1 = \emptyset$ or $N_2 = \emptyset$, then these differences all have the same sign and therefore

$$\sum_{j=1}^{k} |y_j - Y| = \left| \sum_{j=1}^{k} (y_j - Y) \right| = \left| \sum_{j=k+1}^{n} y_j + (k-2)Y \right|,$$

which implies (3.11) for the chosen k.

Now, suppose that $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. Denote by n_1 and n_2 the number of elements of N_1 and N_2, respectively. Then $|n_1 - n_2| \le k - 2$, and therefore

$$\begin{aligned}
\sum_{j=1}^{k} |y_j - Y| &= \sum_{j \in N_1} (y_j - Y) - \sum_{j \in N_2} (y_j - Y) \\
&= \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j - (n_1 - n_2)Y \\
&\le \sum_{j=1}^{k} |y_j| + (k-2)\,|Y|,
\end{aligned}$$

which shows again that (3.11) holds for the chosen k. This completes the proof. $\square$

Proof of Theorem 3.1. From (3.4), we deduce that

$$|\Im z_1| \le \frac{I(f) + |\kappa|}{2}. \tag{3.12}$$

Since $x \mapsto |\Im z_1 + x|$ is a convex function, we find by a convex combination of (3.4) for $\nu = 1$ and (3.12) that

$$\left|\Im z_1 + \lambda\frac{\kappa}{2}\right| \le \lambda\left|\Im z_1 + \frac{\kappa}{2}\right| + (1-\lambda)\left|\Im z_1\right| \le \frac{I(f) + (1-\lambda)\,|\kappa|}{2}$$

for every $\lambda \in [0,1]$. Substituting $\lambda = 2t$, we obtain (3.2).

Next, from Lemma 3.6 in conjunction with (3.5), we deduce that

$$\sum_{j=1}^{k}\left|\Im z_j + \frac{\kappa}{2}\right| \le I(f) + (k-2)\left|\frac{\kappa}{2}\right| \qquad (k = 2,\dots,n).$$

Trivially, by (3.5), we also have

$$\sum_{j=1}^{k}|\Im z_j| \le I(f) \qquad (k = 2,\dots,n).$$

Now, a convex combination of these two inequalities for $k = 2,\dots,n$ leads to the inequalities (3.3) for $\phi(x) \equiv x$.

The results obtained so far say equivalently that the vector

$$\big(\,|\Im z_1 - c|\,,\dots,|\Im z_n - c|\big)$$

is weakly majorized by the vector

$$\left(\frac{I(f) + |\kappa + 2c|}{2}, \frac{I(f) - |\kappa + 2c|}{2}, |c|\,,\dots,|c|\right) \in \mathbb{R}^n$$

(see the definition in [8, p. 10]). Since a weak majorization is preserved under application of a non-decreasing convex function (see [8, p. 116, Theorem A.2]), we obtain (3.3) in its full generality. □

4 Polynomials with Real Coefficients

One of the most perfect and flexible results on estimating the moduli of the zeros of the polynomial (1.1) in terms of the moduli of its coefficients is the theorem of van Vleck–Montel–Ballieu [9, Theorem 9.3.2]. The basic idea of its proof is a polynomial division (see

[9, p. 5] for the principle). Let $z_1, \dots, z_n$ be the zeros of f ordered as $|z_1| \le \cdots \le |z_n|$. Then f has a factorization

$$f(z) = g(z) \prod_{j=k}^{n} (z - z_j) .$$

The coefficients of the standard representation of g can be expressed in terms of $z_k, \dots, z_n$ and some of the coefficients of f. From these relations, an upper bound for $|z_k|$ can be deduced. Unfortunately, for a given orthogonal system, there is no simple relation between the zeros $z_k, \dots, z_n$ and the coefficients of the orthogonal expansions of f and g. However, in the case of polynomials with real coefficients, the can proceed as follows.

Let $z_1, \overline{z}_1, \dots, z_\ell, \overline{z}_\ell$ be ℓ pairs of conjugate zeros of f including the possibility that such a pair constitutes a real double zero. Then, setting

$$w(z) := \prod_{j=1}^{\ell} (z - z_j)(z - \overline{z}_j), \tag{4.1}$$

we have a factorization

$$f(z) := g(z) w(z). \tag{4.2}$$

Clearly,

$$\sigma^{[w]} : x \longmapsto \int_{-\infty}^{x} w(t)\, \mathrm{d}\sigma(t)$$

is also an m-distribution which defines an inner product $\langle \cdot, \cdot \rangle_w$, say. Furthermore, $\mathrm{d}\sigma^{[w]}(x) = w(x)\mathrm{d}\sigma(x)$. Now it follows from (4.2) that $\langle f, h \rangle = \langle g, h \rangle_w$ for any polynomial h. In other words, we can factor out w by letting it become a weight. In order to progress with this idea, we have to study relations between our original orthogonal system and the one with respect to $\sigma^{[w]}$.

From now on, we write $\| \cdot \|_w$ for the norm induced by $\sigma^{[w]}$. Apart from the norm and the inner product, we denote all quantities which are associated with $\sigma^{[w]}$ by attaching a superscript $[w]$ to the corresponding quantities which are associated with σ. Thus, $(P_n^{[w]})_{n \in \mathbb{N}_0}$

is the monic orthogonal system with respect to $\sigma^{[w]}$ and

$$\begin{aligned}\gamma_n^{[w]} &= \left\langle P_n^{[w]}, P_n^{[w]} \right\rangle_w = \left\| P_n^{[w]} \right\|_w^2 \\ &= \int_{-\infty}^{\infty} \left(P_n^{[w]}(x) \right)^2 w(x)\, \mathrm{d}\sigma(x) \qquad (n \in \mathbb{N}_0).\end{aligned}$$

We now state a few auxiliary results involving the weight function (4.1) or the polynomials $P_n^{[w]}$. For the first lemma, see [11, Lemma 2] or [9, Lemma 9.4.2] where a slightly different statement is proved.

Lemma 4.1 *Let g be a polynomial of degree at most n with real coefficients and let w be given by (4.1). Then the orthogonal expansion $g(z) = \sum_{\nu=0}^{n} a_\nu P_\nu(z)$ of g has a continuation to a polynomial $f(z) = \sum_{\nu=0}^{n+2\ell} a_\nu P_\nu(z)$ of degree at most $n + 2\ell$ with real coefficients such that w is a divisor of f.*

In the next lemma, it is remarkable that we have best possible bounds for $\|P_n^{[w]}\|$ which do *not* depend on w. For a reference, see [11, Lemma 3] or [9, Lemma 9.4.4]. However, for the purpose of this paper, it is desirable to have an independent proof for the upper bound (see Remark 4.6 below).

Lemma 4.2 *For w given by (4.1) and $C_{n+\ell,n}$ defined by (2.8), we have*

$$\gamma_n \le \left\| P_n^{[w]} \right\|^2 \le \gamma_n C_{n+\ell,n} \qquad (n \in \mathbb{N}_0). \tag{4.3}$$

These estimates are best possible. Equality is attained in the upper estimate when $w = (P_{n+\ell}/Q_n)^2$, where Q_n is the polynomial which achieves the maximum in the definition of $C_{n+\ell,n}$.

Proof. We only need an alternative proof for (4.3). The first inequality is a simple consequence of a well-known extremal property of orthogonal polynomials [18, Theorem 3.1.3].

Next, let $g = P_n^{[w]} - P_n$, which is a polynomial of degree at most $n - 1$ with real coefficients. By Lemma 4.1 with n replaced by

$n-1$, there exists a polynomial h of degree at most $n-1$ with real coefficients such that

$$f(z) := h(z)w(z) \;=\; \sum_{\nu=0}^{n+2\ell-1} a_\nu P_\nu(z)$$

and $P_n^{[w]}(z) - P_n(z) = \sum_{\nu=0}^{n-1} a_\nu P_\nu(z)$. Obviously, f has at most $n-1$ changes of sign on $\mathbb{R}$. Denoting by $\xi_1, \ldots, \xi_{n+\ell}$ the zeros of $P_{n+\ell}$, we find by a short reflection that there exists a monic divisor Q_n of degree n of $P_{n+\ell}$ such that

$$f(\xi_j)Q_n(\xi_j) \;\geq\; 0 \qquad (j = 1, \ldots, n+\ell);$$

see [9, proof of Theorem 9.4.3] where details of a similar argument are given. Now, denote by $\lambda_1, \ldots, \lambda_{n+\ell}$ the corresponding coefficients of the Gaussian quadrature formula with nodes $\xi_1, \ldots, \xi_{n+\ell}$. This quadrature formula is exact for polynomials up to degree $2n+2\ell-1$ and since its coefficients are positive, we infer that

$$\begin{aligned} 0 \;&\leq\; \sum_{j=1}^{n+\ell} \lambda_j f(\xi_j) Q_n(\xi_j) \;=\; \int_{-\infty}^{\infty} f(x) Q_n(x)\,\mathrm{d}\sigma(x) \\ &=\; \sum_{\nu=0}^{n+2\ell-1} a_\nu \left\langle P_\nu, Q_n \right\rangle \;=\; a_n\gamma_n + \left\langle P_n^{[w]} - P_n, Q_n \right\rangle \\ &=\; a_n\gamma_n + \left\langle P_n^{[w]} - P_n, Q_n - P_n \right\rangle . \end{aligned} \tag{4.4}$$

On the other hand, since h is of degree at most $n-1$, we have

$$\begin{aligned} 0 \;&=\; \left\langle h, P_n^{[w]} \right\rangle_w \;=\; \left\langle f, P_n^{[w]} \right\rangle \;=\; \sum_{\nu=0}^{n+2\ell-1} a_\nu \left\langle P_\nu, P_n^{[w]} \right\rangle \\ &=\; a_n\gamma_n + \left\langle P_n^{[w]} - P_n, P_n^{[w]} \right\rangle \\ &=\; a_n\gamma_n + \left\| P_n^{[w]} - P_n \right\|^2 . \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we find that

$$\left\| P_n^{[w]} - P_n \right\|^2 \;\leq\; \left\langle P_n^{[w]} - P_n, Q_n - P_n \right\rangle .$$

Applying the Cauchy–Schwarz inequality on the right-hand side, we deduce that $\|P_n^{[w]} - P_n\| \le \|Q_n - P_n\|$, which leads to $\|P_n^{[w]}\| \le \|Q_n\|$. Finally, recalling the definition of $C_{n+\ell,n}$, we readily complete the proof of (4.3). □

Obviously, if u and v are two weight functions, then $P_n^{[uv]} = (P_n^{[u]})^{[v]}$, that is, in the case of a product of weights, the factors can be inserted successively. For this technique, the following lemma will be helpful.

Lemma 4.3 *Let*

$$w_0 := 1 \qquad \textit{and} \qquad w_k(z) := \prod_{j=1}^{k} (z - z_j)(z - \overline{z}_j) \qquad (k = 1, \ldots, \ell).$$

If $|\Im z_j| \ge r \ge 0$ *for* $j = 1, \ldots, \ell$, *then*

$$\left\| P_n^{[w_k]} \right\|_{w_i}^2 \ \ge\ \left\| P_{n+1}^{[w_{i-1}]} \right\|_{w_{i-1}}^2 \ + \ r^2 \left\| P_n^{[w_k]} \right\|_{w_{i-1}}^2$$

for $1 \le i \le k \le \ell$.

Proof. Let $\xi_1, \ldots, \xi_{n+1}$ be the zeros of $P_{n+1}^{[w_{i-1}]}$ and let $\sum_{\nu=1}^{n+1} \lambda_\nu f(\xi_\nu)$ be the Gaussian quadrature formula for the integral

$$\int_{-\infty}^{\infty} f(x) w_{i-1}(x) \mathrm{d}\sigma(x).$$

This formula is exact for polynomials f up to degree $2n + 1$. In the following calculation, we apply it two times in order to conclude that

$$\begin{aligned} &\left\| P_n^{[w_k]} \right\|_{w_i}^2 \ - \ \left\| P_{n+1}^{[w_{i-1}]} \right\|_{w_{i-1}}^2 \\ &\qquad = \ \int_{-\infty}^{\infty} \left[\left(P_n^{[w_k]}(x) \right)^2 w_i(x) \ - \ \left(P_{n+1}^{[w_{i-1}]}(x) \right)^2 w_{i-1}(x) \right] \mathrm{d}\sigma(x) \\ &\qquad = \ \sum_{\nu=1}^{n+1} \lambda_\nu \left[\left(P_n^{[w_k]}(\xi_\nu) \right)^2 \frac{w_i(\xi_\nu)}{w_{i-1}(\xi_\nu)} \ - \ \left(P_{n+1}^{[w_{i-1}]}(\xi_\nu) \right)^2 \right] \end{aligned}$$

$$
\begin{aligned}
&\geq \; r^2 \sum_{\nu=1}^{n+1} \lambda_\nu \left(P_n^{[w_k]}(\xi_\nu) \right)^2 \\
&= \; r^2 \int_{-\infty}^{\infty} \left(P_n^{[w_k]}(x) \right)^2 w_{i-1}(x) \, \mathrm{d}\sigma(x) \\
&= \; r^2 \left\| P_n^{[w_k]} \right\|_{w_{i-1}}^2 .
\end{aligned}
$$

This completes the proof. □

Lemma 4.3 can be used successively for constructing a lower bound for $\gamma_n^{[w]} := \|P_n^{[w_\ell]}\|_{w_\ell}^2$. By Lemma 4.2,

$$
\left\| P_n^{[w_k]} \right\|_{w_0}^2 := \left\| P_n^{[w_k]} \right\|^2 \geq \gamma_n
$$

for $k = 0, \ldots, \ell$. Using this estimate, we can deduce from Lemma 4.3 the following result.

Lemma 4.4 *Let w be given by (4.1), and suppose that $|\Im z_j| \geq r$, where $r \geq 0$ and $j = 1, \ldots, \ell$. Then*

$$
\gamma_n^{[w]} \geq \sum_{j=0}^{\ell-1} \binom{\ell}{j} \gamma_{n+\ell-j} r^{2j} + r^{2\ell} \left\| P_n^{[w]} \right\|^2 . \tag{4.6}
$$

On the right-hand side of (4.6), all terms are non-negative. Hence the estimate remains valid if we cancel some of these terms. Moreover, by Lemma 4.2, we may replace $\|P_n^{[w]}\|^2$ by γ_n.

Now we have collected all the auxiliary results for demonstrating how the method of inserting a weight can be used for results on zeros.

Let $f(z) = \sum_{\nu=0}^{n} a_\nu P_\nu(z)$ be an othogonal expansion with real coefficients. Suppose that f has less than k distinct real zeros of odd multiplicity. Then there exists a factorization

$$
f(z) = g(z) w(z), \qquad w(z) = \prod_{j=1}^{\ell} (z - z_j)(z - \overline{z}_j), \tag{4.7}
$$

where $\ell = \lfloor (n-k)/2 \rfloor + 1$ and $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Since g will be of degree less than k, we have

$$0 = \left\langle P_k^{[w]}, g \right\rangle_w = \left\langle P_k^{[w]}, f \right\rangle = \gamma_k a_k + \sum_{\nu=0}^{k-1} a_\nu \left\langle P_k^{[w]}, P_\nu \right\rangle .$$

From this, we conclude by using the Cauchy–Schwarz inequality and Lemma 4.2 that

$$\begin{aligned} \gamma_k^2 a_k^2 &\le \sum_{\nu=0}^{k-1} \gamma_\nu a_\nu^2 \cdot \sum_{\nu=0}^{k-1} \frac{1}{\gamma_\nu} \left\langle P_k^{[w]}, P_\nu \right\rangle^2 \\ &= \sum_{\nu=0}^{k-1} \gamma_\nu a_\nu^2 \cdot \left(\left\| P_k^{[w]} \right\|^2 - \gamma_k \right) \\ &\le \sum_{\nu=0}^{k-1} \gamma_\nu a_\nu^2 \cdot \gamma_k \left(C_{k+\ell,k} - 1 \right) . \end{aligned}$$

Hence, if this inequality is violated, then f must have at least k distinct real zeros of odd multiplicities. This leads us to the following result.

Theorem 4.5 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu P_\nu(z)$ be a polynomial given as an orthogonal expansion with real coefficients. Let $1 \le k \le n$ and set $\ell := \lfloor (n-k)/2 \rfloor + 1$. If*

$$\gamma_k a_k^2 > \left(C_{k+\ell,k} - 1 \right) \sum_{\nu=0}^{k-1} \gamma_\nu a_\nu^2 , \tag{4.8}$$

then f has at least k distinct real zeros of odd multiplicities.

It can be shown that the constant $C_{k+\ell,k}$ is best possible [9, Theorem 9.4.3].

Remark 4.6 In [9, pp. 296–297], a different proof that does not use Lemma 4.2 is given and afterwards Lemma 4.2 is obtained as a consequence of Theorem 4.5.

Now we turn to a second theorem. Suppose that we have again the factorization (4.7) and let $k := n - 2\ell$ be the degree of g. Then g has an expansion $g(z) = \sum_{\nu=0}^{k} c_\nu P_\nu^{[w]}(z)$. Comparing (4.7) with the expansion of f, we infer that $c_k = a_n$. Therefore,

$$a_n \gamma_k^{[w]} = \left\langle P_k^{[w]}, g \right\rangle_w = \left\langle P_k^{[w]}, f \right\rangle = \sum_{\nu=0}^{k} a_\nu \left\langle P_k^{[w]}, P_\nu \right\rangle .$$

Using again the Cauchy–Schwarz inequality on the right-hand side, we obtain

$$\left(a_n \gamma_k^{[w]}\right)^2 \leq \sum_{\nu=0}^{k} \gamma_\nu a_\nu^2 \cdot \left\| P_k^{[w]} \right\|^2$$

and so

$$\frac{\gamma_k^{[w]}}{\sqrt{\gamma_k} \| P_k^{[w]} \|} \leq \sqrt{\sum_{\nu=0}^{k} \frac{\gamma_\nu}{\gamma_k} \left| \frac{a_\nu}{a_n} \right|^2} =: M_k(f) . \tag{4.9}$$

Now suppose that $\min_{1 \leq j \leq \ell} |\Im z_j| = r$ while $|\Im \zeta| \leq r$ for each zero ζ of g. Employing Lemmas 4.2 and 4.4, we conclude that

$$\frac{\gamma_k^{[w]}}{\sqrt{\gamma_k} \| P_k^{[w]} \|} \geq \frac{1}{\gamma_k \sqrt{C_{k+\ell,k}}} \sum_{j=0}^{\ell-1} \binom{\ell}{j} \gamma_{k+\ell-j} r^{2j} + r^{2\ell}.$$

Combining this with (4.9), we arrive at

$$\begin{aligned} \sum_{j=1}^{\ell-1} \binom{\ell}{j} \gamma_{k+\ell-j} r^{2j} &+ \gamma_k \sqrt{C_{k+\ell,k}}\, r^{2\ell} \\ &\leq \gamma_k \sqrt{C_{k+\ell,k}}\, M_k(f) - \gamma_{k+\ell} . \end{aligned} \tag{4.10}$$

It may happen that the right-hand side is negative. Then this inequality cannot be satisfied, which implies that f has at least $k+2$ distinct real zeros of odd multiplicities. However, if the right-hand side is non-negative, then equality in (4.10) defines a uniquely determined positive r which is an upper bound for the moduli of the imaginary parts of $k+2$ zeros of f. This result may be formulated as follows.

Theorem 4.7 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu P_\nu(z)$ be a polynomial of degree n given as an orthogonal expansion with real coefficients. Let k be an integer such that $n-k =: 2\ell$ is even and positive. If*

$$a_n^2 > \frac{\gamma_k C_{k+\ell,k}}{\gamma_{k+\ell}^2} \sum_{\nu=0}^{k} \gamma_\nu a_\nu^2 \,, \tag{4.11}$$

then f has at least $k+2$ distinct real zeros of odd multiplicities. Otherwise, f has at least $k+2$ zeros in the strip $\{z \in \mathbb{C} : |\Im z| \le \rho^{-1/2}\}$, where ρ is the uniquely determined positive root of the equation

$$\Gamma_k + \sum_{j=1}^{\ell-1} \binom{\ell}{j} \gamma_{k+j} x^j = \left(\Gamma_k M_k(f) - \gamma_{k+\ell}\right) x^\ell \tag{4.12}$$

with $M_k(f)$ defined by (4.9) and $\Gamma_k := \gamma_k \sqrt{C_{k+\ell,k}}$.

In (4.11), the constant $C_{k+\ell,k}$ is again best possible [9, Theorem 9.4.6]. In the case of the Hermite expansion, the criterion (4.11) for $k = n-2$ was also obtained by Turán [21, Theorem III]; however, it was not known that this criterion is best possible.

Remark 4.8 The positive root of an equation of the form (4.12) has been extensively studied in the context of the Cauchy bound [9, § 8.1]. For example, defining $A_k := \Gamma_k M_k(f) - \gamma_{k+\ell}$ and employing [9, Proposition 8.1.6], we obtain

$$\rho \ge \max\left\{ \left(\frac{\Gamma_k}{A_k}\right)^{1/\ell}, \left(\frac{\gamma_{k+1}}{A_k}\right)^{1/(\ell-1)}, \ldots, \frac{\gamma_{k+\ell-1}}{A_k} \right\}.$$

If, instead of the maximum, we just take the first term in braces as a lower bound for ρ, we obtain a result in [9, Theorem 9.4.6].

Each of Theorems 4.5 and 4.7 contains a criterion for real zeros. If this criterion fails, then, in the case of Theorem 4.5, we have to go away empty-handed, but, in the case of Theorem 4.7, we obtain at least a bound for the imaginary parts of $k+2$ zeros. This leads us to the following question.

Open Problem Can we extend Theorem 4.5 in such a way that, if (4.8) fails, the data $|a_0/a_k|, \ldots, |a_{k-1}/a_k|$ still provides a bound for the imaginary parts of k zeros?

An open problem by Turán [22, p. 70, Problem LXX] may be seen as a special case of that problem.

5 Applications

5.1 Lower bounds for $\mathbf{L^2}$ norms

The upper bounds for the zeros of f obtained in Sections 3 and 4 can be expressed in terms of $\|f\|^2$. Conversely, these results yield new lower bounds for the L^2 norm of a polynomial under side conditions on the zeros.

As a consequence of Corollary 3.3 in conjunction with Proposition 3.4, or of (3.9) directly, we have the following lower bound.

Corollary 5.1 *Let* $f(z) = \prod_{\nu=1}^{n}(z - z_\nu)$. *Then*

$$\|f\|^2 \geq \gamma_n + \gamma_{n-1}\left[\left(q_{n-1} + \sum_{\nu=1}^{n} \Re z_\nu\right)^2 + \left(\sum_{\nu=1}^{n} |\Im z_\nu|\right)^2\right].$$

Equality is attained for all polynomials of the form

$$f(z) = P_n(z) + a_{n-1}P_{n-1}(z) \qquad (a_{n-1} \in \mathbb{C}) \tag{5.1}$$

and for all polynomials that can be deduced from (5.1) by replacing some of the zeros of f by their conjugates.

The inequality $\|f\|^2 \geq \gamma_n$ is a best possible estimate for monic polynomials of degree n. Even if we restrict ourselves to polynomials with real coefficients, there is no improvement since equality occurs for $f = P_n$ only. However, P_n has n distinct real zeros. Hence, for real monic polynomials of degree n having less than n distinct real zeros there should be an improvement. For $k = n$, Theorem 4.5 implies the following result.

Corollary 5.2 *Let f be a monic polynomial of degree n with real coefficients. If f does not have n distinct real zeros, then*

$$\|f\|^2 \geq \gamma_n \frac{C_{n+1,n}}{C_{n+1,n} - 1} .$$

Unfortunately, this estimate seems to be not sharp.

Now consider the inequality $\|f\|^2 \geq \gamma_n + \gamma_{n-1} |a_{n-1}|^2$. It holds for monic polynomials of the form (1.2) and is sharp, even in the subclass of polynomials with real coefficients. In the latter case, the extremal polynomial has always n distinct real zeros as a consequence of the Hermite–Kakeya theorem [9, Theorem 6.3.8]. Hence, there should be an improvement if f has less than n distinct real zeros. The following results are consequences of Theorems 4.5 and 4.7 for $k = n-1$ and $k = n-2$, respectively. Here we assume f to be given in the standard representation since there is no need for an estimate of the norm when the coefficients of the corresponding orthogonal expansion are known. Note that in the first result, the improvement is by a multiplicative correction while in the second it is by an additive correction.

Corollary 5.3 *Let $f(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0$ be a polynomial with real coefficients. If f has less than n distinct real zeros, then the following sharp estimate holds:*

$$\|f\|^2 \geq \gamma_n + \gamma_{n-1} \left(b_{n-1} - q_{n-1}\right)^2 \frac{C_{n,n-1}}{C_{n,n-1} - 1} .$$

Corollary 5.4 *Let $f(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0$ be a polynomial with real coefficients. If f has less than n distinct real zeros, then the following sharp estimate holds:*

$$\|f\|^2 \geq \gamma_n + \gamma_{n-1} \left(b_{n-1} - q_{n-1}\right)^2 + \frac{\gamma_{n-1}^2}{\gamma_{n-2} C_{n-1,n-2}} .$$

Example To illustrate these corollaries, we may consider the Hermite expansion. In this case,

$$\|f\|^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} |f(x)|^2 \, \mathrm{d}x$$

and

$$q_{n-1} = 0, \quad \gamma_n = \sqrt{\pi}\,\frac{n!}{2^n}\,, \quad C_{n,n-1} = n \qquad (n \in \mathbb{N}),$$

which makes the above bounds concrete.

It is clear that further estimates for $\|f\|^2$ under side conditions on k of the zeros can be deduced from Theorems 4.5 and 4.7 but the results will become somewhat complicated when k is small as compared with the degree of f.

5.1.1 Lower bounds for Vandermonde type determinants of orthogonal polynomials

In Section 4, we did not make any use of the fact that the orthogonal polynomials $P_n^{[w]}$ and their metric constants $\gamma_n^{[w]}$ have explicit representations in terms of Vandermonde type determinants of orthogonal polynomials. Therefore, we can now use the results of Section 4 for establishing lower bounds for this kind of determinants.

For any $z_1, \ldots, z_k \in \mathbb{C}$, define

$$V_n(z_1, \ldots, z_k) \;:=\; \det \begin{pmatrix} P_n(z_1) & \ldots & P_{n+k-1}(z_1) \\ \vdots & & \vdots \\ P_n(z_k) & \ldots & P_{n+k-1}(z_k) \end{pmatrix}. \tag{5.2}$$

Let us now rewrite the weight function (4.1) as

$$w(z) \;=\; \prod_{j=1}^{2\ell}(z - z_j), \qquad z_{\ell+j} = \overline{z}_j \;\text{ for }\; j = 1, \ldots, \ell. \tag{5.3}$$

Then it is known [18, § 2.5] that

$$w(z)P_n^{[w]}(z) \;=\; (-1)^k\,\frac{V_n(z, z_1, \ldots, z_{2\ell})}{V_n(z_1, \ldots, z_{2\ell})} \qquad (n \in \mathbb{N}_0) \tag{5.4}$$

and [12, formula (3.2)]

$$\gamma_n^{[w]} \;=\; \gamma_n \left|\frac{V_{n+1}(z_1, \ldots, z_{2\ell})}{V_n(z_1, \ldots, z_{2\ell})}\right|. \tag{5.5}$$

In (5.4) and (5.5) we admit that some (or even all) of the points $z_1, \ldots, z_{2\ell}$ may coalesce. In that case, we define the quotients by their continuous continuation. More precisely, if $z_{j_0} = z_{j_1} = \cdots = z_{j_m}$, then we replace the polynomials in the j_1-st, j_2-nd, ..., j_m-th row in (5.2) by their first, second, ..., m-th derivative.

Lemma 4.4 with $\|P_n^{[w]}\|^2$ replaced by γ_n, which is admissible by Lemma 4.2, implies the following result.

Corollary 5.5 *Let w be given by (5.3), and suppose that $|\Im z_j| \geq r$, where $r \geq 0$ and $j = 1, \ldots, \ell$. Then*

$$\left|\frac{V_{n+1}(z_1, \ldots, z_{2\ell})}{V_n(z_1, \ldots, z_{2\ell})}\right| \geq \frac{1}{\gamma_n} \sum_{j=0}^{\ell} \binom{\ell}{j} \gamma_{n+\ell-j}\, r^{2j}. \tag{5.6}$$

Note that the right-hand side remains positive as $r \to 0$. Using (5.6) repeatedly and noting that $V_0(z_1, \ldots, z_{2\ell})$ is equal to the classical Vandermonde determinant of $z_1, \ldots, z_{2\ell}$, we can deduce a lower bound for $|V_n(z_1, \ldots, z_{2\ell})|$ itself. In fact, defining

$$\phi_n(x) := \frac{1}{\gamma_n} \sum_{j=0}^{\ell} \binom{\ell}{j} \gamma_{n+\ell-j}\, x^j \qquad (n \in \mathbb{N}_0),$$

we find that

$$|V_n(z_1, \ldots, z_{2\ell})| \geq \prod_{\nu=0}^{n-1} \phi_\nu(r^2) \cdot \prod_{1 \leq i < j \leq 2\ell} |z_j - z_i|.$$

6 Variants and Refinements

Let J_n be the smallest interval containing the zeros of P_n and let J be the smallest interval containing the support of $\mathrm{d}\sigma$. Denote by

$$d_n(z) := \min\left\{ |z - \zeta| \,:\, \zeta \in J_n \right\}$$

the distance of z from J_n and by $d(z)$ the distance of z from J. Then, by a fundamental property of the zeros of orthogonal polynomials, we have

$$d_1(z) \geq d_2(z) \geq \cdots \geq d(z) \geq |\Im z|.$$

The results of Sections 3–5 have analogues or refinements in which the role of the imaginary parts of the zeros $z_1, \ldots, z_n$ is taken by $d_m(z_1), \ldots d_m(z_n)$ for some m; see [9, Lemma 9.1.4] for a general technique. However, there is *one* difference. All the results in Sections 3 and 4 do not depend on $\Re(a_{n-1}/a_n)$. The reason is that, instead of the orthogonal system $(P_\nu)_{\nu\in\mathbb{N}}$, we can tacitly set $\alpha = \Re(a_{n-1}/a_n)$ and use the system $(\widetilde{P}_\nu)_{\nu\in\mathbb{N}}$, described in Proposition 2.1, when we are only interested in the imaginary parts of the zeros. The situation changes when we consider the distance function d_m since, in general, the interval $\widetilde{J}_m$ associated with $\widetilde{P}_m$ will be different from J_m.

A corresponding variant of (3.5) reads as [4], [9, Theorem 9.1.5]:

$$\sum_{j=1}^{n} d_n(z_j) \leq \sqrt{\sum_{\nu=0}^{n-1} \frac{\gamma_\nu}{\gamma_{n-1}} \left|\frac{a_\nu}{a_n}\right|^2}\,.$$

For a variant of Theorem 4.7 in terms of the distance function d_m, where $m = (n+k)/2 - 1$, see [11, Theorem 1]. Correspondingly, the criterion of Theorem 4.5 for the existence k real zeros guarantees that these zeros lie already in the interval J_m, where $m = \lfloor (n+k+2)/2 \rfloor$; see [9, Theorem 9.4.3].

For inequality (5.6) to hold, it is only needed that the points $z_1, \ldots, z_{2\ell}$ have distances at least r from the interval $J_{n+\ell+1}$. Moreover, there is an extension which admits an odd number of points $z_1, \ldots, z_k$ such that $\prod_{j=1}^{k}(z - z_j)$ is a polynomial with real coefficients; see [12, Theorem 1.2].

References

[1] T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.

[2] J. Favard, Sur les polynomes de Tchebycheff, C.R. Acad. Sci. Paris 200 (1935), 2052–2053.

[3] G. Freud, "Orthogonal Polynomials," Pergamon Press, Oxford, 1971.

[4] A. Giroux, Estimates for the imaginary parts of the zeros of a polynomial, Proc. Amer. Math. Soc. 44 (1974), 61–67.

[5] E. M. Gol'berg and V. N. Malozemov, Estimates for the zeros of certain polynomials, Vestnik Leningrad Univ. Math. 6 (1979), 127–135. [Transl. from Vestnik Leningrad Mat. Mekh. Astronom. (1973), No. 7, 18–24].

[6] R. A. Horn and C. A. Johnson, "Topics in Matrix Analysis," Cambridge University Press, Cambridge, 1991.

[7] L. Lajos, Imaginary part bounds on polynomial zeros, Linear Algebra Appl. 44 (1982), 173–180.

[8] A. W. Marshall and I. Olkin, "Inequalities: Theory of Majorization and Its Applications," Academic Press, San Diego, 1979.

[9] Q. I. Rahman and G. Schmeisser, "Analytic Theory of Polynomials," Oxford University Press, Oxford, 2002.

[10] G. Schmeisser, Orthogonal expansion and variations of sign of continuous functions, in "Approximation Theory: In the Memory of A. K. Varma" (N. K. Govil et al., Eds.), Chap. 28, pp. 465–476, Marcel Dekker, New York, 1998.

[11] G. Schmeisser, Orthogonal expansion of real polynomials, location of zeros, and an L^2 inequality, J. Approx. Theory 109, (2001) 126–147.

[12] G. Schmeisser, Bounds for Vandermonde type determinants of orthogonal polynomials, ETNA, Electron. Trans. Num. Anal. 14 (2002), 152–164.

[13] G. Schmeisser, Estimates for the imaginary parts of the zeros of a polynomial, Math. Balkanica 16 (2002), 203–217.

[14] W. Specht, Die Lage der Nullstellen eines Polynoms, Math. Nachr. 15 (1956), 353–374.

[15] W. Specht, Die Lage der Nullstellen eines Polynoms II, Math. Nachr. 16 (1957), 257–263.

[16] W. Specht, Die Lage der Nullstellen eines Polynoms III, Math. Nachr. 16 (1957), 369–389.

[17] W. Specht, Die Lage der Nullstellen eines Polynoms IV, Math. Nachr. 21 (1960), 201–222.

[18] G. Szegő, "Orthogonal Polynomials" (4th edn.), American Mathematical Society, Providence, R. I., 1975.

[19] P. Turán, Sur l'algèbre fonctionelle, in: "Comptes Rendus du Premier Congrès des Mathématiciens Hongrois 1950," pp. 279–290, Akadémiai Kiadó, Budapest, 1952; also see "Collected Papers of Paul Turán, Vol. 1" (P. Erdős, Ed.), pp. 677–688, Akadémiai Kiadó, Budapest, 1990.

[20] P. Turán, Hermite-expansion and strips for zeros of polynomials, Archiv Math. 5 (1954), 148–152.

[21] P. Turán, To the analytic theory of algebraic equations, Bulgar. Akad. Nauk., Otdel. Mat. Fiz. Nauk., Izvestija Mat. Institut 3 (1959), 123–137; also see "Collected Papers of Paul Turán, Vol. 2" (P. Erdős, Ed.), pp. 1080–1090, Akadémiai Kiadó, Budapest, 1990.

[22] P. Turán, On some open problems of approximation theory, J. Approximation Theory 29 (1980), 23–85.

Uniqueness of Tchebycheff Spaces and Their Ideal Relatives

Boris Shekhtman
Department of Mathematics
University of South Florida
4202 East Fowler Ave PHY 114
Tampa, FL 33620, U.S.A.
Email address: boris@math.usf.edu

To the memory of my friend and teacher Ambikeshwar Sharma

Abstract

In the first part of the paper we show that the space of polynomials of degree $n-1$ is the unique n-dimensional Tchebycheff subspace of polynomials. We also define a generalization of Tchebycheff spaces: "Ideal complements" and demonstrate their uniqueness.

In the second part we discuss various analogues of Tchebycheff spaces (minimal interpolating systems) in several variables.

Preface

I first met Professor Sharma twenty seven years ago. I was a young graduate student, my head was filled with " Bourbakisms," my Ph.D. thesis was about interpolation (in Banach spaces, of course) and I was looking forward to learn more from the renowned expert in the field. To my surprise Sharma told me that he didn't understand what a functional was and the only theorem worth knowing in analysis was the Taylor formula and may be "integration by parts" although he had his doubts about the latter. With typical modesty, he told me that he wasn't bright enough for the abstractions. The best he could do was to compute a few "right" examples and hope to get lucky.

That sent me for a spin, that lasted awhile. I tried to “compute” with Sharma only to learn that there is no way for me to keep up with his speed and accuracy. I believe that this was a lesson learned by many of my colleagues. Fortunately “Maple” came about and like “Colt 45,”equalized the playing field.

This paper is about solvability of various interpolation problems and its generalizations, the topic that benefitted greatly by many contributions of Sharma and his collaborators (cf. [4-7], [10], [12], [16-19]).

The paper is divided into two parts. The aim of the first part is to investigate the general form and uniqueness of Tchebycheff and Extended Tchebycheff subspaces as well as “ideal complements” in the spaces of polynomials. In particular we will show that the space of complex polynomials of degree at most $n-1$ is unique Tchebycheff subspace of polynomials. We also introduce a new definition of “an ideal complement” which is formally stronger than that of a Tchebycheff and Extended Tchebycheff subspace and study the form and uniqueness of ideal complements.

In the second part we discuss various generalizations of Tchebycheff subspaces and ideal complements in several variables. In section 2.1 we introduce “minimal k-interpolating spaces” as a generalization of the notion of Tchebycheff spaces and “minimal k-ideal complements.” We investigate the dimension of these spaces. In section 2.2 we introduce another possible generalization of Tchebycheff subspaces and ideal complements in several variables: namely “minimal family of k-interpolating spaces” and “minimal family of k-ideal complements.” While the investigation in section 2.1 has a distinct topological nature, the methods used in this section 2.2 are mostly of combinatorial type. Unlike the Tchebycheff spaces, their analogs in several variables have not received much attention in the literature. Therefore it is not surprising that the ratio of the number of theorems to open problems in this part is rather small.

Going back to my early years, I was convinced that the only obstacle in generalizing results in one variable is unbearable notations in several. By now I know better. Yet the original hindrance remains. Given the survey style of this article, I will take “poetic license” not to dwell on self-evident notations, hence saving trees and not trying the patience of a reader. For the same reason, I will customarily give the simple proofs of a theorem and refer to an original article for more complicated ones. As the wise man said: "a simple example explains the situation much better."

1 Interpolating Spaces in One Variable

All that being said, here are some notations:

Let $\mathbb{F}$ either be the field of real or that of the complex numbers and $\mathbb{F}[x]$ be the ring of polynomials with coefficients in $\mathbb{F}$. As such, $\mathbb{F}[x]$ is a linear space over the field $\mathbb{F}$. We use $\mathbb{F}_{<n}[x]$ to denote the space of polynomials of degree less than n; i.e.

$$\mathbb{F}_{<n}[x] := span[1, x, ..., x^{n-1}] \subset \mathbb{F}[x].$$

An n-dimensional subspace $V \subset \mathbb{F}[x]$ is called Tchebycheff (cf. [13], [14]) if

$$f \in V \text{ and } f(x_j) = 0; j = 1, ..., n$$

for a distinct set of points $\Delta := \{x_1, x_2, ..., x_n\} \subset \mathbb{F}$ implies $f = 0$. That is every non-zero $f \in V$ has at most n distinct zeroes.

Equivalently (cf. [13], [14]) an n-dimensional space $V \subset \mathbb{F}[x]$ is Tchebycheff if and only if it is interpolating:

For any distinct set of points $\Delta := \{x_1, x_2, ..., x_n\} \subset \mathbb{F}$ and any set of values $a_1, a_2, ..., a_n \in \mathbb{F}$ there exists a unique function $f \in V$ such that $f(x_j) = a_j$.

Hence an n-dimensional space $V \subset \mathbb{F}[x]$ is Tchebycheff if and only if for any distinct set of points $\Delta := \{x_1, x_2, ..., x_n\} \subset \mathbb{F}$, the space V is complemented to an ideal

$$J(\Delta) := \{f \in \mathbb{F}[x] : f(x) = 0 \text{ for all } x \in \Delta\}.$$

That is

$$\mathbb{F}[x] = V \oplus J(\Delta) \tag{1.1}$$

for any $\Delta \subset \mathbb{F}$ with cardinality $\#\Delta = n$.

An n-dimensional subspace $V \subset \mathbb{F}[x]$ is called an Extended Tchebycheff space (cf. [14], [15]) if every non-zero $f \in V$ has at most n zeroes, counting multiplicity.

Equivalently (cf. [13]) an n-dimensional space $V \subset \mathbb{F}[x]$ is Extended Tchebycheff if and only if it is Hermite interpolating:

For any distinct set of $m \le n$ points $\Delta := \{x_1, x_2, ..., x_m\} \subset \mathbb{F}$, any set of integers $\mathfrak{N}(m,n) = \{n_1, ...n_m\}$ with $\sum_{j=1}^{m}(n_j - 1) = n$ and any set of n values $\{a_1^{(k_1)}, a_2^{(k_2)}, ..., a_m^{(k_m)} : k_j = 0, ..., n_j - 1\} \subset \mathbb{F}$ there exists a unique function $f \in V$ such that

$$f^{(k_j)}(x_j) = a_j^{(k_j)};$$

where $f^{(k)}$ denotes the k-th derivative of f.

Hence an n-dimensional space $V \subset \mathbb{F}[x]$ is Extended Tchebycheff if and only if for any distinct set of $m \le n$ points $\Delta := \{x_1, x_2, ..., x_m\} \subset \mathbb{F}$ any set of integers $\mathfrak{N}(m,n) = \{n_1, ...n_m\}$ with $\sum_{j=1}^{m}(n_j - 1) = n$, the space V is complemented to an ideal

$$J(\Delta, \mathfrak{N}) := \{f \in \mathbb{F}[x] : f^{(k_j)}(x_j) = 0 \ ; j = 1, ..., m; k_j = 0, ..., n_j\}.$$

That is

$$\mathbb{F}[x] = V \oplus J(\Delta, \mathfrak{N}). \tag{1.2}$$

The reformulations (1.1) and (1.2) of the definitions of Tchebycheff and Extended Tchebycheff spaces motivate the definition of Ideal Complements as the n-dimensional spaces $V \subset \mathbb{F}[x]$ which are complemented to every ideal $J \subset \mathbb{F}[x]$ of codimension n.

An ideal $J \subset \mathbb{F}[x]$ is a subspace of $\mathbb{F}[x]$ such that

$$f \in \mathbb{F}[x], g \in J \Longrightarrow fg \in J.$$

Let $\mathfrak{J}$ be the set of all ideals in $\mathbb{F}[x]$ and let $\mathfrak{J}_n \subset \mathfrak{J}$ be the set of all ideals of codimension n.

Definition 1.1 *An n-dimensional space $V \subset \mathbb{F}[x]$ is called an ideal complement if*

$$\mathbb{F}[x] = V \oplus J \tag{1.3}$$

for every ideal $J \in \mathfrak{J}_n$

Clearly every ideal complement is an Extended Tchebycheff space and every Extended Tchebycheff space is Tchebycheff.

Since $\mathbb{F}[x]$ is a principle ideal domain (cf. [1]),

$$J \in \mathfrak{J} \text{ iff } J = p\mathbb{F}[x] \tag{1.4}$$

for some polynomial $p \in \mathbb{F}[x]$.

Theorem 1.2 *We have*

(1) If p is a polynomial of degree n. Then $p\mathbb{F}[x] \in \mathfrak{J}_n$ and

$$\mathbb{F}[x] = \mathbb{F}_{<n}[x] \oplus (p\mathbb{F}[x]).$$

In particular $\mathbb{F}_{<n}[x]$ is an ideal complement.

(2) An ideal $J \in \mathfrak{J}_n$ iff $J = p\mathbb{F}[x]$ for some $p \in \mathbb{F}[x]$ with $\deg p = n$. In particular every ideal $J \in \mathfrak{J}$ is of finite codimension.

PROOF. If p is a polynomial of degree n, then every non-zero polynomial in $p\mathbb{F}[x]$ has degree at least n. Hence $\mathbb{F}_{<n}[x] \cap (p\mathbb{F}[x]) = \{0\}$. On the other hand every $f \in \mathbb{F}[x]$ can be written as $f = pq + r$ with $\deg r < n$. That proves the first part of the theorem. It also shows that $p\mathbb{F}[x]$ is an ideal of codimension n. To verify the rest of (2), assume that $J \in \mathfrak{J}_n$. Then there exists a polynomial q such that $J = q\mathbb{F}[x]$. If $\deg q \neq n$ then, by part (1), $codim J \neq n$ which gives the contradiction. $\square$

The last theorem shows that $\mathbb{F}_{<n}[x]$ is an ideal complement. In particular $\mathbb{F}_{<n}[x]$ is an Extended Tchebycheff space. Of course this is nothing new, except that the division algorithm used in the proof of the theorem did not employ any determinants or complicated construction of basic polynomials!

In the next section we show $\mathbb{C}_{<n}[x]$ is the unique n-dimensional Tchebycheff subspace in $\mathbb{C}[x]$ and therefore it is the unique ideal complement in $\mathbb{C}[x]$. In particular, Tchebycheff spaces, Extended Tchebycheff spaces and ideal complements coincide.

In section 3, we show that for $n > 1$ the space $\mathbb{R}_{<n}[x]$ is the unique ideal complement in $\mathbb{R}[x]$ but not a unique Tchebycheff or Extended Tchebycheff subspace.

1.1 Complex Case

We start with the quick corollary of Theorem 1.2:

Theorem 1.3 *The space $V = \mathbb{C}_{<n}[x]$ is the unique subspace of $\mathbb{C}[x]$ which complements every $J \in \mathfrak{J}_n$.*

PROOF. Let $V \neq \mathbb{C}_{<n}[x]$ be an n-dimensional subspace of $\mathbb{C}[x]$. Then V contains a polynomial q of degree $\geq n$. Hence $q = pf$ with $\deg p = n$. Let $J = p\mathbb{C}[x]$. By proposition $J \in \mathfrak{J}_n$ and $q \in V \cap J$. Thus V is not complemented to J. □

For the Tchebycheff spaces we have:

Theorem 1.4 *The space $V = \mathbb{C}_{<n}[x]$ is the unique n-dimensional Tchebycheff subspace of $\mathbb{C}[x]$.*

PROOF. Suppose that $V \neq \mathbb{C}_{\leq n}[x]$ is an n-dimensional Tchebycheff subspace of $\mathbb{C}[x]$. Then V contains a polynomial f with $\deg f \geq n$. Since V is Tchebycheff, f has at most $n-1$ zeroes: $\xi_1, ..., \xi_k$ with $k < n$. Once again, since V is Tchebycheff, there exists a polynomial $g \in V$, such that $g(\xi_j) = 1$ for all $j = 1, ..., k$. Hence f and g are relative primes and $(\frac{f}{g})'$ is different from 0. For every $c \in \mathbb{C}$, consider a new polynomial $q(c, x) = f(x) - cg(x)$. We now claim that for all, but a finite many values of $c \in \mathbb{C}$, the polynomial $q(c, x)$ has only simple zeroes. Indeed let $\zeta_1, ..., \zeta_N$ be all the zeroes of the polynomial $fg' - gf'$ and assume that

$$c \neq \frac{f(\zeta_j)}{g(\zeta_j)} \tag{1.5}$$

for those ζ_j, for which $g(\zeta_j) \neq 0$. Then if x_0 is a multiple root of $f(x) - cg(x)$, we have

$$\begin{cases} f(x_0) - cg(x_0) = 0 \\ f'(x_0) - cg'(x_0) = 0 \end{cases}.$$

Since f and g are relative primes, $g(x_0) \neq 0$. From the first of the equations above, we obtain $c = \frac{f(x_0)}{g(x_0)}$ and substituting it into the second equation, we have $f(x_0)g'(x_0) - g(x_0)f'(x_0) = 0$, which contradicts (1.5). □

Corollary 1.5 *The space $V = \mathbb{C}_{<n}[x]$ is the unique n-dimensional Extended Tchebycheff subspace of $\mathbb{C}[x]$.*

1.2 Real Case

Real ideal complements have almost the same description as complex ideal complements.

Theorem 1.6 *Let V be an n-dimensional ideal complement. If $n > 1$ then $V = \mathbb{R}_{<n}[x]$. If $n = 1$, then V is the set of constant multiples of any strictly positive polynomial; $V = span\{p\}$, $p \in \mathbb{R}[x]$ and $p > 0$.*

Proof. Let $n > 1$ and let $\xi_1, ..., \xi_n$ be distinct points in $\mathbb{R}$. Since V is an ideal complement, in particular it is a Tchebycheff subspace. Hence there are n polynomials $p_1, ..., p_n \in V$ such that $p_k(\xi_j) = \delta_{j,k}$. These polynomials are linearly independent and thus span the space V. If $\max\{\deg p : p \in V\} \geq n$ then at least one of the polynomials, say p_1 has degree greater then $n - 1$. Since p_1 has a linear factor, it follows that p_1 has a factor of degree n and hence V is not an ideal complement. The case $n = 1$ is trivial. □

Corollary 1.7 *For $n = 1$ an n-dimensional space V is Tchebycheff if and only if it is an ideal complement. For $n > 1$ there exists an n-dimensional Tchebycheff subspace of $\mathbb{R}[x]$ which is not an ideal complement.*

Proof. Any subspace $V \subset \mathbb{R}[x]$ which is of the form

$$V = r(x)\mathbb{R}_{<n}[s(x)] := span\{r(x), r(x)s(x), ..., r(x)s^{n-1}(x)\} \quad (1.6)$$

where $r(x)$ is a strictly positive polynomial in $\mathbb{R}[x]$ and s is an injective polynomial mapping from $\mathbb{R} \to \mathbb{R}$ is clearly a Tchebycheff space. Yet if $\deg r > 0$ then, as follows from the previous Theorem, it is not an ideal complement. □

This argument leads to the reasonable possibility that a subspace $V \subset \mathbb{R}[x]$ is Tchebycheff if and only if $V = r(x)\mathbb{R}_{<n}[s(x)]$ for some strictly positive polynomial $r(x)$ and an injective polynomial mapping s from $\mathbb{R} \to \mathbb{R}$.

This is clearly true for $n = 1$.

Unfortunately this is not so for $n > 1$. Indeed here is a counterexample:

Let

$$V := span\{1 + x^2, x^3\}. \tag{1.7}$$

We have

$$\begin{aligned} \det \begin{bmatrix} 1+x^2 & x^3 \\ 1+a^2 & a^3 \end{bmatrix} &= -(x-a)\left(x^2a^2 + x^2 + xa + a^2\right) \\ &= -(x-a)\,(x^2a^2 + \frac{1}{2}(x^2 + a^2 + (x+a)^2)) \end{aligned}$$

which is equal to zero if and only if $x = a$. Hence V is a Tchebycheff space that is not of the form (1.6).

Furthermore

$$\det \begin{bmatrix} 1+x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} = 3x^2 + x^4 = 0 \text{ if } x = 0.$$

Hence the Tchebycheff space V defined by (1.7) is not an Extended Tchebycheff space.

On the other hand the space

$$V = span\{x^2 + 1, x^3 + 2x\}$$

is an Extended Tchebycheff space. Indeed

$$\det \begin{bmatrix} x^2+1 & x^3+2x \\ 2x & 3x^2+2 \end{bmatrix} = x^4 + x^2 + 1 > 0.$$

And if $x \neq a$ then

$$\begin{aligned} \det \begin{bmatrix} x^2+1 & x^3+2x \\ a^2+1 & a^3+2a \end{bmatrix} &= -(x-a)\left(x^2a^2 + x^2 - xa + a^2 + 2\right) \\ &= -(x-a)\,(\frac{1}{4}\,(2x-a)^2 + x^2a^2 + \frac{3}{4}a^2 + 2) \\ &\neq 0. \end{aligned}$$

Conclusion 1.8 *In* $\mathbb{R}[x]$ *there are Tchebycheff spaces which are not Extended Tchebycheff spaces and there are Extended Tchebycheff spaces which are not of the form (1.6).*

Problem 1.9 *What is the general form of Tchebycheff spaces in* $\mathbb{R}[x]$*? What is the general form of Extended Tchebycheff spaces in* $\mathbb{R}[x]$*?*

Now suppose that $1 \in V$ and V is a Tchebycheff space. Does that imply that

$$V = \mathbb{R}_{<n}[s(x)] \ ? \tag{1.8}$$

For $n = 2$ it is so. Indeed if $V = span\{1, s(x)\}$ then s is strictly monotone and hence an injection.

Our next example shows that (1.8) fails for $n = 3$:

Let $V = span\{1, x, x^4\}$. Then

$$\det \begin{bmatrix} 1 & x & x^4 \\ 1 & a & a^4 \\ 1 & b & b^4 \end{bmatrix} = ab^4 - a^4b - xb^4 + x^4b + xa^4 - x^4a$$

$$= -\frac{1}{2}(-b+a)(x-b)(x-a)\left((x^2+a^2+b^2)+(x+a+b)^2\right)$$

Hence $span\{1, x, x^4\}$ is a Tchebycheff space that is not an ideal complement.

2 Interpolation Systems in Several Variables

Let $\mathbb{F}[x_1, ..., x_d] = \mathbb{F}[\mathbf{x}]$ be the ring of polynomials of d variables and let $\mathbb{F}_{\leq m}[\mathbf{x}]$ be the space of polynomials of degree at most m. For an ideal $J \subset \mathbb{F}[\mathbf{x}]$ we define

$$Z(J) := \{\mathbf{x} \in \mathbb{F}^d : f(x) = 0 \text{ for all } f \in J\}.$$

If an ideal $J \subset \mathbb{F}$ is generated by polynomials f_1, $f_2 ..., f_n$, we use the standard notation:

$$J = < f_1, f_2 ..., f_n > = < f_j : j = 1, ..., n > .$$

Let, once again, $\mathfrak{J}_n$ denotes the family of ideals of codimension n.

For ideal $J \in \mathfrak{J}_n$, the set $Z(J)$ is finite and moreover

$$\#Z(J) \leq n$$

An ideal $J \subset \mathbb{F}$ is called radical if $f^m \in J$ implies $f \in J$. It is well known (cf. [9]) that an ideal $J \in \mathfrak{J}_n$ is radical if and only if $\#Z(J) = n$.

In several variables there are no Tchebycheff subspaces and therefore there are no ideal complements. For the real field this follows from extremely cute "Mairhuber argument" (cf [15]):

Let $V = span[f_1, f_2, ..., f_n]$. And let $\Delta = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n\}$ be distinct points in $\mathbb{R}^d$ with $d \geq 2$. Position two points $\mathbf{x}_1, \mathbf{x}_2$ on diametrically opposite ends of the unit circle and points $\mathbf{x}_3, ..., \mathbf{x}_n$ outside the circle. If the space V is Tchebycheff, that implies that the determinant

$$D(\Delta) = \det [f_k(\mathbf{x}_j)] \neq 0$$

for any Δ. As we rotate the diameter, the points $\mathbf{x}_1$ and $\mathbf{x}_2$ switch positions and hence $D(\Delta)$ changes sign. By the intermediate value theorem, there exists a pair $\mathbf{x}_1, \mathbf{x}_2$ such that $D(\Delta) = 0$; hence V is not interpolating at these points.

In the absence of an intermediate value theorem, the complex case utilizes different tools. Since this article deals with polynomials, we present an argument based on the attributes from algebraic geometry:

Let

$$Z := \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n) \in \mathbb{C}^{n \cdot d} : D(\Delta) = \det [f_k(\mathbf{x}_j)] = 0\}.$$

Let

$$U_{j,k} := \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n) \in \mathbb{C}^{n \cdot d} : \mathbf{x}_j = \mathbf{x}_k\} \text{ and } U := \bigcup_{j \neq k} U_{j,k}$$

Since Z is the set of solutions of one equation $D(\Delta) = 0$ in $\mathbb{C}^{nd}$, Z is an algebraic variety of codimension one, thus $\dim Z = nd - 1$.

Each $U_{j,k}$ is the zero locus of d equations: $\mathbf{x}_j = \mathbf{x}_k$, and hence it is a variety of codimension d. We conclude that for $d > 1$:

$$\dim U = \max \dim U_{j,k} = nd - d < nd - 1 = \dim Z.$$

Hence there exists an n-tuple $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n) \in Z$ which is not in U. Thus the equation

$$D(\Delta) = \det [f_k(\mathbf{x}_j)] = 0$$

has a solution for some set Δ of distinct points in $\mathbb{C}^d$, which implies that V is not a Tchebycheff space. In the absence of Tchebycheff Spaces in several variables, we have to give something up. We propose two possible analogues of Tchebycheff spaces.

2.1 Interpolating Spaces

Definition 2.1 *A subspace $V \subset \mathbb{F}[\mathbf{x}]$ is called k-interpolating if for every k distinct points $\mathbf{x}_1, ..., \mathbf{x}_k$ in $\mathbb{F}^d$ and for every distinct values $\alpha_1, ..., \alpha_k$ there exists $f \in V$ such that*

$$f(\mathbf{x}_j) = \alpha_j; j = 1, ..., k.$$

Clearly if V is k-interpolating then $\dim V \geq k$. If $\dim V = k$ then V is k-interpolating if and only if V is Tchebycheff. As we mentioned earlier for $d > 1$ Tchebycheff spaces do not exist. The k-interpolating spaces do exist. That means that we give up uniqueness of the interpolating function $f \in V$ but still insist on the existence of one. However we do not want to abandon uniqueness all together. One way of doing so is to ask for a k-interpolating space of minimal dimension. Therefore the problem in several variables can be reformulated as follows:

Problem 2.2 *What is the minimal dimension of k-interpolating spaces in $\mathbb{F}[\mathbf{x}]$? What are the k-interpolating subspaces of $\mathbb{F}[\mathbf{x}]$ of minimal dimension?*

Just as in the last section, we can observe that a space V is k-interpolating if and only if for every radical ideal $J \in \mathbb{F}[\mathbf{x}]$ of codimension k there exists a subspace $E \subset V$ such that

$$E \oplus J = \mathbb{F}[\mathbf{x}].$$

Hence it seems natural to extend this definition to all ideals.

Definition 2.3 *A space $V \subset \mathbb{F}[\mathbf{x}]$ is called a $k-$ideal complement if for every ideal J of codimension k there exists a subspace $E \subset V$ such that*

$$E \oplus J = \mathbb{F}[\mathbf{x}].$$

Once again we have the problem:

Problem 2.4 *What is the minimal dimension of a $k-$ideal complement? What are the k-ideal complements of minimal dimension? Are the minimal k-ideal complements unique? Do the minimal k-ideal complements coincide with the minimal k-interpolating spaces?*

There are some results (cf. [8], [22], [23], [24], [25]) concerning the minimal dimension of k-interpolating subspaces in $\mathbb{R}[\mathbf{x}]$. The most stunning of these is due to F. Cohen and D. Handel [8] (cf. also [24]):

Theorem 2.5 *Let $a(k)$ be the minimal dimension of k-interpolating subspaces in $\mathbb{R}[x, y]$. Then*

$$2k - \eta(k) \leq a(k) \leq 2k - 1,$$

where $\eta(k)$ is the number of 1s in the binary representation of the integer n.

In fact for $k = 3$ the value $a(3) = 4$ as the "unnatural" appearance of $\eta(k)$ in the lower bound would predict (cf[22]). A minimal 3-interpolating subspace is spanned by polynomials $\{1, x, y, x^2 + y^2\}$. For $k = 4$ the lower and the upper bounds coincide. Hence $a(4) = 7$. A minimal 4-interpolating subspace is spanned by polynomials $\{1, x, y, x^2 - y^2, xy, x^3 - 3xy^2, y^3 - 3x^2y\}$.

To the best of my knowledge, the exact value for $a(5)$ is not known. The span of the first $2k - 1$ harmonic polynomials always forms a k-interpolating subspace in $\mathbb{R}[x, y]$. For $d > 2$, the only reasonable bound known to me (cf. [22], [23]) is

$$\frac{1}{2}(d + 1)k \leq a(k) \leq d(k + 1).$$

As far as I know there are no results on minimal k-dimensional interpolating subspaces in the complex case. The ideal complements of minimal dimension have not been studied in either field.

It follows from the Theorem 2.9 mentioned in the next section, that the space $\mathbb{F}_{<k}[\mathbf{x}]$ is a $k-$ideal complement.

Now the standard transversality argument (cf. [21], [23]) gives us a better estimate:

Theorem 2.6 *There exists a k-ideal complement $V \subset \mathbb{F}_{<k}[\mathbf{x}]$ with* $\dim V \leq (d+1)k$.

2.2 Interpolating Families in Several Variables

As we mentioned in the previous section, the four-dimensional space spanned by polynomials $\{1, x, y, x^2+y^2\}$ is 3-interpolating in $\mathbb{R}[x, y]$. Indeed if we have three points $u_1, u_2, u_3 \in \mathbb{R}^2$ that do not lie on the same line, then the three-dimensional space spanned by $\{1, x, y\}$ interpolates at those points. On the other hand if three distinct points $u_1, u_2, u_3 \in \mathbb{R}^2$ do lie on the same line, then either the space spanned by $\{1, x, x^2+y^2\}$ or by $\{1, y, x^2+y^2\}$ interpolate at those points. In other words in order to accomplish the interpolation at arbitrary three points, we do not need all (infinitely many) three-dimensional subspaces of $span\{1, x, y, x^2+y^2\}$. It is sufficient to consider three of them:

$$span\{1, x, y\}, span\{1, x, x^2+y^2\} \text{ and } span\{1, y, x^2+y^2\}.$$

This consideration prompts the following definition:

Definition 2.7 *A family $\mathcal{F}$ of k-dimensional subspaces of $\mathbb{F}[\mathbf{x}]$ is called a family of $k-$ideal complements, if for every ideal $J \subset \mathbb{F}[\mathbf{x}]$ of codimension k there exists a subspace $E \in \mathcal{F}$ such that*

$$E \oplus J = \mathbb{F}[\mathbf{x}].$$

A family $\mathcal{F}$ of k-dimensional subspaces of $\mathbb{F}[\mathbf{x}]$ is called a $k-$interpolating family if for every radical ideal $J \subset \mathbb{F}[\mathbf{x}]$ of codimension k there exists a subspace $E \in \mathcal{F}$ such that

$$E \oplus J = \mathbb{F}[\mathbf{x}].$$

With these definitions come apparent open questions:

Problem 2.8 *What is the minimal number of subspaces in a family of $k-$ideal complements? What is the minimal number of subspaces in a $k-$interpolating family?*

A subspace $V \subset \mathbb{F}[x_1, ..., x_d] = \mathbb{F}[\mathbf{x}]$ is called D-invariant if

$$f \in V \Longrightarrow \frac{\partial}{\partial x_j} f \in V, \forall j = 1, ..., d.$$

The next theorem was first proved in [11]. The introduction of Groebner bases made it a simple theorem (cf [3]):

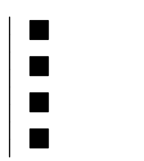
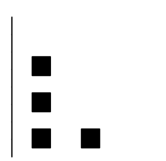
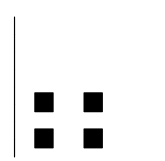
Theorem 2.9 *For every $J \in \mathfrak{J}_n$ there exists a D-invariant subspace $V \subset \mathbb{F}[\mathbf{x}]$ spanned by monomials, such that*

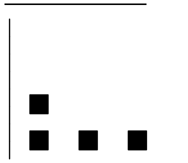
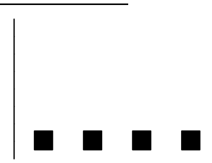
$$V \oplus J = \mathbb{F}[\mathbf{x}].$$

A moment of reflection on D-invariance and monomial nature of this space leads to the conclusion that every such space is a subspace of $\mathbb{F}_{<n}[\mathbf{x}]$ and since there are only finitely many monomials in $\mathbb{F}_{<n}[\mathbf{x}]$, there are only finitely many such spaces.

Corollary 2.10 *There exist a finite $k-$ideal family.*

It is convenient to use Young tables to visualize such subspaces. For instance for $n = 4$ the five subspaces in question are given by tables (staircases):

$\Gamma_1 =$  $\Gamma_2 =$  $\Gamma_3 =$ 

$\Gamma_4 =$  $\Gamma_5 =$ 

These five tables represent all possible D-invariant complements to ideals in $\mathfrak{J}_4$. Thinking of the vertical axes as the number of monomials in y, we can write all five gammas as

$$\begin{aligned}\Gamma_1 &= [1, y, y^2, y^3], \Gamma_2 = [1, y, y^2, x], \Gamma_3 = [1, y, x, xy] \\ \Gamma_4 &= [1, y, x, x^2], \Gamma_5 = [1, x, x^2, x^3].\end{aligned}$$

Now the spaces $G_j := span\Gamma_j$ represent the five complements.

Clearly no four of those subspaces can serve the same purpose, for an ideal generated by, say $< x^4, y > \in \mathfrak{J}_4$ is not complemented to the first four subspaces. It is also easy to see that the minimal 2-interpolating and 2-ideal family is

$$\mathcal{F} = \{span\{1, x\}, span\{1, y\}\},$$

since this family is 2-ideal, by the last theorem, and no one two-dimensional subspace is 2-interpolating, by the results of the previous section.

Theorem 2.11 *The minimal number of subspaces in a family of* 3-*ideal complements in* $\mathbb{C}[x, y]$ *is* 3.

PROOF. It follows from the Theorem 2.9, that the family $\mathcal{F}$ consisting of three spaces:

$$span\{1, x, x^2\}, span\{1, x, y\} \text{ and } span\{1, y, y^2\}$$

is a k-ideal family. Thus it remains to proof that no two three-dimensional spaces form a family of k-ideal complements. Let the subspaces V_1 and V_2 form such a family. Consider several cases:

Case 1: There exists a non-constant polynomial $p \in \mathbb{C}[x, y]$ and polynomials $f_k \in V_k$ such that $f_k = h_k \cdot p$. Then consider the set

$$Z := \{(x, y) \in \mathbb{C}^2 : p(x, y) = 0\}.$$

This is an infinite set and hence contains three distinct points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in Z.$$

Next consider the ideal

$$J := \{f \in \mathbb{C}[x, y] : f(x_j, y_j) = 0, j = 1, 2, 3\}.$$

Clearly, J is a radical ideal in $\mathfrak{J}_3$. Since for $k = 1, 2$ we have $f_k(x_j, y_j) = 0$ and since J is a radical, this implies that $f_k \in J$ and hence $f_k \in V_k \cap J \neq \{0\}$. In particular, neither V_1 nor V_2 complement J and $\{V_1, V_2\}$ is not a k-ideal family.

Case 2.: Suppose that

$$N := \max\deg\{f : f \in V_1\} \cdot \max\deg\{f : f \in V_2\} \geq 3.$$

Let $f_k \in V_k$ be any polynomials, such that

$$\deg f_k = \max\deg\{f : f \in V_k\}, k = 1, 2,$$

then (by Case 1) they do not contain a common non-zero factor. By Bezout's Theorem (cf. [9]), there exist $N \geq 3$ solutions (counting multiplicity) to the set of equations

$$f_k(x, y) = 0, k = 1, 2.$$

Hence, once again, there exist an ideal $J \in \mathfrak{J}_3$ such that $f_k \in V_k \cap J \neq \{0\}$.

Case 3. The last remaining case is when V_1 consists of polynomials of degree one and V_2 consists of polynomials of degree two and does not contain any non-constant linear polynomial. Since $\dim V_k = 3$, hence $V_1 = span\{1, x, y\}$ and V_2 is spanned by three quadratic polynomials and does not contain a non-constant linear function. We claim that at least one polynomial in V_2 has a linear factor, thus reducing this case to Case 1. Indeed, suppose that V_2 is spanned by quadratic polynomials $\{f_j(x, y), j = 1, 2, 3\}$. Consider the polynomial

$$p(x) := \sum_{j=1}^{3} a_j f_j(x, Ax + B) \in V_2.$$

This is a quadratic polynomial with three coefficients that depend on five parameters: A, B, a_j. Setting these coefficients to zero, we obtain three equations in five unknowns, which clearly have a solution in $\mathbb{C}$. Thus the polynomial

$$\sum_{j=1}^{3} a_j f_j(xy)$$

has a linear factor: $y - Ax - B$. $\square$

References

[1] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, pp. 9-10, Addison-Wesley, Reading, MA, 1969.

[2] G. Birkhoff, The Algebra of Multivariate Interpolation, in " Constructive approaches to mathematical models" (C.V. Coffman and G.J.Fix, Eds.), pp. 345-363, Academic Press, New-York, 1979.

[3] C. de Boor, Ideal Interpolation, in "Approximation Theory XI, Gatlinburg 2004" (C. K. Chui, M. Neamtu, and L. L. Schumaker Eds.), pp. 59-91, Nasboro Press, 2005.

[4] Marcel G. de Bruin and A. Sharma, Birkhoff interpolation on nonuniformly distributed roots of unity II, in "Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications, Patras, 1999," J. Comput. Appl. Math. 133 (2001), no. 1-2, 295–303.

[5] Marcel G. de Bruin and A. Sharma, Birkhoff interpolation on non-uniformly distributed roots of unity. Mathematical journey through analysis, matrix theory and scientific computation, Kent, OH, 1999., Numer. Algorithms 25 (2000), no. 1-4, 123–138.

[6] Marcel G. de Bruin, A. Sharma and J. Szabados, Birkhoff type interpolation on perturbed roots of unity. Approximation theory, pp. 167–179, Monogr. Textbooks Pure Appl. Math. 212, Dekker, New York, 1998.

[7] M. A. Bokhari, H. P. Dikshit and A. Sharma, Birkhoff interpolation on some perturbed roots of unity: revisited. Mathematical journey through analysis, matrix theory and scientific computation (Kent, OH, 1999). Numer. Algorithms 25 (2000), no. 1-4, 47–62.

[8] F. R. Cohen and D. Handel, k-regular embeddings of the plane, Proc. Amer. Math. Soc. 72 (1978), 202-204.

[9] D. Cox, J. Little and D. O'Shea, Ideals,Varieties, and Algorithms, Springer-Verlag, New-York-Berlin-Heidelberg, 1997.

[10] J. Fabrykowski, A. Sharma and H. Zassenhaus, Some Birkhoff interpolation problems on the roots of unity, Linear Algebra Appl. 65 (1985), 1–23.

[11] M. Gordan, Les Invariants des Formes Binaries, J. Math. Pures et Appli. (Liuville's J.) 6 (1900), 141-156.

[12] Rong Qing Jia and A. Sharma, Solvability of some multivariate interpolation problems. J. Reine Angew. Math. 421 (1991), 73–81.

[13] S. Karlin and W. J. Studden, Tchebycheff systems: with applications in analysis and statistics, Interscience, New York, 1966.

[14] M. G. Krein and A. A. Nudelman, The Markov Moment Problem and Extremal Problems, Amer. Math. Soc. Transl. Math. Monographs, Providence, RI, 1977.

[15] J. C. Mairhuber, On Haar's Theorem Concerning Chebyshev Approximation Problems Having Unique Solution, Proc. AMS. 7 (1956) 609-615.

[16] A. Sharma and R. S. Varga, On a 2-periodic lacunary trigonometric interpolation problem, in " Approximation theory VI, Vol. II, College Station, TX, 1989," pp. 585–588, Academic Press, Boston, MA, 1989.

[17] A. Sharma and R. S. Varga, On a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes, Results Math. 16 (1989), no. 3-4, 383–404.

[18] A. Sharma, and Z. Ziegler, Hermite interpolation on some perturbed roots of unity, Analysis (Munich) 19 (1999), no. 1, 1–12.

[19] A. Sharma, J. Szabados, B. Underhill and A. K. Varma, On some general lacunary interpolation problems, J. Approx. Theory 87 (1996), no. 2, 194–219.

[20] B. Shekhtman, Case Study in Hermite Interpolation, J. Approx. Theory (submitted)

[21] B. Shekhtman, On Hermite Interpolation in $\mathbb{R}_d$, ETNA 18 (2004), 65-72.

[22] B. Shekhtman, Interpolating subspaces in $\mathbb{R}_n$, in “Approximation Theory, Wavelets and Applications” (S. P. Singh, Ed., pp. 465-471, Kluwer, 1995,.

[23] B. Shekhtman, Interpolation by Polynomials in Several Variables, in “Approximation Theory X” (2001) (C. K. Chui, L. L. Schumaker and J. Stoekler, Eds.), pp. 367-372, Vanderbilt Univ. Press, (2002).

[24] V. A. Vasiliev, On Function Spaces that are Interpolating at every k nodes, Funct. Anal. Appl. (1992), 72-74.

[25] D. Wulbert, Interpolation at a Few Points, J. Approx. Theory, 96(1999), 139-148.

Index