Dictionary on Lie Algebras and Superalgebras

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Prolegomena

Nature is full of symmetries... and of symmetry breakings! The word *symmetry* has a Greek origin: *syn* "with" and *metron* "measure", and means regularity and harmony. The notion of symmetry is important in physics because it relates to the idea of invariance, and therefore of conservation laws and also of selection rules. Indeed, if the notion of symmetry has become so popular today, it is in large part due to the existence of the particularly powerful tool which is group theory.

Group theory has played such a fundamental role in different domains of physics in the twentieth century that it is interesting to consider why the notion was not introduced much earlier. The answer looks simple. The first (or one of the first) general definition of a group, as we know it today, was given at the end of the last century by Weber in his *Traité d'Algèbre Supérieure* (1898). But why did the notion of group come so late in the history of mathematics, while complex numbers were already known in the sixteenth century? Why did not the Greeks, who used numbers and geometry, conceive the idea of group structure? An insight into these questions is given by the mathematician Dieudonné (*La Genèse de la Théorie des Groupes*, La Recherche, **103** (1979) 866; see also *Pour l'Honneur de l'Esprit Humain*, Hachette, Paris (1988)). Based on the fact that the Greeks did not have the notion of negative numbers, Dieudonné explains that they could not imagine the group structure for real numbers with addition as a composition law. And though the Greeks were familiar with the multiplication of positive real numbers, as well as with the notion of fraction, this was the only example of group structure they had. It is clearly difficult to generalize a concept with only one example to hand! The Greeks were also experts in geometry, but they were only studying "static" properties of figures. They lacked the idea of a group of "transformations", or of a group "acting" on a set, which is always the case when group theory is applied to physics. Still following Dieudonné, it seems that mathematicians were led to introduce the notion of group from two sources, both intrinsically related. The first one concerns the composition of applications. Such an idea started slowly around 1770 with the work of Lagrange, among others, dealing with the resolution of

n-th order equations. The study of the variation of an n variable function $f(x_1, x_2, \ldots, x_n)$ when the order in the variables is changed, or "permuted", forced mathematicians of that time to overcome the "static" vision of their science, and led to the discovery of the group of permutations, which was first described in the memoirs of Cauchy (1813) and Galois (1830). Note that the name "group" was introduced by Galois, but owing to his dramatic death in 1832 at the age of twenty-one, his work only came to the attention of the mathematical community in 1846. The second source relates to the notion of equivalence class. Lagrange was probably the first to notice that an equivalence relation may be determined among the integer x, y solutions of the binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2 = n$ with a, b, c integers. But it was Gauss in his *Disquisitiones arithmeticae* (1800) who gave a precise definition of an equivalent class, and wrote explicitly the composition of two classes. If one adds that it was only in 1872 that the general definition of an application from one set to another was given for the first time by Dedekind, who also dared to talk in 1877 about a group structure for the classes of binary quadratic forms, it is easier to understand how difficult and convoluted was the introduction of the mathematical object that we now commonly call a group. This short historical introduction may allow an appreciation of the distance covered by group theory during the twentieth century, after its slow and arduous discovery in the nineteenth century. It is interesting to remark that the notion of transformation group was at the origin of the concept of group. This same notion, so fundamental in physics, also led to the development of group theory, giving rise in particular to the theory of group representations.

A second essential notion concerning symmetries in physics should be added. The physicist is faced with two kinds of symmetries: exact symmetries, for which no kind of violation has been observed, and approached or broken symmetries. In this latter case, one could distinguish between explicit or spontaneous breakings. A typical example of explicit breaking is provided by the so-called Zeeman effect in quantum mechanics in which the introduction of a magnetic field near to an atom leads to the adjunction of a supplementary term in the Hamiltonian H, which breaks the symmetry of H and splits the degenerate energy states. The left-over degeneracy of the energy levels can be understood and classified in terms of the residual symmetry, still present after the introduction of the symmetry breaking term. Examples of spontaneous breakings are found in gauge theory. Here, the Lagrangian is not perturbed when the symmetry of the minima of energy states, or vacua, is broken: one talks about spontaneous symmetry breaking, and this gives rise to the Higgs phenomenon. In both cases it is abolutely necessary to have a deep understanding of the subgroups of a group (and/or of the Lie subalgebras of a Lie algebra), as well as of the decompositions of the group (and/or algebra) representations with respect to the residual subgroup (subalgebra).

It is usually agreed that invariance is of two types: geometrical or dynamical. Geometrical invariance is fundamental in the description of physical systems such as objects in three-dimensional space, or in four-dimensional space–time. For example, for molecules and crystals, it is as important to determine their symmetries to classify them as it is to study their vibratory modes or their electronic properties. And it is hard to think about relativity without referring to the Poincaré group. One of the best examples of dynamical invariance is found in gauge theories in elementary particle physics. The most famous illustration is the Glashow–Weinberg–Salam electroweak theory, which allows us to "classify" members of a same family of quarks and leptons while simultaneously "fixing" their electromagnetic and weak interactions. These gauge symmetries allow us to conceptualize with a unique model two *a priori* different interactions. Including the strong interaction, together with gravitation, in order to obtain a unified theory of the four (known today) fundamental interactions, is of course the next big question! Here, symmetry has led to the notion of unification.

In these examples, two kinds of groups are used: finite groups or more generally discrete groups, and continuous groups or more specifically Lie groups. Both are essential in the study of mathematical physics. Even though Lie groups and Lie algebras are nowadays of greater importance, one cannot avoid the use of finite (discrete) groups to investigate their properties, as for example using the Weyl group to study simple Lie algebras. It should be noted that although Lie algebras have been extensively used throughout the twentieth century, some major developments only occurred in the last twenty years due to the blossoming of supersymmetry and supergravity. Two-dimensional conformal field theories, while federating string theories, models of statistical mechanics and more generally integrable systems, have also strongly contributed to the development of infinite dimensional Lie algebras, in particular Kac–Moody and Virasoro, W algebras and superalgebras. One should not forget the deformations of the universal enveloping algebras known as quantum groups. Other promising structures that are currently studied include hyperbolic algebras, and Yangian and elliptic algebras.

In this book, we will only be concerned with finite dimensional Lie algebras and superalgebras, and consequently efforts will be concentrated on semi-simple algebras and superalgebras. Our aim is to provide the reader with an elementary and easy to use handbook on continuous symmetries and unlike the traditional presentation, this volume is organized as a short dictionary. The main definitions and properties of Lie algebras and superalgebras are given in alphabetical order. Our hope is that in this way the beginner will be able to discover easily the main concepts on algebras and superalgebras, while a more experienced theorist will find quickly the necessary tools and information for any specific use. It follows that the potential readers are researchers working in mathematical and/or theoretical physics, PhD students who need to learn the main elements on symmetries, or experimental physicists looking for formal tools while studying theoretical papers.

It should be noted that Lie algebras and Lie superalgebras are treated on an equal footing in Part I and Part II of the volume. This is done for two main reasons. The first is that superalgebras are becoming increasingly used in theoretical physics. The second is linked to the natural extension of the theory of simple Lie algebras to superalgebras, in spite, of course, of some specific differences, such as, for example, the existence for superalgebras of atypical representations or the indecomposability of products of representations.

We would also like to emphasize two simple and natural guidelines we have followed in the elaboration of this volume. They have been introduced above and concern on the one hand the notion of a group of transformations, or a group acting on a set, and on the other hand the notion of symmetry breaking. In this spirit we have presented a detailed description of the structure of Lie algebras and superalgebras, particularly of the simple and semi-simple ones, and an extensive study of their finite dimensional representation theory.

Since we earnestly desire this book to be of practical use, we have included a large number of tables, which constitute its third part. Such explicit computations often serve to illustrate methods and techniques which are developed in the first two parts and which have never before been considered in text books. As examples, the reader will find in Part I methods, involving generalized Young tableaux, for the decomposition into irreducible representations of the Kronecker product of representations for the orthogonal, symplectic and also G_2 exceptional algebras; techniques for the computation of branching rules of Lie algebras; and the Dynkin method for the classification of $sl(2)$ subalgebras of a simple Lie algebra. In Part II, the special feature of a simple superalgebra of rank larger than one to admit more than one Dynkin diagram and the method for determining $osp(1|2)$ and $sl(1|2)$ sub-superalgebras of a simple superalgebra are developed. Among other results explicit realizations of all simple Lie algebras and superalgebras in terms of bosonic and/or fermionic oscillators are provided.

The discussion of concepts and techniques for supersymmetry would itself require a book. Here again, we have tried to be pragmatic while introducing the supersymmetry algebra, the basic ingredients on superspace and superfields, and spinors in the Lorentz group. We believe that the elementary particle physicist will find here the necessary tools for further computations and developments.

We are aware that this work is far from exhaustive: priority has been given to clarity and to subjects which seem to us of potential interest today as well as for direct applications for the study of more elaborated structures, such as infinite dimensional algebras or quantum groups. The list of references should be viewed as a selection of books and articles in which the reader can find proofs and developments of the presented items, as well as a more detailed bibliography.

Finally, we wish to thank sincerely many colleagues[1] and friends for providing us with useful suggestions and unceasing encouragement.

Annecy and Napoli
November 1999

LUC, NINO and PAUL

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Main Notations

$*$	composition law
$[\,.\,,\,.\,]$	commutator
$\{\,.\,,\,.\,\}$	anticommutator
$[\![\,.\,,\,.\,]\!]$	super or $\mathbb{Z}_2$-graded commutator (Lie superbracket)
$(\,.\,,\,.\,)$	inner product, Killing form
$\cdot$	scalar product (in the root space)
$\ni\!\!\!\!+\,,\ \in\!\!\!\!+$	semi-direct sum
$\ltimes$	semi-direct product
$\overline{0}, \overline{1}$	$\mathbb{Z}_2$-gradation of a superalgebra
$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{H}$	sets of positive integers, of integers, of real numbers, of complex numbers, of quaternions
$\mathbb{K}$	commutative field of characteristic zero
$\mathcal{A}, \mathcal{A}_{\overline{0}}, \mathcal{A}_{\overline{1}}$	(super)algebra, even/odd part of a superalgebra
A_{ij}	Cartan matrix
α^i_j	i-th component of the root α_j
Aut	automorphism group
$\Delta, \Delta_L, \Delta_S$	root system, long root system, short root system
$\Delta_{\overline{0}}, \Delta_{\overline{1}}$	even root system, odd root system
$\Delta^+, \Delta^+_{\overline{0}}, \Delta^+_{\overline{1}}$	positive roots, positive even roots, positive odd roots
Δ^0	simple root system
Der	derivation algebra
G	group, Lie group
$\mathcal{G}, \mathcal{G}_{\overline{0}}, \mathcal{G}_{\overline{1}}$	Lie (super)algebra, even/odd part of a Lie superalgebra
$\mathcal{H}$	Cartan subalgebra
$\mathcal{H}^*$	dual of $\mathcal{H}$, root space
Int	inner automorphism group
Λ, Λ_i	weight, fundamental weight
$[\lambda]$	Young tableau
Out	set of outer automorphisms
$\mathfrak{S}_N$	symmetric group

$\mathcal{V}$	module, representation space, vector space
W	Weyl group
A_n, $sl(n+1)$	unitary simple Lie algebra
B_n, $so(2n+1)$	orthogonal simple Lie algebra
C_n, $sp(2n)$	unitary simple Lie algebra
D_n, $so(2n)$	orthogonal simple Lie algebra
$A(m,n)$, $sl(m+1\|n+1)$	unitary simple Lie superalgebra ($m \neq n$)
$A(n,n)$, $psl(n+1\|n+1)$	unitary simple Lie superalgebra
$B(m,n)$, $osp(2m+1\|2n)$	orthosymplectic simple Lie superalgebra
$C(n+1)$, $osp(2\|2n)$	orthosymplectic simple Lie superalgebra
$D(m,n)$, $osp(2m\|2n)$	orthosymplectic simple Lie superalgebra

Part 1

Lie Algebras

Unless otherwise stated, all Lie algebras considered here are complex and finite dimensional.

1.1 Action of a group

Groups are never considered abstractly in physics, but by their action on some set S: they are called *groups of transformations*.

Definition

Let G be a group and S a set. An action of G on S is an application from $G \times S$ into S such that for all $g \in G$ and $s \in S$ the image is $g(s)$ with the properties

$$g(g'(s)) = (g * g')(s)\,, \qquad \forall\, g, g' \in G,\ \forall\, s \in S$$
$$e(s) = s \quad \text{where } e \text{ is the identity in } G$$

Finally, the group G and its action on the set S need to be "represented" (→ 1.72 Representation of a group and 1.48 Lie group of transformations).

1.2 Adjoint representation

Theorem

The structure constants of a Lie algebra $\mathcal{G}$ (→ 1.85) provide a matrix representation (→ 1.73) for the algebra. This representation is called the *adjoint* or *regular* representation.

In general, this representation is not faithful (that is one to one).

Construction of the adjoint representation

Let X_μ ($\mu = 1, \dots, n$) be a basis of the Lie algebra $\mathcal{G}$ and ${C_{\mu\nu}}^\rho$ the structure constants:

$$\Big[X_\mu, X_\nu\Big] = {C_{\mu\nu}}^\rho X_\rho$$

Let us associate to each X_μ a $n \times n$ matrix M_μ such that

$$(M_\mu)^\beta_\alpha = -{C_{\mu\alpha}}^\beta$$

One can easily check that $\pi : X_\mu \mapsto M_\mu$ is a homomorphism from $\mathcal{G}$ into the group of $n \times n$ matrices, and therefore defines a representation. Indeed:

$$\begin{aligned}
\Big[X_\mu, X_\nu\Big] = {C_{\mu\nu}}^\rho X_\rho \mapsto \Big[M_\mu, M_\nu\Big]^\beta_\alpha &= (M_\mu)^\gamma_\alpha (M_\nu)^\beta_\gamma - (M_\nu)^\gamma_\alpha (M_\mu)^\beta_\gamma \\
&= {C_{\mu\alpha}}^\gamma {C_{\nu\gamma}}^\beta - {C_{\nu\alpha}}^\gamma {C_{\mu\gamma}}^\beta \\
&= -{C_{\mu\nu}}^\gamma {C_{\gamma\alpha}}^\beta \\
&= {C_{\mu\nu}}^\gamma (M_\gamma)^\beta_\alpha
\end{aligned}$$

where the Jacobi identity (→ 1.46) has been used.

Example

The Lie algebra $su(3)$ of the group $SU(3)$ admits as a basis the eight 3×3 hermitian matrices λ_i ($i = 1, \dots, 8$) also called the Gell-Mann matrices (→ 1.90) which satisfy ($i, j, k = 1, \dots, 8$)

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i\, f_{ijk}\, \frac{\lambda_k}{2}$$

The matrices $\pi(\lambda_i/2)$ in the adjoint representation are therefore 8×8 matrices the entries of which are, up to the coefficient i, the structure constants f_{ijk}. More simply, we can say that to the element $\lambda_j/2$ of $su(3)$ will be associated by action of the generator $\lambda_i/2$ the element $\sum_k f_{ijk}\lambda_k/2$. We will write

$$\mathrm{ad}_{\lambda_i/2}\left(\frac{\lambda_j}{2}\right) = \left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i\, f_{ijk}\, \frac{\lambda_k}{2}$$

□

Remark on the adjoint representation

In the case of an Abelian algebra, the adjoint representation is trivial and therefore useless since any structure constant is zero.

In the case of a simple Lie algebra, the adjoint representation is a faithful representation.

Using the notation $\mathrm{ad}_X\,(Y) = [X, Y]$ if $X, Y \in \mathcal{G}$, we deduce

$$\begin{aligned} &\mathrm{ad}_{\alpha X + \beta Y} = \alpha\ \mathrm{ad}_X + \beta\ \mathrm{ad}_Y \\ &\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y] \end{aligned}$$

($\alpha, \beta \in \mathbb{K}$ where $\mathbb{K}$ is the field on which $\mathcal{G}$ is defined).

The last relation is easy to obtain using the Jacobi identity; let $Z \in \mathcal{G}$:

$$\begin{aligned} \mathrm{ad}_{[X,Y]}\,(Z) = \Big[[X, Y], Z\Big] &= \Big[X, [Y, Z]\Big] - \Big[Y, [X, Z]\Big] \\ &= \mathrm{ad}_X\ \mathrm{ad}_Y\,(Z) - \mathrm{ad}_Y\ \mathrm{ad}_X\,(Z) \\ &= \Big[\mathrm{ad}_X, \mathrm{ad}_Y\Big](Z) \end{aligned}$$

Note that we express here in other words what is presented above for the construction of the adjoint representation.

1.3 Algebra

Definition

An algebra $\mathcal{A}$ over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$, fields of real or complex numbers) is a linear vector space with internal law $+$ (addition) endowed with a second internal law $*$ (multiplication), which is right and left distributive with respect to the addition; that is for all $X, Y, Z \in \mathcal{A}$

$$(X+Y)*Z = X*Z + Y*Z$$
$$X*(Y+Z) = X*Y + X*Z$$

The algebra is said to be *associative* if for all $X, Y, Z \in \mathcal{A}$

$$(X*Y)*Z = X*(Y*Z)$$

and *commutative* or *Abelian* if for all $X, Y \in \mathcal{A}$

$$X*Y = Y*X$$

1.4 Automorphisms

Group automorphisms

Definition

Let G be a group. An automorphism of G is a bijective endomorphism of G, that is a bijective homomorphism of G on itself ($\rightarrow$ 1.38).

Property

The set of automorphisms of a group G forms a group denoted $\mathrm{Aut}(G)$. Its internal law is the natural composition of applications. The identity is the identity operator (associating to each element $g \in G$ the element g itself). The inverse of an automorphism of G is the inverse application.

Example

Let g_0 be an element of G. The application

$$\mathrm{Ad}_{g_0} : g \in G \mapsto g_0\, g\, g_0^{-1} \in G$$

is an automorphism. Such an automorphism is called an *inner* automorphism or a conjugation. Moreover the elements g and $g' = g_0\, g\, g_0^{-1}$ will be said to be conjugate with respect to g_0. □

Property

The set of the inner automorphisms of a group G forms a group denoted $\mathrm{Int}(G)$ (which is of course a subgroup of $\mathrm{Aut}(G)$).

It is easy to understand the importance of inner automorphisms. For example, consider the rotation group in three dimensions – orthogonal group $O(3)$ – acting on the three-dimensional space $\mathbb{R}^3$. Then any rotation $R(\theta, \vec{n})$ of angle θ around a fixed axis $\vec{n}$ is conjugate to the rotation $R(\theta, \vec{m})$ of same angle θ around the axis $\vec{m}$. Let us call R' the rotation in the plane $(\vec{n}, \vec{m})$ such that $\vec{m} = R'\vec{n}$; then we will have

$$R' R(\theta, \vec{n}) R'^{-1} = R(\theta, R'\vec{n})$$

Such a conjugation reduces to an orthogonal change of frame.

Property

The inner automorphism group of a group G is isomorphic to the quotient of G by its center $\mathcal{Z}(G)$:

$$\mathrm{Int}(G) \simeq G/\mathcal{Z}(G)$$

This property is obvious since any element in the center of G ($\rightarrow$ 1.10) acts trivially as a conjugation on any element of G. We have in particular the property

$$\mathcal{Z}(G) = \ker(\mathrm{Ad}) = \left\{ g_0 \in G \,\middle|\, \mathrm{Ad}_{g_0} = \mathbb{I} \right\}$$

Remember also that $\mathcal{Z}(G)$ is an invariant subgroup of G.

But any automorphism of a group is not in general an inner one.

Definition

An automorphism of a group G which is not inner is called *outer* automorphism.

The set of outer automorphisms is not a group in general. But one has:

Theorem

The application $\mathrm{Ad} : g \in G \mapsto \mathrm{Ad}_g \in \mathrm{Aut}(G)$ is a homorphism and $\mathrm{Int}(G)$ is a normal (or invariant) subgroup of G. It follows that the quotient $\mathrm{Aut}(G)/\mathrm{Int}(G)$ is a group. Moreover, if the group G is simple ($\rightarrow$ 1.82), the quotient group $\mathrm{Aut}(G)/\mathrm{Int}(G)$ is finite.

Example

Consider the Euclidean group ($\rightarrow$ 1.29) in three dimensions $E(3)$. One can check that the center of $E(3)$ contains only the identity element $\mathcal{Z}(E(3)) = \{(0, \mathbb{I})\}$ and therefore $\mathrm{Int}(G) \simeq G$, but that there exist outer automorphisms, actually dilatations

$$D_\alpha(\vec{a}, R) = (\alpha\vec{a}, R)$$

α being a real number, argument of the dilatation.

These dilatations form a one parameter group $D(1)$ which can be seen as the quotient $\mathrm{Aut}(E(3))/E(3)$. The dilatation group commutes with the rotation part, and therefore $\mathrm{Aut}(E(3))$ can be seen as the semi-direct product of $E(3)$ by $D(1)$ or of $T(3)$ by $SO(3) \times D(1)$ if we denote $E(3)$ as the semi-direct product $T(3) \ltimes SO(3)$. □

Lie algebra automorphisms

Definition

Let $\mathcal{G}$ be a Lie algebra. An automorphism of $\mathcal{G}$ is a bijective homomorphism of $\mathcal{G}$ on itself. The set of automorphisms of a Lie algebra $\mathcal{G}$ forms a group denoted $\mathrm{Aut}(\mathcal{G})$.

The group of inner automorphisms of $\mathcal{G}$, denoted $\mathrm{Int}(\mathcal{G})$, is the group generated by the automorphisms of the form $X \mapsto gXg^{-1}$ with $g = \exp Y$ where $X, Y \in \mathcal{G}$.

Property

Let $\mathcal{G}$ be a *simple* Lie algebra with root system Δ with respect to a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. Let $\mathrm{Aut}(\Delta)$ be the group of automorphisms and $W(\Delta)$ the Weyl group of Δ. One has the following isomorphism:

$$\frac{\mathrm{Aut}(\mathcal{G})}{\mathrm{Int}(\mathcal{G})} \simeq \frac{\mathrm{Aut}(\Delta)}{W(\Delta)} = F(\mathcal{G})$$

$F(\mathcal{G})$ is called the *factor group*; it is isomorphic to $\mathrm{Out}(\mathcal{G})$ if we define $\mathrm{Out}(\mathcal{G})$ as the group of $\mathcal{G}$-automorphisms up to an element of $\mathrm{Int}(\mathcal{G})$, itself isomorphic to $\mathcal{G}$ since $\mathcal{G}$ is simple. $F(\mathcal{G})$ is also isomorphic to the group of symmetry of the Dynkin diagram ($\rightarrow$ 1.27) of $\mathcal{G}$, since an element of $W(\Delta)$ can be lifted to an inner automorphism of $\mathcal{G}$ and a symmetry of the Dynkin diagram is then associated to an outer automorphism of $\mathcal{G}$.

Let us add that one has also

$$\mathrm{Aut}(\Delta) = W(\Delta) \rtimes F(\mathcal{G})$$

where $W(\Delta)$ is a normal subgroup of $\mathrm{Aut}(\Delta)$.

Table 1.1 lists the outer automorphisms of the simple Lie algebras.

Table 1.1: Outer automorphisms of the simple Lie algebras.

simple Lie algebra $\mathcal{G}$	$\mathrm{Out}(\mathcal{G})$	simple Lie algebra $\mathcal{G}$	$\mathrm{Out}(\mathcal{G})$
A_{N-1}, D_N, E_6	$\mathbb{Z}_2$	A_1, B_N, C_N	$\mathbb{I}$
D_4	$\mathfrak{S}_3$	E_7, E_8, F_4, G_2	$\mathbb{I}$

1.5 Branching rules

Definition

Let $\mathcal{G}$ be a (semi-)simple Lie algebra and $\mathcal{K}$ a subalgebra of $\mathcal{G}$. An irreducible representation $R(\mathcal{G})$ of $\mathcal{G}$ is obviously a representation, in general reducible, $R(\mathcal{K})$ of $\mathcal{K}$. So the following formula holds

$$R(\mathcal{G}) = \bigoplus_i m_i \, R_i(\mathcal{K})$$

where $R_i(\mathcal{K})$ is an irreducible representation of $\mathcal{K}$ and $m_i \in \mathbb{Z}_{>0}$ is the number of times (degeneracy) the representation $R_i(\mathcal{K})$ appears in $R(\mathcal{G})$. The determination of the above decomposition for any $R_{\mathcal{G}}$ gives the *branching rules* of $\mathcal{G}$ with respect to $\mathcal{K}$.

One has clearly

$$\dim R(\mathcal{G}) = \sum_i m_i \ \dim R_i(\mathcal{K})$$

In some cases general procedure to determine the branching rules exist, but in most cases one has to work out the result case by case. In the following, we give general procedure for the cases

1. $su(p+q) \supset su(p) \oplus su(q) \oplus U(1)$
2. $su(pq) \supset su(p) \oplus su(q)$
3. $so(2n) \supset su(n)$
4. $sp(2n) \supset su(n)$
5. $so(2n+1) \supset so(2n)$
6. $so(2n) \supset so(2n-1)$

and we report in tables a few results concerning the branching rules for the exceptional Lie algebras with respect to the maximal regular subalgebras and with respect to (singular) subalgebras containing the exceptional Lie algebras (see Tables 3.31–3.37). In refs. [59, 87, 98, 100] tables with branching rules can be found.

Branching rules for $su(p+q) \supset su(p) \oplus su(q) \oplus U(1)$

To any irreducible representation of $su(p)$, $su(q)$ we can associate two Young tableaux ($\rightarrow$ 1.96), say $[\lambda]$ and $[\mu]$. Then an irreducible representation of $su(p+q)$ specified by the Young tableaux $[\rho]$ contains the representation $([\lambda],[\mu])$ if the "outer" product of the Young tableaux $[\lambda]$ and $[\mu]$ contains the Young tableaux $[\rho]$, with the multiplicity given by the times the Young tableaux $[\rho]$ appears in $[\lambda] \otimes [\mu]$ (see ref. [44]). The "outer" product of two Young tableaux is performed using the rules given in $\rightarrow$ 1.96, without imposing any limit on the number of rows. Once the product has been performed, the terms containing Young tableaux with more than $p+q$ rows have to be neglected. As to any Young tableaux of $su(n)$ we can add k columns of n rows, it follows that the same representation $([\lambda],[\mu])$ appears in the representation of $su(p+q)$ with $kp+lq$ extra boxes for $[\rho]$ $(k,l \in \mathbb{Z}_+)$. In general a representation of $([\lambda],[\mu])$ appears in the representation $[\rho]$ of $su(p+q)$ if (n_α being the number of boxes of the Young tableau $[\alpha]$)

$$kp + lq + n_\lambda + n_\mu = n_\rho$$

The value of the charge Y of $U(1)$ can be computed, up to a multiplicative numerical factor which depends on the normalization, by

$$Y = q\, n_\lambda - p\, n_\mu$$

the one-dimensional trivial representation being represented by a Young tableau with zero box.

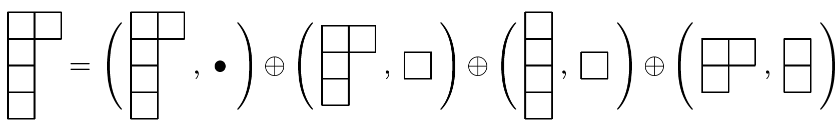
Remarks:

1. in order to apply the above formula, one has to count the number of boxes of the Young tableaux $[\lambda]$ and $[\mu]$ before taking away the columms with, respectively, p and q rows.

2. the sum of the values of the charge Y over all the states of the irreducible representations of $su(p)$, $su(q)$, taken with the appropriate multiplicity, appearing in an irreducible representation of $su(p+q)$ is zero.

Example

The decomposition of the Young tableaux corresponding to the irreducible representation $[2,1,1,1]$ of dimension 504 of $su(8)$ is



(1.1)

$$\oplus \left(\yng(1,1,1)\, , \yng(2) \right) \oplus \left(\yng(1,1,1)\, , \yng(1,1) \right) \oplus \text{terms interchanging the Y.T.} \qquad (1.2)$$

Let us consider the branching rules for $su(8) \supset su(3) \oplus su(5) \oplus U(1)$. In this case, we have to neglect the Young tableau with more than three rows in the first term. Using the table of dimensions, we have, writing the dimension of $su(3)$ and $su(5)$ and the value of the charge Y of $U(1)$:

$$\begin{aligned} 504 \;=\; & (3,5;17) \oplus (8,10;9) \oplus (1,15;9) \oplus (1,10;9) \oplus (6,10;1) \\ & \oplus (3,40;1) \oplus (3,10;1) \oplus (3,45;-7) \oplus (3,5;-7) \oplus (1,24;-15) \end{aligned}$$

Let us consider the branching rule for $su(8) \supset su(4) \oplus su(4) \oplus U(1)$. In this case, all the Young tableaux have to be taken into account and we find

$$\begin{aligned} 504 \;=\; & (4,1;20) \oplus (15,4;12) \oplus (1,4;12) \oplus (20,6;4) \oplus (4,10;4) \\ & \oplus (4,6;4) \oplus (6,4;-4) \oplus (10,4;-4) \oplus (6,20;-4) \oplus (4,1;-12) \\ & \oplus (4,15;-12) \oplus (1,4;-20) \end{aligned}$$

$\square$

Branching rules for $su(pq) \supset su(p) \oplus su(q)$

An irreducible representation of $su(pq)$ labelled by the Young tableau $[\lambda]$ with n_λ boxes contains only the irreducible representations of $(su(p), su(q))$ labelled by Young tableaux containing resp. $n_\lambda - kp$ and $n_\lambda - lq$, boxes with $k, l \in \mathbb{Z}_{>0}$ (see ref. [44]). In order to determine which irreducible representations of $(su(p), su(q))$ are contained in the irreducible representation $[\lambda]$, we have to compute all the possible irreducible representations of $su(q)$ and $su(p)$, with n_λ boxes, such that the "inner" product of such Young tableaux, considered as irreducible representations of the symmetric group $\mathfrak{S}_{n_\lambda}$, contains the Young tableau $[\lambda]$, with the appropriate multiplicity. Rules for performing such "inner" product have been given by several authors, see ref. [38] where references to the original papers can be found. In the following we report some general special formulae and the table of the inner products of $\mathfrak{S}_n$ for $n = 2, 3, 4$ (see Table 3.30). Tables for $n \leq 8$ can be found in ref. [44].

$$\begin{aligned} [n-1,1] \otimes [n-1,1] \;&=\; [n] \oplus [n-1,1] \oplus [n-2,2] \oplus [n-2,1,1] \\ [n-1,1] \otimes [n-2,2] \;&=\; [n-1,1] \oplus [n-2,2] \oplus [n-2,1,1] \oplus [n-3,3] \\ & \quad \oplus [n-3,2,1] \end{aligned}$$

$$\begin{aligned}
[n-1,1] \otimes [n-2,1,1] &= [n-1,1] \oplus [n-2,2] \oplus [n-2,1,1] \\
&\quad \oplus [n-3,2,1] \oplus [n-3,1,1,1] \\
[n-2,2] \otimes [n-2,2] &= [n] \oplus [n-1,1] \oplus 2\,[n-2,2] \oplus [n-2,1,1] \\
&\quad \oplus [n-3,3] \oplus 2\,[n-3,2,1] \oplus [n-3,1,1,1] \\
&\quad \oplus [n-4,4] \oplus [n-4,3,1] \oplus [n-4,2,2]
\end{aligned}$$

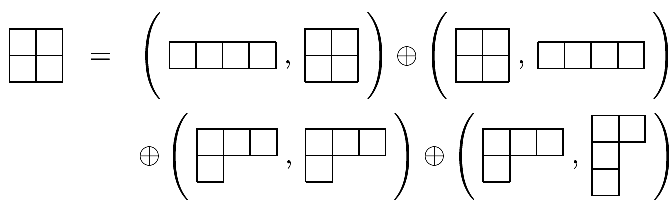
The terms in the r.h.s. which do not correspond to meaningful Young tableaux or to Young tableaux with more than p, resp. q, rows have to be neglected.

$$[\lambda] \otimes [1^n] = [\lambda]^T$$

where $[\lambda]^T$ is the Young tableau obtained by $[\lambda]$ interchanging rows with columns.

Example

Let us consider the branching rule for the irreducible representation $[2,2]$ of $sl(6) \subset sl(2) \oplus sl(3)$. From the table of dimensions, neglecting the Young tableaux with more than two (resp. three) rows in the first term (resp. second term), we find



that is in terms of the dimensions

$$105 = (5,6) \,\oplus\, (1,21) \,\oplus\, (3,15) \,\oplus\, (3,3)$$

□

Branching rules for $so(2n) \supset su(n)$

The Young tableaux associated to the irreducible representations of $su(n)$ appearing in the decomposition of a vector irreducible representation of $so(2n)$ associated to a generalized Young tableau (GYT) $[\lambda]$ ($\rightarrow$ 1.64) can be found by the following two-step procedure:

1. Find the set of generalized Young tableaux $\{[\mu_j]\}$ given by ($0 \le j \le n_\lambda$), n_λ being the number of boxes of $[\lambda]$)

$$\{[\mu_j]\} = (\{L_n^{2j}\} \otimes [\lambda])_A$$

where L_n^{2j} are completely negative Young tableaux with n rows of the form

$$L_n^{2j}(\{a_i\}) = [\ldots, -a_2, -a_2, -a_1, -a_1]$$

where $\{a_i\}$ is a set of non-negative integers satisfying $a_i \geq a_{i+1}$ and $\sum_i a_i = j$. For $n = 5$ we have e.g.

$$L_5^2 = [0,0,0,-1,-1]\,, \quad L_5^4 = [0,-1,-1,-1,-1]\,, \; [0,0,0,-2,-2]\,, \; \ldots$$

The GYTs L_n^{2j} are a generalization of the GYT denoted with the same symbol introduced in → 1.64.

The lower label A means that we have to keep only the GYTs satisfying the following conditions:

(a) Denoting by μ_i the number of boxes in the ith row of a generic GYT in the l.h.s, $\sum_i |\mu_i| \leq n_\lambda$ and $|\mu_i| \leq \lambda_1$.

(b) If some $|\mu_i|$ is equal to λ_1 (which may be either μ_1 or μ_n), then any other label must be not larger than λ_2 and so on.

(c) A GYT appearing more than once in the product with $L_n^{2j}(\{a_i\})$, for the same set $\{a_i\}$, has to be considered only once.

(d) A GYT appearing twice in the product with $L_n^{2j}(\{a_i\})$, for different sets $\{a_i\}$, has to be considered twice only if

$$\sum_i^n |\mu_i| \leq n_\lambda - 2(j-1)$$

For more complicated multiplicity, which however may appear only with large GYT $[\lambda]$, care has to be applied in order to determine how many times the GYT in the l.h.s. has to be counted, the root of the difficulty being the same as the determination of the multiplicity of a state in an irreducible representation of $so(2n)$.

2. Subsequently we have to replace a GYT $[\mu]$ with negative boxes by the *complementary* Young tableau $[m]$ (with positive boxes) obtained from $[\mu]$ by replacing a column with c_i negative boxes by a column with $n-c_i$ positive boxes. E.g. in $so(8)$, the complementary Young tableau of the GYT $[2,0,-1,-2]$ is $[4,2,1,0]$. As a consequence of the definition, the complementary Young tableau satisfies always $m_1 = \mu_1 + |\mu_n|$, $m_n = 0$ and the number of the boxes of $[m]$ is the (algebraic) sum of the boxes of $[\mu]$ plus $n|\mu_n|$.

Example 1

From example 1 of $\rightarrow$ 1.64, the irreducible representation $[1^2]$ of $so(10)$ decomposes with respect to $su(5)$ as

$$\begin{array}{rcl} [1^2] & \rightarrow & [1^2] \oplus [1,0,0,0,-1] \oplus [0] \oplus [0,0,0,-1,-1] \\ & \rightarrow & [1^2] \oplus [2,1,1,1] \oplus [0] \oplus [1^3] \end{array}$$

Using the table of dimensions, we get

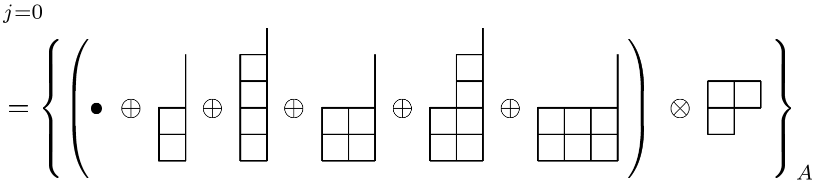
$$45 = 10 \oplus 24 \oplus 1 \oplus 10$$

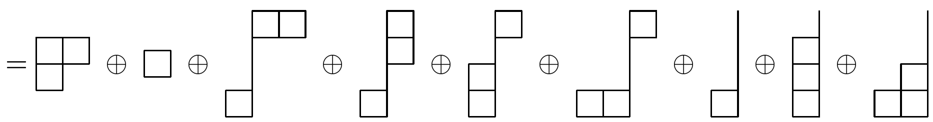
□

Example 2

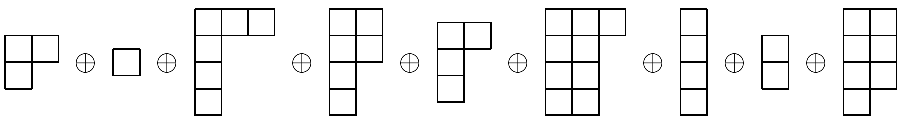
As a second example, we discuss the decomposition of the irreducible representation $[2,1]$ of $so(10)$ with respect to $su(5)$. We get

$$\sum_{j=0}^{3} (L_5^{2j} \otimes [2,1])_A =$$





which leads to



Using the table of dimensions, we have

$$330 = 45 \oplus 5 \oplus 70 \oplus 45 \oplus 45 \oplus 70 \oplus 5 \oplus 10 \oplus 40$$

Note that the GYT $[1,0,0,-1,-1]$ for example appears in the product twice (in $[2,1] \otimes [0,-1,-1,-1,-1]$ and $[2,1] \otimes [0,0,0,-2,-2]$ respectively), but we keep it only once due to rule (c) of step 1. □

Branching rules for $sp(2n) \supset su(n)$

The Young tableaux associated to the irreducible representations of $su(n)$ appearing in the decomposition of an irreducible representation of $sp(2n)$ associated to a generalized Young tableaux $[\lambda]$ (→ 1.66) can be found by the following two-step procedure:

1. Find the set of generalized Young tableaux $\{[\mu_j]\}$ given by $(0 \leq j \leq n_\lambda)$

   $$\{[\mu_j]\} = (\{P_n^{2j}\} \otimes [\lambda])_A$$

   where P_n^{2j} are completely negative Young tableaux with n rows of the form

   $$P_n^{2j}(\{b_i\}) = [\ldots, -b_3, -b_2, -b_1]$$

   where $\{b_i\}$ is a set of non-negative even integers satisfying $b_i \geq b_{i+1}$ and $\sum_i b_i = 2j$. For $n = 4$ we have e.g.

   $$P_4^2 = [0,0,0,-2]\,, \quad P_4^4 = [0,0,-2,-2]\,, \quad [0,0,0,-4]$$

   The GYT P_n^{2j} are a generalization of the GYT denoted with the same symbol introduced in → 1.66.

   The lower label A means that we have to keep only the GYTs satisfying the following conditions:

   (a) Denoting by μ_i the number of boxes in the ith row of a generic GYT in the l.h.s, $\sum_i |\mu_i| \leq n_\lambda$ and $|\mu_i| \leq \lambda_1$.

   (b) If a $|\mu_i|$ is equal to λ_1 (which may be either μ_1 or μ_n), then any other label must be not larger than λ_1 and so on.

   (c) A GYT appearing more than once in the product with $P_n^{2j}(\{b_i\})$, for the same set $\{b_i\}$, has to be considered only once.

   (d) Care has to be applied to determine how many times a GYT appearing more than once has to be counted, the root of the difficulty being the same as the determination of the multiplicity of a state in an irreducible representation of $sp(2n)$. However this happens only for large GYT $[\lambda]$.

2. Subsequently we have to replace a GYT $[\mu]$ with negative boxes by the *complementary* Young tableau $[m]$ (see previous subsection).

Example 1

From example 1 of → 1.66, the irreducible representation $[1^2]$ of $sp(10)$ decomposes with respect to $su(5)$ as

$$\begin{aligned}[1^2] \quad &\to \quad [1^2] \oplus [1,0,0,0,-1] \oplus [0,0,0,-1,-1] \\ &\to \quad [1^2] \oplus [2,1,1,1] \oplus [1^3]\end{aligned}$$

Using the table of dimensions, we get

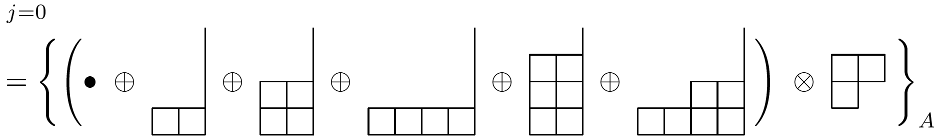
$$44 = 10 \oplus 24 \oplus 10$$

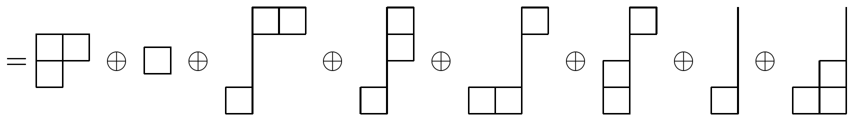
□

Example 2

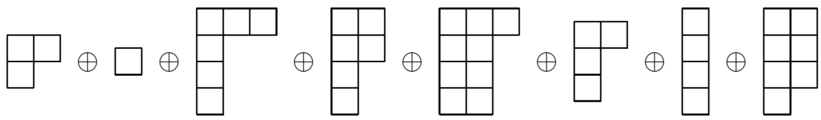
As a second example, we discuss the decomposition of the irreducible representation $[2,1]$ of $sp(10)$ with respect to $su(5)$. We have

$$\sum_{j=0}^{3} (P_5^{2j} \otimes [2,1])_A =$$





which leads to



Using the table of dimensions, we have

$$320 = 40 \oplus 5 \oplus 70 \oplus 45 \oplus 70 \oplus 45 \oplus 5 \oplus 40$$

□

Branching rules for $so(2n+1) \supset so(2n)$

The generalized Young tableaux $\{[\mu_k]\}$ associated to the irreducible representations of $so(2n)$ appearing in the decomposition of an irreducible representation of $so(2n+1)$ associated to a generalized Young tableau $[\lambda] = [\lambda_1, \ldots, \lambda_n]$

(→ 1.65) are given by ($0 \le k \le \lambda_1$)

$$\{[\mu_k]\} = (\{S_n^k\} \otimes [\lambda])_{(+)}$$

where S_n^k are completely negative Young tableaux with n rows of the form

$$S_n^k = [0, \dots, 0, -k] \qquad k = 0, 1, 2, \dots$$

and the lower label (+) means that we have to keep only the completely positive GYTs.

Moreover if the GYT $[\lambda]$ has n non-vanishing rows or in the case of spinor irreducible representations, the conjugate irreducible representations has to be added.

Example

We consider the branching of the irreducible representation $[2, 1, 1, 1]$ of $so(9)$ with respect to $so(8)$. We have

$$\begin{aligned} S_4^0 &\otimes [2,1,1,1] = [2,1,1,1] \\ S_4^1 &\otimes [2,1,1,1] = [1,1,1,1] \oplus [2,1,1] \\ S_4^2 &\otimes [2,1,1,1] = [1,1,1] \end{aligned}$$

So finally we have $[2,1,1,1] = [2,1,1,1] \oplus [2,1,1,-1] \oplus [1,1,1,1] \oplus [1,1,1,-1] \oplus [2,1,1] \oplus [1,1,1]$, that is in terms of the dimensions

$$924 = 224 \oplus \overline{224} \oplus 35 \oplus \overline{35} \oplus 350 \oplus 56$$

□

Branching rules for $so(2n) \supset so(2n-1)$

The generalized Young tableaux $\{[\mu_k]\}$ associated to the irreducible representations of $so(2n-1)$ appearing in the decomposition of an irreducible representation of $so(2n)$ associated to a generalized Young tableau $[\lambda]$ (with $\lambda_n \ge 0$) (→ 1.64) are given by

$$\{[\mu_k]\} = (\{S_n^k\} \otimes [\lambda])_{(+)}$$

where the meaning of S_n^k and of the lower label (+) is the same as in the branching rules for $so(2n+1) \supset so(2n)$. Moreover one has to keep only the GYTs with at most $n-1$ rows. It follows that we have the equality:

$$\dim [p^n] = \dim [p^{n-1}] \qquad (p \in \mathbb{Z}_+)$$

The branching rules for the irreducible representation $[\lambda]$ (with $\lambda_n < 0$) is the same as the complex conjugate irreducible representation, i.e. the one obtained changing the sign of λ_n.

Example

We consider the branching of the spinor irreducible representation $[3]'$ of $so(10)$ with respect to $so(9)$. We have

$$S_5^0 \otimes [3]' = [3]' \qquad S_5^1 \otimes [3]' = [2]'$$
$$S_5^2 \otimes [3]' = [1]' \qquad S_5^3 \otimes [3]' = [0]'$$

So the final result is $[3]' = [3]' \oplus [2]' \oplus [1]' \oplus [0]'$, that is in terms of the dimensions

$$2640 = 1920 \oplus 576 \oplus 128 \oplus 16$$

□

1.6 Cartan–Weyl basis, Cartan subalgebra

Properties of simple Lie algebras, in particular their classification and their representations, are consequences of the structure of the Cartan basis associated to these algebras. In order to better understand the meaning of the Cartan basis, and to keep in mind its properties, we give below the main steps leading to this decomposition.

Cartan's problem

Let $\mathcal{G}$ be a simple Lie algebra of dimension n over $\mathbb{R}$ and $\{X_1, \dots, X_n\}$ a basis of generators of $\mathcal{G}$ such that

$$\Big[X_i, X_j\Big] = C_{ij}{}^k X_k$$

Let $A \in \mathcal{G}$. The Cartan's problem is to find the $X \in \mathcal{G}$ such that

$$\mathrm{ad}_A\,(X) = \Big[A, X\Big] = \rho X$$

that is, if $A = a^i X_i$ and $X = x^j X_j$,

$$a^i x^j C_{ij}{}^k = \rho x^k$$

This is a system of n linear equations, the solution of which is non-trivial if and only if

$$\det(a^i C_{ij}{}^k - \rho\delta_j^k) = 0$$

We have here an nth order equation in ρ, which will admit n solutions $(x^i)_\rho$ and then n operators X_ρ, as soon as one complexifies $\mathcal{G}$, that is one considers $\mathcal{G}$ defined on the complex field $\mathbb{C}$ (remember that an n degree polynomial equation has n solutions on the field of complex numbers $\mathbb{C}$ but not on $\mathbb{R}$: $\mathbb{R}$ is not algebraically closed).

The solution to this problem has been given by Cartan:

Theorem (Cartan)

Let $\mathcal{G}$ be a simple Lie algebra. If A is chosen such that the equation $\det(a^i C_{ij}{}^k - \rho\delta^k_j) = 0$ has a maximal number of different roots, then

1. the root $\rho = 0$ is r times degenerate and there are r linearly independent elements $H_1, \dots, H_r$ such that $(i, j = 1, \dots, r)$

$$\left[A, H_i\right] = \left[H_i, H_j\right] = 0$$

2. the other $n - r$ roots are not degenerate. Denoting E_α the eigenvectors with eigenvalue α, one has

$$\left[A, E_\alpha\right] = \alpha E_\alpha$$

The r elements H_i and the $n - r$ elements E_α are linearly independent and therefore constitute a basis of $\mathcal{G}$, which is called the Cartan basis.

Consequences of Cartan's theorem

Let us note

$$\left[A, \left[H_i, E_\alpha\right]\right] = \left[\left[A, H_i\right], E_\alpha\right] + \left[H_i, \left[A, E_\alpha\right]\right] = \alpha\left[H_i, E_\alpha\right]$$

Since α is a non-degenerate eigenvalue, $\left[H_i, E_\alpha\right]$ is proportional to E_α:

$$\left[H_i, E_\alpha\right] = \alpha^i E_\alpha$$

Moreover, consider

$$\left[A, \left[E_\alpha, E_\beta\right]\right] = \left[\left[A, E_\alpha\right], E_\beta\right] + \left[E_\alpha, \left[A, E_\beta\right]\right] = (\alpha + \beta)\left[E_\alpha, E_\beta\right]$$

We have now three possibilities:

1. $\alpha+\beta$ is not a root: $\Big[E_\alpha, E_\beta\Big] = 0$
2. $\alpha+\beta \neq 0$ $\Big[E_\alpha, E_\beta\Big] = N_{\alpha\beta}\, E_{\alpha+\beta}$
3. $\alpha+\beta = 0$ $\Big[E_\alpha, E_\beta\Big] = \Big[E_\alpha, E_{-\alpha}\Big] = \sum_{i=1}^r \alpha^i H_i$

Actually, one has the following property:

Theorem

If α is a root, then $-\alpha$ is a root.

As a consequence, the number of roots is even and is equal to $n-r$. We have also to note that:

Property

ad_{H_i} can be chosen hermitian. Therefore the eigenvalues α^i are real.

Let us summarize the above discussion with the two definitions:

Cartan–Weyl basis, Cartan subalgebra

Definition

Let $\mathcal{G}$ be a simple complex Lie algebra of dimension n. The *Cartan–Weyl basis* of $\mathcal{G}$ will be constituted by the r generators H_i and the $n-r$ generators E_α satisfying the commutation relations:

$$\Big[H_i, H_j\Big] = 0 \qquad \Big[H_i, E_\alpha\Big] = \alpha^i E_\alpha$$

$$\Big[E_\alpha, E_{-\alpha}\Big] = \sum_{i=1}^r \alpha^i H_i$$

$$\Big[E_\alpha, E_\beta\Big] = N_{\alpha\beta}\, E_{\alpha+\beta} \quad \text{if } \alpha+\beta \text{ is a non-zero root}$$

The r-dimensional vector $\alpha = (\alpha^1, \dots, \alpha^r)$ of $\mathbb{R}^r$ is said to be root associated to the root generator E_α. The coefficients $N_{\alpha\beta}$ satisfy for any pair of roots α and β

$$N_{\alpha\beta}^2 = \tfrac{1}{2} k(k'+1)\alpha^2$$

where k and k' are integers such that $\alpha + k\beta$ and $\alpha - k'\beta$ are roots. Moreover, one has

$$N_{\alpha\beta} = -N_{\beta\alpha} = -N_{-\alpha-\beta}$$

where the roots are normalized such that

$$\sum_{\alpha\neq 0} \alpha^i \alpha^j = \delta_{ij} \Longrightarrow \sum_{\alpha\neq 0} \alpha^2 = r$$

in accordance with the Killing form ($\rightarrow$ 1.44)

$$g_{ij} = \begin{pmatrix} \delta_{ij} & & & & & \\ & 0 & 1 & & 0 & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$
$$\underbrace{\qquad}_{r}\underbrace{\qquad\qquad\qquad}_{n-r=2p}$$

Definition

The set $\mathcal{H}$ generated by the generators $(H_1, \ldots, H_r)$ is called the *Cartan subalgebra* of $\mathcal{G}$. The dimension of the Cartan subalgebra is the *rank* of the Lie algebra $\mathcal{G}$:

$$\operatorname{rank} \mathcal{G} = \dim \mathcal{H}$$

Property

The Cartan subalgebra $\mathcal{H}$ is unique up to conjugation of $\mathcal{G}$. It is maximal as Abelian subalgebra of $\mathcal{G}$, that is any Abelian subalgebra of $\mathcal{G}$ is in $\mathcal{H}$ (up to a conjugation). Because of its uniqueness, one can say that $\mathcal{H}$ is “the” maximal Abelian subalgebra of $\mathcal{G}$.

Property

Any element of a semi-simple algebra $\mathcal{G}$ is conjugate to an element of the Cartan subalgebra $\mathcal{H}$ once given a basis for $\mathcal{G}$ and $\mathcal{H}$ (i.e. for any $X \in \mathcal{G}$, one can choose $\mathcal{H}$ such that $X \in \mathcal{H}$).

For any element $H = \sum_{i=1}^r \xi^i H_i \in \mathcal{H}$, one can write

$$\Big[H, E_\alpha\Big] = \alpha(H)\, E_\alpha$$

where α is a linear functional on $\mathcal{H}$, that is an element of the dual $\mathcal{H}^*$ of $\mathcal{H}$, such that $\alpha(H) = \sum_{i=1}^r \xi^i \alpha^i$.

Since there exists a unique (up to a multiplicative factor) non-degenerate symmetric bilinear form B ($\rightarrow$ 1.44) on the Cartan subalgebra $\mathcal{H}$, one can associate to any functional $\alpha \in \mathcal{H}^*$ an element $H_\alpha \in \mathcal{H}$, such that

$$B(H, H_\alpha) = \alpha(H)$$

Now, one can define the scalar product of two roots α and β by

$$\alpha \cdot \beta = B(H_\alpha, H_\beta)$$

that is $\alpha \cdot \beta = \alpha(H_\beta) = \beta(H_\alpha)$.

Example 1

Consider the Lie algebra $su(2)$ of $SU(2)$, group for 2×2 unitary matrices with determinant equal to one. A basis of generators is J_1, J_2, J_3 such that

$$\Big[J_i, J_j\Big] = i\,\varepsilon_{ijk}\,J_k$$

ε_{ijk} is the completely antisymmetric rank three tensor with $\varepsilon_{123} = 1$. One chooses ($su(2)$ is of rank 1)

$$H = J_3 \qquad \text{and} \qquad E_\pm = J_\pm = J_1 \pm iJ_2$$

then

$$\Big[J_3, J_\pm\Big] = \pm J_\pm \qquad \text{and} \qquad \Big[J_+, J_-\Big] = 2J_3$$

□

Example 2

Consider the Lie algebra $su(3)$ of $SU(3)$, group for 3×3 unitary matrices with determinant equal to one. A basis of generators is given by the Gell-Mann matrices λ_i ($\rightarrow$ 1.90). Choosing ($su(3)$ is of rank 2)

$$H_1 = \lambda_3 \qquad \text{and} \qquad H_2 = \lambda_8$$

the roots can be taken as

$$E_{\pm\alpha} = \lambda_1 \pm i\lambda_2\,, \qquad E_{\pm\beta} = \lambda_4 \pm i\lambda_5\,, \qquad E_{\pm\gamma} = \lambda_6 \pm i\lambda_7$$

The three operators $\lambda_3, \lambda_1 \pm i\lambda_2$ form a Cartan basis of an $su(2)$ subalgebra; so do the three operators $\lambda_3 + \sqrt{3}\lambda_8, \lambda_4 \pm i\lambda_5$ or the three others $\lambda_3 - \sqrt{3}\lambda_8, \lambda_6 \pm i\lambda_7$: we have constructed the so-called I, U and V spin algebras. □

1.7 Cartan classification of simple Lie algebras

From Levi's theorem ($\rightarrow$ 1.47), one knows that any Lie algebra can be seen as the semi-direct sum of a solvable Lie algebra by a semi-simple one. Let us remember that a semi-simple Lie algebra is a direct sum of simple Lie algebras. Owing to the work of Cartan one knows that any simple finite dimensional Lie algebra is one of the following types:

- A_{N-1} or $su(N)$, where $N \geq 2$ is an integer, is the Lie algebra of the group of unitary $N \times N$ matrices of determinant 1. Its rank is $N-1$ and its dimension is $N^2 - 1$.

- B_N or $so(2N+1)$, where $N \geq 1$ is an integer, is the Lie algebra of the group of orthogonal $(2N+1) \times (2N+1)$ matrices of determinant 1. Its rank is N and its dimension is $N(2N+1)$.

- C_N or $sp(2N)$, where $N \geq 1$ is an integer, is the Lie algebra of the group of symplectic $2N \times 2N$ matrices of determinant 1. Its rank is N and its dimension is $N(2N+1)$.

- D_N or $so(2N)$, where $N \geq 3$ is an integer, is the Lie algebra of the group of orthogonal $2N \times 2N$ matrices of determinant 1. Its rank is N and its dimension is $N(2N-1)$. For $N=2$, one finds $so(4)$ which is not simple but semi-simple: $so(4) = so(3) \oplus so(3)$.

To these four series of Lie algebras have to be added five "isolated" simple Lie algebras:

- G_2, F_4, E_6, E_7, E_8, Lie algebras of five exceptional Lie groups of rank 2, 4, 6, 7, 8 and dimension 14, 52, 78, 133, 248 respectively.

For any of these algebras, there exists a positive integer N such that the algebra under consideration is a subalgebra of $gl(N,\mathbb{C})$, the Lie algebra of the group $GL(N,\mathbb{C})$ of $N \times N$ invertible complex matrices.

Based on algebra, this classification can be illustrated geometrically ($\to$ 1.77 Roots, root system).

$\to$ 1.30 Exceptional Lie algebras, 1.58 Orthogonal groups and algebras, 1.89 Symplectic groups and algebras, 1.90 Unitary groups and algebras.

1.8 Cartan matrix

Definition

Let $\mathcal{G}$ be a simple Lie algebra with Cartan subalgebra $\mathcal{H}$ and simple root system $\Delta^0 = (\alpha_1, \ldots, \alpha_r)$. The *Cartan matrix* $A = (A_{ij})$ of the simple Lie algebra $\mathcal{G}$ is the $r \times r$ matrix defined by

$$A_{ij} = 2\,\frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} = \alpha_i^\vee \cdot \alpha_j \qquad 1 \leq i,j \leq r$$

where $\alpha_i^\vee = 2\alpha_i/(\alpha_i \cdot \alpha_i)$ is the coroot ($\to$ 1.83) associated to the root α_i.

This matrix plays a fundamental role in the theory of the simple Lie algebras: in particular, it is the basic ingredient for the description of the Lie algebra in the so-called Serre–Chevalley basis (→ 1.81); this matrix is also encoded in the Dynkin diagram (→ 1.27) of the Lie algebra.

The Cartan matrix satisfies the following properties:

$$\begin{aligned}
&A_{ij} \in \mathbb{Z} \\
&A_{ii} = 2 \qquad \text{and} \qquad A_{ij} \leq 0 \quad (i \neq j) \\
&A_{ij} = 0 \;\Rightarrow\; A_{ji} = 0 \\
&A_{ij}A_{ji} \in \{0, 1, 2, 3\} \quad (i \neq j) \\
&\det A \neq 0 \quad \text{(the Cartan matrix is non-degenerate)}
\end{aligned}$$

In fact, only the following possibilities occur for the simple Lie algebras:
$A_{ij}A_{ji} = 1$ for all pairs $i \neq j$: A_N, D_N, E_6, E_7, E_8.
$A_{ij}A_{ji} = 1$ or 2 for all pairs $i \neq j$: B_N, C_N, F_4.
$A_{ij}A_{ji} = 1$ or 3 for all pairs $i \neq j$: G_2.

The Cartan matrices of the simple Lie algebras are listed in Tables 3.1–3.9.

Generalized Cartan matrix

Let $\mathcal{G}$ be a simple Lie algebra with Cartan subalgebra $\mathcal{H}$ and simple root system $\Delta^0 = (\alpha_1, \ldots, \alpha_r)$. One defines the extended simple root system by $\widehat{\Delta}^0 = \Delta^0 \cup \{\alpha_0\}$ where $-\alpha_0$ is the highest root with respect to Δ^0. The *generalized Cartan matrix* $\widehat{A} = (\widehat{A}_{ij})$ of the simple Lie algebra $\mathcal{G}$ is the $(r+1) \times (r+1)$ matrix defined by

$$\widehat{A}_{ij} = 2\,\frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} \qquad 0 \leq i, j \leq r$$

Obviously one has $\widehat{A}_{ij} = A_{ij}$ for $1 \leq i, j \leq r$ and $\widehat{A}_{00} = 2$, while the 0-th row and 0-th column have the following coefficients ($1 \leq i \leq r$):

$$\widehat{A}_{i0} = -\sum_{j=1}^{r} A_{ij}\, a_j \qquad \text{and} \qquad \widehat{A}_{0i} = -\sum_{j=1}^{r} a_j^\vee\, A_{ji}$$

where a_j and $a_j^\vee$ are the marks and the comarks (→ 1.83), and the roots are normalized such that the longest root has length squared equal to two.

1.9 Casimir invariants

Given a representation π of a semi-simple Lie group G with representation space $\mathcal{V}$, one can form, in the set of polynomials defined on $\mathcal{V}$, quantities which are invariant – or scalar – under G. For each representation π of

G, there exist a finite number of functionally and algebraically independent invariants. As an example, let us consider the three-dimensional representation of the orthogonal group $SO(3)$ and $\vec{v}$ a three-dimensional vector in the representation space $\mathcal{V}$. Then the quantity $\vec{v}^{\,2} = \vec{v}.\vec{v}$ is invariant under $SO(3)$, that is $R\vec{v}.R\vec{v} = \vec{v}.\vec{v}$ for all $R \in SO(3)$. $\vec{v}^{\,2}$ is the only independent invariant of the representation, in the sense that any function of $\vec{v}$ invariant under $SO(3)$ is actually a function of $\vec{v}^{\,2}$.

Specially important invariants are those which can be built in the adjoint representation of G. At the Lie algebra level, one has to find the polynomials in the generators which are invariant under the action of the algebra. A semi-simple Lie algebra $\mathcal{G}$ of rank r has r fundamental invariants (that is any of these invariants cannot be written as the function of other invariants, and any invariant is either one of these r invariants or a function of them). In particular, $\mathcal{G}$ contains one fundamental invariant of degree 2:

$$C_2 = g^{\alpha\beta} X_\alpha X_\beta$$

where $g^{\alpha\beta}$ is the inverse of the Killing form $g_{\alpha\beta} = C_{\alpha\mu}{}^{\nu}\, C_{\beta\nu}{}^{\mu}$ corresponding to the basis $\{X_\alpha\}$ of $\mathcal{G}$ satisfying $[X_\alpha, X_\beta] = i\, C_{\alpha\beta}{}^{\gamma}\, X_\gamma$. C_2 and the other fundamental invariants of $\mathcal{G}$ are called the *Casimir invariants* of $\mathcal{G}$.

Table 1.2 gives the degrees of the Casimir invariants of the simple Lie algebras. The degrees of the Casimir invariants minus one are called the *exponents* of the simple Lie algebra.

Table 1.2: Casimir invariants of the simple Lie algebras.

$\mathcal{G}$	degrees of the Casimirs	$\mathcal{G}$	degrees of the Casimirs
A_{N-1}	2, 3, ..., N	E_6	2, 5, 6, 8, 9, 12
B_N	2, 4, ..., $2N$	E_7	2, 6, 8, 10, 12, 14, 18
C_N	2, 4, ..., $2N$	E_8	2, 8, 12, 14, 18, 20, 24, 30
D_N	2, 4, ..., $2N-2$, N	F_4	2, 6, 8, 12
		G_2	2, 6

Since a Casimir operator C built in the algebra of polynomials of $\mathcal{G}$, or enveloping algebra of $\mathcal{G}$ ($\rightarrow$ 1.91), is $\mathcal{G}$-invariant, one has

$$\Big[C, X\Big] = 0\,, \qquad \forall\, X \in \mathcal{G}$$

It follows, using Schur's theorem, that C is a multiple of the identity in any irreducible representation of G and one has the property:

Property

Let $\mathcal{G}$ be a simple Lie algebra of rank r. The eigenvalues of the r Casimir invariants on an irreducible representation π of $\mathcal{G}$ completely characterize this representation.

Let $\mathcal{G}$ be a simple Lie algebra of rank r. In the Cartan–Weyl basis ($\to$ 1.6) of $\mathcal{G}$, the second order Casimir invariant reads

$$C_2 = \sum_{i=1}^{r} H_i H_i + \sum_{\alpha\in\Delta^+} \frac{\alpha\cdot\alpha}{2}\,(E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha)$$

where Δ^+ is the positive root system, H_i are the Cartan generators and $E_{\pm\alpha}$ the root generators of $\mathcal{G}$.

If π is an irreducible representation of $\mathcal{G}$ with highest weight Λ, the evaluation of C_2 on the corresponding weight vector $|\Lambda\rangle$ is given by

$$C_2|\Lambda\rangle = \Big(\Lambda\cdot\Lambda + \sum_{\alpha\in\Delta^+} \alpha\cdot\Lambda\Big)|\Lambda\rangle = \Lambda\cdot(\Lambda+2\rho)|\Lambda\rangle$$

where ρ is the Weyl vector (half-sum of the positive roots) ($\to$ 1.94).

Since C_2 commutes with all the generators of $\mathcal{G}$, it follows that C_2 has the same eigenvalue on all states of the representation π of highest weight Λ (as indicated in the above property). It has also the same eigenvalue on all states of the conjugate representation π^* of highest weight Λ^*.

Examples

The Lie algebra $sl(2)$ is of rank one. If $\Big[J_i, J_j\Big] = i\,\varepsilon_{ijk}\,J_k$ $(i,j,k = 1,2,3)$, the only Casimir operator is

$$C_2 = J^2 = \sum_{i=1}^{3} J_i^2$$

Its eigenvalue for the representation D_j of dimension $(2j+1)$ is $j(j+1)$.

The Lie algebra $sl(3)$ is of rank two. Using the Gell-Mann matrices λ_i ($i = 1,\dots,8$) satisfying $\Big[\lambda_i,\lambda_j\Big] = 2i\,f_{ijk}\,\lambda_k$ and $\Big\{\lambda_i,\lambda_j\Big\} = \frac{4}{3}\delta_{ij}\,\mathbb{I} + 2d_{ijk}\,\lambda_k$ ($\to$ 1.2), the two Casimir operators can be written as

$$C_2 = \sum_{i=1}^{8} \lambda_i^2 \qquad \text{and} \qquad C_3 = \sum_{i,j,k=1}^{8} d_{ijk}\,\lambda_i\lambda_j\lambda_k$$

and their eigenvalues in the representation $D(j_1, j_2)$ (whose corresponding Young tableau ($\to$ 1.96) has $j_1 + j_2$ boxes in the first row and j_2 boxes in the second row) are:

$$C_2 = \frac{1}{9}\Big[(j_1^2 + j_1 j_2 + j_2^2) + 3(j_1 + j_2)\Big]$$
$$C_3 = 16(j_1 - j_2)\Big[\tfrac{2}{9}(j_1 + j_2)^2 + \tfrac{1}{9}j_1 j_2 + j_1 + j_2 + 1\Big]$$

□

A complete study of the eigenvalues of the Casimir operators for the compact Lie algebras A_N, B_N, C_N, D_N and G_2 can be found in ref. [68].

1.10 Center (of a group, of an algebra)

Definition

Let G be a group; then the center of G is the set of elements of G which commute with any element in G. We will denote it $\mathcal{Z}(G)$:

$$\mathcal{Z}(G) = \Big\{ g \in G \,\Big|\, g * g' = g' * g \,,\ \forall\, g' \in G \Big\}$$

$\mathcal{Z}(G)$ is a normal Abelian subgroup of G.

Definition

Let $\mathcal{G}$ be a Lie algebra; then the center $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ is the set of elements of $\mathcal{G}$ which commute with any element of $\mathcal{G}$:

$$\mathcal{Z}(\mathcal{G}) = \Big\{ X \in \mathcal{G} \,\Big|\, \Big[X, Y\Big] = 0 \,,\ \forall\, Y \in \mathcal{G} \Big\}$$

$\mathcal{Z}(\mathcal{G})$ is an (Abelian) ideal of $\mathcal{G}$.

Property

The center of a simple Lie algebra is empty.

1.11 Centralizer (of a group, of an algebra)

Definition

Let G be a group, S a subset of elements in G. The centralizer $\mathcal{C}_G(S)$ is the subset of G given by

$$\mathcal{C}_G(S) = \Big\{ g \in G \,\Big|\, g * s = s * g \,,\ \forall\, s \in S \Big\}$$

The centralizer $\mathcal{C}_G(S)$ is a subgroup of G.

Definition

Let $\mathcal{G}$ be a Lie algebra, $\mathcal{S}$ a subset of elements in $\mathcal{G}$. The centralizer $\mathcal{C}_{\mathcal{G}}(\mathcal{S})$ is the subset of $\mathcal{G}$ given by

$$\mathcal{C}_{\mathcal{G}}(\mathcal{S}) = \Big\{ X \in \mathcal{G} \,\Big|\, \big[X, Y\big] = 0\,,\ \forall Y \in \mathcal{S} \Big\}$$

The centralizer $\mathcal{C}_{\mathcal{G}}(\mathcal{S})$ is a subalgebra of $\mathcal{G}$.

1.12 Characters

This very important notion is used in many places. It is involved in the determination of dimensions of representations, as well as of partition functions for statistical mechanics models and string theory. Below, and also in the rest of this book, only finite dimensional representations of finite dimensional groups are considered.

Definition

Let G be a group and π a finite dimensional representation of G. The character of π is the function χ from G to $\mathbb{R}$ (or $\mathbb{C}$) defined by

$$\chi(g) = \mathrm{tr}[\pi(g)] \qquad \text{for any } \ g \in G$$

It is obvious that $\chi(g) = \chi(g')$ if g and g' in G are conjugated, that is $\exists\ g_0 \in G$ such that $g = g_0 g' g_0^{-1}$. We note also that two equivalent representations π and π' of G ($\to$ 1.72) have the same character.

Since any element x in a simple Lie algebra $\mathcal{G}$ of rank r is conjugated to an element h in a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ by a element $g \in G$, $x = ghg^{-1}$ with $h = \sum_{j=1}^{r} \phi^j h_j$, the character of a G representation will be a function of the r variables $\phi^1, \dots, \phi^r$.

Example

As an example, let us look at the $SU(2)$ group. Any element in its Lie algebra can be seen as the conjugate of a Cartan element $h = \phi J_3$, referring to traditional notations. It follows that the corresponding G element is written in the $(2j+1)$ dimensional D_j representation as

$$D_j(\phi) = \begin{pmatrix} e^{ij\phi} & & & & \\ & e^{i(j-1)\phi} & & & \\ & & \ddots & & \\ & & & e^{-i(j-1)\phi} & \\ & & & & e^{-ij\phi} \end{pmatrix}$$

Therefore, the character χ_j of D_j depends only of the angle ϕ as follows:

$$\chi_j = \sum_{m=-j}^{j} e^{im\phi} = \frac{\sin(j+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}$$

□

Theorem (Weyl formula)

Considering a (semi-)simple Lie algebra $\mathcal{G}$, the character of its irreducible representation $\pi(\Lambda)$, determined by its highest weight Λ, is given by the Weyl formula:

$$\chi(\Lambda) = \frac{\displaystyle\sum_{w\in W} \varepsilon(w)\, e^{w(\Lambda+\rho)}}{\displaystyle\sum_{w\in W} \varepsilon(w)\, e^{w(\rho)}}$$

where the sum is over all the elements w of the Weyl group W of $\mathcal{G}$, $\varepsilon(w)$ is the parity of the element w ($\rightarrow$ 1.93) and ρ is the Weyl vector, half-sum of the positive roots.

The character function $\chi(\Lambda)$ acts on the Cartan element $h \in \mathcal{H}$ via the root applications defined on its dual $\alpha \in \mathcal{H}^*$, that is

$$\alpha(h) = \sum_{j=1}^{r} \alpha_j \phi^j \quad \text{if } \; h = \sum_{j=1}^{r} \phi^j h_j$$

1.13 Classical Lie groups and Lie algebras

Classical Lie groups

The classical Lie groups can be described as subgroups of the general linear group $GL(n,\mathbb{C})$ (resp. $GL(n,\mathbb{R})$) of order n of invertible $n \times n$ complex (resp. real) matrices.

One sets, if $\mathbb{I}_n$ denotes the $n \times n$ unit matrix,

$$\mathbb{I}_{p,q} = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix} \qquad \text{and} \qquad \mathbb{J}_{2n} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

One defines the classical Lie groups as follows (* denotes the complex conjugation, ^{t} the transposition and † the transconjugation, i.e. transposition and complex conjugation):

Linear groups:

– The special linear group $SL(n,\mathbb{C})$ (resp. $SL(n,\mathbb{R})$) is the group of complex matrices $M \in GL(n,\mathbb{C})$ (resp. real matrices $M \in GL(n,\mathbb{R})$) with determinant 1.

– The unitary group $U(n)$ is the group of complex matrices $M \in GL(n,\mathbb{C})$ such that $M^\dagger M = \mathbb{I}_n$. The special unitary group $SU(n)$ is the subgroup of matrices of $U(n)$ with determinant 1.

– The unitary group $U(p,q)$ with signature (p,q) is the group of complex matrices $M \in GL(p+q,\mathbb{C})$ such that $M^\dagger \mathbb{I}_{p,q} M = \mathbb{I}_{p,q}$. The special unitary group $SU(p,q)$ with signature (p,q) is the subgroup of matrices of $U(p,q)$ with determinant 1.

– The special star unitary group $SU^*(2n)$ is the group of complex matrices $M \in SL(2n,\mathbb{C})$ such that $\mathbb{J}_{2n} M^* = M \mathbb{J}_{2n}$.

Orthogonal groups:

– The orthogonal group $O(n,\mathbb{C})$ is the group of complex matrices $M \in GL(n,\mathbb{C})$ such that $M^t M = \mathbb{I}_n$. The special orthogonal group $SO(n,\mathbb{C})$ is the subgroup of matrices of $O(n,\mathbb{C})$ with determinant 1.

– The orthogonal group $O(n)$ is the group of real matrices $M \in GL(n,\mathbb{R})$ such that $M^t M = \mathbb{I}_n$. The special orthogonal group $SO(n)$ is the subgroup of matrices of $O(n)$ with determinant 1.

– The orthogonal group $O(p,q)$ with signature (p,q) is the group of real matrices $M \in GL(p+q,\mathbb{R})$ such that $M^t \mathbb{I}_{p,q} M = \mathbb{I}_{p,q}$. The special orthogonal group $SO(p,q)$ with signature (p,q) is the subgroup of matrices of $O(p,q)$ with determinant 1.

– The special star orthogonal group $SO^*(2n)$ is the group of complex matrices $M \in SO(2n,\mathbb{C})$ such that $M^t \mathbb{J}_{2n} M^* = \mathbb{J}_{2n}$.

Symplectic groups:

– The symplectic group $Sp(2n,\mathbb{C})$ (resp. $Sp(2n,\mathbb{R})$) is the group of complex matrices $M \in GL(2n,\mathbb{C})$ (resp. real matrices $M \in GL(2n,\mathbb{R})$) such that $M^t \mathbb{J}_{2n} M = \mathbb{J}_{2n}$.

– The symplectic group $Sp(2n)$ is the group of complex matrices $M \in U(2n)$ such that $M^t \mathbb{J}_{2n} M = \mathbb{J}_{2n}$.

– The symplectic group $Sp(2p,2q)$ with signature (p,q) is the group of complex matrices $M \in Sp(2p+2q,\mathbb{C})$ such that $M^\dagger \mathbb{I}_2 \otimes \mathbb{I}_{p,q} M = \mathbb{I}_2 \otimes \mathbb{I}_{p,q}$.

Classical Lie algebras

Since the Lie algebra $gl(n,\mathbb{C})$ of the general linear group $GL(n,\mathbb{C})$ is the Lie algebra of all $n \times n$ complex matrices with the Lie bracket as Lie product ($\rightarrow$ 1.46), the corresponding classical Lie algebras are described in terms of suitable subalgebras of $gl(n,\mathbb{C})$ as follows:

Linear and unitary algebras:
– The special linear algebra $sl(n,\mathbb{C})$ (resp. $sl(n,\mathbb{R})$ is the set of $n\times n$ complex (resp. real) traceless matrices.
– The unitary algebra $u(n)$ is the set of $n\times n$ complex antihermitian matrices: $m^\dagger = -m$. The special unitary Lie algebra $su(n)$ is the subset of $u(n)$ of traceless matrices: $\mathrm{tr}(m) = 0$.
– The unitary algebra $u(p,q)$ with signature (p,q) is the set of $(p+q)\times(p+q)$ complex matrices such that $m^\dagger \mathbb{I}_{p,q} = -m\mathbb{I}_{p,q}$. The special unitary Lie algebra $su(p,q)$ is the subset of $u(p,q)$ of traceless matrices: $\mathrm{tr}(m) = 0$.
– The special star unitary algebra $su^*(2n)$ is the set of $2n \times 2n$ complex traceless matrices such that $\mathbb{J}_{2n} m^* = m\mathbb{J}_{2n}$.

Orthogonal algebras:
– The special orthogonal algebra $so(n,\mathbb{C})$ is the set of $n \times n$ antisymmetric complex matrices: $m^t = -m$.
– The special orthogonal algebra $so(n)$ is the set of $n \times n$ antisymmetric real matrices: $m^t = -m$.
– The special orthogonal algebra $so(p,q)$ with signature (p,q) is the set of $(p+q)\times(p+q)$ real matrices such that $m^t\mathbb{I}_{p,q} = -\mathbb{I}_{p,q}m$.
– The special star orthogonal algebra $so^*(2n)$ is the set of $2n \times 2n$ complex antisymmetric matrices such that $m^\dagger \mathbb{J}_{2n} = -\mathbb{J}_{2n}m$.

Symplectic algebras:
– The symplectic algebra $sp(2n,\mathbb{C})$ (resp. $sp(2n,\mathbb{R})$) is the set of $2n \times 2n$ complex (resp. real) matrices such that $m^t\mathbb{J}_{2n} = -\mathbb{J}_{2n}m$.
– The symplectic algebra $sp(2n)$ is the set of $2n\times 2n$ complex antihermitian matrices ($m^\dagger = -m$) such that $m^t\mathbb{J}_{2n} = -\mathbb{J}_{2n}m$.
– The symplectic algebra $sp(2p,2q)$ with signature (p,q) is the group of matrices of $sp(2p+2q,\mathbb{C})$ such that $m^t\mathbb{J}_{2n} = -\mathbb{J}_{2n}m$ and $m^\dagger\mathbb{I}_2\otimes\mathbb{I}_{p,q} = -\mathbb{I}_2\otimes\mathbb{I}_{p,q}m$.

1.14 Clebsch-Gordan coefficients

The direct product of two irreducible representations $D(j_1)$ and $D(j_2)$ of the rotation group $SO(3)$ decomposes into a direct sum of irreducible representations:

$$D(j_1)\otimes D(j_2) = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} D(J)$$

Let $|j_1m_1\rangle$ with $-j_1 \le m_1 \le j_1$ and $|j_2m_2\rangle$ with $-j_2 \le m_2 \le j_2$ be the canonical bases of the Hilbert spaces $\mathcal{H}(D(j_1))$ and $\mathcal{H}(D(j_2))$ of the irreducible representations $D(j_1)$ and $D(j_2)$. The product space $\mathcal{H}(D(j_1))\otimes \mathcal{H}(D(j_2))$ decomposes into a direct sum of invariant spaces $\mathcal{H}(D(J))$ which transform according to the irreducible representation $D(J)$. Let $|JM\rangle$ with $-J \leq M \leq J$ denote the canonical basis of the space $\mathcal{H}(D(J))$. We have

$$|JM\rangle = \sum_{m_1+m_2=M} \langle j_1 j_2 m_1 m_2 | JM\rangle \; |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

The numerical coefficients $\langle j_1 j_2 m_1 m_2 | JM\rangle$ are called *Clebsch-Gordan coefficients.*

To complete the definition of the Clebsch-Gordan coefficients, once fixed the transformation properties of the canonical basis with respect to the infinitesimal generators, one needs to fix a phase. In general it is required

$$\langle j_1 j_2 m_1 m_2 | JM\rangle \geq 0$$

Since the Clebsch-Gordan coefficients relate two orthonormal bases, they can be considered as matrix elements of an unitary matrix, that is

$$\sum_{m_1} \langle j_1 j_2 m_1 m_2 | JM\rangle \langle j_1 j_2 m_1 m_2 | J'M'\rangle = \delta_{JJ'} \delta_{MM'}$$

The Clebsch-Gordan coefficients have many symmetry and orthogonality relations and can be computed in different ways. For the properties and the explicit calculation of Clebsch-Gordan coefficients, see for example ref. [60]. The concept of Clebsch-Gordan coefficients can be generalized to any direct product of two irreducible representations π_1 and π_2 of any Lie group. In the general definition, Clebsch-Gordan coefficients are the numerical coefficients relating the canonical basis of any irreducible representation π appearing in the direct product to the bases of π_1 and π_2. For the groups of relevant interest for physics they have been explicitly computed: for $SU(3)$ see ref. [15], for $SU(4)$ see ref. [37], for $SU(6)$ see ref. [88].

1.15 Compacity

Definition

A Lie group G of dimension n is *compact* if the domain of variation of its n essential parameters $a_1, \ldots, a_n$ is compact.

As an example, the rotation group in N dimensions $SO(N)$ is compact since the domain of variation of each of its $N(N-1)/2$ parameters is the closed and bounded subset $[0, 2\pi]$ of $\mathbb{R}$.

The Poincaré group $P(3,1)$ is not compact, but contains as a subgroup the rotation group $SO(3)$ which is compact. Actually, $SO(3)$ is maximal as a compact subgroup of $P(3,1)$, any compact subgroup of $P(3,1)$ being a subgroup of $SO(3)$, up to a conjugation. More generally, any Lie group G admits a maximal compact subgroup (which may be the unit element).

The Lorentz group is not compact. Actually, it is isomorphic to $O(3,1)$, which is a non-compact form of $O(4)$. The Lie algebra of $O(3,1)$ can be easily deduced from the Lie algebra of $O(4)$: let M_{ij} $(i,j=0,1,2,3)$ be the generators of $SO(4)$ ($\to$ 1.58), multiply by $i \in \mathbb{C}$ the three generators M_{0j} $(j=1,2,3)$, that is define $M'_{0j} = iM_{0j}$, and keep unchanged the three other generators M_{ij} $(i,j=1,2,3)$. The three generators M_{ij} $(i,j=1,2,3)$ form again the algebra of the rotation group $SO(3)$ while the three generators M'_{0j} $(j=1,2,3)$ generate the three "boosts" in the Minkowski space, or "complex rotations" in the plane tx, ty, tz if the four axis are denoted t,x,y,z.

This method of constructing the non-compact forms from a compact Lie algebra is general, and can be illustrated on the $SO(N)$ Lie algebras as follows:

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline -M_{12}^t & M_{22} \end{array} \right) \in SO(p+q) \to M = \left(\begin{array}{c|c} M_{11} & iM_{12} \\ \hline -iM_{12}^t & M_{22} \end{array} \right) \in SO(p,q)$$

The $SO(p+q)$ group leaves invariant the scalar product $\vec{x}.\vec{y} = \sum_{i=1}^{p+q} x_i y_i$ with $\vec{x}, \vec{y} \in \mathbb{R}^{p+q}$, while the $SO(p,q)$ group leaves invariant the product $\vec{x}.\vec{y} = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{q} x_j y_j$ ($\to$ 1.13 Classical Lie groups and Lie algebras).

More on the general structure of non-compact semi-simple Lie algebra can be found in $\to$ 1.44 Killing form.

1.16 Complex structures

Definition

> A *complex structure* on a vector space $\mathcal{V}$ over $\mathbb{R}$ of finite dimension is an $\mathbb{R}$-linear endomorphism J of $\mathcal{V}$ such that $J^2 = -\mathbb{I}$, $\mathbb{I}$ being the identity mapping of $\mathcal{V}$.

A real vector space $\mathcal{V}$ with a complex structure J can be transformed into a complex vector space $\mathcal{V}_C$ by setting $iX \equiv JX$ for any $X \in \mathcal{V}$. One remarks that $\dim_C \mathcal{V}_C = \frac{1}{2} \dim_R \mathcal{V}$, which implies that $\mathcal{V}$ must be even-dimensional.

Definition

> A *complex structure* on a real Lie algebra $\mathcal{G}$ is a complex structure J on its $\mathcal{G}$ vector space which satisfies in addition
>
> $$[X, JY] = J[X,Y] \qquad \text{for all } X, Y \in \mathcal{G}$$

Referring to the real forms of the simple Lie algebras ($\to$ 1.71), one deduces that the real Lie algebra $\mathcal{G}_R$, called the realification of the complex

Lie algebra $\mathcal{G}$, possesses a canonical complex structure J derived from the multiplication by the complex number i on $\mathcal{G}$.

1.17 Conformal group

In d dimensions, the conformal group is the group of general coordinate transformations which leave the metric invariant, up to a scale factor. Let $\mathbb{R}^d$ be the Euclidean space with the metric $g^{\mu\nu}$ of signature (p,q) where $d = p + q$. Then the conformal transformations can be written as

$$x^\mu \mapsto x'^\mu \qquad \text{such that} \qquad g^{\mu\nu} \mapsto g'^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g^{\alpha\beta} = \Omega(x) g^{\mu\nu}$$

The conformal transformations in d dimensions ($d \geq 3$) contain:

– the d translations $x^\mu \mapsto x^\mu + a^\mu$,
– the $d(d-1)/2$ rotations $x^\mu \mapsto \Omega^\mu_\nu x^\nu$ with $\Omega^\mu_\nu \in SO(p,q)$,
– the dilatation $x^\mu \mapsto \lambda x^\mu$,
– the d special conformal transformations $x^\mu \mapsto \dfrac{x^\mu - b^\mu x^2}{b^2x^2 - 2bx + 1}$.

The corresponding conformal algebra is actually isomorphic to $so(p+1, q+1)$ of dimension $\frac{1}{2}(d+1)(d+2)$.

In the Minkowski space $\mathbb{R}^4$ of signature $(3,1)$, the conformal algebra, isomorphic to $so(4,2)$ is 15-dimensional. Choosing as a basis the matrices $M_{\alpha\beta} = -M_{\beta\alpha}$ satisfying $(\alpha, \beta, \gamma, \delta = 0, 0', 1, 2, 3, 4)$

$$\Big[M_{\alpha\beta}, M_{\gamma\delta}\Big] = i(-g_{\beta\delta}M_{\alpha\gamma} + g_{\beta\gamma}M_{\alpha\delta} + g_{\alpha\delta}M_{\beta\gamma} - g_{\alpha\gamma}M_{\beta\delta})$$

with $g_{00} = g_{0'0'} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, the generators $M_{\mu\nu}$ and $P_\mu = M_{0'\mu} + M_{4\mu}$ with $\mu, \nu = 0, 1, 2, 3$ span the Poincaré algebra, the generators of the special conformal transformations appear as $K_\mu = M_{0'\mu} - M_{4\mu}$, while $D = M_{0'4}$ is the generator of the dilatations. One can check that:

$$\begin{aligned}
&\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma}) \\
&\Big[M_{\mu\nu}, P_\rho\Big] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \\
&\Big[M_{\mu\nu}, K_\rho\Big] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu) \\
&\Big[D, M_{\mu\nu}\Big] = 0 \qquad \Big[D, P_\mu\Big] = -iP_\mu \qquad \Big[D, K_\mu\Big] = iK_\mu \\
&\Big[P_\mu, P_\nu\Big] = 0 \qquad \Big[K_\mu, K_\nu\Big] = 0 \qquad \Big[P_\mu, K_\nu\Big] = 2i(g_{\mu\nu}D - M_{\mu\nu})
\end{aligned}$$

The three Casimir invariants are:

$$C_2 = M_{\alpha\beta} M^{\alpha\beta} \qquad C_3 = M_{\alpha\beta} W^{\alpha\beta} \qquad C_4 = W_{\alpha\beta} W^{\alpha\beta}$$

where $W_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta\rho\sigma} M^{\gamma\delta} M^{\rho\sigma}$.

Since $\left[D, P^2\right] = -2iP^2$, only massless discrete states may exist in a scale invariant theory.

Free electromagnetic theory and wave equation for massless particles exhibit a symmetry for a larger group than the Poincaré group, namely the conformal group $O(4,2)$.

For more details, see refs. [6, 20].

1.18 Connexity

These topological notions are important for at least two reasons. First, they allow us to understand that two Lie groups with the same Lie algebra can differ (example: $SO(3)$ is doubly connected and $SU(2)$ simply connected and as Lie groups are related as follows: $SO(3) \simeq SU(2)/Z_2$). Second, such properties are directly related to the existence of monopoles in gauge theories (see for example ref. [33]).

Definition

A space is said to be *connected* if any two points in the space can be joined by a line, and all the points of the line lie in the space.

Example

Consider the two-dimensional Lie group of transformations G acting on $\mathbb{R}$ such that

$$x \mapsto ax + b \qquad a, b \in G, x \in \mathbb{R}$$

G can actually be defined as the manifold in $\mathbb{R}^2$:

$$G = \left\{ (a,b) \,\middle|\, a, b \in \mathbb{R}, a \neq 0 \text{ and } (a',b') * (a,b) = (aa', b + b') \right\}$$

We realize that G is not connected: the segment joining the two points $(-1,0)$ and $(1,0)$ on the a-axis does not belong to G, since $(0,0) \notin G$. □

Definition

A connected space is said to be *simply connected* if a curve connecting any two points in the space can be continuously deformed into every other curve connecting the same two points.

Example

The torus is not simply connected since there does not exist a continuous mapping of the major circle onto the minor circle. □

The $SO(3) - SU(2)$ case

The two groups $SO(3)$ and $SU(2)$ have isomorphic Lie algebras. Indeed, a basis of the Lie algebra $su(2)$ is provided by the 2×2 hermitian Pauli matrices σ_i $(i = 1, 2, 3)$ which satisfy

$$\Big[\sigma_i, \sigma_j\Big] = 2i\, \varepsilon_{ijk}\, \sigma_k$$

or defining the antihermitian matrices $X_i = i\sigma_i/2$ such that $\Big[X_i, X_j\Big] = -\varepsilon_{ijk} X_k$, while a basis of the Lie algebra $so(3)$ is obtained with the 3×3 antisymmetric real matrices M_i $(i = 1, 2, 3)$ such that $(M_i)_{jk} = \varepsilon_{ijk}$ which also satisfy

$$\Big[M_i, M_j\Big] = -\varepsilon_{ijk}\, M_k$$

The most general elements of $SU(2)$ and $SO(3)$ can be written respectively as

$$\exp(\theta^i X_i) = \mathbb{I}_2 \cos\frac{\theta}{2} + i(\widehat{\theta}^i\, \sigma_i) \sin\frac{\theta}{2}$$

and

$$\exp(\theta^i M_i) = \mathbb{I}_3 + (\widehat{\theta}^i\, M_i) \sin\theta + (\widehat{\theta}^i\, M_i)^2 (1 - \cos\theta)$$

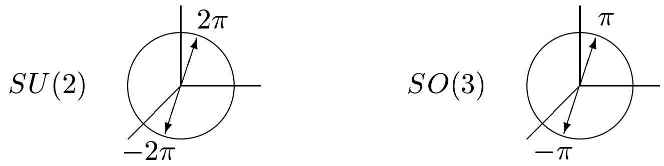
where $\mathbb{I}_2$ and $\mathbb{I}_3$ denote the 2×2 and 3×3 unit matrices, the parameters θ^i $(i = 1, 2, 3)$ determine the angle of rotation $\theta = \sqrt{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^3}$ around the unit vector $\widehat{\theta}$ of components $\widehat{\theta}^i = \theta^i/\theta$. Now, we can see from the above expressions that the two groups are not the same since

$$\text{for } SU(2) \qquad (\widehat{\theta}, \theta) = -(\widehat{\theta}, \theta + 2\pi) = (\widehat{\theta}, \theta + 4\pi)$$

while

$$\text{for } SO(3) \qquad (\widehat{\theta}, \theta) = (\widehat{\theta}, \theta + 2\pi)$$

If we associate, in the three-dimensional space $\mathbb{R}^3$, to each element $(\widehat{\theta}, \theta)$ the end point of the vector of origin O and length θ in the direction $\widehat{\theta} = (\widehat{\theta}^1, \widehat{\theta}^2, \widehat{\theta}^3)$, then the $SU(2)$ group will be represented by a sphere of radius 2π with all the points on its surface identified with $-\mathbb{I}$, while the $SO(3)$ group will be represented by a sphere of radius π with antipodal points (that is a pair of points on the surface on the sphere symmetric with respect to the origin O) identified.



Consider C a closed path AOB cutting the surface of the $SO(3)$ parameter sphere "once", A and B being antipodal points and therefore identified. The closed path C cannot be continuously deformed to the point O. Indeed as soon as one antipodal point is moved one way, the other must move the other way in order to remain antipodal. In fact, it appears that there are two classes of closed paths in $SO(3)$: the class of all paths that cut the surface an even number of times (any path of this kind can be deformed up to the identity), and the class of paths which cut the surface an odd number of times. Considering in $SO(3)$ all the closed paths with a common point, O for instance, one can separate the paths which cut the surface an even number of times from those which cut the surface an odd number of times: the two classes of paths thus obtained are such that two paths can be deformed one into the other if and only if they belong to the same class. One will say that the homotopy group of $SO(3)$ contains two elements, or that $SO(3)$ is doubly connected.

An analogous analysis for $SU(2)$ would show that $SU(2)$ is simply connected, or that its homotopy group reduces to the identity. Moreover, it appears that $SU(2)$ has one and only one (if we exclude the identity) invariant discrete subgroup $Z_2 = \{-\mathbb{I}, \mathbb{I}\}$ and that the quotient group $SU(2)/Z_2$ is isomorphic to $SO(3)$.

Actually, this relationship between homotopy groups and discrete invariant subgroups (or between algebra and topology) in the theory of Lie groups is more general and allows to relate the different Lie groups with the same Lie algebra.

Theorem

Let G be a Lie group with homotopy group H_G and Lie algebra $\mathcal{G}$. Then there exists exactly one simply connected Lie group $\overline{G}$ with Lie algebra $\mathcal{G}$. Moreover

$$\overline{G}/D \simeq G$$

D being one of the discrete subgroup of $\overline{G}$ (actually $H_G \simeq D$). $\overline{G}$ is called the *universal covering group* of G. The Lie group G is said to be multiply connected when D contains more than one element.

Thus the enumeration of all possible Lie groups with the same Lie algebra $\mathcal{G}$ reduces to the problem of finding all possible discrete invariant subgroups of the simply connected group $\overline{G}$ possessing $\mathcal{G}$ as Lie algebra. Such a program can be achieved as follows as soon as $\overline{G}$ is known: determine the center $\mathcal{Z}(\overline{G})$ of $\overline{G}$; then select in $\mathcal{Z}(\overline{G})$ the set of all discrete operations. This set forms a group D. Any subgroup of D is a discrete invariant subgroup of $\overline{G}$ and all discrete invariant subgroups of $\overline{G}$ are subgroups of D.

1.19 Contraction

Let $\mathcal{G}$ be a Lie algebra of dimension n generated by X_i with commutation relations $[X_i, X_j] = C_{ij}{}^k X_k$ where $i, j, k = 1, \dots, n$, and let us define a basis in $\mathcal{G}$ depending on an arbitrary parameter ε such that $\widetilde{X}_i = X_i(\varepsilon)$. Then

$$\left[\widetilde{X}_i, \widetilde{X}_j\right] = C_{ij}{}^k(\varepsilon)\widetilde{X}_k$$

If for any triplet i, j, k, the following limit makes sense:

$$\lim_{\varepsilon\to\infty} C_{ij}{}^k(\varepsilon) = C_{ij}{}^k(\infty)$$

one can define a new Lie algebra $\widetilde{\mathcal{G}}$ of dimension n, called the contraction of $\mathcal{G}$ with the commutation relations

$$\left[\widetilde{X}_i, \widetilde{X}_j\right] = C_{ij}{}^k(\infty)\widetilde{X}_k$$

As an example, consider the Lie algebra $so(3)$:

$$\left[X_i, X_j\right] = \varepsilon_{ijk}\, X_k$$

and define

$$\widetilde{X}_1 = \varepsilon X_1\,, \qquad \widetilde{X}_2 = \varepsilon X_2\,, \qquad \widetilde{X}_3 = X_3$$

Then $[\widetilde{X}_1, \widetilde{X}_2] = \varepsilon \widetilde{X}_3 \to 0$ when $\varepsilon \to 0$, while $[\widetilde{X}_3, \widetilde{X}_1] = \widetilde{X}_2$ and $[\widetilde{X}_3, \widetilde{X}_2] = \widetilde{X}_1$. We therefore obtain the Lie algebra of the Euclidean group $E(2)$ in two dimensions (one rotation and two translations).

Other examples are given with the Galilei group ($\to$ 1.33) and the De Sitter group ($\to$ 1.24).

1.20 Coxeter number

Let $\mathcal{G}$ be a simple Lie algebra, A its Cartan matrix and $\widehat{A}$ its generalized Cartan matrix ($\to$ 1.8). One has $\widehat{A} = (\widehat{A}_{ij})$ with $0 \le i, j \le r$ and $A = (A_{ij})$

with $1 \leq i, j \leq r$. Then there is an unique vector $\vec{a}$ with positive integer components $(a_0, a_1, \dots, a_n)$ that are relatively prime (i.e. their greatest common divisor is one), such that

$$\sum_{j=0}^{n} \widehat{A}_{ij}\, a_j = 0$$

The components of $\vec{a}$ labelled by $j \neq 0$ are just the components of the highest root of the Lie algebra $\mathcal{G}$, that is the marks or Kac labels ($\rightarrow$ 1.83) associated to the Dynkin diagram corresponding to the matrix A. Let us introduce the transposed matrix A^T and the corresponding Dynkin diagram, which is obtained from the previous one by interchanging the direction of the arrows, and the dual vector $\vec{a}^{\vee}$, whose 0-th component is equal to one.

Definition

The *Coxeter number* h and the *dual Coxeter number* $h^{\vee}$ of a simple Lie algebra $\mathcal{G}$ are defined by:

$$h = \sum_{i=0}^{n} a_i \qquad \text{and} \qquad h^{\vee} = \sum_{i=0}^{n} a_i^{\vee}$$

The Coxeter and dual Coxeter numbers of the simple Lie algebras are the following ($h^{\vee} = h$ for the simply-laced Lie algebras):

$\mathcal{G}$	A_N	B_N	C_N	D_N	G_2	F_4	E_6	E_7	E_8
h	$N+1$	$2N$	$2N$	$2N-2$	6	12	12	18	30
$h^{\vee}$	$N+1$	$2N-1$	$N+1$	$2N-2$	4	9	12	18	30

The maximal exponent ($\rightarrow$ 1.9) of a simple Lie algebra is equal to $h-1$.

1.21 Decompositions w.r.t. $sl(2)$ subalgebras

The method for finding the decompositions of the fundamental and the adjoint representations of the classical Lie algebras with respect to their different $sl(2)$ subalgebras is the following:

1. One considers a $sl(2)$ embedding in a classical Lie algebra $\mathcal{G}$, determined by a certain subalgebra $\mathcal{K}$ in $\mathcal{G}$ ($\rightarrow$ 1.28), which is expressed as a direct sum of simple components: $\mathcal{K} = \oplus_i \mathcal{K}_i$.

2. To each couple $(\mathcal{G}, \mathcal{K}_i)$ one associates $sl(2)$ representations given in Table 3.40. Let us recall that the $sl(2)$ representations are labelled by a non-negative integer or half-integer j and denoted by D_j.

3. The decomposition of the fundamental representation of $\mathcal{G}$ with respect to the $sl(2)$ subalgebra under consideration is then given by a direct sum of $sl(2)$ representations.

4. Starting from a decomposition of the fundamental representation of $\mathcal{G}$ of the form
$$\text{fund}_{\mathcal{K}}\ \mathcal{G} = \oplus_i D_{j_i}$$
the decomposition of the adjoint representation $\text{ad}_{\mathcal{K}}\ \mathcal{G}$ is given in the unitary series by
$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\oplus_i D_{j_i} \Big) \otimes \Big(\oplus_i D_{j_i} \Big) - D_0$$
in the symplectic series by
$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\oplus_i D_{j_i} \Big) \otimes \Big(\oplus_i D_{j_i} \Big)\Big|_S$$
and in the orthogonal series by
$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\oplus_i D_{j_i} \Big) \otimes \Big(\oplus_i D_{j_i} \Big)\Big|_A$$

The following formulae, giving the symmetrized (S) and antisymmetrized (A) products of $sl(2)$ representations, are especially convenient:
$$(D_j \otimes D_j)_A = D_{2j-1} \oplus D_{2j-3} \oplus D_{2j-5} \oplus \dots$$
$$(D_j \otimes D_j)_S = D_{2j} \oplus D_{2j-2} \oplus D_{2j-4} \oplus \dots$$
and
$$\begin{aligned}\Big[\underbrace{(D_j \oplus \dots \oplus D_j)}_{m \text{ times}} \otimes \underbrace{(D_j \oplus \dots \oplus D_j)}_{m \text{ times}}\Big]_A &= \frac{m(m+1)}{2}(D_j \otimes D_j)_A \oplus \frac{m(m-1)}{2}(D_j \otimes D_j)_S \\ &= m(D_j \otimes D_j)_A \oplus \frac{m(m-1)}{2}(D_j \otimes D_j)\end{aligned}$$
$$\begin{aligned}\Big[\underbrace{(D_j \oplus \dots \oplus D_j)}_{m \text{ times}} \otimes \underbrace{(D_j \oplus \dots \oplus D_j)}_{m \text{ times}}\Big]_S &= \frac{m(m+1)}{2}(D_j \otimes D_j)_S \oplus \frac{m(m-1)}{2}(D_j \otimes D_j)_A \\ &= m(D_j \otimes D_j)_S \oplus \frac{m(m-1)}{2}(D_j \otimes D_j)\end{aligned}$$

Tables 3.40 and 3.41 give the different decompositions of the fundamental and adjoint representations of the classical Lie algebras with respect to the different $sl(2)$ embeddings.

1.22 Derivation of a Lie algebra

Let $\mathcal{G}$ be a Lie algebra and ϕ an automorphism ($\rightarrow$ 1.4) of $\mathcal{G}$. Then one has for all $X, Y \in \mathcal{G}$ and all $\alpha, \beta \in \mathbb{C}$:

$$\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$$
$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

If ϕ is a continuous automorphism, one can consider its infinitesimal part: $\phi = \mathbb{I} + d + \ldots$ One has then

$$d([X, Y]) = \Big[d(X), Y\Big] + \Big[X, d(Y)\Big]$$

d is called a *derivation* of $\mathcal{G}$. The set of derivations of the Lie algebra $\mathcal{G}$ has the structure of a Lie algebra. Indeed, one can check that if d and d' are derivations of $\mathcal{G}$, then for all $X, Y \in \mathcal{G}$:

$$(dd' - d'd)([X, Y]) = \Big[(dd' - d'd)(X), Y\Big] + \Big[X, (dd' - d'd)(Y)\Big]$$

and denoting $\mathrm{Aut}(\mathcal{G})$ the group of automorphisms of $\mathcal{G}$, its Lie algebra is actually the algebra of the derivations of $\mathcal{G}$ which will be denoted $\mathrm{Der}\,\mathcal{G}$.

In particular,

$$\mathrm{ad}_X : Y \mapsto \mathrm{ad}_X\,(Y) = \Big[X, Y\Big]$$

is a derivation of $\mathcal{G}$. These derivations are called inner derivations of $\mathcal{G}$. They form an ideal $\mathrm{Inder}\,\mathcal{G}$ of $\mathrm{Der}\,\mathcal{G}$. The algebra $\mathrm{Inder}\,\mathcal{G}$ can be identified with the algebra of the group $\mathrm{Int}(\mathcal{G})$, which is also the algebra of the group $\mathrm{Int}(G)$ of inner automorphisms of G, where G is a Lie group whose Lie algebra is $\mathcal{G}$.

Finally, in the same way that $\mathrm{Int}(G) \simeq G/\mathcal{Z}(G)$, we can write $\mathrm{Inder}\,\mathcal{G} \simeq \mathcal{G}/\mathcal{Z}(\mathcal{G})$.

1.23 Derivative of a Lie algebra – Nilpotent and solvable algebras

Definition

Let $\mathcal{G}$ be a Lie algebra. The set of all the elements obtained by commuting elements of $\mathcal{G}$ forms an ideal of $\mathcal{G}$ which is called the *derivative* of $\mathcal{G}$ and denoted $\mathcal{G}'$:

$$\mathcal{G}' = \Big\{ X \in \mathcal{G} \;\Big|\; \exists\, A, B \in \mathcal{G} \text{ such that } X = \Big[A, B\Big] \Big\}$$

and one writes $\mathcal{G}' = \Big[\mathcal{G}, \mathcal{G}\Big]$.

Property

If $\mathcal{G}$ is Abelian, then $\mathcal{G}' = \{0\}$.
If $\mathcal{G}$ is a simple Lie algebra, then $\mathcal{G}' = \mathcal{G}$.

Definition

The Lie algebra $\mathcal{G}$ is said to be *solvable* if, considering the series

$$\Big[\mathcal{G},\mathcal{G}\Big] = \mathcal{G}^{(1)} \ , \quad \Big[\mathcal{G}^{(1)},\mathcal{G}^{(1)}\Big] = \mathcal{G}^{(2)} \ , \quad \dots \ , \quad \Big[\mathcal{G}^{(i-1)},\mathcal{G}^{(i-1)}\Big] = \mathcal{G}^{(i)}$$

then there exists an integer n such that $\mathcal{G}^{(n)} = \{0\}$.
Notice that the $\mathcal{G}^{(i)}$ are ideals of $\mathcal{G}$.

Definition

The Lie algebra $\mathcal{G}$ is said to be *nilpotent* if, considering the series

$$\Big[\mathcal{G},\mathcal{G}\Big] = \mathcal{G}^{[1]} \ , \quad \Big[\mathcal{G},\mathcal{G}^{[1]}\Big] = \mathcal{G}^{[2]} \ , \quad \dots \ , \quad \Big[\mathcal{G},\mathcal{G}^{[i-1]}\Big] = \mathcal{G}^{[i]}$$

then there exists an integer n such that $\mathcal{G}^{[n]} = \{0\}$.
Notice that the $\mathcal{G}^{[i]}$ are ideals of $\mathcal{G}$.

Property

Let $\mathcal{G}$ be a Lie algebra.

– $\mathcal{G}$ is solvable if and only if the derivative $\mathcal{G}'$ is nilpotent.
– If $\mathcal{G}$ is nilpotent, then $\mathcal{G}$ is solvable (but a solvable Lie algebra is not necessary nilpotent: as an example, one can consider the two-dimensional Lie algebra generated by a and b such that $[a,b] = b$).
– If $\mathcal{G}$ is solvable (resp. nilpotent), then any subalgebra $\mathcal{H}$ of $\mathcal{G}$ is a solvable (resp. nilpotent) Lie algebra.

Definition

Let $\mathcal{G}$ be a Lie algebra. The maximal solvable ideal of $\mathcal{G}$ is called the *radical* of $\mathcal{G}$.

1.24 De Sitter group

In a non-flat space–time, the group of invariance of physical laws would no longer be the Poincaré group, but the group of invariance of the space. In a constant curvature space, the De Sitter space, the metric can be written as ($\mu,\nu = 0,1,2,3$)

$$ds^2 = \phi^2(x^2) g^{\mu\nu} dx_\mu dx_\nu$$

where $\phi(x^2) = (1 + x^2/4\pi R^2)^{-1}$, $x^2 = g^{\mu\nu}x_\mu x_\nu$ and $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$. R is a dimensional parameter, the radius of curvature of the space. The x_μ can be seen as stereographic projection coordinates of a five-dimensional pseudo-hypersphere

$$R^2 = g^{\alpha\beta}\xi_\alpha\xi_\beta$$

where $\alpha, \beta = 0, 1, 2, 3, 4$ and $g^{44} = -1$ (we are considering only the case of constant positive curvature). If $g^{44} = 1$, we have the so-called anti-De Sitter space.

The set of linear transformations which map the hypersphere into itself form the De Sitter group $O(4,1)$ (in the case of the anti-De Sitter space $O(3,2)$) and these transformations leave invariant the metric

$$ds^2 = g^{\alpha\beta} d\xi_\alpha d\xi_\beta = \phi^2(x^2) g^{\mu\nu} dx_\mu dx_\nu$$

The element of the group is defined by Λ with

$$\xi'_\alpha = \Lambda^\beta_\alpha \xi_\beta \qquad \text{such that} \qquad g_{\alpha\beta}\Lambda^\alpha_\gamma\Lambda^\beta_\delta = g_{\gamma\delta}$$

The action of the De Sitter group on the Minkowski space is not linear. The relation between the ξ_α and the x_μ is

$$\begin{aligned}
&\xi_\mu = \phi(x^2)x_\mu && \xi_4 = R\phi(x^2)(1 - x^2/4\pi R^2)^{-1} \\
&x_\mu = 2\xi_\mu/(1 + \xi_4/R) && x^2/4R^2 = \frac{1 - \xi_4/R}{1 + \xi_4/R} \\
&\phi(x^2) = \frac{2}{1 + \xi_4/R}
\end{aligned}$$

The De Sitter group is a 10-parameter group and its infinitesimal generators $M_{\alpha\beta}$ with $\alpha, \beta = 0, 1, 2, 3, 4$ satisfy the commutation relations (both for $SO(4,1)$ and $SO(3,2)$ depending on the choice of the metric tensor)

$$\Big[M_{\alpha\beta}, M_{\gamma\delta}\Big] = -i(g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\gamma}M_{\beta\delta} - g_{\alpha\delta}M_{\beta\gamma} - g_{\beta\gamma}M_{\alpha\delta})$$

Defining

$$\pi_\mu = M_{4\mu}/R \qquad \mu = 0, 1, 2, 3$$

we have the commutation relations

$$\begin{aligned}
&\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma}) \\
&\Big[M_{\mu\nu}, \pi_\rho\Big] = i(g_{\nu\rho}\pi_\mu - g_{\mu\rho}\pi_\nu) \\
&\Big[\pi_\mu, \pi_\nu\Big] = iM_{\mu\nu}/R^2
\end{aligned}$$

where $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$ and the $M_{\mu\nu}$ generate the Lorentz algebra.

When $R \to +\infty$, the above commutation relations give the commutation relations of the Poincaré algebra. The Poincaré algebra is a contraction (→ 1.19) of the De Sitter algebra in the limit of infinite radius of curvature of space–time.

The two Casimir operators are

$$C_2 = \frac{1}{2R^2} M_{\alpha\beta} M^{\alpha\beta} = \pi_\mu \pi^\mu + \frac{1}{2R^2} M_{\mu\nu} M^{\mu\nu}$$

and

$$C_4 = W_\alpha W^\alpha = W_\mu W^\mu + W_4^2$$

where $W_\eta = \frac{1}{8R} \varepsilon_{\eta\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta}$.

Taking the limit $R \to +\infty$, one recovers the two Casimir invariants of the Poincaré group (the square of the mass and the square of the Pauli–Lubanski vector). Note that in a curved space–time, the linear momentum and the angular momentum are mixed together. In the limit $R \to +\infty$, they become independent and the mass and the spin become good quantum numbers (see refs. [6, 36]).

1.25 Direct and semi-direct products of groups

Definition

A group G is the direct product of its subgroups H and K, and we note $G = H \times K$, if

1. $h * k = k * h \qquad \forall\, h \in H,\ \forall\, k \in K$
2. $\forall\, g \in G$, the decomposition $g = h * k$ where $h \in H, k \in K$ is unique

Theorem

A group G is the direct product of its subgroups H and K if and only if

1. $G = H * K$
2. $H \cap K = \{e\}$
3. H, K are normal subgroups

Construction: Let A and B be two groups. Defining the groups H and K by the isomorphisms $a \in A \mapsto (a, e'') \in H$ and $b \in B \mapsto (e', b) \in K$ where e' and e'' are the identity elements in A and B respectively, the group $G = H \times K$ can be identified by the set of couples (a, b) with the group law

$$(a, b) * (a', b') = (aa', bb')$$

Example

The translation group $T(2)$ in the two-dimensional Euclidean plane can be seen as the direct product $T_x \times T_y$ of the translation group along the x-axis by the translation group along the y-axis. $T(2)$ is a two-dimensional non-compact Lie group. □

Definition

A group G is the semi-direct product of a subgroup H by a subgroup K, and we note $G = H \ltimes K$, if

1. $G = H * K$
2. $H \cap K = \{e\}$
3. H is a normal subgroup

Then for all $g \in G$, the decomposition $g = h * k$ with $h \in H, k \in K$ is unique.

Construction: Let A and B be two groups. Defining the groups H and K by the isomorphisms $a \in A \mapsto (a, e'') \in H$ and $b \in B \mapsto (e', b) \in K$ where e' and e'' are the identity elements in A and B respectively and choosing a homomorphism $\psi : B \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of automorphisms of A, then we can identify the group G, semi-direct product of H by K, associated with the homomorphism ψ, with the set of couples (a, b) with the group law

$$(a, b) * (a', b') = (a.\psi_b(a'), bb')$$

and we note $G = H \ltimes_\psi K$.

Example

Let us take the case of the Euclidean group $E(3)$. Any element can be written as $(\vec{a}, R)$ (translation, rotation) and we have the group law

$$(\vec{a}, R) * (\vec{a}\,', R') = (\vec{a} + R\vec{a}\,', RR')$$

□

In this example, the homomorphism ψ which is involved associates to each element R of the rotation group $SO(3)$ the rotation R itself acting on the translation group as an automorphism: $R\vec{a}\,' = \vec{a}\,''$. We check immediately from the group law that the translation part $T(3)$ is an invariant subgroup of $E(3)$, indeed:

$$(\vec{a}, R)(\vec{a}_0, \mathbb{I})(\vec{a}, R)^{-1} = (\vec{a} + R\vec{a}_0, R)(-R^{-1}\vec{a}, R^{-1}) = (R\vec{a}_0, \mathbb{I})$$

We note $E(3) = T(3) \ltimes SO(3)$ omitting in general to specify the homomorphism ψ which is given by the group law.

Let us conclude this section by noting that the direct product $G = H \times K$, in which H and K are normal, can be seen as a limit case of the semi-direct product $G' = H \ltimes K$ where only H is normal.

1.26 Direct and semi-direct sums of algebras

Definition

If G is a Lie group, and the direct product of its subgroups H and K, its Lie algebra $\mathcal{G}$ will be the direct sum of the Lie algebra $\mathcal{H}$ and $\mathcal{K}$ associated to H and K respectively:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$$

that is any $X \in \mathcal{G}$ can be written in a unique way $X = Y + Z$ with $Y \in \mathcal{H}$ and $Z \in \mathcal{K}$ and any element of $\mathcal{H}$ commute with any element of $\mathcal{K}$:

$$\Big[\mathcal{H}, \mathcal{K}\Big] = 0$$

Example

The Lie algebra $so(4)$ of the group $SO(4)$ is the direct sum of two $so(3)$ algebras: $so(4) = so(3) \oplus so(3)$ ($\rightarrow$ 1.58). □

To the semi-direct Lie group $G = H \ltimes K$ will correspond the algebra $\mathcal{G}$, semi-direct sum $\mathcal{G} = \mathcal{H} \ni\!\!\!+ \mathcal{K}$. The subgroup H of G being a normal subgroup, its Lie algebra $\mathcal{H}$ will be an ideal in $\mathcal{G}$. It therefore follows:

$$\Big[\mathcal{K}, \mathcal{K}\Big] = \mathcal{K} \qquad \Big[\mathcal{K}, \mathcal{H}\Big] = \mathcal{H} \qquad \Big[\mathcal{H}, \mathcal{H}\Big] = \mathcal{H}$$

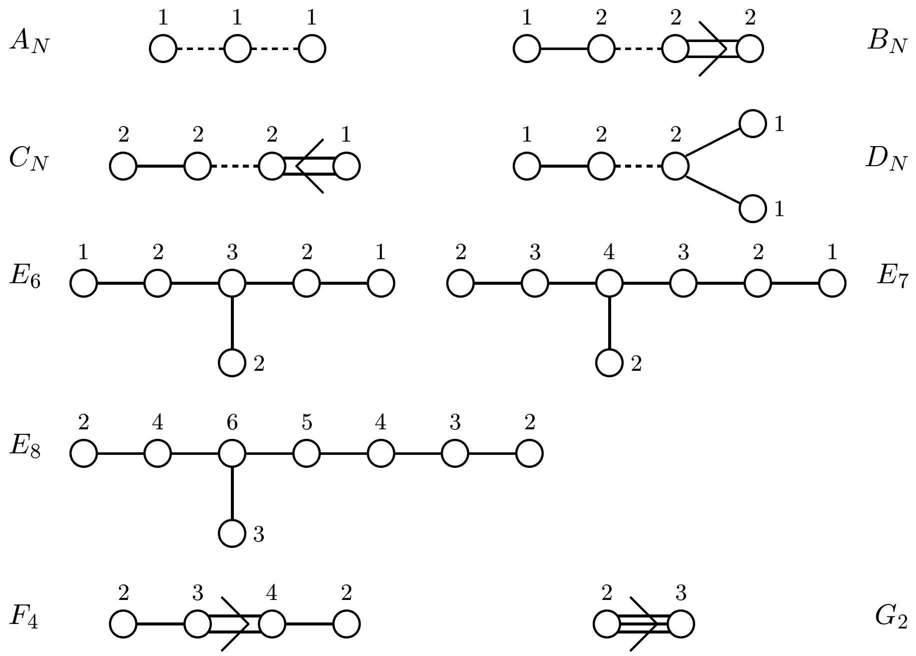
We recall that following Levi's theorem ($\rightarrow$ 1.47) an arbitrary Lie algebra $\mathcal{G}$ has a semi-direct sum structure $\mathcal{G} = \mathcal{S} \in\!\!\!+ \mathcal{R}$ with $\mathcal{R}$ solvable and $\mathcal{S}$ semi-simple.

1.27 Dynkin diagrams

The main properties of the root diagram ($\rightarrow$ 1.77) and therefore of the Cartan matrix ($\rightarrow$ 1.8) of a semi-simple Lie algebra $\mathcal{G}$ of dimension n and of rank r are summarized in the Dynkin diagram of $\mathcal{G}$.

The Dynkin diagram associated to a simple Lie algebra $\mathcal{G}$ is constructed as follows. A simple root ($\rightarrow$ 1.83) will be represented by a circle. Moreover two simple roots α and β will be connected by 0, 1, 2 or 3 lines depending on whether the angle $\theta_{\alpha\beta}$ between them is 90°, 120°, 135° or 150° (→ 1.77). Finally, when two roots α and β are connected by more than one line, one puts an arrow from the longest root towards the shorter one. Then the correspondence simple Lie algebra ↔ Dynkin diagram is unique. The table of Dynkin diagrams associated to the simple Lie algebras can be drawn once one knows for each simple Lie algebra $\mathcal{G}$ the set Δ^0 of the r simple roots which can be computed from its root space (→ 1.77).

We give below the list of the Dynkin diagrams of the simple Lie algebras. The numbers on the diagrams are the marks or Kac labels (→ 1.83).



Remarks on Dynkin diagrams

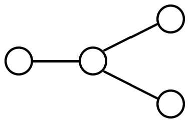
1. The Dynkin diagrams of A_1, B_1 and C_1 are identical, therefore A_1, B_1 and C_1 are isomorphic.

2. The Dynkin diagrams of B_2 and C_2 are identical, therefore B_2 and C_2 are isomorphic.

3. The Dynkin diagrams of A_3 and D_3 are identical, therefore A_3 and D_3

are isomorphic.

4. There is a particular symmetry in the Dynkin diagram of $D_4 = so(8)$. This property is called “triality”.



As a consequence, there are three different representations of dimension 8 (one is called a vector representation 8_V, the others spinor ones 8_S and $8'_S$) that satisfy, when reducing the Kronecker product of two of them: $8_V \otimes 8_S = 8'_S \oplus \ldots$, $8_S \otimes 8'_S = 8_V \oplus \ldots$, $8'_S \otimes 8_V = 8_S \oplus \ldots$ (see ref. [35] for more details).

5. From the form of the Dynkin diagrams of the exceptional algebras E_6, E_7, E_8, one might call E_5 the Lie algebra D_5 and E_4 the Lie algebra A_4.

Extended Dynkin diagram

Let $\Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ be the simple root system of a simple Lie algebra $\mathcal{G}$ of rank r. Let $-\alpha_0$ be the highest root with respect to Δ^0, that is the unique root of maximal height ($\rightarrow$ 1.83). One defines the *extended Dynkin diagram* by adding to the Dynkin diagram of $\mathcal{G}$ a dot associated to the root $+\alpha_0$. This diagram is especially important; in particular it allows us to determine all regular subalgebras of $\mathcal{G}$ ($\rightarrow$ 1.87). Moreover, it is the Dynkin diagram of the so-called affinization $\widehat{\mathcal{G}}$ of the Lie algebra $\mathcal{G}$. For the extended Dynkin diagrams of the simple Lie algebras, see Tables 3.1–3.9.

1.28 Embeddings of $sl(2)$

The problem of the determination of the possible $sl(2)$ subalgebras of a simple Lie algebra $\mathcal{G}$ has been solved by Dynkin (see ref. [17]). The solution uses the notion of principal embedding.

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r with simple root system $\Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ and corresponding simple root generators $e_i^{\pm}$ in the Serre–Chevalley basis ($\rightarrow$ 1.81).

The generators of the $sl(2)$ *principal embedding* in $\mathcal{G}$ are defined by

$$E^+ = \sum_{i=1}^{r} e_i^+ \,, \qquad E^- = \sum_{i=1}^{r}\sum_{j=1}^{r} A^{ij} e_i^-$$

A_{ij} being the Cartan matrix of $\mathcal{G}$ and $A^{ij} = (A^{-1})_{ij}$.

Theorem

1. Any $sl(2)$ embedding in a simple Lie algebra $\mathcal{G}$ can be considered as the principal $sl(2)$ subalgebra of a regular subalgebra $\mathcal{K}$ of $\mathcal{G}$.
2. For $\mathcal{G} = so(2N)$ with $N \geq 4$, besides the $sl(2)$ principal embeddings of item 1, there exist $\left[\frac{N-2}{2}\right]$ $sl(2)$ subalgebras associated to the singular embeddings $so(2k+1) \oplus so(2N-2k-1) \subset so(2N)$ with $1 \leq k \leq N-1$.
3. For $\mathcal{G} = E_6, E_7, E_8$, besides the $sl(2)$ principal embeddings of item 1, there exist $sl(2)$ subalgebras associated to singular embeddings (one for E_6, two for E_7 and E_8, see ref. [17] for more details).

The defining vector of the embedding $sl(2) \subset \mathcal{G}$ corresponding to a principal $sl(2)$ subalgebra of a regular subalgebra $\mathcal{K} \subset \mathcal{G}$ is given by the following rules when $\mathcal{K}$ is simple:

$$A_{2p} \subset A_N \;:\; f = (2p, 2p-2, \dots, 2, \underbrace{0, \dots, 0}_{N+1-2p}, -2, \dots, 2-2p, -2p)$$

$$A_{2p+1} \subset A_N \;:\; f = (2p+1, 2p-1, \dots, 1, \underbrace{0, \dots, 0}_{N-1-2p}, -1, \dots, 1-2p, -1-2p)$$

$$B_p \subset B_N \,, D_{p+1} \subset D_N \,, D_{p+1} \subset B_N \;:\; f = (2p, 2p-2, \dots, 2, 0, \dots, 0)$$

$$A_{2p} \subset D_N \,, A_{2p} \subset B_N \,, A_{2p}^2 \subset C_N \;: \quad f = (2p, 2p, 2p-2, 2p-2, \dots, 2, 2, 0, \dots, 0)$$

$$A_{2p+1} \subset D_N \,, A_{2p+1} \subset B_N \,, A_{2p+1}^2 \subset C_N \;: \quad f = (2p+1, 2p+1, 2p-1, 2p-1, \dots, 1, 1, 0, \dots, 0)$$

$$C_p \subset C_N \;:\; f = (2p-1, 2p-3, \dots, 1, 0, \dots, 0)$$

$$A_1^2 \subset B_N \;:\; f = (2, 0, \dots, 0)$$

$$A_1^1 \subset C_N \;:\; f = (1, 0, \dots, 0)$$

When $\mathcal{K}$ is the sum of simple Lie algebras, $\mathcal{K} = \oplus_i \mathcal{K}_i$, the resulting defining vector f is obtained by merging the different defining vectors f_i corresponding to $\mathcal{K}_i$, in such a way that the number of zeros is minimized and the entries of the defining vector f are put in decreasing order. For example, to the embedding $A_3 \oplus A_2 \subset A_6$ with defining vectors $f_{A_3 \subset A_6} = (3,1,0,0,0,-1,-3)$ and $f_{A_2 \subset A_6} = (2,0,0,0,0,0,-2)$ corresponds the defining vector $f = (3,2,1,0,-1,-2,-3)$.

→ 1.41 Index of an embedding – Defining vector.

1.29 Euclidean group

The Euclidean group $E(n)$ in n dimensions is the group of transformations in the Euclidean space $\mathbb{R}^n$ such that, denoting $(\vec{a}, R)$ its most general element where $\vec{a} \in \mathbb{R}^n$ and $R \in SO(n)$, we have

$$\vec{x}\,' \equiv (\vec{a}, R)\vec{x} = R\vec{x} + \vec{a}\,, \qquad \forall\, \vec{x} \in \mathbb{R}^n$$

$E(n)$ is the semi-direct product of the n-dimensional translation group $T(n)$ by the rotation group $SO(n)$, denoted $T(n) \ltimes SO(n)$. Its product law is as follows:

$$(\vec{a}, R)(\vec{a}\,', R') = (\vec{a} + R\vec{a}\,', RR')$$

$E(n)$ will often be taken as an example to illustrate several concepts.

The Euclidean group is a group with $\frac{1}{2}\,n(n+1)$ parameters. Its Lie algebra is generated by the rotation generators M_{ij} and the translation generators P_i, which satisfy the following commutation relations ($i, j = 1, \dots, n$):

$$\begin{aligned}
\Big[M_{ij}, M_{kl}\Big] &= i(\delta_{jk}M_{il} + \delta_{il}M_{jk} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik}) \\
\Big[M_{ij}, P_k\Big] &= i(\delta_{jk}P_i - \delta_{ik}P_j) \\
\Big[P_i, P_j\Big] &= 0
\end{aligned}$$

An explicit realization of the algebra of the Euclidean group $E(n)$ in terms of differential operators is given by ($1 \le k \ne l \le n$):

for the translation generators $\quad P_k = i\,\dfrac{\partial}{\partial x_k}$

for the rotation generators $\quad M_{kl} = i\Big(x_k \dfrac{\partial}{\partial x_l} - x_l \dfrac{\partial}{\partial x_k}\Big)$

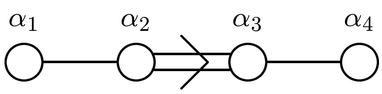
1.30 Exceptional Lie algebras

Looking at the Cartan classification of simple Lie algebras of finite dimension ($\rightarrow$ 1.7), one notes that to the four infinite families denoted A_N, B_N, C_N, D_N, are added five and only five Lie algebras which are called E_6, E_7, E_8, F_4 and G_2 (any attempt to add another simple root to the Dynkin diagram of an exceptional Lie algebra will lead in the best case to an infinite dimensional algebra).

The peculiarity of these algebras, namely their link with octonions ($\rightarrow$ 1.56), their structure – more precisely the type of subalgebras they contain – and of course their property to be only five, have led physicists to think that the exceptional Lie algebras must play a fundamental role in physics (for example Grand Unified models, String theory).

The exceptional Lie algebra F_4

The Lie algebra F_4 of rank 4 has dimension 52. In terms of the orthonormal vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, the root system is given by $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ $(1 \le i \ne j \le 4)$. The simple root system is $\Delta_0 = \{\alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = \varepsilon_4,\ \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ and the corresponding Dynkin diagram and Cartan matrix are



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The algebra F_4 is singularly embedded into $so(26)$. To describe this embedding, it is convenient to use a $so(26)$ basis constructed as follows. Consider the elementary 26×26 matrices m_{ij} such that $(m_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Then the matrices $M_{ij} = m_{ij} - m_{27-j,27-i}$ where $1 \le i \ne j \le 13$ generate $so(26)$: the Cartan subalgebra of $so(26)$ is generated by the matrices M_{ii}, the $\varepsilon_i - \varepsilon_j$ root generators are given by M_{ij}, the $\varepsilon_i + \varepsilon_j$ (resp. $-\varepsilon_i - \varepsilon_j$) root generators by $M_{i,27-j}$ (resp. $M_{27-i,j}$) with $i < j$. In terms of the M_{ij}, the generators of F_4 in the Cartan–Weyl basis are given by (with obvious notations)

– for the Cartan generators:

$$\begin{aligned}
H_1 &= M_{1,1} + \tfrac{1}{2}\,(M_{2,2} + M_{3,3} + M_{4,4} + M_{5,5} + M_{6,6} + M_{8,8} + M_{10,10} + M_{12,12}) \\
H_2 &= M_{7,7} + \tfrac{1}{2}\,(M_{2,2} + M_{3,3} + M_{4,4} + M_{5,5} - M_{6,6} - M_{8,8} - M_{10,10} - M_{12,12}) \\
H_3 &= M_{9,9} + \tfrac{1}{2}\,(M_{2,2} + M_{3,3} - M_{4,4} - M_{5,5} + M_{6,6} + M_{8,8} - M_{10,10} - M_{12,12}) \\
H_4 &= M_{11,11} + \tfrac{1}{2}\,(M_{2,2} - M_{3,3} + M_{4,4} - M_{5,5} + M_{6,6} - M_{8,8} + M_{10,10} - M_{12,12})
\end{aligned}$$

– for the positive long root $so(9)$ generators:

$$E_{\varepsilon_1-\varepsilon_2} = M_{1,7} - M_{6,15} + M_{8,17} \qquad E_{\varepsilon_1+\varepsilon_2} = M_{1,20} - M_{2,22} + M_{3,23}$$
$$E_{\varepsilon_1-\varepsilon_3} = -M_{1,9} - M_{4,15} + M_{5,17} \qquad E_{\varepsilon_1+\varepsilon_3} = M_{1,18} - M_{2,19} + M_{3,21}$$
$$E_{\varepsilon_1-\varepsilon_4} = -M_{1,11} + M_{3,15} - M_{5,19} \qquad E_{\varepsilon_1+\varepsilon_4} = -M_{1,16} + M_{2,17} - M_{4,21}$$
$$E_{\varepsilon_2-\varepsilon_3} = M_{4,6} + M_{5,8} + M_{7,9} \qquad E_{\varepsilon_2+\varepsilon_3} = M_{2,10} + M_{3,12} - M_{7,18}$$
$$E_{\varepsilon_2-\varepsilon_4} = -M_{3,6} + M_{5,10} + M_{7,11} \qquad E_{\varepsilon_2+\varepsilon_4} = M_{2,8} - M_{4,12} + M_{7,16}$$
$$E_{\varepsilon_3-\varepsilon_4} = M_{3,4} + M_{8,10} + M_{9,11} \qquad E_{\varepsilon_3+\varepsilon_4} = -M_{2,5} - M_{6,12} + M_{9,16}$$

– for the positive short root $so(9)$ generators:

$$E_{\varepsilon_1} = \tfrac{1}{\sqrt{2}}\,(M_{1,13} - M_{1,14} + M_{2,15} + M_{3,17} - M_{4,19} - M_{5,21})$$
$$E_{\varepsilon_2} = \tfrac{1}{\sqrt{2}}\,(M_{2,6} - M_{3,8} - M_{4,10} + M_{5,12} + M_{7,13} - M_{7,14})$$
$$E_{\varepsilon_3} = \tfrac{1}{\sqrt{2}}\,(-M_{2,4} + M_{3,5} - M_{6,10} + M_{8,12} + M_{9,13} - M_{9,14})$$
$$E_{\varepsilon_4} = \tfrac{1}{\sqrt{2}}\,(M_{2,3} + M_{4,5} + M_{6,8} + M_{10,12} + M_{11,13} - M_{11,14})$$

– for the positive weights $so(9)$ spinor operators ($j = e^{2i\pi/3}$):

$$E_{\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{1,12} + jM_{2,13} + M_{4,18} - j^2M_{2,14} - M_{3,16} - M_{6,20}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{2,11} + jM_{3,14} + M_{5,18} + M_{7,19} - M_{1,10} - j^2M_{3,13}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{2,9} + j^2M_{4,13} - M_{1,8} - jM_{4,14} - M_{5,15} - M_{7,17}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(jM_{5,13} + M_{7,15} - j^2M_{4,15} - M_{1,6} - M_{3,9} - M_{4,11}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{1,5} + j^2M_{6,13} - M_{2,7} - jM_{6,14} - M_{8,16} - M_{9,17}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{1,4} + M_{3,7} + jM_{8,13} + M_{9,15} - M_{6,11} - j^2M_{8,14}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{4,7} + M_{9,9} + jM_{10,13} + M_{11,15} - M_{1,3} - j^2M_{10,14}\big)$$
$$E_{\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)} = \tfrac{1}{\sqrt{2}}\,\big(M_{1,2} + M_{5,7} + M_{8,9} + M_{10,11} + jM_{12,14} - j^2M_{12,13}\big)$$

The negative root generators are given by $E_{-\alpha} = (E_\alpha)^\dagger$.

The generators in the Serre–Chevalley basis are related to those of the Cartan–Weyl basis by:

$$h_1 = H_2 - H_3 \quad h_2 = H_3 - H_4 \quad h_3 = 2H_4 \quad h_4 = H_1 - H_2 - H_3 - H_4$$
$$e_1 = E_{\varepsilon_2-\varepsilon_3} \quad e_2 = E_{\varepsilon_3-\varepsilon_4} \quad e_3 = \sqrt{2}E_{\varepsilon_4} \quad e_4 = \sqrt{2}E_{\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)}$$
$$f_1 = E_{-\varepsilon_2+\varepsilon_3} \quad f_2 = E_{-\varepsilon_3+\varepsilon_4} \quad f_3 = \sqrt{2}E_{-\varepsilon_4} \quad f_4 = \sqrt{2}E_{\frac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}$$

An irreducible representation of F_4 is characterized by its Dynkin labels (a_1, a_2, a_3, a_4) which are non-negative integers. The dimension of the F_4

irreducible representation with heighest weight $\Lambda(a_1, a_2, a_3, a_4)$ is given by

$$\begin{aligned} N = {} & N_0\, b_1\, b_2\, b_3\, b_4\, (b_1+b_2)(b_2+b_3)(b_3+b_4)(b_1+b_2+b_3)(b_1+2b_2+b_3) \\ & (2b_1+2b_2+b_3)(2b_2+b_3)(b_2+b_3+b_4)(2b_2+b_3+b_4)(2b_1+2b_2+b_3+b_4) \\ & (2b_2+2b_3+b_4)(b_1+b_2+b_3+b_4)(b_1+2b_2+b_3+b_4)(b_1+2b_2+2b_3+b_4) \\ & (b_1+3b_2+2b_3+b_4)(2b_1+2b_2+2b_3+b_4)(2b_1+3b_2+2b_3+b_4) \\ & (2b_1+4b_2+2b_3+b_4)(2b_1+4b_2+3b_3+b_4)(2b_1+4b_2+3b_3+2b_4) \end{aligned}$$

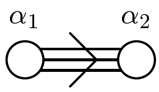
where $b_i = a_i + 1$ and $1/N_0 = 2^{15}\, 3^7\, 5^4\, 7^2\, 11$.

The representations of F_4 are all real ($\rightarrow$ 1.92). The fundamental simple representation (0,0,0,1) is of dimension 26. The adjoint representation (1,0,0,0) is of dimension 52. See Table 3.8 for the complete list of fundamental weights of F_4. The irreducible representations of dimension up to 10^6 are listed in Table 3.21.

F_4 contains as maximal regular subalgebras $so(9)$, $sp(6) \oplus sl(2)$ and $sl(3) \oplus sl(3)$ and as maximal singular subalgebras $G_2 \oplus sl(2)^8$ and $sl(2)^{156}$ ($\rightarrow$ 1.87) where the superscripts are the Dynkin indices ($\rightarrow$ 1.41).

The exceptional Lie algebra G_2

The Lie algebra G_2 of rank 2 has dimension 14. In terms of the orthonormal vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, the root system is given by $\Delta = \{\varepsilon_i - \varepsilon_j,\ \varepsilon_i + \varepsilon_j - 2\varepsilon_k\}$ $(1 \le i \ne j \ne k \le 3)$. The simple root system is $\Delta_0 = \{\alpha_1 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1,\ \alpha_2 = \varepsilon_1 - \varepsilon_2\}$ and the corresponding Dynkin diagram and Cartan matrix are



$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

In order to write the commutation relations of G_2, it is convenient to use a $so(7)$ basis. Consider the $so(7)$ generators $M_{pq} = -M_{qp}$ where $1 \le p \ne q \le 7$. The singular embedding $G_2 \subset so(7)$ is obtained by imposing on the generators M_{pq} the constraints

$$\xi_{ijk} M_{ij} = 0$$

where the tensor ξ_{ijk} is completely antisymmetric and whose non-vanishing components are

$$\xi_{123} = \xi_{145} = \xi_{176} = \xi_{246} = \xi_{257} = \xi_{347} = \xi_{365} = 1$$

The commutation relations of G_2 then read as:

$$\Big[M_{pq}, M_{rs}\Big] = \delta_{qr} M_{ps} + \delta_{ps} M_{qr} - \delta_{pr} M_{qs} - \delta_{qs} M_{pr} + \tfrac{1}{3}\xi_{pqu}\xi_{rsv} M_{uv}$$

In terms of the M_{pq}, the generators of G_2 are given by

$$\begin{array}{ll}
E_1 = i(M_{17} - M_{24}) & E'_1 = i\sqrt{3}(M_{17} + M_{24}) \\
E_2 = i(M_{21} - M_{74}) & E'_2 = -i\sqrt{3}(M_{21} + M_{74}) \\
E_3 = i(M_{72} - M_{14}) & E'_3 = i\sqrt{3}(M_{72} + M_{14}) = -E_8 \\
E_4 = i(M_{43} - M_{16}) & E'_4 = i\sqrt{3}(M_{43} + M_{16}) \\
E_5 = i(M_{31} - M_{46}) & E'_5 = i\sqrt{3}(M_{31} + M_{46}) \\
E_6 = i(M_{62} - M_{73}) & E'_6 = i\sqrt{3}(M_{62} + M_{73}) \\
E_7 = i(M_{32} - M_{67}) & E'_7 = i\sqrt{3}(M_{32} + M_{67})
\end{array}$$

The generators E_a with $a = 1, \dots, 8$ generate $sl(3)$ and satisfy the commutation relations

$$\Big[E_a, E_b\Big] = 2i f_{abc} E_c$$

where f_{abc} are the usual totally antisymmetric Gell-Mann structure constants ($\to$ 1.90). The commutation relations between the G_2 generators E_a and E'_i $(i = 1, 2, 4, 5, 6, 7)$ are

$$\begin{aligned}
\Big[E_a, E'_i\Big] &= 2i c_{aij} E'_j \\
\Big[E'_i, E'_j\Big] &= 2i(c_{aij} E_a + c'_{ijk} E'_k)
\end{aligned}$$

where the structure constants c_{aij} (antisymmetric in the indices i, j) and c'_{ijk} (totally antisymmetric) are

$$\begin{array}{l}
c_{147} = c_{156} = c_{257} = c_{345} = c_{367} = c_{417} = c_{725} = 1/2 \\
c_{246} = c_{426} = c_{516} = c_{527} = c_{615} = c_{624} = c_{714} = -1/2 \\
c_{845} = c_{876} = -1/2\sqrt{3} \qquad c_{812} = -1/\sqrt{3} \\
c'_{147} = c'_{165} = c'_{246} = c'_{257} = -1/\sqrt{3}
\end{array}$$

The generators E_3 and E_8 constitute a Cartan basis of the G_2 algebra. One can also take a basis H_1, H_2, H_3 such that $H_1 + H_2 + H_3 = 0$ given by $H_1 = \frac{1}{2}(E_3 + \frac{\sqrt{3}}{3}E_8)$, $H_2 = \frac{1}{2}(-E_3 + \frac{\sqrt{3}}{3}E_8)$, $H_3 = -\frac{\sqrt{3}}{3}E_8$. The generators in the Cartan–Weyl basis are given by (with obvious notations):

$$\begin{array}{lll}
H_1 = \frac{1}{2}(E_3 + \frac{\sqrt{3}}{3}E_8) & H_2 = \frac{1}{2}(-E_3 + \frac{\sqrt{3}}{3}E_8) & H_3 = -\frac{\sqrt{3}}{3}E_8 \\
E_{\pm(\varepsilon_1-\varepsilon_2)} = E_1 \pm iE_2 & E_{\pm(\varepsilon_2-\varepsilon_3)} = E_6 \pm iE_7 & E_{\pm(\varepsilon_1-\varepsilon_3)} = E_4 \pm iE_5 \\
E_{\pm\varepsilon_1} = E'_7 \mp iE'_6 & E_{\pm\varepsilon_2} = E'_4 \mp iE'_5 & E_{\pm\varepsilon_3} = E'_1 \mp iE'_2
\end{array}$$

An irreducible representation of G_2 is characterized by its Dynkin labels (a_1, a_2) which are non-negative integers. The dimension of the G_2 irreducible representation with heighest weight $\Lambda(a_1, a_2)$ is given by

$$N = \frac{1}{120}\,(a_1+1)(a_2+1)(a_1+a_2+2)(2a_1+a_2+3)(3a_1+a_2+4)(3a_1+2a_2+5)$$

The representations of G_2 are all real ($\rightarrow$ 1.92). The fundamental simple representation (0,1) is of dimension 7. The adjoint representation (1,0) is of dimension 14. See Table 3.9 for the complete list of fundamental weights of G_2. The irreducible representations of dimension up to 20 000 are listed in Table 3.20.

G_2 contains as maximal regular subalgebras $su(3)$ and $su(2) \oplus su(2)$ and as maximal singular subalgebra $sl(2)^{28}$ ($\rightarrow$ 1.87) where the superscripts are the Dynkin indices ($\rightarrow$ 1.41).

The exceptional Lie algebra E_6

The Lie algebra E_6 of rank 6 has dimension 78. The root system of E_6 can be described in two different ways.

Consider the eight-dimensional vector space $\mathbb{R}^8$ with an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_8\}$. Let $\mathcal{V}_6$ be the hyperplane in $\mathbb{R}^8$ orthogonal to the vectors $\varepsilon_7+\varepsilon_8$ and $\varepsilon_6+\varepsilon_7+2\varepsilon_8$. The root system of E_6 is then given by $\Delta(E_6) = \Delta(E_8) \cap \mathcal{V}_6 = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\frac{1}{2}(\pm\varepsilon_1 \pm \dots \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$ $(1 \le i \ne j \le 5)$. The simple root system is $\Delta_0 = \{\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \dots,\ \alpha_5 = \varepsilon_5 - \varepsilon_4,\ \alpha_6 = \varepsilon_1 + \varepsilon_2\}$.

The root system of E_6 can be also described in terms of the orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_6\}$ in a six-dimensional space. The root system of E_6 is then given by $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \frac{1}{2}(\pm\varepsilon_1 \pm \dots \pm \varepsilon_5 \pm \sqrt{3}\,\varepsilon_6)\}$ where $1 \le i \ne j \le 5$ and the total number of minus sign in $\frac{1}{2}(\pm\varepsilon_1 \pm \dots \pm \varepsilon_5 \pm \sqrt{3}\,\varepsilon_6)$ is even.

The corresponding Dynkin diagram and Cartan matrix are given in Table 3.5.

An irreducible representation of E_6 is characterized by its Dynkin labels $(a_1, a_2, a_3, a_4, a_5, a_6)$ which are non-negative integers. The representations such that $a_1 = a_5$ and $a_2 = a_4$ are real, the other ones are complex ($\rightarrow$ 1.92). The fundamental simple representation (1,0,0,0,0,0) is of dimension 27. The adjoint representation (0,0,0,0,0,1) is of dimension 78. See Table 3.5 for the complete list of fundamental weights of E_6. The irreducible representations of dimension up to 10^6 are listed in Table 3.24.

E_6 contains as maximal regular subalgebras $sl(6) \oplus sl(2)$, $sl(3) \oplus sl(3) \oplus sl(3)$ and $so(10) \oplus U(1)$ and as maximal singular subalgebras F_4^1, G_2^3, $G_2^1 \oplus sl(3)^2$, $sp(8)^1$ and $sl(3)^9$ ($\rightarrow$ 1.87) where the superscripts are the Dynkin indices ($\rightarrow$ 1.41).

The exceptional Lie algebra E_7

The Lie algebra E_7 of rank 7 has dimension 133. The root system of E_7 can be described in two different ways.

Consider the eight-dimensional vector space $\mathbb{R}^8$ with an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_8\}$. Let $\mathcal{V}_7$ be the hyperplane in $\mathbb{R}^8$ orthogonal to the vector $\varepsilon_7+\varepsilon_8$. The root system of E_7 is then given by $\Delta(E_7) = \Delta(E_8) \cap \mathcal{V}_7 = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm(\varepsilon_8 - \varepsilon_7),\ \pm\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$ where $1 \leq i \neq j \leq 6$ and the total number of minus signs in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$ is even. The simple root system is $\Delta_0 = \{\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \ldots,\ \alpha_5 = \varepsilon_5 - \varepsilon_4,\ \alpha_6 = \varepsilon_6 - \varepsilon_5,\ \alpha_7 = \varepsilon_1 + \varepsilon_2\}$.

The root system of E_7 can be also described in terms of the orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_7\}$ in a seven-dimensional space. The root system of E_7 is then given by $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\sqrt{2}\,\varepsilon_7,\ \frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 \pm \sqrt{2}\,\varepsilon_7)\}$ $(1 \leq i \neq j \leq 6)$ where the number of minus signs in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 \pm \sqrt{2}\,\varepsilon_7)$ in the first 6 vectors is even.

The corresponding Dynkin diagram and Cartan matrix are given in Table 3.6.

An irreducible representation of E_7 is characterized by its Dynkin labels $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ which are non-negative integers. The representations of E_7 are real if $a_4 + a_6 + a_7$ is even and pseudo-real if $a_4 + a_6 + a_7$ is odd ($\rightarrow$ 1.92). The fundamental simple representation (0,0,0,0,0,1,0) is of dimension 56. The adjoint representation (1,0,0,0,0,0,0) is of dimension 133. See Table 3.6 for the complete list of fundamental weights of E_7. The irreducible representations of dimension up to 10^6 are listed in Table 3.25.

E_7 contains as maximal regular subalgebras $so(12) \oplus sl(2)$, $sl(6) \oplus sl(3)$, $sl(8)$ and $E_6 \oplus U(1)$ and as maximal singular subalgebras $F_4^1 \oplus sl(2)^3$, $G_2^2 \oplus sl(2)^7$, $G_2^1 \oplus sp(6)^1$, $sl(2)^{24} \oplus sl(2)^{15}$, $sl(3)^{21}$, $sl(2)^{231}$ and $sl(2)^{399}$ ($\rightarrow$ 1.87) where the superscripts are the Dynkin indices ($\rightarrow$ 1.41).

The exceptional Lie algebra E_8

The Lie algebra E_8 of rank 8 has dimension 248. In terms of the orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_8\}$ of $\mathbb{R}^8$, the root system of E_8 is given by $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)\}$ where $1 \leq i \neq j \leq 8$ and the total number of minus signs in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)$ is even. The simple root system is $\Delta_0 = \{\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \ldots,\ \alpha_7 = \varepsilon_7 - \varepsilon_6,\ \alpha_8 = \varepsilon_1 + \varepsilon_2\}$. The corresponding Dynkin diagram and Cartan matrix are given in Table 3.7.

An irreducible representation of E_8 is characterized by its Dynkin labels $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ which are non-negative integers. The representations of E_8 are all real ($\rightarrow$ 1.92). The fundamental simple representation is the adjoint representation (0,0,0,0,0,0,1,0) of dimension 248. See Table 3.7 for the complete list of fundamental weights of E_8. The irreducible representations of dimension up to 10^9 are listed in Table 3.26.

E_8 contains as maximal regular subalgebras $so(16)$, $E_7 \oplus sl(2)$ $E_6 \oplus sl(3)$, $sl(5) \oplus sl(5)$ and $sl(9)$ and as singular maximal subalgebras $so(5)^{12}$, $G_2^1 \oplus F_4^1$, $sl(3)^6 \oplus sl(2)^{16}$, $sl(2)^{1240}$, $sl(2)^{760}$ and $sl(2)^{520}$ ($\rightarrow$ 1.87) where the superscripts are the Dynkin indices ($\rightarrow$ 1.41).

1.31 Folding

$\rightarrow$ 1.87 Subalgebras: regular and singular subalgebras.

1.32 Fundamental representation

$\rightarrow$ 1.92 Weights of a representation.

1.33 Galilei group

The Galilei group plays a fundamental role in classical physics since both classical mechanics and non-relativistic quantum mechanics have to satisfy covariance under transformations of the Galilei group.

The Galilei group is the set of linear transformations in Newtonian space–time which leave invariant the time interval between events and the space distance of simultaneous events. Its elements $g = (\tau, \vec{a}, \vec{v}, R)$, where R is a 3×3 orthogonal matrix, $\vec{v}$ and $\vec{a}$ are arbitrary real vectors and τ an arbitrary real number, are defined as follows:

$$x' \equiv (\tau, \vec{a}, \vec{v}, R)x \quad \text{where} \quad x = (t, \vec{r}),\ x' = (t', \vec{r}\,') \quad (t \in \mathbb{R} \ \text{and} \ \vec{r}, \vec{r}\,' \in \mathbb{R}^3)$$

with

$$t' = t + \tau \qquad \text{and} \qquad \vec{r}\,' = R\vec{r} + \vec{v}t + \vec{a}$$

The composition law is

$$(\tau, \vec{a}, \vec{v}, R)(\tau', \vec{a}\,', \vec{v}\,', R') = (\tau + \tau', \vec{a} + R\vec{a}\,' + \tau\vec{v}\,', \vec{v} + R\vec{v}\,', RR')$$

and the inverse element is given by

$$(\tau, \vec{a}, \vec{v}, R)^{-1} = (-\tau, R^{-1}(\vec{a} - \tau\vec{v}), -R^{-1}\vec{v}, R^{-1})$$

The Galilei group is a 10-parameter group. The infinitesimal generators satisfy the Galilei algebra whose commutation relations are given below, where

P_0 is the time translation, P_i are the space translations, J_i the rotations and K_i the pure Galilean transformations ($i = 1, 2, 3$):

$$
\begin{array}{ll}
\big[J_i, J_j\big] = i\,\varepsilon_{ijk}\,J_k & \big[K_i, K_j\big] = \big[K_i, P_j\big] = 0 \\
\big[J_i, K_j\big] = i\,\varepsilon_{ijk}\,K_k & \big[P_i, P_j\big] = \big[P_i, P_0\big] = \big[J_i, P_0\big] = 0 \\
\big[J_i, P_j\big] = i\,\varepsilon_{ijk}\,P_k & \big[K_i, P_0\big] = iP_i
\end{array}
$$

The two Casimir operators are

$$
\begin{aligned}
&\vec{P}^2 = P_i P^i \\
&\vec{N}^2 = (\vec{K} \times \vec{P})^2 = N_i N^i = \varepsilon_{ijk} K^j P^k \varepsilon^{ilm} K_l P_m
\end{aligned}
$$

The Galilei algebra can be obtained as a contraction from the Poincaré algebra. To see this, one rewrites the Poincaré algebra in terms of the infinitesimal generators $J'_i = J_i$, $K'_i = K_i/c$, $P'_i = P_i$, $P'_0 = cP_0$ where c is a parameter to be identified in physics with the speed of light, and apply the limit $c \to \infty$: one obtains the Galilei algebra [54].

The Galilei algebra admits a non-trivial central extension M such that

$$
\begin{aligned}
&\big[K_i, P_j\big] = i\,\delta_{ij}\,M \\
&\big[M, J_i\big] = \big[M, K_i\big] = \big[M, P_i\big] = \big[M, P_0\big] = 0
\end{aligned}
$$

the generator M being interpreted as a mass, and the algebra generated by the K_i, P_i ($i = 1, 2, 3$) and M constituting the Heisenberg algebra $\mathcal{H}_3$.

1.34 Gelfand–Zetlin basis for $su(n)$

An irreducible representation of $su(n)$ can be characterized by a set of $n - 1$ non-negative integers $(m_{1n}, \dots, m_{n-1,n})$ such that $m_{in} \geq m_{i+1,n}$. $[m_{1n}, \dots, m_{n-1,n}]$ corresponds to the Young tableaux ($\to$ 1.96) notation.

Property

The Gelfand–Zetlin basis of the representation space $\mathcal{V}(m_{1n}, \dots, m_{n-1,n})$ is constituted by orthonormal states (m) represented by the triangular patterns

$$(m) = \left| \begin{array}{ccccc} m_{1n} & m_{2n} & \dots & m_{n-1,n} & 0 \\ m_{1,n-1} & m_{2,n-1} & \dots & m_{n-1,n-1} & \\ & \ddots & & \iddots & \\ & & m_{12} \quad m_{22} & & \\ & & m_{11} & & \end{array} \right\rangle$$

where the m_{ij} satisfy the condition

$$m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1} \geq 0$$

Example

Gelfand–Zetlin basis in $su(3)$: The vectors of the Gelfand–Zetlin basis in $su(3)$ take the general form

$$(m) = \left| \begin{array}{ccc} m_{13} & m_{23} & 0 \\ m_{12} & m_{22} & \\ & m_{11} & \end{array} \right\rangle \qquad \text{where} \quad m_{13} \geq m_{12} \geq m_{11}, m_{23} \geq m_{22} \geq 0$$

The fundamental representation of dimension 3 is spanned by the vectors

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & \\ & 1 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & \\ & 0 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & \\ & 0 & \end{array} \right\rangle$$

The adjoint representation of dimension 8 is spanned by the vectors

$$\left| \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & \\ & 2 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 1 & \\ & 1 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 1 & \\ & 1 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 0 & \\ & 2 & \end{array} \right\rangle$$

$$\left| \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 0 & \\ & 1 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 2 & 0 & \\ & 0 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \\ & 1 & \end{array} \right\rangle \quad \left| \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & \\ & 0 & \end{array} \right\rangle$$

□

Introducing the standard generators e_{ij} of $gl(n)$ which satisfy the commutation relations

$$\left[e_{ij}, e_{kl}\right] = \delta_{jk}\, e_{il} - \delta_{il}\, e_{kj}$$

where $i, j, k, l = 1, \dots, n$, the generators e_{ii} are diagonal in the Gelfand–

Zetlin basis

$$e_{ii}|(m)\rangle = \sum_{i<j}(m_{ij} - m_{i,j-1})|(m)\rangle$$

while the generators $e_{i,i+1}$ and $e_{i+1,i}$ act as raising and lowering operators respectively. One has

$$e_{i,i+1}|(m)\rangle = \sum_{\tau=1}^{i} a_{i\tau}^{+}|m_{\tau i} \to m_{\tau i} + 1\rangle$$

$$e_{i+1,i}|(m)\rangle = \sum_{\tau=1}^{i} a_{i\tau}^{-}|m_{\tau i} \to m_{\tau i} - 1\rangle$$

where

$$a_{i\tau}^{+} = \left[(-1)^i \frac{\prod_{s=1}^{i+1}(m_{s,i+1} - m_{\tau,i} - s + \tau)\prod_{s=1}^{i-1}(m_{s,i-1} - m_{\tau,i} - s + \tau - 1)}{\prod_{s=1,s\neq\tau}^{i} (m_{s,i} - m_{\tau,i} - s + \tau)\ (m_{s,i} - m_{\tau,i} - s + \tau - 1)} \right]^{1/2}$$

and

$$a_{i\tau}^{-} = \left[(-1)^i \frac{\prod_{s=1}^{i+1}(m_{s,i+1} - m_{\tau,i} - s + \tau + 1)\prod_{s=1}^{i-1}(m_{s,i-1} - m_{\tau,i} - s + \tau)}{\prod_{s=1,s\neq\tau}^{i} (m_{s,i} - m_{\tau,i} - s + \tau + 1)\ (m_{s,i} - m_{\tau,i} - s + \tau)} \right]^{1/2}$$

In this framework, it is specially easy to decompose a $su(n)$ representation $\pi(m_{1n}, \dots, m_{n-1,n})$ into irreducible representations of $su(n-1)$:

$$\begin{aligned} \pi(m_{1n}, \dots, m_{n-1,n}) &= \\ \bigoplus_{m_{in} \geq m_{i,n-1} \geq m_{i+1,n}} & \pi(m_{1,n-1} - m_{n-1,n-1}, \dots, m_{n-2,n-1} - m_{n-1,n-1}) \end{aligned}$$

the basis vectors of the $su(n-1)$ representation $\pi(m_{1,n-1} - m_{n-1,n-1}, \dots, m_{n-2,n-1} - m_{n-1,n-1})$ being obtained from the above vectors in which the first row has been discarded. Iterating this reduction, one sees that the construction of the Gelfand–Zetlin basis for an irreducible representation π of $su(n)$ stands on the reduction of π with respect to the following chain of embeddings:

$$su(n) \supset su(n-1) \supset su(n-2) \supset \dots \supset su(2) \supset U(1)$$

1.35 Gelfand–Zetlin basis for $so(n)$

An irreducible representation of $so(n)$ with $n = 2p$ or $n = 2p+1$ is characterized by a set of p integers or p half-integers $(m_1, \dots, m_p)$ such that

$$\begin{array}{ll} \text{for } n = 2p, & m_1 \geq m_2 \geq \dots \geq m_{p-1} \geq |m_p| \\ \text{for } n = 2p+1, & m_1 \geq m_2 \geq \dots \geq m_{p-1} \geq m_p \geq 0 \end{array}$$

In the case of $so(2p+2)$, the Gelfand–Zetlin tableaux are of the form

$$(m) = \left| \begin{array}{cccccccccc} m_1 & & m_2 & & \dots & & \dots & & m_p & & m_{p+1} \\ & m_{1,2p} & & m_{2,2p} & & \dots & & m_{p-1,2p} & & m_{p,2p} & \\ & m_{1,2p-1} & & m_{2,2p-1} & & \dots & & m_{p-1,2p-1} & & m_{p,2p-1} & \\ & & m_{1,2p-2} & & & \dots & & & m_{p-1,2p-2} & & \\ & & m_{1,2p-3} & & & \dots & & & m_{p-1,2p-3} & & \\ & & & & \ddots & & \iddots & & & & \\ & & & & m_{14} & & m_{24} & & & & \\ & & & & m_{13} & & m_{23} & & & & \\ & & & & & m_{12} & & & & & \\ & & & & & m_{11} & & & & & \end{array} \right\rangle$$

where the numbers m_{ij} are simultaneously all integers or all half-integers, and satisfy the conditions

$$\begin{array}{ll} m_1 \geq m_{1,2p} \geq m_2 \geq m_{2,2p} \geq \dots \geq m_p \geq m_{p,2p} \geq |m_{p+1}| & \\ m_{i,2t+1} \geq m_{i,2t} \geq m_{i+1,2t+1} & (1 \leq i \leq t \leq p-1) \\ m_{i,2t} \geq m_{i,2t-1} \geq m_{i+1,2t} & (1 \leq i \leq t-1 \leq p-1) \end{array}$$

with the particular conditions on the right boundary

$$m_{t,2t} \geq |m_{t+1,2t+1}| \qquad \text{and} \qquad m_{t,2t-1} \geq -m_{t,2t}$$

In the case of $so(2p+1)$, the Gelfand–Zetlin tableaux are of the form

$$(m) = \left| \begin{array}{ccccccccc} m_1 & & m_2 & & \dots & & m_{p-1} & & m_p \\ m_{1,2p-1} & & m_{2,2p-1} & & \dots & & m_{p-1,2p-1} & & m_{k,2p-1} \\ & m_{1,2p-2} & & & \dots & & & m_{p-1,2p-2} & \\ & m_{1,2p-3} & & & \dots & & & m_{p-1,2p-3} & \\ & & & \ddots & & \iddots & & & \\ & & & m_{14} & & m_{24} & & & \\ & & & m_{13} & & m_{23} & & & \\ & & & & m_{12} & & & & \\ & & & & m_{11} & & & & \end{array} \right\rangle$$

satisfying the conditions

$$\begin{array}{ll} m_1 \geq m_{1,2p-1} \geq m_2 \geq m_{2,2p-1} \geq \ldots \geq m_p \geq m_{p,2p-1} \geq -m_p & \\ m_{i,2t+1} \geq m_{i,2t} \geq m_{i+1,2t+1} & (1 \leq i \leq t \leq p-1) \\ m_{i,2t} \geq m_{i,2t-1} \geq m_{i+1,2t} & (1 \leq i \leq t-1 \leq p-1) \end{array}$$

and on the right boundary

$$m_{t,2t} \geq |m_{t+1,2t+1}| \qquad \text{and} \qquad m_{t,2t-1} \geq -m_{t,2t}$$

As in the $su(n)$ case ($\rightarrow$ 1.34), the Gelfand–Zetlin basis for $so(n)$ makes transparent the canonical decomposition $so(n) \supset so(n-1) \supset so(n-2) \supset \ldots$ Indeed, if the first line $m_1, \ldots, m_{p+1}$ of the Gelfand–Zetlin pattern for $so(2p+2)$ determines an irreducible representation $\pi(m_1, \ldots, m_{p+1})$ of $so(2p+2)$, then the second lines allowed by the above conditions label the $so(2p+1)$ representations contained in $\pi(m_1, \ldots, m_{p+1})$. So does the third row for the representations of $so(2p)$, and so on, up to the last lines in which m_{12} and m_{11} characterize the $so(3)$ and $so(2)$ representations respectively. It is because the rank of $so(2p)$ and $so(2p+1)$ are the same, and equal to p, that the number of m_{ij}'s in a row decreases by one every other two lines.

The action of the $so(n)$ generators on the vectors in the Gelfand–Zetlin basis is much more involved than in the $su(n)$ case. Let us introduce the standard generators M_{ij} of $so(n)$ which satisfy the commutation relations (where $i,j,k,l = 1, \ldots, n$)

$$\Big[M_{ij}, M_{kl}\Big] = \delta_{il}\, M_{jk} + \delta_{jk}\, M_{il} - \delta_{ik}\, M_{jl} - \delta_{jl}\, M_{ik}$$

The generator $M_{i,i+1}$ acting on a Gelfand–Zetlin basis vector will change only the line $(i-1)$ and will leave the other rows unchanged. Denoting by $|m_{ij}^{\pm}\rangle$ the vector obtained from the vector $|(m)\rangle$ by $m_{ij} \rightarrow m_{ij} \pm 1$, one gets

$$\begin{aligned} M_{2t+1,2t}|(m)\rangle &= \sum_{j=1}^{t} A(m_{j,2t-1})|m_{j,2t-1}^{+}\rangle - \sum_{j=1}^{t} A(m_{j,2t-1}-1)|m_{j,2t-1}^{-}\rangle \\ M_{2t+2,2t+1}|(m)\rangle &= \sum_{j=1}^{t} B(m_{j,2t})|m_{j,2t}^{+}\rangle - \sum_{j=1}^{t} B(m_{j,2t-1}-1)|m_{j,2t}^{-}\rangle \\ &\quad + iC_{2t}|(m)\rangle \end{aligned}$$

where the expressions of A, B, C are given by, setting $l_{\tau,2t-1} = m_{\tau,2t-1} + t - \tau$

and $l_{\tau,2t} = m_{\tau,2t} + t - \tau + 1$:

$$A(m_{j,2t-1}) = \left[\frac{\prod_{\tau=1}^{t-1} (l_{\tau,2t-2} - l_{j,2t-1} - 1)(l_{\tau,2t-2} + l_{j,2t-1})}{\prod_{\tau \neq j} (l_{\tau,2t-1}^2 - l_{j,2t-1}^2)(l_{\tau,2t-1}^2 - (l_{j,2t-1} + 1)^2)} \right]^{1/2} \times \left[\prod_{\tau=1}^{t} (l_{\tau,2t} - l_{j,2t-1} - 1)(l_{\tau,2t} + l_{j,2t-1}) \right]^{1/2}$$

$$B(m_{j,2t}) = \left[\frac{\prod_{\tau=1}^{t} (l_{\tau,2t-1}^2 - l_{j,2t}^2) \prod_{\tau=1}^{t+1} (l_{\tau,2t+1}^2 - l_{j,2t}^2)}{l_{j,2t}^2 (4l_{j,2t}^2 - 1) \prod_{\tau \neq j} (l_{\tau,2t}^2 - l_{j,2t}^2)((l_{\tau,2t} - 1)^2 - l_{j,2t}^2)} \right]^{1/2}$$

$$C_{2t} = \frac{\prod_{\tau=1}^{t} l_{\tau,2t-1} \prod_{\tau=1}^{t+1} l_{\tau,2t+1}}{\prod_{\tau=1}^{t} l_{\tau,2t}(l_{\tau,2t} - 1)}$$

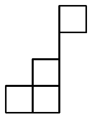
Note that the Gelfand–Zetlin basis vectors are not eigenvectors of the p commuting generators M_{12}, M_{34}, ..., $M_{2p-1,2p}$, but only of the first one M_{12}.

For more details, see refs. [27, 29].

1.36 Generalized Young tableaux

It is useful to introduce generalized Young tableaux for performing the product of representations for orthogonal, symplectic and even exceptional groups (→ 1.64, 1.65, 1.66 and 1.63).

So, following ref. [30], we define in the context of a rank n simple Lie group G, in particular with $G = SO(2n)$, $SO(2n+1)$, $Sp(2n)$ or G_2, a generalized Young tableau (GYT) as a tableau associated with the ordered set of positive, null and negative numbers $[\alpha_1, \ldots, \alpha_n]$ satisfying $\alpha_1 \geq \ldots \geq \alpha_n$, the considered tableau being constituted by $|\alpha_i|$ boxes in the ith row, these boxes being arranged on the right hand side (resp. left hand side) of a virtual vertical axis following $\alpha_i > 0$ (resp. $\alpha_i < 0$). For example, the GYT $[1, 0, -1, -2]$ will be drawn as follows:



One can add that the introduction of such tableaux is related to the study of the different weights in an irreducible representation of G.

Finally, we point out that a GYT relative to a rank n Lie group G cannot have more than n rows.

Let us define the rules for the product of two GYTs.

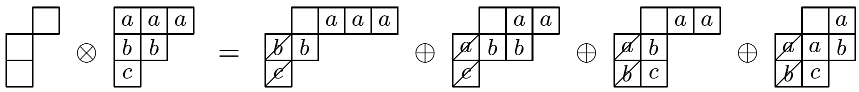
If the product we have to consider concerns two GYTs with only positive rows (standard Young tableaux), then the product has to be performed according to the product law of $SU(n)$ Young tableaux, the only difference being that a GYT with n positive rows $[m_1, \ldots, m_n]$ is not equivalent to the simplified GYT $[m_1 - m_n, \ldots, m_{n-1} - m_n, 0]$.

Otherwise the multiplication law we define is a direct generalization of the product of $SU(n)$ Young tableaux, and it is convenient to consider three kinds of products:

1. A completely arbitrary GYT $[\alpha]$ with $\alpha_1 \geq \ldots \geq \alpha_n$ multiplying a positive GYT $[\mu]$ with $\mu_1 \geq \ldots \geq \mu_n \geq 0$.

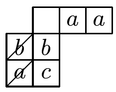
   Call the boxes in the first line a, those of the second line b and so on up to the n-th line of the GYT $[\mu]$. Add to the GYT $[\alpha]$ one box $\boxed{a}$ of $[\mu]$ using all different ways so that one always gets a GYT. A box added to a negative row of $[\alpha]$ will cancel the negative box furthest left in this row. Then add a second box $\boxed{a}$ to the obtained tableaux and so on using the usual $SU(n)$ prescriptions.

   As an illustration, let us consider the following product relative to $SO(6)$:



$$[1,-1,-1] \otimes [3,2,1] \quad = \quad [4,1,0] \ \oplus \ [3,2,0] \ \oplus \ [3,1,1] \ \oplus \ [2,2,1]$$

   Notice that the tableau

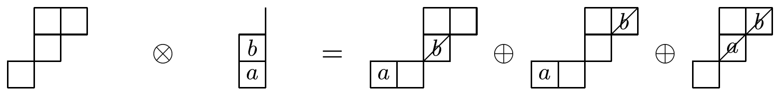


   does not exist since, before adding the b boxes, we would get the tableau $[3,-1,0]$, which is not a GYT.

2. A completely arbitrary GYT $[\alpha]$ multiplying a negative GYT $[\mu]$ with $0 \geq \mu_1 \geq \ldots \geq \mu_n$.

   Call the boxes in the last line a, those of the line just above b and so on up to the first line of GYT $[\mu]$. Add to the GYT $[\alpha]$ one box $\boxed{a}$ of $[\mu]$ in all the possible ways giving always a GYT. A negative box added to a positive row of $[\alpha]$ will cancel the positive box furthest right in the row. Then add a second box $\boxed{a}$ and so on using the usual $SU(n)$ prescriptions, but reading from left to right and from bottom to top, to satisfy $n_i(a) \geq n_i(b) \geq \ldots$ (instead of counting from right to left and from top to bottom).

   As an example:

   $$[2,1,-1] \otimes [0,-1,-1] \quad = \quad [2,0,-2] \,\oplus\, [1,1,-2] \,\oplus\, [1,0,-1]$$

3. Finally, the product of two arbitrary GYTs can be done by combining the rules of cases 1 and 2 and the following recurrence formula:

   $$[\alpha] \otimes [\mu] = ([\alpha] \otimes [\mu]_-) \otimes [\mu]_+ \;\ominus\; [\alpha] \otimes \{[\mu]\}$$

   where

   $$[\mu]_- = [0, \ldots, 0, \mu_{k+1}, \ldots, \mu_n] \quad \text{and} \quad [\mu]_+ = [\mu_1, \ldots, \mu_k, 0, \ldots, 0]$$

   with $\mu_1, \ldots, \mu_k \geq 0$, $\mu_{k+1}, \ldots, \mu_n < 0$ and $\{[\mu]\}$ denotes the set of GYTs obtained from $[\mu]$ cancelling in all the possible ways one or more negative boxes with one or more positive boxes ("contraction") with the following prescription: one labels by a the boxes on the first positive row, b on the second row and so on; in the *contracted* diagram, denoting $n_i(a)$ the number of boxes labelled by a in the i-th first columns starting from the right and $n'_i(a)$ the number of boxes labelled by a in the i-th first rows starting from the top, and identical definitions for the other labels $b, c, \ldots$, one must have $n_i(a) \geq n_i(b) \geq n_i(c) \geq \ldots$ and $n'_i(a) \geq n'_i(b) \geq n'_i(c) \geq \ldots$.

   As an example:

   $$\begin{aligned}\{[2,2,-1,-2]\} \;&=\; [2,1,0,-2] \oplus [2,1,-1,-1] \oplus [1,1,0,-1] \\ &\quad \oplus [1,0,0,0] \oplus [2,0,0,-1]\end{aligned}$$

   Each element of $\{[\mu]\}$ has again to be decomposed into a positive and a negative part. It is clear that the decomposition of $[\alpha] \otimes [\mu]$ will be obtained by repeating the formula a finite number of times.

Remark:
A procedure to perform the product of two GYTs, say $[\lambda] \otimes [\mu]$, just bringing it back to the product of two standard Young tableaux, is to add to each row of a GYT $[\lambda]$ (resp. $[\mu]$) $|\lambda_n|$ boxes (resp. $|\mu_n|$ boxes). In this way we get two standard Young tableaux with no boxes in the last row. We perform the product according to the Young tableaux rules ($\rightarrow$ 1.67) and we subtract from each row of the obtained Young tableaux $|\lambda_n| + |\mu_n|$ boxes. This procedure justifies the above set of rules.

1.37 Group – Subgroup

Definition

A *group* G is a set of elements together with a composition law (denoted here multiplicatively) such that

1. $\forall\, g, g' \in G, \quad g * g' \in G$ — internal law
2. $\forall\, g, g', g'' \in G, \quad (g * g') * g'' = g * (g' * g'')$ — associativity
3. $\exists\, e \in G$ such that $\forall\, g \in G, \quad g * e = e * g$ — identity element
4. $\forall\, g \in G,\ \exists\, g^{-1} \in G$ such that $g * g^{-1} = e$ — inverse

A group is said to be *finite* if it contains a finite number of elements, this number being called the order of the group.

A group is said to be *commutative* or *Abelian* if $g * g' = g' * g$ for all $g, g' \in G$.

One can prove the following theorem:

Theorem

Let G be a group. Then

1. $\forall\, g \in G,\ g^{-1} * g = e$
2. $\forall\, g \in G,\ e * g = g$
3. e is unique, and for any $g \in G$, g^{-1} is unique

Definition

A *subgroup* of a group G is a non-empty part $H \subset G$ which is a group with the composition law induced by G. H is a *proper* subgroup of G if $H \neq G$ and $H \neq \{e\}$.

Theorem

The non-empty part $H \subset G$ of a group G is a subgroup of G if and only if $x * y^{-1} \in H$ for all $x, y \in H$.

Definition

Let G be a group and H a subgroup of G. H is called a *normal* or *invariant* subgroup of G if it is globally invariant by conjugation of any element $g \in G$, that is

$$gHg^{-1} = H\,, \qquad \forall\, g \in G$$

Example

Consider the Euclidean group $E(3)$ in three dimensions ($\rightarrow$ 1.29). Its three-dimensional translation subgroup $T(3) = \{(\vec{a}, \mathbb{I})\}$ is a normal subgroup of $E(3)$. □

The importance of the notion of normal subgroup is developed in → 1.68 Quotient group and → 1.25 Direct and semi-direct products of groups.

1.38 Group morphisms

Let G and G' be two groups with internal laws $*$ and $\star$ respectively. A *homomorphism* Φ of the group G on the group G' is an application from G to G' which respects the group laws, that is

$$\Phi(g * g') = \Phi(g) \star \Phi(g')\,, \qquad \forall\, g, g' \in G$$

• An *isomorphism* is a bijective homomorphism.
• An *endomorphism* is a homomorphism of the group G on itself. An important example of endomorphism is given by the conjugation by an element g_0 of G: $\Phi_{g_0}(g) = g_0 * g * g_0^{-1}$.
• An *automorphism* is a bijective endomorphism.

1.39 Group parameter

→ 1.48 Lie group and 1.49 Lie group of transformations.

1.40 Ideal

→ 1.51 Lie subalgebra – Ideal.

1.41 Index of an embedding – Defining vector

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r and $\widetilde{\mathcal{G}}$ a subalgebra of $\mathcal{G}$ of rank ℓ. Denoting by $\mathcal{H} = \{H_1, \dots, H_r\}$ and $\widetilde{\mathcal{H}} = \{\widetilde{H}_1, \dots, \widetilde{H}_\ell\}$ the Cartan subalgebras of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ respectively, the embedding $\widetilde{\mathcal{G}} \subset \mathcal{G}$ is completely defined by the mapping f from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$:

$$f(\widetilde{H}_i) = \sum_{j=1}^{r} f_{ij} \, H_j \qquad 1 \le i \le \ell$$

The matrix (f_{ij}) is called the *defining matrix* of the embedding $\widetilde{\mathcal{G}} \subset \mathcal{G}$. When $\widetilde{\mathcal{G}} = sl(2)$, this matrix becomes a vector $(f_1, \dots, f_r)$ called the *defining vector* of the embedding $sl(2) \subset \mathcal{G}$.

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r and $\widetilde{\mathcal{G}}$ a subalgebra of $\mathcal{G}$ of rank ℓ with the defining matrix (f_{ij}) ($1 \le i \le \ell$ and $1 \le j \le r$). If K and $\widetilde{K}$ denote the Killing forms ($\rightarrow$ 1.44) on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, one has

$$K(f(\widetilde{X}), f(\widetilde{Y})) = j_f \, \widetilde{K}(\widetilde{X}, \widetilde{Y}) \qquad \widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{G}}$$

The number j_f, which is independent of $\widetilde{X}, \widetilde{Y}$, is called the Dynkin *index* of the embedding $\widetilde{\mathcal{G}} \subset \mathcal{G}$.

Theorem

The index j_f of an embedding $\widetilde{\mathcal{G}} \subset \mathcal{G}$ is an integer.

Property

Let $f_1, \dots, f_n$ be embeddings of a simple Lie algebra $\widetilde{\mathcal{G}}$ into the simple Lie algebra $\mathcal{G}$, such that

$$[f_i(\widetilde{X}), f_j(\widetilde{Y})] = 0 \qquad i \neq j \text{ and } \widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{G}}$$

Then $f = f_1 + \ldots + f_n$ is also an embedding and $j_f = j_{f_1} + \ldots + j_{f_n}$.

Property

Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ be simple Lie algebras such that $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3$. Then

$$j_{\mathcal{G}_1 \subset \mathcal{G}_3} = j_{\mathcal{G}_1 \subset \mathcal{G}_2} \, j_{\mathcal{G}_2 \subset \mathcal{G}_3}$$

Property

Let $\mathcal{G}$ be a simple Lie algebra of rank r and $(f_1, \dots, f_r)$ the defining vector of an embedding $sl(2) \subset \mathcal{G}$. Then the index of the embedding is given by

$$j_f = \tfrac{1}{2} \sum_{i=1}^{r} f_i^2$$

The case of the embedding $sl(2) \subset \mathcal{G}$ is treated in detail in → 1.28 Embeddings of $sl(2)$.

1.42 Index of a representation

Definition

Let $\mathcal{G}$ be a simple Lie algebra and π a representation of $\mathcal{G}$. The bilinear form associated to π is defined by $B_\pi(X,Y) = \mathrm{tr}(\pi(X)\,\pi(Y))$ and the Killing form by $K(X,Y) = \mathrm{tr}(\mathrm{ad}_X\ \mathrm{ad}_Y)$ where $X, Y \in \mathcal{G}$ (→ 1.44). One has

$$B_\pi(X,Y) = j_\pi\, K(X,Y)$$

where j_π is independent of the elements X and Y of $\mathcal{G}$. The number j_π is called the *index* of the representation π.

Property

Let $\mathcal{G}$ be a simple Lie algebra and π a representation of $\mathcal{G}$ of dimension N and highest weight Λ. The value of the index j_π is given by

$$j_\pi = \frac{N}{\dim \mathcal{G}}\, C_2(\Lambda)$$

where $C_2(\Lambda)$ is the value of the second order Casimir in the representation π, properly normalized (i.e. $C_2 = 1$ in the adjoint representation).

1.43 Iwasawa decomposition

Let $\mathcal{G}$ a semi-simple real Lie algebra. One knows (→ 1.44) that $\mathcal{G}$ satisfies the Cartan decomposition

$$\mathcal{G} = \mathcal{C} \oplus \mathcal{P}$$

where $\mathcal{C}$ is compact.

Through the complex extension $\mathcal{G}_C$ of $\mathcal{G}$, one can consider to each root α of $\mathcal{G}_C$ the root subspace

$$\mathcal{G}_\alpha = \Big\{ x \in \mathcal{G}_C \;\Big|\; \big[h, x\big] = \alpha(h)x,\ h \in \mathcal{H} \Big\}$$

where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$.

ϕ being the involution defined on $\mathcal{G}$ ($\rightarrow$ 1.44) such that

$$\phi(X) = X \quad \text{for} \quad X \in \mathcal{C}$$
$$\phi(X) = -X \quad \text{for} \quad X \in \mathcal{P}$$

it can be shown that the application α^ϕ defined from α by

$$\alpha^\phi(h) = \alpha(\phi h) \qquad \forall\, h \in \mathcal{H}$$

is also a root for the algebra $\mathcal{G}_C$.

Then defining

$$\mathcal{N} = \bigoplus_{\alpha \in P_+} \mathcal{G}_\alpha \qquad \text{where} \ \ P_+ = \left\{ \alpha \in \mathcal{H}^* \,\middle|\, \alpha \in \Delta\,, \ \ \alpha \neq \alpha^\phi \right\}$$

and

$$\mathcal{N}_0 = \mathcal{G} \cap \mathcal{N}$$

one has the following theorem:

Theorem (Iwasawa)

The semi-simple real Lie algebra $\mathcal{G}$ admits the direct vector space decomposition

$$\mathcal{G} = \mathcal{C} \oplus \mathcal{H}_\mathcal{P} \oplus \mathcal{N}_0$$

where C is compact, $\mathcal{N}_0 = \mathcal{G} \cap \mathcal{N}$ is as defined above and $\mathcal{H}_\mathcal{P}$ is a maximal Abelian subalgebra of $\mathcal{P}$.

The spaces $\mathcal{N}$ and $\mathcal{N}_0$ are nilpotent Lie algebras and $\mathcal{S}_0 = \mathcal{H}_\mathcal{P} \oplus \mathcal{N}_0$ is a solvable algebra ($\rightarrow$ 1.23).

1.44 Killing form

The importance of the Killing form stands in its properties of providing criteria for the semi-simplicity of a Lie algebra and for the compacity of semi-simple groups.

Definition

Let $\mathcal{G}$ be a Lie algebra. One defines the bilinear form B_π associated to a representation π of $\mathcal{G}$ as a bilinear form from $\mathcal{G} \times \mathcal{G}$ into the field of real numbers $\mathbb{R}$ such that

$$B_\pi(X, Y) = \mathrm{tr}(\pi(X)\, \pi(Y))\,, \qquad \forall\ X, Y \in \mathcal{G}$$

$\pi(X)$ are the matrices of the generators $X \in \mathcal{G}$ in the representation π.

Definition

The bilinear form associated to the adjoint representation of $\mathcal{G}$ is called the *Killing form* on $\mathcal{G}$ and is denoted $K(X,Y)$:

$$K(X,Y) = \mathrm{tr}(\mathrm{ad}_X\ \mathrm{ad}_Y)\,, \qquad \forall\ X,Y \in \mathcal{G}$$

We recall that $\mathrm{ad}_X\,(Y) = [X,Y]$ and $(\mathrm{ad}_{X_\mu})^\beta_\alpha = -C_{\mu\alpha}{}^\beta$ (→ 1.2) where $C_{\mu\alpha}{}^\beta$ are the structure constants for the basis $\{X_\mu\}$ of $\mathcal{G}$ ($\mu = 1,\ldots,\dim\mathcal{G}$). We can therefore write

$$K(X_\mu, X_\nu) = C_{\mu\alpha}{}^\beta C_{\nu\beta}{}^\alpha = g_{\mu\nu}$$

Cartan's criterion

$\mathcal{G}$ is semi-simple if and only if K is non-degenerate, that is $\det g \neq 0$.

Theorem

$\mathcal{G}$ is semi-simple if and only if $\mathcal{G}$ is a direct sum of ideals which are simple as Lie algebras.

Theorem (Weyl)

Let G be a semi-simple connected Lie group. Then G is compact if and only if the Killing form on its Lie algebra $\mathcal{G}$ is negative definite, that is $\forall\,X \in \mathcal{G}\,, K(X,X) < 0$.

We remark that in this last case, $-K$ defines a scalar product on $\mathcal{G}$.

Finally, for a general semi-simple Lie algebra, we have the following result:

Theorem (Cartan)

Any semi-simple Lie algebra $\mathcal{G}$ over $\mathbb{R}$ can be seen as the direct sum of two vector spaces:

$$\mathcal{G} = \mathcal{C} \oplus \mathcal{P}$$

where:

1. $\mathcal{C}$ is a compact subalgebra of $\mathcal{G}$, that is $\left[\mathcal{C},\mathcal{C}\right] \subset \mathcal{C}$.

   $\mathrm{ad}_{\mathcal{C}}$ leaves $\mathcal{P}$ invariant, that is $\left[\mathcal{C},\mathcal{P}\right] \subset \mathcal{P}$ and $\left[\mathcal{P},\mathcal{P}\right] \subset \mathcal{C}$.

2. The Killing form K of $\mathcal{G}$ restricted to $\mathcal{C}$ is negative definite, while that restricted to $\mathcal{P}$ is positive definite. Furthermore, any subalgebra $\mathcal{C}'$ of $\mathcal{G}$ such that the Killing form of $\mathcal{G}$ restricted to $\mathcal{C}'$ is negative definite is conjugate under G to a subalgebra of $\mathcal{C}$.

   The coset space G/C is called a *symmetric space* (→ 1.71).

The subalgebra $\mathcal{C}$ is called a symmetric subalgebra because of the existence of the automorphism ϕ of $\mathcal{G}$ defined as follows:

$$\begin{aligned} \phi(X) &= X \quad \text{for} \quad X \in \mathcal{C} \\ \phi(X) &= -X \quad \text{for} \quad X \in \mathcal{P} \end{aligned}$$

Compactification of $\mathcal{G}$: from above, one can note that $\mathcal{G}$ can be "compactified" by replacing any element $X \in \mathcal{P}$ by iX. Then $K(iX,iX) = -K(X,X) < 0$.

1.45 Lattices

Definition

Let $\mathcal{V}$ be an N-dimensional vector space on $\mathbb{R}$ with basis $(\varepsilon_1,\dots,\varepsilon_N)$. $\mathcal{V}$ is endowed with a symmetric bilinear form (inner product), denoted $u \cdot v$ for $u,v \in \mathcal{V}$. A lattice $\mathfrak{L}$ is defined as a set of points in $\mathcal{V}$ such that

$$\mathfrak{L} = \Big\{ \sum_{i=1}^{N} n_i\, \varepsilon_i \,,\ n_i \in \mathbb{Z} \Big\}$$

In the following, we assume $\mathcal{V} = \mathbb{R}^{p,q}$ where $p+q = N$, with pseudo-Euclidean inner product of signature (p,q). In the case $p = N, q = 0$, the lattice $\mathfrak{L}$ is said to be Euclidean. In the case $p = N-1, q = 1$, the lattice $\mathfrak{L}$ is said to be Lorentzian.

Definition

The dual lattice $\mathfrak{L}^*$ of an n-dimensional lattice $\mathfrak{L}$ is defined by

$$\mathfrak{L}^* = \Big\{ y \in \mathcal{V} \ \Big|\ y \cdot x \in \mathbb{Z},\ \forall\, x \in \mathfrak{L} \Big\}$$

A basis of $\mathfrak{L}^*$ is given by the dual basis $(\varepsilon_1^*,\dots,\varepsilon_N^*)$ to the basis $(\varepsilon_1,\dots,\varepsilon_N)$, that is $\varepsilon_i^* \cdot \varepsilon_j = \delta_{ij}$.

Definition

A lattice $\mathfrak{L}$ is called:

– *unimodular* if it has one point per unit volume, that is $\text{vol}(\mathfrak{L}) = \sqrt{|\det g|} = 1$ where $g_{ij} = \varepsilon_i \cdot \varepsilon_j$.
– *integral* if $x \cdot y \in \mathbb{Z}$ for all $x,y \in \mathfrak{L}$. One has then $\mathfrak{L} \subset \mathfrak{L}^*$. Furthermore an integral lattice is called even if all lattice vectors have even squared length: $x^2 \in 2\mathbb{Z}$ for all $x \in \mathfrak{L}$. It is called odd otherwise.
– *self-dual* if $\mathfrak{L} = \mathfrak{L}^*$.

Property

A lattice $\mathfrak{L}$ is self-dual if and only if it is unimodular and integral.

Among the lattices related to the simple Lie algebras, the root lattice Q, the coroot lattice $Q^\vee$ and the weight lattice P are particularly relevant. For a given Lie algebra $\mathcal{G}$ of rank r, they are defined as follows: $\{\alpha_1, \ldots, \alpha_r\}$ being the simple root system, $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ the simple coroot system where $\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i \cdot \alpha_i)$ and $\{\Lambda_1, \ldots, \Lambda_r\}$ the fundamental weight system, one has:

$$\begin{aligned} Q &= \Big\{ \sum_{i=1}^{r} n_i \, \alpha_i \, , \ n_i \in \mathbb{Z} \Big\} \\ Q^\vee &= \Big\{ \sum_{i=1}^{r} n_i \, \alpha_i^\vee \, , \ n_i \in \mathbb{Z} \Big\} \\ P &= \Big\{ \sum_{i=1}^{r} n_i \, \Lambda_i \, , \ n_i \in \mathbb{Z} \Big\} \end{aligned}$$

From the relation $2 \dfrac{\Lambda_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \Lambda_i \cdot \alpha_j^\vee = \delta_{ij}$, it follows that the weight lattice P is dual to the coroot lattice $Q^\vee$. The integer numbers specifying the position of a point in P are the eigenvalues of the Cartan–Chevalley generators h_i. The effects on P of the other generators is to shift the point in the lattice by an element of Q. Since the roots are the weights ($\rightarrow$ 1.92) of the adjoint representation, one has the inclusion $Q \subseteq P$. In particular for the algebras E_8, F_4 and G_2, the lattices Q and P coincide. The quotient P/Q is a finite group whose order $|P/Q|$ is equal to the determinant of the Cartan matrix of the algebra. This finite group is isomorphic to the center $\mathcal{Z}(G)$ of the group G corresponding to the algebra $\mathcal{G}$. The elements of P/Q define the conjugacy or congruence classes. Of course, the weight of an irreducible representation of $\mathcal{G}$ belongs to exactly one conjugacy class. For instance, for the algebra $sl(n)$ there are n conjugacy classes which are defined by

$$\sum_{i=1}^{n-1} i \, \Lambda_i \quad \bmod P/Q$$

For any simple Lie algebra $\mathcal{G}$ the conjugacy classes are defined by

$$\vec{\lambda} \cdot \vec{C} = \sum_{i=1}^{r} C_i \, \Lambda_i \quad \bmod P/Q \text{ and } \bmod \mathbb{Z}/2\mathbb{Z} \text{ for } D_{2N}$$

where $\vec{C}$ is the so-called conjugacy or congruence vector.

For a simple Lie algebra $\mathcal{G}$, one has

$$\text{vol}(Q) = \sqrt{\det(A_{ij})} = \sqrt{N_c}$$

where (A_{ij}) is the Cartan matrix ($\rightarrow$ 1.8) and N_c is the number of conjugacy classes of $\mathcal{G}$.

Table 1.3 gives for the simple Lie algebras the root lattice Q, the coroot lattice $Q^\vee$, the weight lattice P, the discrete factor group P/Q and the conjugacy vectors (for non-simply-laced algebras: $P \neq Q$). In this table, $\mathfrak{L}_N$ is the cubic lattice $\mathbb{Z}^N$ with basis $(\varepsilon_1, \dots, \varepsilon_N)$. The lattice $\mathfrak{L}'_N$ is the sublattice of $\mathfrak{L}_N$ constituted by the points $x = \sum_{i=1}^N x_i \varepsilon_i$ such that $\sum_i x_i \in 2\mathbb{Z}$. One defines also the lattices $\widetilde{\mathfrak{L}}_N = \mathfrak{L}_N \cup \varpi\mathbb{Z}$ and $\widetilde{\mathfrak{L}}'_N = \mathfrak{L}'_N \cup \varpi\mathbb{Z}$ where $\varpi = \frac{1}{2}\sum_{i=1}^N \varepsilon_i$. Clearly $\mathfrak{L}_N/\mathfrak{L}'_N \simeq \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{\mathfrak{L}}_N/\mathfrak{L}_N \simeq \mathbb{Z}/2\mathbb{Z}$. $\mathcal{V}_N$ is the hyperplane in $\mathbb{R}^N$ orthogonal to ϖ and $\Lambda_1 = \varepsilon_1 - \frac{2}{N}\,\varpi$ is the first fundamental weight of A_{N-1}; $\mathcal{V}_7$ is the hyperplane in $\mathbb{R}^8$ orthogonal to the fundamental weight Λ_8 of E_8 and $\Lambda'_7 = \varepsilon_6 + \frac{1}{2}\,(\varepsilon_8 - \varepsilon_7)$ is the last fundamental weight of E_7; $\mathcal{V}_6$ is the hyperplane in $\mathbb{R}^8$ orthogonal to the fundamental weights Λ_8 and Λ_7 of E_8 and $\Lambda''_6 = \varepsilon_5 + \frac{1}{3}\,(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)$ is the last fundamental weight of E_6; $\mathfrak{L}[\varepsilon_{12}, \varepsilon_{13}]$ is the lattice with basis $\varepsilon_{12} \equiv \varepsilon_1 - \varepsilon_2, \varepsilon_{13} \equiv \varepsilon_1 - \varepsilon_3$ in the hyperplane of $\mathbb{R}^3$ orthogonal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

For more details, see also Tables 3.1–3.9.

Table 1.3: Root and weight lattices of the simple Lie algebras.

algebra	Q	P	P/Q	conjugacy vectors
A_{N-1}	$\mathfrak{L}_N \cap \mathcal{V}_N$	$Q \cup \Lambda_1\mathbb{Z}$	$\mathbb{Z}/N\mathbb{Z}$	$1, 2, \dots, N-1, N$
B_N	$\mathfrak{L}_N$	$\widetilde{\mathfrak{L}}_N$	$\mathbb{Z}/2\mathbb{Z}$	$0, 0, \dots, 0, 1$
C_N	$\mathfrak{L}'_N$	$\mathfrak{L}_N$	$\mathbb{Z}/2\mathbb{Z}$	$1, 2, \dots, N-1, N$
D_{2N}	$\mathfrak{L}'_{2N}$	$\widetilde{\mathfrak{L}}_{2N}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$2, 4, \dots, 2N-2, N$
D_{2N+1}	$\mathfrak{L}'_{2N+1}$	$\widetilde{\mathfrak{L}}_{2N+1}$	$\mathbb{Z}/4\mathbb{Z}$	$0, 0, \dots, 1, 1$
E_6	$\widetilde{\mathfrak{L}}'_N \cap \mathcal{V}_6$	$\widetilde{\mathfrak{L}}'_N \cup \Lambda''_6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$1, 2, 0, 1, 2, 0$
E_7	$\widetilde{\mathfrak{L}}'_N \cap \mathcal{V}_7$	$\widetilde{\mathfrak{L}}'_N \cup \Lambda'_7\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0, 0, 0, 1, 0, 1, 1
E_8	$\widetilde{\mathfrak{L}}'_N$	$\widetilde{\mathfrak{L}}'_N$	$\mathbb{I}$	
F_4	$\widetilde{\mathfrak{L}}_N$	$\widetilde{\mathfrak{L}}_N$	$\mathbb{I}$	
G_2	$\mathfrak{L}[\varepsilon_{12}, \varepsilon_{13}]$	Q	$\mathbb{I}$	

Self-dual lattices play a particular role in certain theories, for example in string theory. Very few self-dual lattices exist for small dimensions. For Euclidean lattices, the first examples are the root lattice of the Lie algebra E_8 (in dimension 8) which is even, and the weight lattice of the Lie algebra D_{12} (in dimension 12) which is odd. Even self-dual Euclidean lattices only occur in dimension $8k$ with k positive integer; there are two in dimension 16 (the root lattice of $E_8 \times E_8$ and the weight lattice of $SO(32)/\mathbb{Z}_2$), and 24 in dimension 24, in which appears the Leech lattice whose minimum non-zero squared length is 4. For Lorentzian lattices, there is exactly one odd self-dual lattice in each dimension, which is $\mathbb{Z}^{N-1,1}$. Even self-dual Lorentzian lattices only occur in dimension $8k+2$ with k positive integer; they are denoted by $\Pi^{N-1,1} = \{x \in \mathbb{Z}^{N-1,1} \text{ or } x \in \mathbb{Z}^{N-1,1} + \varpi \mid x \cdot \varpi \in \mathbb{Z}\}$ where $\varpi = (\frac{1}{2}, \dots, \frac{1}{2})$.

1.46 Lie algebra: definition

Definition

$\mathcal{G}$ is a Lie algebra over a field $\mathbb{K}$ of characteristic zero (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) if $\mathcal{G}$ is an algebra over $\mathbb{K}$ with its second internal law $*$ satisfying $\forall\, X, Y, Z \in \mathcal{G}$:

$$X * Y = -Y * X$$
$$X * (Y * Z) + Y * (Z * X) + Z * (X * Y) = 0$$

The first equation means that the product $*$ is antisymmetric and the second equation is called the Jacobi identity.

An associative algebra $\mathcal{A}$ over the field $\mathbb{K}$ (with product $*$) acquires the structure of a Lie algebra by taking for the product the Lie bracket $[\ ,\ ]$ (also called commutator):

$$\Big[X, Y\Big] = X * Y - Y * X$$

for two elements $X, Y \in \mathcal{A}$.
It is the case for example of Lie algebras of linear operators (or matrices).

Example 1

Consider the Lie algebras of dimension 2 over $\mathbb{C}$, generated by two elements a and b. There exist only two such Lie algebras: the Abelian Lie algebra with commutation relation $[a, b] = 0$ and the non-Abelian one with commutation relation $[a, b] = b$. □

Example 2

Consider the Lie algebras of dimension 3 over $\mathbb{C}$, generated by three elements a, b and c. Let us denote the derivative ($\rightarrow$ 1.23) of the Lie algebra $\mathcal{G}$ by $\mathcal{G}'$ and the center ($\rightarrow$ 1.10) of $\mathcal{G}$ by $\mathcal{Z}(\mathcal{G})$. The Lie algebras of dimension 3 over $\mathbb{C}$ are defined by one of the following sets of commutation relations ($\alpha, \beta \in \mathbb{C}\backslash\{0\}$):

$$
\begin{array}{ll}
[a,b] = [b,c] = [c,a] = 0 & (\dim \mathcal{G}' = 0, \mathcal{G} \text{ is Abelian}) \\
[a,b] = c, [c,a] = [c,b] = 0 & (\dim \mathcal{G}' = 1 \text{ and } \mathcal{G}' \subset \mathcal{Z}(\mathcal{G})) \\
[a,b] = b, [c,a] = [c,b] = 0 & (\dim \mathcal{G}' = 1 \text{ and } \mathcal{G}' \not\subset \mathcal{Z}(\mathcal{G})) \\
[a,b] = 0, [c,a] = a, [c,b] = \alpha b & (\dim \mathcal{G}' = 2) \\
[a,b] = 0, [c,a] = a + \beta b, [c,b] = b & (\dim \mathcal{G}' = 2) \\
[a,b] = c, [c,a] = \alpha a, [c,b] = \beta b & (\dim \mathcal{G}' = 3)
\end{array}
$$

□

1.47 Lie algebra: general decomposition

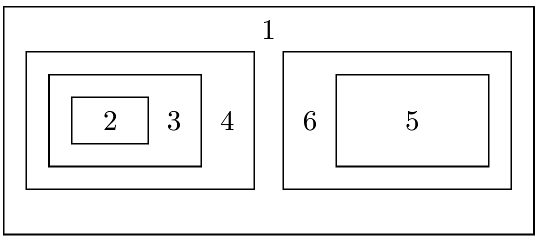
Theorem (Levi)

An arbitrary Lie algebra $\mathcal{G}$ has a semi-direct sum structure $\mathcal{G} = \mathcal{S} \mathbin{\in\!\!\!\!+} \mathcal{R}$ with $\mathcal{R}$ solvable Lie algebra (called the *radical* of the Lie algebra), ideal of $\mathcal{G}$ and $\mathcal{S}$ semi-simple Lie algebra. It follows

$$\left[\mathcal{R},\mathcal{R}\right] \subset \mathcal{R}\,, \qquad \left[\mathcal{R},\mathcal{S}\right] \subset \mathcal{R}\,, \qquad \left[\mathcal{S},\mathcal{S}\right] \subset \mathcal{S}$$

Theorem (Malcev)

Any semi-simple subalgebra of $\mathcal{G}$ is conjugate (by an inner automorphism) to a subalgebra of $\mathcal{S}$, that is $\mathcal{S}$ is unique. $\mathcal{S}$ is called the Levi's factor of the Lie algebra $\mathcal{G}$.



1 Lie Algebras (L.A.)
2 Abelian L.A.
3 Nilpotent L.A.
4 Solvable L.A.
5 Simple L.A.
6 Semi-simple L.A.

Example

Consider the Lie algebra of the Euclidean group $E(3)$. $E(3)$ is the semi-direct product of the three-dimensional Abelian translation group $T(3)$ by the rotation group in three dimensions $SO(3)$. Its Lie algebra is therefore the semi-direct sum $t(3) \ni\!\!\!+ so(3)$, with the generators J_i and P_i $(i = 1, 2, 3)$ of $so(3)$ and $t(3)$ respectively satisfying:

$$\left[J_i, J_j\right] = i\,\varepsilon_{ijk}\,J_k\,, \qquad \left[J_i, P_j\right] = i\,\varepsilon_{ijk}\,P_k\,, \qquad \left[P_i, P_j\right] = 0$$

The subalgebra generated by $\{J_z, J_x + P_y, J_y - P_x\}$ is isomorphic to $so(3)$. From Malcev's theorem, it must be conjugated to the algebra generated by $\{J_x, J_y, J_z\}$. By action of the element $\exp(P_z)$, we obtain:

$$\begin{aligned}
\exp(P_z) J_x \exp(-P_z) &= J_x + \left[P_z, J_x\right] = J_x + P_y \\
\exp(P_z) J_y \exp(-P_z) &= J_y + \left[P_z, J_y\right] = J_y - P_x \\
\exp(P_z) J_z \exp(-P_z) &= J_z
\end{aligned}$$

□

1.48 Lie group

Definition

Let G be a group. G is a Lie group of dimension n if it is an n-dimensional analytic manifold, that is:

1. For all $g \in G$, the element x can be parametrized, at least locally, with the help of n parameters $a_1, \dots, a_n \in \mathbb{R}$ (or $\mathbb{C}$).
2. For all $g, g' \in G$ such that $g = g(a_1, \dots, a_n)$ and $g' = g'(a'_1, \dots, a'_n)$, the elements $g.g' = g''(a''_1, \dots, a''_n)$ and $g^{-1} = \overline{g}(\overline{a}_1, \dots, \overline{a}_n)$ are such that the functions $a''_i(a_1, \dots, a_n, a'_1, \dots, a'_n)$ and $\overline{a}_i(a_1, \dots, a_n)$ are analytic functions of the arguments.

1.49 Lie group of transformations

Let G be a Lie group of dimension n. G is a Lie group of transformations on the m-dimensional manifold M if there exists an application $f : G \times M \to M$, that is

$$x'^i = f^i(x^1, \dots, x^m; a_1, \dots, a_n) \qquad i = 1, \dots, m$$

which is analytic in its $(n + m)$ arguments and in addition satisfies

1. For all $x' = f(x;a)$ and $x'' = f(x';a')$, there exists $a'' = a''(a,a')$ analytic in a and a' with $x'' = f(x;a'')$.
2. For all a, there exists a unique $\overline{a}$ such that $x' = f(x;a) \Rightarrow x = f(x';\overline{a})$.
3. There exists a_0 such that $x = f(x;a_0)$ for all $x \in G$.

Example of a Lie group of transformation

Consider the rotation group $O(3)$ in three dimensions. Let g be the element which rotates the frame (Ox, Oy, Oz) into the new frame (Ox', Oy', Oz'). Such a rotation will be completely determined once one knows for example the Euler angles ψ, θ, ϕ. We can represent $g = g(\psi, \theta, \phi)$ by the matrix

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & -\cos\psi\sin\phi - \cos\theta\sin\psi\cos\phi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & -\cos\psi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

where $0 \le \phi, \psi \le 2\pi$ and $0 \le \theta \le \pi$. One can remark that

$$g(\psi, \theta, \phi) = g_\phi \, g_\theta \, g_\psi$$

that is $g(\psi, \theta, \phi)$ can be decomposed as the product of three rotations: a rotation g_ψ around the axis Oz, followed by a rotation g_θ around the axis OL (which determines the intersection of the two planes xOy and $x'Oy'$), and finally a rotation g_ϕ around the axis Oz'. Moreover, the inverse of g can be written $g(\psi, \theta, \phi)^{-1} = g(\pi - \phi, \theta, \pi - \psi)$.

Let us emphasize that:

1. this group contains three essential parameters (a fourth parameter would be redundant and it is impossible to describe any rotation of the three-dimensional space $\mathbb{R}^3$ with only two parameters).
2. one has $g(\psi, \theta, \phi) \cdot g'(\psi', \theta', \phi') = g''(\psi'', \theta'', \phi'')$ for any $g(\psi, \theta, \phi)$ and $g'(\psi', \theta', \phi')$, where ψ'', θ'', ϕ'' are continuous functions in ψ, θ, ϕ, ψ', θ', ϕ'.
3. the domain of variation of each parameter is compact (since it is a closed set of the real axis: $0 \le \phi, \psi \le 2\pi$ and $0 \le \theta \le \pi$).

It follows that the group $O(3)$ is a continuous three-dimensional group: it is a three-dimensional Lie group of transformations. Moreover, it is compact since the domain of variation of each parameter of $O(3)$ is compact. Topologically, $O(3)$ is a manifold: indeed there is a one-to-one correspondence between the elements of $O(3)$ and the elements in the subspace $S = \{0 \le \psi, \phi \le 2\pi, \ 0 \le \theta \le \pi\}$ of $\mathbb{R}^3$:

$$g \in O(3) \quad \mapsto \quad (\psi, \theta, \phi) \in S$$

in which concepts and methods in $\mathbb{R}^3$ are valid (S is locally Euclidean).

1.50 Lie group of transformation generator

From the above definition of a Lie group of transformations, one has $x^i = f^i(x;0)$ for any $x \in M$. Considering an infinitesimal transformation $x \mapsto x + dx$ of parameters $da = (da^1, \dots, da^n)$, one obtains $x + dx = f(x; da)$, thus

$$dx^i = \left.\frac{\partial f^i(x;a)}{\partial a^j}\right|_{a=0} da^j = \sum_j u^i_j(x)\, da^j \qquad (i = 1, \dots, m \text{ and } j = 1, \dots, n)$$

Let F be a form from M into $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

$$dF = \frac{\partial F}{\partial x}\, dx = \sum_{ij} u^i_j(x)\, \frac{\partial F}{\partial x^i}\, da^j = \sum_j (da^j X_j)\, F$$

The n operators

$$X_j = \sum_{i=1}^m u^i_j(x)\, \frac{\partial}{\partial x^i}$$

are called the *generators* of the Lie group of transformations G.

Example

Let G be the three-dimensional rotation group $SO(3)$. Then $x' = f(x;a) = Ax$, where A is a 3×3 orthogonal matrix (that is $AA^t = \mathbb{I}$ or infinitesimally $dA = -dA^t$), thus:

$$dA = \begin{pmatrix} 0 & da^3 & -da^2 \\ -da^3 & 0 & da^1 \\ da^2 & -da^1 & 0 \end{pmatrix}$$

Then

$$\begin{aligned} dx^1 &= x^2 da^3 - x^3 da^2 \\ dx^2 &= x^3 da^1 - x^1 da^3 \\ dx^3 &= x^1 da^2 - x^2 da^1 \end{aligned}$$

and since $dx^i = \sum_{i=1}^3 u^i_j(x)\, da^j$, we deduce the representation of the three generators of $SO(3)$:

$$\begin{aligned} X_1 &= x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} \\ X_2 &= x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} \\ X_3 &= x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \end{aligned}$$

and we can check the commutation relations

$$\Big[X_1, X_2\Big] = -X_3\,, \qquad \Big[X_2, X_3\Big] = -X_1\,, \qquad \Big[X_3, X_1\Big] = -X_2$$

Let us also consider the functions F^k $(k = 1, 2, 3)$ which associate to any vector $\vec{x} \in \mathbb{R}^3$ its components $x^k \in \mathbb{R}$, $F^j : \vec{x} \mapsto x^j$. Then

$$dF^k(\vec{x}) = \sum_{ij} da^j \left(u^i_j(x) \frac{\partial}{\partial x^i} \right) x^k = \sum_j da^j \, u^k_j(x)$$

that is

$$d\vec{x} = \left(\sum_j da^j X_j \right) \vec{x}$$

But since $dx^i = dA^i_j \, x^j$ and due to the form of the matrix dA, we get the representation of X_j by 3×3 matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

□

1.51 Lie subalgebra – Ideal

Definition

Let $\mathcal{G}$ be a Lie algebra over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A subset $\mathcal{H}$ of $\mathcal{G}$ is called a *subalgebra* of $\mathcal{G}$ if $\mathcal{H}$ is a vector subspace of $\mathcal{G}$ which is itself a Lie algebra. A subalgebra $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{H} \neq \mathcal{G}$ is called a *proper* subalgebra of $\mathcal{G}$.

Definition

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{I}$ a subalgebra of $\mathcal{G}$. $\mathcal{I}$ is called an *ideal* of $\mathcal{G}$ if $\Big[\mathcal{G}, \mathcal{I}\Big] \subset \mathcal{I}$, that is

$$X \in \mathcal{G}, Y \in \mathcal{I} \;\Rightarrow\; \Big[X, Y\Big] \in \mathcal{I}$$

An ideal $\mathcal{I}$ of $\mathcal{G}$ such that $\mathcal{I} \neq \mathcal{G}$ is called a proper ideal of $\mathcal{G}$.

If H is a normal subgroup of the Lie group G, then its Lie algebra $\mathcal{H}$ is an ideal of the Lie algebra $\mathcal{G}$ of G.

Property

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{I}, \mathcal{I}'$ two ideals of $\mathcal{G}$. Then $\left[\mathcal{I}, \mathcal{I}'\right]$, $\mathcal{I} \cap \mathcal{I}'$ and $\mathcal{I} \cup \mathcal{I}'$ are ideals of $\mathcal{G}$.

1.52 Lie theorems

Let G be an n-dimensional Lie group of transformations on the m-dimensional manifold M.

Theorem

Let $x \in M$. In an infinitesimal transformation of the G-parameters, one has $dx^i = \sum_j u^i_j(x)\, da^j$ (→ 1.50). Then the functions u^i_j are analytic.

Theorem

If X_μ $(\mu = 1, \dots, n)$ are the generators of an n-dimensional Lie group G, then the coefficients ${C_{\mu\nu}}^\lambda$ given by

$$\left[X_\mu, X_\nu\right] = X_\mu X_\nu - X_\nu X_\mu = {C_{\mu\nu}}^\lambda X_\lambda$$

are constants and called the *structure constants* of G (note that they do not depend on the representation chosen for G: they are intrinsic quantities of G once given a basis of generators of G).

Theorem

The structure constants satisfy:

$$\begin{aligned} &{C_{\mu\nu}}^\lambda = -{C_{\nu\mu}}^\lambda \\ &{C_{\lambda\mu}}^\sigma {C_{\sigma\nu}}^\alpha + {C_{\mu\nu}}^\sigma {C_{\sigma\lambda}}^\alpha + {C_{\nu\lambda}}^\sigma {C_{\sigma\mu}}^\alpha = 0 \end{aligned}$$

The first property comes from the antisymmetry of the Lie bracket and the last one can be easily deduced from the Jacobi identity

$$\left[X_\lambda, \left[X_\mu, X_\nu\right]\right] + \left[X_\mu, \left[X_\nu, X_\lambda\right]\right] + \left[X_\nu, \left[X_\lambda\, X_\mu\right]\right] = 0$$

for any triplet X_λ, X_μ, X_ν of generators.

Property

The infinitesimal generators of an n-dimensional Lie group generate an n-dimensional Lie algebra (→ 1.46).

Example

Consider the Lie algebra $su(3)$ of $SU(3)$ and the basis constituted by the matrices λ_i $(i = 1, \ldots, 8)$ $(\rightarrow 1.90)$

$$\Big[\lambda_i, \lambda_j\Big] = \lambda_i\lambda_j - \lambda_i\lambda_j = 2i\, f_{ij}{}^k\, \lambda_k$$

In this basis, the structure constants $f_{ij}{}^k$ are not only antisymmetric in the indices i, j $(f_{ij}{}^k = -f_{ji}{}^k)$ but completely antisymmetric in the three indices i, j, k, that is

$$f_{ij}{}^k \equiv f_{ijk} = -f_{jik} = -f_{kji} = -f_{ikj} = f_{jki} = f_{kij}$$

that last property being obviously not the case in any basis of $su(3)$.

Converse of the Lie theorems

We have just seen that to each Lie group can be associated one and only one Lie algebra. Is the converse true? The answer is *no*. More precisely, in general it is not the case that only one Lie group corresponds to a Lie algebra. Let us state the "converse of the Lie theorem":

Theorem

Let $\mathcal{G}$ be an abstract Lie algebra of dimension n over the field $\mathbb{R}$ of real numbers. Then there is a simply connected Lie group G of dimension n $(\rightarrow 1.18)$ whose Lie algebra is isomorphic to $\mathcal{G}$. The group G is uniquely determined by $\mathcal{G}$ up to a local analytic isomorphism. G is called the *universal covering group.*

Let us emphasize that there is not a one-to-one correspondence between Lie algebras and Lie groups, since several Lie groups may have the same Lie algebra.

The above theorem leads to the following questions: How to construct a Lie group from a Lie algebra? How to get all Lie groups with the same Lie algebra? The answer to the last question has been given in $\rightarrow$ 1.18 (Connexity). The answer to the first question is given by the Taylor's theorem:

Theorem (Taylor)

The Lie group operation corresponding to the Lie algebra element $\alpha^\mu X_\mu$ is

$$\alpha^\mu X_\mu \quad \mapsto \quad \exp(\alpha^\mu X_\mu)$$

Example

Let us consider in $\mathbb{R}^3$ a rotation of angle θ around the Oz axis:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\downarrow \quad (\theta \to 0)$$

$$\begin{pmatrix} 1 - \frac{1}{2}\theta^2 + \dots & \theta + \dots & 0 \\ -\theta + \dots & 1 - \frac{1}{2}\theta^2 + \dots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the lower matrix is obtained from the upper matrix by a Taylor expansion. We realize that $R_{Oz}(\theta)$ can be rewritten as an exponential of the generator X_3 already introduced in the previous paragraph, that is

$$R_{Oz}(\theta) = \exp(\theta X_3) \quad \longrightarrow \quad \mathbb{I} + \theta X_3 + \dots$$

□

Taylor's theorem does not insure that we can obtain any Lie group element by taking the exponential of some element in the Lie algebra. Considering for example the Euclidean group in three dimensions ($\rightarrow$ 1.29). It appears that the most general element can only be written as the product of two exponentials $\exp(\beta^i P_i)\ \exp(\alpha^i J_i)$ if P_i and J_i are respectively the translation and the rotation generators ($i = 1, 2, 3$), but not as a single exponential of a Lie algebra element. However, we have the following property:

Property

Every element of a compact Lie group G lies on a one-dimensional Abelian subgroup of G and can be obtained by exponentiating some element in the Lie algebra.

1.53 Lorentz group

The Lorentz invariance of physical laws implies that physical equations have to be covariant under transformations of the Lorentz group.

The Lorentz group L or $O(3,1)$ is the set of linear transformations which leave invariant the scalar product of two four-vectors in the Minkowski space ($\mu, \nu = 0, 1, 2, 3$):

$$xy = g^{\mu\nu} x_\mu x_\nu \qquad \text{with} \qquad g^{\mu\nu} = 2\delta^{\mu 0}\delta^{\nu 0} - \delta^{\mu\nu}$$

Its elements are defined by $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ where the 4×4 matrices Λ satisfy

$$g_{\mu\nu}\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu = g_{\alpha\beta} \qquad \text{or} \qquad \Lambda^t g\Lambda = g \text{ with } \det\Lambda = \pm 1$$

The Lorentz group is a 6-parameter group. Its infinitesimal generators $M_{\mu\nu} = -M_{\nu\mu}$ satisfy the commutation relations (Lorentz algebra):

$$\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma})$$

In the 4×4 matrix representation they can be written as $(i, j = 1, 2, 3)$

$$\begin{aligned}(M_{ij})_{\alpha\beta} &= -i(\delta_{i\alpha}\,\delta_{j\beta} - \delta_{i\beta}\,\delta_{j\alpha}) \\ (M_{0i})_{\alpha\beta} &= i(\delta_{0\alpha}\,\delta_{i\beta} + \delta_{0\beta}\,\delta_{i\alpha})\end{aligned}$$

where the row and column indices α and β run from 0 to 3.

If we define

$$J_i = \tfrac{1}{2}\,\varepsilon_{ijk}M_{jk} \qquad \text{and} \qquad K_i = M_{0i}$$

we have

$$\Big[J_i, J_j\Big] = i\,\varepsilon_{ijk}\,J_k\,, \qquad \Big[J_i, K_j\Big] = i\,\varepsilon_{ijk}\,K_k\,, \qquad \Big[K_i, K_j\Big] = -i\,\varepsilon_{ijk}\,J_k$$

The Casimir operators are given by

$$\begin{aligned}C_1 &= \tfrac{1}{2}\,M^{\mu\nu}M_{\mu\nu} = \vec{J}^{\,2} - \vec{K}^{\,2} \\ C_2 &= \tfrac{1}{4}\,\varepsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma} = 2\vec{J}\cdot\vec{K}\end{aligned}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the complete antisymmetric tensor such that $\varepsilon_{0123} = +1$.

The finite (not unitary) irreducible representations of the proper Lorentz group $SO(3,1)$ ($\det\Lambda = 1$), or more correctly of its covering group $SL(2,\mathbb{C})$, are labelled by two non-negative integers (tensor representations) or half-integers (spinor representations) (j_1, j_2) and have dimension $(2j_1{+}1)(2j_2{+}1)$. The unitary irreducible representations, infinite dimensional, are labelled by two numbers (c, M) where *i*) c is an arbitrary imaginary number and M an arbitrary non-negative integer or half-integer (principal series) or *ii*) c is a real number so that $0 < c < 1$ and $M = 0$ (complementary series).

For references, see ref. [66].

1.54 Module of a Lie algebra

→ 1.73 Representation of a Lie algebra.

1.55 Normalizer (of a group, of an algebra)

Definition

Let G be a group, S a subset of elements in G. The normalizer $\mathcal{N}_G(S)$ is the subset of G given by

$$\mathcal{N}_G(S) = \Big\{ g \in G \,\Big|\, g * s * g^{-1} \in S,\ \forall\, s \in S \Big\}$$

It is a subgroup of G.

Definition

Let $\mathcal{G}$ be a Lie algebra, $\mathcal{S}$ a subset of elements in $\mathcal{G}$. The normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{S})$ is the subset of $\mathcal{G}$ given by

$$\mathcal{C}_{\mathcal{G}}(\mathcal{S}) = \Big\{ X \in \mathcal{G} \,\Big|\, \Big[X, Y\Big] \in \mathcal{S},\ \forall\, Y \in \mathcal{S} \Big\}$$

It is a subalgebra of $\mathcal{G}$.

Notice that if $\mathcal{S}$ is an ideal of $\mathcal{G}$, then $\mathcal{N}_{\mathcal{G}}(\mathcal{S}) = \mathcal{G}$.

1.56 Octonions – Quaternions

Definition

Let $\mathcal{A}$ be an unital algebra (that is with an identity element).
$\mathcal{A}$ is a *composition algebra* if it has a norm N such that

$$N(x)N(y) = N(x * y)\,, \qquad \forall\, x, y \in \mathcal{A}$$

$\mathcal{A}$ is a *division algebra* if

$$x * y = 0 \;\Rightarrow\; x = 0 \;\text{ or }\; y = 0\,, \qquad \forall\, x, y \in \mathcal{A}$$

According to Hurwitz theorem, there are only four different algebras which are division and composition algebras: these are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

In the same way any complex number is written $z = a + ib$ with a, b real numbers and i satisfies $i^2 = -1, \bar{\imath} = -i$, a quaternion q is obtained by defining a secondary imaginary unit j such that

$$j^2 = -1\,, \quad \bar{\jmath} = -j\,, \quad ij + ji = 0\,, \quad \bar{\imath}\bar{\jmath} = \bar{\jmath}\bar{\imath} = -ij$$

It follows that a quaternion can be written in terms of the real numbers a, b, c, d as

$$q = a + ib + jc + ijd = (a + jc) + i(b + jd)$$

The norm for complex numbers is defined as $N^2(z) = z\overline{z} = a^2 + b^2$. The norm for quaternions will be defined as

$$N^2(q) = q\overline{q} = a^2 + b^2 + c^2 + d^2$$

The algebra $\mathbb{H}$ is not a commutative algebra, but it is still an associative one. Setting $e_1 \equiv i$, $e_2 \equiv j$ and $e_3 \equiv ij$, the multiplication table for quaternionic units can be summarized as $(\alpha, \beta, \gamma = 1, 2, 3)$:

$$e_\alpha * e_\beta = -\delta_{\alpha\beta} + \varepsilon_{\alpha\beta\gamma}\, e_\gamma$$

We note the isomorphism between these units and the Pauli matrices via the identification $e_\alpha = i\sigma_\alpha$, $\alpha = 1, 2, 3$.

The octonion algebra $\mathbb{O}$ can be defined by adding to the quaternion algebra another imaginary unit k such that

$$k^2 = -1\,, \quad ik + ki = jk + kj = 0\,, \quad (ij)k + (ji)k = i(jk) + j(ik)$$

We obtain now seven octonionic units and any element of $\mathbb{O}$ will be written as

$$\omega = a_0 + \sum_{\alpha=1}^{7} a_\alpha e_\alpha$$

where $a_0, \dots, a_7$ are real numbers and

$$e_1 \equiv j\,, \quad e_2 \equiv k\,, \quad e_3 \equiv jk\,, \quad e_4 \equiv ij\,, \quad e_5 \equiv ik\,, \quad e_6 \equiv i(jk)\,, \quad e_7 \equiv i\,,$$

the e_α satisfying the multiplication law

$$e_\alpha * e_\beta = \delta_{\alpha\beta} + \mathrm{sign}(f_{\alpha\beta\gamma})\, e_\gamma$$

$f_{\alpha\beta\gamma}$ being the $su(3)$ structure constants in the so-called Gell-Mann basis ($\rightarrow$ 1.90).

Notice that with the octonion algebra $\mathbb{O}$, one loses also the associativity. The norm in $\mathbb{O}$ is defined by

$$N^2(\omega) = \omega\overline{\omega} = a_0^2 + \sum_{\alpha=1}^{7} a_\alpha^2$$

The norm is left invariant under an $SO(8)$ transformation. A subgroup of $SO(8)$ is the automorphism group of the octonions algebra (that is the group of transformations leaving the above multiplication table invariant): it is the exceptional group G_2.

For more details on the structure of octonions and the embeddings $G_2 \subset SO(7)$ and $SO(7) \subset SO(8)$, see ref. [35].

1.57 Orbit and stratum

Let G be a group and π a representation of G with representation space $\mathcal{V}$.

Definition

The *orbit* of $m \in \mathcal{V}$ is the set of elements $m' \in \mathcal{V}$ one can reach by action of G on m, that is

$$G(m) = \Big\{ m' = g(m) \,\Big|\, g \in G \Big\}$$

denoting $g(m) = \pi(g(m))$.

Definition

The *little group* or stabilizer of $m \in \mathcal{V}$ is the set

$$G_m = \Big\{ g \in G \,\Big|\, g(m) = m \Big\}$$

G_m is a subgroup of G.

Property

If m and m' are on the same orbit, their little groups are conjugate. More precisely, if $m' = gm$ then $G_{m'} = gG_m g^{-1}$.

Now, two points m and m' need not be on the same orbit to have conjugate little groups; by definition they are on the same stratum.

Definition

The stratum $S(m)$ is the union of all orbits such that the little groups of their points are all conjugate.

Property

The orbits form a partition of $\mathcal{V}$. The strata also form a partition of $\mathcal{V}$ (that is the union of the orbits (resp. strata) is $\mathcal{V}$ and the intersection of two different orbits (resp. strata) is zero).

Therefore the decomposition of $\mathcal{V}$ into orbits and strata is equivalent to the classification of little groups for the considered representation. When G is compact, the number of little groups is finite and several nice properties can be proved (see refs. [30, 61, 62]).

1.58 Orthogonal groups and algebras

The orthogonal group in the n-dimensional real space $\mathbb{R}^n$ is the group of $n \times n$ real matrices O leaving invariant the scalar product

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = O\vec{x} \cdot O\vec{y}\,, \qquad \forall\, \vec{x}, \vec{y} \in \mathbb{R}^n$$

This group is denoted $O(n)$. One can also define it as the group of $n \times n$ real orthogonal matrices: $O^t = O^{-1}$. Note that any matrix $O \in O(n)$ satisfies $\det O = \pm 1$. The set of $n \times n$ orthogonal matrices such that $\det O = 1$ forms a subgroup of $O(n)$ denoted $SO(n)$ or special orthogonal group in n dimensions.

The group $O(n)$ with $n \geq 2$ is compact and its Lie algebra $so(n)$ is simple for $n \geq 2, n \neq 4$. Any element $O \in O(n)$ can be written

$$O = e^M \qquad \text{with } M \text{ antisymmetric: } M^t = -M$$

There exist $n(n-1)/2$ independent $n \times n$ antisymmetric matrices. Therefore the Lie algebra of $O(n)$ has $\frac{1}{2}n(n-1)$ generators. Note that the Lie algebra of $SO(n)$ has the same number of generators as that of $O(n)$ (contrary to the case of the $U(n)$ and $SU(n)$ groups).

As a basis of $so(n)$, one can choose the set of matrices M_{ij} with all entries equal to 0 zero except the i-th row and j-th column equal to 1 and the j-th row and i-th column equal to -1 $(i,j = 1, \dots, n)$. The matrices M_{ij} are antisymmetric, $M_{ij} = -M_{ji}$, and the commutation relations are

$$\Big[M_{ij}, M_{kl}\Big] = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik}$$

In the particular case of $so(3)$, we have

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad M_{31} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The algebra $so(4)$ is not simple, but is the direct sum of two $so(3)$ Lie algebras. One can easily convince oneself of this property by building from the six generators M_{ij} $(i,j = 1,2,3,4)$ the two subsets $\{M_{12} + M_{34}, M_{23} + M_{14}, M_{31} + M_{24}\}$ and $\{M_{12} - M_{34}, M_{23} - M_{14}, M_{31} - M_{24}\}$, each of them constituting a basis of $so(3)$ and any generator of the first set commuting with the three generators of the other. As Lie groups, we have

$$SO(4) \simeq \frac{SU(2) \times SU(2)}{Z_2}$$

Note also that $SO(3) \simeq SU(2)/Z_2$ and $SO(6) \simeq SU(4)/Z_2$ ($\to$ 1.18): $SU(2)$ is the covering group of $SO(3)$ and $SU(4)$ is the covering group of $SO(6)$).

Let us also note that if M is antisymmetric real, then $\pm iM$ is hermitian. Therefore, we can write any element $O \in O(n)$ as

$$O = e^{iN} \qquad \text{with } N \text{ hermitian: } N^\dagger = N$$

Considering again the $so(3)$ Lie algebra, we can choose as a basis the three hermitian matrices

$$J_1 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \sqrt{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which satisfy $(i,j,k = 1,2,3)$

$$\Big[J_i, J_j\Big] = i\,\varepsilon_{ijk}\,J_k$$

An irreducible representation of $SO(n)$ with $n = 2p$ or $n = 2p+1$ is either characterized by the Dynkin labels $(a_1, \dots, a_p)$ where a_i are positive or null integers, or by a set of p integers or p half-integers $(m_1, \dots, m_p)$ such that

$$\begin{aligned} &\text{for } n = 2p, && m_1 \geq m_2 \geq \ldots \geq m_{p-1} \geq |m_p| \\ &\text{for } n = 2p+1, && m_1 \geq m_2 \geq \ldots \geq m_{p-1} \geq m_p \geq 0 \end{aligned}$$

In the case of a representation of $SO(2p)$ the last index can be negative. Actually the representations $(m_1, \dots, m_{p-1}, m_p)$ and $(m_1, \dots, m_{p-1}, -m_p)$ are conjugate representations.

The $SO(n)$ representations $(m_1, \dots, m_p)$ with $m_1, \dots, m_p$ integers (resp. half-integers) are called *vector* (resp. *spinor*) representations; in $SO(3)$ for example, the spinor representations are the representations D_j with $j = 1/2, 3/2, \ldots$

The correspondence between the two labellings is the following:

$$\text{for } n = 2p \qquad \begin{aligned} &a_i = m_i - m_{i+1} \quad (1 \leq i \leq p-1) \\ &a_p = m_{p-1} + m_p \end{aligned}$$

$$\text{for } n = 2p+1 \qquad \begin{aligned} &a_i = m_i - m_{i+1} \quad (1 \leq i \leq p-1) \\ &a_p = 2m_p \end{aligned}$$

The dimension of the $SO(n)$ irreducible representation $\pi(m_1,\dots,m_p)$ is given by

$$\text{for } n=2p, \qquad N(m_1,\dots,m_p) = \frac{\displaystyle\prod_{1\le i<j\le p} (l_i-l_j)(l_i+l_j)}{\displaystyle\prod_{1\le i<j\le p} (\tau_i-\tau_j)(\tau_i+\tau_j)}$$

$$\text{for } n=2p+1, \qquad N(m_1,\dots,m_p) = \frac{\displaystyle\prod_{1\le i\le p} l_i \prod_{1\le i<j\le p} (l_i-l_j)(l_i+l_j)}{\displaystyle\prod_{1\le i\le p} \tau_i \prod_{1\le i<j\le p} (\tau_i-\tau_j)(\tau_i+\tau_j)}$$

where $l_i = m_i + \tau_i$ with $\tau_i = p-i$ in the case $n=2p$ and $\tau_i = p-i+\frac{1}{2}$ in the case $n=2p+1$, that is:

for $SO(2p)$

$$N = \prod_{1\le i<j\le p} \frac{m_i-m_j+j-i}{j-i} \prod_{1\le i<j\le p} \frac{m_i+m_j+2p-i-j}{2p-i-j}$$

and for $SO(2p+1)$

$$N = \prod_{1\le i\le p} \frac{2m_i+2p-2i+1}{2p-2i+1} \prod_{1\le i<j\le p} \frac{m_i-m_j+j-i}{j-i} \times \prod_{1\le i<j\le p} \frac{m_i+m_j+2p-i-j+1}{2p-i-j+1}$$

1.59 Oscillator realizations: classical Lie algebras

Let us consider a set of $2N$ bosonic oscillators b_i^- and b_i^+ with commutation relations:

$$\left[b_i^-, b_j^-\right] = \left[b_i^+, b_j^+\right] = 0 \quad \text{and} \quad \left[b_i^-, b_j^+\right] = \delta_{ij}$$

and a set of $2N$ fermionic oscillators a_i^- and a_i^+ with anticommutation relations:

$$\left\{a_i^-, a_j^-\right\} = \left\{a_i^+, a_j^+\right\} = 0 \quad \text{and} \quad \left\{a_i^-, a_j^+\right\} = \delta_{ij}$$

the two sets commuting each other:

$$\left[b_i^-, a_j^-\right] = \left[b_i^-, a_j^+\right] = \left[b_i^+, a_j^-\right] = \left[b_i^+, a_j^+\right] = 0$$

Oscillator realization of A_{N-1}

Let $\Delta = \{ \varepsilon_i - \varepsilon_j \}$ be the root system of A_{N-1} expressed in terms of the orthonormal vectors $\varepsilon_1, \dots, \varepsilon_N$. The Lie algebra A_{N-1} admits two different natural oscillator realizations, a bosonic one and a fermionic one. A bosonic oscillator realization of the generators of A_{N-1} in the Cartan–Weyl basis is given by

$$\begin{aligned} &H_i = b_i^+ b_i^- - b_{i+1}^+ b_{i+1}^- \quad (1 \le i \le N-1) \\ &E_{\varepsilon_i - \varepsilon_j} = b_i^+ b_j^- \end{aligned}$$

and a fermionic oscillator realization by

$$\begin{aligned} &H_i = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- \quad (1 \le i \le N-1) \\ &E_{\varepsilon_i - \varepsilon_j} = a_i^+ a_j^- \end{aligned}$$

Oscillator realization of B_N

Let $\Delta = \{ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \}$ be the root system of B_N expressed in terms of the orthonormal vectors $\varepsilon_1, \dots, \varepsilon_N$. An oscillator realization of the generators of B_N in the Cartan–Weyl basis is given by

$$\begin{aligned} &H_i = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- \quad (1 \le i \le N-1)\,, \qquad H_N = 2a_N^+ a_N^- - 1 \\ &E_{\pm\varepsilon_i \pm \varepsilon_j} = a_i^\pm a_j^\pm \\ &E_{\pm\varepsilon_i} = \pm(-1)^N a_i^\pm \end{aligned}$$

where $N = \sum_{k=1}^m a_k^+ a_k^-$.

Oscillator realization of C_N

Let $\Delta = \{ \pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i \}$ be the root system of C_N expressed in terms of the orthonormal vectors $\varepsilon_1, \dots, \varepsilon_N$. An oscillator realization of the generators of C_N in the Cartan–Weyl basis is given by

$$\begin{aligned} &H_i = b_i^+ b_i^- - b_{i+1}^+ b_{i+1}^- \quad (1 \le i \le N-1)\,, \qquad H_N = 2b_N^+ b_N^- + 1 \\ &E_{\pm\varepsilon_i \pm \varepsilon_j} = b_i^\pm b_j^\pm \\ &E_{\pm 2\varepsilon_i} = (b_i^\pm)^2 \end{aligned}$$

Oscillator realization of D_N

Let $\Delta = \{ \pm\varepsilon_i \pm \varepsilon_j \}$ be the root system of D_N expressed in terms of the orthonormal vectors $\varepsilon_1, \dots, \varepsilon_N$. An oscillator realization of the generators of D_N in the Cartan–Weyl basis is given by

$$
\begin{aligned}
&H_i = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- \quad (1 \le i \le N-1) \\
&H_N = a_{N-1}^+ a_{N-1}^- + a_N^+ a_N^- - 1 \\
&E_{\pm\varepsilon_i \pm\varepsilon_j} = a_i^\pm a_j^\pm
\end{aligned}
$$

1.60 Oscillator realizations: exceptional Lie algebras

Oscillator realization of E_8

The realization of E_8 in terms of fermionic oscillators [82] is based on the following two key points:

1. The embedding
$$E_8 \supset so(8) \oplus so'(8)$$
with the corresponding decomposition of the adjoint of E_8
$$248 = (28, 1') \oplus (1, 28') \oplus (8_v, 8'_v) \oplus (8_s, 8'_s) \oplus (8_c, 8'_c)$$
where we have denoted with 28, 8_v, 8_s and 8_c, respectively the adjoint, the vector, the spinor and the complex conjugate spinor representations of $so(8)$ and with a prime the corresponding representations of $so'(8)$.
2. The definition of fermionic operators which transform respectively as 8_v, 8_s and 8_c and the triality property.

Let us consider

1. A set of 8 fermionic oscillators a_i^+ and a_i transforming as 8_v $(i = 1, 2, 3, 4)$ with standard anticommutation relations.
2. A set of 16 fermionic oscillators labelled by a quadruple of numbers $l_1,\ l_2,\ l_3,\ l_4$, $(l_i \in \{-1/2, +1/2\},\ i = 1, 2, 3, 4)$. We introduce an hermitian conjugation (h.c.), denoted by "+", which changes the signs of l_i. We divide this set in two 8-dimensional subsets, denoted by α and β, transforming respectively as 8_s and 8_c. We denote each element by lower labels ranging between 1 and 4 , written in increasing order and identifying the labels $l_i = +1/2$. It is convenient to introduce the notation (no sum over repeated labels)

$$
\begin{aligned}
&\alpha_{1234} = (\alpha_{1234}^+)^+ && \beta_i = \varepsilon_{ijkl}\, \beta_{jkl}^+ \\
&\alpha_{ij} = \epsilon_{ijkl}\, \alpha_{kl}^+ && \beta_{ijk}^+ = \eta\, \beta_{P(ijk)}^+ \\
&\alpha_{ij}^+ = -\alpha_{ji}^+ \ \ (i < j) && (\eta \equiv \text{parity of permutation P})
\end{aligned}
$$

The sets satisfy the following anticommutation relations:

$$\left\{\alpha,\alpha\right\} = \left\{\alpha^+,\alpha^+\right\} = \left\{\beta,\beta\right\} = \left\{\beta^+,\beta^+\right\} = 0$$

and

$$\left\{\alpha_{1234},\alpha^+_{1234}\right\} = 1 \qquad \left\{\alpha_{ij},\alpha^+_{kl}\right\} = \delta_{ik}\delta_{jl} \qquad \left\{\beta_i,\beta^+_j\right\} = \delta_{ij}$$

Due to the triality property we can realize the algebra $so(8)$ by using one of the three sets introduced above. So we identify

$$\begin{aligned} a_i^+\, a_j^+ &= \alpha^+_{1234}\,\alpha^+_{ij} = \beta^+_{ijk}\,\beta^+_{ijl} \qquad (k<l) \\ a_i^+\, a_j^- &= \alpha^+_{ik}\,\alpha^-_{jl} = \beta^+_i\,\beta^-_j \qquad (i<j, k<l) \end{aligned}$$

and (with $h_i = a_i^+\, a_i$)

$$\alpha^+_{1234}\,\alpha_{1234} \equiv \tfrac{1}{2}\sum_{i=1}^{4} h_i\,, \quad \alpha^+_{ij}\,\alpha_{ij} \equiv \tfrac{1}{2}\Big(h_i + h_j - \sum_{k\neq i,j} h_k\Big)\,, \quad \beta^+_i\,\beta_i \equiv h_i - \sum_{l\neq i} h_l$$

From consistency relations we require the three sets to satisfy the following commutation relations:

$$\begin{array}{lll}
\left[a_i,\alpha^+_{1234}\right] = \beta_j & \left[a_j,\beta^+_{jkl}\right] = \alpha^+_{kl} & \left[a_i,\alpha^+_{jk}\right] = \delta_{ij}\,\beta^+_k - \delta_{jk}\,\beta^+_i \\
\left[a_i^+,\alpha^+_{1234}\right] = 0 & \left[a_i^+,\beta^+_{jkl}\right] = \alpha_{ijkl} & \left[a_i^+,\alpha^+_{jk}\right] = \beta^+_{ijk} \\
\left[\beta_j,\alpha_{1234}\right] = a_j & \left[\beta^+_i,\alpha_{1234}\right] = 0 & \left[\beta_i,\alpha^+_{jk}\right] = \delta_{ij}\,a^+_k - \delta_{ik}\,a^+_j \\
\left[a^+_i,\beta^+_j\right] = \alpha^+_{ij} & \left[\beta^+_i,\alpha^+_{jk}\right] = \varepsilon_{ijkl}\,a_l & \left[a^+_i,\beta_j\right] = \delta_{ij}\,\alpha^+_{1234}
\end{array}$$

and the ones obtained by hermitian conjugation.

In the following we denote the sets of fermionic operators relative to $so'(8)$ by a four-units shift in the labels.

We can now write the explicit realization of E_8 in terms of bilinears in the previous fermionic operators:

$$\begin{array}{ll}
(28,1') & a_i^+\,a_j^+\,,\quad a_i\,a_j\,,\quad a_i^+\,a_j\,,\quad a_j^+\,a_i\,,\quad h_i = a_i^+\,a_i\,, \\
(1,28') & a_m^+\,a_n^+\,,\quad a_m\,a_n\,,\quad a_m^+\,a_n\,,\quad a_n^+\,a_m\,,\quad h_m = a_m^+\,a_m\,, \\
(8_v,8'_v) & a_i^+\,a_m^+\,,\quad a_i\,a_m\,,\quad a_i^+\,a_m\,,\quad a_i\,a_m^+\,, \\
(8_s,8'_s) & \alpha^+_{1234}\,\alpha^+_{mn}\,,\quad \alpha^+_{ij}\,\alpha_{5678}\,,\quad \alpha^+_{1234}\,\alpha^+_{5678}\,,\quad \alpha^+_{1234}\,\alpha_{5678}\,,\quad \alpha^+_{ij}\,\alpha^+_{mn}\,, \\
 & \alpha_{1234}\,\alpha_{mn}\,,\quad \alpha_{ij}\,\alpha^+_{5678}\,,\quad \alpha_{1234}\,\alpha_{5678}\,,\quad \alpha_{1234}\,\alpha^+_{5678}\,, \\
(8_c,8'_c) & \beta^+_{ijk}\,\beta^+_{mnp}\,,\quad \beta^+_{ijk}\,\beta^+_m\,,\quad \beta^+_i\,\beta^+_{mnp}\,,\quad \beta^+_i\,\beta^+_m
\end{array}$$

where $1 \leq i < j < k \leq 4$ and $5 \leq m < n < p \leq 8$.

Each line corresponds to a term in the decomposition of the adjoint of E_8 with respect to $so(8) \oplus so(8)$. The simple positive generators are, with reference to the labelling of Table 3.7,

$$e_1^+ = \beta_1^+ \beta_8^+ , \quad e_r^+ = a_r^+ a_{r-1} \ (r = 2, .., 7) , \quad e_8^+ = a_1^+ a_2^+ ,$$

the simple negative ones e_i^- are obtained by hermitian conjugation and the corresponding Cartan generators are

$$H_1 = \tfrac{1}{2}\Big(h_1 + h_8 - \sum_{k=2}^{7} h_k\Big) , \quad H_8 = h_1 + h_2 - 1 , \quad H_r = h_r - h_{r-1}$$

The negative generators are obtained from the positive ones by hermitian conjugation. Let us remark that the generators bilinear in the set $\{a\}$ correspond to the roots of the form $\pm\varepsilon_i \pm \varepsilon_j$ (i.e. the roots of $so(16) \subset E_8$), while the generators bilinear in the sets $\{\alpha\}$ and $\{\beta\}$ correspond to the remaining roots (i.e. to the 128-dimensional spinor representation of $so(16)$).

Oscillator realization of E_7

From the embedding $E_8 \supset E_7 \oplus su(2)$ with the corresponding decomposition of the adjoint representation of E_8, $248 = (133, 1) \oplus (1, 3) \oplus (56, 2)$, we obtain

$$\begin{array}{l}
a_i^+ a_j^+ , \quad a_i a_j , \quad a_i^+ a_j , \quad a_j^+ a_i , \quad h_i , \\
a_5^+ a_6^+ , \quad a_5 a_6 , \quad a_5^+ a_6 , \quad a_6^+ a_5 , \quad a_7^+ a_8 , \quad a_8^+ a_7 , \quad h_7 - h_8 , \\
a_i^+ a_m^+ , \quad a_i a_m , \quad a_i^+ a_m , \quad a_m^+ a_i , \quad h_m , \\
\alpha_{1234}^+ \alpha_{mp}^+ , \quad \alpha_{1234} \alpha_{mp}^+ , \quad \alpha_{ij}^+ \alpha_{mp}^+ \\
\beta_{ijk}^+ \beta_{56p}^+ , \quad \beta_{ijk}^+ \beta_p^+ , \quad \beta_i^+ \beta_{56p}^+ , \quad \beta_i^+ \beta_p^+
\end{array}$$

where $1 \leq i < j < k \leq 4$, $m = 5, 6$ and $p = 7, 8$.

The simple generators are the same as E_8 omitting the value $r = 7$.

Oscillator realization of E_6

From the embedding $E_7 \supset E_6 \oplus U(1)$ with the corresponding decomposition of the adjoint representation of E_7, $133 = (78, 0) \oplus (1, 0) \oplus (27, -1) \oplus (\overline{27}, 1)$, we obtain (where $1 \leq i < j < k \leq 4$)

$$\begin{array}{l}
a_i^+ a_j^+ , \quad a_i a_j , \quad a_i^+ a_j , \quad a_j^+ a_i , \quad h_i , \\
a_i^+ a_5^+ , \quad a_i a_5 , \quad a_i^+ a_5 , \quad a_5^+ a_i , \quad h_5 , 2h_6 + h_7 - h_8 , \\
\alpha_{1234}^+ \alpha_{67}^+ , \quad \alpha_{1234} \alpha_{67}^+ , \quad \alpha_{ij}^+ \alpha_{67}^+ , \quad \alpha_{1234}^+ \alpha_{58}^+ , \quad \alpha_{1234} \alpha_{58}^+ , \quad \alpha_{ij}^+ \alpha_{58}^+ \\
\beta_{ijk}^+ \beta_{567}^+ , \quad \beta_{ijk}^+ \beta_8^+ , \quad \beta_i^+ \beta_{567}^+ , \quad \beta_i^+ \beta_8^+
\end{array}$$

The simple generators are the same as E_8 omitting the values $r = 6$ and $r = 7$.

Oscillator realization of F_4

From the embedding $E_6 \supset F_4$ with the corresponding decomposition of the adjoint representation of E_6, $78 = 52 \oplus 26$, we obtain, for the simple positive generators

$$e_1^+ = a_1^+ a_2^+ , \quad e_2^+ = a_3^+ a_2 , \quad e_3^+ = a_2^+ a_1 + a_4^+ a_3 , \quad e_4^+ = \beta_1^+ \beta_8^+ + a_5^+ a_4$$

and for the Cartan generators

$$H_1 = h_1 + h_2 , \quad H_2 = h_3 - h_2 , \quad H_3 = h_2 + h_4 - h_1 - h_3$$
$$H_4 = \tfrac{1}{2}\Big(h_1 + h_8 + 2h_5 - 2h_4 - \sum_{k=2}^{7} h_k\Big)$$

the remaining twenty positive generators are given by:

$$\begin{array}{l}
a_3^+ a_4^+ , \ a_4^+ a_1 , \ a_3 a_5^+ - \beta_2^+ \beta_8^+ , \ a_1 a_5^+ - \beta_4^+ \beta_8^+ , \ a_2 a_5^+ + \beta_3^+ \beta_8^+ \\
a_1^+ a_3^+ , \ a_2^+ a_4^+ , \ a_1^+ a_5^+ - \beta_{123}^+ \beta_8^+ , \ a_2^+ a_5^+ - \beta_{124}^+ \beta_8^+ , \ a_3^+ a_5^+ + \beta_{134}^+ \beta_8^+ \\
\alpha_{12}^+ \alpha_{58}^+ , \ \alpha_{13}^+ \alpha_{58}^+ , \ \alpha_{1234} \alpha_{58}^+ , \ \alpha_{23}^+ \alpha_{58}^+ + \alpha_{14}^+ \alpha_{58}^+ , \ a_2^+ a_3^+ + a_3^+ a_4^+ \\
\alpha_{24}^+ \alpha_{58}^+ , \ \alpha_{34}^+ \alpha_{58}^+ , \ \alpha_{1234}^+ \alpha_{58}^+ , \ a_1^+ a_2 + \alpha_{13}^+ \alpha_{58}^+ , \ a_3^+ a_1 - a_4^+ a_2
\end{array}$$

Oscillator realization of G_2

From the embedding $so(8) \supset G_2$ with the corresponding decomposition of the adjoint representation of $so(8)$, $28 = 14 \oplus 7 \oplus 7$, we obtain for the simple generators

$$e_1 = a_2^+ a_3 , \qquad e_2 = a_1^+ a_2 + a_3^+ a_4 + a_3^+ a_4^+$$

and for the Cartan generators

$$H_1 = h_2 - h_3 , \qquad H_2 = h_1 - h_2 + 2h_3 - 1$$

The remaining four positive generators are

$$a_1^+ a_3 - a_2^+ a_4 - a_2^+ a_4^+ , \quad a_1^+ a_4 + a_1^+ a_4^+ + a_2^+ a_3^+ , \quad a_1^+ a_3^+ , \quad a_1^+ a_2^+$$

Another realization, using only six fermionic operators, can be obtained from the embedding $G_2 \supset su(3)$ with the corresponding decomposition of the adjoint representation of G_2, $14 = 8 \oplus 3 \oplus \overline{3}$. We obtain for the simple generators

$$e_1 = a_1^+ a_2 , \quad e_2 = a_2^+ + a_3^+ a_1$$

and for the Cartan generators

$$H_1 = h_1 - h_2 , \quad H_2 = 2h_2 - h_1 + h_3 - 1$$

1.61 Pauli matrices

→ 1.90 Unitary groups and algebras.

1.62 Poincaré group

The relativistic quantum theory has to satisfy Poincaré invariance, which implies that the Hilbert space of physical states is a representation space of the Poincaré group.

The Poincaré group P – or inhomogeneous Lorentz group – is the set of linear transformations which leave invariant the interval $(x-y)^2$ with $x = (x_0 = ct, x_1, x_2, x_3)$ in the Minkowski space. The general element of the Poincaré group will be denoted (a, Λ) with

$$x'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu$$

where Λ is an element of the Lorentz group L and a is a four-vector in the Minkowski space.

The Poincaré group appears as the semi-direct product of the four-dimensional translation group $T(4)$ in the Minkowski space by the Lorentz group L: $P = T(4) \ltimes L$. The composition law is

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda')$$

and the inverse of an element is

$$(a, \Lambda)^{-1} = (-\Lambda^{-1}a, \Lambda^{-1})$$

The Poincaré group is a 10-parameter group. Its Lie algebra, the Poincaré algebra, is generated by $M_{\mu\nu}$ (Lorentz generators) and P_μ (translation generators), which satisfy the following commutation relations:

$$\begin{aligned}
\left[M_{\mu\nu}, M_{\rho\sigma}\right] &= i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma}) \\
\left[M_{\mu\nu}, P_\rho\right] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \\
\left[P_\mu, P_\nu\right] &= 0
\end{aligned}$$

where the metric tensor $g^{\mu\nu}$ is given by $g^{\mu\nu} = 2\delta^{\mu 0}\delta^{\nu 0} - \delta^{\mu\nu}$.

An explicit realization of the Poincaré algebra in terms of differential operators is given by $(\mu, \nu = 0, 1, 2, 3)$:

for the translation generators $P_\mu = i\,\partial_\mu$

for the Lorentz generators $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} = g_{\mu\nu}\dfrac{\partial}{\partial x_\nu}$.

The Casimir operators are

$$\begin{aligned} &P^2 = P_\mu P^\mu \\ &W^2 = W_\mu W^\mu \qquad \text{with} \qquad W^\sigma = \tfrac{1}{2}\varepsilon^{\sigma\rho\mu\nu} P_\rho M_{\mu\nu} \end{aligned}$$

where

$$\begin{aligned} &\Big[W_\mu, P_\nu\Big] = 0 \qquad \text{and} \qquad W_\mu P^\mu = 0 \\ &\Big[W_\mu, W_\nu\Big] = i\,\varepsilon_{\mu\nu}{}^{\rho\sigma}\, W_\rho W_\sigma \\ &\Big[M_{\mu\nu}, W_\rho\Big] = i(g_{\nu\rho} W_\mu - g_{\mu\rho} W_\nu) \end{aligned}$$

The unitary irreducible representations of the proper Poincaré group (that is $\det\Lambda = 1$), or more correctly of its covering group $T(4) \ltimes SL(2,\mathbb{C})$, can be essentially classified into three classes:

1. $P^2 > 0$. The unitary irreducible representations are characterized by a real positive number m (the mass) and by an integer or half-integer non-negative number s (the spin). To each spin s there are $(2s+1)$ states. These representations describe physical particles of non-vanishing mass.

2. $P^2 < 0$ (representations of imaginary mass). The unitary irreducible representations are characterized by a continuous spectrum corresponding to the unitary irreducible representations of the three-dimensional Lorentz group $SU(1,1)$ (→ 1.13).

3. $P^2 = 0$ (and $P_\mu \neq 0$). The unitary irreducible representations are characterized by $m = 0$ and by:
   (a) if $W^2 = 0$, an arbitrary integer or half-integer number h called the helicity;
   (b) if $W^2 < 0$, a continuous spectrum corresponding to the unitary irreducible representations of the three-dimensional Euclidean group $E(2)$.

   These representations describe physical particles of zero mass.

For references, see ref. [97].

1.63 Products of G_2 representations

Rules for performing the Kronecker product of two irreducible representations of G_2 have been given by R.C. King in ref. [51]. Here we present a method, see ref. [85], which allows us to perform the product of any irreducible representation with a fundamental one. Despite its apparent limitation, the repeated application of this method allows us to perform any pair of G_2 irreducible representations, since any irreducible representation can be constructed by Kronecker products of the fundamental ones. The advantage of the following method is its simplicity; in fact the rules to perform the Kronecker product of two G_2 irreducible representations are brought back to the rules of the Kronecker product of two $SU(3)$ irreducible representations.

We specify a G_2 irreducible representation characterized by the Dynkin labels (a_1, a_2), where the label a_1 corresponds to the long root, by a two-row Young tableau $[\lambda] = [\lambda_1, \lambda_2]$ with

$$\lambda_1 = 3a_1 + 2a_2 \qquad \lambda_2 = a_2$$

Note that the most common definition of a G_2 Young tableau $[\mu] = [\mu_1, \mu_2]$ is different from ours:

$$\mu_1 = a_1 + a_2 \qquad \mu_2 = a_1$$

As the Dynkin labels a_1, a_2 are non-negative integers, it follows from the inverse relations

$$a_1 = \tfrac{1}{3}(\lambda_1 - 2\lambda_2) \qquad a_2 = \lambda_1$$

that the *meaningful* Young tableaux $[\lambda_1, \lambda_2]$ have to satisfy the following constraints:

$$\lambda_1 + \lambda_2 = 3n \quad \text{where } n \in \mathbb{N} \qquad \text{and} \qquad \lambda_1 \geq 2\lambda_2 \qquad\qquad (*)$$

The two fundamental G_2 irreducible representations $(a_1 = 1, a_2 = 0)$ and $(a_1 = 0, a_2 = 1)$ of respective dimensions 7 and 14 can therefore be represented by the $[\lambda]$ Young tableaux $[2, 1]$ and $[3, 0]$. Note that the 14-dimensional representation $[3, 0]$ is also the adjoint G_2 representation.

So, let us consider separately the product of any G_2 representation $[\lambda]$ by the representation $[2, 1]$ and by the representation $[3, 0]$.

<u>Case 1</u>: The rules specifying the irreducible representations which appear in the decomposition of the Kronecker product $[\lambda_1, \lambda_2] \otimes [2, 1]$ are specially simple:
– act as in the $SU(3)$ case (with the tableau $[1^3]$ corresponding to the trivial G_2 representation) and draw away any tableau not satisfying the constraints (*).
– subtract from the sum one tableau $[\lambda_1, \lambda_2]$.

<u>Case 2</u>: The rules specifying the irreducible representations which appear in the decomposition of the Kronecker product $[\lambda_1, \lambda_2] \otimes [3, 0]$ are the following:
– act as in the $SU(3)$ case (with the tableau $[1^3]$ corresponding to the trivial G_2 representation) and draw away any tableau not satisfying the constraints (*).
– if $\lambda_2 = 0$, take away the tableau $[\lambda_1 + 2, 1]$.
– if $\lambda_1 = 2\lambda_2$ and $\lambda_2 \geq 3$, add the tableau $[\lambda_1, \lambda_2 - 3]$.
– if $\lambda_1 > 2\lambda_2$, add the following tableaux if meaningful: $[\lambda_1, \lambda_2]$, $[\lambda_1 + 3, \lambda_2 + 3]$, $[\lambda_1 - 3, \lambda_2 + 3]$ and $[\lambda_1 - 3, \lambda_2]$.
– finally, for the special case $\lambda_1 = 2, \lambda_2 = 1$, we have to use the conditions given in case 1.

Let us illustrate the method on the following example. We consider the tensor product of the irreducible representation $[6, 0]$ (of dimension 77) with the representation $[4, 2]$ (of dimension 27). First we remark that $[4, 2]$ appears in the tensor product of the representation $[2, 1]$ by itself:

$$[2,1] \otimes [2,1] = [4,2] \oplus [3,0] \oplus [2,1] \oplus [0]$$

It follows that:

$$[6,0] \otimes [4,2] = ([6,0] \otimes [2,1]) \otimes [2,1] \ominus ([6,0] \otimes [3,0]) \ominus ([6,0] \otimes [2,1]) \ominus [6,0]$$

which necessitates decomposition, using the rules given above, of the following products:

$$\begin{array}{l}
[6,0] \otimes [2,1] = [8,1] \oplus [7,2] \oplus [5,1] \\
[8,1] \otimes [2,1] = [10,2] \oplus [9,3] \oplus [9,0] \oplus [8,1] \oplus [7,2] \oplus [6,0] \\
[7,2] \otimes [2,1] = [9,3] \oplus [8,4] \oplus [8,1] \oplus [7,2] \oplus [6,3] \oplus [6,0] \oplus [5,1] \\
[5,1] \oplus [2,1] = [7,2] \oplus [6,3] \oplus [6,0] \oplus [5,1] \oplus [4,2] \oplus [3,0] \\
[6,0] \otimes [3,0] = [9,0] \oplus [9,3] \oplus [7,2] \oplus [6,3] \oplus [6,0] \oplus [3,0]
\end{array}$$

Finally, we get

$$\begin{aligned}
[6,0] \otimes [4,2] &= [10,2] \oplus [9,3] \oplus [8,4] \oplus [8,1] \oplus [7,2] \\
&\quad \oplus [6,3] \oplus [6,0] \oplus [5,1] \oplus [4,2]
\end{aligned}$$

that is, in terms of the dimensions:

$$77 \otimes 27 = 729 \oplus 448 \oplus 182 \oplus 286 \oplus 189 \oplus 77 \oplus 77 \oplus 64 \oplus 27$$

A list of dimensions and explicit products of G_2 representations is also available in the table section (see Tables 3.20 and 3.22).

For more details, see ref. [85].

1.64 Products of $SO(2n)$ representations

We present here a simplified version of the method introduced in ref. [31] for performing the Kronecker product of two irreducible representations of the orthogonal group $SO(2n)$ $(n > 3)$ by means of product of generalized Young tableaux or GYT (→ 1.36). We identify an irreducible representation of $SO(2n)$ by n integer (half-integer) numbers m_i, respectively for vector (V) and spinor (S) irreducible representations, such that $m_1 \geq \ldots \geq |m_n| \geq 0$, so the last one m_n can also be negative. The correspondence between the set of non-negative integers $(a_1, \ldots, a_n)$ identifying an irreducible representation $D(a_1, \ldots, a_n)$ (Dynkin labels) and the n-uple $(m_1, \ldots, m_n)$ is

$$a_j = m_j - m_{j+1} \quad \text{for } j = 1, \ldots, n-1$$
$$a_n = m_{n-1} + m_n$$

It may be useful to point out that:

- the n-th label m_n can be not vanishing only if the first $n-1$ labels are all not vanishing; m_n is always non-zero for spinor irreducible representations.
- two irreducible representations which differ by the sign of m_n are conjugate to one another.
- the irreducible representations of $SO(4q + 2)$ with $m_{2q+1} \neq 0$ are complex representations, while the irreducible representations with $m_{2q+1} = 0$ as well as all the irreducible representations of $SO(4q)$ are real. We recall that an irreducible representation is real (complex) if in the Kronecker product with itself (resp. its conjugate) the identity representation appears.

To such an n-uple $(m_1, \ldots, m_n)$, we associate a GYT $[\mu]$ made of $\mu_i = |m_i|$ boxes if $m_i \in \mathbb{Z}$ and by $\mu_i = |m_i - \frac{1}{2}|$ boxes if $m_i \in \mathbb{Z} + \frac{1}{2}$, in the i-th row, all these rows being on the right hand side of a vertical axis, except the n-th one which will stand on the left hand side when $m_n < 0$. Note that this prescription leads to association to the fundamental spinor irreducible representation $m_1 = \ldots = m_n = \frac{1}{2}$, the null GYT, and to the complex conjugate one $m_1 = \ldots = m_{n-1} = -m_n = \frac{1}{2}$, the GYT $[\mu] = [0, \ldots, 0, -1]$. Let us add that associating the GYT [0] to the trivial vector representation $(0, \ldots, 0)$ as well as to the fundamental spinor one $(\frac{1}{2}, \ldots, \frac{1}{2})$ will not bring any confusion in our product computation if we keep in mind that simple rule, (V) and (S) denoting vector and spinor representations respectively:

$$(V) \otimes (V) = (V)\,, \qquad (V) \otimes (S) = (S)\,, \qquad (S) \otimes (S) = (V)$$

However in the following the GYTs associated to the spinor representations will be denoted by a prime in order to avoid any misunderstanding.

Below, we present the method which allows us to achieve the product of any irreducible representation, denoted by the GYT $[\mu]$, with any $SO(2n)$ fundamental representation labelled by the Dynkin labels $a_k = 1$ and $a_j = 0$ for $j \neq k$. The GYT associated to such a fundamental representation will be $[1^k]$, $(k = 1, \dots, n-2)$ in the vector case and $[\eta] = [0]'$ or $[0^{n-1}, -1]'$ in the spinor case. Indeed any irreducible representation can be obtained by performing repeated products of the fundamental irreducible representations: we will illustrate this property by an example at the end of this entry. A list of dimensions and explicit products of $SO(2n)$ representations is also available in the table section (see Tables 3.14, 3.16 and 3.18).

We successively consider the products $[\mu] \otimes [1^k]$ and $[\mu] \otimes [\eta]$ where $[\mu]$ is a vector or a spinor representation.

<u>Case 1</u>: The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product $[\mu] \otimes [1^k]$ with $[\mu]$ associated to a vector representation are given by

$$[\mu]\otimes[1^k] = \sum_{j=0}^{k} (L_n^{2j}\otimes[1^k])_A \otimes [\mu] \;\ominus \sum_{\substack{i=1,j=0\\ i+j\leq k}}^{k} (L_n^{2i}\otimes[\mu])_{NA} \otimes (L_n^{2j}\otimes[1^k])_A$$

where:

– the symbol L_n^{2j} denotes a negative GYT of the following type:

$$L_n^{2j} = [0, \dots, 0, -\xi_q, -\xi_q, \dots, -\xi_2, -\xi_2, -\xi_1, -\xi_1]$$

where $q = [[n/2]]$ (integer part of $n/2$), $\xi_i \in \{0,1,2\}$ satisfying $\xi_i - \xi_j \in \{0,1\}$ with $i < j$ and $\sum_i \xi_i = 2j$. Explicitly, one has

$$L_{2n}^{2j} = \begin{cases} (\underbrace{0,\dots,0}_{2n-2j},\underbrace{-1,\dots,-1}_{2j}) & \text{if } 0 \leq j \leq n \\ (\underbrace{-1,\dots,-1}_{4n-2j},\underbrace{-2,\dots,-2}_{2j-2n}) & \text{if } n \leq j \leq 2n \end{cases} \quad \text{for } SO(4n)$$

$$L_{2n+1}^{2j} = \begin{cases} (0,\underbrace{0,\dots,0}_{2n-2j},\underbrace{-1,\dots,-1}_{2j}) & \text{if } 0 \leq j \leq n \\ (0,\underbrace{-1,\dots,-1}_{4n-2j},\underbrace{-2,\dots,-2}_{2j-2n}) & \text{if } n \leq j \leq 2n \end{cases} \quad \text{for } SO(4n+2)$$

– the suffix A means that in the product of the two GYTs we have to keep only the "allowed" GYTs $[\lambda]$, which satisfy the two conditions $|\lambda_i| \leq 1$ for all i and $\sum_i |\lambda_i| \leq k$.
– the suffix NA means that in the product of the two GYTs we have to keep

only the (not allowed) GYTs $[\zeta]$ such that $\sum_i |\zeta_i| > \sum_i |\mu_i|$.
– in the r.h.s. of the above equation, after performing the product one has to keep only the GYTs $[\lambda]$ with at most one negative row, the last one satisfying $\lambda_{n-1} > |\lambda_n|$.
– a GYT appearing only in the subtracting part of $[\mu] \otimes [1^k]$ has to be omitted.
– in the subtracting part of $[\mu] \otimes [1^k]$, if the same GYT appears for different values of i and j with $i+j$ fixed, it has to be taken into account only once.

Finally, let us add that in case of the product $[1^p] \otimes [1^k]$ $(p \neq k)$, it is more convenient to choose for $[\mu]$ the longer tableau.

In the general formula for $[\mu] \otimes [1^k]$, the second term of the r.h.s. in most cases reduces to $\sum_{j=1}^{k} (L_n^{2j} \otimes [\mu])_{NA} \otimes [1^k]$. In fact, the $j \geq 1$ contributions may appear only for $k \geq 3$ and $n \geq 4$.

<u>Case 2</u>: The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with $[1^k]$ with $[\mu]$ associated to a spinor representation are given as follows. The labels μ_i are related to the $(m_1, \dots, m_n)$ by the relations $\mu_i = m_i - \frac{1}{2}$. Then, one proceeds as in Case 1, with the following modifications:

– the suffix NA is relative to a GYT $[\zeta]$ such that $\sum_i |\zeta_i + \frac{1}{2}| > \sum_i |\mu_i + \frac{1}{2}|$.
– in the final result, one has to keep only the GYTs $[\lambda]$ such that, after adding $\frac{1}{2}$ to each label λ_i, at most the last row is negative and satisfies $\lambda_{n-1} \geq |\lambda_n|$.

<u>Case 3</u>: The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with $[\eta]$ with $[\mu]$ associated to a vector representation are given by

$$[\mu] \otimes [\eta] = \sum_{j=0}^{k} (\widetilde{L}_n^{2j} \otimes [\eta]) \ \otimes \ [\mu]$$

Let us recall that $[\eta]$ is of the form $[0^n]'$ or $[0^{n-1}, -1]'$. The $\widetilde{L}_n^{2j}$'s are a subset of the L_n^{2j}'s, the ξ_i labels being now restricted to the values $\{0, 1\}$. Explicitly, one has (with $0 \leq j \leq 2n$)

$$\begin{aligned} \widetilde{L}_{2n}^{2j} &= (\underbrace{0, \dots, 0}_{2n-2j}, \underbrace{-1, \dots, -1}_{2j}) \qquad \text{for } SO(4n) \\ \widetilde{L}_{2n+1}^{2j} &= (0, \underbrace{0, \dots, 0}_{2n-2j}, \underbrace{-1, \dots, -1}_{2j}) \qquad \text{for } SO(4n+2) \end{aligned}$$

As in Case 2, one has to keep in the final result only the GYTs $[\lambda]$ such that, after adding $\frac{1}{2}$ to each label λ_i, at most the last row is negative and satisfies $\lambda_{n-1} \geq |\lambda_n|$.

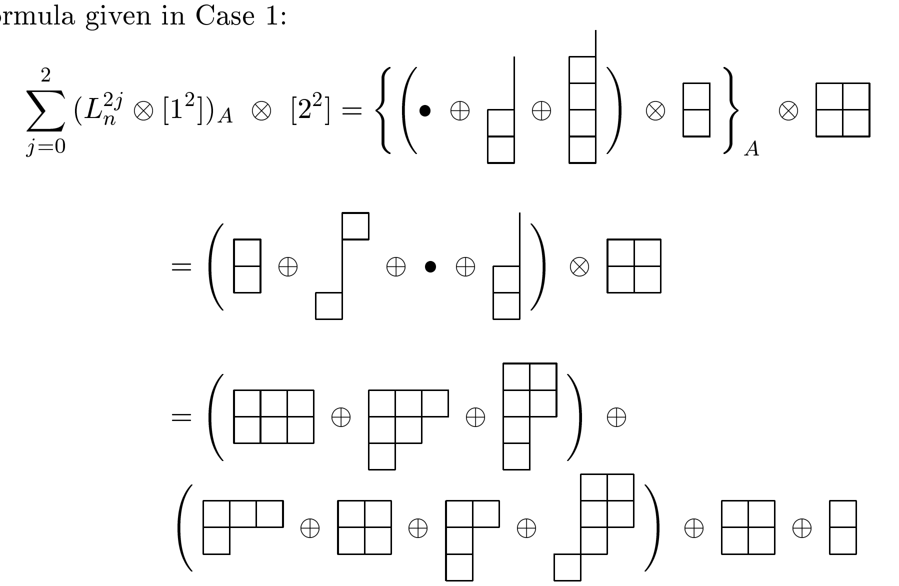
<u>Case 4</u>: To obtain the GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with $[\eta]$ with $[\mu]$ associated to a spinor representation, one proceeds as in Case 3, and then one has to add 1 to all the rows of the obtained GYTs $[\lambda]$, which must now satisfy $\lambda_{n-1} \geq |\lambda_n| \geq 0$. Again, let us recall that the product of two spinor representations produces only vector representations.

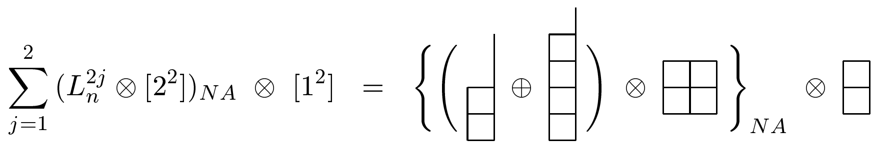
Finally, we have to consider the product $[\mu] \otimes [\nu]$ where neither $[\mu]$ nor $[\nu]$ is a fundamental $SO(2n)$ irreducible representation. Then it is always possible to rewrite $[\nu]$ with the help of products involving fundamental representations, and to use the above described formulae. As a illustration, let us consider the product $[5] \otimes [3]$. One can use the decomposition $[2] \otimes [1] = [3] \oplus [2,1] \oplus [1]$ and $[1^2] \otimes [1] = [2,1] \oplus [1^3] \oplus [1]$ to rewrite $[5] \otimes [3]$ as $([5] \otimes [2] \otimes [1]) \ominus ([5] \otimes [1^2] \otimes [1]) \ominus 2([5] \otimes [1]) \ominus ([5] \otimes [1^3])$.

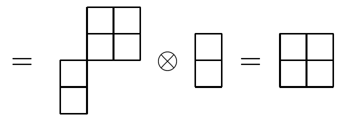
Note: The Kronecker product is of course commutative, notwithstanding the fact that the above formula deals with the two irrreducible representations on a different basis.

Example 1

As a first example of the above formulae, let us discuss the Kronecker product of the representations $[2^2]$ and $[1^2]$ in $SO(8)$. Using the simplified formula given in Case 1:







leading to

$$[2^2] \otimes [1^2] \;=\; [3^2] \oplus [3,2,1] \oplus [3,1] \oplus [2,2,1,1] \\ \oplus [2,2,1,-1] \oplus [2,2] \oplus [2,1,1] \oplus [1^2]$$

that is, making the dimensions of these $SO(8)$ representations explicit:

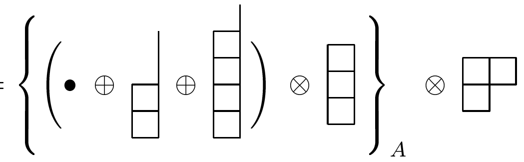
$$300 \otimes 28 = 1925 \oplus 4096 \oplus 567 \oplus 567' \oplus 567'' \oplus 300 \oplus 350 \oplus 28$$

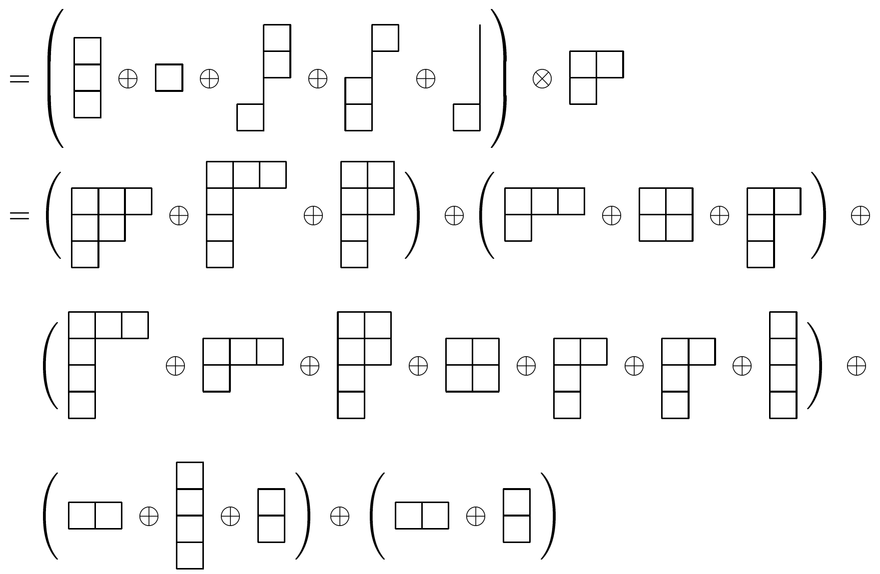
The product $[2^2] \otimes [1^2]$ in $SO(2n)$ with $n > 4$ is obtained by simply disgarding in the r.h.s. of the above decomposition the tableau $[2,2,1,-1]$, which obviously does not correspond to a representation in $SO(2n)$ when $n \neq 4$. □

Example 2

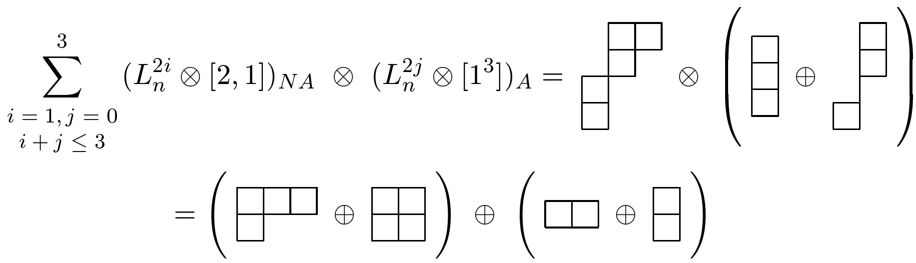
Now, consider the case of the product $[2,1] \otimes [1^3]$ in $SO(8)$ for which it is needed to apply the full general formula of Case 1.

$$\sum_{j=0}^{2} (L_n^{2j} \otimes [1^3])_A \;\otimes\; [2,1] =$$





and



leading to

$$\begin{aligned}[2,1] \otimes [1^3] \;=\;& [3,2,1] \oplus [3,1,1,1] \oplus [3,1,1,-1] \oplus [3,1] \\ & \oplus [2,2,1,1] \oplus [2,2,1,-1] \oplus [2,2] \oplus 3\,[2,1,1] \oplus [2] \\ & \oplus [1,1,1,1] \oplus [1,1,1,-1] \oplus [1^2]\end{aligned}$$

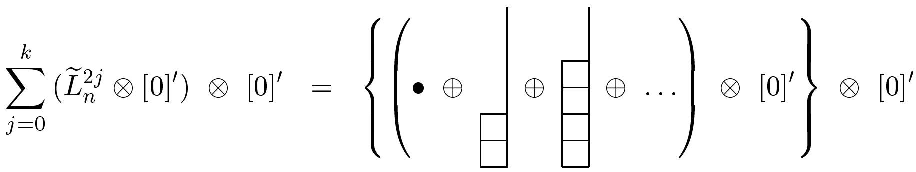
that is, making the dimensions explicit:

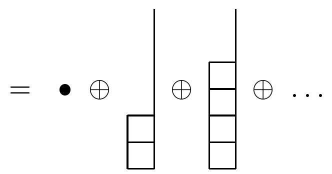
$$\begin{aligned}160 \otimes 56 \;=\;& 4096 \oplus 840 \oplus 840^* \oplus 567 \oplus 567' \oplus 567'^* \oplus 300 \\ & \oplus 350 \oplus 350 \oplus 350 \oplus 35 \oplus 35' \oplus 35'^* \oplus 28\end{aligned}$$

□

Example 3

Finally, consider in $SO(2n)$ the product of the spinor fundamental representation $(\frac{1}{2}, \ldots, \frac{1}{2}) = [0]'$ by itself. Using the appropriate formula:





Since we started from two spinor representations, we have to add one box to each row of the obtained GYTs, that is finally:

$$[0]' \otimes [0]' = [1^n] \oplus [1^{n-2}] \oplus [1^{n-4}] \oplus \ldots \oplus [\kappa]$$

where $\kappa = 0$ or 1 depending on whether n is even or odd, that is also in the m_i notation:

$$(\tfrac{1}{2}^n) \otimes (\tfrac{1}{2}^n) = (1^n) \oplus (1^{n-2}) \oplus (1^{n-4}) \oplus \ldots \oplus (\kappa)$$

with again $\kappa = 0$ for n even and $\kappa = 1$ for n odd.

In particular, for $SO(8)$ we get $8_S \otimes 8_S = 35 \oplus 28 \oplus 1$. □

More generally, some compact formulae can be obtained for product of representations with special symmetry. Using the m_i notation in order to consider simultaneously vector and spinor representations, we can write:

– in the $SO(4q)$ case:

$$[m, \ldots, m] \otimes [m', \ldots, m'] = \bigoplus_{\{k_i\}} [m + m' - k_1, m + m' - k_1, \ldots, m + m' - k_q, m + m' - k_q]$$

$$[m, \ldots, m] \otimes [m', \ldots, m', -m'] = \bigoplus_{\{k_i\}} [m + m', m + m' - k_1, m + m' - k_1, \ldots, m + m' - k_{q-1}, m + m' - k_{q-1}, m - m']$$

where the $k_i \in \mathbb{Z}$ satisfy $0 \le k_1 \le \ldots \le k_{q-1} \le 2n$.

– in the $SO(4q + 2)$ case:

$$[m, \ldots, m] \otimes [m', \ldots, m'] = \bigoplus_{\{k_i\}} [m + m', m + m' - k_1, m + m' - k_1, \ldots, m + m' - k_q, m + m' - k_q]$$

$$[m, \ldots, m] \otimes [m', \ldots, m', -m'] = \bigoplus_{\{k_i\}} [m + m' - k_1, m + m' - k_1, \ldots, m + m' - k_q, m + m' - k_q, m - m']$$

where the $k_i \in \mathbb{Z}$ satisfy $0 \le k_1 \le \ldots \le k_q \le 2n$.

Finally for the product of two completely $SO(2n)$ symmetric representation we get:

$$[m, 0, \ldots, 0] \otimes [m', 0, \ldots, 0] = \bigoplus_{l=0}^{m'} \bigoplus_{k=0}^{m'-l} [m + m' - k - 2l, k, \ldots, 0]$$

with $m \ge m'$, $m + m' - 2l \ge 2k \ge 0$.

For more details, see ref. [31].

1.65 Products of $SO(2n+1)$ representations

We present here a simplified version of the method introduced in ref. [31] for performing the Kronecker product of two irreducible representations of the orthogonal group $SO(2n+1)$ $(n > 1)$ by means of the product of generalized Young tableaux or GYTs ($\rightarrow$ 1.36). We identify an irreducible representation of $SO(2n+1)$ by n integer (half-integer) numbers m_i, respectively for vector (V) and spinor (S) irreducible representations, such that $m_1 \geq \ldots \geq m_n \geq 0$. The correspondence between the set of non-negative integers $(a_1, \ldots, a_n)$ identifying an irreducible representation $D(a_1, \ldots, a_n)$ (Dynkin labels) and the n-uple $(m_1, \ldots, m_n)$ is

$$\begin{aligned} a_j &= m_j - m_{j+1} \quad \text{for } j = 1, \ldots, n-1 \\ a_n &= 2m_n \end{aligned}$$

To such an n-uple $(m_1, \ldots, m_n)$, we associate a GYT $[\mu]$ made of $\mu_i = m_i$ boxes if $m_i \in \mathbb{Z}$ and by $\mu_i = m_i - \frac{1}{2}$ boxes if $m_i \in \mathbb{Z} + \frac{1}{2}$, in the i-th row. The rules for the product of two GYTs have been given in $\rightarrow$ 1.36 Generalized Young tableaux.

Below, we present the method which allows us to achieve the product of a general irreducible representation, denoted by the GYT $[\mu]$, with any $SO(2n+1)$ fundamental representation labelled by the Dynkin labels $a_k = 1$ and $a_j = 0$ for $j \neq k$, denoted by $[1^k]$,$(k = 1, \ldots, n-2)$ for the vector representation and by $[\eta] = [0]'$ for the spinor one (the GYT $[0]$ denotes the trivial vector representation). Indeed any irreducible representation can be obtained by products of the fundamental irreducible representations by repeated applications of the rules given below. For examples, see more details in $\rightarrow$ 1.64.

A list of dimensions and explicit products of $SO(2n+1)$ representations is also available in the table section (see Tables 3.15, 3.17 and 3.19).

We have to introduce a class of completely negative GYTs made of two columns with i boxes in the first column to the left of the vertical axis and j boxes in the second one with $i \geq j$. We denote such a tableau by L_n^{ij}. Explicitly, one has

$$L_n^{ij} = [\underbrace{0, \ldots, 0}_{n-i}, \underbrace{-1, \ldots, -1}_{i-j}, \underbrace{-2, \ldots, -2}_{j}]$$

For necessity, we complete this set of GYTs with the trivial one L_n^{00}.

As in $SO(2n)$, we discuss separately the product of a general irreducible representation $[\mu]$ by a vector fundamental one, and the product of $[\mu]$ by a spinor fundamental one, taking in both cases $[\mu]$ either as a vector or as a spinor representation.

Case 1: The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with $[1^k]$ $(k = 1, \dots, n-1)$ are given by

$$[\mu] \otimes [1^k] = \sum_{i,j=0,\, j\leq i}^{i,j=k} (L_n^{ij} \otimes [1^k])_A \;\otimes\; [\mu] \;\ominus\; \sum_{i,j=0,\, j\leq i}^{i,j=k} (L_n^{ij} \otimes [\mu])_{NA} \;\otimes\; [1^k]$$

where:

– the suffix A means that in the product of the two GYTs we have to keep only the "allowed" GYTs $[\lambda]$, which satisfy the two conditions $|\lambda_i| \leq 1$ for all i and $\sum_i |\lambda_i| \leq k$.
– if the same GYT appears in the product with two different L_n^{ij}, it has to be taken into account only once.
– the suffix NA means that in the product of the two GYTs we have to keep only the (not allowed) GYTs $[\zeta]$ such that $\sum_i |\zeta_i| > \sum_i |\mu_i|$ if $[\mu]$ corresponds to a vector representation and $\sum_i |\zeta_i + \frac{1}{2}| > \sum_i |\mu_i + \frac{1}{2}|$ if $[\mu]$ corresponds to a spinor one.
– in the r.h.s. of the above equation, after performing the product one has to keep only the GYTs $[\lambda]$ with positive rows.
– a GYT appearing only in the subtracting part of $[\mu] \otimes [1^k]$ has to be omitted.
– if $[\mu]$ is a GYT of the form $[1^p]$, one has to assume $p \leq k$.
– in the case where $[\mu]$ is spinor, do not forget to add $\frac{1}{2}$ to each row of each obtained tableau after computation.

Case 2: The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with the fundamental spinor representation $[0]'$ are given by

$$[\mu] \otimes [0]' = \sum_{i=0}^{n} (L_n^{i0} \otimes [0]') \;\otimes\; [\mu]$$

with the same prescriptions as above, and adding either $\frac{1}{2}$ or 1 to each row of the obtained tableaux depending on whether $[\mu]$ is relative to a vector or a spinor representation.

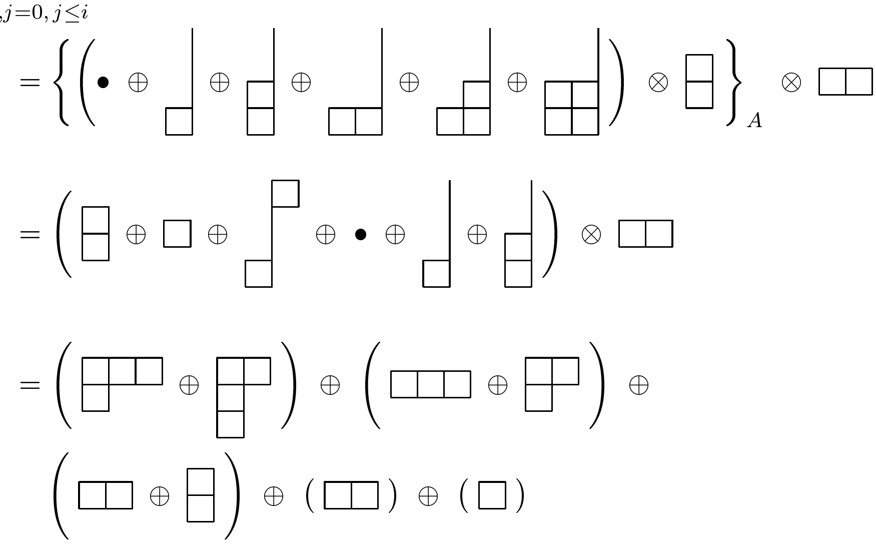
As for the $SO(2n)$ case, in order to perform the product $[\mu] \otimes [\nu]$ when neither $[\mu]$ nor $[\nu]$ corresponds to a fundamental representation, one has first to rewrite the smaller of these two GYTs as a sum of products involving the fundamental representations and operate as above.

Note: The Kronecker product is of course commutative, notwithstanding the fact that the above formula deals with the two irrreducible representations on a different basis.

Example

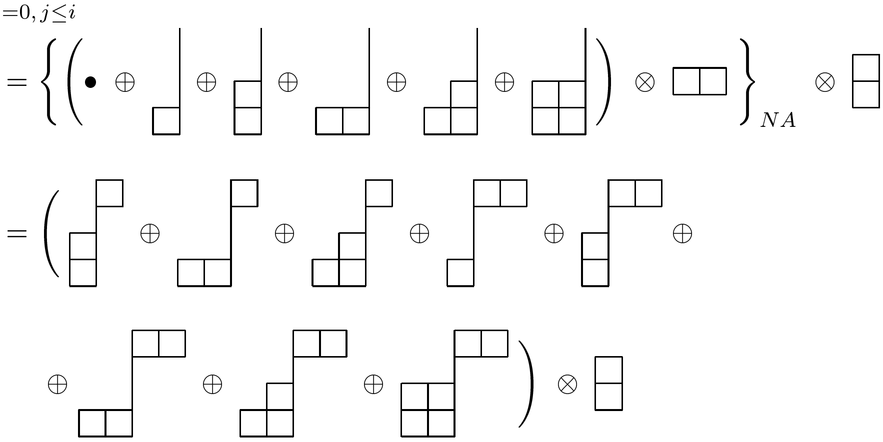
As an example of the above formulae, we discuss the Kronecker product of [2] with $[1^2]$ for $SO(9)$. Let us summarize the different steps of the computation:

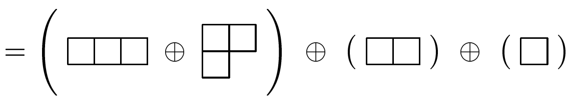
$$\sum_{i,j=0,\, j\le i}^{i,j=2} (L_4^{ij} \otimes [1^2])_A \;\otimes\; [2] =$$



and

$$\sum_{i,j=0,\, j\le i}^{i,j=2} (L_4^{ij} \otimes [2])_{NA} \;\otimes\; [1^2] =$$





that is, after subtraction:

$$[2] \otimes [1^2] = [3,1] \oplus [2,1,1] \oplus [1^2] \oplus [2]$$

that is also, making the dimensions of these $SO(9)$ representations explicit:

$$44 \otimes 36 = 910 \oplus 594 \oplus 36 \oplus 44$$

□

Finally, we can give compact formulae for the product of two completely symmetric or antisymmetric GYTs:

$$[\mu, 0, \dots, 0] \otimes [\nu, 0, \dots, 0] = \sum_{l=0}^{\nu} \sum_{k=0}^{\nu - l} [\mu + \nu - k - 2l, k, \dots, 0] \quad (\mu \geq \nu)$$

$$[\mu, \dots, \mu] \otimes [\nu, \dots, \nu] = \sum_{k_1=0}^{\mu-\nu} \sum_{k_2=0}^{\mu-\nu-k_1} \dots \sum_{k_n=0}^{\mu-\nu-k_1-\dots-k_{n-1}} [\mu + \nu - k_1, \mu + \nu - k_1 - k_2, \dots, \mu + \nu - k_1 - \dots - k_n]$$

1.66 Products of $Sp(2n)$ representations

We present here a simplified version of the method introduced in ref. [32] for performing the Kronecker product of two irreducible representations of the symplectic group $Sp(2n)$ by means of product of generalized Young tableaux or GYT (→ 1.36). An irreducible representation of $Sp(2n)$ is either labelled by $D(a_1, \dots, a_n)$ where the positive or null integers a_i are the Dynkin labels, or by the set of n non-negative integers $(m_1, \dots, m_n)$ such that $m_i \geq m_{i+1}$, the correspondence between the two notations being given by:

$$\begin{aligned} a_j &= m_j - m_{j+1} \quad \text{for } j = 1, \dots, n-1 \\ a_n &= m_n \end{aligned}$$

To such an n-uple $(m_1, \dots, m_n)$, we associate a GYT $[\mu]$ made of $\mu_i = m_i$ boxes in the i-th row, that is a tableau which is drawn as a standard Young tableau $[\mu_1, \dots, \mu_n]$. However, in order to perform the product of two $Sp(2n)$ irreducible representations, negative GYTs have to be introduced and the product of a negative GYT with a positive GYT is considered (→ 1.36).

First we present the method to perform the product of any irreducible representation, denoted by the GYT $[\mu]$, with a fundamental representation denoted by the GYT $[1^k]$, that is a GYT with k rows of one box or one column with k boxes ($k \leq n$), or equivalently, by the Dynkin labels $a_k = 1$ and $a_j = 0$ for $j \neq k$.

The GYTs specifying the irreducible representations which appear in the decomposition of the Kronecker product of $[\mu]$ with $[1^k]$ are given by

$$[\mu] \otimes [1^k] = \sum_{j=0}^{k} (P_n^{2j} \otimes [1^k])_A \ \otimes \ [\mu]$$

where:

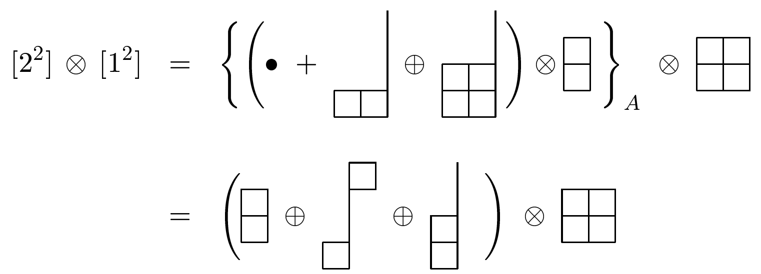
– the symbol P_n^{2j} denotes a negative GYT with n rows $[-\alpha_1, \ldots, -\alpha_n]$, where the $\alpha_i \in \{0, 2\}$ are constrained by the condition $\sum_i \alpha_i = 2j$. Explicitly, one has (with $0 \leq j \leq n$)

$$P_n^{2j} = [\underbrace{0, \ldots, 0}_{n-j}, \underbrace{-2, \ldots, -2}_{j}]$$

– the suffix A means that in the product of the two GYTs we have to keep only the "allowed" GYTs $[\lambda]$ which satisfy the two conditions $|\lambda_i| \leq 1$ for all i and $\sum_i |\lambda_i| \leq k$.

Note that at each level j, the product $(P_n^{2j} \otimes [1^k])_A$ provides only one tableau.

Let us illustrate this procedure with the example $[2^2] \otimes [1^2]$ in $Sp(2n)$ with $n \geq 3$. Then:



that is

$$\begin{aligned}
[1^2] \otimes [2^2] &= [3,3] \oplus [3,2,1] \oplus [2,2,1,1] \\
[1,0,\ldots,0,-1] \otimes [2^2] &= [3,1] \oplus [2,2] \oplus [2,1,1] \\
[0,\ldots,0,-1,-1] \otimes [2^2] &= [1^2]
\end{aligned}$$

and finally

$$[2^2] \otimes [1^2] = [3,3] \oplus [3,2,1] \oplus [3,1] \oplus [2,2,1,1] \oplus [2,2] \oplus [2,1,1] \oplus [1^2]$$

We note that the representation $[2,2,1,1]$ does not exist for $Sp(6)$. More generally, a tableau with more than n rows has to be rejected as soon as it is the group $Sp(2n)$ which is considered.

– in the r.h.s. of the above equation, after performing the product, one has to keep only positive GYTs.
– if $[\mu]$ is a GYT of the form $[1^p]$, one has to assume $p \geq k$.

Now, in order to perform the Kronecker product of any two $Sp(2n)$ irreducible representations $[\mu] \otimes [\nu]$, we need to apply the above considered formula several times, using the property of any irreducible represention $[\nu]$ to arise from the product of fundamental irreducible ones. For example, considering the product $[5] \otimes [2]$, one will first compute $[1] \otimes [1] = [2] \oplus [1^2] \oplus [0]$, where $[0]$ denotes the (one-dimensional) trivial $Sp(2n)$ irreducible representation, and one rewrites $[5] \otimes [2] = ([5] \otimes [1]) \otimes [1] \ominus ([5] \otimes [1^2]) \ominus [5]$. As a second example, if $[\nu] = [3,1]$, one could proceed as follows: $[2,1] \otimes [1] = [3,1] \oplus [2^2] \oplus [2,1,1] \oplus [2] \oplus [1^2]$, $[1^2] \otimes [1] = [2,1] \oplus [1^3] \oplus [1]$ and then $[1^2] \otimes [1^2] = [2^2] \oplus [2,1,1] \oplus [1^4] \oplus [2] \oplus [1^2] \oplus [0]$, providing $[3,1] = ([1^2] \otimes [1] \otimes [1]) \ominus ([1^3] \otimes [1]) \ominus ([1] \otimes [1]) \ominus ([1^2] \otimes [1^2]) \oplus [1^4] \otimes [0]$. It is this last expression for $[3,1]$, written as a sum of products of fundamental $[1^k]$ representations, that we will put into our canonical formula.

Note: The Kronecker product is of course commutative, notwithstanding the fact that the above formula deals with the two irrreducible representations on a different basis.

We refer to Tables 3.12 and 3.13 for the dimensions and the explicit computation of $Sp(2n)$ representation products.

Finally we can give compact formulae for the product of two completely symmetric or antisymmetric GYTs:

$$[\mu, 0, \dots, 0] \otimes [\nu, 0, \dots, 0] = \sum_{l=0}^{\nu} \sum_{k=0}^{\nu - l} [\mu + \nu - k - 2l, k, \dots, 0] \quad (\mu \geq \nu)$$

$$[\mu, \dots, \mu] \otimes [\nu, \dots, \nu] = \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_n} [\mu + \nu - 2k_1, \dots, \mu + \nu - 2k_n]$$

For more details, see ref. [32].

1.67 Products of $SU(n)$ representations

Let G be a Lie group, and π and π' be two representations of G. Then the tensor product $\pi \otimes \pi'$ is also a representation of G ($\rightarrow$ 1.75). But if π and π' are irreducible, the representation $\pi \otimes \pi'$ will be in general a reducible representation of G. When G is compact, it can be decomposed into a direct sum of irreducible representations as follows:

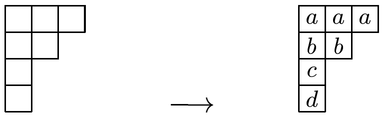
$$\pi \otimes \pi' = \bigoplus_i n_i\,\pi_i$$

We provide below the decomposition method based on the use of Young tableaux ($\rightarrow$ 1.96) in the case where $G = SU(n)$.

Method

Let $\mathcal{T}$ and $\mathcal{T}'$ be the Young tableaux associated to the representations π and π' respectively.

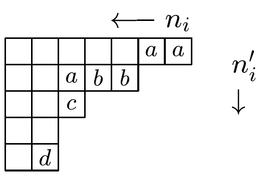
1. Choose the simpler tableau, say $\mathcal{T}'$, and call a each box in the first row, b those of the second row, and so on.



2. Add to $\mathcal{T}$ one box marked with a of $\mathcal{T}'$ using all possible ways such that one always gets a Young tableau. Then add to the obtained tableau a second box marked with a with the prescription that two boxes marked with a must not be in the same column.

3. When all the boxes marked with a are used, add the boxes marked with b, then the boxes marked with c, and so on in the same way, but with the conditions:

   – two boxes with the same label must not be in the same column,
   – denoting $n_i(a)$ the number of boxes labelled by a in the i-th first columns starting from the right and $n'_i(a)$ the number of boxes labelled by a in the i-th first rows starting from the top, and identical definitions for the other labels $b, c, \ldots$, one must have



$$n_i(a) \geq n_i(b) \geq n_i(c) \geq \ldots$$
$$n'_i(a) \geq n'_i(b) \geq n'_i(c) \geq \ldots$$

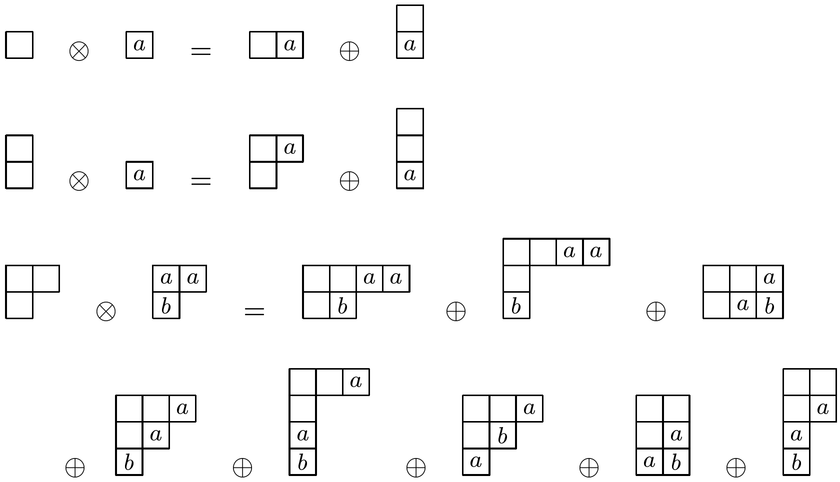
4. Any obtained tableau with more than n rows will be suppressed. Moreover, any tableau with n rows will be replaced by the corresponding tableau in which the n box columns are suppressed.

   For example, in the case of $SU(5)$, $[3, 2, 2, 1, 1]$ does not exist and $[3, 2, 2, 1]$ has to be replaced by $[2, 1, 1, 0]$.

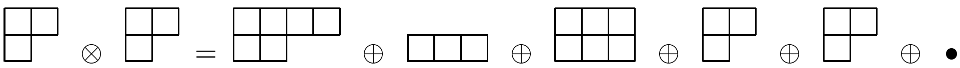
5. The same dummy (that is without the labels $a, b, c, \ldots$) tableau may appear several times. Suppose it appears twice:

   – if the distribution of $a, b, c, \ldots$ in the tableaux is the same, then one of the tableaux must be suppressed.
   – if the distribution of $a, b, c, \ldots$ in the tableaux is different, then the irreducible representation associated with the dummy tableaux appears twice.

Examples



In this last example, following prescription (5), the representation associated to the Young tableau $[3, 2, 1]$ appears twice since the distribution of the labels a, b in the two corresponding tableaux is different. If we consider $SU(3)$, the two tableaux with four rows have to be thrown away and the four tableaux with three rows have to be simplified. We will finally get in $SU(3)$:



that is

$$8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$$

□

A list of dimensions and explicit products of $SU(n)$ representations is also available in the table section (see Tables 3.10 and 3.11).

The reduction of the product of two representations of the orthogonal groups and of the symplectic groups can be done using generalized Young tableaux [31, 32].

1.68 Quotient group

Let G be a group and H a subgroup of G. One can define the equivalence relation

$$x \sim y \;\leftrightarrow\; x^{-1}y \in H\,, \qquad \forall\, x, y \in G$$

(indeed this relation is reflexive: $x \sim x\,,\ \forall\, x \in G$, symmetric: $x \sim y \Rightarrow y \sim x\,,\ \forall\, x, y \in G$, transitive: $x \sim y$ and $y \sim z \Rightarrow x \sim z\,,\ \forall\, x, y, z \in G$).

The set of elements in G which are equivalent to the element $x \in G$ will be called the *left coset (modulo H)* associated to x and denoted

$$\dot{x} \equiv xH = \{y \in G \mid x \sim y\} = \{xh \,|\, h \in H\}$$

Such a class can be defined by any of its elements (indeed if $x \sim y$ then $y = xh$ or $x = yh^{-1}$ and therefore $\dot{x} = \dot{y}$). The set of left cosets (modulo H) G/H is a partition of G (that is it covers all G and the intersection of two different classes is empty). In the same way, one can define a second equivalence relation by

$$x \sim y \;\leftrightarrow\; xy^{-1} \in H\,, \qquad \forall\, x, y \in G$$

and therefore the *right coset (modulo H)* associated to x:

$$\ddot{x} \equiv Hx = \{y \in G \mid x \sim y\} = \{hx \,|\, h \in H\}$$

and the set of right cosets $H \setminus G$.

When do the partitions in G/H and $H \setminus G$ coincide? One must have $xH = Hx\,, \forall\, x \in G$, that means that H must be an invariant subgroup of G ($\rightarrow$ 1.37). In this last case, G/H ($H \setminus G$) has canonically a group structure: indeed, given that xH and yH are two elements of G/H, one can define the product $(xH) * (yH) = xyH$ (this product being uniquely defined since if $z \in yH$, that is $z = yh$ with $h \in H$, one has $(xH)(yH) = xzH = xyH$). The identity element is $eH = H$ and the inverse of xH is $(xH)^{-1} = x^{-1}H$. Actually, one has a canonical homomorphism

$$\pi : G \rightarrow G/H\,, \quad x \mapsto xH$$

G/H is called the *quotient group* of G by H.

Example

From the Euclidean group in n dimensions $E(n) = T(n) \ltimes SO(n)$, one can construct the quotient group $E(n)/T(n)$ since the translation group $T(n)$ is an invariant group of $E(n)$. Such a quotient group is isomorphic to the rotation group $SO(n)$. □

More generally, for any group which is the semi-direct product $A \ltimes B$ of two groups A and B with A invariant subgroup, one can form the quotient group $(A \ltimes B)/A \simeq B$.

1.69 Quotient Lie algebra

We can use the concepts above discussed (→ 1.68) to define a quotient algebra.

Theorem

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ an ideal of $\mathcal{G}$, that is $\Big[\mathcal{G}, \mathcal{H}\Big] \subset \mathcal{H}$. Then the quotient space $\mathcal{G}/\mathcal{H}$ made of elements $x + \mathcal{H} = \{y \in \mathcal{G} \mid y = x + h\,,\ h \in \mathcal{H}\}$ is a Lie algebra with the composition laws:

$$(x + H) + (y + H) = (x + y)H$$
$$\Big[x + H, y + H\Big] = [x, y]H$$

1.70 Racah coefficients

The tensor product of three irreducible representations $D(j_1)$, $D(j_2)$ and $D(j_3)$ of the rotation group $SO(3)$ decomposes into a direct sum of irreducible representations:

$$D(j_1) \otimes D(j_2) \otimes D(j_3) = \bigoplus_{J=|j_1 \pm j_2 \pm j_3|}^{j_1+j_2+j_3} D(J)$$

The following two different coupling schemes allow two different decomposition of the space $\mathcal{H}(D(j_1)) \otimes \mathcal{H}(D(j_2)) \otimes \mathcal{H}(D(j_3))$ into irreducible spaces $\mathcal{H}(D(J))$ (from the associativity of the tensor product):

$$D(j_1) \otimes D(j_2) \otimes D(j_3) = \bigoplus_{J_{12}} D(J_{12}) \otimes D(j_3) = \bigoplus_{J_{12},J} D_{J_{12},j_3}(J)$$
$$D(j_1) \otimes D(j_2) \otimes D(j_3) = \bigoplus_{J_{23}} D(j_1) \otimes D(J_{23}) = \bigoplus_{J_{23},J} D_{j_1,J_{23}}(J)$$

Let $|j_i m_i\rangle$ with $-j_i \le m_i \le j_i$ and $i = 1, 2, 3$ be the canonical bases of the Hilbert spaces $\mathcal{H}(D(j_i))$ of the irreducible representations $D(j_i)$. We have ($\to$ 1.14):

$$|J_{12} j_3; JM\rangle = \sum_{m_1,m_2,m_3} \langle j_1 j_2 m_1 m_2 | J_{12} M_{12}\rangle \, \langle J_{12} j_3 M_{12} m_3 | JM\rangle \, |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle$$

and

$$|j_1 J_{23}; JM\rangle = \sum_{m_1,m_2,m_3} \langle j_2 j_3 m_2 m_3 | J_{23} M_{23}\rangle \, \langle j_1 J_{23} m_1 M_{23} | JM\rangle \, |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle$$

It is possible to go from one basis to the other one by means of unitary transformation. The Wigner coefficients (or $6j$-symbols) are defined in terms of the coefficients of this unitary transformation and one has

$$|J_{12} j_3; JM\rangle = \sum_{J_{23}} \sqrt{(2J_{12}+1)(2J_{23}+1)} \; (-1)^{j_1+j_2+j_3+J} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} |j_1 J_{23}; JM\rangle$$

The Racah coefficients are defined in terms of the $6j$-symbols

$$W(j_1 j_2 J_1 J_2; j_3 J_3) = (-1)^{j_1+j_2+J_1+J_2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix}$$

The Wigner and Racah coefficients are related to the Clebsch-Gordan coefficients ($\to$ 1.14) and satisfy several symmetry relations (see ref. [60] for references). They are used in physics when one has to couple three angular momenta j_1, j_2, j_3.

1.71 Real forms

Let us recall that a Lie algebra is in particular a vector space defined on a field $\mathbb{K}$: the Lie algebra is called *real* when $\mathbb{K}$ is the field of real numbers $\mathbb{R}$ and *complex* when $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$. Although real Lie algebras (and real Lie groups) are intensively used, the notion of complex Lie algebra is fundamental: for example, the Cartan theory of simple Lie algebras could not be obtained by restricting to real Lie algebras, because the field of real numbers is not algebraically closed ($\to$ 1.7).

Starting from the real Lie algebra $\mathcal{G}$, one defines the Lie algebra $\mathcal{G}_C$ as the *complexification* of $\mathcal{G}$ by applying on $\mathcal{G}$ a sort of "analytic continuation" process, i.e.

$$\mathcal{G}_C = \Big\{ Z = X + iY \;\Big|\; X, Y \in \mathcal{G} \Big\} = \mathcal{G} \oplus i\mathcal{G}$$

Note that $\mathcal{G}_C$ can be considered as a real Lie algebra of twice the real dimension of $\mathcal{G}$ (X and iX being considered as two independent elements), although it is also a complex Lie algebra of the same dimension as the real Lie algebra $\mathcal{G}$. If $\{e_1, \dots, e_n\}$ denotes a basis of $\mathcal{G}_C$ of dimension n, the Lie algebra over $\mathbb{R}$ with basis $\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ will be called the *realification* $\mathcal{G}_R$ of $\mathcal{G}_C$ and will be of dimension $2n$. One may note that the real Lie algebra $\mathcal{G}_R$ possesses a complex structure J ($\rightarrow$ 1.16) derived from multiplication by the complex number i on $\mathcal{G}$.

Example

Let us consider the real Lie algebra $gl(n, \mathbb{R})$, generated by the elementary $n \times n$ matrices e_{ij} such that $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Its complex extension is $gl(n, \mathbb{C})$ with matrices $M = a^{ij} e_{ij}$, the a^{ij} being complex numbers.

Consider now the Lie algebra $u(n, \mathbb{C})$. It is constituted by $n \times n$ matrices M with complex entries satisfying the condition $M^\dagger = -M$, that is, if $M = a^{ij} e_{ij}$, such that $(a^{ij})^* = -a^{ji}$. The complexification of the real algebra $u(n, \mathbb{C})$ provides the algebra $gl(n, \mathbb{C})$. □

We now come to the definition of a real form:

Definition

Let $\mathcal{G}_C$ be a complex Lie algebra and $\mathcal{G}_R$ its realification. A real subalgebra $\mathcal{S}$ of $\mathcal{G}_R$ is called a *real form* of $\mathcal{G}_C$ if $\mathcal{S}_C = \mathcal{G}_C$.

The two following theorems are of special importance:

Theorem

Let $\mathcal{G}$ be a simple real Lie algebra. Then $\mathcal{G}$ is either isomorphic to the realification of a simple complex Lie algebra, or $\mathcal{G}$ is isomorphic to a real form of a simple complex Lie algebra.

Theorem

Every realification of a simple complex Lie algebra is a simple real Lie algebra.

As an example, let us think at the real simple algebra $sl(2, \mathbb{R})$. Its complexification is $sl(2, \mathbb{C})$. The realification of $sl(2, \mathbb{C})$ is a six-dimensional algebra isomorphic to $so(3, 1)$ ($\rightarrow$ 1.53 Lorentz group).

It is also important to recall the Cartan's theorem (→ 1.44) ensuring that any (semi)simple Lie algebra $\mathcal{G}$ has a vector space decomposition $\mathcal{G} = \mathcal{C} \oplus \mathcal{P}$ such that $\mathcal{C}$ is a compact subalgebra and $\mathcal{P}$ satisfies $\Big[\mathcal{C}, \mathcal{P}\Big] \subset \mathcal{P}$ and $\Big[\mathcal{P}, \mathcal{P}\Big] \subset \mathcal{C}$. From the property of the Killing form to be negative on a compact simple Lie algebra, the possibility of transforming $\mathcal{G}$ into a compact algebra by acting on the $\mathcal{P}$ generators by multiplication by i appears rather clearly in this framework: $K(X,X) > 0 \Rightarrow K(iX,iX) < 0$ for any $X \in \mathcal{P}$. Therefore, one can state:

Theorem

|| Every (semi)simple complex Lie algebra $\mathcal{G}$ has a compact real form.

Morevover, the existence of an involutive automorphism ϕ (i.e. $\phi^2 = 1$) of $\mathcal{G}$ acting as 1 on $\mathcal{C}$ elements and as -1 on $\mathcal{P}$ elements is obvious. Actually, all the real forms of a semi(simple) complex Lie algebra $\mathcal{G}_C$ will be obtained by determining all the involutive automorphisms of its compact form.

The real forms of the simple Lie algebras are given in Table 3.38. One usually defines the character of a real form of a complex semi-simple Lie algebra $\mathcal{G}$ as the difference between the number of non-compact generators and the number of compact ones. This explains the index written in parentheses after each real form of the exceptional Lie algebras appearing in Table 3.38. For example, the compact E_6 algebra of dimension 78 is denoted by $E_{6(-78)}$ (zero non-compact generators and 78 compact ones), while the E_6 real form with F_4 as maximal compact subalgebra is denoted $E_{6(-26)}$ (26 non-compact generators and 52 compact ones).

Let us add that the involutive automorphisms involved in the determination of the real forms are inner except for the five following cases: $sl(N,\mathbb{R})$, $su^*(2N)$, $so(2N-2p-1, 2p+1)$, $E_{6(-26)}$ and $E_{6(+6)}$, which are related to outer automorphisms (see ref. [13] for more details).

Finally, we must add that the spaces G/C, where G is a connected Lie group whose Lie algebra is a real form of a simple Lie algebra over G and C a maximal compact subgroup of G, determine symmetric spaces: a detailed study is offered in ref. [39].

1.72 Representation of a group

We know the explicit action of a group of transformations G on a set S if we know the representation of G on S. Actually, S will be in general a linear vector space on the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers (sometimes on the quaternions $\mathbb{H}$ or octonions $\mathbb{O}$) and we will be interested in the "linear representations" of G.

Definition

Let G be a group and $\mathcal{V}$ a linear vector space on the field $\mathbb{K}$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A linear representation of G in $\mathcal{V}$ is a homomorphism π of G into the group of the linear and invertible operators of $\mathcal{V}$, that is

$$\pi(g)\pi(g') = \pi(gg') \,, \qquad \forall\, g, g' \in G$$

In particular, $\pi(e) = \mathbb{I}_{\mathcal{V}}$ where e is the identity element of G and $\pi(g^{-1}) = \pi(g)^{-1}$.

The vector space $\mathcal{V}$ is the representation space. The dimension of the representation π is the dimension of the vector space $\mathcal{V}$.

Consider for example the orthogonal group in 3 dimensions. The three-dimensional representation of $O(3)$ is immediately given by the definition itself of $O(3)$ (group of 3×3 orthogonal matrices). But one can imagine that there exist representations of $O(3)$ on vector spaces of a different dimension: actually any irreducible representation of $O(3)$ is labelled by an integer j and is of dimension $2j+1$ ($\rightarrow$ 1.92), for the explicit form of the infinitesimal matrices of the representation j of $O(3)$).

Definition

The representation π of G on $\mathcal{V}$ is said to be:
– faithful if $\forall\, g \in \mathcal{G}$, $\pi(g) \neq \mathbb{I}_{\mathcal{V}}$ (identity on $\mathcal{V}$).
– trivial if $\forall\, g \in \mathcal{G}$, $\pi(g) = \mathbb{I}_{\mathcal{V}}$.

One can define an equivalence relation among representations as follows:

Definition

Let $\pi : \mathcal{V} \rightarrow \mathcal{V}$ and $\pi' : \mathcal{V}' \rightarrow \mathcal{V}'$ be two linear representations of G in the linear vector space $\mathcal{V}$ and $\mathcal{V}'$ respectively. π and π' are called *equivalent representations* if there exists a one-to-one linear mapping $A : \mathcal{V} \rightarrow \mathcal{V}'$ such that

$$A\, \pi(g)\, A^{-1} = \pi'(g) \,, \qquad \forall\, g \in G$$

If $\mathcal{V}$ and $\mathcal{V}'$ are Hilbert spaces and A unitary then π and π' will be called unitarily equivalent representations.

Let π be a linear representation of a Lie group G. One defines ($\forall\, g \in G$):

– the complex conjugate representation π^* by $\pi^*(g) = [\pi(g)]^*$,
– the transposed representation π^t by $\pi^t(g) = [\pi(g)]^t$,
– the hermitian conjugate representation $\pi^\dagger$ by $\pi^\dagger(g) = [\pi(g)]^\dagger \equiv [\pi^*(g)]^t$.

One can check that π^*, π^t and $\pi^\dagger$ are indeed representations.

Property

1. The representations π, π^*, π^t and $\pi^\dagger$ are simultaneously irreducible or reducible ($\to$ 1.74).

2. If the representation π is unitary, so are π^*, π^t and $\pi^\dagger$.

3. The representation π is real if and only if $\pi = \pi^*$.

4. If the representation π is real, then the character ($\to$ 1.12) $\chi(g)$ is real.

5. If $\chi(g)$ is real, then π and π^* are equivalent (there exists a matrix A such that $\pi^* = A\pi A^{-1}$). The representations π and π^* are then called self-conjugated. Moreover, if the representation π is unitary, the matrix A is either symmetric or antisymmetric.

Finally, let us note that a representation may be of finite or infinite dimension. However:

Theorem (Peter–Weyl)

Any unitary irreducible representation of a compact group is finite dimensional.

1.73 Representation of a Lie algebra

Definition

Let $\mathcal{G}$ be a Lie algebra over the field $\mathbb{K}$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $\mathcal{V}$ be a vector space on $\mathbb{K}$ and consider the algebra $\operatorname{End}\mathcal{V}$ of endomorphisms of $\mathcal{V}$. A linear representation π of $\mathcal{G}$ in $\mathcal{V}$ is a homomorphism of $\mathcal{G}$ into $\operatorname{End}\mathcal{V}$, that is for all $X, Y \in \mathcal{G}$, $\alpha \in \mathbb{K}$, and $[\ ,\ ]$ denoting the Lie bracket

$$\begin{aligned}
\pi(\alpha X) &= \alpha\pi(X) \\
\pi(X + Y) &= \pi(X) + \pi(Y) \\
\pi([X, Y]) &= [\pi(X), \pi(Y)]
\end{aligned}$$

The vector space $\mathcal{V}$ is the representation space. The dimension of the representation π is the dimension of the vector space $\mathcal{V}$.

As an example, $\to$ 1.2 Adjoint representation.

Definition

Let $\mathcal{G}$ be a Lie algebra over the field $\mathbb{K}$ and $\mathcal{V}$ a vector space on $\mathbb{K}$. $\mathcal{V}$ is called a *$\mathcal{G}$-module* if there exists an action of $\mathcal{G}$ on $\mathcal{V}$ such that, for all $X, Y \in \mathcal{G}$, $\alpha, \beta \in \mathbb{K}$ and $\vec{v}, \vec{w} \in \mathcal{V}$:

$$(\alpha X + \beta Y)(\vec{v}) = \alpha X(\vec{v}) + \beta Y(\vec{v})$$
$$X(\alpha \vec{v} + \beta \vec{w}) = \alpha X(\vec{v}) + \beta X(\vec{w})$$
$$[X, Y](\vec{v}) = X(Y(\vec{v})) - Y(X(\vec{v}))$$

Let π be a representation of the Lie algebra $\mathcal{G}$ with representation space $\mathcal{V}$. Then the vector space $\mathcal{V}$ has the structure of a $\mathcal{G}$-module by $X(\vec{v}) = \pi(X)\vec{v}$.

Theorem (Ado)

Every finite dimensional Lie algebra $\mathcal{G}$ over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ has a faithful finite dimensional representation.

1.74 Representation: reducibility

Definition

Let G be a Lie group and π a representation of G in the vector space $\mathcal{V}$. The part $\mathcal{V}' \subset \mathcal{V}$ is an invariant subspace of $\mathcal{V}$ under π if $\forall\, g \in G$

$$[\pi(g)](\mathcal{V}') \subset \mathcal{V}'$$

Let $\mathcal{V}$ be a finite dimensional representation space of G. Then $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$, $\mathcal{V}''$ being a complementary subspace of $\mathcal{V}'$ in $\mathcal{V}$, and the invariance of $\mathcal{V}'$ under π implies $\forall\, g \in G$:

$$\pi(g) = \left(\begin{array}{c|c} \pi^{(1)}(g) & \pi^{(12)}(g) \\ \hline 0 & \pi^{(2)}(g) \end{array}\right) \quad \text{acting on vectors} \quad \left(\begin{array}{c} \\ \hline \\ \end{array}\right) \begin{array}{l} \}\dim \mathcal{V}' \\ \}\dim \mathcal{V}'' \end{array}$$

It follows that the restriction $\pi^{(1)}$ of π to $\mathcal{V}'$ as well as the restriction $\pi^{(2)}$ of π to a complementary subspace of $\mathcal{V}'$ in $\mathcal{V}$ are themselves representations of G.

Definition

The representation π of G in $\mathcal{V}$ is called *irreducible* if there are no invariant subspaces except trivial ones. Otherwise the representation π is said *reducible*.

Definition

The representation π of G in $\mathcal{V}$ is called *completely reducible* if for any invariant subspace, there exists a complementary subspace which is itself invariant. A representation which is reducible but not completely reducible is called *indecomposable*.

In this case, the representation π can be decomposed as follows:

$$\pi(g) = \begin{pmatrix} \pi^{(1)}(g) & & 0 \\ & \ddots & \\ 0 & & \pi^{(k)}(g) \end{pmatrix}$$

the representations $\pi^{(1)}, \dots, \pi^{(k)}$ of G, acting on $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(k)}$ with $\mathcal{V} = \mathcal{V}^{(1)} \oplus \dots \oplus \mathcal{V}^{(k)}$, being irreducible.

Definition

The same definitions hold for the representations of Lie algebras. Moreover, if a representation π of a Lie algebra $\mathcal{G}$ with representation space $\mathcal{V}$ is irreducible (resp. completely reducible), the $\mathcal{G}$-module $\mathcal{V}$ is called a *simple* (resp. *semi-simple*) module.

Now we have the following important theorem:

Theorem

Any unitary representation U of G in a Hilbert space $\mathcal{H}$ is completely reducible.

Let $\mathcal{V}$ be a $\mathcal{H}$ subspace invariant by the unitary representation U, and $\mathcal{V}^\perp$ the supplementary subspace. Then for all $x \in \mathcal{V}$, $y \in \mathcal{V}^\perp$, $g \in G$, one has $(U(g)x, y) = 0 = (x, U(g^{-1})y)$, that is $[U(g)]\mathcal{V}^\perp \subset \mathcal{V}^\perp$ for all $g \in G$.

Corollary

Any representation π of a compact group G in a linear vector space $\mathcal{V}$ is completely reducible.

Two important cases for the reduction of the representation of a compact group G in physics are the following:

1. Let π be an irreducible representation of G and S a subgroup of G. π is therefore a representation of S but in general reducible under S.

2. Let π and π' be two irreducible representations of G. Then the tensor product $\pi \otimes \pi'$ is a representation of G which is in general reducible under G.

Let us conclude this entry by emphasizing the properties of a linear representation of a compact group: there always exists a scalar product making the representation unitary. This unitary representation is completely reducible and finite dimensional ($\rightarrow$ 1.72). Note that these properties are valid for finite groups which can be seen as particular cases of compact groups. Actually, it appears that most of the properties on representations of finite groups are valid for compact Lie groups.

1.75 Representation: sums and products

Sum and products: general definitions

Let us first recall that if $\mathcal{V}$ and $\mathcal{V}'$ are two linear vector spaces on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ of respective dimensions n and n' with $\{\vec{e}_1, \dots, \vec{e}_n\}$ basis of $\mathcal{V}$ and $\{\vec{e}_{n+1}, \dots, \vec{e}_{n+n'}\}$ basis of $\mathcal{V}'$, the direct sum of $\mathcal{V}$ and $\mathcal{V}'$ is the linear vector space $\mathcal{V} \oplus \mathcal{V}'$ of dimension $n + n'$ such that any $\vec{v} \in \mathcal{V} \oplus \mathcal{V}'$ can be written as $\vec{v} = \sum_{i=1}^{n+n'} v^i \vec{e}_i$ where $v^i \in \mathbb{K}$, and the tensor product of $\mathcal{V}$ and $\mathcal{V}'$ is the linear vector space $\mathcal{V} \otimes \mathcal{V}'$ of dimension nn' such that any $\vec{v} \in \mathcal{V} \otimes \mathcal{V}'$ can be written as $\vec{v} = \sum_{i,j=1}^{nn'} v^{ij} \vec{e}_i \otimes \vec{e}_j$ where $v^{ij} \in \mathbb{K}$.

We can then give the definitions:

Definition

Let π and π' be two representations of G with representation spaces $\mathcal{V}$ and $\mathcal{V}'$. One defines the direct sum representation $\pi \oplus \pi'$ with representation space $\mathcal{V} \oplus \mathcal{V}'$, the action of which on $\mathcal{V} \oplus \mathcal{V}'$ being given by, for $g \in G$, $\vec{v} \in \mathcal{V}$ and $\vec{v}\,' \in \mathcal{V}'$:

$$(\pi \oplus \pi')(g)\vec{v} \oplus \vec{v}\,' = \pi(g)\vec{v} \;\oplus\; \pi'(g)\vec{v}\,'$$

that is

$$(\pi \oplus \pi')(g) : \begin{pmatrix} \vec{v} \\ \vec{v}\,' \end{pmatrix} \mapsto \begin{pmatrix} \pi(g) & 0 \\ 0 & \pi'(g) \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{v}\,' \end{pmatrix} = \begin{pmatrix} \pi(g)\vec{v} \\ \pi'(g)\vec{v}\,' \end{pmatrix}$$

Definition

Let π and π' be two representations of G with representation spaces $\mathcal{V}$ and $\mathcal{V}'$. One defines the tensor product representation $\pi \otimes \pi'$ with representation space $\mathcal{V} \otimes \mathcal{V}'$, the action of which on $\mathcal{V} \otimes \mathcal{V}'$ being given by, for $g \in G$, $\vec{v} \in \mathcal{V}$ and $\vec{v}\,' \in \mathcal{V}'$:

$$(\pi \otimes \pi')(g)\vec{v} \otimes \vec{v}\,' = \pi(g)\vec{v} \;\otimes\; \pi'(g)\vec{v}\,'$$

The tensor product of representations is associative. It follows that if π, π' and π'' are three representations of the Lie group G, the tensor products $(\pi \otimes \pi') \otimes \pi''$ and $\pi \otimes (\pi' \otimes \pi'')$ are isomorphic. For fixed bases in each of the representations π, π', π'', the entries of the invertible matrix which relates the two products $(\pi \otimes \pi') \otimes \pi''$ and $\pi \otimes (\pi' \otimes \pi'')$ are called the *Wigner coefficients* or *6j-symbols* ($\rightarrow$ 1.70).

The same definitions hold for the representations of Lie algebras.

If $\pi(g) = e^X$ and $\pi'(g) = e^{X'}$ with X and X' representing the same element in the Lie algebra $\mathcal{G}$ of G for the two representations π and π', then for all

$\vec{v} \in \mathcal{V}$ and $\vec{v}\,' \in \mathcal{V}'$

$$\begin{aligned}\vec{v} \otimes \vec{v}\,' \mapsto (e^X \vec{v}) \otimes (e^{X'} \vec{v}\,') &= [(\mathbb{I} + X + \ldots)\vec{v}] \otimes [(\mathbb{I} + X' + \ldots)\vec{v}\,'] \\ &= [\mathbb{I}_{\mathcal{V}} \otimes \mathbb{I}_{\mathcal{V}'} + (X \otimes \mathbb{I}_{\mathcal{V}'} + \mathbb{I}_{\mathcal{V}} \otimes X') + \ldots]\vec{v} \otimes \vec{v}\,'\end{aligned}$$

from which we deduce that

$$(\pi \otimes \pi')(g) = \exp(X \otimes \mathbb{I}_{\mathcal{V}'} + \mathbb{I}_{\mathcal{V}} \otimes X')$$

More generally:

Definition

Let $G = H \times K$ be the direct product of the groups H and K, and $\pi^{(H)}$ and $\pi^{(K)}$ be representations of H and K in $\mathcal{V}_H$ and $\mathcal{V}_K$ respectively. The Kronecker product $\pi^{(H)} \otimes \pi^{(K)}$ of the representations $\pi^{(H)}$ and $\pi^{(K)}$ is the representation such that for $h \in H$ and $k \in K$:

$$(\pi^{(H)} \otimes \pi^{(K)})(h, k) = \pi^{(H)}(h) \otimes \pi^{(K)}(k)$$

One can check that the applications $\pi \oplus \pi'$, $\pi \otimes \pi'$ and $\pi^{(H)} \otimes \pi^{(K)}$ defined above are representations.

The same definition holds for Lie algebras, replacing the direct product of groups by direct sums of algebras.

Example

Let G be the group $SU(3) \times SU(2)$ and let us call (u, d, s) a basis of the three-dimensional fundamental representation space $\mathcal{V}$ of $SU(3)$ and $(\uparrow, \downarrow)$ a basis of the two-dimensional fundamental representation space $\mathcal{V}'$ of $SU(2)$. The direct product $\mathcal{V} \otimes \mathcal{V}'$ contains three quarks u, d, s with their spin states $(\uparrow, \downarrow)$: $u^\uparrow, d^\uparrow, s^\uparrow, u^\downarrow, d^\downarrow, s^\downarrow$. The generators of $SU(3)$ and $SU(2)$ being denoted by λ_i $(i = 1, \ldots, 8)$ and σ_j $(j = 1, 2, 3)$ respectively, we can define the tensor product of representations $\pi \otimes \pi'$ $\forall\, g \in G$ by

$$\begin{aligned}(\pi \otimes \pi')(g) q \otimes s &= \exp(i\alpha^i \lambda_i) q \otimes \exp(i\beta^j \sigma_j) s \\ &= \exp(i(\alpha^i \lambda_i \otimes \mathbb{I}_2 + \mathbb{I}_3 \otimes \beta^j \sigma_j)) q \otimes s\end{aligned}$$

Actually, one can embed $SU(3) \times SU(2)$ into the group $SU(6)$ by adding to the eleven generators $\lambda_i \otimes \mathbb{I}_2$ and $\mathbb{I}_3 \otimes \sigma_j$ the 24 generators $\lambda_i \otimes \sigma_j$. Notice that this 6-dimensional representation of $SU(6)$ will reduce with respect to this subgroup $SU(3) \times SU(2)$ as $6 = (3, 2)$. □

Symmetric and antisymmetric products

Very often it is useful to know the symmetric and antisymmetric spaces contained in the tensor product of k irreducible representations π of a compact Lie group G. A useful tool to solve the problem is the knowledge of the formula which connects the character ($\rightarrow$ 1.12), denoted by $\widehat{\chi}_k$, of the (generally reducible) symmetric representation π_S (resp. antisymmetric π_A) ($\pi_S, \pi_A \subset \otimes^k \pi$), in terms of the character χ of the irreducible representation π of the compact Lie group G. These formulae can be obtained from the character theory of the symmetric group $\mathfrak{S}_k$ [57]. The general formulae are ($g \in G$):

– for the symmetric representation:

$$\widehat{\chi}_k(g) = \sum \frac{\chi^{q_1}(g^{p_1})\,\chi^{q_2}(g^{p_2})\ldots\chi^{q_j}(g^{p_j})}{q_1!\,p_1^{q_1}\;q_2!\,p_2^{q_2}\ldots q_j!\,p_j^{q_j}}$$

where the sum is over all possible different partitions of k

$$k = q_1p_1 \;+\; q_2p_2 \;+\; \ldots + \; q_jp_j$$

– for the antisymmetric representation:

$$\widehat{\chi}_k(g) = \sum (-1)^{q_1+q_2\ldots+q_j-k}\, \frac{\chi^{q_1}(g^{p_1})\,\chi^{q_2}(g^{p_2})\ldots\chi^{q_j}(g^{p_j})}{q_1!\,p_1^{q_1}\;q_2!\,p_2^{q_2}\ldots q_j!\,p_j^{q_j}}$$

In particular for $k = 2, 3, 4$ we obtain:

– for the symmetric representation:

$$\begin{aligned}
\widehat{\chi}_2(g) &= \tfrac{1}{2}\,\chi(g^2) + \tfrac{1}{2}\,\chi^2(g) \\
\widehat{\chi}_3(g) &= \tfrac{1}{3}\,\chi(g^3) + \tfrac{1}{2}\,\chi(g^2)\chi(g) + \tfrac{1}{6}\,\chi^3(g) \\
\widehat{\chi}_4(g) &= \tfrac{1}{4}\,\chi(g^4) + \tfrac{1}{3}\,\chi(g^3)\chi(g) + \tfrac{1}{8}\,\chi^2(g^2) + \tfrac{1}{4}\,\chi^2(g)\chi(g^2) + \tfrac{1}{24}\,\chi^4(g)
\end{aligned}$$

– for the antisymmetric representation:

$$\begin{aligned}
\widehat{\chi}_2(g) &= \tfrac{1}{2}\,\chi^2(g) - \tfrac{1}{2}\,\chi(g^2) \\
\widehat{\chi}_3(g) &= \tfrac{1}{3}\,\chi(g^3) - \tfrac{1}{2}\,\chi(g^2)\chi(g) + \tfrac{1}{6}\,\chi^3(g) \\
\widehat{\chi}_4(g) &= -\tfrac{1}{4}\,\chi(g^4) + \tfrac{1}{3}\,\chi(g^3)\chi(g) + \tfrac{1}{8}\,\chi^2(g^2) - \tfrac{1}{4}\,\chi^2(g)\chi(g^2) + \tfrac{1}{24}\,\chi^4(g)
\end{aligned}$$

Choosing as element $g \in G$ the identity e, the above formulae give the dimension of the symmetric representation π_S, resp. antisymmetric representation π_A, contained in the k-fold tensor product of π. This knowledge is useful to obtain the decomposition of the symmetric, resp. antisymmetric, space into its irreducible subspaces. Recall that the highest irreducible representation contained in the tensor product, i.e. the irreducible representation labelled by the Dynkin labels sum of k times the Dynkin labels of π, is always symmetric.

Example

Find the symmetric irreducible representation contained in the square product of the irreducible representation $[2,1]$ of dimension 8 of $SU(3)$. Using the rules to perform the Kronecker product of two irreducible representations of $SU(n)$ ($\to$ 1.67) we obtain

$$[2,1] \otimes [2,1] = [4,2] \oplus [3,3] \oplus 2\,[2,1] \oplus [3] \oplus [0]$$

Using the above formula for $k = 2$, we find: $\dim\,([2,1] \otimes [2,1])_S = \frac{1}{2}\,8^2 + \frac{1}{2}\,8 = 36$. As the highest irreducible representation $[4,2]$ has dimension 27, it is immediately possible to conclude that

$$\begin{aligned} ([2,1] \otimes [2,1])_S &= [4,2] \oplus [2,1] \oplus [0] \\ ([2,1] \otimes [2,1])_A &= [3,3] \oplus [2,1] \oplus [3] \end{aligned}$$

1.76 Representation: unitarity

Definition

Let $\mathcal{H}$ be a Hilbert space. A unitary representation of a group G in $\mathcal{H}$ is a homomorphism U of G in the group of unitary operators of $\mathcal{H}$.

In particular, $U(e) = \mathbb{I}_{\mathcal{H}}$ and for all $g \in G$, $U(g^{-1}) = U(g)^{-1} = U(g)^\dagger$ from the definition of a unitary operator in $\mathcal{H}$: $(\,U(g)x\,,\,U(g)y\,) = (\,x\,,\,y\,)$ for all $x, y \in \mathcal{H}$ where $(\,.\,,\,.\,)$ is the scalar product in $\mathcal{H}$.

Theorem

For any linear representation π of a compact Lie group on $\mathcal{V}$, there exists a scalar product making the representation unitary.

Indeed, if $(\,.\,,\,.\,)$ is a scalar product in $\mathcal{V}$, then the hermitian form h such that, for all $\vec{v}, \vec{v}\,' \in \mathcal{V}$,

$$h(\vec{v}, \vec{v}\,') = \int_G (\,\pi(g)\vec{v}\,,\,\pi(g)\vec{v}\,'\,)\,d\mu(g)$$

where μ is the Haar measure, defines a new scalar product on $\mathcal{V}$ which is invariant with respect to π (that is $h(\vec{v}, \vec{v}\,') = h(\pi(g)\vec{v}\,\pi(g)\vec{v}\,')$ for all $g \in G$). Therefore π is unitary with the scalar product defined by h.

See for example ref. [73].

1.77 Roots, root system

Definition

Let $\mathcal{G}$ be a simple Lie algebra of dimension n and rank r. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ with basis of generators $\{H_1, \ldots, H_r\}$. From Cartan's theorem, one can complete the Cartan basis of $\mathcal{G}$ with $n - r$ generators E_α that are simultaneous eigenvectors of H_i with eigenvalues α^i:

$$\Big[H_i, E_\alpha\Big] = \alpha^i E_\alpha$$

The r-dimensional vector $\alpha = (\alpha^1, \ldots, \alpha^r)$ is called the *root (vector)* associated to the root generator E_α.

Hence the algebra $\mathcal{G}$ can be decomposed as follows:

$$\mathcal{G} = \bigoplus_\alpha \mathcal{G}_\alpha$$

where

$$\mathcal{G}_\alpha = \Big\{ x \in \mathcal{G} \,\Big|\, \Big[h, x\Big] = \alpha(h)\, x,\ h \in \mathcal{H} \Big\}$$

The set

$$\Delta = \Big\{ \alpha \in \mathcal{H}^* \,\Big|\, \mathcal{G}_\alpha \neq 0 \Big\}$$

is by definition the *root system* of $\mathcal{G}$.

Property

The roots have the following properties:

1. $\alpha \in \Delta \ \Rightarrow\ k\alpha \in \Delta$ if and only if $k = -1, 0, 1$.
2. If α and β are roots, then $2\,\dfrac{\alpha \cdot \beta}{\alpha \cdot \alpha}$ is an integer.
3. If α and β are roots, then $\gamma = \beta - 2\,\dfrac{\alpha \cdot \beta}{\alpha \cdot \alpha}\,\alpha$ is a root.

Two roots α and β being given, γ is the symmetric of β with respect to the hyperplane orthogonal to α. Such a geometrical transformation, transforming the root β into the root γ, is called a Weyl reflection ($\rightarrow$ 1.93). The Weyl reflections are particularly important to construct the root diagram of a Lie algebra $\mathcal{G}$.

As a consequence of the above properties, one can deduce the following result:

$$\frac{\alpha \cdot \beta}{\alpha \cdot \alpha} \in \tfrac{1}{2}\mathbb{Z} \ \Rightarrow\ \alpha \cdot \beta = \frac{m}{2}\,\alpha \cdot \alpha$$

$$\frac{\alpha \cdot \beta}{\alpha.\alpha} \in \tfrac{1}{2}\mathbb{Z} \;\Rightarrow\; \alpha \cdot \beta = \frac{n}{2}\, \beta \cdot \beta$$

where m, n are integers. It follows that

$$\cos^2 \theta_{\alpha\beta} = \frac{(\alpha \cdot \beta)^2}{(\alpha \cdot \alpha)(\beta \cdot \beta)} = \frac{mn}{4} \qquad \text{and} \qquad \frac{\alpha \cdot \alpha}{\beta \cdot \beta} = \frac{n}{m}$$

Choosing the angle $\theta_{\alpha\beta}$ between the roots α and β to be such that $0 \leq \theta_{\alpha\beta} \leq 90^\circ$ (for $\theta_{\alpha\beta} > 90^\circ$ change α and/or β in $-\alpha$ and/or $-\beta$) we can draw the following tableau where all the possibilities are present:

$\cos^2 \theta_{\alpha\beta}$	1	3/4	1/2	1/4	0
$\theta_{\alpha\beta}$	0°	30°	45°	60°	90°
α^2/β^2	1	3	2	1	undetermined

A simple Lie algebra for which all roots have the same length is called *simply-laced.*

Root diagram – Root space

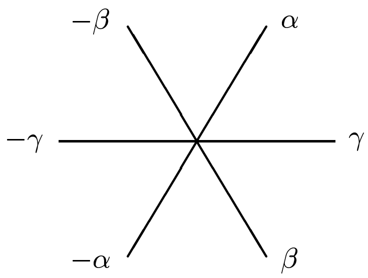
The vector space spanned by all the possible roots is called the root space. It is the dual $\mathcal{H}^*$ of the Cartan subalgebra $\mathcal{H}$ as vector space. Owing to the above geometrical properties of the roots, the set of $n - r$ roots of the Lie algebra $\mathcal{G}$ of dimension n and rank r can be represented without difficulty – at least for small values of r – in the r-dimensional Euclidean space: such a representation will constitute the root diagram of $\mathcal{G}$. The different types of simple Lie algebras could then be selected and classified, that completes the Cartan classification already discussed above.

Examples

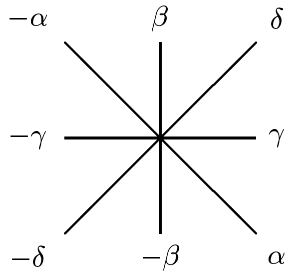
Case $r = 1$: The only possibility is $sl(2) = A_1$. The root space is one-dimensional and the Weyl hyperplane is an axis orthogonal to this axis.

Case $r = 2$: There are three possibilities corresponding to the Lie algebras $sl(3) = A_2$, $so(5) = B_2 \simeq sp(4) = C_2$ and G_2 (the corresponding root systems are drawn below). Let α and β be two roots.

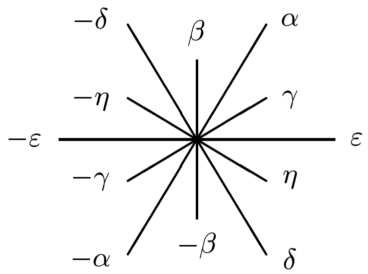
Consider first the situation where $\theta_{\alpha\beta} = 60^\circ$. Then $\alpha^2 = \beta^2$. By a Weyl reflection with respect to the axis orthogonal to α, we get the root γ. Then we construct the opposite roots $-\alpha, -\beta, -\gamma$. By Weyl reflections, no new root is obtained. We therefore get six roots, hence six root generators, which with the two Cartan generators form an eight-dimensional Lie algebra, the Lie algebra $sl(3)$.



Consider then the situation where $\theta_{\alpha\beta} = 45°$. Then one can choose $\alpha^2/\beta^2 = 2$ for instance. By a Weyl reflection with respect to the axis orthogonal to α, we get the root γ. Then by a Weyl reflection with respect to the axis orthogonal to β, we get the root δ. Then we construct the opposite roots $-\alpha, -\beta, -\gamma, -\delta$. By Weyl reflections, no new root is obtained. We therefore get eight roots, hence eight root generators, which with the two Cartan generators form a ten-dimensional Lie algebra, the Lie algebra $so(5) \simeq sp(4)$.



Consider now the situation where $\theta_{\alpha\beta} = 30°$ and one takes $\alpha^2/\beta^2 = 3$. By a Weyl reflection with respect to the axis orthogonal to α, we get the root γ. Then by a Weyl reflection with respect to the axis orthogonal to β, we get the root δ. Repeating the Weyl reflection with respect to the axis orthogonal to α, we get the root ϵ and finally the Weyl reflection with respect to the axis orthogonal to γ leads to the root η. Then we construct the opposite roots $-\alpha, -\beta, -\gamma, -\delta, -\epsilon, -\eta$. By Weyl reflections, no new root is obtained. We therefore get twelve roots, hence twelve root generators, which with the two Cartan generators form a fourteen-dimensional Lie algebra, the Lie algebra G_2.



Finally, the case $\theta_{\alpha\beta} = 90°$ leads to the semi-simple algebra $so(4) = A_1 \oplus A_1$ and the case $\theta_{\alpha\beta} = 0°$ is a special case: since α and β are proportional, we will not find there a (semi)simple Lie algebra. □

General case: In the same way one can construct the two-dimensional root spaces from the one-dimensional one, one can build the $(r+1)$-dimensional root spaces Δ_{r+1} from the r-dimensional ones Δ_r by:

– adding to each space Δ_r an additional vector such that it coincides with none of the vectors in Δ_r and the condition $2\,\dfrac{\alpha \cdot \beta}{\alpha \cdot \alpha} \in \mathbb{Z}$ is satisfied.
– completing then the space by Weyl reflections.
– if all vectors resulting from the completion still obey the condition $2\,\dfrac{\alpha \cdot \beta}{\alpha \cdot \alpha} \in \mathbb{Z}$, then we have obtained a new root space; if not, it is not.

Table 1.4: Root systems of the simple Lie algebras.

algebra $\mathcal{G}$	root system Δ	dim Δ
A_{N-1}	$\varepsilon_i - \varepsilon_j$	$N(N-1)$
B_N	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i$	$2N^2$
C_N	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i$	$2N^2$
D_N	$\pm\varepsilon_i \pm \varepsilon_j$	$2N(N-1)$
E_6	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$	72
E_7	$\pm\,\varepsilon_i \pm \varepsilon_j,\ \pm(\varepsilon_8 - \varepsilon_7),$ $\pm\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$	126
E_8	$\pm\varepsilon_i \pm \varepsilon_j,\ \frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)$	240
F_4	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$	48
G_2	$\varepsilon_i - \varepsilon_j,\ \pm(\varepsilon_i + \varepsilon_j) \mp 2\varepsilon_k$	12

For the algebras A_{N-1}, B_N, C_N, D_N, the indices $i \neq j$ run from 1 to N. For the algebras E_6, E_7, E_8, F_4, G_2, the indices $i \neq j$ run from 1 to 5, 6, 8, 4, 3 respectively, with in the case of G_2, the condition $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ and i, j, k is a permutation of (1,2,3). The total number of $+$ signs (or $-$ signs) is even in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)$ for E_8, $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$ for E_7, $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$ for E_6.

The description of the root space can be simplified owing to Dynkin diagrams. We have then to introduce the notions of positive and simple roots (→ 1.27, 1.83).

1.78 Schrödinger algebra

The Schrödinger equation [67] in 3+1 dimensions is left invariant under a group of space–time transformations which is bigger than the Galilei group ($\rightarrow$ 1.33). These extra transformations are a dilatation

$$\vec{r}\,' = \exp(\lambda)\,\vec{r} \quad \text{and} \quad t' = \exp(2\lambda)\,t$$

and a conformal transformation

$$\vec{r}\,' = \frac{\vec{r}}{1-\alpha t} \quad \text{and} \quad t' = \frac{t}{1-\alpha t}$$

where λ and α are arbitrary real parameters.

A basis of the Schrödinger algebra is therefore obtained by adding to the Galilei algebra two extra generators D and C satisfying the following non-zero commutation relations:

$$\begin{array}{lll} \big[D, P_j\big] = i\,P_j & \big[D, K_j\big] = i\,K_j & \big[D, P_0\big] = -2i\,P_0 \\ \big[C, P_j\big] = i\,K_j & \big[C, P_0\big] = -i\,D & \big[C, D\big] = -2i\,C \end{array}$$

It has to be noted that the three generators P_0, D, C form the $sl(2,\mathbb{R})$ algebra and naturally commute with the rotation algebra generated by the J_i ($i = 1,2,3$).

The Schrödinger algebra acting on the 3+1 dimensional space–time is therefore twelve dimensional. It admits, as the Galilei algebra, a central extension. The extended Schrödinger algebra is obtained by adding to the eleven-dimensional extended Galilei algebra the two generators D and C.

It is of physical interest to remark [11] that the conformal algebra ($\rightarrow$ 1.17) associated with 3+1 dimensional space–time contains as a subalgebra not only the Galilei algebra in 2+1 dimensions, but also the Schrödinger algebra in 2+1 dimensions. Using the notations of section 1.17, one has the following correspondence:

$$\begin{array}{l} P_0 = \frac{1}{2}\,(M_{0'0} + M_{40} - M_{0'3} - M_{43}) = \frac{1}{2}\,(p_0 - p_3) \\ P_j = M_{0'j} + M_{4j} = p_j\,, \qquad K_j = -M_{0j} - M_{3j} \qquad (j = 1,2) \\ M = M_{0'0} + M_{40} + M_{0'3} + M_{43} = p_0 + p_3 \\ C = \frac{1}{2}\,(M_{0'0} - M_{40} + M_{0'3} - M_{43}) = \frac{1}{2}\,(k_0 + k_3) \\ D = M_{0'4} - M_{03}\,, \qquad J_3 = M_{12} \end{array}$$

where the $M_{\alpha\beta}$ ($\alpha, \beta = 0, 0', 1, 2, 3, 4$) are the generators of the $so(4,2)$ algebra and we have denoted by $p_\mu = M_{0'\mu} + M_{4\mu}$ and $k_\mu = M_{0'\mu} - M_{4\mu}$ the translations and special conformal transformations of section 1.17. One has as expected the following non-zero commutation relations ($j, k = 1, 2$):

$$\Big[J_3, P_j\Big] = i\,\varepsilon_{3jk}\,P_k \qquad \Big[J_3, K_j\Big] = i\,\varepsilon_{3jk}\,K_j$$
$$\Big[P_0, K_j\Big] = -i\,P_j \qquad \Big[K_j, P_k\Big] = i\,\delta_{jk}\,M$$

1.79 Schur function

Schur functions (S-functions) are special functions of the roots of the matrices characterizing the classical groups, which allow the study of the weights of the irreducible representations of this class of groups. Before introducing the S-functions we need the following definitions:

Definition

A set (λ) of l integer numbers λ_i ordered from greatest to smallest is called an *ordered partition* of order l of the integer number $N = \sum_i^l \lambda_i$. A Young tableau ($\rightarrow$ 1.96) can be associated to any ordered partition.

Definition

A *symmetric function* of n variables x_i is a function left unchanged by any permutation of the variables x_i.

Definition

Given a partition (λ) of N, the associated *Schur function*, denoted usually by $\{\lambda\}$, is a symmetric function of n variables ($N \leq n$) defined as the ratio of two determinants:

$$\{\lambda\} = \frac{\det\left|x_j^{\lambda_i+n-i}\right|}{\det\left|x_j^{n-i}\right|}$$

where (x_j^{k-i}) ($k \geq n$) is an $n \times n$ matrix whose ij-entry is x_j^{k-i} and $\lambda_i = 0$ for $i \geq l$.

Note that the determinant in the denominator of the above equation is just the Vandermonde determinant which can also be written as

$$\det\left|x_j^{n-i}\right| = \prod_{1\leq i\leq j\leq n} (x_i - x_j)$$

To give an alternative expression of the Schur function, we introduce the definition of a symmetric monomial function which is associated to any partition $\{\lambda\}$ by the formula

$$S_{(\lambda)} = \sum_{P(\{m_j\})} x_{m_1}^{\lambda_1} \dots x_{m_l}^{\lambda_l}$$

where the sum is over the different permutations of x_i and $m_j \in \{1, 2, \dots, n\}$, $j = 1, 2, \dots, l$.

A homogeneous polynomial symmetric function of degree N, denoted p_N, is the sum of all the symmetric monomial functions associated with the partitions of N

$$p_N = \sum_{\{\lambda\}} S_{(\lambda)}$$

where the sum is over all the partitions of N.

For example, for $n = 4$, we have one monomial symmetric function for $N = 1$

$$S_{(1)} \; = \; x_1 + x_2 + x_3 + x_4$$

for $N = 2$ there are two monomial symmetric functions

$$\begin{aligned} S_{(11)} &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ S_{(2)} &= x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{aligned}$$

and for $N = 3$ one finds three monomial symmetric functions

$$\begin{aligned} S_{(111)} &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\ S_{(21)} &= x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_1 + x_2^2x_3 + x_2^2x_4 \\ &\quad + x_3^2x_1 + x_3^2x_2 + x_3^2x_4 + x_4^2x_1 + x_4^2x_2 + x_4^2x_3 \\ S_{(3)} &= x_1^3 + x_2^3 + x_3^3 + x_4^3 \end{aligned}$$

The Schur function can be rewritten in the form

$$\{\lambda\} = \det |p_{\lambda_i - i + j}|$$

where i, j are respectively the row and column indices and it is assumed that $p_0 = 1$ and $p_q = 0$ for $q < 0$. So the order of the matrix is given by the order l of the partition. For instance, the S-function associated to the partition $\{221\}$ is

$$\{221\} = \begin{pmatrix} p_2 & p_3 & p_4 \\ p_1 & p_2 & p_3 \\ p_{-1} & p_0 & p_1 \end{pmatrix} = p_1p_2^2 + p_1p_4 - p_1^2p_3 - p_2p_3$$

One can show that

$$\{N, 0, \ldots, 0\} = p_N \quad \text{and} \quad \{1, 1, \ldots, 1\} = x_1 x_2 \ldots x_N = S_{11\ldots1}$$

Littlewood has shown that there is a complete equivalence between the above defined S-function and a function which can be defined on the symmetric group $\mathfrak{S}_N$ and which characterizes the irreducible representation of $\mathfrak{S}_N$. From the connection between S_N and $GL(N)$, many properties of the classical Lie groups can be expressed in terms of the S-function. It is possible to define inner and outer multiplication and division of S-function. Then characters (→ 1.12), Kronecker products of representations (→ 1.96), and branching rules (→ 1.5) of classical groups can be expressed by use of S-functions. We refer to refs. [55, 98, 99] for detailed discussion and for reference to the original works.

Example

As an illustration we derive a formula for the dimension of the irreducible representations of $sl(N)$ labelled by the Young tableau $[\lambda]$ or highest weight Λ. The Weyl character formula (→ 1.12) reads

$$\chi(\Lambda) = \frac{\displaystyle\sum_{w \in W} \varepsilon(w)\, e^{w(\Lambda+\rho)}}{\displaystyle\sum_{w \in W} \varepsilon(w)\, e^{w(\rho)}}$$

where the sum is over all the elements w of the Weyl group W (→ 1.93), $\varepsilon(w)$ is the parity of w, and ρ is the Weyl vector, half-sum of the positive roots. Writing the roots of $sl(N)$ in the orthogonal basis $\{\varepsilon_i\}$ (→ Table 3.1), the Weyl group becomes the permutation group $\mathfrak{S}_N$ acting on ε_i. Introducing the formal variables

$$x_i = e^{\varepsilon_i} \;\text{ with }\; \prod_i^N x_i = 1$$

using the following expression for the determinant

$$\det \left| x_j^{k-i} \right| = \sum_P \varepsilon(P) \prod_{i=1}^n x_{e_i}^{k-i}$$

where the sum is over all the permutations and the set $\{e_1, \ldots, e_n\}$ is obtained from $\{1, 2, \ldots, n\}$ by action of the permutation P of parity $\varepsilon(P)$, the Weyl character formula can be written in the form of a Schur function (as in the definition above).

Performing the limit $x_i \to 1$ in the formula we finally get

$$\dim \pi(\lambda) = \prod_{1 \le i \le j \le n} \frac{(\lambda_j - \lambda_i + j - i)}{(j - i)}$$

the λ_i being the components of the Young tableau $[\lambda]$. □

1.80 Schur's Lemma

Lemma (Schur)

Let π be an irreducible representation of a group G in a complex linear vector space $\mathcal{V}$. Let

$$\mathcal{C}(\pi) = \Big\{\phi : \mathcal{V} \to \mathcal{V} \Big| \Big[\pi(g), \phi\Big] = 0\,,\ \forall\, g \in G\Big\}$$

where $\phi \in \operatorname{End} \mathcal{V}$. Then $\mathcal{C}(\pi)$ is a multiple of the identity operator $\mathbb{I}$.

This lemma is specially important. In particular it allows us to deduce:

Properties

1. Any irreducible complex representation of an Abelian group G is unidimensional.
   [the proof is straightforward: for a given $g_0 \in G$, $[\pi(g), \pi(g_0)] = 0$ $\forall\, g \in G$ since G is Abelian and therefore $\pi(g_0) = \lambda_{g_0}\mathbb{I}$, $\lambda_{g_0} \in \mathbb{C}$, if π is irreducible.]

2. Let π be an irreducible complex representation of a Lie group G. Then all Casimir invariants ($\to$ 1.9) are multiple of the identity (or even zero).

Finally, the Schur's Lemma has a converse theorem which gives a criterion for the irreducibility of an unitary representation.

Theorem (converse of the Schur's Lemma)

Let U be an unitary representation of the group G in a complex vector space $\mathcal{V}$. If for any operator ϕ such that $\forall\, g \in G$, $\Big[U(g), \phi\Big] = 0$, ϕ is proportional to the identity operator, then U is irreducible.

Let us recall that for any linear representation of a compact Lie group on a vector space, there exists a scalar product making the representation unitary ($\to$ 1.76).

1.81 Serre–Chevalley basis

The Serre–Chevalley presentation of a Lie algebra consists to describe the algebra in terms of simple root and Cartan generators, the only parameters being the entries of the Cartan matrix of the algebra.

Let $\mathcal{G}$ be a Lie algebra of rank r with Cartan subalgebra $\mathcal{H} = \{h_1, \dots, h_r\}$ and simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ and denote by $e_i^{\pm}$ $(1 \le i \le r)$ the corresponding simple root generators. Let $A = (A_{ij})$ be the Cartan matrix of $\mathcal{G}$. The Serre–Chevalley basis of $\mathcal{G}$ is the basis in which the commutation relations take the form

$$\Big[h_i, h_j\Big] = 0 \qquad \Big[h_i, e_j^{\pm}\Big] = \pm A_{ij}\, e_j^{\pm} \qquad \Big[e_i^{+}, e_j^{-}\Big] = \delta_{ij}\, h_i$$

The remaining commutation relations are given by the so-called Serre relations:

$$\big(\mathrm{ad}\, e_i^{\pm}\big)^{1-A_{ij}}\, e_j^{\pm} = \sum_{n=0}^{1-A_{ij}} (-1)^n (e_i^{\pm})^{1-A_{ij}-n} e_j^{\pm} (e_i^{\pm})^n = 0$$

Example

Consider the Lie algebra $sl(3)$ with simple root system $\{\alpha_1, \alpha_2\}$ and simple root generators $e_1^{\pm}, e_2^{\pm}$. The Serre relations read as

$$\Big[e_1^{\pm}, \Big[e_1^{\pm}, e_2^{\pm}\Big]\Big] = 0 \quad \text{and} \quad \Big[e_2^{\pm}, \Big[e_2^{\pm}, e_1^{\pm}\Big]\Big] = 0$$

It follows that $\pm(\alpha_1 + \alpha_2)$ are roots of height 2 (since $[e_1^{\pm}, e_2^{\pm}] \neq 0$) but $\pm(2\alpha_1 + \alpha_2)$ and $\pm(\alpha_1 + 2\alpha_2)$ are not roots. Thus, the root system of $sl(3)$ is given by $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$ (for the notion of height $\rightarrow$ 1.77). □

Relation between the Serre–Chevalley and the Cartan–Weyl bases

Let Δ be the root system of $\mathcal{G}$. The commutation relations among the generators of $\mathcal{G}$ in the Cartan–Weyl basis read as

$$\begin{aligned}
&\Big[H_i, H_j\Big] = 0 \\
&\Big[H_i, E_\alpha\Big] = \alpha^i E_\alpha \\
&\Big[E_\alpha, E_{-\alpha}\Big] = \sum_{i=1}^{r} \alpha^i H_i \\
&\Big[E_\alpha, E_\beta\Big] = N_{\alpha\beta}\, E_{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a non-zero root}
\end{aligned}$$

Restricting oneself to the simple roots $\{\alpha_1, \dots, \alpha_r\}$ of $\mathcal{G}$, one gets

$$\begin{aligned}
\Big[H_i, H_j\Big] &= 0 \\
\Big[H_i, E_{\alpha_j}\Big] &= \alpha_j^i \, E_{\alpha_j} \\
\Big[E_{\alpha_i}, E_{-\alpha_j}\Big] &= \alpha_j \cdot H \, \delta_{ij}
\end{aligned}$$

The relation between the two bases is then given by

$$e_i^{\pm} = \sqrt{\frac{2}{\alpha_i^2}}\, E_{\pm\alpha_i} \quad \text{and} \quad h_i = \frac{2}{\alpha_i^2}\, \alpha_i \cdot H = \frac{2}{\alpha_i^2} \sum_{j=1}^{r} \alpha_i^j H_j$$

1.82 Simple and semi-simple Lie groups and Lie algebras

Definition

The group G is a *simple* group if it does not contain any invariant subgroups other than the group G itself and the identity element $\{e\}$.

Definition

The group G is a *semi-simple* group if it does not contain any Abelian invariant subgroups except the identity element $\{e\}$.

As an example, the group $SO(3)$ is simple and the group $SO(4)$ is semi-simple. The group $SU(3)$ is not simple since its center is Z_3 – group with three elements, the three cubic roots of unity – but $SU(3)/Z_3$ is simple.

Since to an invariant Lie subgroup H of a group G there corresponds in the Lie algebra $\mathcal{G}$ an ideal (or invariant subalgebra) $\mathcal{H}$, one can state:

Definition

The Lie algebra $\mathcal{G}$ is a *simple* algebra if it does not contain any ideal other than $\mathcal{G}$ itself and $\{0\}$.

Definition

The Lie algebra $\mathcal{G}$ is a *semi-simple* algebra if it does not contain any Abelian ideal except $\{0\}$.

An important property is the following:

Theorem

The Lie algebra $\mathcal{G}$ is a semi-simple Lie algebra if and only if it is a direct sum of ideals that are simple as Lie algebras.

1.83 Simple root systems

Definition

Let $\mathcal{G}$ be a simple Lie algebra with Cartan subalgebra $\mathcal{H}$. Then $\mathcal{G}$ admits subalgebras $\mathcal{N}^+$ and $\mathcal{N}^-$ such that $\left[\mathcal{H},\mathcal{N}^+\right] \subset \mathcal{N}^+$ and $\left[\mathcal{H},\mathcal{N}^-\right] \subset \mathcal{N}^-$ with $\dim \mathcal{N}^+ = \dim \mathcal{N}^-$. The decomposition $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ is called a Borel decomposition. The subalgebras $\mathcal{B} = \mathcal{H} \oplus \mathcal{N}^\pm$ are called *Borel subalgebras* of $\mathcal{G}$ (that is maximal solvable subalgebras of $\mathcal{G}$).

Example

Consider the Lie algebra $sl(3)$. A basis of generators is given by the Gell-Mann matrices λ_i with $i = 1, \dots, 8$ ($\rightarrow$ 1.90). A Cartan subalgebra $\mathcal{H}$ is generated by λ_3 and λ_8. The subalgebras $\mathcal{N}^+$ and $\mathcal{N}^-$ can be chosen as $\mathcal{N}^+ = \{\lambda_1+i\lambda_2, \lambda_4+i\lambda_5, \lambda_6+i\lambda_7\}$ and $\mathcal{N}^- = \{\lambda_1-i\lambda_2, \lambda_4-i\lambda_5, \lambda_6-i\lambda_7\}$.
□

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r with Cartan subalgebra $\mathcal{H}$ and root system Δ. Denote by $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ a Borel decomposition of $\mathcal{G}$. The root decomposition of $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha\in\Delta} \mathcal{G}_\alpha$$

A root α is called *positive* if $\mathcal{G}_\alpha \cap \mathcal{N}^+ \neq \emptyset$ and negative if $\mathcal{G}_\alpha \cap \mathcal{N}^- \neq \emptyset$. A root is called *simple* if it cannot be decomposed into a sum of positive roots. The set of all simple roots is called a *simple root system* of $\mathcal{G}$ and is denoted here by Δ^0. The number of simple roots of $\mathcal{G}$ is equal to the rank r.

Example

Consider the Lie algebra $sl(3)$. A Cartan subalgebra $\mathcal{H}$ being generated by λ_3 and λ_8, the root generators can be taken as

$$E_{\pm\alpha} = \lambda_1 \pm i\lambda_2\,, \qquad E_{\pm\beta} = \lambda_4 \pm i\lambda_5\,, \qquad E_{\pm\gamma} = \lambda_6 \pm i\lambda_7$$

The set of positive roots is $\Delta^+ = \{\alpha, \beta, \gamma\}$. A system of simple roots is given by $\Delta^0 = \{\alpha, \beta\}$ and $\gamma = \alpha + \beta$. □

Let us remark that the choice of the Borel decomposition is far from being unique. However, in the case of the simple Lie algebras, all Borel subalgebras are conjugate and hence all simple root systems are equivalent. More precisely, one can get a simple root system Δ'^0 from another one Δ^0 by means of Weyl transformations on the simple roots of Δ^0.

$\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ being a simple root system of $\mathcal{G}$, each root $\alpha \in \Delta$ can be decomposed uniquely in terms of the simple roots α_i:

$$\alpha = \sum_{i=1}^{r} \alpha^i \alpha_i$$

where all the α^i are positive (resp. negative) integers if α is a positive (resp. negative) root. The quantity $h_\alpha = \sum_{i=1}^{r} \alpha^i$ is called the *height* of the root α (obviously, the roots of height 1 are the simple roots).

When the Lie algebra $\mathcal{G}$ is not simply-laced ($\rightarrow$ 1.77), it is useful to define the simple coroots $\alpha_i^\vee$ as follows:

$$\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i \cdot \alpha_i)$$

$\Delta^{0\vee} = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ is called the simple coroot system.

When the Lie algebra is simply-laced, choosing the normalization of the roots such that their squared length is equal to 2, the roots and the coroots coincide so that $\Delta^{0\vee} = \Delta^0$.

Among the roots, there is a unique root of maximal height: it is called the *highest root*. Denoting the highest root by $-\alpha_0$, one has

$$-\alpha_0 = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee$$

The numbers a_i and $a_i^\vee$ are called respectively the *marks* or *Kac labels* and the *comarks* or *dual Kac labels*. There are usually written on the Dynkin diagram of $\mathcal{G}$ ($\rightarrow$ 1.27).

Note that the sum of the marks of a simple Lie algebra, that is the height of its highest root, is equal to the maximal exponent ($\rightarrow$ 1.9) of the algebra.

Example

In the Lie algebra $so(5)$, the eight non-zero roots are $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm\varepsilon_i$ $(i, j = 1, 2,\ i \neq j)$ where $(\varepsilon_1, \varepsilon_2)$ forms an orthonormal basis in the vector space $\mathbb{R}^2$ and $1 \leq i, j \leq 2$. The simple roots are $\alpha_1 = \varepsilon_1$ and $\alpha_2 = \varepsilon_2 - \varepsilon_1$. The positive roots are α_1 and α_2 of height 1, $\alpha_1 + \alpha_2 = \varepsilon_2$ of height 2, $2\alpha_1 + \alpha_2 = \varepsilon_1 + \varepsilon_2$ of height 3, which is the highest root. □

Table 1.5 provides a list of the simple root systems of the simple Lie algebras in terms of the orthonormal vectors ε_i.

Table 1.5: Simple root systems of the simple Lie algebras.

algebra $\mathcal{G}$	simple root system Δ^0
A_{N-1}	$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{N-1} - \varepsilon_N$
B_N	$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{N-1} - \varepsilon_N, \varepsilon_N$
C_N	$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{N-1} - \varepsilon_N, 2\varepsilon_N$
D_N	$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{N-1} - \varepsilon_N, \varepsilon_{N-1} + \varepsilon_N$
E_6	$\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j), \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \dots, \varepsilon_5 - \varepsilon_4$
E_7	$\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j), \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \dots, \varepsilon_6 - \varepsilon_5$
E_8	$\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j), \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \dots, \varepsilon_7 - \varepsilon_6$
F_4	$\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$
G_2	$\varepsilon_1 - \varepsilon_2, \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1$

1.84 Spinors

The representation $j = 1/2$ of the group $SU(2)$ is called a spinor representation and the elements of the two-dimensional representation space are called spinors under $SU(2)$. By extension, any representation $j = (2p+1)/2$ is called a spinor representation of $SU(2)$. Moreover, any representation of an algebra $so(2n)$ or $so(2n+1)$, the components of the highest weight of which are n half-integers, is also called a spinor representation. We recall that the representations $j = (2p+1)/2$ of the algebra $so(3)$ are not representations of the group $SO(3)$ but of its covering group $SU(2)$ ($\rightarrow$ 1.18). In the same way, the spinor representations of $so(n)$ ($\rightarrow$ 1.58) are not representations of the group $SO(n)$ but of its covering group called $Spin(n)$ for $n \geq 7$.

Now, let us consider the case of the Lorentz group. Its Lie algebra is also the Lie algebra of the group $SO(3,1)$, non-compact form of $SO(4)$, and also of the $SL(2,\mathbb{C})$ group, group of 2×2 complex matrices of determinant 1, generated by the six matrices $\sigma_j, i\sigma_j$ where the σ_j are the Pauli matrices. There are two fundamental representations, each of dimension 2, of $SL(2,\mathbb{C})$, denoted $D(1/2,0)$ and $D(0,1/2)$. The Dirac representation corresponds to the representation $D(1/2,0) \oplus D(0,1/2)$. From the four basic Dirac matrices γ_μ ($\mu = 0,1,2,3$) satisfying the Clifford algebra ($\rightarrow$ 2.10), one can form the spinor representation of the Lie algebra of the Lorentz group:

$$\sigma_{\mu\nu} = \frac{1}{4}\Big[\gamma_\mu, \gamma_\nu\Big]$$

The elements in the four-dimensional representation space are the Dirac spinors. Using the projectors $\frac{1}{2}(\mathbb{I} + \gamma_5)$ where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ one decomposes a four-component Dirac spinor into two two-component Weyl spinors on the subspaces of $D(1/2, 0)$ and $D(0, 1/2)$ respectively. Finally, if the γ matrices are chosen real, one obtains a Majorana representation in which the Majorana spinors are real. See for example ref. [94].

The spinors in the Lorentz group are treated in detail in section 2.47.

1.85 Structure constants

→ 1.52 Lie theorems.

1.86 Subalgebra – Subgroup

Definition

A subalgebra $\mathcal{A}'$ of an algebra $\mathcal{A}$ is a non-empty subset $\mathcal{A}' \subset \mathcal{A}$ which is an algebra with the two composition laws induced by $\mathcal{A}$.

$\mathcal{A}'$ will be called a proper subalgebra of $\mathcal{A}$ if $\mathcal{A}' \neq \mathcal{A}$ and $\mathcal{A}' \neq \{0\}$.

Theorem

The part $\mathcal{A}'$ of the real Lie algebra $\mathcal{A}$ is a Lie subalgebra of $\mathcal{A}$ if and only if

$$\alpha A + \beta B \in \mathcal{A}' \quad \text{and} \quad \Big[A, B\Big] \in \mathcal{A}' \qquad \forall\, \alpha, \beta \in \mathbb{R},\ \forall\, A, B \in \mathcal{A}'$$

Definition

A subgroup G' of a group G is a non-empty subset $G' \subset G$ which is a group with the composition law induced by G.

G' will be called a proper subgroup of G if $G' \neq G$ and $G' \neq \{e\}$ where $\{e\}$ is the identity element of G.

Theorem

The non-empty subset $G' \subset G$ of a group G is a subgroup of G if and only if, for all $x, y \in G'$, $xy^{-1} \in G'$.

One is often faced with the problem of symmetry breaking in physics: explicit symmetry breaking (for example the Zeeman effect) or spontaneous symmetry breaking (Higgs mechanism). Then the knowledge of the subgroups of a group is sometimes useful. If one is interested only in the Lie subgroups of a Lie group, the following theorem is specially important.

Theorem

Let G be a connected Lie group. Then:

1. If H is a Lie subgroup of G, its Lie algebra $\mathcal{H}$ is a subalgebra of the Lie algebra $\mathcal{G}$ of G.

2. Each subalgebra of $\mathcal{G}$ is the Lie algebra of exactly one connected Lie subgroup of G.

Consequence: Classifying all the subalgebras $\mathcal{H}$ of $\mathcal{G}$ is equivalent to classifying all the connected subgroups of the connected group G.

Of course, one is interested in the classification of the subgroups of a group up to a conjugation, or inner automorphism ($\rightarrow$ 1.4). One can add:

Property

The classification of the connected subgroups of the connected Lie group G up to a conjugation of G is equivalent to the classification of the Lie subalgebras of $\mathcal{G}$ up to an inner automorphism of $\mathcal{G}$.

General theorems on the classification of the simple Lie algebras have been given in ref. [17] ($\rightarrow$ 1.87). An explicit classification of the semi-simple Lie subalgebras of the simple Lie algebras is given for algebras up to rank 6 in ref. [56]. Tables of branching rules (that is reduction with respect to subalgebras) for representations of simple Lie algebras can be found in ref. [59].

1.87 Subalgebras: regular and singular subalgebras

Regular subalgebras

Definition

Let $\mathcal{G}$ be a simple Lie algebra, and consider its canonical root decomposition

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha$$

where $\mathcal{H}$ is the Cartan subalgebra $\mathcal{G}$ and Δ its corresponding root system. A subalgebra $\mathcal{G}'$ of $\mathcal{G}$ is called *regular* if $\mathcal{G}'$ has the root decomposition

$$\mathcal{G}' = \mathcal{H}' \oplus \bigoplus_{\alpha' \in \Delta'} \mathcal{G}'_{\alpha'}$$

where $\mathcal{H}' \subset \mathcal{H}$ and $\Delta \subset \Delta'$.

Property

Let $\mathcal{G}'$ be a regular subalgebra of $\mathcal{G}$. Then $\mathcal{G}'$ is semi-simple if and only if $\alpha \in \Delta' \Rightarrow -\alpha \in \Delta'$ and $\mathcal{H}'$ is the linear closure of Δ'.

The method for finding the regular semi-simple subalgebras of a given simple Lie algebra $\mathcal{G}$ is based on the use of the *extended* Dynkin diagram of $\mathcal{G}$ (→ 1.27). A simple root system Δ^0 of $\mathcal{G}$ and its associated Dynkin diagram being given, one defines the corresponding extended simple root system by $\widehat{\Delta}^0 = \Delta^0 \cup \{\alpha_0\}$ where $-\alpha_0$ is the highest root with respect to Δ^0, to which is associated the extended Dynkin diagram.

Deleting arbitrarily some dot(s) of the extended diagram will yield to some connected Dynkin diagram or a set of disjointed Dynkin diagrams corresponding to a regular semi-simple subalgebra of $\mathcal{G}$. Indeed, taking away one or more roots from $\widehat{\Delta}^0$, one is left with a set of independent roots which constitute the simple root system of a regular semi-simple subalgebra of $\mathcal{G}$. Then repeating the same operation on the obtained Dynkin diagrams – that is adjunction of a dot associated to the lowest root of a simple part and cancellation of one arbitrary dot (or two in the unitary case) – as many time as necessary, one obtains all the Dynkin diagrams associated with regular semi-simple subalgebras. One gets the maximal regular semi-simple subalgebras of the same rank r of $\mathcal{G}$ by carrying out only the first step. The other possible maximal regular subalgebras of $\mathcal{G}$ if they exist will be obtained by deleting only one dot in the non-extended Dynkin diagram of $\mathcal{G}$ and will be therefore of rank $r-1$.

Let us emphasize that for the exceptional Lie algebras, deleting one dot of the extended Dynkin diagram may not lead to a maximal regular subalgebra: in that case, the obtained algebra is contained in a maximal regular subalgebra reached by the deletion of another dot from the same extended Dynkin diagram. Denoting by $\widehat{\mathcal{G}}/\alpha_i$ the subalgebra of $\mathcal{G}$ obtained by deleting the simple root α_i, one has

$$\widehat{F}_4/\alpha_3 \subset \widehat{F}_4/\alpha_4 \qquad \widehat{E}_7/\alpha_3 \subset \widehat{E}_7/\alpha_1$$
$$\widehat{E}_8/\alpha_2 \subset \widehat{E}_8/\alpha_7 \qquad \widehat{E}_8/\alpha_3 \subset \widehat{E}_8/\alpha_6 \qquad \widehat{E}_8/\alpha_5 \subset \widehat{E}_8/\alpha_1$$

The regular semi-simple subalgebras of the simple Lie algebras A_{N-1}, B_N, C_N and D_N are of the following type:

algebra	regular subalgebra	dimension
A_{N-1}	$A_{k_1} \oplus \ldots \oplus A_{k_p}$	$\sum_{i=1}^{p}(k_i+1) = N$
B_N	$A_{k_1} \oplus \ldots \oplus A_{k_p} \oplus$ $D_{m_1} \oplus \ldots \oplus D_{m_q} \oplus B_r$	$\sum_{i=1}^{p}(k_i+1) + \sum_{i=1}^{q} m_i + r = N$

algebra	regular subalgebra	dimension
C_N	$A_{k_1} \oplus \ldots \oplus A_{k_p} \oplus$ $C_{n_1} \oplus \ldots \oplus C_{n_q}$	$\sum_{i=1}^{p}(k_i+1) + \sum_{i=1}^{q} n_i = N$
D_N	$A_{k_1} \oplus \ldots \oplus A_{k_p} \oplus$ $D_{m_1} \oplus \ldots \oplus D_{m_q}$	$\sum_{i=1}^{p}(k_i+1) + \sum_{i=1}^{q} m_i = N$

where $k_1 \geq \ldots \geq k_p \geq 0$, $l_1 \geq \ldots \geq l_q > 0$, $m_1 \geq \ldots \geq m_p > 1$ and $r \geq 0$.

Table 1.6 presents the list of the maximal regular semi-simple subalgebras of the simple Lie algebras.

Table 1.6: Maximal regular subalgebras of the simple Lie algebras

algebras	subalgebras				
A_{N-1}	$A_{i-1} \oplus A_{N-i-1}$				
B_N	$B_i \oplus D_{N-i}$	B_{N-1}			
C_N	$C_i \oplus C_{N-i}$	A_{N-1}			
D_N	$D_i \oplus D_{N-i}$	D_{N-1}	A_{N-1}		
E_6	$A_2 \oplus A_2 \oplus A_2$	$D_5 \oplus U(1)$	$A_5 \oplus A_1$		
E_7	$D_6 \oplus A_1$	$A_5 \oplus A_2$	$E_6 \oplus U(1)$	A_7	
E_8	$E_7 \oplus A_1$	$E_6 \oplus A_2$	$A_4 \oplus A_4$	D_8	A_8
F_4	$C_3 \oplus A_1$	$A_2 \oplus A_2$	B_4		
G_2	$A_1 \oplus A_1$	A_2			

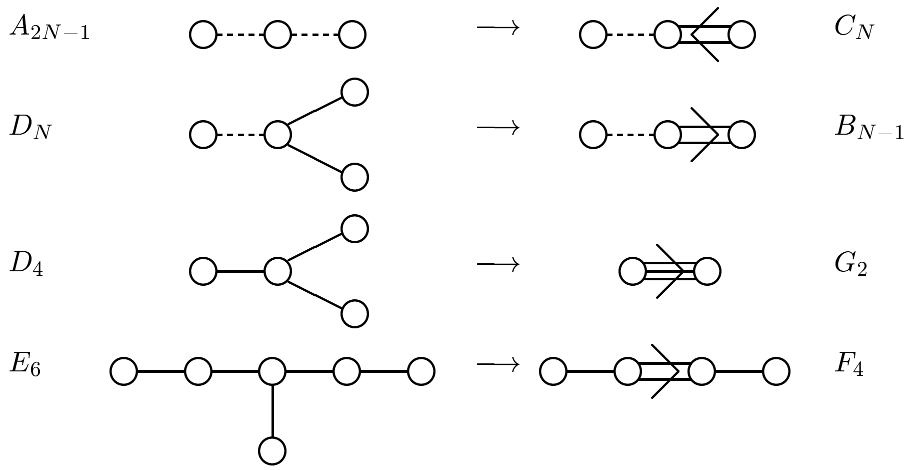
Singular subalgebras

Definition

Let $\mathcal{G}$ be a simple Lie algebra. A subalgebra $\mathcal{G}'$ of $\mathcal{G}$ is called *singular* if it is not regular.

The *folding method* allows us to obtain some singular subalgebras of the simple Lie algebras. Let $\mathcal{G}$ be a simple Lie algebra, with non-trivial outer automorphism, that is $\text{Out}(\mathcal{G})$ does not reduce to the identity. Then, the Dynkin diagram of $\mathcal{G}$ has a symmetry τ of order N ($\tau^N = 1$) given by $\text{Out}(\mathcal{G})$. This symmetry induces a direct construction of the subalgebra $\mathcal{G}'$

invariant under the $\mathcal{G}$ outer automorphism associated to τ: if the simple root α_i is transformed into $\tau(\alpha_i)$, then $\alpha_i + \tau(\alpha_i) + \ldots + \tau^{N-1}(\alpha_i)$ is τ-invariant since $\tau^N = 1$, and appears as a simple root of $\mathcal{G}'$ associated to the generator $E_{\alpha_i} + E_{\tau(\alpha_i)} + \ldots + E_{\tau^{N-1}(\alpha_i)}$, where $E_{\tau^k(\alpha_i)}$ is the generator corresponding to the root $\tau^k(\alpha_i)$ ($k = 0, \ldots, N-1$). A Dynkin diagram of $\mathcal{G}'$ will therefore be obtained by folding the $\mathbb{Z}_N$-symmetric Dynkin diagram of $\mathcal{G}$, that is by transforming each N-uple $(\alpha_i, \tau(\alpha_i), \ldots, \tau^{N-1}(\alpha_i))$ into the root $\alpha_i + \tau(\alpha_i) + \ldots + \tau^{N-1}(\alpha_i)$ of $\mathcal{G}'$. One obtains the following invariant subalgebras, which are singular by construction:



Moreover, suppose that a Lie algebra $\mathcal{G}$ has an N-dimensional representation π with highest weight Λ and invariant bilinear form B_π (→ 1.44). Then $\mathcal{G}$ can be embedded in $so(N)$ (resp. $sp(N)$) if and only if the bilinear form B_π is symmetric (resp. antisymmetric). If the representation π does not admit an invariant bilinear form, then $\mathcal{G}$ can be embedded into $sl(N)$. A necessary condition for a representation to admit an invariant form is to be real or self-conjugate (→ 1.92). An invariant form is symmetric (resp. antisymmetric) if

$$\Lambda \cdot L = 0 \quad (\text{resp. } 1) \quad \bmod 2$$

where L is the *level vector* (→ 1.92) of $\mathcal{G}$ (see Tables 3.1–3.9).

For more details, see ref. [17].

1.88 Symmetric spaces

→ 1.71 Real forms.

1.89 Symplectic groups and algebras

The symplectic group $Sp(2n)$ in the $2n$-dimensional real space $\mathbb{R}^{2n}$ is the group of $2n \times 2n$ real matrices S leaving invariant the scalar product

$$\vec{x} \cdot \vec{y} = \vec{x}\, g\, \vec{y} = S\vec{x} \cdot S\vec{y}\,, \qquad \forall\, \vec{x}, \vec{y} \in \mathbb{R}^{2n}$$

with the metric g defined as follows

$$g = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & 0 & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & 0 & \\ -1 & & & & & \end{array} \right)$$

The group $Sp(2n)$ is compact and its Lie algebra simple. Any element S of $Sp(2n)$ satisfies

$$S^t\, g\, S = g$$

and can be written

$$S = e^K \qquad \text{with } K \text{ such that } \; K^t\, g = -g\, K$$

We will rewrite K in the form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

where the four $n \times n$ K_{ij} submatrices satisfy

$$K_{11} = -\widetilde{K}_{22} \qquad K_{12} = \widetilde{K}_{12} \qquad K_{21} = \widetilde{K}_{21}$$

where $\widetilde{K}_{ij}$ means the matrix obtained from K_{ij} by transposing its elements with respect to the minor diagonal.

It follows that the number of independent K matrices and therefore the dimension of the Lie algebra $sp(2n)$ is

$$n^2 + 2\left(n + \frac{n-1}{2}\right) = n(2n+1) = \frac{2n(2n+1)}{2}$$

Note that $Sp(2) \simeq SU(2)$ and $Sp(4)/Z_2 \simeq SO(5)$.

With respect to the basis $(\vec{e}_n, \dots, \vec{e}_1, \vec{e}_{-1}, \dots, \vec{e}_{-n})$, the $sp(2n)$ algebra can be seen as generated by the matrices

$$M_{ij} = E^{(2n)}_{ij} - \sigma(i)\sigma(j) E^{(2n)}_{-j,-i}$$
$$M_{i,-i} = E^{(2n)}_{i,-i}$$

where $\sigma(i) = i/|i| = sign(i)$ and $i, j = \pm 1, \dots, \pm n$, $i \neq j$. The matrices $E_{ij}^{(2n)}$ are the $2n \times 2n$ elementary matrices with entry 1 in the i-th row and j-th column, and zeros elsewhere. We have then commutation relations

$$\Big[M_{ij}, M_{kl}\Big] = sign(jk)\ (M_{il}\delta_{kj} + M_{-j,k}\delta_{il} + M_{i,-k}\delta_{-j,l} + M_{-j,l}\delta_{k,-i})$$

and the property

$$M_{ij} = -sign(ij)\, M_{-j,-i}$$

Relabelling the generators M_{ij} as follows

$$M_{i'j'} \longrightarrow M_{ij} \quad \text{with} \quad \begin{cases} i = 2i' & (i' > 0) \\ i = 2(-i') - 1 & (i' < 0) \end{cases}$$

the n generators $H_i = M_{2i,2i}$ $(i = 1, \dots, n)$ span the Cartan subalgebra.

An irreducible representation of $Sp(2n)$ is either characterized by the Dynkin labels $(a_1, \dots, a_n)$ where a_i are positive or null integers, or by the set of n non-negative integers $(m_1, \dots, m_n)$ such that $m_i \geq m_{i+1}$, with $a_i = m_i - m_{i+1}$ for $i = 1, \dots, n-1$ and $a_n = m_n$.

The dimension of the $Sp(2n)$ irreducible representation $\pi(m_1, \dots, m_n)$ is given by

$$N(m_1, \dots, m_n) = \frac{\displaystyle\prod_{1 \leq i \leq n} l_i \prod_{1 \leq i < j \leq n} (l_i - l_j)(l_i + l_j)}{\displaystyle\prod_{1 \leq i \leq n} \tau_i \prod_{1 \leq i < j \leq n} (\tau_i - \tau_j)(\tau_i + \tau_j)}$$

where $\tau_i = n - i + 1$ and $l_i = m_i + \tau_i$, that is

$$N = \prod_{1 \leq i \leq n} \frac{m_i + n - i + 1}{n - i + 1} \prod_{1 \leq i < j \leq n} \frac{m_i - m_j + j - i}{j - i} \times \prod_{1 \leq i < j \leq n} \frac{m_i + m_j + 2n - i - j + 2}{2n - i - j + 2}$$

1.90 Unitary groups and algebras

The unitary group in the n-dimensional complex space $\mathbb{C}^n$ is the group of $n \times n$ complex matrices U leaving invariant the scalar product

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i^* y_i = U\vec{x} \cdot U\vec{y}\,, \qquad \forall\, \vec{x}, \vec{y} \in \mathbb{C}^n$$

This group is denoted $U(n)$. One can also define it as the group of $n \times n$ complex unitary matrices U such that $U^\dagger = U^{-1}$. Imposing on U the condition $\det U = 1$, we obtain the subgroup $SU(n)$ of $U(n)$ or special unitary group in n dimensions. The group $U(n)$ is compact and its Lie algebra $su(n)$ is simple. Any element U of $U(n)$ can be written

$$U = e^{iM} \qquad \text{with } M \text{ hermitian: } M^\dagger = M$$

or

$$U = e^{N} \qquad \text{with } N \text{ antihermitian: } N^\dagger = -N$$

One can form n^2 independent $n \times n$ hermitian matrices. Therefore the number of generators of $U(n)$ is n^2. Imposing the condition $\det U = 1$ implies, since $\det U = \exp(\mathrm{tr}(\ln U))$, that $\mathrm{tr}(M) = 0$, and therefore $SU(n)$ has $n^2 - 1$ generators.

An irreducible representation of $SU(n)$ is either characterized by the Dynkin labels $(a_1, \ldots, a_{n-1})$ where a_i are positive or null integers, or by the set of $n-1$ non-negative integers $(m_1, \ldots, m_{n-1})$ such that $m_i \geq m_{i+1}$, with $a_i = m_i - m_{i+1}$ for $i = 1, \ldots, n-2$ and $a_{n-1} = m_{n-1}$. $[m_1, \ldots, m_{n-1}]$ corresponds to the Young tableaux ($\rightarrow$ 1.96) notation.

The dimension of the $SU(n)$ irreducible representation $\pi(m_1, \ldots, m_{n-1})$ is given by

$$N(m_1, \ldots, m_{n-1}) = \frac{\displaystyle\prod_{1 \leq i < j \leq n-1} (l_i - l_j)}{\displaystyle\prod_{1 \leq i < j \leq n-1} (\tau_i - \tau_j)} \prod_{1 \leq i \leq n-1} \frac{l_i}{\tau_i}$$

where $\tau_i = n - i$ and $l_i = m_i + \tau_i$, that is

$$N = \prod_{1 \leq i < j \leq n-1} \frac{m_i - m_j + j - i}{j - i} \prod_{1 \leq i \leq n-1} \frac{m_i + n - i}{n - i}$$

In terms of the Dynkin labels $a_1, \ldots, a_{n-1}$, it can be written as

$$N = \frac{\displaystyle\prod_{i=1}^{n-1} (a_i + 1) \prod_{i=1}^{n-2} (a_i + a_{i+1} + 2) \ldots (a_1 + \ldots + a_{n-1} + n - 1)}{1!\ 2!\ \ldots\ (n-1)!}$$

In particular, the completely symmetric representations are characterized by $m_1 = m$ and $m_i = 0$ for $i \neq 1$ (the Young tableau reduces to one row of m boxes). The dimension of the representation $\pi(m, 0, \ldots, 0)$ is given by

$$N(m, 0, \ldots, 0) = \binom{n + m - 1}{m} = \frac{(n + m - 1)!}{m!\,(n - 1)!}$$

The completely antisymmetric representations are characterized by $m_i = 1$ for $1 \leq i \leq j \leq n-1$ and $m_i = 0$ for $i > j$ (the Young tableau reduces to one column of j boxes). The dimension of the representation $\pi(1^j, 0, \ldots, 0)$ is given by

$$N(1^j, 0, \ldots, 0) = \binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

The Lie algebra $su(2)$

For the $su(2)$ Lie algebra, one can choose as a basis the well-known Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying

$$\Big[\sigma_i, \sigma_j\Big] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

Adding to them the identity matrix $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ one obtains a basis for the Lie algebra $u(2) = su(2) \oplus u(1)$ of $U(2)$.

The Lie algebra $su(3)$

A basis of the Lie algebra $su(3)$ is given by the Gell-Mann matrices λ_i $(i = 1, \ldots, 8)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = 1/\sqrt{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Adding the generator $\lambda_0 = \sqrt{2/3}\,\mathbb{I}_3$ ($\mathbb{I}_3$ is the 3×3 unit matrix) we obtain a basis for the Lie algebra $u(3) = su(3) \oplus u(1)$ of $U(3)$. Note that the generators λ_i are normalized such that $(i, j = 0, 1, \ldots, 8)$

$$\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$$

The communtation relations among the Gell-Mann matrices read

$$\Big[\lambda_i, \lambda_j\Big] = \lambda_i\lambda_j - \lambda_i\lambda_j = 2i\, f_{ijk}\, \lambda_k$$

where the structure constants f_{ijk} are completely antisymmetric in the three indices i, j, k. Their non-vanishing values are

$$f_{123} = 1, f_{458} = f_{678} = \sqrt{3}/2$$
$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2$$

In the same way, one can define completely symmetric coefficients for the Lie algebra $su(3)$, since

$$\Big\{\lambda_i, \lambda_j\Big\} = \lambda_i\lambda_j + \lambda_i\lambda_j = \tfrac{4}{3}\, \delta_{ij}\, \mathbb{I} + 2\, d_{ijk}\, \lambda_k$$

with d_{ijk} completely symmetric in its three indices, and such that

$$d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3}$$
$$d_{147} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2$$
$$d_{448} = d_{558} = d_{668} = d_{778} = -1/2\sqrt{3}$$

Such completely symmetric and antisymmetric quantities can be defined only for any $su(n)$ Lie algebra (see ref. [62]).

1.91 Universal enveloping algebra

Definition

Let $\mathcal{G}$ be a Lie algebra of dimension n. $\mathcal{G}^{\otimes}$ being the tensor algebra over $\mathcal{G}$, and $\mathcal{I}$ the ideal of $\mathcal{G}^{\otimes}$ generated by $[X, Y] - (X \otimes Y - Y \otimes X)$ where $X, Y \in \mathcal{G}$, the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is the quotient $\mathcal{G}^{\otimes}/\mathcal{I}$.

Theorem (Poincaré–Birkhoff–Witt)

Let $b_1, \dots, b_n$ be a basis of $\mathcal{G}$. Then the elements

$$b_1^{i_1} \dots b_B^{i_n} \quad \text{with} \quad i_1, \dots, i_n \geq 0$$

form a basis of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$.

The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is endowed with a natural coproduct Δ defined by

$$\begin{aligned} \Delta : \mathcal{U}(\mathcal{G}) \ &\rightarrow \ \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \\ g \ &\mapsto \ \Delta(g) = g \otimes 1 + 1 \otimes g \end{aligned}$$

Together with the antipode $S : \mathcal{U}(\mathcal{G}) \rightarrow \mathcal{U}(\mathcal{G})$, $g \mapsto S(g) = -g$ such that $S(g_1 g_2) = S(g_2)S(g_1)$ and the counit $\varepsilon : \mathcal{U}(\mathcal{G}) \rightarrow \mathbb{C}$, $g \mapsto \varepsilon(g) = 0$, the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ acquires the structure of a Hopf algebra.

1.92 Weights of a representation

Notion of weight – Basic properties

Let $\mathcal{G}$ be a semi-simple Lie algebra of dimension n and rank r. Denote by H_i $(i = 1, \dots, r)$ the Cartan generators of $\mathcal{G}$, which can be taken to be hermitian. Let π be a representation of $\mathcal{G}$: one can choose a system of common eigenvectors for the H_i which will provide a basis of the representation space $\mathcal{V}$. If we denote $|\Lambda\rangle$ such a vector, we have

$$H_i|\Lambda\rangle = \Lambda^i|\Lambda\rangle$$

where the H_i appear as $N \times N$ matrices if $\dim \mathcal{V} = N$.

Definition

The r-dimensional vector $\Lambda = (\Lambda^1, \dots, \Lambda^r)$ associated to $|\Lambda\rangle$ such that $H_i|\Lambda\rangle = \Lambda^i|\Lambda\rangle$ is called the *weight* of $|\Lambda\rangle$ in π.

The number of eigenvectors $|\Lambda\rangle$ with the same weight Λ is called the *multiplicity* of Λ. A weight is said to be *simple* if its multiplicity is one.

Property

The knowledge of the weight diagram of a representation completely characterizes the representation, that is it allows us to obtain explicitly the matrices H_i and E_α.

Example

Considering the Cartan's basis of $\mathcal{G}$ such that $\left[H_i, E_\alpha\right] = \alpha^i E_\alpha$, the root diagram of $\mathcal{G}$ is the weight diagram of the adjoint representation of $\mathcal{G}$. □

Definition

Two weights Λ and Λ' are said to be *equivalent* if they are related by an element of the Weyl reflection group.

Properties

1. If π is a representation of $\mathcal{G}$ with representation space $\mathcal{V}$ of dimension N, then there exist at most N weights (indeed the eigenvectors corresponding to the different weights are linearly independent and there are at most N independent vectors in $\mathcal{V}$).

2. Let $|\Lambda\rangle$ be an eigenvector with weight Λ, then $E_\alpha|\Lambda\rangle$ is either zero or is an eigenvector with weight $\Lambda + \alpha$.

3. Let α be a root and Λ be a weight. Then $2\,\dfrac{\Lambda \cdot \alpha}{\alpha \cdot \alpha}$ is an integer and $\Lambda' = \Lambda - 2\,\dfrac{\Lambda \cdot \alpha}{\alpha \cdot \alpha}\,\alpha$ is a weight with the same multiplicity as Λ.

Highest weights – Fundamental weights

Definition

– A weight Λ is said *higher* than the weight Λ' if the vector $\Lambda - \Lambda'$ is positive, that is its first non-zero component is strictly positive.
– The highest weight of a set of equivalent weights is said to be *dominant*.

Theorem

In any irreducible representation of a simple Lie algebra $\mathcal{G}$, there is a highest weight. This highest weight is simple. Moreover, two irreducible representations of $\mathcal{G}$ are equivalent if and only if they have the same highest weight.

It follows from this theorem that *the highest weight of an irreducible representation of a semi-simple Lie algebra fully characterizes this representation.*

Example

Let us take the simplest non-trivial case, that is the representations of the $SO(3)$ group. The rank being one, each weight in a given representation is a one-dimensional vector. Using the basis J_1, J_2, J_3 ($\rightarrow$ 1.58) the weights in the three-dimensional or adjoint representation are the eigenvalues of J_3, that is $1, 0, -1$ associated to the eigenvectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in the three-dimensional representation space. Then 1 is the highest weight and one obtains the other weights by action of the ladder operator $J_- = J_1 - iJ_2$. The adjoint representation is denoted $j = 1$.

More generally, in the representation j of dimension $2j+1$, the highest weight is j and the others weights $j-1, j-2, \ldots, -j+1, -j$. It will be possible to choose J_3 diagonal with eigenvalues $j, \ldots, -j$ and to construct J_1 and J_2 from the relations

$$\begin{aligned} J_-|\Lambda\rangle &= (J_1 - iJ_2)\,|\Lambda\rangle = \sqrt{(j+\Lambda)(j-\Lambda+1)}\ |\Lambda - 1\rangle \\ J_+|\Lambda\rangle &= (J_1 + iJ_2)\,|\Lambda\rangle = \sqrt{(j-\Lambda)(j+\Lambda+1)}\ |\Lambda + 1\rangle \end{aligned}$$

For any representation of $SO(3)$ there is no degeneracy of the states. The situation is more complicated for representations of groups of rank $r > 1$ (see ref. [78] for example).

Moreover, the representations of the Lie algebra $so(3) \simeq su(2)$ are labelled by j integer or half-integer. At the group level, $SO(3)$ admits only representations with j integer while $SU(2)$ admits also j half-integer or spin representations ($\rightarrow$ 1.18 and 1.84). □

The classification of the irreducible representations of a simple Lie algebra is made possible by a theorem due to Cartan:

Theorem (Cartan)

For every simple Lie algebra of rank r, there are r dominant weights called *fundamental dominant weights* Λ_i ($i = 1, \ldots, r$) such that any other dominant weight Λ is

$$\Lambda = \sum_{i=1}^{r} a_i \Lambda_i \equiv \Lambda(a_1, \ldots, a_r)$$

where the a_i are non-negative integers. The numbers a_i are called the *Dynkin labels*.

Furthermore, there exist r so-called *fundamental irreducible representations* which have the r fundamental dominant weights as their highest weights.

Finally, to any dominant weight $\Lambda(a_1, \ldots, a_r)$ there corresponds one and only one (up to an equivalence) irreducible representation with $\Lambda(a_1, \ldots, a_r)$ as its highest weight: this representation will be denoted by $D(a_1, \ldots, a_r)$.

The r fundamental representations of a simple Lie algebra of rank r are denoted $D^{(i)} = D(0, \ldots, 1, \ldots, 0)$ (the 1 being at the ith place) while the representation $D(0, \ldots, 0)$ will be the trivial representation.

The fundamental weights are determined as follows: let $\Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ be the simple root system of $\mathcal{G}$ and $\Delta^{0\vee} = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ the corresponding simple coroot system ($\rightarrow$ 1.83). The fundamental dominant weights Λ_j are dual to the simple coroots:

$$2\,\frac{\Lambda_j \cdot \alpha_i}{\alpha_i \cdot \alpha_i} = \Lambda_j \cdot \alpha_i^\vee = \delta_{ij}$$

The Dynkin labels $(a_1, \ldots, a_r)$ corresponding to a dominant weight Λ are given by

$$a_i = \Lambda \cdot \alpha_i^\vee = 2\,\frac{\Lambda \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$$

It follows that the Dynkin labels are in a one-to-one correspondence with the dots of the Dynkin diagram of the Lie algebra $\mathcal{G}$. The fundamental representations for which the non-zero Dynkin label corresponds to an endpoint of the Dynkin diagram are called the *simple fundamental representations.* All other finite dimensional irreducible representations of $\mathcal{G}$ can be constructed by tensor products of the simple fundamental ones.

The fundamental weights of the simple Lie algebras are displayed in Tables 3.1–3.9.

Examples

Consider the Lie algebra $sl(2)$ of rank 1. From the root diagram and the above relation, we deduce

$$\Lambda_i \equiv \Lambda = \tfrac{1}{2}\alpha$$

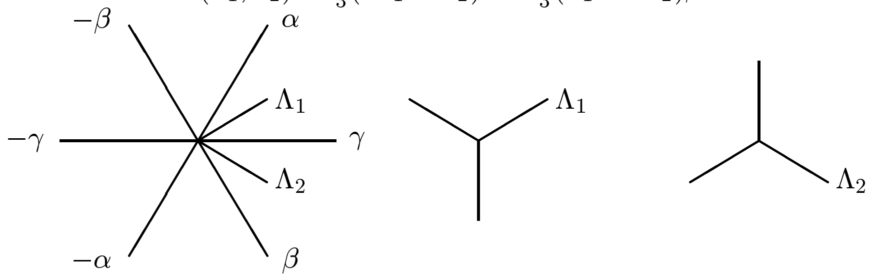
By Weyl reflection with respect to the axis orthogonal to α at the origin, we get the other weight $-\Lambda$ and thus the two-dimensional fundamental representation of $sl(2)$.

Consider now the Lie algebra $sl(3)$ of rank two. The two simple roots being α and β, we deduce Λ_1 and Λ_2 using again the above relation:

$$\Lambda_1 = \tfrac{1}{3}(2\alpha + \beta) \quad \text{and} \quad \Lambda_2 = \tfrac{1}{3}(\alpha + 2\beta)$$

By Weyl reflections, we deduce from Λ_1 the first fundamental $sl(3)$ representation $D(1,0)$ and from Λ_2 the second fundamental representation $D(0,1)$, both of dimension 3.

Moreover, in terms of the Dynkin labels λ_1 and λ_2, the dominant weights take the form $\Lambda(\lambda_1, \lambda_2) = \frac{1}{3}(2\lambda_1 + \lambda_2)\alpha + \frac{1}{3}(\lambda_1 + 2\lambda_2)\beta$.



□

Complex, real and pseudo-real representations

Definition

Let $\mathcal{G}$ be a simple Lie algebra and π be an irreducible representation of $\mathcal{G}$ of highest weight Λ. The *level* of a weight of the representation π is the number of simple roots one has to substract from Λ to obtain this weight.

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r and π be an irreducible representation of $\mathcal{G}$ of highest weight Λ with Dynkin labels $(a_1, \ldots, a_r)$.

The highest level of the irreducible representation π is called the *height* of the representation. It is given by

$$h(\Lambda) = \sum_{i=1}^{r} L_i a_i$$

where $L = (L_1, \dots, L_r)$ is the *level vector* or *height vector* of $\mathcal{G}$ (see Tables 3.1–3.9).

Let π be a highest weight irreducible representations (HWIR) of a simple Lie algebra $\mathcal{G}$ with representation space $\mathcal{V}$. The HWIRs of a simple Lie algebra $\mathcal{G}$ fall into one of two classes:

1. The *complex* representations: the weights at level ℓ are not the negative of those at level $h(\Lambda) - \ell$. This case occurs when the Dynkin diagram of $\mathcal{G}$ has a non-trivial symmetry (A_N, D_N, E_6), see below.

2. The *self-conjugate* representations: the weights at level ℓ are the negative of those at level $h(\Lambda) - \ell$. In that case, there exists a bilinear form B on $\mathcal{V}$ such that

$$B(Lx, y) = B(x, Ly) \qquad L \in \pi(\mathcal{G}) \,, \;\; x, y \in \mathcal{V}$$

   One has then to distinguish according to the parity of $h(\Lambda)$.

   (a) the *real* representations: if $h(\Lambda)$ is even, one can set the entries of the representation matrices to real values. The bilinear form on $\mathcal{V}$ is then symmetric: $(x, y) = (y, x)$.

   (b) the *pseudo-real* representations: if $h(\Lambda)$ is odd, one cannot set the entries of the representation matrices to real values. The bilinear form on $\mathcal{V}$ is then antisymmetric: $(x, y) = -(y, x)$.

   For example, the representations D_j of $su(2)$ with j integer are real, while the representations D_j with j half-integer are pseudo-real.

The classification of the HWIR of the simple Lie algebras is the following:

– For A_N, the representation $D(a_1, \dots, a_N)$ is:
complex if $(a_1, \dots, a_N) \neq (a_N, \dots, a_1)$,
otherwise

- when $N \neq 4q + 1$: real if $(a_1, \dots, a_N) = (a_N, \dots, a_1)$.
- when $N = 4q + 1$: real if $(a_1, \dots, a_N) = (a_N, \dots, a_1)$ and $a_{(N+1)/2}$ is even, pseudo-real if $(a_1, \dots, a_N) = (a_N, \dots, a_1)$ and $a_{(N+1)/2}$ is odd.

– For B_N, the representation $D(a_1, \dots, a_N)$ is:

- when $N = 4q+1$ or $4q+2$: real if a_N is even, pseudo-real if a_N is odd.
- when $N = 4q$ or $4q+3$: always real.

– For C_N, the representation $D(a_1, \dots, a_N)$ is:
real if $a_1 + a_3 + a_5 + \dots$ is even, pseudo-real if $a_1 + a_3 + a_5 + \dots$ is odd.

– For D_N, the representation $D(a_1, \dots, a_N)$ is:

- when $N = 4q$: always real.
- when $N = 4q+2$:
  real if $a_{4q+1} + a_{4q+2}$ is even, pseudo-real if $a_{4q+1} + a_{4q+2}$ is odd.
- when $N = 4q+1$ or $4q+3$:
  complex if $(a_1, \dots, a_{N-2}, a_{N-1}, a_N) \neq (a_1, \dots, a_{N-2}, a_N, a_{N-1})$,
  real if $(a_1, \dots, a_{N-2}, a_{N-1}, a_N) = (a_1, \dots, a_{N-2}, a_N, a_{N-1})$.

– For E_6, the representation $D(a_1, \dots, a_N)$ is:
complex if $(a_1, a_2, a_3, a_4, a_5, a_6) \neq (a_5, a_4, a_3, a_2, a_1, a_6)$,
real if $(a_1, a_2, a_3, a_4, a_5, a_6) = (a_5, a_4, a_3, a_2, a_1, a_6)$.

– For E_7, the representation $D(a_1, \dots, a_N)$ is:
real if $a_4 + a_6 + a_7$ is even, pseudo-real if $a_4 + a_6 + a_7$ is odd.

– The exceptional Lie algebras E_8, F_4 and G_2 have only real representations (without any conditions on the Dynkin labels).

Weyl formula

Theorem (Weyl dimension formula)

Let $\mathcal{G}$ be a simple Lie algebra of rank r, Δ^+ the set of positive roots of $\mathcal{G}$ and ρ the Weyl vector (half-sum of the positive roots). The dimension of an irreducible representation of highest weight Λ is given by the Weyl formula:

$$N(\Lambda) = \prod_{\alpha \in \Delta^+} \frac{(\Lambda + \rho) \cdot \alpha}{\rho \cdot \alpha}$$

Expressing the highest weight Λ in terms of the Dynkin labels $(a_1, \dots, a_r)$ and decomposing the positive roots in the simple coroot system $\Delta^\vee$, that is $\alpha = \sum_{i=1}^r \alpha^{i\vee} \alpha_i^\vee$, the Weyl formula takes the following form:

$$N(\Lambda) = \prod_{\alpha \in \Delta^+} \frac{\sum_{i=1}^r (a_i + 1) \alpha^{i\vee}}{\sum_{i=1}^r \alpha^{i\vee}}$$

Example

Consider the Lie algebra $sl(3)$ with positive roots α_1, α_2 and $\alpha_1 + \alpha_2$. The representation $D(a_1, a_2)$ has dimension

$$N(a_1, a_2) = \tfrac{1}{2}\,(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2)$$

□

Freudenthal recursion formula

When the dimension of an irreducible representation is sufficiently large, the multiplicities of the weight are not so easy to determine and the construction of the weight diagram can be cumbersome. The problem is solved by the Freudenthal recursion formula that allows us to determine the multiplicity of a weight λ from the multiplicities of the weights of lower level.

Theorem (Freudenthal recursion formula)

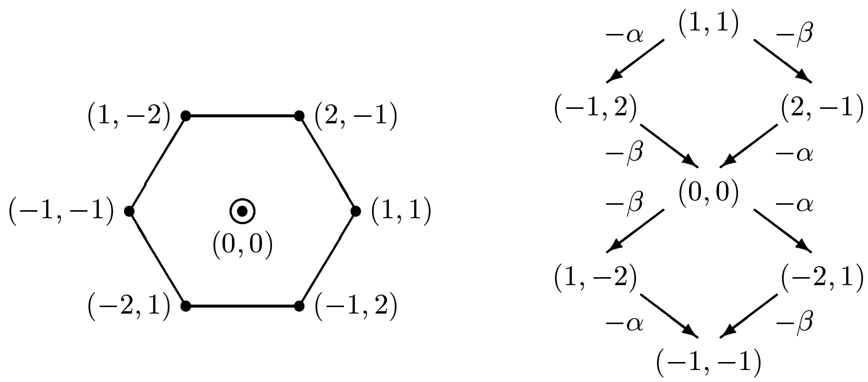
Let $\mathcal{G}$ be a simple Lie algebra, π an irreducible representation of highest weight Λ and ρ the Weyl vector. Denoting by $\mathrm{mult}_\Lambda(\lambda)$ the multiplicity of a weight λ of the representation π, one has:

$$[(\Lambda+\rho)\cdot(\Lambda+\rho)-(\lambda+\rho)\cdot(\lambda+\rho)]\,\mathrm{mult}_\Lambda(\lambda) = 2\sum_{\alpha\in\Delta^+}\sum_{k>0}(\lambda+k\alpha)\cdot\alpha\,\mathrm{mult}_\Lambda(\lambda+k\alpha)$$

the sums over $\alpha \in \Delta^+$ and $k > 0$ being such that $\lambda + k\alpha$ is still a weight.

Example

Let us illustrate the previous notions on the adjoint representation of $sl(3)$. Denote by $\Delta^0 = \{\alpha, \beta\}$ the simple root system of $sl(3)$. The highest weight of the adjoint representation is the highest root $\alpha+\beta$. The non-zero weights of the representations are the roots of the Lie algebra, with multiplicity one. The zero weight appears with multiplicity equal to the rank of the algebra, as can be checked by the Freudenthal formula. Finally, the descending scheme and the weight diagram are the following:



□

Symmetric quadratic form

Any weight w of an irreducible representation can be written as a linear combination of the simple (co)roots ($\rightarrow$ 1.83) as

$$w = \sum_i w_i \, \alpha_i^\vee = \sum_i w_i \, \frac{2}{\alpha_i \cdot \alpha_i} \, \alpha_i$$

The Dynkin labels for any weight w are defined as

$$a_i = w \cdot \alpha_i^\vee = 2 \, \frac{w \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$$

If w is the highest weight Λ we find the (non-negative) integers which label the irreducible representation. For a generic weight the Dynkin labels can be negative. One can define the scalar product of two weights by

$$w^1 \cdot w^2 = \sum_{i,j} w_i^1 \, \frac{2}{\alpha_i \cdot \alpha_i} \, \alpha_i \; w_j^2 \, \frac{2}{\alpha_j \cdot \alpha_j} \, \alpha_j = \sum_{i,j} a_i^1 \, G_{ij} \, a_j^2$$

where (G_{ij}) is called the *symmetric metric tensor* or *symmetric quadratic form* and is given by

$$G_{ij} = (A^{-1})_{ij} \, \frac{\alpha_i \cdot \alpha_i}{2}$$

where (A_{ij}) is the Cartan matrix ($\rightarrow$ 1.8) and the roots are normalized such that the longest root has length squared equal to two.

Introducing a basis η_i in the weight space as $\eta_i = \sum_j (A^{-1})_{ji} \, \alpha_j$ the symmetric metric tensor can be written as

$$G_{ij} = \eta_i \cdot \eta_j$$

For the explicit form of (G_{ij}) for the simple Lie algebras, see Tables 3.1–3.9.

1.93 Weyl group

Definition

Let $\mathcal{G}$ be a simple Lie algebra of rank r with root system Δ and coroot system $\Delta^\vee$. For any root $\alpha \in \Delta$ there is a transformation w_α in the weight space (therefore, in particular, in the root space), called *Weyl reflection*, such that if λ is a weight:

$$w_\alpha(\lambda) = \lambda - 2 \, \frac{\alpha \cdot \lambda}{\alpha \cdot \alpha} \, \alpha = \lambda - (\alpha^\vee \cdot \lambda) \, \alpha$$

The set of Weyl reflections with respect to all the roots of $\mathcal{G}$ forms a finite group W called the *Weyl group* of $\mathcal{G}$.

The Weyl reflection w_α leaves fixed any vector in the hyperplane orthogonal to α and transforms α into $-\alpha$. The Weyl group is generated by the r reflections w_{α_i} with respect to the simple positive roots α_i $(1 \leq i \leq r)$ and by the identity. The *length* $l(w)$ of $w \in W$ is the minimun number of reflections w_{α_i} such that $w = \prod_i w_{\alpha_i}$. The *parity* $\varepsilon(w)$ of w is defined as $\varepsilon(w) = (-1)^{l(w)}$.

W is a normal subgroup ($\to$ 1.37) of the automorphism group ($\to$ 1.4) of Δ. It is often interesting to know the conjugacy class of W as conjugate automorphisms correspond to a different choice of the simple root systems. A complete list of the conjugacy classes can be found in ref. [12].

Definition

Denote by $\{\alpha_1, \dots, \alpha_r\}$ the simple root system of $\mathcal{G}$. The element

$$c = w_{\alpha_1} \dots w_{\alpha_r}$$

of W is called the *Coxeter element* of W. Its order is equal to the Coxeter number h of $\mathcal{G}$. It is the element of the Weyl group of maximal length.

Table 1.7: Weyl group of the simple Lie algebras.

simple Lie algebra $\mathcal{G}$	Weyl group W	$\dim W$
A_{N-1}	$\mathfrak{S}_N$	$N!$
B_N	$\mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N$	$2^N N!$
C_N	$\mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N$	$2^N N!$
D_N	$\mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^{N-1}$	$2^{N-1} N!$
E_6		$2^7.3^4.5$
E_7		$2^{10}.3^4.5.7$
E_8		$2^{14}.3^5.5^2.7$
F_4	$\mathfrak{S}_3 \ltimes (\mathfrak{S}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^2)$	$2^7.3^2$
G_2	dihedral group	12

1.94 Weyl vector

Definition

Let $\mathcal{G}$ be a simple Lie algebra, Δ the root system and $\Delta^\vee$ the coroot system of $\mathcal{G}$. The vectors

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \qquad \text{and} \qquad \rho^\vee = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha^\vee$$

are called respectively the *Weyl vector* and the *dual Weyl vector* of $\mathcal{G}$.

Properties

1. The Weyl vector of a simple Lie algebra $\mathcal{G}$ is equal to the sum of the fundamental weights ($\to$ 1.92) of $\mathcal{G}$ (r is the rank of $\mathcal{G}$):

$$\rho = \sum_{i=1}^{r} \Lambda_i$$

2. The dual Weyl vector $\rho^\vee$ and the level vector L ($\to$ 1.92) of $\mathcal{G}$ are related by $\rho^\vee = \frac{1}{2} L$.

3. Let $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ and $\Delta^{0\vee} = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ be the simple root and simple coroot systems of $\mathcal{G}$. The Weyl vector and the dual Weyl vector satisfy

$$\rho \cdot \alpha_i^\vee = 1 \qquad \text{and} \qquad \rho^\vee \cdot \alpha_i = 1$$

Theorem (strange formula)

Let $\mathcal{G}$ be a simple Lie algebra, ρ is the Weyl vector, $-\alpha_0$ is the highest root, $h^\vee$ is the dual Coxeter number. Then the following formula, called the *strange formula*, holds:

$$\frac{\rho \cdot \rho}{\alpha_0 \cdot \alpha_0} = \tfrac{1}{24}\, h^\vee \dim \mathcal{G}$$

1.95 Wigner-Eckart theorem

A family of operators A_λ is called an irreducible λ-tensor operator family under a compact group G, if each operator of the family transforms under the action of the group as follows:

$$\pi(g)\, A_\lambda\, \pi^{-1}(g) = \sum_{\lambda'} \pi^{\Lambda}_{\lambda\lambda'}(g)\, A_{\lambda'}$$

where $\pi^{\Lambda}_{\lambda\lambda'}(g)$ are the matrix elements of the unitary irreducible representation π with highest weight Λ of G. Note that Λ and λ, λ' are short hand notations for the set of labels which specify the unitary irreducible representation and its space states.

The matrix element of an operator A_λ between the states of the unitary irreducible representations of G satisfies the following identity, due to Wigner-Eckart:

$$\langle \Lambda'\lambda' | A_\lambda | \Lambda''\lambda'' \rangle = \sum \langle \Lambda' \| A_\lambda \| \Lambda'' \rangle \langle \Lambda''\lambda''\Lambda\lambda | \Lambda'\lambda' \rangle$$

where $\langle \Lambda' \| A_\lambda \| \Lambda'' \rangle$ is called the reduced matrix element and $\langle \Lambda''\lambda''\Lambda\lambda | \Lambda'\lambda' \rangle$ are the Clebsch-Gordan coefficients.

1.96 Young tableaux: $SU(N)$ representations

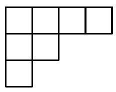
The Young tableau technique is a powerful tool for the study of the irreducible representations of the groups $GL(N, \mathbb{C})$ ($\rightarrow$ 1.13) and of its subgroups, in particular $SU(N)$. Hereafter we will be concerned only with the case of the $SU(N)$ group.

Any representation of $SU(N)$ can be obtained from the tensor product of its fundamental representation $D(1, 0, \ldots, 0)$ – using the notation of weights ($\rightarrow$ 1.92) – by itself a certain number of times. The elements in the representation space of the tensor representation thus obtained are tensors satisfying symmetry properties under the permutation group S_N. These properties are contained in the Young tableau formalism which allows us to obtain explicit forms for the vectors in a representation space and also gives a method for the reduction into irreducible representations of the product of two $SU(N)$ representations.

Let us first define a Young tableau:

Definition

A *Young tableau* is constituted by a certain number of "boxes" set on one or more rows such that, if the rows are numbered from the top to the bottom, the number of boxes in the i-th row is bigger or equal to the number of boxes in the $(i+1)$-th row.

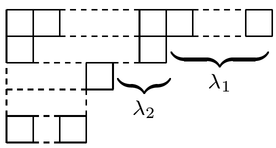
For example, the Young tableau $[4, 2, 1]$ is given by  .

To the irreducible representation $D(\lambda_1, \ldots, \lambda_{N-1})$ of $SU(N)$ will be associated the $(N-1)$ row Young tableau $[m_{1N}, m_{2N}, \ldots, m_{N-2,N}, m_{N-1,N}]$ with

$$\begin{aligned} m_{1N} &= \lambda_1 + \ldots + \lambda_{N-1} \\ m_{2N} &= \lambda_2 + \ldots + \lambda_{N-1} \\ &\ldots \\ m_{N-2,N} &= \lambda_{N-2} + \lambda_{N-1} \\ m_{N-1,N} &= \lambda_{N-1} \end{aligned}$$

that is

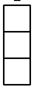
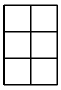
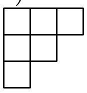
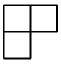
$D(\lambda_1, \ldots, \lambda_{N-1}) \quad \longrightarrow$



Then the fundamental representation $D(1, 0, \ldots, 0)$ of $SU(N)$ of dimension N will be represented by

$D(1, 0, \ldots, 0) \quad \longrightarrow$ 

Let us remark that if to a given representation $D(\lambda_1, \ldots, \lambda_{N-1})$ of $SU(N)$ there corresponds one and only one Young tableau, the converse is not true, since to the tableau $[m_1, \ldots, m_{N-1}]$ will correspond the representations $D(m_1 - m_2, \ldots, m_{N-1})$ of $SU(N)$, $D(m_1 - m_2, \ldots, m_{N-1}, 0)$ of $SU(N+1)$, $D(m_1 - m_2, \ldots, m_{N-1}, 0, 0)$ of $SU(N+2)$, and so on. In order to avoid mistakes, one has just to keep in mind that any irreducible representation of $SU(N)$ will be represented by a Young tableau with at most N rows. If the Young tableau has more than N rows, it will correspond to the representation 0 of $SU(N)$. If $\mathcal{T}$ has exactly N rows, $[m_1, \ldots, m_N]$ will be replaced by $[m_1 - m_N, \ldots, m_{N-1} - m_N]$, since the m_N columns of length N are in fact related to the $U(1)$ generator commuting with $SU(N)$ in the $U(N)$ group. Therefore the trivial representation (of dimension 1) $D(0, \ldots, 0)$ of $SU(N)$ will be represented by no box, or one or several columns of N boxes.

As an example, the trivial representation $D(0,0)$ of the group $SU(3)$ will admit as Young tableau $\bullet$ or  or  etc., while the representation  will be considered the same as  .

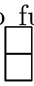
The two representations $D(\lambda_1, \ldots, \lambda_{N-1})$ and $D(\lambda_{N-1}, \ldots, \lambda_1)$ will be represented by two Young tableaux $[m_{1N}, \ldots, m_{N-1,N}]$ and $[m'_{1N}, \ldots, m'_{N-1,N}]$ such that $m_{1N} = m'_{1N} = m_{2N} + m'_{N-1,N} = m_{3N} + m'_{N-2,N} = \ldots = m_{iN} + m'_{N+1-i,N} = m_{N-1,N} + m'_{2N}$. That is $[m'_{1N}, \ldots, m'_{N-1,N}]$ can be seen as the upside down complementary subspace of $[m_{1N}, \ldots, m_{N-1,N}]$ in a rectangle of $N.m_{1N}$ boxes with m_{1N} boxes in each row and N boxes in each column.

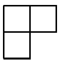
Two such representations $D(\lambda_1, \ldots, \lambda_{N-1})$ and $\overline{D} = D(\lambda_{N-1}, \ldots, \lambda_1)$ are called *contragredient representations* and can be obtained one from the other as follows:

$$\overline{D}(g) = D(g^{-1})^t \,, \qquad \forall\, g \in G$$

that is, if $D = U$ unitary, $\overline{U}(g) = U(g)^\dagger$. We can deduce that if $U = \overline{U}$ (that is $m_{iN} = m'_{iN}$ for $i = 1, \ldots, N-1$) then the representation is real.

As an example, let us consider the group $SU(3)$.

– The two fundamental representations $D(1,0)$ or  also denoted 3, and $D(0,1)$ or  also denoted $\overline{3}$, are contragredient.

– The 8-dimensional (adjoint) representation  is equal to its contragredient and is real.

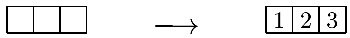
Now let us define the Young operator associated to an irreducible $SU(N)$ representation.

Definition

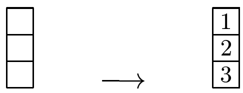
Let us consider the Young tableau $\mathcal{T} = [m_1, \dots, m_{N-1}]$ and let us label $1, \dots, m_1$ the boxes in the first row from left to right, $m_1+1, \dots, m_1+m_2$ the boxes in the second row, and so on up to the last box of the last row. Then denoting $\sigma_\mathcal{T}$ (resp. $\tau_\mathcal{T}$) any permutation leaving the rows (resp. the columns) globally invariant, the operator $\mathcal{Y} = \mathcal{Q}\mathcal{P}$ with $\mathcal{P} = \sum \sigma_\mathcal{T}$ and $\mathcal{Q} = \sum \varepsilon(\tau_\mathcal{T})\tau_\mathcal{T}$ where $\varepsilon(\tau_\mathcal{T})$ is the parity of the permutation $\tau_\mathcal{T}$ will be called the Young operator associated with $\mathcal{T}$ while $\mathcal{P}$ and $\mathcal{Q}$ will be called respectively the symmetrizer and antisymmetrizer of $\mathcal{T}$.

Example

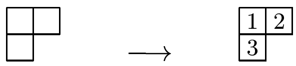
Let us consider the following 3-box diagrams $[3]$, $[1,1,1]$ and $[2,1]$. Using the permutation group S_3 we will have:



Here, $\mathcal{Q}$ is the identity while $\mathcal{P}$ is the sum of all permutations of 1,2,3. It follows that $\mathcal{Y} = e + (1,2) + (2,3) + (3,1) + (1,2,3) + (1,3,2)$.



Here, $\mathcal{P}$ is the identity while $\mathcal{Q}$ is the sum of all graded permutations of 1,2,3. It follows that $\mathcal{Y} = e - (1,2) - (2,3) - (3,1) + (1,2,3) + (1,3,2)$.



Here, $\mathcal{P} = e+(1,2)$, $\mathcal{Q} = e-(1,3)$ and therefore $\mathcal{Y} = [e+(1,2)][e-(1,3)] = e + (1,2) - (1,3) - (1,2,3)$. □

Property

The knowledge of the operator $\mathcal{Y}$ associated to the Young tableau $\mathcal{T}$ allows the construction of all the states of the $SU(n)$ representation D whose Young tableau is $\mathcal{T}$. All the states of D are simply obtained by action of $\mathcal{Y}$ on the most general vector of the tensor vector space $\mathcal{V}_n^{\otimes k}$ where $\mathcal{V}_n$ is the fundamental representation space of $SU(n)$ and k the number of boxes in $\mathcal{T}$.

Example

Consider the group $SU(3)$. Its fundamental representation is of dimension 3: let us denote a, b, c an orthonormal basis in $\mathcal{V}_3$.

The 10-dimensional representation space of $D(3,0)$ or $\square\!\square\!\square$ will be generated by the vectors aaa, bbb, ccc, $Sym(aab) = aab+aba+baa$, $Sym(aac)$, $Sym(abb)$, $Sym(acc)$, $Sym(bbc)$, $Sym(bcc)$, $Sym(abc)$. $\square$

Part 2

Lie Superalgebras

Unless otherwise stated, all Lie superalgebras considered here are complex and finite dimensional.

2.1 Automorphisms

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a simple Lie superalgebra. An automorphism Φ of $\mathcal{G}$ is a bijective homomorphism from $\mathcal{G}$ into itself which respects the $\mathbb{Z}_2$-gradation, that is $\Phi(\mathcal{G}_{\overline{0}}) \subset \mathcal{G}_{\overline{0}}$ and $\Phi(\mathcal{G}_{\overline{1}}) \subset \mathcal{G}_{\overline{1}}$. The automorphisms of $\mathcal{G}$ form a group denoted by $\mathrm{Aut}(\mathcal{G})$. The group $\mathrm{Int}(\mathcal{G})$ of inner automorphisms of $\mathcal{G}$ is the group generated by the automorphisms of the form $X \mapsto gXg^{-1}$ with $g = \exp Y$ where $X \in \mathcal{G}$ and $Y \in \mathcal{G}_{\overline{0}}$. Every inner automorphism of $\mathcal{G}_{\overline{0}}$ can be extended to an inner automorphism of $\mathcal{G}$. The automorphisms of $\mathcal{G}$ which are not inner are called outer automorphisms.

In the case of a simple Lie algebra $\mathcal{A}$, the quotient of the automorphism group by the inner automorphism group $\mathrm{Aut}(\mathcal{A})/\mathrm{Int}(\mathcal{A})$ – called the factor group $F(\mathcal{A})$ – is finite and isomorphic to the group of symmetries of the Dynkin diagram of $\mathcal{A}$ ($\rightarrow$ 1.4).

The situation is slightly different for the superalgebras: the result is as follows. In most of the cases, the outer automorphisms of a basic Lie superalgebra $\mathcal{G}$ can be studied from the Dynkin diagrams of $\mathcal{G}$. More precisely, defining $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$, it is possible to reconstruct $\mathrm{Out}(\mathcal{G})$ by looking at the symmetries of the Dynkin diagrams of $\mathcal{G}$. Two exceptions have to be mentioned:

– Such a diagrammatic approach does not hold for $A(2m, 2n)$ with $m \neq n$ or $m = n$.
– For $A(n, n)$, in addition to the discrete set of outer automorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2$, which in particular acts non-trivially on the $\mathcal{G}_{\overline{0}}$ part, has to be added a continuous one-parameter set of automorphisms $\{\delta_\lambda \,|\, \lambda \in \mathbb{R}\}$ acting as follows on the general element $M \in A(n, n)$ ($\rightarrow$ 2.25):

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \longrightarrow \quad \delta_\lambda(M) = \begin{pmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{pmatrix}$$

One recognizes the generator of this transformation in the diagonal $(2n \times 2n)$ matrix $\mathrm{diag}(\mathbb{I}_n, -\mathbb{I}_n)$, and notices that the automorphism δ_λ acts as a dilatation on the fermionic part $\mathcal{G}_{\overline{1}}$ while it leaves each element of $\mathcal{G}_{\overline{0}}$ invariant.

As a conclusion, $\mathrm{Out}(\mathcal{G})$ is a finite group for any basic superalgebra $\mathcal{G}$ except for $A(n, n)$ where a continuous part has to be added: these discrete groups, which in particular will allow us to construct the twisted affine simple Lie superalgebras, are given in Table 2.1.

For more details, see refs. [19, 23, 86].

$\rightarrow$ 2.15 Dynkin diagram, 2.42 Roots, root systems, 2.63 Weyl group.

Table 2.1: Discrete outer automorphisms of the basic Lie superalgebras.

basic superalgebra $\mathcal{G}$	Out($\mathcal{G}$)	basic superalgebra $\mathcal{G}$	Out($\mathcal{G}$)
$A(m,n)$ $(m \neq n \neq 0)$	$\mathbb{Z}_2$	$C(n+1)$	$\mathbb{Z}_2$
$A(1,1)$	$\mathbb{Z}_2$	$D(m,n)$	$\mathbb{Z}_2$
$A(0,2n-1)$	$\mathbb{Z}_2$	$D(2,1;e^{2i\pi/3})$	$\mathbb{Z}_3$
$A(n,n)$ $(n \neq 0,1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D(2,1;\alpha)$ for generic α	$\mathbb{I}$
$A(0,2n)$	$\mathbb{Z}_4$	$F(4), G(3)$	$\mathbb{I}$
$B(m,n)$	$\mathbb{I}$		

2.2 Cartan matrix

Let $\mathcal{G}$ be a basic Lie superalgebra with Cartan subalgebra $\mathcal{H}$ and simple root system $\Delta^0 = (\alpha_1, \dots, \alpha_r)$. The following commutation relations hold:

$$\Big[H_i, H_j\Big] = 0$$
$$\Big[H_i, E_{\pm\alpha_j}\Big] = \alpha_j^i \, E_{\pm\alpha_j}$$
$$\Big[\!\Big[E_{\alpha_j}, E_{-\alpha_j}\Big]\!\Big] = \alpha_j \cdot H$$

For any root $\alpha \in \Delta$, one can write

$$\Big[H, E_{\pm\alpha}\Big] = \pm\alpha(H) \, E_{\pm\alpha}$$

where α is a linear functional on $\mathcal{H}$, that is an element of the dual $\mathcal{H}^*$ of $\mathcal{H}$, such that $\alpha(H_i) = \alpha^i$ (α^i being the i-th component of the root α).

For all basic Lie superalgebras, there exists a supersymmetric bilinear form B, which is non-degenerate on the Cartan subalgebra $\mathcal{H}$, and which coincides (up to a multiplicative factor) with the Killing form ($\rightarrow$ 2.23) except for $A(n,n)$, $D(n+1,n)$ and $D(2,1;\alpha)$ for which the Killing form vanishes. Therefore, one can associate to any functional $\alpha \in \mathcal{H}^*$ an element $H_\alpha \in \mathcal{H}$, such that

$$B(H, H_\alpha) = \alpha(H)$$

Now, one can define the scalar product of two roots α and β by

$$\alpha \cdot \beta = B(H_\alpha, H_\beta)$$

that is $\alpha \cdot \beta = \alpha(H_\beta) = \beta(H_\alpha)$.

Rescaling the generators H_i, one defines

$$h_i = \frac{2}{(\alpha_i, \alpha_i)} H_i \qquad \text{if } \alpha_i \cdot \alpha_i \neq 0$$
$$h_i = \frac{1}{(\alpha_i, \alpha_i')} H_i \qquad \text{if } \alpha_i \cdot \alpha_i = 0$$

where $\alpha_i' \in \{\alpha_k'\}$ such that $\alpha_i \cdot \alpha_k' \neq 0$ and $\alpha_i \cdot \alpha_i'$ has the smallest value. Then, one can write

$$\Big[h_i, E_{\pm\alpha_j}\Big] = \pm A_{ij} \, E_{\pm\alpha_j}$$

where one defines the *Cartan matrix* of $\mathcal{G}$ as:

$$A_{ij} = 2 \, \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} \qquad \text{if } \alpha_i \cdot \alpha_i \neq 0$$
$$A_{ij} = \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i'} \qquad \text{if } \alpha_i \cdot \alpha_i = 0$$

If $\alpha_i \cdot \alpha_i \neq 0$, then $A_{ii} = 2$. If $\alpha_i \cdot \alpha_i = 0$, then $A_{ii} = 0$ and it may happen that some off-diagonal entries of the i-th row of the matrix take non-integer values. In this case, one has to rescale the corresponding row such that all the entries assume the smallest integer values.

Definition

For each basic Lie superalgebra, there exists a particular simple root system for which the number of odd roots is equal to one. Such a simple root system is called a distinguished simple root system (→ 2.45). All equivalent distinguished simple root systems lead to the same Cartan matrix as defined above. Such a Cartan matrix is called the *distinguished Cartan matrix*.

The distinguished Cartan matrices of the basic Lie superalgebras can be found in Tables 3.52–3.60.

One can also use symmetric Cartan matrices, which is equivalent to rescaling the Cartan generators h_i. The symmetric Cartan matrices are computed as follows. One considers a distinguished simple root system $\Delta^0 = (\alpha_1, \dots, \alpha_r)$ such that $\alpha_i \cdot \alpha_j \in \mathbb{Z}$ and $\min |\alpha_i \cdot \alpha_j| = 1$ if $\alpha_i \cdot \alpha_j \neq 0$. This can always be done since the scalar product on the root space is defined up to a multiplicative factor. Then, for any simple root system, one defines the *symmetric Cartan matrix* A' with integer entries as:

$$A'_{ij} = \alpha_i \cdot \alpha_j$$

If one defines the matrix $D_{ij} = d_i \delta_{ij}$ where the rational coefficients d_i satisfy $d_i A_{ij} = d_j A_{ji}$, the distinguished symmetric Cartan matrix is given from the distinguished Cartan matrix A by $A' = DA$, where

$$d_i = (\underbrace{1, \ldots, 1}_{m+1}, \underbrace{-1, \ldots, -1}_{n}) \text{ for } A(m,n),$$
$$d_i = (\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m-1}, -1/2) \text{ for } B(m,n),$$
$$d_i = (\underbrace{1, \ldots, 1}_{n-1}, 1/2) \text{ for } B(0,n),$$
$$d_i = (1, \underbrace{-1, \ldots, -1}_{n-1}, -2) \text{ for } C(n+1),$$
$$d_i = (\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m}) \text{ for } D(m,n),$$
$$d_i = (1, -1, -2, -2) \text{ for } F(4),$$
$$d_i = (1, -1, -3) \text{ for } G(3).$$

Such a diagonal matrix D_{ij} can be defined for any simple root system.

Example

The distinguished Cartan matrix A and the distinguished symmetric Cartan matrix A' for $F(4)$ are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

associated to the distinguished simple root system $\Delta^0 = \{\alpha_1 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_2 = \varepsilon_3, \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_1 - \varepsilon_2\}$ ($\rightarrow$ 2.18) with the scalar product $\varepsilon_i \cdot \varepsilon_j = -2\delta_{ij}$, $\delta \cdot \delta = 6$, $\varepsilon_i \cdot \delta = 0$ (which should be compared with the scalar product used in section 2.18).

$\rightarrow$ 2.23 Killing form, 2.45 Simple root systems.

2.3 Cartan subalgebras

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra. A Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ is defined as the maximal nilpotent ($\rightarrow$ 2.26) subalgebra of $\mathcal{G}$ coinciding with its own normalizer, that is

$$\mathcal{H} \text{ nilpotent and } \Big\{X \in \mathcal{G} \,\Big|\, \Big[X, \mathcal{H}\Big] \subseteq \mathcal{H}\Big\} = \mathcal{H}$$

In most cases (for basic Lie superalgebras e.g.), a Cartan subalgebra $\mathcal{H}$ reduces to the Cartan subalgebra of the even part $\mathcal{G}_{\overline{0}}$ (then the Cartan subalgebras of a Lie superalgebra are conjugate since the Cartan subalgebras of a Lie algebra are conjugate and any inner automorphism of the even part $\mathcal{G}_{\overline{0}}$ can be extended to an inner automorphism of $\mathcal{G}$).

In the case of the strange superalgebra $Q(n)$, the Cartan subalgebra $\mathcal{H}$ does not coincide with the Cartan subalgebra of the even part $sl(n)$, but admits also an odd part: $\mathcal{H} \cap \mathcal{G}_{\overline{1}} \neq \emptyset$. Since the odd generators of $\mathcal{H}$ change the gradation of the generators on which they act, it is rather convenient to give the root decomposition of $Q(n)$ with respect to $\mathcal{H}_{\overline{0}} = \mathcal{H} \cap \mathcal{G}_{\overline{0}}$ instead of $\mathcal{H}$.

From what precedes, all Cartan subalgebras of a classical superalgebra $\mathcal{G}$ have the same dimension. By definition, the dimension of a Cartan subalgebra $\mathcal{H}$ is the rank of $\mathcal{G}$:

$$\operatorname{rank} \mathcal{G} = \dim \mathcal{H}$$

2.4 Cartan type superalgebras

The Cartan type Lie superalgebras are the simple Lie superalgebras in which the representation of the even subalgebra on the odd part is not completely reducible (→ 2.9). The Cartan type simple Lie superalgebras are classified into four infinite families called $W(n)$ with $n \geq 2$, $S(n)$ with $n \geq 3$, $\widetilde{S}(n)$ and $H(n)$ with $n \geq 4$. $S(n)$ and $\widetilde{S}(n)$ are called special Cartan type Lie superalgebras and $H(n)$ Hamiltonian Cartan type Lie superalgebras. Strictly speaking, $W(2)$, $S(3)$ and $H(4)$ are not Cartan type superalgebras since they are isomorphic to classical ones (see below).

Cartan type superalgebras $W(n)$

Consider $\Gamma(n)$ the Grassmann algebra (→ 2.22) of order n with generators $\theta_1, \dots, \theta_n$ and relations $\theta_i\theta_j = -\theta_j\theta_i$. The $\mathbb{Z}_2$-gradation is induced by setting $\deg \theta_i = \overline{1}$. Let $W(n)$ be the derivation superalgebra of $\Gamma(n)$: $W(n) = \operatorname{Der}\Gamma(n)$. Any derivation $D \in W(n)$ is written as

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \theta_i}$$

where $P_i \in \Gamma(n)$ and the action of the θ-derivative is defined by

$$\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij}$$

The $\mathbb{Z}_2$-gradation of $\Gamma(n)$ induces a consistent $\mathbb{Z}$-gradation of $W(n)$ by

$$W(n)_k = \Big\{ \sum_{i=1}^{n} P_i \frac{\partial}{\partial \theta_i}, \;\; P_i \in \Gamma(n), \;\; \deg P_i = k+1 \Big\} \qquad (-1 \leq k \leq n-1)$$

One has

$$W(n) = \bigoplus_{k=-1}^{n-1} W(n)_k$$

where

$$\left[\!\left[W(n)_i, W(n)_j \right]\!\right] \subset W(n)_{i+j}$$

The superalgebra $W(n)$ has the following properties:

- $W(n)$ has dimension $n2^n$, the number of even generators being equal to the number of odd generators.
- The superalgebra $W(n)$ is simple for $n \geq 2$.
- The semi-simple part of $W(n)_0$ is isomorphic to $gl(n)$.
- The superalgebra $W(2)$ is isomorphic to $A(1,0)$.
- Every automorphism of $W(n)$ with $n \geq 3$ is induced by an automorphism of $\Gamma(n)$.
- The superalgebra $W(n)$ is transitive.
- $W(n)$ is universal as a $\mathbb{Z}$-graded Lie superalgebra. More precisely, if $\mathcal{G} = \oplus_{i\geq -1}\mathcal{G}_i$ is a transitive $\mathbb{Z}$-graded superalgebra with $\dim \mathcal{G}_{-1} = n$, then there is an embedding of $\mathcal{G}$ in $W(n)$ preserving the $\mathbb{Z}$-gradation.
- The representations of $sl(n)$ in the subspace $W(n)_i$ ($i = -1, 0, \dots, n-1$) are in Young tableaux notation $[2^{i+1}1^{n-2-i}] \oplus [1^i]$ where the second representation appears only for $i \geq 0$ and $[1^0]$ has to be read as the singlet. For example we have (the subscripts stand for the $\mathbb{Z}$-gradation indices i):

for $W(3)$	$(\overline{3})_{-1} \oplus (8 \oplus 1)_0 \oplus (\overline{6} \oplus 3)_1 \oplus (\overline{3})_2$
for $W(4)$	$(\overline{4})_{-1} \oplus (15 \oplus 1)_0 \oplus (\overline{20} \oplus 4)_1 \oplus (\overline{10} \oplus 6)_2 \oplus (\overline{4})_3$
for $W(5)$	$(\overline{5})_{-1} \oplus (24 \oplus 1)_0 \oplus (\overline{45} \oplus 5)_1 \oplus (\overline{40} \oplus 10)_2 \oplus (\overline{15} \oplus \overline{10})_3$ $\oplus (\overline{5})_4$

Cartan type superalgebras $S(n)$ and $\widetilde{S}(n)$

The Cartan type Lie superalgebras $S(n)$ and $\widetilde{S}(n)$, called special Lie superalgebras, are constructed as follows. Consider $\Theta(n)$ the associative superalgebra over $\Gamma(n)$ with generators denoted by $\xi\theta_1, \dots, \xi\theta_n$ and relations

$\xi\theta_i \wedge \xi\theta_j = -\xi\theta_j \wedge \xi\theta_i$. A $\mathbb{Z}_2$-gradation is induced by setting $\deg \xi\theta_i = \overline{1}$. Any element of $\Theta(n)$ is written as

$$\omega_k = \sum_{i_1<\ldots<i_k} a_{i_1\ldots i_k}\, \xi\theta_{i_1} \wedge \ldots \wedge \xi\theta_{i_k}$$

where $a_{i_1\ldots i_k} \in \Gamma(n)$.

One then defines the volume form superalgebra $S(\omega)$ as a $W(n)$ subsuperalgebra by

$$S(\omega) = \Big\{ D \in W(n) \,\Big|\, D(\omega) = 0 \Big\}$$

where $\omega = a(\theta_1,\ldots,\theta_n)\,\xi\theta_1 \wedge \ldots \wedge \xi\theta_n$ and $a \in \Gamma(n)_{\overline{0}}$, $a(0) \neq 0$.

Any element of $S(\omega)$ has the form

$$\sum_{i=1}^n P_i \frac{\partial}{\partial\theta_i} \qquad \text{with} \qquad \sum_{i=1}^n \frac{\partial(aP_i)}{\partial\theta_i} = 0$$

One sets also

$$S(n) = S\Big(\omega = \xi\theta_1 \wedge \ldots \wedge \xi\theta_n\Big) = \Big\{ D \in W(n) \,\Big|\, D\Big(\xi\theta_1 \wedge \ldots \wedge \xi\theta_n\Big) = 0 \Big\}$$

and for n even

$$\begin{aligned} \widetilde{S}(n) &= S\Big(\omega = (1+\theta_1\ldots\theta_n)\,\xi\theta_1 \wedge \ldots \wedge \xi\theta_n\Big) \\ &= \Big\{ D \in W(n) \,\Big|\, D\Big((1+\theta_1\ldots\theta_n)\,\xi\theta_1 \wedge \ldots \wedge \xi\theta_n\Big) = 0 \Big\} \end{aligned}$$

Elements of $S(n)$ are thus divergenceless derivations of $W(n)$:

$$S(n) = \Big\{ \sum_{i=1}^n P_i \frac{\partial}{\partial\theta_i} \in W(n) \,\Big|\, \sum_{i=1}^n \frac{\partial P_i}{\partial\theta_i} = 0 \Big\}$$

The Lie superalgebras $S(n)$ and $\widetilde{S}(n)$ have the following properties:

- $S(n)$ and $\widetilde{S}(n)$ have dimension $(n-1)2^n + 1$, the number of even generators being less (resp. greater) by 1 than the number of odd generators for n even (resp. odd).
- The superalgebra $S(n)$ is simple for $n \geq 3$ and $\widetilde{S}(n)$ is simple for $n \geq 4$.
- The semi-simple part of $S(n)_0$ and $\widetilde{S}(n)_0$ is isomorphic to $sl(n)$.
- The superalgebra $S(3)$ is isomorphic to $P(3)$.

- the $\mathbb{Z}$-graded Lie superalgebra $S(n)$ is transitive.
- Every automorphism of $S(n)$ with $n \geq 3$ and $\widetilde{S}(n)$ with $n \geq 4$ is induced by an automorphism of $\Gamma(n)$.
- Every superalgebra $S(\omega)$ is isomorphic either to $S(n)$ or $\widetilde{S}(n)$.
- The representation of $sl(n)$ in the subspace $S(n)_i$ ($i = -1, 0, \dots, n-2$) is in Young tableaux notation $[2^{i+1}1^{n-2-i}]$. For example we have (the subscripts stand for the $\mathbb{Z}$-gradation indices i):

for $S(4)$ $(\overline{4})_{-1} \oplus (15)_0 \oplus (\overline{20})_1 \oplus (\overline{10})_2$

for $S(5)$ $(\overline{5})_{-1} \oplus (24)_0 \oplus (\overline{45})_1 \oplus (\overline{40})_2 \oplus (\overline{15})_3$

for $S(6)$ $(\overline{6})_{-1} \oplus (35)_0 \oplus (\overline{84})_1 \oplus (\overline{105})_2 \oplus (\overline{70})_3 \oplus (\overline{21})_4$

Cartan type superalgebras $H(n)$

The Cartan type Lie superalgebras $H(n)$ and $\widetilde{H}(n)$, called Hamiltonian Lie superalgebras, are constructed as follows. Consider $\Omega(n)$ the associative superalgebra over $\Gamma(n)$ with generators denoted by $d\theta_1, \dots, d\theta_n$ and relations $d\theta_i \circ d\theta_j = d\theta_j \circ d\theta_i$. The $\mathbb{Z}_2$-gradation is induced by setting $\deg d\theta_i = \overline{0}$. Any element of $\Omega(n)$ is written as

$$\omega_k = \sum_{i_1 \leq \ldots \leq i_k} a_{i_1 \ldots i_k} \, d\theta_{i_1} \circ \ldots \circ d\theta_{i_k}$$

where $a_{i_1 \ldots i_k} \in \Gamma(n)$.

Among them are the Hamiltonian forms defined by

$$\omega = \sum_{i,j=1}^{n} a_{ij} \, d\theta_i \circ d\theta_j$$

where $a_{ij} \in \Gamma(n)$, $a_{ij} = a_{ji}$ and $\det(a_{ij}(0)) \neq 0$. One then defines for each Hamiltonian form ω the Hamiltonian form superalgebra $\widetilde{H}(\omega)$ as a $W(n)$ subsuperalgebra by

$$\widetilde{H}(\omega) = \Big\{ D \in W(n) \,\Big|\, D(\omega) = 0 \Big\}$$

and

$$H(\omega) = \Big[\!\Big[\widetilde{H}(\omega), \widetilde{H}(\omega) \Big]\!\Big]$$

Any element of $\widetilde{H}(\omega)$ has the form

$$\sum_{i=1}^{n} P_i \frac{\partial}{\partial \theta_i} \qquad \text{with} \qquad \frac{\partial}{\partial \theta_j} \sum_{t=1}^{n} a_{it} P_t + \frac{\partial}{\partial \theta_i} \sum_{t=1}^{n} a_{jt} P_t = 0$$

One sets also

$$\widetilde{H}(n) = \widetilde{H}\Big((d\theta_1)^2 + \ldots + (d\theta_n)^2\Big)$$
$$H(n) = \Big[\!\!\Big[\widetilde{H}(n), \widetilde{H}(n)\Big]\!\!\Big]$$

The Lie superalgebra $H(n)$ has the following properties:

- $H(n)$ has dimension $2^n - 2$, the number of even generators being equal (resp. less by 2) to (than) the number of odd generators for n odd (resp. even).
- The superalgebra $H(n)$ is simple for $n \geq 4$.
- The semi-simple part of $\widetilde{H}(n)_0$ is isomorphic to $so(n)$.
- The superalgebra $H(4)$ is isomorphic to $A(1,1)$.
- The $\mathbb{Z}$-graded Lie superalgebras $H(n)$ and $\widetilde{H}(n)$ are transitive.
- Every automorphism of $H(n)$ with $n \geq 4$ and of $\widetilde{H}(n)$ with $n \geq 3$ is induced by an automorphism of $\Gamma(n)$.
- The representation of $so(n)$ in the subspace $H(n)_i$ ($i = -1, 0, \ldots, n-3$) is given by the antisymmetric tensor of rank $i+2$. For example we have (the subscripts stand for the $\mathbb{Z}$-gradation indices i):

$$\begin{array}{ll} \text{for } H(4) & (4)_{-1} \oplus (6)_0 \oplus (4)_1 \\ \text{for } H(5) & (5)_{-1} \oplus (10)_0 \oplus (10)_1 \oplus (5)_2 \\ \text{for } H(10) & (10)_{-1} \oplus (45)_0 \oplus (120)_1 \oplus (210)_2 \oplus (252)_3 \oplus (210)_4 \\ & \oplus (45)_6 \oplus (10)_7 \oplus (120)_5 \end{array}$$

For more details, see refs. [46, 47, 48].

2.5 Casimir invariants

The study of Casimir invariants plays a great role in the representation theory of simple Lie algebras since their eigenvalues on a finite dimensional highest weight irreducible representation completely characterize this representation. In the case of Lie superalgebras, the situation is different. In fact, the eigenvalues of the Casimir invariants *do not* always characterize the finite dimensional highest weight irreducible representations of a Lie superalgebra. More precisely, their eigenvalues on a *typical* representation completely characterize this representation while they are identically vanishing on an *atypical* representation ($\rightarrow$ 2.40).

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra and $\mathcal{U}(\mathcal{G})$ its universal enveloping superalgebra ($\rightarrow$ 2.62). An element $C \in \mathcal{U}(\mathcal{G})$ such that $[\![C, X]\!] = 0$ for all $X \in \mathcal{U}(\mathcal{G})$ is called a *Casimir element* of $\mathcal{G}$ ($[\![\ ,\ ]\!]$ denotes the $\mathbb{Z}_2$-graded commutator). The algebra of the Casimir elements of $\mathcal{G}$ is the $\mathbb{Z}_2$-center of $\mathcal{U}(\mathcal{G})$, denoted by $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$. It is a ($\mathbb{Z}_2$-graded) subalgebra of $\mathcal{U}(\mathcal{G})$.

Standard sequences of Casimir elements of the basic Lie superalgebras can be constructed as follows. Let $\mathcal{G} = sl(m|n)$ with $m \neq n$ or $osp(m|n)$ be a basic Lie superalgebra with non-degenerate bilinear form. Let $\{E_{IJ}\}$ be a matrix basis of generators of $\mathcal{G}$ where $I, J = 1, \ldots, m+n$ with $\deg I = 0$ for $1 \leq I \leq m$ and $\deg I = 1$ for $m+1 \leq I \leq m+n$. Then defining $(\overline{E}^0)_{IJ} = \delta_{IJ}$ and $(\overline{E}^{p+1})_{IJ} = (-1)^{\deg K} E_{IK} (\overline{E}^p)_{KJ}$, a standard sequence of Casimir operators is given by

$$\begin{aligned} C_p &= \operatorname{str}(\overline{E}^p) = (-1)^{\deg I} (\overline{E}^p)_{II} \\ &= E_{II_1} (-1)^{\deg I_1} \ldots E_{I_k I_{k+1}} (-1)^{\deg I_{k+1}} \ldots E_{I_{p-1} I} \end{aligned}$$

Consider the $(m+n)^2$ elementary matrices e_{IJ} of order $m+n$ satisfying $(e_{IJ})_{KL} = \delta_{IL} \delta_{JK}$.

In the case of $sl(m|n)$ with $m \neq n$, a basis $\{E_{IJ}\}$ is given by the matrices

$$E_{ij} = e_{ij} - \frac{1}{m} \delta_{ij} \sum_{q=1}^{q=m} e_{qq}$$

$$E_{kl} = e_{kl} - \frac{1}{n} \delta_{kl} \sum_{q=m+1}^{q=m+n} e_{qq}$$

$$Y = -\frac{1}{m-n} \left(n \sum_{q=1}^{q=m} e_{qq} + m \sum_{q=m+1}^{q=m+n} e_{qq} \right)$$

for the even part and $E_{ik} = e_{ik}$, $E_{kj} = e_{kj}$ for the odd part, where $1 \le i,j \le m$ and $m+1 \le k,l \le m+n$. One finds for example $C_1 = 0$ and

$$C_2 = \sum_{i,j=1}^{m} E_{ij}E_{ji} - \sum_{k,l=m+1}^{m+n} E_{kl}E_{lk} + \sum_{i=1}^{m}\sum_{k=m+1}^{m+n} (E_{ki}E_{ik} - E_{ik}E_{ki}) - \frac{m-n}{mn}Y^2$$

In the case of $osp(m|n)$, a basis $\{E_{IJ}\}$ is given by the matrices

$$E_{IJ} = G_{IK}e_{KJ} + (-1)^{(1+\deg I)(1+\deg J)}G_{JK}e_{KI}$$

where the matrix G_{IJ} is defined in $\to$ 2.27. One finds for example $C_1 = 0$ and

$$C_2 = \sum_{i,j=1}^{m} E_{ij}E_{ji} - \sum_{i',j'=m+1}^{m+n} E_{i'j'}E_{j'i'} + \sum_{i=1}^{m}\sum_{i'=m+1}^{m+n} (E_{i'i}E_{ii'} - E_{ii'}E_{i'i})$$

where $1 \le i,j \le m$ and $m+1 \le i',j' \le m+n$.

One has to stress that unlike the algebraic case, the center $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ for the classical Lie superalgebras is in general *not finitely generated*. More precisely, the only classical Lie superalgebras for which the center $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ is finitely generated are $osp(1|2n)$. In that case, $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ is generated by n Casimirs invariants of degree $2, 4, \ldots, 2n$.

Example 1

Consider the superalgebra $sl(1|2)$ with generators H, Z, E^+, E^-, F^+, F^-, $\overline{F}^+$, $\overline{F}^-$ ($\to$ 2.53). Then one can prove that a generating system of the center $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ is given by, for $p \in \mathbb{N}$ and $H_\pm \equiv H \pm Z$:

$$\begin{aligned} C_{p+2} \;=\; & H_+H_-Z^p + E^-E^+(Z-\tfrac{1}{2})^p \\ & + \overline{F}^-F^+\Big(H_+Z^p - (H_+ + 1)(Z+\tfrac{1}{2})^p\Big) \\ & + F^-\overline{F}^+\Big((H_- + 1)(Z-\tfrac{1}{2})^p - H_-Z^p\Big) \\ & + (E^-\overline{F}^+F^+ + \overline{F}^-F^-E^+)\Big(Z^p - (Z-\tfrac{1}{2})^p\Big) \\ & + \overline{F}^-F^-\overline{F}^+F^+\Big((Z+\tfrac{1}{2})^p + (Z-\tfrac{1}{2})^p - 2Z^p\Big) \end{aligned}$$

In that case, the Casimir elements C_p satisfy the polynomial relations $C_pC_q = C_rC_s$ for $p+q = r+s$ where $p,q,r,s \ge 2$. □

Example 2

Consider the superalgebra $osp(1|2)$ with generators H, $E^{\pm}$, $F^{\pm}$ ($\rightarrow$ 2.52). In that case, the center $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ is finitely generated by

$$C_2 = H^2 + \tfrac{1}{2}(E^-E^+ + E^+E^-) - (F^+F^- - F^-F^+)$$

Moreover, there exists in the universal enveloping superalgebra $\mathcal{U}$ of $osp(1|2)$ an even operator S which is a square root of the Casimir operator C_2 that commutes with the even generators and anticommutes with the odd ones, given by

$$S = 2(F^+F^- - F^-F^+) + \tfrac{1}{4}$$

More precisely, it satisfies $S^2 = C_2 + \frac{1}{16}$. Such an operator exists for any superalgebra of the type $osp(1|2n)$ [1]). □

Harish–Chandra homomorphism

Consider a Borel decomposition $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ of $\mathcal{G}$ ($\rightarrow$ 2.45) where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ and set $\rho = \rho_0 - \rho_1$ where ρ_0 is the half-sum of positive even roots and ρ_1 the half-sum of positive odd roots. The universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$ can be decomposed as follows:

$$\mathcal{U}(\mathcal{G}) = \mathcal{U}(\mathcal{H}) \oplus (\mathcal{N}^-\mathcal{U}(\mathcal{G}) + \mathcal{U}(\mathcal{G})\mathcal{N}^+)$$

Then any element of the center $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ can be written as $z = z_0 + z'$ where $z_0 \in \mathcal{U}(\mathcal{H})$ and $z' \in \mathcal{N}^-\mathcal{U}(\mathcal{G}) + \mathcal{U}(\mathcal{G})\mathcal{N}^+$. Let $S(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$ be the symmetric algebra over $\mathcal{H}$. Consider the projection $\overline{h}\,:\ \mathcal{Z}_{\mathcal{U}}(\mathcal{G}) \rightarrow S(\mathcal{H})$, $z \mapsto z_0$ and γ the automorphism of $S(\mathcal{H})$ such that for all $H \in \mathcal{H}$ and $\lambda \in \mathcal{H}^*$, $\gamma(H(\lambda)) = H(\lambda - \rho)$. The mapping

$$h = \gamma \circ \overline{h}:\ \mathcal{Z}_{\mathcal{U}}(\mathcal{G}) \rightarrow S(\mathcal{H}),\ z \mapsto \gamma(z_0)$$

is called the Harish–Chandra homomorphism [49, 50].

Property

Let $S(\mathcal{H})^W$ be the subset of elements of $S(\mathcal{H})$ invariant under the Weyl group of $\mathcal{G}$ ($\rightarrow$ 2.63). Then the image of $\mathcal{Z}_{\mathcal{U}}(\mathcal{G})$ by the Harish–Chandra homomorphism is a subset of $S(\mathcal{H})^W$.

Example

Consider the Casimir elements C_p of $sl(1|2)$ given above. In the fermionic basis of $sl(1|2)$ ($\rightarrow$ 2.45), the positive (resp. negative) root generators are $E^+, F^+, \overline{F}^+$ (resp. $E^-, F^-, \overline{F}^-$) and $\rho = 0$.

It follows that the image of C_p by the Harish–Chandra homomorphism is given by

$$h(C_{p+2}) = H_+H_-Z^p = 2^{-p}H_+H_-(H_+ - H_-)^p$$

which is obviously invariant under the action of the Weyl group $H_+ \leftrightarrow -H_-$. □

For more details, see refs. [34, 45, 77, 80].

2.6 Centralizer, center, normalizer of a Lie superalgebra

The definitions of the centralizer, the center, the normalizer of a Lie superalgebra follow those of a Lie algebra.

Definition

Let $\mathcal{G}$ be a Lie superalgebra and $\mathcal{S}$ a subset of elements in $\mathcal{G}$.

– The centralizer $\mathcal{C}_\mathcal{G}(\mathcal{S})$ is the subset of $\mathcal{G}$ given by

$$\mathcal{C}_\mathcal{G}(\mathcal{S}) = \Big\{ X \in \mathcal{G} \,\Big|\, [\![X, Y]\!] = 0, \ \forall Y \in \mathcal{S} \Big\}$$

– The center $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ is the set of elements of $\mathcal{G}$ which commute with any element of $\mathcal{G}$ (in other words, it is the centralizer of $\mathcal{G}$ in $\mathcal{G}$):

$$\mathcal{Z}(\mathcal{G}) = \Big\{ X \in \mathcal{G} \,\Big|\, [\![X, Y]\!] = 0, \ \forall Y \in \mathcal{G} \Big\}$$

– The normalizer $\mathcal{N}_\mathcal{G}(\mathcal{S})$ is the subset of $\mathcal{G}$ given by

$$\mathcal{N}_\mathcal{G}(\mathcal{S}) = \Big\{ X \in \mathcal{G} \,\Big|\, [\![X, Y]\!] \in \mathcal{S}, \ \forall Y \in \mathcal{S} \Big\}$$

2.7 Characters and supercharacters

Let $\mathcal{G}$ be a basic Lie superalgebra with Cartan subalgebra $\mathcal{H}$. Consider $\mathcal{V}(\Lambda)$ a highest weight representation (→ 2.35) of $\mathcal{G}$ with highest weight Λ, the weight decomposition of $\mathcal{V}$ with respect to $\mathcal{H}$ is

$$\mathcal{V}(\Lambda) = \bigoplus_\lambda \mathcal{V}_\lambda \quad \text{where} \quad \mathcal{V}_\lambda = \Big\{ \vec{v} \in \mathcal{V} \,\Big|\, h(\vec{v}) = \lambda(h)\vec{v}, \ h \in \mathcal{H} \Big\}$$

Let e^λ be the formal exponential, function on $\mathcal{H}^*$ (dual of $\mathcal{H}$) such that $e^\lambda(\mu) = \delta_{\lambda,\mu}$ for two elements $\lambda, \mu \in \mathcal{H}^*$, which satisfies $e^\lambda e^\mu = e^{\lambda+\mu}$.

The *character* and *supercharacter* of $\mathcal{V}(\Lambda)$ are defined by

$$\operatorname{ch}\mathcal{V}(\Lambda) = \sum_{\lambda} (\dim \mathcal{V}_\lambda) e^{\lambda}$$
$$\operatorname{sch}\mathcal{V}(\Lambda) = \sum_{\lambda} (-1)^{\deg\lambda} (\dim \mathcal{V}_\lambda) e^{\lambda}$$

Let $W(\mathcal{G})$ be the Weyl group ($\rightarrow$ 2.63) of $\mathcal{G}$, Δ the root system of $\mathcal{G}$, $\Delta_{\overline{0}}^{+}$ the set of positive even roots, $\Delta_{\overline{1}}^{+}$ the set of positive odd roots, $\overline{\Delta}_{\overline{0}}^{+}$ the subset of roots $\alpha \in \Delta_{\overline{0}}^{+}$ such that $\alpha/2 \notin \Delta_{\overline{1}}^{+}$. We set for an element $w \in W(\mathcal{G})$, $\varepsilon(w) = (-1)^{\ell(w)}$ and $\varepsilon'(w) = (-1)^{\ell'(w)}$ where $\ell(w)$ is the number of reflections in the expression of $w \in W(\mathcal{G})$ and $\ell'(w)$ is the number of reflections with respect to the roots of $\overline{\Delta}_{\overline{0}}^{+}$ in the expression of $w \in W(\mathcal{G})$. We denote by ρ_0 and ρ_1 the half-sums of positive even roots and positive odd roots, and $\rho = \rho_0 - \rho_1$. The characters and supercharacters of the typical finite dimensional representations $\mathcal{V}(\Lambda)$ ($\rightarrow$ 2.40) of the basic Lie superalgebras are given by

$$\operatorname{ch}\mathcal{V}(\Lambda) = L^{-1} \sum_{w} \varepsilon(w) e^{w(\Lambda+\rho)}$$
$$\operatorname{sch}\mathcal{V}(\Lambda) = L'^{-1} \sum_{w} \varepsilon'(w) e^{w(\Lambda+\rho)}$$

where

$$L = \frac{\prod_{\alpha\in\Delta_{\overline{0}}^{+}} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha\in\Delta_{\overline{1}}^{+}} (e^{\alpha/2} + e^{-\alpha/2})} \qquad \text{and} \qquad L' = \frac{\prod_{\alpha\in\overline{\Delta}_{\overline{0}}^{+}} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha\in\Delta_{\overline{1}}^{+}} (e^{\alpha/2} - e^{-\alpha/2})}$$

In the case of the superalgebra $B(0,n)$ all the representations are typical. One finds then explicitly

$$\operatorname{ch}\mathcal{V}(\Lambda) = \frac{\sum_w \varepsilon(w) e^{w(\Lambda+\rho)}}{\sum_w \varepsilon(w) e^{w(\rho)}} \qquad \text{and} \qquad \operatorname{sch}\mathcal{V}(\Lambda) = \frac{\sum_w \varepsilon'(w) e^{w(\Lambda+\rho)}}{\sum_w \varepsilon'(w) e^{w(\rho)}}$$

In the case of the superalgebra $A(m,n)$, the character of the typical representation $\mathcal{V}(\Lambda)$ is given by

$$\operatorname{ch}\mathcal{V}(\Lambda) = \frac{1}{L_0} \sum_{w} \varepsilon(w) w \left(e^{\Lambda+\rho_0} \prod_{\beta\in\Delta_{\overline{1}}^{+}} (1 + e^{-\beta}) \right)$$

and the character of the singly atypical representation by (see ref. [93])

$$\mathrm{ch}\,\mathcal{V}(\Lambda) = \frac{1}{L_0} \sum_w \varepsilon(w) w \left(e^{\Lambda+\rho_0} \prod_{\beta \in \Delta_{\overline{1}}^+, \langle \Lambda+\rho|\beta\rangle \neq 0} (1+e^{-\beta}) \right)$$

where L_0 is defined as

$$\prod_{\alpha \in \Delta_{\overline{0}}^+} (e^{\alpha/2} - e^{-\alpha/2})$$

In the case of the superalgebra $C(n+1)$, the highest weight irreducible representations are either typical or singly atypical. It follows that the character formulae of the typical and atypical representations of $C(n+1)$ are the same as for $A(m,n)$ above (with the symbols being those of $C(n+1)$).

$\rightarrow$ 2.35 Representations: highest weight representations, 2.36 Representations: induced modules, 2.40 Representations: typicality and atypicality.

For more details, see refs. [49, 93].

2.8 Classical Lie superalgebras

Definition

A simple Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ is called *classical* if the representation of the even subalgebra $\mathcal{G}_{\overline{0}}$ on the odd part $\mathcal{G}_{\overline{1}}$ is completely reducible.

Theorem

A simple Lie superalgebra $\mathcal{G}$ is classical if and only if its even part $\mathcal{G}_{\overline{0}}$ is a reductive Lie algebra.

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra. Then the representation of $\mathcal{G}_{\overline{0}}$ on $\mathcal{G}_{\overline{1}}$ is either:

1. irreducible; the superalgebra is said to be of type II.

or

2. the direct sum of two irreducible representations of $\mathcal{G}_{\overline{0}}$; the superalgebra is said to be of type I. In that case, one has $\mathcal{G}_{\overline{1}} = \mathcal{G}_{-1} \oplus \mathcal{G}_1$ with

$$\left\{\mathcal{G}_{-1}, \mathcal{G}_1\right\} = \mathcal{G}_{\overline{0}} \qquad \text{and} \qquad \left\{\mathcal{G}_1, \mathcal{G}_1\right\} = \left\{\mathcal{G}_{-1}, \mathcal{G}_{-1}\right\} = 0$$

Table 2.2 resumes the classification, while Table 2.3 gives the $\mathbb{Z}_2$-gradation ($\mathcal{G}_{\overline{0}}$ and $\mathcal{G}_{\overline{1}}$ structure) of the classical Lie superalgebras $\mathcal{G}$. In Table 2.3, the parentheses denote the dimension of the representations of the non-Abelian part of $\mathcal{G}_{\overline{0}}$ and the value of the $U(1)$, and the brackets [ ] are a Young tableau notation (→ 1.96).

Table 2.2: Classical Lie superalgebras.

	type I	type II
BASIC (non-degenerate Killing form)	$A(m,n)$ $m > n \geq 0$ $C(n+1)$ $n \geq 1$	$B(m,n)$ $m \geq 0, n \geq 1$ $D(m,n)$ $\begin{cases} m \geq 2, n \geq 1 \\ m \neq n+1 \end{cases}$ $F(4)$ $G(3)$
BASIC (zero Killing form)	$A(n,n)$ $n \geq 1$	$D(n+1,n)$ $n \geq 1$ $D(2,1;\alpha)$ $\alpha \notin \{0,-1\}$
STRANGE	$P(n)$ $n \geq 2$	$Q(n)$ $n \geq 2$

Table 2.3: $\mathbb{Z}_2$-gradation of the classical Lie superalgebras.

superalgebra $\mathcal{G}$	even part $\mathcal{G}_{\overline{0}}$	odd part $\mathcal{G}_{\overline{1}}$
$A(m,n)$	$A_m \oplus A_n \oplus U(1)$	$(m,\overline{n},1) \oplus (\overline{m},n,-1)$
$A(n,n)$	$A_n \oplus A_n$	$(n,\overline{n}) \oplus (\overline{n},n)$
$C(n+1)$	$C_n \oplus U(1)$	$(2n,1) \oplus (2n,-1)$
$B(m,n)$	$B_m \oplus C_n$	$(2m+1,2n)$
$D(m,n)$	$D_m \oplus C_n$	$(2m,2n)$
$F(4)$	$A_1 \oplus B_3$	$(2,8)$
$G(3)$	$A_1 \oplus G_2$	$(2,7)$
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus A_1$	$(2,2,2)$
$P(n)$	A_n	$[2] \oplus [1^{n-1}]$
$Q(n)$	A_n	$\text{ad}(A_n)$

Definition

A classical Lie superalgebra $\mathcal{G}$ is called *basic* if there exists a non-degenerate invariant bilinear form on $\mathcal{G}$ (→ 2.23). The classical Lie superalgebras which are not basic are called *strange*.

Theorem

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical basic Lie superalgebra. Then there exists a consistent $\mathbb{Z}$-gradation (→ 2.24) $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\, \mathcal{G}_i$ of $\mathcal{G}$, called the distinguished $\mathbb{Z}$-gradation, such that

– for superalgebras of type I, $\mathcal{G}_i = 0$ for $|i| > 1$ and $\mathcal{G}_{\overline{0}} = \mathcal{G}_0$, $\mathcal{G}_{\overline{1}} = \mathcal{G}_{-1} \oplus \mathcal{G}_1$.
– for superalgebras of type II, $\mathcal{G}_i = 0$ for $|i| > 2$ and $\mathcal{G}_{\overline{0}} = \mathcal{G}_{-2} \oplus \mathcal{G}_0 \oplus \mathcal{G}_2$, $\mathcal{G}_{\overline{1}} = \mathcal{G}_{-1} \oplus \mathcal{G}_1$.

Table 2.4 gives the $\mathbb{Z}$-gradation structure of the classical basic Lie superalgebras (same notations as for Table 2.3).

Table 2.4: $\mathbb{Z}$-gradation of the classical basic Lie superalgebras.

$\mathcal{G}$	$\mathcal{G}_0$	$\mathcal{G}_1 \oplus \mathcal{G}_{-1}$	$\mathcal{G}_2 \oplus \mathcal{G}_{-2}$
$A(m,n)$	$A_m \oplus A_n \oplus U(1)$	$(m,\overline{n}) \oplus (\overline{m},n)$	$\emptyset$
$A(n,n)$	$A_n \oplus A_n$	$(n,\overline{n}) \oplus (\overline{n},n)$	$\emptyset$
$C(n+1)$	$C_n \oplus U(1)$	$(2n)_+ \oplus (2n)_-$	$\emptyset$
$B(m,n)$	$B_m \oplus A_{n-1} \oplus U(1)$	$(2m+1,n) \oplus (2m+1,\overline{n})$	$[2] \oplus [2^{n-1}]$
$D(m,n)$	$D_m \oplus A_{n-1} \oplus U(1)$	$(2m,n) \oplus (2m,\overline{n})$	$[2] \oplus [2^{n-1}]$
$F(4)$	$B_3 \oplus U(1)$	$8_+ \oplus 8_-$	$1_+ \oplus 1_-$
$G(3)$	$G_2 \oplus U(1)$	$7_+ \oplus 7_-$	$1_+ \oplus 1_-$
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus U(1)$	$(2,2)_+ \oplus (2,2)_-$	$1_+ \oplus 1_-$

→ 2.18–2.20 Exceptional Lie superalgebras, 2.27 Orthosymplectic superalgebras, 2.48–2.49 Strange superalgebras, 2.61 Unitary superalgebras.

For more details, see refs. [46, 47, 48, 79].

2.9 Classification of simple Lie superalgebras

Among Lie superalgebras appearing in the classification of simple Lie superalgebras, one distinguishes two general families: the classical Lie superalgebras in which the representation of the even subalgebra on the odd part is completely reducible, and the Cartan type superalgebras in which such a property is no longer valid. Among the classical superalgebras (→ 2.8), one naturally separates the basic series from the strange ones.

The basic (or contragredient) Lie superalgebras split into four infinite families denoted by $A(m,n)$ or $sl(m+1|n+1)$ for $m \neq n$ and $A(n,n)$ or $sl(n+1|n+1)/\mathcal{Z} = psl(n+1|n+1)$ where $\mathcal{Z}$ is a one-dimensional center for $m = n$ (unitary series), $B(m,n)$ or $osp(2m+1|2n)$, $C(n)$ or $osp(2|2n)$, $D(m,n)$ or $osp(2m|2n)$ (orthosymplectic series) and three exceptional superalgebras $F(4)$, $G(3)$ and $D(2,1;\alpha)$, the last one being actually a one-parameter family of superalgebras. Two infinite families denoted by $P(n)$ and $Q(n)$ constitute the strange (or non-contragredient) superalgebras.

The Cartan type superalgebras (→ 2.4) are classified into four infinite families, $W(n)$, $S(n)$, $\widetilde{S}(n)$ and $H(n)$.

The following scheme resumes this classification:

Simple Lie superalgebras

↙ ↘

Classical Lie superalgebras — Cartan type superalgebras $W(n), S(n)$, $\widetilde{S}(n), H(n)$

↙ ↘

Basic Lie superalgebras $A(m,n), B(m,n)$, $C(n+1), D(m,n)$, $F(4), G(3), D(2,1;\alpha)$ — Strange superalgebras $P(n), Q(n)$

→ 2.4 Cartan type superalgebras, 2.8 Classical Lie superalgebras.

For more details, see refs. [46, 47, 48, 64, 69].

2.10 Clifford algebra

Let $\{\gamma_i\}$ $(i = 1, \dots, n)$ be a set of square matrices such that

$$\Big\{\gamma_i, \gamma_j\Big\} = \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbb{I}$$

where $\mathbb{I}$ is the unit matrix. The algebra spanned by the n matrices γ_i is called the Clifford algebra. These relations can be satisfied by matrices of order 2^p when $n = 2p$ or $n = 2p + 1$.

Consider the 2×2 Pauli matrices σ_1, σ_2, σ_3:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the matrices γ_i can be expressed in terms of a p-fold tensor product of the Pauli matrices.

Property

There exists a representation such that

1. if n is even, the matrices γ_i are hermitian, half of them being symmetric, half of them being antisymmetric.
2. if n is odd, the matrices γ_i with $i = 1, \dots, 2p$ are hermitian, half of them being symmetric, half of them being antisymmetric and the matrix γ_{2p+1} is diagonal.

In this representation, the matrices γ can be written as $(i = 1, \dots, p-1)$

$$\begin{aligned}
\gamma_1 &= \sigma_1^{(1)} \otimes \dots \otimes \sigma_1^{(p)} \\
\gamma_{2i} &= \sigma_1^{(1)} \otimes \dots \otimes \sigma_1^{(p-i)} \otimes \sigma_2^{(p-i+1)} \otimes \mathbb{I}^{(p-i+2)} \otimes \dots \otimes \mathbb{I}^{(p)} \\
\gamma_{2i+1} &= \sigma_1^{(1)} \otimes \dots \otimes \sigma_1^{(p-i)} \otimes \sigma_3^{(p-i+1)} \otimes \mathbb{I}^{(p-i+2)} \otimes \dots \otimes \mathbb{I}^{(p)} \\
\gamma_{2p} &= \sigma_2^{(1)} \otimes \mathbb{I}^{(2)} \otimes \dots \otimes \mathbb{I}^{(p)} \\
\gamma_{2p+1} &= \sigma_3^{(1)} \otimes \mathbb{I}^{(2)} \otimes \dots \otimes \mathbb{I}^{(p)}
\end{aligned}$$

One can check that with this representation, one has $(i = 1, \dots, p)$

$$\gamma_{2i}^t = -\gamma_{2i} \qquad \text{and} \qquad \gamma_{2i+1}^t = \gamma_{2i+1} \,, \gamma_{2p+1}^t = \gamma_{2p+1}$$

Definition

The matrix $C = \prod_{i=1}^{p} \gamma_{2i-1}$ for $n = 2p$ and $C = \prod_{i=1}^{p+1} \gamma_{2i-1}$ for $n = 2p+1$ is called the charge conjugation matrix.

Property

The charge conjugation matrix satisfies

- $C^t C = 1$
- for $n = 2p$
  $$C^t = (-1)^{p(p-1)/2} C = \begin{cases} C & \text{for } p = 0, 1 \pmod 4 \\ -C & \text{for } p = 2, 3 \pmod 4 \end{cases}$$
  $$C\gamma_i = (-1)^{p+1} \gamma_i^t C \quad (i = 1, \dots, 2p)$$
- for $n = 2p + 1$
  $$C^t = (-1)^{p(p+1)/2} C = \begin{cases} C & \text{for } p = 0, 3 \pmod 4 \\ -C & \text{for } p = 1, 2 \pmod 4 \end{cases}$$
  $$C\gamma_i = (-1)^{p} \gamma_i^t C \quad (i = 1, \dots, 2p+1)$$

2.11 Decompositions w.r.t. $osp(1|2)$ subalgebras

The method for finding the decompositions of the fundamental and the adjoint representations of the basic Lie superalgebras with respect to their different $osp(1|2)$ subsuperalgebras is the following:

1. One considers an $osp(1|2)$ embedding in a basic Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$, determined by a certain subsuperalgebra $\mathcal{K}$ in $\mathcal{G}$ ($\rightarrow$ 2.16), which is expressed as a direct sum of simple components: $\mathcal{K} = \oplus_i \mathcal{K}_i$.

2. To each couple $(\mathcal{G}, \mathcal{K}_i)$ one associates $osp(1|2)$ representations given in Table 3.76; the notations $\mathcal{R}$ and $\mathcal{R}''$ are explained below.

3. The decomposition of the fundamental representation of $\mathcal{G}$ with respect to the $osp(1|2)$ subalgebra under consideration is then given by a direct sum of $osp(1|2)$ representations.

4. Starting from a decomposition of the fundamental representation of $\mathcal{G}$ of the form

$$\text{fund}_{\mathcal{K}}\, \mathcal{G} = \Big(\underset{i}{\oplus} \mathcal{R}_{j_i}\Big) \oplus \Big(\underset{k}{\oplus} \mathcal{R}''_{j_k}\Big)$$

the decomposition of the adjoint representation $\text{ad}_{\mathcal{K}}\, \mathcal{G}$ is given in the unitary series by

$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\underset{i}{\oplus} \mathcal{R}_{j_i} \underset{k}{\oplus} \mathcal{R}''_{j_k}\Big) \otimes \Big(\underset{i}{\oplus} \mathcal{R}_{j_i} \underset{k}{\oplus} \mathcal{R}''_{j_k}\Big) - \mathcal{R}_0 \quad \text{for } \ sl(m|n)\,, m \neq n$$

$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\underset{i}{\oplus} \mathcal{R}_{j_i} \underset{k}{\oplus} \mathcal{R}''_{j_k}\Big) \otimes \Big(\underset{i}{\oplus} \mathcal{R}_{j_i} \underset{k}{\oplus} \mathcal{R}''_{j_k}\Big) - 2\mathcal{R}_0 \quad \text{for } \ psl(n|n)$$

and in the orthosymplectic series by

$$\text{ad}_{\mathcal{K}}\ \mathcal{G} = \Big(\underset{i}{\oplus} \mathcal{R}_{j_i}\Big)^{\otimes 2}\Big|_A \oplus \Big(\underset{k}{\oplus} \mathcal{R}''_{j_k}\Big)^{\otimes 2}\Big|_S \oplus \Big(\underset{i}{\oplus} \mathcal{R}_{j_i}\Big) \otimes \Big(\underset{k}{\oplus} \mathcal{R}''_{j_k}\Big)$$

The symmetrized and antisymmetrized products of $osp(1|2)$ representations $\mathcal{R}_j$ are expressed, by analogy with the Lie algebra case, by (in the following formulae j and q are integer)

$$\mathcal{R}_j \otimes \mathcal{R}_j\Big|_A = \bigoplus_{q=1}^{j} \Big(\mathcal{R}_{2q-1} \oplus \mathcal{R}_{2q-1/2}\Big)$$

$$\mathcal{R}_j \otimes \mathcal{R}_j\Big|_S = \bigoplus_{q=0}^{j-1} \Big(\mathcal{R}_{2q} \oplus \mathcal{R}_{2q+1/2}\Big) \oplus \mathcal{R}_{2j}$$

$$\mathcal{R}_{j-1/2} \otimes \mathcal{R}_{j-1/2}\Big|_A = \bigoplus_{q=0}^{j-1} \Big(\mathcal{R}_{2q} \oplus \mathcal{R}_{2q+1/2}\Big)$$

$$\mathcal{R}_{j-1/2} \otimes \mathcal{R}_{j-1/2}\Big|_S = \bigoplus_{q=1}^{j-1} \Big(\mathcal{R}_{2q-1} \oplus \mathcal{R}_{2q-1/2}\Big) \oplus \mathcal{R}_{2j-1}$$

together with (for j, k integer or half-integer)

$$(\mathcal{R}_j \oplus \mathcal{R}_k)^{\otimes 2}\Big|_A = \Big(\mathcal{R}_j \otimes \mathcal{R}_j\Big)\Big|_A \oplus \Big(\mathcal{R}_k \otimes \mathcal{R}_k\Big)\Big|_A \oplus (\mathcal{R}_j \oplus \mathcal{R}_k)$$

$$(\mathcal{R}_j \oplus \mathcal{R}_k)^{\otimes 2}\Big|_S = \Big(\mathcal{R}_j \otimes \mathcal{R}_j\Big)\Big|_S \oplus \Big(\mathcal{R}_k \otimes \mathcal{R}_k\Big)\Big|_S \oplus (\mathcal{R}_j \oplus \mathcal{R}_k)$$

and (n integer)

$$\begin{aligned}(n\mathcal{R}_j \otimes n\mathcal{R}_j)\Big|_A &= \frac{n(n+1)}{2}(\mathcal{R}_j \otimes \mathcal{R}_j)\Big|_A \oplus \frac{n(n-1)}{2}(\mathcal{R}_j \otimes \mathcal{R}_j)\Big|_S \\ (n\mathcal{R}_j \otimes n\mathcal{R}_j)\Big|_S &= \frac{n(n+1)}{2}(\mathcal{R}_j \otimes \mathcal{R}_j)\Big|_S \oplus \frac{n(n-1)}{2}(\mathcal{R}_j \otimes \mathcal{R}_j)\Big|_A\end{aligned}$$

The same formulae also hold for the $\mathcal{R}''$ representations.

Let us stress that one has to introduce here two different notations for the $osp(1|2)$ representations which enter into the decomposition of the fundamental representation of $\mathcal{G}$, depending on the origin of the two factors $\mathcal{D}_j$ and $\mathcal{D}_{j-1/2}$ of a representation $\mathcal{R}_j$ (we recall that an $osp(1|2)$ representation $\mathcal{R}_j$ decomposes under the $sl(2)$ part as $\mathcal{R}_j = \mathcal{D}_j \oplus \mathcal{D}_{j-1/2}$). For $\mathcal{G} = sl(m|n)$ (resp. $\mathcal{G} = osp(m|n)$), an $osp(1|2)$ representation is denoted $\mathcal{R}_j$ if the representation $\mathcal{D}_j$ comes from the decomposition of the fundamental of $sl(m)$ (resp. $so(m)$), and $\mathcal{R}''_j$ if the representation $\mathcal{D}_j$ comes from the decomposition of the fundamental of $sl(n)$ (resp. $sp(n)$).

In the same way, considering the tensor products of $\mathcal{R}$ and $\mathcal{R}''$ representations given above, one has to distinguish the $osp(1|2)$ representations in the decomposition of the adjoint representations: the $\mathcal{R}_j$ representations are such that the $\mathcal{D}_j$ comes from the decomposition of the even part $\mathcal{G}_{\overline{0}}$ for j integer or of the odd part $\mathcal{G}_{\overline{1}}$ for j half-integer and the $\mathcal{R}'_j$ representations are such that $\mathcal{D}_j$ comes from the decomposition of the even part $\mathcal{G}_{\overline{0}}$ for j half-integer or of the odd part $\mathcal{G}_{\overline{1}}$ for j integer.

Finally, the products between unprimed and primed representations obey the following rules

$$\mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} = \begin{cases} \oplus\mathcal{R}_{j_3} & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus\mathcal{R}'_{j_3} & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

$$\mathcal{R}''_{j_1} \otimes \mathcal{R}''_{j_2} = \begin{cases} \oplus\mathcal{R}_{j_3} & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus\mathcal{R}'_{j_3} & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

$$\mathcal{R}_{j_1} \otimes \mathcal{R}''_{j_2} = \begin{cases} \oplus\mathcal{R}'_{j_3} & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus\mathcal{R}_{j_3} & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

Tables 3.76–3.85 give the different decompositions of the fundamental and adjoint representations of the basic Lie superalgebras with respect to the different $osp(1|2)$ embeddings. For more details, see ref. [22].

2.12 Decompositions w.r.t. $sl(1|2)$ subalgebras

The method for finding the decompositions of the fundamental and the adjoint representations of the basic Lie superalgebras with respect to their different $sl(1|2)$ subsuperalgebras is the following:

1. One considers a $sl(1|2)$ embedding in a basic Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$, determined by a certain subsuperalgebra $\mathcal{K}$ in $\mathcal{G}$ ($\rightarrow$ 2.17), which is expressed as a direct sum of simple components: $\mathcal{K} = \oplus_i \mathcal{K}_i$.

2. To each pair $(\mathcal{G}, \mathcal{K}_i)$ one associates (atypical) $sl(1|2)$ representations $\pi(\pm j_i, j_i) \equiv \pi_\pm(j_i)$ or $osp(2|2)$ representations $\pi(0, \frac{1}{2})$ ($\rightarrow$ 2.53) given in the following table:

$\mathcal{G}$	$\mathcal{K}$	$\text{fund}_{\mathcal{K}}\, \mathcal{G}$
$sl(m\|n)$	$sl(p+1\|p)$	$\pi_+(\frac{p}{2})$
	$sl(p\|p+1)$	$\pi''_+(\frac{p}{2})$
$osp(m\|2n)$	$sl(p+1\|p)$	$\pi_+(\frac{p}{2}) \oplus \pi_-(\frac{p}{2})$
	$sl(p\|p+1)$	$\pi''_+(\frac{p}{2}) \oplus \pi''_-(\frac{p}{2})$
	$osp(2\|2)$	$\pi''(0, \frac{1}{2})$

   (The notation π or π'' is just to distinguish between the superalgebras $sl(p+1|p)$ or $sl(p|p+1)$ they come from. This will be used below).

   In the case of $sl(m|n)$, one could also use π_- and π''_- representations as well, leading to different but equivalent decompositions of the adjoint representation of $\mathcal{G}$. This fact is related to the existence of non-trivial outer automorphisms for $sl(1|2)$.

3. The decomposition of the fundamental representation of $\mathcal{G}$ with respect to the $sl(1|2)$ subalgebra under consideration is then given by a direct sum of $sl(1|2)$ representations of the above type, eventually completed by trivial representations.

4. Starting from a decomposition of the fundamental representation of $\mathcal{G}$ of the form

$$\text{fund}_{\mathcal{K}}\, \mathcal{G} = \Big(\oplus_i\, \pi_\pm(j_i) \Big) \oplus \Big(\oplus_k\, \pi''_\pm(j_k) \Big)$$

   the decomposition of the adjoint representation $\text{ad}_{\mathcal{K}}\, \mathcal{G}$ is given in the unitary series by

$$\text{ad}_{\mathcal{K}}\, \mathcal{G} = \Big(\oplus_i\, \pi_\pm(j_i) \oplus_k \pi''_\pm(j_k) \Big)^{\otimes 2} - \pi(0,0) \qquad \text{for } sl(m|n)\,, m \neq n$$

$$\text{ad}_{\mathcal{K}}\, \mathcal{G} = \Big(\oplus_i\, \pi_\pm(j_i) \oplus_k \pi''_\pm(j_k) \Big)^{\otimes 2} - 2\pi(0,0) \qquad \text{for } psl(n|n)$$

   and in the orthosymplectic series by

$$\text{ad}_{\mathcal{K}}\, \mathcal{G} = \Big(\oplus_i\, \pi_\pm(j_i) \Big)^{\otimes 2}\Big|_A \oplus \Big(\oplus_k\, \pi''_\pm(j_k) \Big)^{\otimes 2}\Big|_S \oplus \Big(\oplus_i\, \pi_\pm(j_i) \oplus_i \pi''_\pm(j_k) \Big)$$

The symmetrized and antisymmetrized products of atypical $sl(1|2)$ representations are given by

$$\begin{aligned}
\Big(\pi_\pm(j)\oplus\pi_\pm(k)\Big)^{\otimes 2}\Big|_A &= \Big(\pi_\pm(j)\otimes\pi_\pm(j)\Big)\Big|_A \oplus \Big(\pi_\pm(k)\otimes\pi_\pm(k)\Big)\Big|_A \\
&\quad \oplus\Big(\pi_\pm(j)\otimes\pi_\pm(k)\Big) \\
\Big(\pi_\pm(j)\oplus\pi_\pm(k)\Big)^{\otimes 2}\Big|_S &= \Big(\pi_\pm(j)\otimes\pi_\pm(j)\Big)\Big|_S \oplus \Big(\pi_\pm(k)\otimes\pi_\pm(k)\Big)\Big|_S \\
&\quad \oplus\Big(\pi_\pm(j)\otimes\pi_\pm(k)\Big)
\end{aligned}$$

and (n integer)

$$\begin{aligned}
\Big(n\pi_\pm(j)\otimes n\pi_\pm(j)\Big)\Big|_A &= \frac{n(n+1)}{2}\Big(\pi_\pm(j)\otimes\pi_\pm(j)\Big)\Big|_A \\
&\quad \oplus \frac{n(n-1)}{2}\Big(\pi_\pm(j)\otimes\pi_\pm(j)\Big)\Big|_S \\
\Big(n\pi_\pm(j)\otimes n\pi_\pm(j)\Big)\Big|_S &= \frac{n(n+1)}{2}\Big(\pi_\pm(j)\oplus\pi_\pm(j)\Big)\Big|_S \\
&\quad \oplus \frac{n(n-1)}{2}\Big(\pi_\pm(j)\otimes\pi_\pm(j)\Big)\Big|_A
\end{aligned}$$

where (in the following formulae j and q are integer)

$$\begin{aligned}
\Big(\pi_+(j)\oplus\pi_-(j)\Big)^{\otimes 2}\Big|_A &= \bigoplus_{q=0}^{2j}\pi(0,q)\ \bigoplus_{q=1}^{j}\pi(2j+\tfrac{1}{2},2q-\tfrac{1}{2}) \\
&\quad \bigoplus_{q=1}^{j}\pi(-2j-\tfrac{1}{2},2q-\tfrac{1}{2}) \\
\Big(\pi_+(j)\oplus\pi_-(j)\Big)^{\otimes 2}\Big|_S &= \bigoplus_{q=0}^{2j}\pi(0,q)\ \bigoplus_{q=0}^{j-1}\pi(2j+\tfrac{1}{2},2q+\tfrac{1}{2}) \\
&\quad \bigoplus_{q=0}^{j}\pi(-2j-\tfrac{1}{2},2q+\tfrac{1}{2})\oplus\pi_+(2j)\oplus\pi_-(2j)
\end{aligned}$$

$$\begin{aligned}
\Big(\pi_+(j+\tfrac{1}{2})\oplus\pi_-(j+\tfrac{1}{2})\Big)^{\otimes 2}\Big|_A &= \bigoplus_{q=0}^{2j+1}\pi(0,q)\ \bigoplus_{q=0}^{j}\pi(2j+\tfrac{3}{2},2q+\tfrac{1}{2}) \\
&\quad \bigoplus_{q=0}^{j}\pi(-2j-\tfrac{3}{2},2q+\tfrac{1}{2}) \\
\Big(\pi_+(j+\tfrac{1}{2})\oplus\pi_-(j+\tfrac{1}{2})\Big)^{\otimes 2}\Big|_S &= \bigoplus_{q=0}^{2j+1}\pi(0,q)\ \bigoplus_{q=1}^{j}\pi(2j+\tfrac{3}{2},2q-\tfrac{1}{2}) \\
&\quad \bigoplus_{q=1}^{j}\pi(-2j-\tfrac{3}{2},2q-\tfrac{1}{2}) \\
&\quad \oplus\pi_+(2j+1)\oplus\pi_-(2j+1)
\end{aligned}$$

Finally, in the case of $osp(2|2)$ embeddings, the product of the $\pi(0, \frac{1}{2})$ representation by itself is not fully reducible but gives rise to the indecomposable $sl(1|2)$ representation of the type $\pi(0; -\frac{1}{2}, \frac{1}{2}; 0)$ ($\rightarrow$ 2.53).

Considering the tensor products of π and π'' representations given above, one has to distinguish the $sl(1|2)$ representations in the decomposition of the adjoint representations. Let us recall that a $sl(1|2)$ representation $\pi(b, j)$ decomposes under the $sl(2) \oplus U(1)$ part as $\pi(b,j) = D_j(b) \oplus D_{j-1/2}(b-1/2) \oplus D_{j-1/2}(b+1/2) \oplus D_{j-1}(b)$ and $\pi_\pm(j) = D_j(\pm j) \oplus D_{j-1/2}(\pm j \pm 1/2)$. The $\pi(b,j)$ representations are such that the $\mathcal{D}_j$ comes from the decomposition of the even part $\mathcal{G}_{\overline{0}}$ for j integer or of the odd part $\mathcal{G}_{\overline{1}}$ for j half-integer and the $\pi'(b,j)$ representations are such that $\mathcal{D}_j$ comes from the decomposition of the even part $\mathcal{G}_{\overline{0}}$ for j half-integer or of the odd part $\mathcal{G}_{\overline{1}}$ for j integer. Finally, the products between unprimed and primed representations obey the following rules

$$\pi(b_1, j_1) \otimes \pi(b_2, j_2) = \begin{cases} \oplus \pi(b_3, j_3) & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus \pi'(b_3, j_3) & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

$$\pi''(b_1, j_1) \otimes \pi''(b_2, j_2) = \begin{cases} \oplus \pi(b_3, j_3) & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus \pi'(b_3, j_3) & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

$$\pi(b_1, j_1) \otimes \pi''(b_2, j_2) = \begin{cases} \oplus \pi'(b_3, j_3) & \text{if } j_1 + j_2 \text{ is integer} \\ \oplus \pi(b_3, j_3) & \text{if } j_1 + j_2 \text{ is half-integer} \end{cases}$$

Tables 3.86–3.89 give the different decompositions of the fundamental and adjoint representations of the basic Lie superalgebras with respect to the different $sl(1|2)$ embeddings.

For more details, see ref. [74].

2.13 Derivation of a Lie superalgebra

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a Lie superalgebra. A *derivation* D of degree $\deg D \in \mathbb{Z}_2$ of the superalgebra $\mathcal{G}$ is an endomorphism of $\mathcal{G}$ such that

$$D[\![X, Y]\!] = [\![D(X), Y]\!] + (-1)^{\deg D . \deg X} [\![X, D(Y)]\!]$$

If $\deg D = \overline{0}$, the derivation is even, otherwise $\deg D = \overline{1}$ and the derivation is odd.

The space of all the derivations of $\mathcal{G}$ is denoted by $\mathrm{Der}\,\mathcal{G} = \mathrm{Der}_{\overline{0}}\,\mathcal{G} \oplus \mathrm{Der}_{\overline{1}}\,\mathcal{G}$. If D and D' are two derivations of $\mathcal{G}$, then $[\![D, D']\!] \in \mathrm{Der}\,\mathcal{G}$, that is the space $\mathrm{Der}\,\mathcal{G}$ closes under the Lie superbracket.

The space Der $\mathcal{G}$ is called the *superalgebra of derivations* of $\mathcal{G}$. In particular,

$$\mathrm{ad}_X : Y \mapsto \mathrm{ad}_X(Y) = [\![X, Y]\!]$$

is a derivation of $\mathcal{G}$. These derivations are called inner derivations of $\mathcal{G}$. They form an ideal Inder $\mathcal{G}$ of Der $\mathcal{G}$. Every derivation of a simple Lie superalgebra with non-degenerate Killing form is inner.

2.14 Dirac matrices

→ 2.10 Clifford algebra, 2.47 Spinors (in the Lorentz group), 2.59 Supersymmetry algebra: definition, 2.54 Superconformal algebra.

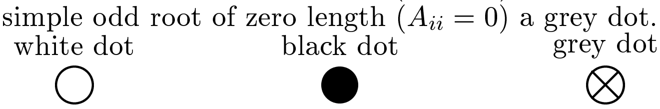
2.15 Dynkin diagrams

Let $\mathcal{G}$ be a basic Lie superalgebra of rank r and dimension n with Cartan subalgebra $\mathcal{H}$. Let $\Delta^0 = (\alpha_1, \dots, \alpha_r)$ be a simple root system (→ 2.45) of $\mathcal{G}$, A be the associated Cartan matrix and $A' = (A'_{ij})$ be the corresponding symmetric Cartan matrix (→ 2.2), defined by $A'_{ij} = \alpha_i \cdot \alpha_j$.

One can associate to Δ^0 a Dynkin diagram according to the following rules.

1. Using the Cartan matrix A:

   (a) One associates to each simple even root a white dot, to each simple odd root of non-zero length ($A_{ii} \neq 0$) a black dot and to each simple odd root of zero length ($A_{ii} = 0$) a grey dot.

   (b) The i-th and j-th dots will be joined by η_{ij} lines where

   $$\eta_{ij} = \max\Big(|A_{ij}|, |A_{ji}|\Big)$$

   (c) We add an arrow on the lines connecting the i-th and j-th dots when $\eta_{ij} > 1$ and $|A_{ij}| \neq |A_{ji}|$, pointing from j to i if $|A_{ij}| > 1$.

   (d) For $D(2,1;\alpha)$, $\eta_{ij} = 1$ if $A_{ij} \neq 0$ and $\eta_{ij} = 0$ if $A_{ij} = 0$. No arrow is put on the Dynkin diagram.

2. Using the symmetric Cartan matrix A':

   (a) One associates to each simple even root a white dot, to each simple odd root of non-zero length ($A'_{ii} \neq 0$) a black dot and to each simple odd root of zero length ($A'_{ii} = 0$) a grey dot (see pictures above).

(b) The i-th and j-th dots will be joined by η_{ij} lines where

$$\eta_{ij} = \frac{2|A'_{ij}|}{\min(|A'_{ii}|, |A'_{jj}|)} \quad \text{if } A'_{ii}.A'_{jj} \neq 0$$
$$\eta_{ij} = \frac{2|A'_{ij}|}{\min(|A'_{ii}|, 2)} \quad \text{if } A'_{ii} \neq 0 \text{ and } A'_{jj} = 0$$
$$\eta_{ij} = |A'_{ij}| \quad \text{if } A'_{ii} = A'_{jj} = 0$$

(c) We add an arrow on the lines connecting the i-th and j-th dots when $\eta_{ij} > 1$, pointing from i to j if $A'_{ii}.A'_{jj} \neq 0$ and $|A'_{ii}| > |A'_{jj}|$ or if $A'_{ii} = 0$, $A'_{jj} \neq 0$, $|A'_{jj}| < 2$, and pointing from j to i if $A'_{ii} = 0$, $A'_{jj} \neq 0$, $|A'_{jj}| > 2$.

(d) For $D(2,1;\alpha)$, $\eta_{ij} = 1$ if $A'_{ij} \neq 0$ and $\eta_{ij} = 0$ if $A'_{ij} = 0$. No arrow is put on the Dynkin diagram.

Although the rules seem more complicated when using the symmetric Cartan matrix A', note that the computation of the Cartan matrix A is more involved than the symmetric Cartan matrix A'.

Since a basic Lie superalgebra possesses many inequivalent simple root systems (→ 2.45), for a basic Lie superalgebra there will be many inequivalent Dynkin diagrams. For each basic Lie superalgebra, there is a particular Dynkin diagram which can be considered as canonical. Its characteristic is that it contains exactly one odd root. Such a Dynkin diagram is called distinguished.

Definition

The *distinguished Dynkin diagram* is the Dynkin diagram associated to a distinguished simple root system (→ 2.45) (note that the Dynkin diagrams corresponding to equivalent simple root systems are the same). It is constructed as follows. Consider the distinguished $\mathbb{Z}$-gradation of $\mathcal{G}$ (→ 2.8 and Table 2.4): $\mathcal{G} = \oplus_{i\in\mathbb{Z}} \mathcal{G}_i$. The even dots are given by the Dynkin diagram of $\mathcal{G}_0$ (it may be not connected) and the odd dot corresponds to the lowest weight of the representation $\mathcal{G}_1$ of $\mathcal{G}_0$.

Lists of the distinguished Dynkin diagrams of the basic Lie superalgebras are given in Tables 3.52–3.60), while in Table 3.61 are displayed all Dynkin diagrams of the basic Lie superalgebras of rank $r \leq 4$.

→ 2.2 Cartan matrix, 2.45 Simple root systems.

For more details, see refs. [23, 47, 48].

2.16 Embeddings of $osp(1|2)$

The determination of the possible $osp(1|2)$ subsuperalgebras of a basic Lie superalgebra $\mathcal{G}$ can be seen as the supersymmetric version of the Dynkin classification of $sl(2)$ subalgebras in a simple Lie algebra. Interest for this problem appears in the framework of supersymmetric integrable models (for instance super-Toda theories) and super-W algebras [22, 52]. As in the algebraic case, it uses the notion of principal (here superprincipal) embedding.

Definition

Let $\mathcal{G}$ be a basic Lie superalgebra of rank r with simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ and corresponding simple root generators $e_i^\pm$ in the Serre–Chevalley basis ($\to$ 2.44). The generators of the $osp(1|2)$ *superprincipal embedding* in $\mathcal{G}$ are defined by

$$F^+ = \sum_{i=1}^{r} e_i^+ \,, \qquad F^- = \sum_{i=1}^{r}\sum_{j=1}^{r} A^{ij} e_i^-$$

A_{ij} being the Cartan matrix of $\mathcal{G}$ and $A^{ij} = (A^{-1})_{ij}$. The even generators of the superprincipal $osp(1|2)$ are given by anticommutation of the odd generators F^+ and F^-:

$$H = 2\Big\{F^+, F^-\Big\} \qquad E^+ = 2\Big\{F^+, F^+\Big\} \qquad E^- = -2\Big\{F^-, F^-\Big\}$$

Not all the basic Lie superalgebras admit an $osp(1|2)$ superprincipal embedding. It is clear from the expression of the $osp(1|2)$ generators that a superprincipal embedding can be defined only if the superalgebra under consideration admits a completely odd simple root system (which corresponds to a Dynkin diagram with no white dot). This condition is however necessary but not sufficient (the superalgebra $A(n|n)$ does not admit a superprincipal embedding although it has a completely odd simple root system). The basic Lie superalgebras admitting a superprincipal $osp(1|2)$ are the following:

$$\begin{array}{l} sl(n \pm 1|n), \;\; osp(2n \pm 1|2n), \;\; osp(2n|2n), \\ osp(2n+2|2n), \;\; D(2,1;\alpha) \;\; (\alpha \neq 0, \pm 1) \end{array}$$

The classification of the $osp(1|2)$ embeddings of a basic Lie superalgebra $\mathcal{G}$ is given by the following theorem.

Theorem

1. Any $osp(1|2)$ embedding in a basic Lie superalgebra $\mathcal{G}$ can be considered as the superprincipal $osp(1|2)$ subsuperalgebra of a regular subsuperalgebra $\mathcal{K}$ of $\mathcal{G}$.

2. For $\mathcal{G} = osp(2n \pm 2|2n)$ with $n \geq 2$, besides the $osp(1|2)$ superprincipal embeddings of item 1, there exist $osp(1|2)$ embeddings associated to the singular embeddings $osp(2k \pm 1|2k) \oplus osp(2n - 2k \pm 1|2n - 2k) \subset osp(2n \pm 2|2n)$ with $1 \leq k \leq \left[\frac{n-1}{2}\right]$.

3. For $\mathcal{G} = osp(2n|2n)$ with $n \geq 2$, besides the $osp(1|2)$ superprincipal embeddings of item 1, there exist $osp(1|2)$ embeddings associated to the singular embeddings $osp(2k \pm 1|2k) \oplus osp(2n-2k \mp 1|2n-2k) \subset osp(2n|2n)$ with $1 \leq k \leq \left[\frac{n-2}{2}\right]$.

2.17 Embeddings of $sl(2|1)$

In the same way one can consider $osp(1|2)$ embeddings of a basic Lie superalgebra, it is possible to determine the $sl(2|1)$ subsuperalgebras of a basic Lie superalgebra $\mathcal{G}$. This problem was recently considered for an exhaustive classification and characterization of all extended $N = 2$ superconformal algebras and all string theories obtained by gauging $N = 2$ Wess–Zumino–Witten models [74]. Let us consider the basic Lie superalgebra $sl(n+1|n)$ with completely odd simple root system Δ^0:

$$\Delta^0 = \Big\{ \varepsilon_1 - \delta_1,\ \delta_1 - \varepsilon_2,\ \varepsilon_2 - \delta_2,\ \dots,\ \delta_{n-1} - \varepsilon_n,\ \varepsilon_n - \delta_n,\ \delta_n - \varepsilon_{n+1} \Big\}$$

Denoting by $E_{\pm(\varepsilon_i - \delta_i)}$, $E_{\pm(\delta_i - \varepsilon_{i+1})}$ $(1 \leq i \leq n)$ the simple root generators in the Serre–Chevalley basis ($\to$ 2.44), the $sl(2|1)$ *superprincipal embedding* in $sl(n+1|n)$ is defined as follows. The odd part of the superprincipal $sl(2|1)$ is generated by $F_{\pm\alpha}$ and $F_{\pm\beta}$ where

$$F_{+\alpha} = \sum_{i=1}^{n} E_{\delta_i - \varepsilon_{i+1}} \qquad F_{-\alpha} = \sum_{i=1}^{n} \sum_{j=1}^{2n} A^{j,2i} E_{-\delta_i + \varepsilon_{i+1}}$$
$$F_{+\beta} = \sum_{i=1}^{n} E_{\varepsilon_i - \delta_i} \qquad F_{-\beta} = \sum_{i=1}^{n} \sum_{j=1}^{2n} A^{j,2i-1} E_{-\varepsilon_i + \delta_i}$$

A_{ij} being the Cartan matrix of $sl(n+1|n)$, $A^{ij} = (A^{-1})_{ij}$ and A_{ij} is chosen as

$$A = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

The even generators of the superprincipal $sl(2|1)$ are given by anticommutation of the odd generators $F_{\pm\alpha}$ and $F_{\pm\beta}$:

$$\Big\{F_{+\alpha}, F_{-\alpha}\Big\} = H_+ + H_- \qquad \Big\{F_{+\beta}, F_{-\beta}\Big\} = H_+ - H_-$$

$$\Big\{F_{\pm\alpha}, F_{\pm\alpha}\Big\} = \Big\{F_{\pm\beta}, F_{\pm\beta}\Big\} = 0 \qquad \Big\{F_{\pm\alpha}, F_{\pm\beta}\Big\} = E_\pm$$

One obtains finally

$$\begin{aligned} &\Big[E_\pm, F_{\pm\alpha}\Big] = 0 && \Big[E_\pm, F_{\pm\beta}\Big] = 0 \\ &\Big[E_\pm, F_{\mp\alpha}\Big] = \mp F_{\pm\beta} && \Big[E_\pm, F_{\mp\beta}\Big] = \mp F_{\pm\alpha} \\ &\Big[H_\pm, F_{+\alpha}\Big] = \pm\tfrac{1}{2} F_{+\alpha} && \Big[H_\pm, F_{-\alpha}\Big] = \mp\tfrac{1}{2} F_{-\alpha} \\ &\Big[H_\pm, F_{+\beta}\Big] = \tfrac{1}{2} F_{+\beta} && \Big[H_\pm, F_{-\beta}\Big] = -\tfrac{1}{2} F_{-\beta} \\ &\Big[H_+, E_\pm\Big] = \pm E_\pm && \Big[H_-, E_\pm\Big] = 0 \end{aligned}$$

where

$$H_\pm = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{2n} (a^{j,2i} H_{2i} \pm a^{j,2i-1} H_{2i-1})$$

This $sl(2|1)$ superprincipal embedding contains as maximal subsuperalgebra the superprincipal $osp(1|2)$ with generators $H_+, E_\pm$ and $F_\pm = F_{\pm\alpha} + F_{\pm\beta}$.

The classification of the $sl(2|1)$ embeddings of a basic Lie superalgebra $\mathcal{G}$ is given by the following theorem.

Theorem

Let $\mathcal{G}$ be a basic Lie superalgebra. Any $sl(2|1)$ embedding into $\mathcal{G}$ can be seen as the principal embedding of a (sum of) regular $sl(n|n \pm 1)$ subsuperalgebra of $\mathcal{G}$, except in the case of $osp(m|n)$ $(m > 1)$, $F(4)$ and $D(2,1;\alpha)$ where the (sum of) regular $osp(2|2)$ has also to be considered.

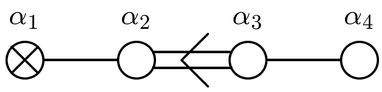
2.18 Exceptional Lie superalgebra $F(4)$

The Lie superalgebra $F(4)$ of rank 4 has dimension 40. The even part (of dimension 24) is a non-compact form of $sl(2) \oplus so(7)$ and the odd part (of dimension 16) is the spinor representation $(2, 8)$ of $sl(2) \oplus so(7)$. In terms of the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and δ such that $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$, $\delta \cdot \delta = -3$, $\varepsilon_i \cdot \delta = 0$, the root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ is given by

$$\Delta_{\overline{0}} = \Big\{ \pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \tfrac{1}{2}(\pm\delta \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3) \Big\}$$

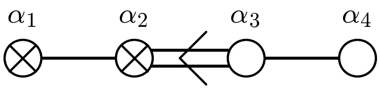
The different simple root systems of $F(4)$ with the corresponding Dynkin diagrams and Cartan matrices are the following:

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_2 = \varepsilon_3, \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_1 - \varepsilon_2\Big\}$



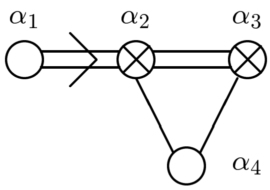
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \frac{1}{2}(-\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \alpha_2 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 + \varepsilon_3), \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_1 - \varepsilon_2\Big\}$



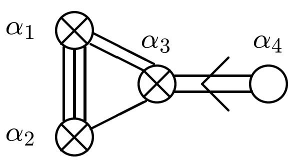
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta + \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \alpha_4 = \varepsilon_3\Big\}$



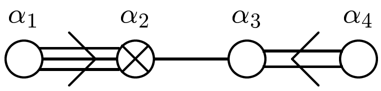
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_2 = \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3), \alpha_4 = \varepsilon_2 - \varepsilon_3\Big\}$



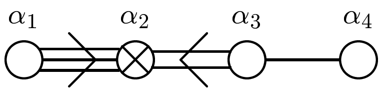
$$\begin{pmatrix} 0 & 3 & -2 & 0 \\ 3 & 0 & -1 & 0 \\ -2 & -1 & 0 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_3 = \varepsilon_3, \alpha_4 = \varepsilon_2 - \varepsilon_3\Big\}$



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

- Simple root system $\Delta^0 = \Big\{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \alpha_3 = \varepsilon_1 - \varepsilon_2, \alpha_4 = \varepsilon_2 - \varepsilon_3\Big\}$



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The superalgebra $F(4)$ is of type II. Denoting by $\mathcal{H}$ the distinguished Cartan subalgebra, the distinguished $\mathbb{Z}$-gradation ($\rightarrow$ 2.8) has the following structure:

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where $(1 \le i < j \le 3)$

$$\mathcal{G}_0 = \mathcal{H} \cup \{E_{\pm\varepsilon_i \pm\varepsilon_j},\, E_{\pm\varepsilon_i}\} \qquad \mathcal{G}_2 = \{E_\delta\} \qquad \mathcal{G}_{-2} = \{E_{-\delta}\}$$

$$\mathcal{G}_1 = \{E_{\frac{1}{2}(\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3)}\} \qquad \mathcal{G}_{-1} = \{E_{\frac{1}{2}(\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3)}\}$$

Denoting by T_i where $i = 1, 2, 3$ the generators of $sl(2)$, by $M_{pq} = -M_{qp}$ where $1 \le p \ne q \le 7$ the generators of $so(7)$ and by $F_{\alpha\mu}$ where $\alpha = +, -$ and $1 \le \mu \le 8$ the generators of the odd part, the commutation relations of $F(4)$ read as:

$$\Big[T_i, T_j\Big] = i\varepsilon_{ijk} T_k \qquad \Big[T_i, M_{pq}\Big] = 0$$

$$\Big[M_{pq}, M_{rs}\Big] = \delta_{qr} M_{ps} + \delta_{ps} M_{qr} - \delta_{pr} M_{qs} - \delta_{qs} M_{pr}$$

$$\Big[T_i, F_{\alpha\mu}\Big] = \tfrac{1}{2}\sigma^i_{\beta\alpha} F_{\beta\mu} \qquad \Big[M_{pq}, F_{\alpha\mu}\Big] = \tfrac{1}{2}(\gamma_p\gamma_q)_{\nu\mu} F_{\alpha\nu}$$

$$\Big\{F_{\alpha\mu}, F_{\beta\nu}\Big\} = 2C^{(8)}_{\mu\nu}(C^{(2)}\sigma^i)_{\alpha\beta} T_i + \tfrac{1}{3}\, C^{(2)}_{\alpha\beta}(C^{(8)}\gamma_p\gamma_q)_{\mu\nu} M_{pq}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices and $C^{(2)}$ ($= i\sigma^2$) is the 2×2 charge conjugation matrix. The 8-dimensional matrices γ_p form a Clifford algebra $\{\gamma_p, \gamma_q\} = 2\delta_{pq}$ and $C^{(8)}$ is the 8×8 charge conjugation matrix. They can be chosen, $\mathbb{I}$ being the 2×2 unit matrix, as ($\rightarrow$ 2.10):

$$\gamma_1 = \sigma^1 \otimes \sigma^3 \otimes \mathbb{I}\,, \qquad \gamma_2 = \sigma^1 \otimes \sigma^1 \otimes \sigma^3\,, \qquad \gamma_3 = \sigma^1 \otimes \sigma^1 \otimes \sigma^1$$

$$\gamma_4 = \sigma^2 \otimes \mathbb{I} \otimes \mathbb{I}\,, \qquad \gamma_5 = \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}\,, \qquad \gamma_6 = \sigma^1 \otimes \sigma^1 \otimes \sigma^2$$

$$\gamma_7 = \sigma^3 \otimes \mathbb{I} \otimes \mathbb{I}$$

The generators in the Cartan–Weyl basis are given by (with obvious notations):

$$
\begin{array}{llll}
H_2 = iM_{41} & H_3 = iM_{52} & H_4 = iM_{63} & H_1 = T_3 \\
E_{\pm\varepsilon_1} = \frac{i}{\sqrt{2}}(M_{17} \pm iM_{47}) & & E_{\pm\varepsilon_2} = \frac{i}{\sqrt{2}}(M_{27} \pm iM_{57}) & \\
E_{\pm\varepsilon_3} = \frac{i}{\sqrt{2}}(M_{37} \pm iM_{67}) & & E_{\pm\delta} = T_1 \pm iT_2 &
\end{array}
$$

$$E_{\pm(\varepsilon_1+\varepsilon_2)} = \tfrac{i}{2}(M_{12} \pm iM_{42} + M_{54} \pm iM_{15})$$

$$E_{\pm(\varepsilon_1-\varepsilon_2)} = \tfrac{i}{2}(M_{12} \pm iM_{42} - M_{54} \mp iM_{15})$$

$$E_{\pm(\varepsilon_2+\varepsilon_3)} = \tfrac{i}{2}(M_{23} \pm iM_{53} + M_{65} \pm iM_{26})$$

$$E_{\pm(\varepsilon_2-\varepsilon_3)} = \tfrac{i}{2}(M_{23} \pm iM_{53} - M_{65} \mp iM_{26})$$

$$E_{\pm(\varepsilon_1+\varepsilon_3)} = \tfrac{i}{2}(M_{13} \pm iM_{43} + M_{64} \pm iM_{16})$$

$$E_{\pm(\varepsilon_1-\varepsilon_3)} = \tfrac{i}{2}(M_{13} \pm iM_{43} - M_{64} \mp iM_{16})$$

$$E_{\frac{1}{2}(\pm\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3)} = F_{\alpha,\mu}$$

where in the last equation the index α and the sign in $\pm\delta$ are in one-to-one correspondence and the correspondence between the index μ and the signs in $\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$ is given by $(1,2,3,4,5,6,7,8) = (+++,+--,--+,-+-,-++,---,+-+,++-)$.

For more details, see refs. [16, 47, 64].

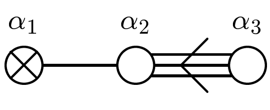
2.19 Exceptional Lie superalgebra $G(3)$

The Lie superalgebra $G(3)$ of rank 3 has dimension 31. The even part (of dimension 17) is a non-compact form of $sl(2) \oplus G_2$ and the odd part (of dimension 14) is the representation $(2,7)$ of $sl(2) \oplus G_2$. In terms of the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ and δ such that $\varepsilon_i \cdot \varepsilon_j = 1 - 3\delta_{ij}$, $\delta \cdot \delta = 2$, $\varepsilon_i \cdot \delta = 0$, the root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ is given by

$$\Delta_{\overline{0}} = \Big\{ \pm 2\delta,\ \varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \pm \varepsilon_i \pm \delta,\ \pm\delta \Big\}$$

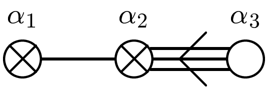
The different simple root systems of $G(3)$ with the corresponding Dynkin diagrams and Cartan matrices are the following:

- Simple root system $\Delta^0 = \Big\{\alpha_1 = \delta + \varepsilon_3, \alpha_2 = \varepsilon_1, \alpha_3 = \varepsilon_2 - \varepsilon_1\Big\}$



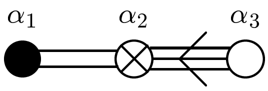
$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = -\delta - \varepsilon_3, \alpha_2 = \delta - \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1\Big\}$



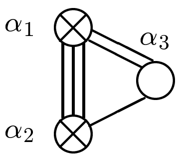
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \delta, \alpha_2 = -\delta + \varepsilon_1, \alpha_3 = \varepsilon_2 - \varepsilon_1\Big\}$



$$\begin{pmatrix} 2 & -2 & 0 \\ 2 & 0 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{\alpha_1 = \delta - \varepsilon_1, \alpha_2 = -\delta + \varepsilon_2, \alpha_3 = \varepsilon_1\Big\}$



$$\begin{pmatrix} 0 & 3 & -2 \\ 3 & 0 & -1 \\ -2 & -1 & 2 \end{pmatrix}$$

The superalgebra $G(3)$ is of type II. Denoting by $\mathcal{H}$ the distinguished Cartan subalgebra, the distinguished $\mathbb{Z}$-gradation ($\to$ 2.8) has the following structure:

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where $(1 \le i \ne j \le 3)$

$$\mathcal{G}_0 = \mathcal{H} \cup \{E_{\varepsilon_i - \varepsilon_j},\, E_{\pm\varepsilon_i}\} \qquad \mathcal{G}_2 = \{E_{2\delta}\} \qquad \mathcal{G}_{-2} = \{E_{-2\delta}\}$$

$$\mathcal{G}_1 = \{E_{\delta\pm\varepsilon_i},\, E_\delta\} \qquad \mathcal{G}_{-1} = \{E_{-\delta\pm\varepsilon_i},\, E_{-\delta}\}$$

In order to write the commutation relations of $G(3)$, it is convenient to use a $so(7)$ basis. Consider the $so(7)$ generators $M_{pq} = -M_{qp}$ where $1 \le p \ne q \le 7$. The singular embedding $G_2 \subset so(7)$ is obtained by imposing to the generators M_{pq} the constraints ($\to$ 1.30)

$$\xi_{ijk} M_{ij} = 0$$

where the symbol ξ_{ijk} is completely antisymmetric and whose non-vanishing components are

$$\xi_{123} = \xi_{145} = \xi_{176} = \xi_{246} = \xi_{257} = \xi_{347} = \xi_{365} = 1$$

Denoting by T_i where $i = 1,2,3$ the generators of $sl(2)$, by $F_{\alpha p}$ where $\alpha = +,-$ and $1 \le p \le 7$ the generators of the odd part, the commutation relations of $G(3)$ read as:

$$\Big[T_i, T_j\Big] = i\varepsilon_{ijk}T_k \qquad \Big[T_i, M_{pq}\Big] = 0$$

$$\Big[M_{pq}, M_{rs}\Big] = \delta_{qr}M_{ps} + \delta_{ps}M_{qr} - \delta_{pr}M_{qs} - \delta_{qs}M_{pr} + \tfrac{1}{3}\xi_{pqu}\xi_{rsv}M_{uv}$$

$$\Big[T_i, F_{\alpha p}\Big] = \tfrac{1}{2}\sigma^i_{\alpha\beta}F_{\beta p} \qquad \Big[M_{pq}, F_{\alpha r}\Big] = \tfrac{2}{3}\delta_{qr}F_{\alpha p} - \tfrac{2}{3}\delta_{pr}F_{\alpha q} + \tfrac{1}{3}\zeta_{pqrs}F_{\alpha s}$$

$$\Big\{F_{\alpha p}, F_{\beta q}\Big\} = 2\delta_{pq}(C\sigma^i)_{\alpha\beta}T_i + \tfrac{3}{2}C_{\alpha\beta}M_{pq}$$

where the symbol ζ_{pqrs} is completely antisymmetric and whose non-vanishing components are

$$\zeta_{1247} = \zeta_{1265} = \zeta_{1364} = \zeta_{1375} = \zeta_{2345} = \zeta_{2376} = \zeta_{4576} = 1$$

It can be written as

$$\zeta_{pqrs} = \delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + \sum_{u=1}^{7}\xi_{pqu}\xi_{rsu}$$

The σ^i are the Pauli matrices and C $(= i\sigma^2)$ is the 2×2 charge conjugation matrix.

In terms of the M_{pq}, the generators of G_2, denoted by E_i and E'_i where $1 \le i \le 7$, are given in section 1.30. The generators E_a with $a = 1,\dots,8$ $(E_8 \equiv E'_3)$ generate $sl(3)$ while the generators E'_i with $1 \le i \le 7$, $i \ne 3$ transform as the 3 and $\overline{3}$ representations of $sl(3)$. The generators E_3 and E_8 constitute a Cartan basis of the G_2 algebra. One can also take a basis H_1, H_2, H_3 such that $H_1 + H_2 + H_3 = 0$ given by $H_1 = \frac{1}{2}(E_3 + \frac{\sqrt{3}}{3}E_8)$, $H_2 = \frac{1}{2}(-E_3 + \frac{\sqrt{3}}{3}E_8)$, $H_3 = -\frac{\sqrt{3}}{3}E_8$. The generators of $G(3)$ in the Cartan–Weyl basis are then given by (with obvious notations):

$H_1 = \frac{1}{2}(E_3 + \frac{\sqrt{3}}{3}E_8)$	$H_2 = \frac{1}{2}(-E_3 + \frac{\sqrt{3}}{3}E_8)$	$H_3 = -\frac{\sqrt{3}}{3}E_8$
$E_{\pm(\varepsilon_1-\varepsilon_2)} = E_1 \pm iE_2$	$E_{\pm(\varepsilon_2-\varepsilon_3)} = E_6 \pm iE_7$	$E_{\pm(\varepsilon_1-\varepsilon_3)} = E_4 \pm iE_5$
$E_{\pm\varepsilon_1} = E'_7 \mp iE'_6$	$E_{\pm\varepsilon_2} = E'_4 \mp iE'_5$	$E_{\pm\varepsilon_3} = E'_1 \mp iE'_2$
$E_{\pm\delta+\varepsilon_1} = F_{\pm1} + iF_{\pm4}$	$E_{\pm\delta+\varepsilon_2} = F_{\pm7} + iF_{\pm2}$	$E_{\pm\delta+\varepsilon_3} = F_{\pm3} + iF_{\pm6}$
$E_{\pm\delta-\varepsilon_1} = F_{\pm1} - iF_{\pm4}$	$E_{\pm\delta-\varepsilon_2} = F_{\pm7} - iF_{\pm2}$	$E_{\pm\delta-\varepsilon_3} = F_{\pm3} - iF_{\pm6}$
$H_4 = T_3$	$E_{\pm2\delta} = T_1 \pm iT_2$	$E_{\pm\delta} = F_{\pm5}$

For more details, see refs. [16, 47, 64].

2.20 Exceptional Lie superalgebras $D(2,1;\alpha)$

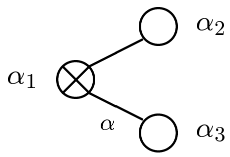
The Lie superalgebras $D(2,1;\alpha)$ with $\alpha \neq 0, -1, \infty$ form a one-parameter family of superalgebras of rank 3 and dimension 17. The even part (of dimension 9) is a non-compact form of $sl(2) \oplus sl(2) \oplus sl(2)$ and the odd part (of dimension 8) is the spinor representation $(2,2,2)$ of the even part. In terms of the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1^2 = -(1+\alpha)/2$, $\varepsilon_2^2 = 1/2$, $\varepsilon_3^2 = \alpha/2$ and $\varepsilon_i \cdot \varepsilon_j = 0$ if $i \neq j$, the root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ is given by

$$\Delta_{\overline{0}} = \Big\{ \pm 2\varepsilon_i \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \Big\}$$

$D(2,1;\alpha)$ is actually a deformation of the superalgebra $D(2,1)$ which corresponds to the case $\alpha = 1$.

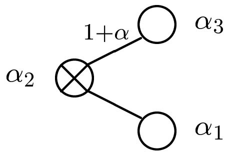
The different simple root systems of $D(2,1;\alpha)$ with the corresponding Dynkin diagrams and Cartan matrices are the following:

• Simple root system $\Delta^0 = \Big\{ \alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3, \alpha_2 = 2\varepsilon_2, \alpha_3 = 2\varepsilon_3 \Big\}$



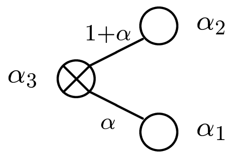
$$\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{ \alpha_1 = 2\varepsilon_2, \alpha_2 = -\varepsilon_1 - \varepsilon_2 + \varepsilon_3, \alpha_3 = 2\varepsilon_1 \Big\}$



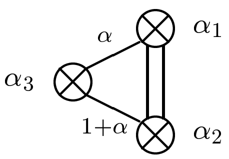
$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1+\alpha \\ 0 & -1 & 2 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{ \alpha_1 = 2\varepsilon_3, \alpha_2 = 2\varepsilon_1, \alpha_3 = -\varepsilon_1 + \varepsilon_2 - \varepsilon_3 \Big\}$



$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -\alpha & 1+\alpha & 0 \end{pmatrix}$$

• Simple root system $\Delta^0 = \Big\{ \alpha_1 = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \alpha_2 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \Big\}$



$$\begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & -1-\alpha \\ \alpha & -1-\alpha & 0 \end{pmatrix}$$

(the labels on the links are equal to the absolute values of the scalar products of the simple roots which are linked.)

The superalgebra $D(2,1;\alpha)$ is of type II. Denoting by $\mathcal{H}$ the distinguished Cartan subalgebra, the distinguished $\mathbb{Z}$-gradation ($\rightarrow$ 2.8) has the following structure:

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where

$$\mathcal{G}_0 = \mathcal{H} \cup \{E_{\pm 2\varepsilon_1}, E_{\pm 2\varepsilon_2}\} \qquad \mathcal{G}_2 = \{E_{2\varepsilon_3}\} \qquad \mathcal{G}_{-2} = \{E_{-2\varepsilon_3}\}$$

$$\mathcal{G}_1 = \{E_{\varepsilon_3 \pm \varepsilon_1 \pm \varepsilon_2}\} \qquad \mathcal{G}_{-1} = \{E_{-\varepsilon_3 \pm \varepsilon_1 \pm \varepsilon_2}\}$$

Denoting by $T_i^{(a)}$ where $i = 1,2,3$ and $a = 1,2,3$ the generators of the three $sl(2)$ and by $F_{\beta\beta'\beta''}$ where $\beta, \beta', \beta'' = +, -$, the generators of the odd part, the commutation relations of $D(2,1;\alpha)$ read as:

$$\left[T_i^{(a)}, T_j^{(b)}\right] = i\delta_{ab}\varepsilon_{ijk}T_k^{(a)} \qquad \left[T_i^{(1)}, F_{\beta\beta'\beta''}\right] = \tfrac{1}{2}\sigma^i_{\gamma\beta}F_{\gamma\beta'\beta''}$$

$$\left[T_i^{(2)}, F_{\beta\beta'\beta''}\right] = \tfrac{1}{2}\sigma^i_{\gamma'\beta'}F_{\beta\gamma'\beta''} \qquad \left[T_i^{(3)}, F_{\beta\beta'\beta''}\right] = \tfrac{1}{2}\sigma^i_{\gamma''\beta''}F_{\beta\beta'\gamma''}$$

$$\left\{F_{\beta\beta'\beta''}, F_{\gamma\gamma'\gamma''}\right\} = s_1 C_{\beta'\gamma'}C_{\beta''\gamma''}(C\sigma^i)_{\beta\gamma}T_i^{(1)} + s_2 C_{\beta''\gamma''}C_{\beta\gamma}(C\sigma^i)_{\beta'\gamma'}T_i^{(2)}$$
$$+ s_3 C_{\beta\gamma}C_{\beta'\gamma'}(C\sigma^i)_{\beta''\gamma''}T_i^{(3)}$$

where $s_1 + s_2 + s_3 = 0$ is imposed by the generalized Jacobi identity. The σ^i are the Pauli matrices and C $(= i\sigma^2)$ is the 2×2 charge conjugation matrix. Since the superalgebras defined by the triplets $\lambda s_1, \lambda s_2, \lambda s_3$ ($\lambda \in \mathbb{C}$) are isomorphic, one can set $s_2/s_1 = \alpha$ and $s_3/s_1 = -1-\alpha$ (the normalization of the roots given above corresponds to the choice $s_1 = 1$, $s_2 = \alpha$ and $s_3 = -1-\alpha$). One can deduce after some simple calculation that:

Property

The superalgebras defined by the parameters $\alpha^{\pm 1}$, $-(1+\alpha)^{\pm 1}$ and $\left(\dfrac{-\alpha}{1+\alpha}\right)^{\pm 1}$ are isomorphic. Moreover, for the values 1, -2 and $-1/2$ of the parameter α, the superalgebra $D(2,1;\alpha)$ is isomorphic to $D(2,1)$.

In the Cartan–Weyl basis, the generators are given by:

$$H_1 = T_3^{(1)} \qquad H_2 = T_3^{(2)} \qquad H_3 = T_3^{(3)}$$

$$E_{\pm 2\varepsilon_1} = T_1^{(1)} \pm iT_2^{(1)} \qquad E_{\pm 2\varepsilon_2} = T_1^{(2)} \pm iT_2^{(2)} \qquad E_{\pm 2\varepsilon_3} = T_1^{(3)} \pm iT_2^{(3)}$$

$$E_{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3} = F_{\beta\beta'\beta''}$$

where in the last equation the signs in the indices $\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$ and the indices $\beta\beta'\beta''$ are in one-to-one correspondence.

For more details, see refs. [47, 64].

2.21 Gelfand–Zetlin basis

Let us recall that the finite dimensional representations of the basic Lie superalgebras are either irreducible representations or indecomposable ones or direct sums of these. The irreducible representations are called atypical when there exists a weight vector, different from the highest one, that is annihilated by all the positive root generators. Otherwise, the representation is called typical. The atypical representations therefore can be seen as the building blocks for the indecomposable ones, and will not be considered in this section.

Definition

Consider the $gl(m|n)$ basic Lie superalgebra with the chain of embeddings

$$gl(m|n) \supset gl(m|n-1) \supset \ldots \supset gl(m|k) \supset \ldots \supset gl(m|1) \supset gl(m)$$

Let π be a finite dimensional typical irreducible representation of $gl(m|n)$ and consider the reduction of π with respect to the previous chain of subalgebras. The representation π is called *essentially typical* if there is no atypical $gl(m|k)$-module for any k in the above chain.

From the concept of Gelfand–Zetlin basis, it follows that one can construct such a basis if and only if the representation is essentially typical. Note that if the representation is not essentially typical, some $gl(m|k)$-module is atypical. In that case, the highest weight is not sufficient to cope with this fact and one needs additional labels. However it has not been clear up to now how to introduce them.

We give in the following some ideas in the case of the Lie superalgebras $gl(n|m)$ and in particular $gl(n|1)$, and we refer the reader to the bibliography for more details, see refs. [70, 71].

Gelfand–Zetlin basis for $gl(n|m)$ Lie superalgebras

Let e_{ij} be the standard generators of $gl(n|m)$ which satisfy the commutation relations $(1 \le i,j,k,l \le r = n+m)$

$$\left[\!\left[e_{ij}, e_{kl} \right]\!\right] = \delta_{jk}\, e_{il} - (-1)^{[\deg(i)+\deg(j)][\deg(k)+\deg(l)]}\, \delta_{il}\, e_{kj}$$

where $\deg(i) = 0$ for $i = 1, \ldots, n$ and $\deg(i) = 1$ for $i = n+1, \ldots, n+m$. A basis of the Cartan subalgebra is provided by the generators $(e_{11}, \ldots, e_{rr})$.

A highest weight finite dimensional irreducible representation π of $gl(n|m)$ is determined by the components $m_{1r}, \ldots, m_{rr}$ of the highest weight Λ in the basis dual to the basis $(e_{11}, \ldots, e_{rr})$, such that $m_{ir} - m_{i+1,r} \in \mathbb{Z}_+$ for $i = 1, \ldots, r$ and $i \neq n$.

Property

The representation $\pi(m_{1r},\ldots,m_{rr})$ of $gl(n|m)$ is typical if and only if the numbers

$$l_{ir} = (-1)^{\deg(i)} m_{ir} + |n-i| + 1 - \deg(i) \qquad (i=1,\ldots,r)$$

are all different.

Property

The representation $\pi(m_{1r},\ldots,m_{rr})$ of $gl(n|m)$ is essentially typical if and only if

$$l_{ir} \notin \{l_{n+1,r}, l_{n+1,r}+1, l_{n+1,r}+2, \ldots, l_{rr}\} \qquad (i=1,\ldots,r)$$

Property

A Gelfand–Zetlin basis of the essentially typical $gl(n|m)$ representation $\pi(m_{1r},\ldots,m_{rr})$ can be chosen as follows (with $r=n+m$):

$$(m) = \left| \begin{array}{ccccc} m_{1r} & m_{2r} & \ldots & m_{r-1,r} & m_{rr} \\ m_{1,r-1} & m_{2,r-1} & \ldots & m_{r-1,r-1} & \\ \ddots & & & \iddots & \\ & m_{12} & m_{22} & & \\ & & m_{11} & & \end{array} \right\rangle$$

where the numbers m_{ij} are constrained by the following conditions:

$$\begin{aligned} &m_{jk} - m_{j,k-1} \in \{0,1\} \\ &m_{jk} - m_{j+1,k} \in \mathbb{Z}_+ \\ &m_{i,j+1} - m_{ij} \in \mathbb{Z}_+ \\ &m_{ij} - m_{i+1,j+1} \in \mathbb{Z}_+ \end{aligned}$$

with $1 \le i \le j \le r-1$ and $1 \le k \le r-1$.

The $gl(n)$-generators e_{ii} $(1 \le i \le n)$ are diagonal in the Gelfand–Zetlin basis:

$$e_{ii}|(m)\rangle = \sum_{j<i} (m_{ji} - m_{j,i-1}) |(m)\rangle$$

while the $gl(n)$-generators $e_{i,i+1}$ and $e_{i+1,i}$ act as raising and lowering operators respectively:

$$e_{i,i+1}|(m)\rangle = \sum_{j\le i} a^+_{ji}\, |m_{ji} \to m_{ji}+1\rangle$$

$$e_{i+1,i}|(m)\rangle = \sum_{j \le i} a_{ji}^- \, |m_{ji} \to m_{ji} - 1\rangle$$

where

$$a_{ji}^+ = \left[\frac{\prod_{k=1}^{i+1} (l_{k,i+1} - l_{ji}) \prod_{k=1}^{i-1} (l_{k,i-1} - l_{ji} - 1)}{\prod_{k=1, k \neq j}^{i} (l_{ki} - l_{ji}) \, (l_{ki} - l_{ji} - 1)} \right]^{1/2}$$

and

$$a_{ji}^- = \left[\frac{\prod_{k=1}^{i+1} (l_{k,i+1} - l_{ji} + 1) \prod_{k=1}^{i-1} (l_{k,i-1} - l_{ji})}{\prod_{k=1, k \neq j}^{i} (l_{ki} - l_{ji} + 1) \, (l_{ki} - l_{ji})} \right]^{1/2}$$

the coefficients l_{ij} being given by $l_{ij} = (-1)^{\deg(i)} m_{ij} + |n-i| + 1 - \deg(i)$ for any $1 \le i \le j \le r$.

Gelfand–Zetlin basis for $gl(n|1)$ Lie superalgebras

Property

Every finite dimensional typical representation of $gl(n|1)$ is essentially typical. More precisely, any finite dimensional typical representation of $gl(n|1)$ decomposes as a direct sum of $gl(n)$ representations with multiplicity one. It follows that any finite dimensional typical representation of $gl(n|1)$ admits a Gelfand–Zetlin basis.

The vectors of the Gelfand–Zetlin basis are obtained from the expression given for $gl(n|m)$ with $m = 1$. The action of the even generators of $gl(n|1)$ is determined by the above equations while the action of the odd generators is given by

$$e_{n,n+1}|(m)\rangle = \sum_{j \le n} a_{jn}^+ \, |m_{jn} \to m_{jn} + 1\rangle$$

$$e_{n+1,n}|(m)\rangle = \sum_{j \le n} a_{jn}^- \, |m_{jn} \to m_{jn} - 1\rangle$$

where

$$a_{jn}^{+} = \theta_{jn}(-1)^{\sum\limits_{k=1}^{j-1}(1+\theta_{kn})} \left[\frac{\prod\limits_{k=1}^{n-1}(l_{k,n-1} - l_{jn} - 1)}{\prod\limits_{k=1,k\neq j}^{n}(l_{k,n+1} - l_{j,n+1})} \right]^{1/2}$$

and

$$a_{jn}^{-} = (1-\theta_{jn})(-1)^{\sum\limits_{k=1}^{j-1}(1+\theta_{kn})}(l_{j,n+1} - l_{n+1,n+1}) \left[\frac{\prod\limits_{k=1}^{n-1}(l_{k,n-1} - l_{jn})}{\prod\limits_{k=1,k\neq j}^{n}(l_{k,n+1} - l_{j,n+1})} \right]^{1/2}$$

For more details on the Gelfand–Zetlin basis for Lie superalgebras, see refs. [70, 71] and references therein.

2.22 Grassmann algebras

Definition

The real (resp. complex) Grassmann algebra $\Gamma(n)$ of order n is the algebra over $\mathbb{R}$ (resp. $\mathbb{C}$) generated from the unit element 1 and the n quantities θ_i (called Grassmann variables) which satisfy the anticommutation relations

$$\left\{\theta_i, \theta_j\right\} = 0$$

This algebra has 2^n generators 1, θ_i, $\theta_i\theta_j$, $\theta_i\theta_j\theta_k$, $\dots$, $\theta_1 \dots \theta_n$.

Putting $\deg\theta_i = \overline{1}$, the algebra $\Gamma(n)$ acquires the structure of a superalgebra: $\Gamma(n) = \Gamma(n)_{\overline{0}} \oplus \Gamma(n)_{\overline{1}}$, where $\Gamma(n)_{\overline{0}}$ is generated by the monomials in θ_i with an even number of θ_i (even generators) and $\Gamma(n)_{\overline{1}}$ by the monomials in θ_i with an odd number of θ_i (odd generators). Since $\dim\Gamma(n)_{\overline{0}} = \dim\Gamma(n)_{\overline{1}} = 2^{n-1}$, the superalgebra $\Gamma(n)$ is supersymmetric. The Grassmann superalgebra is associative and commutative (in the sense of the superbracket).

It is possible to define the complex conjugation on the Grassmann variables. However, there are two possibilities for doing so. If c is a complex number and c^* its complex conjugate, θ_i, θ_j being Grassmann variables, the star operation, denoted by *, is defined by

$$(c\theta_i)^* = c^*\theta_i^*\,, \qquad \theta_i^{**} = \theta_i\,, \qquad (\theta_i\theta_j)^* = \theta_j^*\theta_i^*$$

and the superstar operation, denoted by $^\#$, is defined by

$$(c\theta_i)^\# = c^*\theta_i^\# \,, \qquad \theta_i^{\#\#} = -\theta_i \,, \qquad (\theta_i\theta_j)^\# = \theta_i^\#\theta_j^\#$$

Let us mention that the derivation superalgebra (→ 2.13) $\mathrm{Der}\,\Gamma(n)$ of $\Gamma(n)$ is the Cartan type (→ 2.4) simple Lie superalgebra $W(n)$.

2.23 Killing form

Definition

Let $\mathcal{G}$ be a Lie superalgebra. One defines the bilinear form B_π associated to a representation π of $\mathcal{G}$ as a bilinear form from $\mathcal{G} \times \mathcal{G}$ into the field of real numbers $\mathbb{R}$ such that

$$B_\pi(X,Y) = \mathrm{str}(\pi(X)\,\pi(Y))\,, \qquad \forall\; X,Y \in \mathcal{G}$$

$\pi(X)$ are the matrices of the generators $X \in \mathcal{G}$ in the representation π and str denotes the supertrace (→ 2.57).

If $\{X_i\}$ is the basis of generators of $\mathcal{G}$ ($i = 1,\ldots,\dim\mathcal{G}$), one has therefore

$$B_\pi(X_i,Y_j) = \mathrm{str}(\pi(X_i)\,\pi(Y_j)) = g_{ij}^\pi$$

Definition

A bilinear form B on $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ is called
– consistent if $B(X,Y) = 0$ for all $X \in \mathcal{G}_{\overline{0}}$ and all $Y \in \mathcal{G}_{\overline{1}}$.
– supersymmetric if $B(X,Y) = (-1)^{\deg X.\deg Y} B(Y,X)$, for all $X,Y \in \mathcal{G}$.
– invariant if $B([\![X,Y]\!],Z) = B(X,[\![Y,Z]\!])$, for all $X,Y,Z \in \mathcal{G}$.

Property

An invariant form on a simple Lie superalgebra $\mathcal{G}$ is either non-degenerate (that is its kernel is zero) or identically zero, and two invariant forms on $\mathcal{G}$ are proportional.

Definition

A bilinear form on $\mathcal{G}$ is called an *inner product* on $\mathcal{G}$ if it is consistent, supersymmetric and invariant.

Definition

The bilinear form associated to the adjoint representation of $\mathcal{G}$ is called the *Killing form* on $\mathcal{G}$ and is denoted $K(X,Y)$:

$$K(X,Y) = \mathrm{str}(\mathrm{ad}(X)\;\mathrm{ad}(Y))\,, \qquad \forall\, X,Y \in \mathcal{G}$$

We recall that $\mathrm{ad}(X)Z = [\![X, Z]\!]$ and $\Big(\mathrm{ad}(X_i)\Big)_j^{\ k} = -C_{ij}^{\ \ k}$ where $C_{ij}^{\ \ k}$ are the structure constants for the basis $\{X_i\}$ of generators of $\mathcal{G}$. We can therefore write

$$K(X_i, X_j) = (-1)^{\deg X_m} C_{mi}^{\ \ \ n} C_{nj}^{\ \ m} = g_{ij}$$

Property

The Killing form K of a Lie superalgebra $\mathcal{G}$ is consistent, supersymmetric and invariant (in other words, it is an inner product).

Property

The Killing form K of a Lie superalgebra $\mathcal{G}$ satisfies

$$K(\phi(X), \phi(Y)) = K(X, Y)$$

for all $\phi \in \mathrm{Aut}(\mathcal{G})$ and $X, Y \in \mathcal{G}$.

The following theorems give the fundamental results concerning the Killing form of the (simple) Lie superalgebras:

Theorem

1. A Lie superalgebra $\mathcal{G}$ with a non-degenerate Killing form is a direct sum of simple Lie superalgebras each having a non-degenerate Killing form.

2. A simple finite dimensional Lie superalgebra $\mathcal{G}$ with a non-degenerate Killing form is of the type $A(m,n)$ where $m \neq n$, $B(m,n)$, $C(n+1)$, $D(m,n)$ where $m \neq n+1$, $F(4)$ or $G(3)$.

3. A simple finite dimensional Lie superalgebra $\mathcal{G}$ with a zero Killing form is of the type $A(n,n)$, $D(n+1,n)$, $D(2,1;\alpha)$, $P(n)$ or $Q(n)$.

For more details, see refs. [47, 48].

2.24 Lie superalgebra, subalgebra, ideal

Definition

A *Lie superalgebra* $\mathcal{G}$ over a field $\mathbb{K}$ of characteristic zero (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a $\mathbb{Z}_2$-graded algebra, that is a vector space, direct sum of two vector spaces $\mathcal{G}_{\overline{0}}$ and $\mathcal{G}_{\overline{1}}$, in which a product $[\![\ ,\ ]\!]$ is defined as follows:

- $\mathbb{Z}_2$-gradation:

$$\left[\!\left[\mathcal{G}_i, \mathcal{G}_j\right]\!\right] \subset \mathcal{G}_{i+j} \qquad (i, j \in \mathbb{Z}/2\mathbb{Z})$$

- graded-antisymmetry:

$$\left[\!\left[X_i, X_j\right]\!\right] = -(-1)^{\deg X_i . \deg X_j} \left[\!\left[X_j, X_i\right]\!\right]$$

where $\deg X_i$ is the degree of the vector space. $\mathcal{G}_{\overline{0}}$ is called the even space and $\mathcal{G}_{\overline{1}}$ the odd space. If $\deg X_i . \deg X_j = 0$, the bracket $[\![\ ,\ ]\!]$ defines the usual commutator, otherwise it is an anticommutator.

- generalized Jacobi identity:

$$(-1)^{\deg X_i . \deg X_k} \left[\!\left[X_i, \left[\!\left[X_j, X_k\right]\!\right]\right]\!\right] + (-1)^{\deg X_j . \deg X_i} \left[\!\left[X_j, \left[\!\left[X_k, X_i\right]\!\right]\right]\!\right]$$
$$+ (-1)^{\deg X_k . \deg X_j} \left[\!\left[X_k, \left[\!\left[X_i, X_j\right]\!\right]\right]\!\right] = 0$$

Notice that $\mathcal{G}_{\overline{0}}$ is a Lie algebra – called the even or bosonic part of $\mathcal{G}$ – while $\mathcal{G}_{\overline{1}}$ – called the odd or fermionic part of $\mathcal{G}$ – is not an algebra.

An associative superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ over the field $\mathbb{K}$ acquires the structure of a Lie superalgebra by taking for the product $[\![\ ,\ ]\!]$ the *Lie superbracket* or *supercommutator* (also called generalized or graded commutator)

$$\left[\!\left[X, Y\right]\!\right] = XY - (-1)^{\deg X . \deg Y} YX$$

for two elements $X, Y \in \mathcal{G}$.

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a Lie superalgebra. $\mathcal{G}$ is a $\mathbb{Z}$-graded Lie superalgebra if it can be written as a direct sum of finite dimensional $\mathbb{Z}_2$-graded subspaces $\mathcal{G}_i$ such that

$$\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i \qquad \text{where} \qquad \left[\!\left[\mathcal{G}_i, \mathcal{G}_j\right]\!\right] \subset \mathcal{G}_{i+j}$$

The $\mathbb{Z}$-gradation is said to be *consistent* with the $\mathbb{Z}_2$-gradation if

$$\mathcal{G}_{\overline{0}} = \sum_{i \in \mathbb{Z}} \mathcal{G}_{2i} \qquad \text{and} \qquad \mathcal{G}_{\overline{1}} = \sum_{i \in \mathbb{Z}} \mathcal{G}_{2i+1}$$

It follows that G_0 is a Lie subalgebra and that each G_i $(i \neq 0)$ is a G_0-module.

Definition

A $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G} = \underset{i\in\mathbb{Z}}{\oplus} \mathcal{G}_i$ is called *transitive* if for $a \in \mathcal{G}_{i\geq 0}$

$$\left[\!\left[a, \mathcal{G}_{-1}\right]\!\right] = 0 \quad \Rightarrow \quad a = 0$$

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{0}}$ be a Lie superalgebra.

– A *subalgebra* $\mathcal{K} = \mathcal{K}_{\overline{0}} \oplus \mathcal{K}_{\overline{1}}$ of $\mathcal{G}$ is a subset of elements of $\mathcal{G}$ which forms a vector subspace of $\mathcal{G}$ that is closed with respect to the Lie product of $\mathcal{G}$ such that $\mathcal{K}_{\overline{0}} \subset \mathcal{G}_{\overline{0}}$ and $\mathcal{K}_{\overline{1}} \subset \mathcal{G}_{\overline{1}}$. A subalgebra $\mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{K} \neq \mathcal{G}$ is called a proper subalgebra of $\mathcal{G}$.

– An *ideal* $\mathcal{I}$ of $\mathcal{G}$ is a subalgebra of $\mathcal{G}$ such that $\left[\!\left[\mathcal{G}, \mathcal{I}\right]\!\right] \subset \mathcal{I}$, that is

$$X \in \mathcal{G}, Y \in \mathcal{I} \;\Rightarrow\; \left[\!\left[X, Y\right]\!\right] \in \mathcal{I}$$

An ideal $\mathcal{I}$ of $\mathcal{G}$ such that $\mathcal{I} \neq \mathcal{G}$ is called a proper ideal of $\mathcal{G}$.

Property

Let $\mathcal{G}$ be a Lie superalgebra and $\mathcal{I}$, $\mathcal{I}'$ two ideals of $\mathcal{G}$. Then $\left[\!\left[\mathcal{I}, \mathcal{I}'\right]\!\right]$ is an ideal of $\mathcal{G}$.

2.25 Matrix realizations of the classical Lie superalgebras

The classical Lie superalgebras can be described as matrix superalgebras as follows. Consider the $\mathbb{Z}_2$-graded vector space $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$ with $\dim \mathcal{V}_{\overline{0}} = m$ and $\dim \mathcal{V}_{\overline{1}} = n$. Then the algebra $\mathrm{End}\,\mathcal{V}$ acquires naturally a superalgebra structure by

$$\mathrm{End}\,\mathcal{V} = \mathrm{End}_{\overline{0}}\mathcal{V} \oplus \mathrm{End}_{\overline{1}}\mathcal{V} \quad \text{where} \quad \mathrm{End}_i\mathcal{V} = \left\{\phi \in \mathrm{End}\,\mathcal{V} \;\middle|\; \phi(\mathcal{V}_j) \subset \mathcal{V}_{i+j}\right\}$$

The Lie superalgebra $gl(m|n)$ is defined as the superalgebra $\mathrm{End}\,\mathcal{V}$ supplied with the Lie superbracket ($\rightarrow$ 2.24). $gl(m|n)$ is spanned by matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and D are $gl(m)$ and $gl(n)$ matrices, B and C are $m \times n$ and $n \times m$ rectangular matrices.

One defines on $gl(m|n)$ the supertrace function denoted by str:

$$\mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D)$$

The unitary superalgebra $A(m-1,n-1) = sl(m|n)$ is defined as the superalgebra of matrices $M \in gl(m|n)$ satisfying the supertrace condition $\mathrm{str}(M) = 0$. In the case $m = n$, $sl(n|n)$ contains a one-dimensional ideal $\mathcal{I}$ generated by $\mathbb{I}_{2n}$ and one sets $A(n-1,n-1) = sl(n|n)/\mathcal{I} = psl(n|n)$.

The orthosymplectic superalgebra $osp(m|2n)$ is defined as the superalgebra of matrices $M \in gl(m|n)$ satisfying the conditions

$$A^t = -A\,, \qquad D^tG = -GD\,, \qquad B = C^tG$$

where ^{t} denotes the usual sign of transposition and the matrix G is given by

$$G = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

The strange superalgebra $P(n)$ is defined as the superalgebra of matrices $M \in gl(n|n)$ satisfying the conditions

$$A^t = -D\,, \qquad B^t = B\,, \qquad C^t = -C\,, \qquad \mathrm{tr}(A) = 0$$

The strange superalgebra $\widetilde{Q}(n)$ is defined as the superalgebra of matrices $M \in gl(n|n)$ satisfying the conditions

$$A = D\,, \qquad B = C\,, \qquad \mathrm{tr}(B) = 0$$

The superalgebra $\widetilde{Q}(n)$ has a one-dimensional center $\mathcal{Z}$. The simple superalgebra $Q(n)$ is given by $Q(n) = \widetilde{Q}(n)/\mathcal{Z}$.

$\rightarrow$ 2.27 Orthosymplectic superalgebras, 2.48–2.49 Strange superalgebras, 2.61 Unitary superalgebras.

For more details, see refs. [48, 75].

2.26 Nilpotent and solvable Lie superalgebras

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a Lie superalgebra. $\mathcal{G}$ is said to be *nilpotent* if, considering the series

$$[\![\mathcal{G},\mathcal{G}]\!] = \mathcal{G}^{[1]}\ , \quad [\![\mathcal{G},\mathcal{G}^{[1]}]\!] = \mathcal{G}^{[2]}\ , \quad \dots\ , \quad [\![\mathcal{G},\mathcal{G}^{[i-1]}]\!] = \mathcal{G}^{[i]}$$

then there exists an integer n such that $\mathcal{G}^{[n]} = \{0\}$.

Definition

$\mathcal{G}$ is said to be *solvable* if, considering the series

$$\left[\!\left[\mathcal{G},\mathcal{G}\right]\!\right] = \mathcal{G}^{(1)} \quad , \quad \left[\!\left[\mathcal{G}^{(1)},\mathcal{G}^{(1)}\right]\!\right] = \mathcal{G}^{(2)} \quad , \quad \dots \quad , \quad \left[\!\left[\mathcal{G}^{(i-1)},\mathcal{G}^{(i-1)}\right]\!\right] = \mathcal{G}^{(i)}$$

then there exists an integer n such that $\mathcal{G}^{(n)} = \{0\}$.

Theorem

The Lie superalgebra $\mathcal{G}$ is solvable if and only if $\mathcal{G}_{\overline{0}}$ is solvable.

Property

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a solvable Lie superalgebra. Then the irreducible representations of $\mathcal{G}$ are one-dimensional if and only if $\left\{\mathcal{G}_{\overline{1}},\mathcal{G}_{\overline{1}}\right\} \subset \left[\mathcal{G}_{\overline{0}},\mathcal{G}_{\overline{0}}\right]$ (let us recall that in the case of a solvable Lie algebra, the irreducible finite dimensional representations are one-dimensional).

Property

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a solvable Lie superalgebra and let $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$ be the space of irreducible finite dimensional representations. Then either $\dim \mathcal{V}_{\overline{0}} = \dim \mathcal{V}_{\overline{1}}$ and $\dim \mathcal{V} = 2^s$ with $1 \le s \le \dim \mathcal{G}_{\overline{1}}$, or $\dim \mathcal{V} = 1$.

2.27 Orthosymplectic superalgebras

The orthosymplectic superalgebras form three infinite families of basic Lie superalgebras. The superalgebra $B(m,n)$ or $osp(2m+1|2n)$ defined for $m \ge 0, n \ge 1$ has as even part the Lie algebra $so(2m+1) \oplus sp(2n)$ and as odd part the $(2m+1,2n)$ representation of the even part; it has rank $m+n$ and dimension $2(m+n)^2+m+3n$. The superalgebra $C(n+1)$ or $osp(2|2n)$ where $n \ge 1$ has as even part the Lie algebra $so(2) \oplus sp(2n)$ and the odd part is twice the fundamental representation $(2n)$ of $sp(2n)$; it has rank $n+1$ and dimension $2n^2+5n+1$. The superalgebra $D(m,n)$ or $osp(2m|2n)$ defined for $m \ge 2, n \ge 1$ has as even part the Lie algebra $so(2m) \oplus sp(2n)$ and its odd part is the $(2m,2n)$ representation of the even part; it has rank $m+n$ and dimension $2(m+n)^2-m+n$.

The root systems can be expressed in terms of the orthogonal vectors ε_1, ..., ε_m and δ_1, ..., δ_n as follows.
– for $B(m,n)$ with $m \ne 0$:

$$\Delta_{\overline{0}} = \Big\{ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_i \pm \delta_j,\ \pm 2\delta_i \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \pm\varepsilon_i \pm \delta_j,\ \pm\delta_j \Big\},$$

– for $B(0,n)$:

$$\Delta_{\overline{0}} = \Big\{ \pm\delta_i \pm \delta_j,\ \pm 2\delta_i \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \pm\delta_j \Big\},$$

– for $C(n+1)$:

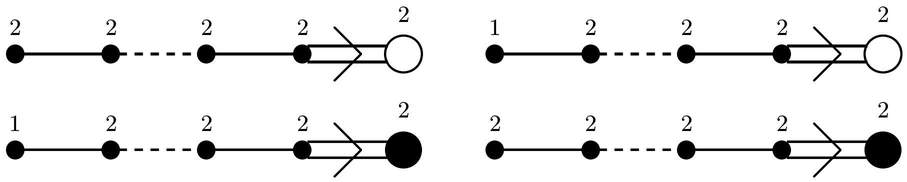
$$\Delta_{\overline{0}} = \left\{ \pm \delta_i \pm \delta_j, \ \pm 2\delta_i \right\} \quad \text{and} \quad \Delta_{\overline{1}} = \left\{ \pm \varepsilon \pm \delta_j \right\},$$

– for $D(m,n)$:

$$\Delta_{\overline{0}} = \left\{ \pm \varepsilon_i \pm \varepsilon_j, \ \pm\delta_i \pm \delta_j, \ \pm 2\delta_i \right\} \quad \text{and} \quad \Delta_{\overline{1}} = \left\{ \pm \varepsilon_i \pm \delta_j \right\}.$$

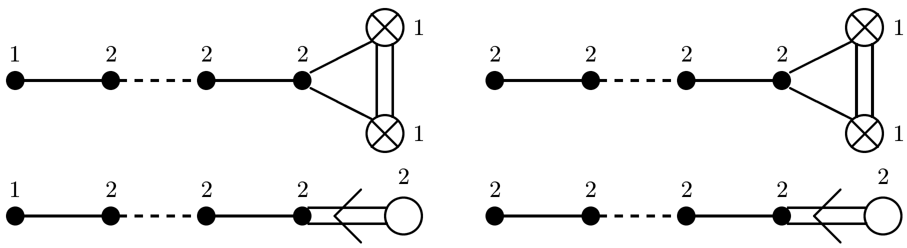
The Dynkin diagrams (→ 2.15) of the orthosymplectic superalgebras are of the following types:

– for the superalgebra $B(m,n)$



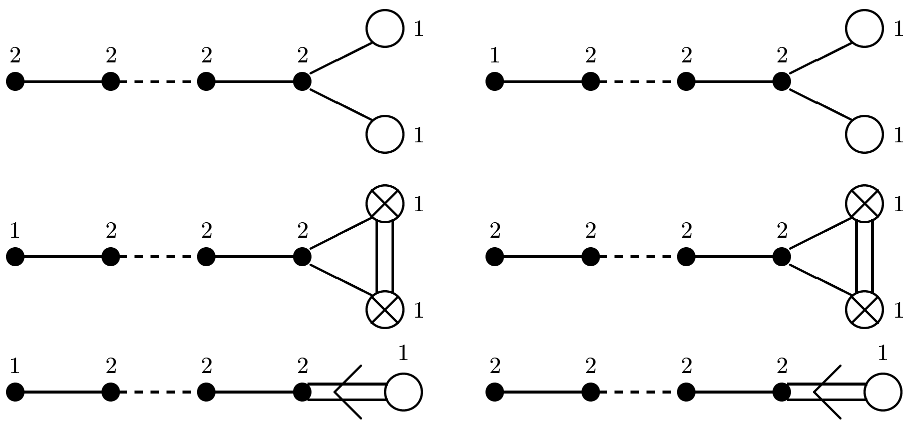
$K = 1$ $K = 0$

– for the superalgebra $C(n+1)$



$K = 1$ $K = 0$

–for the superalgebra $D(m,n)$



$K = 1$ $K = 0$

In these diagrams, the labels are the Kac labels which give the decomposition of the highest root in terms of the simple roots. The small black dots represent either white dots (associated to even roots) or grey dots (associated to odd roots of zero length), and K is the parity of the number of grey dots. The Dynkin diagrams of the orthosymplectic Lie superalgebras up to rank 4 are given in Table 3.61.

The superalgebras $B(m,n)$ and $D(m,n)$ are of type II while the superalgebras $C(n+1)$ are of type I ($\rightarrow$ 2.8). Denoting by $\mathcal{H}$ the distinguished Cartan subalgebra, the distinguished $\mathbb{Z}$-gradation ($\rightarrow$ 2.8) of the type II orthosymplectic superalgebras has the following structure:

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where:
– for the superalgebras $B(m,n)$ (with $1 \le i < j \le m$, $1 \le k < l \le n$)

$$\begin{aligned}
&\mathcal{G}_0 = \mathcal{H} \cup \{E_{\pm\varepsilon_i \pm\varepsilon_j},\, E_{\pm\varepsilon_i},\, E_{\delta_k-\delta_l}\} && \\
&\mathcal{G}_2 = \{E_{\delta_k+\delta_l},\, E_{2\delta_k}\} && \mathcal{G}_{-2} = \{E_{-\delta_k-\delta_l},\, E_{-2\delta_k}\} \\
&\mathcal{G}_1 = \{E_{\pm\varepsilon_i+\delta_k},\, E_{\delta_k}\} && \mathcal{G}_{-1} = \{E_{\pm\varepsilon_i-\delta_k},\, E_{-\delta_k}\}
\end{aligned}$$

– for the superalgebras $D(m,n)$ (with $1 \le i < j \le m$, $1 \le k < l \le n$)

$$\begin{aligned}
&\mathcal{G}_0 = \mathcal{H} \cup \{E_{\pm\varepsilon_i \pm\varepsilon_j},\, E_{\delta_k-\delta_l}\} && \\
&\mathcal{G}_2 = \{E_{\delta_k+\delta_l},\, E_{2\delta_k}\} && \mathcal{G}_{-2} = \{E_{-\delta_k-\delta_l},\, E_{-2\delta_k}\} \\
&\mathcal{G}_1 = \{E_{\pm\varepsilon_i+\delta_k}\} && \mathcal{G}_{-1} = \{E_{\pm\varepsilon_i-\delta_k}\}
\end{aligned}$$

The orthosymplectic superalgebras $osp(M|N)$ (with $M = 2m$ or $2m+1$ and $N = 2n$) can be generated as matrix superalgebras by taking a basis of $(M+N)^2$ elementary matrices e_{IJ} of order $M+N$ satisfying $(e_{IJ})_{KL} = \delta_{IL}\delta_{JK}$ ($I,J,K,L = 1,\dots,M+N$). One defines the following graded matrices

$$G_{IJ} = \left(\begin{array}{cc|cc} 0 & \mathbb{I}_m & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \mathbb{I}_m & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & \mathbb{I}_n \\ & & -\mathbb{I}_n & 0 \end{array}\right) \quad \text{if } M = 2m$$

$$G_{IJ} = \left(\begin{array}{ccc|cc} 0 & \mathbb{I}_m & 0 & & \\ \mathbb{I}_m & 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & 1 & & \\ \hline & & & 0 & \mathbb{I}_n \\ \multicolumn{3}{c|}{0} & -\mathbb{I}_n & 0 \end{array}\right) \quad \text{if } M = 2m+1$$

where $\mathbb{I}_m$ and $\mathbb{I}_n$ are the $m \times m$ and $n \times n$ identity matrices respectively.

Dividing the capital indices $I, J, \dots$ into small unprimed indices $i, j, \dots$ running from 1 to M and small primed indices $i', j', \dots$ running from $M+1$ to $M+N$, the generators of $osp(M|N)$ are given by

$$\begin{aligned}
E_{ij} &= G_{ik}e_{kj} - G_{jk}e_{ki} \\
E_{i'j'} &= G_{i'k'}e_{k'j'} + G_{j'k'}e_{k'i'} \\
E_{ij'} &= E_{j'i} = G_{ik}e_{kj'}
\end{aligned}$$

Then the E_{ij} (antisymmetric in the indices i, j) generate the $so(M)$ part, the $E_{i'j'}$ (symmetric in the indices i', j') generate the $sp(N)$ part and the $E_{ij'}$ transform as the (M, N) representation of $osp(M|N)$. They satisfy the following (super)commutation relations:

$$\begin{aligned}
&\Big[E_{ij}, E_{kl}\Big] = G_{jk}E_{il} + G_{il}E_{jk} - G_{ik}E_{jl} - G_{jl}E_{ik} \\
&\Big[E_{i'j'}, E_{k'l'}\Big] = -G_{j'k'}E_{i'l'} - G_{i'l'}E_{j'k'} - G_{j'l'}E_{i'k'} - G_{i'k'}E_{j'l'} \\
&\Big[E_{ij}, E_{k'l'}\Big] = 0 \\
&\Big[E_{ij}, E_{kl'}\Big] = G_{jk}E_{il'} - G_{ik}E_{jl'} \\
&\Big[E_{i'j'}, E_{kl'}\Big] = -G_{i'l'}E_{kj'} - G_{j'l'}E_{ki'} \\
&\Big\{E_{ij'}, E_{kl'}\Big\} = G_{ik}E_{j'l'} - G_{j'l'}E_{ik}
\end{aligned}$$

In the case of the superalgebra $osp(1|N)$, the commutation relations greatly simplify. One obtains

$$\begin{aligned}
&\Big[E_{i'j'}, E_{k'l'}\Big] = -G_{j'k'}E_{i'l'} - G_{i'l'}E_{j'k'} - G_{j'l'}E_{i'k'} - G_{i'k'}E_{j'l'} \\
&\Big[E_{i'j'}, E_{k'}\Big] = -G_{i'k'}E_{j'} - G_{j'k'}E_{i'} \\
&\Big\{E_{i'}, E_{j'}\Big\} = E_{i'j'}
\end{aligned}$$

where $E_{i'}$ denote the odd generators.

2.28 Oscillator realizations: Cartan type superalgebras

Oscillator realizations of the Cartan type superalgebras can be obtained as follows. Take a set of $2n$ fermionic oscillators a_i^- and a_i^+ with standard anticommutation relations

$$\left\{a_i^-, a_j^-\right\} = \left\{a_i^+, a_j^+\right\} = 0 \quad \text{and} \quad \left\{a_i^+, a_j^-\right\} = \delta_{ij}$$

In the case of the $W(n)$ superalgebra, one defines the following subspaces:

$$\begin{aligned}
&\mathcal{G}_{-1} = \left\{a_{i_0}^-\right\} \qquad \text{and} \qquad \mathcal{G}_0 = \left\{a_{i_0}^+ a_{i_1}^-\right\} \\
&\mathcal{G}_1 = \left\{a_{i_0}^+ a_{i_1}^+ a_{i_2}^-\right\} \quad i_0 \neq i_1 \\
&\dots \\
&\mathcal{G}_{n-1} = \left\{a_{i_0}^+ a_{i_1}^+ \dots a_{i_{n-1}}^+ a_{i_n}^-\right\} \quad i_0 \neq i_1 \neq \dots \neq i_{n-1}
\end{aligned}$$

the superalgebra $W(n)$ is given by

$$W(n) = \bigoplus_{i=-1}^{n-1} \mathcal{G}_i$$

with $\mathbb{Z}$-gradation $\left[\mathcal{G}_i, \mathcal{G}_j\right] \subset \mathcal{G}_{i+j}$.

In the case of $S(n)$ and $\widetilde{S}(n)$, defining the following subspaces:

$$\begin{aligned}
&\mathcal{G}_{-1} = \left\{a_{i_0}^-\right\} \qquad \text{and} \qquad \mathcal{G}'_{-1} = \left\{(1 + a_1^+ \dots a_n^+) a_{i_0}^-\right\} \\
&\mathcal{G}_0 = \left\{a_1^+ a_1^- - a_{i_0}^+ a_{i_0}^- \quad (i_0 \neq 1), \quad a_{i_0}^+ a_{i_1}^- \quad (i_1 \neq i_0)\right\} \\
&\mathcal{G}_1 = \Big\{a_{i_1}^+ (a_1^+ a_1^- - a_{i_0}^+ a_{i_0}^-) \quad (i_1 \neq i_0 \neq 1), \\
&\qquad a_1^+ (a_2^+ a_2^- - a_{i_0}^+ a_{i_0}^-) \quad (i_0 \neq 1, 2), \\
&\qquad a_{i_2}^+ a_{i_1}^+ a_{i_0}^- \quad (i_2 \neq i_1 \neq i_0)\Big\} \\
&\mathcal{G}_2 = \Big\{a_{i_2}^+ a_{i_1}^+ (a_1^+ a_1^- - a_{i_0}^+ a_{i_0}^-) \quad (i_2 \neq i_1 \neq i_0 \neq 1), \\
&\qquad a_{i_1}^+ a_1^+ (a_2^+ a_2^- - a_{i_0}^+ a_{i_0}^-) \quad (i_1 \neq i_0 \neq 1, 2), \\
&\qquad a_1^+ a_2^+ (a_3^+ a_3^- - a_{i_0}^+ a_{i_0}^-) \quad (i_0 \neq 1, 2, 3), \\
&\qquad a_{i_3}^+ a_{i_2}^+ a_{i_1}^+ a_{i_0}^- \quad (i_3 \neq i_2 \neq i_1 \neq i_0)\Big\} \\
&\dots
\end{aligned}$$

the superalgebra $S(n)$ is given by

$$S(n) = \bigoplus_{i=0}^{n-2} \mathcal{G}_i \oplus \mathcal{G}_{-1}$$

and the superalgebra $\widetilde{S}(n)$ by

$$\widetilde{S}(n) = \bigoplus_{i=0}^{n-2} \mathcal{G}_i \oplus \mathcal{G}'_{-1}$$

Finally, in the case of $H(n)$ one defines the following subspaces:

$$\mathcal{G}_{-1} = \left\{ a^-_{i_0} \right\} \qquad \mathcal{G}_0 = \left\{ a^+_{i_0} a^-_{i_1} - a^+_{i_1} a^-_{i_0} \right\}$$

$$\mathcal{G}_1 = \left\{ a^+_{i_0} a^+_{i_1} a^-_{i_2} - a^+_{i_0} a^+_{i_2} a^-_{i_1} - a^+_{i_2} a^+_{i_0} a^-_{i_1} + a^+_{i_1} a^+_{i_2} a^-_{i_0} + a^+_{i_2} a^+_{i_0} a^-_{i_1} - a^+_{i_2} a^+_{i_1} a^-_{i_0} \right\}$$

$\dots$

The superalgebra $H(n)$ is given by

$$H(n) = \bigoplus_{i=-1}^{n-3} \mathcal{G}_i$$

For more details, see ref. [75].

2.29 Oscillator realizations: exceptional Lie superalgebras

In terms of the 24 fermionic oscillators denoted by a, α and β ($\rightarrow$ 1.60 for the notations), but imposing now that

$$\{a\,,\,\alpha\} = \{a\,,\,\beta\} = \{\beta\,,\,\alpha\} = 0$$

and of six bosonic oscillators b^+_i, b_i, which commute with the operators a, α and β, we can write oscillator realizations of the exceptional Lie superalgebras. See ref. [14] for more details.

Oscillator realization of $F(4)$

With reference to the Dynkin diagram in the distinguished basis ($\rightarrow$ 2.18), we can write the generators corresponding to the simple positive roots as

$$\begin{aligned}
E_1 &= \sqrt{\tfrac{3}{2}}\,(\beta^+_4\, b^+_1 + \alpha_{1234}\, b^+_2) \\
E_2 &= \alpha^+_{31}\,\alpha^+_{32} + \alpha^+_{1234}\,\alpha^+_{34} + \beta^+_3\,\beta_4 + \beta_1\,\beta_2 \\
E_3 &= \alpha^+_{21}\,\alpha^+_{24} + \beta^+_2\,\beta_3 \\
E_4 &= \alpha^+_{13}\,\alpha^+_{14} + \beta^+_1\,\beta_2
\end{aligned}$$

The generators corresponding to the simple negative roots are obtained by hermitian conjugation. The Cartan generators are

$$\begin{aligned}
H_1 &= \tfrac{3}{2}\,(\alpha^+_{1234}\,\alpha_{1234} - \beta^+_4\,\beta_4 + b^+_1\,b_1 + b^+_2\,b_2) \\
H_2 &= \alpha^+_{31}\,\alpha_{31} + \alpha^+_{32}\,\alpha_{32} + \alpha^+_{1234}\,\alpha_{1234} + \alpha^+_{34}\,\alpha_{34} \\
&\quad + \beta^+_3\,\beta_3 - \beta^+_4\,\beta_4 - \beta^+_1\,\beta_1 - \beta^+_2\,\beta_2 \\
H_3 &= \alpha^+_{21}\,\alpha_{21} + \alpha^+_{24}\,\alpha_{24} + \beta^+_2\,\beta_2 - \beta^+_3\,\beta_3 \\
H_4 &= \alpha^+_{13}\,\alpha_{13} + \alpha^+_{14}\,\alpha_{14} + \beta^+_1\,\beta_1 - \beta^+_2\,\beta_2
\end{aligned}$$

The remaining generators are obtained by the supercommutators of the simple generators. The Cartan generator of the $sl(2)$ hidden by the fermionic root α_1 is given by

$$K = \tfrac{1}{3}\,(2H_1 - 3H_2 - 4H_3 - 2H_4)$$

Note that we have made use of only 16 fermionic (α, β) and four bosonic oscillators.

Oscillator realization of $G(3)$

With reference to the Dynkin diagram in the distinguished basis ($\rightarrow$ 2.19), we can write the generators corresponding to the simple positive roots as

$$\begin{aligned}
E_1 &= \sqrt{2}\,(\beta^+_4\,b^+_1 + \alpha_{1234}\,b^+_1 + a_1\,b^+_3) \\
E_2 &= a^+_1\,a_2 + a^+_3\,a_4 + a^+_3\,a^+_4 + \alpha^+_{13}\,\alpha^+_{14} + \alpha^+_{31}\,\alpha^+_{32} + \alpha^+_{1234}\,\alpha^+_{34} \\
&\quad + \beta^+_1\,\beta_2 + \beta^+_3\,\beta_4 + \beta_1\,\beta_2 \\
E_3 &= a^+_2\,a_3 + \alpha^+_{21}\,\alpha^+_{24} + \beta^+_2\,\beta_3
\end{aligned}$$

The generators corresponding to the simple negative roots are obtained by hermitian conjugation. The Cartan generators are

$$\begin{aligned}
H_1 &= 2\,(a^+_1\,a_1 + \alpha^+_{1234}\,\alpha_{1234} - \beta^+_4\,\beta_4 + b^+_1\,b_1 + b^+_2\,b_2 + b^+_3\,b_3 + 1) \\
H_2 &= a^+_1\,a_1 - a^+_2\,a_2 + 2\,a^+_3\,a_3 + 2\alpha^+_{13}\,\alpha_{13} + \alpha^+_{34}\,\alpha_{34} + \alpha^+_{1234}\,\alpha_{1234} \\
&\quad - 2\beta^+_2\,\beta_2 + \beta^+_3\,\beta_3 - \beta^+_4\,\beta_4 \\
H_3 &= a^+_1\,a_1 - a^+_2\,a_2 + \alpha^+_{21}\,\alpha_{21} + \alpha^+_{24}\,\alpha_{24} + \beta^+_2\,\beta_2 - \beta^+_3\,\beta_3
\end{aligned}$$

while the Cartan generator of the $sl(2)$ hidden by the fermionic root α_1 is given by

$$K = \tfrac{1}{2}\,(H_1 - 2H_2 - 3H_3)$$

Oscillator realization of $D(2,1;\alpha)$

With reference to the Dynkin diagram in the distinguished basis (→ 2.20), we can write the generators corresponding to the simple positive roots as

$$E_1 = \tfrac{1}{2}\, b_1^+\, b_2\, b_3 \qquad E_2 = -\tfrac{1}{2}\,(b_2^+)^2 \qquad E_3 = -\tfrac{1}{2}\,(b_3^+)^2$$

Note that we have made use only of the six bosonic oscillators. The generators corresponding to negative simple roots are obtained by hermitian conjugation. Moreover as we have written the fermionic generator E_1 as a trilinear in the bosonic operators, we cannot obtain the Cartan generators E_1 by means of the usual anticommutator of E_1 with F_1, which, indeed, would give for H_1 an expression as a quadrilinear in the bosonic oscillators. So we define the following symmetric composition rule

$$\begin{aligned} b_1^+\, b_2\, b_3 \diamond b_1\, b_2^+\, b_3^+ \;\; &\equiv \;\; \sigma_1\, \{b_1^+, b_1\}\, [b_2, b_2^+]\, [b_3, b_3^+] \\ &+ \;\; \sigma_2\, \{b_2^+, b_2\}\, [b_1^+, b_1]\, [b_3, b_3^+] \\ &+ \;\; \sigma_3\, \{b_3^+, b_3\}\, [b_1^+, b_1]\, [b_2, b_2^+] \end{aligned}$$

where [,] and { , } are the usual commutator and anticommutator and σ_i are three complex numbers satisfying the constraint, required by the Jacobi identity,

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

Choosing $(\sigma_1, \sigma_2, \sigma_3) = (1+\alpha, -1, -\alpha)$, we get

$$H_1 = \tfrac{1}{2}\, \{(1+\alpha)K + H_2 + \alpha H_3\}$$

where $(j = 2, 3)$

$$H_j = \tfrac{1}{4}\, [(b_j^+)^2, (b_j)^2] = \tfrac{1}{2}\, \{b_j^+, b_j\} \qquad \text{and} \qquad K = \tfrac{1}{2}\, \{b_1^+, b_1\}$$

K is the Cartan generator of the $sl(2)$ hidden by the fermionic root α_1. Notice that for $\alpha = 1$ (resp. $\alpha = -1$) we get a new realization of $D(2,1)$ (resp. $A(1,1)$) in terms that are bilinear and trilinear in bosonic operators.

2.30 Oscillator realizations: orthosymplectic and unitary series

Let us consider a set of $2n$ bosonic oscillators b_i^- and b_i^+ with commutation relations:

$$\left[b_i^-, b_j^-\right] = \left[b_i^+, b_j^+\right] = 0 \quad \text{and} \quad \left[b_i^-, b_j^+\right] = \delta_{ij}$$

and a set of $2m$ fermionic oscillators a_i^- and a_i^+ with anticommutation relations:

$$\left\{a_i^-, a_j^-\right\} = \left\{a_i^+, a_j^+\right\} = 0 \quad \text{and} \quad \left\{a_i^-, a_j^+\right\} = \delta_{ij}$$

the two sets commuting each other:

$$\left[b_i^-, a_j^-\right] = \left[b_i^-, a_j^+\right] = \left[b_i^+, a_j^-\right] = \left[b_i^+, a_j^+\right] = 0$$

In the case of $B(m,n)$, one needs also a supplementary real fermionic oscillator e such that $e^2 = 1$ with (anti)commutation relations

$$\left\{a_i^\pm, e\right\} = 0 \quad \text{and} \quad \left[b_i^\pm, e\right] = 0$$

Oscillator realization of $A(m-1, n-1)$

Let

$$\Delta = \left\{ \ \varepsilon_i - \varepsilon_j, \ \delta_k - \delta_l, \ \varepsilon_i - \delta_k, \ -\varepsilon_i + \delta_k \ \right\}$$

be the root system of $A(m-1, n-1)$ expressed in terms of the orthogonal vectors $\varepsilon_1, \ldots, \varepsilon_m$ and $\delta_1, \ldots, \delta_n$. An oscillator realization of the simple generators in the distinguished basis is given by (where $1 \le i \le m-1$ and $1 \le k \le n-1$)

$$\begin{array}{lll}
H_i = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- & E_{\varepsilon_i - \varepsilon_{i+1}} = a_i^+ a_{i+1}^- & E_{\varepsilon_{i+1} - \varepsilon_i} = a_{i+1}^+ a_i^- \\
H_m = a_m^+ a_m^- + b_1^+ b_1^- & E_{\varepsilon_m - \delta_1} = a_m^+ b_1^- & E_{\delta_1 - \varepsilon_m} = b_1^+ a_m^- \\
H_{m+k} = b_k^+ b_k^- - b_{k+1}^+ b_{k+1}^- & E_{\delta_k - \delta_{k+1}} = b_k^+ b_{k+1}^- & E_{\delta_{k+1} - \delta_k} = b_{k+1}^+ b_k^-
\end{array}$$

By commutation relation, one finds the realization of the whole set of root generators ($1 \le i, j \le m$ and $1 \le k, l \le n$):

$$E_{\varepsilon_i - \varepsilon_j} = a_i^+ a_j^- \ , \quad E_{\delta_k - \delta_l} = b_k^+ b_l^- \ , \quad E_{\varepsilon_i - \delta_k} = a_i^+ b_k^- \ , \quad E_{-\varepsilon_i + \delta_k} = b_k^+ a_i^-$$

Oscillator realization of $B(m, n)$

Let

$$\Delta = \left\{ \ \pm\varepsilon_i \pm \varepsilon_j, \ \pm\varepsilon_i, \ \pm\delta_k \pm \delta_l, \ \pm 2\delta_k, \ \pm\varepsilon_i \pm \delta_k, \ \pm\delta_k \ \right\}$$

be the root system of $B(m,n)$ expressed in terms of the orthogonal vectors $\varepsilon_1, \ldots, \varepsilon_m$ and $\delta_1, \ldots, \delta_n$. An oscillator realization of the simple generators in the distinguished basis is given by (where $1 \le i \le m-1$ and $1 \le k \le n-1$)

$$\begin{array}{lll}
H_k = b_k^+ b_k^- - b_{k+1}^+ b_{k+1}^- & E_{\delta_k - \delta_{k+1}} = b_k^+ b_{k+1}^- & E_{\delta_{k+1} - \delta_k} = b_{k+1}^+ b_k^- \\
H_n = b_n^+ b_n^- + a_1^+ a_1^- & E_{\delta_n - \varepsilon_1} = b_n^+ a_1^- & E_{\varepsilon_1 - \delta_n} = a_1^+ b_n^- \\
H_{n+i} = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- & E_{\varepsilon_i - \varepsilon_{i+1}} = a_i^+ a_{i+1}^- & E_{\varepsilon_{i+1} - \varepsilon_i} = a_{i+1}^+ a_i^- \\
H_{n+m} = 2a_m^+ a_m^- - 1 & E_{\varepsilon_m}^+ = a_m^+ e & E_{\varepsilon_m}^- = e a_m^-
\end{array}$$

By commutation relation, one finds the realization of the whole set of root generators ($1 \le i,j \le m$ and $1 \le k,l \le n$):

$$E_{\pm\varepsilon_i\pm\varepsilon_j} = a_i^\pm a_j^\pm\,, \qquad E_{\pm\varepsilon_i\pm\delta_k} = a_i^\pm b_k^\pm\,, \qquad E_{\varepsilon_i} = a_i^+ e\,, \qquad E_{-\varepsilon_i} = e a_i^-$$
$$E_{\pm\delta_k\pm\delta_l} = b_k^\pm b_l^\pm\,, \qquad E_{\pm 2\delta_k} = (b_k^\pm)^2\,, \qquad E_{\delta_k} = b_k^+ e\,, \qquad E_{-\delta_k} = e b_k^-$$

Oscillator realization of $B(0,n)$

The case $B(0,n)$ requires special attention. The root system of $B(0,n)$ can be expressed in terms of the orthogonal vectors $\delta_1,\dots,\delta_n$ and reduces to

$$\Delta = \Big\{ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\delta_k \Big\}$$

An oscillator realization of the generators of $B(0,n)$ can be obtained only with the help of bosonic oscillators. It is given for the simple generators by (where $1 \le k \le n-1$)

$$H_k = b_k^+ b_k^- - b_{k+1}^+ b_{k+1}^- \qquad E_{\delta_k-\delta_{k+1}} = b_k^+ b_{k+1}^- \qquad E_{\delta_{k+1}-\delta_k} = b_{k+1}^+ b_k^-$$
$$H_n = b_n^+ b_n^- + \tfrac{1}{2} \qquad E_{\delta_n} = \tfrac{1}{\sqrt{2}}\, b_n^+ \qquad E_{-\delta_n} = \tfrac{1}{\sqrt{2}}\, b_n^-$$

By commutation relation, one finds the realization of the whole set of root generators ($1 \le k,l \le n$):

$$E_{\pm\delta_k\pm\delta_l} = b_k^\pm b_l^\pm\,, \qquad E_{\pm 2\delta_k} = (b_k^\pm)^2\,, \qquad E_{\pm\delta_k} = \tfrac{1}{\sqrt{2}}\, b_k^\pm$$

Oscillator realization of $C(n+1)$

Let

$$\Delta = \Big\{ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\varepsilon \pm \delta_k \Big\}$$

be the root system of $C(n+1)$ expressed in terms of the orthogonal vectors $\varepsilon, \delta_1, \dots, \delta_n$. An oscillator realization of the simple generators in the distinguished basis is given by (where $2 \le k \le n$)

$$H_1 = a_1^+ a_1^- + b_1^+ b_1^- \qquad E_{\varepsilon-\delta_1} = a_1^+ b_1^- \qquad E_{\delta_1-\varepsilon} = b_1^+ a_1^-$$
$$H_k = b_k^+ b_k^- - b_{k+1}^+ b_{k+1}^- \qquad E_{\delta_k-\delta_{k+1}} = b_k^+ b_{k+1}^- \qquad E_{\delta_{k+1}-\delta_k} = b_{k+1}^+ b_k^-$$
$$H_{n+1} = -b_n^+ b_n^- - 1/2 \qquad E_{2\delta_n} = \tfrac{1}{2}(b_n^+)^2 \qquad E_{-2\delta_n} = \tfrac{1}{2}(b_n^-)^2$$

By commutation relation, one finds the realization of the whole set of root generators ($1 \le k,l \le n$):

$$E_{\pm\delta_k\pm\delta_l} = b_k^\pm b_l^\pm\,, \qquad E_{\pm 2\delta_k} = \tfrac{1}{2}(b_k^\pm)^2\,, \qquad E_{\varepsilon\pm\delta_l} = a_1^+ b_l^\pm\,, \qquad E_{-\varepsilon\pm\delta_l} = b_l^\pm a_1^-$$

Oscillator realization of $D(m,n)$

Let

$$\Delta = \Big\{ \ \pm\varepsilon_i \pm \varepsilon_j, \ \pm\delta_k \pm \delta_l, \ \pm 2\delta_k, \ \pm\varepsilon_i \pm \delta_k \ \Big\}$$

be the root system of $D(m,n)$ expressed in terms of the orthogonal vectors $\varepsilon_1, \dots, \varepsilon_m$ and $\delta_1, \dots, \delta_n$. An oscillator realization of the simple generators in the distinguished basis is given by (where $1 \le i \le m-1$ and $1 \le k \le n-1$)

$$\begin{array}{lll}
H_k = b_k^+ b_k^- - b_{k+1}^+ b_{k+1}^- & E_{\delta_k-\delta_{k+1}} = b_k^+ b_{k+1}^- & E_{\delta_{k+1}-\delta_k} = b_{k+1}^+ b_k^- \\
H_n = b_n^+ b_n^- + a_1^+ a_1^- & E_{\delta_n-\varepsilon_1} = b_n^+ a_1^- & E_{\varepsilon_1-\delta_n} = a_1^+ b_n^- \\
H_{n+i} = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- & E_{\varepsilon_i-\varepsilon_{i+1}} = a_i^+ a_{i+1}^- & E_{\varepsilon_{i+1}-\varepsilon_i} = a_{i+1}^+ a_i^- \\
H_{n+m} = a_{m-1}^+ a_{m-1}^- + a_m^+ a_m^- - 1 & & \\
 & E_{\varepsilon_{m-1}+\varepsilon_m} = a_{m-1}^+ a_m^+ & E_{-\varepsilon_m-\varepsilon_{m-1}} = a_m^- a_{m-1}^-
\end{array}$$

By commutation relation, one finds the realization of the whole set of root generators ($1 \le i,j \le m$ and $1 \le k,l \le n$):

$$E_{\pm\varepsilon_i\pm\varepsilon_j} = a_i^\pm a_j^\pm \ , \quad E_{\pm\varepsilon_i\pm\delta_k} = a_i^\pm b_k^\pm \ , \quad E_{\pm\delta_k\pm\delta_l} = b_k^\pm b_l^\pm \ , \quad E_{\pm 2\delta_k} = (b_k^\pm)^2$$

For more details, see ref. [5]. In ref. [5] oscillator realizations were used to analyze supersymmetric structure in the spectra of complex nuclei; the first reference to this interesting approach is [43].

2.31 Oscillator realizations: strange series

Let us consider a set of $2n$ bosonic oscillators b_i^- and b_i^+ with commutation relations:

$$\Big[b_i^-, b_j^-\Big] = \Big[b_i^+, b_j^+\Big] = 0 \quad \text{and} \quad \Big[b_i^-, b_j^+\Big] = \delta_{ij}$$

and a set of $2n$ fermionic oscillators a_i^- and a_i^+ with anticommutation relations:

$$\Big\{a_i^-, a_j^-\Big\} = \Big\{a_i^+, a_j^+\Big\} = 0 \quad \text{and} \quad \Big\{a_i^-, a_j^+\Big\} = \delta_{ij}$$

the two sets commuting each other:

$$\Big[b_i^-, a_j^-\Big] = \Big[b_i^-, a_j^+\Big] = \Big[b_i^+, a_j^-\Big] = \Big[b_i^+, a_j^+\Big] = 0$$

Oscillator realization of $P(n)$

An oscillator realization of the generators of $P(n)$ is obtained as follows:
– the generators of the even $sl(n)$ part are given by

$$\begin{aligned}
H_i &= a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- + b_i^+ b_i^- - b_{i+1}^+ b_{i+1}^- \quad \text{with} \quad 1 \le i \le n-1 \\
E_{ij}^+ &= a_i^+ a_j^- + b_i^+ b_j^- \quad \text{with} \quad 1 \le i < j \le n \\
E_{ij}^- &= a_i^+ a_j^- + b_i^+ b_j^- \quad \text{with} \quad 1 \le j < i \le n
\end{aligned}$$

– the generators of the odd symmetric part $\mathcal{G}_S$ of $P(n)$ by

$$\begin{aligned}
F_{ij}^+ &= b_i^+ a_j^+ + b_j^+ a_i^+ \quad \text{with} \quad 1 \le i \neq j \le n \\
F_i^+ &= b_i^+ b_j^+ \quad \text{with} \quad 1 \le i \le n
\end{aligned}$$

– the generators of the odd antisymmetric part $\mathcal{G}_A$ of $P(n)$ by

$$F_{ij}^- = b_i^- a_j^- + b_j^- a_i^- \quad \text{with} \quad 1 \le i \neq j \le n$$

Oscillator realization of $Q(n)$

An oscillator realization of the generators of $Q(n)$ is obtained as follows:
– the generators of the even $sl(n)$ part are given by

$$\begin{aligned}
H_i &= a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^- + b_i^+ b_i^- - b_{i+1}^+ b_{i+1}^- \\
E_{ij} &= a_i^+ a_j^- + b_i^+ b_j^-
\end{aligned}$$

– the generator of the $U(1)$ part by

$$Z = \sum_{i=1}^{n} a_i^+ a_i^- + b_i^+ b_i^-$$

– the generators of the odd $sl(n)$ part by

$$\begin{aligned}
K_i &= a_i^+ b_i^- - a_{i+1}^+ b_{i+1}^- + b_i^+ a_i^- - b_{i+1}^+ a_{i+1}^- \\
F_{ij} &= a_i^+ b_j^- + b_i^+ a_j^-
\end{aligned}$$

For more details, see ref. [24].

2.32 Real forms

Definition

Let $\mathcal{G}$ be a classical Lie superalgebra over $\mathbb{C}$. A semi-morphism ϕ of $\mathcal{G}$ is a semi-linear transformation of $\mathcal{G}$ which preserves the gradation, that is such that

$$\begin{aligned} \phi(\alpha X + \beta Y) &= \alpha^* \phi(X) + \beta^* \phi(Y) \\ [\![\phi(X), \phi(Y)]\!] &= \phi([\![X, Y]\!]) \end{aligned}$$

for all $X, Y \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$ (α^* denotes the complex conjugate of α).

If ϕ is an involutive semi-morphism of $\mathcal{G}$, the superalgebra $\mathcal{G}^\phi = \{X + \phi(X) \mid X \in \mathcal{G}\}$ is a real classical Lie superalgebra. Moreover, two involutive semi-morphisms ϕ and ϕ' of $\mathcal{G}$ being given, the real forms $\mathcal{G}^\phi$ and $\mathcal{G}^{\phi'}$ are isomorphic if and only if ϕ and ϕ' are conjugate by an automorphism ($\rightarrow$ 2.1) of $\mathcal{G}$.

It follows that the real classical Lie superalgebras are either the complex classical Lie superalgebras regarded as real superalgebras or the real forms obtained as subsuperalgebras of fixed points of the involutive semi-morphisms of a complex classical Lie superalgebra. The real forms of a complex classical Lie superalgebra $\mathcal{G}$ are thus classified by the involutive semi-morphisms of $\mathcal{G}$ in the automorphism group of $\mathcal{G}$. One can prove that the real forms of the complex classical Lie superalgebras are uniquely determined by the real forms $\mathcal{G}_{\overline{0}}^\phi$ of the even part $\mathcal{G}_{\overline{0}}$ of $\mathcal{G}$. These are displayed in Table 3.75.

The integers m, n have to be even for $sl(m|n, \mathbb{H})$, $psl(n|n, \mathbb{H})$ and $HQ(n)$. We recall that $su^*(2n)$ is the set of $2n \times 2n$ matrices of the form $\begin{pmatrix} X_n & Y_n \\ -Y_n^* & X_n^* \end{pmatrix}$ such that X_n, Y_n are matrices of order n and and $\mathrm{tr}(X_n) + \mathrm{tr}(X_n^*) = 0$ and $so^*(2n)$ is the set of $2n \times 2n$ matrices of the form $\begin{pmatrix} X_n & Y_n \\ -Y_n^* & X_n^* \end{pmatrix}$ such that X_n and Y_n are antisymmetric and hermitian complex matrices of order n respectively.

For more details, see refs. [47, 72].

2.33 Representations: basic definitions

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra. Let $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space and consider the superalgebra $\mathrm{End}\,\mathcal{V} = \mathrm{End}_{\overline{0}}\mathcal{V} \oplus \mathrm{End}_{\overline{1}}\mathcal{V}$ of endomorphisms of $\mathcal{V}$.

A linear representation π of $\mathcal{G}$ is a homomorphism of $\mathcal{G}$ into $\mathrm{End}\,\mathcal{V}$, that is, $[\![\ ,\ ]\!]$ denoting the superbracket,

$$\pi(\alpha X + \beta Y) = \alpha\pi(X) + \beta\pi(Y)$$
$$\pi([\![X, Y]\!]) = [\![\pi(X), \pi(Y)]\!]$$
$$\pi(\mathcal{G}_{\overline{0}}) \subset \mathrm{End}_{\overline{0}}\mathcal{V} \quad \text{and} \quad \pi(\mathcal{G}_{\overline{1}}) \subset \mathrm{End}_{\overline{1}}\mathcal{V}$$

for all $X, Y \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$.

The vector space $\mathcal{V}$ is the representation space. The vector space $\mathcal{V}$ has the structure of a $\mathcal{G}$-module by $X(\vec{v}) = \pi(X)\vec{v}$ for $X \in \mathcal{G}$ and $\vec{v} \in \mathcal{V}$.

The dimension (resp. superdimension) of the representation π is the dimension (resp. graded dimension) of the vector space $\mathcal{V}$:

$$\dim \pi = \dim \mathcal{V}_{\overline{0}} + \dim \mathcal{V}_{\overline{1}}$$
$$\mathrm{sdim}\,\pi = \dim \mathcal{V}_{\overline{0}} - \dim \mathcal{V}_{\overline{1}}$$

Definition

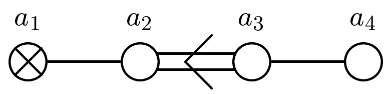
The representation π is said to be
– faithful if $\pi(X) \neq 0$ for all $X \in \mathcal{G}$.
– trivial if $\pi(X) = 0$ for all $X \in \mathcal{G}$.

Every classical Lie superalgebra has a finite dimensional faithful representation. In particular, the representation $\mathrm{ad} : \mathcal{G} \to \mathrm{End}\,\mathcal{G}$ ($\mathcal{G}$ being considered as a $\mathbb{Z}_2$-graded vector space) such that $\mathrm{ad}(X)Y = [\![X, Y]\!]$ is called the *adjoint* representation of $\mathcal{G}$.

2.34 Representations: exceptional superalgebras

Representations of $F(4)$

A highest weight irreducible representation of $F(4)$ is characterized by its Dynkin labels ($\to$ 2.35) drawn on the distinguished Dynkin diagram:



where a_2, a_3, a_4 are positive or null integers.

For the $so(7)$ part, a_2 is the shorter root. The $sl(2)$ representation label is hidden by the odd root and its value is given by $b = \frac{1}{3}(2a_1 - 3a_2 - 4a_3 - 2a_4)$. Since b has to be a non-negative integer, this relation implies a_1 to be a positive integer or half-integer. Finally, a $F(4)$ representation with $b < 4$ has to satisfy a consistency condition, that is

$b = 0$	$a_1 = a_2 = a_3 = a_4 = 0$
$b = 1$	not possible
$b = 2$	$a_2 = a_4 = 0$
$b = 3$	$a_2 = 2a_4 + 1$

The eight atypicality conditions for the $F(4)$ representations are the following:

$a_1 = 0$	or	$b = 0$
$a_1 = a_2 + 1$	or	$b = \frac{1}{3}(2 - a_2 - 4a_3 - 2a_4)$
$a_1 = a_2 + 2a_3 + 3$	or	$b = \frac{1}{3}(6 - a_2 - 2a_4)$
$a_1 = a_2 + 2a_3 + 2a_4 + 5$	or	$b = \frac{1}{3}(10 - a_2 + 2a_4)$
$a_1 = 2a_2 + 2a_3 + 4$	or	$b = \frac{1}{3}(8 + a_2 - 2a_4)$
$a_1 = 2a_2 + 2a_3 + 2a_4 + 6$	or	$b = \frac{1}{3}(12 + a_2 + 2a_4)$
$a_1 = 2a_2 + 4a_3 + 2a_4 + 8$	or	$b = \frac{1}{3}(16 + a_2 + 4a_3 + 2a_4)$
$a_1 = 3a_2 + 4a_3 + 2a_4 + 9$	or	$b = \frac{1}{3}(18 + 3a_2 + 4a_3 + 2a_4)$

Moreover, a necessary (but not sufficient) condition for a representation to be typical is that $b \geq 4$.

The dimension of a typical representation with highest weight $\Lambda = (a_1, a_2, a_3, a_4)$ is given by

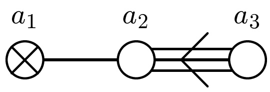
$$\begin{aligned}\dim \mathcal{V}(\Lambda) \;\; = \;\; & \frac{32}{45}(a_2+1)(a_3+1)(a_4+1)(a_2+a_3+2)(a_3+a_4+2) \\ & (a_2+2a_3+3)(a_2+a_3+a_4+3)(a_2+2a_3+2a_4+5) \\ & (a_2+2a_3+a_4+4)(2a_1-3a_2-4a_3-2a_4-9)\end{aligned}$$

Moreover, the reader will find in Table 3.73 dimensions of representations for the exceptional Lie superalgebra $F(4)$. See page 364 for explanations.

For more details, see refs. [50, 84].

Representations of $G(3)$

A highest weight irreducible representation of $G(3)$ is characterized by its Dynkin labels ($\rightarrow$ 2.35) drawn on the distinguished Dynkin diagram:



where a_2, a_3 are positive or null integers.

For the $G(2)$ part, a_2 is the shorter root. The $sl(2)$ representation label is hidden by the odd root and its value is given by $b = \frac{1}{2}(a_1 - 2a_2 - 3a_3)$. Since b has to be a non-negative integer, this relation implies a_1 to be a positive integer. Finally, a $G(3)$ representation with $b < 3$ has to satisfy a consistency condition, that is

$b = 0$	$a_1 = a_2 = a_3 = 0$
$b = 1$	not possible
$b = 2$	$a_2 = 0$

The six atypicality conditions for the $G(3)$ representations are the following:

$a_1 = 0$	or	$b = 0$
$a_1 = a_2 + 1$	or	$b = \frac{1}{2}(1 - a_2 - 3a_3)$
$a_1 = a_2 + 3a_3 + 4$	or	$b = \frac{1}{2}(4 - a_2)$
$a_1 = 3a_2 + 3a_3 + 6$	or	$b = \frac{1}{2}(6 + a_2)$
$a_1 = 3a_2 + 6a_3 + 9$	or	$b = \frac{1}{2}(4 + a_2 + 3a_3)$
$a_1 = 4a_2 + 6a_3 + 10$	or	$b = \frac{1}{2}(10 + 2a_2 + 3a_3)$

Let us remark that the first condition corresponds to the trivial representation and the second one is never satisfied.

Moreover, a necessary (but not sufficient) condition for a representation to be typical is that $b \geq 3$.

The dimension of a typical representation with highest weight $\Lambda = (a_1, a_2, a_3)$ is given by

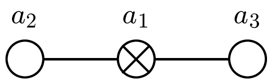
$$\begin{aligned} \dim \mathcal{V}(\Lambda) \;=\; & \frac{8}{15}(a_2+1)(a_3+1)(a_2+a_3+2)(a_2+3a_3+4) \\ & (a_2+2a_3+3)(2a_2+3a_3+5)(a_1-2a_2-3a_3-5) \end{aligned}$$

Moreover, the reader will find in Table 3.74 dimensions of representations for the exceptional Lie superalgebra $G(3)$. See page 364 for explanations.

For more details, see refs. [50, 85].

Representations of $D(2,1;\alpha)$

A highest weight irreducible representation of $D(2,1;\alpha)$ is characterized by its Dynkin labels ($\rightarrow$ 2.35) drawn on the distinguished Dynkin diagram:



where a_2, a_3 are positive or null integers.

The $sl(2)$ representation label is hidden by the odd root and its value is given by $b = \frac{1}{1+\alpha}(2a_1 - a_2 - \alpha a_3)$, which has to be a non-negative integer. Finally, a $D(2,1;\alpha)$ representation with $b < 2$ has to satisfy a consistency condition, that is

$$b = 0 \qquad a_1 = a_2 = a_3 = 0$$
$$b = 1 \qquad \alpha(a_3 + 1) = \pm(a_2 + 1)$$

The four atypicality conditions for the $D(2,1;\alpha)$ representations are the following:

$$a_1 = 0 \quad \text{or} \quad b = 0$$
$$a_1 = a_2 + 1 \quad \text{or} \quad b = \tfrac{1}{1+\alpha}(2 + 2a_2 - \alpha a_3)$$
$$a_1 = \alpha(a_3 + 1) \quad \text{or} \quad b = \tfrac{1}{1+\alpha}(2\alpha - a_2 - \alpha a_3)$$
$$a_1 = a_2 + \alpha a_3 + 1 + \alpha \quad \text{or} \quad b = \tfrac{1}{1+\alpha}(2 + 2\alpha + a_2 + \alpha a_3)$$

The dimension of a typical representation with highest weight $\Lambda = (a_1, a_2, a_3)$ is given by

$$\dim \mathcal{V}(\Lambda) = \frac{16}{1+\alpha}(a_2 + 1)(a_3 + 1)(2a_1 - a_2 - \alpha a_3 - 1 - \alpha)$$

For more details, see refs. [50, 91].

2.35 Representations: highest weight representations

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a basic Lie superalgebra with Cartan subalgebra $\mathcal{H}$ and $\mathcal{H}^*$ be the dual of $\mathcal{H}$. We assume that $\mathcal{G} \neq A(n,n)$ but the following results still hold for $sl(n+1|n+1)$. Let $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ be a Borel decomposition ($\rightarrow$ 2.45) of $\mathcal{G}$ where $\mathcal{N}^+$ (resp. $\mathcal{N}^-$) is spanned by the positive (resp. negative) root generators of $\mathcal{G}$ ($\rightarrow$ 2.45).

Definition

A representation $\pi : \mathcal{G} \to \operatorname{End}\mathcal{V}$ with representation space $\mathcal{V}$ is called a *highest weight* representation with highest weight $\Lambda \in \mathcal{H}^*$ if there exists a non-zero vector $\vec{v}_\Lambda \in \mathcal{V}$ such that

$$\mathcal{N}^+\vec{v}_\Lambda = 0$$
$$h(\vec{v}_\Lambda) = \Lambda(h)\vec{v}_\Lambda \qquad (h \in \mathcal{H})$$

The $\mathcal{G}$-module $\mathcal{V}$ is called a highest weight module, denoted by $\mathcal{V}(\Lambda)$, and the vector $\vec{v}_\Lambda \in \mathcal{V}$ a highest weight vector.

From now on, $\mathcal{H}$ is the *distinguished* Cartan subalgebra ($\to$ 2.3) of $\mathcal{G}$ with basis of generators $(H_1, \dots, H_r)$ where $r = \operatorname{rank}\mathcal{G}$ and H_s denotes the Cartan generator associated to the odd simple root. The Dynkin labels are defined by

$$a_i = 2\,\frac{\Lambda \cdot \alpha_i}{\alpha_i \cdot \alpha_i} \;\text{ for }\; i \neq s \qquad \text{and} \qquad a_s = \Lambda \cdot \alpha_s$$

A weight $\Lambda \in \mathcal{H}^*$ is called a dominant weight if $a_i \geq 0$ for all $i \neq s$, integral if $a_i \in \mathbb{Z}$ for all $i \neq s$, and integral dominant if $a_i \in \mathbb{N}$ for all $i \neq s$.

Property

A necessary condition for the highest weight representation of $\mathcal{G}$ with highest weight Λ to be finite dimensional is that Λ be an integral dominant weight.

Following V.G. Kac (see ref. [50]), one defines the Kac's module:

Definition

Let $\mathcal{G}$ be a basic Lie superalgebra with the distinguished $\mathbb{Z}$-gradation $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\,\mathcal{G}_i$ ($\to$ 2.8). Let $\Lambda \in \mathcal{H}^*$ be an integral dominant weight and $\mathcal{V}_0(\Lambda)$ be the $\mathcal{G}_0$-module with highest weight $\Lambda \in \mathcal{H}^*$. Consider the $\mathcal{G}$-subalgebra $\mathcal{K} = \mathcal{G}_0 \oplus \mathcal{N}^+$ where $\mathcal{N}^+ = \oplus_{i>0}\mathcal{G}_i$. The $\mathcal{G}_0$-module $\mathcal{V}_0(\Lambda)$ is extended to a $\mathcal{K}$-module by setting $\mathcal{N}^+\mathcal{V}_0(\Lambda) = 0$. The Kac module $\overline{\mathcal{V}}(\Lambda)$ is defined as follows:

1. if the superalgebra $\mathcal{G}$ is of type I (the odd part is the direct sum of two irreducible representations of the even part), the Kac module is the induced module ($\to$ 2.36)

$$\overline{\mathcal{V}}(\Lambda) = \operatorname{Ind}_{\mathcal{K}}^{\mathcal{G}}\, \mathcal{V}_0(\Lambda)$$

2. if the superalgebra $\mathcal{G}$ is of type II (the odd part is an irreducible representation of the even part), the induced module $\mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V}_0(\Lambda)$ contains a submodule $\mathcal{M}(\Lambda) = \mathcal{U}(\mathcal{G})\mathcal{G}_{-\psi}^{b+1}\mathcal{V}_0(\Lambda)$ where ψ is the longest simple root of $\mathcal{G}_{\overline{0}}$ which is hidden behind the odd simple root (that is the longest simple root of $sp(2n)$ in the case of $osp(m|2n)$ and the simple root of $sl(2)$ in the case of $F(4)$, $G(3)$ and $D(2,1;\alpha)$) and $b = 2\Lambda \cdot \psi / \psi \cdot \psi$ is the component of Λ with respect to ψ ($\rightarrow$ 2.34 and 2.37 for explicit expressions of b). The Kac module is then defined as the quotient of the induced module $\mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V}_0(\Lambda)$ by the submodule $\mathcal{M}(\Lambda)$:

$$\overline{\mathcal{V}}(\Lambda) = \mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V}_0(\Lambda)/\mathcal{U}(\mathcal{G})\mathcal{G}_{-\psi}^{b+1}\mathcal{V}_0(\Lambda)$$

In the case where the Kac module is not simple, it contains a maximal submodule $\mathcal{I}(\Lambda)$ and the quotient module $\mathcal{V}(\Lambda) = \overline{\mathcal{V}}(\Lambda)/\mathcal{I}(\Lambda)$ is a simple module.

The fundamental result concerning the representations of basic Lie superalgebras is the following:

Theorem

- Any finite dimensional irreducible representation of $\mathcal{G}$ is of the form $\mathcal{V}(\Lambda) = \overline{\mathcal{V}}(\Lambda)/\mathcal{I}(\Lambda)$ where Λ is an integral dominant weight.

- Any finite dimensional simple $\mathcal{G}$-module is uniquely characterized by its integral dominant weight Λ: two $\mathcal{G}$-modules $\mathcal{V}(\Lambda)$ and $\mathcal{V}(\Lambda')$ are isomorphic if and only if $\Lambda = \Lambda'$.

- The finite dimensional simple $\mathcal{G}$-module $\mathcal{V}(\Lambda) = \overline{\mathcal{V}}(\Lambda)/\mathcal{I}(\Lambda)$ has the weight decomposition

$$\mathcal{V}(\Lambda) = \bigoplus_{\lambda \le \Lambda} \mathcal{V}_\lambda \quad \text{with} \quad \mathcal{V}_\lambda = \Big\{ \vec{v} \in \mathcal{V} \,\Big|\, h(\vec{v}) = \lambda(h)\vec{v},\ h \in \mathcal{H} \Big\}$$

2.36 Representations: induced modules

The method of induced representations is an elegant and powerful way to construct the highest weight representations ($\rightarrow$ 2.35) of the basic Lie superalgebras. This section is quite formal compared to the rest of the text but is fundamental for the representation theory of the Lie superalgebras.

Let $\mathcal{G}$ be a basic Lie superalgebra and $\mathcal{K}$ be a subalgebra of $\mathcal{G}$. Denote by $\mathcal{U}(\mathcal{G})$ and $\mathcal{U}(\mathcal{K})$ the corresponding universal enveloping superalgebras ($\rightarrow$

2.62). From a $\mathcal{K}$-module $\mathcal{V}$ ($\rightarrow$ 2.33), it is possible to construct a $\mathcal{G}$-module in the following way. The vector space $\mathcal{V}$ is naturally extended to a $\mathcal{U}(\mathcal{K})$-module. One considers the factor space $\mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathcal{K})} \mathcal{V}$ consisting of elements of $\mathcal{U}(\mathcal{G}) \otimes \mathcal{V}$ such that the elements $h \otimes \vec{v}$ and $1 \otimes h(\vec{v})$ have been identified for $h \in \mathcal{K}$ and $\vec{v} \in \mathcal{V}$. This space acquires the structure of a $\mathcal{G}$-module by setting $g(u \otimes \vec{v}) = gu \otimes \vec{v}$ for $u \in \mathcal{U}(\mathcal{G})$, $g \in \mathcal{G}$ and $\vec{v} \in \mathcal{V}$.

Definition

The $\mathcal{G}$-module $\mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathcal{K})} \mathcal{V}$ is called the *induced module* from the $\mathcal{K}$-module $\mathcal{V}$ and denoted by $\mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V}$.

Theorem

Let $\mathcal{K}'$ and $\mathcal{K}''$ be subalgebras of $\mathcal{G}$ such that $\mathcal{K}'' \subset \mathcal{K}' \subset \mathcal{G}$. If $\mathcal{V}$ is a $\mathcal{K}''$-module, then

$$\mathrm{Ind}_{\mathcal{K}'}^{\mathcal{G}}\ (\mathrm{Ind}_{\mathcal{K}''}^{\mathcal{K}'}\ \mathcal{V}) = \mathrm{Ind}_{\mathcal{K}''}^{\mathcal{G}}\ \mathcal{V}$$

Theorem

Let $\mathcal{G}$ be a basic Lie superalgebra, $\mathcal{K}$ be a subalgebra of $\mathcal{G}$ such that $\mathcal{G}_{\overline{0}} \subset \mathcal{K}$ and $\mathcal{V}$ a $\mathcal{K}$-module. If $\{f_1, \dots, f_d\}$ denotes a basis of odd generators of $\mathcal{G}/\mathcal{K}$, then

$$\mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V} = \bigoplus_{1 \le i_1 < \dots < i_k \le d} f_{i_1} \dots f_{i_k} \mathcal{V}$$

is a direct sum of subspaces and $\dim \mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}\ \mathcal{V} = 2^d \dim \mathcal{V}$.

Example

Consider a basic Lie superalgebra $\mathcal{G}$ of type I (the odd part is the direct sum of two irreducible representations of the even part, that is $\mathcal{G} = sl(m|n)$ or $osp(2|2n)$) with $\mathbb{Z}$-gradation $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ ($\rightarrow$ 2.8). Take for $\mathcal{K}$ the subalgebra $\mathcal{G}_0 \oplus \mathcal{G}_1$. Let $\mathcal{V}_0(\Lambda)$ be a $\mathcal{G}_0$-module with highest weight Λ, which is extended to a $\mathcal{K}$-module by setting $\mathcal{G}_1 \mathcal{V}_0(\Lambda) = 0$. Since $\left\{\mathcal{G}_{-1}, \mathcal{G}_{-1}\right\} = 0$, only the completely antisymmetric combinations of the generators of $\mathcal{G}_{-1}$ can apply on $\mathcal{V}_0(\Lambda)$. In other words, the $\mathcal{G}$-module $\mathcal{V}(\Lambda)$ is obtained by

$$\mathcal{V} = \bigwedge(\mathcal{G}_{-1}) \otimes \mathcal{V}_0 \simeq \mathcal{U}(\mathcal{G}_{-1}) \otimes \mathcal{V}_0$$

where

$$\bigwedge(\mathcal{G}_{-1}) = \bigoplus_{k=0}^{\dim \mathcal{G}_{-1}} \wedge^k(\mathcal{G}_{-1})$$

is the exterior algebra over $\mathcal{G}_{-1}$ of dimension 2^d if $d = \dim \mathcal{G}_{-1}$. It follows that $\mathcal{V}(\Lambda)$ is built from $\mathcal{V}_0(\Lambda)$ by induction of the generators of $\mathcal{G}/\mathcal{K}$:

$$\mathcal{V} = \mathcal{U}(\mathcal{G}_{-1}) \otimes \mathcal{V}_0 = \mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathcal{G}_0 \oplus \mathcal{G}_1)} \mathcal{V}_0 = \mathrm{Ind}_{\mathcal{G}_0 \oplus \mathcal{G}_1}^{\mathcal{G}}\ \mathcal{V}_0$$

Since $\dim \wedge^k(\mathcal{G}_{-1}) = \binom{d}{k}$, the dimension of $\mathcal{V}$ is given by

$$\dim \mathcal{V}(\Lambda) = \sum_{k=0}^{d} \binom{d}{k} \dim \mathcal{V}_0(\Lambda) = 2^d \dim \mathcal{V}_0(\Lambda)$$

while its superdimension (→ 2.33) is identically zero

$$\mathrm{sdim}\, \mathcal{V}(\Lambda) = \sum_{k=0}^{d} (-1)^k \binom{d}{k} \dim \mathcal{V}_0(\Lambda) = 0$$

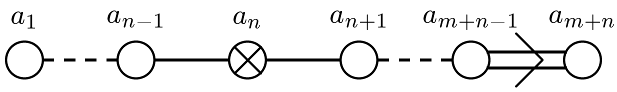
Let us stress that such a $\mathcal{G}$-module is not always an irreducible one. □

For more details, see refs. [49, 50, 93].

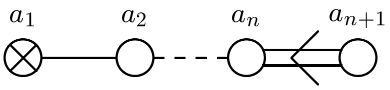
2.37 Representations: orthosymplectic superalgebras

A highest weight irreducible representation of $osp(M|N)$ is characterized by its Dynkin labels (→ 2.35) drawn on the distinguished Dynkin diagram. The different diagrams are the following:

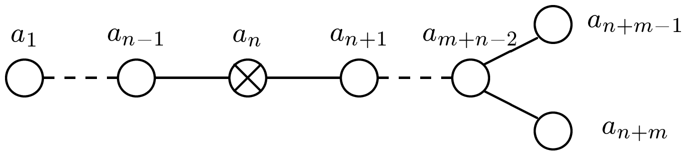
- $osp(2m+1|2n)$ with $\Lambda = (a_1, \dots, a_{m+n})$



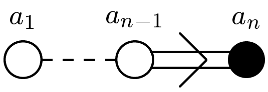
- $osp(2|2n)$ with $\Lambda = (a_1, \dots, a_{n+1})$



- $osp(2m|2n)$ with $\Lambda = (a_1, \dots, a_{m+n})$



- $osp(1|2n)$ with $\Lambda = (a_1, \dots, a_n)$



The superalgebra $osp(2|2n)$ is of type I, while the superalgebras $osp(2m+1|2n)$ and $osp(2m|2n)$ are of type II: in the first case, the odd part is the direct sum of two irreducible representations of the even part, in the second case it is an irreducible representation of the even part. The numbers a_i are constrained to satisfy the following conditions:

a_n is integer or half-integer for $osp(2m+1|2n)$ and $osp(2m|2n)$,
a_1 is an arbitrary complex number for $osp(2|2n)$.

The coordinates of Λ in the root space characterize a $so(M) \oplus sp(2n)$ representation ($M = 2m$ or $M = 2m+1$). The $so(M)$ representation can be directly read on the Kac–Dynkin diagram, but the longest simple root of $sp(2n)$ is hidden behind the odd simple roots. From the knowledge of $(a_n, \dots, a_{m+n})$, it is possible to deduce the component b that Λ would have with respect to the longest simple root:

in the $osp(2m+1|2n)$ case

$$b = a_n - a_{n+1} - \dots - a_{m+n-1} - \tfrac{1}{2} a_{m+n}$$

in the $osp(2m|2n)$ case

$$b = a_n - a_{n+1} - \dots - a_{m+n-2} - \tfrac{1}{2}(a_{m+n-1} + a_{m+n})$$

The number b has to be a non-negative integer.

The highest weight of a finite representation of $osp(M|2n)$ belongs therefore to a $so(M) \oplus sp(2n)$ representation and thus one must have the following consistency conditions:

$$\begin{array}{ll} b \geq 0 & \\ \text{for } osp(2m+1|2n), & a_{n+b+1} = \dots = a_{n+m} = 0 \text{ if } b \leq m-1 \\ \text{for } osp(2m|2n), & a_{n+b+1} = \dots = a_{n+m} = 0 \text{ if } b \leq m-2 \\ & \text{and } a_{n+m-1} = a_{n+m} \text{ if } b = m-1 \end{array}$$

We give below the atypicality conditions of the representations for the superalgebras of the orthosymplectic series. If at least one of these conditions is satisfied, the representation is an atypical one. Otherwise, the representation is typical, the dimension of which is given by the number $\dim \mathcal{V}(\Lambda)$.

The reader will find in Part 3 tables of dimensions of representations for the orthosymplectic superalgebras of small rank (see Tables 3.62–3.68). See page 364 for explanations.

Lie superalgebras $osp(2m+1|2n)$

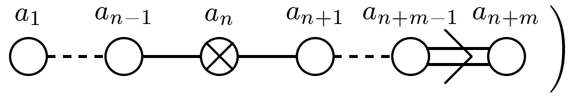
The atypicality conditions are

$$\sum_{q=i}^{n} a_q - \sum_{q=n+1}^{j} a_q + 2n - i - j = 0$$

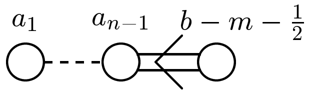
$$\sum_{q=i}^{n} a_q - \sum_{q=n+1}^{j} a_q - 2\sum_{q=j+1}^{m+n-1} a_q - a_{m+n} - 2m - i + j + 1 = 0$$

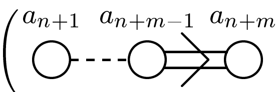
$$\text{with} \quad 1 \leq i \leq n \leq j \leq m+n-1$$

The dimensions of the typical representations are given by

dim (

) $= 2^{(2m+1)n} \times$

dim (

) × dim (

)

that is

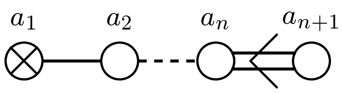
$$\begin{aligned}
\dim \mathcal{V}(\Lambda) \;&=\; 2^{(2m+1)n} \\
&\times \prod_{1\leq i\leq j\leq n-1} \frac{\sum_{q=i}^{j} a_q + j - i + 1}{j - i + 1} \prod_{n+1\leq i\leq j\leq n+m-1} \frac{\sum_{q=i}^{j} a_q + j - i + 1}{j - i + 1} \\
&\times \prod_{1\leq i\leq j\leq n} \frac{(\sum_{q=i}^{j-1} + 2\sum_{q=j}^{n} - 2\sum_{q=n+1}^{n+m-1})a_q - a_{n+m} + 2n - 2m - i - j + 1}{2n - i - j + 2} \\
&\times \prod_{n+1\leq i\leq j\leq n+m} \frac{\sum_{q=i}^{j-1} a_q + 2\sum_{q=j}^{m+n-1} a_q + a_{m+n} + 2m - i - j + 1}{2m - i - j + 1}
\end{aligned}$$

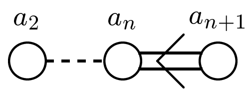
Lie superalgebras $osp(2|2n)$

The atypicality conditions are

$$\left.\begin{aligned}
&a_1 - \sum_{q=2}^{i} a_q - i + 1 = 0 \\
&a_1 - \sum_{q=2}^{i} a_q - 2\sum_{q=i+1}^{n+1} a_q - 2n + i - 1 = 0
\end{aligned}\right. \qquad \text{with} \quad 1 \leq i \leq n$$

The dimensions of the typical representations are given by

dim (

) $= 2^{2n} \times$ dim (

)

that is

$$\dim \mathcal{V}(\Lambda) = 2^{2n} \prod_{2 \le i \le j \le n} \frac{\sum_{q=i}^{j} a_q + j - i + 1}{j - i + 1} \times \prod_{2 \le i \le j \le n+1} \frac{\sum_{q=i}^{j-1} a_q + 2 \sum_{q=j}^{n+1} a_q + 2n - i - j + 4}{2n - i - j + 4}$$

Lie superalgebras $osp(2m|2n)$

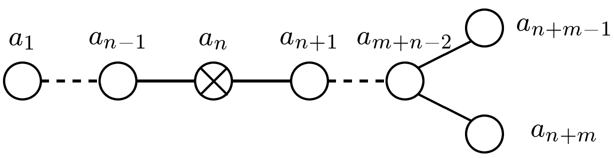
The atypicality conditions are

$$\sum_{q=i}^{n} a_q - \sum_{q=n+1}^{j} a_q + 2n - i - j = 0 \quad \text{with} \quad 1 \le i \le n \le j \le m + n - 1$$

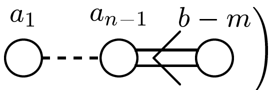
$$\sum_{q=i}^{n} a_q - \sum_{q=n+1}^{m+n-2} a_q - a_{m+n} + n - m - i + 1 = 0 \quad \text{with} \quad 1 \le i \le n$$

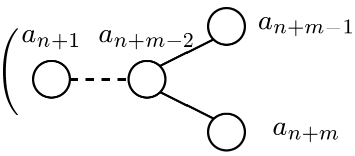
$$\sum_{q=i}^{n} a_q - \sum_{q=n+1}^{j} a_q - 2 \sum_{q=j+1}^{m+n-2} a_q - a_{m+n-1} - a_{m+n} - 2m - i + j + 2 = 0$$
$$\text{with} \quad 1 \le i \le n \le j \le m + n - 2$$

The dimensions of the typical representations are given by

dim (

) $= 2^{2mn} \times$

$\times$ dim (

) $\times$ dim (

)

that is

$$\dim \mathcal{V}(\Lambda) = 2^{2mn}$$
$$\times \prod_{1 \le i \le j \le n-1} \frac{\sum_{q=i}^{j} a_q + j - i + 1}{j - i + 1} \prod_{n+1 \le i \le j \le n+m-1} \frac{\sum_{q=i}^{j} a_q + j - i + 1}{j - i + 1}$$

$$\times \prod_{n+1 \le i \le j \le n+m-2} \frac{\sum_{q=i}^{j-1} a_q + 2 \sum_{q=j}^{n+m-2} a_q + a_{n+m-1} + a_{n+m} + 2m + 2n - i - j}{2m + 2n - i - j}$$

$$\times \prod_{1 \le i \le j \le n} \frac{(\sum_{q=i}^{j-1} + 2 \sum_{q=j}^{n} - 2 \sum_{q=n+1}^{n+m-2}) a_q - a_{n+m-1} - a_{n+m} - 2m + 2n - i - j + 1}{2n - i - j + 2}$$

$$\times \prod_{n+1 \le i \le n+m-1} \frac{\sum_{q=i}^{n+m-2} a_q + a_{n+m} + m - i}{m - i}$$

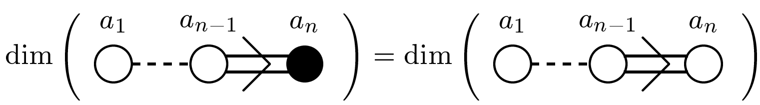
Lie superalgebras $osp(1|2n)$

The superalgebras $osp(1|2n)$ carry the property of having only typical representation (the Dynkin diagram of $osp(1|2n)$ does not contain any grey dot). One has

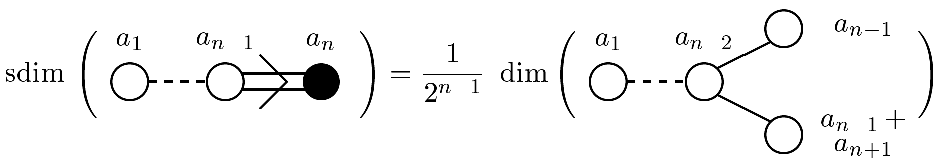
$$\dim \mathcal{V}(\Lambda) = \prod_{1 \le i < j \le n} \frac{\sum_{q=i}^{j-1} a_q + 2 \sum_{q=j}^{n-1} a_q + a_n + 2n - j - i + 2}{2n - j - i + 2}$$

$$\times \prod_{1 \le i < j \le n} \frac{\sum_{q=i}^{j-1} a_q + j - i}{j - i} \prod_{1 \le i \le n} \frac{2 \sum_{q=i}^{n-1} a_q + a_n + 2n - 2i + 1}{2n - 2i + 1}$$

Moreover, the representations of $osp(1|2n)$ can be put in a one-to-one correspondence with those of $so(2n+1)$ [77]. More precisely, one has



as well as



(let us recall that $\dim \mathcal{V} = \dim \mathcal{V}_{\overline{0}} + \dim \mathcal{V}_{\overline{1}}$ while $\text{sdim}\, \mathcal{V} = \dim \mathcal{V}_{\overline{0}} - \dim \mathcal{V}_{\overline{1}}$).

For more details, see refs. [40, 41, 50, 63, 77].

2.38 Representations: reducibility

Definition

Let $\mathcal{G}$ be a classical Lie superalgebra. A representation $\pi : \mathcal{G} \to \mathrm{End}\,\mathcal{V}$ is called *irreducible* if the $\mathcal{G}$-module $\mathcal{V}$ has no $\mathcal{G}$-submodules except trivial ones. The $\mathcal{G}$-module $\mathcal{V}$ is then called a *simple* module.

Otherwise the representation π is said to be *reducible*. In that case, one has $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$, $\mathcal{V}''$ being a complementary subspace of $\mathcal{V}'$ in $\mathcal{V}$ and the $\mathcal{G}$-submodule $\mathcal{V}'$ is an invariant subspace under π. If the subspace $\mathcal{V}''$ is also an invariant subspace under π, the representation π is said to be *completely reducible*. The $\mathcal{G}$-module $\mathcal{V}$ is then called a *semi-simple* module.

A representation which is reducible but not completely reducible is called *indecomposable*.

Definition

Two representations π and π' of $\mathcal{G}$ being given, with representation spaces $\mathcal{V}$ and $\mathcal{V}'$, one defines the direct sum $\pi \oplus \pi'$ with representation space $\mathcal{V} \oplus \mathcal{V}'$ and the direct (or tensor) product $\pi \otimes \pi'$ with representation space $\mathcal{V} \otimes \mathcal{V}'$ of the two representations. The action of the representations $\pi \oplus \pi'$ and $\pi \otimes \pi'$ on the corresponding representation spaces is given by, for $X \in \mathcal{G}$, $\vec{v} \in \mathcal{V}$ and $\vec{v}\,' \in \mathcal{V}'$:

$$(\pi \oplus \pi')(X)\vec{v} \oplus \vec{v}\,' = \pi(X)\vec{v} \;\oplus\; \pi'(X)\vec{v}\,'$$
$$(\pi \otimes \pi')(X)\vec{v} \otimes \vec{v}\,' = \pi(X)\vec{v} \;\otimes\; \vec{v}\,' + \vec{v} \;\otimes\; \pi'(X)\vec{v}\,'$$

The representations π and π' of $\mathcal{G}$ being irreducible, the tensor product $\pi \otimes \pi'$ is a representation which is in general reducible. Notice however that, contrary to the Lie algebra case, in the Lie superalgebra case the tensor product of two irreducible representations is not necessarily completely reducible. In fact, one has the following theorem:

Theorem (Djokovic–Hochschild)

The only Lie superalgebras for which all finite dimensional representations are completely reducible are the direct products of $osp(1|2n)$ superalgebras and semi-simple Lie algebras.

2.39 Representations: star and superstar representations

The star and superstar representations of a classical Lie superalgebra are the generalization of the hermitian representations of a simple Lie algebra. The importance of the hermitian representations for simple Lie algebras comes from the fact that the finite dimensional representations of a compact simple Lie algebra are equivalent to hermitian representations.

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra. One can define two different adjoint operations as follows.

Definition

An adjoint operation in $\mathcal{G}$, denoted by $\dagger$, is a mapping from $\mathcal{G}$ into $\mathcal{G}$ such that:

- $X \in \mathcal{G}_i \;\Rightarrow\; X^\dagger \in \mathcal{G}_i$ for $i = \overline{0}, \overline{1}$,
- $(\alpha X + \beta Y)^\dagger = \alpha^* X^\dagger + \beta^* Y^\dagger$,
- $[\![X, Y]\!]^\dagger = [\![Y^\dagger, X^\dagger]\!]$,
- $(X^\dagger)^\dagger = X$,

where $X, Y \in \mathcal{G}$, $\alpha, \beta \in \mathbb{C}$ and α^*, β^* are the complex conjugate of α, β.

Definition

A superadjoint operation in $\mathcal{G}$, denoted by $\ddagger$, is a mapping from $\mathcal{G}$ into $\mathcal{G}$ such that:

- $X \in \mathcal{G}_i \;\Rightarrow\; X^\ddagger \in \mathcal{G}_i$ for $i = \overline{0}, \overline{1}$,
- $(\alpha X + \beta Y)^\ddagger = \alpha^* X^\ddagger + \beta^* Y^\ddagger$,
- $[\![X, Y]\!]^\ddagger = (-1)^{\deg X . \deg Y} [\![Y^\ddagger, X^\ddagger]\!]$,
- $(X^\ddagger)^\ddagger = (-1)^{\deg X} X$,

where $X, Y \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$.

The definitions of the star and superstar representations follow immediately.

Definition

Let $\mathcal{G}$ be a classical Lie superalgebra and π a representation of $\mathcal{G}$ acting in a $\mathbb{Z}_2$-graded vector space $\mathcal{V}$. Then π is a star representation of $\mathcal{G}$ if $\pi(X^\dagger) = \pi(X)^\dagger$ and a superstar representation of $\mathcal{G}$ if $\pi(X^\ddagger) = \pi(X)^\ddagger$ for all $X \in \mathcal{G}$.

The following properties hold:

Property

1. Any star representation π of $\mathcal{G}$ in a graded Hilbert space $\mathcal{V}$ is completely reducible.
2. Any superstar representation π of $\mathcal{G}$ in a graded Hilbert space $\mathcal{V}$ is completely reducible.
3. The tensor product $\pi \otimes \pi'$ of two star representations (resp. to superstar representations) π and π' is a star representation (resp. a superstar representation).
4. The tensor product $\pi \otimes \pi'$ of two star representations π and π' is completely reducible.

Let us emphasize that the last property does not hold for superstar representations (that is the tensor product of two superstar representations is in general not completely reducible).

The classes of star and superstar representations of the classical Lie superalgebras are the following:

– the superalgebra $A(m,n)$ has two classes of star representations and two classes of superstar representations.
– the superalgebras $B(m,n)$ and $D(m,n)$ have two classes of superstar representations.
– the superalgebra $C(n+1)$ has either two classes of star representations and two classes of superstar representations, or one class of superstar representations, depending on the definition of the adjoint operation in the Lie algebra part.
– the superalgebras $F(4)$ and $G(3)$ have two classes of superstar representations.
– the superalgebra $P(n)$ has neither star nor superstar representations.
– the superalgebra $Q(n)$ has two classes of star representations.

For more details, see ref. [65].

2.40 Representations: typicality and atypicality

Any representation of a basic Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ can be decomposed into a direct sum of irreducible representations of the even subalgebra $\mathcal{G}_{\overline{0}}$. The generators associated to the odd roots will transform a vector basis belonging to a certain representation of $\mathcal{G}_{\overline{0}}$ into a vector in another representation of $\mathcal{G}_{\overline{0}}$ (or into the null vector), while the generators associated to the even roots will operate inside an irreducible representation of $\mathcal{G}_{\overline{0}}$.

The presence of odd roots will have another important consequence in the representation theory of superalgebras. Indeed, one might find that in certain representations weight vectors, different from the highest one specifying the representation, are annihilated by all the generators corresponding to positive roots. Such vector have, of course, to be decoupled from the representation. Representations of this kind are called *atypical*, while the other irreducible representations not suffering this pathology are called *typical*.

More precisely, let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a basic Lie superalgebra with distinguished Cartan subalgebra $\mathcal{H}$. Let $\Lambda \in \mathcal{H}^*$ be an integral dominant weight. Denote the root system of $\mathcal{G}$ by $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$. One defines $\overline{\Delta}_{\overline{0}}$ as the subset of roots $\alpha \in \Delta_{\overline{0}}$ such that $\alpha/2 \notin \Delta_{\overline{1}}$ and $\overline{\Delta}_{\overline{1}}$ as the subset of roots $\alpha \in \Delta_{\overline{1}}$ such that $2\alpha \notin \Delta_{\overline{0}}$. Let ρ_0 be the half-sum of the roots of $\Delta_{\overline{0}}^+$, $\overline{\rho}_0$ the half-sum of the roots of $\overline{\Delta}_{\overline{0}}^+$, ρ_1 the half-sum of the roots of $\Delta_{\overline{1}}^+$, and $\rho = \rho_0 - \rho_1$.

Definition

The representation π with highest weight Λ is called *typical* if

$$(\Lambda + \rho) \cdot \alpha \neq 0 \qquad \text{for all } \alpha \in \overline{\Delta}_{\overline{1}}^+$$

The highest weight Λ is then called typical.
If there exists some $\alpha \in \overline{\Delta}_{\overline{1}}^+$ such that $(\Lambda + \rho) \cdot \alpha = 0$, the representation π and the highest weight Λ are called *atypical*. The number of distinct elements $\alpha \in \overline{\Delta}_{\overline{1}}^+$ for which Λ is atypical is the degree of atypicality of the representation π. If there exists one and only one $\alpha \in \overline{\Delta}_{\overline{1}}^+$ such that $(\Lambda + \rho) \cdot \alpha = 0$, the representation π and the highest weight Λ are called *singly atypical*.

Denoting as before $\overline{\mathcal{V}}(\Lambda)$ the Kac module ($\rightarrow$ 2.35) corresponding to the integral dominant weight Λ, one has the following theorem:

Theorem

The Kac module $\overline{\mathcal{V}}(\Lambda)$ is a simple $\mathcal{G}$-module if and only if the highest weight Λ is typical.

Properties

1. All the finite dimensional representations of $B(0, n)$ are typical.
2. All the finite dimensional representations of $C(n+1)$ are either typical or singly atypical.

Let $\mathcal{V}$ be a typical finite dimensional representation of $\mathcal{G}$. Then the dimension of $\mathcal{V}(\Lambda)$ is given by

$$\dim \mathcal{V}(\Lambda) = 2^{\dim \Delta_{\overline{1}}^{+}} \prod_{\alpha \in \Delta_{\overline{0}}^{+}} \frac{(\Lambda+\rho)\cdot\alpha}{\rho_0 \cdot \alpha}$$

and

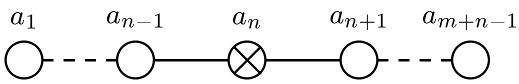
$$\dim \mathcal{V}_{\overline{0}}(\Lambda) - \dim \mathcal{V}_{\overline{1}}(\Lambda) = 0 \quad \text{if} \quad \mathcal{G} \neq B(0,n)$$
$$\dim \mathcal{V}_{\overline{0}}(\Lambda) - \dim \mathcal{V}_{\overline{1}}(\Lambda) = \prod_{\alpha \in \overline{\Delta}_{\overline{0}}^{+}} \frac{(\Lambda+\rho)\cdot\alpha}{\overline{\rho}_0 \cdot \alpha} \quad \text{if} \quad \mathcal{G} = B(0,n)$$

The fundamental representations of the basic Lie superalgebras $sl(m|n)$ and $osp(m|n)$ (of dimension $m+n$) are atypical ones as well as the adjoint representations of the basic Lie superalgebras $\mathcal{G} \neq sl(n \pm 1|n)$, $osp(2n \pm 1|2n)$, $osp(2n|2n)$, $osp(2n+2|2n)$, $osp(1|2n)$ (of dimension $\dim \mathcal{G}$) (since $\dim \mathcal{V}_{\overline{0}} - \dim \mathcal{V}_{\overline{1}} \neq 0$).

For more details, see refs. [49, 50].

2.41 Representations: unitary superalgebras

A highest weight irreducible representation of $sl(m|n)$ is characterized by its Dynkin labels (→ 2.35) drawn on the distinguished Dynkin diagram. The different diagrams are the following:



The numbers a_i are constrained: a_i are non-negative integer for $i = 1, \dots, n-1, n+1, \dots, m+n-1$, and a_n is an arbitrary real number.

For the atypical representations, the numbers a_i have to satisfy one of the following atypicality conditions (where $1 \leq i \leq n \leq j \leq m+n-1$):

$$\sum_{k=i}^{n-1} a_k - \sum_{k=n+1}^{j} a_k + a_n + 2n - i - j = 0$$

Otherwise, the representation under consideration is a typical one. Then its dimension is given by

$$\dim \mathcal{V}(\Lambda) = 2^{mn} \prod_{1 \leq i \leq j \leq n-1} \frac{\sum\limits_{q=i}^{q=j} a_q + j - i + 1}{j - i + 1} \prod_{n+1 \leq i \leq j \leq m+n-1} \frac{\sum\limits_{q=i}^{q=j} a_q + j - i + 1}{j - i + 1}$$

The reader will find in Part 3 tables of dimensions of representations for the unitary superalgebras of small rank (see Tables 3.69–3.72). See page 364 for explanations.

For more details, see refs. [40, 42, 50].

2.42 Roots, root systems

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra of dimension n. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. The superalgebra $\mathcal{G}$ can be decomposed as follows:

$$\mathcal{G} = \bigoplus_{\alpha} \mathcal{G}_\alpha$$

where

$$\mathcal{G}_\alpha = \Big\{ x \in \mathcal{G} \,\Big|\, \big[h, x\big] = \alpha(h)\, x,\ h \in \mathcal{H} \Big\}$$

The set

$$\Delta = \Big\{ \alpha \in \mathcal{H}^* \,\Big|\, \mathcal{G}_\alpha \neq 0 \Big\}$$

is by definition the *root system* of $\mathcal{G}$. A root α is called even (resp. odd) if $\mathcal{G}_\alpha \cap \mathcal{G}_{\overline{0}} \neq \emptyset$ (resp. $\mathcal{G}_\alpha \cap \mathcal{G}_{\overline{1}} \neq \emptyset$). The set of even roots is denoted by $\Delta_{\overline{0}}$: it is the root system of the even part $\mathcal{G}_{\overline{0}}$ of $\mathcal{G}$. The set of odd roots is denoted by $\Delta_{\overline{1}}$: it is the weight system of the representation of $\mathcal{G}_{\overline{0}}$ in $\mathcal{G}_{\overline{1}}$. One has $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$. Notice that a root can be both even and odd (however this only occurs in the case of the superalgebra $Q(n)$). The vector space spanned by all the possible roots is called the root space. It is the dual $\mathcal{H}^*$ of the Cartan subalgebra $\mathcal{H}$ as vector space.

Except for $A(1,1)$, $P(n)$ and $Q(n)$, using the invariant bilinear form defined on the superalgebra $\mathcal{G}$, one can define a bilinear form on the root space $\mathcal{H}^*$ by $(\alpha_i, \alpha_j) = (H_i, H_j)$ where the H_i form a basis of $\mathcal{H}$ ($\to$ 2.2 and 2.23). One has the following properties.

Properties

1. $\mathcal{G}_{(\alpha=0)} = \mathcal{H}$ except for $Q(n)$.
2. $\dim \mathcal{G}_\alpha = 1$ when $\alpha \neq 0$ except for $A(1,1)$, $P(2)$, $P(3)$ and $Q(n)$.
3. Except for $A(1,1)$, $P(n)$ and $Q(n)$, one has
   - $\big[\!\big[\mathcal{G}_\alpha, \mathcal{G}_\beta\big]\!\big] \neq 0$ if and only if $\alpha, \beta, \alpha+\beta \in \Delta$
   - $(\mathcal{G}_\alpha, \mathcal{G}_\beta) = 0$ for $\alpha + \beta \neq 0$

- if $\alpha \in \Delta$ (resp. $\Delta_{\overline{0}}$, $\Delta_{\overline{1}}$), then $-\alpha \in \Delta$ (resp. $\Delta_{\overline{0}}$, $\Delta_{\overline{1}}$)
- $\alpha \in \Delta \Longrightarrow 2\alpha \in \Delta$ if and only if $\alpha \in \Delta_{\overline{1}}$ and $\alpha \cdot \alpha \neq 0$
- $\Delta_{\overline{0}}$ and $\Delta_{\overline{1}}$ are invariant under the action of the Weyl group of $\mathcal{G}_{\overline{0}}$

The roots of a basic Lie superalgebra do not satisfy many properties of the roots of a simple Lie algebra. In particular, the bilinear form on $\mathcal{H}^*$ has in general pseudo-Euclidean signature (except in the case of $B(0,n)$). The roots of a basic Lie superalgebra can be classified into three classes:

– roots α such that $\alpha \cdot \alpha \neq 0$ and 2α is not a root. Such roots will be called even or bosonic roots.
– roots α such that $\alpha \cdot \alpha \neq 0$ and 2α is still a root (of bosonic type). Such roots will be called odd or fermionic roots of non-zero length.
– roots α such that $\alpha \cdot \alpha = 0$. Such roots will be called odd or fermionic roots of zero length (or also isotropic odd roots).

The root systems of the basic Lie superalgebras are given in Table 2.5.

Table 2.5: Root systems of the basic Lie superalgebras.

superalgebra $\mathcal{G}$	even root system $\Delta_{\overline{0}}$	odd root system $\Delta_{\overline{1}}$
$A(m-1,n-1)$	$\varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j$	$\pm(\varepsilon_i - \delta_j)$
$B(m,n)$	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_i \pm \delta_j,\ \pm 2\delta_i$	$\pm\varepsilon_i \pm \delta_j,\ \pm\delta_i$
$B(0,n)$	$\pm\delta_i \pm \delta_j,\ \pm 2\delta_i$	$\pm\delta_i$
$C(n+1)$	$\pm\delta_i \pm \delta_j,\ \pm 2\delta_i$	$\pm\varepsilon \pm \delta_i$
$D(m,n)$	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_i \pm \delta_j,\ \pm 2\delta_i$	$\pm\varepsilon_i \pm \delta_j$
$F(4)$	$\pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i$	$\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)$
$G(3)$	$\pm 2\delta,\ \pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j$	$\pm\delta,\ \pm\varepsilon_i \pm \delta$
$D(2,1;\alpha)$	$\pm 2\varepsilon_i$	$\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$

For the superalgebras $A(m-1,n-1)$, $B(m,n)$, $D(m,n)$, the indices $i \neq j$ run from 1 to m for the vectors ε and from 1 to n for the vectors δ. For the superalgebras $C(n+1)$, the indices $i \neq j$ run from 1 to n for the vectors δ. For the superalgebras $F(4)$, $G(3)$, $D(2,1;\alpha)$, the indices $i \neq j$ run from 1 to 3 for the vectors ε, with the condition $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ in the case of $G(3)$ (see Tables 3.52–3.60 for more details). For the superalgebras $A(n,n)$, one has to add the condition $\sum_{i=1}^{n} \varepsilon_i = \sum_{i=1}^{n} \delta_i$.

→ 2.2 Cartan matrix, 2.23 Killing form, 2.45 Simple root systems.

For more details, see refs. [47, 48].

2.43 Schur's Lemma

The Schur's Lemma is of special importance. Let us stress however that in the superalgebra case it takes a slightly different form than in the algebra case [48].

Lemma

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a basic Lie superalgebra and π be an irreducible representation of $\mathcal{G}$ in a complex linear vector space $\mathcal{V}$. Let

$$\mathcal{C}(\pi) = \Big\{\phi : \mathcal{V} \to \mathcal{V} \Big| \Big[\!\!\Big[\pi(X), \phi\Big]\!\!\Big] = 0\,,\ \forall\, X \in \mathcal{G}\Big\}$$

where $\phi \in \operatorname{End}\mathcal{V}$. Then either

• $\mathcal{C}(\pi)$ is a multiple of the identity operator $\mathbb{I}$.

or

• If $\dim \mathcal{G}_{\overline{0}} = \dim \mathcal{G}_{\overline{1}}$, $\mathcal{C}(\pi) = \Big\{\mathbb{I}, \sigma\Big\}$ where σ is a non-singular operator in $\mathcal{G}$ permuting $\mathcal{G}_{\overline{0}}$ and $\mathcal{G}_{\overline{1}}$.

2.44 Serre–Chevalley basis

The Serre presentation of a Lie algebra consists in describing the algebra in terms of simple generators and relations, the only parameters being the entries of the Cartan matrix of the algebra. For the basic Lie superalgebras, the presentation is quite similar to the Lie algebra case but with some subtleties.

Let $\mathcal{G}$ be a basic Lie superalgebra of rank r with Cartan subalgebra $\mathcal{H} = \{h_1, \dots, h_r\}$ and simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_r\}$ and denote by $e_i^\pm$ $(1 \le i \le r)$ the corresponding simple root generators. If τ is a subset of $I = \Big\{1, \dots, r\Big\}$, the $\mathbb{Z}_2$-gradation is defined by $\deg e_i^\pm = \overline{0}$ if $i \notin \tau$ and $\deg e_i^\pm = \overline{1}$ if $i \in \tau$. The defining (super)commutation relations are

$$\Big[h_i, h_j\Big] = 0 \qquad\qquad \Big[h_i, e_j^\pm\Big] = \pm A_{ij}\, e_j^\pm$$

$$\Big[\!\!\Big[e_i^+, e_j^-\Big]\!\!\Big] = h_i\, \delta_{ij} \qquad\qquad \Big\{e_i^\pm, e_i^\pm\Big\} = 0 \quad \text{if} \quad A_{ii} = 0$$

and the Serre relations read as

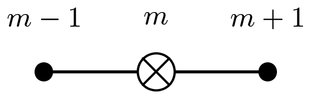
$$\big(\operatorname{ad} e_i^\pm\big)^{1-\widetilde{A}_{ij}}\ e_j^\pm = 0$$

where the matrix $\widetilde{A} = (\widetilde{A}_{ij})$ is deduced from the Cartan matrix $A = (A_{ij})$ of $\mathcal{G}$ by replacing all its positive off-diagonal entries by -1. Here ad denotes the adjoint action:

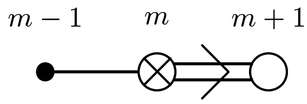
$$(\operatorname{ad} X)\, Y = XY - (-1)^{\deg X . \deg Y}\, YX$$

In the case of superalgebras however, the description given by these Serre relations leads in general to a bigger superalgebra than the superalgebra under consideration. It is necessary to write supplementary relations involving more than two generators, in order to quotient the bigger superalgebra and recover the original one. As one can imagine, these supplementary conditions appear when one deals with odd roots of zero length (that is $A_{ii} = 0$).

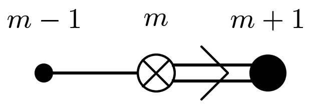
The supplementary conditions depend on the different kinds of vertices which appear in the Dynkin diagrams. For the superalgebras $A(m,n)$ with $m,n \geq 1$ and $B(m,n)$, $C(n+1)$, $D(m,n)$, the vertices can be of the following type:



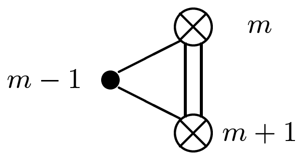
type I



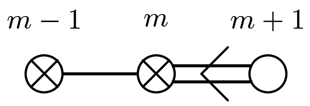
type IIa



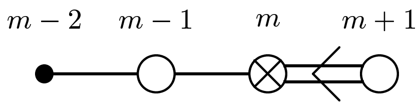
type IIb



type III



type IV



type V

where the small black dots represent either white dots associated to even roots or grey dots associated to isotropic odd roots.

The supplementary conditions take the following form:

- for the type I, IIa and IIb vertices:
$$(\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m+1})(\text{ad}\, e^\pm_m)e^\pm_{m-1} = (\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m-1})(\text{ad}\, e^\pm_m)e^\pm_{m+1} = 0$$

- for the type III vertex:
$$(\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m+1})e^\pm_{m-1} - (\text{ad}\, e^\pm_{m+1})(\text{ad}\, e^\pm_m)e^\pm_{m-1} = 0$$

- for the type IV vertex:
$$(\text{ad}\, e^\pm_m)\left(\Big[(\text{ad}\, e^\pm_{m+1})(\text{ad}\, e^\pm_m)e^\pm_{m-1}, (\text{ad}\, e^\pm_m)e^\pm_{m-1}\Big]\right) = 0$$

- for the type V vertex:
$$(\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m-1})(\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m+1})(\text{ad}\, e^\pm_m)(\text{ad}\, e^\pm_{m-1})e^\pm_{m-2} = 0$$

For $A(m,n)$ with $m = 0$ or $n = 0$, $F(4)$ and $G(3)$, it is not necessary to impose supplementary conditions.

For more details, see refs. [21, 53, 81, 101].

2.45 Simple root systems

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a basic Lie superalgebra with Cartan subalgebra $\mathcal{H}$ and root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$. Then $\mathcal{G}$ admits a Borel decomposition $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ where $\mathcal{N}^+$ and $\mathcal{N}^-$ are subalgebras such that $\left[\mathcal{H}, \mathcal{N}^+\right] \subset \mathcal{N}^+$ and $\left[\mathcal{H}, \mathcal{N}^-\right] \subset \mathcal{N}^-$ with $\dim \mathcal{N}^+ = \dim \mathcal{N}^-$.

If $\mathcal{G} = \mathcal{H} \bigoplus_\alpha \mathcal{G}_\alpha$ is the root decomposition of $\mathcal{G}$, a root α is called positive if $\mathcal{G}_\alpha \cap \mathcal{N}^+ \neq \emptyset$ and negative if $\mathcal{G}_\alpha \cap \mathcal{N}^- \neq \emptyset$. A root is called simple if it cannot be decomposed into a sum of positive roots. The set of all simple roots is called a *simple root system* of $\mathcal{G}$ and is denoted here by Δ^0.

Let ρ_0 be the half-sum of the positive even roots, ρ_1 the half-sum of the positive odd roots and $\rho = \rho_0 - \rho_1$. Then one has for a simple root α_i, $\rho \cdot \alpha_i = \frac{1}{2}\, \alpha_i \cdot \alpha_i$. In particular, one has $\rho \cdot \alpha_i = 0$ if $\alpha_i \in \Delta^0_{\overline{1}}$ with $\alpha_i \cdot \alpha_i = 0$.

We will call $\mathcal{B} = \mathcal{H} \oplus \mathcal{N}^+$ a *Borel subalgebra* of $\mathcal{G}$. Notice that such a Borel subalgebra is solvable but not maximal solvable. Indeed, adding to such a Borel subalgebra $\mathcal{B}$ a negative simple isotropic root generator (that is a generator associated to an odd root of zero length, → 2.42), the obtained subalgebra is still solvable since the superalgebra $sl(1|1)$ is solvable. However, $\mathcal{B}$ contains a maximal solvable subalgebra $\mathcal{B}_{\overline{0}}$ of the even part $\mathcal{G}_{\overline{0}}$.

In general, for a basic Lie superalgebra $\mathcal{G}$, there are many inequivalent classes of conjugacy of Borel subalgebras (while for the simple Lie algebras, all Borel subalgebras are conjugate). To each class of conjugacy of Borel subalgebras of $\mathcal{G}$ is associated a simple root system Δ^0. Hence, contrary to the Lie algebra case, to a given basic Lie superalgebra $\mathcal{G}$ will be associated in general many inequivalent simple root systems, up to a transformation of the Weyl group $W(\mathcal{G})$ of $\mathcal{G}$ (under a transformation of $W(\mathcal{G})$, a simple root system will be transformed into an equivalent one with the same Dynkin diagram).

The generalization of the Weyl group (→ 2.63) for a basic Lie superalgebra $\mathcal{G}$ gives a method for constructing all the simple root systems of $\mathcal{G}$ and hence all the inequivalent Dynkin diagrams (→ 2.15) of $\mathcal{G}$. A simple root system Δ^0 being given, from any root $\alpha \in \Delta^0$ such that $(\alpha, \alpha) = 0$, one constructs the simple root system $w_\alpha(\Delta^0)$ where w_α is the generalized Weyl reflection with respect to α and one repeats the procedure on the obtained system until no new basis arises.

In the set of all inequivalent simple root systems of a basic Lie superalgebra, there is one simple root system that plays a particular role: the distinguished simple root system.

Definition

For each basic Lie superalgebra, there exists a particular simple root system for which the number of odd roots is equal to one. It is constructed as follows. Consider the distinguished $\mathbb{Z}$-gradation of $\mathcal{G}$ ($\rightarrow$ 2.8): $\mathcal{G} = \oplus_{i\in\mathbb{Z}} \mathcal{G}_i$. The even simple roots are given by the simple root system of the Lie subalgebra $\mathcal{G}_0$ and the odd simple root is the lowest weight of the representation $\mathcal{G}_1$ of $\mathcal{G}_0$. Such a simple root system is called a *distinguished simple root system*. Two different distinguished simple root systems are related by means of Weyl reflections ($\rightarrow$ 1.93) with respect to the even roots; hence all distinguished simple root systems are equivalent.

Table 2.6: Distinguished simple root systems of the basic Lie superalgebras.

superalgebra $\mathcal{G}$	distinguished simple root system Δ^0
$A(m-1,n-1)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\varepsilon_1,\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m$
$B(m,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\varepsilon_1,$ $\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m$
$B(0,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n$
$C(n)$	$\varepsilon-\delta_1,\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,2\delta_n$
$D(m,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\varepsilon_1,$ $\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_{m-1}+\varepsilon_m$
$F(4)$	$\frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3),\varepsilon_3,\varepsilon_2-\varepsilon_3,\varepsilon_1-\varepsilon_2$
$G(3)$	$\delta+\varepsilon_3,\varepsilon_1,\varepsilon_2-\varepsilon_1$
$D(2,1;\alpha)$	$\varepsilon_1-\varepsilon_2-\varepsilon_3,2\varepsilon_2,2\varepsilon_3$

Example

Consider the basic Lie superalgebra $sl(2|1)$ with Cartan generators H, Z and root generators $E^\pm$, $F^\pm$, $\overline{F}^\pm$. The root system is given by $\Delta = \{\pm(\varepsilon_1-\varepsilon_2), \pm(\varepsilon_1-\delta), \pm(\varepsilon_2-\delta)\}$. One can find two inequivalent Borel subalgebras, namely $\mathcal{B}' = \{H, Z, E^+, \overline{F}^+, \overline{F}^-\}$ and $\mathcal{B}'' = \{H, Z, E^+, \overline{F}^+, F^+\}$, with positive root systems $\Delta'^+ = \{\varepsilon_1-\varepsilon_2, \varepsilon_1-\delta, \varepsilon_2-\delta\}$ and $\Delta''^+ = \{\varepsilon_1-\varepsilon_2, \varepsilon_1-\delta, -\varepsilon_2+\delta\}$ respectively.

The corresponding simple root systems are $\Delta'^0 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta\}$ (called distinguished simple root system) and $\Delta''^0 = \{\varepsilon_1 - \delta, -\varepsilon_2 + \delta\}$ (called fermionic simple root system). The fermionic simple root system Δ''^0 is obtained from the distinguished one Δ'^0 by the Weyl transformation associated to the odd root $\varepsilon_2 - \delta$: $w_{\varepsilon_2-\delta}(\varepsilon_2 - \delta) = -\varepsilon_2 + \delta$ and $w_{\varepsilon_2-\delta}(\varepsilon_1 - \varepsilon_2) = \varepsilon_1 - \delta$. □

Table 2.6 lists the distinguished simple root systems of the basic Lie superalgebras in terms of the orthogonal vectors ε_i and δ_i. For more details, see ref. [47]. See also Tables 3.52–3.60 for the corresponding distinguished Dynkin diagrams and Table 3.61 for the list of Dynkin diagrams of the basic Lie superalgebras of rank $r \leq 4$.

2.46 Simple and semi-simple Lie superalgebras

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a non-Abelian Lie superalgebra. The Lie superalgebra $\mathcal{G}$ is called *simple* if it does not contain any non-trivial ideal. The Lie superalgebra $\mathcal{G}$ is called *semi-simple* if it does not contain any non-trivial solvable ideal.

A necessary condition for a Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ (with $\mathcal{G}_{\overline{1}} \neq \emptyset$) to be simple is that the representation of $\mathcal{G}_{\overline{0}}$ on $\mathcal{G}_{\overline{1}}$ is faithful and $\left\{\mathcal{G}_{\overline{1}}, \mathcal{G}_{\overline{1}}\right\} = \mathcal{G}_{\overline{0}}$. If the representation of $\mathcal{G}_{\overline{0}}$ on $\mathcal{G}_{\overline{1}}$ is irreducible, then $\mathcal{G}$ is simple.

Recall that if $\mathcal{A}$ is a semi-simple Lie algebra, then it can be written as the direct sum of simple Lie algebras $\mathcal{A}_i$: $\mathcal{A} = \oplus_i \mathcal{A}_i$. *This is not the case for superalgebras.* However, the following properties hold.

Properties

1. If $\mathcal{G}$ is a Lie superalgebra and $\mathcal{I}$ is the maximal solvable ideal, then the quotient $\mathcal{G}/\mathcal{I}$ is a semi-simple Lie superalgebra. However, unlike the case of Lie algebras, one cannot write here $\mathcal{G} = \overline{\mathcal{G}} \in\!\!\!\!\! \supset \mathcal{I}$ where $\overline{\mathcal{G}}$ is a direct sum of simple Lie superalgebras.

2. If $\mathcal{G}$ is a Lie superalgebra with a non-singular Killing form, then $\mathcal{G}$ is a direct sum of simple Lie superalgebras with non-singular Killing form.

3. If $\mathcal{G}$ is a Lie superalgebra, all of whose finite dimensional representations are completely reducible, then $\mathcal{G}$ is a direct sum of simple Lie algebras and $osp(1|n)$ simple superalgebras.

4. Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a Lie superalgebra such that its even part $\mathcal{G}_{\overline{0}}$ is a semi-simple Lie algebra. Then $\mathcal{G}$ is an elementary extension of a direct sum of Lie algebras or one of the Lie superalgebras $A(n,n)$, $B(m,n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$, $\mathrm{Der}\, Q(n)$ or $G(S_1,\dots,S_r;L)$. (For the definition of $G(S_1,\dots,S_r;L)$, see ref. [47]).

   The elementary extension of a Lie superalgebra $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ is defined as $\mathcal{G} \in\!\!\!\!\!\!\in \mathcal{I}$ where $\mathcal{I}$ is an odd commutative ideal and $\left\{\mathcal{G}_{\overline{1}}, \mathcal{I}\right\} = 0$.

For more details, see refs. [47, 75].

2.47 Spinors (in the Lorentz group)

The algebra of the Lorentz group is $o(1,3)$ whose generators $M_{\mu\nu} = -M_{\nu\mu}$ satisfy the commutation relations ($\mu, \nu = 0,1,2,3$)

$$\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma})$$

where the metric is $g^{\mu\nu} = 2\delta^{\mu 0}\delta^{\nu 0} - \delta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $g^{\mu\sigma}g_{\sigma\nu} = \delta^{\mu}_{\nu}$.

If we define $J_i = \frac{1}{2}\,\varepsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$, we have

$$\Big[J_i, J_j\Big] = i\,\varepsilon_{ijk}\,J_k\,, \qquad \Big[J_i, K_j\Big] = i\,\varepsilon_{ijk}\,K_k\,, \qquad \Big[K_i, K_j\Big] = -i\,\varepsilon_{ijk}\,J_k$$

where $i,j,k = 1,2,3$ and ε_{ijk} is the completely antisymmetric rank three tensor, $\varepsilon_{123} = 1$.

Defining $M_i = \frac{1}{2}\,(J_i + iK_i)$ and $N_i = \frac{1}{2}\,(J_i - iK_i)$, the Lorentz algebra can be rewritten as:

$$\Big[M_i, M_j\Big] = i\,\varepsilon_{ijk}\,M_k\,, \qquad \Big[N_i, N_j\Big] = i\,\varepsilon_{ijk}\,N_k\,, \qquad \Big[M_i, N_j\Big] = 0$$

The finite dimensional irreducible representations of the Lorentz group are labelled by a pair of integers or half-integers (m,n); the representation (m,n) is of dimension $(2m+1)(2n+1)$. These representations are non-unitary since the generators M_i and N_i can be represented by finite dimensional hermitian matrices, hence J_i is hermitian ($J_i^\dagger = J_i$) while K_i is antihermitian ($K_i^\dagger = -K_i$). Because of the relation $J_i = M_i + N_i$, the combination $m+n$ is the spin of the representation. Representations with half-integer spin (resp. integer spin) are called spinor (resp. tensor) representations. The two representations $(1/2,0)$ and $(0,1/2)$ are the fundamental spinor representations: all the spinor and tensor representations of the Lorentz group can be obtained by tensorization and symmetrization of these.

The σ^i being the Pauli matrices, one has in the representation $(1/2,0)$

$$M_i = \tfrac{1}{2}\,\sigma^i \quad \text{and} \quad N_i = 0 \qquad \text{that is} \qquad J_i = \tfrac{1}{2}\,\sigma^i \quad \text{and} \quad K_i = -\tfrac{i}{2}\,\sigma^i$$

and in the representation $(0,1/2)$

$$M_i = 0 \quad \text{and} \quad N_i = \tfrac{1}{2}\,\sigma^i \qquad \text{that is} \qquad J_i = \tfrac{1}{2}\,\sigma^i \quad \text{and} \quad K_i = \tfrac{i}{2}\,\sigma^i$$

The vectors of the representation spaces of the spinor representations are called (Weyl) *spinors* under the Lorentz group. Define $\sigma^\mu = (\mathbb{I},\sigma^i)$ and $\overline{\sigma}^\mu = (\mathbb{I},-\sigma^i)$.

Under a Lorentz transformation Λ^μ_ν, a *covariant undotted* spinor ψ_α (resp. *contravariant undotted* spinor ψ^α) transforms as

$$\psi_\alpha \quad\mapsto\quad S_\alpha{}^\beta\,\psi_\beta \qquad \text{and} \qquad \psi^\alpha \quad\mapsto\quad \psi^\beta\,(S^{-1})_\beta{}^\alpha$$

where the matrix S is related to the matrix Λ^μ_ν by

$$\Lambda^\mu_\nu = \tfrac{1}{2}\,\mathrm{tr}(S\sigma_\nu S^\dagger\overline{\sigma}^\mu)$$

The spinors ψ_α (or ψ^α) transform as the $(1/2,0)$ representation of the Lorentz group.

The generators of the Lorentz group in the spinor representations $(1/2,0)$ are given by

$$\tfrac{1}{2}\,\sigma^{\mu\nu} = \tfrac{i}{4}\,(\sigma^\mu\overline{\sigma}\nu - \sigma^\nu\overline{\sigma}\mu)$$

$\sigma^{\mu\nu}$ is a self-dual rank two tensor:

$$\sigma^{\mu\nu} = \tfrac{1}{2i}\,\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric rank four tensor with $\varepsilon_{0123}=1$ and $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma}$.

Under a Lorentz transformation Λ^μ_ν, a *covariant dotted* spinor $\overline{\psi}_{\dot\alpha}$ (resp. *contravariant dotted* spinor $\overline{\psi}^{\dot\alpha}$) transforms as

$$\overline{\psi}_{\dot\alpha} \quad\mapsto\quad \overline{\psi}_{\dot\beta}\,(S^\dagger)^{\dot\beta}{}_{\dot\alpha} \qquad \text{and} \qquad \overline{\psi}^{\dot\alpha} \quad\mapsto\quad (S^{\dagger^{-1}})^{\dot\alpha}{}_{\dot\beta}\,\overline{\psi}^{\dot\beta}$$

where the matrix S is related to the matrix Λ^μ_ν by

$$\Lambda^\mu_\nu = \tfrac{1}{2}\,\mathrm{tr}((S^\dagger)^{-1}\overline{\sigma}_\nu S^{-1}\sigma^\mu)$$

The spinors $\overline{\psi}_{\dot\alpha}$ (or $\overline{\psi}^{\dot\alpha}$) transform as the $(0,1/2)$ representation of the Lorentz group.

The generators of the Lorentz group in the spinor representations $(0,1/2)$ are given by

$$\tfrac{1}{2}\,\overline{\sigma}^{\mu\nu} = \tfrac{i}{4}\,(\overline{\sigma}^{\mu}\sigma\nu - \overline{\sigma}^{\nu}\sigma\mu)$$

$\overline{\sigma}^{\mu\nu}$ is an antiself-dual rank two tensor:

$$\overline{\sigma}^{\mu\nu} = -\tfrac{1}{2i}\,\varepsilon^{\mu\nu\rho\sigma}\overline{\sigma}_{\rho\sigma}$$

The relation between covariant and contravariant spinors is given by means of the two-dimensional Levi-Cività undotted tensors $\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}$ and dotted tensors $\varepsilon_{\dot\alpha\dot\beta}, \varepsilon^{\dot\alpha\dot\beta}$ such that $\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = -\varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\dot\alpha\dot\beta}$ and $\varepsilon_{12} = 1$:

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\,\psi_{\beta}\,, \quad \psi_{\alpha} = \psi^{\beta}\,\varepsilon_{\beta\alpha}\,, \quad \overline{\psi}^{\dot\alpha} = \overline{\psi}_{\dot\beta}\,\varepsilon^{\dot\beta\dot\alpha}\,, \quad \overline{\psi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\overline{\psi}^{\dot\beta}$$

Notice that $\overline{\psi}^{\dot\alpha} = (\psi^{\alpha})^*$ and $\overline{\psi}_{\dot\alpha} = (\psi_{\alpha})^*$ where the star denotes the complex conjugation, and also $\varepsilon_{\dot\alpha\dot\beta} = -(\varepsilon_{\alpha\beta})^*$.

Finally, the rule for contracting undotted and dotted spinor indices is the following:

$$\psi\zeta \equiv \psi^{\alpha}\zeta_{\alpha} = -\psi_{\alpha}\zeta^{\alpha} \qquad \text{and} \qquad \overline{\psi\zeta} \equiv \overline{\psi}_{\dot\alpha}\overline{\zeta}^{\dot\alpha} = -\overline{\psi}^{\dot\alpha}\overline{\zeta}_{\dot\alpha}$$

Under a Lorentz transformation Λ^{μ}_{ν}, a *covariant* vector ψ_{μ} (resp. *contravariant* vector ψ^{μ}) transforms as

$$\psi_{\mu} \quad \mapsto \quad \psi_{\nu}\,\Lambda^{\nu}{}_{\mu} \qquad \text{and} \qquad \psi^{\mu} \quad \mapsto \quad \Lambda^{\mu}{}_{\nu}\,\psi^{\nu}$$

The relation between covariant and contravariant vectors is given by means of the metric tensor $g^{\mu\nu}$: $\psi^{\mu} = g^{\mu\nu}\psi_{\nu}$ and $\psi_{\mu} = g_{\mu\nu}\psi^{\nu}$.

From the relations $\sigma^{\mu} = \Lambda^{\mu}{}_{\nu}S\sigma^{\nu}S^{\dagger}$ and $\overline{\sigma}^{\mu} = \Lambda^{\mu}{}_{\nu}(S^{\dagger})^{-1}\overline{\sigma}^{\nu}S^{-1}$, one can assign to the matrices σ^{μ} and $\overline{\sigma}^{\mu}$ two spinor indices, one dotted and one undotted, as follows: $\sigma^{\mu}{}_{\alpha\dot\alpha}$ and $\overline{\sigma}^{\mu\dot\alpha\alpha}$.

There is a one-to-one correspondence between a vector ψ^{μ} and a bispinor $\psi^{\dot\alpha\alpha}$ or $\psi^{\alpha\dot\alpha}$:

$$\begin{aligned} \psi^{\dot\alpha\alpha} &= \psi^{\mu}\,\overline{\sigma}_{\mu}{}^{\dot\alpha\alpha} \qquad \text{and} \qquad \psi^{\mu} = \tfrac{1}{2}\,\psi^{\dot\alpha\alpha}\,\sigma^{\mu}{}_{\alpha\dot\alpha} \\ \psi^{\alpha\dot\alpha} &= \psi^{\mu}\,\sigma_{\mu}{}^{\alpha\dot\alpha} \qquad \text{and} \qquad \psi^{\mu} = \tfrac{1}{2}\,\psi^{\alpha\dot\alpha}\,\overline{\sigma}^{\mu}{}_{\dot\alpha\alpha} \end{aligned}$$

The vectors ψ_{μ} (or ψ^{μ}) transform as the $(1/2,1/2)$ representation.

The space inversion leaves the rotation generators J_i invariant but changes the sign of the boost generators K_i. It follows that under the space inversion, the undotted Weyl spinors are transformed into dotted ones and vice versa. On the (reducible) representation $(1/2,0)\oplus(0,1/2)$, the space inversion acts in a well-defined way. The corresponding vectors in the representation space are called *Dirac spinors*. In the Weyl representation, the Dirac spinors are given by

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix}$$

Under a Lorentz transformation Λ^μ_ν, a Dirac spinor Ψ_D transforms as

$$\Psi_D \quad \mapsto \quad L\,\Psi_D = \begin{pmatrix} S(\Lambda^\mu_\nu) & 0 \\ 0 & S(\Lambda^{\mu\dagger}_\nu)^{-1} \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} S(\Lambda^\mu_\nu)\,\psi_\alpha \\ S(\Lambda^{\mu\dagger}_\nu)^{-1}\,\overline{\chi}^{\dot\alpha} \end{pmatrix}$$

The generators of the Lorentz group in the $(1/2,0)\oplus(0,1/2)$ representation are given by

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} = \tfrac{i}{2} \begin{pmatrix} \sigma^\mu\overline{\sigma}\nu - \sigma^\nu\overline{\sigma}\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma\nu - \overline{\sigma}^\nu\sigma\mu \end{pmatrix} = \tfrac{i}{2}\Big[\gamma^\mu,\gamma^\nu\Big]$$

where the matrices γ^μ, called the *Dirac matrices*, are given by (in the Weyl representation)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$

They satisfy the Clifford algebra ($\to$ 2.10) in four dimensions:

$$\Big\{\gamma^\mu,\gamma^\nu\Big\} = \gamma^\mu\,\gamma^\nu + \gamma^\nu\,\gamma^\mu = 2g^{\mu\nu}$$

One defines also the γ_5 matrix by $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ such that $\Big\{\gamma_5,\gamma^\mu\Big\} = 0$, $\gamma_5^2 = \mathbb{I}$ and $\gamma_5^\dagger = \gamma_5$.

The adjoint spinor $\overline{\Psi}$ and the charge conjugate spinors Ψ^c and $\overline{\Psi}^c$ of a Dirac spinor $\Psi = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix}$ are defined by $\overline{\Psi} = \big(\chi^\alpha\,\overline{\psi}_{\dot\alpha}\big)$, $\Psi^c = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}$ and $\overline{\Psi}^c = (\psi^\alpha\,\overline{\chi}_{\dot\alpha})$. The spinors Ψ and Ψ^c are related through the charge conjugation matrix C by $\Psi^c = C\overline{\Psi}^t$. The six matrices $C, \gamma^\mu\gamma_5 C, \gamma_5 C$ are antisymmetric and the ten matrices $\gamma^\mu C, \Sigma^{\mu\nu}C$ are symmetric. They form a set of 16 linearly independent matrices.

The charge conjugation matrix C satisfies the following relations:

$$\begin{aligned} &C^2 = -\mathbb{I}\,, \qquad CC^\dagger = C^\dagger C = \mathbb{I}\,, \\ &C\gamma^\mu C^{-1} = -(\gamma^\mu)^t\,, \qquad C\gamma_5 C^{-1} = (\gamma_5)^t\,, \qquad C\gamma_5\gamma^\mu C^{-1} = (\gamma_5\gamma^\mu)^t \end{aligned}$$

A Majorana spinor is a Dirac spinor such that $\Psi = \Psi^c$. For such a spinor, there is a relation between the two Weyl components: a Majorana spinor Ψ has the form $\Psi = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}$. In the Majorana representation of the γ matrices, the components of a Majorana spinor are all real and the γ matrices are all purely imaginary.

The γ matrices are given, in the Weyl representation, by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

Another used representation of the γ matrices is the Dirac representation:

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$$

Finally, in the Majorana representation, one has:

$$\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$$

The γ matrices satisfy the following relations (independent of the chosen representation):

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}\,, \quad \gamma^0\gamma_5\gamma^0 = -\gamma_5{}^\dagger\,, \quad \gamma^0\gamma_5\gamma^\mu\gamma^0 = (\gamma_5\gamma^\mu)^\dagger$$

There exist also very nice contraction and trace formulae that are extremely useful for example in the calculation of Feynman graphs:

$$\gamma^\lambda\gamma_\lambda = 4\,, \quad \gamma^\lambda\gamma^\mu\gamma_\lambda = -2\gamma^\mu\,, \quad \gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\lambda = 4g^{\mu\nu}\,, \quad \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\lambda = -2\gamma^\rho\gamma^\nu\gamma^\mu$$

and

$$\begin{aligned} &\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\gamma^\mu) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0 \\ &\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \\ &\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ &\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\,\varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$

2.48 Strange superalgebras $P(n)$

We consider the superalgebra $A(n-1,n-1)$ and $P(n-1)$ the subalgebra of $A(n-1,n-1)$ generated by the $2n \times 2n$ matrices of the form

$$\begin{pmatrix} \lambda & S \\ A & -\lambda^t \end{pmatrix}$$

where λ are $sl(n)$ matrices, S and A are $n \times n$ symmetric and antisymmetric complex matrices which can be seen as elements of the twofold symmetric representation ([2] in Young tableau notation) of dimension $n(n+1)/2$ and of the $(n-2)$-fold antisymmetric representation ($[1^{n-2}]$ in Young tableau notation) of dimension $n(n-1)/2$ respectively. The $\mathbb{Z}$-gradation of the superalgebra $P(n-1)$ being $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ where $\mathcal{G}_0 = sl(n)$, $G_1 = [2]$ and $G_{-1} = [1^{n-2}]$, the subspaces $\mathcal{G}_i$ satisfy the following commutation relations

$$\Big[\mathcal{G}_0, \mathcal{G}_0\Big] \subset \mathcal{G}_0 \qquad \Big[\mathcal{G}_0, \mathcal{G}_{\pm 1}\Big] \subset \mathcal{G}_{\pm 1}$$
$$\Big\{\mathcal{G}_1, \mathcal{G}_1\Big\} = \Big\{\mathcal{G}_{-1}, \mathcal{G}_{-1}\Big\} = 0 \qquad \Big\{\mathcal{G}_1, \mathcal{G}_{-1}\Big\} \subset \mathcal{G}_0$$

The $\mathbb{Z}$-gradation is consistent: $\mathcal{G}_{\overline{0}} = \mathcal{G}_0$ and $\mathcal{G}_{\overline{1}} = \mathcal{G}_{-1} \oplus \mathcal{G}_1$.

Defining the Cartan subalgebra $\mathcal{H}$ as the Cartan subalgebra of the even part, the root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ of $P(n-1)$ can be expressed in terms of the orthogonal vectors $\varepsilon_1, \dots, \varepsilon_n$ as

$$\Delta_{\overline{0}} = \Big\{ \ \alpha_{ij} = \varepsilon_i - \varepsilon_j \ \Big\}$$

and

$$\Delta_{\overline{1}} = \Big\{ \ \pm \beta_{ij} = \pm \Big(\varepsilon_i + \varepsilon_j - \frac{2}{n} \sum_{k=1}^{n} \varepsilon_k \Big) \, , \ \gamma_i = 2\varepsilon_i - \frac{2}{n} \sum_{k=1}^{n} \varepsilon_k \ \Big\}$$

Denoting by H_i the Cartan generators, by E_α the even root generators and by E_β, E_γ the odd root generators of $P(n-1)$, the commutation relations in the Cartan–Weyl basis are the following:

$$\Big[H_k, E_{\alpha_{ij}}\Big] = (\delta_{ik} - \delta_{jk} - \delta_{i,k+1} + \delta_{j,k+1}) E_{\alpha_{ij}}$$
$$\Big[H_k, E_{\beta_{ij}}\Big] = (\delta_{ik} + \delta_{jk} - \delta_{i,k+1} - \delta_{j,k+1}) E_{\beta_{ij}}$$
$$\Big[H_k, E_{-\beta_{ij}}\Big] = -(\delta_{ik} + \delta_{jk} - \delta_{i,k+1} - \delta_{j,k+1}) E_{-\beta_{ij}}$$
$$\Big[H_k, E_{\gamma_i}\Big] = 2(\delta_{ik} - \delta_{i,k+1}) E_{\gamma_i}$$

$$\Big[E_{\alpha_{ij}}, E_{\alpha_{kl}}\Big] = \delta_{jk} E_{\alpha_{il}} - \delta_{il} E_{\alpha_{kj}}$$

$$\Big[E_{\alpha_{ij}}, E_{-\alpha_{ij}}\Big] = \sum_{k=i}^{j-1} H_k$$

$$\Big[E_{\alpha_{ij}}, E_{\beta_{kl}}\Big] = \begin{cases} \delta_{jk} E_{\beta_{il}} + \delta_{jl} E_{\beta_{ik}} & \text{if } (i,j) \neq (k,l) \\ E_{\gamma_i} & \text{if } (i,j) = (k,l) \end{cases}$$

$$\Big[E_{\alpha_{ij}}, E_{-\beta_{kl}}\Big] = \begin{cases} -\delta_{ik} E_{-\beta_{jl}} + \delta_{il} E_{-\beta_{jk}} & \text{if } (i,j) \neq (k,l) \\ 0 & \text{if } (i,j) = (k,l) \end{cases}$$

$$\Big[E_{\alpha_{ij}}, E_{\gamma_k}\Big] = \delta_{jk} E_{\beta_{ik}}$$

$$\Big\{E_{-\beta_{ij}}, E_{\gamma_k}\Big\} = -\delta_{ik} E_{\alpha_{kj}} + \delta_{jk} E_{\alpha_{ki}}$$

$$\Big\{E_{\beta_{ij}}, E_{-\beta_{kl}}\Big\} = \begin{cases} -\delta_{ik} E_{\alpha_{jl}} + \delta_{il} E_{\alpha_{jk}} - \delta_{jk} E_{\alpha_{il}} + \delta_{jl} E_{\alpha_{ik}} \\ \qquad \text{if } (i,j) \neq (k,l) \\ \sum_{k=i}^{j-1} H_k \ \text{ if } (i,j) = (k,l) \end{cases}$$

$$\Big\{E_{\beta_{ij}}, E_{\beta_{kl}}\Big\} = \Big\{E_{-\beta_{ij}}, E_{-\beta_{kl}}\Big\} = \Big\{E_{\beta_{ij}}, E_{\gamma_k}\Big\} = 0$$

Let us emphasize that $P(n)$ is a non-contragredient simple Lie superalgebra, that is the number of positive roots and the number of negative roots are not equal. Moreover, since every bilinear form is identically vanishing in $P(n)$, it is impossible to define a non-degenerate scalar product on the root space. It follows that the notions of Cartan matrix and Dynkin diagram are not defined for $P(n)$. However, using an extension of $P(n)$ by suitable diagonal matrices, one can construct a non-vanishing bilinear form on the Cartan subalgebra of this extension and therefore one can generalize to this case the notions of Cartan matrix and Dynkin diagram.

→ 2.31 Oscillator realization: strange series.

For more details, see ref. [24].

2.49 Strange superalgebras $Q(n)$

We consider the superalgebra $sl(n|n)$ and $\widetilde{Q}(n-1)$ the subalgebra of $sl(n|n)$ generated by the $2n \times 2n$ matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where A and B are $sl(n)$ matrices. The even part of the superalgebra $\widetilde{Q}(n-1)$ is the Lie algebra $\mathcal{G}_{\overline{0}} = sl(n) \oplus U(1)$ of dimension n^2 and the odd part is the adjoint representation $\mathcal{G}_{\overline{1}}$ of $sl(n)$ of dimension $n^2 - 1$. The even generators of $\mathcal{G}_{\overline{0}}$ are divided into the $sl(n)$ Cartan generators H_i with $1 \le i \le n-1$, the $U(1)$ generator Z and the $n(n-1)$ root generators E_{ij} with $1 \le i \ne j \le n$ of $sl(n)$. The odd root generators of $\mathcal{G}_{\overline{1}}$ are also divided into two parts, F_{ij} with $1 \le i \ne j \le n$ and K_i with $1 \le i \le n-1$. This superalgebra $\widetilde{Q}(n-1)$ is not a simple superalgebra: in order to obtain a simple superalgebra, one should factor out the one-dimensional center, as in the case of the $sl(n|n)$ superalgebra. We will denote by $Q(n-1)$ the simple superalgebra $\widetilde{Q}(n-1)/U(1)$.

Following the definition of the Cartan subalgebra (→ 2.3), the strange superalgebra $Q(n-1)$ has the property that the Cartan subalgebra $\mathcal{H}$ does not coincide with the Cartan subalgebra of the even part $sl(n)$, but admits also an odd part: $\mathcal{H} \cap \mathcal{G}_{\overline{1}} \ne \emptyset$. More precisely, one has

$$\mathcal{H} = \mathcal{H}_{\overline{0}} \oplus \mathcal{H}_{\overline{1}}$$

where $\mathcal{H}_{\overline{0}}$ is spanned by the H_i generators and $\mathcal{H}_{\overline{1}}$ by the K_i generators ($1 \le i \le n-1$). However, since the K_i generators are odd, the root generators E_{ij} and F_{ij} are not eigenvectors of $\mathcal{H}_{\overline{1}}$. It is convenient in this case to give the root decomposition with respect to $\mathcal{H}_{\overline{0}} = \mathcal{H} \cap \mathcal{G}_{\overline{0}}$ instead of $\mathcal{H}$. The root system Δ of $Q(n-1)$ coincide then with the root system of $sl(n)$. One has

$$\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}} = \mathcal{H}_{\overline{0}} \oplus \Big(\bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha \Big) \quad \text{with} \quad \dim \mathcal{G}_{(\alpha \ne 0)} = 2 \quad \text{and} \quad \dim \mathcal{G}_{(\alpha = 0)} = n$$

Moreover, since $\dim \mathcal{G}_\alpha \cap \mathcal{G}_{\overline{0}} \ne \emptyset$ and $\dim \mathcal{G}_\alpha \cap \mathcal{G}_{\overline{1}} \ne \emptyset$ for any non-zero root α, the non-zero roots of $Q(n-1)$ are both even and odd.

Denoting by H_i the Cartan generators, by E_{ij} the even root generators and by F_{ij} the odd root generators of $\widetilde{Q}(n)$, the commutation relations in the Cartan–Weyl basis are the following:

$$\begin{aligned}
&\Big[H_i, H_j\Big] = \Big[H_i, K_j\Big] = 0 \\
&\Big\{K_i, K_j\Big\} = \frac{2}{n}\Big(2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}\Big)\Big(Z - \sum_{k=1}^{n-1} kH_k\Big) \\
&\qquad\qquad + 2(\delta_{ij} - \delta_{i,j+1}) \sum_{k=i}^{n-1} H_k + 2(\delta_{ij} - \delta_{i,j-1}) \sum_{k=i+1}^{n-1} H_k \\
&\Big[H_k, E_{ij}\Big] = (\delta_{ik} - \delta_{jk} - \delta_{i,k+1} + \delta_{j,k+1})E_{ij}
\end{aligned}$$

$$\Big[H_k, F_{ij}\Big] = (\delta_{ik} - \delta_{jk} - \delta_{i,k+1} + \delta_{j,k+1})F_{ij}$$
$$\Big[K_k, E_{ij}\Big] = (\delta_{ik} - \delta_{jk} - \delta_{i,k+1} + \delta_{j,k+1})F_{ij}$$
$$\Big\{K_k, F_{ij}\Big\} = (\delta_{ik} + \delta_{jk} - \delta_{i,k+1} - \delta_{j,k+1})E_{ij}$$
$$\Big[E_{ij}, E_{kl}\Big] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \quad (i,j) \neq (k,l)$$
$$\Big[E_{ij}, E_{ji}\Big] = \sum_{k=i}^{j-1} H_k$$
$$\Big[E_{ij}, F_{kl}\Big] = \delta_{jk}F_{il} - \delta_{il}F_{kj} \quad (i,j) \neq (k,l)$$
$$\Big[E_{ij}, F_{ji}\Big] = \sum_{k=i}^{j-1} K_k$$
$$\Big\{F_{ij}, F_{kl}\Big\} = \delta_{jk}E_{il} + \delta_{il}E_{kj} \quad (i,j) \neq (k,l)$$
$$\Big\{F_{ij}, F_{ji}\Big\} = \frac{2}{n}\,Z + \frac{n-2}{n}\left(2\sum_{k=i}^{n-1} kH_k - n\sum_{k=i}^{n-1} H_k - n\sum_{k=j}^{n-1} H_k\right)$$

→ 2.3 Cartan subalgebras, 2.31 Oscillator realization: strange series.

2.50 Subsuperalgebras: regular and singular subsuperalgebras

Regular subsuperalgebras

Definition

Let $\mathcal{G}$ be a basic Lie superalgebra. Consider its canonical root decomposition, where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ and Δ its corresponding root system (→ 2.42):

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha$$

A subsuperalgebra $\mathcal{G}'$ of $\mathcal{G}$ is called *regular* (by analogy with the algebra case) if it has the root decomposition

$$\mathcal{G}' = \mathcal{H}' \oplus \bigoplus_{\alpha' \in \Delta'} \mathcal{G}'_{\alpha'}$$

where $\mathcal{H}' \subset \mathcal{H}$ and $\Delta \subset \Delta'$. The semi-simplicity of $\mathcal{G}'$ will be insured if to each $\alpha' \in \Delta'$ then $-\alpha' \in \Delta'$ and $\mathcal{H}'$ is the linear closure of Δ'.

The method for finding the regular semi-simple sub(super)algebras of a given basic Lie superalgebra $\mathcal{G}$ is completely analogous to the usual one for Lie algebras by means of extended Dynkin diagrams. However, one has now to consider all the Dynkin diagrams associated to the inequivalent simple root systems. For a given simple root system Δ^0 of $\mathcal{G}$, one considers the associated Dynkin diagram. The corresponding extended simple root system is $\widehat{\Delta}^0 = \Delta^0 \cup \{\Psi\}$ where Ψ is the lowest root with respect to Δ^0, to which is associated the extended Dynkin diagram. Now, if one deletes arbitrarily some dot(s) of the extended diagram, one obtains some connected Dynkin diagram or a set of disjointed Dynkin diagrams corresponding to a regular semi-simple sub(super)algebra of $\mathcal{G}$. Indeed, taking away one or more roots from $\widehat{\Delta}^0$, one is left with a set of independent roots which constitute the simple root system of a regular semi-simple subsuperalgebra of $\mathcal{G}$. Then repeating the same operation on the obtained Dynkin diagrams – that is adjunction of a dot associated to the lowest root of a simple part and cancellation of one arbitrary dot (or two in the unitary case) – as many time as necessary, one obtains all the Dynkin diagrams associated with regular semi-simple basic Lie sub(super)algebras. In order to get the maximal regular semi-simple sub(super)algebras of the same rank r of $\mathcal{G}$, only the first step has to be achieved. The other possible maximal regular subsuperalgebras of $\mathcal{G}$, if they exist, will be obtained by deleting only one dot in the non-extended Dynkin diagram of $\mathcal{G}$ and will be therefore of rank $r-1$.

Table 2.7 lists the maximal regular semi-simple sub(super)algebras for the basic Lie superalgebras.

Singular subsuperalgebras

Definition

Let $\mathcal{G}$ be a basic Lie superalgebra. A subsuperalgebra $\mathcal{G}'$ of $\mathcal{G}$ is called *singular* if it is not regular.

Some of the singular subsuperalgebras of the basic Lie superalgebras can be found by the folding technique. Let $\mathcal{G}$ be a basic Lie superalgebra, with non-trivial outer automorphism ($\mathrm{Out}(\mathcal{G})$ does not reduce to the identity). Then, there exists at least one Dynkin diagram of $\mathcal{G}$ which has the symmetry given by $\mathrm{Out}(\mathcal{G})$. It can be seen that each symmetry τ exhibited by the Dynkin diagram induces a direct construction of the subsuperalgebra $\mathcal{G}'$ invariant under the $\mathcal{G}$ outer automorphism associated to τ. Indeed, if the simple root α is transformed into $\tau(\alpha)$, then $\frac{1}{2}(\alpha + \tau(\alpha))$ is τ-invariant since $\tau^2 = 1$, and appears as a simple root of $\mathcal{G}'$ associated to the generators $E_\alpha + E_{\tau(\alpha)}$, the generator E_α (resp. $E_{\tau(\alpha)}$ corresponding to the root α (resp. $\tau(\alpha)$). A Dynkin diagram of $\mathcal{G}'$ will therefore be obtained by folding the $\mathbb{Z}_2$-symmetric Dynkin diagram of $\mathcal{G}$, that is by transforming each couple $(\alpha, \tau(\alpha))$ into the

Table 2.7: Maximal regular sub(super)algebras of the basic Lie superalgebras.

superalgebras	subsuperalgebras		
$A(m,n)$	$A(i,j) \oplus A(i',j')$	$(i+i'=m-1$ and $j+j'=n-1)$	
	$A_m \oplus A_n$		
$B(m,n)$	$B(i,j) \oplus D(i',j')$	$(i+i'=m$ and $j+j'=n)$	
	$B_m \oplus C_n$		
	$D(m,n)$		
$C(n+1)$	$C_i \oplus C(n-i+1)$	$A(m-1,n-1)$ C_n	
$D(m,n)$	$D(i,j) \oplus D(i',j')$	$(i+i'=m$ and $j+j'=n)$	
	$D_m \oplus C_n$		
F_4	$A_1 \oplus B_3$	$A_2 \oplus A(0,1)$	$A_1 \oplus D(2,1;2)$
	$A(0,3)$	$C(3)$	
G_2	$A_1 \oplus G_2$	$A_1 \oplus B(1,1)$	$A_2 \oplus B(0,1)$
	$A(0,2)$	$D(2,1;3)$	
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus A_1$	$A(0,1)$	

root $\frac{1}{2}(\alpha+\tau(\alpha))$ of $\mathcal{G}'$. One obtains the following invariant subsuperalgebras (which are singular):

superalgebra $\mathcal{G}$	singular subsuperalgebra $\mathcal{G}'$
$sl(2m+1\|2n)$	$osp(2m+1\|2n)$
$sl(2m\|2n)$	$osp(2m\|2n)$
$osp(2m\|2n)$	$osp(2m-1\|2n)$
$osp(2\|2n)$	$osp(1\|2n)$

→ 2.3 Cartan subalgebras, 2.15 Dynkin diagrams, 2.42 Roots, root systems, 2.46 Simple and semi-simple Lie superalgebras.

For more details, see ref. [92].

2.51 Superalgebra, subsuperalgebra

Definition

Let $\mathcal{A}$ be an algebra over a field $\mathbb{K}$ of characteristic zero (usually $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) with internal laws $+$ and $*$. One sets $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\overline{0},\overline{1}\}$. $\mathcal{A}$ is called a superalgebra or $\mathbb{Z}_2$-graded algebra if $\mathcal{A}$ can be written into a direct sum of two spaces $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$, such that

$$\mathcal{A}_{\overline{0}} * \mathcal{A}_{\overline{0}} \subset \mathcal{A}_{\overline{0}}, \qquad \mathcal{A}_{\overline{0}} * \mathcal{A}_{\overline{1}} \subset \mathcal{A}_{\overline{1}}, \qquad \mathcal{A}_{\overline{1}} * \mathcal{A}_{\overline{1}} \subset \mathcal{A}_{\overline{0}}$$

Elements $X \in \mathcal{A}_{\overline{0}}$ are called even or of degree $\deg X = 0$ while elements $X \in \mathcal{A}_{\overline{1}}$ are called odd or of degree $\deg X = 1$.

One defines the *Lie superbracket* or *supercommutator* of two elements X and Y by

$$[\![X, Y]\!] = X * Y - (-1)^{\deg X . \deg Y} Y * X$$

A superalgebra $\mathcal{A}$ is called associative if $(X * Y) * Z = X * (Y * Z)$ for all elements $X, Y, Z \in \mathcal{A}$.

A superalgebra $\mathcal{A}$ is called commutative if $X * Y = Y * X$ for all elements $X, Y \in \mathcal{A}$.

Definition

A (graded) subalgebra $\mathcal{K} = \mathcal{K}_{\overline{0}} \oplus \mathcal{K}_{\overline{1}}$ of a superalgebra $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$ is a non-empty set $\mathcal{K} \subset \mathcal{A}$ which is a superalgebra with the two composition laws induced by $\mathcal{A}$ such that $\mathcal{K}_{\overline{0}} \subset \mathcal{A}_{\overline{0}}$ and $\mathcal{K}_{\overline{1}} \subset \mathcal{A}_{\overline{1}}$.

Definition

A homomorphism Φ from a superalgebra $\mathcal{A}$ into a superalgebra $\mathcal{A}'$ is a linear application from $\mathcal{A}$ into $\mathcal{A}'$ which respects the $\mathbb{Z}_2$-gradation, that is $\Phi(\mathcal{A}_{\overline{0}}) \subset \mathcal{A}'_{\overline{0}}$ and $\Phi(\mathcal{A}_{\overline{1}}) \subset \mathcal{A}'_{\overline{1}}$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two superalgebras. One defines the tensor product $\mathcal{A} \otimes \mathcal{A}'$ of the two superalgebras by

$$(X_1 \otimes X'_1)(X_2 \otimes X'_2) = (-1)^{\deg X_2 . \deg X'_1} (X_1 X_2 \otimes X'_1 X'_2)$$

if $X_1, X_2 \in \mathcal{A}$ and $X'_1, X'_2 \in \mathcal{A}'$.

$\rightarrow$ 2.24 Lie superalgebra, superalgebra, ideal.

2.52 Superalgebra $osp(1|2)$

The superalgebra $osp(1|2)$ is the simplest one and can be viewed as the supersymmetric version of $sl(2)$. It contains three bosonic generators E^+, E^-, H which form the Lie algebra $sl(2)$ and two fermionic generators F^+, F^-, whose non-vanishing commutation relations in the Cartan–Weyl basis read as

$$\begin{aligned} &\Big[H, E^{\pm}\Big] = \pm E^{\pm} && \Big[E^{+}, E^{-}\Big] = 2H \\ &\Big[H, F^{\pm}\Big] = \pm\tfrac{1}{2}F^{\pm} && \Big\{F^{+}, F^{-}\Big\} = \tfrac{1}{2}H \\ &\Big[E^{\pm}, F^{\mp}\Big] = -F^{\pm} && \Big\{F^{\pm}, F^{\pm}\Big\} = \pm\tfrac{1}{2}E^{\pm} \end{aligned}$$

The three-dimensional matrix representation (fundamental representation) is given by

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^{+} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F^{+} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad F^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

The quadratic Casimir operator is

$$C_2 = H^2 + \tfrac{1}{2}(E^{+}E^{-} + E^{-}E^{+}) - (F^{+}F^{-} - F^{-}F^{+})$$

The superalgebra $osp(1|2)$ reveals many features which make it very close to the Lie algebras. In particular, one has the following results for the representation theory:

1. All finite dimensional representations of $osp(1|2)$ are completely reducible.

2. Any irreducible representation of $osp(1|2)$ is typical.

3. An irreducible representation $\mathcal{R}$ of $osp(1|2)$ is characterized by a non-negative integer or half-integer $j = 0, 1/2, 1, 3/2, \ldots$ and decomposes under the even part $sl(2)$ into two multiplets $\mathcal{R}_j = D_j \oplus D_{j-1/2}$ for $j \neq 0$, the case $j = 0$ reducing to the trivial one-dimensional representation. The dimension of an irreducible representation $\mathcal{R}_j$ of $osp(1|2)$ is $4j+1$. The eigenvalue of the quadratic Casimir C_2 in the representation $\mathcal{R}_j$ is $j(j+\frac{1}{2})$.

4. The product of two irreducible $osp(1|2)$ representations decomposes as follows:

$$\mathcal{R}_{j_1} \otimes \mathcal{R}_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j=j_1+j_2} \mathcal{R}_j$$

   j taking integer and half-integer values.

→ 2.5 Casimir invariants, 2.11 Decomposition w.r.t. $osp(1|2)$ subalgebras, 2.16 Embeddings of $osp(1|2)$.

For more details, see refs. [9, 65].

2.53 Superalgebra $sl(1|2)$

The superalgebra $sl(1|2) \simeq sl(2|1)$ is the $N = 2$ extended supersymmetric version of $sl(2)$ and contains four bosonic generators E^+, E^-, H, Z which form the Lie algebra $sl(2) \oplus U(1)$ and four fermionic generators F^+, F^-, $\overline{F}^+$, $\overline{F}^-$, whose commutation relations in the Cartan–Weyl basis read as

$$\begin{array}{lll}
\Big[H, E^{\pm}\Big] = \pm E^{\pm} & \Big[H, F^{\pm}\Big] = \pm\frac{1}{2}F^{\pm} & \Big[H, \overline{F}^{\pm}\Big] = \pm\frac{1}{2}\overline{F}^{\pm} \\
\Big[Z, H\Big] = \Big[Z, E^{\pm}\Big] = 0 & \Big[Z, F^{\pm}\Big] = \frac{1}{2}F^{\pm} & \Big[Z, \overline{F}^{\pm}\Big] = -\frac{1}{2}\overline{F}^{\pm} \\
\Big[E^{\pm}, F^{\pm}\Big] = \Big[E^{\pm}, \overline{F}^{\pm}\Big] = 0 & \Big[E^{\pm}, F^{\mp}\Big] = -F^{\pm} & \Big[E^{\pm}, \overline{F}^{\mp}\Big] = \overline{F}^{\pm} \\
\Big[E^{+}, E^{-}\Big] = 2H & \Big\{F^{\pm}, \overline{F}^{\mp}\Big\} = Z \mp H & \Big\{F^{\pm}, \overline{F}^{\pm}\Big\} = E^{\pm}
\end{array}$$

$$\Big\{F^{\pm}, F^{\pm}\Big\} = \Big\{\overline{F}^{\pm}, \overline{F}^{\pm}\Big\} = 0 \qquad \Big\{F^{\pm}, F^{\mp}\Big\} = \Big\{\overline{F}^{\pm}, \overline{F}^{\mp}\Big\} = 0$$

The three-dimensional matrix representation (fundamental representation) is given by

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \overline{F}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \overline{F}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The quadratic and cubic Casimir operators are

$$\begin{aligned}
C_2 &= H^2 - Z^2 + E^-E^+ + F^-\overline{F}^+ - \overline{F}^-F^+ \\
C_3 &= (H^2 - Z^2)Z + E^-E^+(Z - \tfrac{1}{2}) - \tfrac{1}{2}F^-\overline{F}^+(H - 3Z + 1) \\
&\quad - \tfrac{1}{2}\overline{F}^-F^+(H + 3Z + 1) + \tfrac{1}{2}E^-\overline{F}^+F^+ + \tfrac{1}{2}\overline{F}^-F^-E^+
\end{aligned}$$

The irreducible representations of $sl(1|2)$ are characterized by the pair of labels (b,j) where j is a non-negative integer or half-integer and b an arbitrary complex number. The representations $\pi(b,j)$ with $b \neq \pm j$ are typical and have dimension $8j$. The representations $\pi(\pm j, j)$ are atypical and have dimension $4j+1$. In the typical representation $\pi(b,j)$, the Casimir operators C_2 and C_3 have the eigenvalues $C_2 = j^2 - b^2$ and $C_3 = b(j^2 - b^2)$ while they are identically zero in the atypical representations $\pi(\pm j, j)$.

The typical representation $\pi(b,j)$ of $sl(1|2)$ decomposes under the even part $sl(2) \oplus U(1)$ for $j \geq 1$ as

$$\pi(b,j) = D_j(b) \oplus D_{j-1/2}(b-1/2) \oplus D_{j-1/2}(b+1/2) \oplus D_{j-1}(b)$$

the case $j = \frac{1}{2}$ reducing to

$$\pi(b, \tfrac{1}{2}) = D_{1/2}(b) \oplus D_0(b-1/2) \oplus D_0(b+1/2)$$

where $D_j(b)$ denotes the representation of $sl(2) \oplus U(1)$ with isospin j and hypercharge b.

The irreducible atypical representations $\pi_\pm(j) \equiv \pi(\pm j, j)$ of $sl(1|2)$ decompose under the even part $sl(2) \oplus U(1)$ as

$$\begin{aligned} \pi_+(j) &= D_j(j) \oplus D_{j-1/2}(j+1/2) \\ \pi_-(j) &= D_j(-j) \oplus D_{j-1/2}(-j-1/2) \end{aligned}$$

The not completely reducible atypical representations of $sl(1|2)$ decompose as *semi-direct* sums of $sl(1|2)$ irreducible (atypical) representations. More precisely, they are of the following types:

$$\begin{aligned}
&\pi_\pm(j; j-1/2) \equiv \pi_\pm(j) \ni \pi_\pm(j-1/2) \\
&\pi_\pm(j-1/2; j) \equiv \pi_\pm(j-1/2) \ni \pi_\pm(j) \\
&\pi_\pm(j-1/2, j+1/2; j) \equiv \pi_\pm(j-1/2) \ni \pi_\pm(j) \in \pi_\pm(j+1/2) \\
&\pi_\pm(j; j-1/2, j+1/2) \equiv \pi_\pm(j-1/2) \in \pi_\pm(j) \ni \pi_\pm(j+1/2) \\
&\pi_\pm(j, j\pm 1; j\pm 1/2; j\pm 3/2) \\
&\qquad\qquad \equiv \pi_\pm(j) \ni \pi_\pm(j\pm 1/2) \in \pi_\pm(j\pm 1) \ni \pi_\pm(j\pm 3/2) \\
&\pi_\pm(j; j-1/2, j+1/2; j) \equiv \pi_\pm(j) \begin{array}{l} \ni \pi_\pm(j-1/2) \ni \\ \ni \pi_\pm(j+1/2) \ni \end{array} \pi_\pm(j)
\end{aligned}$$

where the semi-direct sum symbol $\ni$ (resp. $\in$) means that the representation space on the left (resp. on the right) is an invariant subspace of the whole representation space.

It is also possible to decompose the $sl(1|2)$ representations under the superprincipal $osp(1|2)$ subsuperalgebra of $sl(1|2)$ ($\rightarrow$ 2.16). One obtains for the typical representations $\pi(b,j) = \mathcal{R}_j \oplus \mathcal{R}_{j-1/2}$ and for the irreducible atypical representations $\pi_\pm(j) = \mathcal{R}_j$ where $\mathcal{R}_j$ denotes an irreducible $osp(1|2)$ representation.

We give now the formulae of the tensor products of two $sl(1|2)$ representations $\pi(b_1,j_1)$ and $\pi(b_2,j_2)$. In what follows, we set $b = b_1 + b_2$, $j = j_1 + j_2$ and $\overline{\jmath} = |j_1 - j_2|$. Moreover, the product of two irreducible representations will be called non-degenerate if it decomposes into a direct sum of irreducible representations; otherwise it is called degenerate.

Product of two typical representations

The product of two typical representations $\pi(b_1,j_1)$ and $\pi(b_2,j_2)$ is non-degenerate when $b \neq \pm(j-n)$ for $n = 0,1,\dots,2\min(j_1,j_2)$. It is then given by

$$\begin{aligned}
\pi(b_1,j_1) \otimes \pi(b_2,j_2) &= \bigoplus_{n=0}^{2\min(j_1,j_2)-\delta_{j_1,j_2}} \pi(b,j-n) \bigoplus_{n=1}^{2\min(j_1,j_2)-1} \pi(b,j-n) \\
&\quad \bigoplus_{n=0}^{2\min(j_1,j_2)-1} \pi(b+\tfrac{1}{2},j-\tfrac{1}{2}-n) \oplus \pi(b-\tfrac{1}{2},j-\tfrac{1}{2}-n) \\
\pi(b_1,j_1) \otimes \pi(b_2,\tfrac{1}{2}) &= \pi(b,j_1+\tfrac{1}{2}) \oplus \pi(b,j_1-\tfrac{1}{2}) \oplus \pi(b+\tfrac{1}{2},j_1) \\
&\quad \oplus \pi(b-\tfrac{1}{2},j_1) \\
\pi(b_1,\tfrac{1}{2}) \otimes \pi(b_2,\tfrac{1}{2}) &= \pi(b,1) \oplus \pi(b+\tfrac{1}{2},\tfrac{1}{2}) \oplus \pi(b-\tfrac{1}{2},\tfrac{1}{2})
\end{aligned}$$

When the product is degenerate, one has

1. if $b = \pm j$:
   $\pi(b,j) \oplus \pi(b \mp 1/2, j-1/2)$ is replaced by $\pi_\pm(j-1/2; j-1, j; j-1/2)$

2. if $b = \pm\overline{\jmath} \neq 0$:
   $\pi(b,\overline{\jmath}) \oplus \pi(b \pm 1/2, \overline{\jmath}+1/2)$ is replaced by $\pi_\pm(\overline{\jmath}; \overline{\jmath}-1/2, \overline{\jmath}+1/2; \overline{\jmath})$

3. if $b = \overline{\jmath} = 0$:
   $\pi(1/2,1/2) \oplus \pi(-1/2,1/2)$ is replaced by $\pi(0; -1/2, 1/2; 0)$

4. if $b = \pm(j-n)$ for $n = 1,\dots,2\min(j_1,j_2)$:
   $\pi(b \pm 1/2, j+1/2-n) \oplus \pi(b,j-n) \oplus \pi(b,j-n) \oplus \pi(b \mp 1/2, j-1/2-n)$ is replaced by $\pi_\pm(j-1/2-n; j-1-n, j-n; j-1/2-n) \oplus \pi_\pm(j-n; j-1/2-n, j+1/2-n; j-n)$

Product of a typical with an atypical representation

The non-degenerate product of a typical representation $\pi(b_1, j_1)$ with an atypical one $\pi_\pm(j_2)$ ($b_2 = \pm j_2$) is given by

if $j_1 \le j_2$

$$\pi(b_1, j_1) \otimes \pi_\pm(j_2) = \bigoplus_{n=0}^{2\min(j_1,j_2)-1} \pi(b, j-n) \oplus \pi(b \pm \tfrac{1}{2}, j - \tfrac{1}{2} - n)$$

if $j_1 > j_2$

$$\pi(b_1, j_1) \otimes \pi_\pm(j_2) = \bigoplus_{n=0}^{2\min(j_1,j_2)-1} \pi(b, j-n) \oplus \pi(b \pm \tfrac{1}{2}, j - \tfrac{1}{2} - n) \oplus \pi(b, |j_1 - j_2|)$$

When the product $\pi(b_1, j_1) \otimes \pi_+(j_2)$ is degenerate, one has

1. if $b = -(j-n)$ for $n = 0, 1, \ldots, 2\min(j_1, j_2) - 1$:
   $\pi(b, j-n) \oplus \pi(b + 1/2, j - 1/2 - n)$ is replaced by $\pi_\pm(j - 1/2 - n; j - 1 - n, j - n; j - 1/2 - n)$

2. if $b = j - n$ for $n = 1, \ldots, 2\min(j_1, j_2)$:
   $\pi(b, j-n) \oplus \pi(b + 1/2, j + 1/2 - n)$ is replaced by $\pi_\pm(j - n; j - 1/2 - n, j + 1/2 - n; j - n)$

The case of the degenerate product $\pi(b_1, j_1) \otimes \pi_-(j_2)$ is similar.

Product of two atypical representations

The product of two atypical representations $\pi_\pm(j_1)$ and $\pi_\pm(j_2)$ is always non-degenerate. It is given by

$$\pi_\pm(j_1) \otimes \pi_\pm(j_2) = \pi_\pm(j) \oplus \bigoplus_{n=0}^{2\min(j_1,j_2)-1} \pi(\pm(j + \tfrac{1}{2}), j - \tfrac{1}{2} - n)$$

$$\pi(j_1, j_1) \otimes \pi(-j_2, j_2) = \bigoplus_{n=0}^{2\min(j_1,j_2)-1} \pi(j_1 - j_2, j - n)$$

$$\oplus \begin{cases} \pi(j_1 - j_2, j_1 - j_2) & \text{if } j_1 > j_2 \\ \pi(j_1 - j_2, j_2 - j_1) & \text{if } j_1 < j_2 \\ (0) & \text{if } j_1 = j_2 \end{cases}$$

→ 2.5 Casimir invariants, 2.12 Decomposition w.r.t. $sl(1|2)$ subalgebras, 2.17 Embeddings of $sl(1|2)$.

For more details, see refs. [58, 65].

2.54 Superconformal algebra

For massless theory the concept of Fermi-Bose symmetry or supersymmetry requires the extension of the conformal Lie algebra including the generators of the supersymmetry transformations Q_α, S_α which transform bosonic fields into fermionic ones and vice versa. The conformal algebra in four space–time dimensions is spanned by the 15 generators $M_{\mu\nu}$, P_μ, K_μ and D (with the Greek labels running from 0 to 3). The generators $M_{\mu\nu}$ and P_μ span the Poincaré algebra and their commutation relations are given in → 2.59 (Supersymmetry algebra: definition), while K_μ and D are respectively the generators of the conformal transformations and of the dilatation (→ 1.17). The commutation relations of the $N = 1$ superconformal algebra read as (the metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$):

$$\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma})$$

$$\Big[M_{\mu\nu}, P_\rho\Big] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \qquad \Big[P_\mu, P_\nu\Big] = 0$$

$$\Big[M_{\mu\nu}, K_\rho\Big] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu) \qquad \Big[K_\mu, K_\nu\Big] = 0$$

$$\Big[P_\mu, K_\nu\Big] = 2i(g_{\mu\nu}D - M_{\mu\nu}) \qquad \Big[D, M_{\mu\nu}\Big] = 0$$

$$\Big[D, P_\mu\Big] = -iP_\mu \qquad \Big[D, K_\mu\Big] = iK_\mu$$

$$\Big[M_{\mu\nu}, Q_a\Big] = -\tfrac{1}{2}(\Sigma_{\mu\nu})_a{}^b Q_b \qquad \Big[M_{\mu\nu}, S_a\Big] = -\tfrac{1}{2}(\Sigma_{\mu\nu})_a{}^b S_b$$

$$\Big[P_\mu, Q_a\Big] = 0 \qquad \Big[P_\mu, S_a\Big] = -(\gamma_\mu)_a{}^b Q_b$$

$$\Big[K_\mu, Q_a\Big] = -(\gamma_\mu)_a{}^b S_b \qquad \Big[K_\mu, S_a\Big] = 0$$

$$\Big[D, Q_a\Big] = -\tfrac{1}{2}iQ_a \qquad \Big[D, S_a\Big] = \tfrac{1}{2}iS_a$$

$$\Big[Y, Q_a\Big] = i(\gamma_5)_a{}^b Q_b \qquad \Big[Y, S_a\Big] = -i(\gamma_5)_a{}^b S_b$$

$$\Big[Y, M_{\mu\nu}\Big] = \Big[Y, D\Big] = 0 \qquad \Big[Y, P_\mu\Big] = \Big[Y, K_\mu\Big] = 0$$

$$\Big\{Q_a, Q_b\Big\} = 2(\gamma_\mu C)_{ab}P^\mu \qquad \Big\{S_a, S_b\Big\} = 2(\gamma_\mu C)_{ab}K^\mu$$

$$\Big\{Q_a, S_b\Big\} = (\Sigma_{\mu\nu}C)_{ab}M^{\mu\nu} + 2iC_{ab}D + 3i(\gamma_5 C)_{ab}Y$$

where the Dirac indices a, b run from 1 to 4, γ^μ are the Dirac matrices in Majorana representation, C is the charge conjugation matrix, the $\Sigma^{\mu\nu}$ are the generators of the Lorentz group in the representation $(1/2, 0) \oplus (0, 1/2)$: $\Sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ (→ 2.47) and Y is the generator of the (chiral)

$U(1)$. The transformations of Q_a and S_a under $M_{\mu\nu}$ show that the Q_a and S_a are spinors. The superconformal algebra contains the super-Poincaré as subsuperalgebra, however in the conformal case the are no central charges for $N > 1$.

Let us emphasize that the superconformal algebra is isomorphic to the simple Lie superalgebra $su(2,2|N)$, real form of $sl(4|N)$.

→ 2.47 Spinors (in the Lorentz group), 2.59 Supersymmetry algebra: definition.

For more details, see refs. [89, 96].

2.55 Supergroups

In order to construct the supergroup or group with Grassmann structure associated to a (simple) superalgebra $\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$, one starts from the complex Grassmann algebra (→ 2.22) $\Gamma(n)$ of order n with n generators 1, $\theta_1, \dots, \theta_n$ satisfying $\left\{\theta_i, \theta_j\right\} = 0$. The element

$$\eta = \sum_{m \geq 0} \sum_{i_1 < \dots < i_m} \eta_{i_1 \dots i_m} \theta_{i_1} \dots \theta_{i_m}$$

is called even (resp. odd) if each complex coefficient $\eta_{i_1 \dots i_m}$ in the above expression of η corresponds to an even (resp. odd) value of m. As a vector space, one decomposes $\Gamma(n)$ as $\Gamma(n) = \Gamma(n)_{\overline{0}} \oplus \Gamma(n)_{\overline{1}}$ with $\Gamma(n)_{\overline{0}}$ (resp. $\Gamma(n)_{\overline{1}}$) made of homogeneous even (resp. odd) elements.

The Grassmann envelope $A(\Gamma)$ of $\mathcal{A}$ consists of formal linear combinations $\sum_i \eta_i a_i$ where $\{a_i\}$ is a basis of $\mathcal{A}$ and $\eta_i \in \Gamma(n)$ such that for a fixed index i, the elements a_i and η_i are both even or odd. The commutator between two arbitrary elements $X = \sum_i \eta_i a_i$ and $Y = \sum_j \eta'_j a_j$ is naturally defined by $[X, Y] = \sum_{ij} \eta_i \eta'_j [\![a_i, a_j]\!]$ where $[\![a_i, a_j]\!]$ means the supercommutator in $\mathcal{A}$. This commutator confers to the Grassmann envelope $A(\Gamma)$ of $\mathcal{A}$ a Lie algebra structure.

The relation between a supergroup and its superalgebra is analogous to the Lie algebra case: the supergroup A associated to the superalgebra $\mathcal{A}$ is the exponential mapping of the Grassmann envelope $A(\Gamma)$ of $\mathcal{A}$, the even generators of the superalgebra $\mathcal{A}$ corresponding to even parameters (that is even elements of the Grassmann algebra) and the odd generators of $\mathcal{A}$ to odd parameters (that is odd elements of the Grassmann algebra).

The above approach is due to Berezin. In particular, the case of $osp(1|2)$ is worked out explicitly in ref. [9]. On classical supergroups, see also refs. [47, 48, 76].

→ 2.22 Grassmann algebras.

2.56 Supergroups of linear transformations

Let $\Gamma = \Gamma_{\overline{0}} \oplus \Gamma_{\overline{1}}$ be a Grassmann algebra (→ 2.22) over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and consider the set of $(m+n) \times (m+n)$ even supermatrices (→ 2.57) of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ submatrices respectively, with even entries in $\Gamma_{\overline{0}}$ for A, D and odd entries in $\Gamma_{\overline{1}}$ for B, C.

The *general linear supergroup* $GL(m|n;\mathbb{K})$ is the supergroup of even invertible supermatrices M, the product law being the usual matrix multiplication.

The transposition and adjoint operations allow us to define the classical subsupergroups of $GL(m|n;\mathbb{K})$ corresponding to the classical superalgebras.

The special linear supergroup $SL(m|n;\mathbb{K})$ is the subsupergroup of supermatrices $M \in GL(m|n;\mathbb{K})$ such that $\operatorname{sdet} M = 1$.

The unitary and superunitary supergroups $U(m|n)$ and $sU(m|n)$ are the subsupergroups of supermatrices $M \in GL(m|n;\mathbb{C})$ such that $MM^\dagger = 1$ and $MM^\ddagger = 1$ respectively (for the notations † and ‡, → 2.57).

The orthosymplectic supergroup $OSP(m|n;\mathbb{K})$ is the subsupergroup of supermatrices $M \in GL(m|n;\mathbb{K})$ such that $M^{st}HM = H$ where $(n = 2p)$

$$H = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & \mathbb{J}_{2p} \end{pmatrix} \qquad \text{and} \qquad \mathbb{J}_{2p} = \begin{pmatrix} 0 & \mathbb{I}_p \\ -\mathbb{I}_p & 0 \end{pmatrix}$$

The compact forms are $USL(m|n)$ and $sOSP(m|n)$, subsupergroups of supermatrices $M \in GL(m|n;\mathbb{C})$ such that $\operatorname{sdet} M = 1$, $MM^\dagger = 1$ and $M^{st}HM = H$, $MM^\ddagger = 1$ respectively.

Finally the strange supergroups are defined as follows. The supergroup $P(n)$ is the subsupergroup of supermatrices $M \in GL(n|n;\mathbb{K})$ such that $\operatorname{sdet} M = 1$ and $M\mathbb{J}_{2n}M^{st} = \mathbb{J}_{2n}$ with $\mathbb{J}_{2n}$ defined above. The supergroup $Q(n)$ is the subsupergroup of supermatrices $M \in GL(n|n;\mathbb{K})$ with $A = D$ and $B = C$ such that $\operatorname{tr}\ln((A-B)^{-1}(A+B)) = 0$.

For more details, see ref. [75].

2.57 Supermatrices

Definition

A matrix M is called a complex (resp. real) *supermatrix* if its entries have values in a complex (resp. real) Grassmann algebra $\Gamma = \Gamma_{\overline{0}} \oplus \Gamma_{\overline{1}}$ (→ 2.22). More precisely, consider the set of $(m+n) \times (p+q)$ supermatrices M of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $m \times p$, $n \times p$, $m \times q$ and $n \times q$ submatrices respectively. The supermatrix M is called *even* (or of degree 0) if $A, D \in \Gamma_{\overline{0}}$ and $B, C \in \Gamma_{\overline{1}}$, while it is called *odd* (or of degree 1) if $A, D \in \Gamma_{\overline{1}}$ and $B, C \in \Gamma_{\overline{0}}$.

The product of supermatrices is defined as the product of matrices: if M and M' are two $(m+n) \times (p+q)$ and $(p+q) \times (r+s)$ supermatrices, then the entries of the $(m+n) \times (r+s)$ supermatrix MM' are given by

$$(MM')_{ij} = \sum_{k=1}^{p+q} M_{ik} M'_{kj}$$

Since the Grassmann algebra Γ is associative, the product of supermatrices is also associative.

From now on, we will consider only square supermatrices, that is such that $m = p$ and $n = q$. The set of $(m+n) \times (m+n)$ complex (resp. real) square supermatrices is denoted by $M(m|n;\mathbb{C})$ (resp. $M(m|n;\mathbb{R})$).

A square supermatrix M is said to be invertible if there exists a square supermatrix M' such that $MM' = M'M = I$ where I is the unit supermatrix (even supermatrix with zero off-diagonal entries and diagonal entries equal to the unit 1 of the Grassmann algebra Γ).

Definition

The *general linear supergroup* $GL(m|n;\mathbb{C})$ (resp. $GL(m|n;\mathbb{R})$) is the supergroup of even invertible complex (resp. real) supermatrices, the group law being the product of supermatrices.

The usual operations of transposition, determinant, trace, adjoint are defined as follows in the case of supermatrices.

Let $M \in M(m|n;\mathbb{C})$ be a complex square supermatrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The transpose and supertranspose of M are defined by:

$$M^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \qquad \text{transpose}$$

$$M^{st} = \begin{pmatrix} A^t & (-1)^{\deg M} C^t \\ -(-1)^{\deg M} B^t & D^t \end{pmatrix} \qquad \text{supertranspose}$$

Explicitly, one finds

$$M^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} \quad \text{if } M \text{ is even}$$

$$M^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} \quad \text{if } M \text{ is odd}$$

It follows that

$$((M)^{st})^{st} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$
$$((((M)^{st})^{st})^{st})^{st} = M$$
$$(MN)^{st} = (-1)^{\deg M.\deg N} N^{st} M^{st}$$

but $(MN)^t \neq N^t M^t$.

The supertrace of M is defined by

$$\mathrm{str}(M) = \mathrm{tr}(A) - (-1)^{\deg M}\mathrm{tr}(D) = \begin{cases} \mathrm{tr}(A) - \mathrm{tr}(D) & \text{if } M \text{ is even} \\ \mathrm{tr}(A) + \mathrm{tr}(D) & \text{if } M \text{ is odd} \end{cases}$$

One has the following properties for the supertrace:

$$\mathrm{str}(M+N) = \mathrm{str}(M) + \mathrm{str}(N) \quad \text{if } \deg M = \deg N$$
$$\mathrm{str}(MN) = (-1)^{\deg M.\deg N}\mathrm{str}(M)\mathrm{str}(N)$$
$$\mathrm{str}(M^{st}) = \mathrm{str}(M)$$

If M is even invertible, one defines the superdeterminant (or Berezinian) of M by

$$\mathrm{sdet}(M) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)}$$

Notice that M being an even invertible matrix, the inverse matrices A^{-1} and D^{-1} exist.

One has the following properties for the superdeterminant:

$$\mathrm{sdet}(MN) = \mathrm{sdet}(M)\ \mathrm{sdet}(N)$$
$$\mathrm{sdet}(M^{st}) = \mathrm{sdet}(M)$$
$$\mathrm{sdet}(\exp(M)) = \exp(\mathrm{str}(M))$$

The adjoint operations on the supermatrix M are defined by

$$M^\dagger = (M^t)^* \quad \text{adjoint}$$
$$M^\ddagger = (M^{st})^\# \quad \text{superadjoint}$$

One has

$$(MN)^\dagger = N^\dagger M^\dagger$$
$$(MN)^\ddagger = N^\ddagger M^\ddagger$$
$$(M^\dagger)^\dagger = M \qquad \text{and} \qquad (M^\ddagger)^\ddagger = M$$
$$\operatorname{sdet} M^\dagger = \overline{\operatorname{sdet} M} = (\operatorname{sdet} M)^*$$

where the bar denotes the usual complex conjugation and the star the Grassmann complex conjugation (→ 2.22).

→ 2.56 Supergroups of linear transformations.

For more details, see refs. [4, 75].

2.58 Superspace and superfields

It is fruitful to consider the supergroup associated to the supersymmetry algebra, the super-Poincaré group. A group element g is then given by the exponential of the supersymmetry algebra generators. However, since Q_α and $\overline{Q}_{\dot\alpha}$ are fermionic, the corresponding parameters have to be anticommuting (→ 2.22). More precisely, a group element g with parameters $x^\mu, \omega^{\mu\nu}, \theta^\alpha, \overline{\theta}^{\dot\alpha}$ is given by

$$g(x^\mu, \omega^{\mu\nu}, \theta^\alpha, \overline{\theta}^{\dot\alpha}) = \exp i(x^\mu P_\mu + \tfrac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \theta^\alpha Q_\alpha + \overline{Q}_{\dot\alpha}\overline{\theta}^{\dot\alpha})$$

One defines the superspace as the coset space of the super-Poincaré group by the Lorentz group, parametrized by the coordinates $x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}$ subject to the condition $\overline{\theta}^{\dot\alpha} = (\theta^\alpha)^*$. The multiplication of group elements is induced by the supersymmetry algebra:

$$g(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})\, g(y^\mu, \zeta^\alpha, \overline{\zeta}^{\dot\alpha}) = g(x^\mu + y^\mu + i\theta\sigma^\mu\overline{\zeta} - i\zeta\sigma^\mu\overline{\theta}, \theta + \zeta, \overline{\theta} + \overline{\zeta})$$

If group element multiplication is considered as a left action, one can write infinitesimally

$$g(y^\mu, \zeta^\alpha, \overline{\zeta}^{\dot\alpha})\, g(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}) = \left[1 - iy^\mu P_\mu - i\zeta^\alpha Q_\alpha - i\overline{\zeta}_{\dot\alpha}\overline{Q}^{\dot\alpha}\right] g(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$$

where the differential operators

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\overline{\theta})_\alpha\partial_\mu \qquad \text{and} \qquad \overline{Q}_{\dot\alpha} = -i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

are the supersymmetry generators of the supersymmetry algebra (→ 2.59).

If group element multiplication is considered as a right action, one has infinitesimally

$$g(x^\mu,\theta^\alpha,\overline{\theta}^{\dot\alpha})\,g(y^\mu,\zeta^\alpha,\overline{\zeta}^{\dot\alpha}) = \left[1 - iy^\mu P_\mu - i\zeta^\alpha D_\alpha - i\overline{\zeta}_{\dot\alpha}\overline{D}^{\dot\alpha}\right] g(x^\mu,\theta^\alpha,\overline{\theta}^{\dot\alpha})$$

where the differential operators

$$D_\alpha = i\frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\overline{\theta})_\alpha\partial_\mu \qquad \text{and} \qquad \overline{D}_{\dot\alpha} = -i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

satisfy the following algebra

$$\begin{aligned} \left\{D_\alpha,\overline{D}_{\dot\beta}\right\} &= -2i\sigma^\mu{}_{\alpha\dot\beta}\,\partial_\mu \\ \left\{D_\alpha,D_\beta\right\} &= \left\{\overline{D}_{\dot\alpha},\overline{D}_{\dot\beta}\right\} = 0 \end{aligned}$$

and anticommute with the Q_α and $\overline{Q}_{\dot\alpha}$ generators.

Unlike the Q generators, the D generators behave like covariant derivatives under the super-Poincaré group.

One defines a *superfield* $\mathcal{F}$ as a function of the superspace. Since the parameters $\theta^\alpha,\overline{\theta}^{\dot\alpha}$ are Grassmann variables, a Taylor expansion of $\mathcal{F}$ in $\theta,\overline{\theta}$ has a finite number of terms:

$$\begin{aligned} \mathcal{F}(x,\theta,\overline{\theta}) \;=\;& f(x) + \theta\phi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) \\ & +\theta\sigma^\mu\overline{\theta}A_\mu(x) + \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\theta\lambda'(x) + \theta\theta\overline{\theta\theta}d(x) \end{aligned}$$

Notice the very important property that the product of two superfields is again a superfield.

Under a superspace transformation, the variation of the superfield $\mathcal{F}$ is given by the action of the supersymmetry generators Q_α and $\overline{Q}_{\dot\alpha}$:

$$\delta\mathcal{F}(x,\theta,\overline{\theta}) = -i(\zeta Q + \overline{Q\zeta})\mathcal{F}$$

The superfield $\mathcal{F}$ forms a representation of the supersymmetry algebra. However, this representation is not irreducible. Irreducible representations can be obtained by imposing constraints on the superfields. The two main examples are the scalar (chiral or antichiral) and the vector superfields.

– The *chiral* superfield $\mathcal{F}$ is defined by the covariant constraint $\overline{D}_{\dot\alpha}\mathcal{F} = 0$. It follows that the chiral superfield $\mathcal{F}$ can be expressed, in terms of $y^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}$ and θ, as

$$\mathcal{F} = A(y) + 2\theta\psi(y) + \theta\theta F(y)$$

The transformation law for the chiral superfield is therefore

$$\begin{aligned} \delta A &= 2\zeta\psi \\ \delta\psi &= -i\sigma^\mu\overline{\zeta}\partial_\mu A + \zeta F \\ \delta F &= 2i\partial_\mu\psi\sigma^\mu\overline{\zeta} \end{aligned}$$

– In the same way, the *antichiral* superfield $\mathcal{F}$ is defined by the covariant constraint $D_\alpha\mathcal{F} = 0$. The antichiral superfield $\mathcal{F}$ can be expressed, in terms of $(y^\mu)^\dagger = x^\mu + i\theta\sigma^\mu\overline{\theta}$ and $\overline{\theta}$, as

$$\mathcal{F} = A^*(y^\dagger) + 2\overline{\theta\psi}(y^\dagger) + \overline{\theta\theta}F^*(y^\dagger)$$

and the transformation law for the antichiral superfield is

$$\begin{aligned} \delta A^\dagger &= 2\overline{\psi\zeta} \\ \delta\overline{\psi} &= i\zeta\sigma^\mu\partial_\mu A^\dagger + F^\dagger\overline{\zeta} \\ \delta F^\dagger &= -2i\zeta\sigma^\mu\partial_\mu\overline{\psi} \end{aligned}$$

– The *vector* superfield $\mathcal{F}$ is defined by the reality constraint $\mathcal{F}^\dagger = \mathcal{F}$. In terms of $x, \theta, \overline{\theta}$, it takes the form (with standard notations)

$$\begin{aligned} \mathcal{F}(x,\theta,\overline{\theta}) \;\;=\;\; & C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \tfrac{i}{2}\theta\theta\Big(M(x) + iN(x)\Big) \\ & -\tfrac{i}{2}\overline{\theta\theta}\Big(M(x) - iN(x)\Big) - \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta}\Big(\overline{\lambda}(x) + \tfrac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\Big) \\ & -i\overline{\theta\theta}\theta\Big(\lambda(x) + \tfrac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\Big) + \tfrac{1}{2}\theta\theta\overline{\theta\theta}\Big(D(x) + \tfrac{1}{2}\Box C(x)\Big) \end{aligned}$$

where C, M, N, D are real scalar fields, A_μ is a real vector field and χ, λ are spinor fields.

→ 2.22 Grassmann algebras, 2.47 Spinors (in the Lorentz group), 2.59–2.60 Supersymmetry algebra: definition, representations.

For more details, see refs. [3, 89, 96].

2.59 Supersymmetry algebra: definition

The concept of Fermi-Bose symmetry or supersymmetry requires the extension of the Poincaré Lie algebra including the generators of the supersymmetry transformations Q_α and $\overline{Q}_{\dot\alpha}$, which are fermionic, that is they transform bosonic fields into fermionic ones and vice versa. The supersymmetry generators Q_α and $\overline{Q}_{\dot\alpha}$ behave like (1/2,0) and (0,1/2) spinors under the Lorentz group (→ 2.47).

The metric being $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, the $N=1$ supersymmetry algebra takes the following form in two-spinor notation (the indices $\mu, \nu, \ldots = 0,1,2,3$ are space–time indices while the indices $\alpha, \beta = 1,2$ and $\dot\alpha, \dot\beta = \dot 1, \dot 2$ are spinor ones):

$$\begin{aligned}
&\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i(-g_{\nu\sigma}M_{\mu\rho} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma}) \\
&\Big[M_{\mu\nu}, P_\rho\Big] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \\
&\Big[P_\mu, P_\nu\Big] = 0 \\
&\Big[M_{\mu\nu}, Q_\alpha\Big] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \qquad \Big[M_{\mu\nu}, \overline{Q}_{\dot\alpha}\Big] = -\tfrac{1}{2}\overline{Q}_{\dot\beta}(\overline{\sigma}_{\mu\nu})^{\dot\beta}{}_{\dot\alpha} \\
&\Big[P_\mu, Q_\alpha\Big] = \Big[P_\mu, \overline{Q}_{\dot\alpha}\Big] = 0 \\
&\Big\{Q_\alpha, Q_\beta\Big\} = \Big\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\Big\} = 0 \qquad \Big\{Q_\alpha, \overline{Q}_{\dot\beta}\Big\} = 2\sigma^\mu{}_{\alpha\dot\beta}P_\mu
\end{aligned}$$

where the σ^i are the Pauli matrices, $\overline{\sigma}^i = -\sigma^i$ for $i = 1,2,3$ and $\sigma^0 = \overline{\sigma}^0 = \mathbb{I}$. The matrices $\frac{1}{2}\sigma^{\mu\nu}$ and $\frac{1}{2}\overline{\sigma}^{\mu\nu}$ are the generators of the Lorentz group in the two fundamental spinor representations: $\sigma^{\mu\nu} = \frac{i}{2}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$ and $\overline{\sigma}^{\mu\nu} = \frac{i}{2}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)$.

In four-spinor notation, the $N=1$ supersymmetry algebra reads as:

$$\begin{aligned}
&\Big[M_{\mu\nu}, Q_a\Big] = -\tfrac{1}{2}(\Sigma_{\mu\nu})_a{}^b Q_b \\
&\Big[P_\mu, Q_a\Big] = 0 \\
&\Big\{Q_a, Q_b\Big\} = 2(\gamma_\mu C)_{ab}P^\mu
\end{aligned}$$

where $Q_a = \begin{pmatrix} Q_\alpha \\ \overline{Q}^{\dot\alpha} \end{pmatrix}$ is a Majorana spinor ($a = 1,2,3,4$), γ^μ are the Dirac matrices in the Majorana representation, C is the charge conjugation matrix and the $\Sigma^{\mu\nu}$ are the generators of the Lorentz group in the representation $(1/2,0) \oplus (0,1/2)$: $\Sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ ($\rightarrow$ 2.47).

It is also useful to write the full supersymmetry algebra in two-spinor notation including the Poincaré generators. One has

$$\begin{aligned}
&P_\mu = \tfrac{1}{2}\,\overline{\sigma}_\mu{}^{\dot\beta\alpha}\,P_{\alpha\dot\beta} \\
&M_{\mu\nu} = \tfrac{i}{4}\left((\varepsilon\sigma_{\mu\nu})^{\alpha\beta}\,J_{\alpha\beta} + (\overline{\sigma}_{\mu\nu}\varepsilon)^{\dot\alpha\dot\beta}\,\overline{J}_{\dot\alpha\dot\beta}\right)
\end{aligned}$$

or

$$J_{\alpha\beta} = -\tfrac{i}{2}\,(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\,M_{\mu\nu}$$
$$\overline{J}_{\dot\alpha\dot\beta} = -\tfrac{i}{2}\,(\varepsilon\overline{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}\,M_{\mu\nu} = J^{\dagger}_{\alpha\beta}$$
$$P_{\alpha\dot\beta} = \sigma^{\mu}{}_{\alpha\dot\beta}\,P_\mu$$

The Poincaré algebra then reads

$$\Big[J_{\alpha\beta}, \overline{J}_{\dot\gamma\dot\delta}\Big] = 0$$
$$\Big[J_{\alpha\beta}, P_{\gamma\dot\delta}\Big] = -i(\varepsilon_{\alpha\gamma}P_{\beta\dot\delta} + \varepsilon_{\beta\gamma}P_{\alpha\dot\delta})$$
$$\Big[J_{\alpha\beta}, J^{\gamma\delta}\Big] = -i(\delta^\gamma_\alpha J^\delta_\beta + \delta^\gamma_\beta J^\delta_\alpha + \delta^\delta_\alpha J^\gamma_\beta + \delta^\delta_\beta J^\gamma_\alpha)$$

The remaining commutation relations of the supersymmetry algebra can be written in the form

$$\Big[J_{\alpha\beta}, Q_\gamma\Big] = i(\varepsilon_{\gamma\alpha}Q_\beta + \varepsilon_{\gamma\beta}Q_\alpha)$$
$$\Big[P_{\alpha\dot\beta}, Q_\gamma\Big] = \Big[\overline{J}_{\dot\alpha\dot\beta}, Q_\gamma\Big] = 0$$
$$\Big\{Q_\alpha, Q_\beta\Big\} = 0 \qquad \Big\{Q_\alpha, Q_{\dot\beta}\Big\} = P_{\alpha\dot\beta}$$

There is an extended version of this algebra if one considers many supersymmetry generators Q^A_α, $\overline{Q}^A_{\dot\alpha}$ with $A = 1, \ldots, N$ transforming under some symmetry group. The extended N-supersymmetry algebra then becomes in two-spinor notation:

$$\Big[M_{\mu\nu}, Q^A_\alpha\Big] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta \qquad \Big[M_{\mu\nu}, \overline{Q}^A_{\dot\alpha}\Big] = -\tfrac{1}{2}\overline{Q}^A_{\dot\beta}(\overline{\sigma}_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$$
$$\Big[P_\mu, Q^A_\alpha\Big] = 0 \qquad \Big[P_\mu, \overline{Q}^A_{\dot\alpha}\Big] = 0$$
$$\Big\{Q^A_\alpha, Q^B_\beta\Big\} = 2\varepsilon_{\alpha\beta}Z^{AB} \qquad \Big\{\overline{Q}^A_{\dot\alpha}, \overline{Q}^B_{\dot\beta}\Big\} = -2\varepsilon_{\dot\alpha\dot\beta}(Z^{AB})^\dagger$$
$$\Big\{Q^A_\alpha, \overline{Q}^B_{\dot\beta}\Big\} = 2\sigma^\mu{}_{\alpha\dot\beta}P_\mu\,\delta_{AB}$$
$$\Big[T_i, T_j\Big] = i\,f_{ij}{}^k\,T_k \qquad \Big[T_i, M_{\mu\nu}\Big] = \Big[T_i, P_\mu\Big] = 0$$
$$\Big[T_i, Q^A_\alpha\Big] = (\zeta_i)^A{}_B Q^B_\alpha \qquad \Big[T_i, \overline{Q}^A_{\dot\alpha}\Big] = -\overline{Q}^B_{\dot\alpha}(\zeta_i)_B{}^A$$
$$\Big[Z^{AB}, \text{anything}\Big] = 0$$

where Z^{AB} are central charges, $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are (un)dotted tensors given in section 2.47.

In four-spinor notation it takes the form (for the relations involving the supersymmetry generators):

$$\begin{aligned}
&\Big[M_{\mu\nu}, Q^A_a\Big] = -\tfrac{1}{2}(\Sigma_{\mu\nu})_a{}^b Q^A_b \\
&\Big[P_\mu, Q^A_a\Big] = 0 \\
&\Big\{Q^A_a, Q^B_b\Big\} = 2(\gamma^\mu C)_{ab} P_\mu \delta^{AB} + C_{ab} U^{AB} + (\gamma_5 C)_{ab} V^{AB} \\
&\Big[T_i, Q^A_a\Big] = (\xi_i)^A{}_B Q^B_a + (i\zeta_i)^A{}_B (\gamma_5)_a{}^b Q^B_b \\
&\Big[U^{AB}, \text{anything}\Big] = \Big[V^{AB}, \text{anything}\Big] = 0
\end{aligned}$$

U^{AB} and V^{AB} being central charges and the matrices ξ_i, ζ_i having to satisfy $(\xi_i + i\zeta_i) + (\xi_i + i\zeta_i)^\dagger = 0$.

In fact, the number of central charges $U^{AB} = -U^{BA}$ and $V^{AB} = -V^{BA}$ present in the algebra imposes constraints on the symmetry group of the matrices ξ_i and ζ_i. If there is no central charge this symmetry group is $U(N)$, otherwise it is $USp(2N)$, compact form of $Sp(2N)$.

For more details, see refs. [3, 18, 25, 89, 96].

2.60 Supersymmetry algebra: representations

We will only consider the finite dimensional representations of the N-supersymmetry algebra ($\rightarrow$ 2.59). Since the translation generators P^μ commute with the supersymmetry generators Q^A_α and $\overline{Q}^A_{\dot{\alpha}}$, the representations of the N-supersymmetry algebra are labelled by the mass M if M^2 is the eigenvalue of the Casimir operator $P^2 = P^\mu P_\mu$.

If N_F is the fermion number operator, the states $|B\rangle$ such that $(-1)^{N_F}|B\rangle = |B\rangle$ are bosonic states while the states $|F\rangle$ such that $(-1)^{N_F}|F\rangle = -|F\rangle$ are fermionic ones. In a finite dimensional representation, one has $\mathrm{tr}(-1)^{N_F} = 0$, from which it follows that *the finite dimensional representations of the supersymmetry algebra contain an equal number of bosonic and fermionic states*.

For the massive representations ($M \neq 0$), the supersymmetry algebra in the rest frame, where $P^\mu = (M, 0, 0, 0)$, takes the form (with vanishing central charges)

$$\Big\{Q^A_\alpha, \overline{Q}^B_{\dot{\beta}}\Big\} = 2M\delta_{\alpha\dot{\beta}}\delta_{AB}$$

$$\left\{Q_\alpha^A, Q_\beta^B\right\} = \left\{\overline{Q}_{\dot\alpha}^A, \overline{Q}_{\dot\beta}^B\right\} = 0$$

with $A, B = 1, \ldots, N$.

The rescaled operators $a_\alpha^A = Q_\alpha^A/\sqrt{2M}$ and $(a_\alpha^A)^\dagger = \overline{Q}_{\dot\alpha}^A/\sqrt{2M}$ satisfy the Clifford algebra (→ 2.10) in $2N$ dimensions. The states of a representation can be arranged into spin multiplets of some ground state – or vacuum – $|\Omega\rangle$ of given spin s, annihilated by the a_α^A operators. The other states of the representation are given by

$$|a_{\alpha_1}^{A_1} \ldots a_{\alpha_n}^{A_n}\rangle = (a_{\alpha_1}^{A_1})^\dagger \ldots (a_{\alpha_n}^{A_n})^\dagger |\Omega\rangle$$

When the ground state $|\Omega\rangle$ has spin s, the maximal spin state has spin $s+\frac{1}{2}N$ and the minimal spin state has spin 0 if $s \le \frac{1}{2}N$ or $s - \frac{1}{2}N$ if $s \ge \frac{1}{2}N$.

When the ground state $|\Omega\rangle$ has spin zero, the total number of states is equal to 2^{2N} with 2^{2N-1} fermionic states (constructed with an odd number of $(a_\alpha^A)^\dagger$ operators) and 2^{2N-1} bosonic states (constructed with an even number of $(a_\alpha^A)^\dagger$ operators). The maximal spin is $\frac{1}{2}N$ and the minimal spin is 0.

In the case $N = 1$, when the ground state $|\Omega\rangle$ has spin j, the states of the multiplet have spins $(j, j+\frac{1}{2}, j-\frac{1}{2}, j)$. When the ground state $|\Omega\rangle$ has spin 0, the multiplet has two states of spin 0 and one state of spin $\frac{1}{2}$.
The following table gives the dimensions of the massive representations with ground states Ω_s (of spin s) for $N = 1, 2, 3, 4$.

$N = 1$

spin	Ω_0	$\Omega_{1/2}$	Ω_1	$\Omega_{3/2}$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1

$N = 4$

spin	Ω_0
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1

$N = 2$

spin	Ω_0	$\Omega_{1/2}$	Ω_1
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$\frac{3}{2}$		1	4
2			1

$N = 3$

spin	Ω_0	$\Omega_{1/2}$
0	14	14
$\frac{1}{2}$	14	20
1	6	15
$\frac{3}{2}$	1	6
2		1

We consider now the massless representations corresponding to $P^2 = 0$. In a reference frame where $P^\mu = (E, 0, 0, E)$, the supersymmetry algebra become

$$\left\{Q_\alpha^A, \overline{Q}_{\dot\beta}^B\right\} = 4E\delta^{AB}\delta_{\alpha\dot\beta,1\dot1}$$

$$\left\{Q^A_\alpha, Q^B_\beta\right\} = \left\{\overline{Q}^A_{\dot\alpha}, \overline{Q}^B_{\dot\beta}\right\} = 0$$

The rescaled operators $a^A = Q^A_1/\sqrt{4E}$ and $(a^A)^\dagger = \overline{Q}^A_{\dot 1}/\sqrt{4E}$ satisfy the Clifford algebra in N dimensions while the operators $a'^A = Q^A_2/\sqrt{4E}$ and $(a'^A)^\dagger = \overline{Q}^A_{\dot 2}/\sqrt{4E}$ mutually anticommute and act as zero on the representation states. A representation of the supersymmetry algebra is therefore characterized by a Clifford ground state $|\Omega\rangle$ labelled by the energy E and the helicity λ and annihilated by the a^A operators. The other states of the representation are given by

$$|a^{A_1} \dots a^{A_n}\rangle = (a^{A_1})^\dagger \dots (a^{A_n})^\dagger |\Omega\rangle$$

The number of states with helicity $\lambda + n$ with $0 \le n \le \frac{1}{2}N$ is $\binom{N}{2n}$. The total number of states is therefore 2^N with 2^{N-1} bosonic states and 2^{N-1} fermionic states.

For more details on the supersymmetry representations (in particular when the central charges are not zero), see refs. [3, 18, 89, 96].

2.61 Unitary superalgebras

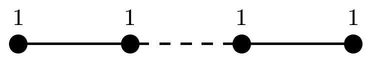
The superalgebras $A(m-1, n-1)$ with $m \neq n$

The unitary superalgebra $A(m-1,n-1)$ or $sl(m|n)$ with $m \neq n$ defined for $m > n \geq 0$ has as even part the Lie algebra $sl(m) \oplus sl(n) \oplus U(1)$ and as odd part the $(\overline{m}, n) + (m, \overline{n})$ representation of the even part; it has rank $m+n-1$ and dimension $(m+n)^2 - 1$. One has $A(m-1, n-1) \simeq A(n-1, m-1)$.

The root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ of $A(m-1, n-1)$ can be expressed in terms of the orthogonal vectors $\varepsilon_1, \dots, \varepsilon_m$ and $\delta_1, \dots, \delta_n$ such that $\varepsilon_i^2 = 1$ and $\delta_i^2 = -1$ as

$$\Delta_{\overline{0}} = \left\{ \varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j \right\} \quad \text{and} \quad \Delta_{\overline{1}} = \left\{ \varepsilon_i - \delta_j,\ -\varepsilon_i + \delta_j \right\}$$

The Dynkin diagrams of the unitary superalgebras $A(m-1, n-1)$ are of the following types:



where the small black dots represent either white dots (associated to even roots) or grey dots (associated to odd roots of zero length). The diagrams are drawn with their Dynkin labels which give the decomposition of the highest root in terms of the simple ones. The Dynkin diagrams of the unitary Lie superalgebras up to rank 4 are given in Table 3.61.

The superalgebra $A(m-1,n-1)$ can be generated as a matrix superalgebra by taking matrices of the form

$$M = \begin{pmatrix} X_{mm} & T_{mn} \\ T_{nm} & X_{nn} \end{pmatrix}$$

where X_{mm} and X_{nn} are $gl(m)$ and $gl(n)$ matrices, T_{mn} and T_{nm} are $m \times n$ and $n \times m$ matrices respectively, with the supertrace condition

$$\mathrm{str}(X) = \mathrm{tr}(X_{mm}) - \mathrm{tr}(X_{nn}) = 0$$

A basis of matrices can be constructed as follows. Consider $(m+n)^2$ elementary matrices e_{IJ} of order $m+n$ such that $(e_{IJ})_{KL} = \delta_{IL}\delta_{JK}$ $(I,J,K,L = 1,\dots,m+n)$ and define the $(m+n)^2 - 1$ generators

$$\begin{array}{ll} E_{ij} = e_{ij} - \frac{1}{m-n}\delta_{ij}\Big(\sum_{k=1}^{m} e_{kk} + \sum_{k'=m+1}^{m+n} e_{k'k'}\Big) & E_{ij'} = e_{ij'} \\ E_{i'j'} = e_{i'j'} + \frac{1}{m-n}\delta_{i'j'}\Big(\sum_{k=1}^{m} e_{kk} + \sum_{k'=m+1}^{m+n} e_{k'k'}\Big) & E_{i'j} = e_{i'j} \\ Y = -\frac{1}{m-n}\Big(n\sum_{k=1}^{m} e_{kk} + m\sum_{k'=m+1}^{m+n} e_{k'k'}\Big) & \end{array}$$

where the indices $i,j,\dots$ run from 1 to m and $i',j',\dots$ from $m+1$ to $m+n$. Then the generator Y generates the $U(1)$ part, the generators $E_{ij} - \frac{1}{m}\delta_{ij}Y$ generate the $sl(m)$ part and the generators $E_{i'j'} + \frac{1}{n}\delta_{i'j'}Y$ generate the $sl(n)$ part, while $E_{ij'}$ and $E_{i'j}$ transform as the $(\overline{m},n)$ and $(m,\overline{n})$ representations of $sl(m) \oplus sl(n)$ with $U(1)$ values $+1$ and -1 respectively. In all these expressions, summation over repeated indices is understood.

The generators in the Cartan–Weyl basis are given by:

- for the Cartan subalgebra

$$\begin{array}{l} H_i = E_{ii} - E_{i+1,i+1} \;\text{ with }\; 1 \le i \le m-1 \\ H_{i'} = E_{i'i'} - E_{i'+1,i'+1} \;\text{ with }\; m+1 \le i' \le m+n-1 \\ H_m = E_{mm} + E_{m+1,m+1} \end{array}$$

- for the raising operators

$$\begin{array}{l} E_{ij} \;\text{ with }\; i<j \;\text{ for }\; sl(m) \\ E_{i'j'} \;\text{ with }\; i'<j' \;\text{ for }\; sl(n) \\ E_{ij'} \;\text{ for the odd part} \end{array}$$

- for the lowering operators

$$\begin{array}{l} E_{ji} \;\text{ with }\; i<j \;\text{ for }\; sl(m) \\ E_{j'i'} \;\text{ with }\; i'<j' \;\text{ for }\; sl(n) \\ E_{i'j} \;\text{ for the odd part} \end{array}$$

The commutation relations in the Cartan–Weyl basis read as:

$$\Big[H_I, H_J\Big] = 0$$

$$\Big[H_K, E_{IJ}\Big] = \delta_{IK}E_{KJ} - \delta_{I,K+1}E_{K+1,J} - \delta_{KJ}E_{IK} + \delta_{K+1,J}E_{I,K+1}$$

$$\Big[H_m, E_{IJ}\Big] = \delta_{Im}E_{mJ} + \delta_{I,m+1}E_{m+1,J} - \delta_{mJ}E_{Im} - \delta_{m+1,J}E_{I,m+1}$$

$$\Big[E_{IJ}, E_{KL}\Big] = \delta_{JK}E_{IL} - \delta_{IL}E_{KJ} \quad \text{for } E_{IJ} \text{ and } E_{KL} \text{ even } (IJ \neq LK)$$

$$\Big[E_{IJ}, E_{KL}\Big] = \delta_{JK}E_{IL} - \delta_{IL}E_{KJ} \quad \text{for } E_{IJ} \text{ even and } E_{KL} \text{ odd}$$

$$\Big\{E_{IJ}, E_{KL}\Big\} = \delta_{JK}E_{IL} + \delta_{IL}E_{KJ} \quad \text{for } E_{IJ} \text{ and } E_{KL} \text{ odd } (IJ \neq LK)$$

$$\Big[E_{ij}, E_{ji}\Big] = \sum_{k=1}^{j-1} H_k \qquad \Big[E_{i'j'}, E_{j'i'}\Big] = \sum_{k'=m+1}^{j'-1} H_{k'}$$

$$\Big\{E_{ij'}, E_{j'i}\Big\} = \sum_{k=1}^{m} H_k - \sum_{k'=m+1}^{j'-1} H_{k'}$$

The superalgebras $A(n-1,n-1)$ with $n > 1$

The unitary superalgebra $A(n-1,n-1)$ or $psl(n|n)$ defined for $n > 1$ has as even part the Lie algebra $sl(n) \oplus sl(n)$ and as odd part the $(\overline{n}, n) + (n, \overline{n})$ representation of the even part; it has rank $2n-2$ and dimension $4n^2 - 2$. Note that the superalgebra $A(0,0)$ is not simple.

The root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$ of $A(n-1,n-1)$ can be expressed in terms of the orthogonal vectors $\varepsilon_1, \dots, \varepsilon_n$ and $\delta_1, \dots, \delta_n$ such that $\varepsilon_i^2 = 1$, $\delta_i^2 = -1$ and $\sum_{i=1}^n \varepsilon_i = \sum_{i=1}^n \delta_i$ as

$$\Delta_{\overline{0}} = \Big\{ \ \varepsilon_i - \varepsilon_j, \ \delta_i - \delta_j \ \Big\} \quad \text{and} \quad \Delta_{\overline{1}} = \Big\{ \ \varepsilon_i - \delta_j, \ -\varepsilon_i + \delta_j \ \Big\}$$

The Dynkin diagrams of the unitary superalgebras $A(n-1,n-1)$ are of the same type as those of the $A(m-1,n-1)$ case.

The superalgebra $A(n-1,n-1)$ can be generated as a matrix superalgebra by taking matrices of $sl(n|n)$. However, $sl(n|n)$ contains a one-dimensional ideal $\mathcal{I}$ generated by $\mathbb{I}_{2n}$ and one sets $A(n-1,n-1) \equiv sl(n|n)/\mathcal{I} = psl(n|n)$, hence the rank and dimension of $A(n-1,n-1)$.

It should be stressed that the rank of the superalgebra is $2n-2$ although the Dynkin diagram has $2n-1$ dots: the $2n-1$ associated simple roots are *not linearly independent* in that case.

Moreover, in the case of $A(1,1)$, one has the relations $\varepsilon_1 + \varepsilon_2 = \delta_1 + \delta_2$ from which it follows that there are only four distinct odd roots α such that $\dim \mathcal{G}_\alpha = 2$ and each odd root is *both positive and negative*.

2.62 Universal enveloping superalgebra

Definition

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a Lie superalgebra over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The definition of the universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$ is similar to the definition in the algebraic case. If $\mathcal{G}^{\otimes}$ is the tensor algebra over $\mathcal{G}$ with $\mathbb{Z}_2$-graded tensor product ($\rightarrow$ 2.51) and $\mathcal{I}$ the ideal of $\mathcal{G}$ generated by $[\![X,Y]\!] - (X \otimes Y - (-1)^{\deg X.\deg Y} Y \otimes X)$ where $X, Y \in \mathcal{G}$, the universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$ is the quotient $\mathcal{G}^{\otimes}/\mathcal{I}$.

Theorem (Poincaré–Birkhoff–Witt)

Let $b_1, \dots, b_B$ ($B = \dim \mathcal{G}_{\overline{0}}$) be a basis of the even part $\mathcal{G}_{\overline{0}}$ and $f_1, \dots, f_F$ ($F = \dim \mathcal{G}_{\overline{1}}$) be a basis of the odd part $\mathcal{G}_{\overline{1}}$. Then the elements

$$b_1^{i_1} \dots b_B^{i_B} f_1^{j_1} \dots f_F^{j_F} \quad \text{with} \quad i_1, \dots, i_B \geq 0 \ \text{and} \ j_1, \dots, j_F \in \{0,1\}$$

form a basis of the universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$, called the Poincaré–Birkhoff–Witt (PBW) basis.

The universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$ contains in general zero divisors (let us recall that $\mathcal{U}(\mathcal{G}_{\overline{0}})$ never contains zero divisors). In fact, if $F \in \mathcal{G}_{\overline{1}}$ is a generator associated to an isotropic root, one has $F^2 = \{F,F\} = 0$ in $\mathcal{U}(\mathcal{G})$. More precisely, one has the following property:

Property

The universal enveloping superalgebra $\mathcal{U}(\mathcal{G})$ does not contain any zero divisors if and only if $\mathcal{G} = ops(1|2n)$. In that case, $\mathcal{U}(\mathcal{G})$ is said to be *entire*.

Filtration of $\mathcal{G}$: $\mathcal{U}(\mathcal{G})$ can be naturally filtered as follows. Let $\mathcal{U}_n$ be the subspace of $\mathcal{U}(\mathcal{G})$ generated by the PBW-basis monomials of degree $\leq n$ (e.g. $\mathcal{U}_0 = \mathbb{K}$ and $\mathcal{U}_1 = \mathbb{K} + \mathcal{G}$). Then one has the following filtration, with $\mathcal{U}_i\, \mathcal{U}_j \subset \mathcal{U}_{i+j}$:

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \dots \subset \mathcal{U}_n \subset \dots \subset \mathcal{U}(\mathcal{G}) = \bigcup^{\infty} \mathcal{U}_n$$

Defining the quotient subspaces $\overline{\mathcal{U}}_0 = \mathcal{U}_0$ and $\overline{\mathcal{U}}_i = \mathcal{U}_i/\mathcal{U}_{i-1}$ for $i \geq 1$, one can associate to $\mathcal{U}(\mathcal{G})$ the following graded algebra $\mathrm{Gr}(\mathcal{U}(\mathcal{G}))$:

$$\mathrm{Gr}(\mathcal{U}(\mathcal{G})) = \overline{\mathcal{U}}_0 \oplus \overline{\mathcal{U}}_1 \oplus \dots \oplus \overline{\mathcal{U}}_n \oplus \dots$$

Then, one can show that

$$\mathrm{Gr}(\mathcal{U}(\mathcal{G})) \simeq \mathbb{K}[b_1, \dots, b_B] \otimes \Lambda(f_1, \dots, f_F)$$

where $\mathbb{K}[b_1, \dots, b_B]$ is the ring of polynomials in the indeterminates $b_1, \dots, b_B$ with coefficients in $\mathbb{K}$ and $\Lambda(f_1, \dots, f_F)$ is the exterior algebra over $\mathcal{G}$.

For more details, see ref. [47].

2.63 Weyl group

Let $\mathcal{G} = \mathcal{G}_{\overline{0}} \oplus \mathcal{G}_{\overline{1}}$ be a classical Lie superalgebra with root system $\Delta = \Delta_{\overline{0}} \cup \Delta_{\overline{1}}$. $\Delta_{\overline{0}}$ is the set of even roots and $\Delta_{\overline{1}}$ the set of odd roots. The Weyl group $W(\mathcal{G})$ of $\mathcal{G}$ is generated by the Weyl reflections w with respect to the even roots:

$$w_\alpha(\beta) = \beta - 2\,\frac{\alpha \cdot \beta}{\alpha \cdot \alpha}\,\alpha$$

where $\alpha \in \Delta_{\overline{0}}$ and $\beta \in \Delta$.

The properties of the Weyl group are the following.

Properties

1. The Weyl group $W(\mathcal{G})$ leaves Δ, $\Delta_{\overline{0}}$, $\Delta_{\overline{1}}$, $\overline{\Delta}_{\overline{0}}$, $\overline{\Delta}_{\overline{1}}$ invariant, where Δ, $\Delta_{\overline{0}}$, $\Delta_{\overline{1}}$ are defined above, $\overline{\Delta}_{\overline{0}}$ is the subset of roots $\alpha \in \Delta_{\overline{0}}$ such that $\alpha/2 \notin \Delta_{\overline{1}}$ and $\overline{\Delta}_{\overline{1}}$ is the subset of roots $\alpha \in \Delta_{\overline{1}}$ such that $2\alpha \notin \Delta_{\overline{0}}$.

2. Let e^λ be the formal exponential, function on $\mathcal{H}^*$ such that $e^\lambda(\mu) = \delta_{\lambda,\mu}$ for two elements $\lambda, \mu \in \mathcal{H}^*$, which satisfies $e^\lambda e^\mu = e^{\lambda+\mu}$. One defines

$$L = \frac{\prod_{\alpha \in \Delta_{\overline{0}}^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in \Delta_{\overline{1}}^+} (e^{\alpha/2} + e^{-\alpha/2})} \qquad \text{and} \qquad L' = \frac{\prod_{\alpha \in \overline{\Delta}_{\overline{0}}^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in \Delta_{\overline{1}}^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

where $\Delta_{\overline{0}}^+$ and $\Delta_{\overline{1}}^+$ are the sets of positive even roots and positive odd roots respectively. Then one has

$$w(L) = \varepsilon(w) L \quad \text{and} \quad w(L') = \varepsilon'(w) L' \quad \text{where} \quad w \in W(\mathcal{G})$$

with $\varepsilon(w) = (-1)^{\ell(w)}$ and $\varepsilon'(w) = (-1)^{\ell'(w)}$ where $\ell(w)$ is the number of reflections in the expression of $w \in W(\mathcal{G})$ and $\ell'(w)$ is the number of reflections with respect to the roots of $\overline{\Delta}_{\overline{0}}^+$ in the expression of $w \in W(\mathcal{G})$.

For the basic Lie superalgebras, one can extend the Weyl group $W(\mathcal{G})$ to a larger group by adding the following transformations (called generalized Weyl transformations) associated to the odd roots of $\mathcal{G}$ [23, 52]. For $\alpha \in \Delta_{\bar{1}}$, one defines:

$$
\begin{aligned}
w_\alpha(\beta) &= \beta - 2\,\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}\,\alpha && \text{if } \alpha\cdot\alpha \neq 0 \\
w_\alpha(\beta) &= \beta + \alpha && \text{if } \alpha\cdot\alpha = 0 \text{ and } \alpha\cdot\beta \neq 0 \\
w_\alpha(\beta) &= \beta && \text{if } \alpha\cdot\alpha = 0 \text{ and } \alpha\cdot\beta = 0 \\
w_\alpha(\alpha) &= -\alpha
\end{aligned}
$$

Notice that the transformation associated to an odd root α of zero length cannot be lifted to an automorphism of the superalgebra since w_α transforms even roots into odd ones, and vice versa, and the $\mathbb{Z}_2$-gradation would not be respected.

For more details, see ref. [47].

Part 3

Tables

Tables on Lie algebras

Table 3.1: The simple Lie algebra $A_{N-1} = sl(N)$.

Rank: $N-1$, dimension: N^2-1.
Root system ($1 \le i \ne j \le N$):

$$\Delta = \{\varepsilon_i - \varepsilon_j\}$$

Simple root system:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N$$

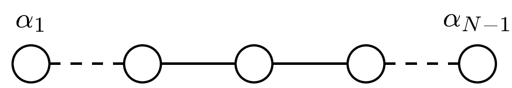
Positive roots ($1 \le i < j \le N$):

$$\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$$

Sum of positive roots:

$$2\rho \ = \ \sum_{i=1}^{N} (N-2i+1)\varepsilon_i \ = \ \sum_{i=1}^{N-1} i(N-i)\alpha_i$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Fundamental weights ($1 \le i \le N-1$):

$$\Lambda_i \ = \ \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{N}\sum_{j=1}^{N}\varepsilon_j \ = \ \frac{1}{N}\sum_{j=1}^{N-1} \min(i,j)\ \min(N-i, N-j)\,\alpha_j$$

(continued)

Table 3.1 (continued)

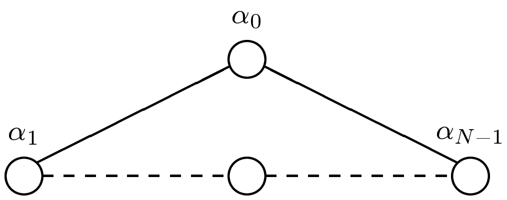
Level vector: $L = (N, 2(N-1), 3(N-2), \dots, (N-1)2, N)$
Weyl group: $W = \mathfrak{S}_N$ (permutation group), $\dim W = N!$.
Root lattice Q, weight lattice P:

$$Q = \sum_{i=1}^{N} x_i \varepsilon_i \ \text{ where } \ \sum_i x_i = 0\,, \qquad P = Q \cup \Lambda_1 \mathbb{Z}\,, \qquad P/Q \simeq \mathbb{Z}/N\mathbb{Z}$$

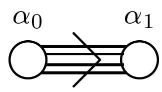
Highest root: $-\alpha_0 \ = \ \alpha_1 + \cdots + \alpha_N \ = \ \varepsilon_1 - \varepsilon_N$.
Coxeter number $h = N$, dual Coxeter number $h^\vee = N$.
Quadratic matrix form G:

$$G = \frac{1}{N} \begin{pmatrix} 1.(N-1) & 1.(N-2) & 1.(N-3) & \cdots & 1.2 & 1.1 \\ 1.(N-2) & 2.(N-2) & 2.(N-3) & \cdots & 2.2 & 2.1 \\ 1.(N-3) & 2.(N-3) & 3.(N-3) & \cdots & 3.2 & 3.1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1.2 & 2.2 & 3.2 & \cdots & (N-2).2 & (N-2).1 \\ 1.1 & 2.1 & 3.1 & \cdots & (N-2).2 & (N-1).1 \end{pmatrix}$$

Extended Dynkin diagram:



$N \geq 3$



$N = 2$

Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{Z}_2$.

Table 3.2: The simple Lie algebra $B_N = so(2N+1)$.

Rank: N, dimension: $N(2N+1)$.
Root system $(1 \le i \ne j \le N)$:

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i\}\,, \qquad \Delta_L = \{\pm\varepsilon_i \pm \varepsilon_j\}\,, \qquad \Delta_S = \{\pm\varepsilon_i\}$$

$\dim\Delta = 2N^2$, $\dim\Delta_L = 2N(N-1)$, $\dim\Delta_S = 2N$.
Simple root system:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N,\ \alpha_N = \varepsilon_N$$

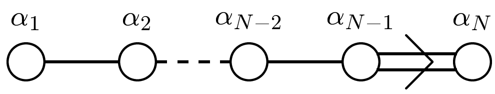
Positive roots $(1 \le i < j \le N)$:

$$\begin{aligned} \varepsilon_i - \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} \\ \varepsilon_i + \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_N) \\ \varepsilon_i &= \alpha_i + \cdots + \alpha_N \end{aligned}$$

Sum of positive roots:

$$2\rho = \sum_{i=1}^{N} (2N - 2i + 1)\varepsilon_i = \sum_{i=1}^{N} i(2N - i)\alpha_i$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}$$

(continued)

Table 3.2 (continued)

Fundamental weights:

$$\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i = \sum_{j=1}^{N} \min(i,j)\, \alpha_j \quad (1 \le i \le N-1)$$

$$\Lambda_N = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_N) = \tfrac{1}{2}\sum_{j=1}^{N} j\alpha_j$$

Level vector: $L = (2N, 2(2N-1), 3(2N-2), \ldots, (N-1)(N+2), \frac{1}{2}N(N+1))$
Weyl group: $W = \mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N$, $\dim W = 2^N N!$.
Root lattice Q, weight lattice P:

$$Q = \sum_{i=1}^{N} x_i\varepsilon_i \quad \text{where all } x_i \in \mathbb{Z}$$

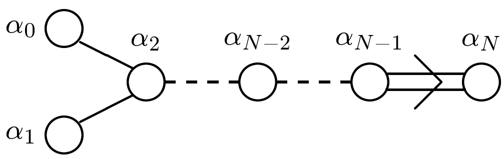
$$P = \sum_{i=1}^{N} x_i\varepsilon_i \quad \text{where all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2}$$

$$P/Q \simeq \mathbb{Z}/2\mathbb{Z}$$

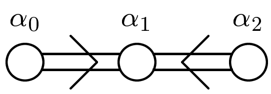
Highest root: $-\alpha_0 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_N = \varepsilon_1 + \varepsilon_2$.
Coxeter number $h = 2N$, dual Coxeter number $h^\vee = 2N-1$.
Quadratic matrix form G:

$$G = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(N-1) & N-1 \\ 1 & 2 & 3 & \cdots & N-1 & N/2 \end{pmatrix}$$

Extended Dynkin diagram:



$N \ge 3$



$N = 2$

Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.3: The simple Lie algebra $C_N = sp(2N)$.

Rank: N, dimension: $N(2N+1)$.
Root system ($1 \le i \ne j \le N$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i\}\,, \qquad \Delta_L = \{\pm 2\varepsilon_i\}\,, \qquad \Delta_S = \{\pm\varepsilon_i \pm \varepsilon_j\}$$

$\dim\Delta = 2N^2$, $\dim\Delta_L = 2N$, $\dim\Delta_S = 2N(N-1)$.
Simple root system:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N,\ \alpha_N = 2\varepsilon_N$$

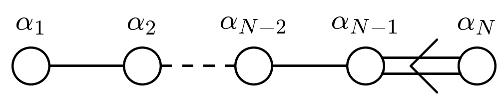
Positive roots ($1 \le i < j \le N$):

$$\begin{array}{rcl}
\varepsilon_i - \varepsilon_j &=& \alpha_i + \cdots + \alpha_{j-1} \\
\varepsilon_i + \varepsilon_j &=& \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{N-1}) + \alpha_N \quad (j \ne N) \\
\varepsilon_i + \varepsilon_N &=& \alpha_i + \cdots + \alpha_N \\
2\varepsilon_i &=& 2(\alpha_i + \cdots + \alpha_{N-1}) + \alpha_N \quad (i \ne N) \\
2\varepsilon_N &=& \alpha_N
\end{array}$$

Sum of positive roots:

$$2\rho \;=\; \sum_{i=1}^{N} (2N - 2i + 2)\varepsilon_i \;=\; \sum_{i=1}^{N-1} i(2N - i + 1)\alpha_i + \tfrac{1}{2}N(N+1)\alpha_N$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 & -1 & 0 \\
\vdots & & \ddots & -1 & 2 & -2 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}$$

(continued)

Table 3.3 (continued)

Fundamental weights ($1 \le i \le N$):

$$\Lambda_i \;=\; \varepsilon_1 + \cdots + \varepsilon_i \;=\; \alpha_1 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{N-1} + \tfrac{1}{2}\alpha_N)$$

Level vector: $L = (2N-1, 2(2N-2), 3(2N-3), \ldots, (N-1)(N+1), N^2)$
Weyl group: $W = \mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N$, $\dim W = 2^N N!$.
Root lattice Q, weight lattice P:

$$Q = \sum_{i=1}^{N} x_i \varepsilon_i \quad \text{where } \textstyle\sum_i x_i \in 2\mathbb{Z}$$

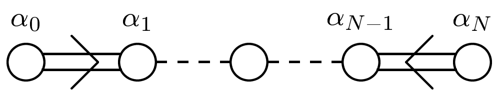
$$P = \sum_{i=1}^{N} x_i \varepsilon_i \quad \text{where all } x_i \in \mathbb{Z}$$

$$P/Q \simeq \mathbb{Z}/2\mathbb{Z}$$

Highest root: $-\alpha_0 \;=\; 2\alpha_1 + \cdots + 2\alpha_{N-1} + \alpha_N \;=\; 2\varepsilon_1$.
Coxeter number $h = 2N$, dual Coxeter number $h^\vee = N+1$.
Quadratic matrix form G:

$$G = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & N-1 & N-1 \\ 1 & 2 & 3 & \cdots & N-1 & N \end{pmatrix}$$

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.4: The simple Lie algebra $D_N = so(2N)$.

Rank: N, dimension: $N(2N-1)$.
Root system ($1 \leq i \neq j \leq N$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j\}$$

Simple root system:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \; \ldots, \; \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N, \; \alpha_N = \varepsilon_{N-1} + \varepsilon_N$$

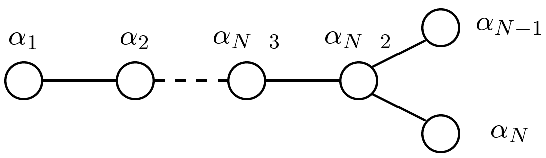
Positive roots ($1 \leq i < j \leq N$):

$$\begin{aligned}
&\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} \\
&\varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N \quad (j \leq N-2) \\
&\varepsilon_i + \varepsilon_{N-1} = \alpha_i + \cdots + \alpha_{N-1} \\
&\varepsilon_i + \varepsilon_N = \alpha_i + \cdots + \alpha_N
\end{aligned}$$

Sum of positive roots:

$$2\rho \;=\; \sum_{i=1}^{N}(2N-2i)\varepsilon_i \;=\; \sum_{i=1}^{N-2} i(N-i-1)\alpha_i + \tfrac{1}{2}N(N-1)(\alpha_{N-1}+\alpha_N)$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & 2 & -1 & -1 \\
\vdots & & 0 & -1 & 2 & 0 \\
0 & \cdots & 0 & -1 & 0 & 2
\end{pmatrix}$$

(continued)

Table 3.4 (continued)

Fundamental weights:

$$\begin{aligned}
\Lambda_i &= \varepsilon_1 + \cdots + \varepsilon_i \\
&= \sum_{j=1}^{N-2} \min(i,j)\,\alpha_j + \tfrac{1}{2} i(\alpha_{N-1} + \alpha_N) \quad (1 \le i \le N-2) \\
\Lambda_{N-1} &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{N-1} - \varepsilon_N) \\
&= \tfrac{1}{2} \sum_{j=1}^{N-2} j\,\alpha_j + \tfrac{1}{4} N \alpha_{N-1} + \tfrac{1}{4}(N-2)\alpha_N \\
\Lambda_N &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{N-1} + \varepsilon_N) \\
&= \tfrac{1}{2} \sum_{j=1}^{N-2} j\,\alpha_j + \tfrac{1}{4}(N-2)\alpha_{N-1} + \tfrac{1}{4} N \alpha_N
\end{aligned}$$

Level vector: $L = (2N-2, 2(2N-3), 3(2N-4), \ldots, (N-2)(N+1), \frac{1}{2}\,N(N-1), \frac{1}{2}\,N(N-1))$

Weyl group: $W = \mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^{N-1}$, $\dim W = 2^{N-1} N!$.

Root lattice Q, weight lattice P:

$$Q = \sum_{i=1}^{N} x_i \varepsilon_i \quad \text{where } \textstyle\sum_i x_i \in 2\mathbb{Z}$$

$$P = \sum_{i=1}^{N} x_i \varepsilon_i \quad \text{where all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2}$$

$$P/Q \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{for } N \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{for } N \text{ even} \end{cases}$$

Highest root: $-\alpha_0 \;=\; \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \alpha_N \;=\; \varepsilon_1 + \varepsilon_2$.

Coxeter number $h = 2N-2$, dual Coxeter number $h^\vee = 2N-2$.

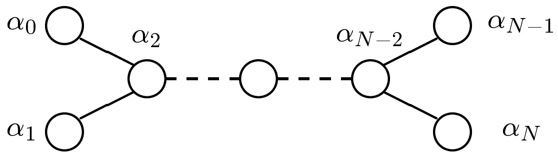
Quadratic matrix form G:

$$G = \frac{1}{2} \begin{pmatrix}
2 & 2 & 2 & \cdots & 2 & 1 & 1 \\
2 & 4 & 4 & \cdots & 4 & 2 & 2 \\
2 & 4 & 6 & \cdots & 6 & 3 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & 4 & 6 & \cdots & 2(N-2) & N-2 & N-2 \\
1 & 2 & 3 & \cdots & N-2 & N/2 & (N-2)/2 \\
1 & 2 & 3 & \cdots & N-2 & (N-2)/2 & N/2
\end{pmatrix}$$

(continued)

Table 3.4 (continued)

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$:

$$\mathrm{Out}(\mathcal{G}) = \begin{cases} \mathfrak{S}_3 & \text{for } N = 4 \\ \mathbb{Z}_2 & \text{for } N \geq 5 \end{cases}$$

Table 3.5: The simple Lie algebra E_6.

Rank: 6, dimension: 78.
Root system ($1 \leq i \neq j \leq 5$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\tfrac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$$

The total number of $+$ signs (or $-$ signs) is even in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$.
Simple root system:

$$\alpha_1 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \ldots,\ \alpha_5 = \varepsilon_5 - \varepsilon_4,\ \alpha_6 = \varepsilon_1 + \varepsilon_2$$

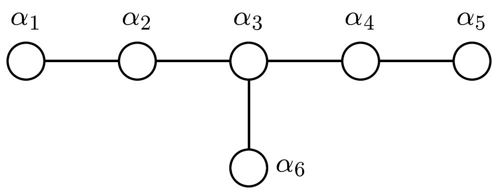
Positive roots ($1 \leq j < i \leq 5$):

$$\varepsilon_i - \varepsilon_j\,, \quad \varepsilon_i + \varepsilon_j\,, \quad \tfrac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{j=1}^{5} \pm\varepsilon_j)$$

Sum of positive roots:

$$\begin{aligned} 2\rho &= 2\varepsilon_2 + 4\varepsilon_3 + 6\varepsilon_4 + 8\varepsilon_5 + 8(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\ &= 16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 30\alpha_4 + 16\alpha_5 + 22\alpha_6 \end{aligned}$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

(continued)

Table 3.5 (continued)

Fundamental weights:

$$
\begin{aligned}
\Lambda_1 &= \tfrac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
&= \tfrac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6) \\
\Lambda_2 &= \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \tfrac{5}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
&= \tfrac{1}{3}(5\alpha_1 + 10\alpha_2 + 12\alpha_3 + 8\alpha_4 + 4\alpha_5 + 6\alpha_6) \\
\Lambda_3 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_8 - \varepsilon_7 - \varepsilon_6 \\
&= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 \\
\Lambda_4 &= \varepsilon_4 + \varepsilon_5 + \tfrac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
&= \tfrac{1}{3}(4\alpha_1 + 8\alpha_2 + 10\alpha_3 + 12\alpha_4 + 5\alpha_5 + 6\alpha_6) \\
\Lambda_5 &= \varepsilon_5 + \tfrac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
&= \tfrac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6) \\
\Lambda_6 &= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\
&= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6
\end{aligned}
$$

Level vector: $L = (16, 30, 42, 30, 16, 22)$
Weyl group: $\dim W = 2^7.3^4.5 = 51\,840$.
Root lattice Q, weight lattice P:

$Q = Q(E_8) \cap \mathcal{V}_6$

($\mathcal{V}_6$ hyperplane in $\mathbb{R}^8$ orthogonal to $\varepsilon_7 + \varepsilon_8$ and $\varepsilon_6 + \varepsilon_7 + 2\varepsilon_8$)

$P = Q(E_8) \cup \left(\varepsilon_5 + \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)\right)\mathbb{Z}$

$P/Q \simeq \mathbb{Z}/3\mathbb{Z}$

Highest root:

$$
\begin{aligned}
-\alpha_0 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \\
&= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)
\end{aligned}
$$

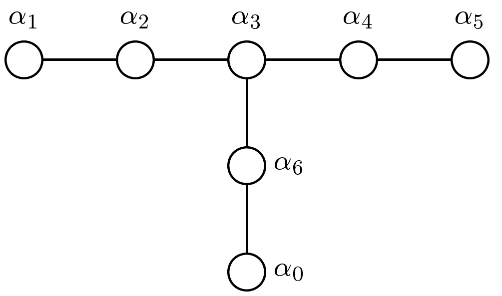
Coxeter number $h = 12$, dual Coxeter number $h^\vee = 12$.
Quadratic matrix form G:

$$
G = \frac{1}{3}\begin{pmatrix}
4 & 5 & 6 & 4 & 2 & 3 \\
5 & 10 & 12 & 8 & 4 & 6 \\
6 & 12 & 18 & 12 & 6 & 9 \\
4 & 8 & 12 & 10 & 5 & 6 \\
2 & 4 & 6 & 5 & 4 & 3 \\
3 & 6 & 9 & 6 & 3 & 6
\end{pmatrix}
$$

(continued)

Table 3.5 (continued)

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.6: The simple Lie algebra E_7.

Rank: 7, dimension: 133.
Root system ($1 \leq i \neq j \leq 6$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm(\varepsilon_8 - \varepsilon_7),\ \pm\tfrac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$$

The total number of $+$ signs (or $-$ signs) is even in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$.
Simple root system:

$$\alpha_1 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \ldots,\ \alpha_6 = \varepsilon_6 - \varepsilon_5,\ \alpha_7 = \varepsilon_1 + \varepsilon_2$$

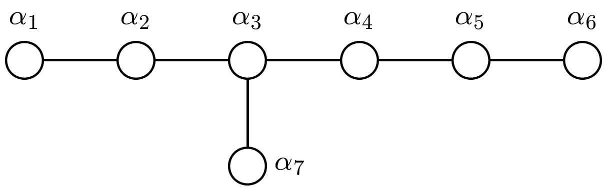
Positive roots ($1 \leq j < i \leq 6$):

$$\varepsilon_i - \varepsilon_j\,, \quad \varepsilon_i + \varepsilon_j\,, \quad \varepsilon_8 - \varepsilon_7\,, \quad \tfrac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{j=1}^{6} \pm\varepsilon_j)$$

Sum of positive roots:

$$\begin{aligned} 2\rho &= 2\varepsilon_2 + 4\varepsilon_3 + 6\varepsilon_4 + 8\varepsilon_5 + 10\varepsilon_6 + 17(\varepsilon_8 - \varepsilon_7) \\ &= 34\alpha_1 + 66\alpha_2 + 96\alpha_3 + 75\alpha_4 + 52\alpha_5 + 27\alpha_6 + 49\alpha_7 \end{aligned}$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

(continued)

Table 3.6 (continued)

Fundamental weights:

$$\begin{aligned}
\Lambda_1 &= \varepsilon_8 - \varepsilon_7 \\
&= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 \\
\Lambda_2 &= \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6) + \tfrac{3}{2}(\varepsilon_8 - \varepsilon_7) \\
&= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 4\alpha_7 \\
\Lambda_3 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + 2(\varepsilon_8 - \varepsilon_7) \\
&= 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 6\alpha_7 \\
\Lambda_4 &= \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \tfrac{3}{2}(\varepsilon_8 - \varepsilon_7) \\
&= \tfrac{1}{2}(6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 15\alpha_4 + 10\alpha_5 + 5\alpha_6 + 9\alpha_7) \\
\Lambda_5 &= \varepsilon_5 + \varepsilon_6 + \varepsilon_8 - \varepsilon_7 \\
&= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 2\alpha_6 + 3\alpha_7 \\
\Lambda_6 &= \varepsilon_6 + \tfrac{1}{2}(\varepsilon_8 - \varepsilon_7) \\
&= \tfrac{1}{2}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 3\alpha_7) \\
\Lambda_7 &= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6) + \varepsilon_8 - \varepsilon_7 \\
&= \tfrac{1}{2}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 7\alpha_7)
\end{aligned}$$

Level vector: $L = (34, 66, 96, 75, 52, 27, 49)$
Weyl group: $\dim W = 2^{10}.3^4.5.7 = 2\,903\,040$.
Root lattice Q, weight lattice P:

$$\begin{aligned}
&Q = Q(E_8) \cap \mathcal{V}_7\,, \quad \mathcal{V}_7 \text{ hyperplane in } \mathbb{R}^8 \text{ orthogonal to } \varepsilon_7 + \varepsilon_8 \\
&P = Q(E_8) \cup \left(\varepsilon_6 + \tfrac{1}{2}\left(\varepsilon_8 - \varepsilon_7\right)\right)\mathbb{Z} \\
&P/Q \simeq \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

Highest root:

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 = \varepsilon_8 - \varepsilon_7$$

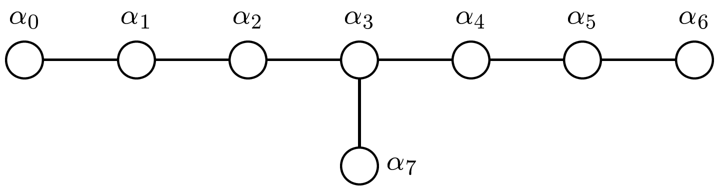
Coxeter number $h = 18$, dual Coxeter number $h^\vee = 18$.
Quadratic matrix form G:

$$G = \frac{1}{2}\begin{pmatrix}
4 & 6 & 8 & 6 & 4 & 2 & 4 \\
6 & 12 & 16 & 12 & 8 & 4 & 8 \\
8 & 16 & 24 & 18 & 12 & 6 & 12 \\
6 & 12 & 18 & 15 & 10 & 5 & 9 \\
4 & 8 & 12 & 10 & 8 & 4 & 6 \\
2 & 4 & 6 & 5 & 4 & 3 & 3 \\
4 & 8 & 12 & 9 & 6 & 3 & 7
\end{pmatrix}$$

(continued)

Table 3.6 (continued)

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.7: The simple Lie algebra E_8.

Rank: 8, dimension: 248.
Root system ($1 \leq i \neq j \leq 8$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \tfrac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)\}$$

The total number of + signs (or − signs) is even in $\frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8)$.
Simple root system:

$$\alpha_1 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_8 - \sum_{j=2}^{7} \varepsilon_j),\ \alpha_2 = \varepsilon_2 - \varepsilon_1,\ \ldots,\ \alpha_7 = \varepsilon_7 - \varepsilon_6,\ \alpha_8 = \varepsilon_1 + \varepsilon_2$$

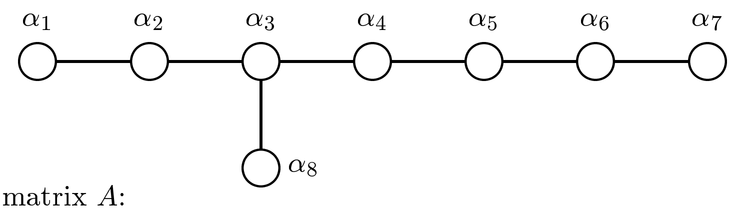
Positive roots ($1 \leq j < i \leq 8$):

$$\varepsilon_i - \varepsilon_j\,, \quad \varepsilon_i + \varepsilon_j\,, \quad \tfrac{1}{2}(\varepsilon_8 + \sum_{j=1}^{7} \pm\varepsilon_j)$$

Sum of positive roots:

$$\begin{aligned} 2\rho &= 2\varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8 \\ &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \end{aligned}$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

(continued)

Table 3.7 (continued)

Fundamental weights:

$$\begin{array}{lcl}
\Lambda_1 & = & 2\varepsilon_8 \;=\; 4\alpha_1 + 7\alpha_2 + 10\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 5\alpha_8 \\
\Lambda_2 & = & \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 7\varepsilon_8) \\
 & = & 7\alpha_1 + 14\alpha_2 + 20\alpha_3 + 16\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7 + 10\alpha_8 \\
\Lambda_3 & = & \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8 \\
 & = & 10\alpha_1 + 20\alpha_2 + 30\alpha_3 + 24\alpha_4 + 18\alpha_5 + 12\alpha_6 + 6\alpha_7 + 15\alpha_8 \\
\Lambda_4 & = & \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 4\varepsilon_8 \\
 & = & 8\alpha_1 + 16\alpha_2 + 24\alpha_3 + 20\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7 + 12\alpha_8 \\
\Lambda_5 & = & \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 3\varepsilon_8 \\
 & = & 6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 15\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7 + 9\alpha_8 \\
\Lambda_6 & = & \varepsilon_6 + \varepsilon_7 + 2\varepsilon_8 \\
 & = & 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 3\alpha_7 + 6\alpha_8 \\
\Lambda_7 & = & \varepsilon_7 + \varepsilon_8 \\
 & = & 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8 \\
\Lambda_8 & = & \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8) \\
 & = & 5\alpha_1 + 10\alpha_2 + 15\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7 + 8\alpha_8
\end{array}$$

Level vector: $L = (92, 182, 270, 220, 168, 114, 58, 136)$
Weyl group: $\dim W = 2^{14}.3^5.5^2.7 = 696\,729\,600$.
Root lattice Q, weight lattice P:

$$Q = P = \sum_{i=1}^{N} x_i\varepsilon_i \quad \text{where } 2x_i \in \mathbb{Z}, x_i - x_j \in \mathbb{Z}, \sum_i x_i \in 2\mathbb{Z}.$$

Highest root:

$$-\alpha_0 \;=\; 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8 \;=\; \varepsilon_8 + \varepsilon_7$$

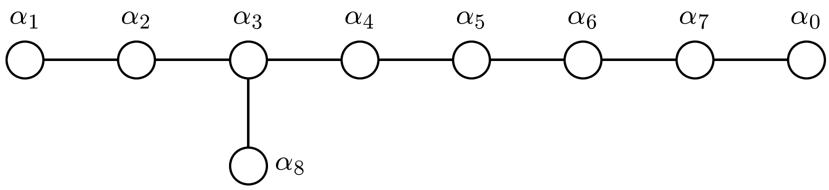
Coxeter number $h = 30$, dual Coxeter number $h^\vee = 30$.
Quadratic matrix form G:

$$G = \begin{pmatrix}
4 & 7 & 10 & 8 & 6 & 4 & 2 & 5 \\
7 & 14 & 20 & 16 & 12 & 8 & 4 & 10 \\
10 & 20 & 30 & 24 & 18 & 12 & 6 & 15 \\
8 & 16 & 24 & 20 & 15 & 10 & 5 & 12 \\
6 & 12 & 18 & 15 & 12 & 8 & 4 & 9 \\
4 & 8 & 12 & 10 & 8 & 6 & 3 & 6 \\
2 & 4 & 6 & 5 & 4 & 3 & 2 & 3 \\
5 & 10 & 15 & 12 & 9 & 6 & 3 & 8
\end{pmatrix}$$

(continued)

Table 3.7 (continued)

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.8: The simple Lie algebra F_4.

Rank: 4, dimension: 52.
Root system ($1 \leq i \neq j \leq 4$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$$
$$\Delta_L = \{\pm\varepsilon_i \pm \varepsilon_j\}\,, \qquad \Delta_S = \{\pm\varepsilon_i,\ \tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$$

$\dim\Delta = 48$, $\dim\Delta_L = 24$, $\dim\Delta_S = 24$.
Simple root system:

$$\alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = \varepsilon_4,\ \alpha_4 = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$

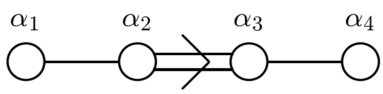
Positive roots ($1 \leq i < j \leq 4$):

$$\varepsilon_i - \varepsilon_j\,, \quad \varepsilon_i + \varepsilon_j\,, \quad \varepsilon_i\,, \quad \tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$$

Sum of positive roots:

$$2\rho \ = \ 11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4 \ = \ 16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4$$

Dynkin diagram:



Cartan matrix A:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Fundamental weights:

$$\begin{aligned}
\Lambda_1 &= \varepsilon_1 + \varepsilon_2 \ = \ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \\
\Lambda_2 &= 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \ = \ 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4 \\
\Lambda_3 &= \tfrac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \ = \ 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 \\
\Lambda_4 &= \varepsilon_1 \ = \ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4
\end{aligned}$$

Level vector: $L = (22, 42, 30, 16)$
Weyl group: $W = \mathfrak{S}_3 \ltimes (\mathfrak{S}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^2)$, $\dim W = 2^7.3^2 = 1\,152$.
Root lattice Q, weight lattice P:

$$Q = P = \sum_{i=1}^{N} x_i\varepsilon_i \quad \text{where all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2}$$

(continued)

Table 3.8 (continued)

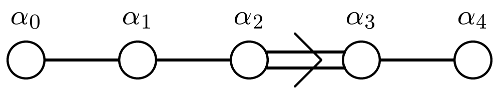
Highest root:

$$-\alpha_0 \;=\; 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \;=\; \varepsilon_1 + \varepsilon_2$$

Coxeter number $h = 12$, dual Coxeter number $h^\vee = 9$.
Quadratic matrix form G:

$$G = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} & 1 \end{pmatrix}$$

Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.9: The simple Lie algebra G_2.

Rank: 2, dimension: 14.
Root system ($1 \leq i \neq j \neq k \leq 3$):

$\Delta = \{\varepsilon_i - \varepsilon_j,\ \varepsilon_i + \varepsilon_j - 2\varepsilon_k\}$, $\Delta_L = \{\varepsilon_i - \varepsilon_j\}$, $\Delta_S = \{\varepsilon_i + \varepsilon_j - 2\varepsilon_k\}$

$\dim \Delta = 12$, $\dim \Delta_L = 6$, $\dim \Delta_S = 6$.
Simple root system:

$$\alpha_1 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1,\ \alpha_2 = \varepsilon_1 - \varepsilon_2$$

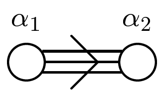
Positive roots:

$$\begin{aligned} &\varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 = \alpha_1,\ \varepsilon_1 - \varepsilon_2 = \alpha_2,\ \varepsilon_3 - \varepsilon_1 = \alpha_1 + \alpha_2, \\ &\varepsilon_3 - \varepsilon_2 = \alpha_1 + 2\alpha_2,\ \varepsilon_1 + \varepsilon_3 - 2\varepsilon_2 = \alpha_1 + 3\alpha_2, \\ &-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 = 2\alpha_1 + 3\alpha_2 \end{aligned}$$

Sum of positive roots:

$$2\rho \ = \ -2\varepsilon_1 - 4\varepsilon_2 + 6\varepsilon_3 \ = \ 6\alpha_1 + 10\alpha_2$$

Dynkin diagram:



Cartan matrix A and Quadratic matric form G:

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \qquad G = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

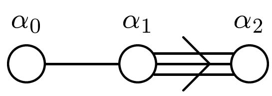
Fundamental weights:

$$\begin{aligned} \Lambda_1 &= -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 \ = \ 2\alpha_1 + 3\alpha_2 \\ \Lambda_2 &= \varepsilon_3 - \varepsilon_2 \ = \ \alpha_1 + 2\alpha_2 \end{aligned}$$

Level vector: $L = (10, 6)$
Weyl group: W is the dihedral group, $\dim W = 12$.
Root lattice Q, weight lattice P:

$$Q = P = \Lambda[\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3]$$

Highest root: $-\alpha_0 \ = \ 2\alpha_1 + 3\alpha_2 \ = \ -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3$
Coxeter number $h = 6$, dual Coxeter number $h^\vee = 4$.
Extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.10: Dimensions of $SU(n)$ irreducible representations.

YT	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$	$SU(8)$
[1]	3	4	5	6	8
[2]	6	10	15	21	36
[2, 1]	8	20	40	70	168
[3]	10	20	35	56	120
[3, 1]	15	45	105	210	630
[4]	15	35	70	126	330
[4, 1]	24	84	224	504	1848
[4, 2]	27	126	420	1134	5544
[5]	21	56	126	252	792
[5, 1]	35	140	420	1050	4620
[5, 2]	42	224	840	2520	14784
[6]	28	84	210	462	1716
[6, 1]	48	216	720	1980	10296
[6, 2]	60	360	1500	4950	34320
[6, 3]	64	480	2400	9240	82368
[7]	36	120	330	792	3432
[7, 1]	63	315	1155	3465	21021
[7, 2]	81	540	2475	8910	72072
[8]	45	165	495	1287	6435
[8, 1]	80	440	1760	5720	40040
[9]	55	220	715	2002	11440
[9, 1]	99	594	2574	9009	72072

YT	$SU(3)$	$SU(4)$
[8, 3]	120	1100
[8, 4]	125	1375
[9, 2]	132	1056
[9, 3]	154	1540
[9, 4]	165	1980
[10, 2]	162	1404
[10, 3]	192	2080
[11, 2]	195	1820
[12, 1]	168	1260
[13, 1]	195	1560
[17]	171	1140
[18]	190	1330

YT	$SU(6)$	$SU(8)$
$[1^3]$	20	56
$[2, 1^4]$	35	420
[2, 2, 1, 1]	189	1512
[2, 2, 2]	175	1176
$[3, 1^4]$	120	1800

YT	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
[10]	66	286	1001	3003
[11]	78	364	1365	4368
[12]	91	455	1820	6188
[13]	105	560	2380	8568
[14]	120	680	3060	11628
[16]	153	969	4845	

YT	$SU(3)$	$SU(4)$	$SU(5)$
[7, 3]	90	750	4125
[8, 2]	105	770	3850
[10, 1]	120	780	3640
[11, 1]	143	1001	5005
[15]	136	816	3876

(continued)

Table 3.10 (continued)

YT	$SU(4)$	$SU(5)$	$SU(6)$	$SU(8)$	YT	$SU(5)$	$SU(6)$	$SU(8)$
[1, 1]	6	10	15	28	[2, 1, 1, 1]	24	84	504
[2, 1, 1]	15	45	105	378	[2, 2, 1]	75	210	1008
[2, 2]	20	50	105	336	[3, 1, 1, 1]	70	280	2100
[3, 1, 1]	36	126	336	1512	[3, 2, 1, 1]	175	840	8400
[3, 2]	60	175	420	1680	[3, 2, 2]	210	840	7056
[3, 2, 1]	64	280	896	5376	[3, 3, 1]	315	1176	9072
[3, 3]	50	175	490	2520	[4, 1, 1, 1]	160	720	6600
[4, 1, 1]	70	280	840	4620	[4, 2, 1, 1]	450	2430	29700
[4, 2, 1]	140	700	2520	18480	[4, 2, 2, 1]	480	3240	55440
[4, 2, 2]	84	560	2520	25872	[4, 2, 2, 2]	200	1800	46200
[4, 3]	140	560	1764	11088	[4, 3, 1, 1]	720	4536	71280
[4, 3, 1]	175	1050	4410	41580	[4, 3, 2, 1]	1024	8064	177408
[4, 4]	105	490	1764	13860	[4, 3, 2]	1120	5880	77616
[5, 1, 1]	120	540	1800	11880	[4, 3, 3]	700	4410	77616
[5, 2, 1]	256	1440	5760	50688	[4, 4, 1]	980	4704	55440
[5, 2, 2]	160	1200	6000	73920	[5, 1, 1, 1]	315	1575	17325
[5, 3]	280	1260	4410	33264	[5, 2, 1, 1]	945	5670	83160
[5, 4]	280	1470	5880	55440	[5, 2, 2, 1]	1050	7875	161700
[5, 5]	196	1176	5292	60984	[5, 2, 2, 2]	450	4500	138600
[6, 1, 1]	189	945	3465	27027	[6, 1, 1, 1]	560	3080	40040
[7, 1, 1]	280	1540	6160	56056	[6, 2, 2, 2]	875	9625	350350
					[7, 1, 1, 1]	924	5544	84084

YT	$SU(4)$	$SU(5)$	YT	$SU(4)$	$SU(5)$	YT	$SU(5)$	$SU(6)$
[5, 3, 1]	360	2430	[7, 2, 2]	420	3850	[3, 2, 2, 2]	560	11760
[5, 3, 2]	300	2700	[7, 3, 1]	1000	8250	[3, 3, 3]	980	14112
[5, 4, 1]	384	3024	[7, 3, 2]	875	9625	$[4, 1^4]$	315	5775
[6, 2, 1]	420	2625	[7, 3, 3]	500	6875	[4, 4, 2]	1176	7056
[6, 2, 2]	270	2250	[7, 4]	900	5775	[5, 3, 2, 2]	1215	14175
[6, 3, 1]	630	4725	[7, 5]	945	6930			
[6, 3, 2]	540	5400	[7, 6]	840	6930			
[6, 3, 3]	300	3750	[7, 7]	540	4950			
[6, 4]	540	3150	[8, 1, 1]	396	2376			
[6, 4, 1]	756	6615	[9, 1, 1]	540	3510			
[6, 4, 2]	729	8505	[8, 2, 1]	924	6930			
[6, 5]	504	3360	[8, 2, 2]	616	6160			
[6, 5, 1]	735	7350	[9, 2, 2]	864	9360			
[6, 6]	336	2520	[10, 1, 1]	715	5005			
[7, 2, 1]	640	4400	[11, 1, 1]	924	6930			

Table 3.11: Products of $SU(n)$ irreducible representations.

$$[p] \otimes [1^k] = [p+1, 1^{k-1}] \oplus [p, 1^k]$$

$$[1^p] \otimes [1^k] = \bigoplus_{i=0}^{k} [2^k, 1^{p+k-2i}] \qquad (p \geq k)$$
$$= [2^k, 1^{p-k}] \oplus [2^{k-1}, 1^{p-k+2}] \oplus \ldots \oplus [2, 1^{p+k-2}] \oplus [1^{p+k}]$$

$$[p] \otimes [k] = \bigoplus_{i=0}^{k} [p+k-i, i] \qquad (p \geq k)$$
$$= [p+k] \oplus [p+k-1, 1] \oplus \ldots \oplus [p+1, k-1] \oplus [p, k]$$

$$[p,1] \otimes [1^k] = [p+1, 2, 1^{k-2}] \oplus [p+1, 1^k] \oplus [p, 2, 1^{k-1}] \oplus [p, 1^{k+1}]$$

$$[p,1] \otimes [k] = \bigoplus_{0 \leq i \leq \min(k,p-1)} [p+k-i, i+1]$$
$$\bigoplus_{0 \leq i \leq \min(k-1,p-1)} [p+k-i-1, i+1, 1]$$

$$[p,1] \otimes [k,1] = \bigoplus_{0 \leq i \leq \min(k,p-1)} ([p+k-i+1, i+1] \oplus [p+k-i, i+1, 1])$$
$$\bigoplus_{0 \leq i \leq \min(k-1,p-1)} ([p+k-i, i+1, 1] \oplus [p+k-i-1, i+1, 1, 1])$$
$$\bigoplus_{0 \leq i \leq \min(k,p-2)} [p+k-i, i+2] \quad \bigoplus_{0 \leq i \leq \min(k-1,p-2)} [p+k-i-1, i+2, 1]$$
$$\bigoplus_{1 \leq i \leq \min(k-1,p-1)} [p+k-i-1, i+1, 2]$$

N.B. One has to avoid in the r.h.s. tableaux with a number of rows larger than the rank n of the algebra under consideration.
Moreover, any tableau with n rows will be replaced by the corresponding tableau in which the n box columns are suppressed.

Table 3.12: Dimensions of $Sp(2n)$ irreducible representations.

Young Tabl. $[m_1, \ldots, m_n]$	$Sp(6)$	$Sp(8)$	$Sp(10)$	$Sp(12)$
$[0]$	1	1	1	1
$[1]$ (fond.)	6	8	10	12
$[1^2]$	14	27	44	65
$[1^3]$	14	48	110	208
$[1^4]$	–	42	165	429
$[1^5]$	–	–	132	572
$[2]$ (adjoint)	21	36	55	78
$[2,1]$	64	160	320	560
$[2,1,1]$	70	315	891	2002
$[2,1,1,1]$	–	288	1408	4368
$[2,2]$	90	308	780	1650
$[2,2,1]$	126	792	2860	7800
$[2,2,2]$	84	825	4004	13650
$[3]$	56	120	220	364
$[3,1]$	189	594	1430	2925
$[3,1,1]$	216	1232	4212	11088
$[3,2]$	350	1512	4620	11440
$[3,2,1]$	512	4096	17920	57344
$[3,2,2]$	378	4752	28028	112320
$[3,3]$	385	2184	8250	24310
$[3,3,1]$	616	6552	35640	136136
$[3,3,2]$	594	10010	73710	353430
$[3,3,3]$	330	8008	76440	448800
$[4]$	126	330	715	1365
$[4,1]$	448	1728	4928	11648
$[4,1,1]$	525	3696	15015	45760
$[4,2]$	924	4914	17820	51051
$[5]$	252	792	2002	4368
$[5,1]$	924	4290	14300	38675
$[6]$	462	1716	5005	12376
$[7]$	792	3432	11440	31824

Table 3.13: Products of $Sp(2n)$ irreducible representations ($n \geq 3$).

$$[p] \otimes [k] = \bigoplus_{l=0}^{k} \bigoplus_{m=l}^{k} [p+k-m-l, m-l] \qquad (p \geq k)$$

$$[p] \otimes [1] = [p+1] \oplus [p,1] \oplus [p-1]$$

$$[p] \otimes [1^k] = [p+1, 1^{k-1}] \oplus [p, 1^k] \oplus [p, 1^{k-2}] \oplus [p-1, 1^{k-1}]$$

$$[p,k] \otimes [1] = [p+1,k] \oplus [p,k+1] \oplus [p,k,1] \oplus [p,k-1] \oplus [p-1,k] \qquad (p \geq k)$$

$$\begin{aligned}[p,1] \otimes [1^k] = {} & [p+1,2,1^{k-2}] \oplus [p+1,1^k] \oplus [p+1,1^{k-2}] \\ & \oplus [p,2,1^{k-1}] \oplus [p,2,1^{k-3}] \oplus [p,1^{k+1}] \oplus [p,1^{k-1}] \\ & \oplus [p,1^{k-1}]^* \oplus [p,1^{k-3}] \oplus [p-1,2,1^{k-2}] \\ & \oplus [p-1,1^k] \oplus [p-1,1^{k-2}] \qquad (p,k \geq 2)\end{aligned}$$

$$\begin{aligned}[p,2] \otimes [1^k] = {} & [p+1,3,1^{k-2}] \oplus [p+1,2,1^{k-1}] \oplus [p+1,2,1^{k-3}] \\ & \oplus [p+1,1^{k-1}] \oplus [p,3,1^{k-1}] \oplus [p,3,1^{k-3}] \\ & \oplus [p,2,1^k] \oplus [p,2,1^{k-2}] \oplus [p,2,1^{k-4}] \oplus [p,1^{k-2}] \\ & \oplus [p-1,3,1^{k-2}] \oplus [p-1,2,1^{k-1}] \oplus [p-1,2,1^{k-3}] \\ & \oplus [p-1,1^{k-1}] \qquad (p,k \geq 2)\end{aligned}$$

$$[1^p] \otimes [1^k] = \bigoplus_{l=0}^{k} \bigoplus_{m=l}^{k} [2^{m-l}, 1^{p+k-2m}] \qquad (p \geq k)$$

where the GYTs such that $p+k-m > n$ do not appear for $Sp(2n)$

$$[\lambda] \otimes [1] = \{[\lambda]_+\} \oplus \{[\lambda]_-\}$$

N.B. One has to avoid in the r.h.s. tableaux with a number of rows larger than the rank n of the algebra under consideration or any non-meaningful GYT. The * representations do not appear for $Sp(2k)$.
In the last product, $[\lambda]$ is any GYT, $\{[\lambda]_+\}$ (respectively $\{[\lambda]_-\}$) is the set of GYTs obtained by adding (respectively subtracting) a box in any meaningful way to any row of $[\lambda]$.

Table 3.14: Dimensions of $SO(2n)$ irreducible vector representations.

Young Tabl.	$SO(8)$	$SO(10)$	$SO(12)$	$SO(14)$	$SO(16)$	$SO(24)$	$SO(32)$
[1]	8	10	12	14	16	24	32
$[1^2]$	28	45	66	91	120	276	496
$[1^3]$	56	120	220	364	560	2024	4960
$[1^4]$	35	210	495	1001	1820	10626	35960
$[1^5]$	–	126	792	2002	4368	42504	201376
$[1^6]$	–	–	462	3003	8008	134596	906192
[2]	35	54	77	104	135	299	527
[2, 1]	160	320	560	896	1344	4576	10880
[2, 1, 1]	350	945	2079	4004	7020	37674	122264
[2, 1, 1, 1]	224	1728	4928	11648	24192	210496	944384
[2, 1, 1, 1, 1]	–	1050	8085	24024	60060	874874	5501880
[2, 2]	300	770	1638	3080	5304	27300	86768
[2, 2, 1]	840	2970	8008	18200	36720	299000	1298528
[2, 2, 2]	840	4125	14014	38220	89760	1136200	6678144
[3]	112	210	352	546	800	2576	5952
[3, 1]	567	1386	2860	5265	8925	44275	138105
[3, 1, 1]	1296	4312	11232	24948	49504	388080	1653696
[3, 2]	1400	4410	11088	24024	46800	351624	1467168
[3, 3]	1925	7644	23100	58344	129675	1450449	8023575
[4]	294	660	1287	2275	3740	17250	51832
[4, 1]	1568	4608	11088	23296	44352	315744	1281664
[5]	672	1782	4004	8008	14688	95680	371008
[5, 1]	3696	12870	35750	85085	180800	1821600	9548000
[6]	1386	4290	11011	24752	50388	457470	2272424

Young Tabl.	$SO(8)$	$SO(10)$	$SO(12)$	$SO(14)$	$SO(16)$	$SO(24)$
[2, 2, 1, 1]	567	5940	21021	58968	141372	1874730
[2, 2, 1, 1, 1]	–	3696	36036	128128	371280	8288280
[2, 2, 2, 1]	672	10560	48048	163072	456960	9472320
[2, 2, 2, 2]	294	8910	55055	231868	771120	
[3, 1, 1, 1]	840	8085	27456	75075	176800	2254000
[3, 1, 1, 1, 1]	–	4950	45760	157950	448800	9614000
[3, 2, 1]	4096	17920	57344	150528	344064	4100096
[3, 2, 1, 1]	2800	36750	155232	504504	1372800	
[3, 2, 2]	4536	27720	112112	353808	942480	
[3, 3, 1]	6160	34398	133056	408408	1067040	
[4, 1, 1]	3675	14784	45045	114400	255255	2877875
[4, 1, 1, 1]	2400	28160	112320	352000	933504	
[4, 2]	4312	16380	48114	119119	260832	2816856
[4, 3]	7840	37632	133056	384384	960960	
[5, 1, 1]	8800	42120	148500	427856	1067040	
[7]	2640	9438	27456	68952	155040	1937520
[8]	4719	19305	63206	176358	436050	7413705
[9]	8008	37180	136136	419900	1136960	

Table 3.15: Dimensions of $SO(2n+1)$ irreducible vector representations.

Young Tabl.	$SO(5)$	$SO(7)$	$SO(9)$	$SO(11)$	$SO(13)$	$SO(15)$
[1]	5	7	9	11	13	15
$[1^2]$	10	21	36	55	78	105
$[1^3]$	–	35	84	165	286	455
$[1^4]$	–	–	126	330	715	1365
$[1^5]$	–	–	–	462	1287	3003
[2]	14	27	44	65	90	119
[2, 1]	35	105	231	429	715	1105
[2, 1, 1]	–	189	594	1430	2925	5355
[2, 1, 1, 1]	–	–	924	3003	7722	17017
[2, 2]	35	168	495	1144	2275	4080
[2, 2, 1]	–	378	1650	5005	12285	26180
[2, 2, 1, 1]	–	–	2772	11583	36036	92820
[2, 2, 2]	–	294	1980	7865	23660	59500
[2, 2, 2, 1]	–	–	4158	23595	91091	278460
[2, 2, 2, 2]	–	–	2772	23595	117117	433160
[3]	30	77	156	275	442	665
[3, 1]	81	330	910	2025	3927	6916
[3, 1, 1]	–	616	2457	7128	17017	35568
[3, 1, 1, 1]	–	–	3900	15400	46410	117040
[3, 2]	105	693	2574	7150	16575	33915
[3, 2, 1]	–	1617	9009	33033	94809	230945
[3, 2, 1, 1]	–	–	15444	78650	287300	847875
[3, 2, 2]	–	1386	12012	57915	204204	587860
[3, 3]	84	825	4004	13650	37400	88179
[3, 3, 1]	–	2079	15444	70070	238680	671517
[4]	55	182	450	935	1729	2940
[4, 1]	154	819	2772	7293	16302	32487
[4, 1, 1]	–	1560	7700	26520	73150	173264
[4, 1, 1, 1]	–	–	12375	58344	203775	583440
[4, 2]	220	1911	8748	28798	77064	178605
[4, 2, 1]	–	4550	31500	137445	456456	1261260
[4, 3]	231	3003	18018	72930	230945	617253
[4, 4]	165	3003	22932	112200	415701	1270815
[5]	91	378	1122	2717	5733	10948
[5, 1]	260	1750	7140	21945	56056	125580
[5, 1, 1]	–	3375	20196	81510	257400	686205
[5, 2]	390	4312	23868	91960	281554	734160
[6]	140	714	2508	7007	16744	35700
[6, 1]	405	3366	16302	57915	167739	419900
[7]	204	1254	5148	16445	44200	104652
[7, 1]	595	5985	33957	138138	450450	1253070
[8]	285	2079	9867	35750	107406	281010
[9]	385	3289	17875	72930	243542	700910

Table 3.16: Dimensions of $SO(2n)$ irreducible spinor representations.

Young Tabl.	$SO(8)$	$SO(10)$	$SO(12)$	$SO(14)$	$SO(16)$	$SO(24)$	$SO(32)$
$[0]'$	8	16	32	64	128	2048	32768
$[1]'$	56	144	352	832	1920	47104	1015808
$[1^2]'$	160	560	1728	4928	13312	516096	
$[1^3]'$	224	1200	4928	17472	56320	3579904	
$[1^4]'$	112	1440	8800	40768	161280		
$[1^5]'$	–	672	9504	64064	326144		
$[1^6]'$	–	–	4224	64064	465920		
$[2]'$	224	720	2112	5824	15360	565248	
$[2,1]'$	840	3696	13728	45760	141440	8243200	
$[2,1,1]'$	1296	8800	43680	181440	670208		
$[2,1,1,1]'$	672	11088	82368	448448	2036736		
$[2,2]'$	1400	8064	36960	146432	524160		
$[3]'$	672	2640	9152	29120	87040	4710400	
$[3,1]'$	2800	15120	66528	256256	898560		
$[4]'$	1680	7920	32032	116480	391680		

Table 3.17: Dimensions of $SO(2n+1)$ irreducible spinor representations.

Young Tabl.	$SO(5)$	$SO(7)$	$SO(9)$	$SO(11)$	$SO(13)$	$SO(15)$
$[0]'$	4	8	16	32	64	128
$[1]'$	16	48	128	320	768	1792
$[1^2]'$	20	112	432	1408	4160	11520
$[1^3]'$	–	112	768	3520	13312	44800
$[1^4]'$	–	–	672	5280	27456	116480
$[2]'$	40	168	576	1760	4992	13440
$[2,1]'$	64	512	2560	10240	35840	114688
$[2,1,1]'$	–	560	5040	28512	128128	499200
$[2,2]'$	56	720	4928	24960	105600	396032
$[2,2,1]'$	–	1008	12672	91520	499200	2284800
$[2,2,2]'$	–	672	13200	128128	873600	4787200
$[3]'$	80	448	1920	7040	23296	71680
$[3,1]'$	140	1512	9504	45760	187200	685440
$[3,2]'$	160	2800	24192	147840	732160	3144960
$[3,3]'$	120	3080	34944	264000	1555840	7745920
$[4]'$	140	1008	5280	22880	87360	304640
$[5']$	224	2016	12672	64064	279552	1096704

Table 3.18: Products of $SO(2n)$ irreducible representations ($n \geq 4$).

$$[p] \otimes [p'] = \bigoplus_{l=0}^{p'} \bigoplus_{k=0}^{p'-l} [p+p'-k-2l, k] \qquad (p \geq p')$$

$$[p] \otimes [1] = [p+1] \oplus [p,1] \oplus [p-1]$$

$$[p,1] \otimes [1] = [p+1,1] \oplus [p,2] \oplus [p,1,1] \oplus [p] \oplus [p-1,1]$$

$$[p,2] \otimes [1] = [p+1,2] \oplus [p,3] \oplus [p,2,1] \oplus [p,1] \oplus [p-1,2]$$

$$[p,1,1] \otimes [1] = [p+1,1,1] \oplus [p,2,1] \oplus [p,1,1,1] \oplus [p,1,1,-1]^* \oplus [p,1] \oplus [p-1,1,1]$$

$$[p,2,1] \otimes [1] = [p+1,2,1] \oplus [p,3,1] \oplus [p,2,2] \oplus [p,2,1,1] \oplus [p,2,1,-1]^* \oplus [p,2] \oplus [p,1,1] \oplus [p-1,2,1]$$

$$[p] \otimes [1^2] = [p+1,1] \oplus [p,1,1] \oplus [p] \oplus [p-1,1]$$

$$[p,1] \otimes [1^2] = [p+1,2] \oplus [p+1,1,1] \oplus [p+1] \oplus [p,2,1] \oplus [p,1,1,1] \oplus [p,1,1,-1]^* \oplus 2\,[p,1] \oplus [p-1,2] \oplus [p-1,1,1] \oplus [p-1] \qquad (p \geq 2)$$

$$[1^2] \otimes [1^2] = [2,2] \oplus [2,1,1] \oplus [2] \oplus [1^4] \oplus [1^3,-1]^* \oplus [1^2] \oplus [0]$$

$$[1^3] \otimes [1^2] = [2,2,1] \oplus [2,1,1,1] \oplus [2,1] \oplus [1^3] \oplus [1] \oplus \begin{cases} [2,1,1,-1] \oplus [1^3] & \text{for } n=4 \\ [1^5] \oplus [1^4,-1] & \text{for } n=5 \\ [1^5] & \text{for } n \geq 6 \end{cases}$$

$$[1^3] \otimes [1^3] = [2,2,2] \oplus [2,2,1,1] \oplus [2,2] \oplus [2,1,1] \oplus [2] \oplus [1^4] \oplus [1^2] \oplus [0] \oplus \begin{cases} [2,2,1,-1] \oplus [2,1,1] \oplus [1^3,-1] \oplus [1^2] & \text{for } n=4 \\ [2,1,1,1,1] \oplus [2,1,1,1,-1] \oplus [1^4] & \text{for } n=5 \\ [2,1,1,1,1] \oplus [1^6] \oplus [1^5,-1] & \text{for } n=6 \\ [2,1,1,1,1] \oplus [1^6] & \text{for } n \geq 7 \end{cases}$$

$$[2] \otimes [1^3] = [3,1,1] \oplus [2,1,1,1] \oplus [2,1,1,-1]^* \oplus [2,1] \oplus [1^3]$$

(continued)

Table 3.18 (continued)

$$
\begin{aligned}
[p] \otimes [0]' &= [p]' \oplus [p-1]'' \\
[1^k] \otimes [0]' &= [1^k]' \oplus [1^{k-1}]'' \oplus [1^{k-2}]' \oplus [1^{k-3}]'' \oplus \ldots \\
&\quad \oplus \begin{cases} [0]' & \text{for } k \text{ even} \\ [0]'' & \text{for } k \text{ odd} \end{cases} \\
[1] \otimes [1]' &= [2]' \oplus [1^2]' \oplus [1]'' \oplus [0]' \\
[1^2] \otimes [1]' &= [2,1]' \oplus [1^3]' \oplus [2]'' \oplus [1^2]'' \oplus 2\,[1]' \oplus [0]'' \\
[1^3] \otimes [1]' &= [2,1,1]' \oplus [2,1]'' \oplus [2]' \oplus [1^4]' \oplus [1^3]'' \oplus 2\,[1^2]' \\
&\quad \oplus 2\,[1]'' \oplus [0]' \\
[0]' \otimes [0]' &= [1^n] \oplus [1^{n-2}] \oplus \ldots \oplus [\tfrac{1}{2}(1-(-1)^n)] \\
[0]' \otimes [0]'' &= [1^{n-1}] \oplus [1^{n-3}] \oplus \ldots \oplus [\tfrac{1}{2}(1+(-1)^n)] \\
[0]' \otimes [1]' &= [21^{n-1}] \oplus [21^{n-3}] \oplus \ldots \oplus [2, \tfrac{1}{2}(1+(-1)^n)] \\
&\quad \oplus [1^{n-1}] \oplus [1^{n-3}] \oplus \ldots \oplus [1, \tfrac{1}{2}(1-(-1)^n)] \\
[0]'' \otimes [1]' &= [21^{n-2}] \oplus [21^{n-4}] \oplus \ldots \oplus [2, \tfrac{1}{2}(1-(-1)^n)] \\
&\quad \oplus [1^{n-2}] \oplus [1^{n-4}] \oplus \ldots \oplus [1, \tfrac{1}{2}(1+(-1)^n)] \\
[2] \otimes [1]' &= [3]' \oplus [2,1]' \oplus [2]'' \oplus [1^2]'' \oplus [1]' \oplus [0]''
\end{aligned}
$$

N.B. One has to avoid in the r.h.s. tableaux with a number of rows larger than the rank n of the algebra under consideration or any non-meaningful GYT. The GYTs with * exist only for $SO(8)$.
The notation $[\underbrace{\ldots,\ldots,\ldots}_{p}]''$ means $[\underbrace{\ldots,\ldots,\ldots}_{p}, 0^{n-p-1}, -1]'$.
The spinor representations $[u,v,w,\ldots]'$ and $[u,v,w,\ldots]''$ have the same dimension (they are conjugate to each other).

One has also the general formula

$$[\lambda] \otimes [1] = \{[\lambda]_+\} \oplus \{[\lambda]_-\}$$

where $[\lambda]$ is any GYT, $\{[\lambda]_+\}$ (respectively $\{[\lambda]_-\}$) is the set of GYTs obtained by adding (respectively subtracting) a box in any meaningful way to any row of $[\lambda]$.

Table 3.19: Products of $SO(2n+1)$ irreducible representations ($n \geq 3$).

$$[p] \otimes [1] = [p+1] \oplus [p,1] \oplus [p-1]$$

$$[1^2] \otimes [1] = [2,1] \oplus [1^3] \oplus [1]$$

$$[1^2] \otimes [2] = [3,1] \oplus [2,1,1] \oplus [2] \oplus [1^2]$$

$$[2] \otimes [2] = [4] \oplus [3,1] \oplus [2,2] \oplus [2] \oplus [1^2] \oplus [0]$$

$$[1^2] \otimes [1^2] = \begin{cases} [2,2] \oplus [2,1,1] \oplus [1^3] \oplus [2] \oplus [1^2] \oplus [0] & \text{for } n=3 \\ [2,2] \oplus [2,1,1] \oplus [1^4] \oplus [2] \oplus [1^2] \oplus [0] & \text{for } n\geq 4 \end{cases}$$

$$[1^3] \otimes [1] = \begin{cases} [2,1,1] \oplus [1^3] \oplus [1^2] & \text{for } n=3 \\ [2,1,1] \oplus [1^4] \oplus [1^2] & \text{for } n\geq 4 \end{cases}$$

$$[1^3] \otimes [1^2] = \begin{cases} [2,2,1] \oplus [2,1,1] \oplus [2,1] \oplus [1^3] \oplus [1^2] \oplus [1] & \text{for } n=3 \\ [2,2,1] \oplus [2,1,1,1] \oplus [2,1] \oplus [1^4] \oplus [1^3] \oplus [1] & \text{for } n=4 \\ [2,2,1] \oplus [2,1,1,1] \oplus [2,1] \oplus [1^5] \oplus [1^3] \oplus [1] & \text{for } n\geq 5 \end{cases}$$

$$[1^4] \otimes [1] = \begin{cases} [2,1,1,1] \oplus [1^4] \oplus [1^3] & \text{for } n=4 \\ [2,1,1,1] \oplus [1^5] \oplus [1^3] & \text{for } n\geq 5 \end{cases}$$

$$[2] \otimes [1^3] = \begin{cases} [3,1,1] \oplus [2,1,1] \oplus [2,1] \oplus [1^3] & \text{for } n=3 \\ [3,1,1] \oplus [2,1,1,1] \oplus [2,1] \oplus [1^3] & \text{for } n\geq 4 \end{cases}$$

$$[2] \otimes [1^4] = \begin{cases} [3,1,1,1] \oplus [2,1,1,1] \oplus [2,1,1] \oplus [1^4] & \text{for } n=4 \\ [3,1,1,1] \oplus [2,1,1,1,1] \oplus [2,1,1] \oplus [1^4] & \text{for } n\geq 5 \end{cases}$$

(continued)

Table 3.19 (continued)

$$
\begin{aligned}
[p] \otimes [0]' &= [p]' \oplus [p-1]' \\
[1^k] \otimes [0]' &= [1^k]' \oplus [1^{k-1}]' \oplus [1^{k-2}]' \oplus \dots \oplus [0]' \\
[1] \otimes [1]' &= [2]' \oplus [1^2]' \oplus [1]' \oplus [0]' \\
[1^2] \otimes [1]' &= [2,1]' \oplus [2]' \oplus [1^3]' \oplus [1^2]' \oplus 2\,[1]' \oplus [0]' \\
[1^3] \otimes [1]' &= [2,1,1]' \oplus [2,1]' \oplus [2]' \oplus [1^4]' \oplus [1^3]' \oplus 2\,[1^2]' \\
&\quad \oplus 2\,[1]' \oplus [0]' \\
[2] \otimes [1]' &= [3]' \oplus [2,1]' \oplus [2]' \oplus [1^2]' \oplus [1]' \oplus [0]' \\
[0]' \otimes [0]' &= [1^n] \oplus [1^{n-1}] \oplus \dots \oplus [0] \\
[0]' \otimes [1]' &= [21^{n-1}] \oplus [21^{n-2}] \oplus \dots \oplus [2] \\
&\quad \oplus [1^n] \oplus [1^{n-1}] \oplus \dots \oplus [0]
\end{aligned}
$$

N.B. One has to avoid in the r.h.s. tableaux with a number of rows larger than the rank n of the algebra under consideration or any non-meaningful GYT.

One has also the general formula

$$
[\lambda] \otimes [1] = \begin{cases} \{[\lambda]_+\} \oplus \{[\lambda]_-\} & \text{if } \lambda_n = 0 \\ \{[\lambda]_+\} \oplus \{[\lambda]_-\} \oplus [\lambda] & \text{if } \lambda_n \neq 0 \end{cases}
$$

where $[\lambda]$ is any GYT, $\{[\lambda]_+\}$ (respectively $\{[\lambda]_-\}$) is the set of GYTs obtained by adding (respectively subtracting) a box in any meaningful way to any row of $[\lambda]$.

Table 3.20: Dimensions of G_2 irreducible representations.

Dynkin	dim.	Dynkin	dim.	Dynkin	dim.	Dynkin	dim.
0 1	7	4 0	748	3 3	4096	0 12	10556
1 0	14	3 1	896	4 2	4914	8 0	11571
0 2	27	1 4	924	5 1	4928	1 9	11648
1 1	64	0 7	1254	1 7	$4928'$	3 5	12096
2 0	77	2 3	1547	0 10	5005	2 7	13090
0 3	$77'$	1 5	1728	2 5	5103	0 13	14756
0 4	182	5 0	1729	7 0	6630	4 4	15625
1 2	189	3 2	2079	3 4	7293	1 10	17017
3 0	273	0 8	$2079'$	0 11	7371	7 1	17472
2 1	286	4 1	2261	1 8	7722	5 3	18304
0 5	378	2 4	2926	2 6	8372	3 6	19019
1 3	448	1 6	3003	4 3	9177	9 0	19096
0 6	714	0 9	3289	6 1	9660	6 2	19278
2 2	729	6 0	3542	5 2	10206	2 8	19683

Table 3.21: Dimensions of F_4 irreducible representations.

Dynkin	dimension	Dynkin	dimension	Dynkin	dimension
0001	26	2001	17901	3001	184756
1000	52	0101	19278	0021	205751
0010	273	0020	19448	0013	212992
0002	324	1100	29172	0200	226746
1001	1053	0012	34749	2100	340119
2000	$1053'$	1003	76076	0006	342056
0100	1274	0005	81081	1101	379848
0003	2652	4000	100776	1004	412776
0011	4096	1011	106496	1020	420147
1010	8424	0110	107406	5000	627912
1002	10829	2010	119119	0030	629356
3000	12376	0102	160056	1012	787644
0004	16302	2002	$160056'$	0103	952952

Table 3.22: Products of G_2 irreducible representations.

$7 \otimes 7$	$=$	$(1 \oplus 27)_s \oplus (7 \oplus 14)_a$
$7 \otimes 14$	$=$	$7 \oplus 27 \oplus 64$
$14 \otimes 14$	$=$	$(1 \oplus 27 \oplus 77)_s \oplus (14 \oplus 77')_a$
$7 \otimes 27$	$=$	$7 \oplus 14 \oplus 27 \oplus 64 \oplus 77'$
$14 \otimes 27$	$=$	$7 \oplus 14 \oplus 27 \oplus 64 \oplus 77' \oplus 189$
$27 \otimes 27$	$=$	$(1 \oplus 27 \oplus 27 \oplus 64 \oplus 77 \oplus 182)_s$ $\oplus (7 \oplus 14 \oplus 64 \oplus 77' \oplus 189)_a$
$7 \otimes 64$	$=$	$14 \oplus 27 \oplus 64 \oplus 77 \oplus 77' \oplus 189$
$14 \otimes 64$	$=$	$7 \oplus 27 \oplus 64 \oplus 64 \oplus 77' \oplus 182 \oplus 189 \oplus 286$
$27 \otimes 64$	$=$	$7 \oplus 14 \oplus 27 \oplus 27 \oplus 64 \oplus 64 \oplus 77 \oplus 77' \oplus 77' \oplus 182 \oplus 189$ $\oplus 189 \oplus 286 \oplus 448$
$64 \otimes 64$	$=$	$1 \oplus 7 \oplus 14 \oplus 14 \oplus 27 \oplus 27 \oplus 64 \oplus 64 \oplus 77 \oplus 77 \oplus 77'$ $77' \oplus \oplus 77' \oplus 182 \oplus 182 \oplus 189 \oplus 189 \oplus 189 \oplus 273 \oplus 286$ $\oplus 378 \oplus 448 \oplus 448 \oplus 729$
$7 \otimes 77$	$=$	$64 \oplus 189 \oplus 286$
$14 \otimes 77$	$=$	$14 \oplus 77 \oplus 77' \oplus 189 \oplus 273 \oplus 448$
$27 \otimes 77$	$=$	$27 \oplus 64 \oplus 77 \oplus 77' \oplus 182 \oplus 189 \oplus 286 \oplus 448 \oplus 729$
$7 \otimes 77'$	$=$	$27 \oplus 64 \oplus 77' \oplus 182 \oplus 189$
$14 \otimes 77'$	$=$	$14 \oplus 27 \oplus 64 \oplus 77 \oplus 77' \oplus 182 \oplus 189 \oplus 448$
$27 \otimes 77'$	$=$	$7 \oplus 14 \oplus 27 \oplus 64 \oplus 64 \oplus 77 \oplus 77' \oplus 77' \oplus 182 \oplus 189$ $\oplus 189 \oplus 286 \oplus 378 \oplus 448$
$7 \otimes 182$	$=$	$77' \oplus 182 \oplus 189 \oplus 378 \oplus 448$
$14 \otimes 182$	$=$	$64 \oplus 77' \oplus 182 \oplus 189 \oplus 286 \oplus 378 \oplus 448 \oplus 924$
$7 \otimes 189$	$=$	$64 \oplus 77 \oplus 77' \oplus 182 \oplus 189 \oplus 286 \oplus 448$
$14 \otimes 189$	$=$	$27 \oplus 64 \oplus 77 \oplus 77' \oplus 182 \oplus 189 \oplus 189 \oplus 286 \oplus 378 \oplus 448$ $\oplus 729$
$7 \otimes 273$	$=$	$286 \oplus 729 \oplus 896$
$14 \otimes 273$	$=$	$77 \oplus 273 \oplus 448 \oplus 729 \oplus 748 \oplus 1547$

Table 3.23: Products of F_4 irreducible representations.

$26 \otimes 26$	$=$	$(1 \oplus 26 \oplus 324)_s \oplus (52 \oplus 273)_a$
$26 \otimes 52$	$=$	$26 \oplus 273 \oplus 1053$
$52 \otimes 52$	$=$	$(1 \oplus 324 \oplus 1053')_s \oplus (52 \oplus 1274)_a$
$26 \otimes 273$	$=$	$26 \oplus 52 \oplus 273 \oplus 324 \oplus 1053 \oplus 1274 \oplus 4096$
$52 \otimes 273$	$=$	$26 \oplus 273 \oplus 324 \oplus 1053 \oplus 4096 \oplus 8424$
$273 \otimes 273$	$=$	$(1 \oplus 26 \oplus 324 \oplus 324 \oplus 1053 \oplus 1053' \oplus 2652 \oplus 4096$ $\oplus 8424 \oplus 19448)_s \oplus (52 \oplus 273 \oplus 273 \oplus 1053 \oplus 1274$ $\oplus 4096 \oplus 10829 \oplus 19278)_a$
$26 \otimes 324$	$=$	$26 \oplus 273 \oplus 324 \oplus 1053 \oplus 2652 \oplus 4096$
$52 \otimes 324$	$=$	$52 \oplus 273 \oplus 324 \oplus 1274 \oplus 4096 \oplus 10829$
$273 \otimes 324$	$=$	$26 \oplus 52 \oplus 273 \oplus 273 \oplus 324 \oplus 1053 \oplus 1053 \oplus 1274 \oplus 2652$ $\oplus 4096 \oplus 4096 \oplus 8424 \oplus 10829 \oplus 19278 \oplus 34749$
$324 \otimes 324$	$=$	$(1 \oplus 26 \oplus 324 \oplus 324 \oplus 1053' \oplus 2652 \oplus 4096 \oplus 8424$ $\oplus 16302 \oplus 19448)_s \oplus (52 \oplus 273 \oplus 1053 \oplus 1274 \oplus 4096$ $\oplus 10829 \oplus 34749)_a$

Table 3.24: Dimensions of E_6 irreducible representations.

Dynkin label	dimension	Dynkin label	dimension	Dynkin label	dimension
100000	27	000003	43758 R	011000	386100
000001	78 R	100002	46332	000102	393822
000100	351	101000	51975	000210	412776
000020	$351'$	210000	54054	600000	442442
100010	650 R	100030	61425	200002	459459
100001	1728	010100	70070 R	001020	494208
000002	2430 R	010020	78975	000004	537966 R
001000	2925 R	200020	85293 R	010030	579150
300000	3003	500000	100386	020010	600600
000110	5824	001001	105600 R	200030	638820
010010	7371	100110	112320	000013	741312
200010	7722	300001	146432	100012	812175 R
000101	17550	000111	252252	101010	852930 R
000021	19305	100101	314496	000041	853281
400000	$19305'$	000130	359424	010110	967680
020000	34398	200011	359424	100120	972972
100011	34749 R	100040	371800		

The real representations are labelled by (R).

Table 3.25: Dimensions of E_7 irreducible representations.

Dynkin label	dimension	Dynkin label	dimension	Dynkin label	dimension
0000010	56	0001000	27664	200010	320112
1000000	133	0000011	40755	0100010	362880
0000001	912	0000110	51072	0010000	365750
0000020	1463	1000001	86184	1100000	573440
0000100	1539	1000020	150822	0000200	617253
1000010	6480	1000100	152152	0000101	861840
2000000	7371	3000000	238602	0000021	885248
0100000	8645	0000002	253935	0000120	915705
0000030	24320	0000040	293930	0001010	980343

Table 3.26: Dimensions of E_8 irreducible representations.

Dynkin label	dimension	Dynkin label	dimension	Dynkin label	dimension
00000010	248	00001000	2450240	00000040	79143000
10000000	3875	00000110	4096000	00010000	146325270
00000020	27000	20000000	4881384	00000200	203205000
00000100	30380	01000000	6696000	00000120	281545875
00000001	147250	00000011	26411008	10000001	301694976
10000010	779247	10000020	70680000	00001010	344452500
00000030	1763125	10000100	76271625	20000010	820260000

Table 3.27: Products of E_6 irreducible representations.

$27 \otimes 27$	$=$	$(\overline{27} \oplus \overline{351}')_s \oplus \overline{351}_a$
$27 \otimes \overline{27}$	$=$	$1 \oplus 78 \oplus 650$
$27 \otimes 78$	$=$	$27 \oplus 351 \oplus 1728$
$78 \otimes 78$	$=$	$(1 \oplus 650 \oplus 2430)_s \oplus (78 \oplus 2925)_a$
$\overline{27} \otimes 351$	$=$	$78 \oplus 650 \oplus 2925 \oplus 5824$
$\overline{27} \otimes \overline{351}$	$=$	$27 \oplus 351 \oplus 1728 \oplus 7371$
$27 \otimes \overline{351}'$	$=$	$650 \oplus 3003 \oplus \overline{5824}$
$\overline{27} \otimes \overline{351}'$	$=$	$27 \oplus 1728 \oplus 7722$
$78 \otimes 351$	$=$	$27 \oplus 351 \oplus 351' \oplus 1728 \oplus 7371 \oplus 17550$
$78 \otimes 351'$	$=$	$351 \oplus 351' \oplus 7371 \oplus 19305$
$351 \otimes \overline{351}$	$=$	$1 \oplus 78 \oplus 650 \oplus 650 \oplus 2430 \oplus 2925 \oplus 5824 \oplus \overline{5824}$ $\oplus 34749 \oplus 70070$
$\overline{351} \otimes \overline{351}$	$=$	$(27 \oplus 351' \oplus 1728 \oplus 7722 \oplus 17550 \oplus 34398)_s$ $\oplus (351 \oplus 1728 \oplus 7371 \oplus 51975)_a$
$351' \otimes \overline{351}'$	$=$	$1 \oplus 78 \oplus 650 \oplus 2430 \oplus 34749 \oplus 85293$
$\overline{351}' \otimes \overline{351}'$	$=$	$(351' \oplus 7722 \oplus 19305' \oplus 34398)_s \oplus (7371 \oplus 54054)_a$
$\overline{351} \otimes 351'$	$=$	$78 \oplus 650 \oplus 2925 \oplus 5824 \oplus 34749 \oplus 78975$
$\overline{351} \otimes \overline{351}'$	$=$	$351 \oplus 1728 \oplus 7371 \oplus 7722 \oplus 51975 \oplus 54054$
$27 \otimes 650$	$=$	$27 \oplus 351 \oplus 351' \oplus 1728 \oplus 7371 \oplus 7722$
$78 \otimes 650$	$=$	$78 \oplus 650 \oplus 650 \oplus 2925 \oplus 5824 \oplus \overline{5824} \oplus 34749$
$27 \otimes \overline{1728}$	$=$	$78 \oplus 650 \oplus 2430 \oplus 2925 \oplus 5824 \oplus 34749$
$\overline{27} \otimes \overline{1728}$	$=$	$351 \oplus 351' \oplus 1728 \oplus 7371 \oplus 17550 \oplus 19305$
$78 \otimes 1728$	$=$	$27 \oplus 351 \oplus 1728 \oplus 1728 \oplus 7371 \oplus 7722 \oplus 17550 \oplus 46332$ $\oplus 51975$
$650 \otimes 650$	$=$	$(1 \oplus 78 \oplus 650 \oplus 650 \oplus 2430 \oplus 3003 \oplus \overline{3003} \oplus 5824$ $\oplus \overline{5824} \oplus 34749 \oplus 70070 \oplus 85293)_s \oplus (78 \oplus 650 \oplus 2925$ $\oplus 2925 \oplus 5824 \oplus \overline{5824} \oplus 34749 \oplus 78975 \oplus \overline{78975})_a$

Table 3.28: Products of E_7 irreducible representations.

$56 \otimes 56$	$=$	$(133 \oplus 1463)_s \oplus (1 \oplus 1539)_a$
$56 \otimes 133$	$=$	$56 \oplus 912 \oplus 6480$
$133 \otimes 133$	$=$	$(1 \oplus 1539 \oplus 7371)_s \oplus (133 \oplus 8645)_a$
$56 \otimes 912$	$=$	$133 \oplus 1539 \oplus 8645 \oplus 40755$
$133 \otimes 912$	$=$	$56 \oplus 912 \oplus 6480 \oplus 27664 \oplus 86184$
$912 \otimes 912$	$=$	$(133 \oplus 1463 \oplus 8645 \oplus 152152 \oplus 253935)_s$ $\oplus (1 \oplus 1539 \oplus 7371 \oplus 40755 \oplus 365750)_a$
$56 \otimes 1463$	$=$	$56 \oplus 6480 \oplus 24320 \oplus 51072$
$133 \otimes 1463$	$=$	$1463 \oplus 1539 \oplus 40755 \oplus 150822$
$912 \otimes 1463$	$=$	$912 \oplus 6480 \oplus 27664 \oplus 51072 \oplus 362880 \oplus 885248$
$1463 \otimes 1463$	$=$	$(1 \oplus 1539 \oplus 7371 \oplus 150822 \oplus 293930 \oplus 617253)_s$ $\oplus (133 \oplus 1463 \oplus 152152 \oplus 915705)_a$
$56 \otimes 1539$	$=$	$56 \oplus 912 \oplus 6480 \oplus 27664 \oplus 51072$
$133 \otimes 1539$	$=$	$133 \oplus 1463 \oplus 1539 \oplus 8645 \oplus 40755 \oplus 152152$
$912 \otimes 1539$	$=$	$56 \oplus 912 \oplus 6480 \oplus 6480 \oplus 27664 \oplus 51072 \oplus 86184$ $\oplus 362880 \oplus 861840$
$1463 \otimes 1539$	$=$	$133 \oplus 1463 \oplus 1539 \oplus 8645 \oplus 40755 \oplus 150822 \oplus 152152$ $\oplus 915705 \oplus 980343$
$1539 \otimes 1539$	$=$	$(133 \oplus 1463 \oplus 8645 \oplus 40755 \oplus 152152 \oplus 980343)_a \oplus$ $(1 \oplus 1539 \oplus 1539 \oplus 7371 \oplus 40755 \oplus 150822 \oplus 365750$ $\oplus 617253)_s$

Table 3.29: Products of E_8 irreducible representations.

$248 \otimes 248$	$=$	$(1 \oplus 3875 \oplus 27000)_s \oplus (248 \oplus 30380)_a$
$248 \otimes 3875$	$=$	$248 \oplus 3875 \oplus 30380 \oplus 147250 \oplus 779247$
$3875 \otimes 3875$	$=$	$(1 \oplus 3875 \oplus 27000 \oplus 147250 \oplus 2450240 \oplus 4881384)_s$ $\oplus (248 \oplus 30380 \oplus 779247 \oplus 6696000)_a$
$248 \otimes 27000$	$=$	$248 \oplus 27000 \oplus 30380 \oplus 779247 \oplus 1763125 \oplus 4096000$
$248 \otimes 30380$	$=$	$248 \oplus 3875 \oplus 27000 \oplus 30380 \oplus 147250 \oplus 779247$ $\oplus 2450240 \oplus 4096000$

Table 3.30: Inner products for the symmetric groups.

Inner products for the symmetric group $\mathfrak{S}_2$

$[2] \otimes [2] = [2]$	$[1^2] \otimes [1^2] = [2]$
$[1^2] \otimes [2] = [1^2]$	

Inner products for the symmetric group $\mathfrak{S}_3$

$[3] \otimes [3] = [3]$	$[1^3] \otimes [1^3] = [3]$
$[3] \otimes [2,1] = [2,1]$	$[1^3] \otimes [2,1] = [2,1]$
$[2,1] \otimes [2,1] = [3] \oplus [2,1] \oplus [1^3]$	$[3] \otimes [1^3] = [1^3]$

Inner products for the symmetric group $\mathfrak{S}_4$

$[4] \otimes [4] = [4]$	$[1^4] \otimes [1^4] = [4]$
$[4] \otimes [3,1] = [3,1]$	$[1^4] \otimes [2,1^2] = [3,1]$
$[4] \otimes [2^2] = [2^2]$	$[1^4] \otimes [2^2] = [2^2]$
$[3,1] \otimes [2^2] = [3,1] \oplus [2,1^2]$	$[2,1^2] \otimes [2^2] = [3,1] \oplus [2,1^2]$
$[1^4] \otimes [3,1] = [3,1]$	$[4] \otimes [2,1^2] = [3,1]$
$[2^2] \otimes [2^2] = [4] \oplus [2^2] \oplus [1^4]$	$[4] \otimes [1^4] = [1^4]$

$[3,1] \otimes [3,1] = [4] \oplus [3,1] \oplus [2^2] \oplus [2,1^2]$
$[2,1^2] \otimes [2,1^2] = [4] \oplus [3,1] \oplus [2^2] \oplus [2,1^2]$
$[3,1] \otimes [2,1^2] = [3,1] \oplus [2^2] \oplus [2,1^2] \oplus [1^4]$

Table 3.31: Branching rules for F_4.

Branching rules for $F_4 \supset so(9)$

26	=	$1 \oplus 9 \oplus 16$
52	=	$16 \oplus 36$
273	=	$9 \oplus 16 \oplus 36 \oplus 84 \oplus 128$
324	=	$1 \oplus 9 \oplus 16 \oplus 44 \oplus 126 \oplus 128$
1053	=	$16 \oplus 36 \oplus 84 \oplus 126 \oplus 128 \oplus 231 \oplus 432$
1053′	=	$126 \oplus 432 \oplus 495$
1274	=	$36 \oplus 84 \oplus 128 \oplus 432 \oplus 594$
2652	=	$1 \oplus 9 \oplus 16 \oplus 44 \oplus 126 \oplus 128 \oplus 156 \oplus 576 \oplus 672 \oplus 924$
4096	=	$9 \oplus 16 \oplus 36 \oplus 44 \oplus 84 \oplus 126 \oplus 128 \oplus 128 \oplus 231 \oplus 432 \oplus 576$ $\oplus 594 \oplus 768 \oplus 924$
8424	=	$84 \oplus 126 \oplus 128 \oplus 231 \oplus 432 \oplus 432 \oplus 495 \oplus 594 \oplus 768 \oplus 924$ $\oplus 1650 \oplus 2560$

Branching rules for $F_4 \supset sl(3) \oplus sl(3)$

26	=	$(8,1) \oplus (3,3) \oplus (\overline{3},\overline{3})$
52	=	$(8,1) \oplus (1,8) \oplus (6,\overline{3}) \oplus (\overline{6},3)$
273	=	$(1,1) \oplus (8,1) \oplus (3,3) \oplus (\overline{3},\overline{3}) \oplus (10,1) \oplus (\overline{10},1) \oplus (6,\overline{3}) \oplus (\overline{6},3)$ $\oplus (3,\overline{6}) \oplus (\overline{3},6) \oplus (15,3) \oplus (\overline{15},\overline{3}) \oplus (8,8)$
324	=	$(1,1) \oplus (8,1) \oplus (1,8) \oplus (3,3) \oplus (\overline{3},\overline{3}) \oplus (27,1) \oplus (6,\overline{3}) \oplus (\overline{6},3)$ $\oplus (6,6) \oplus (\overline{6},\overline{6}) \oplus (15,3) \oplus (\overline{15},\overline{3}) \oplus (8,8)$

Branching rules for $F_4 \supset sl(2) \oplus sp(6)$

26	=	$(2,6) \oplus (1,14)$
52	=	$(3,1) \oplus (1,21) \oplus (2,14)$

Branching rules for $F_4 \supset sl(2) \oplus G_2$

26	=	$(5,1) \oplus (3,7)$
52	=	$(3,1) \oplus (1,14) \oplus (5,7)$

Table 3.32: Branching rules for $so(26) \supset F_4$.

26	=	26
325	=	$273 \oplus 52$
350	=	$324 \oplus 26$
2600	=	$1274 \oplus 1053 \oplus 273$
3250	=	$2652 \oplus 324 \oplus 273 \oplus 1$
5824	=	$4096 \oplus 1053 \oplus 324 \oplus 273 \oplus 52 \oplus 26$
14950	=	$8424 \oplus 4096 \oplus 1053 \oplus 1053' \oplus 324$
23400	=	$16302 \oplus 4096 \oplus 2652 \oplus 324 \oplus 26$
37674	=	$19448 \oplus 8424 \oplus 4096 \oplus 2652 \oplus 1053 \oplus 1053' \oplus 324 \oplus 324 \oplus 273$ $\oplus 26 \oplus 1$
52325	=	$19278 \oplus 10829 \oplus 8424 \oplus 2\,(4096 \oplus 1274 \oplus 1053 \oplus 273) \oplus 324$ $\oplus 52 \oplus 26$
60750	=	$34749 \oplus 10829 \oplus 2652 \oplus 1274 \oplus 2\,(4096 \oplus 1053 \oplus 273) \oplus 324$ $\oplus 52 \oplus 26$

Table 3.33: Branching rules for $so(7) \supset G_2$.

7	=	7
21	=	$7 \oplus 14$
27	=	27
35	=	$1 \oplus 7 \oplus 27$
48	=	$7 \oplus 14 \oplus 27$
77	=	$77'$
105	=	$14 \oplus 27 \oplus 64$
168	=	$27 \oplus 64 \oplus 77$
182	=	182
189	=	$7 \oplus 14 \oplus 27 \oplus 64 \oplus 77'$
330	=	$64 \oplus 77' \oplus 189$
378	=	$7 \oplus 14 \oplus 27 \oplus 64 \oplus 77' \oplus 189$
616	=	$27 \oplus 64 \oplus 77 \oplus 77' \oplus 182 \oplus 189$

Table 3.34: Branching rules for E_6.

Branching rules for $E_6 \supset F_4$

$$\begin{aligned}
27 &= 27 \oplus 1 \\
78 &= 52 \oplus 26 \\
351 &= 273 \oplus 52 \oplus 26 \\
351' &= 324 \oplus 26 \oplus 1 \\
650 &= 324 \oplus 273 \oplus 26 \oplus 26 \oplus 1 \\
1728 &= 1053 \oplus 324 \oplus 273 \oplus 52 \oplus 26 \\
2430 &= 1053 \oplus 1053' \oplus 324 \\
2925 &= 1274 \oplus 1053 \oplus 273 \oplus 273 \oplus 52
\end{aligned}$$

Branching rules for $E_6 \supset so(10)$

$$\begin{aligned}
27 &= 16 \oplus 10 \oplus 1 \\
78 &= 45 \oplus 16 \oplus \overline{16} \oplus 1 \\
351 &= 144 \oplus 120 \oplus 45 \oplus 16 \oplus \overline{16} \oplus 10 \\
351' &= 144 \oplus \overline{126} \oplus 54 \oplus \overline{16} \oplus 10 \oplus 1 \\
650 &= 210 \oplus 144 \oplus \overline{144} \oplus 54 \oplus 45 \oplus 16 \oplus \overline{16} \oplus 10 \oplus 10 \oplus 1 \\
1728 &= 560 \oplus 320 \oplus 210 \oplus 144 \oplus \overline{144} \oplus 126 \oplus 120 \oplus 45 \oplus 16 \oplus 16 \oplus \overline{16} \\
&\quad \oplus 10 \oplus 1 \\
2430 &= 770 \oplus 560 \oplus \overline{560} \oplus 210 \oplus 126 \oplus \overline{126} \oplus 45 \oplus 16 \oplus \overline{16} \oplus 1 \\
2925 &= 945 \oplus 560 \oplus \overline{560} \oplus 210 \oplus 144 \oplus \overline{144} \oplus 120 \oplus 120 \oplus 45 \oplus 45 \oplus 16 \oplus \overline{16}
\end{aligned}$$

Branching rules for $E_6 \supset sl(6) \oplus sl(2)$

$$\begin{aligned}
27 &= (15,1) \oplus (\overline{6},2) \\
78 &= (35,1) \oplus (20,2) \oplus (1,3) \\
351 &= (105,1) \oplus (\overline{84},2) \oplus (21,1) \oplus (15,3) \oplus (\overline{6},2) \\
351' &= (105',1) \oplus (\overline{84},2) \oplus (21,3) \oplus (15,1) \\
650 &= (189,1) \oplus (70,2) \oplus (\overline{70},2) \oplus (35,3) \oplus (35,1) \oplus (20,2) \oplus (1,1) \\
1728 &= (384,1) \oplus (\overline{210},2) \oplus (\overline{120},2) \oplus (105,3) \oplus (105,1) \oplus (\overline{84},2) \oplus (15,3) \\
&\quad \oplus (15,1) \oplus (\overline{6},4) \oplus (\overline{6},2) \\
2430 &= (540,2) \oplus (405,1) \oplus (189,1) \oplus (175,3) \oplus (35,3) \oplus (20,4) \oplus (20,2) \\
&\quad \oplus (1,5) \oplus (1,1) \\
2925 &= (540,2) \oplus (280,1) \oplus (\overline{280},1) \oplus (189,3) \oplus (175,1) \oplus (70,2) \oplus (\overline{70},2) \\
&\quad \oplus (35,3) \oplus (35,1) \oplus (20,4) \oplus (20,2) \oplus (1,3)
\end{aligned}$$

(continued)

Table 3.34 (continued)

Branching rules for $E_6 \supset sl(3) \oplus sl(3) \oplus sl(3)$

$$\begin{aligned}
27 &= (\overline{3},3,1) \oplus (3,1,3) \oplus (1,\overline{3},\overline{3}) \\
78 &= (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,3,\overline{3}) \oplus (\overline{3},\overline{3},3) \\
351 &= (\overline{3},3,1) \oplus (\overline{3},\overline{6},1) \oplus (6,3,1) \oplus (3,1,3) \oplus (\overline{6},1,3) \oplus (3,8,3) \\
&\oplus (1,\overline{3},\overline{3}) \oplus (1,6,\overline{3}) \oplus (8,\overline{3},\overline{3}) \oplus (3,1,\overline{6}) \oplus (1,\overline{3},6) \oplus (\overline{3},3,8) \\
351' &= (1,\overline{3},\overline{3}) \oplus (3,1,3) \oplus (\overline{3},3,1) \oplus (1,6,6) \oplus (\overline{6},1,\overline{6}) \oplus (6,\overline{6},1) \\
&\oplus (8,\overline{3},\overline{3}) \oplus (3,8,3) \oplus (\overline{3},3,8) \\
650 &= 2\,[(1,1,1) \oplus (3,3,\overline{3}) \oplus (\overline{3},\overline{3},3)] \oplus (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \\
&(8,8,1) \oplus (8,1,8) \oplus (1,8,8) \oplus \oplus (6,\overline{3},3) \oplus (\overline{6},3,\overline{3}) \oplus (\overline{3},6,3) \\
&\oplus (3,\overline{6},\overline{3}) \oplus (3,3,6) \oplus (\overline{3},\overline{3},\overline{6}) \\
1728 &= (15,1,3) \oplus (3,1,15) \oplus (\overline{3},15,1) \oplus (\overline{15},3,1) \oplus (1,\overline{3},\overline{15}) \oplus (1,\overline{15},\overline{3}) \\
&\oplus (8,6,\overline{3}) \oplus (8,\overline{3},6) \oplus (6,3,8) \oplus (\overline{6},8,3) \oplus (3,8,\overline{6}) \oplus (\overline{3},\overline{6},8) \\
&\oplus 2\,[(8,\overline{3},\overline{3}) \oplus (\overline{3},3,8) \oplus (3,8,3) \oplus (\overline{3},3,1) \oplus (1,\overline{3},\overline{3}) \oplus (3,1,3)] \\
&\oplus (6,3,1) \oplus (1,\overline{3},6) \oplus (1,6,\overline{3}) \oplus (\overline{6},1,3) \oplus (3,1,\overline{6}) \oplus (\overline{3},\overline{6},1) \\
2430 &= (27,1,1) \oplus (1,27,1) \oplus (1,1,27) \oplus (8,8,8) \oplus (6,6,\overline{6}) \oplus (\overline{6},\overline{6},6) \\
&\oplus (15,3,\overline{3}) \oplus (\overline{15},\overline{3},3) \oplus (3,15,\overline{3}) \oplus (\overline{3},\overline{15},3) \oplus (\overline{3},\overline{3},15) \oplus (3,3,\overline{15}) \\
&\oplus (8,8,1) \oplus (8,1,8) \oplus (1,8,8) \oplus (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \\
&\oplus (6,\overline{3},3) \oplus (\overline{6},3,\overline{3}) \oplus (\overline{3},6,3) \oplus (3,\overline{6},\overline{3}) \oplus (3,3,6) \oplus (\overline{3},\overline{3},\overline{6}) \\
&\oplus (3,3,\overline{3}) \oplus (\overline{3},\overline{3},3) \oplus (1,1,1) \\
2925 &= (15,3,\overline{3}) \oplus (\overline{15},\overline{3},3) \oplus (3,15,\overline{3}) \oplus (\overline{3},\overline{15},3) \oplus (\overline{3},\overline{3},15) \oplus (3,3,\overline{15}) \\
&\oplus (10,1,1) \oplus (1,10,1) \oplus (1,1,10) \oplus (\overline{10},1,1) \oplus (1,\overline{10},1) \oplus (1,1,\overline{10}) \\
&\oplus (8,8,8) \oplus 2\,[(8,8,1) \oplus (8,1,8) \oplus (1,8,8)] \oplus 3\,[(3,3,\overline{3}) \oplus (\overline{3},\overline{3},3)] \\
&\oplus (6,6,3) \oplus (\overline{6},\overline{6},\overline{3}) \oplus (6,\overline{3},\overline{6}) \oplus (\overline{6},3,6) \oplus (\overline{3},6,\overline{6}) \oplus (3,\overline{6},6) \\
&\oplus (6,\overline{3},3) \oplus (\overline{6},3,\overline{3}) \oplus (\overline{3},6,3) \oplus (3,\overline{6},\overline{3}) \oplus (3,3,6) \oplus (\overline{3},\overline{3},\overline{6}) \\
&\oplus (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (1,1,1)
\end{aligned}$$

Branching rules for $E_6 \supset sl(2) \oplus G_2$

$$\begin{aligned}
27 &= (\overline{6},1) \oplus (3,7) \\
78 &= (8,1) \oplus (1,14) \oplus (8,7)
\end{aligned}$$

Branching rules for $E_6 \supset G_2$

$$\begin{aligned}
27 &= 27 \\
78 &= 14 \oplus 64
\end{aligned}$$

Table 3.35: Branching rules for E_7.

Branching rules for $E_7 \supset E_6$

$$\begin{aligned}
56 &= 27 \oplus \overline{27} \oplus 1 \oplus 1 \\
133 &= 78 \oplus 27 \oplus \overline{27} \oplus 1 \\
912 &= 351 \oplus \overline{351} \oplus 78 \oplus 78 \oplus 27 \oplus \overline{27} \\
1463 &= 650 \oplus 351' \oplus \overline{351}' \oplus 27 \oplus 27 \oplus \overline{27} \oplus \overline{27} \oplus 1 \oplus 1 \oplus 1 \\
1539 &= 650 \oplus 351 \oplus \overline{351} \oplus 78 \oplus 27 \oplus 27 \oplus \overline{27} \oplus \overline{27} \oplus 1
\end{aligned}$$

Branching rules for $E_7 \supset sl(8)$

$$\begin{aligned}
56 &= 28 \oplus \overline{28} \\
133 &= 70 \oplus 63 \\
912 &= 420 \oplus \overline{420} \oplus 36 \oplus \overline{36} \\
1463 &= 720 \oplus 336 \oplus \overline{336} \oplus 70 \oplus 1 \\
1539 &= 720 \oplus 378 \oplus \overline{378} \oplus 63
\end{aligned}$$

Branching rules for $E_7 \supset sl(6) \oplus sl(3)$

$$\begin{aligned}
56 &= (20,1) \oplus (6,3) \oplus (\overline{6},\overline{3}) \\
133 &= (35,1) \oplus (15,\overline{3}) \oplus (\overline{15},3) \oplus (1,8) \\
912 &= (84,3) \oplus (\overline{84},\overline{3}) \oplus (70,1) \oplus (\overline{70},1) \oplus (20,8) \oplus (6,\overline{6}) \oplus (\overline{6},6) \\
&\quad \oplus (6,3) \oplus (\overline{6},\overline{3}) \\
1463 &= (175,1) \oplus (105,\overline{3}) \oplus (\overline{105},3) \oplus (35,8) \oplus (35,1) \oplus (21,6) \oplus (\overline{21},\overline{6}) \\
&\quad \oplus (15,\overline{3}) \oplus (\overline{15},3) \oplus (1,1) \\
1539 &= (189,1) \oplus (105,\overline{3}) \oplus (\overline{105},3) \oplus (35,8) \oplus (35,1) \oplus (21,\overline{3}) \oplus (\overline{21},3) \\
&\quad \oplus (15,6) \oplus (\overline{15},\overline{6}) \oplus (15,\overline{3}) \oplus (\overline{15},3) \oplus (1,8) \oplus (1,1)
\end{aligned}$$

Branching rules for $E_7 \supset so(12) \oplus sl(2)$

$$\begin{aligned}
56 &= (32,1) \oplus (12,2) \\
133 &= (66,1) \oplus (32',2) \oplus (1,3) \\
912 &= (352,1) \oplus (220,2) \oplus (32,3) \oplus (12,2) \\
1463 &= (462,1) \oplus (352',2) \oplus (77,3) \oplus (66,1) \\
1539 &= (495,1) \oplus (352',2) \oplus (77,1) \oplus (66,3) \oplus (32',2) \oplus (1,1)
\end{aligned}$$

Branching rules for $E_7 \supset G_2 \oplus sp(6)$

$$\begin{aligned}
56 &= (1,14') \oplus (7,6) \\
133 &= (14,1) \oplus (1,21) \oplus (7,14)
\end{aligned}$$

Table 3.36: Branching rules for E_8.

Branching rules $E_8 \supset so(16)$

$$248 = 128 \oplus 120$$
$$3875 = 1920 \oplus 1820 \oplus 135$$

Branching rules $E_8 \supset sl(9)$

$$248 = 84 \oplus \overline{84} \oplus 80$$
$$3875 = 1215 \oplus 1050 \oplus \overline{1050} \oplus 240 \oplus \overline{240} \oplus 80$$

Branching rules $E_8 \supset E_7 \oplus sl(2)$

$$248 = (133,1) \oplus (56,2) \oplus (1,3)$$
$$3875 = (1539,1) \oplus (912,2) \oplus (133,3) \oplus (56,2) \oplus (1,1)$$

Branching rules $E_8 \supset E_6 \oplus sl(3)$

$$248 = (78,1) \oplus (27,3) \oplus (\overline{27},\overline{3}) \oplus (1,8)$$
$$3875 = (650,1) \oplus (351,3) \oplus (\overline{351},\overline{3}) \oplus (78,8) \oplus (27,\overline{6}) \oplus (\overline{27},6) \oplus (27,3) \oplus (\overline{27},\overline{3}) \oplus (1,8) \oplus (1,1)$$

Branching rules $E_8 \supset sl(5) \oplus sl(5)$

$$248 = (1,24) \oplus (24,1) \oplus (5,\overline{10}) \oplus (\overline{5},10) \oplus (10,5) \oplus (\overline{10},\overline{5})$$
$$3875 = (1,24) \oplus (24,1) \oplus (5,\overline{10}) \oplus (\overline{5},10) \oplus (10,5) \oplus (\overline{10},\overline{5}) \oplus (1,1) \oplus (5,\overline{15}) \oplus (\overline{5},15) \oplus (15,5) \oplus (\overline{15},\overline{5}) \oplus (1,75) \oplus (75,1) \oplus (5,\overline{40}) \oplus (\overline{5},40) \oplus (40,5) \oplus (\overline{40},\overline{5}) \oplus (45,\overline{10}) \oplus (\overline{45},10) \oplus (10,45) \oplus (\overline{10},\overline{45}) \oplus (24,24)$$

Branching rules $E_8 \supset G_2 \oplus F_4$

$$248 = (14,1) \oplus (1,52) \oplus (7,26)$$

Table 3.37: Branching rules for G_2.

Branching rules for $G_2 \supset sl(3)$

$$
\begin{aligned}
7 &= 3 \oplus \overline{3} \oplus 1 \\
14 &= 8 \oplus 3 \oplus \overline{3} \\
27 &= 8 \oplus 6 \oplus \overline{6} \oplus 3 \oplus \overline{3} \oplus 1 \\
64 &= 15 \oplus \overline{15} \oplus 8 \oplus 8 \oplus 6 \oplus \overline{6} \oplus 3 \oplus \overline{3} \oplus 1 \\
77 &= 27 \oplus 15 \oplus \overline{15} \oplus 8 \oplus 6 \oplus \overline{6} \\
77' &= 15 \oplus \overline{15} \oplus 10 \oplus \overline{10} \oplus 6 \oplus \overline{6} \\
182 &= 27 \oplus 24 \oplus \overline{24} \oplus 15 \oplus \overline{15} \oplus 15' \oplus \overline{15}' \oplus 10 \oplus \overline{10} \oplus 8 \oplus 6 \oplus \overline{6} \oplus 3 \oplus \overline{3} \oplus 1 \\
189 &= 27 \oplus 24 \oplus \overline{24} \oplus 2\,(15 \oplus \overline{15} \oplus 8) \oplus 10 \oplus \overline{10} \oplus 6 \oplus \overline{6} \oplus 3 \oplus \overline{3} \\
273 &= 64 \oplus 42 \oplus \overline{42} \oplus 27 \oplus 24 \oplus \overline{24} \oplus 15 \oplus \overline{15} \oplus 10 \oplus \overline{10} \\
286 &= 42 \oplus \overline{42} \oplus 2\,(27 \oplus 15 \oplus \overline{15}) \oplus 24 \oplus \overline{24} \oplus 10 \oplus \overline{10} \oplus 8 \oplus 6 \oplus \overline{6}
\end{aligned}
$$

Branching rules for $G_2 \supset sl(2) \oplus sl(2)$

$$
\begin{aligned}
7 &= (2,2) \oplus (1,3) \\
14 &= (3,1) \oplus (2,4) \oplus (1,3) \\
27 &= (3,3) \oplus (2,4) \oplus (2,2) \oplus (1,5) \oplus (1,1) \\
64 &= (4,2) \oplus (3,5) \oplus (3,3) \oplus (2,6) \oplus (2,4) \oplus (2,2) \oplus (1,5) \oplus (1,3) \\
77 &= (5,1) \oplus (4,4) \oplus (3,7) \oplus (3,3) \oplus (2,6) \oplus (2,4) \oplus (1,5) \oplus (1,1) \\
77' &= (4,4) \oplus (3,5) \oplus (3,3) \oplus (3,1) \oplus (2,6) \oplus (2,4) \oplus (2,2) \oplus (1,7) \oplus (1,3) \\
182 &= (5,5) \oplus (4,6) \oplus (4,4) \oplus (4,2) \oplus (3,7) \oplus (3,5) \oplus (3,3) \oplus (3,3) \oplus (2,8) \\
&\quad \oplus (2,6) \oplus (2,4) \oplus (2,2) \oplus (1,9) \oplus (1,5) \oplus (1,1) \\
189 &= (5,3) \oplus (4,6) \oplus (4,4) \oplus (4,2) \oplus (3,7) \oplus (3,5) \oplus (3,5) \oplus (3,3) \oplus (3,1) \\
&\quad \oplus (2,8) \oplus (2,6) \oplus (2,4) \oplus (2,4) \oplus (2,2) \oplus (1,7) \oplus (1,5) \oplus (1,3) \\
273 &= (7,1) \oplus (6,4) \oplus (5,7) \oplus (5,3) \oplus (4,10) \oplus (4,6) \oplus (4,4) \oplus (3,9) \oplus (3,7) \\
&\quad \oplus (3,5) \oplus (3,1) \oplus (2,8) \oplus (2,6) \oplus (2,4) \oplus (1,7) \oplus (1,3) \\
286 &= (6,2) \oplus (5,5) \oplus (5,3) \oplus (4,8) \oplus (4,6) \oplus (4,4) \oplus (4,2) \oplus (3,9) \oplus (3,7) \\
&\quad \oplus (3,5) \oplus (3,5) \oplus (3,3) \oplus (2,8) \oplus (2,6) \oplus (2,6) \oplus (2,4) \oplus (2,2) \\
&\quad \oplus (1,7) \oplus (1,5) \oplus (1,3)
\end{aligned}
$$

Table 3.38: Real forms of the simple Lie algebras.

$\mathcal{G}$	Compact form	Associated non-compact form	Maximal compact subalgebra
A_{N-1}	$su(N)$	$sl(N,\mathbb{R})$	$so(N)$
	$su(2N)$	$su^*(2N)$	$sp(2N)$
	$su(p+q)$	$su(p,q)$	$su(p) \oplus su(q) \oplus U(1)$
B_N	$so(p+q)$	$so(p,q)$	$so(p) \oplus so(q)$
C_N	$sp(2N)$	$sp(2N,\mathbb{R})$	$su(N) \oplus U(1)$
	$sp(2p+2q)$	$sp(2p,2q)$	$sp(2p) \oplus sp(2q)$
D_N	$so(p+q)$	$so(p,q)$	$so(p) \oplus so(q)$
	$so(2N)$	$so^*(2N)$	$su(N) \oplus U(1)$
E_6	$E_{6(-78)}$	$E_{6(-26)}$	F_4
	$E_{6(-78)}$	$E_{6(-14)}$	$D_5 \oplus \mathbb{R}$
	$E_{6(-78)}$	$E_{6(+2)}$	$A_5 \oplus A_1$
	$E_{6(-78)}$	$E_{6(+6)}$	C_4
E_7	$E_{7(-133)}$	$E_{7(-25)}$	$E_6 \oplus \mathbb{R}$
	$E_{7(-133)}$	$E_{7(-5)}$	$D_6 \oplus A_1$
	$E_{7(-133)}$	$E_{7(+7)}$	A_7
E_8	$E_{8(-248)}$	$E_{8(-24)}$	$E_7 \oplus A_1$
	$E_{8(-248)}$	$E_{8(+8)}$	D_8
F_4	$F_{4(-52)}$	$F_{4(-20)}$	B_4
	$F_{4(-52)}$	$F_{4(+4)}$	$C_3 \oplus A_1$
G_2	$G_{2(-14)}$	$G_{2(+2)}$	$A_1 \oplus A_1$

Table 3.39: Singular subalgebras of the exceptional simple Lie algebras.

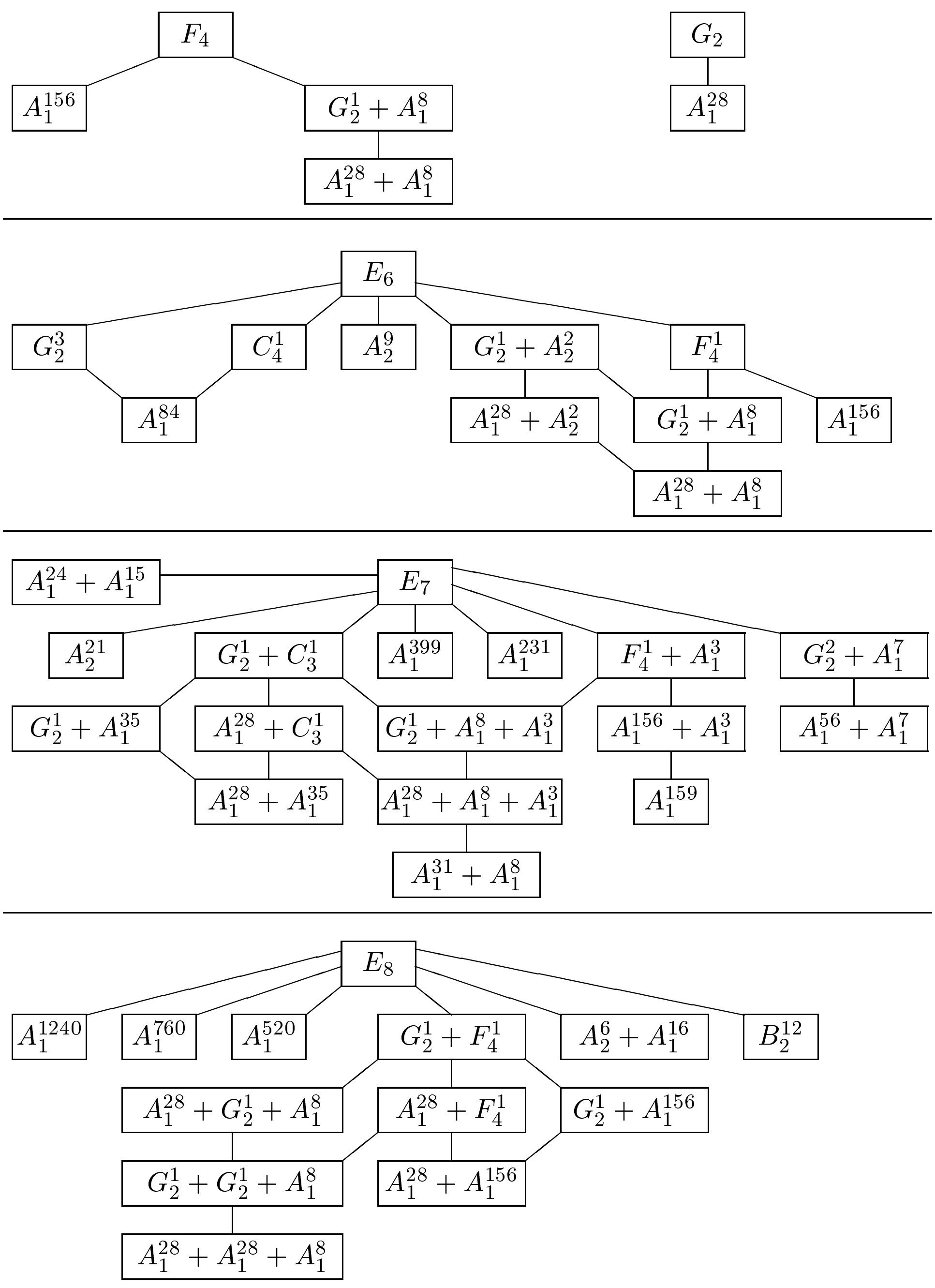


Table 3.40: $sl(2)$ decompositions of the fundamental representations of the classical Lie algebras.

$\mathcal{G}$	$\mathcal{K}$	$\mathrm{fund}\,\mathcal{G}\,/\,\mathcal{K}$
sl_N	sl_p	$D_{(p-1)/2} \oplus (N-p)D_0$
$sp(2N)$	$sp(2p)$	$D_{p-1/2} \oplus (2N-2p+1)D_0$
	$sl(p)$	$2D_{(p-1)/2} \oplus (2N-2p)D_0$
	$sl(2)_2$	$2D_{1/2} \oplus (2N-4)D_0$
	$sl(2)_1$	$D_{1/2} \oplus (2N-2)D_0$
$so(N)$	$so(2p+1)$ $so(2p+2)$	$D_p \oplus (N-2p-1)D_0$
	$sl(p)\ \ (p \neq 2)$	$2D_{(p-1)/2} \oplus (N-2p)D_0$
	$sl(2)_1$	$2D_{1/2} \oplus (N-4)D_0$
	$2sl(2)$	$4D_{1/2} \oplus (N-8)D_0$
	$2sl(2)'$	$D_1 \oplus (N-3)D_0$
$so(2N+1)$	$sl(2)_2$	$D_1 \oplus (2N-2)D_0$
$so(2N)$	$so(2k+1)\oplus$ $so(2N-2k-1)$	$D_k \oplus D_{N-k-1}$

Table 3.41: $sl(2)$ decompositions of the adjoint representations of the classical Lie algebras.

$\mathcal{G}$	$\mathcal{K}$	$\mathrm{ad}\,\mathcal{G}\,/\,\mathcal{K}$
sl_N	sl_p	$\overset{p-1}{\underset{j=1}{\oplus}} D_j \oplus 2(N-p)D_{(p-1)/2} \oplus (N-p)^2 D_0$
$sp(2N)$	$sp(2p)$	$\overset{p-1}{\underset{j=0}{\oplus}} D_j \oplus (2N-2p+1)D_{p-1/2}$ $\oplus(N-p)(2N-2p+1)D_0$
	$sl(p)$	$2\overset{[p/2]}{\underset{j=1}{\oplus}} D_{p-2j} \oplus 2(2N-2p)D_{(p-1)/2}$ $\overset{p-1}{\underset{j=0}{\oplus}} D_j \oplus (N-p)(2N-2p-1)D_0$
	$sl(2)_2$	$3D_1 \oplus (4N-8)D_{1/2} \oplus (2N^2-7N+7)D_0$
	$sl(2)_1$	$D_1 \oplus (2N-2)D_{1/2} \oplus (N-1)(2N-1)D_0$
$so(N)$	$so(2p+1)$ $so(2p+2)$	$2\overset{p}{\underset{j=1}{\oplus}} D_{2j-1} \oplus (N-2p-1)D_p$ $\oplus\frac{1}{2}(N-2p-1)(N-2p-2)D_0$
	$sl(p)\ \ (p \neq 2)$	$\overset{[p/2]}{\underset{j=1}{\oplus}} D_{p-2j} \oplus \overset{p-1}{\underset{j=0}{\oplus}} D_j \oplus 2(N-2p)D_{(p-1)/2}$ $\oplus\frac{1}{2}(N-2p)(N-2p-1)D_0$
	$sl(2)_1$	$D_1 \oplus 2(N-4)D_{1/2}$ $\oplus(3+\frac{1}{2}(N-4)(N-5))D_0$
	$2sl(2)$	$6D_1 \oplus 4(N-8)D_{1/2}$ $\oplus(10+\frac{1}{2}(N-8)(N-9))D_0$
	$2sl(2)'$	$(N-2)D_1 \oplus \frac{1}{2}(N-3)(N-4)D_0$
$so(2N+1)$	$sl(2)_2$	$(2N-1)D_1 \oplus (N-1)(2N-3)D_0$

Table 3.42: $sl(2)$ decompositions of the A_N Lie algebras up to rank 4.

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
A_1	A_1	$D_{1/2}$	D_1
A_2	A_1	$D_{1/2} \oplus D_0$	$D_1 \oplus 2D_{1/2} \oplus D_0$
	A_2	D_1	$D_2 \oplus D_1$
A_3	A_1	$D_{1/2} \oplus 2D_0$	$D_1 \oplus 4D_{1/2} \oplus 4D_0$
	$2A_1$	$2D_{1/2}$	$4D_1 \oplus 3D_0$
	A_2	$D_1 \oplus D_0$	$D_2 \oplus 3D_1 \oplus D_0$
	A_3	$D_{3/2}$	$D_3 \oplus D_2 \oplus D_1$
A_4	A_1	$D_{1/2} \oplus 3D_0$	$D_1 \oplus 6D_{1/2} \oplus 9D_0$
	$2A_1$	$2D_{1/2} \oplus D_0$	$4D_1 \oplus 4D_{1/2} \oplus 4D_0$
	A_2	$D_1 \oplus 2D_0$	$D_2 \oplus 5D_1 \oplus 4D_0$
	$A_2 \oplus A_1$	$D_1 \oplus D_{1/2}$	$D_2 \oplus 2D_{3/2} \oplus 2D_1$ $\oplus 2D_{1/2} \oplus D_0$
	A_3	$D_{3/2} \oplus D_0$	$D_3 \oplus D_2 \oplus 2D_{3/2} \oplus D_1 \oplus D_0$
	A_4	D_2	$D_4 \oplus D_3 \oplus D_2 \oplus D_1$

Table 3.43: $sl(2)$ decompositions of the D_N Lie algebras up to rank 4.

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
D_3	A_1	$2D_{1/2} \oplus 2D_0$	$D_1 \oplus 4D_{1/2} \oplus 4D_0$
	$2A_1$	$D_1 \oplus 3D_0$	$4D_1 \oplus 3D_0$
	A_2	$2D_1$	$D_2 \oplus 3D_1 \oplus D_0$
	D_3	$D_2 \oplus D_0$	$D_3 \oplus D_2 \oplus D_1$
D_4	A_1	$2D_{1/2} \oplus 4D_0$	$D_1 \oplus 8D_{1/2} \oplus 9D_0$
	$2A_1$	$D_1 \oplus 5D_0$	$6D_1 \oplus 10D_0$
	$(2A_1)'$	$4D_{1/2}$	$6D_1 \oplus 10D_0$
	$3A_1$	$D_1 \oplus 2D_{1/2} \oplus D_0$	$2D_{3/2} \oplus 3D_1$ $\oplus 4D_{1/2} \oplus 3D_0$
	A_2 , $4A_1$	$2D_1 \oplus 2D_0$	$D_2 \oplus 7D_1 \oplus 2D_0$
	A_3	$2D_{3/2}$	$D_3 \oplus 3D_2 \oplus D_1 \oplus 3D_0$
	D_3	$D_2 \oplus 3D_0$	$D_3 \oplus 3D_2 \oplus D_1 \oplus 3D_0$
	$B_2 \oplus B_1$	$D_2 \oplus D_1$	$2D_3 \oplus D_2 \oplus 3D_1$
	D_4	$D_3 \oplus D_0$	$D_5 \oplus 2D_3 \oplus D_1$

Table 3.44: $sl(2)$ decompositions of the B_N Lie algebras up to rank 4.

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
B_2	A_1	$2D_{1/2} \oplus D_0$	$D_1 \oplus 2D_{1/2} \oplus 3D_0$
	A_1^2 , $2A_1$	$D_1 \oplus 2D_0$	$3D_1 \oplus D_0$
	B_2	D_2	$D_3 \oplus D_1$
B_3	A_1	$2D_{1/2} \oplus 3D_0$	$D_1 \oplus 6D_{1/2} \oplus 6D_0$
	A_1^2 , $2A_1$	$D_1 \oplus 4D_0$	$5D_1 \oplus 6D_0$
	$A_1 \oplus A_1^2$	$D_1 \oplus 2D_{1/2}$	$2D_{3/2} \oplus 2D_1 \oplus 2D_{1/2} \oplus 3D_0$
	A_2 $2A_1 \oplus A_1^2$	$2D_1 \oplus D_0$	$D_2 \oplus 5D_1 \oplus D_0$
	A_3 , B_2	$D_2 \oplus 2D_0$	$D_3 \oplus 2D_2 \oplus D_1 \oplus D_0$
	B_3	D_3	$D_5 \oplus D_3 \oplus D_1$
B_4	A_1	$2D_{1/2} \oplus 5D_0$	$D_1 \oplus 10D_{1/2} \oplus 13D_0$
	A_1^2 , $2A_1$	$D_1 \oplus 6D_0$	$7D_1 \oplus 15D_0$
	$(2A_1)'$	$4D_{1/2} \oplus D_0$	$6D_1 \oplus 4D_{1/2} \oplus 10D_0$
	$A_1 \oplus A_1^2$ $3A_1$	$D_1 \oplus 2D_{1/2} \oplus 2D_0$	$2D_{3/2} \oplus 4D_1 \oplus 6D_{1/2} \oplus 4D_0$
	A_2 $4A_1$ $2A_1 \oplus A_1^2$	$2D_1 \oplus 3D_0$	$D_2 \oplus 9D_1 \oplus 4D_0$
	$A_2 \oplus A_1^2$	$3D_1$	$3D_2 \oplus 6D_1 \oplus 3D_0$
	A_3	$2D_{3/2} \oplus D_0$	$D_3 \oplus 3D_2 \oplus 2D_{3/2}$ $\oplus D_1 \oplus 3D_0$
	A_3 , B_2	$D_2 \oplus 4D_0$	$D_3 \oplus 4D_2 \oplus D_1 \oplus 6D_0$
	$B_2 \oplus A_1$	$D_2 \oplus 2D_{1/2}$	$D_3 \oplus 2D_{5/2} \oplus 2D_{3/2}$ $\oplus 2D_1 \oplus 3D_0$
	$B_2 \oplus 2A_1$ $A_3 \oplus A_1^2$	$D_2 \oplus D_1 \oplus D_0$	$2D_3 \oplus 2D_2 \oplus 4D_1$
	B_3 , D_4	$D_3 \oplus 2D_0$	$D_5 \oplus 3D_3 \oplus D_1 \oplus D_0$
	B_4	D_4	$D_7 \oplus D_5 \oplus D_3 \oplus D_1$

Table 3.45: $sl(2)$ decompositions of the C_N Lie algebras up to rank 4.

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
C_2	A_1	$D_{1/2} \oplus 2D_0$	$D_1 \oplus 2D_{1/2} \oplus 3D_0$
	A_1^2 , $2A_1$	$2D_{1/2}$	$3D_1 \oplus D_0$
	C_2	$D_{3/2}$	$D_3 \oplus D_1$
C_3	A_1	$D_{1/2} \oplus 4D_0$	$D_1 \oplus 4D_{1/2} \oplus 10D_0$
	A_1^2 , $2A_1$	$2D_{1/2} \oplus 2D_0$	$3D_1 \oplus 4D_{1/2} \oplus 4D_0$
	A_2^2	$2D_1$	$3D_2 \oplus D_1 \oplus 3D_0$
	C_2	$D_{3/2} \oplus 2D_0$	$D_3 \oplus 2D_{3/2} \oplus D_1 \oplus 3D_0$
	$A_1 \oplus A_1^2$, $3A_1$	$3D_{1/2}$	$6D_1 \oplus 3D_0$
	$C_2 \oplus A_1$	$D_{3/2} \oplus D_{1/2}$	$D_3 \oplus D_2 \oplus 3D_1$
	C_3	$D_{5/2}$	$D_5 \oplus D_3 \oplus D_1$
C_4	A_1	$D_{1/2} \oplus 6D_0$	$D_1 \oplus 6D_{1/2} \oplus 21D_0$
	A_1^2 , $2A_1$	$2D_{1/2} \oplus 4D_0$	$3D_1 \oplus 8D_{1/2} \oplus 11D_0$
	$A_1 \oplus A_1^2$, $3A_1$	$3D_{1/2} \oplus 2D_0$	$6D_1 \oplus 6D_{1/2} \oplus 6D_0$
	$2A_1^2$, $4A_1$, $2A_1 \oplus A_1^2$	$4D_{1/2}$	$10D_1 \oplus 6D_0$
	A_2^2	$2D_1 \oplus 2D_0$	$3D_2 \oplus 5D_1 \oplus 6D_0$
	$A_2^2 \oplus A_1$	$2D_1 \oplus D_{1/2}$	$3D_2 \oplus 2D_{3/2} \oplus 2D_1 \oplus 2D_{1/2} \oplus 3D_0$
	C_2	$D_{3/2} \oplus 4D_0$	$D_3 \oplus 4D_{3/2} \oplus D_1 \oplus 10D_0$
	$C_2 \oplus A_1$	$D_{3/2} \oplus D_{1/2} \oplus 2D_0$	$D_3 \oplus D_2 \oplus 2D_{3/2} \oplus 3D_1 \oplus 2D_{1/2} \oplus 3D_0$
	$C_2 \oplus A_1^2$, $C_2 \oplus 2A_1$	$D_{3/2} \oplus 2D_{1/2}$	$D_3 \oplus 2D_2 \oplus 6D_1 \oplus D_0$
	A_3^2 , $2C_2$	$2D_{3/2}$	$3D_3 \oplus D_2 \oplus 3D_1 \oplus D_0$
	C_3	$D_{5/2} \oplus 2D_0$	$D_5 \oplus D_3 \oplus 2D_{5/2} \oplus D_1 \oplus 3D_0$
	$C_3 \oplus A_1$	$D_{5/2} \oplus D_{1/2}$	$D_5 \oplus 2D_3 \oplus D_2 \oplus 2D_1$
	C_4	$D_{7/2}$	$D_7 \oplus D_5 \oplus D_3 \oplus D_1$

Table 3.46: $sl(2)$ decompositions of the exceptional Lie algebras G_2 and F_4 (fundamental representation).

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$
G_2	A_1	$2D_{1/2} \oplus 3D_0$
	A_1^2	$D_1 \oplus 2D_{1/2}$
	$A_1 \oplus A_1^2$	$2D_1 \oplus D_0$
	G_2	D_3
F_4	A_1	$6D_{1/2} \oplus 14D_0$
	A_1^2 , $2A_1$	$D_1 \oplus 8D_{1/2} \oplus 7D_0$
	$A_1 \oplus A_1^2$ $3A_1$	$3D_1 \oplus 6D_{1/2} \oplus 5D_0$
	$4A_1$ $2A_1 \oplus A_1^2$ A_2	$6D_1 \oplus 8D_0$
	A_2^2	$D_2 \oplus 7D_1$
	$A_2 \oplus A_1^2$	$D_2 \oplus 2D_{3/2} \oplus 3D_1 \oplus 2D_{1/2}$
	$A_1 \oplus A_2^2$	$2D_{3/2} \oplus 3D_1 \oplus 4D_{1/2} \oplus D_0$
	$A_2^2 \oplus A_2$ $A_3 \oplus A_1^2$ $B_2 \oplus 2A_1$	$3D_2 \oplus 3D_1 \oplus 2D_0$
	B_2 , A_3	$D_2 \oplus 4D_{3/2} \oplus 5D_0$
	$B_2 \oplus A_1$	$2D_2 \oplus 2D_{3/2} \oplus D_1 \oplus 2D_{1/2} \oplus D_0$
	B_3 , D_4	$3D_3 \oplus 5D_0$
	B_4	$D_5 \oplus D_4 \oplus D_2 \oplus D_0$
	C_3	$D_4 \oplus 2D_{5/2} \oplus D_2$
	$C_3 \oplus A_1$	$D_4 \oplus D_3 \oplus 2D_2$
	F_4	$D_8 \oplus D_4$

Table 3.47: $sl(2)$ decompositions of the exceptional Lie algebras G_2 and F_4 (adjoint representation).

$\mathcal{G}$	Subalgebra in $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
G_2	A_1	$D_1 \oplus 4D_{1/2} \oplus 3D_0$
	A_1^2	$2D_{3/2} \oplus D_1 \oplus 3D_0$
	$A_1 \oplus A_1^2$	$D_2 \oplus 3D_1$
	G_2	$D_5 \oplus D_1$
F_4	A_1	$D_1 \oplus 14D_{1/2} \oplus 21D_0$
	A_1^2 , $2A_1$	$7D_1 \oplus 10D_{1/2} \oplus 15D_0$
	$A_1 \oplus A_1^2$, $3A_1$	$2D_{3/2} \oplus 6D_1 \oplus 10D_{1/2} \oplus 6D_0$
	$4A_1$, $2A_1 \oplus A_1^2$, A_2	$D_2 \oplus 13D_1 \oplus 8D_0$
	A_2^2	$7D_2 \oplus D_1 \oplus 14D_0$
	$A_2 \oplus A_1^2$	$2D_3 \oplus 3D_2 \oplus 6D_1 \oplus 2D_{1/2} \oplus D_0$
	$A_1 \oplus A_2^2$	$3D_2 \oplus 2D_{3/2} \oplus 6D_1 \oplus 4D_{1/2} \oplus 3D_0$
	$A_2^2 \oplus A_2$, $A_3 \oplus A_1^2$, $B_2 \oplus 2A_1$	$2D_3 \oplus 4D_2 \oplus 6D_1$
	B_2 , A_3	$D_3 \oplus 4D_2 \oplus 4D_{3/2} \oplus D_1 \oplus 6D_0$
	$B_2 \oplus A_1$	$D_3 \oplus 2D_{5/2} \oplus D_2 \oplus 4D_{3/2} \oplus 3D_1 \oplus 3D_0$
	B_3 , D_4	$D_5 \oplus 5D_3 \oplus D_1 \oplus 3D_0$
	B_4	$D_7 \oplus 2D_5 \oplus D_3 \oplus D_2 \oplus D_1$
	C_3	$D_5 \oplus 2D_{9/2} \oplus D_3 \oplus 2D_{3/2} \oplus D_1 \oplus 3D_0$
	$C_3 \oplus A_1$	$2D_5 \oplus D_4 \oplus D_3 \oplus D_2 \oplus 3D_1$
	F_4	$D_{11} \oplus D_7 \oplus D_5 \oplus D_1$

Tables on Lie superalgebras

Table 3.48: Classification of the simple Lie superalgebras.

Simple Lie superalgebras
↙ ↘

Classical Lie superalgebras
↙ ↘

Cartan type superalgebras
$W(n), S(n)$
$\widetilde{S}(n), H(n)$

Basic Lie superalgebras
$A(m,n), B(m,n)$
$C(n+1), D(m,n)$
$F(4), G(3), D(2,1;\alpha)$

Strange superalgebras
$P(n), Q(n)$

Table 3.49: Classical Lie superalgebras.

	type I	type II
BASIC (non-degenerate Killing form)	$A(m,n)$ $m > n \geq 0$ $C(n+1)$ $n \geq 1$	$B(m,n)$ $m \geq 0, n \geq 1$ $D(m,n)$ $\begin{cases} m \geq 2, n \geq 1 \\ m \neq n+1 \end{cases}$ $F(4)$ $G(3)$
BASIC (zero Killing form)	$A(n,n)$ $n \geq 1$	$D(n+1,n)$ $n \geq 1$ $D(2,1;\alpha)$ $\alpha \notin \{0,-1\}$
STRANGE	$P(n)$ $n \geq 2$	$Q(n)$ $n \geq 2$

Table 3.50: $\mathbb{Z}_2$-gradation of the classical Lie superalgebras.

superalgebra $\mathcal{G}$	even part $\mathcal{G}_{\overline{0}}$	odd part $\mathcal{G}_{\overline{1}}$
$A(m,n)$	$A_m \oplus A_n \oplus U(1)$	$(m,\overline{n}) \oplus (\overline{m},n)$
$A(n,n)$	$A_n \oplus A_n$	$(n,\overline{n}) \oplus (\overline{n},n)$
$C(n+1)$	$C_n \oplus U(1)$	$(2n) \oplus (2n)$
$B(m,n)$	$B_m \oplus C_n$	$(2m+1,2n)$
$D(m,n)$	$D_m \oplus C_n$	$(2m,2n)$
$F(4)$	$A_1 \oplus B_3$	$(2,8)$
$G(3)$	$A_1 \oplus G_2$	$(2,7)$
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus A_1$	$(2,2,2)$
$P(n)$	A_n	$[2] \oplus [1^{n-1}]$
$Q(n)$	A_n	$\mathrm{ad}(A_n)$

Table 3.51: $\mathbb{Z}$-gradation of the classical basic Lie superalgebras.

$\mathcal{G}$	$\mathcal{G}_0$	$\mathcal{G}_1 \oplus \mathcal{G}_{-1}$	$\mathcal{G}_2 \oplus \mathcal{G}_{-2}$
$A(m,n)$	$A_m \oplus A_n \oplus U(1)$	$(m,\overline{n}) \oplus (\overline{m},n)$	$\emptyset$
$A(n,n)$	$A_n \oplus A_n$	$(n,\overline{n}) \oplus (\overline{n},n)$	$\emptyset$
$C(n+1)$	$C_n \oplus U(1)$	$(2n)_+ \oplus (2n)_-$	$\emptyset$
$B(m,n)$	$B_m \oplus A_{n-1} \oplus U(1)$	$(2m+1,n) \oplus (2m+1,\overline{n})$	$[2] \oplus [2^{n-1}]$
$D(m,n)$	$D_m \oplus A_{n-1} \oplus U(1)$	$(2m,n) \oplus (2m,\overline{n})$	$[2] \oplus [2^{n-1}]$
$F(4)$	$B_3 \oplus U(1)$	$8_+ \oplus 8_-$	$1_+ \oplus 1_-$
$G(3)$	$G_2 \oplus U(1)$	$7_+ \oplus 7_-$	$1_+ \oplus 1_-$
$D(2,1;\alpha)$	$A_1 \oplus A_1 \oplus U(1)$	$(2,2)_+ \oplus (2,2)_-$	$1_+ \oplus 1_-$

Table 3.52: The basic Lie superalgebra $A(m-1,n-1) = sl(m|n)$.

Structure: $\mathcal{G}_{\overline{0}} = sl(m) \oplus sl(n) \oplus U(1)$ and $\mathcal{G}_{\overline{1}} = (\overline{m}, n, 1) \oplus (m, \overline{n}, -1)$, type I.
Rank: $m+n-1$, dimension: $(m+n)^2-1$.
Root system ($1 \le i \ne j \le m$ and $1 \le k \ne l \le n$):

$$\Delta = \{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_l,\ \varepsilon_i - \delta_k,\ \delta_k - \varepsilon_i\}$$
$$\Delta_{\overline{0}} = \overline{\Delta}_{\overline{0}} = \{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_l\}, \quad \Delta_{\overline{1}} = \overline{\Delta}_{\overline{1}} = \{\varepsilon_i - \delta_k,\ \delta_k - \varepsilon_i\}$$

$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = m^2 + n^2 - m - n + 1$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 2mn$.
Distinguished simple root system:

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m$$

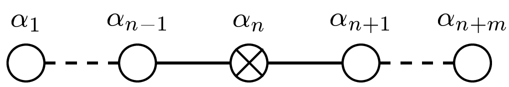
Distinguished positive roots ($1 \le i < j \le m$ and $1 \le k < l \le n$):

$$\begin{aligned} \delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\ \varepsilon_i - \varepsilon_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\ \delta_k - \varepsilon_i &= \alpha_k + \cdots + \alpha_{n+i-1} \end{aligned}$$

Sums of even/odd distinguished positive roots:

$$\begin{aligned} 2\rho_0 &= (m-1)\varepsilon_1 + (m-3)\varepsilon_2 + \cdots - (m-3)\varepsilon_{m-1} - (m-1)\varepsilon_m \\ &\quad +(n-1)\delta_1 + (n-3)\delta_2 + \cdots - (n-3)\delta_{n-1} - (n-1)\delta_n \\ &= \sum_{i=1}^{n}(n-2i+1)\delta_i + \sum_{j=1}^{m}(m-2j+1)\varepsilon_j \\ &= \sum_{i=1}^{n-1} i(n-i)\alpha_i + \sum_{j=1}^{m-1} j(m-j)\alpha_{n+j} \\ 2\rho_1 &= n(\varepsilon_1 + \cdots + \varepsilon_m) - m(\delta_1 + \cdots + \delta_n) \end{aligned}$$

Distinguished Dynkin diagram:



(continued)

Table 3.52 (continued)

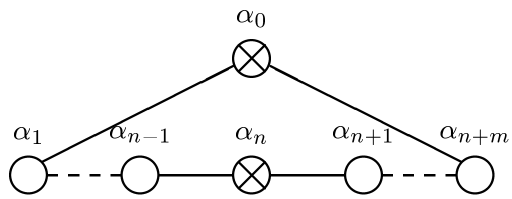
Distinguished Cartan matrix:

$$
\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
& & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
& & & & & & 0 & \ddots & & \ddots & 0 \\
& & & & & & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{array}\right)
$$

Highest distinguished root:

$$-\alpha_0 = \alpha_1 + \cdots + \alpha_{m+n-1} = \delta_1 - \varepsilon_m$$

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$:

$\mathrm{Out}(\mathcal{G}) = \mathbb{Z}_2$ for $A(m,n)$ with $m \neq n \neq 0$ and $A(0,2n-1)$
$\mathrm{Out}(\mathcal{G}) = \mathbb{Z}_4$ for $A(0,2n)$

Table 3.53: The basic Lie superalgebra $A(n-1,n-1) = psl(n|n)$ $(n>1)$.

Structure: $\mathcal{G}_{\overline{0}} = sl(n) \oplus sl(n)$ and $\mathcal{G}_{\overline{1}} = (\overline{n}, n) \oplus (n, \overline{n})$, type I.
Rank: $2n-2$, dimension: $4n^2-2$.
Root system $(1 \le i \ne j \le n)$:

$$\Delta = \{\varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j,\ \varepsilon_i - \delta_j,\ \delta_j - \varepsilon_i\}$$
$$\Delta_{\overline{0}} = \overline{\Delta}_{\overline{0}} = \{\varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j\}, \quad \Delta_{\overline{1}} = \overline{\Delta}_{\overline{1}} = \{\varepsilon_i - \delta_j,\ \delta_j - \varepsilon_i\}$$

where $\sum_{i=1}^n \varepsilon_i = \sum_{i=1}^n \delta_i$.
$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = 2n^2 - 2n$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 2n^2$.
Distinguished simple root system:

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{2n-1} = \varepsilon_{n-1} - \varepsilon_n$$

Number of simple roots $= 2n-1$ ($\ne$ rank); the simple roots are not independent:

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n + (n-1)\alpha_{n+1} + \cdots + 2\alpha_{2n-2} + \alpha_{2n-1} = 0$$

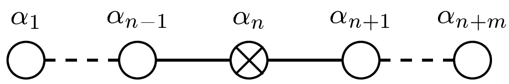
Distinguished positive roots $(1 \le i < j \le n)$:

$$\begin{aligned} \delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\ \varepsilon_i - \varepsilon_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\ \delta_k - \varepsilon_i &= \alpha_k + \cdots + \alpha_{n+i-1} \end{aligned}$$

Sums of even/odd distinguished positive roots:

$$\begin{aligned} 2\rho_0 &= (n-1)\varepsilon_1 + (n-3)\varepsilon_2 + \cdots - (n-3)\varepsilon_{n-1} - (n-1)\varepsilon_n \\ &\quad +(n-1)\delta_1 + (n-3)\delta_2 + \cdots - (n-3)\delta_{n-1} - (n-1)\delta_n \\ &= \sum_{i=1}^n (n-2i+1)(\varepsilon_i + \delta_i) \\ 2\rho_1 &= 0 \end{aligned}$$

Distinguished Dynkin diagram:



(continued)

Table 3.53 (continued)

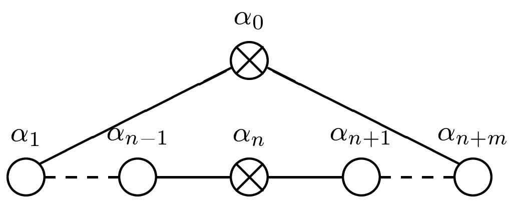
Distinguished Cartan matrix:

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & & \ddots & 0 \\
 & & & & & & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{array}\right)$$

Highest distinguished root:

$$-\alpha_0 = \alpha_1 + \cdots + \alpha_{2n-1} = \delta_1 - \varepsilon_n$$

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$:

$$\begin{array}{l}
\mathrm{Out}(\mathcal{G}) \supset \mathbb{Z}_2 \times \mathbb{Z}_2 \ \text{ for } \ A(n,n) \ \text{ with } \ n \neq 1 \\
\mathrm{Out}(\mathcal{G}) \supset \mathbb{Z}_2 \ \text{ for } \ A(1,1)
\end{array}$$

(see section 2.1 for more details.)

Table 3.54: The basic Lie superalgebra $B(m,n) = osp(2m+1|2n)$.

Structure: $\mathcal{G}_{\overline{0}} = so(2m+1) \oplus sp(2n)$ and $\mathcal{G}_{\overline{1}} = (2m+1, 2n)$, type II.
Rank: $m+n$, dimension: $2(m+n)^2 + m + 3n$.
Root system ($1 \le i \ne j \le m$ and $1 \le k \ne l \le n$):

$$\begin{array}{l}\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\varepsilon_i \pm \delta_k,\ \pm\delta_k\} \\ \Delta_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k\}, \quad \Delta_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_k,\ \pm\delta_k\} \\ \overline{\Delta}_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_l\}, \quad \overline{\Delta}_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_k\}\end{array}$$

$\dim \Delta_{\overline{0}} = 2m^2 + 2n^2$, $\dim \Delta_{\overline{1}} = 4mn + 2n$, $\dim \overline{\Delta}_{\overline{0}} = 2m^2 + 2n^2 - 2n$, $\dim \overline{\Delta}_{\overline{1}} = 4mn$.
Distinguished simple root system:

$$\begin{array}{l}\alpha_1 = \delta_1 - \delta_2,\ \dots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1, \\ \alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \dots,\ \alpha_{n+m-1} = \varepsilon_{m-1} - \varepsilon_m,\ \alpha_{n+m} = \varepsilon_m\end{array}$$

Distinguished positive roots ($1 \le i < j \le m$ and $1 \le k < l \le n$):

$$\begin{array}{rcl}\delta_k - \delta_l &=& \alpha_k + \cdots + \alpha_{l-1} \\ \delta_k + \delta_l &=& \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n+m} \\ 2\delta_k &=& 2\alpha_k + \cdots + 2\alpha_{n+m} \\ \varepsilon_i - \varepsilon_j &=& \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\ \varepsilon_i + \varepsilon_j &=& \alpha_{n+i} + \cdots + \alpha_{n+j-1} + 2\alpha_{n+j} + \cdots + 2\alpha_{n+m} \\ \varepsilon_i &=& \alpha_{n+i} + \cdots + \alpha_{n+m} \\ \delta_k - \varepsilon_i &=& \alpha_k + \cdots + \alpha_{n+i-1} \\ \delta_k + \varepsilon_i &=& \alpha_k + \cdots + \alpha_{n+i-1} + 2\alpha_{n+i} + \cdots + 2\alpha_{n+m} \\ \delta_k &=& \alpha_k + \cdots + \alpha_{n+m}\end{array}$$

Sums of even/odd distinguished positive roots:

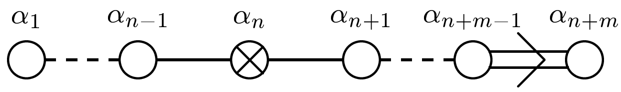
$$\begin{array}{rcl}2\rho_0 &=& (2m-1)\varepsilon_1 + (2m-3)\varepsilon_2 + \cdots + 3\varepsilon_{m-1} + \varepsilon_m \\ && +2n\delta_1 + (2n-2)\delta_2 + \cdots + 4\delta_{n-1} + 2\delta_n \\ &=& \displaystyle\sum_{i=1}^{m}(2m-2i+1)\varepsilon_i + \sum_{k=1}^{n}(2n-2i+2)\delta_k \\ &=& \displaystyle\sum_{k=1}^{n-1} k(2n-k+1)\alpha_k + n(n+1)\sum_{k=n}^{n+m}\alpha_k + \sum_{i=1}^{m} i(2m-i)\alpha_i\end{array}$$

(continued)

Table 3.54 (continued)

$$\begin{aligned} 2\rho_1 &= (2m+1)(\delta_1 + \cdots + \delta_n) \\ &= (2m+1)\sum_{k=1}^{n} k\alpha_k + n(2m+1)\sum_{i=1}^{m} \alpha_{n+j} \end{aligned}$$

Distinguished Dynkin diagram:



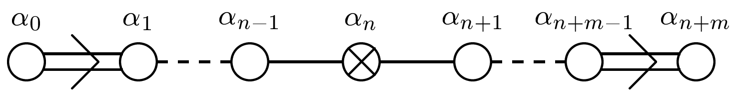
Distinguished Cartan matrix:

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & \ddots & -1 & 0 \\
 & & & & & & & \ddots & -1 & 2 & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -2 & 2
\end{array}\right)$$

Highest distinguished root:

$$-\alpha_0 = 2\alpha_1 + \cdots + 2\alpha_{n+m} = 2\delta_1$$

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.55: The basic Lie superalgebra $B(0,n) = osp(1|2n)$.

Structure: $\mathcal{G}_{\overline{0}} = sp(2n)$ and $\mathcal{G}_{\overline{1}} = (2n)$, type II.
Rank: n, dimension: $2n^2 + 3n$.
Root system $(1 \le k \ne l \le n)$:

$$\begin{aligned}
&\Delta = \{\pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\delta_k\} \\
&\Delta_{\overline{0}} = \{\pm\delta_k \pm \delta_l,\ \pm 2\delta_k\}, \quad \Delta_{\overline{1}} = \{\pm\delta_k\} \\
&\overline{\Delta}_{\overline{0}} = \{\pm\delta_k \pm \delta_l\}, \quad \overline{\Delta}_{\overline{1}} = \emptyset
\end{aligned}$$

$\dim \Delta_{\overline{0}} = 2n^2$, $\dim \Delta_{\overline{1}} = 2n$, $\dim \overline{\Delta}_{\overline{0}} = 2n^2 - 2n$, $\dim \overline{\Delta}_{\overline{1}} = 0$.
Simple root system:

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n$$

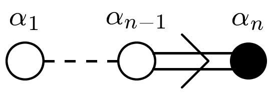
Positive roots $(1 \le k < l \le n)$:

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\
\delta_k + \delta_l &= \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n+m} \\
2\delta_k &= 2\alpha_k + \cdots + 2\alpha_{n+m} \\
\delta_k &= \alpha_k + \cdots + \alpha_{n+m}
\end{aligned}$$

Sums of even/odd positive roots:

$$\begin{aligned}
2\rho_0 &= 2n\delta_1 + (2n-2)\delta_2 + \cdots + 4\delta_{n-1} + 2\delta_n = \sum_{k=1}^{n} (2n - 2i + 2)\delta_k \\
2\rho_1 &= \delta_1 + \cdots + \delta_n
\end{aligned}$$

Dynkin diagram:



Cartan matrix:

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 & -1 & 0 \\
\vdots & & \ddots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -2 & 2
\end{pmatrix}$$

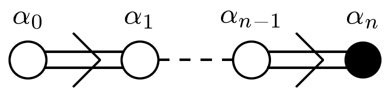
(continued)

Table 3.55 (continued)

Highest root:

$$-\alpha_0 = 2\alpha_1 + \cdots + 2\alpha_n = 2\delta_1$$

Extended Dynkin diagram:



Factor group $\text{Out}(\mathcal{G}) = \text{Aut}(\mathcal{G})/\text{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.56: The basic Lie superalgebra $C(n+1) = osp(2|2n)$.

Structure: $\mathcal{G}_{\overline{0}} = so(2) \oplus sp(2n)$ and $\mathcal{G}_{\overline{1}} = (2n) \oplus (2n)$, type I.
Rank: $n+1$, dimension: $2n^2 + 5n + 1$.
Root system ($1 \le k \ne l \le n$):

$$\Delta = \{\pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\varepsilon \pm \delta_k\}$$
$$\Delta_{\overline{0}} = \overline{\Delta}_{\overline{0}} = \{\pm\delta_k \pm \delta_l,\ \pm 2\delta_k\}, \quad \Delta_{\overline{1}} = \overline{\Delta}_{\overline{1}} = \{\pm\varepsilon \pm \delta_k\}$$

$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = 2n^2$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 4n$.
Distinguished simple root system:

$$\alpha_1 = \varepsilon - \delta_1,\ \alpha_2 = \delta_1 - \delta_2,\ \dots,\ \alpha_n = \delta_{n-1} - \delta_n,\ \alpha_{n+1} = 2\delta_n$$

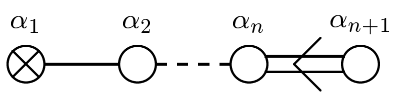
Distinguished positive roots ($1 \le k < l \le n$):

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_{k+1} + \cdots + \alpha_l \\
\delta_k + \delta_l &= \alpha_{k+1} + \cdots + \alpha_l + 2\alpha_{l+1} + \cdots + 2\alpha_n + \alpha_{n+1} \quad (l < n) \\
\delta_k + \delta_n &= \alpha_{k+1} + \cdots + \alpha_{n+1} \\
2\delta_k &= 2\alpha_{k+1} + \cdots + 2\alpha_n + \alpha_{n+1} \quad (k < n) \\
2\delta_n &= \alpha_{n+1} \\
\varepsilon - \delta_k &= \alpha_1 + \cdots + \alpha_k \\
\varepsilon + \delta_k &= \alpha_1 + \cdots + \alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_n + \alpha_{n+1} \quad (k < n) \\
\varepsilon + \delta_n &= \alpha_1 + \cdots + \alpha_{n+1}
\end{aligned}$$

Sums of even/odd distinguished positive roots:

$$\begin{aligned}
2\rho_0 &= 2n\delta_1 + (2n-2)\delta_2 + \cdots + 4\delta_{n-1} + 2\delta_n = \sum_{k=1}^{n} (2n - 2i + 2)\delta_k \\
2\rho_1 &= 2n\varepsilon
\end{aligned}$$

Distinguished Dynkin diagram:



(continued)

Table 3.56 (continued)

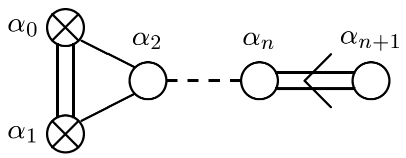
Distinguished Cartan matrix:

$$\left(\begin{array}{c|cccccc} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{array}\right)$$

Highest distinguished root:

$$-\alpha_0 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n+1} + \alpha_n = \varepsilon + \delta_1$$

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{Z}_2$.

Table 3.57: The basic Lie superalgebra $D(m,n) = osp(2m|2n)$.

Structure: $\mathcal{G}_{\overline{0}} = so(2m) \oplus sp(2n)$ and $\mathcal{G}_{\overline{1}} = (2m, 2n)$, type II.
Rank: $m+n$, dimension: $2(m+n)^2 - m + n$.
Root system ($1 \le i \ne j \le m$ and $1 \le k \ne l \le n$):

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k,\ \pm\varepsilon_i \pm \delta_k\}$$
$$\Delta_{\overline{0}} = \overline{\Delta}_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_k \pm \delta_l,\ \pm 2\delta_k\}, \quad \Delta_{\overline{1}} = \overline{\Delta}_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_k\}$$

$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = 2m^2 + 2n^2 - 2m$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 4mn$.
Distinguished simple root system:

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n+m-1} = \varepsilon_{m-1} - \varepsilon_m,\ \alpha_{n+m} = \varepsilon_{m-1} + \varepsilon_m$$

Distinguished positive roots ($1 \le i < j \le m$ and $1 \le k < l \le n$):

$$\begin{aligned}
\delta_k - \delta_l &= \alpha_k + \cdots + \alpha_{l-1} \\
\delta_k + \delta_l &= \alpha_k + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
2\delta_k &= 2\alpha_k + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
\varepsilon_i - \varepsilon_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} \\
\varepsilon_i + \varepsilon_j &= \alpha_{n+i} + \cdots + \alpha_{n+j-1} + 2\alpha_{n+j} + \cdots + 2\alpha_{n+m-2} \\
&\quad + \alpha_{n+m-1} + \alpha_{n+m}\ (j < m-1) \\
\varepsilon_i + \varepsilon_{m-1} &= \alpha_{n+i} + \cdots + \alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
\varepsilon_i + \varepsilon_m &= \alpha_{n+i} + \cdots + \alpha_{n+m-2} + \alpha_{n+m} \\
\delta_k - \varepsilon_i &= \alpha_k + \cdots + \alpha_{n+i-1} \\
\delta_k + \varepsilon_i &= \alpha_k + \cdots + \alpha_{n+i-1} + 2\alpha_{n+i} + \cdots + 2\alpha_{n+m-2} \\
&\quad + \alpha_{n+m-1} + \alpha_{n+m}\ (i < m-1) \\
\delta_k + \varepsilon_{m-1} &= \alpha_k + \cdots + \alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} \\
\delta_k + \varepsilon_m &= \alpha_k + \cdots + \alpha_{n+m-2} + \alpha_{n+m}
\end{aligned}$$

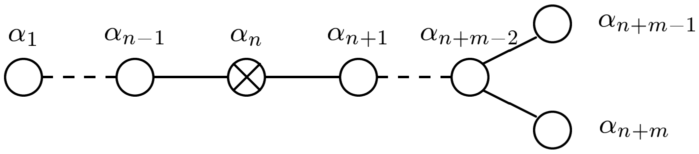
Sums of even/odd distinguished positive roots:

$$\begin{aligned}
2\rho_0 &= (2m-2)\varepsilon_1 + (2m-4)\varepsilon_2 + \cdots + 2\varepsilon_{m-1} \\
&\quad + 2n\delta_1 + (2n-2)\delta_2 + \cdots + 4\delta_{n-1} + 2\delta_n \\
&= \sum_{i=1}^{m} (2m-2i)\varepsilon_i + \sum_{k=1}^{n} (2n-2k+2)\delta_k \\
2\rho_1 &= 2m(\delta_1 + \cdots + \delta_n)
\end{aligned}$$

(continued)

Table 3.57 (continued)

Distinguished Dynkin diagram:



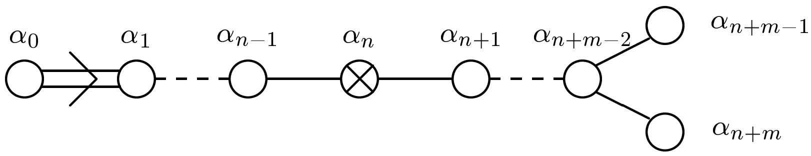
Distinguished Cartan matrix:

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
& & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
& & & & & & 0 & \ddots & \ddots & -1 & -1 \\
& & & & & & & \ddots & -1 & 2 & 0 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & -1 & 0 & 2
\end{array}\right)$$

Highest distinguished root:

$$-\alpha_0 = 2\alpha_1 + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} = 2\delta_1$$

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{Z}_2$.

Table 3.58: The basic Lie superalgebra $F(4)$.

Structure: $\mathcal{G}_{\overline{0}} = sl(2) \oplus so(7)$ and $\mathcal{G}_{\overline{1}} = (2,8)$, type II.
Rank: 4, dimension: 40.
Root system ($1 \le i \ne j \le 3$):

$$\Delta = \{\pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\}$$
$$\Delta_{\overline{0}} = \overline{\Delta}_{\overline{0}} = \{\pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i\}, \quad \Delta_{\overline{1}} = \overline{\Delta}_{\overline{1}} = \{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\}$$

$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = 20$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 16$.
Distinguished simple root system:

$$\alpha_1 = \tfrac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3),\ \alpha_2 = \varepsilon_3,\ \alpha_3 = \varepsilon_2 - \varepsilon_3,\ \alpha_4 = \varepsilon_1 - \varepsilon_2$$

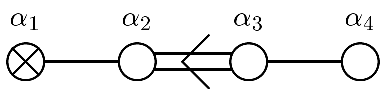
Distinguished positive roots ($1 \le i < j \le 3$):

$$\begin{array}{rcl}
\varepsilon_i - \varepsilon_j &=& \alpha_3,\ \alpha_4,\ \alpha_3 + \alpha_4 \\
\varepsilon_i + \varepsilon_j &=& 2\alpha_2 + \alpha_3,\ 2\alpha_2 + \alpha_3 + \alpha_4,\ 2\alpha_2 + 2\alpha_3 + \alpha_4 \\
\varepsilon_i &=& \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4 \\
\delta &=& 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 \\
\tfrac{1}{2}(\delta \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3) &=& \alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
&& \alpha_1 + 2\alpha_2 + \alpha_3,\ \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\
&& \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4
\end{array}$$

Sums of even/odd distinguished positive roots:

$$2\rho_0 = 5\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 + \delta$$
$$2\rho_1 = 4\delta$$

Distinguished Dynkin diagram:



Distinguished Cartan matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

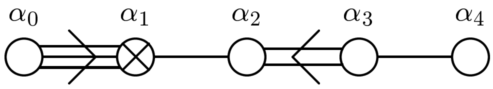
Highest distinguished root:

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = \delta$$

(continued)

Table 3.58 (continued)

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.59: The basic Lie superalgebra $G(3)$.

Structure: $\mathcal{G}_{\overline{0}} = sl(2) \oplus G_2$ and $\mathcal{G}_{\overline{1}} = (2,7)$, type II.
Rank: 3, dimension: 31.
Root system ($1 \le i \ne j \le 3$):

$$\begin{aligned}
&\Delta = \{\pm 2\delta,\ \pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j,\ \pm\delta,\ \pm\varepsilon_i \pm \delta\} \\
&\Delta_{\overline{0}} = \{\pm 2\delta,\ \pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j\}\,, \quad \Delta_{\overline{1}} = \{\pm\delta,\ \pm\varepsilon_i \pm \delta\} \\
&\overline{\Delta}_{\overline{0}} = \{\pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j\}\,, \quad \overline{\Delta}_{\overline{1}} = \{\pm\varepsilon_i \pm \delta\}
\end{aligned}$$

where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$.
$\dim \Delta_{\overline{0}} = 14$, $\dim \Delta_{\overline{1}} = 14$, $\dim \overline{\Delta}_{\overline{0}} = 12$, $\dim \overline{\Delta}_{\overline{1}} = 12$.
Distinguished simple root system:

$$\alpha_1 = \delta + \varepsilon_3,\ \alpha_2 = \varepsilon_1,\ \alpha_3 = \varepsilon_2 - \varepsilon_1$$

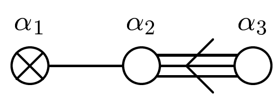
Distinguished positive roots:

$$\begin{aligned}
&\text{even roots: } \alpha_2,\ \alpha_3,\ \alpha_2+\alpha_3,\ 2\alpha_2+\alpha_3,\ 3\alpha_2+\alpha_3,\ 3\alpha_2+2\alpha_3, \\
&\qquad 2\alpha_1+4\alpha_2+2\alpha_3 \\
&\text{odd roots: } \alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+\alpha_3, \\
&\qquad \alpha_1+3\alpha_2+\alpha_3,\ \alpha_1+3\alpha_2+2\alpha_3,\ \alpha_1+4\alpha_2+2\alpha_3
\end{aligned}$$

Sums of even/odd distinguished positive roots:

$$\begin{aligned}
2\rho_0 &= 2\varepsilon_1 + 4\varepsilon_2 - 2\varepsilon_3 + 2\delta \\
2\rho_1 &= 7\delta
\end{aligned}$$

Distinguished Dynkin diagram:



Distinguished Cartan matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

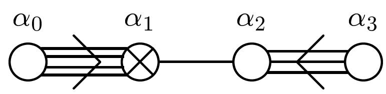
Highest distinguished root:

$$-\alpha_0 = 2\alpha_1 + 4\alpha_2 + 2\alpha_3 = 2\delta$$

(continued)

Table 3.59 (continued)

Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G}) = \mathbb{I}$.

Table 3.60: The basic Lie superalgebra $D(2,1;\alpha)$.

Structure: $\mathcal{G}_{\overline{0}} = sl(2) \oplus sl(2) \oplus sl(2)$ and $\mathcal{G}_{\overline{1}} = (2,2,2)$, type II.
Rank: 3, dimension: 17.
Root system ($1 \le i \le 3$):

$$\begin{array}{l} \Delta = \{\pm 2\varepsilon_i,\ \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\} \\ \Delta_{\overline{0}} = \{\pm 2\varepsilon_i\}, \quad \Delta_{\overline{1}} = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\} \\ \overline{\Delta}_{\overline{0}} = \Delta_{\overline{0}}, \quad \overline{\Delta}_{\overline{1}} = \Delta_{\overline{1}} \end{array}$$

$\dim \Delta_{\overline{0}} = \dim \overline{\Delta}_{\overline{0}} = 6$ and $\dim \Delta_{\overline{1}} = \dim \overline{\Delta}_{\overline{1}} = 8$.
Distinguished simple root system:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3, \alpha_2 = 2\varepsilon_2, \alpha_3 = 2\varepsilon_3$$

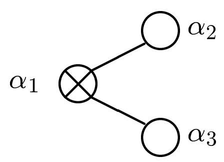
Distinguished positive roots ($1 \le j \le 3$):

$$\begin{array}{l} \text{even roots: } \alpha_2,\ \alpha_3,\ 2\alpha_1 + \alpha_2 + \alpha_3 \\ \text{odd roots: } \alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3 \end{array}$$

Sums of even/odd distinguished positive roots:

$$\begin{array}{l} 2\rho_0 = 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 \\ 2\rho_1 = 4\varepsilon_1 \end{array}$$

Distinguished Dynkin diagram:



Distinguished Cartan matrix:

$$\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

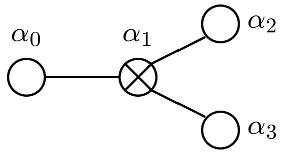
Highest distinguished root:

$$-\alpha_0 = 2\alpha_1 + \alpha_2 + \alpha_3 = 2\varepsilon_1$$

(continued)

Table 3.60 (continued)

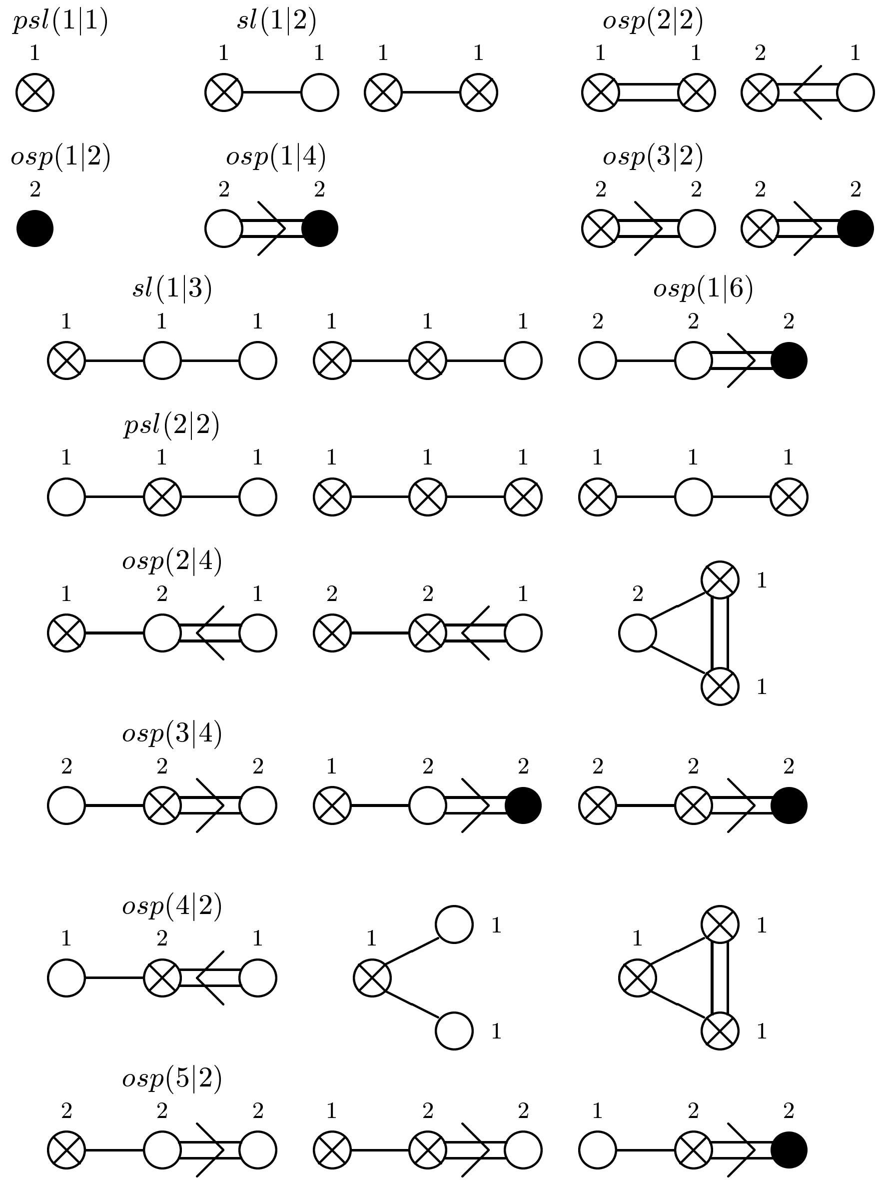
Distinguished extended Dynkin diagram:



Factor group $\mathrm{Out}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$:

$$\mathrm{Out}(\mathcal{G}) = \mathbb{I} \text{ for generic } \alpha$$
$$\mathrm{Out}(\mathcal{G}) = \mathbb{Z}_2 \text{ for } \alpha = 1, -2, -1/2$$
$$\mathrm{Out}(\mathcal{G}) = \mathbb{Z}_3 \text{ for } \alpha = e^{2i\pi/3}, e^{4i\pi/3}$$

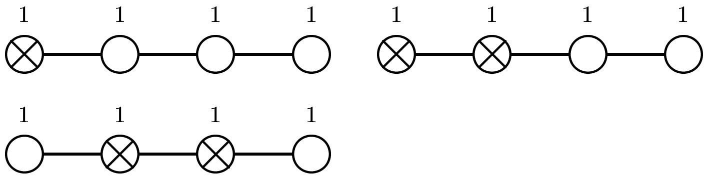
Table 3.61: Dynkin diagrams of the basic Lie superalgebras of rank ≤ 4.



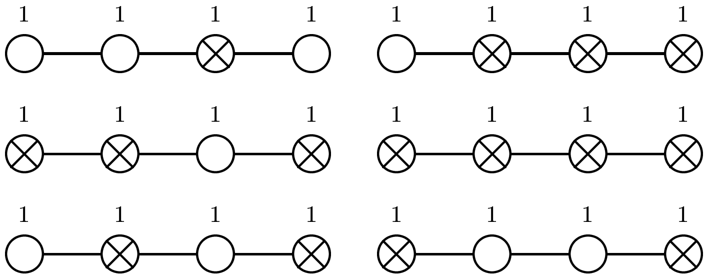
(continued)

Table 3.61 (continued)

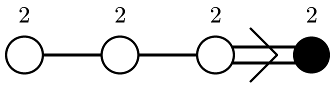
$sl(1|4)$



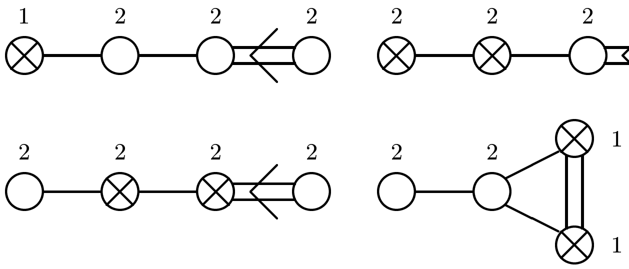
$sl(2|3)$



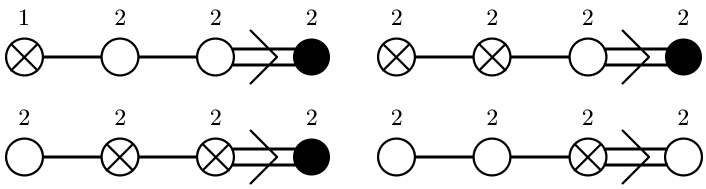
$osp(1|8)$



$osp(2|6)$



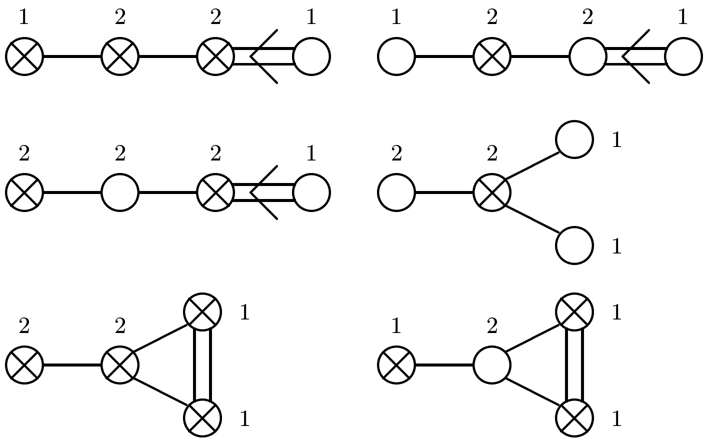
$osp(3|6)$



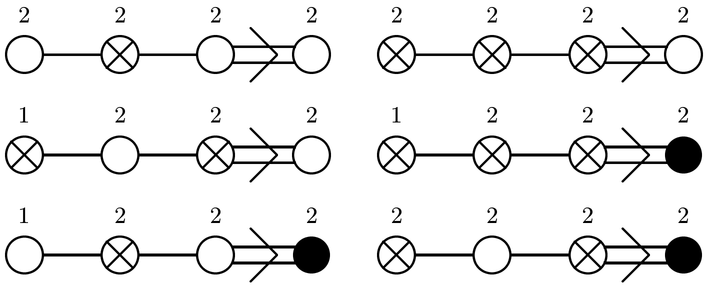
(continued)

Table 3.61 (continued)

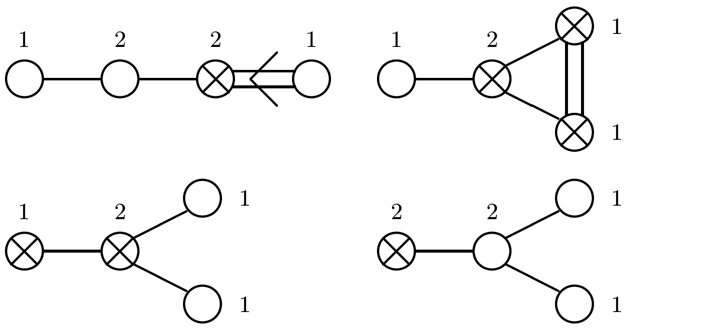
$osp(4|4)$



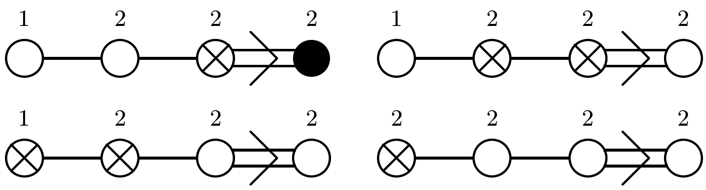
$osp(5|4)$



$osp(6|2)$



$osp(7|2)$



Tables 3.62–3.74 provide dimensions of representations for the basic superalgebras of small rank: $osp(1|4)$, $osp(1|6)$, $osp(3|2)$, $osp(4|2)$, $osp(5|2)$ and $osp(2|4)$ for the orthosymplectic series, $sl(1|3)$, $sl(2|2)$, $sl(1|4)$ and $sl(2|3)$ for the unitary series, $F(4)$ and $G(3)$. For the superalgebras $sl(1|2)$ and $osp(1|2)$, see sections 2.52 and 2.53. For each representation these tables provide the Dynkin labels, the type of representation (typical or atypical, in the latter case the number of the atypicality condition is given), the dimension of the representation space $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$, the dimension of the bosonic part $\mathcal{V}_{\overline{0}}$, the dimension of the fermionic part $\mathcal{V}_{\overline{1}}$ and the decomposition under the bosonic subalgebra $\mathcal{G}_{\overline{0}}$. The superscript $\pm$ appearing in the terms of the decomposition indicates whether the corresponding representation belongs to the bosonic space $\mathcal{V}_{\overline{0}}$ $(+)$ or the fermionic one $\mathcal{V}_{\overline{1}}$ $(-)$.

Table 3.62: Dimensions of $osp(1|4)$ irreducible representations.

labels	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\bar{0}}$	$\dim \mathcal{R}_{\bar{1}}$	decomposition under $sp(4)$
0;0	1	1	1	1^+
0;1	5	4	1	4^+ / 1^-
1;0	10	6	4	5^+ / 4^- / 1^+
0;2	14	10	4	10^+ / 4^-
0;3	30	20	10	20^+ / 10^-
1;1	35	20	15	16^+ / 10^- 5^- / 4^+
2;0	35	19	16	14^+ / 16^- / 5^+
0;4	55	35	20	35^+ / 20^-
1;2	81	45	36	35^+ / 20^- 16^- / 10^+
3;0	84	44	40	30^+ / 40^- / 14^+
0;5	91	56	35	56^+ / 35^-
2;1	105	56	49	40^+ / 35^- 14^- / 16^+
0;6	140	84	56	84^+ / 56^-
1;3	154	84	70	64^+ / 35^- 35^- / 20^+
4;0	165	85	80	55^+ / 80^- / 30^+
2;2	220	116	104	81^+ / 64^- 40^- / 35^+
3;1	231	120	111	80^+ / 81^- 30^- / 40^+
1;4	260	140	120	105^+ / 64^- 56^- / 35^+

Table 3.63: Dimensions of $osp(1|6)$ irreducible representations.

labels	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\bar{0}}$	$\dim \mathcal{R}_{\bar{1}}$	decomposition under $sp(6)$
0;0,0	1	1	0	1^+
0;0,1	7	6	1	6^+ / 1^-
0;1,0	21	15	6	14^+ / 6^- / 1^+
0;0,2	27	21	6	21^+ / 6^-
1;0,0	35	20	15	14^+ / 14^- / 6^+ / 1^-
0;0,3	77	56	21	56^+ / 21^-
0;1,1	105	70	35	64^+ / 21^- 14^- / 6^+
0;2,0	168	104	64	90^+ / 64^- / 14^+
0;0,4	182	126	56	126^+ / 56^-
1;0,1	189	105	84	70^+ / 64^- 14^- / 21^+ 14^+ / 6^-
2;0,0	294	154	140	84^+ / 126^- / 70^+ / 14^-
0;1,2	330	210	120	189^+ / 64^- 56^- / 21^+
1;1,0	378	204	174	126^+ / 90^- 70^- / 64^+ 14^+ / 14^-
1;0,2	616	336	280	216^+ / 189^- 70^- / 64^+ 56^+ / 21^-
0;2,1	693	414	279	350^+ / 189^- 90^- / 64^+
0;1,3	819	504	315	448^+ / 189^- 126^- / 56^+
0;3,0	825	475	350	385^+ / 350^- / 90^+

Table 3.64: Dimensions of $osp(2|4)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sp(4) \oplus U(1)$
0;0,0	atp-1	1	1	0	$(1,0)^+$
1;0,0	atp-2	6	2	4	$(1,1)^+$ / $(4,0)^-$ / $(1,-1)^+$
3;0,0	atp-3	10	6	4	$(1,3)^+$ / $(4,2)^-$ / $(5,1)^+$
−1;0,1	atp-1	10	6	4	$(5,-1)^+$ / $(4,-2)^-$ / $(1,-3)^+$
4;0,0	atp-4	15	7	8	$(1,4)^+$ / $(4,3)^-$ / $(5,2)^+$ $(1,2)^+$ / $(4,1)^-$
−1;1,0	atp-1	15	8	7	$(4,-1)^+$ / $(5,-2)^-$ $(1,-2)^-$ / $(4,-3)^+$ / $(1,-4)^-$
k;0,0	typ	16	8	8	$(1,k)^+$ / $(4,k-1)^-$ / $(5,k-2)^+$ $(1,k-2)^+$ / $(4,k-3)^-$ / $(1,k-4)^+$
1;1,0	atp-2	19	8	11	$(4,1)^+$ / $(10,0)^-$ $(1,0)^-$ / $(4,-1)^+$
4;0,1	atp-3	35	19	16	$(5,4)^+$ / $(16,3)^-$ / $(14,2)^+$
−2;0,2	atp-1	35	19	16	$(14,-2)^+$ / $(16,-3)^-$ / $(5,-4)^+$
1;2,0	atp-2	44	20	24	$(10,1)^+$ / $(20,0)^-$ $(4,0)^-$ / $(10,-1)^+$
0;0,1	atp-2	45	21	24	$(5,0)^+$ / $(16,-1)^-$ $(4,-1)^-$ / $(10,-2)^+$ $(5,-2)^+$ $(1,-2)^+$ / $(4,-3)^-$
3;1,0	atp-3	45	24	21	$(4,3)^+$ / $(10,2)^-$ $(5,2)^-$ $(1,2)^-$ / $(16,1)^+$ $(4,1)^+$ / $(5,0)^-$
5;1,0	atp-4	49	24	25	$(4,5)^+$ / $(10,4)^-$ $(5,4)^-$ / $(16,3)^+$ $(4,3)^+$ / $(10,2)^-$
−2;2,0	atp-1	49	25	24	$(10,-2)^+$ / $(16,-3)^-$ $(4,-3)^-$ / $(10,-4)^+$ $(5,-4)^+$ / $(4,-5)^-$
k;1,0	typ	64	32	32	$(4,k)^+$ / $(10,k-1)^-$ $(5,k-1)^-$ $(1,k-1)^-$ / $(16,k-2)^+$ $(24,k-2)^+$ / $(10,k-3)^-$ $(5,k-3)^-$ $(1,k-3)^-$ / $(4,k-4)^+$
5;0,1	atp-4	70	34	36	$(5,5)^+$ / $(16,4)^-$ $(4,4)^-$ / $(14,3)^+$ $(10,3)^+$ $(5,3)^+$ / $(16,2)^-$
−2;1,1	atp-1	70	36	34	$(16,-2)^+$ / $(14,-3)^-$ $(10,-3)^-$ $(5,-3)^-$ / $(16,-4)^+$ $(4,-4)^+$ / $(5,-5)^-$
k;0,1	typ	80	40	40	$(5,k)^+$ / $(16,k-1)^-$ $(4,k-1)^-$ / $(14,k-2)^+$ $(10,k-2)^+$ $(5,k-2)^+$ $(1,k-2)^+$ / $(16,k-3)^-$ $(4,k-3)^-$ / $(5,k-4)^+$
5;0,2	atp-3	84	44	40	$(14,5)^+$ / $(40,4)^-$ / $(30,3)^+$
−3;0,3	atp-1	84	44	40	$(30,-3)^+$ / $(40,-4)^-$ / $(14,-5)^+$
1;3,0	atp-2	85	40	45	$(20,1)^+$ / $(35,0)^-$ $(10,0)^-$ / $(20,-1)^+$
k;2,0	typ	160	80	80	$(10,k)^+$ / $(20,k-1)^-$ $(16,k-1)^-$ $(4,k-1)^-$ / $(210,k-2)^+$ $(35,k-2)^+$ $(5,k-2)^+$ / $(20,k-3)^-$ / $(16,k-3)^-$ $(4,k-3)^-$ / $(10,k-4)^+$
k;0,2	typ	224	112	112	$(14,k)^+$ / $(40,k-1)^-$ $(16,k-1)^-$ / $(35,k-2)^+$ $(30,k-2)^+$ $(14,k-2)^+$ $(5,k-2)^+$ / $(40,k-3)^-$ $(16,k-3)^-$ / $(14,k-4)^+$

Table 3.65: Dimensions of $osp(3|2)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sp(2) \oplus so(3)$
0;0	atp-1	1	1	0	$(1,1)^+$
1;0	atp-2	5	2	3	$(2,1)^+$ / $(1,3)^-$
1;1	typ	8	4	4	$(2,2)^+$ / $(1,4)^-$
1;2	typ	12	6	6	$(2,3)^+$ / $(1,5)^-$ $(1,1)^-$
2;0	typ	12	6	6	$(3,1)^+$ / $(2,3)^-$ $(1,3)^-$
1;3	typ	16	8	8	$(2,4)^+$ / $(1,6)^-$ $(1,2)^-$
1;4	typ	20	10	10	$(2,5)^+$ / $(1,7)^-$ $(1,3)^-$
3;0	typ	20	10	10	$(4,1)^+$ / $(3,3)^-$ / $(2,3)^+$ / $(1,1)^-$
1;5	typ	24	12	12	$(2,6)^+$ / $(1,8)^-$ $(1,4)^-$
2;1	typ	24	12	12	$(3,2)^+$ / $(2,4)^-$ $(2,2)^-$ / $(1,4)^+$ $(1,2)^+$
1;6	typ	28	14	14	$(2,7)^+$ / $(1,9)^-$ $(1,5)^-$
4;0	typ	28	14	14	$(5,1)^+$ / $(4,3)^-$ / $(3,3)^+$ / $(2,1)^-$
2;2	atp-2	30	14	16	$(3,3)^+$ / $(2,5)^-$ $(2,3)^-$ / $(1,5)^+$
1;7	typ	32	16	16	$(2,8)^+$ / $(1,10)^-$ $(1,6)^-$
5;0	typ	36	18	18	$(6,1)^+$ / $(5,3)^-$ / $(4,3)^+$ / $(3,1)^-$
3;1	typ	40	20	20	$(4,2)^+$ / $(3,4)^-$ $(3,2)^-$ / $(2,4)^+$ $(2,2)^+$ / $(1,2)^-$
6;0	typ	44	22	22	$(7,1)^+$ / $(6,3)^-$ / $(5,3)^+$ / $(4,1)^-$
2;3	typ	48	24	24	$(3,4)^+$ / $(2,6)^-$ $(2,4)^-$ $(2,2)^-$ / $(1,6)^+$ $(1,4)^+$ $(1,2)^+$
7;0	typ	52	26	26	$(8,1)^+$ / $(7,3)^-$ / $(6,3)^+$ / $(5,1)^-$
4;1	typ	56	28	28	$(5,2)^+$ / $(4,4)^-$ $(4,2)^-$ / $(3,4)^+$ $(3,2)^+$ / $(2,2)^-$
2;4	typ	60	30	30	$(3,5)^+$ / $(2,7)^-$ $(2,5)^-$ $(2,3)^-$ / $(1,7)^+$ $(1,5)^+$ $(1,3)^+$
3;2	typ	60	30	30	$(4,3)^+$ / $(3,5)^-$ $(3,3)^-$ $(3,1)^-$ / $(2,5)^+$ $(2,3)^+$ $(2,1)^+$ / $(1,3)^-$
8;0	typ	60	30	30	$(9,1)^+$ / $(8,3)^-$ / $(7,3)^+$ / $(6,1)^-$
3;4	atp-2	70	34	36	$(4,5)^+$ / $(3,7)^-$ $(3,5)^-$ / $(2,7)^+$
2;5	typ	72	36	36	$(3,6)^+$ / $(2,8)^-$ $(2,6)^-$ $(2,4)^-$ / $(1,8)^+$ $(1,6)^+$ $(1,4)^+$
5;1	typ	72	36	36	$(6,2)^+$ / $(5,4)^-$ $(5,2)^-$ / $(4,4)^+$ $(4,2)^+$ / $(3,2)^-$
3;3	typ	80	40	40	$(4,4)^+$ / $(3,6)^-$ $(3,4)^-$ $(3,2)^-$ / $(2,6)^+$ $(2,4)^+$ $(2,2)^+$ / $(1,4)^-$
2;6	typ	84	42	42	$(3,7)^+$ / $(2,9)^-$ $(2,7)^-$ $(2,5)^-$ / $(1,9)^+$ $(1,7)^+$ $(1,5)^+$
4;2	typ	84	42	42	$(5,3)^+$ / $(4,5)^-$ $(4,3)^-$ $(4,1)^-$ / $(3,5)^+$ $(3,3)^+$ $(3,1)^+$ / $(2,3)^-$
6;1	typ	88	44	44	$(7,2)^+$ / $(6,4)^-$ $(6,2)^-$ / $(5,4)^+$ $(5,2)^+$ / $(4,2)^-$

Table 3.66: Dimensions of $osp(4|2)$ tensor irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomp. under $sl(2) \oplus sl(2) \oplus sl(2)$
0;0,0	atp-1	1	1	0	$(1,0)^+$
1;0,0	atp-2,3	6	2	4	$(2,1,1)^+$ / $(1,2,2)^-$
2;0,0	atp-4	17	9	8	$(3,1,1)^+$ / $(2,2,2)^-$ / $(1,3,1)^+$ $(1,1,3)^+$
1;1,1	atp-2,3	18	8	10	$(2,2,2)^+$ / $(1,3,3)^-$ $(1,1,1)^-$
2;2,0	atp-2	30	14	16	$(3,3,1)^+$ / $(2,4,2)^-$ / $(1,5,1)^+$
3;0,0	typ	32	16	16	$(4,1,1)^+$ / $(3,2,2)^-$ / $(2,3,1)^+$ $(2,1,3)^+$ / $(1,2,2)^-$
1;2,2	atp-2,3	38	18	20	$(2,3,3)^+$ / $(1,4,4)^-$ $(1,2,2)^-$
4;0,0	typ	48	24	24	$(5,1,1)^+$ / $(4,2,2)^-$ / $(3,3,1)^+$ $(3,1,3)^+$ / $(2,2,2)^-$ / $(1,1,1)^+$
2;1,1	typ	64	32	32	$(3,2,2)^+$ / $(2,3,3)^-$ $(2,3,1)^-$ $(2,1,3)^-$ $(2,1,1)^-$ / $(1,4,2)^+$ $(1,2,4)^+$ $(1,2,2)^+$
5;0,0	typ	64	32	32	$(6,1,1)^+$ / $(5,2,2)^-$ / $(4,3,1)^+$ $(4,1,3)^+$ / $(3,2,2)^-$ / $(2,1,1)^+$
1;3,3	atp-2,3	66	32	34	$(2,4,4)^+$ / $(1,5,5)^-$ $(1,3,3)^-$
3;4,0	atp-2	70	34	36	$(4,5,1)^+$ / $(3,6,2)^-$ / $(2,7,1)^+$
2;4,0	typ	80	40	40	$(3,5,1)^+$ / $(2,6,2)^-$ $(2,4,2)^-$ / $(1,7,1)^+$ $(1,5,3)^+$ $(1,3,1)^+$
6;0,0	typ	80	40	40	$(7,1,1)^+$ / $(6,2,2)^-$ / $(5,3,1)^+$ $(5,1,3)^+$ / $(4,2,2)^-$ / $(3,1,1)^+$
2;3,1	atp-2	90	44	46	$(3,4,2)^+$ / $(2,5,3)^-$ $(2,5,1)^-$ $(2,3,1)^-$ / $(1,6,2)^+$ $(1,4,2)^+$
3;2,0	atp-4	90	46	44	$(4,3,1)^+$ / $(3,4,2)^-$ $(3,2,2)^-$ / $(2,5,1)^+$ $(2,3,3)^+$ $(2,3,1)^+$ / $(1,4,2)^-$
7;0,0	typ	96	48	48	$(8,1,1)^+$ / $(7,2,2)^-$ / $(6,3,1)^+$ $(6,1,3)^+$ / $(5,2,2)^-$ / $(4,1,1)^+$
3;1,1	atp-4	110	56	54	$(4,2,2)^+$ / $(3,3,3)^-$ $(3,3,1)^-$ $(3,1,3)^-$ / $(2,4,2)^+$ $(2,2,4)^+$ $(2,2,2)^+$ / $(1,3,3)^-$
2;6,0	typ	112	56	56	$(3,7,1)^+$ / $(2,8,2)^-$ $(2,6,2)^-$ / $(1,9,1)^+$ $(1,7,3)^+$ $(1,5,1)^+$
8;0,0	typ	112	56	56	$(9,1,1)^+$ / $(8,2,2)^-$ / $(7,3,1)^+$ $(7,1,3)^+$ / $(6,2,2)^-$ / $(5,1,1)^+$
2;2,2	typ	144	72	72	$(3,3,3)^+$ / $(2,4,4)^-$ $(2,4,2)^-$ $(2,2,4)^-$ $(2,2,2)^-$ / $(1,5,3)^+$ $(1,3,5)^+$ $(1,3,3)^+$ $(1,3,1)^+$ $(1,1,3)^+$
4;2,0	typ	144	72	72	$(5,3,1)^+$ / $(4,4,2)^-$ $(4,2,2)^-$ / $(3,5,1)^+$ $(3,3,3)^+$ $(3,3,1)^+$ $(3,1,1)^+$ / $(2,4,2)^-$ $(2,2,2)^-$ / $(1,3,1)^+$

The decompositions under $sl(2) \oplus sl(2) \oplus sl(2)$ of the representations labelled by $(k;l,m)$ and $(k;m,l)$ are obtained by interchanging the last two $sl(2)$ representations.

Table 3.67: Dimensions of $osp(4|2)$ spinor irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus sl(2) \oplus sl(2)$
2;1,0	typ	32	16	16	$(3,2,1)^+$ / $(2,3,2)^-$ $(2,1,2)^-$ / $(1,4,1)^+$ $(1,2,3)^+$
2;3,0	typ	64	32	32	$(3,4,1)^+$ / $(2,5,2)^-$ $(2,3,2)^-$ / $(1,6,1)^+$ $(1,4,3)^+$ $(1,2,1)^+$
3;1,0	typ	64	32	32	$(4,2,1)^+$ / $(3,3,2)^-$ $(3,1,2)^-$ / $(2,4,1)^+$ $(2,2,3)^+$ $(2,2,1)^+$ / $(1,3,2)^-$ $(1,1,2)^-$
2;1,2	typ	96	48	48	$(3,2,3)^+$ / $(2,3,4)^-$ $(2,3,2)^-$ $(2,1,4)^-$ $(2,1,2)^-$ / $(1,4,3)^+$ $(1,2,5)^+$ $(1,2,3)^+$ $(1,2,1)^+$
2;5,0	typ	96	48	48	$(3,6,1)^+$ / $(2,7,2)^-$ $(2,5,2)^-$ / $(1,8,1)^+$ $(1,6,3)^+$ $(1,4,1)^+$
4;1,0	typ	96	48	48	$(5,2,1)^+$ / $(4,3,2)^-$ $(4,1,2)^-$ / $(3,4,1)^+$ $(3,2,3)^+$ $(3,2,1)^+$ / $(2,3,2)^-$ $(2,1,2)^-$ / $(1,2,1)^+$
3;3,0	typ	128	64	64	$(4,4,1)^+$ / $(3,5,2)^-$ $(3,3,2)^-$ / $(2,6,1)^+$ $(2,4,3)^+$ $(2,4,1)^+$ $(2,2,1)^+$ / $(1,5,2)^-$ $(1,3,2)^-$
5;1,0	typ	128	64	64	$(6,2,1)^+$ / $(5,3,2)^-$ $(5,1,2)^-$ / $(4,4,1)^+$ $(4,2,3)^+$ $(4,2,1)^+$ / $(3,3,2)^-$ $(3,1,2)^-$ / $(2,2,1)^+$
2;1,4	typ	160	80	80	$(3,2,5)^+$ / $(2,3,6)^-$ $(2,3,4)^-$ $(2,1,6)^-$ $(2,1,4)^-$ / $(1,4,5)^+$ $(1,2,7)^+$ $(1,2,5)^+$ $(1,2,3)^+$
6;1,0	typ	160	80	80	$(7,2,1)^+$ / $(6,3,2)^-$ $(6,1,2)^-$ / $(5,4,1)^+$ $(5,2,3)^+$ $(5,2,1)^+$ / $(4,3,2)^-$ $(4,1,2)^-$ / $(3,2,1)^+$

The spinor representations $(k;l,m)$ and $(k;m,l)$ are conjugate to each other. Their decompositions under $sl(2) \oplus sl(2) \oplus sl(2)$ are obtained by interchanging the last two $sl(2)$ representations.

Table 3.68: Dimensions of $osp(5|2)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sp(2) \oplus so(5)$
0;0,0	atp-1	1	1	0	$(1,1)^+$
1;0,0	atp-2	7	2	5	$(2,1)^+$ / $(1,5)^-$
2;0,0	atp-3	23	13	10	$(3,1)^+$ / $(2,5)^+$ $(1,10)^-$
1;1,0	atp-2	25	10	15	$(2,5)^+$ / $(1,14)^-$ $(1,1)^-$
3;0,0	atp-4	49	24	25	$(4,1)^+$ / $(3,5)^-$ / $(2,10)^+$ / $(1,10)^-$
1;2,0	atp-2	63	28	35	$(2,14)^+$ / $(1,30)^-$ $(1,5)^-$
2;0,1	typ	64	32	32	$(3,4)^+$ / $(2,16)^-$ / $(1,20)^+$
4;0,0	typ	80	40	40	$(5,1)^+$ / $(4,5)^-$ / $(3,10)^+$ / $(2,10)^-$ / $(1,5)^+$
2;1,0	atp-3	105	55	50	$(3,5)^+$ / $(2,14)^-$ $(2,10)^-$ $(2,1)^-$ / $(1,35)^+$ $(1,5)^+$
5;0,0	typ	112	56	56	$(6,1)^+$ / $(5,5)^-$ / $(4,10)^+$ / $(3,10)^-$ / $(2,5)^+$ / $(1,1)^-$
1;3,0	atp-2	129	60	69	$(2,30)^+$ / $(1,55)^-$ $(1,14)^-$
6;0,0	typ	144	72	72	$(7,1)^+$ / $(6,5)^-$ / $(5,10)^+$ / $(4,10)^-$ / $(3,5)^+$ / $(2,1)^-$
2;0,2	typ	160	80	80	$(3,10)^+$ / $(2,35)^-$ $(2,5)^-$ / $(1,35)^+$ $(1,14)^+$ $(1,1)^+$
3;0,1	typ	192	96	96	$(4,4)^+$ / $(3,16)^-$ $(3,4)^-$ / $(2,20)^+$ $(2,16)^+$ $(2,4)^+$ / $(1,20)^-$ $(1,16)^-$
1;4,0	atp-2	231	110	121	$(2,55)^+$ / $(1,91)^-$ $(1,30)^-$
3;1,0	typ	240	120	120	$(4,5)^+$ / $(3,14)^-$ $(3,10)^-$ $(3,1)^-$ / $(2,35)^+$ $(2,10)^+$ $(2,5)^+$ / $(1,35)^-$ $(1,10)^-$
2;1,1	typ	256	128	128	$(3,16)^+$ / $(2,40)^-$ $(2,20)^-$ $(2,4)^-$ / $(1,64)^+$ $(1,16)^+$
2;2,0	atp-3	287	147	140	$(3,14)^+$ / $(2,35)^-$ $(2,30)^-$ $(2,5)^-$ / $(1,81)^+$ $(1,14)^+$ $(1,10)^+$
2;0,3	typ	320	160	160	$(3,20)^+$ / $(2,64)^-$ $(2,16)^-$ / $(1,56)^+$ $(1,40)^+$ $(1,4)^+$
4;0,1	typ	320	160	160	$(5,4)^+$ / $(4,16)^-$ $(4,4)^-$ / $(3,20)^+$ $(3,16)^+$ $(3,4)^+$ / $(2,20)^-$ $(2,16)^-$ $(2,4)^-$ / $(1,16)^+$ $(1,4)^+$
3;0,2	atp-3	350	180	170	$(4,10)^+$ / $(3,35)^-$ $(3,10)^-$ / $(2,35)^+$ $(2,35)^+$ / $(1,35)^-$
4;1,0	atp-4	350	174	176	$(5,5)^+$ / $(4,14)^-$ $(4,10)^-$ / $(3,35)^+$ $(3,10)^+$ / $(2,35)^-$ $(2,5)^-$ / $(1,14)^+$
1;5,0	atp-2	377	182	195	$(2,91)^+$ / $(1,140)^-$ $(1,55)^-$
5;0,1	typ	448	224	224	$(6,4)^+$ / $(5,16)^-$ $(5,4)^-$ / $(4,20)^+$ $(4,16)^+$ $(4,4)^+$ / $(3,20)^-$ $(3,16)^-$ $(3,4)^-$ / $(2,16)^+$ $(2,4)^+$ / $(1,4)^-$

Table 3.69: Dimensions of $sl(1|3)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(3) \oplus U(1)$
0;0,0	atp-1	1	1	0	$(1,0)^+$
3/2;0,0	atp-2	4	1	3	$(1,3/2)^+$ / $(3,1/2)^-$
−1/2;0,1	atp-1	4	3	1	$(3',-1/2)^+$ / $(1,-3/2)^-$
3;0,0	atp-3	7	4	3	$(1,3)^+$ / $(3,2)^-$ / $(3',1)^+$
−1;1,0	atp-1	7	4	3	$(3,-1)^+$ / $(3',-2)^-$ / $(1,-3)^+$
k;0,0	typ	8	4	4	$(1,k)^+$ / $(3,k-1)^-$ / $(3',k-2)^+$ / $(1,k-3)^-$
−1;0,2	atp-1	9	6	3	$(6',-1)^+$ / $(3',-2)^-$
2;1,0	atp-2	9	3	6	$(3,2)^+$ / $(6,1)^-$
1;0,1	atp-2	15	6	9	$(3',1)^+$ / $(8,0)^-$ $(1,0)^-$ / $(3,-1)^+$
−3/2;0,3	atp-1	16	10	6	$(10',-3/2)^+$ / $(6',-5/2)^-$
5/2;2,0	atp-2	16	6	10	$(6,5/2)^+$ / $(10,3/2)^-$
4;0,1	atp-3	17	9	8	$(3',4)^+$ / $(8,3)^-$ / $(6',2)^+$
−2;2,0	atp-1	17	9	8	$(6,-2)^+$ / $(8,-3)^-$ / $(3,-4)^+$
7/2;1,0	atp-3	20	11	9	$(3,7/2)^+$ / $(6,5/2)^-$ $(3',5/2)^-$ / $(8,3/2)^+$
−3/2;1,1	atp-1	20	9	11	$(8,-3/2)^+$ / $(6',-5/2)^-$ $(3,-5/2)^-$ / $(3',-7/2)^+$
k;0,1	typ	24	12	12	$(3',k)^+$ / $(8,k-1)^-$ $(1,k-1)^-$ / $(6',k-2)^+$ $(3,k-2)^+$ / $(3',k-3)^-$
k;1,0	typ	24	12	12	$(3,k)^+$ / $(6,k-1)^-$ $(3',k-1)^-$ / $(8,k-2)^+$ $(1,k-2)^+$
−2;0,4	atp-1	25	15	10	$(15',-2)^+$ / $(10',-3)^-$
3;3,0	atp-2	25	10	15	$(10,3)^+$ / $(15,2)^-$
5;0,2	atp-3	31	16	15	$(6',5)^+$ / $(15',4)^-$ / $(10',3)^+$
−3;3,0	atp-1	31	16	15	$(10,-3)^+$ / $(15,-4)^-$ / $(6,-5)^+$
1/2;0,2	atp-2	32	14	18	$(6',1/2)^+$ / $(15',-1/2)^-$ $(3',-1/2)^-$ / $(8,-3/2)^+$
3/2;1,1	atp-2	32	14	18	$(8,3/2)^+$ / $(15,1/2)^-$ $(3,1/2)^-$ / $(6,-1/2)^+$
7/2;4,0	atp-2	36	15	21	$(15,7/2)^+$ / $(21,5/2)^-$
−2;1,2	atp-1	39	21	18	$(15',-2)^+$ / $(10',-3)^-$ $(8,-3)^-$ / $(6',-4)^+$
4;2,0	atp-3	39	21	18	$(6,4)^+$ / $(10,3)^-$ $(8,3)^-$ / $(15,2)^+$
9/2;1,1	atp-3	44	23	21	$(8,9/2)^+$ / $(15,7/2)^-$ $(6',7/2)^-$ / $(15',5/2)^+$
−5/2;2,1	atp-1	44	23	21	$(15,-5/2)^+$ / $(15',-7/2)^-$ $(6,-7/2)^-$ / $(8,-9/2)^+$
k;0,2	typ	48	24	24	$(6',k)^+$ / $(15',k-1)^-$ $(3',k-1)^-$ / $(10',k-2)^+$ $(8,k-2)^+$ / $(6',k-3)^-$
k;2,0	typ	48	24	24	$(6,k)^+$ / $(10,k-1)^-$ $(8,k-1)^-$ / $(15,k-2)^+$ $(3,k-2)^+$ / $(6,k-3)^-$
6;0,3	atp-3	49	25	24	$(10',6)^+$ / $(24',5)^-$ / $(15',4)^+$
−4;4,0	atp-1	49	25	24	$(15,-4)^+$ / $(24,-5)^-$ / $(10,-6)^+$

Table 3.70: Dimensions of $psl(2|2)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus sl(2)$
0,0	atp-1,4	1	1	0	$(1,1)^+$
1,1	atp-1,4	14	8	6	$(2,2)^+ \;/\; (3,1)^- \; (1,3)^- \;/\; (2,2)^+$
0,1	typ	32	16	16	$(1,2)^+ \;/\; (2,3)^- \; (2,1)^- \;/\; (3,2)^+ \; (1,4)^+$ $(1,2)^+ \;/\; (2,3)^- \; (2,1)^- \;/\; (1,2)^+$
2,2	atp-1,4	34	18	16	$(3,3)^+ \;/\; (4,2)^- \; (2,4)^- \;/\; (3,3)^+$
0,2	typ	48	24	24	$(1,3)^+ \;/\; (2,4)^- \; (2,2)^- \;/\; (3,3)^+ \; (1,5)^+$ $(1,3)^+ \; (1,1)^+ \;/\; (2,4)^- \; (2,2)^- \;/\; (1,3)^+$
3,3	atp-1,4	62	32	30	$(4,4)^+ \;/\; (5,3)^- \; (3,5)^- \;/\; (4,4)^+$
0,3	typ	64	32	32	$(1,4)^+ \;/\; (2,5)^- \; (2,3)^- \;/\; (3,4)^+ \; (1,6)^+$ $(1,4)^+ \; (1,2)^+ \;/\; (2,5)^- \; (2,3)^- \;/\; (1,4)^+$
0,4	typ	80	40	40	$(1,5)^+ \;/\; (2,6)^- \; (2,4)^- \;/\; (3,5)^+ \; (1,7)^+$ $(1,5)^+ \; (1,3)^+ \;/\; (2,6)^- \; (2,4)^- \;/\; (1,5)^+$
0,5	typ	96	48	48	$(1,6)^+ \;/\; (2,7)^- \; (2,5)^- \;/\; (3,6)^+ \; (1,8)^+$ $(1,6)^+ \; (1,4)^+ \;/\; (2,7)^- \; (2,5)^- \;/\; (1,6)^+$
1,2	typ	96	48	48	$(2,3)^+ \;/\; (3,4)^- \; (3,2)^- \; (1,4)^- \; (1,2)^-$ $/\; (4,3)^+ \; (2,5)^+ \; 2\,(2,3)^+ \; (2,1)^+ \;/$ $(3,4)^- \; (3,2)^- \; (1,4)^- \; (1,2)^- \;/\; (2,3)^+$
4,4	atp-1,4	98	50	48	$(5,5)^+ \;/\; (6,4)^- \; (4,6)^- \;/\; (5,5)^+$
0,6	typ	112	56	56	$(1,7)^+ \;/\; (2,8)^- \; (2,6)^- \;/\; (3,7)^+ \; (1,9)^+$ $(1,7)^+ \; (1,5)^+ \;/\; (2,8)^- \; (2,6)^- \;/\; (1,7)^+$
0,7	typ	128	64	64	$(1,8)^+ \;/\; (2,9)^- \; (2,7)^- \;/\; (3,8)^+ \; (1,10)^+$ $(1,8)^+ \; (1,6)^+ \;/\; (2,9)^- \; (2,7)^- \;/\; (1,8)^+$
1,3	typ	128	64	64	$(2,4)^+ \;/\; (3,5)^- \; (3,3)^- \; (1,5)^- \; (1,3)^-$ $/\; (4,4)^+ \; (2,6)^+ \; 2\,(2,4)^+ \; (2,2)^+ \;/$ $(3,5)^- \; (3,3)^- \; (1,5)^- \; (1,3)^- \;/\; (2,4)^+$
0,8	typ	144	72	72	$(1,9)^+ \;/\; (2,10)^- \; (2,8)^- \;/\; (3,9)^+$ $(1,11)^+ \; (1,9)^+ \; (1,7)^+ \;/\; (2,10)^-$ $(2,8)^- \;/\; (1,9)^+$
1,4	typ	160	80	80	$(2,5)^+ \;/\; (3,6)^- \; (3,4)^- \; (1,6)^- \; (1,4)^-$ $/\; (4,5)^+ \; (2,7)^+ \; 2\,(2,5)^+ \; (2,3)^+ \;/$ $(3,6)^- \; (3,4)^- \; (1,6)^- \; (1,4)^- \;/\; (2,5)^+$
1,5	typ	192	96	96	$(2,6)^+ \;/\; (3,7)^- \; (3,5)^- \; (1,7)^- \; (1,5)^-$ $/\; (4,6)^+ \; (2,8)^+ \; 2\,(2,6)^+ \; (2,4)^+ \;/$ $(3,7)^- \; (3,5)^- \; (1,7)^- \; (1,5)^- \;/\; (2,6)^+$
2,3	typ	192	96	96	$(3,4)^+ \;/\; (4,5)^- \; (4,3)^- \; (2,5)^- \; (2,3)^- \;/$ $(5,4)^+ \; (3,6)^+ \; 2\,(3,4)^+ \; (3,2)^+ \; (1,4)^+ \;/$ $(4,5)^- \; (4,3)^- \; (2,5)^- \; (2,3)^- \;/\; (3,4)^+$
1,6	typ	224	112	112	$(2,7)^+ \;/\; (3,8)^- \; (3,6)^- \; (1,8)^- \; (1,6)^-$ $/\; (4,7)^+ \; (2,9)^+ \; 2\,(2,7)^+ \; (2,5)^+ \;/$ $(3,8)^- \; (3,6)^- \; (1,8)^- \; (1,6)^- \;/\; (2,7)^+$
2,4	typ	240	120	120	$(3,5)^+ \;/\; (4,6)^- \; (4,4)^- \; (2,6)^- \; (2,4)^- \;/$ $(5,5)^+ \; (3,7)^+ \; 2\,(3,5)^+ \; (3,3)^+ \; (1,5)^+ \;/$ $(4,6)^- \; (4,4)^- \; (2,6)^- \; (2,4)^- \;/\; (3,5)^+$

Table 3.71: Dimensions of $sl(1|4)$ irreducible representations.

labels	type	dim $\mathcal{R}$	dim $\mathcal{R}_{\overline{0}}$	dim $\mathcal{R}_{\overline{1}}$	decomposition under $sl(4) \oplus U(1)$
0;0,0,0	atp-1	1	1	0	$(1,0)^+$
4/3;0,0,0	atp-2	5	1	4	$(1,4/3)^+$ / $(4,1/3)^-$
−1/3;0,0,1	atp-1	5	4	1	$(4',-1/3)^+$ / $(1,-4/3)^-$
8/3;0,0,0	atp-3	11	7	4	$(1,8/3)^+$ / $(4,5/3)^-$ / $(6,2/3)^+$
−2/3;0,1,0	atp-1	11	7	4	$(6,-2/3)^+$ / $(4',-5/3)^-$ / $(1,-8/3)^-$
−2/3;0,0,2	atp-1	14	10	4	$(10',-2/3)^+$ / $(4',-5/3)^-$
5/3;1,0,0	atp-2	14	4	10	$(4,5/3)^+$ / $(10,2/3)^-$
4;0,0,0	atp-4	15	7	8	$(1,4)^+$ / $(4,3)^-$ / $(6,2)^+$ / $(4',1)^-$
−1;1,0,0	atp-1	15	8	7	$(4,-1)^+$ / $(6,-2)^-$ / $(4',-3)^+$ / $(1,-4)^-$
k;0,0,0	typ	16	8	8	$(1,k)^+$ / $(4,k-1)^-$ / $(6,k-2)^+$ / / $(4',k-3)^-$ / $(1,k-4)^+$
1;0,0,1	atp-2	24	8	16	$(4',1)^+$ / $(15,0)^-$ $(1,0)^-$ / $(4,-1)^+$
2;2,0,0	atp-2	30	10	20	$(10,2)^+$ / $(20,1)^-$
−1;0,1,1	atp-1	40	24	16	$(20',-1)^+$ / $(10',-2)^-$ $(6,-2)^-$ / $(4',-3)^+$
3;1,0,0	atp-3	40	24	16	$(4,3)^+$ / $(10,2)^-$ $(6,2)^-$ / $(20,1)^+$
10/3;0,1,0	atp-3	46	26	20	$(6,10/3)^+$ / $(20,7/3)^-$ / $(20,4/3)^+$
−4/3;0,2,0	atp-1	46	26	20	$(20,-4/3)^+$ / $(20',-7/3)^-$ / $(6,-10/3)^+$
5;0,0,1	atp-4	49	24	25	$(4',5)^+$ / $(15,4)^-$ / $(20',3)^+$ / $(10',2)^-$
−2;2,0,0	atp-1	49	25	24	$(10,-2)^+$ / $(20,-3)^-$ / $(15,-4)^+$ $(4,-5)^-$
7/3;0,0,1	atp-3	50	28	22	$(4',7/3)^+$ / $(15,4/3)^-$ $(1,4/3)^-$ / $(20',1/3)^+$ $(4,1/3)^+$ / $(6,-2/3)^-$
2/3;0,1,0	atp-2	50	22	28	$(6,2/3)^+$ / $(20,-1/3)^-$ $(4',-1/3)^-$ / $(15,-4/3)^+$ $(1,-4/3)^+$ / $(4,-7/3)^-$
13/3;1,0,0	atp-4	59	28	31	$(4,13/3)^+$ / $(10,10/3)^-$ $(6,10/3)^-$ / $(20,7/3)^+$ $(4',7/3)^+$ / $(15,4/3)^-$
−4/3;1,0,1	atp-1	59	31	28	$(15,-4/3)^+$ / $(20',-7/3)^-$ $(4,-7/3)^-$ / $(10',-10/3)^+$ $(6,-10/3)^+$ / $(4',-13/3)^-$
k;0,0,1	typ	64	32	32	$(4',k)^+$ / $(15,k-1)^-$ $(1,k-1)^-$ / $(20',k-2)^+$ $(4,k-2)^+$ / $(10,k-3)^-$ $(6,k-3)^-$ / $(4',k-4)^+$
k;1,0,0	typ	64	32	32	$(4,k)^+$ / $(10,k-1)^-$ $(6,k-1)^-$ / $(20,k-2)^+$ $(4',k-2)^+$ / $(15,k-3)^-$ $(1,k-3)^-$ / $(4,k-4)^+$
2/3;0,0,2	atp-2	65	25	40	$(10',2/3)^+$ / $(36',-1/3)^-$ $(4',-1/3)^-$ / $(15,-4/3)^+$
4/3;1,0,1	atp-2	65	25	40	$(15,4/3)^+$ / $(36,1/3)^-$ $(4,1/3)^-$ / $(10,-2/3)^+$

Table 3.72: Dimensions of $sl(2|3)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomp. under $sl(2) \oplus sl(3) \oplus U(1)$
0;0;0,0	atp-1,4	1	1	0	$(1,1,0)^+$
1;3;0,0	atp-1,6	5	2	3	$(2,1,3)^+$ / $(1,3,2)^-$
2;−2;0,1	atp-1,4	5	3	2	$(1,3',-2)^+$ / $(2,1,-3)^-$
0;−4;1,0	atp-1	12	6	6	$(1,3,-4)^+$ /$(2,3',-5)^-$ / $(3,1,-6)^+$
2;6;0,0	atp-1	12	6	6	$(3,1,6)^+$ / $(2,3,5)^-$ / $(1,3',4)^+$
0;6;0,0	atp-3,6	13	7	6	$(1,1,6)^+$ / $(2,3,5)^-$ / $(1,6,4)^+$
0;−4;0,2	atp-1,4	13	7	6	$(1,6',-4)^+$ / $(2,3',-5)^-$ / $(1,1-6)^+$
0;−6;0,0	atp-2	20	10	10	$(1,1-6)^+$ / $(2,3,-7)^-$ / $(3,3',-8)^+$ / $(4,1,-9)^-$
1;1;0,1	atp-1	24	12	12	$(2,3',1)^+$ / $(1,8,0)^-$ $(3,1,0)^-$ $(1,1,0)^-$ / $(2,3,-1)^+$
0;8;1,0	atp-3,6	25	13	12	$(1,3,8)^+$ / $(2,6,7)^-$ / $(1,10,6)^+$
1;−9;0,0	atp-2	28	14	14	$(2,1-9)^+$ / $(3,3,-10)^-$ / $(4,3',-11)^+$ / $(5,1,-12)^-$
2;−12;0,0	atp-2	36	18	18	$(3,1,-12)^+$ / $(4,3,-13)^-$ / $(5,3',-14)^+$ / $(6,1,-15)^-$
0;−6;1,1	atp-1	40	20	20	$(1,8,-6)^+$ / $(2,6',-7)^-$ $(2,3,-7)^-$ / $(3,3',-8)^+$ $(1,3',-8)^+$ / $(2,1,-9)^-$
1;9;0,0	atp-3	40	20	20	$(2,1,9)^+$ / $(3,3,8)^-$ $(1,3,8)^-$ / $(2,6,7)^+$ $(2,3',7)^+$ / $(1,8,6)^-$
0;10;2,0	atp-3,6	41	21	20	$(1,6,10)^+$ / $(2,10,9)^-$ / $(1,15,8)^+$
0;12,0,0	atp-5	44	22	22	$(1,1,12)^+$ / $(2,3,11)^-$ / $(1,6,10)^+$ $(3,3,',10)^+$ / $(2,8,9)^-$ / $(1,6',8)^+$
0;−8;2,0	atp-1	44	22	22	$(1,6,-8)^+$ / $(2,8,-9)^-$ / $(1,6',-10)^+$ $(3,3,-10)^+$ / $(2,3',-11)^-$ / $(1,1,-12)^+$
1;−1;1,0	atp-1,4	50	26	24	$(2,3,-1)^+$ / $(1,6,-2)^-$ $(3,3',-2)^-$ / $(2,8,-3)^+$ $(4,1,-3)^+$ / $(3,3,-4)^-$
0;4;0,1	atp-3	60	30	30	$(1,3',4)^+$ / $(2,8,3)^-$ $(2,1,3)^-$ / $(1,15,2)^+$ $(3,3,2)^+$ $(1,3,2)^+$ / $(2,6,1)^-$
0;k;0,0	typ	64	32	32	$(1,1,k)^+$ / $(2,3,k-1)^-$ / $(1,6,k-2)^+$ $(3,3',k-2)^+$ / $(2,8,k-3)^-$ $(4,1,k-3)^-$ / $(1,6',k-4)^+$ $(3,3,k-4)^+$ / $(2,3',k-5)^-$ / $(1,1,k-6)^+$
0;−8;0,1	atp-2	72	36	36	$(1,3',-8)^+$ / $(2,8,-9)^-$ $(2,1,-9)^-$ / $(3,6',-10)^+$ $(3,3,-10)^+$ $(1,3,-10)^+$ / $(4,3',-11)^-$ $(2,3',-11)^-$/ $(3,1,-12)^+$
2;12;0,0	atp-3	72	36	36	$(3,1,12)^+$ / $(4,3,11)^-$ $(2,3,11)^-$ / $(3,6,10)^+$ $(3,3',10)^+$ $(1,3',10)^+$ / $(2,8,9)^-$ $(2,1,9)^-$ / $(1,3,8)^+$

Table 3.73: Dimensions of $F(4)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus so(7)$
0;0,0,0	atp-1	1	1	0	$(1,1)^+$
2;0,0,0	atp-3	40	24	16	$(3,1)^+$ / $(2,8)^-$ / $(1,21)^+$
4;0,0,0	atp-6	296	152	144	$(5,1)^+$ / $(4,8)^-$ / $(3,21)^+$ $(3,7)^+$ / $(2,48)^-$ $(2,8)^-$ / $(1,35)^+$ $(1,27)^+$ $(1,1)^+$
2;0,1,0	atp-3	507	267	240	$(3,21)^+$ / $(2,112)^-$ $(2,8)^-$ / $(1,168)^+$ $(1,35)^+$ $(1,1)^+$
5;0,0,0	typ	512	256	256	$(6,1)^+$ / $(5,8)^-$ / $(4,21)^+$ $(4,7)^+$ / $(3,48)^-$ $(3,8)^-$ / $(2,35)^+$ $(2,27)^+$ $(2,7)^+$ / $(1,48)^-$
3;1,0,0	atp-4,5	756	368	364	$(4,8)^+$ / $(3,35)^-$ $(3,21)^-$ / $(2,112)^+$ $(2,48)^+$ $(2,8)^+$ / $(1,189)^-$ $(1,7)^-$
6;0,0,0	atp-8	769	385	384	$(7,1)^+$ / $(6,8)^-$ / $(5,21)^+$ $(5,7)^+$ / $(4,48)^-$ $(4,8)^-$ / $(3,35)^+$ $(3,27)^+$ $(3,7)^+$ $(3,1)^+$ / $(2,48)^-$ $(2,8)^-$ / $(1,21)^+$ $(1,7)^+$
4;0,0,1	atp-5	1036	508	528	$(5,7)^+$ / $(4,48)^-$ / $(3,105)^+$ $(3,27)^+$ / $(2,168)^-$ / $(1,77)^+$
4;1,0,0	typ	2048	1024	1024	$(5,8)^+$ / $(4,35)^-$ $(4,21)^-$ $(4,7)^-$ $(4,1)^-$ / $(3,112)^+$ $2\,(3,48)^+$ $(3,8)^+$ / $(2,189)^-$ $(2,105)^-$ $(2,35)^-$ $(2,27)^-$ $(2,21)^-$ $(2,7)^-$ / $(1,168)^+$ $(1,112)^+$ $(1,48)^+$ $(1,8)^+$
2;0,2,0	atp-3	3392	1728	1664	$(3,168)^+$ / $(2,720)^-$ $(2,112)^-$ / $(1,825)^+$ $(1,378)^+$ $(1,21)^+$
5;0,0,1	typ	3584	1792	1792	$(6,7)^+$ / $(5,48)^-$ $(5,8)^-$ / $(4,105)^+$ $(4,35)^+$ $(4,27)^+$ $(4,21)^+$ $(4,7)^+$ $(4,1)^+$ / $(3,168)^-$ $(3,112)^-$ $2\,(3,48)^-$ $2\,(3,8)^-$ / $(2,189)^+$ $(2,105)^+$ $(2,77)^+$ $(2,35)^+$ $(2,27)^+$ $2\,(2,21)^+$ $(2,7)^+$ $(2,1)^+$ / $(1,168)^-$ $(1,112)^-$ $(1,48)^-$ $(1,8)^-$
5;1,0,0	typ	4096	2048	2048	$(6,8)^+$ / $(5,35)^-$ $(5,21)^-$ $(5,7)^-$ $(5,1)^-$ / $(4,112)^+$ $2\,(4,48)^+$ $2\,(4,8)^+$ / $(3,1)^-$ $(3,189)^-$ $(3,105)^-$ $2\,(3,35)^-$ $(3,27)^-$ $2\,(3,21)^-$ $2\,(3,7)^-$ / $(2,168)^+$ $2\,(2,8)^+$ $2\,(2,112)^+$ $3\,(2,48)^+$ / $(1,189)^-$ $(1,7)^-$ $(1,105)^-$ $(1,35)^-$ $(1,27)^-$ $(1,21)^-$
4;0,0,2	typ	6912	3456	3456	$(5,27)^+$ / $(4,168)^-$ $(4,48)^-$ / $(3,330)^+$ $(3,189)^+$ $(3,105)^+$ $(3,77)^+$ $(3,21)^+$ $(3,7)^+$ / $(2,512)^-$ $(2,448)^-$ $(2,168)^-$ $(2,112)^-$ $(2,48)^-$ $(2,8)^-$ / $(1,616)^+$ $(1,182)^+$ $(1,168)^+$ $(1,105)^+$ $(1,35)^+$ $(1,27)^+$ $(1,1)^+$

Table 3.74: Dimensions of $G(3)$ irreducible representations.

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus G(2)$
0;0,0	atp-1	1	1	0	$(1,1)^+$
2;0;0	atp-3	31	17	14	$(3,1)^+$ / $(2,7)^-$ / $(1,14)^+$
3;0,0	atp-4	95	46	49	$(4,1)^+$ / $(3,7)^-$ / $(2,14)^+$ $(2,7)^+$ / $(1,27)^-$ $(1,1)^-$
4;0,0	typ	192	96	96	$(5,1)^+$ / $(4,7)^-$ / $(3,14)^+$ $(3,7)^+$ / $(2,27)^-$ $(2,7)^-$ / $(1,27)^+$ $(1,1)^+$
2;0,1	atp-3	289	147	142	$(3,14)^+$ / $(2,64)^-$ $(2,7)^-$ / $(1,77)^+$ $(1,27)^+$ $(1,1)^+$
5;0,0	atp-6	321	160	161	$(6,1)^+$ / $(5,7)^-$ / $(4,14)^+$ $(4,7)^+$ / $(3,27)^-$ $(3,7)^-$ $(3,1)^-$ / $(2,27)^+$ $(2,7)^+$ $(2,1)^+$ / $(1,14)^-$ $(1,7)^-$
6;0,0	typ	448	224	224	$(7,1)^+$ / $(6,7)^-$ / $(5,14)^+$ $(5,7)^+$ / $(4,27)^-$ $(4,7)^-$ $(4,1)^-$ / $(3,27)^+$ $(3,7)^+$ $(3,1)^+$ / $(2,14)^-$ $(2,7)^-$ / $(1,7)^+$
3;1,0	typ	448	224	224	$(4,7)^+$ / $(3,27)^-$ $(3,14)^-$ $(3,1)^-$ / $(2,64)^+$ $(2,27)^+$ $(2,7)^+$ / $(1,77)^-$ $(1,14)^-$ $(1,7)^-$
7;0,0	typ	576	288	288	$(8,1)^+$ / $(7,7)^-$ / $(6,14)^+$ $(6,7)^+$ / $(5,27)^-$ $(5,7)^-$ $(5,1)^-$ / $(4,27)^+$ $(4,7)^+$ $(4,1)^+$ / $(3,14)^-$ $(3,7)^-$ / $(2,7)^+$ / $(1,1)^-$
3;0,1	atp-4	1185	590	595	$(4,14)^+$ / $(3,64)^-$ $(3,27)^-$ $(3,7)^-$ / $(2,77)^+$ $(2,77)^+$ $(2,64)^+$ $(2,27)^+$ $(2,14)^+$ $(2,7)^+$ $(2,1)^+$ / $(1,189)^-$ $(1,64)^-$ $(1,27)^-$ $(1,14)^-$ $(1,7)^-$
4;1,0	typ	1344	672	672	$(5,7)^+$ / $(4,27)^-$ $(4,14)^-$ $(4,7)^-$ $(4,1)^-$ / $(3,64)^+$ $2\,(3,27)^+$ $(3,14)^+$ $2\,(3,7)^+$ $(3,1)^+$ / $(2,77)^-$ $(2,64)^-$ $2\,(2,27)^-$ $2\,(2,14)^-$ $2\,(2,7)^-$ $(2,1)^-$ / $(1,77)^+$ $(1,64)^+$ $(1,27)^+$ $(1,14)^+$ $2\,(1,7)^+$
2;0,2	atp-3	1407	707	700	$(3,77)^+$ / $(2,286)^-$ $(2,64)^-$ / $(1,273)^+$ $(1,189)^+$ $(1,14)^+$
3;2,0	typ	1728	864	864	$(4,27)^+$ / $(3,77)^-$ $(3,64)^-$ $(3,14)^-$ $(3,7)^-$ / $(2,189)^+$ $(2,77)^+$ $(2,64)^+$ $(2,27)^+$ $(2,14)^+$ $(2,7)^+$ / $(1,182)^-$ $(1,77)^-$ $(1,64)^-$ $2\,(1,27)^-$ $(1,1)^-$
5;1,0	atp-5	2114	1060	1054	$(6,7)^+$ / $(5,27)^-$ $(5,14)^-$ $(5,7)^-$ $(5,1)^-$ / $(4,64)^+$ $2\,(4,27)^+$ $(4,14)^+$ $2\,(4,7)^+$ / $(3,77)^-$ $(3,64)^-$ $2\,(3,27)^-$ $2\,(3,14)^-$ $2\,(3,7)^-$ / $(2,77)^+$ $(2,64)^+$ $2\,(2,27)^+$ $(2,14)^+$ $(2,7)^+$ $(2,1)^+$ / $(1,64)^-$ $(1,27)^-$ $(1,7)^-$

(continued)

Table 3.74 (continued)

labels	type	$\dim \mathcal{R}$	$\dim \mathcal{R}_{\overline{0}}$	$\dim \mathcal{R}_{\overline{1}}$	decomposition under $sl(2) \oplus G(2)$
4;0,1	typ	2688	1344	1344	$(5,14)^+$ / $(4,64)^-$ $(4,27)^-$ $(4,7)^-$ / $(3,77)^+$ $(3,77)^+$ $(3,64)^+$ $2\,(3,27)^+$ $(3,7)^+$ $(3,14)^+$ $(3,1)^+$ / $(2,77)^-$ $(2,14)^-$ $2\,(2,7)^-$ $(2,189)^-$ $2\,(2,64)^-$ $2\,(2,27)^-$ / $(1,189)^+$ $(1,77)^+$ $(1,64)^+$ $(1,27)^+$ $2\,(1,14)^+$ $(1,7)^+$
6;1,0	atp-6	2814	1406	1408	$(7,7)^+$ / $(6,27)^-$ $(6,14)^-$ $(6,7)^-$ / $(5,64)^+$ $2\,(5,27)^+$ $(5,14)^+$ $(5,7)^+$ $(5,1)^+$ / $(4,77)^-$ $(4,64)^-$ $2\,(4,27)^-$ $(4,14)^-$ $(4,1)^-$ $2\,(4,7)^-$ / $(3,77)^+$ $(3,64)^+$ $(3,27)^+$ $2\,(3,14)^+$ $2\,(3,7)^+$ / $(2,64)^-$ $(2,27)^-$ $(2,14)^-$ $(2,7)^-$ / $(1,27)^+$
4;2,0	atp-4	3710	1850	1860	$(5,27)^+$ / $(4,77)^-$ $(4,64)^-$ $(4,27)^-$ $(4,7)^-$ / $(3,189)^+$ $2\,(3,77)^+$ $(3,64)^+$ $(3,27)^+$ $(3,14)^+$ $(3,7)^+$ / $(2,189)^-$ $(2,182)^-$ $(2,77)^-$ $(2,64)^-$ $2\,(2,27)^-$ $(2,14)^-$ / $(1,182)^+$ $(1,77)^+$ $(1,64)^+$ $(1,27)^+$
3;1,1	typ	4096	2048	2048	$(4,64)^+$ / $(3,189)^-$ $(3,77)^-$ $(3,77)^-$ $(3,27)^-$ $(3,14)^-$ / $(2,286)^+$ $(2,189)^+$ $(2,182)^+$ $(2,77)^+$ $2\,(2,64)^+$ $(2,27)^+$ $(2,7)^+$ / $(1,448)^-$ $(1,189)^-$ $(1,77)^-$ $(1,77)^-$ $(1,64)^-$ $(1,27)^-$ $(1,14)^-$
6;0,1	atp-5	4158	2082	2076	$(7,14)^+$ / $(6,64)^-$ $(6,27)^-$ / $(5,77)^+$ $(5,77)^+$ $(5,64)^+$ $(5,27)^+$ / $(4,189)^-$ $(4,77)^-$ $(4,64)^-$ $(4,14)^-$ / $(3,189)^+$ $(3,64)^+$ / $(2,77)^-$
5;0,1	typ	4480	2240	2240	$(6,14)^+$ / $(5,64)^-$ $(5,27)^-$ $(5,7)^-$ / $(4,77)^+$ $(4,77)^+$ $(4,64)^+$ $2\,(4,27)^+$ $(4,1)^+$ $(4,14)^+$ $(4,7)^+$ / $(3,189)^-$ $(3,77)^-$ $2\,(3,64)^-$ $2\,(3,27)^-$ $2\,(3,14)^-$ $2\,(3,7)^-$ / $(2,189)^+$ $(2,77)^+$ $2\,(2,64)^+$ $2\,(2,27)^+$ $2\,(2,14)^+$ $2\,(2,7)^+$ / $(1,77)^-$ $(1,77)^-$ $(1,64)^-$ $2\,(1,27)^-$ $(1,7)^-$ $(1,1)^-$
2;0,3	atp-3	4737	2373	2364	$(3,273)^+$ / $(2,896)^-$ $(2,286)^-$ / $(1,748)^+$ $(1,729)^+$ $(1,77)^+$
3;3,0	typ	4928	2464	2464	$(4,77)^+$ / $(3,189)^-$ $(3,182)^-$ $(3,64)^-$ $(3,27)^-$ / $(2,448)^+$ $(2,189)^+$ $(2,182)^+$ $(2,77)^+$ $(2,77)^+$ $(2,64)^+$ $(2,27)^+$ $(2,14)^+$ / $(1,378)^-$ $(1,286)^-$ $(1,189)^-$ $2\,(1,77)^-$ $(1,64)^-$ $(1,7)^-$
3;0,2	atp-4	6335	3164	3171	$(4,77)^+$ / $(3,286)^-$ $(3,189)^-$ $(3,64)^-$ / $(2,448)^+$ $(2,286)^+$ $(2,273)^+$ $(2,189)^+$ $(2,77)^+$ $(2,77)^+$ $(2,64)^+$ $(2,14)^+$ / $(1,729)^-$ $(1,286)^-$ $(1,189)^-$ $(1,182)^-$ $(1,77)^-$ $(1,64)^-$ $(1,27)^-$

Table 3.75: Real forms of the classical Lie superalgebras.

$\mathcal{G}$	$\mathcal{G}_{\overline{0}}$	$\mathcal{G}^{\phi}$	$\mathcal{G}_{\overline{0}}^{\phi}$
$A(m,n)$	$sl(m) \oplus sl(n) \oplus U(1)$	$sl(m\|n;\mathbb{R})$	$sl(m,\mathbb{R}) \oplus sl(n,\mathbb{R}) \oplus \mathbb{R}$
		$sl(m\|n;\mathbb{H})$	$su^*(m) \oplus su^*(n) \oplus \mathbb{R}$
		$su(p,m-p\|q,n-q)$	$su(p,m-p) \oplus su(q,n-q) \oplus i\mathbb{R}$
$A(n,n)$	$sl(n) \oplus sl(n)$	$psl(n\|n;\mathbb{R})$	$sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$
		$psl(n\|n;\mathbb{H})$	$su^*(n) \oplus su^*(n)$
		$su(p,n-p\|q,n-q)$	$su(p,n-p) \oplus su(q,n-q)$
$B(m,n)$	$so(2m+1) \oplus sp(2n)$	$osp(p,2m+1-p\|2n;\mathbb{R})$	$so(p,2m+1-p) \oplus sp(2n,\mathbb{R})$
$B(0,n)$	$sp(2n)$	$osp(1\|2n;\mathbb{R})$	$sp(2n,\mathbb{R})$
$C(n+1)$	$so(2) \oplus sp(2n)$	$osp(2\|2n;\mathbb{R})$	$so(2) \oplus sp(2n,\mathbb{R})$
		$osp(2\|2q,2n-2q;\mathbb{H})$	$so^*(2) \oplus sp(2q,2n-2q)$
$D(m,n)$	$so(2m) \oplus sp(2n)$	$osp(p,2m-p\|2n;\mathbb{R})$	$so(p,2m-p) \oplus sp(2n,\mathbb{R})$
		$osp(2m\|2q,2n-2q;\mathbb{H})$	$so^*(2m) \oplus sp(2q,2n-2q)$
$F(4)$	$sl(2) \oplus so(7)$	$F(4;0)$	$sl(2,\mathbb{R}) \oplus so(7)$
		$F(4;3)$	$sl(2,\mathbb{R}) \oplus so(1,6)$
		$F(4;2)$	$sl(2,\mathbb{R}) \oplus so(2,5)$
		$F(4;1)$	$sl(2,\mathbb{R}) \oplus so(3,4)$
$G(3)$	$sl(2) \oplus G_2$	$G(3;0)$	$sl(2,\mathbb{R}) \oplus G_{2,0}$
		$G(3;1)$	$sl(2,\mathbb{R}) \oplus G_{2,2}$
$D(2,1;\alpha)$	$sl(2) \oplus sl(2) \oplus sl(2)$	$D(2,1;\alpha;0)$	$sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$
		$D(2,1;\alpha;1)$	$su(2) \oplus su(2) \oplus sl(2,\mathbb{R})$
		$D(2,1;\alpha;2)$	$sl(2,\mathbb{C}) \oplus sl(2,\mathbb{R})$

Table 3.76: $osp(1|2)$ decompositions of the fundamental representations of the basic Lie superalgebras.

$\mathcal{G}$	$\mathcal{K}$	$\mathrm{fund}\,\mathcal{G}\,/\,\mathcal{K}$
$sl(m\|n)$	$sl(p+1\|p)$	$\mathcal{R}_{p/2} \oplus (m-p-1)\mathcal{R}_0 \oplus (n-p)\mathcal{R}''_0$
	$sl(p\|p+1)$	$\mathcal{R}''_{p/2} \oplus (m-p)\mathcal{R}_0 \oplus (n-p-1)\mathcal{R}''_0$
$osp(2m\|2n)$	$osp(2k\|2k)$	$\mathcal{R}''_{k-1/2} \oplus (2n-2k)\mathcal{R}''_0 \oplus (2m-2k+1)\mathcal{R}_0$
	$osp(2k+2\|2k)$	$\mathcal{R}_k \oplus (2m-2k-1)\mathcal{R}_0 \oplus (2n-2k)\mathcal{R}''_0$
	$sl(p+1\|p)$	$2\mathcal{R}_{p/2} \oplus 2(m-p-1)\mathcal{R}_0 \oplus 2(n-p)\mathcal{R}''_0$
	$sl(p\|p+1)$	$2\mathcal{R}''_{p/2} \oplus 2(n-p-1)\mathcal{R}''_0 \oplus 2(m-p)\mathcal{R}_0$
$osp(2m+1\|2n)$	$osp(2k\|2k)$, $osp(2k-1\|2k)$	$\mathcal{R}''_{k-1/2} \oplus (2n-2k)\mathcal{R}''_0 \oplus (2m-2k+2)\mathcal{R}_0$
	$osp(2k+2\|2k)$, $osp(2k+1\|2k)$	$\mathcal{R}_k \oplus (2m-2k)\mathcal{R}_0 \oplus (2n-2k)\mathcal{R}''_0$
	$sl(p+1\|p)$	$2\mathcal{R}_{p/2} \oplus 2(m-p-1)\mathcal{R}_0 \oplus \mathcal{R}_0 \oplus 2(n-p)\mathcal{R}''_0$
	$sl(p\|p+1)$	$2\mathcal{R}''_{p/2} \oplus 2(n-p-1)\mathcal{R}''_0 \oplus \mathcal{R}_0 \oplus 2(m-p)\mathcal{R}_0$
$osp(2\|2n)$	$osp(2\|2)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}_0 \oplus (2n-2)\mathcal{R}''_0$
	$sl(1\|2)$	$2\mathcal{R}''_{1/2} \oplus (2n-4)\mathcal{R}''_0$
$osp(2n+2\|2n)$	$osp(2k+1\|2k) \oplus osp(2n-2k+1\|2n-2k)$	$\mathcal{R}_k \oplus \mathcal{R}_{n-k}$
$osp(2n-2\|2n)$	$osp(2k-1\|2k) \oplus osp(2n-2k-1\|2n-2k)$	$\mathcal{R}''_{k-1/2} \oplus \mathcal{R}''_{n-k-1/2}$
$osp(2n\|2n)$	$osp(2k+1\|2k) \oplus osp(2n-2k-1\|2n-2k)$	$\mathcal{R}_k \oplus \mathcal{R}''_{n-k-1/2}$
	$osp(2k-1\|2k) \oplus osp(2n-2k+1\|2n-2k)$	$\mathcal{R}_{n-k} \oplus \mathcal{R}''_{k-1/2}$

Table 3.77: $osp(1|2)$ decompositions of the adjoint representations of the basic Lie superalgebras (regular cases).

$$\frac{\mathrm{ad}\, sl(m|n)}{sl(p+1|p)} = \mathcal{R}_p \oplus \mathcal{R}_{p-1/2} \oplus \mathcal{R}_{p-1} \oplus \ldots \oplus \mathcal{R}_{1/2}$$
$$\oplus\, 2(n-p)\mathcal{R}_{p/2} \oplus 2(m-p-1)\mathcal{R}'_{p/2}$$
$$\oplus\, [(m-p-1)^2 + (n-p)^2]\mathcal{R}_0 \oplus 2(m-p-1)(n-p)\mathcal{R}'_0$$

$$\frac{\mathrm{ad}\, sl(m|n)}{sl(p|p+1)} = \mathcal{R}_p \oplus \mathcal{R}_{p-1/2} \oplus \mathcal{R}_{p-1} \oplus \ldots \oplus \mathcal{R}_{1/2}$$
$$\oplus\, 2(m-p)\mathcal{R}_{p/2} \oplus 2(n-p-1)\mathcal{R}'_{p/2}$$
$$\oplus\, [(m-p)^2 + (n-p-1)^2]\mathcal{R}_0 \oplus 2(m-p)(n-p-1)\mathcal{R}'_0$$

$$\frac{\mathrm{ad}\, psl(n|n)}{sl(p+1|p)} = \mathcal{R}_p \oplus \mathcal{R}_{p-1/2} \oplus \mathcal{R}_{p-1} \oplus \ldots \oplus \mathcal{R}_{1/2}$$
$$\oplus\, 2(n-p)\mathcal{R}_{p/2} \oplus 2(n-p-1)\mathcal{R}'_{p/2}$$
$$\oplus\, 2(n-p-1)(n-p)\mathcal{R}_0 \oplus 2(n-p-1)(n-p)\mathcal{R}'_0$$

$$\frac{\mathrm{ad}\, psl(n|n)}{sl(p|p+1)} = \mathcal{R}_p \oplus \mathcal{R}_{p-1/2} \oplus \mathcal{R}_{p-1} \oplus \ldots \oplus \mathcal{R}_{1/2}$$
$$\oplus\, 2(n-p-1)\mathcal{R}_{p/2} \oplus 2(n-p)\mathcal{R}'_{p/2}$$
$$\oplus\, 2(n-p-1)(n-p)\mathcal{R}_0 \oplus 2(n-p-1)(n-p)\mathcal{R}'_0$$

$$\frac{\mathrm{ad}\, osp(2m|2n)}{osp(2k|2k)} = \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-3} \oplus \mathcal{R}_{2k-9/2} \oplus \ldots$$
$$\oplus\, \mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus (2m-2k+1)\mathcal{R}_{k-1/2} \oplus 2(n-k)\mathcal{R}'_{k-1/2}$$
$$\oplus\, 2(2m-2k+1)(n-k)\mathcal{R}'_0$$
$$\oplus\, [(2m-2k+1)(m-k) + (2n-2k+1)(n-k)]\mathcal{R}_0$$

(continued)

Table 3.77 (continued)

$$\frac{\mathrm{ad}\, osp(2m|2n)}{osp(2k+2|2k)} = \mathcal{R}_{2k-1/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-3} \oplus \dots$$
$$\oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus (2m-2k-1)\mathcal{R}_k \oplus 2(n-k)\mathcal{R}'_k$$
$$\oplus 2(2m-2k-1)(n-k)\mathcal{R}'_0$$
$$\oplus [(2m-2k-1)(m-k-1) + (2n-2k+1)(n-k)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2m|2n)}{sl(2k+1|2k)} = \mathcal{R}_{2k} \oplus 3\mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-2} \oplus \dots \oplus \mathcal{R}_2 \oplus 3\mathcal{R}_1 \oplus \mathcal{R}_0$$
$$\oplus 3\mathcal{R}_{2k-1/2} \oplus \mathcal{R}_{2k-3/2} \oplus 3\mathcal{R}_{2k-5/2} \oplus \dots \oplus 3\mathcal{R}_{3/2} \oplus \mathcal{R}_{1/2}$$
$$\oplus 4(m-2k-1)\mathcal{R}_k \oplus 4(n-2k)\mathcal{R}'_k \oplus 4(m-2k-1)(n-2k)\mathcal{R}'_0$$
$$\oplus [(2m-4k-3)(m-2k-1) + (2n-4k+1)(n-2k)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2m|2n)}{sl(2k-1|2k)} = 3\mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-2} \oplus 3\mathcal{R}_{2k-3} \oplus \dots \oplus \mathcal{R}_2 \oplus 3\mathcal{R}_1$$
$$\oplus \mathcal{R}_0 \oplus \mathcal{R}_{2k-3/2} \oplus 3\mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-7/2} \oplus \dots \oplus 3\mathcal{R}_{3/2} \oplus \mathcal{R}_{1/2}$$
$$\oplus 4(m-2k+1)\mathcal{R}_{k-1/2} \oplus 4(n-2k)\mathcal{R}'_{k-1/2}$$
$$\oplus 4(m-2k+1)(n-2k)\mathcal{R}'_0$$
$$\oplus [(2m-4k+1)(m-2k+1) + (2n-4k+1)(n-2k)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2m|2n)}{sl(2k|2k+1)} = 3\mathcal{R}_{2k} \oplus \mathcal{R}_{2k-1} \oplus 3\mathcal{R}_{2k-2} \oplus \dots \oplus 3\mathcal{R}_2 \oplus \mathcal{R}_1$$
$$\oplus 3\mathcal{R}_0 \oplus \mathcal{R}_{2k-1/2} \oplus 3\mathcal{R}_{2k-3/2} \oplus \mathcal{R}_{2k-5/2} \oplus \dots \oplus \mathcal{R}_{3/2} \oplus 3\mathcal{R}_{1/2}$$
$$\oplus 4(m-2k)\mathcal{R}'_k \oplus 4(n-2k-1)\mathcal{R}_k \oplus 4(m-2k)(n-2k-1)\mathcal{R}'_0$$
$$\oplus [(2m-4k-1)(m-2k) + (2n-4k-1)(n-2k-1)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2m|2n)}{sl(2k|2k-1)} = \mathcal{R}_{2k-1} \oplus 3\mathcal{R}_{2k-2} \oplus \mathcal{R}_{2k-3} \oplus \dots \oplus 3\mathcal{R}_2 \oplus \mathcal{R}_1$$
$$\oplus 3\mathcal{R}_0 \oplus 3\mathcal{R}_{2k-3/2} \oplus \mathcal{R}_{2k-5/2} \oplus 3\mathcal{R}_{2k-7/2} \oplus \dots \oplus \mathcal{R}_{3/2}$$
$$\oplus 3\mathcal{R}_{1/2} \oplus 4(m-2k)\mathcal{R}'_{k-1/2} \oplus 4(n-2k+1)\mathcal{R}_{k-1/2}$$
$$\oplus 4(m-2k)(n-2k+1)\mathcal{R}'_0$$
$$\oplus [(2m-4k-1)(m-2k) + (2n-4k+3)(n-2k+1)]\mathcal{R}_0$$

(continued)

Table 3.77 (continued)

$$\frac{\mathrm{ad}\, osp(2m+1|2n)}{osp(2k|2k)} = \frac{\mathrm{ad}\, osp(2m+1|2n)}{osp(2k-1|2k)} =$$
$$\mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-3} \oplus \mathcal{R}_{2k-9/2} \oplus \ldots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1$$
$$\oplus\, 2(m-k+1)\mathcal{R}_{k-1/2} \oplus 2(n-k)\mathcal{R}'_{k-1/2} \oplus 4(m-k+1)(n-k)\mathcal{R}'_0$$
$$\oplus\, [(2m-2k+1)(m-k+1)+(2n-2k+1)(n-k)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2m+1|2n)}{osp(2k+2|2k)} = \frac{\mathrm{ad}\, osp(2m+1|2n)}{osp(2k+1|2k)} =$$
$$\mathcal{R}_{2k-1/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-3} \oplus \ldots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1$$
$$\oplus\, 2(n-k)\mathcal{R}'_k \oplus 2(m-k)\mathcal{R}_k \oplus 4(m-k)(n-k)\mathcal{R}'_0$$
$$\oplus\, [(2m-2k-1)(m-k)+(2n-2k+1)(n-k)]\mathcal{R}_0$$

$$\frac{\mathrm{ad}\, osp(2|2n)}{osp(2|2)} = \mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus (2n-2)\mathcal{R}'_{1/2} \oplus (2n^2-3n+1)\mathcal{R}_0 \oplus$$
$$(2n-2)\mathcal{R}'_0$$

$$\frac{\mathrm{ad}\, osp(2|2n)}{sl(1|2)} = 3\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus (4n-8)\mathcal{R}'_{1/2} \oplus (2n^2-7n+7)\mathcal{R}_0$$

Table 3.78: $osp(1|2)$ decompositions of the adjoint representations of the basic Lie superalgebras (singular cases).

$$\frac{\mathrm{ad}\, osp(2n+2|2n)}{osp(2k+1|2k) \oplus osp(2n-2k+1|2n-2k)} = \mathcal{R}_{2n-2k-1}$$
$$\oplus \mathcal{R}_{2n-2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2n-2k-1/2} \oplus \mathcal{R}_{2n-2k-3/2} \oplus \dots$$
$$\oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2k-1/2} \oplus \mathcal{R}_{2k-3/2}$$
$$\oplus \dots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_n \oplus \mathcal{R}_{n-1} \oplus \dots \oplus \mathcal{R}_{n-2k}$$
$$\oplus \mathcal{R}_{n-1/2} \oplus \mathcal{R}_{n-3/2} \oplus \dots \oplus \mathcal{R}_{n-2k+1/2}$$

$$\frac{\mathrm{ad}\, osp(2n-2|2n)}{osp(2k-1|2k) \oplus osp(2n-2k-1|2n-2k)} = \mathcal{R}_{2n-2k-1}$$
$$\oplus \mathcal{R}_{2n-2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2n-2k-5/2} \oplus \mathcal{R}_{2n-2k-7/2} \oplus \dots$$
$$\oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-7/2}$$
$$\oplus \dots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{n-1} \oplus \mathcal{R}_{n-2} \oplus \dots \oplus \mathcal{R}_{n-2k}$$
$$\oplus \mathcal{R}_{n-3/2} \oplus \mathcal{R}_{n-5/2} \oplus \dots \oplus \mathcal{R}_{n-2k+1/2}$$

$$\frac{\mathrm{ad}\, osp(2n|2n)}{osp(2k+1|2k) \oplus osp(2n-2k-1|2n-2k)} = \mathcal{R}_{2n-2k-1}$$
$$\oplus \mathcal{R}_{2n-2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2n-2k-5/2} \oplus \mathcal{R}_{2n-2k-7/2} \oplus \dots$$
$$\oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2k-1/2} \oplus \mathcal{R}_{2k-3/2}$$
$$\oplus \dots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{n-1} \oplus \mathcal{R}_{n-2} \oplus \dots \oplus \mathcal{R}_{n-2k}$$
$$\oplus \mathcal{R}_{n-1/2} \oplus \mathcal{R}_{n-3/2} \oplus \dots \oplus \mathcal{R}_{n-2k-1/2}$$

$$\frac{\mathrm{ad}\, osp(2n|2n)}{osp(2k-1|2k) \oplus osp(2n-2k+1|2n-2k)} = \mathcal{R}_{2n-2k-1}$$
$$\oplus \mathcal{R}_{2n-2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2n-2k-1/2} \oplus \mathcal{R}_{2n-2k-3/2} \oplus \dots$$
$$\oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-3} \oplus \dots \oplus \mathcal{R}_1 \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-7/2}$$
$$\oplus \dots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_{n-1} \oplus \mathcal{R}_{n-2} \oplus \dots \oplus \mathcal{R}_{n-2k+1}$$
$$\oplus \mathcal{R}_{n-1/2} \oplus \mathcal{R}_{n-3/2} \oplus \dots \oplus \mathcal{R}_{n-2k+1/2}$$

Table 3.79: $osp(1|2)$ decompositions of the $A(m,n)$ and $C(n+1)$ superalgebras up to rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$A(0,1)$	$A(0,1)$	$\mathcal{R}''_{1/2}$	$\mathcal{R}_1 \oplus \mathcal{R}_{1/2}$
$A(0,2)$	$A(0,1)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}''_0$	$\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus \mathcal{R}_0$
$A(1,1)$	$A(0,1)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2}$
$A(0,3)$	$A(0,1)$	$\mathcal{R}''_{1/2} \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 4\mathcal{R}_0$
$A(1,2)$	$A(1,2)$	$\mathcal{R}''_1$	$\mathcal{R}_2 \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus \mathcal{R}_{1/2}$
	$A(0,1)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}_0 \oplus \mathcal{R}''_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 2\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
	$A(1,0)$	$\mathcal{R}_{1/2} \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 5\mathcal{R}_{1/2} \oplus 4\mathcal{R}_0$
$C(3)$	$A(0,1)$	$2\mathcal{R}''_{1/2}$	$3\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus \mathcal{R}_0$
	$C(2)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 3\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
$C(4)$	$A(0,1)$	$2\mathcal{R}''_{1/2} \oplus 2\mathcal{R}''_0$	$3\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 4\mathcal{R}_0$
	$C(2)$	$\mathcal{R}''_{1/2} \oplus \mathcal{R}_0 \oplus 4\mathcal{R}''_0$	$\mathcal{R}_1 \oplus \mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 10\mathcal{R}_0 \oplus 4\mathcal{R}'_0$

Table 3.80: $osp(1|2)$ decompositions of the $B(m,n)$ superalgebras of rank 2 and 3.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$B(0,2)$	$B(0,1)$	$\mathcal{R}''_{1/2} \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 2\mathcal{R}'_{1/2} \oplus 3\mathcal{R}_0$
$B(1,1)$	$B(1,1)$	$\mathcal{R}_1$	$\mathcal{R}_{3/2} \oplus \mathcal{R}_1$
	$C(2)\,,\,B(0,1)$	$\mathcal{R}''_{1/2} \oplus 2\mathcal{R}_0$	$\mathcal{R}_1 \oplus 2\mathcal{R}_{1/2} \oplus \mathcal{R}_0$
$B(0,3)$	$B(0,1)$	$\mathcal{R}''_{1/2} \oplus 4\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 4\mathcal{R}'_{1/2} \oplus 10\mathcal{R}_0$
$B(1,2)$	$B(1,2)$	$\mathcal{R}''_{3/2}$	$\mathcal{R}_3 \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1$
	$B(1,1)$	$\mathcal{R}_1 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus 2\mathcal{R}'_1 \oplus 3\mathcal{R}_0$
	$C(2)\,,\,B(0,1)$	$\mathcal{R}''_{1/2} \oplus 2\mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 2\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 4\mathcal{R}_0 \oplus 4\mathcal{R}'_0$
	$C(2) \oplus B(0,1)\,,\,A(0,1)$	$2\mathcal{R}''_{1/2} \oplus \mathcal{R}_0$	$3\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus \mathcal{R}_0$
$B(2,1)$	$D(2,1)\,,\,B(1,1)$	$\mathcal{R}_1 \oplus 2\mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 3\mathcal{R}_1 \oplus \mathcal{R}_0$
	$C(2)\,,\,B(0,1)$	$\mathcal{R}''_{1/2} \oplus 4\mathcal{R}_0$	$\mathcal{R}_1 \oplus 4\mathcal{R}_{1/2} \oplus 6\mathcal{R}_0$
	$A(1,0)$	$2\mathcal{R}_{1/2} \oplus \mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 3\mathcal{R}_0$

Table 3.81: $osp(1|2)$ decompositions of the $B(m,n)$ superalgebras of rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$B(0,4)$	$B(0,1)$	$\mathcal{R}''_{1/2} \oplus 6\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 6\mathcal{R}'_{1/2} \oplus 21\mathcal{R}_0$
$B(1,3)$	$B(1,2)$	$\mathcal{R}''_{3/2} \oplus 2\mathcal{R}''_0$	$\mathcal{R}_3 \oplus \mathcal{R}_{3/2} \oplus 2\mathcal{R}'_{3/2} \oplus \mathcal{R}_1 \oplus 3\mathcal{R}_0$
	$B(1,1)$	$\mathcal{R}_1 \oplus 4\mathcal{R}''_0$	$\mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus 4\mathcal{R}'_1 \oplus 10\mathcal{R}_0$
	$C(2) \oplus B(0,1)\,,\ A(0,1)$	$2\mathcal{R}''_{1/2} \oplus \mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$3\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 4\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
	$C(2)\,,\ B(0,1)$	$\mathcal{R}''_{1/2} \oplus 4\mathcal{R}''_0 \oplus 2\mathcal{R}_0$	$\mathcal{R}_1 \oplus 2\mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 11\mathcal{R}_0 \oplus 9\mathcal{R}'_0$
$B(2,2)$	$B(2,2)$	$\mathcal{R}_2$	$\mathcal{R}_{7/2} \oplus \mathcal{R}_3 \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1$
	$D(2,2)\,,\ B(1,2)$	$\mathcal{R}''_{3/2} \oplus 2\mathcal{R}_0$	$\mathcal{R}_3 \oplus 3\mathcal{R}_{3/2} \oplus \mathcal{R}_1 \oplus \mathcal{R}_0$
	$D(2,1)\,,\ B(1,1)$	$\mathcal{R}_1 \oplus 2\mathcal{R}''_0 \oplus 2\mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 3\mathcal{R}_1 \oplus 2\mathcal{R}'_1 \oplus 4\mathcal{R}_0 \oplus 4\mathcal{R}'_0$
	$D(2,1) \oplus B(0,1)\,,\ B(1,1) \oplus C(2)$	$\mathcal{R}_1 \oplus \mathcal{R}''_{1/2} \oplus \mathcal{R}_0$	$2\mathcal{R}_{3/2} \oplus 4\mathcal{R}_1 \oplus 2\mathcal{R}_{1/2}$
	$C(2) \oplus C(2)\,,\ A(0,1)$	$2\mathcal{R}''_{1/2} \oplus 3\mathcal{R}_0$	$3\mathcal{R}_1 \oplus 7\mathcal{R}_{1/2} \oplus 4\mathcal{R}_0$
	$C(2)\,,\ B(0,1)$	$\mathcal{R}''_{1/2} \oplus 4\mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 4\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 9\mathcal{R}_0 \oplus 8\mathcal{R}'_0$
	$A(1,0)$	$2\mathcal{R}_{1/2} \oplus \mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 7\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 6\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
$B(3,1)$	$D(2,1)\,,\ B(1,1)$	$\mathcal{R}_1 \oplus 4\mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 5\mathcal{R}_1 \oplus 6\mathcal{R}_0$
	$C(2)\,,\ B(0,1)$	$\mathcal{R}''_{1/2} \oplus 6\mathcal{R}_0$	$\mathcal{R}_1 \oplus 6\mathcal{R}_{1/2} \oplus 15\mathcal{R}_0$
	$A(1,0)$	$2\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 6\mathcal{R}'_{1/2} \oplus 6\mathcal{R}_0$

Table 3.82: $osp(1|2)$ decompositions of the $D(m,n)$ superalgebras up to rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$D(2,1)$	$D(2,1)$	$\mathcal{R}_1 \oplus \mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 2\mathcal{R}_1$
	$C(2)$	$\mathcal{R}''_{1/2} \oplus 3\mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$
	$A(1,0)$	$2\mathcal{R}_{1/2}$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$
$D(2,2)$	$D(2,2)$	$\mathcal{R}''_{3/2} \oplus \mathcal{R}_0$	$\mathcal{R}_3 \oplus 2\mathcal{R}_{3/2} \oplus \mathcal{R}_1$
	$D(2,1)$	$\mathcal{R}_1 \oplus \mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_{3/2} \oplus 2\mathcal{R}_1 \oplus 2\mathcal{R}'_1 \oplus 5\mathcal{R}_0$
	$C(2)$	$\mathcal{R}''_{1/2} \oplus 3\mathcal{R}_0 \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 2\mathcal{R}'_{1/2} \oplus 6\mathcal{R}_0 \oplus 6\mathcal{R}'_0$
	$C(2) \oplus C(2)\,,\ A(0,1)$	$\mathcal{R}''_{1/2} \oplus 2\mathcal{R}_0$	$3\mathcal{R}_1 \oplus 5\mathcal{R}_{1/2} \oplus 2\mathcal{R}_0$
	$B(1,1) \oplus B(0,1)$	$\mathcal{R}_1 \oplus \mathcal{R}''_{1/2}$	$2\mathcal{R}_{3/2} \oplus 3\mathcal{R}_1 \oplus \mathcal{R}_{1/2}$
	$A(1,0)$	$2\mathcal{R}_{1/2} \oplus 2\mathcal{R}''_0$	$\mathcal{R}_1 \oplus 7\mathcal{R}_{1/2} \oplus 6\mathcal{R}_0$
$D(3,1)$	$D(2,1)$	$\mathcal{R}_1 \oplus 3\mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 4\mathcal{R}_1 \oplus 3\mathcal{R}_0$
	$C(2)$	$\mathcal{R}''_{1/2} \oplus 5\mathcal{R}_0$	$\mathcal{R}_1 \oplus 5\mathcal{R}_{1/2} \oplus 10\mathcal{R}_0$
	$A(1,0)$	$2\mathcal{R}_{1/2} \oplus 2\mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 4\mathcal{R}_0$

Table 3.83: $osp(1|2)$ decompositions of the superalgebra $F(4)$.

SSA in $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$A(1,0)$	$\mathcal{R}_1 \oplus 7\mathcal{R}_{1/2} \oplus 14\mathcal{R}_0$
$A(0,1)$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 6\mathcal{R}'_{1/2} \oplus 6\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
$C(2)$	$5\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 6\mathcal{R}_0$
$D(2,1;2)$	$\mathcal{R}_{3/2} \oplus 2\mathcal{R}'_{3/2} \oplus 2\mathcal{R}_1 \oplus 2\mathcal{R}'_{1/2} \oplus 3\mathcal{R}_0$

Table 3.84: $osp(1|2)$ decompositions of the superalgebra $G(3)$.

SSA in $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$A(1,0)$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 4\mathcal{R}'_{1/2} \oplus 3\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
$A(1,0)'$	$2\mathcal{R}'_{3/2} \oplus \mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$
$B(0,1)$	$\mathcal{R}_1 \oplus 6\mathcal{R}_{1/2} \oplus 8\mathcal{R}_0$
$B(1,1)$	$\mathcal{R}_{3/2} \oplus 2\mathcal{R}'_{3/2} \oplus \mathcal{R}_1 \oplus 3\mathcal{R}_0 \oplus 2\mathcal{R}'_0$
$D(2,1;3)$	$\mathcal{R}_2 \oplus \mathcal{R}_{3/2} \oplus 3\mathcal{R}_1$

Table 3.85: $osp(1|2)$ decompositions of the superalgebra $D(2,1;\alpha)$.

SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$D(2,1)$	$\mathcal{R}_1 \oplus \mathcal{R}_0$	$\mathcal{R}_{3/2} \oplus 2\mathcal{R}_1$
$C(2)$	$\mathcal{R}''_{1/2} \oplus 3\mathcal{R}_0$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$
$A(1,0)$	$2\mathcal{R}_{1/2}$	$\mathcal{R}_1 \oplus 3\mathcal{R}_{1/2} \oplus 3\mathcal{R}_0$

Table 3.86: $sl(1|2)$ decompositions of the $A(m,n)$ and $C(n+1)$ superalgebras up to rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$A(0,1)$	$A(0,1)$	$\pi''(\frac{1}{2},\frac{1}{2})$	$\pi(0,1)$
$A(0,2)$	$A(0,1)$	$\pi''(\frac{1}{2},\frac{1}{2}) \oplus \pi''(0,0)$	$\pi(0,1) \oplus \pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$
$A(1,1)$	$A(0,1)$	$\pi''(\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{1}{2},\frac{1}{2})$
$A(0,3)$	$A(0,1)$	$\pi''(\frac{1}{2},\frac{1}{2}) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus 2\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
$A(1,2)$	$A(1,2)$	$\pi''(1,1)$	$\pi(0,2) \oplus \pi(0,1)$
	$A(0,1)$	$\pi''(\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0) \oplus \pi''(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi(0,0) \oplus 2\pi'(0,0)$
	$A(1,0)$	$\pi(\frac{1}{2},\frac{1}{2}) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus 2\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
$C(3)$	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2})$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus \pi(0,0)$
	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus 2\pi'(0,\frac{1}{2}) \oplus 3\pi(0,0)$
$C(4)$	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus 2\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 4\pi''(0,0)$	$\pi(0,1) \oplus 4\pi'(0,\frac{1}{2}) \oplus 10\pi(0,0)$

Table 3.87: $sl(1|2)$ decompositions of the $B(m,n)$ superalgebras up to rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$B(1,1)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus \pi(0,0)$	$\pi(0,1) \oplus \pi(0,\frac{1}{2})$
$B(1,2)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus \pi(0,0) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus \pi(0,\frac{1}{2}) \oplus 2\pi'(0,\frac{1}{2}) \oplus 3\pi(0,0) \oplus 2\pi'(0,0)$
	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus \pi(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$
$B(2,1)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 3\pi(0,0)$	$\pi(0,1) \oplus 3\pi(0,\frac{1}{2}) \oplus 3\pi(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus \pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$
$B(1,3)$	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus \pi(\pm\frac{1}{2},\frac{1}{2})$ $\oplus 2\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0) \oplus 2\pi'(0,0)$
	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 4\pi''(0,0) \oplus \pi(0,0)$	$\pi(0,1) \oplus \pi(0,\frac{1}{2}) \oplus 4\pi'(0,\frac{1}{2}) \oplus 10\pi(0,0) \oplus 4\pi'(0,0)$
$B(2,2)$	$2\,C(2)$	$2\pi''(0,\frac{1}{2}) \oplus \pi(0,0)$	$3\pi(0,1) \oplus 2\pi(0,\frac{1}{2}) \oplus \pi(0;-\frac{1}{2},\frac{1}{2};0)$
	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2}) \oplus 3\pi(0,0)$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus 3\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 3\pi(0,0) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus 3\pi(0,\frac{1}{2}) \oplus 2\pi'(0,\frac{1}{2}) \oplus 6\pi(0,0) \oplus 6\pi'(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus 2\pi(\pm\frac{1}{2},\frac{1}{2})$ $\oplus \pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0) \oplus 2\pi'(0,0)$
$B(3,1)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 5\pi(0,0)$	$\pi(0,1) \oplus 5\pi(0,\frac{1}{2}) \oplus 10\pi(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 3\pi(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus 3\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$

Table 3.88: $sl(1|2)$ decompositions of the $D(m,n)$ superalgebras up to rank 4.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$D(2,1)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 2\pi(0,0)$	$\pi(0,1) \oplus 2\pi(0,\frac{1}{2}) \oplus \pi(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2})$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus \pi(0,0)$
$D(2,2)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 2\pi(0,0) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus 2\pi(0,\frac{1}{2}) \oplus 2\pi'(0,\frac{1}{2}) \oplus 4\pi(0,0) \oplus 4\pi'(0,0)$
	$2C(2)$	$2\pi''(0,\frac{1}{2})$	$3\pi(0,1) \oplus \pi(0;-\frac{1}{2},\frac{1}{2};0)$
	$A(0,1)$	$\pi''(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi(0,0)$	$\pi(0,1) \oplus \pi(\pm 1,1) \oplus 2\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi''(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus 2\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
$D(3,1)$	$C(2)$	$\pi''(0,\frac{1}{2}) \oplus 4\pi(0,0)$	$\pi(0,1) \oplus 4\pi(0,\frac{1}{2}) \oplus 6\pi(0,0)$
	$A(1,0)$	$\pi(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi(0,0)$	$\pi(0,1) \oplus \pi(\pm\frac{3}{2},\frac{1}{2}) \oplus 2\pi'(\pm\frac{1}{2},\frac{1}{2}) \oplus 2\pi(0,0)$

Table 3.89: $sl(1|2)$ decompositions of the exceptional superalgebras.

$\mathcal{G}$	SSA in $\mathcal{G}$	Decomposition of the adjoint of $\mathcal{G}$
$F(4)$	$A(1,0)$	$(0,1) \oplus 3\pi(\frac{1}{6},\frac{1}{2}) \oplus 3\pi(-\frac{1}{6},\frac{1}{2}) \oplus 8\pi(0,0)$
	$A(0,1)$	$\pi(0,1) \oplus \pi(1,\frac{1}{2}) \oplus \pi(-1,\frac{1}{2}) \oplus 4\pi(0,0)$ $\oplus 2\pi'(\frac{1}{2},\frac{1}{2}) \oplus 2\pi'(-\frac{1}{2},\frac{1}{2}) \oplus 2\pi'(0,\frac{1}{2})$
	$C(2)$	$\pi(0,1) \oplus 2\pi(1,1) \oplus 2\pi(-1,1)$ $\oplus \pi(\frac{5}{2},\frac{1}{2}) \oplus \pi(-\frac{5}{2},\frac{1}{2}) \oplus 4\pi(0,0)$
$G(3)$	$A(1,0)$	$\pi(0,1) \oplus \pi(\frac{5}{6},\frac{1}{2}) \oplus \pi(-\frac{5}{6},\frac{1}{2}) \oplus \pi'(\frac{1}{6},\frac{1}{2})$ $\oplus \pi'(-\frac{1}{6},\frac{1}{2}) \oplus \pi'(\frac{1}{2},\frac{1}{2}) \oplus \pi'(-\frac{1}{2},\frac{1}{2}) \oplus \pi(0,0)$
	$A(1,0)'$	$\pi(0,1) \oplus \pi(\frac{7}{2},\frac{1}{2}) \oplus \pi(-\frac{7}{2},\frac{1}{2})$ $\oplus \pi'(\frac{3}{2},\frac{3}{2}) \oplus \pi'(-\frac{3}{2},\frac{3}{2}) \oplus \pi(0,0)$
	$C(2)$	$\pi(0,1) \oplus 2\pi(\frac{1}{4},\frac{1}{2}) \oplus 2\pi(-\frac{1}{4},\frac{1}{2}) \oplus \pi(0,\frac{1}{2})$
$D(2,1;\alpha)$	$A(1,0)$	$\pi(0,1) \oplus \pi(\alpha+\frac{1}{2},\frac{1}{2}) \oplus \pi(-\alpha-\frac{1}{2}),\frac{1}{2}) \oplus \pi(0,0)$
	$A(1,0)'$	$\pi(0,1) \oplus \pi(\frac{1}{2}\left(\frac{1-\alpha}{1+\alpha}\right),\frac{1}{2}) \oplus \pi(-\frac{1}{2}\left(\frac{1-\alpha}{1+\alpha}\right),\frac{1}{2}) \oplus \pi(0,0)$
	$C(2)$	$\pi(0,1) \oplus \pi(\frac{1}{2}\left(\frac{2+\alpha}{\alpha}\right),\frac{1}{2}) \oplus \pi(-\frac{1}{2}\left(\frac{2+\alpha}{\alpha}\right),\frac{1}{2}) \oplus \pi(0,0)$

Let us remark that for $D(2,1;\alpha)$ from any $sl(1|2)$ decomposition one gets the two others by replacing α by one of the values α^{-1}, $-1-\alpha$, $\dfrac{-\alpha}{1+\alpha}$. This corresponds to isomorphic versions of the exceptional superalgebra $D(2,1;\alpha)$ ($\rightarrow$ 2.20). One can check this triality-like property, which certainly deserves some developments, by the studying the completely odd Dynkin diagram of $D(2,1;\alpha)$.

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