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BY

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Preface

This book is designed to be used for a one-semester course in differential geometry for senior undergraduates or first year graduate students. It presents the fundamental concepts of the differential geometry of curves and surfaces in three-dimensional Euclidean space and applies these concepts to many examples and solved problems.

The basic theory of vectors and vector calculus of a single variable is given in Chapters 1 and 2. The concept of a curve is presented in Chapter 3, and Chapters 4 and 5 discuss the theory of curves in E^3, including selected topics in the theory of contact, a very natural approach to the classical theory of curves.

Considerable care is given to the definition of a surface so as to provide the reader with a firm foundation for the treatment of global problems and for further study in modern differential geometry. In order to accomplish this, background material in analysis and point set topology in Euclidean spaces is presented in Chapters 6 and 7. The surface is then defined in Chapter 8 and Chapters 9 and 10 are devoted to the theory of the non-intrinsic geometry of a surface, including an introduction to tensor methods and selected topics in the global geometry of surfaces. The final chapter presents the basic theory of the intrinsic geometry of surfaces in E^3.

Numerous illustrations are presented throughout the book to help the reader visually, and many graded supplementary problems are included at the end of each chapter to help the reader test his understanding of the subject matter.

It is a pleasure to acknowledge the help of Martin Silverstein and Jih-Shen Chiu who made many useful suggestions and criticisms. I am also grateful to Daniel Schaum and Nicola Monti for their splendid editorial cooperation and to Henry Hayden for typographical arrangement and art work for the figures. Finally I wish to express my appreciation to my wife Sarah for carefully typing the manuscript.

MARTIN M. LIPSCHUTZ

Bridgeport, Conn.
March 1969

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Chapter 1

Vectors

INTRODUCTION

Differential geometry is the study of geometric figures using the methods of calculus. In particular the introductory theory investigates curves and surfaces embedded in three dimensional Euclidean space E^3.

Properties of curves and surfaces which depend only upon points close to a particular point of the figure are called *local* properties. The study of local properties is called differential geometry *in the small.* Those properties which involve the entire geometric figure are called *global* properties. The study of global properties, in particular as they relate to local properties, is called differential geometry *in the large.*

Example 1.1.

Let Q and R be two points near a point P on a curve Γ in a plane and let C_{QR} be the circle through P, Q and R, as shown in Fig. 1-1. Now consider the limiting position of the circles C_{QR} as Q and R approach P. In general, the limiting position will be a circle C tangent to Γ at P. The radius of C is the *radius of curvature* of Γ at P. The radius of curvature is an example of a local property of the curve, for it depends only on the points on Γ near P.

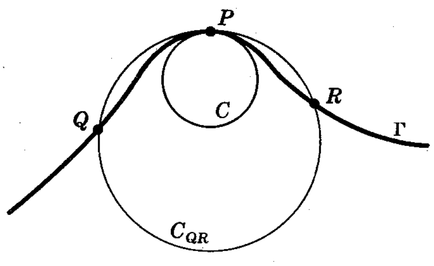


Fig. 1-1

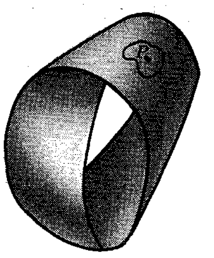


Fig. 1-2

Example 1.2.

The Moebius strip shown in Fig. 1-2 is an example of a one-sided surface. One-sidedness is an example of a global property of a figure, for it depends on the nature of the entire surface. Observe that a small part of the surface surrounding an arbitrary point P is a regular two-sided surface, i.e. *locally* the Moebius strip is two-sided.

We first investigate local properties of curves and surfaces and then apply the results to problems of differential geometry in the large. We begin with a review of vectors in E^3.

VECTORS

By Euclidean space E^3 we mean the set of ordered triplets $\mathbf{a} = (a_1, a_2, a_3)$ with a_1, a_2, a_3 real. A *vector* is a point in E^3 and in general will be denoted by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}, \mathbf{y}, \ldots$ or $P, Q, R, \ldots$. The *negative* of a vector $\mathbf{a}$ is the vector $-\mathbf{a}$ defined by $-\mathbf{a} = (-a_1, -a_2, -a_3)$. The *zero* vector is the vector $\mathbf{0} = (0, 0, 0)$. The *length* or *magnitude* of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is the real number $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. Clearly $|\mathbf{a}| \geqq 0$ and $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

ADDITION OF VECTORS

Given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in E^3, their *sum* $\mathbf{a}+\mathbf{b}$ is the vector defined by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

The difference of two vectors **a** and **b** is the vector $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$. In Problem 1.1 we prove that vector addition satisfies

[$\mathbf{A}_1$] $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ (Commutative Law)

[$\mathbf{A}_2$] $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$ (Associative Law)

[$\mathbf{A}_3$] $\mathbf{0}+\mathbf{a} = \mathbf{a}$ for all **a**

[$\mathbf{A}_4$] $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$ for all **a**

Example 1.3.

Let $\mathbf{a} = (1,-2,0)$ and $\mathbf{b} = (0,1,1)$. Then $\mathbf{a}+\mathbf{b} = (1,-1,1)$, $-\mathbf{a} = (-1,2,0)$, $\mathbf{b}-\mathbf{a} = (-1,3,1)$, $|\mathbf{a}| = \sqrt{5}$.

Example 1.4.

Using [$\mathbf{A}_1$] through [$\mathbf{A}_4$] we see that for any **a** and **b**,

$$\mathbf{a} + (\mathbf{b}-\mathbf{a}) = \mathbf{a} + (\mathbf{b}+(-\mathbf{a})) = \mathbf{a} + (-\mathbf{a}) + \mathbf{b} = \mathbf{0} + \mathbf{b} = \mathbf{b}$$

Thus the vector equation $\mathbf{a}+\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = \mathbf{b}-\mathbf{a}$. It is also the only solution. For if $\mathbf{a}+\mathbf{y} = \mathbf{b}$, then

$$(-\mathbf{a}) + \mathbf{a} + \mathbf{y} = (-\mathbf{a}) + \mathbf{b} = \mathbf{b}-\mathbf{a}, \quad \text{or} \quad \mathbf{0}+\mathbf{y} = \mathbf{b}-\mathbf{a}, \quad \text{or} \quad \mathbf{y} = \mathbf{b}-\mathbf{a}$$

Given two points P and Q in E^3 (that is, two vectors P and Q) we introduce the special notation **PQ** for their difference $Q-P$ and we picture **PQ** as an arrow drawn from P to Q as shown in Fig. 1-3. By the distance from P to Q we mean the length $|\mathbf{PQ}|$. Evidently $\mathbf{PQ} = -\mathbf{QP}$, $|\mathbf{PQ}| = |\mathbf{QP}|$, $\mathbf{PQ} = \mathbf{P'Q'}$ if and only if $Q - P = Q' - P'$, and $\mathbf{PP} = \mathbf{0}$ for all P.

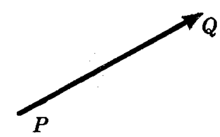


Fig. 1-3

Example 1.5.

Let $\mathbf{a} = \mathbf{PQ}$, $\mathbf{b} = \mathbf{QR}$ and $\mathbf{c} = \mathbf{RS}$, $\mathbf{d} = \mathbf{SP}$ as shown in Fig. 1-4. Then

$$\mathbf{a}+\mathbf{b} = \mathbf{PQ}+\mathbf{QR} = Q-P+R-Q = R-P = \mathbf{PR}$$

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{PR}+\mathbf{RS} = R-P+S-R = S-P = \mathbf{PS} = -\mathbf{d}$$

$$\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \mathbf{PS}+\mathbf{SP} = S-P+P-S = \mathbf{0}$$

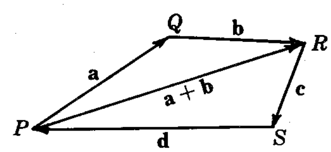


Fig. 1-4

MULTIPLICATION OF A VECTOR BY A SCALAR

If k is a real number and $\mathbf{a} = (a_1, a_2, a_3)$ a vector, we define the product $k\mathbf{a}$ to be the vector

$$k\mathbf{a} = (ka_1, ka_2, ka_3)$$

Clearly $0\mathbf{a} = k\mathbf{0} = \mathbf{0}$ for all k and **a**.

In the study of vectors we usually refer to the real numbers as *scalars*. The product $k\mathbf{a}$ is called *multiplication of a vector by a scalar*.

In Problem 1.4 we prove that multiplication of vectors by scalars satisfies

[$\mathbf{B}_1$] $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a} = k_1k_2\mathbf{a}$

[$\mathbf{B}_2$] $(k_1+k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$, $k(\mathbf{a}+\mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ (Distributive Laws)

[$\mathbf{B}_3$] $1\mathbf{a} = \mathbf{a}$

Finally, if $\mathbf{a} = (a_1, a_2, a_3)$, then

$$|k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} = \sqrt{k^2}\sqrt{a_1^2 + a_2^2 + a_3^2}$$

Thus for all k and $\mathbf{a}$,
$$|k\mathbf{a}| = |k|\,|\mathbf{a}| \tag{1.1}$$

Example 1.6.

Let $\mathbf{a} = (1, \pi, 0)$ and $\mathbf{b} = (0, 2, -1)$. Then $2\mathbf{a} = (2, 2\pi, 0)$, $(-1)\mathbf{a} = (-1, -\pi, 0) = -\mathbf{a}$, and $\mathbf{a} - 3\mathbf{b} = (1, \pi - 6, 3)$.

Example 1.7.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be given vectors and let $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2$, $\mathbf{b} = -\mathbf{u}_2 + 2\mathbf{u}_3$, and $\mathbf{c} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$. Then

$$\begin{aligned} \mathbf{a} - 2\mathbf{b} - \mathbf{c} &= (\mathbf{u}_1 - 2\mathbf{u}_2) - 2(-\mathbf{u}_2 + 2\mathbf{u}_3) - (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \\ &= \mathbf{u}_1 - 2\mathbf{u}_2 + 2\mathbf{u}_2 - 4\mathbf{u}_3 - \mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 = -\mathbf{u}_2 - 5\mathbf{u}_3 \end{aligned}$$

A vector $\mathbf{a}$ is said to have the same *direction* as a nonzero vector $\mathbf{b}$ if for some $k \geq 0$, $\mathbf{a} = k\mathbf{b}$. If $\mathbf{a}$ has the same direction as $\mathbf{b}$ and also the same length as $\mathbf{b}$, then from equation (*1.1*), $|\mathbf{a}| = |k|\,|\mathbf{b}| = k|\mathbf{b}| = |\mathbf{b}|$. Thus $k = 1$, and $\mathbf{a}$ equals $\mathbf{b}$. A vector is thus uniquely determined by its direction and length. If $\mathbf{a} = k\mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ and $k \leq 0$, then $\mathbf{a}$ has the *opposite direction* to $\mathbf{b}$. If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ has the same or opposite direction to $\mathbf{b}$, i.e. $\mathbf{a} = k\mathbf{b}$ for some real k, then $\mathbf{a}$ is *parallel* to $\mathbf{b}$.

We call a vector $\mathbf{u}$ of unit length a *unit* vector. In general $\mathbf{u}_\mathbf{a}$ shall denote the unit vector in the direction of a nonzero vector $\mathbf{a}$. Clearly this is obtained by multiplying $\mathbf{a}$ by $1/|\mathbf{a}|$, i.e.

$$\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| \tag{1.2}$$

Example 1.8.

Let $\mathbf{a} = (1, -1, 3)$, $\mathbf{b} = (2, -2, 6)$ and $\mathbf{c} = (-3, 3, -9)$. Since $\mathbf{a} = \frac{1}{2}\mathbf{b}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ have the same direction. The vectors $\mathbf{b}$ and $\mathbf{c}$ have opposite directions since $\mathbf{b} = -(2/3)\mathbf{c}$. The unit vector in the direction of $\mathbf{a}$ is the vector $\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| = (1/\sqrt{11}, -1/\sqrt{11}, 3/\sqrt{11})$.

Example 1.9.

In the triangle OAB shown in Fig. 1-5, let $\mathbf{a} = \mathbf{OA}$ and $\mathbf{b} = \mathbf{OB}$, and let M be the midpoint of side AB. Then the vector $\mathbf{OM}$ can be expressed in terms of $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\begin{aligned} \mathbf{OM} &= \mathbf{a} + \mathbf{AM} = \mathbf{a} + \tfrac{1}{2}\mathbf{AB} \\ &= \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} \\ &= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} \end{aligned}$$

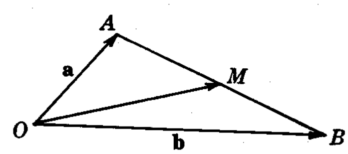


Fig. 1-5

LINEAR DEPENDENCE AND INDEPENDENCE

We now define the very important concepts of linear dependence and linear independence. Namely, the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are said to be *linearly dependent* if there exist scalars $k_1, k_2, \ldots, k_n$ *not all zero* such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = \mathbf{0} \tag{1.3}$$

The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are said to be *linearly independent* if they are not linearly dependent. That is, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent if (*1.3*) implies all $k_1 = k_2 = \cdots = k_n = 0$.

Note that a set of vectors which includes the zero vector is dependent; for we can always write $1\mathbf{0} + 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_n = \mathbf{0}$.

Example 1.10.

The vectors $\mathbf{a} = (1, -1, 0)$, $\mathbf{b} = (0, 2, -1)$, $\mathbf{c} = (2, 0, -1)$ are linearly dependent, since $2\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$.

Example 1.11.

Suppose $\mathbf{a}$ is parallel to $\mathbf{b}$. Then $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} = k\mathbf{b}$, i.e. $\mathbf{a} - k\mathbf{b} = \mathbf{0}$. Thus $\mathbf{a}$ and $\mathbf{b}$ are dependent. Conversely, suppose $\mathbf{a}$ and $\mathbf{b}$ are dependent. Then $k_1\mathbf{a} + k_2\mathbf{b} = \mathbf{0}$ where, say, $k_1 \neq 0$. But then $\mathbf{a} = -(k_2/k_1)\mathbf{b}$. Thus two vectors are dependent if and only if they are parallel.

In Problem 1.10 we prove the following important property of linearly *independent* vectors:

Theorem 1.1. If a vector is expressed as a linear function of independent vectors, then it is expressed so *uniquely*. That is, if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are independent, and if

$$\mathbf{u} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = k_1'\mathbf{u}_1 + k_2'\mathbf{u}_2 + \cdots + k_n'\mathbf{u}_n$$

then $k_1 = k_1',\ k_2 = k_2',\ \ldots,\ k_n = k_n'$.

BASES AND COMPONENTS

The three vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ are independent. For $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = (k_1,k_2,k_3)$ and so if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{0}$, then $k_1 = k_2 = k_3 = 0$. Also any vector $\mathbf{a} = (a_1,a_2,a_3)$ can be written $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and by Theorem 1.1 this representation is unique.

In general we call a set of vectors B a *basis* for E^3 if (i) *every* vector in E^3 can be written as a linear combination of the vectors in B, (ii) B is a linearly independent set of vectors.

In Problem 1.11 we prove

Theorem 1.2. Any three linearly independent vectors form a basis in E^3. Conversely, every basis in E^3 consists of three linearly independent vectors.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis in space and let $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$. The scalars a_1, a_2, a_3, for short, a_i, $i = 1,2,3$, are called the *components* of $\mathbf{a}$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

It follows from Theorem 1.1 that the components of a vector with respect to a given basis are unique. However, note that the components of a vector depend upon the basis chosen and in general the components will change if there is a change in basis. An exception is the vector $\mathbf{0}$ whose components are always 0, 0, 0.

In general we shall denote the components of vectors $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \ldots$ with respect to some prescribed basis by $a_i, b_i, x_i, y_i, u_i, \ldots$.

Example 1.12.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{a} = 2\mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{b} = \mathbf{u}_2 - 2\mathbf{u}_3$, and $\mathbf{c} = 3\mathbf{u}_1 + \mathbf{u}_3$. We will show that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent and hence also form a basis. For, suppose

$$k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} = (2k_1 + 3k_3)\mathbf{u}_1 + (-k_1 + k_2)\mathbf{u}_2 + (-2k_2 + k_3)\mathbf{u}_3 = \mathbf{0}$$

Since the $\mathbf{u}_i$ are independent, it follows that

$$2k_1 + 3k_3 = 0, \quad -k_1 + k_2 = 0, \quad -2k_2 + k_3 = 0$$

This is a system of three homogeneous linear equations in k_1, k_2, k_3. Since the determinant of the coefficients

$$\det\begin{pmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = 8 \neq 0$$

the *only* solution is $k_1 = k_2 = k_3 = 0$. Hence the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are independent. Observe that the components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ appear as the columns in the above determinant.

As suggested in the above example, we have in general

Theorem 1.3. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let

$$\begin{aligned} \mathbf{v}_1 &= a_{11}\mathbf{u}_1 + a_{21}\mathbf{u}_2 + a_{31}\mathbf{u}_3 \\ \mathbf{v}_2 &= a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + a_{32}\mathbf{u}_3 \\ \mathbf{v}_3 &= a_{13}\mathbf{u}_1 + a_{23}\mathbf{u}_2 + a_{33}\mathbf{u}_3 \end{aligned}$$

or, in short, $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i$, $j = 1, 2, 3$. Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0$$

SCALAR PRODUCT OF VECTORS

The *dot* or *scalar product* of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the real number

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

In particular, for $\mathbf{a} = \mathbf{b}$ we have the formula

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \tag{1.4}$$

In Problem 1.14 we prove that scalar multiplication satisfies

[C₁] $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (Symmetric Law)

[C₂] $(k\mathbf{a})\cdot\mathbf{b} = k(\mathbf{a}\cdot\mathbf{b})$ (k = scalar)

[C₃] $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ (Distributive Law)

[C₄] Scalar multiplication is *positive definite*; that is,

(i) $\mathbf{a}\cdot\mathbf{a} \geq 0$ for all $\mathbf{a}$

(ii) $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$

Clearly, from the definition, $\mathbf{a}\cdot\mathbf{0} = 0$ for all $\mathbf{a}$. Also, if $\mathbf{a}\cdot\mathbf{b} = 0$ for all $\mathbf{a}$, then $\mathbf{b}\cdot\mathbf{b} = 0$, and hence from [C₄](ii), $\mathbf{b} = \mathbf{0}$.

Example 1.13.

Let $\mathbf{a} = (-2, 1, 0)$ and $\mathbf{b} = (2, 1, 1)$. Then $\mathbf{a}\cdot\mathbf{b} = -3$ and $\mathbf{a}\cdot\mathbf{a} = 5 = |\mathbf{a}|^2$.

Example 1.14.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be given vectors and let $\mathbf{a} = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{b} = 2\mathbf{u}_1 + \mathbf{u}_2$. Then

$$\mathbf{a}\cdot\mathbf{b} = (\mathbf{u}_1-\mathbf{u}_2)\cdot(2\mathbf{u}_1+\mathbf{u}_2) = 2\mathbf{u}_1\cdot\mathbf{u}_1 - 2\mathbf{u}_1\cdot\mathbf{u}_2 + \mathbf{u}_1\cdot\mathbf{u}_2 - \mathbf{u}_2\cdot\mathbf{u}_2 = 2|\mathbf{u}_1|^2 - \mathbf{u}_1\cdot\mathbf{u}_2 - |\mathbf{u}_2|^2$$

In Problem 1.16 we prove the *Cauchy-Schwarz inequality*

$$|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$$

with *equality holding if and only if* **a** *and* **b** *are linearly dependent.* The *angle between two nonzero vectors* **a** *and* **b**, denoted by $\theta = \measuredangle(\mathbf{a},\mathbf{b})$, is the unique solution of

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{1.5}$$

satisfying $0 \leq \theta \leq \pi$.

Example 1.15.

In the triangle ABC shown in Fig. 1-6, let $\mathbf{a} = \mathbf{BC}$, $\mathbf{b} = \mathbf{AC}$, $\mathbf{c} = \mathbf{BA} = \mathbf{a} - \mathbf{b}$, and $\theta = \measuredangle ACB = \measuredangle(\mathbf{a},\mathbf{b})$. If we consider

$$|\mathbf{c}|^2 = |\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}$$

we have the law of cosines

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta + |\mathbf{b}|^2$$

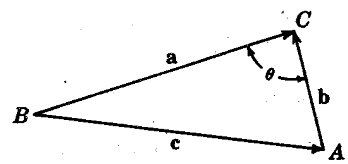


Fig. 1-6

Let **b** be a nonzero vector. The *scalar projection of* **a** *onto* **b**, denoted by $P_\mathbf{b}(\mathbf{a})$, is the scalar $P_\mathbf{b}(\mathbf{a}) \equiv (\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|$. The vector $P_\mathbf{b}(\mathbf{a})\mathbf{u}_\mathbf{b}$, where $\mathbf{u}_\mathbf{b}$ is the *unit vector in the direction of* **b**, is called the *vector projection of* **a** *onto* **b** and is denoted by $\mathbf{P}_\mathbf{b}(\mathbf{a})$. It follows that

$$\mathbf{P_b(a)} = P_\mathbf{b}(\mathbf{a})\mathbf{u_b} = ((\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|)(\mathbf{b}/|\mathbf{b}|) = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} \tag{1.6}$$

Clearly $P_\mathbf{b}(0) = 0$ and $\mathbf{P_b}(\mathbf{0}) = \mathbf{0}$. If $\mathbf{a} \neq \mathbf{0}$, then from equation (*1.5*), $P_\mathbf{b}(\mathbf{a}) = |\mathbf{a}| \cos\theta$ and $\mathbf{P_b}(\mathbf{a}) = |\mathbf{a}|\cos\theta\,\mathbf{u_b}$, where $\theta = \measuredangle(\mathbf{a},\mathbf{b})$. It follows that $P_\mathbf{b}(\mathbf{a})$ and $\mathbf{P_b}(\mathbf{a})$ are independent of the length of **b** but depend only on its direction as indicated in Fig. 1-7. In fact, the vector $\mathbf{P_b}(\mathbf{a})$ is also independent of the sense of **b**; that is, $\mathbf{P_{-b}}(\mathbf{a}) = \mathbf{P_b}(\mathbf{a})$. For

$$\mathbf{P_{-b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot(-\mathbf{b})}{|-\mathbf{b}|^2}(-\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \mathbf{P_b}(\mathbf{a})$$

The scalar $P_\mathbf{b}(\mathbf{a})$ changes sign with a change in the sense of **b**.

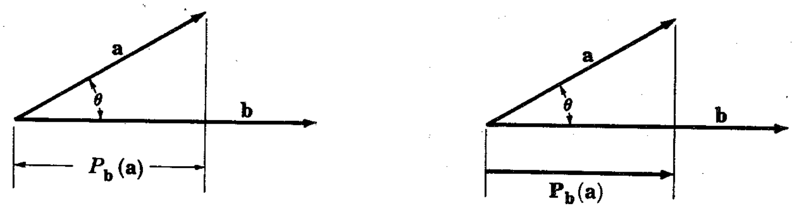


Fig. 1-7

ORTHOGONAL VECTORS

Two vectors **a** and **b** are said to be orthogonal, written $\mathbf{a} \perp \mathbf{b}$, if $\mathbf{a}\cdot\mathbf{b} = 0$. It follows from equation (*1.5*) that **a** and **b** are orthogonal if and only if either $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\theta = \measuredangle(\mathbf{a},\mathbf{b}) = \pi/2$.

Example 1.16.

Let **a** and **b** be linearly independent and let $\mathbf{c} = \mathbf{a} - \mathbf{P_b}(\mathbf{a})$. Then **c** is a nonzero vector orthogonal to **b**. For suppose $\mathbf{c} = \mathbf{0}$; then from equation (*1.6*), $\mathbf{0} = 1\mathbf{a} - \mathbf{P_b}(\mathbf{a}) = 1\mathbf{a} - k\mathbf{b}$, where $k = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|^2$, which is impossible since **a** and **b** are independent. Hence $\mathbf{c} \neq \mathbf{0}$. Finally,

$$\mathbf{c}\cdot\mathbf{b} = \left(\mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}\right)\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b})}{|\mathbf{b}|^2} = (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b}) = 0$$

Thus $\mathbf{c} \perp \mathbf{b}$.

ORTHONORMAL BASES

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be three mutually orthogonal unit vectors as shown in Fig. 1-8. These vectors are independent; for if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{0}$, then $0 = \mathbf{e}_i\cdot\mathbf{0} = \mathbf{e}_i\cdot(k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3) = \mathbf{e}_i\cdot k_i\mathbf{e}_i = k_i$ or $k_i = 0$ for each i. Therefore they form a basis called an *orthonormal* basis.

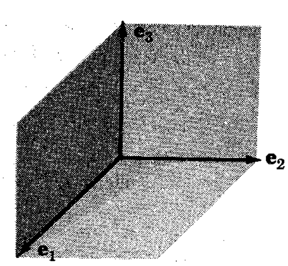


Fig. 1-8

We observe that $\mathbf{e}_i$, $i = 1,2,3$, is an orthonormal basis if and only if

$$\mathbf{e}_1\cdot\mathbf{e}_1 = \mathbf{e}_2\cdot\mathbf{e}_2 = \mathbf{e}_3\cdot\mathbf{e}_3 = 1 \quad \text{(Unit vectors)}$$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_3 = \mathbf{e}_1\cdot\mathbf{e}_3 = 0 \quad \text{(Mutually orthogonal)}$$

or, in short,

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases} \qquad (i,j = 1,2,3) \tag{1.7}$$

The quantity δ_{ij} is called the *Kronecker* symbol and will be used repeatedly.

In Problem 1.23 we prove

Theorem 1.4. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis and let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. Then

(i) $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_ib_i$

(ii) $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{i=1}^{3} a_i^2}$

(iii) $a_i = \mathbf{a}\cdot\mathbf{e}_i, \quad (i = 1, 2, 3)$.

Example 1.17.

Let $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_3$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$, and $\mathbf{c} = -2\mathbf{e}_2 + \mathbf{e}_3$. Then

(*a*) $\mathbf{a}\cdot\mathbf{b} = (1)(2) + (0)(1) + (2)(-2) = -2$

(*b*) $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = [(1)(0) + (0)(-2) + (2)(1)]\,(2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3) = 4\mathbf{e}_1 + 2\mathbf{e}_2 - 4\mathbf{e}_3$

(*c*) $|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

(*d*) $\mathbf{u}_\mathbf{a} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = (1/\sqrt{5})\mathbf{e}_1 + (2/\sqrt{5})\mathbf{e}_3$

(*e*) $\cos \measuredangle(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-2}{3\sqrt{5}}$

Let a nonzero vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and let $\theta_i = \measuredangle(\mathbf{a}, \mathbf{e}_i)$, $i = 1, 2, 3$, as shown in Fig. 1-9. The scalars $\cos\theta_1, \cos\theta_2, \cos\theta_3$ are called the *direction cosines* of $\mathbf{a}$. Since $\mathbf{a}\cdot\mathbf{e}_i = |\mathbf{a}|\cos\theta_i = a_i$, we have

$$\cos\theta_i = a_i/|\mathbf{a}|, \quad i = 1, 2, 3$$

Note that

$$\mathbf{u}_\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{a_1}{|\mathbf{a}|}\mathbf{e}_1 + \frac{a_2}{|\mathbf{a}|}\mathbf{e}_2 + \frac{a_3}{|\mathbf{a}|}\mathbf{e}_3$$
$$= (\cos\theta_1)\mathbf{e}_1 + (\cos\theta_2)\mathbf{e}_2 + (\cos\theta_3)\mathbf{e}_3$$

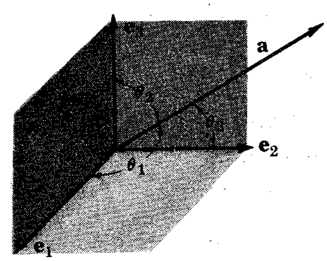


Fig. 1-9

That is, the direction cosines of **a** are the components of the unit vector in the direction of **a**.

ORIENTED BASES

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ be *ordered* orthonormal bases and imagine that the triad $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is rotated to make $\mathbf{g}_1$ and $\mathbf{g}_2$ coincide with $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. Then $\mathbf{g}_3$ will either coincide with $\mathbf{e}_3$ in which case we say that $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ has the same orientation as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, or $\mathbf{g}_3$ will point in the direction opposite to $\mathbf{e}_3$, in which case the bases are said to have *opposite orientation.* To formulate this concept of orientation in a precise manner, not only for orthonormal bases but for arbitrary bases, we proceed as follows:

Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be ordered bases and let $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i$. Then $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ if $\det(a_{ij}) > 0$. In Problem 1.27 we show that this defines an equivalence relation on the set of all ordered bases in E^3. This relation partitions the bases into exactly two equivalence classes. Ordered bases in the same class have the same orientation and ordered bases in different classes have opposite orientation.

In order to distinguish graphically one orientation of an ordered basis, we say $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is a *right-handed* basis if the vectors assume the same directions in space as the thumb, index finger and middle finger of the right hand; otherwise the basis is said to be a *left-handed* system.

Example 1.18.

The triplets $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ in Fig. 1-10(*a*) and (*c*) are right-handed bases. In Fig. 1-10(*b*) and (*d*) they are left-handed bases.

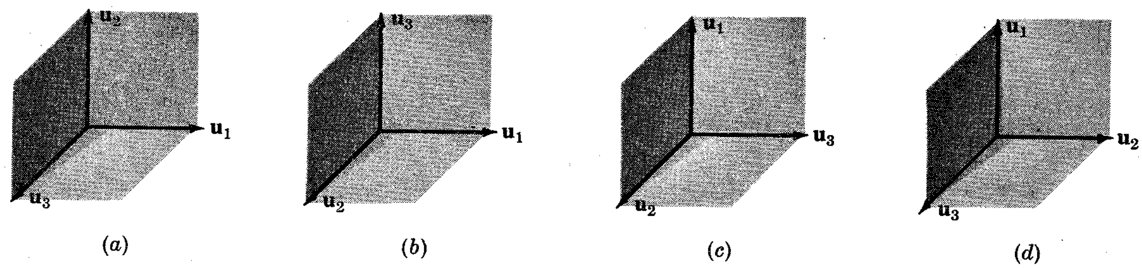


Fig. 1-10

Note. Unless stated otherwise, our bases shall be right-handed orthonormal bases.

VECTOR PRODUCT OF VECTORS

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a right-handed orthonormal basis and let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. The *cross* or *vector* product of **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is the vector

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

As an aid in computing the above, we observe that it can be obtained as the expansion of the determinant

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \det\begin{pmatrix}\mathbf{e}_1 & a_1 & b_1\\ \mathbf{e}_2 & a_2 & b_2\\ \mathbf{e}_3 & a_3 & b_3\end{pmatrix} = \mathbf{e}_1\det\begin{pmatrix}a_2 & b_2\\ a_3 & b_3\end{pmatrix} - \mathbf{e}_2\det\begin{pmatrix}a_1 & b_1\\ a_3 & b_3\end{pmatrix} + \mathbf{e}_3\det\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}\\ &= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3\end{aligned}$$

Example 1.19.

Let $\mathbf{a} = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{c} = -2\mathbf{e}_1 - \mathbf{e}_3$. Then

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix}\mathbf{e}_1 & 1 & 0\\ \mathbf{e}_2 & -1 & 1\\ \mathbf{e}_3 & 0 & 2\end{pmatrix} = -2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$$

In Problem 1.32 we prove that the vector product is in fact independent of the right-handed orthonormal basis chosen. Also in Problem 1.31 we prove

Theorem 1.5. (i) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, where $\theta = \measuredangle(\mathbf{a},\mathbf{b})$

(ii) a. $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$

b. If $\mathbf{a}\times\mathbf{b} \neq 0$, then $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ is a right-handed linearly independent triplet.

Since $|\mathbf{a}|\,|\mathbf{b}|\sin\theta = 0$ if and only if $|\mathbf{a}| = 0$, $|\mathbf{b}| = 0$, $\theta = 0$, or $\theta = \pi$, we have from (i) above and the strict form of the Schwarz inequality (that is, $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$ if and only if **a** and **b** are linearly dependent),

Theorem 1.6. $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if and only if **a** and **b** are linearly dependent.

If $\mathbf{a}$ and $\mathbf{b}$ are not linearly dependent, i.e. if $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$, Theorem 1.5(ii) states that $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$ and such that $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ is a right-handed triplet, as shown in Fig. 1-11(*a*).

Observe that the vector product is in general not commutative. Although $\mathbf{b}\times\mathbf{a}$ has the same magnitude as $\mathbf{a}\times\mathbf{b}$ (Theorem 1.5(i)) and is parallel to $\mathbf{a}\times\mathbf{b}$ (Theorem 1.5(ii)a), it has the opposite direction (Theorem 1.5(ii)b). Thus $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$, as shown in Fig. 1.11(*b*).

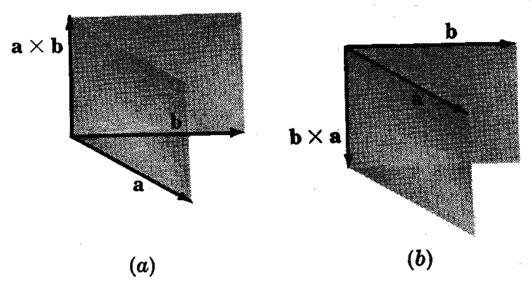


Fig. 1-11

Example 1.20.

For an orthonormal basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ shown in Fig. 1-12, it follows from Theorem 1.5 that

$$\begin{array}{lll} \mathbf{g}_1\times\mathbf{g}_1 = \mathbf{0} & \mathbf{g}_2\times\mathbf{g}_1 = -\mathbf{g}_3 & \mathbf{g}_3\times\mathbf{g}_1 = \mathbf{g}_2 \\ \mathbf{g}_1\times\mathbf{g}_2 = \mathbf{g}_3 & \mathbf{g}_2\times\mathbf{g}_2 = \mathbf{0} & \mathbf{g}_3\times\mathbf{g}_2 = -\mathbf{g}_1 \\ \mathbf{g}_1\times\mathbf{g}_3 = -\mathbf{g}_2 & \mathbf{g}_2\times\mathbf{g}_3 = \mathbf{g}_1 & \mathbf{g}_3\times\mathbf{g}_3 = \mathbf{0} \end{array}$$

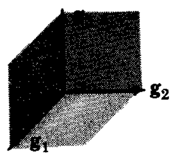


Fig. 1-12

In Problem 1.29 we prove that the vector product satisfies

[E₁] $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$ (Anticommutative Law)

[E₂] $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (Distributive Law)

[E₃] $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$ (k = scalar)

[E₄] $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

Note that the vector product is not only not commutative but also not associative; that is, in general $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. For as shown in Example 1.20, $\mathbf{g}_1\times(\mathbf{g}_1\times\mathbf{g}_2) = \mathbf{g}_1\times\mathbf{g}_3 = -\mathbf{g}_2$, whereas $(\mathbf{g}_1\times\mathbf{g}_1)\times\mathbf{g}_2 = \mathbf{0}\times\mathbf{g}_2 = \mathbf{0}$.

Example 1.21.

Consider the triangle ABC shown in Fig. 1-13. Let $\mathbf{a} = \mathbf{BC}$, $\mathbf{b} = \mathbf{AC}$, $\mathbf{c} = \mathbf{AB} = \mathbf{b}-\mathbf{a}$, $\alpha = \measuredangle(\mathbf{b}, \mathbf{c})$, $\beta = \measuredangle(\mathbf{c}, \mathbf{a})$, and $\gamma = \measuredangle(\mathbf{a}, \mathbf{b})$. Now,

$$\mathbf{0} = \mathbf{c}\times\mathbf{c} = \mathbf{c}\times(\mathbf{b}-\mathbf{a}) = \mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}$$

or

$$\mathbf{c}\times\mathbf{b} = \mathbf{c}\times\mathbf{a}$$

Similarly,

$$\mathbf{c}\times\mathbf{b} = (\mathbf{b}-\mathbf{a})\times\mathbf{b} = \mathbf{b}\times\mathbf{b}-\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{a}$$

Hence

$$\mathbf{c}\times\mathbf{b} = \mathbf{c}\times\mathbf{a} = \mathbf{b}\times\mathbf{a}$$

But then

$$|\mathbf{c}\times\mathbf{b}| = |\mathbf{c}\times\mathbf{a}| = |\mathbf{b}\times\mathbf{a}|$$

or

$$|\mathbf{c}|\,|\mathbf{b}|\sin\alpha = |\mathbf{c}|\,|\mathbf{a}|\sin\beta = |\mathbf{b}|\,|\mathbf{a}|\sin\gamma$$

which gives the law of sines

$$\frac{\sin\alpha}{|\mathbf{a}|} = \frac{\sin\beta}{|\mathbf{b}|} = \frac{\sin\gamma}{|\mathbf{c}|}$$

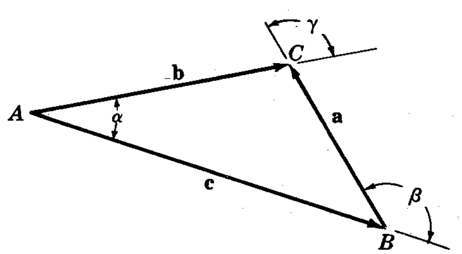


Fig. 1-13

TRIPLE PRODUCTS AND VECTOR IDENTITIES

The product $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is called the *mixed or triple scalar product*. Note that parentheses are not required; for this can only mean $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, the scalar product of the vector $\mathbf{a}$ and the vector $\mathbf{b}\times\mathbf{c}$. This product is also conveniently given in terms of a determinant. For, let

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \quad \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3, \quad \mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$$

Then

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot\det\begin{pmatrix}\mathbf{e}_1 & b_1 & c_1\\ \mathbf{e}_2 & b_2 & c_2\\ \mathbf{e}_3 & b_3 & c_3\end{pmatrix}\\ &= a_1(b_2c_3 - b_3c_2) - a_2(c_3b_1 - c_1b_3) + a_3(b_1c_2 - b_2c_1) \qquad (1.8)\\ &= \det\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\end{aligned}$$

It follows from properties of the determinant that

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = -(\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}) = -(\mathbf{c}\cdot\mathbf{b}\times\mathbf{a}) = -(\mathbf{a}\cdot\mathbf{c}\times\mathbf{b}) \qquad (1.9)$$

In particular, it follows that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$. Thus we can drop the dot and cross in the notation of the triple scalar product and use instead the notation

$$[\mathbf{abc}] = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$$

As immediate consequence of Theorem 1.3 and equation (*1.8*) we have

Theorem 1.7. $[\mathbf{abc}] = 0$ if and only if **a**, **b**, **c** are linearly dependent.

A number of useful identities relate vector and scalar products of vectors. A basic identity, which is derived in Problem 1.35, is

Theorem 1.8. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Others, easily derived from the above, are

[$\mathbf{F}_1$] $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

[$\mathbf{F}_2$] $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}$

Example 1.22.

Let $\mathbf{u} = \mathbf{c}\times\mathbf{d}$. Then

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{u} = \mathbf{a}\cdot\mathbf{b}\times\mathbf{u} = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}]$$

where we used (*1.9*) and Theorem 1.8. It follows that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

which proves [$\mathbf{F}_1$] above.

Solved Problems

VECTOR ADDITION

1.1. Prove properties [$\mathbf{A}_1$] through [$\mathbf{A}_4$] for vector addition. That is, prove that [$\mathbf{A}_1$] $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$, [$\mathbf{A}_2$] $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$, [$\mathbf{A}_3$] $\mathbf{a}+\mathbf{0} = \mathbf{a}$, [$\mathbf{A}_4$] $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$.

[$\mathbf{A}_1$]: $\mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2, a_3+b_3) = (b_1+a_1, b_2+a_2, b_3+a_3) = \mathbf{b}+\mathbf{a}$

[$\mathbf{A}_2$]: $(\mathbf{a}+\mathbf{b})+\mathbf{c} = [(a_1+b_1)+c_1, (a_2+b_2)+c_2, (a_3+b_3)+c_3]$
$= [a_1+(b_1+c_1), a_2+(b_2+c_2), a_3+(b_3+c_3)] = \mathbf{a}+(\mathbf{b}+\mathbf{c})$

[$\mathbf{A}_3$]: $\mathbf{a}+\mathbf{0} = (a_1+0, a_2+0, a_3+0) = (a_1, a_2, a_3) = \mathbf{a}$

[$\mathbf{A}_4$]: $\mathbf{a}+(-\mathbf{a}) = (a_1-a_1, a_2-a_2, a_3-a_3) = (0, 0, 0) = \mathbf{0}$

1.2. In the parallelepiped shown in Fig. 1-14 let $\mathbf{a} = \mathbf{OP}$, $\mathbf{b} = \mathbf{OR}$, $\mathbf{c} = \mathbf{OS}$. Find **OV**, **VQ** and **RT** in terms of **a**, **b**, **c**.

$$\mathbf{OV} = \mathbf{OR} + \mathbf{RV} = \mathbf{OR} + \mathbf{OS} = \mathbf{b} + \mathbf{c}$$

$$\mathbf{VQ} = \mathbf{VR} + \mathbf{RQ} = -\mathbf{RV} + \mathbf{RQ} = -\mathbf{OS} + \mathbf{OP} = -\mathbf{c} + \mathbf{a}$$

$$\mathbf{RT} = \mathbf{RS} + \mathbf{ST} = \mathbf{RO} + \mathbf{OS} + \mathbf{ST} = -\mathbf{b} + \mathbf{c} + \mathbf{a}$$

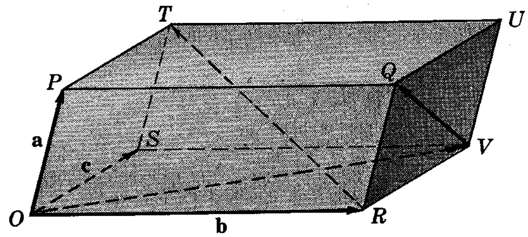


Fig. 1-14

1.3. It has been shown (Example 1.4.) using the properties [$\mathbf{A}_1$] through [$\mathbf{A}_4$] that the vector equation $\mathbf{a} + \mathbf{x} = \mathbf{b}$ has a *unique* solution $\mathbf{x} = \mathbf{b} + (-\mathbf{a}) = \mathbf{b} - \mathbf{a}$. Using this result, show that:

(*a*) the vector **0** is unique, that is, if $\mathbf{0}' + \mathbf{a} = \mathbf{a}$, then $\mathbf{0}' = \mathbf{0}$;

(*b*) the vector $-\mathbf{a}$ is unique, that is if $\mathbf{a}' + \mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = -\mathbf{a}$;

(*c*) $-(-\mathbf{a}) = \mathbf{a}$ for all **a**.

(*a*) follows from the uniqueness of the solution to the equation $\mathbf{x} + \mathbf{a} = \mathbf{a}$.

(*b*) follows from the uniqueness of the solution to the equation $\mathbf{x} + \mathbf{a} = \mathbf{0}$.

(*c*) follows when we consider the equation $-\mathbf{a} + \mathbf{x} = \mathbf{0}$. This has the solution $\mathbf{x} = \mathbf{0} - (-\mathbf{a}) = -(-\mathbf{a})$. But also $-\mathbf{a} + \mathbf{a} = \mathbf{0}$; thus $-(-\mathbf{a}) = \mathbf{a}$, by uniqueness of the solution to the equation.

MULTIPLICATION BY A SCALAR

1.4. Prove properties [$\mathbf{B}_1$] through [$\mathbf{B}_3$] for multiplication of a vector by a scalar. That is, prove that: [$\mathbf{B}_1$] $k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$; [$\mathbf{B}_2$] $(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$, $k(\mathbf{a}+\mathbf{b}) = k\mathbf{a} + k\mathbf{b}$; [$\mathbf{B}_3$] $1\mathbf{a} = \mathbf{a}$.

$$[\mathbf{B}_1]:\quad k_1(k_2\mathbf{a}) = (k_1(k_2a_1), k_1(k_2a_2), k_1(k_2a_3)) = ((k_1k_2)a_1, (k_1k_2)a_2, (k_1k_2)a_3) = (k_1k_2)\mathbf{a}$$

$$[\mathbf{B}_2]:\quad (k_1+k_2)\mathbf{a} = ((k_1+k_2)a_1, (k_1+k_2)a_2, (k_1+k_2)a_3) = (k_1a_1 + k_2a_1, k_1a_2 + k_2a_2, k_1a_3 + k_2a_3) = k_1\mathbf{a} + k_2\mathbf{a}$$

$$k(\mathbf{a}+\mathbf{b}) = (k(a_1+b_1), k(a_2+b_2), k(a_3+b_3)) = (ka_1 + kb_1, ka_2 + kb_2, ka_3 + kb_3) = k\mathbf{a} + k\mathbf{b}$$

$$[\mathbf{B}_3]:\quad 1\mathbf{a} = (1a_1, 1a_2, 1a_3) = (a_1, a_2, a_3) = \mathbf{a}$$

1.5. If $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2 + 3\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{c} = \mathbf{u}_1 + 2\mathbf{u}_2$, find $2\mathbf{a} - 3(\mathbf{b} - \mathbf{c})$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$2\mathbf{a} - 3(\mathbf{b}-\mathbf{c}) = 2\mathbf{a} - 3\mathbf{b} + 3\mathbf{c} = 2(\mathbf{u}_1 - 2\mathbf{u}_2 + 3\mathbf{u}_3) - 3(\mathbf{u}_2 - \mathbf{u}_3) + 3(\mathbf{u}_1 + 2\mathbf{u}_2)$$
$$= 2\mathbf{u}_1 - 4\mathbf{u}_2 + 6\mathbf{u}_3 - 3\mathbf{u}_2 + 3\mathbf{u}_3 + 3\mathbf{u}_1 + 6\mathbf{u}_2 = 5\mathbf{u}_1 - \mathbf{u}_2 + 9\mathbf{u}_3$$

1.6. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and has one-half its magnitude.

Let M and M' be the midpoints of the sides AB and AC respectively of a triangle ABC shown in Fig. 1-15. Then $\mathbf{AM} = \frac{1}{2}\mathbf{AB}$, $\mathbf{AM}' = \frac{1}{2}\mathbf{AC}$ and $\mathbf{MM}' = \mathbf{AM}' - \mathbf{AM} = \frac{1}{2}(\mathbf{AC} - \mathbf{AB}) = \frac{1}{2}\mathbf{BC}$. Thus MM' is parallel to **BC** and has half the magnitude.

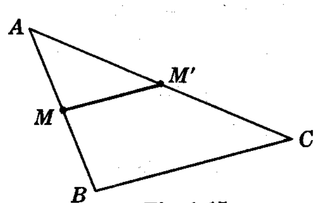


Fig. 1-15

1.7. Let $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$, $\mathbf{b} \neq \mathbf{a}$ and $\mathbf{c} = \mathbf{OC}$ as shown in Fig. 1-16. Show that C lies on the line L determined by A and B if and only if $\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$ where $k_1 + k_2 = 1$.

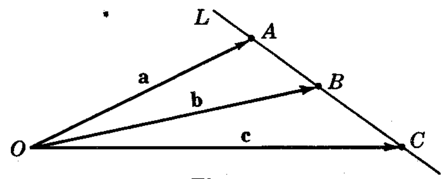


Fig. 1-16

If C is on L, then $\mathbf{BA} = \mathbf{a} - \mathbf{b}$ and $\mathbf{BC} = \mathbf{c} - \mathbf{b}$ are parallel. Hence there exists k such that

$$\mathbf{c} - \mathbf{b} = k(\mathbf{a} - \mathbf{b}) \quad \text{or} \quad \mathbf{c} = k\mathbf{a} + (1-k)\mathbf{b} = k_1\mathbf{a} + k_2\mathbf{b}$$

where $k_1 + k_2 = k + 1 - k = 1$. Conversely, if $\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$, where $k_1 + k_2 = 1$, $\mathbf{b} \neq \mathbf{a}$, then

$$\mathbf{c} - \mathbf{b} = k_1\mathbf{a} + k_2\mathbf{b} - \mathbf{b} = k_1\mathbf{a} - (1-k_2)\mathbf{b} = k_1\mathbf{a} - k_1\mathbf{b} = k_1(\mathbf{a} - \mathbf{b})$$

That is, $\mathbf{c} - \mathbf{b} = \mathbf{BC}$ and $\mathbf{a} - \mathbf{b} = \mathbf{BA}$ are parallel, so that C is on the line determined by A and B.

LINEAR DEPENDENCE AND INDEPENDENCE

1.8. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are dependent if and only if one of the vectors is a linear combination of the others.

Suppose, say, $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \ldots, \mathbf{u}_n$; i.e. $\mathbf{u}_1 = k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n$. Then $\mathbf{u}_1 - k_2\mathbf{u}_2 - \cdots - k_n\mathbf{u}_n = \mathbf{0}$ where at least the coefficient 1 of $\mathbf{u}_1$ is not zero. Hence $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are dependent.

Conversely, if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are dependent, then there exists $k_1, \ldots, k_n$ not all zero such that $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = \mathbf{0}$. Suppose, say, $k_1 \neq 0$; then $\mathbf{u}_1 = -(k_2/k_1)\mathbf{u}_2 - \cdots - (k_n/k_1)\mathbf{u}_n$ and so $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \ldots, \mathbf{u}_n$.

1.9. Prove that a set of vectors which contains a linearly dependent subset is linearly dependent.

Suppose, say, the subset $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is dependent. Then there exist $k_1, \ldots, k_k$ not all zero such that $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_k\mathbf{u}_k = \mathbf{0}$. But then

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_n = 0$$

and so $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are dependent.

1.10. Prove Theorem 1.1: If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are independent and

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = k_1'\mathbf{u}_1 + k_2'\mathbf{u}_2 + \cdots + k_n'\mathbf{u}_n$$

then $k_1 = k_1'$, $k_2 = k_2'$, $\ldots$, $k_n = k_n'$.

Suppose some $k_j \neq k_j'$; then

$$(k_1 - k_1')\mathbf{u}_1 + (k_2 - k_2')\mathbf{u}_2 + \cdots + (k_j - k_j')\mathbf{u}_j + \cdots + (k_n - k_n')\mathbf{u}_n = \mathbf{0}$$

where $k_j - k_j' \neq 0$. But this implies $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are dependent, which is a contradiction.

BASES AND COMPONENTS

1.11. Show that any vector in E^3 can be written as a linear combination of three independent vectors; hence three independent vectors form a basis in E^3.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, then the equation

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$$

has only the solution $x = y = z = 0$. Equivalently, the system

$$xa_1 + yb_1 + zc_1 = 0$$
$$xa_2 + yb_2 + zc_2 = 0$$
$$xa_3 + yb_3 + zc_3 = 0$$

has only the trivial solution $x = y = z = 0$, which is the case only if the matrix of coefficients has determinant not zero, that is,

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0$$

But then for any vector $\mathbf{u} = (u_1, u_2, u_3)$ in E^3 the system

$$xa_1 + yb_1 + zc_1 = u_1$$
$$xa_2 + yb_2 + zc_2 = u_2$$
$$xa_3 + yb_3 + zc_3 = u_3$$

has a solution $x = k_1$, $y = k_2$, $z = k_3$ which means that $\mathbf{u} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c}$ as required.

1.12. Show that any four or more vectors in E^3 are linearly dependent.

Consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$. We can assume that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, are independent. For otherwise, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$, having a dependent subset, would be dependent. But if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent, they form a basis and so $\mathbf{u}_4 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3$, which implies $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are dependent. Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n$ are dependent.

1.13. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{a} = \mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{c} = -\mathbf{u}_2$. Find the components of $2\mathbf{a} - \mathbf{b} - 2\mathbf{c}$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$\begin{aligned} 2\mathbf{a} - \mathbf{b} - 2\mathbf{c} &= 2(\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3) - (\mathbf{u}_2 - \mathbf{u}_3) - 2(-\mathbf{u}_2) \\ &= 2\mathbf{u}_1 - 2\mathbf{u}_2 + 4\mathbf{u}_3 - \mathbf{u}_2 + \mathbf{u}_3 + 2\mathbf{u}_2 = 2\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3 \end{aligned}$$

Thus the components of $2\mathbf{a} - \mathbf{b} - 2\mathbf{c}$ with respect to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are $2, -1, 5$.

SCALAR PRODUCT

1.14. Prove properties $[\mathbf{C}_1]$ through $[\mathbf{C}_4]$, page 5, for the scalar product of vectors.

$[\mathbf{C}_1]$: $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b}\cdot\mathbf{a}$

$[\mathbf{C}_2]$: $(k\mathbf{a})\cdot\mathbf{b} = ka_1b_1 + ka_2b_2 + ka_3b_3 = k(a_1b_1 + a_2b_2 + a_3b_3) = k(\mathbf{a}\cdot\mathbf{b})$

$[\mathbf{C}_3]$: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$
$\qquad = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

$[\mathbf{C}_4]$: Clearly $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 \geq 0$ and $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = 0$ iff $a_1 = a_2 = a_3 = 0$.

1.15. In the triangle OAB shown in Fig. 1-17, let $\mathbf{a} = \mathbf{OA}$ and $\mathbf{b} = \mathbf{OB}$. If $|\mathbf{OA}| = 2$, $|\mathbf{OB}| = 3$ and $\measuredangle AOB = 30°$, find (a) $\mathbf{a}\cdot\mathbf{b}$, (b) $P_\mathbf{a}(\mathbf{b})$, (c) $\mathbf{P}_\mathbf{a}(\mathbf{b})$.

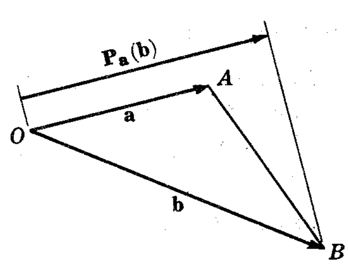


Fig. 1-17

(a) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\measuredangle(\mathbf{a},\mathbf{b})$
$\qquad = (2)(3)\cos 30° = 3\sqrt{3}$

(b) $P_\mathbf{a}(\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}| = 3\sqrt{3}/2$

(c) $\mathbf{P}_\mathbf{a}(\mathbf{b}) = P_\mathbf{a}(\mathbf{b})\dfrac{\mathbf{a}}{|\mathbf{a}|} = (3\sqrt{3}/4)\mathbf{a}$

1.16. Prove the *Cauchy-Schwarz* inequality, $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$, with equality holding if and only if **a** and **b** are linearly dependent.

The inequality is clearly valid if $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$, and so we may assume that $\mathbf{a},\mathbf{b}\neq\mathbf{0}$. Then from $[C_4]$,

$$0 \;\le\; \left(\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} \pm \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b}\right)\cdot\left(\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} \pm \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b}\right) \;\le\; 2|\mathbf{a}|\,|\mathbf{b}| \;\pm\; 2\mathbf{a}\cdot\mathbf{b}$$

or $\pm 2\mathbf{a}\cdot\mathbf{b} \le 2|\mathbf{a}|\,|\mathbf{b}|$ which gives the desired inequality $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$. The remaining statement follows from the fact that equality can hold if and only if either $\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} + \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b} = \mathbf{0}$ or $\sqrt{\frac{|\mathbf{b}|}{|\mathbf{a}|}}\,\mathbf{a} - \sqrt{\frac{|\mathbf{a}|}{|\mathbf{b}|}}\,\mathbf{b} = \mathbf{0}$ which is the case if and only if **a** and **b** are linearly dependent.

1.17. Prove the *triangle inequality*, $|\mathbf{a}\pm\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$.

$$|\mathbf{a}\pm\mathbf{b}|^2 = (\mathbf{a}\pm\mathbf{b})\cdot(\mathbf{a}\pm\mathbf{b}) = |\mathbf{a}|^2+|\mathbf{b}|^2 \pm 2(\mathbf{a}\cdot\mathbf{b}) \le |\mathbf{a}|^2+|\mathbf{b}|^2+2|\mathbf{a}|\,|\mathbf{b}| \le (|\mathbf{a}|+|\mathbf{b}|)^2$$

and the desired result follows upon taking square roots.

1.18. Show that $||\mathbf{a}|-|\mathbf{b}|| \le |\mathbf{a}\pm\mathbf{b}|$ for all **a** and **b**.

From the triangle inequality,

$$|\mathbf{a}| = |\mathbf{a}\pm\mathbf{b}\mp\mathbf{b}| \le |\mathbf{a}\pm\mathbf{b}|+|\mathbf{b}| \quad\text{or}\quad |\mathbf{a}|-|\mathbf{b}| \le |\mathbf{a}\pm\mathbf{b}|$$

Also
$$|\mathbf{b}| = |\pm\mathbf{b}| = |\mathbf{a}\pm\mathbf{b}-\mathbf{a}| \le |\mathbf{a}\pm\mathbf{b}|+|\mathbf{a}| \quad\text{or}\quad |\mathbf{b}|-|\mathbf{a}| \le |\mathbf{a}\pm\mathbf{b}|$$

Thus $||\mathbf{a}|-|\mathbf{b}|| = \text{Max}\,(|\mathbf{a}|-|\mathbf{b}|,\,|\mathbf{b}|-|\mathbf{a}|) \le |\mathbf{a}\pm\mathbf{b}|$, which is the required result.

ORTHOGONAL VECTORS

1.19. Let **c** be orthogonal to **a** and **b**. Show that **c** is orthogonal to $k_1\mathbf{a}+k_2\mathbf{b}$ for all k_1, k_2.

Since **c** is orthogonal to **a** and **b**, $\mathbf{c}\cdot\mathbf{a}=0$ and $\mathbf{c}\cdot\mathbf{b}=0$. Hence

$$\mathbf{c}\cdot(k_1\mathbf{a}+k_2\mathbf{b}) = k_1(\mathbf{c}\cdot\mathbf{a})+k_2(\mathbf{c}\cdot\mathbf{b}) = 0$$

Thus **c** is orthogonal to $k_1\mathbf{a}+k_2\mathbf{b}$.

1.20. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and define

$$\mathbf{a}=\mathbf{u}_1,\quad \mathbf{b}=\mathbf{u}_2-\mathbf{P_a}(\mathbf{u}_2),\quad \mathbf{c}=\mathbf{u}_3-\mathbf{P_a}(\mathbf{u}_3)-\mathbf{P_b}(\mathbf{u}_3)$$

Show that **a**, **b**, **c** are nonzero mutually orthogonal vectors.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot(\mathbf{u}_2-\mathbf{P_a}(\mathbf{u}_2)) = \mathbf{a}\cdot[\mathbf{u}_2-(\mathbf{a}\cdot\mathbf{u}_2)\mathbf{a}/|\mathbf{a}|^2]\\ &= \mathbf{a}\cdot\mathbf{u}_2-(\mathbf{a}\cdot\mathbf{u}_2)(\mathbf{a}\cdot\mathbf{a})/|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{u}_2-\mathbf{a}\cdot\mathbf{u}_2 = 0\end{aligned}$$

and so $\mathbf{a}\perp\mathbf{b}$. Also,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{c} &= \mathbf{a}\cdot[\mathbf{u}_3-\mathbf{P_a}(\mathbf{u}_3)-\mathbf{P_b}(\mathbf{u}_3)] = \mathbf{a}\cdot[\mathbf{u}_3-(\mathbf{a}\cdot\mathbf{u}_3)\mathbf{a}/|\mathbf{a}|^2-(\mathbf{b}\cdot\mathbf{u}_3)\mathbf{b}/|\mathbf{b}|^2]\\ &= \mathbf{a}\cdot\mathbf{u}_3-\mathbf{a}\cdot\mathbf{u}_3-(\mathbf{b}\cdot\mathbf{u}_3)(\mathbf{a}\cdot\mathbf{b})/|\mathbf{b}|^2\end{aligned}$$

Since $\mathbf{a}\cdot\mathbf{b}=0$, $\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{u}_3-\mathbf{a}\cdot\mathbf{u}_3=0$ and so $\mathbf{a}\perp\mathbf{c}$. Finally,

$$\begin{aligned}\mathbf{b}\cdot\mathbf{c} &= \mathbf{b}\cdot[\mathbf{u}_3-(\mathbf{a}\cdot\mathbf{u}_3)\mathbf{a}/|\mathbf{a}|^2-(\mathbf{b}\cdot\mathbf{u}_3)\mathbf{b}/|\mathbf{b}|^2]\\ &= (\mathbf{b}\cdot\mathbf{u}_3)-(\mathbf{a}\cdot\mathbf{u}_3)(\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}|^2-(\mathbf{b}\cdot\mathbf{u}_3)(\mathbf{b}\cdot\mathbf{b})/|\mathbf{b}|^2 = (\mathbf{b}\cdot\mathbf{u}_3)-(\mathbf{b}\cdot\mathbf{u}_3) = 0\end{aligned}$$

Thus **a**, **b**, **c** are mutually orthogonal. They are also nonzero vectors, because: $\mathbf{a}=\mathbf{u}_1\neq\mathbf{0}$; if $\mathbf{b}=\mathbf{0}$,

$$\mathbf{0} = \mathbf{b} = \mathbf{u}_2-\mathbf{P_a}(\mathbf{u}_2) = \mathbf{u}_2-k\mathbf{a} = \mathbf{u}_2-k\mathbf{u}_1$$

which is impossible since $\mathbf{u}_1$ and $\mathbf{u}_2$ are independent; if $\mathbf{c}=\mathbf{0}$,

$$\mathbf{0} = \mathbf{c} = \mathbf{u}_3-\mathbf{P_a}(\mathbf{u}_3)-\mathbf{P_a}(\mathbf{u}_3) = \mathbf{u}_3-k_1\mathbf{a}-k_2\mathbf{b} = \mathbf{u}_3-k_1\mathbf{u}_1-k_2(\mathbf{u}_2-k\mathbf{u}_1) = \mathbf{u}_3-k_3\mathbf{u}_1-k_2\mathbf{u}_2$$

which is impossible since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent.

ORTHONORMAL BASES

1.21. Show that (a) $2\delta_{21} + 3\delta_{22} + 4\delta_{23} = 3$, (b) $\sum_{j=1}^{3} \delta_{ij}\mathbf{b}_j = \mathbf{b}_i$.

(a) $\delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$. Hence $2\delta_{21} + 3\delta_{22} + 4\delta_{23} = 2(0) + 3(1) + 4(0) = 3$.

(b) As above, the only contribution comes from δ_{ii}. Thus $\sum_{j=1}^{3} \delta_{ij}\mathbf{b}_j = \delta_{ii}\mathbf{b}_i = \mathbf{b}_i$.

1.22. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{v}_j = \sum_{i=1}^{3} a_{ij}\mathbf{u}_i$ and $\mathbf{u}_j = \sum_{i=1}^{3} b_{ij}\mathbf{v}_i$. Show that $\sum_{k=1}^{3} a_{ik}b_{kj} = \delta_{ij}$.

We write $\mathbf{u}_j = \sum_{i=1}^{3} \delta_{ij}\mathbf{u}_i = \sum_{k=1}^{3} b_{kj}\mathbf{v}_k$ where we have changed the name of the index from i to k. Also, $\mathbf{v}_k = \sum_{i=1}^{3} a_{ik}\mathbf{u}_i$. Substituting,

$$\sum_{i=1}^{3} \delta_{ij}\mathbf{u}_i = \sum_{k=1}^{3} b_{kj} \sum_{i=1}^{3} a_{ik}\mathbf{u}_i = \sum_{i=1}^{3}\left[\sum_{k=1}^{3} a_{ik}b_{kj}\right]\mathbf{u}_i$$

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are independent equate components and obtain $\delta_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj}$.

1.23. Prove Theorem 1.4: If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, then

(a) $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, (b) $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, (c) $a_i = \mathbf{a}\cdot\mathbf{e}_i$, $i = 1, 2, 3$.

(a)
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ &= a_1b_1(\mathbf{e}_1\cdot\mathbf{e}_1) + a_1b_2(\mathbf{e}_1\cdot\mathbf{e}_2) + a_1b_3(\mathbf{e}_1\cdot\mathbf{e}_3) \\ &\quad + a_2b_1(\mathbf{e}_2\cdot\mathbf{e}_1) + a_2b_2(\mathbf{e}_2\cdot\mathbf{e}_2) + a_2b_3(\mathbf{e}_2\cdot\mathbf{e}_3) \\ &\quad + a_3b_1(\mathbf{e}_3\cdot\mathbf{e}_1) + a_3b_2(\mathbf{e}_3\cdot\mathbf{e}_2) + a_3b_3(\mathbf{e}_3\cdot\mathbf{e}_3) \\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

In short, we have

$$\mathbf{a}\cdot\mathbf{b} = \left(\sum_{i=1}^{3} a_i\mathbf{e}_i\right)\cdot\left(\sum_{j=1}^{3} b_j\mathbf{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j(\mathbf{e}_i\cdot\mathbf{e}_j)$$

or

$$\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j\delta_{ij} = \sum_{i=1}^{3} a_ib_i$$

(b) $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(c) $\mathbf{a}\cdot\mathbf{e}_i = \left(\sum_j a_j\mathbf{e}_j\right)\cdot\mathbf{e}_i = \sum_j a_j(\mathbf{e}_j\cdot\mathbf{e}_i) = \sum_j a_j\delta_{ji} = a_i$

1.24. Let $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_2 - \mathbf{e}_3$. Find (a) $\mathbf{a}\cdot\mathbf{b}$, (b) $|\mathbf{a}|$, (c) $\mathbf{u}_\mathbf{a}$, (d) $P_\mathbf{a}(\mathbf{b})$, (e) $\mathbf{P}_\mathbf{a}(\mathbf{b})$, (f) $\cos\measuredangle(\mathbf{a},\mathbf{b})$, (g) $\mathbf{a}\cdot\mathbf{e}_1$, $\mathbf{a}\cdot\mathbf{e}_2$, $\mathbf{a}\cdot\mathbf{e}_3$, (h) direction cosines of $\mathbf{a}$.

(a) $\mathbf{a}\cdot\mathbf{b} = (1)(0) + (-2)(1) + (3)(-1) = -5$

(b) $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$

(c) $\mathbf{u}_\mathbf{a} = \mathbf{a}/|\mathbf{a}| = (1/\sqrt{14})(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3)$

(d) $P_\mathbf{a}(\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}| = -5/\sqrt{14}$

(e) $\mathbf{P}_\mathbf{a}(\mathbf{b}) = P_\mathbf{a}(\mathbf{b})\mathbf{u}_\mathbf{a} = -(5/14)(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3)$

(f) $\cos\measuredangle(\mathbf{a},\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})/|\mathbf{a}|\,|\mathbf{b}| = (-5/\sqrt{14}\sqrt{2}) = -5/(2\sqrt{7})$

(g) $\mathbf{a}\cdot\mathbf{e}_1 = 1$, $\mathbf{a}\cdot\mathbf{e}_2 = -2$, $\mathbf{a}\cdot\mathbf{e}_3 = 3$

(h) $\cos\measuredangle(\mathbf{a},\mathbf{e}_1) = a_1/|\mathbf{a}| = 1/\sqrt{14}$, $\cos\measuredangle(\mathbf{a},\mathbf{e}_2) = a_2/|\mathbf{a}| = -2/\sqrt{14}$, $\cos\measuredangle(\mathbf{a},\mathbf{e}_3) = a_3/|\mathbf{a}| = 3/\sqrt{14}$

1.25. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be an arbitrary basis. Show that there exists a unique basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that

$$\begin{array}{lll} \mathbf{v}_1 \cdot \mathbf{u}_1 = 1 & \mathbf{v}_2 \cdot \mathbf{u}_1 = 0 & \mathbf{v}_3 \cdot \mathbf{u}_1 = 0 \\ \mathbf{v}_1 \cdot \mathbf{u}_2 = 0 & \mathbf{v}_2 \cdot \mathbf{u}_2 = 1 & \mathbf{v}_3 \cdot \mathbf{u}_2 = 0 \\ \mathbf{v}_1 \cdot \mathbf{u}_3 = 0 & \mathbf{v}_2 \cdot \mathbf{u}_3 = 0 & \mathbf{v}_3 \cdot \mathbf{u}_3 = 1 \end{array}$$

or $\mathbf{v}_i \cdot \mathbf{u}_j = \delta_{ij}$, $i, j = 1, 2, 3$. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is called the *dual or reciprocal basis* to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Accordingly if $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ and $\mathbf{b} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$, then

$$\mathbf{a} \cdot \mathbf{b} = \left(\sum_i a_i\mathbf{u}_i\right) \cdot \left(\sum_j b_j\mathbf{v}_j\right) = \sum_i \sum_j a_ib_j(\mathbf{u}_i \cdot \mathbf{v}_j) = \sum_i \sum_j a_ib_j\delta_{ij} = \sum_i a_ib_i$$

Observe that an orthonormal basis is its own dual.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis and let

$$\begin{aligned} \mathbf{u}_1 &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3 \\ \mathbf{u}_2 &= a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3 \\ \mathbf{u}_3 &= a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3 \end{aligned}$$

and

$$\mathbf{v}_1 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

Then

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{u}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 1 \\ \mathbf{v}_1 \cdot \mathbf{u}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ \mathbf{v}_1 \cdot \mathbf{u}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{aligned}$$

is a system of three equations for x_1, x_2, x_3. Since $\det(a_{ij}) \neq 0$, there exists a unique solution $\mathbf{v}_1 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Similarly we have unique solutions for $\mathbf{v}_2$ and $\mathbf{v}_3$. It remains to show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent and hence form a basis. We consider

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \sum_{i=1}^{3} k_i\mathbf{v}_i = \mathbf{0}$$

If we multiply by $\mathbf{u}_j$, $j = 1, 2, 3$, we obtain

$$\left[\sum_{i=1}^{3} k_i\mathbf{v}_i\right] \cdot \mathbf{u}_j = \sum_{i=1}^{3} k_i(\mathbf{v}_i \cdot \mathbf{u}_j) = \sum_{i=1}^{3} k_i\delta_{ij} = k_j = 0, \quad j = 1, 2, 3$$

Thus $k_1 = k_2 = k_3 = 0$ and so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent and form a basis.

ORIENTATION

1.26. Show that the triplet $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where $\mathbf{v}_1 = 2\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3$ and $\mathbf{v}_3 = -\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$, has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

The determinant of the components is $\det\begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} = 1 > 0.$

Hence $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

1.27. Show that orientation is an *equivalence relation* on the set of all ordered bases in E^3. That is, show that:

(*a*) $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for all $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*b*) If $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(*c*) If $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

(a) We write $\mathbf{v}_j = \sum_{i=1}^{3} \delta_{ij}\mathbf{v}_i$, $j = 1,2,3$. Since $\det(\delta_{ij}) = 1$, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(b) Let $\mathbf{v}_j = \sum_i a_{ij}\mathbf{u}_i$ and $\mathbf{u}_j = \sum_i b_{ij}\mathbf{v}_i$. From Problem 1.22, $\sum_{k=1}^{3} a_{ik}b_{kj} = \delta_{ij}$. It is also easily verified by expanding that $\det\left(\sum_{k=1}^{3} a_{ik}b_{kj}\right) = \det(a_{ij})\det(b_{ij})$, and so

$$\det(b_{ij}) = \frac{\det(\delta_{ij})}{\det(a_{ij})} = \frac{1}{\det(a_{ij})}$$

Since $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, $\det(a_{ij}) > 0$. Hence $\det(b_{ij}) > 0$, and thus $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(c) Let $\mathbf{v}_k = \sum_{i=1}^{3} a_{ik}\mathbf{u}_i$ and $\mathbf{w}_j = \sum_{k=1}^{3} b_{kj}\mathbf{v}_k$. Substituting, we obtain

$$\mathbf{w}_j = \sum_{k=1}^{3} b_{kj} \sum_{i=1}^{3} a_{ik}\mathbf{u}_i = \sum_{i=1}^{3}\left(\sum_{k=1}^{3} a_{ik}b_{kj}\right)\mathbf{u}_i$$

Thus $\mathbf{w}_j = \sum_{i=1}^{3} c_{ij}\mathbf{u}_i$ where $c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj}$, $i,j = 1,2,3$. Also,

$$\det(c_{ij}) = \det\left(\sum a_{ik}b_{kj}\right) = \det(a_{ij})\det(b_{ij})$$

Since $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $\det(b_{ij}) > 0$ and $\det(a_{ij}) > 0$; hence $\det(c_{ij}) > 0$. Thus $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

VECTOR PRODUCT

1.28. Let $\mathbf{a} = 2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = \mathbf{e}_2 + 2\mathbf{e}_3$. Determine (a) $\mathbf{a} \times \mathbf{b}$, (b) $\mathbf{b} \times \mathbf{a}$, (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, (d) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, (e) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, (f) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.

(a)
$$\mathbf{a} \times \mathbf{b} = \det\begin{pmatrix} \mathbf{e}_1 & 2 & 1 \\ \mathbf{e}_2 & -1 & 2 \\ \mathbf{e}_3 & 1 & -1 \end{pmatrix} = \mathbf{e}_1 \det\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} - \mathbf{e}_2 \det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} + \mathbf{e}_3 \det\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$
$$= -\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3$$

(b)
$$\mathbf{b} \times \mathbf{a} = \det\begin{pmatrix} \mathbf{e}_1 & 1 & 2 \\ \mathbf{e}_2 & 2 & -1 \\ \mathbf{e}_3 & -1 & 1 \end{pmatrix} = \mathbf{e}_1 - 3\mathbf{e}_2 - 5\mathbf{e}_3.$$
Observe that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

(c)
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) \times \det\begin{pmatrix} \mathbf{e}_1 & 1 & 0 \\ \mathbf{e}_2 & 2 & 1 \\ \mathbf{e}_3 & -1 & 2 \end{pmatrix}$$
$$= (2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) \times (5\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3) = \det\begin{pmatrix} \mathbf{e}_1 & 2 & 5 \\ \mathbf{e}_2 & -1 & -2 \\ \mathbf{e}_3 & 1 & 1 \end{pmatrix} = \mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3$$

(d)
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) \times (\mathbf{e}_2 + 2\mathbf{e}_3) = \det\begin{pmatrix} \mathbf{e}_1 & -1 & 0 \\ \mathbf{e}_2 & 3 & 1 \\ \mathbf{e}_3 & 5 & 2 \end{pmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$$

Observe that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(e)
$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) \cdot (\mathbf{e}_2 + 2\mathbf{e}_3) = (-1)(0) + (3)(1) + (5)(2) = 13$$

(*f*) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \det\begin{pmatrix}\mathbf{e}_1 & 2 & 1\\ \mathbf{e}_2 & -1 & 3\\ \mathbf{e}_3 & 1 & 1\end{pmatrix} = -4\mathbf{e}_1 - \mathbf{e}_2 + 7\mathbf{e}_3$, $\mathbf{a}\times\mathbf{b} = -\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3$, and $\mathbf{a}\times\mathbf{c} = -3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3$. Then

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) - \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} = (-4\mathbf{e}_1 - \mathbf{e}_2 + 7\mathbf{e}_3) - (-\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3) - (-3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3) = \mathbf{0}$$

1.29. Prove that (*a*) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$, (*b*) $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$.

(*a*)
$$\begin{aligned}\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= [a_2(b_3+c_3) - a_3(b_2+c_2)]\mathbf{e}_1 + [a_3(b_1+c_1) - a_1(b_3+c_3)]\mathbf{e}_2\\ &\quad + [a_1(b_2+c_2) - a_2(b_1+c_1)]\mathbf{e}_3\\ &= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3\\ &\quad + (a_2c_3 - a_3c_2)\mathbf{e}_1 + (a_3c_1 - a_1c_3)\mathbf{e}_2 + (a_1c_2 - a_2c_1)\mathbf{e}_3\\ &= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}\end{aligned}$$

(*b*)
$$\begin{aligned}(k\mathbf{a})\times\mathbf{b} &= (ka_2b_3 - ka_3b_2)\mathbf{e}_1 + (ka_3b_1 - ka_1b_3)\mathbf{e}_2 + (ka_1b_2 - ka_2b_1)\mathbf{e}_3\\ &= k[(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3] = k(\mathbf{a}\times\mathbf{b})\end{aligned}$$

1.30. Show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$.

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) &= [(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3]\\ &\quad\cdot[(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3]\\ &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2\\ &= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2\\ &\quad - 2a_2b_2a_3b_3 - 2a_1b_1a_3b_3 - 2a_1b_1a_2b_2\end{aligned}$$

$$\begin{aligned}|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2 &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})\\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2\\ &= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - 2a_1b_1a_2b_2 - 2a_1b_1a_3b_3 - 2a_2b_2a_3b_3\end{aligned}$$

The required identity follows by comparing the above.

1.31. Prove Theorem 1.5:

(i) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, $\theta = \measuredangle(a, b)$

(ii) a. $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$

b. If $(\mathbf{a}\times\mathbf{b}) \neq \mathbf{0}$, then $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ has the same orientation as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

(i) Using the preceding problem,

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta\\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta) = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = (|\mathbf{a}|\,|\mathbf{b}|\sin\theta)^2\end{aligned}$$

Since $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$, we have $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$.

(ii) a. Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= [(a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3]\cdot(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1 = 0\end{aligned}$$

Similarly, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. Hence $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a}$ and $(\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$.

b. The determinant of the components of $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ is

$$\det\begin{pmatrix}a_1 & b_1 & (a_2b_3 - a_3b_2)\\ a_2 & b_2 & (a_3b_1 - a_1b_3)\\ a_3 & b_3 & (a_1b_2 - a_2b_1)\end{pmatrix} = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = |\mathbf{a}\times\mathbf{b}|^2$$

If $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$, then $|\mathbf{a}\times\mathbf{b}|^2 > 0$ and $(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$ has the same orientation as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

1.32. Prove that the definition of the vector product is independent of the basis.

Let $\mathbf{c}$ and $\mathbf{c}'$ be the vector products of $\mathbf{a}$ and $\mathbf{b}$ with respect to two different right-handed orthonormal bases. We may assume that $\mathbf{a}$ and $\mathbf{b}$ are independent. Otherwise, from Theorem 1.6, $\mathbf{c} = \mathbf{c}' = \mathbf{0}$. From Theorem 1.5(ii), $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis and we can write $\mathbf{c}' = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$. Also from Theorem 1.5(ii), $\mathbf{a}\cdot\mathbf{c}' = \alpha|\mathbf{a}|^2 + \beta(\mathbf{a}\cdot\mathbf{b}) = 0$ and $\mathbf{b}\cdot\mathbf{c}' = \alpha(\mathbf{b}\cdot\mathbf{a}) + \gamma|b|^2 = 0$. Since $\mathbf{a}$ and $\mathbf{b}$ are independent, $|a|^2\,|b|^2 - |a\cdot b|^2 \neq 0$. Hence $\alpha = \beta = 0$ and $\mathbf{c}' = \gamma\mathbf{c}$. Since $(\mathbf{a}, \mathbf{b}, \mathbf{c}')$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are both right-handed, $\gamma > 0$. From Theorem 1.5(i), $|\mathbf{c}'| = |\mathbf{c}| = \gamma|\mathbf{c}'|$. Thus $\gamma = 1$ and $\mathbf{c} = \mathbf{c}'$.

TRIPLE PRODUCTS

1.33. Let $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{b} = -\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{c} = -\mathbf{e}_2 + 2\mathbf{e}_3$. Find $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \det\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix} = 5$$

1.34. Show that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. Then

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = -\det\begin{pmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{pmatrix} = \det\begin{pmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{pmatrix} = \mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$$

1.35. Prove Theorem 1.8: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. Then

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\times[(b_2c_3 - b_3c_2)\mathbf{e}_1 + (b_3c_1 - c_3b_1)\mathbf{e}_2 + (b_1c_2 - b_2c_1)\mathbf{e}_3] \\ &= (a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)\mathbf{e}_1 \\ &\quad + (a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1)\mathbf{e}_2 \\ &\quad + (a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\mathbf{e}_3\end{aligned}$$

Thus comparing with the above,

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ &\quad - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3) \\ &= (a_2b_1c_2 + a_3b_1c_3 - a_2c_1b_2 - a_3c_1b_3)\mathbf{e}_1 \\ &\quad + (b_2a_1c_1 + b_2a_3c_3 - c_2a_1b_1 - c_2a_3b_3)\mathbf{e}_2 \\ &\quad + (b_3a_1c_1 + b_3a_2c_2 - c_3a_1b_1 - c_3a_2b_2)\mathbf{e}_3 \\ &= \mathbf{a}\times(\mathbf{b}\times\mathbf{c})\end{aligned}$$

Supplementary Problems

1.36. In the tetrahedron $OPQR$ shown in Fig. 1-18, let $\mathbf{a} = \mathbf{OP}$, $\mathbf{b} = \mathbf{OQ}$, $\mathbf{c} = \mathbf{OR}$ and let M be the midpoint of edge RQ. Find $\mathbf{PM}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. *Ans.* $\mathbf{PM} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \mathbf{a}$

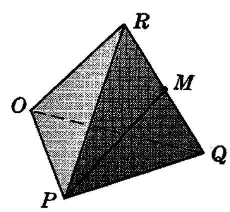


Fig. 1-18

1.37. Let $\mathbf{a} = 2\mathbf{u}_1 + \mathbf{u}_2 - 3\mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{c} = -\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3$. Find $3\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. *Ans.* $3\mathbf{u}_1 + 9\mathbf{u}_2 - 12\mathbf{u}_3$

1.38. Show that $|\mathbf{a}\pm\mathbf{b}\pm\mathbf{c}| \leq |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|$.

1.39. Show that the midpoints of the lines joining the midpoints of opposite sides of a quadrilateral coincide.

1.40. Show that the angle bisectors of a triangle meet at a point.

1.41. Show that the medians of a triangle meet at a point.

1.42. Prove that a subset of a linearly independent set of vectors is linearly independent.

1.43. Prove that two linearly independent vectors in E^2 form a basis in E^2.

1.44. Prove that three or more vectors in E^2 are linearly dependent.

1.45. Prove that if a_i, b_i, c_i are the components of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with respect to a basis, then (i) $\mathbf{a} = \mathbf{b}$ iff $a_i = b_i$, (ii) $\mathbf{c} = \mathbf{a} + \mathbf{b}$ iff $c_i = a_i + b_i$, (iii) $\mathbf{b} = k\mathbf{a}$ iff $b_i = ka_i$.

1.46. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis. Determine whether $\mathbf{a} = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{b} = \mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{c} = 2\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3$ are linearly independent. *Ans.* Yes

1.47. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis and let $\mathbf{v}_1 = -\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3$, $\mathbf{v}_3 = 2\mathbf{u}_1 + \mathbf{u}_3$. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis and find the components of $\mathbf{a} = 2\mathbf{u}_1 - \mathbf{u}_3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
Ans. $\mathbf{u}_1 = -2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{u}_2 = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{u}_3 = 4\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3$, $\mathbf{a} = -8\mathbf{v}_1 + 4\mathbf{v}_2 - 5\mathbf{v}_3$

1.48. Let $\mathbf{a} = -\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$. Find (*a*) $\mathbf{a} \cdot \mathbf{b}$, (*b*) $|\mathbf{a}|$, (*c*) $\cos \measuredangle(\mathbf{a}, \mathbf{b})$, (*d*) $P_\mathbf{b}(\mathbf{a})$, (*e*) $\mathbf{P}_\mathbf{b}(\mathbf{a})$.
Ans. (*a*) -4, (*b*) $\sqrt{6}$, (*c*) $-4/(3\sqrt{2})$, (*d*) $-4/\sqrt{3}$, (*e*) $-(4/3)(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$

1.49. Find the direction cosines of the vector $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$. *Ans.* $2/\sqrt{14}, 1/\sqrt{14}, -3/\sqrt{14}$

1.50. Determine x so that $\mathbf{a} = x\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = 2\mathbf{e}_1 - x\mathbf{e}_2 + \mathbf{e}_3$ are orthogonal. *Ans.* $x = 1$

1.51. Factor $\alpha\gamma|\mathbf{a}|^2 - (\alpha\delta + \beta\gamma)(\mathbf{a} \cdot \mathbf{b}) + \beta\delta|\mathbf{b}|^2$. *Ans.* $(\alpha\mathbf{a} - \beta\mathbf{b}) \cdot (\gamma\mathbf{a} - \delta\mathbf{b})$

1.52. Let $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = -\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$. Find a vector $\mathbf{c}$ so that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form the sides of a right triangle. *Ans.* $\mathbf{c} = \pm(2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$

1.53. Show that $\mathbf{g}_1 = (1/3)(2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)$, $\mathbf{g}_2 = (1/3)(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3)$ and $\mathbf{g}_3 = (1/3)(2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3)$ form an orthonormal basis and find $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in terms of $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$.
Ans. $\mathbf{e}_1 = (1/3)(2\mathbf{g}_1 + \mathbf{g}_2 + 2\mathbf{g}_3)$, $\mathbf{e}_2 = (1/3)(-2\mathbf{g}_1 + 2\mathbf{g}_2 + \mathbf{g}_3)$, $\mathbf{e}_3 = (1/3)(\mathbf{g}_1 + 2\mathbf{g}_2 - 2\mathbf{g}_3)$

1.54. Show that the sum of the squares of all the sides of a parallelogram is equal to the sum of the squares of its diagonals.

1.55. If $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$, $\mathbf{b} = 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{c} = \mathbf{e}_2 + \mathbf{e}_3$, find (*a*) $\mathbf{a} \times \mathbf{b}$, (*b*) $\mathbf{b} \times \mathbf{a}$, (*c*) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{abc}]$, (*d*) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. *Ans.* (*a*) $5\mathbf{e}_1 + 7\mathbf{e}_2 + 3\mathbf{e}_3$, (*b*) $-5\mathbf{e}_1 - 7\mathbf{e}_2 - 3\mathbf{e}_3$, (*c*) 10, (*d*) $2\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3$

1.56. Find a unit vector orthogonal to $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{b} = -\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$. *Ans.* $\pm(1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_3)$

1.57. Find the distance d from the point P to the plane S where $\mathbf{a} = \mathbf{OP} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ is the vector from a point O on S to P and $\mathbf{b} = -\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2$ are along S. *Ans.* $d = |P_{\mathbf{b} \times \mathbf{c}}(\mathbf{a})|$

1.58. Prove that
$$[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3][\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3] = \det\begin{pmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 & \mathbf{u}_1 \cdot \mathbf{v}_3 \\ \mathbf{u}_2 \cdot \mathbf{v}_1 & \mathbf{u}_2 \cdot \mathbf{v}_2 & \mathbf{u}_2 \cdot \mathbf{v}_3 \\ \mathbf{u}_3 \cdot \mathbf{v}_1 & \mathbf{u}_3 \cdot \mathbf{v}_2 & \mathbf{u}_3 \cdot \mathbf{v}_3 \end{pmatrix}.$$

1.59. Prove that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) = 0$.

1.60. Prove that $[(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d})(\mathbf{e} \times \mathbf{f})] = [\mathbf{abd}][\mathbf{cef}] - [\mathbf{abc}][\mathbf{def}]$.

1.61. Show that if $\mathbf{a}$ and $\mathbf{b}$ lie in a plane normal to a plane containing $\mathbf{c}$ and $\mathbf{d}$, then $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0$.

1.62. Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be an arbitrary basis and let $\mathbf{v}_1 = \dfrac{\mathbf{u}_2 \times \mathbf{u}_3}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$, $\mathbf{v}_2 = \dfrac{\mathbf{u}_3 \times \mathbf{u}_1}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$, $\mathbf{v}_3 = \dfrac{\mathbf{u}_1 \times \mathbf{u}_2}{[\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]}$. Show that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is dual to $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, i.e. $\mathbf{u}_i \cdot \mathbf{v}_j = \delta_{ij}$, $i, j = 1, 2, 3$.

1.63. Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be dual bases. Show that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ has the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

1.64. Show that there exist two equivalent classes of oriented bases. Namely, show that if $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ do not have the same orientation as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, then $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ have the same orientation. Thus we can say that two ordered bases have the same or opposite orientation.

Chapter 2

Vector Functions of a Real Variable

LINES AND PLANES

Let **a** and **u** be vectors in E^3 with $\mathbf{u} \neq \mathbf{0}$. By the *straight line through* **a** *parallel to* **u** we mean the set of **x** in E^3 which can be represented by

$$\mathbf{x} = k\mathbf{u} + \mathbf{a}, \qquad -\infty < k < \infty \tag{2.1}$$

or in component form,

$$x_1 = ku_1 + a_1, \quad x_2 = ku_2 + a_2, \quad x_3 = ku_3 + a_3, \qquad -\infty < k < \infty \tag{2.2}$$

The equations (*2.1*) or (*2.2*) are called the parametric equations of the line. We say that the point **x** *generates* the line as the *parameter k* varies over the real line. Any vector which is linearly dependent on **u** will be said to be *parallel* to this line. Two lines will be said to be parallel if their respective vectors **u** are linearly dependent.

Example 2.1.

The parametric equation of the line through $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$ parallel to $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_3$ is

$$\mathbf{x} = k\mathbf{u} + \mathbf{a} = k(\mathbf{e}_1 - \mathbf{e}_3) + (\mathbf{e}_1 + 2\mathbf{e}_2) = (k+1)\mathbf{e}_1 + 2\mathbf{e}_2 - k\mathbf{e}_3$$

or $x_1 = k+1, \ x_2 = 2, \ x_3 = -k$.

Example 2.2.

If **a** and **b** are distinct points on a line, then $\mathbf{b} - \mathbf{a}$ is a nonzero vector parallel to the line. Thus the equation of the line through **a** and **b** is $\mathbf{x} = k(\mathbf{b} - \mathbf{a}) + \mathbf{a}$, or

$$x_1 = k(b_1 - a_1) + a_1, \quad x_2 = k(b_2 - a_2) + a_2, \quad x_3 = k(b_3 - a_3) + a_3$$

By the *plane through* **a** *parallel to two independent vectors* **u** *and* **v** we mean the set of **x** in E^3 which can be represented by

$$\mathbf{x} = h\mathbf{u} + k\mathbf{v} + \mathbf{a}, \qquad -\infty < h < \infty, \ -\infty < k < \infty \tag{2.3}$$

or, equating components,

$$x_1 = hu_1 + kv_1 + a_1, \quad x_2 = hu_2 + kv_2 + a_2, \quad x_3 = hu_3 + kv_3 + a_3 \tag{2.4}$$

The equations (*2.3*) and (*2.4*) are called the parametric equations of the plane. We say that **x** generates the plane as the parameters h and k vary independently over the real numbers. A vector will be said to be parallel to the plane if it is linearly dependent upon **u** and **v**, and it will be said to be normal to the plane if it is orthogonal to both **u** and **v**.

If **n** is a nonzero vector normal to the plane $\mathbf{x} = h\mathbf{u} + k\mathbf{v} + \mathbf{a}$, then the point **x** lies on the plane if and only if $\mathbf{x} - \mathbf{a}$ is orthogonal to **n**, or

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0 \tag{2.5}$$

In terms of the components of **x**, **a** and **u**, this becomes

$$(x_1 - a_1)n_1 + (x_2 - a_2)n_2 + (x_3 - a_3)n_3 = 0 \tag{2.6}$$

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are noncollinear points on a plane, then $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are linearly independent vectors parallel to the plane and $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})$ is a nonzero vector normal to the plane as indicated in Fig. 2-1. It follows from equation (*2.5*) that the equation of the plane through $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$[(\mathbf{x}-\mathbf{a})(\mathbf{b}-\mathbf{a})(\mathbf{c}-\mathbf{a})] \;=\; 0 \qquad (2.7)$$

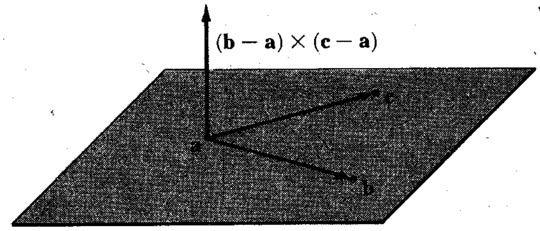


Fig. 2-1

Example 2.3.

The parametric equation of the plane through $\mathbf{a} = \mathbf{e}_2$ parallel to $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{v} = -\mathbf{e}_1 + \mathbf{e}_3$ is

$$\mathbf{x} \;=\; h\mathbf{u} + k\mathbf{v} + \mathbf{a} \;=\; (h-k)\mathbf{e}_1 + \mathbf{e}_2 + k\mathbf{e}_3 \quad \text{or} \quad x_1 \;=\; h-k, \;\; x_2 \;=\; 1, \;\; x_3 \;=\; k$$

The plane is also given by

$$[(\mathbf{x}-\mathbf{a})\mathbf{u}\mathbf{v}] \;=\; \det\begin{pmatrix} x_1 & 1 & -1 \\ x_2-1 & 0 & 0 \\ x_3 & 0 & 1 \end{pmatrix} \;=\; 0 \quad \text{or} \quad x_2 - 1 \;=\; 0$$

NEIGHBORHOODS

Local properties of functions are conveniently described in terms of the concept of a spherical neighborhood. Namely, the *ϵ-open sphere* or *ϵ-spherical neighborhood* of a vector $\mathbf{a}$, denoted by $S_\epsilon(\mathbf{a})$, is the set of $\mathbf{x}$ satisfying $|\mathbf{x}-\mathbf{a}| < \epsilon$. As shown in Fig. 2-2, a point $\mathbf{x}$ is in $S_\epsilon(\mathbf{a})$ if and only if $\mathbf{x}$ is in the interior of the sphere of radius ϵ about $\mathbf{a}$. In E^2, $S_\epsilon(\mathbf{a})$ is the interior of the circle of radius ϵ about $\mathbf{a}$; and in E^1, $S_\epsilon(\mathbf{a})$ is the open interval of length 2ϵ with $\mathbf{a}$ at its center, as shown in Fig. 2-3.

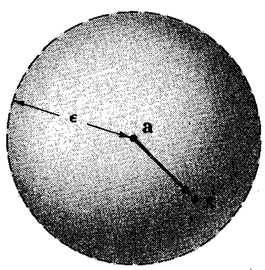


Fig. 2-2

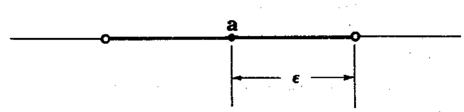


Fig. 2-3

It is also convenient to consider a spherical neighborhood of $\mathbf{a}$ less $\mathbf{a}$ itself. The set $S_\epsilon(\mathbf{a})$ excluding $\mathbf{a}$ is called the *ϵ-deleted spherical neighborhood* of $\mathbf{a}$ and is denoted by $S'_\epsilon(\mathbf{a})$. Since $|\mathbf{x}-\mathbf{a}| = 0$ if and only if $\mathbf{x} = \mathbf{a}$, $S'_\epsilon(\mathbf{a})$ consists of the vectors $\mathbf{x}$ satisfying $0 < |\mathbf{x}-\mathbf{a}| < \epsilon$.

Example 2.4.

The 1/10 spherical neighborhood of the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, i.e. $S_{1/10}(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$, consists of the vectors $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ satisfying

$$|\mathbf{x}-\mathbf{a}| \;=\; [(x_1-1)^2 + (x_2-2)^2 + (x_3-3)^2]^{1/2} \;<\; 1/10 \quad \text{or} \quad (x_1-1)^2 + (x_2-2)^2 + (x_3-3)^2 \;<\; 1/100$$

Example 2.5.

$S_{1/100}(5)$ on E^1 is the set of numbers x satisfying $|x-5| < 1/100$ or $5 - 1/100 < x < 5 + 1/100$. Note that $S_{1/100}$ is the open interval of length 1/50 centered about 5.

VECTOR FUNCTIONS

The assignment of a vector $\mathbf{f}(t)$ to each real number t of a set of real numbers S defines a *vector function* $\mathbf{f}$ *of the single real variable* t *in* S. As in the case of scalar functions of a real variable, the set S is called the *domain of definition* of $\mathbf{f}$; and the set of assigned vectors, denoted by $\mathbf{f}(S)$, is called the *image* of $\mathbf{f}$.

Example 2.6.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be fixed vectors in space. The equation

$$\mathbf{f}(t) = \mathbf{a} - 2t\mathbf{b} + t^2\mathbf{c}, \qquad -2 \leq t \leq 2$$

defines a vector function of t with domain $-2 \leq t \leq 2$. A table of some assigned vectors is

t	-2	-1	0	1	2
$\mathbf{f}(t)$	$\mathbf{a} + 4\mathbf{b} + 4\mathbf{c}$	$\mathbf{a} + 2\mathbf{b} + \mathbf{c}$	$\mathbf{a}$	$\mathbf{a} - 2\mathbf{b} + \mathbf{c}$	$\mathbf{a} - 4\mathbf{b} + 4\mathbf{c}$

Example 2.7.

In Example 2.6 suppose that $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_3$. Then

$$\mathbf{f}(t) = (\mathbf{e}_1 + 2\mathbf{e}_2) - 2t(\mathbf{e}_2 - \mathbf{e}_3) + t^2(\mathbf{e}_1 - \mathbf{e}_3) = (1 + t^2)\mathbf{e}_1 + (2 - 2t)\mathbf{e}_2 + (2t - t^2)\mathbf{e}_3$$

Here $\mathbf{f}$ is expressed in terms of the three scalar functions $f_1(t) = 1 + t^2$, $f_2(t) = 2 - 2t$, $f_3(t) = 2t - t^2$, its components with respect to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

As indicated in the example above, $\mathbf{f}(t)$ uniquely determines three scalar functions $f_1(t), f_2(t), f_3(t)$, its components with respect to the basis. Conversely, three scalar functions $f_1(t), f_2(t), f_3(t)$ on a common domain S uniquely define a vector function

$$\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$$

on S whose components with respect to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are f_1, f_2, f_3.

Vector functions will be used to define curves. Let $\mathbf{x} = \mathbf{f}(t)$; then as t varies, the point $\mathbf{x}$ will trace out a curve, as shown in Fig. 2-4. The equation $\mathbf{x} = \mathbf{f}(t)$ or, componentwise, the three scalar equations

$$x_1 = f_1(t), \quad x_2 = f_2(t), \quad x_3 = f_3(t)$$

will be called *a parametric representation* of the curve, and the variable t will be the *parameter*.

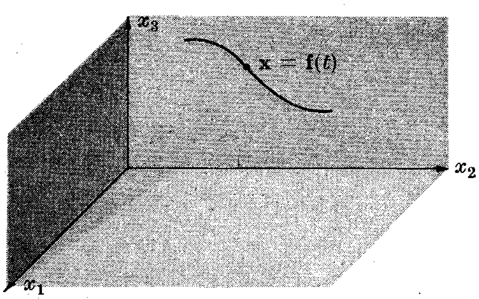


Fig. 2-4

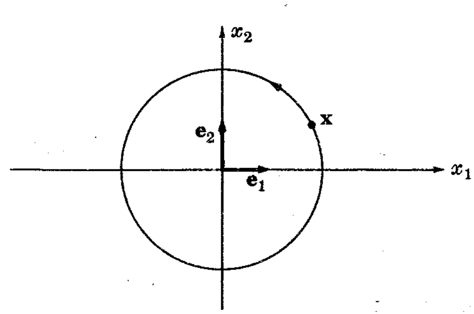


Fig. 2-5

Example 2.8.

The equation $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$ or $x_1 = a \cos t$, $x_2 = a \sin t$, $a > 0$, $0 \leq t \leq 2\pi$ is a parametric representation of the circle of radius a about the origin. As t increases through the interval $0 \leq t \leq 2\pi$, the circle is traced in a counterclockwise direction, as shown in Fig. 2-5 above.

For the most part we shall assume that our functions are defined on intervals. These consist of the finite open and closed intervals $a < t < b$ and $a \leq t \leq b$, the finite half-open intervals $a \leq t < b$ and $a < t \leq b$, and the infinite intervals such as $-\infty < t < \infty$, $a \leq t < \infty$, $-\infty < t < a$, etc.

BOUNDED FUNCTIONS

A function $\mathbf{f}(t)$ is said to be *bounded* on the interval I if there exists a scalar $M > 0$ such that $|\mathbf{f}(t)| \leq M$ for t in I. Observe in Fig. 2-6 that if $\mathbf{x} = \mathbf{f}(t)$, then $\mathbf{f}(t)$ is bounded on I if and only if there exists a sphere of radius M about the origin such that the point $\mathbf{x}$ is in the sphere for t in I.

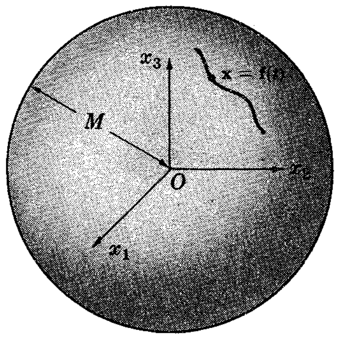


Fig. 2-6

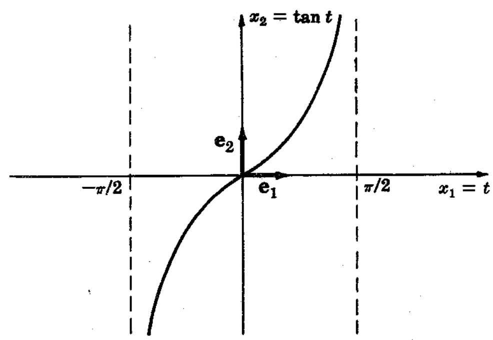


Fig. 2-7

Example 2.9.

The curve traced by $\mathbf{x} = t\mathbf{e}_1 + (\tan t)\mathbf{e}_2$ on $-\pi/2 < t < \pi/2$ is shown in Fig. 2-7. Observe that $|\mathbf{x}|$ becomes arbitrarily large for t close to $\pi/2$. Thus $\mathbf{x}$ is not bounded on $-\pi/2 < t < \pi/2$. Note, however, that $\mathbf{x}$ is bounded on the interval $-\pi/2 + \epsilon < t < \pi/2 - \epsilon$ for any $\epsilon > 0$. For these t,

$$|\mathbf{x}| = |t\mathbf{e}_1 + (\tan t)\mathbf{e}_2| \leq |t|\,|\mathbf{e}_1| + |\tan t|\,|\mathbf{e}_2| \leq |t| + |\tan t| \leq M$$

where $M = \pi/2 - \epsilon + \tan(\pi/2 - \epsilon)$.

A function $\mathbf{f}(t)$ is said to be bounded at $t = t_0$ if there exists an $\epsilon > 0$ such that $\mathbf{f}(t)$ is bounded for t in $S_\epsilon(t_0)$; or, equivalently, $\mathbf{f}(t)$ is bounded at t_0 if there exists an $M > 0$ and an $\epsilon > 0$ such that $|\mathbf{f}(t)| \leq M$ for $|t - t_0| < \epsilon$.

Clearly if $\mathbf{f}(t)$ is defined and bounded on an interval I, then it is bounded at each t_0 in I. However, the converse is not true, as shown by the example above, where $\mathbf{f}(t)$ is bounded at each t_0 in $-\pi/2 < t < \pi/2$ but not on the whole interval.

LIMITS

A vector function $\mathbf{f}(t)$ has a limit $\mathbf{L}$ as t approaches t_0, written

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{L}$$

or $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$, if for every $\epsilon > 0$, one can find a $\delta > 0$, depending on ϵ, such that the vectors $\mathbf{f}(t)$ are in $S_\epsilon(\mathbf{L})$ for t in $S'_\delta(t_0)$. Observe in Fig. 2-8 that $\mathbf{x} = \mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$ if and only if for every open sphere $S_\epsilon(\mathbf{L})$ about the point $\mathbf{L}$, one can find a deleted $S'_\delta(t_0)$ such that the points $\mathbf{x}$ are in $S_\epsilon(\mathbf{L})$ for t in $S'_\delta(t_0)$.

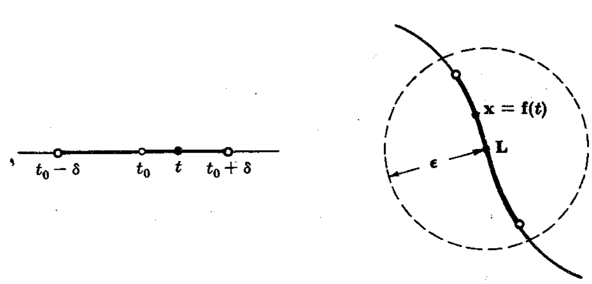


Fig. 2-8

Note that the existence of a limit at t_0 is a *local* property of a function, depending only on the nature of the function in a *deleted neighborhood* of t_0. Moreover, $\mathbf{f}(t)$ need not be defined at t_0. For example, its domain could be the open interval $a < t < t_0$.

Example 2.10.

Let $\mathbf{f}(t) = \mathbf{a} = \text{constant}$. Then for any t_0, $\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{a}$. For $\mathbf{f}(t) \equiv \mathbf{a}$ is in every $S_\epsilon(\mathbf{a})$ for *all* t and hence for all $S_\delta(t_0)$ for all t_0.

Example 2.11.

The function $\mathbf{x} = \mathbf{f}(t) = \begin{cases} t\mathbf{e}_1 + \mathbf{e}_2, & t \geq 0 \\ t\mathbf{e}_1 - \mathbf{e}_2, & t < 0 \end{cases}$,

or $x_1 = f_1(t) = t, \quad x_2 = f_2(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$,

shown in Fig. 2-9, does not have a limit as $t \to 0$; for any point $\mathbf{L}$ has a neighborhood $S_\epsilon(\mathbf{L})$ which does not intersect both the line $x_2 = 1$ and the line $x_2 = -1$. (For example, as shown, $S_{1/2}(0,1)$ will not include points on $x_2 = -1$.) For these $S_\epsilon(\mathbf{L})$ there will *not* exist a $\delta > 0$ such that for *all* $0 < |t| < \delta$ the points $\mathbf{x} = \mathbf{f}(t)$ are in $S_\epsilon(\mathbf{L})$. Since $\mathbf{L}$ is arbitrary, there is no limit. On the other hand, the function does have a limit for any other choice of t_0. For example, as $t \to \frac{1}{2}$ the limit is $\frac{1}{2}\mathbf{e}_1 + \mathbf{e}_2$.

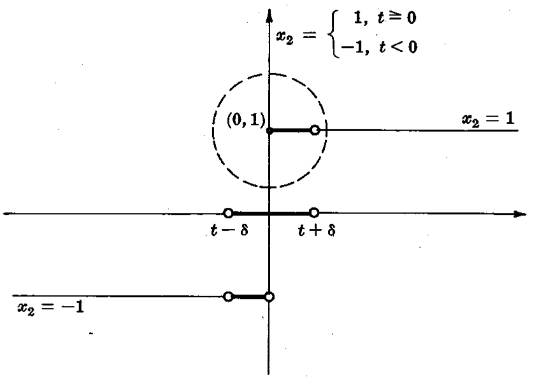


Fig. 2-9

Now, we recall that a scalar function $g(t) \to 0$ as $t \to t_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|g(t)| < \epsilon$ for t in $S'(t_0)$. If we let $g(t) = |\mathbf{f}(t) - \mathbf{L}|$, then $|g(t)| = |\mathbf{f}(t) - \mathbf{L}| < \epsilon$ if and only if $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{L})$. Thus we have the important

Theorem 2.1. $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$ iff $|\mathbf{f}(t) - \mathbf{L}| \to 0$ as $t \to t_0$.

Example 2.12.

$\lim_{t \to 1} (t^2\mathbf{e}_1 - (t+1)\mathbf{e}_2) = \mathbf{e}_1 - 2\mathbf{e}_2$, since

$$\lim_{t \to 1} |\mathbf{f}(t) - \mathbf{L}| = \lim_{t \to 1} |(t^2 - 1)\mathbf{e}_1 - (t-1)\mathbf{e}_2| = \lim_{t \to 1} [(t^2-1)^2 + (t-1)^2]^{1/2} = 0$$

Finally, suppose $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$. Then for an arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{f}(t) - \mathbf{L}| < \epsilon$ for t in $S'_\delta(t_0)$. Hence for t in $S_\delta(t_0)$,

$$|\mathbf{f}(t)| = |\mathbf{f}(t) - \mathbf{L} + \mathbf{L}| \leq |\mathbf{f}(t) - \mathbf{L}| + |\mathbf{L}| \leq M$$

where $M = \text{Max}(\epsilon, |\mathbf{f}(t_0) - \mathbf{L}|) + |\mathbf{L}|$. Thus we have

Theorem 2.2. If $\mathbf{f}(t)$ has a limit as $t \to t_0$, then $\mathbf{f}(t)$ is bounded at t_0.

PROPERTIES OF LIMITS

Suppose $\lim_{t \to t_0} f_i(t) = L_i$, $i = 1, 2, 3$; then

$$\lim_{t \to t_0} [f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3] = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$$

For, let $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ and $\mathbf{L} = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$; then

$$\begin{aligned} \lim_{t \to t_0} |\mathbf{f}(t) - \mathbf{L}| &= \lim_{t \to t_0} |(f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3) - (L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3)| \\ &= \lim_{t \to t_0} [(f_1(t) - L_1)^2 + (f_2(t) - L_2)^2 + (f_3(t) - L_3)^2]^{1/2} \\ &= 0 \end{aligned}$$

The converse of the above is also true. Namely, we have

Theorem 2.3. The function $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ has a limit as $t \to t_0$ if and only if $f_i(t)$, $i = 1,2,3$, have limits as $t \to t_0$, in which case

$$\lim_{t\to t_0} \mathbf{f}(t) = \left(\lim_{t\to t_0} f_1(t)\right)\mathbf{e}_1 + \left(\lim_{t\to t_0} f_2(t)\right)\mathbf{e}_2 + \left(\lim_{t\to t_0} f_3(t)\right)\mathbf{e}_3$$

Example 2.13.

$$\lim_{t\to 0} ((\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + t\mathbf{e}_3) = \left(\lim_{t\to 0} \sin t\right)\mathbf{e}_1 + \left(\lim_{t\to 0} \cos t\right)\mathbf{e}_2 + \left(\lim_{t\to 0} t\right)\mathbf{e}_3 = \mathbf{e}_2$$

Example 2.14.

Let $\mathbf{f}(t) = t^2\mathbf{e}_1 + t\mathbf{e}_2$. Then

$$\lim_{h\to 0} \frac{\mathbf{f}(2+h) - \mathbf{f}(2)}{h} = \lim_{h\to 0} \frac{((2+h)^2\mathbf{e}_1 + (2+h)\mathbf{e}_2) - (4\mathbf{e}_1 + 2\mathbf{e}_2)}{h}$$

$$= \lim_{h\to 0}\left[\frac{((2+h)^2 - 4)\mathbf{e}_1}{h} + \frac{h\mathbf{e}_2}{h}\right] = 4\mathbf{e}_1 + \mathbf{e}_2$$

Now suppose $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$; then $|\mathbf{f}(t)| \to |\mathbf{L}|$ as $t \to t_0$. For, letting $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ and $\mathbf{L} = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$,

$$\lim_{t\to t_0} |\mathbf{f}(t)| = \lim_{t\to t_0} [f_1^2(t) + f_2^2(t) + f_3^2(t)]^{1/2}$$

$$= \left[\left(\lim_{t\to t_0} f_1(t)\right)^2 + \left(\lim_{t\to t_0} f_2(t)\right)^2 + \left(\lim_{t\to t_0} f_3(t)\right)^2\right]^{1/2}$$

$$= [L_1^2 + L_2^2 + L_3^2]^{1/2} = |\mathbf{L}|$$

Note, however, that the converse of the above is not true. That is, $|\mathbf{f}(t)|$ may have a limit even though $\mathbf{f}(t)$ does not. This happens in Example 2.11 at $t_0 = 0$.

We state the above result formally as

Theorem 2.4. If $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$, then $|\mathbf{f}(t)| \to |\mathbf{L}|$ as $t \to t_0$.

Finally, we have: If $\lim_{t\to t_0} \mathbf{f}(t) = \mathbf{L}$, $\lim_{t\to t_0} \mathbf{g}(t) = \mathbf{M}$ and $\lim_{t\to t_0} h(t) = N$, then

[**H**$_1$] $\lim_{t\to t_0} (\mathbf{f}(t) + \mathbf{g}(t)) = \lim_{t\to t_0} \mathbf{f}(t) + \lim_{t\to t_0} \mathbf{g}(t) = \mathbf{L} + \mathbf{M}$

[**H**$_2$] $\lim_{t\to t_0} (h(t)\mathbf{g}(t)) = \lim_{t\to t_0} h(t) \lim_{t\to t_0} \mathbf{g}(t) = N\mathbf{M}$

[**H**$_3$] If $N \neq 0$, then $\lim_{t\to t_0} (\mathbf{f}(t)/h(t)) = \lim_{t\to t_0} \mathbf{f}(t)/\lim_{t\to t_0} h(t) = \mathbf{L}/N$.

[**H**$_4$] $\lim_{t\to t_0} (\mathbf{f}(t)\cdot\mathbf{g}(t)) = \lim_{t\to t_0} \mathbf{f}(t)\cdot \lim_{t\to t_0} \mathbf{g}(t) = \mathbf{L}\cdot\mathbf{M}$

[**H**$_5$] $\lim_{t\to t_0} (\mathbf{f}(t)\times\mathbf{g}(t)) = \lim_{t\to t_0} \mathbf{f}(t)\times \lim_{t\to t_0} \mathbf{g}(t) = \mathbf{L}\times\mathbf{M}$.

[**H**$_6$] If $\lim_{t\to t_0} \mathbf{f}(t) = \mathbf{f}(t_0)$ and $\lim_{\theta\to\theta_0} h(\theta) = t_0$, then $\lim_{\theta\to\theta_0} \mathbf{f}(h(\theta)) = \mathbf{f}\left(\lim_{\theta\to\theta_0} h(\theta)\right) = \mathbf{f}(t_0)$.

Example 2.15.

Let $\lim_{t\to t_0} \mathbf{f}(t) = \mathbf{L}$, $\lim_{t\to t_0} \mathbf{g}(t) = \mathbf{M}$, $\lim_{t\to t_0} \mathbf{h}(t) = \mathbf{N}$. Then

$$\lim_{t\to t_0} [\mathbf{f}(t)\,\mathbf{g}(t)\,\mathbf{h}(t)] = \lim_{t\to t_0} (\mathbf{f}(t)\cdot\mathbf{g}(t)\times\mathbf{h}(t)) = \lim_{t\to t_0} \mathbf{f}(t)\cdot \lim_{t\to t_0} (\mathbf{g}(t)\times\mathbf{h}(t))$$

$$= \lim_{t\to t_0} \mathbf{f}(t)\cdot \lim_{t\to t_0} \mathbf{g}(t) \times \lim_{t\to t_0} \mathbf{h}(t) = [\mathbf{LMN}]$$

CONTINUITY

A vector function $\mathbf{f}(t)$ defined at t_0 is *continuous* at t_0 if for every $\epsilon > 0$ there exists a $\delta > 0$, depending on ϵ, such that $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{f}(t_0))$ for all t in $S_\delta(t_0)$; or, equivalently, $\mathbf{f}(t)$ is continuous at t_0 if

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{f}(t_0) \tag{2.8}$$

The function $\mathbf{f}(t)$ is said to be continuous on I if it is continuous at all $t = t_0$ in I.

It follows from Theorem 2.3 that $\mathbf{f}(t)$ is continuous if and only if its components $f_i(t)$, $i = 1, 2, 3$, are continuous. Also it follows from [$\mathbf{H}_1$] through [$\mathbf{H}_6$] that the sum, product, and scalar and vector products of continuous functions are continuous and that a continuous function of a continuous function is continuous.

Finally, we note that *(2.8)* is equivalent to

$$\lim_{t \to t_0} (\mathbf{f}(t) - \mathbf{f}(t_0)) = 0$$

or, if we let $h = t - t_0$,

$$\lim_{h \to 0} (\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)) = 0$$

Example 2.16.

Let $\mathbf{f}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2$ with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ = constants. Then

$$\lim_{t \to t_0} \mathbf{f}(t) = \lim_{t \to t_0} (\mathbf{a} + \mathbf{b}t + \mathbf{c}t^2) = \mathbf{a} + \mathbf{b}t_0 + \mathbf{c}t_0^2 = \mathbf{f}(t_0)$$

Hence $\mathbf{f}(t)$ is continuous for all t.

Example 2.17.

Let $\mathbf{f}(t) = \begin{cases} \dfrac{t^2 - 1}{t - 1}\mathbf{e}_1 + t^3\mathbf{e}_2, & t \neq 1 \\ 2\mathbf{e}_1 + \mathbf{e}_2, & t = 1 \end{cases}$. Then $\mathbf{f}(t)$ is continuous for all t. For $t_0 \neq 1$,

$$\lim_{t \to t_0} \mathbf{f}(t) = \lim_{t \to t_0} \left(\frac{t^2 - 1}{t - 1}\mathbf{e}_1 + t^3\mathbf{e}_2 \right) = \frac{t_0^2 - 1}{t_0 - 1}\mathbf{e}_1 + t_0^3\mathbf{e}_2 = \mathbf{f}(t_0)$$

For $t_0 = 1$,

$$\lim_{t \to 1} \mathbf{f}(t) = \lim_{t \to 1} \left(\frac{t^2 - 1}{t - 1}\mathbf{e}_1 + t^3\mathbf{e}_2 \right) = \lim_{t \to 1} ((t+1)\mathbf{e}_1 + t^3\mathbf{e}_2) = 2\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{f}(1)$$

Example 2.18.

The function $\mathbf{f}(t) = \begin{cases} t\mathbf{e}_1 + \mathbf{e}_2, & t \geq 0 \\ t\mathbf{e}_1 - \mathbf{e}_2, & t < 0 \end{cases}$ in Example 2.11 is continuous at all t except $t = 0$ where the limit does not exist.

DIFFERENTIATION

The limit

$$\mathbf{f}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} \tag{2.9}$$

if it exists, defines the *derivative* of $\mathbf{f}(t)$ at $t = t_0$. If $\mathbf{f}'(t_0)$ exists, we say $\mathbf{f}(t)$ is *differentiable* at t_0.

Observe that if we substitute $t = t_0 + \Delta t$ in the above, the derivative at t_0 is also given as

$$\mathbf{f}'(t_0) = \lim_{\Delta t \to 0} \frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} \tag{2.10}$$

Example 2.19.

Let $\mathbf{f}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2$, with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ = constants. Then

$$\mathbf{f}'(t_0) = \lim_{\Delta t \to 0} \frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[\mathbf{a} + \mathbf{b}(t_0 + \Delta t) + \mathbf{c}(t_0 + \Delta t)^2] - (\mathbf{a} + \mathbf{b}t_0 + \mathbf{c}t_0^2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\mathbf{b}\,\Delta t + 2\mathbf{c}t_0\,\Delta t + \mathbf{c}(\Delta t)^2}{\Delta t} = \lim_{\Delta t \to 0} (\mathbf{b} + 2\mathbf{c}t_0 + \mathbf{c}\,\Delta t) = \mathbf{b} + 2\mathbf{c}t_0$$

Thus $\mathbf{f}(t)$ is differentiable at t_0, with derivative $\mathbf{f}'(t_0) = \mathbf{b} + 2\mathbf{c}t_0$.

Now, if $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$, then it follows from Theorem 2.3 that

$$\begin{aligned}
\mathbf{f}'(t_0) &= \lim_{t\to t_0} \frac{\mathbf{f}(t)-\mathbf{f}(t_0)}{t-t_0} \\
&= \lim_{t\to t_0}\left[\frac{f_1(t)-f_1(t_0)}{t-t_0}\mathbf{e}_1 + \frac{f_2(t)-f_2(t_0)}{t-t_0}\mathbf{e}_2 + \frac{f_3(t)-f_3(t_0)}{t-t_0}\mathbf{e}_3\right] \\
&= \left[\lim_{t\to t_0}\frac{f_1(t)-f_1(t_0)}{t-t_0}\right]\mathbf{e}_1 + \left[\lim_{t\to t_0}\frac{f_2(t)-f_2(t_0)}{t-t_0}\right]\mathbf{e}_2 + \left[\lim_{t\to t_0}\frac{f_3(t)-f_3(t_0)}{t-t_0}\right]\mathbf{e}_3 \\
&= f_1'(t_0)\mathbf{e}_1 + f_2'(t_0)\mathbf{e}_2 + f_3'(t_0)\mathbf{e}_3
\end{aligned}$$

Thus we have

Theorem 2.5. A function $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ is differentiable at t_0 if and only if each component $f_i(t)$, $i = 1,2,3$, is differentiable at t_0, in which case

$$\mathbf{f}'(t_0) = f_1'(t_0)\mathbf{e}_1 + f_2'(t_0)\mathbf{e}_2 + f_3'(t_0)\mathbf{e}_3$$

If $\mathbf{f}(t)$ is differentiable on an interval I, then $\mathbf{f}'(t)$ is again a vector function on I which may again be differentiable. This will give the second order derivative of $\mathbf{f}(t)$, denoted by $\mathbf{f}''(t)$. Higher order derivatives are defined similarly.

As with scalar functions, if $\mathbf{u} = \mathbf{f}(t)$, we use the notation

$$\mathbf{u}' = \frac{d\mathbf{u}}{dt} = \mathbf{f}'(t), \quad \mathbf{u}'' = \frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\right) = \frac{d^2\mathbf{u}}{dt^2} = \mathbf{f}''(t), \quad \text{etc.}$$

Example 2.20.

If $\mathbf{u} = (t^3+2t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3$, then

$$\begin{aligned}
\mathbf{u}' &= \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(t^3+2t)\mathbf{e}_1 + \frac{d}{dt}(\sin t)\mathbf{e}_2 + \frac{d}{dt}(e^t)\mathbf{e}_3 = (3t^2+2)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + e^t\mathbf{e}_3 \\
\mathbf{u}'' &= \frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\right) = \frac{d}{dt}(3t^2+2)\mathbf{e}_1 + \frac{d}{dt}(\cos t)\mathbf{e}_2 + \frac{d}{dt}(e^t)\mathbf{e}_3 = 6t\mathbf{e}_1 - (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3 \\
\mathbf{u}''' &= \frac{d}{dt}\frac{d^2\mathbf{u}}{dt^2} = \frac{d}{dt}(6t)\mathbf{e}_1 - \frac{d}{dt}(\sin t)\mathbf{e}_2 - \frac{d}{dt}(e^t)\mathbf{e}_3 = 6\mathbf{e}_1 - (\cos t)\mathbf{e}_2 + e^t\mathbf{e}_3
\end{aligned}$$

Example 2.21.

$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$ traces the circle of radius a about the origin, as shown in Fig. 2-10. The derivative $\mathbf{x}' = d\mathbf{x}/dt = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2$ is tangent to the circle at $\mathbf{x}$ and, as we expect, orthogonal to $\mathbf{x}$, since $\mathbf{x}\cdot\mathbf{x}' = 0$.

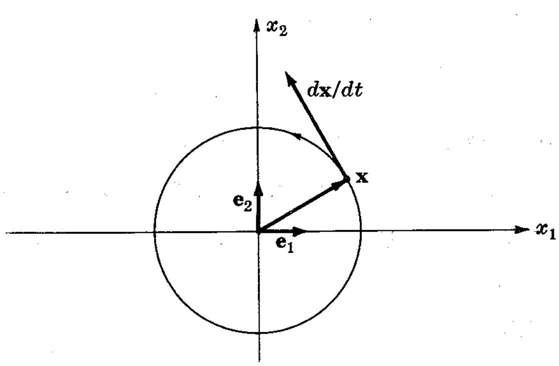


Fig. 2-10

Many properties of scalar functions carry over to vectors. For example, in Problem 2.26 we prove

Theorem 2.6. If $\mathbf{f}(t)$ is differentiable at t_0, then $\mathbf{f}(t)$ is continuous at t_0.

DIFFERENTIATION FORMULAS

If $\mathbf{u}$, $\mathbf{v}$, h are differentiable functions of t on I, then

[J_1] $\mathbf{u}+\mathbf{v}$ is differentiable on I and $\dfrac{d}{dt}(\mathbf{u}+\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}+\dfrac{d\mathbf{v}}{dt}$

[J_2] $h\mathbf{u}$ is differentiable on I and $\dfrac{d}{dt}(h\mathbf{u}) = h\dfrac{d\mathbf{u}}{dt}+\dfrac{dh}{dt}\mathbf{u}$

[J_3] $\mathbf{u}\cdot\mathbf{v}$ is differentiable on I and $\dfrac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}+\dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v}$

[J_4] $\mathbf{u}\times\mathbf{v}$ is differentiable on I and $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times\dfrac{d\mathbf{v}}{dt}+\dfrac{d\mathbf{u}}{dt}\times\mathbf{v}$

Finally we have the *chain rule*:

[J_5] If $\mathbf{u}=\mathbf{f}(t)$ is differentiable on I_t and $t=h(\theta)$ is differentiable on I_θ, where the image $h(I_\theta)$ is contained in I_t, then $\mathbf{u}=\mathbf{g}(\theta)=\mathbf{f}(h(\theta))$ is differentiable on I_θ and

$$\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt}\frac{dt}{d\theta}$$

Example 2.22.

Let $\mathbf{u} = a(\cos t)\mathbf{e}_1 - a(\sin t)\mathbf{e}_2$, $\theta = (1+t^2)^{1/2}$, $t>0$. Then

$$\begin{aligned}\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt}\frac{dt}{d\theta} = \frac{d\mathbf{u}}{dt}\Big/\frac{d\theta}{dt} &= (-a(\sin t)\mathbf{e}_1 - a(\cos t)\mathbf{e}_2)/[t(1+t^2)^{-1/2}] \\ &= -(a/t)(1+t^2)^{1/2}((\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2)\end{aligned}$$

where we used the fact that for scalar functions $\theta = h(t)$ such that $d\theta/dt \neq 0$, we have $dt/d\theta = 1/(d\theta/dt)$.

Example 2.23.

$$\frac{d}{dt}\left(\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\right) = \mathbf{u}\cdot\frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\right)+\frac{d\mathbf{u}}{dt}\cdot\frac{d\mathbf{u}}{dt} = \mathbf{u}\cdot\frac{d^2\mathbf{u}}{dt^2}+\left|\frac{d\mathbf{u}}{dt}\right|^2$$

Example 2.24.

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}\ \frac{d\mathbf{u}}{dt}\ \frac{d^2\mathbf{u}}{dt^2}\right] &= \frac{d}{dt}\left(\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\times\frac{d^2\mathbf{u}}{dt^2}\right) = \mathbf{u}\cdot\frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\times\frac{d^2\mathbf{u}}{dt^2}\right)+\frac{d\mathbf{u}}{dt}\cdot\left(\frac{d\mathbf{u}}{dt}\times\frac{d^2\mathbf{u}}{dt}\right) \\ &= \mathbf{u}\cdot\left[\left(\frac{d\mathbf{u}}{dt}\times\frac{d^3\mathbf{u}}{dt^3}\right)+\left(\frac{d^2\mathbf{u}}{dt^2}\times\frac{d^2\mathbf{u}}{dt^2}\right)\right]+\mathbf{0} = \mathbf{u}\cdot\left(\frac{d\mathbf{u}}{dt}\times\frac{d^3\mathbf{u}}{dt^3}\right) = \left[\mathbf{u}\ \frac{d\mathbf{u}}{dt}\ \frac{d^3\mathbf{u}}{dt^3}\right]\end{aligned}$$

Finally, if $\mathbf{u}$ is a vector function of constant magnitude, i.e. if $|\mathbf{u}| = \text{constant}$, then $\mathbf{u}\cdot\mathbf{u} = \text{constant}$, and, differentiating, we obtain

$$\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}+\frac{d\mathbf{u}}{dt}\cdot\mathbf{u} = 0 \quad\text{or}\quad \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0$$

Hence $\mathbf{u}$ is orthogonal to $d\mathbf{u}/dt$. In particular we have

Theorem 2.7. If $\mathbf{u}$ is a unit vector function, then $d\mathbf{u}/dt$ is orthogonal to $\mathbf{u}$.

This theorem is an important result which will be used often.

FUNCTIONS OF CLASS C^m

In general we require that our functions can be differentiated at least once and usually twice, or more often. Also we will want to know the largest class of functions for which a result will be valid. Accordingly we say a scalar or vector valued function $\mathbf{f}$ belongs to class C^m on an interval I if the mth order derivative of $\mathbf{f}$ exists and is continuous on I. We denote the class of continuous functions by C^0 and the class of functions which have derivatives of all orders by C^∞.

Since a vector function is continuous or has a derivative if and only if all components are continuous or have derivatives, we have

Theorem 2.8. A vector function $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ belongs to C^m on I if and only if its components $f_1(t)$, $f_2(t)$ and $f_3(t)$ belong to C^m on I.

Note that since a differentiable function is continuous, if a function belongs to C^m it belongs to C^j for all $j \leqq m$.

Example 2.25.

Consider the vector function $\mathbf{f}(t) = t^3\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t^{8/3}\mathbf{e}_3$, $-\infty < t < \infty$. Here

$$\mathbf{f}'(t) = 3t^2\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + (8/3)t^{5/3}\mathbf{e}_3 \quad \text{and} \quad \mathbf{f}''(t) = 6t\mathbf{e}_1 - (\sin t)\mathbf{e}_2 + (40/9)t^{2/3}\mathbf{e}_3$$

are continuous for all t. However, $\mathbf{f}'''(t) = 6\mathbf{e}_1 - (\cos t)\mathbf{e}_2 + (80/27)t^{-1/3}\mathbf{e}_3$ does not exist at $t = 0$, since $t^{1/3}$ appears in the denominator. Hence $\mathbf{f}(t)$ is in C^2 on $-\infty < t < \infty$ but not in C^3. In any interval not containing the origin, $\mathbf{f}$ has continuous derivatives of all orders and hence in such an interval $\mathbf{f}$ belongs to class C^∞.

As a consequence of the differentiation formulas $[\mathrm{J}_1]$ through $[\mathrm{J}_5]$ we have

Theorem 2.9. If $\mathbf{f}$, $\mathbf{g}$, h belong to C^m on I, then $h\mathbf{f}$, $\mathbf{f}+\mathbf{g}$, $\mathbf{f}\cdot\mathbf{g}$ and $\mathbf{f}\times\mathbf{g}$ belong to C^m on I.

Theorem 2.10. If $\mathbf{f}(t)$ belongs to C^m on I_t and if $t(\theta)$ belongs to C^m on I_θ, where $t(I_\theta)$ is contained in I_t, then the composite function $\mathbf{g}(\theta) = \mathbf{f}(t(\theta))$ belongs to class C^m on I_θ. In other words, a function of class C^m of a function of class C^m is a function of class C^m.

TAYLOR'S FORMULA

Let $f(t)$ be of class C^m on I. Then (Taylor's formula) for every t and t_0 in I,

$$f(t) = f(t_0) + \frac{f'(t_0)}{1}(t-t_0) + \cdots + \frac{f^{(m)}(t_0)}{m!}(t-t_0)^m + R_m(t,t_0)$$

where the *remainder* $R_m(t, t_0)$ has the property that

$$\frac{R_m(t,t_0)}{(t-t_0)^m} \to 0 \quad \text{as} \quad t \to t_0$$

Clearly, by applying the formula to the components of a vector function $\mathbf{f}(t)$ we have

Theorem 2.11. Taylor's Formula. Let $\mathbf{f}(t)$ belong to C^m on I; then for every t and t_0 in I,

$$\mathbf{f}(t) = \mathbf{f}(t_0) + \frac{\mathbf{f}'(t_0)}{1}(t-t_0) + \cdots + \frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{R}_m(t,t_0)$$

where

$$\frac{\mathbf{R}_m(t,t_0)}{(t-t_0)^m} \to \mathbf{0} \quad \text{as} \quad t \to t_0$$

Example 2.26.

If $\mathbf{f}(t) = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2$, then $\mathbf{f}(0) = \mathbf{e}_1$, $\mathbf{f}'(0) = \mathbf{e}_2$, $\mathbf{f}''(0) = -\mathbf{e}_1$, $\mathbf{f}'''(0) = -\mathbf{e}_2$, $\mathbf{f}^{(4)}(0) = \mathbf{e}_1$. Hence about $t_0 = 0$ we have

$$(\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2 t - (\mathbf{e}_1/2!)t^2 - (\mathbf{e}_2/3!)t^3 + (\mathbf{e}_1/4!)t^4 + \mathbf{R}_4(t)$$

where $\mathbf{R}_4(t)/t^4 \to \mathbf{0}$ as $t \to 0$.

It is often convenient to use the Landau symbols **o** and **O**, to investigate the behavior of a function in the neighborhood of a point. Namely, let a scalar function $g(t)$ be different from zero in some deleted neighborhood of t_0. A scalar or vector function $\mathbf{f}(t)$ is said to be *"small oh"* of $g(t)$ at t_0, denoted by $\mathbf{f}(t) = \mathbf{o}(g(t))$, if $\mathbf{f}(t)/g(t) \to \mathbf{0}$ as $t \to t_0$. A scalar or vector function $\mathbf{f}(t)$ is said to be *"big oh"* of $g(t)$ at t_0, denoted by $\mathbf{f}(t) = \mathbf{O}(g(t))$, if $\mathbf{f}(t)/g(t)$ is bounded at t_0.

Example 2.27.

If $\mathbf{f}(t) = \mathbf{a}t^4 + \mathbf{b}t^5 + \mathbf{c}t^6$, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ = constants, then $\mathbf{f}(t) = \mathbf{o}(t^3)$ at $t = 0$. For,

$$\lim_{t\to 0} \mathbf{f}(t)/t^3 = \lim_{t\to 0} (\mathbf{a}t + \mathbf{b}t^2 + \mathbf{c}t^3) = \mathbf{0}$$

Note also that $\mathbf{f}(t) = \mathbf{o}(t^2)$. However, $\mathbf{f}(t) \neq \mathbf{o}(t^n)$ for integers $n > 3$.

Example 2.28.

If $\mathbf{f}(t) = (\sin^2 t)\mathbf{e}_1 + (t^2 + t^3)\mathbf{e}_2 + t^4\mathbf{e}_3$, then $\mathbf{f}(t) = \mathbf{O}(t^2)$. For,

$$\lim_{t\to 0} \mathbf{f}(t)/t^2 = \lim_{t\to 0}\left[\frac{\sin^2 t}{t^2}\mathbf{e}_1 + (1+t)\mathbf{e}_2 + t^2\mathbf{e}_3\right] = \mathbf{e}_1 + \mathbf{e}_2$$

Since the limit exists, $\mathbf{f}(t)/t^2$ is bounded; thus $\mathbf{f}(t) = \mathbf{O}(t^2)$. Note that $\mathbf{f}(t)/t \to \mathbf{0}$ as $t \to 0$; thus also $\mathbf{f}(t) = \mathbf{O}(t)$. But $\mathbf{O}(t^2)$ is the best estimate, since $|\mathbf{f}(t)/t^\alpha| \to \infty$ as $t \to 0$ for $\alpha > 2$.

Example 2.29.

Let $\mathbf{f}(t)$ be of class C^m on I. It follows from Taylor's formula that at t_0

$$\mathbf{f}(t) = \mathbf{f}(t_0) + \frac{\mathbf{f}'(t_0)}{1}(t-t_0) + \cdots + \frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{o}[(t-t_0)^m]$$

ANALYTIC FUNCTIONS

Suppose $\mathbf{f}(t)$ is of class C^∞ on I. Then for every m and all t and t_0 in I, we have

$$\mathbf{f}(t) = \mathbf{f}(t_0) + \frac{\mathbf{f}'(t_0)}{1} + \cdots + \frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{R}_m(t, t_0)$$

Now, if *in addition*, $\lim_{m\to\infty} \mathbf{R}_m(t, t_0) = \mathbf{0}$, then $\mathbf{f}(t)$ can be expressed in I as a power series

$$\mathbf{f}(t) = \sum_{n=0}^{\infty} \frac{\mathbf{f}^{(n)}(t_0)}{n!}(t-t_0)^n$$

When such is the case, $\mathbf{f}(t)$ is said to be *analytic* in I, or, more generally, $\mathbf{f}(t)$ is analytic in I if for every t_0 in I, there exists a neighborhood $S_\delta(t_0)$ such that $\mathbf{f}(t)$ has a power series expansion

$$\mathbf{f}(t) = \sum_{n=0}^{\infty} \mathbf{a}_n(t-t_0)^n$$

which converges to $\mathbf{f}(t)$ for all t in $S_\delta(t_0)$. The class of analytic functions on I shall be denoted by C^A.

As shown in the example below, a function of class C^∞ need not be analytic. However, it can be shown that any function represented by a power series can be differentiated in the interior of the interval of convergence and the derivative is represented by the power series formed by differentiating the original power series term by term. Thus every analytic function is of class C^∞. Moreover, $\mathbf{f}^{(n)}(t_0) = \mathbf{a}_n n!$

It follows from the sum, product and substitution theorems for power series that the sum, product and scalar and vector products of analytic functions are analytic and that an analytic function of an analytic function is again analytic.

Example 2.30.

The function $f(t) = e^{-1/t^2}$ is continuous for all t except $t = 0$. If we define $f(0) = 0$, then $f(t)$ will be continuous and in fact will belong to C^∞ for all $-\infty < t < \infty$. However, $f(t)$ is not analytic in any interval containing $t = 0$. For it can be shown that $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, etc., so that if $f(t)$ did have a power series expansion in some $S_\delta(0)$, the series would converge to zero for every t in $S_\delta(0)$, which is impossible since $f(t)$ is not identically zero in any $S_\delta(0)$.

Finally we note that the elementary functions, i.e. polynomials, rational functions, trigonometric functions and exponential functions, are analytic in any interval in which they are continuous and that their inverses are analytic in any interval in which they are differentiable.

Solved Problems

LINES AND PLANES

2.1. Find a unit vector normal to the plane S containing $P(0,1,1)$, $Q(1,0,-1)$, $R(1,-1,0)$.

$$\mathbf{PQ}\times\mathbf{PR} = \begin{pmatrix} \mathbf{e}_1 & 1 & 1 \\ \mathbf{e}_2 & -1 & -2 \\ \mathbf{e}_3 & -2 & -1 \end{pmatrix} = -3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$$

is normal to S, and $\pm\dfrac{\mathbf{PQ}\times\mathbf{PR}}{|\mathbf{PQ}\times\mathbf{PR}|} = \pm(1/\sqrt{11})(3\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$ are unit normals to S.

2.2. Find the equation of the line through $A(1,-1,2)$ and parallel to the x_3 axis.

Let $\mathbf{x} = \mathbf{OP}$, $\mathbf{a} = \mathbf{OA}$. Then P is on the line if and only if $\mathbf{AP} = (\mathbf{x}-\mathbf{a}) = k\mathbf{e}_3$ or

$$(x_1-1)\mathbf{e}_1 + (x_2+1)\mathbf{e}_2 + (x_3-2)\mathbf{e}_3 = k\mathbf{e}_3$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = k+2$ $(-\infty < k < \infty)$.

2.3. Let $d \geqq 0$ and let $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ be a unit vector. Show that

$$n_1x_1 + n_2x_2 + n_3x_3 = d$$

is the equation of the plane S whose distance from the origin is d and whose unit normal vector directed away from the origin is $\mathbf{n}$.

Let $\mathbf{x}$ be a general point on S. Then

$$n_1x_1 + n_2x_2 + n_3x_3 = \mathbf{n}\cdot\mathbf{x} = |\mathbf{x}|\cos\measuredangle(\mathbf{n},\mathbf{x}) = d \geqq 0$$

implies $\cos\measuredangle(\mathbf{n},\mathbf{x}) \geqq 0$ or $0 \leqq \measuredangle(\mathbf{n},\mathbf{x}) \leqq \pi/2$. That is, $\mathbf{n}$ is directed from the origin to S. As indicated in Fig. 2-11, the distance from the origin to S is

$$|P_{\mathbf{n}}(\mathbf{x})| = |\mathbf{n}\cdot\mathbf{x}|/|\mathbf{n}| = |\mathbf{n}\cdot\mathbf{x}| = |d| = d$$

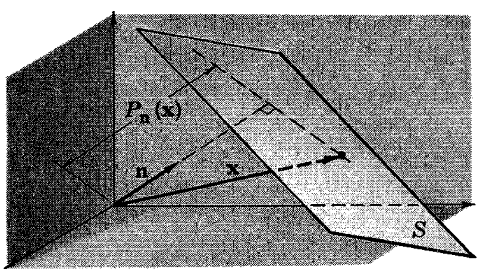


Fig. 2-11

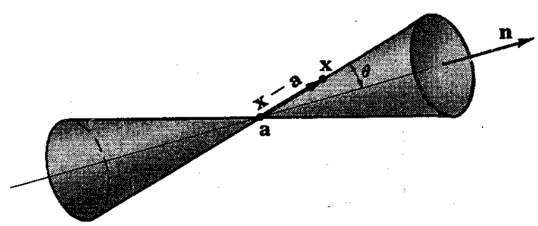


Fig. 2-12

2.4. Derive the equation of the sphere of radius r about $\mathbf{a}$.

$$|\mathbf{x}-\mathbf{a}| = r \quad \text{or} \quad (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a}) = r^2$$

2.5. Let $\mathbf{a}$ be the vertex of a right circular cone with axis in the direction of a unit vector $\mathbf{n}$, and half angle $\theta = \cos^{-1}k$, $k > 0$. Show that the equation of the cone is

$$[(\mathbf{x}-\mathbf{a})\cdot\mathbf{n}]^2 - k^2(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a}) = 0$$

As shown in Fig. 2-12, $\mathbf{x}$ lies on the cone if and only if $\measuredangle(\mathbf{x}-\mathbf{a},\mathbf{n}) = \theta$ or $(\pi-\theta)$, i.e. iff

$$|\cos\measuredangle(\mathbf{x}-\mathbf{a},\mathbf{n})| = |\cos\theta| = k \quad \text{or} \quad |(\mathbf{x}-\mathbf{a})\cdot\mathbf{n}| = k|\mathbf{x}-\mathbf{a}|$$

or, squaring, $[(\mathbf{x}-\mathbf{a})\cdot\mathbf{n}]^2 = k^2(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})$ from which the result follows.

2.6. Let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be an *arbitrary basis* in E^3, let O be a fixed point in E^3, and let the coordinates (x_1, x_2, x_3) of P be defined as the components of the vector

$$\mathbf{x} = \mathbf{OP} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$$

with respect to the basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. A coordinate system established in this way is called an *affine* coordinate system. Show that in an affine coordinate system the square of the distance between $P(x_1, x_2, x_3)$ and $Q(y_1, y_2, y_3)$ is given by

$$\begin{aligned}|\mathbf{PQ}|^2 = {} & g_{11}(x_1-y_1)^2 + g_{12}(x_1-y_1)(x_2-y_2) + g_{13}(x_1-y_1)(x_3-y_3) \\ & + g_{21}(x_2-y_2)(x_1-y_1) + g_{22}(x_2-y_2)^2 + g_{23}(x_2-y_2)(x_3-y_3) \\ & + g_{31}(x_3-y_3)(x_1-y_1) + g_{32}(x_3-y_3)(x_2-y_2) + g_{33}(x_3-y_3)^2\end{aligned}$$

or, in short, $$|\mathbf{PQ}|^2 = \sum_i \sum_j g_{ij}(x_i-y_i)(x_j-y_j), \qquad i,j = 1,2,3$$

where the g_{ij} satisfy (*a*) $g_{ij} = g_{ji}$, (*b*) $\det(g_{ij}) > 0$.

$$\begin{aligned}|\mathbf{PQ}|^2 &= |\mathbf{QP}|^2 = |\mathbf{OP}-\mathbf{OQ}|^2 = |\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\ &= \left[\sum_i (x_i-y_i)\mathbf{u}_i\right]\cdot\left[\sum_j (x_j-y_j)\mathbf{u}_j\right] = \sum_i\sum_j (\mathbf{u}_i\cdot\mathbf{u}_j)(x_i-y_i)(x_j-y_j)\end{aligned}$$

or $$|\mathbf{PQ}|^2 = \sum_i\sum_j g_{ij}(x_i-y_i)(x_j-y_j) \quad \text{where} \quad g_{ij} = \mathbf{u}_i\cdot\mathbf{u}_j,\ i,j=1,2,3.$$

Clearly (*a*) $g_{ij} = \mathbf{u}_i\cdot\mathbf{u}_j = \mathbf{u}_j\cdot\mathbf{u}_i = g_{ji}$, $i,j = 1,2,3$. Also, from Problem 1.58,

$$(b) \quad \det(g_{ij}) = \det\begin{pmatrix}\mathbf{u}_1\cdot\mathbf{u}_1 & \mathbf{u}_1\cdot\mathbf{u}_2 & \mathbf{u}_1\cdot\mathbf{u}_3 \\ \mathbf{u}_2\cdot\mathbf{u}_1 & \mathbf{u}_2\cdot\mathbf{u}_2 & \mathbf{u}_2\cdot\mathbf{u}_3 \\ \mathbf{u}_3\cdot\mathbf{u}_1 & \mathbf{u}_3\cdot\mathbf{u}_2 & \mathbf{u}_3\cdot\mathbf{u}_3\end{pmatrix} = [\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]^2 > 0$$

FUNCTIONS

2.7. Compute the vectors $\mathbf{x} = t^2\mathbf{e}_1 + (1-t)\mathbf{e}_2$ for t any integer between -4 and 4 and sketch the curve traced by the terminal points of $\mathbf{x}$.

t	$\mathbf{x}$
-4	$16\mathbf{e}_1 + 5\mathbf{e}_2$
-3	$9\mathbf{e}_1 + 4\mathbf{e}_2$
-2	$4\mathbf{e}_1 + 3\mathbf{e}_2$
-1	$\mathbf{e}_1 + 2\mathbf{e}_2$
0	$\mathbf{e}_2$
1	$\mathbf{e}_1$
2	$4\mathbf{e}_1 - \mathbf{e}_2$
3	$9\mathbf{e}_1 - 2\mathbf{e}_2$
4	$16\mathbf{e}_1 - 3\mathbf{e}_2$

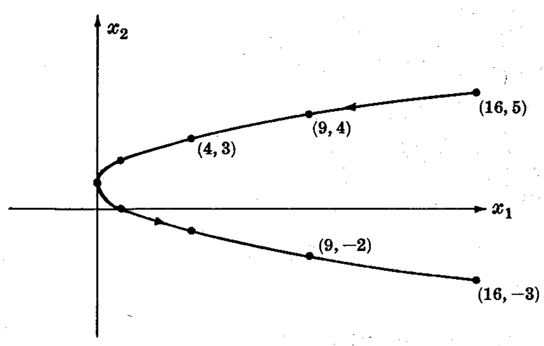


Fig. 2-13

2.8. Let $\mathbf{f}(t) = (1+t^3)\mathbf{e}_1 + (2t-t^2)\mathbf{e}_2 + t\mathbf{e}_3$, $\mathbf{g}(t) = (1+t^2)\mathbf{e}_1 + t^3\mathbf{e}_2$, $h(t) = (2t-1)$. Find (*a*) $h(2)(\mathbf{f}(1)+\mathbf{g}(-1))$, (*b*) $|\mathbf{g}(2)|$, (*c*) $\mathbf{f}(a)\cdot\mathbf{g}(b)$, (*d*) $\mathbf{f}(t)\times\mathbf{g}(t)$, (*e*) $\mathbf{g}(2a-b)$, (*f*) $\mathbf{f}(t_0+\Delta t) - \mathbf{f}(t_0)$, (*g*) $\mathbf{f}(h(t))$.

(*a*) $h(2)(\mathbf{f}(1)+\mathbf{g}(-1)) = (3)[(2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) + (2\mathbf{e}_1-\mathbf{e}_2)] = 12\mathbf{e}_1 + 3\mathbf{e}_3$

(*b*) $|\mathbf{g}(2)| = |5\mathbf{e}_1 + 8\mathbf{e}_2| = \sqrt{89}$

(*c*) $\mathbf{f}(a)\cdot\mathbf{g}(b) = [(1+a^3)\mathbf{e}_1 + (2a-a^2)\mathbf{e}_2 + a\mathbf{e}_3]\cdot[(1+b^2)\mathbf{e}_1 + b^3\mathbf{e}_2]$
$= (1+a^3)(1+b^2) + b^3(2a-a^2)$

$$(d)\quad \mathbf{f}(t)\times\mathbf{g}(t) \;=\; \det\begin{pmatrix} \mathbf{e}_1 & (1+t^3) & (1+t^2) \\ \mathbf{e}_2 & (2t-t^2) & t^3 \\ \mathbf{e}_3 & t & 0 \end{pmatrix}$$
$$= \; -t^4\mathbf{e}_1 + (t+t^3)\mathbf{e}_2 + (t^6+t^4-t^3+t^2-2t)\mathbf{e}_3$$

$$(e)\quad \mathbf{g}(2a-b) \;=\; (1+(2a-b)^2)\mathbf{e}_1 + (2a-b)^3\mathbf{e}_2$$

$$(f)\quad \mathbf{f}(t_0+\Delta t) - \mathbf{f}(t_0) \;=\; [1+(t_0+\Delta t)^3]\mathbf{e}_1 + [2(t_0+\Delta t) - (t_0+\Delta t)^2]\mathbf{e}_2 + (t_0+\Delta t)\mathbf{e}_3 - (1+t_0^3)\mathbf{e}_1 - (2t_0 - t_0^2)\mathbf{e}_2 - t_0\mathbf{e}_3$$
$$= \; (3t_0^2\,\Delta t + 3t_0\,\Delta t^2 + \Delta t^3)\mathbf{e}_1 + (2\,\Delta t - 2t_0\,\Delta t - \Delta t^2)\mathbf{e}_2 + \Delta t\,\mathbf{e}_3$$

$$(g)\quad \mathbf{f}(h(t)) \;=\; \mathbf{f}(2t-1) \;=\; (1+(2t-1)^3)\mathbf{e}_1 + (2(2t-1)-(2t-1)^2)\mathbf{e}_2 + (2t-1)\mathbf{e}_3$$
$$= \; (8t^3-12t^2+6t)\mathbf{e}_1 + (-4t^2+8t-3)\mathbf{e}_2 + (2t-1)\mathbf{e}_3$$

2.9. Show that the curve generated by
$$\mathbf{x} \;=\; (-1+\sin 2t\cos 3t)\mathbf{e}_1 + (2+\sin 2t\sin 3t)\mathbf{e}_2 + (-3+\cos 2t)\mathbf{e}_3$$
lies on the sphere of radius 1 about $\mathbf{a} = -\mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3$.

$$|\mathbf{x}-\mathbf{a}| \;=\; |(\sin 2t\cos 3t)\mathbf{e}_1 + (\sin 2t\sin 3t)\mathbf{e}_2 + (\cos 2t)\mathbf{e}_3|$$
$$= \; (\sin^2 2t\cos^2 3t + \sin^2 2t\sin^2 3t + \cos^2 2t)^{1/2} \;=\; (\sin^2 2t + \cos^2 2t)^{1/2} \;=\; 1$$

from which the result follows.

2.10. Show that the curve generated by
$$\mathbf{x} \;=\; (-2+\sin t)\mathbf{e}_1 + (t^2+2)\mathbf{e}_2 + (t^2-1+2\sin t)\mathbf{e}_3$$
lies on the plane through $\mathbf{a} = \mathbf{e}_2 + 2\mathbf{e}_3$ and normal to $\mathbf{N} = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$.

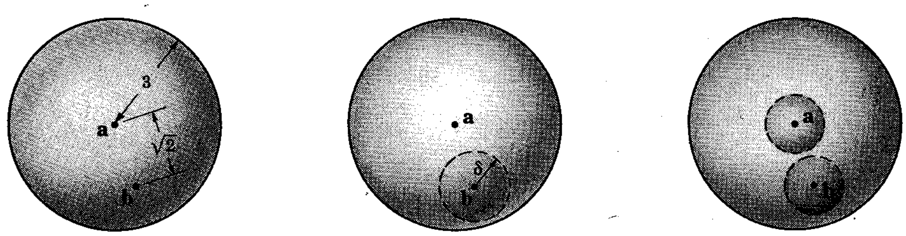
$$(\mathbf{x}-\mathbf{a})\cdot\mathbf{N} \;=\; [(-2+\sin t)\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 + (t^2-3+2\sin t)\mathbf{e}_3]\cdot[2\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3] \;=\; 0$$

It follows that $\mathbf{x}$ lies on the plane through $\mathbf{a}$ and normal to $\mathbf{N}$.

2.11. Let $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{b} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$. (*a*) Show that $\mathbf{b}$ is in $S_3(\mathbf{a})$. (*b*) Find a $\delta > 0$ such that $S_\delta(\mathbf{b})$ is contained in $S_3(\mathbf{a})$. (*c*) Find ϵ_1 and ϵ_2 such that $S_{\epsilon_1}(\mathbf{a})$ and $S_{\epsilon_2}(\mathbf{b})$ are disjoint.

(*a*) Since $|\mathbf{b}-\mathbf{a}| = \sqrt{2} < 3$, $\mathbf{b}$ is in $S_3(\mathbf{a})$.

(*b*) Let $\delta \leqq 3 - |\mathbf{b}-\mathbf{a}| = 3-\sqrt{2}$. If $\mathbf{x}$ belongs to $S_\delta(\mathbf{b})$, i.e. if $|\mathbf{x}-\mathbf{b}| < \delta$, then
$$|\mathbf{x}-\mathbf{a}| \;=\; |\mathbf{x}-\mathbf{b}+\mathbf{b}-\mathbf{a}| \;\leqq\; |\mathbf{x}-\mathbf{b}| + |\mathbf{b}-\mathbf{a}| \;<\; \delta + \sqrt{2} \;\leqq\; 3-\sqrt{2}+\sqrt{2} \;=\; 3$$
that is, $|\mathbf{x}-\mathbf{a}| < 3$ and $\mathbf{x}$ is in $S_3(\mathbf{a})$. Since any $\mathbf{x}$ in $S_\delta(\mathbf{b})$ is in $S_3(\mathbf{a})$, $S_\delta(\mathbf{b})$ is contained in $S_3(\mathbf{a})$. See Fig. 2-14.



$\delta \leqq 3-\sqrt{2}$ $\qquad$ $\epsilon_1 = \epsilon_2 \leqq \sqrt{2}/2$

Fig. 2-14

(*c*) Let $\epsilon_1 = \epsilon_2 \leqq \frac{1}{2}|\mathbf{b}-\mathbf{a}| = \sqrt{2}/2$; then $S_{\epsilon_1}(\mathbf{a})$ and $S_{\epsilon_2}(\mathbf{b})$ are disjoint. For suppose otherwise, i.e. suppose $\mathbf{y}$ is in $S_{\epsilon_1}(\mathbf{a})$ and $S_{\epsilon_2}(\mathbf{b})$; then $|\mathbf{y}-\mathbf{a}| < \sqrt{2}/2$ and $|\mathbf{y}-\mathbf{b}| < \sqrt{2}/2$. But
$$\sqrt{2} \;=\; |\mathbf{b}-\mathbf{a}| \;=\; |\mathbf{b}-\mathbf{y}+\mathbf{y}-\mathbf{a}| \;\leqq\; |\mathbf{y}-\mathbf{b}| + |\mathbf{y}-\mathbf{a}| \;<\; \sqrt{2}/2+\sqrt{2}/2$$
which is impossible. Thus $S_{\epsilon_1}(\mathbf{a})$ and $S_{\epsilon_2}(\mathbf{b})$ are disjoint.

2.12. Show that the points $P(t^2, -t, 2t)$ lie in $S_{1/3}(1,-1,2)$ for all t in $S_{1/10}(1)$.

If t is in $S_{1/10}(1)$, then $|t-1| < 1/10$ or $(t-1)^2 < 1/100$. Also,

$$(t+1)^2 = ((t-1)+2)^2 = (t-1)^2 + 4(t-1) + 4$$
$$\leqq (t-1)^2 + 4|t-1| + 4 \leqq (1/100) + (4/10) + 4 \leqq 5$$

Now, for these t, the distance between $P(t^2, -t, 2t)$ and $(1,-1,2)$ is

$$[(t^2-1)^2 + (-t+1)^2 + (2t-2)^2]^{1/2} = [(t+1)^2(t-1)^2 + (t-1)^2 + 4(t-1)^2]^{1/2}$$
$$\leqq [(5/100) + (1/100) + (4/100)]^{1/2} \leqq 1/\sqrt{10} < 1/3$$

Thus P is in $S_{1/3}(1,-1,2)$ for all t in $S_{1/10}(1)$.

2.13. Find a $\delta > 0$ such that the vectors $\mathbf{x} = t^2\mathbf{e}_1 - t\mathbf{e}_2 + 2t\mathbf{e}_3$ are in $S_{1/100}(\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)$ for all t in $S_\delta(1)$.

Let $\mathbf{a} = \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$. Then

$$|\mathbf{x}-\mathbf{a}| = |(t^2-1)\mathbf{e}_1 - (t-1)\mathbf{e}_2 + (2t-2)\mathbf{e}_3|$$
$$\leqq |t^2-1|\,|\mathbf{e}_1| + |t-1|\,|\mathbf{e}_2| + |2t-2|\,|\mathbf{e}_2| \leqq |t-1|\,|t+1| + |t-1| + 2|t-1|$$
$$\leqq |t-1|(|t+1|+3) = |t-1|(|t-1+2|+3) \leqq |t-1|(|t-1|+5)$$

Now suppose $|t-1| < 1$; then $|\mathbf{x}-\mathbf{a}| \leqq |t-1|6 < 1/100$ if $|t-1| < 1/600$. Thus if $|t-1| < 1/600$, i.e. if t is in $S_\delta(1)$ where $\delta = 1/600$, then certainly $|t-1| < 1$ and

$$|\mathbf{x}-\mathbf{a}| \leqq |t-1|(|t-1|+5) \leqq |t-1|6 < (1/600)6 = 1/100$$

that is, $\mathbf{x}$ is in $S_{1/100}(\mathbf{a})$, which is the required result.

LIMITS AND CONTINUITY

2.14. Evaluate $\lim_{t\to 2} [(3t^2+1)\mathbf{e}_1 - t^3\mathbf{e}_2 + \mathbf{e}_3]$.

$$\lim_{t\to 2} [(3t^2+1)\mathbf{e}_1 - t^3\mathbf{e}_2 + \mathbf{e}_3] = \left(\lim_{t\to 2}(3t^2+1)\right)\mathbf{e}_1 - \left(\lim_{t\to 2} t^3\right)\mathbf{e}_2 + \left(\lim_{t\to 2} 1\right)\mathbf{e}_3 = 13\mathbf{e}_1 - 8\mathbf{e}_2 + \mathbf{e}_3$$

2.15. Let $\mathbf{f}(t) = (\sin t)\mathbf{e}_1 + t\mathbf{e}_3$ and $\mathbf{g}(t) = (t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2$. Find (*a*) $\lim_{t\to 0}(\mathbf{f}(t)\cdot\mathbf{g}(t))$, (*b*) $\lim_{t\to 0}(\mathbf{f}(t)\times\mathbf{g}(t))$.

(*a*)
$$\lim_{t\to 0}(\mathbf{f}(t)\cdot\mathbf{g}(t)) = \lim_{t\to 0}\mathbf{f}(t)\cdot\lim_{t\to 0}\mathbf{g}(t)$$
$$= \lim_{t\to 0}((\sin t)\mathbf{e}_1 + t\mathbf{e}_3)\cdot\lim_{t\to 0}((t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2)$$
$$= \mathbf{0}\cdot(\mathbf{e}_1+\mathbf{e}_2) = 0$$

(*b*)
$$\lim_{t\to 0}(\mathbf{f}(t)\times\mathbf{g}(t)) = \lim_{t\to 0}\mathbf{f}(t)\times\lim_{t\to 0}\mathbf{g}(t)$$
$$= \lim_{t\to 0}((\sin t)\mathbf{e}_1 + t\mathbf{e}_3)\times\lim_{t\to 0}[(t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2]$$
$$= \mathbf{0}\times(\mathbf{e}_1+\mathbf{e}_2) = \mathbf{0}$$

2.16. Define the function $\mathbf{f}(t) = \dfrac{\sin t}{t}\mathbf{e}_1 + (\cos t)\mathbf{e}_2$ at $t = 0$ so that $\mathbf{f}(t)$ is continuous at $t = 0$.

$$\lim_{t\to 0}\mathbf{f}(t) = \lim_{t\to 0}\left(\frac{\sin t}{t}\mathbf{e}_1 + (\cos t)\mathbf{e}_2\right) = \mathbf{e}_1 + \mathbf{e}_2$$

Thus if we define $\mathbf{f}(0) = \mathbf{e}_1 + \mathbf{e}_2$, then $\lim_{t\to 0}\mathbf{f}(t) = \mathbf{f}(0)$ and $\mathbf{f}(t)$ will be continuous at $t = 0$.

2.17. Prove that if $\mathbf{f}(t)$, $\mathbf{g}(t)$ and $\mathbf{h}(t)$ are continuous at t_0, then $[\mathbf{f}(t)\,\mathbf{g}(t)\,\mathbf{h}(t)]$ is continuous at t_0.

It is given that $\lim_{t\to t_0}\mathbf{f}(t) = \mathbf{f}(t_0)$, $\lim_{t\to t_0}\mathbf{g}(t) = \mathbf{g}(t_0)$, and $\lim_{t\to t_0}\mathbf{h}(t) = \mathbf{h}(t_0)$. It follows from Example 2.15 that

$$\lim_{t\to t_0}[\mathbf{f}(t)\,\mathbf{g}(t)\,\mathbf{h}(t)] = [\mathbf{f}(t_0)\,\mathbf{g}(t_0)\,\mathbf{h}(t_0)]$$

Hence $[\mathbf{f}(t)\,\mathbf{g}(t)\,\mathbf{h}(t)]$ is continuous at t_0.

2.18. Using the definition of the limit, show that

$$\lim_{t\to 1} (t^2\mathbf{e}_1 + (t+1)\mathbf{e}_2) = \mathbf{e}_1 + 2\mathbf{e}_2$$

Let $\mathbf{f}(t) = t^2\mathbf{e}_1 + (t+1)\mathbf{e}_2$ and $\mathbf{L} = \mathbf{e}_1 + 2\mathbf{e}_2$, and consider

$$|\mathbf{f}(t) - \mathbf{L}| = |(t^2-1)\mathbf{e}_1 + (t-1)\mathbf{e}_2| \leqq |t^2-1|\,|\mathbf{e}_1| + |t-1|\,|\mathbf{e}_2|$$
$$\leqq |t-1|\,|t+1| + |t-1| \leqq |t-1|(|t-1|+2+1) \leqq |t-1|(|t-1|+3)$$

If we take $|t-1| < 1$, it follows further that

$$|\mathbf{f}(t) - \mathbf{L}| \leqq |t-1|4 < \epsilon \quad \text{if} \quad |t-1| < \epsilon/4$$

Thus given an arbitrary $\epsilon > 0$, we are led to choose $\delta = \min(1, \epsilon/4)$. Then if $|t-1| < \delta$, i.e. if t is in $S_\delta(1)$, we have both $|t-1| \leq 1$ and $|t-1| < \epsilon/4$; and for these t,

$$|\mathbf{f}(t) - \mathbf{L}| \leqq |t-1|(|t-1|+3) = |t-1|4 < \epsilon$$

i.e. $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{L})$, which is the required result.

2.19. If $\mathbf{f}(t)$ is bounded at t_0 and $\mathbf{g}(t) \to \mathbf{0}$ as $t \to t_0$, show that $\mathbf{f}(t) \times \mathbf{g}(t) \to \mathbf{0}$ as $t \to t_0$.

Let $\epsilon > 0$ be arbitrary. Since $\mathbf{f}(t)$ is bounded at t_0, there exist $M > 0$ and $\delta_1 > 0$ such that $|\mathbf{f}(t)| \leqq M$ for all $0 < |t-t_0| < \delta_1$. Also, since $\mathbf{g}(t) \to 0$ as $t \to t_0$, there exists $\delta_2 > 0$ such that $|\mathbf{g}(t)| < \epsilon/M$ for $0 < |t-t_0| < \delta_2$. Now choose $\delta = \min(\delta_1, \delta_2)$; then for $0 < |t-t_0| < \delta$ we have $0 < |t-t_0| < \delta_1$ and $0 < |t-t_0| < \delta_2$. Hence

$$|\mathbf{f}(t) \times \mathbf{g}(t) - \mathbf{0}| = |\mathbf{f}(t) \times \mathbf{g}(t)| = |\mathbf{f}(t)|\,|\mathbf{g}(t)|\,|\sin \measuredangle(\mathbf{f}, \mathbf{g})| \leqq |\mathbf{f}(t)|\,|\mathbf{g}(t)| < M(\epsilon/M) = \epsilon$$

Thus $\mathbf{f}(t) \times \mathbf{g}(t) \to 0$ as $t \to t_0$.

2.20. If $\mathbf{f}(t) \to \mathbf{L}$ and $\mathbf{g}(t) \to \mathbf{M}$ as $t \to t_0$, show that $\mathbf{f}(t) \times \mathbf{g}(t) \to \mathbf{L} \times \mathbf{M}$ as $t \to t_0$.

Now let $\epsilon > 0$ be arbitrary. Since $\mathbf{g}(t) \to \mathbf{M}$ as $t \to t_0$, there exists a $\delta_1 > 0$ such that $|\mathbf{g}(t) - \mathbf{M}| < \epsilon/(2|\mathbf{L}|)$ for $0 < |t-t_0| < \delta_1$. Also $\mathbf{g}(t)$ is bounded at t_0, and so there exist $\delta_2 > 0$ and $K > 0$ such that $|\mathbf{g}(t)| \leqq K$ for $0 < |t-t_0| < \delta_2$. Finally, since $\mathbf{f}(t) \to \mathbf{L}$ as $t \to t_0$ there exists a $\delta_3 > 0$ such that $|\mathbf{f}(t) - \mathbf{L}| < \epsilon/2K$ whenever $0 < |t-t_0| < \delta_3$. Then if $0 < |t-t_0| < \delta = \min(\delta_1, \delta_2, \delta_3)$, we have $0 < |t-t_0| < \delta_1$, $0 < |t-t_0| < \delta_2$ and $0 < |t-t_0| < \delta_3$ and so

$$|(\mathbf{f} \times \mathbf{g}) - (\mathbf{L} \times \mathbf{M})| \leqq |\mathbf{f} - \mathbf{L}|\,|\mathbf{g}| + |\mathbf{L}|\,|\mathbf{g} - \mathbf{M}| < (\epsilon/2K)(K) + |\mathbf{L}|(\epsilon/2|\mathbf{L}|) = \epsilon$$

DIFFERENTIATION

2.21. Let $\mathbf{u} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a, b \neq 0$. Find (a) $\dfrac{d\mathbf{u}}{dt}$, (b) $\left|\dfrac{d\mathbf{u}}{dt}\right|$, (c) $\dfrac{d^2\mathbf{u}}{dt^2}$, (d) $\left|\dfrac{d^2\mathbf{u}}{dt^2}\right|$.

(a) $$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}a(\cos t)\mathbf{e}_1 + \frac{d}{dt}a(\sin t)\mathbf{e}_2 + \frac{d}{dt}(bt)\mathbf{e}_3 = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3$$

(b) $$\left|\frac{d\mathbf{u}}{dt}\right| = (a^2\sin^2 t + a^2\cos^2 t + b^2)^{1/2} = (a^2 + b^2)^{1/2}$$

(c) $$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}(-a(\sin t))\mathbf{e}_1 + \frac{d}{dt}a(\cos t)\mathbf{e}_2 + \frac{d}{dt}b\mathbf{e}_3 = -a(\cos t)\mathbf{e}_1 - a(\sin t)\mathbf{e}_2$$

(d) $$\left|\frac{d^2u}{dt^2}\right| = (a^2\cos^2 t + a^2\sin^2 t)^{1/2} = |a|$$

2.22. Find the equation of the line tangent to the curve generated by $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$.

A vector tangent to the curve at the point $\mathbf{x}$ is $d\mathbf{x}/dt = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$. If $\mathbf{y}$ denotes a generic point on the tangent line, then the equation of the tangent line (see Fig. 2-15) is

$$(\mathbf{y} - \mathbf{x}) = k\frac{d\mathbf{x}}{dt} \quad \text{or} \quad \mathbf{y} = k\frac{d\mathbf{x}}{dt} + \mathbf{x}, \quad -\infty < k < \infty$$

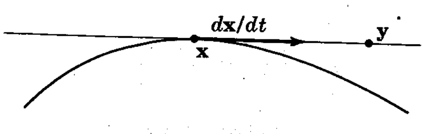


Fig. 2-15

At $t = 1$, $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $d\mathbf{x}/dt = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$. Hence the tangent line at $t = 1$ is

$$\mathbf{y} = k(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \qquad -\infty < k < \infty$$

or

$$\mathbf{y} = (k+1)\mathbf{e}_1 + (2k+1)\mathbf{e}_2 + (3k+1)\mathbf{e}_3, \qquad -\infty < k < \infty$$

2.23. If $\mathbf{u} = (3t^2+1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2$ and $\mathbf{v} = (\cos t)\mathbf{e}_1 + e^t\mathbf{e}_3$, find (a) $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})$, (b) $\frac{d}{dt}(\mathbf{u}\times\mathbf{v})$, (c) $\frac{d}{dt}|\mathbf{u}|$.

(a)
$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} = ((3t^2+1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)\cdot(-(\sin t)\mathbf{e}_1 + e^t\mathbf{e}_3) \\ &\qquad + (6t\mathbf{e}_1 + (\cos t)\mathbf{e}_2)\cdot((\cos t)\mathbf{e}_1 + e^t\mathbf{e}_3) \\ &= -(3t^2+1)\sin t + 6t\cos t\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v} \\ &= \det\begin{pmatrix}\mathbf{e}_1 & 3t^2+1 & -\sin t \\ \mathbf{e}_2 & \sin t & 0 \\ \mathbf{e}_3 & 0 & e^t\end{pmatrix} + \det\begin{pmatrix}\mathbf{e}_1 & 6t & \cos t \\ \mathbf{e}_2 & \cos t & 0 \\ \mathbf{e}_3 & 0 & e^t\end{pmatrix} \\ &= (\sin t)e^t\mathbf{e}_1 - (3t^2+1)e^t\mathbf{e}_2 + (\sin^2 t)\mathbf{e}_3 + (\cos t)e^t\mathbf{e}_1 - 6te^t\mathbf{e}_2 - (\cos^2 t)\mathbf{e}_3 \\ &= (\sin t + \cos t)e^t\mathbf{e}_1 - (3t^2 + 6t + 1)e^t\mathbf{e}_2 + (\sin^2 t - \cos^2 t)\mathbf{e}_3\end{aligned}$$

Another method:

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix}\mathbf{e}_1 & 3t^2+1 & \cos t \\ \mathbf{e}_2 & \sin t & 0 \\ \mathbf{e}_3 & 0 & e^t\end{pmatrix} = (\sin t)e^t\mathbf{e}_1 - (3t^2+1)e^t\mathbf{e}_2 - (\sin t\cos t)\mathbf{e}_3$$

and

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= [(\sin t)e^t + (\cos t)e^t]\mathbf{e}_1 - [(3t^2+1)e^t + 6te^t]\mathbf{e}_2 - [-\sin^2 t + \cos^2 t]\mathbf{e}_3 \\ &= (\sin t + \cos t)e^t\mathbf{e}_1 - (3t^2+6t+1)e^t\mathbf{e}_2 + (\sin^2 t - \cos^2 t)\mathbf{e}_3\end{aligned}$$

(c)
$$\begin{aligned}\frac{d}{dt}|\mathbf{u}| &= \frac{d}{dt}(\mathbf{u}\cdot\mathbf{u})^{1/2} = \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{u})^{-1/2}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) = \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{u})^{-1/2}2\left(\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\right) = (\mathbf{u}/|\mathbf{u}|)\cdot\frac{d\mathbf{u}}{dt} \\ &= [(3t^2+1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2]/[(3t^2+1)^2 + \sin^2 t]^{1/2}\cdot(6t\mathbf{e}_1 + (\cos t)\mathbf{e}_2) \\ &= (18t^3 + 6t + \sin t\cos t)/[(3t^2+1)^2 + \sin^2 t]^{1/2}\end{aligned}$$

2.24. Let $\mathbf{u} = (\sin t)\mathbf{e}_1 + 2t^2\mathbf{e}_2 + t\mathbf{e}_3$, $(t > 0)$, and $t = \log\theta$. Find $d\mathbf{u}/d\theta$ as a (a) function of θ, (b) function of t.

(a) $\dfrac{d\mathbf{u}}{d\theta} = \dfrac{d\mathbf{u}}{dt}\dfrac{dt}{d\theta} = ((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)(1/\theta)$. Substituting for t,

$$d\mathbf{u}/d\theta = (1/\theta)((\cos\log\theta)\mathbf{e}_1 + 4(\log\theta)\mathbf{e}_2 + \mathbf{e}_3)$$

Another method:

Substituting for t, we have $\mathbf{u} = (\sin\log\theta)\mathbf{e}_1 + 2(\log^2\theta)\mathbf{e}_2 + (\log\theta)\mathbf{e}_3$. Hence

$$\begin{aligned}d\mathbf{u}/d\theta &= (\cos\log\theta)(1/\theta)\mathbf{e}_1 + 4(\log\theta)(1/\theta)\mathbf{e}_2 + (1/\theta)\mathbf{e}_3 \\ &= (1/\theta)((\cos\log\theta)\mathbf{e}_1 + 4(\log\theta)\mathbf{e}_2 + \mathbf{e}_3)\end{aligned}$$

(b) $\dfrac{d\mathbf{u}}{d\theta} = \dfrac{d\mathbf{u}}{dt}\dfrac{dt}{d\theta} = ((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)(1/\theta) = e^{-t}((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)$

Another method:

Since $\theta = e^t$, $d\theta/dt = e^t$. Hence

$$\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt}\frac{dt}{d\theta} = \frac{d\mathbf{u}}{dt}\Big/\frac{d\theta}{dt} = e^{-t}((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)$$

2.25. Using the definition of the derivative, show that:
(*a*) If $\mathbf{f}(t) = \mathbf{a}$, $\mathbf{a} = \text{constant}$, then $\mathbf{f}'(t) = \mathbf{0}$.
(*b*) If $\mathbf{f}(t) = \mathbf{a}h(t)$, $\mathbf{a} = \text{constant}$, then $\mathbf{f}'(t) = \mathbf{a}h'(t)$.

(*a*) $$\mathbf{f}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{f}(t+\Delta t) - \mathbf{f}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{a} - \mathbf{a}}{\Delta t} = \lim_{\Delta t \to 0} \mathbf{0} = \mathbf{0}$$

(*b*) $$\mathbf{f}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{f}(t+\Delta t) - \mathbf{f}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{a}h(t+\Delta t) - \mathbf{a}h(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \mathbf{a} \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} = \mathbf{a}h'(t)$$

2.26. Prove Theorem 2.6: If $\mathbf{f}(t)$ is differentiable at t_0, then $\mathbf{f}(t)$ is continuous at t_0.

We consider

$$\lim_{t \to t_0} [\mathbf{f}(t) - \mathbf{f}(t_0)] = \lim_{t \to t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0}(t - t_0) = \lim_{t \to t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} \lim_{t \to t_0} (t - t_0) = [\mathbf{f}'(t_0)]0 = \mathbf{0}$$

and so $\mathbf{f}(t)$ is continuous at t_0.

2.27. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t, show that

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$

We write $\mathbf{w}(t) = \mathbf{u}(t) + \mathbf{v}(t)$. Then

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d\mathbf{w}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{w}(t+\Delta t) - \mathbf{w}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) + \mathbf{v}(t+\Delta t) - \mathbf{u}(t) - \mathbf{v}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) - \mathbf{u}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$

2.28. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t, show that $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}$.

Let $\mathbf{w}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$. Then

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{w}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{w}(t+\Delta t) - \mathbf{w}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) \times \mathbf{v}(t+\Delta t) - \mathbf{u}(t) \times \mathbf{v}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \left[\frac{\mathbf{u}(t+\Delta t) \times (\mathbf{v}(t+\Delta t) - \mathbf{v}(t))}{\Delta t} + \frac{(\mathbf{u}(t+\Delta t) - \mathbf{u}(t)) \times \mathbf{v}(t)}{\Delta t}\right]$$
$$= \lim_{\Delta t \to 0} \mathbf{u}(t+\Delta t) \times \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{u}(t+\Delta t) - \mathbf{u}(t)}{\Delta t} \times \lim_{\Delta t \to 0} \mathbf{v}(t)$$
$$= \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}$$

where we used the fact that $\mathbf{u}(t)$, since it is differentiable, is continuous. Then $\lim_{\Delta t \to 0} \mathbf{u}(t+\Delta t) = \mathbf{u}(t)$, and $\mathbf{v}(t)$ is independent of Δt; hence $\lim_{\Delta t \to 0} \mathbf{v}(t) = \mathbf{v}(t)$. Another method is to find $\mathbf{u}$ and $\mathbf{v}$ in terms of a basis and to differentiate componentwise.

2.29. Show that $\mathbf{u} = \mathbf{a}\cos kt + \mathbf{b}\sin kt$, $\mathbf{a},\mathbf{b}$ = constant, is a solution to $d^2\mathbf{u}/dt^2 = -k^2\mathbf{u}$.

$$d\mathbf{u}/dt \;=\; \mathbf{a}\frac{d}{dt}\cos kt + \mathbf{b}\frac{d}{dt}\sin kt \;=\; -\mathbf{a}k\sin kt + \mathbf{b}k\cos kt$$

$$d^2\mathbf{u}/dt^2 \;=\; -\mathbf{a}k^2\cos kt - \mathbf{b}k^2\sin kt \;=\; -k^2(\mathbf{a}\cos kt + \mathbf{b}\sin kt) \;=\; -k^2\mathbf{u}$$

2.30. Show that $\mathbf{u}\cdot\dfrac{d\mathbf{u}}{dt} = |\mathbf{u}|\dfrac{d|\mathbf{u}|}{dt}$.

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) \;=\; \frac{d}{dt}|\mathbf{u}|^2, \qquad \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{u} \;=\; 2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} \;=\; 2|\mathbf{u}|\frac{d|\mathbf{u}|}{dt}, \quad \text{or} \quad \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} \;=\; |\mathbf{u}|\frac{d|\mathbf{u}|}{dt}$$

2.31. If $\mathbf{f}(t)$ is differentiable at t_0, prove that

$$\mathbf{f}(t_0+\Delta t) \;=\; \mathbf{f}(t_0) + \mathbf{f}'(t_0)\Delta t + \mathbf{R}(t_0,\Delta t)$$

where $(\mathbf{R}(t_0,\Delta t)/\Delta t) \to \mathbf{0}$ as $\Delta t \to 0$.

Define $\mathbf{R} = \mathbf{f}(t_0+\Delta t) - \mathbf{f}(t_0) - \mathbf{f}'(t_0)\,\Delta t$. Then

$$\begin{aligned}\lim_{\Delta t\to 0}\mathbf{R}/\Delta t \;&=\; \lim_{\Delta t\to 0}\,[\mathbf{f}(t_0+\Delta t) - \mathbf{f}(t_0) - \mathbf{f}'(t_0)\,\Delta t]/\Delta t\\ &=\; \lim_{\Delta t\to 0}\left[\frac{\mathbf{f}(t_0+\Delta t)-\mathbf{f}(t_0)}{\Delta t} - \mathbf{f}'(t_0)\right] \;=\; \mathbf{f}'(t_0) - \mathbf{f}'(t_0) \;=\; \mathbf{0}\end{aligned}$$

which is the required result.

2.32. Prove the converse of Problem 2.31. That is, if there exists a linear function $\mathbf{a}\,\Delta t$ of Δt, such that $\mathbf{f}(t_0+\Delta t) = \mathbf{f}(t_0) + \mathbf{a}\,\Delta t + \mathbf{R}$, where $\lim_{\Delta t\to 0}\mathbf{R}/\Delta t = \mathbf{0}$, then $\mathbf{f}(t)$ is differentiable at t_0 and $\mathbf{a} = \mathbf{f}'(t_0)$.

$$\lim_{\Delta t\to 0}\frac{\mathbf{f}(t_0+\Delta t)-\mathbf{f}(t_0)}{\Delta t} \;=\; \lim_{\Delta t\to 0}\frac{\mathbf{a}\,\Delta t + \mathbf{R}}{\Delta t} \;=\; \lim_{\Delta t\to 0}\mathbf{a} + \lim_{\Delta t\to 0}\mathbf{R}/\Delta t \;=\; \mathbf{a}$$

Hence $\mathbf{f}(t)$ is differentiable at t_0 and $\mathbf{f}'(t_0) = \mathbf{a}$.

TAYLOR'S THEOREM AND ANALYTIC FUNCTIONS

2.33. Show that

$$(\sin t)\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 \;=\; \mathbf{e}_1 + \tfrac{1}{4}(\pi^2+4)\mathbf{e}_2 + \pi\mathbf{e}_2(t-\pi/2) + \tfrac{1}{2}(-\mathbf{e}_1+2\mathbf{e}_2)(t-\pi/2)^2 + \mathbf{o}[(t-\pi/2)^2]$$

We find the first three terms of the Taylor expansion of $\mathbf{f}(t) = (\sin t)\mathbf{e}_1 + (t^2+1)\mathbf{e}_2$ about $t=\pi/2$. Namely,

$$\begin{aligned}\mathbf{f}(t) &= (\sin t)\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 & \mathbf{f}(\pi/2) &= \mathbf{e}_1 + \tfrac{1}{4}(\pi^2+4)\mathbf{e}_2\\ \mathbf{f}'(t) &= \cos t\mathbf{e}_1 + 2t\mathbf{e}_2 & \mathbf{f}'(\pi/2) &= \pi\mathbf{e}_2\\ \mathbf{f}''(t) &= -\sin t\mathbf{e}_1 + 2\mathbf{e}_2 & \mathbf{f}''(\pi/2) &= -\mathbf{e}_1 + 2\mathbf{e}_2\end{aligned}$$

Thus $\quad (\sin t)\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 = \mathbf{e}_1 + \tfrac{1}{4}(\pi^2+4)\mathbf{e}_2 + \pi\mathbf{e}_2(t-\pi/2) + \tfrac{1}{2}(-\mathbf{e}_1+2\mathbf{e}_2)(t-\pi/2)^2 + \mathbf{R}$

where $\lim_{t\to\pi/2}\mathbf{R}/(t-\pi/2)^2 = \mathbf{0}$.

2.34. Show that at $t=0$: (a) $t\mathbf{o}(t^2) = \mathbf{o}(t^3)$, (b) $\mathbf{o}(t^2) + \mathbf{o}(t^3) = \mathbf{o}(t^2)$, (c) $\mathbf{o}(t^2)\cdot\mathbf{o}(t^3) = \mathbf{o}(t^5)$.

(a) $\lim_{t\to 0} t\mathbf{o}(t^2)/t^3 \;=\; \lim_{t\to 0}(t/t)(\mathbf{o}(t^2)/t^2) \;=\; \lim_{t\to 0}\mathbf{o}(t^2)/t^2 \;=\; \mathbf{0}$

(b) $\lim_{t\to 0}(\mathbf{o}(t^2)+\mathbf{o}(t^3))/t^2 \;=\; \lim_{t\to 0}\mathbf{o}(t^2)/t^2 + \lim_{t\to 0}t(\mathbf{o}(t^3)/t^3) \;=\; \mathbf{0}$

(c) $\lim_{t\to 0}\mathbf{o}(t^2)\cdot\mathbf{o}(t^3)/t^5 \;=\; \lim_{t\to 0}\mathbf{o}(t^2)/t^2\,\lim_{t\to 0}\mathbf{o}(t^3)/t^3 \;=\; 0$

2.35. If $\mathbf{f}(t)$ is of class C^m on I, show that

$$\mathbf{f}(t) = \mathbf{f}(t_0) + \frac{\mathbf{f}'(t_0)}{1}(t-t_0) + \cdots + \frac{\mathbf{f}^{(m-1)}(t_0)}{(m-1)!}(t-t_0)^{m-1} + \mathbf{O}[(t-t_0)^m]$$

By Taylor's theorem,

$$\mathbf{f}(t) = \mathbf{f}(t_0) + \frac{\mathbf{f}'(t_0)}{1}(t-t_0) + \cdots + \frac{\mathbf{f}^{(m-1)}(t_0)}{(m-1)!}(t-t_0)^{m-1} + \frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{o}[(t-t_0)^m]$$

Hence it remains to show that

$$\frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{o}[(t-t_0)^m] = \mathbf{O}[(t-t_0)^m]$$

But
$$\lim_{t\to t_0}\left\{\left[\frac{\mathbf{f}^{(m)}(t_0)}{m}(t-t_0)^m + \mathbf{o}[(t-t_0)^m]\right]\Big/(t-t_0)^m\right\}$$
$$= \lim_{t\to t_0}\frac{\mathbf{f}^m(t_0)}{m!} + \lim_{t\to t_0}\mathbf{o}[(t-t_0)^m]/(t-t_0)^m = \frac{\mathbf{f}^m(t_0)}{m!}$$

Hence $\left[\frac{\mathbf{f}^{(m)}(t_0)}{m!}(t-t_0)^m + \mathbf{o}[(t-t_0)^m]\right]\Big/(t-t_0)^m$ is bounded, which gives the required result.

2.36. If $\mathbf{f}(t) = \mathbf{o}(g(t))$ at t_0, show that $\mathbf{f}(t) = \mathbf{O}(g(t))$ at t_0.

Since $\mathbf{f}(t) = \mathbf{o}(g(t))$, $\mathbf{f}(t)/g(t) \to 0$ as $t \to t_0$. Hence by Theorem 2.2, $\mathbf{f}(t)/g(t)$ is bounded at t_0. Thus $\mathbf{f}(t) = \mathbf{O}(g(t))$ at t_0.

2.37. If $\mathbf{f}_1(t) = \mathbf{o}(g_1(t))$ and $\mathbf{f}_2(t) = \mathbf{O}(g_2(t))$ at t_0, show that

$$\mathbf{f}_1(t) \times \mathbf{f}_2(t) = \mathbf{o}(g_1(t)g_2(t)) \quad \text{at } t_0$$

or, as we say, $\mathbf{o}(g_1(t)) \times \mathbf{O}(g_1(t)) = \mathbf{o}(g_1(t)g_2(t))$.

Since $\mathbf{f}_1(t)/g_1(t) \to \mathbf{0}$ and $\mathbf{f}_2(t)/g_2(t)$ is bounded at $t = t_0$, it follows from Problem 2.19 that

$$\frac{\mathbf{f}_1(t) \times \mathbf{f}_2(t)}{g_1(t)\,g_2(t)} = \frac{\mathbf{f}_1(t)}{g_1(t)} \times \frac{\mathbf{f}_2(t)}{g_2(t)} \to \mathbf{0}$$

Hence $\mathbf{f}_1(t) \times \mathbf{f}_2(t) = \mathbf{o}(g_1(t)g_2(t))$.

2.38. If $|g_1(t)| \leqq |g_2(t)|$ in some $S_\delta(t_0)$, show that at t_0

$$\mathbf{o}(g_1(t)) + \mathbf{o}(g_2(t)) = \mathbf{o}(g_2(t))$$

Consider

$$\left|\frac{\mathbf{o}(g_1(t)) + \mathbf{o}(g_2(t))}{g_2(t)}\right| \leqq \left|\frac{\mathbf{o}(g_1(t))}{g_2(t)}\right| + \left|\frac{\mathbf{o}(g_2(t))}{g_2(t)}\right| \leqq \left|\frac{\mathbf{o}(g_1(t))}{g_1(t)}\right| + \left|\frac{\mathbf{o}(g_2(t))}{g_2(t)}\right|$$

where we used $|g_1(t)| \leqq |g_2(t)|$ and thus $\left|\frac{\mathbf{o}(g_1(t))}{g_2(t)}\right| \leqq \left|\frac{\mathbf{o}(g_1(t))}{g_1(t)}\right|$. Since $\mathbf{o}(g_1(t))/g_1(t) \to \mathbf{0}$ and $\mathbf{o}(g_2(t))/g_2(t) \to \mathbf{0}$, for a sufficiently small neighborhood of t_0,

$$\left|\frac{\mathbf{o}(g_1(t))}{g_1(t)}\right| < \epsilon/2 \quad \text{and} \quad \left|\frac{\mathbf{o}(g_2(t))}{g_2(t)}\right| < \epsilon/2$$

for arbitrary $\epsilon > 0$. Hence $\left|\frac{\mathbf{o}(g_1(t)) + \mathbf{o}(g_2(t))}{g_2(t)}\right| < \epsilon$ and so

$$\left|\frac{\mathbf{o}(g_1(t)) + \mathbf{o}(g_2(t))}{g_2(t)}\right| \to 0 \quad \text{or} \quad \mathbf{o}(g_1(t)) + \mathbf{o}(g_2(t)) = \mathbf{o}(g_2(t))$$

Supplementary Problems

2.39. Find the equation of the plane through $A(1,0,-1)$, $B(0,0,1)$, $C(-1,-1,0)$.
Ans. $2x_1 - 3x_2 + x_3 = 1$

2.40. Find the equation of the plane through $A(1,-1,0)$ and normal to the line $x_1 = -k+1$, $x_2 = k+1$, $x_3 = 3$. *Ans.* $x_1 - x_2 = 2$

2.41. Find the equations of the line which is the intersection of the planes $3x_1 - 2x_2 + x_3 = 5$ and $2x_1 + 3x_2 - x_3 = -1$. *Ans.* $x_1 = -k+1$, $x_2 = 5k-1$, $x_3 = 13k$, $(-\infty < k < \infty)$

2.42. Prove that the equation of the line through $\mathbf{a}$ and normal to the plane $\mathbf{x}\cdot\mathbf{n} = d$, $|n| \neq 0$, is $\mathbf{x} = k\mathbf{n} + \mathbf{a}$, $-\infty < k < \infty$.

2.43. Prove that the equation of the line through $\mathbf{a}$ and orthogonal to $\mathbf{c}$ and $\mathbf{d}$, $\mathbf{c}\times\mathbf{d} \neq 0$, is $\mathbf{x} = k(\mathbf{c}\times\mathbf{d}) + \mathbf{a}$.

2.44. Find the equation of the cone with vertex at $A(0,1,1)$, with axis parallel to the x_1 axis, and half angle $\theta = 60°$. *Ans.* $3x_1^2 - (x_2-1)^2 - (x_3-1)^2 = 0$

2.45. Calculate the vectors $\mathbf{x} = (t^3+1)\mathbf{e}_1 + (1-t^2)\mathbf{e}_2$ for integer t in $-4 \leq t \leq 4$ and sketch.

2.46. Let $\mathbf{f}(t) = (t^2+1)\mathbf{e}_1 + t^3\mathbf{e}_3$ and $\mathbf{g}(t) = (\sin t)\mathbf{e}_1 - (\cos t)\mathbf{e}_2$. Find (*a*) $\mathbf{f}(a+b)$, (*b*) $\mathbf{g}(t+\Delta t)$, (*c*) $\mathbf{f}(\sin t)\times\mathbf{g}(t^2+1)$.
Ans. (*a*) $(a^2+2ab+b^2+1)\mathbf{e}_1 + (a^3+3a^2b+3b^2a+b^3)\mathbf{e}_3$
(*b*) $\sin(t+\Delta t)\mathbf{e}_1 - \cos(t+\Delta t)\mathbf{e}_2$
(*c*) $(\cos(t^2+1)\sin^3 t)\mathbf{e}_1 + (\sin(t^2+1)\sin^3 t)\mathbf{e}_2 - (\cos(t^2+1)(1+\sin^2 t))\mathbf{e}_3$

2.47. If $\mathbf{a} = 2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, show that $\mathbf{b}$ is in $S_4(\mathbf{a})$ and find $\delta > 0$ such that $S_\delta(\mathbf{b})$ is contained in $S_4(\mathbf{a})$.

2.48. Evaluate $\lim_{t\to -1} [(t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2 + [(t^2-1)/(t+1)]\mathbf{e}_3]$. *Ans.* $2\mathbf{e}_1 + (1/e)\mathbf{e}_2 - 2\mathbf{e}_3$

2.49. Determine the values of t for which $\mathbf{f}(t) = [(t^2+1)/(t^2-1)]\mathbf{e}_1 + (\tan t)\mathbf{e}_2$ is discontinuous.
Ans. $t = 1, -1, \frac{1}{2}\pi \pm n\pi$, $n = 0,1,2,\ldots$

2.50. Let $\mathbf{f}(t) = (t^2-1)\mathbf{e}_2 + (\cos t)\mathbf{e}_3$ and $\mathbf{g}(t) = (\sin t)\mathbf{e}_1 + e^t\mathbf{e}_2$. Find (*a*) $\lim_{t\to 0}(\mathbf{f}(t)\cdot\mathbf{g}(t))$, (*b*) $\lim_{t\to 0}(\mathbf{f}(t)\times\mathbf{g}(t))$. *Ans.* (*a*) -1, (*b*) $-\mathbf{e}_1$

2.51. If $\mathbf{f}(t)$, $\mathbf{g}(t)$ and $\mathbf{h}(t)$ are continuous on I, show that $\mathbf{f}(t)\times(\mathbf{g}(t)\times\mathbf{h}(t))$ is continuous on I.

2.52. If $\mathbf{u} = (t^2+1)\mathbf{e}_1 - te^t\mathbf{e}_2 + (\log t)\mathbf{e}_3$, $t > 0$, find (*a*) $d\mathbf{u}/dt$, (*b*) $d^2\mathbf{u}/dt^2$.
Ans. (*a*) $2t\mathbf{e}_1 - (t+1)e^t\mathbf{e}_2 + (1/t)\mathbf{e}_3$, (*b*) $2\mathbf{e}_1 - (t+2)e^t\mathbf{e}_2 - (1/t^2)\mathbf{e}_3$

2.53. Find the equation of the line tangent to the curve traced by $\mathbf{x} = (t^2-2)\mathbf{e}_1 + (t+3)\mathbf{e}_2 + (t^4+4t+1)\mathbf{e}_3$ at $t = 1$. *Ans.* $\mathbf{x} = (2k-1)\mathbf{e}_1 + (k+4)\mathbf{e}_2 + (8k+6)\mathbf{e}_3$, $-\infty < k < \infty$

2.54. If $\mathbf{u} = (2+t)\mathbf{e}_2 + (\log t)\mathbf{e}_3$ and $\mathbf{v} = (\sin t)\mathbf{e}_1 - (\cos t)\mathbf{e}_2$, $t > 0$, find (*a*) $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})$, (*b*) $\frac{d}{dt}(\mathbf{u}\times\mathbf{v})$.
Ans. (*a*) $(2+t)\sin t - \cos t$
(*b*) $[(1/t)\cos t - \log t\sin t]\mathbf{e}_1 + [(1/t)\sin t + \log t\cos t]\mathbf{e}_2 - [(2+t)\cos t + \sin t]\mathbf{e}_3$

2.55. Let $\mathbf{u} = e^t\mathbf{e}_1 + 2(\sin t)\mathbf{e}_2 + (t^2+1)\mathbf{e}_3$ and $t = \theta^2 + 2$, $t \geq 2$. Find $d\mathbf{u}/d\theta$ and $d^2\mathbf{u}/d\theta^2$ as functions of t.
Ans. $d\mathbf{u}/d\theta = 2(t-2)^{1/2}(e^t\mathbf{e}_1 + 2(\cos t)\mathbf{e}_2 + 2t\mathbf{e}_3)$
$d^2\mathbf{u}/d\theta^2 = (4t-6)e^t\mathbf{e}_1 + [4\cos t - (8t-16)\sin t]\mathbf{e}_2 + (12t-16)\mathbf{e}_3$

2.56. Show that $\frac{d}{dt}\left(\mathbf{u}\cdot\frac{d\mathbf{v}}{dt} - \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\right) = \mathbf{u}\cdot\frac{d^2\mathbf{v}}{dt^2} - \frac{d^2\mathbf{u}}{dt^2}\cdot\mathbf{v}$.

2.57. Find the first three terms of the Taylor expansion of $\mathbf{f}(t) = (\cos t)\mathbf{e}_1 + (t^2+2t+1)\mathbf{e}_2$ about $t = 0$.
Ans. $(\mathbf{e}_1 + \mathbf{e}_2) + 2\mathbf{e}_2 t - \mathbf{e}_1 t^2/2 + t^2\mathbf{e}_2$

2.58. Using the definition of the derivative (Δ process), show that

$$\frac{d}{dt}[(t^2+1)\mathbf{e}_1 + (1/(t+1))\mathbf{e}_2 + \mathbf{e}_3] = 2t\mathbf{e}_1 - (1/(t+1)^2)\mathbf{e}_2$$

2.59. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t, show that $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}$.

2.60. Find all $\mathbf{u}$ such that $d\mathbf{u}/dt = (3t^2+1)\mathbf{e}_1 + t^3\mathbf{e}_2 - (\sin t)\mathbf{e}_3$.
Ans. $\mathbf{u} = (t^3+t+C_1)\mathbf{e}_1 + (t^4/4+C_2)\mathbf{e}_2 + (\cos t + C_3)\mathbf{e}_3$

2.61. Find all $\mathbf{u}$ such that $d^2\mathbf{u}/dt^2 = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$, $\mathbf{a},\mathbf{b},\mathbf{c}$ = constants.
Ans. $\mathbf{u} = \frac{1}{12}\mathbf{a}t^4 + \frac{1}{6}\mathbf{b}t^3 + \frac{1}{2}\mathbf{c}t^2 + \mathbf{C}_1 t + \mathbf{C}_2$

2.62. Show that $\mathbf{u}\times d\mathbf{u}/dt = \mathbf{0}$ if and only if $\mathbf{u}$ has constant direction.

2.63. Show that $(\tan^2 t)\mathbf{e}_1 + (2t^3+t^4)\mathbf{e}_2 = \mathbf{O}(t^2)$ at $t = 0$.

2.64. Let $\mathbf{f}(t)$ be analytic at $t = t_0$ and let

$$\mathbf{f}'(t_0) = \mathbf{0},\quad \mathbf{f}''(t_0) = \mathbf{0},\quad \ldots,\quad \mathbf{f}^{(n)}(t_0) = \mathbf{0},\quad \mathbf{f}^{(n+1)}(t_0) \neq \mathbf{0}$$

Show that $\mathbf{f}^{(n+1)}(t_0)$ represents a vector tangent to the curve $\mathbf{x} = \mathbf{f}(t)$ at $\mathbf{f}(t_0)$.

2.65. If $\mathbf{f}(t) = \begin{cases} e^{-(1/t)^2} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}$, show that $\mathbf{f}^{(n)}(0) = 0$ for all n.

2.66. Show that the vectors $\mathbf{x} = (t^2+1)\mathbf{e}_1 + (t+1)\mathbf{e}_2 - t\mathbf{e}_3$ lie in $S_{1/10}(5\mathbf{e}_1 + 3\mathbf{e}_2 - 2\mathbf{e}_3)$ for all t in $S_{1/100}(2)$.

2.67. If $\lim_{t\to t_0} [f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3] = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$, show that $\lim_{t\to t_0} f_i(t) = L_i$, $i = 1,2,3$.

2.68. If $\mathbf{f}(t)\to\mathbf{L}$ and $\mathbf{g}(t)\to\mathbf{M}$ as $t\to t_0$, show that $\mathbf{f}(t)\cdot\mathbf{g}(t)\to\mathbf{L}\cdot\mathbf{M}$ as $t\to t_0$.

2.69. Show that $[\mathbf{a}t^2 + \mathbf{o}(t^3)]\cdot[\mathbf{b}t + \mathbf{o}(t^2)] = \mathbf{a}\cdot\mathbf{b}t^3 + \mathbf{o}(t^4)$, $\mathbf{a},\mathbf{b}$, = constants.

2.70. Show that $g(t) = \mathrm{O}(g(t))$.

2.71. Show that $\mathbf{o}(g_1(t))\cdot\mathbf{O}(g_2(t)) = \mathbf{o}(g_1(t)g_2(t))$.

2.72. Show that $\mathbf{O}(g_1(t))\times\mathbf{O}(g_2(t)) = \mathbf{O}(g_1(t)g_2(t))$.

2.73. If $\mathbf{f}(t)\to\mathbf{f}(t_0)$ as $t\to t_0$, and $h(\theta)\to t_0$ as $\theta\to\theta_0$, show that $\mathbf{f}(h(\theta))\to\mathbf{f}(t_0)$ as $\theta\to\theta_0$.

2.74. Prove the chain rule: If $\mathbf{u} = \mathbf{f}(t)$ and $t = h(\theta)$ are differentiable functions of t and θ respectively, then $\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt}\frac{dt}{d\theta}$.

Chapter 3

Concept of a Curve

REGULAR REPRESENTATIONS

By a *regular parametric representation* we mean a vector function

$$\mathbf{x} = \mathbf{x}(t), \qquad t \in I \tag{3.1}$$

of t in an interval I with the property that

(i) $\mathbf{x}(t)$ is of class C^1 in I

(ii) $\mathbf{x}'(t) \neq \mathbf{0}$ for all t in I

The variable t is called the *parameter* of the representation.

If a basis is chosen in E^3, the equation $\mathbf{x} = \mathbf{x}(t)$ is equivalent to three scalar equations

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = x_3(t), \qquad t \in I \tag{3.2}$$

the components of $\mathbf{x} = \mathbf{x}(t)$ with respect to the basis. Evidently $\mathbf{x} = \mathbf{x}(t)$ is a regular parametric representation if and only if each $x_i(t)$ is in class C^1 and if for each t in I at least one of the $x_i'(t) \neq 0$.

Example 3.1.

The function $\qquad \mathbf{x} = (t+1)\mathbf{e}_1 + (t^2+3)\mathbf{e}_2, \qquad -\infty < t < \infty$

is a regular parametric representation, since $\mathbf{x}' = \mathbf{e}_1 + 2t\mathbf{e}_2$ is continuous and $\mathbf{x}' \neq \mathbf{0}$ for all t. The image of the function is the parabola shown in Fig. 3-1.

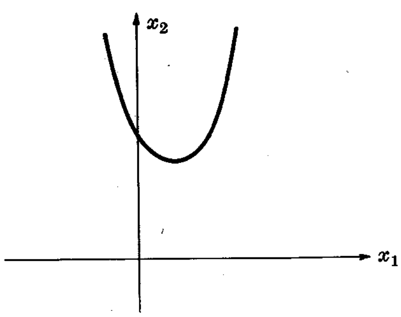


Fig. 3-1

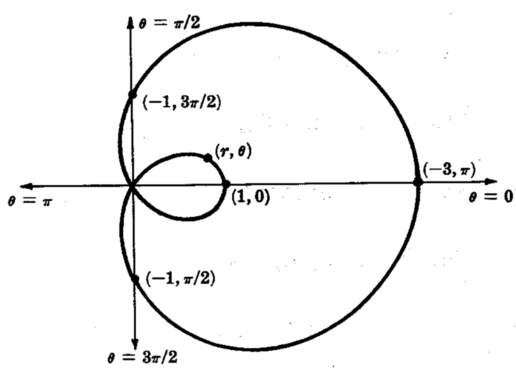


Fig. 3-2

Example 3.2.

The graph of the equation $r = 2\cos\theta - 1$, $0 \leq \theta \leq 2\pi$, in polar coordinates is shown in Fig. 3-2. Polar and rectangular coordinates are related by the equations $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Upon substitution for r, we obtain the representation

$$x_1 = (\cos\theta)(2\cos\theta - 1), \quad x_2 = (\sin\theta)(2\cos\theta - 1), \qquad 0 \leq \theta \leq 2\pi$$

or

$$\mathbf{x} = (\cos\theta)(2\cos\theta - 1)\mathbf{e}_1 + (\sin\theta)(2\cos\theta - 1)\mathbf{e}_2$$

This representation is regular, since

$$\mathbf{x}' = [-4 \sin\theta \cos\theta + \sin\theta]\mathbf{e}_1 + [2\cos^2\theta - 2\sin^2\theta - \cos\theta]\mathbf{e}_2$$

is continuous and it can be computed that $|\mathbf{x}'| = \sqrt{5 - 4\cos\theta} \neq 0$ for all θ and hence $\mathbf{x}' \neq \mathbf{0}$ for all θ.

A regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I can have multiple points, i.e. $t_1 \neq t_2$ in I for which $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. However, *locally* this will not be the case. In Problem 3.7 we prove

Theorem 3.1. If $\mathbf{x} = \mathbf{x}(t)$ is a regular parametric representation on I, then for each t_0 in I there exists a neighborhood of t_0 in which $\mathbf{x}(t)$ is one-to-one.

Example 3.3.

The function $\mathbf{x} = a(\cos\theta)\mathbf{e}_1 + a(\sin\theta)\mathbf{e}_2$, $a \neq 0$, $(-\infty < \theta < \infty)$ is a regular representation of the circle of radius $|a|$ about the origin, since $d\mathbf{x}/d\theta = -a(\sin\theta)\mathbf{e}_1 + a(\cos\theta)\mathbf{e}_2$ is continuous for all θ and

$$|d\mathbf{x}/d\theta| = |-a(\sin\theta)\mathbf{e}_1 + a(\cos\theta)\mathbf{e}_2| = |a| \neq 0$$

Observe that every point on this representation is a multiple point, since for any θ_0,

$$a\cos(\theta_0 + 2\pi)\mathbf{e}_1 + a\sin(\theta_0 + 2\pi)\mathbf{e}_2 = a(\cos\theta_0)\mathbf{e}_1 + a(\sin\theta_0)\mathbf{e}_2$$

However, restricted to, say, the interval $\theta_0 - \frac{1}{2}\pi < \theta < \theta_0 + \frac{1}{2}\pi$, the function $\mathbf{x} = a(\cos\theta)\mathbf{e}_1 + a(\sin\theta)\mathbf{e}_2$ is one-to-one.

Example 3.4.

The function
$$x_1 = t^2, \qquad x_2 = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2 \sin 1/t, & \text{if } t > 0 \end{cases} \qquad -\infty < t < \infty$$

shown in Fig. 3-3 has continuous derivatives for all t; however, at $t = 0$, $dx_1/dt = dx_2/dt = 0$ and hence it is not a regular representation. Note that this function has multiple points in *every* neighborhood of $t = 0$. For consider an arbitrary $\delta > 0$. Select an integer $N > 0$ so that $1/2\pi N < \delta$ and consider $t_1 = -(1/2\pi N)$ and $t_2 = +(1/2\pi N)$. Clearly, $-\delta < t_1 < t_2 < \delta$; moreover,

$$x_1(t_1) = 1/4\pi^2 N^2 = x_1(t_2)$$

and

$$x_2(t_1) = 0 = (1/4\pi^2 N^2)\sin 2\pi N = x_2(t_2)$$

Thus we have a multiple point in $-\delta < t < \delta$.

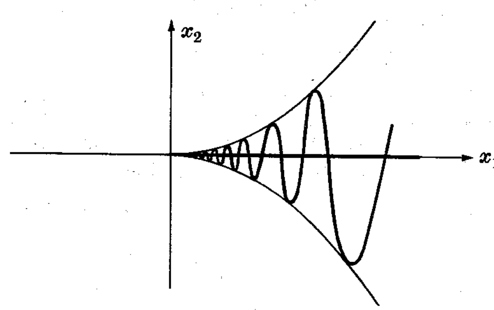


Fig. 3-3

REGULAR CURVES

A real valued function $t = t(\theta)$ on an interval I_θ is an *allowable change of parameter* if

(i) $t(\theta)$ is of class C^1 in I_θ, (ii) $dt/d\theta \neq 0$ for all θ in I_θ

Note that if $t = t(\theta)$ is an allowable change of parameter on I_θ, then $dt/d\theta$ is continuous and $dt/d\theta \neq 0$. Hence either $dt/d\theta > 0$ on I_θ, in which case $t(\theta)$ is a smooth increasing function, or $dt/d\theta < 0$ on I_θ and $t(\theta)$ is a smooth decreasing function. This will enable us to prove (Problem 3.13)

Theorem 3.2. If $t = t(\theta)$ is an allowable change of parameter on I_θ, then

(i) $t = t(\theta)$ is a one-to-one mapping of I_θ onto an interval $I_t = t(I_\theta)$.

(ii) The inverse function $\theta = \theta(t)$ is an allowable change of parameter on I_t.

Example 3.5.

(a) The function $t = (b-a)\theta + a$, $0 \leqq \theta \leqq 1$, $a < b$, is an allowable change of parameter which takes the interval $0 \leqq \theta \leqq 1$ onto $a \leqq t \leqq b$. The inverse $\theta = (t-a)/(b-a)$ is an allowable change of parameter which takes $a \leqq t \leqq b$ onto $0 \leqq \theta \leqq 1$.

(b) The function $t = \tan(\pi\theta/2)$, $0 \leqq \theta < 1$, is an allowable change of parameter which takes $0 \leqq \theta < 1$ onto $0 \leqq t < \infty$. The inverse is $\theta = (2/\pi)\,\mathrm{Tan}^{-1}\,t$ which takes $0 \leqq t < \infty$ onto $0 \leqq \theta < 1$.

A regular parametric representation $\mathbf{x} = \mathbf{x}(t)$, $t \in I_t$, is *equivalent* to a regular parametric representation $\mathbf{x} = \mathbf{x}^*(\theta)$, $\theta \in I_\theta$, if there exists an allowable change of parameter $t = t(\theta)$ on I_θ such that

$$\text{(i)}\ t(I_\theta) = I_t, \qquad \text{(ii)}\ \mathbf{x}(t(\theta)) = \mathbf{x}^*(\theta)$$

In Problem 3.14 we show that this defines an equivalence relation on the set of regular representations. *We define a regular curve to be an equivalence class of regular parametric representations.*

Note that a representation $\mathbf{x} = \mathbf{x}(t)$ uniquely determines a curve C consisting of all representations related to it by an allowable change of parameter. Thus we may say "the curve C given by $\mathbf{x} = \mathbf{x}(t)\ldots$". However, a property of $\mathbf{x} = \mathbf{x}(t)$ may not necessarily be a property of the curve. Any property of the curve must be common to all representations or, as we say, "independent of the parameter."

Example 3.6.

Suppose we introduce the allowable change in parameter $\theta = t+1$, $-1 \leqq t \leqq 2\pi - 1$, in the representation

$$\mathbf{x} = (\cos\theta)(2\cos\theta - 1)\mathbf{e}_1 + (\sin\theta)(2\cos\theta - 1)\mathbf{e}_2, \qquad 0 \leqq \theta \leqq 2\pi$$

of Example 3.2. This gives the equivalent parametric representation

$$\mathbf{x} = [\cos(t+1)][2\cos(t+1) - 1]\mathbf{e}_1 + [\sin(t+1)][2\cos(t+1) - 1]\mathbf{e}_2, \qquad -1 \leqq t \leqq 2\pi - 1$$

As t increases through the interval $-1 \leqq t \leqq 2\pi - 1$, the quantity $\theta = t+1$ increases smoothly through $0 \leqq \theta \leqq 2\pi$, and the transformed equation traces the same set of points in the same direction as before, as shown in Fig. 3-4(a). If we introduce the allowable change in parameter $\theta = -t$, $-2\pi \leqq t \leqq 0$, we obtain the equivalent representation

$$\mathbf{x} = (\cos t)(2\cos t - 1)\mathbf{e}_1 - (\sin t)(2\cos t - 1)\mathbf{e}_2, \qquad -2\pi \leqq t \leqq 0$$

Here as t increases through $-2\pi \leqq t \leqq 0$, the quantity $\theta = -t$ decreases through the interval $0 \leqq \theta \leqq 2\pi$ and the set of points is traced in the opposite direction or *sense* as shown in Fig. 3-4(b). Thus the sense in which a curve is traced is a property of the representation and not of the curve. If we introduce the change of parameter

$$\theta = \theta(t) = \begin{cases} t, & \text{for } 0 \leqq t \leqq \pi/3 \\ -t + 2\pi, & \text{for } \pi/3 < t < 5\pi/3 \\ t, & \text{for } 5\pi/3 \leqq t \leqq 2\pi \end{cases}$$

which we note is not an allowable change in parameter, the transformed equation

$$\mathbf{x} = [\cos\theta(t)](2\cos\theta(t) - 1)\mathbf{e}_1 + [\sin\theta(t)](2\cos\theta(t) - 1)\mathbf{e}_2, \qquad 0 \leqq t \leqq 2\pi$$

will trace the same set of points but in the direction shown in Fig. 3-4(c). With our definition this curve is *not* the same as the one above. Thus a curve should be thought of not simply as a set of points in E^3 but as a general method of traversing the set of points specified by a collection of equivalent parametric representations.

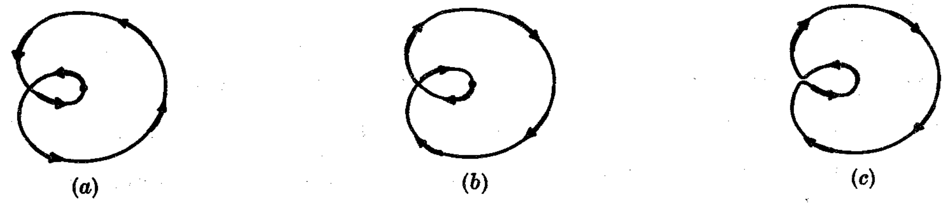


Fig. 3-4

Example 3.7.

An important example of a space curve is the circular helix

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3, \quad a, b \neq 0, \quad -\infty < t < \infty$$

or

$$x_1 = a\cos t, \quad x_2 = a\sin t, \quad x_3 = bt, \quad a, b \neq 0, \quad -\infty < t < \infty$$

shown in Fig. 3-5. The curve lies on the right circular cylinder of radius $|a|$: $x_1 = a\cos t$, $x_2 = a\sin t$, $-\infty < x_3 < \infty$. The equation $x_3 = bt$ "moves" the points of the curve uniformly in the x_3 direction. When t increases by 2π, then x_1 and x_2 return to their original values, while x_3 increases $(b > 0)$ or decreases $(b < 0)$ by $2\pi|b|$, the *pitch* of the helix.

Example 3.8.

It is interesting to note that the unit interval $0 \leq t \leq 1$ can be mapped *continuously onto* the unit square Q: $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$ in E^2, giving a "curve" which fills a two dimensional region. Such a mapping, called a Peano curve, is constructed as follows. We divide Q into four equal squares which together with their boundaries are denoted by Q_0, Q_1, Q_2, Q_3. Suppose further that each of the Q_i is again subdivided into four equal squares $Q_{i0}, Q_{i1}, Q_{i2}, Q_{i3}$, and each of the latter subdivided again, etc. We also suppose the squares are indexed so that if we pass through the squares in the order of increasing subscripts, we obtain an arc which does not cross itself, as shown in Fig. 3-6. That this can be done is left to the reader as an exercise.

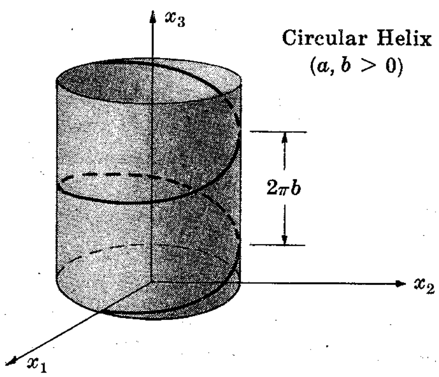


Fig. 3-5

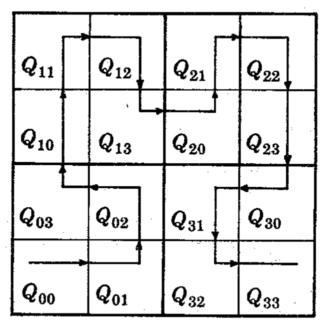


Fig. 3-6

Every t_0 in $0 \leq t \leq 1$ can be expressed uniquely as an infinite decimal:

$$t_0 = .a_1a_2a_3\ldots = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$$

The same is true with base four, that is, each t_0 can be expressed uniquely as a series

$$t_0 = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \cdots$$

with integers $0 \leq a_i \leq 3$. $\left(\text{Uniqueness is obtained by omitting series ending in .3's, i.e. } \frac{1}{4} + \frac{1}{4^2} + \frac{3}{4^3} + \frac{3}{4^4} + \cdots = \frac{1}{4} + \frac{2}{4^2}.\right)$

To each $t_0 = \sum_i a_i/4^i$ we now assign the unique point P_0 in Q which is common to the infinite sequence of closed nested squares $Q_{a_1}, Q_{a_1a_2}, Q_{a_1a_2a_3}, \ldots$. This mapping is onto Q, for it can be shown that every P in Q will be a point common to an allowable sequence of nested intervals.

Finally, this mapping is continuous. For let $S_\epsilon(P_0)$ be an arbitrary neighborhood of P_0. As shown in Fig. 3-7, we choose $Q_{a_1a_2\ldots a_n}$ containing P_0 so small that it and its adjacent squares of the same size

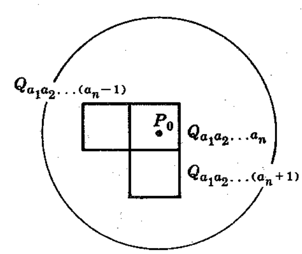


Fig. 3-7

are contained in $S_\epsilon(P_0)$. But then all t in the following open interval containing t_0

$$\frac{a_1}{4} + \frac{a_2}{4^2} + \cdots + \frac{a_n - 1}{4^n} < t < \frac{a_1}{4} + \frac{a_2}{4^2} + \cdots + \frac{a_n + 1}{4^n}$$

will be mapped into $S_\epsilon(P_0)$. Hence the mapping is continuous.

This mapping is not one-to-one since points on the boundaries of the squares are common to more than one sequence of nested squares. In fact it can be shown that no continuous mapping of the line onto the square can be one-to-one.

A regular curve $\mathbf{x} = \mathbf{x}(t)$, $t \in I$, is said to be *simple* if there are no multiple points; that is, if $t_1 \neq t_2$ implies $\mathbf{x}(t_1) \neq \mathbf{x}(t_2)$. This is clearly a property of the curve, not of the representation.

A regular curve $\mathbf{x} = \mathbf{x}(t)$, $t \in I$, is called a *regular arc* if I is a closed interval $a \leq t \leq b$. The points $\mathbf{x}(a)$ and $\mathbf{x}(b)$ are called the *end points* of the arc. An *arc segment* of a curve $\mathbf{x} = \mathbf{x}(t)$ on I is an arc $\mathbf{x} = \mathbf{x}^*(t)$, $t \in I^*$, where I^* is any closed interval contained in I and $\mathbf{x}^*(t)$ is the restriction of $\mathbf{x}(t)$ to I^*.

Example 3.9.

The curve in Example 3.2 is a regular arc, since

$$x_1 = (\cos\theta)(2\cos\theta - 1), \quad x_2 = (\sin\theta)(2\cos\theta - 1), \qquad 0 \leq \theta \leq 2\pi$$

is a regular representation on the closed interval $0 \leq \theta \leq 2\pi$. Observe that here the end points of the curve are equal. The part of the curve restricted to, say, the interval $0 \leq \theta \leq \pi$ is a simple arc segment of the curve and is shown in Fig. 3-8.

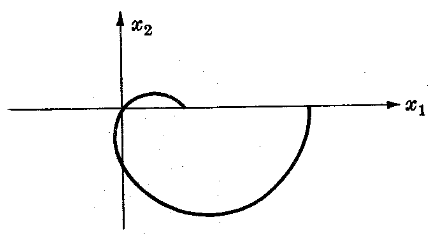


Fig. 3-8

Finally, let $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ be two representations of a regular curve. If $dt/d\theta > 0$, then t increases with increasing θ, and $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ trace the curve in the same direction. If $dt/d\theta < 0$, then t decreases with increasing θ, and $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ will trace the curve in opposite directions. A *regular oriented* curve is a curve along which a specific direction is chosen. That is, a regular oriented curve is a collection of regular parametric representations any two of which are related by an allowable change of parameter having a positive derivative.

ORTHOGONAL PROJECTIONS

Let $\mathbf{x} = \mathbf{x}(t)$ represent a curve C, as shown in Fig. 3-9. For a fixed t_0 the equation

$$\mathbf{x} = x_1(t_0)\mathbf{e}_1 + x_2(t_0)\mathbf{e}_2 + k\mathbf{e}_3, \qquad -\infty < k < \infty$$

or

$$x_1 = x_1(t_0), \quad x_2 = x_2(t_0), \quad x_3 = k, \qquad -\infty < k < \infty$$

is the equation of the line orthogonal to the x_1x_2 plane and passing through the point $\mathbf{x}(t_0)$. It follows that the family of lines

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = k, \qquad -\infty < k < \infty \tag{3.3}$$

generate a cylindrical surface orthogonal to the x_1x_2 plane and containing the curve C.

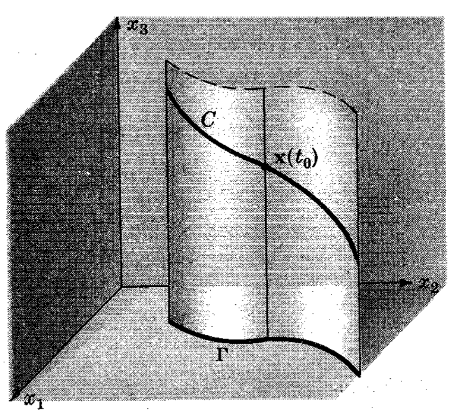


Fig. 3-9

The intersection of the cylinder (*3.3*) with the x_1x_2 plane, $x_3 = 0$, is the *orthogonal projection* Γ of $\mathbf{x} = \mathbf{x}(t)$ onto the x_1x_2 plane. Hence the orthogonal projection of $\mathbf{x} = \mathbf{x}(t)$ is given by

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = 0$$

The orthogonal projections of $\mathbf{x} = \mathbf{x}(t)$ onto the x_2x_3 and x_1x_3 planes are respectively

$$x_1 = 0, \quad x_2 = x_2(t), \quad x_3 = x_3(t)$$

and

$$x_1 = x_1(t), \quad x_2 = 0, \quad x_3 = x_3(t)$$

Example 3.10.

The orthogonal projection of the space curve $x_1 = t$, $x_2 = t^2$, $x_3 = t^3$, $-\infty < t < \infty$, onto the x_1x_2 plane is the parabola $x_1 = t$, $x_2 = t^2$, $x_3 = 0$. The projection onto the x_1x_3 plane is the cubic $x_1 = t$, $x_2 = 0$, $x_3 = t^3$. The curve itself is the intersection of the two cylinders

$$x_1 = t,\ x_2 = t^2,\ -\infty < x_3 < \infty \qquad \text{and} \qquad x_1 = t,\ x_3 = t^3,\ -\infty < x_2 < \infty$$

as shown in Fig. 3-10.

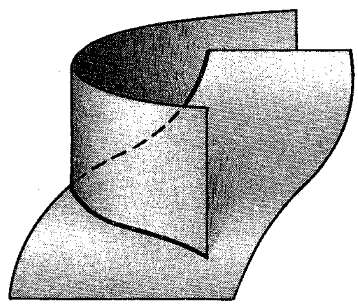


Fig. 3-10

IMPLICIT REPRESENTATIONS OF CURVES

A curve in space can be determined as the intersection of two surfaces, i.e. as those points (x_1, x_2, x_3) satisfying two relations of the form

$$F_1(x_1, x_2, x_3) = 0 \quad \text{and} \quad F_2(x_1, x_2, x_3) = 0 \tag{3.4}$$

If at a point (x_1, x_2, x_3) satisfying the above,

$$\det\begin{pmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 \\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 \end{pmatrix} \neq 0$$

then it follows from the implicit function theorem that for some neighborhood of x_3 we can solve (*3.4*) for x_1 and x_2 as functions of x_3, obtaining a representation of the form

$$x_1 = x_1(x_3), \quad x_2 = x_2(x_3), \quad x_3 = x_3$$

with x_3 itself the parameter. This defines at least locally a regular curve.

Example 3.11.

The intersection of the two second degree surfaces $x_2 - x_3^2 = 0$ and $x_3x_1 - x_2^2 = 0$ is the third degree curve $x_1 = t^3$, $x_2 = t^2$, $x_3 = t$ together with the x_1 axis, $x_1 = t$, $x_2 = 0$, $x_3 = 0$. These are obtained as follows. For $x_3 \neq 0$ we can solve the given relations for x_1 and x_2 in terms of x_3, obtaining

$$x_2 = x_3^2, \quad x_1 = x_2^2/x_3 = x_3^4/x_3 = x_3^3$$

or, if we let $x_3 = t$,

$$x_1 = t^3, \quad x_2 = t^2, \quad x_3 = t$$

If $x_3 = 0$, then $x_2 = x_3^2 = 0$ and x_1 can be arbitrary. This gives the x_1 axis, $x_1 = t$, $x_2 = 0$, $x_3 = 0$. Observe that the point $(0, 0, 0)$ is the intersection of the two curves.

REGULAR CURVES OF CLASS C^m

We define a regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I to be a *regular parametric representation of class* C^m ($m \geq 1$) if $\mathbf{x}(t)$ is of class C^m on I. Similarly, an allowable change of parameter $t = t(\theta)$ on I_θ shall be called an *allowable change of parameter of class* C^m if $t(\theta)$ is of class C^m on I_θ. Finally, two regular representations of class C^m define the same *regular curve of class* C^m if they are related by an allowable change of parameter of class C^m. Thus a regular curve of class C^m is a collection of representations of class C^m any two of which are related by an allowable change of parameter of class C^m.

Although a representation $\mathbf{x} = \mathbf{x}(t)$ of class C^m is also of class C^j for all $j \leq m$, the *curve* $\mathbf{x} = \mathbf{x}(t)$ of class C^m is not a curve of class C^j for $j < m$, since the curve $\mathbf{x} = \mathbf{x}(t)$ of class C^m contains only representations related to $\mathbf{x} = \mathbf{x}(t)$ by allowable changes of parameter of class C^m, whereas the curve $\mathbf{x} = \mathbf{x}(t)$ of class C^j for $j < m$ contains in addition representations related to $\mathbf{x} = \mathbf{x}(t)$ by allowable changes of parameter which are, say, of class C^j but not of class C^m.

Example 3.12.

The vector function $\mathbf{w}(t) = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $-\infty < t < \infty$, is analytic. Thus the helix $\mathbf{x} = \mathbf{w}(t)$ can be considered as a regular analytic curve provided we consider only those representations which are related to it by an analytic change in parameter.

Example 3.13.

The representation

$$\mathbf{x} = \begin{cases} t\mathbf{e}_1 + e^{-1/t^2}\,\mathbf{e}_3 & \text{for } t < 0 \\ 0 & \text{for } t = 0 \\ t\mathbf{e}_1 + e^{-1/t^2}\,\mathbf{e}_2 & \text{for } t > 0 \end{cases}$$

is of class C^∞ (see Example 2.30, page 31), and together with all related representations of class C^∞ defines the curve of class C^∞ shown in Fig. 3-11. Observe that for all $t < 0$ the curve lies in the x_1x_3 plane, and for all $t > 0$ the curve lies in the x_1x_2 plane.

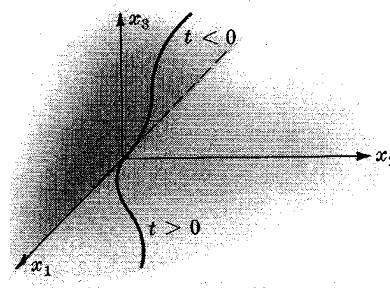


Fig. 3-11

DEFINITION OF ARC LENGTH

The length of an arc is defined in terms of the lengths of approximating polygonal arcs. Namely, let an arc C, not necessarily regular, be given by $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, and consider a subdivision

$$a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $a \leq t \leq b$. This determines a sequence of points in E^3

$$\mathbf{x}_0 = \mathbf{x}(t_0), \quad \mathbf{x}_1 = \mathbf{x}(t_1), \quad \ldots, \quad \mathbf{x}_n = \mathbf{x}(t_n)$$

which are joined in sequence to form an *approximating polygonal arc* P as shown in Fig. 3-12. The length of the line between two adjacent points $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ is $|\mathbf{x}_i - \mathbf{x}_{i-1}|$. Hence the length of P is

$$s(P) = \sum_{i=1}^{n} |\mathbf{x}_i - \mathbf{x}_{i-1}| = \sum_{i=1}^{n} |\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})| \qquad (3.5)$$

Now suppose we introduce a better approximating polygonal arc P' by introducing additional points as shown. Since the length of one side of a polygon is less than or equal to the sum of the lengths of the other sides, it follows that the length of P is less than or equal to the length of P', i.e. $s(P) \leq s(P')$.

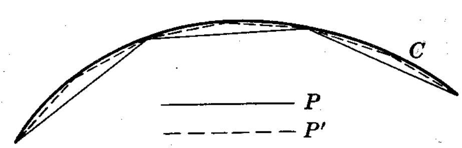


Fig. 3-12

Thus we are led to define the length of the arc C as the greatest of lengths of all possible approximating polygonal arcs P. Specifically, an arc $\mathbf{x} = \mathbf{x}(t)$, $a \leqq t \leqq b$, is said to be *rectifiable* if the set S of all possible $s(P)$ is bounded from above. In this case the set S has a *supremum* which is defined to be the *length* of the arc.

A set of S of real numbers is said to be bounded from above if there is a real number M such that $x \leqq M$ for all x in S. In this case the number M is called an upper bound for S. Note that if M is an upper bound of S, then any L such that $M \leqq L$ is also an upper bound. One of the basic properties of the real numbers is that if S has an upper bound M, then it has a *least upper bound* or *supremum,* that is, an upper bound s such that if L is any upper bound, then $L \geqq s$.

Note that the length of an arc C is independent of the parameter. For let $\mathbf{x} = \mathbf{x}(t)$ on I_t and $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ be two representations of C such that $t = t(\theta)$ is one-to-one. To every subdivision $\theta_0 < \theta_1 < \cdots < \theta_n$ of I_θ there corresponds a unique subdivision $t_0 < t_1 < \cdots < t_n$ of I_t, or, depending upon orientation, $t_n < t_{n-1} < \cdots < t_0$, where $t_i = t_i(\theta)$, $i = 1, \ldots, n$, which gives the same polygonal arc P, and conversely. Thus the set S of the lengths of all approximating polygonal arcs is independent of the parameter and hence so is the supremum of S, which is the length of C.

Example 3.14.

The arc $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2$, $0 \leqq t \leqq 1$, is rectifiable. For consider a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$. The length of the approximating polygonal arc is

$$\begin{aligned} s(P) &= \sum_{i=1}^{n} |(t_i\mathbf{e}_1 + t_i^2\mathbf{e}_2) - (t_{i-1}\mathbf{e}_1 + t_{i-1}^2\mathbf{e}_2)| \\ &= \sum_i |(t_i - t_{i-1})\mathbf{e}_1 + (t_i^2 - t_{i-1}^2)\mathbf{e}_2| \\ &\leqq \sum_i [\,|t_i - t_{i-1}|\,|\mathbf{e}_1| + |t_i^2 - t_{i-1}^2|\,|\mathbf{e}_2|\,] \\ &\leqq \sum_i [(t_i - t_{i-1}) + (t_i - t_{i-1})(t_i + t_{i-1})] \\ &\leqq \sum_i (t_i - t_{i-1})(1 + t_i + t_{i-1}) \\ &\leqq 3\sum_i (t_i - t_{i-1}) \;=\; 3 \end{aligned}$$

where we used the fact that for $0 \leqq t_{i-1} < t_i \leqq 1$, $1 + t_{i-1} + t_i \leqq 3$, and $\sum_i (t_i - t_{i-1}) = t_n - t_0 = 1$.

Thus for any P the quantity $s(P)$ is bounded by 3. Hence the arc is rectifiable and has a length equal to the supremum of the $s(P)$.

Example 3.15.

The curve

$$x_1 = t$$
$$x_2 = \begin{cases} t\cos(1/t) & \text{for } 0 < t \leqq 1 \\ 0 & \text{for } t = 0 \end{cases} \qquad (0 \leqq t \leqq 1)$$

shown in Fig. 3-13 below is not rectifiable. For, using the subdivision 0, $1/(N-1)\pi$, ..., $1/2\pi$, $1/\pi$, 1, we have

$$\begin{aligned} s(P) = {} & \left| \frac{1}{(N-1)\pi}\mathbf{e}_1 + \frac{1}{(N-1)\pi}[\cos(N-1)\pi]\mathbf{e}_2 \right| \\ & + \left| \left[\frac{1}{(N-2)\pi} - \frac{1}{(N-1)\pi}\right]\mathbf{e}_1 + \left[\frac{1}{(N-2)\pi}\cos(N-2)\pi - \frac{1}{(N-1)\pi}\cos(N-1)\pi\right]\mathbf{e}_2 \right| \\ & + \cdots + \left| \left[1 - \frac{1}{\pi}\right]\mathbf{e}_1 + \left[\cos 1 - \frac{1}{\pi}\cos\pi\right]\mathbf{e}_2 \right| \end{aligned}$$

If we drop the first and last terms, then

$$s(P) \geq \sum_{n=1}^{N-2} \left| \left[\frac{1}{n\pi} - \frac{1}{(n+1)\pi}\right]\mathbf{e}_1 + \left[\frac{1}{n\pi}\cos n\pi - \frac{1}{n+1}\cos(n+1)\pi\right]\mathbf{e}_2 \right|$$

$$\geq \sum_{n=1}^{N-2} \left| \left[\frac{1}{n\pi}\cos n\pi - \frac{1}{n+1}\cos(n+1)\pi\right]\mathbf{e}_2 \right|$$

$$= \sum_{n=1}^{N-2} \left| (-1)^n \frac{1}{n\pi} - (-1)^{n+1}\frac{1}{(n+1)\pi} \right|$$

$$= \sum_{n=1}^{N-2} \left| \frac{1}{n\pi} + \frac{1}{(n+1)\pi} \right| \geq \frac{2}{\pi}\sum_{n=1}^{N-2}\frac{1}{n+1}$$

where in going from the first to the second line we have used the inequality $|a\mathbf{e}_1 + b\mathbf{e}_2| \geq |b\mathbf{e}_2|$. But the sum $\sum_{n=1}^{N-2}\frac{1}{n+1}$ diverges to infinity. That is, $s(P)$ can be made arbitrarily large by making N sufficiently large. Thus the curve is not rectifiable.

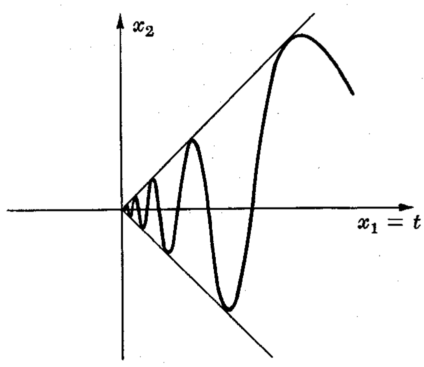


Fig. 3-13

In Problem 3.24 we prove

Theorem 3.3. A regular arc $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, is rectifiable and its length is given by the integral

$$s = \int_a^b \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2}\, dt \qquad (3.6)$$

Example 3.16.

The length of the arc segment of the helix $\mathbf{x} = (a\cos t)\mathbf{e}_1 + (a\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $0 \leq t \leq 2\pi$, is

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\, dt = \int_0^{2\pi} (a^2+b^2)^{1/2}\, dt = 2\pi(a^2+b^2)^{1/2}$$

ARC LENGTH AS A PARAMETER

Let $\mathbf{x} = \mathbf{x}(t)$ be a regular curve on I, and consider the function

$$s = s(t) = \int_{t_0}^{t} \left|\frac{d\mathbf{x}}{dt}\right| dt \qquad (3.7)$$

If $t \geq t_0$, then $s \geq 0$ and is equal to the length of the arc segment of the curve between $\mathbf{x}(t_0)$ and $\mathbf{x}(t)$. If $t < t_0$, then $s < 0$ and is equal to minus the length of the arc segment between $\mathbf{x}(t_0)$ and $\mathbf{x}(t)$.

Now, it follows from the Fundamental Theorem of Calculus that (3.7) has a continuous nonvanishing derivative given by

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t}\left|\frac{d\mathbf{x}}{dt}\right| dt = \left|\frac{d\mathbf{x}}{dt}\right|$$

Hence $s = s(t)$ is an allowable change of parameter on I. Also $s(t)$ is of class C^m on I if $\mathbf{x}(t)$ is of class C^m. Thus the arc length s can be introduced along the curve as a parameter.

Note that a representation in terms of arc length is not unique, for it depends upon the chosen initial point t_0 (where $s = 0$) and on orientation; that is, we could take

$$s(t) = \int_{t}^{t_0}\left|\frac{d\mathbf{x}}{dt}\right| dt = -\int_{t_0}^{t}\left|\frac{d\mathbf{x}}{dt}\right| dt$$

To be precise, therefore, we define a representation $\mathbf{x} = \mathbf{x}(s)$ on I_s to be a *representation in terms of arc length* or a *natural representation* if $|d\mathbf{x}/ds| = 1$. From this we prove (Problems 3.19 and 3.20)

Theorem 3.4. If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of a curve C, then

(i) $|s_2 - s_1|$ is the length of the arc segment of C between $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$.

(ii) If $\mathbf{x} = \mathbf{x}^*(s^*)$ is any other natural representation of C, then $s = \pm s^* +$ constant.

(iii) If $\mathbf{x} = \mathbf{x}^*(t)$ is any representation of C of the same orientation as $\mathbf{x} = \mathbf{x}(s)$, then $ds/dt = |d\mathbf{x}/dt|$. Otherwise $ds/dt = -|d\mathbf{x}/dt|$.

Note finally that if $s = s(t)$ is defined by the integral in *(3.7)*, then $\mathbf{x} = \mathbf{x}(t(s))$ is a natural representation since

$$\left|\frac{d\mathbf{x}}{ds}\right| = \left|\frac{d\mathbf{x}}{dt}\right|\left|\frac{dt}{ds}\right| = \left|\frac{d\mathbf{x}}{dt}\right| \Big/ \left|\frac{ds}{dt}\right| = \left|\frac{d\mathbf{x}}{dt}\right| \Big/ \left|\frac{d\mathbf{x}}{dt}\right| = 1$$

Example 3.17.

To obtain a natural representation of the helix

$$\mathbf{x} = (a\cos t)\mathbf{e}_1 + (a\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$$

we consider

$$s = \int_0^t \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_0^t (a^2+b^2)^{1/2}\,dt = (a^2+b^2)^{1/2}t$$

If we substitute $t = (a^2+b^2)^{-1/2}s$ into the above, we obtain the natural representation

$$\mathbf{x} = a\cos[(a^2+b^2)^{-1/2}s]\mathbf{e}_1 + a\sin[(a^2+b^2)^{-1/2}s]\mathbf{e}_2 + b(a^2+b^2)^{-1/2}s\mathbf{e}_3$$

Unless otherwise stated, differentiation with respect to a natural parameter s will be denoted by dots and differentiation with respect to any other parameter will be denoted by primes; for example,

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds}, \quad \ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{ds^2}, \quad \mathbf{x}' = \frac{d\mathbf{x}}{dt}, \quad \mathbf{x}'' = \frac{d^2\mathbf{x}}{dt^2}, \text{ etc.}$$

Solved Problems

REGULAR REPRESENTATIONS

3.1. Show that $\mathbf{x} = t\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 + (t-1)^3\mathbf{e}_3$ is a regular parametric representation for all t and find the projections onto the x_1x_2 and x_1x_3 planes.

$d\mathbf{x}/dt = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3(t-1)^2\mathbf{e}_3$ is continuous and $|d\mathbf{x}/dt| = [1 + 4t^2 + 9(t-1)^4]^{1/2} \neq 0$ for all t. Hence $\mathbf{x}$ is regular for all t. The projection onto the x_1x_2 plane is the parabola $x_1 = t$, $x_2 = t^2+1$, $x_3 = 0$, or $x_2 = x_1^2+1$, $x_3 = 0$. The projection onto the x_1x_3 plane is the cubic $x_1 = t$, $x_3 = (t-1)^3$, $x_2 = 0$, or $x_3 = (x_1-1)^3$, $x_2 = 0$. The curve is the intersection of the cylinders $x_2 = x_1^2+1$ and $x_3 = (x_1-1)^3$.

3.2. Show that the representation $x_1 = (1+\cos\theta)$, $x_2 = \sin\theta$, $x_3 = 2\sin(\theta/2)$, $-2\pi \leqq \theta \leqq 2\pi$, is regular and lies on the sphere of radius 2 about the origin and the cylinder $(x_1-1)^2 + x_2^2 = 1$.

$dx_1/d\theta = -\sin\theta,\ dx_2/d\theta = \cos\theta,\ dx_3/d\theta = \cos(\theta/2)$ are continuous and

$$\left[\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dx_3}{d\theta}\right)^2\right]^{1/2} = [1+\cos^2(\theta/2)]^{1/2} \neq 0$$

Hence the representation is regular. Since

$$x_1^2 + x_2^2 + x_3^2 = (1+\cos\theta)^2 + \sin^2\theta + 4\sin^2(\theta/2) = (1+\cos\theta)^2 + \sin^2\theta + 2(1-\cos\theta) = 4$$

and $(x_1-1)^2 + x_2^2 = \cos^2\theta + \sin^2\theta = 1$, the curve lies on the sphere of radius 2 and the circular cylinder $(x_1-1)^2 + x_2^2 = 1$. It is the intersection of these surfaces as shown in Fig. 3-14.

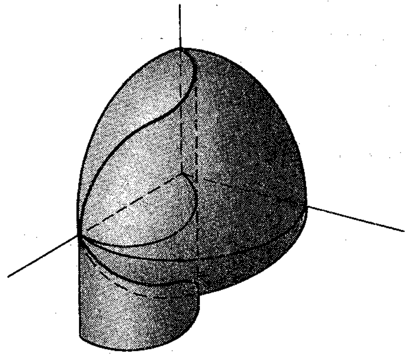


Fig. 3-14

3.3. The equation of the *cissoid of Diocles* in polar coordinates is $r = 2\sin\theta\tan\theta$, $-\pi/2 < \theta < \pi/2$. Sketch the curve and find a parametric representation in rectangular coordinates.

θ	$2\sin\theta$	$\tan\theta$	r
$-\pi/2^+$	-2	$-\infty$	∞
$-\pi/4$	$-\sqrt{2}$	-1	$\sqrt{2}$
0	0	0	0
$\pi/4$	$\sqrt{2}$	1	$\sqrt{2}$
$\pi/2^-$	2	$+\infty$	∞

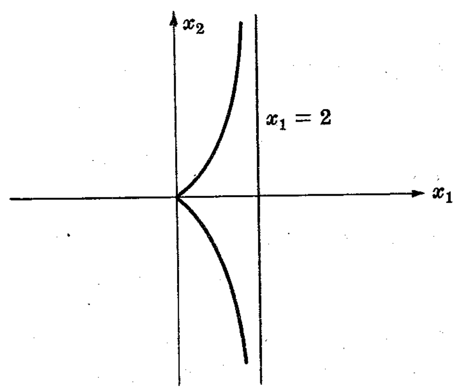


Fig. 3-15

Since $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, we have the parametric representation

$$x_1 = 2\sin^2\theta, \quad x_2 = 2\sin^2\theta\tan\theta, \qquad -\pi/2 < \theta < \pi/2$$

Observe that $x_1 \to 2$ as $\theta \to -\pi/2$ or as $\theta \to \pi/2$. Note that the representation above is not regular at $\theta = 0$, since both

$$dx_1/d\theta = 4\sin\theta\cos\theta \quad \text{and} \quad dx_2/d\theta = 2\sin^2\theta\sec^2\theta + 4\sin\theta\cos\theta\tan\theta$$

vanish at $\theta = 0$.

3.4. The *epicycloid* is a plane curve generated by a point P on the circumference of a circle C as C rolls without sliding on the exterior of a fixed circle C_0 as shown in Fig. 3-16. Find a parametric representation of the epicycloid when C has radius r, C_0 is at the origin with radius r_0, and P is initially located at $(r_0, 0)$.

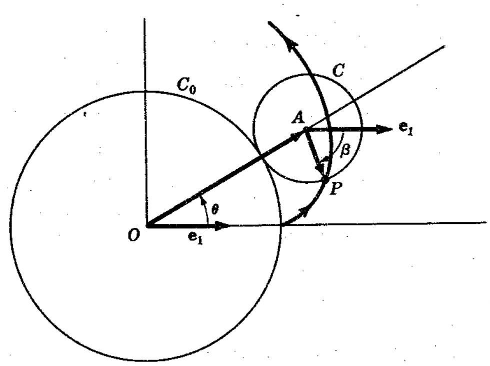


Fig. 3-16

Let A denote the center of C and θ the angle that OA makes with $\mathbf{e}_1$. Then

$$\mathbf{OA} = |\mathbf{OA}|(\cos\theta)\mathbf{e}_1 + |\mathbf{OA}|(\sin\theta)\mathbf{e}_2 = (r_0+r)(\cos\theta)\mathbf{e}_1 + (r_0+r)(\sin\theta)\mathbf{e}_2$$

If β is the angle that **AP** makes with $\mathbf{e}_1$, then

$$\beta = \measuredangle OAP + \theta - \pi \quad \text{or} \quad \beta = \theta\frac{r_0}{r} + \theta - \pi = \frac{r_0 + r}{r}\theta - \pi$$

It follows that

$$\mathbf{AP} = |\mathbf{AP}|(\cos\beta)\mathbf{e}_1 + |\mathbf{AP}|(\sin\beta)\mathbf{e}_2 = r\left[\cos\left(\frac{r_0+r}{r}\theta - \pi\right)\right]\mathbf{e}_1 + r\left[\sin\left(\frac{r_0+r}{r}\theta - \pi\right)\right]\mathbf{e}_2$$

$$= -r\left[\cos\left(\frac{r_0+r}{r}\theta\right)\right]\mathbf{e}_1 - r\left[\sin\left(\frac{r_0+r}{r}\theta\right)\right]\mathbf{e}_2$$

Hence

$$\mathbf{x} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP} = \left[(r_0+r)\cos\theta - r\cos\left(\frac{r_0+r}{r}\theta\right)\right]\mathbf{e}_1 + \left[(r_0+r)\sin\theta - r\sin\left(\frac{r_0+r}{r}\theta\right)\right]\mathbf{e}_2$$

which is the required result.

3.5. If in the preceding problem $r_0 = 3$ and $r = 1$, the equation of the epicycloid is

$$x_1 = 4\cos\theta - \cos 4\theta, \qquad x_2 = 4\sin\theta - \sin 4\theta$$

Determine the singular (non-regular) points and sketch.

$dx_1/d\theta = -4\sin\theta + 4\sin 4\theta = 0$ iff $\sin\theta = \sin 4\theta$, or $\theta = 2n\pi/3, (2n+1)\pi/5$, $n = 0, \pm 1, \ldots$.

$dx_2/d\theta = 4\cos\theta - 4\cos 4\theta = 0$ iff $\cos\theta = \cos 4\theta$, or $\theta = 2n\pi/3, 2n\pi/5$, $n = 0, \pm 1, \ldots$.

It follows that both derivatives vanish if and only if $\theta = 2n\pi/3$, $n = 0, \pm 1, \ldots$. Observe that the curve has period 2π.

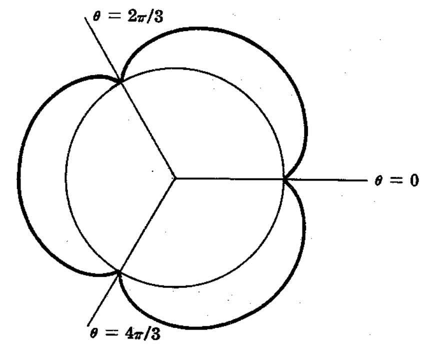


Fig. 3-17

3.6. Determine a parametric representation for the intersection of the cylinder $x_1^2 + x_2^2 = 1$ and the plane $x_1 + x_2 + x_3 = 1$ that does not involve radicals.

We are led to take $x_1 = \cos\theta$ and $x_2 = \sin\theta$, $0 \leqq \theta \leqq 2\pi$. Then $x_3 = 1 - x_1 - x_2 = 1 - \cos\theta - \sin\theta$. Thus

$$\mathbf{x} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 + (1 - \cos\theta - \sin\theta)\mathbf{e}_3, \qquad 0 \leqq \theta \leqq 2\pi$$

is a parametric representation of the intersection.

3.7. If $g(t)$ is continuous at $t = t_0$ and $g(t_0) \neq 0$, show there exists a $\delta > 0$ such that $g(t) \neq 0$ for t in $S_\delta(t_0)$. Use this to prove Theorem 3.1: If $\mathbf{x} = \mathbf{x}(t)$ is a regular representation on I, then for every t_0 in I there exists a neighborhood of t_0 in which $\mathbf{x}(t)$ is one-to-one.

Take $\epsilon = \frac{1}{2}|g(t_0)|$. Since $g(t)$ is continuous at t_0, there exists a $\delta > 0$ such that $|g(t) - g(t_0)| < \epsilon$ for t in $S_\delta(t_0)$. Hence for t in $S_\delta(t_0)$,

$$|g(t_0)| = |g(t_0) - g(t) + g(t)| \leqq |g(t) - g(t_0)| + |g(t)| = \epsilon + |g(t)| \leqq \tfrac{1}{2}|g(t_0)| + |g(t)|$$

or $|g(t)| \geqq \frac{1}{2}|g(t_0)|$. Since $g(t_0) \neq 0$, then $g(t) \neq 0$ for t in $S_\delta(t_0)$.

Since $\mathbf{x} = \mathbf{x}(t)$ is regular on I and t_0 is in I, at least one of the derivatives, say $x_1'(t_0) \neq 0$. Also $x_1'(t)$ is continuous at t_0. It follows that there exists a $\delta > 0$ such that $x_1'(t) \neq 0$ for all t in $S_\delta(t_0)$. Now, in $S_\delta(t_0)$, $\mathbf{x}(t)$ is one-to-one, for otherwise there would exist $t_1 \neq t_2$ in $S_\delta(t_0)$ such that $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. Hence $x_1(t_1) = x_1(t_2)$. But applying the mean value theorem,

$$0 = \frac{x_1(t_1) - x_1(t_2)}{t_1 - t_2} = x_1'(t'), \qquad t_1 < t' < t_2$$

which is impossible since $x_1'(t) \neq 0$ on $S_\delta(t_0)$. Thus the theorem is proved.

3.8. If $\mathbf{x} = \mathbf{x}(t)$ is a regular representation on I with $x_1'(t_0) \neq 0$, show that there exists a neighborhood of t_0 in which $\mathbf{x} = \mathbf{x}(t)$ can be represented implicitly in the form $x_2 = F_1(x_1)$, $x_3 = F_2(x_1)$.

Since $x_1'(t_0) \neq 0$, in some $S_\delta(t_0)$, $x_1 = x_1(t)$ is one-to-one and has an inverse $t = t(x_1)$. Substitute into the parametric equations $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$ to obtain $x_2 = x_2(t(x_1))$, $x_3 = x_3(t(x_1))$ or $x_2 = F_1(x_1)$, $x_3 = F_2(x_1)$.

REGULAR CURVES

3.9. Show that $t = \theta^2/(\theta^2+1)$ is an allowable change of parameter on $0 < \theta < \infty$ and takes the interval $0 < \theta < \infty$ onto $0 < t < 1$.

$dt/d\theta = 2\theta/(\theta^2+1)^2$ is continuous and $dt/d\theta \neq 0$ on $0 < \theta < \infty$. Hence it is an allowable change of parameter on $0 < \theta < \infty$. Since $\theta^2/(\theta^2+1)|_{\theta=0} = 0$ and $\lim_{\theta\to\infty} \theta^2/(\theta^2+1) = 1$, it takes the interval $0 < \theta < \infty$ onto $0 < t < 1$.

3.10. Introduce $t = \text{Tan}^{-1}(\theta/4)$ as a parameter along the circle $x_1 = a\cos\theta$, $x_2 = a\sin\theta$, $-\pi \leqq \theta \leqq \pi$.

The half angle identities give

$$\begin{aligned}\cos\theta &= \cos^4(\theta/4) - 6\cos^2(\theta/4)\sin^2(\theta/4) + \sin^4(\theta/4)\\ &= \frac{1}{(t^2+1)^2} - 6\frac{t^2}{(t^2+1)^2} + \frac{t^4}{(t^2+1)^2} = \frac{t^4-6t^2+1}{(t^2+1)^2}\end{aligned}$$

and

$$\sin\theta = 4(\sin(\theta/4)\cos^3(\theta/4) - \sin^3(\theta/4)\cos(\theta/4)) = 4\frac{t}{(t^2+1)^2} - \frac{t^3}{(t^2+1)^2} = \frac{4t(1-t^2)}{(t^2+1)^2}$$

Thus $x_1 = a\dfrac{(t^4-6t^2+1)}{(t^2+1)^2}$, $x_2 = \dfrac{4at(t-1)}{(t^2+1)^2}$, $-1 \leqq t \leqq 1$, is the desired representation.

3.11. Introduce the parameter $t = 2\sin\theta$ along the cissoid

$$x_1 = 2\sin^2\theta, \quad x_2 = 2\sin^2\theta\tan\theta, \qquad -\pi/2 < \theta < \pi/2$$

and obtain the first two nonzero terms of the power series expansion of x_1 and x_2 about the singular point $t = 0$. (See Problem 3.3.)

$$x_1 = 2(\tfrac{1}{2}t)^2 = \tfrac{1}{2}t^2, \quad x_2 = \tfrac{1}{2}t^2\tan(\text{Sin}^{-1}\tfrac{1}{2}t) = \tfrac{1}{2}t^3(4-t^2)^{-1/2}$$

The expansion of $(4-t^2)^{-1/2}$ is $\frac{1}{2} + \frac{1}{16}t^2 + o(t^2)$. Thus $x_1 = \frac{1}{2}t^2$, $x_2 = \frac{1}{4}t^3 + \frac{1}{32}t^5 + o(t^5)$. Observe that in the neighborhood of $t = 0$ the curve is shaped like the cusp $x_2 = \dfrac{1}{\sqrt{2}}x_1^{3/2} + o(x_1^2)$.

3.12. Show that there exists an allowable change of parameter $t = t(\theta)$ which takes any interval I onto one of the following three intervals: (i) $0 \leqq t \leqq 1$, (ii) $0 < t < 1$, (iii) $0 \leqq t < 1$. Hence every regular curve has a representation defined in one of the above three intervals.

As shown in Example 3.5, page 45, the linear function $t = (\theta - a)/(b-a)$ is an allowable change of parameter which takes $a \leqq \theta \leqq b$ onto $0 \leqq t \leqq 1$, $a < \theta < b$ onto $0 < t < 1$, and $a \leqq \theta < b$ onto $0 \leqq t < 1$. The linear function $t = -(\theta - a)/(b-a) + 1$ will take $a < \theta \leqq b$ onto $0 \leqq t < 1$. It remains to consider the infinite intervals. The function $\theta = \text{Tan}^{-1}s$ takes the interval $-\infty < s < \infty$ onto $-\pi/2 < \theta < \pi/2$, and $t = (\theta + \frac{1}{2}\pi)/\pi$ takes the interval $-\pi/2 < \theta < \pi/2$ onto $0 < t < 1$. Hence the composite function $t = (\pi/2 + \text{Tan}^{-1}s)/\pi$ takes $-\infty < s < \infty$ onto $0 < t < 1$. The function $\theta = \text{Tan}^{-1}s$ takes the interval $a \leqq s < \infty$ onto $\text{Tan}^{-1}a \leqq \theta < \pi/2$ and $t = (\theta - \text{Tan}^{-1}a)/(\pi/2 - \text{Tan}^{-1}a)$ takes $\text{Tan}^{-1}a \leqq \theta < \pi/2$ onto $0 \leqq t < 1$. Hence the composite function $t = \dfrac{\text{Tan}^{-1}s - \text{Tan}^{-1}a}{\pi/2 - \text{Tan}^{-1}a}$ takes $a \leqq s < \infty$ onto $0 \leqq t < 1$. We leave the remaining cases as exercises for the reader.

Note that all the above functions are *analytic*. Thus, in fact, there are allowable changes of parameter of any class which give the above results.

3.13. Prove Theorem 3.2: If $t = t(\theta)$ is an allowable change of parameter on I_θ, then $t(\theta)$ is one-to-one and its inverse $\theta = \theta(t)$ is an allowable change of parameter on $I_t = t(I_\theta)$.

Since $dt/d\theta$ is continuous and $dt/d\theta \neq 0$, it follows that $dt/d\theta > 0$ or $dt/d\theta < 0$ on I_θ. Suppose $dt/d\theta > 0$ on I_θ; then $t(\theta)$ is strictly increasing. For if otherwise, i.e. if $t(\theta_1) \geqq t(\theta_2)$ with $\theta_1 < \theta_2$, then, using the mean value theorem,

$$0 \geqq \frac{t(\theta_1) - t(\theta_2)}{\theta_1 - \theta_2} = t'(\theta')$$

which is impossible, since $dt/d\theta > 0$ on I_θ. Since $t(\theta)$ is strictly increasing, it is one-to-one and has an inverse $\theta(t)$. Now, since $t(\theta)$ is increasing and continuous, it follows that its inverse $\theta(t)$ is increasing and continuous. This we leave to the reader as an exercise. But then $\theta(t)$ also has a derivative

$$\frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = 1\Big/\lim_{\Delta\theta \to 0} \frac{\Delta t}{\Delta\theta} = 1\Big/\frac{dt}{d\theta}$$

which is continuous and different from zero since $dt/d\theta$ is continuous and different from zero, which completes the proof.

3.14. We recall that a regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I_t is defined to be equivalent to a regular parametric representation $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ if there exists an allowable change of parameter $t = t(\theta)$ such that $t(I_\theta) = I_t$ and $\mathbf{x}(t(\theta)) = \mathbf{x}^*(\theta)$. Show that this defines an equivalence relation on the set of regular representations.

$\mathbf{x} = \mathbf{x}(t)$ is equivalent to itself under the identity change of parameter $t = \theta$. If $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^*(\theta)$ under the change of parameter $t = t(\theta)$, then $\mathbf{x} = \mathbf{x}^*(\theta)$ is equivalent to $\mathbf{x} = \mathbf{x}(t)$ under the inverse function $\theta = \theta(t)$, since $\theta(I_t) = I_\theta$ and $\mathbf{x}^*(\theta(t)) = \mathbf{x}(t(\theta(t))) = \mathbf{x}(t)$. Finally, suppose $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^*(\theta)$ under $t = t(\theta)$, and $\mathbf{x} = \mathbf{x}^*(\theta)$ is equivalent to $\mathbf{x} = \mathbf{x}^{**}(\phi)$ under $\theta = \theta(\phi)$. Consider the composite function $t = t(\theta(\phi))$. It follows that $\dfrac{dt}{d\phi} = \dfrac{dt}{d\theta}\dfrac{d\theta}{d\phi}$ is continuous and $dt/d\phi \neq 0$ on I_ϕ. Hence $t = t(\theta(\phi))$ is an allowable change of parameter on I_ϕ. Also, $t(\theta(I_\phi)) = t(I_\theta) = I_t$ and $\mathbf{x}(t(\theta(\phi))) = \mathbf{x}^*(\theta(\phi)) = \mathbf{x}^{**}(\phi)$. Thus $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^{**}(\phi)$, which completes the proof.

ARC LENGTH

3.15. Compute the length of the arc $\mathbf{x} = 3(\cosh 2t)\mathbf{e}_1 + 3(\sinh 2t)\mathbf{e}_2 + 6t\mathbf{e}_3$, $0 \leqq t \leqq \pi$.

$$\begin{aligned} s &= \int_0^\pi \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_0^\pi |6 \sinh 2t\mathbf{e}_1 + 6 \cosh 2t\mathbf{e}_2 + 6\mathbf{e}_3|\, dt \\ &= \int_0^\pi 6[\sinh^2 2t + \cosh^2 2t + 1]^{1/2}\, dt \\ &= \int_0^\pi 6[2\cosh^2 2t]^{1/2}\, dt = \int_0^\pi 6\sqrt{2}\cosh 2t\, dt = 3\sqrt{2}\sinh 2\pi \end{aligned}$$

3.16. Find the arc length as a function of θ along the epicycloid

$$x_1 = (r_0 + r)\cos\theta - r\cos\left(\frac{r_0 + r}{r}\theta\right), \quad x_2 = (r_0 + r)\sin\theta - r\sin\left(\frac{r_0 + r}{r}\theta\right)$$

See Problem 3.4.

$$\begin{aligned} s &= \int_0^\theta \left[\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dx_2}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \int_0^\theta (r_0 + r)\left[\left(-\sin\theta + \sin\left(\frac{r_0 + r}{r}\theta\right)\right)^2 + \left(\cos\theta - \cos\left(\frac{r_0 + r}{r}\theta\right)\right)^2\right]^{1/2} d\theta \\ &= (r_0 + r)\int_0^\theta [2 - 2\cos(r_0\theta/r)]^{1/2}\, d\theta = 2(r_0 + r)\int_0^\theta \sin(r_0\theta/2r)\, d\theta \\ &= 4\frac{(r_0 + r)r}{r_0}\cos(r_0\theta/2r)\Big]_0^\theta = 4\frac{(r_0 + r)r}{r_0}[\cos(r_0\theta/2r) - 1] \end{aligned}$$

3.17. Introduce arc length as a parameter along

$$\mathbf{x} \;=\; (e^t \cos t)\mathbf{e}_1 + (e^t \sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \qquad -\infty < t < \infty$$

$$s \;=\; \int_0^t \left|\frac{d\mathbf{x}}{dt}\right| dt \;=\; \int_0^t |(e^t \cos t - e^t \sin t)\mathbf{e}_1 \;+\; (e^t \sin t + \theta^t \cos t)\mathbf{e}_2 \;+\; e^t\mathbf{e}_3|\,dt$$

$$= \int_0^t [e^{2t}(-2\cos t \sin t + 1) + e^{2t}(2\cos t \sin t + 1) + e^{2t}]^{1/2}\,dt \;=\; \sqrt{3}\int_0^t e^t\,dt \;=\; \sqrt{3}\,(e^t - 1)$$

Solving, $t = \log(s/\sqrt{3} + 1)$, $-\sqrt{3} < s < \infty$. Introducing arc length s as a parameter,

$$\mathbf{x} \;=\; (s/\sqrt{3} + 1)(\cos\log(s/\sqrt{3} + 1)\mathbf{e}_1 \;+\; \sin\log(s/\sqrt{3} + 1)\mathbf{e}_2 + \mathbf{e}_3)$$

3.18. Show that

$$\mathbf{x} \;=\; \tfrac{1}{2}(s + \sqrt{s^2+1})\mathbf{e}_1 \;+\; \tfrac{1}{2}(s + \sqrt{s^2+1})^{-1}\mathbf{e}_2 \;+\; \tfrac{1}{2}\sqrt{2}\,(\log(s + \sqrt{s^2+1}))\mathbf{e}_3$$

is a natural representation, i.e. $|d\mathbf{x}/ds| = 1$.

Let $u = s + \sqrt{s^2+1}$. Then $\mathbf{x} = \frac{1}{2}u\mathbf{e}_1 + \frac{1}{2}u^{-1}\mathbf{e}_2 + \frac{1}{2}\sqrt{2}\,(\log u)\mathbf{e}_3$ and

$$\frac{d\mathbf{x}}{ds} \;=\; \frac{d\mathbf{x}}{du}\frac{du}{ds} \;=\; (\tfrac{1}{2}\mathbf{e}_1 - \tfrac{1}{2}u^{-2}\mathbf{e}_2 \;+\; \tfrac{1}{2}\sqrt{2}\,u^{-1}\mathbf{e}_3)\left(1 + \frac{s}{\sqrt{s^2+1}}\right)$$

Hence
$$\left|\frac{d\mathbf{x}}{ds}\right| \;=\; \left|\frac{d\mathbf{x}}{du}\right|\left|\frac{du}{ds}\right| \;=\; \tfrac{1}{2}(1 + u^{-4} + 2u^{-2})^{1/2}\frac{s + \sqrt{s^2+1}}{\sqrt{s^2+1}}$$
$$=\; \frac{1}{2}(1 + u^{-2})\frac{u}{\sqrt{s^2+1}} \;=\; \frac{1}{2}\,\frac{u^2+1}{u\sqrt{s^2+1}} \;=\; \frac{s^2 + s\sqrt{s^2+1} + 1}{(s + \sqrt{s^2+1})\sqrt{s^2+1}} \;=\; 1$$

Since $|d\mathbf{x}/ds| = 1$, s is a natural parameter.

3.19. Prove Theorem 3.4(i): If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation on I_s, then $|s_2 - s_1|$ is the length of the arc $\mathbf{x} = \mathbf{x}(s)$ between the points corresponding to $\mathbf{f}(s_1)$ and $\mathbf{f}(s_2)$.

If $s_1 \leq s_2$, the length of the arc is $\displaystyle\int_{s_1}^{s_2}\left|\frac{d\mathbf{x}}{ds}\right| ds \;=\; \int_{s_1}^{s_2} 1\,ds \;=\; s_2 - s_1 \;=\; |s_2 - s_1|$.

If $s_1 > s_2$, the length of the arc is $\displaystyle\int_{s_2}^{s_1}\left|\frac{d\mathbf{x}}{ds}\right| ds \;=\; \int_{s_2}^{s_1} ds \;=\; s_1 - s_2 \;=\; |s_2 - s_1|$.

3.20. Prove Theorem 3.4(ii): If $\mathbf{x} = \mathbf{x}(s)$ and $\mathbf{x} = \mathbf{x}^*(s^*)$ are natural representations of the same curve, then $s = \pm s^* + \text{constant}$.

Let $s = s(s^*)$. Then $\dfrac{d\mathbf{x}}{ds^*} = \dfrac{d\mathbf{x}}{ds}\dfrac{ds}{ds^*}$ and $\left|\dfrac{d\mathbf{x}}{ds^*}\right| = \left|\dfrac{d\mathbf{x}}{ds}\right|\left|\dfrac{ds}{ds^*}\right|$.

But $\left|\dfrac{d\mathbf{x}}{ds}\right| = \left|\dfrac{d\mathbf{x}}{ds^*}\right| = 1$. Hence $\left|\dfrac{ds}{ds^*}\right| = 1$ or $\dfrac{ds}{ds^*} = \pm 1$ or $s = \pm s^* + \text{constant}$.

3.21. Show that the arc $\mathbf{x} = t^2\mathbf{e}_1 + \sin t\,\mathbf{e}_2$, $0 \leq t \leq \pi/2$, is rectifiable.

We consider an arbitrary subdivision $0 = t_0 < t_1 < t_2 < \cdots < t_n = \pi/2$ and compute the length of the polygonal arc:

$$s(P) \;=\; \sum_{i=1}^{n} |\mathbf{x}_i - \mathbf{x}_{i-1}| \;=\; \sum_i |(t_i^2\mathbf{e}_1 + (\sin t_i)\mathbf{e}_2) - (t_{i-1}^2\mathbf{e}_1 + (\sin t_{i-1})\mathbf{e}_2)|$$
$$\leq\; \sum_i [(t_i^2 - t_{i-1}^2)\,|\mathbf{e}_1| + |\sin t_i - \sin t_{i-1}|\,|\mathbf{e}_2|]$$

Using the mean value theorem, we have

$$\begin{aligned} s(P) &\leqq \sum_i [(t_i - t_{i-1})(t_i + t_{i-1}) + |\cos\theta_i|(t_i - t_{i-1})] \\ &\leqq \sum_i (t_i - t_{i-1})[t_i + t_{i-1} + |\cos\theta_i|] \qquad t_{i-1} < \theta_i < t_i \end{aligned}$$

Since $|\cos\theta_i| \leqq 1$ and $(t_i + t_{i-1}) \leqq \pi$ for $0 \leqq t_{i-1} < t_i \leqq \pi/2$,

$$s(P) \leqq (\pi + 1)\sum_i (t_i - t_{i-1}) \leqq (\pi/2)(\pi + 1)$$

Since the $s(P)$ are bounded, the arc is rectifiable.

3.22. If $\mathbf{x} = \mathbf{f}(t)$, $a \leqq t \leqq b$, is a rectifiable arc, show that given an arbitrary $\delta > 0$ and $\epsilon > 0$, there exists a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ with polygonal approximation P such that

$$\text{(i) } t_i - t_{i-1} < \delta,\ i = 1, \ldots, n \qquad \text{(ii) } |s - s(P)| < \epsilon$$

where s and $s(P)$ are lengths of $\mathbf{x} = \mathbf{f}(t)$ and P respectively.

Since s is the supremum of all possible $s(P)$, there exists a subdivision $a = t_0' < t_1' < \cdots < t_n' = b$ with polygonal approximation P' such that $s(P') > s - \epsilon$. For otherwise, $s(P) \leqq s - \epsilon$ for all $s(P)$, so that $s - \epsilon$ is an upper bound of the $s(P)$ less than the supremum s, which is impossible. Now, if the above subdivision does not satisfy (i), a finer subdivision $a = t_0 < t_1 < \cdots < t_n = b$ satisfying $(t_i - t_{i-1}) < \delta$ can be obtained by introducing additional points. But the new polygonal arc P' obtained this way satisfies $s(P) \leqq s(P') \leqq s$ and therefore also $|s - s(P)| < \epsilon$, as required.

3.23. Show that a regular arc $\mathbf{x} = \mathbf{f}(t)$, $a \leqq t \leqq b$, is rectifiable.

Consider an arbitrary subdivision $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Then

$$\begin{aligned} s(P) &= \sum_i |\mathbf{x}_i - \mathbf{x}_{i-1}| = \sum_i |\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})| \\ &= \sum_i |(f_1(t_i) - f_1(t_{i-1}))\mathbf{e}_1 + (f_2(t_i) - f_2(t_{i-1}))\mathbf{e}_2 + (f_3(t_i) - f_3(t_i))\mathbf{e}_3| \\ &\leqq \sum_i [|f_1(t_i) - f_1(t_{i-1})| + |f_2(t_i) - f_2(t_{i-1})| + |f_3(t_i) - f_3(t_{i-1})|] \\ &\leqq \sum_i [|f_1'(\theta_i)|(t_i - t_{i-1}) + |f_2'(\theta_i')|(t_i - t_{i-1}) + |f_3'(\theta_i'')|(t_i - t_{i-1})] \end{aligned}$$

where we used the mean value theorem for the $f_i(t)$. Since the $f_i'(t)$ are continuous on the closed interval $a \leqq t \leqq b$, they are bounded on $a \leqq t \leqq b$, say by M_i. Hence

$$s(P) \leqq (M_1 + M_2 + M_3)\sum_i (t_i - t_{i-1}) \leqq (M_1 + M_2 + M_3)(b - a)$$

Thus the $s(P)$ are all bounded by $(M_1 + M_2 + M_3)(b - a)$ and so the arc is rectifiable.

3.24. Show that the length of a regular arc $\mathbf{x} = \mathbf{f}(t)$, $a \leqq t \leqq b$, is given by the integral $s = \int_a^b \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_a^b |\mathbf{f}'(t)|\, dt$. This together with Problem 3.23 proves Theorem 3.3, page 51.

Let ϵ be arbitrary. Since the $f_i'(t)$, $i = 1, 2, 3$, are continuous on a closed interval $a \leqq t \leqq b$, they are uniformly continuous on the interval. Namely, there exists $\delta_1 > 0$ such that

$$\text{(i)} \qquad |f_i'(t) - f_i'(t')| < \frac{\epsilon}{9(b - a)}, \quad i = 1, 2, 3$$

for all $|t - t'| < \delta_1$. Also, by definition of the integral, there exists a δ_2 such that for $|t_i - t_{i-1}| < \delta_2$ we have

$$\text{(ii)} \qquad \left| \int_a^b |\mathbf{f}'(t)|\, dt - \sum_{i=1}^n |\mathbf{f}'(\theta_i)|(t_i - t_{i-1}) \right| < \epsilon/3 \qquad t_{i-1} \leqq \theta_i \leqq t_i$$

Now let $\delta = \min(\delta_i, \delta_2)$. It follows from Problem 3.22 that there exists a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ and polygonal approximation P such that $(t_i - t_{i-1}) < \delta$ and

$$\text{(iii)} \qquad |s - s(P)| < \epsilon/3$$

Now consider the quantity

$$I = \left| s - \int_a^b |\mathbf{f}'(t)|\,dt \right| \leq |s - s(P)| + \left| s(P) - \int_a^b |\mathbf{f}'(t)|\,dt \right|$$

$$\leq \frac{\epsilon}{3} + \left| \sum_i |\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})| - \int_a^b |\mathbf{f}'(t)|\,dt \right|$$

$$\leq \frac{\epsilon}{3} + \left| \sum_i |(f_1(t_i) - f_1(t_{i-1}))\mathbf{e}_1 + (f_2(t_i) - f_2(t_{i-1}))\mathbf{e}_2 + (f_3(t_i) - f_3(t_{i-1}))\mathbf{e}_3| - \int_a^b |\mathbf{f}'(t)|\,dt \right|$$

It follows from the mean value theorem that

$$I \leq \epsilon/3 + \left| \sum_i |f_1'(t_i')\mathbf{e}_1 + f_2'(t_i'')\mathbf{e}_2 + f_3'(t_i''')\mathbf{e}_3|\,(t_i - t_{i-1}) - \int_a^b |\mathbf{f}'(t)|\,dt \right|$$

and, adding and subtracting $\sum_i |\mathbf{f}'(t_i)|\,(t_i - t_{i-1})$,

$$I \leq \epsilon/3 + \left| \sum_i \mathbf{f}'(t_i)(t_i - t_{i-1}) - \int_a^b \mathbf{f}'(t)\,dt \right|$$
$$+ \left| \sum_i [|f_1'(t_i')\mathbf{e}_1 + f_2'(t_i'')\mathbf{e}_2 + f_3'(t_i''')\mathbf{e}_3| - |\mathbf{f}'(t_i)|\,(t_i - t_{i-1})] \right|$$

Using (ii) above and the inequality $\big|\,|a| - |b|\,\big| \leq |a + b|$, we have

$$I < \epsilon/3 + \epsilon/3 + \sum [|f_1'(t_i') - f_1'(t_i)| + |f_2'(t_i'') - f_2'(t_i)| + |f_3'(t_i''') - f_3'(t_i)|](t_i - t_{i-1})$$

Finally, using (i), we obtain

$$I < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3(b-a)} \sum_i (t_i - t_{i-1}) < \epsilon$$

Since ϵ is arbitrary,

$$I = \left| s - \int_a^b |\mathbf{f}'(t)|\,dt \right| = 0 \quad \text{or} \quad s = \int_a^b |\mathbf{f}'(t)|\,dt$$

Supplementary Problems

3.25. Show that the representation

$$\mathbf{x} = t\mathbf{e}_1 + (t^2 + 2)\mathbf{e}_2 + (t^3 + t)\mathbf{e}_3$$

is regular for all t and sketch the projections on the x_1x_3 and x_1x_2 planes.

3.26. The conchoid of Nicomedes in polar coordinates is $r = \dfrac{a}{\cos\theta} + c$, $a \neq 0$, $c \neq 0$, $-\pi \leq \theta \leq \pi$. Sketch and find a representation in rectangular coordinates.
Ans. $x_1 = a + c\cos\theta$, $x_2 = a\tan\theta + c\sin\theta$

3.27. Find a representation of the intersection of the cylinders $x_3^2 = x_1$, $x_2^2 = 1 - x_1$ that does not involve radicals. *Hint.* $x_2^2 + x_3^2 = 1$.
Ans. $x_1 = \cos^2\theta$, $x_2 = \sin\theta$, $x_3 = \cos\theta$, $0 \leq \theta \leq 2\pi$

3.28. The *hypocycloid* is the plane curve generated by a point P on the circumference of a circle C as C rolls without sliding on the *interior* of a fixed circle C_0, as shown in Fig. 3-18. If C has radius r and C_0 is at the origin with radius r_0 and P is initially located at $(r_0, 0)$, find a representation of the hypocycloid.

$$Ans.\quad x_1 = (r_0 - r)\cos\theta + r\cos\left(\frac{r_0 - r}{r}\theta\right)$$

$$x_2 = (r_0 - r)\sin\theta - r\sin\left(\frac{r_0 - r}{r}\theta\right)$$

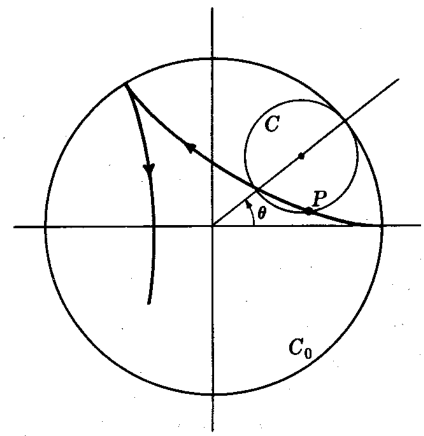


Fig. 3-18

3.29. If in the preceding problem $r_0 = 5$ and $r = 2$, the equation of the hypocycloid is

$$x_1 = 3\cos\theta + 2\cos 3\theta/2, \quad x_2 = 3\sin\theta - 2\sin 3\theta/2$$

Find the singular points and sketch.

Ans. $\theta = (4/5)n\pi, \ n = 0, \pm 1, \ldots$

3.30. Show that $\theta = 3t^5 + 10t^3 + 15t + 1$ is an allowable change of parameter for all t.

3.31. Find an allowable change of parameter which maps the interval $0 < t \leq 2$ onto $-\infty < \theta \leq 0$.

3.32. Compute the length of the arc $\mathbf{x} = e^t(\cos t)\mathbf{e}_1 + e^t(\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \quad 0 \leq t \leq \pi$. *Ans.* $3(e^\pi - 1)$

3.33. Referring to Problem 3.28, find arc length as a function of θ along the hypocycloid.

$$x_1 = (r_0 - r)\cos\theta + r\cos\left(\frac{r_0 - r}{r}\theta\right), \qquad x_2 = (r_0 - r)\sin\theta - r\sin\left(\frac{r_0 - r}{r}\theta\right), \qquad r_0 > r$$

$$Ans.\quad s = \frac{4r(r_0 - r)}{r_0}[1 - \cos(r_0\theta/2r)]$$

3.34. Show that

$$\mathbf{x} = t\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \qquad -\infty < t < \infty$$

and

$$\mathbf{x} = (\log t)\mathbf{e}_1 + \sin(\log t)\,\mathbf{e}_2 + t\mathbf{e}_3, \qquad 0 < t < \infty$$

are representations of the same oriented curve.

3.35. Show that an oriented regular curve has a representation defined on one of the following four intervals: (i) $0 \leq t \leq 1$, (ii) $0 < t < 1$, (iii) $0 \leq t < 1$, (iv) $0 < t \leq 1$.

3.36. Suppose two regular representations $\mathbf{x} = \mathbf{x}(t)$ on I_t and $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ are said to be equivalent if they represent the same oriented curve; i.e. if there exists an allowable change of parameter $t = t(\theta)$ such that $dt/d\theta > 0$, $t(I_\theta) = I_t$ and $x(t(\theta)) = x^*(\theta)$. Show that this is an equivalence relation on the set of regular representations. Thus a regular oriented curve is an equivalence class of regular representations related by allowable changes of parameter with a positive derivative.

3.37. Show that an arc segment $\mathbf{x} = \mathbf{x}(t)$ on I^* of a rectifiable arc $\mathbf{x} = \mathbf{x}(t)$ on I is rectifiable.

3.38. If $\mathbf{x} = \mathbf{x}(t)$ on $a \leq t \leq b$ is a rectifiable arc with length s and $a < t_0 < b$, show that the arc segments $\mathbf{x} = \mathbf{x}(t)$ on $a \leq t \leq t_0$ and $\mathbf{x} = \mathbf{x}(t)$ on $t_0 \leq t \leq b$ are both rectifiable with lengths s_1 and s_2 respectively and $s = s_1 + s_2$.

3.39. Prove Theorem 3.4(iii): If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of an oriented curve C and $\mathbf{x} = \mathbf{x}^*(t)$ is any other representation of C, then $ds/dt = |d\mathbf{x}/dt|$.

Chapter 4

Curvature and Torsion

INTRODUCTION

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. We will see that a regular curve is uniquely determined by two scalar quantities, called curvature and torsion, as functions of the natural parameter.

UNIT TANGENT VECTOR

Let $\mathbf{x} = \mathbf{x}(s)$ be a natural representation of a regular curve C. The derivative $d\mathbf{x}/ds = \dot{\mathbf{x}}(s)$ will be used to define the *tangent* direction to C at the point $\mathbf{x}(s)$. This agrees with our geometric intuition, since

$$\dot{\mathbf{x}}(s) = \lim_{\Delta s \to 0} \frac{\mathbf{x}(s+\Delta s) - \mathbf{x}(s)}{\Delta s}$$

and $\dfrac{\mathbf{x}(s+\Delta s)-\mathbf{x}(s)}{\Delta s}$ is a secant to C as shown in Fig. 4-1. The vector $\dot{\mathbf{x}}$ is also of unit length, for in a natural representation $|d\mathbf{x}/ds| = |\dot{\mathbf{x}}| = 1$.

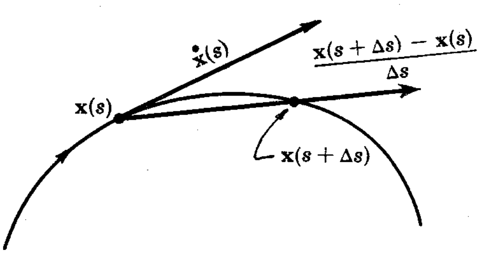


Fig. 4-1

If $\mathbf{x} = \mathbf{x}(s^*)$ is any other natural representation of C, then from Theorem 3.4, page 52, $s = \pm s^* + \text{constant}$ and

$$\frac{d\mathbf{x}}{ds^*} = \frac{d\mathbf{x}}{ds}\frac{ds}{ds^*} = \pm\frac{d\mathbf{x}}{ds}$$

That is, $d\mathbf{x}/ds^*$ has the same or opposite sense as $d\mathbf{x}/ds$ depending on the orientation of $\mathbf{x} = \mathbf{x}(s^*)$. Thus $\dot{\mathbf{x}}$ is an *oriented* quantity. As shown in Fig. 4-1, it is in the direction of increasing s.

The vector $\dot{\mathbf{x}}(s)$ is called the *unit tangent vector* to the oriented curve $\mathbf{x} = \mathbf{x}(s)$ at $\mathbf{x}(s)$ and shall be denoted by $\mathbf{t} = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$.

Example 4.1.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a, b \neq 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3 \quad \text{and} \quad \left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{1/2}$$

Then

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{d\mathbf{x}}{dt}\Big/\frac{ds}{dt} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

where we used the fact that $ds/dt = |d\mathbf{x}/dt|$ (Theorem 3.4). Observe that along the helix the unit tangent $\mathbf{t}$ makes a constant angle $\theta = \cos^{-1}(\mathbf{t}\cdot\mathbf{e}_3) = \cos^{-1} b(a^2+b^2)^{-1/2}$ with the x_3 axis.

As in the case of the unit tangent, other geometric quantities along the curve will be defined in terms of a natural representation. However, by using the chain rule and the relation $ds/dt = |d\mathbf{x}/dt|$, these quantities can all be derived in terms of an arbitrary parameter, as in the example above.

If $\mathbf{x} = \mathbf{x}(t)$ is an arbitrary representation of C with the same orientation as $\mathbf{x} = \mathbf{x}(s)$, then

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \mathbf{t}\left|\frac{d\mathbf{x}}{dt}\right| = \mathbf{t}|\mathbf{x}'|$$

where again we used $ds/dt = |d\mathbf{x}/dt|$. Thus as we expect, the derivative $\mathbf{x}'$ has the same direction as $\mathbf{t}$, namely, it is also a tangent vector to the curve and we have the formula

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| \tag{4.1}$$

TANGENT LINE AND NORMAL PLANE

The straight line through a point $\mathbf{x}$ on a regular curve C parallel to the tangent vectors at $\mathbf{x}$ is called the *tangent line* to C at $\mathbf{x}$. See Fig. 4-2. It follows from equation (2.1), page 21, that the tangent line at the point $\mathbf{x}_0 = \mathbf{x}(t_0)$ is given by

$$\mathbf{x} = \mathbf{x}_0 + k\mathbf{t}_0, \qquad -\infty < k < \infty$$

where $\mathbf{t}_0 = \mathbf{t}(t_0)$ is a unit tangent at $\mathbf{x}_0$.

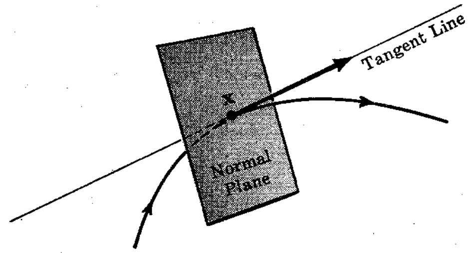


Fig. 4-2

The plane through $\mathbf{x}$ orthogonal to the tangent line at $\mathbf{x}$ is called the *normal plane* to C at $\mathbf{x}$. It follows from equation (2.5), page 21, the normal plane at $\mathbf{x}_0$ is given by

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{t}_0 = 0$$

It is convenient to introduce a second variable, say $\mathbf{y}$, to denote a generic point on a figure related to $\mathbf{x} = \mathbf{x}(t)$. Using this, we can express the equation of the tangent line at an arbitrary point $\mathbf{x}$ on C as

$$\mathbf{y} = \mathbf{x} + k\mathbf{t}, \qquad -\infty < k < \infty \tag{4.2}$$

and the normal plane at $\mathbf{x}$ as

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{t} = 0 \tag{4.3}$$

Finally we note that $\mathbf{x}'$ is parallel to $\mathbf{t}$, so that the tangent line and normal plane are also given by

$$\mathbf{y} = \mathbf{x} + k\mathbf{x}', \qquad -\infty < k < \infty$$

and

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{x}' = 0$$

respectively.

Example 4.2.

The tangent line to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (1+k)\mathbf{e}_1 + (1+2k)\mathbf{e}_2 + (1+3k)\mathbf{e}_3, \quad -\infty < k < \infty$$

The normal plane at $t = 1$ is

$$(\mathbf{y} - \mathbf{x}(1)) \cdot \mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1 - 1) + (y_2 - 1)2 + (y_3 - 1)3 = 0$$

or $y_1 + 2y_2 + 3y_3 = 6$.

CURVATURE

We now assume that $\mathbf{x} = \mathbf{x}(s)$ is a regular curve of class $\geqq 2$. Then the tangent vector $\mathbf{t} = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$ is of class C^1, and we can consider its derivative

$$d\mathbf{t}/ds = \dot{\mathbf{t}}(s) = \ddot{\mathbf{x}}(s)$$

Whereas the sense of the unit tangent $\mathbf{t}$ depends upon the orientation of C, $\dot{\mathbf{t}}$ is independent of orientation. For let $\mathbf{x} = \mathbf{x}(s^*)$ be any other natural representation of C with, say, unit tangent $\mathbf{t}^* = d\mathbf{x}/ds^*$. Then $s = \pm s^* + \text{constant}$ and

$$\frac{d\mathbf{t}^*}{ds^*} = \frac{d}{ds^*}\left(\frac{d\mathbf{x}}{ds^*}\right) = \frac{d}{ds^*}\left(\pm\frac{d\mathbf{x}}{ds}\right) = \pm\frac{d}{ds}\left(\frac{d\mathbf{x}}{ds}\right)\frac{ds}{ds^*} = (\pm 1)^2\frac{d}{ds}\left(\frac{d\mathbf{x}}{ds}\right) = \frac{d\mathbf{t}}{ds}$$

Thus $\dot{\mathbf{t}}$ is independent of orientation.

The vector $\dot{\mathbf{t}}(s)$ is called the *curvature vector* on C at the point $\mathbf{x}(s)$ and is denoted by $\mathbf{k} = \mathbf{k}(s) = \dot{\mathbf{t}}(s)$.

Since $\mathbf{t}$ is a unit vector, it follows from Theorem 2.7, page 29, that $\mathbf{k} = \dot{\mathbf{t}}$ is orthogonal to $\mathbf{t}$ and hence parallel to the normal plane. When different from zero, it is in the direction in which the curve is turning, as indicated in Fig. 4-3.

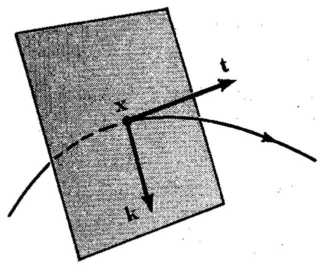


Fig. 4-3

The magnitude of the curvature vector is denoted by

$$|\kappa| = |\mathbf{k}(s)| \tag{4.4}$$

and is called the *curvature* of C at $\mathbf{x}(s)$. The reciprocal of the curvature is denoted by

$$\rho = \frac{1}{|\kappa|} = \frac{1}{|\mathbf{k}(s)|} \tag{4.5}$$

and is called the *radius of curvature* at $\mathbf{x}(s)$.

A point on C where the curvature vector $\mathbf{k} = \mathbf{0}$ is called a *point of inflection.* Thus at a point of inflection the curvature $|\kappa|$ is zero and the radius of curvature ρ is infinite.

In Problem 4.9, page 73, we prove that the curvature is equal to the rate of change of the direction of the tangent with respect to arc length. Thus along a curve which has a rapidly changing tangent direction with respect to arc length, such as a circle with a small radius, the curvature is relatively large, or, equivalently, the radius of curvature is small.

Example 4.3.

Along the circle of radius a, $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$, $a > 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2, \qquad \left|\frac{d\mathbf{x}}{dt}\right| = a$$

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -(\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2$$

and $$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \frac{d\mathbf{t}}{dt}\Big/\frac{ds}{dt} = \frac{d\mathbf{t}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -\frac{1}{a}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

Note that $\mathbf{k}$ is directed towards the origin. The curvature is constant, equal to $|\kappa| = |\mathbf{k}| = 1/a$ and the radius of curvature is $\rho = 1/|\kappa| = a$. Hence as we expect, the radius of curvature of a circle is simply its radius.

Example 4.4.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a > 0$, $b \neq 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3, \qquad \left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{1/2}$$

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

and

$$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{dt}\bigg/\left|\frac{d\mathbf{x}}{dt}\right|$$
$$= (a^2+b^2)^{-1/2}(-a(\cos t)\mathbf{e}_1 - a(\sin t)\mathbf{e}_2)/(a^2+b^2)^{1/2}$$
$$= -\frac{a}{a^2+b^2}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

Observe that $\mathbf{k}$ is parallel to the x_1x_2 plane and directed toward the origin, as shown in Fig. 4-4. The curvature is constant, equal to $|\kappa| = |\mathbf{k}| = a/(a^2+b^2)$.

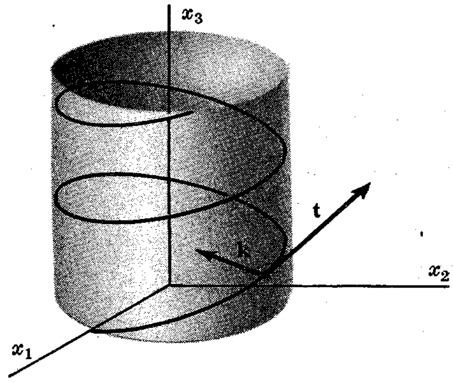


Fig. 4-4

If the curvature is identically zero along a curve C, i.e. if $|\mathbf{k}| \equiv 0$, then $\dot{\mathbf{t}} \equiv \mathbf{0}$ and, integrating,

$$\mathbf{t} = \mathbf{a}, \qquad \mathbf{a} = \text{constant} \neq \mathbf{0}$$

Since $\mathbf{t} = \dot{\mathbf{x}}$, we have, integrating again,

$$\mathbf{x} = \mathbf{a}s + \mathbf{b}, \qquad \mathbf{b} = \text{constant}$$

that is, C is a straight line through (b_1, b_2, b_3) parallel to $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. Conversely, if C is the straight line

$$\mathbf{x} = \mathbf{a}t + \mathbf{b}, \qquad \mathbf{a} \neq \mathbf{0}$$

then

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\bigg/\left|\frac{d\mathbf{x}}{dt}\right| = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{and} \quad |\mathbf{k}| = |\dot{\mathbf{t}}| = 0$$

Thus we have

Theorem 4.1. A regular curve of class ≥ 2 is a straight line if and only if its curvature is identically zero.

A formula can be derived which expresses the curvature directly in term of the derivatives of an arbitrary representation. Namely, in Problem 4.7, page 73, we prove

Theorem 4.2. If $\mathbf{x} = \mathbf{x}(t)$ is an arbitrary representation of a curve of class ≥ 2, then

$$|\kappa| = |\mathbf{x}' \times \mathbf{x}''|/|\mathbf{x}'|^3$$

PRINCIPAL NORMAL UNIT VECTOR

Since C is of class ≥ 2, the curvature vector $\mathbf{k} = \dot{\mathbf{t}} = \ddot{\mathbf{x}}$ varies continuously along C; however, the unit vector in the direction of $\mathbf{k}$,

$$\mathbf{u_k} = \mathbf{k}/|\mathbf{k}|$$

is not defined where $\mathbf{k} = \mathbf{0}$, and may jump, as in the examples below. Thus we are led to consider not $\mathbf{u_k}$ itself but a unit vector parallel to $\mathbf{k}$ with its sense chosen arbitrarily but in such a way as to be continuous along C *wherever possible.* This vector shall be denoted by $\mathbf{n} = \mathbf{n}(s)$ and is called the *principal normal unit vector* to C at the point $\mathbf{x}(s)$.

Note that if there are no points of inflection on C, i.e. if $\mathbf{k}(s) \neq \mathbf{0}$ for all s, we can simply choose

$$\mathbf{n} = \mathbf{k}(s)/|\mathbf{k}(s)|$$

the unit vector in the direction of $\mathbf{k}$. Note also that along a straight line we have $\mathbf{k} \equiv \mathbf{0}$, so that the principal normal unit vector $\mathbf{n}$ is undetermined.

Having selected $\mathbf{n}(s)$, there will be a continuous function $\kappa(s)$ along C such that

$$\mathbf{k}(s) \quad = \quad \kappa(s)\,\mathbf{n}(s) \tag{4.6}$$

At a point on the curve where $\mathbf{n}$ has the same direction as $\mathbf{k}$, we have $\kappa = |\mathbf{k}|$; where $\mathbf{n}$ is opposite to $\mathbf{k}$, $\kappa = -|\mathbf{k}|$; at a point of inflection, $\mathbf{k} = 0$ and $\kappa = 0$.

The quantity $\kappa(s)$ defined by equation (4.6) is *also* called the *curvature* of C at the point $\mathbf{x}(s)$. Note, however, since the sense of $\mathbf{n}$ is initially arbitrary, the function $\kappa(s)$ is determined only within a sign; and locally only its absolute value $|\kappa| = |\mathbf{k}|$, the curvature as defined previously, is an intrinsic property of the curve.

If we multiply (4.6) by $\mathbf{n}$ and use $\mathbf{n}\cdot\mathbf{n} = |\mathbf{n}|^2 = 1$, we have the formula

$$\kappa \quad = \quad \mathbf{k}(s)\cdot\mathbf{n}(s) \tag{4.7}$$

Example 4.5.

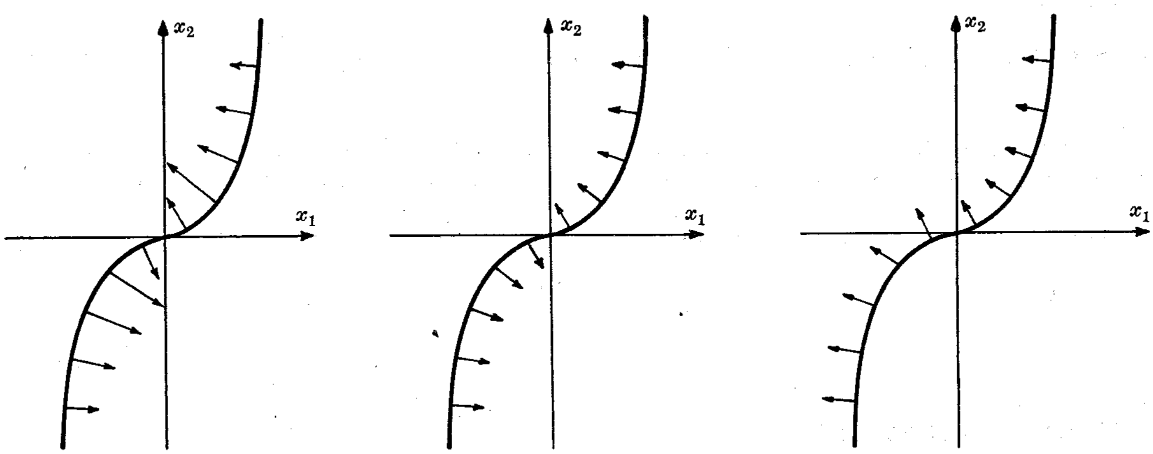
Along the third degree curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1}{3}t^3\mathbf{e}_2$ shown in Fig. 4-5,

$$\frac{d\mathbf{x}}{dt} = \mathbf{e}_1 + t^2\mathbf{e}_2, \quad \left|\frac{d\mathbf{x}}{dt}\right| = (1+t^4)^{1/2}, \quad \mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (1+t^4)^{-1/2}(\mathbf{e}_1 + t^2\mathbf{e}_2)$$

and

$$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -2t(1+t^4)^{-2}(t^2\mathbf{e}_1 - \mathbf{e}_2)$$

The direction of $\mathbf{k}$ is shown in Fig. 4-5(*a*).



(*a*) The vector $\mathbf{k}$ (*b*) The vector $\mathbf{u_k} = \mathbf{k}/|\mathbf{k}|$ (*c*) The vector $\mathbf{n} = \begin{cases} -\mathbf{u_k} & \text{for } t<0 \\ \mathbf{e}_2 & \text{for } t=0 \\ \mathbf{u_k} & \text{for } t>0 \end{cases}$

Fig. 4-5

At $t = 0$, $\mathbf{k} = 0$ and we have a point of inflection. Here $\mathbf{u_k}$ has a jump in sense as shown in Fig. 4-5(*b*). For, the limit of $\mathbf{u_k}$ as t approaches 0 through positive t is

$$\lim_{t\to 0+}\mathbf{u_k} = \lim_{t\to 0+}\frac{\mathbf{k}}{|\mathbf{k}|} = \lim_{t\to 0+}\frac{-t}{|t|\,(1+t^4)^{1/2}}(t^2\mathbf{e}_1 - \mathbf{e}_2) = \lim_{t\to 0}\frac{-(t^2\mathbf{e}_1 - \mathbf{e}_2)}{(1+t^4)^{1/2}} = \mathbf{e}_2$$

whereas the limit of $\mathbf{u_k}$ as t approaches 0 through negative t is

$$\lim_{t\to 0-}\mathbf{u_k} = \lim_{t\to 0-}\frac{\mathbf{k}}{|\mathbf{k}|} = \lim_{t\to 0-}\frac{-t}{|t|\,(1+t^4)^{1/2}}(t^2\mathbf{e}_1 - \mathbf{e}_2) = \lim_{t\to 0}\frac{t^2\mathbf{e}_1 - \mathbf{e}_2}{(1+t^4)^{1/2}} = -\mathbf{e}_2$$

where we used the fact that $\dfrac{t}{|t|} = \begin{cases} 1 & \text{for } t>0 \\ -1 & \text{for } t<0 \end{cases}$. If we choose

$$\mathbf{n} = \begin{cases} -\mathbf{k}/|\mathbf{k}| & \text{for } t<0 \\ \mathbf{e}_2 & \text{for } t=0 \\ \mathbf{k}/|\mathbf{k}| & \text{for } t>0 \end{cases} = -(1+t^4)^{-1/2}(t^2\mathbf{e}_1-\mathbf{e}_2)$$

n will vary continuously along the curve as shown in Fig. 4-5(*c*). For this **n** we have, from equation (*4.7*),

$$\kappa = \mathbf{k}\cdot\mathbf{n} = [-2t(1+t^4)^{-2}(t^2\mathbf{e}_1-\mathbf{e}_2)]\cdot[-(1+t^4)^{-1/2}(t^2\mathbf{e}_1-\mathbf{e}_2)] = 2t(1+t^4)^{-3/2}$$

Example 4.6.

We consider the curve of class C^∞ (see Example 3.13, page 49):

$$\mathbf{x} = \begin{cases} t\mathbf{e}_1 + e^{-1/t^2}\mathbf{e}_3 & \text{for } t<0 \\ 0 & \text{for } t=0 \\ t\mathbf{e}_1 + e^{-1/t^2}\mathbf{e}_2 & \text{for } t>0 \end{cases}$$

As shown in Fig. 4-6, the curve lies in the x_1x_3 plane for $t<0$ and in the x_1x_2 plane for $t>0$. It follows that **k** lies in the x_1x_3 plane for $t<0$ and in the x_1x_2 plane for $t>0$. Here it is impossible to define **n** so that it is continuous at $t=0$, since **k** jumps from the x_1x_3 plane to the x_1x_2 plane.

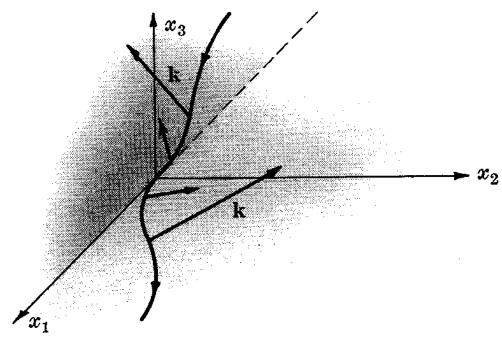


Fig. 4-6

As shown in the example above, even a curve of class C^∞ may not have a definite principal normal at a point of inflection. However, if the curve is analytic, a continuous principal normal will always exist. Namely, in Problem 4.15, page 75, we prove

Theorem 4.3. An analytic curve which is not a straight line will have a definite continuous principal normal unit vector in a neighborhood of a point of inflection.

PRINCIPAL NORMAL LINE AND OSCULATING PLANE

The straight line passing through a point **x** on a curve C parallel to the principal unit normal as shown in Fig. 4-7 is called the *principal normal line* to C at **x**. It follows that the equation of the principal normal line at **x** is

$$\mathbf{y} = \mathbf{x} + k\mathbf{n}, \quad -\infty < k < \infty \tag{4.8}$$

The plane parallel to the unit tangent and the principal unit normal is called the osculating plane to C at **x**. It follows from Example 2.3, page 22, that the equation of the osculating plane at **x** is given by the triple scalar product

$$[(\mathbf{y}-\mathbf{x})\mathbf{t}\mathbf{n}] = 0 \tag{4.9}$$

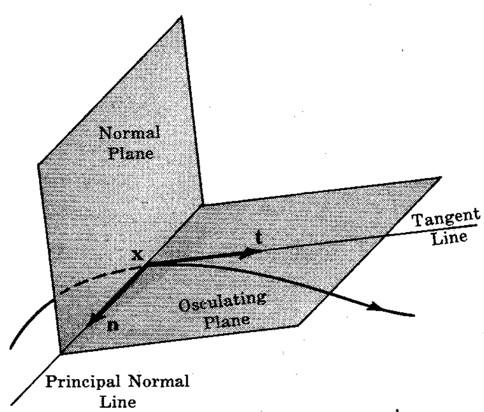


Fig. 4-7

If we use the fact that $\mathbf{t}=\dot{\mathbf{x}}$ and that $\dot{\mathbf{t}}=\ddot{\mathbf{x}}$ is parallel to **n**, then at a point where $k \neq 0$, the equation of the osculating plane is also given by

$$[(\mathbf{y}-\mathbf{x})\dot{\mathbf{x}}\ddot{\mathbf{x}}] = 0 \tag{4.10}$$

We recall that the tangent line at a point on a curve can be defined as the limiting position of a line passing through two neighboring points on the curve as the two points approach the given point. In this way a line is obtained that in a sense best fits the curve at a point. Similarly, the osculating plane at a point can be defined as the limiting position of a plane passing through three neighboring points on the curve as the points approach the given point. The tangent line and osculating plane are examples of geometric figures which have a certain order of *contact* with the curve. The theory of contact between curves and surfaces will be considered in the next chapter and the osculating plane will be considered again from this point of view.

Example 4.7.

Consider the helix $\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$.

$$\mathbf{x}' = (-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3, \quad |\mathbf{x}'| = \sqrt{2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1/\sqrt{2})((-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3)$$

and

$$\mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| = -(\tfrac{1}{2})((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

and, since $\mathbf{k} \neq 0$ for all t,

$$\mathbf{n} = \mathbf{k}/|\mathbf{k}| = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

The equation of the principal normal at $t = \pi/2$ is

$$\mathbf{y} = \mathbf{x}(\pi/2) + k\mathbf{n}(\pi/2) \quad \text{or} \quad \mathbf{y} = (1-k)\mathbf{e}_2 + \pi/2\mathbf{e}_3, \qquad -\infty < k < \infty$$

and the equation of the osculating plane at $t = \pi/2$ is

$$[(\mathbf{y} - \mathbf{x}(\pi/2))t(\pi/2)n(\pi/2)] = 0$$

or

$$\det\begin{pmatrix} y_1 & -1/\sqrt{2} & 0 \\ y_2 - 1 & 0 & -1 \\ y_3 - \pi/2 & 1/\sqrt{2} & 0 \end{pmatrix} = 0 \quad \text{or} \quad y_1 + y_3 = \pi/2$$

BINORMAL. MOVING TRIHEDRON

Let $\mathbf{x} = \mathbf{x}(s)$ be a regular curve C of class ≥ 2 along which $\mathbf{n}$ is continuous. Then at each point on C we have two orthogonal and continuous unit vectors: the unit tangent $\mathbf{t}$ and the unit principal normal $\mathbf{n}$. Now consider the vector

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$$

We observe that $\mathbf{b}$ is continuous and of unit length, and that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ forms a right-handed orthonormal triplet as shown in Fig. 4-8. The vector $\mathbf{b}(s)$ is called the *unit binormal vector* to C at the point $\mathbf{x}(s)$ and the triplet $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ is called the *moving trihedron* of C.

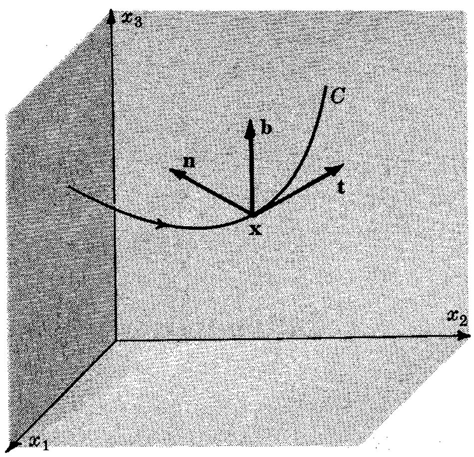


Fig. 4-8

The straight line through $\mathbf{x}$ parallel to $\mathbf{b}$ is called the *binormal line* to C at $\mathbf{x}$. It follows that the equation of the binormal at $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{b}, \quad -\infty < k < \infty \qquad (4.11)$$

The plane through a point $\mathbf{x}$ on C parallel to $\mathbf{t}$ and $\mathbf{b}$ is called the *rectifying* plane at $\mathbf{x}$. The equation of the rectifying plane at $\mathbf{x}$ is

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n} = 0 \qquad (4.12)$$

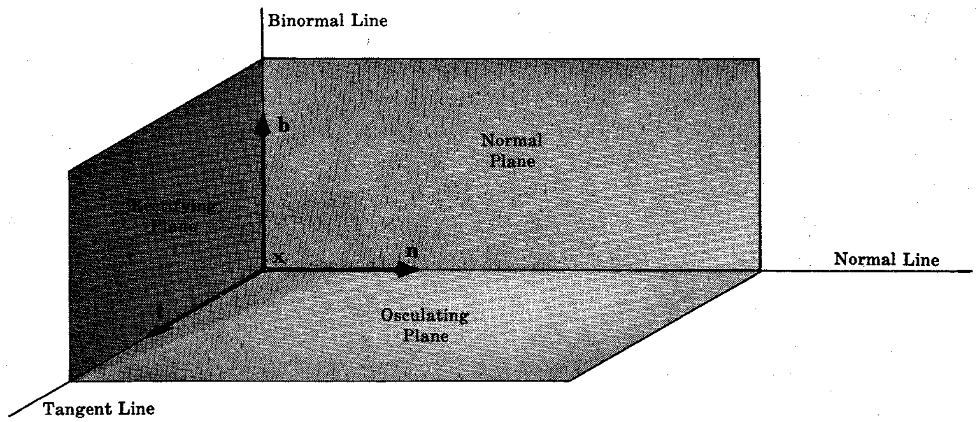


Fig. 4-9

Thus at each point $\mathbf{x}$ on C we have the following three characteristic lines and planes:

Tangent line:	$\mathbf{y} = \mathbf{x} + k\mathbf{t}$
Principal normal line:	$\mathbf{y} = \mathbf{x} + k\mathbf{n}$
Binormal line:	$\mathbf{y} = \mathbf{x} + k\mathbf{b}$
Normal plane:	$(\mathbf{y}-\mathbf{x})\cdot\mathbf{t} = 0$
Rectifying plane:	$(\mathbf{y}-\mathbf{x})\cdot\mathbf{n} = 0$
Osculating plane:	$(\mathbf{y}-\mathbf{x})\cdot\mathbf{b} = 0$

Example 4.8.

Referring to the helix in Example 4.4, we have

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3, \quad a > 0, \quad b \neq 0$$

$$\mathbf{t} = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

$$\mathbf{k} = -\frac{a}{a^2+b^2}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2), \quad \mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} = \det\begin{pmatrix} \mathbf{e}_1 & -a(a^2+b^2)^{-1/2}\sin t & -\cos t \\ \mathbf{e}_2 & a(a^2+b^2)^{-1/2}\cos t & -\sin t \\ \mathbf{e}_3 & b(a^2+b^2)^{-1/2} & 0 \end{pmatrix}$$

$$= (a^2+b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

The equation of the binormal line at $t = t_0$ is

$$\mathbf{y} = \mathbf{x}(t_0) + k\mathbf{b}(t_0)$$

or

$$\mathbf{y} = (a\cos t_0 + kb(a^2+b^2)^{-1/2}\sin t_0)\mathbf{e}_1 + (a\sin t_0 - kb(a^2+b^2)^{-1/2}\cos t_0)\mathbf{e}_2 + (bt_0 + ak(a^2+b^2)^{-1/2})\mathbf{e}_3, \quad -\infty < k < \infty$$

Or, if we introduce the change in parameter $\theta = k(a^2+b^2)^{-1/2}$,

$$\mathbf{y} = (a\cos t_0 + \theta b\sin t_0)\mathbf{e}_1 + (a\sin t_0 - \theta b\cos t_0)\mathbf{e}_2 + (bt_0 + a\theta)\mathbf{e}_3, \quad -\infty < \theta < \infty$$

The equation of the rectifying plane at $t = t_0$ is

$$(\mathbf{y} - \mathbf{x}(t_0))\cdot\mathbf{n}(t_0) = 0$$

or

$$(y_1 - a\cos t_0)(-\cos t_0) + (y_2 - a\sin t_0)(-\sin t_0) = 0$$

or

$$y_1\cos t_0 + y_2\sin t_0 = a$$

Observe that rectifying planes are parallel to the x_3 axis.

TORSION

We now suppose that $\mathbf{x} = \mathbf{x}(s)$ is a regular curve of class ≥ 3 along which $\mathbf{n}(s)$ is of class C^1. Then we can differentiate $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$, obtaining

$$\dot{\mathbf{b}}(s) \;=\; \dot{\mathbf{t}}(s) \times \mathbf{n}(s) + \mathbf{t}(s) \times \dot{\mathbf{n}}(s) \;=\; \kappa(s)[\mathbf{n}(s) \times \mathbf{n}(s)] + \mathbf{t}(s) \times \dot{\mathbf{n}}(s) \;=\; \mathbf{t}(s) \times \dot{\mathbf{n}}(s) \qquad (4.13)$$

where we used equation (*4.6*) and the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all $\mathbf{a}$. Since $\mathbf{n}$ is a unit vector, $\dot{\mathbf{n}}$ is orthogonal to $\mathbf{n}$ and therefore is parallel to the rectifying plane. It follows that $\dot{\mathbf{n}}$ is a linear combination of $\mathbf{t}$ and $\mathbf{b}$, say

$$\dot{\mathbf{n}}(s) \;=\; \mu(s)\,\mathbf{t}(s) + \tau(s)\,\mathbf{b}(s)$$

Substituting into (*4.13*),

$$\dot{\mathbf{b}}(s) \;=\; \mathbf{t}(s) \times [\mu(s)\,\mathbf{t}(s) + \tau(s)\,\mathbf{b}(s)] \;=\; \tau(s)[\mathbf{t}(s) \times \mathbf{b}(s)]$$

or

$$\dot{\mathbf{b}}(s) \;=\; -\tau(s)\,\mathbf{n}(s) \qquad (4.14)$$

where we use the fact that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is a right-handed orthonormal triplet and hence $\mathbf{t} \times \mathbf{b} = -\mathbf{n}$.

The continuous function $\tau(s)$ defined by (*4.14*) is called the *second curvature* or *torsion* of C at $\mathbf{x}(s)$. Note that if we take the scalar product of (*4.14*) with $\mathbf{n}$, we obtain the formula

$$\tau \;=\; -\dot{\mathbf{b}}(s)\,\mathbf{n}(s) \qquad (4.15)$$

Note that the sign of τ is independent of the sense of $\mathbf{n}$ and the orientation of C, and hence is an intrinsic property of the curve. For suppose first that we change the sense of $\mathbf{n}$, i.e. suppose $\mathbf{n}^* = -\mathbf{n}$, then $\mathbf{b}^* = \mathbf{t} \times \mathbf{n}^* = \mathbf{t} \times (-\mathbf{n}) = -\mathbf{b}$ and, from equation (*4.15*),

$$\tau^* \;=\; -\dot{\mathbf{b}}^* \cdot \mathbf{n}^* \;=\; -(-\dot{\mathbf{b}}) \cdot (-\mathbf{n}) \;=\; -\dot{\mathbf{b}} \cdot \mathbf{n} \;=\; \tau$$

Thus τ is independent of the sense of $\mathbf{n}$. Next, suppose we introduce a change in orientation; that is, suppose $s = -s^* + \text{constant}$. Then $\mathbf{t}^* = -\mathbf{t}$. Also,

$$\mathbf{b}^* \;=\; \mathbf{t}^* \times \mathbf{n} \;=\; -(\mathbf{t} \times \mathbf{n}) \;=\; -\mathbf{b}$$

and

$$\frac{d\mathbf{b}^*}{ds^*} \;=\; \frac{d\mathbf{b}^*}{ds}\,\frac{ds}{ds^*} \;=\; -\frac{d\mathbf{b}}{ds}(-1) \;=\; \frac{d\mathbf{b}}{ds}$$

so that again

$$\tau^* \;=\; -\frac{d\mathbf{b}^*}{ds^*} \cdot \mathbf{n} \;=\; -\frac{d\mathbf{b}}{ds} \cdot \mathbf{n} \;=\; \tau$$

which is the required result.

Example 4.9.

We consider again the helix

$$\mathbf{x} \;=\; a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3, \qquad a > 0,\; b \neq 0$$

Referring to Example 4.8, we have

$$\mathbf{b} \;=\; (a^2 + b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

$$\dot{\mathbf{b}} \;=\; \frac{d\mathbf{b}}{ds} \;=\; \frac{d\mathbf{b}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| \;=\; (a^2 + b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2)$$

The torsion is constant, equal to

$$\tau \;=\; -\dot{\mathbf{b}} \cdot \mathbf{n} \;=\; -(a^2 + b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2) \cdot ((-\cos t)\mathbf{e}_1 - (\sin t)\mathbf{e}_2) \;=\; b/(a^2 + b^2)$$

Note that if $b > 0$ (so that $\tau > 0$) the helix is a right-handed curve as shown in Fig. 4-10(*a*) below. If $b < 0$ (so that $\tau < 0$) the curve is a left-handed helix as shown in Fig. 4-10(*b*). Since the sign of τ is an intrinsic property, we conclude that these two curves cannot be superimposed.

(a) Right-handed helix, $\tau > 0$ (b) Left-handed helix, $\tau < 0$

Fig. 4-10

If the torsion vanishes identically along a curve $\mathbf{x} = \mathbf{x}(s)$, i.e. if $\tau \equiv 0$, then $\dot{\mathbf{b}} = -\tau\mathbf{n} \equiv \mathbf{0}$. Hence $\mathbf{b} = \text{constant} = \mathbf{b}_0$. Now consider

$$\frac{d}{ds}(\mathbf{x}\cdot\mathbf{b}_0) \quad = \quad \dot{\mathbf{x}}\cdot\mathbf{b}_0 \quad = \quad \mathbf{t}\cdot\mathbf{b}_0$$

Since $\mathbf{t}$ and $\mathbf{b}_0$ are orthogonal, then $\frac{d}{ds}(\mathbf{x}\cdot\mathbf{b}_0) \equiv 0$ and, integrating again,

$$\mathbf{x}\cdot\mathbf{b}_0 \quad = \quad \text{constant} \qquad (4.16)$$

That is, $\mathbf{x} = \mathbf{x}(s)$ is a *plane* curve confined to the plane $\mathbf{x}\cdot\mathbf{b}_0 = \text{constant}$. In particular, $\mathbf{x} = \mathbf{x}(s)$ lies in its osculating plane as shown in Fig. 4-11. The converse is also true. Thus

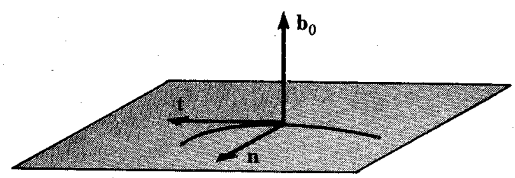


Fig. 4-11

Theorem 4.4. A curve of class ≥ 3 along which $\mathbf{n}$ is of class C^1 is a plane curve if and only if its torsion vanishes identically.

Unless otherwise stated we shall assume that our curves are regular curves of class ≥ 3 and that along the curve $\mathbf{n}$ is of class C^1. In this case it follows that τ is continuous and that $\kappa, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are of class C^1.

A convenient formula is also available for torsion in terms of an arbitrary representation. In Problem 4.19, page 77, we prove

Theorem 4.5. At a point on a curve $\mathbf{x} = \mathbf{x}(t)$ at which $\kappa \neq 0$,

$$\tau \quad = \quad \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

SPHERICAL INDICATRICES

The unit tangent vectors along a curve C generate a curve Γ on the sphere of radius 1 about the origin as shown in Fig. 4-12. The curve Γ is called the *spherical indicatrix* of $\mathbf{t}$.

If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of C, then $\mathbf{x}_1 = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$ will be a representation of Γ. However, in general s will not be a natural parameter along $\mathbf{x}_1 = \mathbf{t}(s)$, since

$$\left|\frac{d\mathbf{x}_1}{ds}\right| \quad = \quad \left|\frac{d\mathbf{t}}{ds}\right| \quad = \quad |\dot{\mathbf{t}}| \quad = \quad |\kappa|$$

In fact $\mathbf{x}_1 = \mathbf{t}(s)$ is a natural representation of Γ iff the curvature $|\kappa| \equiv 1$ along $\mathbf{x} = \mathbf{x}(s)$.

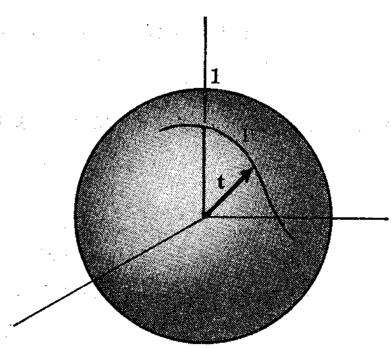


Fig. 4-12

Similarly one considers the spherical indicatrix of the unit normal $\mathbf{x}_2 = \mathbf{n}(s)$ and the spherical indicatrix $\mathbf{x}_3 = \mathbf{b}(s)$ of the unit binormal.

Example 4.10.

Referring to the helix in Examples 4.8 and 4.9, we have

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3 \qquad a > 0,\ b \neq 0$$

$$\mathbf{t} = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

$$\mathbf{n} = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

$$\mathbf{b} = (a^2+b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

Observe that $\mathbf{t}, \mathbf{n}, \mathbf{b}$ all have constant components with respect to $\mathbf{e}_3$, so that their spherical images are circles about the x_3 axis.

The radius of the spherical indicatrix of $\mathbf{t}$ and hence its radius of curvature, is

$$\rho_{\mathbf{t}} = \left(1 - \frac{b^2}{a^2+b^2}\right)^{1/2} = \frac{a}{(a^2+b^2)^{1/2}}$$

The radii of curvature of the spherical indicatrix of $\mathbf{n}$ and $\mathbf{b}$ are $\rho_{\mathbf{n}} = 1$ and $\rho_{\mathbf{b}} = \dfrac{b}{(a^2+b^2)^{1/2}}$ respectively.

Solved Problems

TANGENT LINE AND NORMAL PLANE

4.1. Find the equations of the tangent line and normal plane to the curve

$$\mathbf{x} = (1+t)\mathbf{e}_1 - t^2\mathbf{e}_2 + (1+t^3)\mathbf{e}_3$$

at $t = 1$.

$\mathbf{x}' = \mathbf{e}_1 - 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$, $\quad \mathbf{x}(1) = 2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$, $\quad \mathbf{x}'(1) = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$.

The equation of the tangent line at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (2+k)\mathbf{e}_1 - (1+2k)\mathbf{e}_2 + (2+3k)\mathbf{e}_3$$

The equation of the normal plane is

$$(\mathbf{y}-\mathbf{x}(1))\cdot\mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1-2) + (y_2+1)(-2) + (y_3-2)3 = 0 \quad \text{or} \quad y_1 - 2y_2 + 3y_3 = 10$$

4.2. Find the intersection of the x_1x_2 plane and the tangent lines to the helix

$$\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3 \qquad (t > 0)$$

The tangent line at an arbitrary point $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{x}' \quad \text{or} \quad \mathbf{y} = (\cos t - k\sin t)\mathbf{e}_1 + (\sin t + k\cos t)\mathbf{e}_2 + (t+k)\mathbf{e}_3$$

or, using $\mathbf{x}$ as the position vector,

$$x_1 = \cos t - k\sin t, \quad x_2 = \sin t + k\cos t, \quad x_3 = t + k$$

The equation of the x_1x_2 plane is $x_3 = 0$. Hence along the intersection, $t + k = 0$ or $k = -t$. Thus the intersection is the curve

$$x_1 = \cos t + t\sin t, \quad x_2 = \sin t - t\cos t, \quad x_3 = 0$$

4.3. Show that the tangent vectors along the curve $\mathbf{x} = at\mathbf{e}_1 + bt^2\mathbf{e}_2 + t^3\mathbf{e}_3$ where $2b^2 = 3a$, make a constant angle with the vector $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$.

$$\mathbf{x}' = a\mathbf{e}_1 + 2bt\mathbf{e}_2 + 3t^2\mathbf{e}_3$$

$$|\mathbf{x}'| = (a^2 + 4b^2t^2 + 9t^4)^{1/2} = (a^2 + 6at^2 + 9t^4)^{1/2} = a + 3t^2$$

where we have used $2b^2 = 3a$. Then the angle between the tangent $\mathbf{x}'$ and $\mathbf{a}$ is

$$\cos^{-1}\left\{\frac{(\mathbf{x}' \cdot \mathbf{a})}{|\mathbf{x}'|\,|\mathbf{a}|}\right\} = \cos^{-1}\left\{\frac{a + 3t^2}{(a + 3t^2)\sqrt{2}}\right\} = \cos^{-1}(1/\sqrt{2}) = \pi/4$$

4.4. A curve is called a *general* or *cylindrical helix* or a curve of *constant slope*, if, as in the preceding problem, there is a fixed vector in space, called the *axis* of the helix, such that the angle α between the tangent vectors and the axis is constant. We exclude the case $\alpha = 0$ for which the tangent vectors are all parallel, in which case (see Problem 4.28) the curve is a straight line. Show that a general helix has a natural representation of the form

$$\mathbf{x} = x_1(s^*)\mathbf{e}_1 + x_2(s^*)\mathbf{e}_2 + s^*(\cos\alpha)\mathbf{e}_3$$

We suppose that the helix is positioned so that the origin is on the curve and $\mathbf{e}_3$ is parallel to the axis of the helix. Then

$$\cos\alpha = \cos\measuredangle(\mathbf{t}, \mathbf{e}_3) = \mathbf{t}\cdot\mathbf{e}_3 = \dot{\mathbf{x}}\cdot\mathbf{e}_3 = \dot{x}_3$$

Integrating $\dot{x}_3 = \cos\alpha$ gives

$$x_3 = s\cos\alpha + c, \qquad c = \text{constant}$$

If $\alpha \neq \pi/2$, set $s^* = s + c/(\cos\alpha)$. Then $x_3 = s^*\cos\alpha$ and we have a natural representation of the form

$$\mathbf{x} = x_1(s^*)\mathbf{e}_1 + x_2(s^*)\mathbf{e}_3 + s^*(\cos\alpha)\mathbf{e}_3$$

as required. If $\alpha = \pi/2$, then $x_3 = c = 0$ since the origin is on the curve. Thus

$$\mathbf{x} = x_1(s)\mathbf{e}_1 + x_2(s)\mathbf{e}_3$$

In this case the curve is confined to the x_1x_2 plane.

CURVATURE

4.5. Find the curvature vector $\mathbf{k}$ and curvature $|\kappa|$ on the curve

$$\mathbf{x} = t\mathbf{e}_1 + \tfrac{1}{2}t^2\mathbf{e}_2 + \tfrac{1}{3}t^3\mathbf{e}_3$$

at the point $t = 1$.

$$\mathbf{x}' = \mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3, \qquad |\mathbf{x}'| = (1 + t^2 + t^4)^{1/2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1 + t^2 + t^4)^{-1/2}(\mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3)$$

$$\begin{aligned}\mathbf{t}' &= (1 + t^2 + t^4)^{-1/2}(\mathbf{e}_2 + 2t\mathbf{e}_3) - (\mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3)(1 + t^2 + t^4)^{-3/2}(t + 2t^3)\\ &= -(1 + t^2 + t^4)^{-3/2}[(2t^3 + t)\mathbf{e}_1 + (t^4 - 1)\mathbf{e}_2 - (t^3 + 2t)\mathbf{e}_3]\\ \mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| &= -(1 + t^2 + t^4)^{-2}[(2t^3 + t)\mathbf{e}_1 + (t^4 - 1)\mathbf{e}_2 - (t^3 + 2t)\mathbf{e}_3]\end{aligned}$$

At $t = 1$ we have $\mathbf{k} = -\frac{1}{3}(\mathbf{e}_1 - \mathbf{e}_3)$ and $|\kappa| = |\mathbf{k}| = \frac{1}{3}\sqrt{2}$.

4.6. Show that the curvature $|\kappa^*|$ of the projection of a general helix (see Problem 4.4) onto a plane normal to its axis is given by $|\kappa^*| = |\kappa|/\sin^2\alpha$ where $\alpha \neq 0$ is the angle between the axis and the tangent vectors of the helix and $|\kappa|$ is the curvature of the helix.

Let $\mathbf{x} = \mathbf{x}(s)$ be the helix and $\mathbf{u}$, a vector of unit length, its axis. As indicated in Fig. 4-13, the projection on a plane perpendicular to $\mathbf{u}$ and passing through the origin is the curve

$$\mathbf{x}^* \;=\; \mathbf{x}(s) - (\mathbf{u} \cdot \mathbf{x}(s))\mathbf{u}$$

Note that in general s will not be a natural parameter for the projection $\mathbf{x}^* = \mathbf{x}^*(s)$. Differentiating gives

$$\frac{d\mathbf{x}^*}{ds} \;=\; \mathbf{t} - (\mathbf{u} \cdot \mathbf{t})\mathbf{u} \;=\; \mathbf{t} - (\cos\alpha)\mathbf{u}$$

and

$$\left|\frac{d\mathbf{x}^*}{ds}\right| \;=\; \left[\frac{d\mathbf{x}^*}{ds} \cdot \frac{d\mathbf{x}^*}{ds}\right]^{1/2}$$
$$=\; [(\mathbf{t} \cdot \mathbf{t}) - 2(\cos\alpha)(\mathbf{t} \cdot \mathbf{u}) + (\cos^2\alpha)(\mathbf{u} \cdot \mathbf{u})]^{1/2}$$
$$=\; [1 - 2\cos^2\alpha + \cos^2\alpha]^{1/2} \;=\; \sin\alpha, \qquad 0 < \alpha < \pi$$

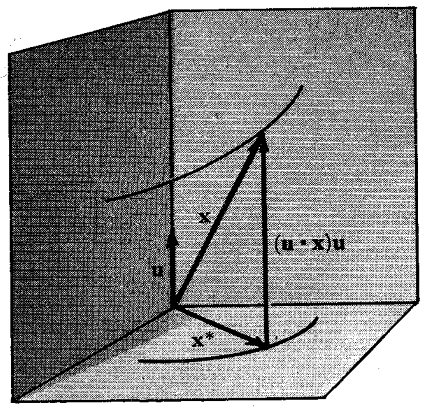


Fig. 4-13

Thus $$\mathbf{t}^* \;=\; \frac{d\mathbf{x}^*}{ds}\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| \;=\; \frac{\mathbf{t} - (\cos\alpha)\mathbf{u}}{\sin\alpha} \quad\text{and}\quad \frac{d\mathbf{t}^*}{ds} \;=\; \dot{\mathbf{t}}/\sin\alpha$$

and finally $$|\kappa^*| \;=\; |\dot{\mathbf{t}}^*| \;=\; \left|\frac{d\mathbf{t}^*}{ds}\right|\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| \;=\; |\dot{\mathbf{t}}|/\sin^2\alpha \;=\; |\kappa|/\sin^2\alpha$$

4.7. Prove Theorem 4.2: Along a curve $\mathbf{x} = \mathbf{x}(t)$ of class $\geqq 2$, $|\kappa| = |\mathbf{x}' \times \mathbf{x}''|/|\mathbf{x}'|^3$.

$$\mathbf{x}' \;=\; \frac{d\mathbf{x}}{dt} \;=\; \frac{d\mathbf{x}}{ds}\frac{ds}{dt} \;=\; \dot{\mathbf{x}}s', \qquad \mathbf{x}'' \;=\; \frac{d}{dt}(\dot{\mathbf{x}}s') \;=\; \dot{\mathbf{x}}\frac{ds'}{dt} + s'\frac{d\dot{\mathbf{x}}}{dt} \;=\; \dot{\mathbf{x}}s'' + (s')^2\ddot{\mathbf{x}}$$

$$\mathbf{x}' \times \mathbf{x}'' \;=\; (\dot{\mathbf{x}}s') \times (\dot{\mathbf{x}}s'' + \ddot{\mathbf{x}}(s')^2) \;=\; (s')^3(\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) \;=\; |\mathbf{x}'|^3(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})$$

where we have used $s' = ds/dt = |\mathbf{x}'|$. Then

$$|\mathbf{x}' \times \mathbf{x}''| \;=\; |\mathbf{x}'|^3\,|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}| \;=\; |\mathbf{x}'|^3\,|\dot{\mathbf{x}}|\,|\ddot{\mathbf{x}}| \sin\measuredangle(\dot{\mathbf{x}}, \ddot{\mathbf{x}})$$

But $\dot{\mathbf{x}} = \mathbf{t}$ and $\ddot{\mathbf{x}} = \dot{\mathbf{t}}$ are orthogonal, $|\dot{\mathbf{x}}| = 1$, and $|\ddot{\mathbf{x}}| = |\dot{\mathbf{t}}| = |\kappa|$. Hence $|\kappa| = |\mathbf{x}' \times \mathbf{x}''|/|\mathbf{x}'|^3$.

4.8. Show that a curve $\mathbf{x} = \mathbf{x}(t)$ of class $\geqq 2$ is a straight line if $\mathbf{x}(t)$ and $\mathbf{x}''(t)$ are linearly dependent for all t.

Since $|\mathbf{x}' \times \mathbf{x}''| = 0$ if $\mathbf{x}'$ and $\mathbf{x}''$ are dependent, we have $|\kappa| = |\mathbf{x}' \times \mathbf{x}''|/|\mathbf{x}'|^3 = 0$ for all t. Hence from Theorem 4.1, page 63, $\mathbf{x} = \mathbf{x}(t)$ is a straight line.

4.9. Let $\mathbf{x} = \mathbf{x}(s)$ be of class $\geqq 2$ and let $\Delta\theta$ denote the angle between the unit tangent $\mathbf{t}(s)$ at $\mathbf{x}(s)$ and $\mathbf{t}(s + \Delta s)$ at a neighboring point $\mathbf{x}(s + \Delta s)$, $\Delta s > 0$, as shown in Fig. 4-14(*a*). Show that the curvature

$$|\kappa| \;=\; \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} \;=\; \frac{d\theta}{ds}$$

Namely, $|\kappa|$ is a measure of rate of change of direction of the tangent with respect to arc length.

(*a*) (*b*)

Fig. 4-14

Since t is a unit vector, $|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)|$ is the base of an isosceles triangle with sides of length 1, as shown in Fig. 4-14(*b*). Hence

$$|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)| \;=\; 2\sin(\tfrac{1}{2}\Delta\theta) \;=\; \Delta\theta + o(\Delta\theta)$$

where we have used the Taylor expansion for the sine function. Then

$$|\kappa| \;=\; |\dot{\mathbf{t}}| \;=\; \left|\lim_{\Delta s\to 0}\frac{\mathbf{t}(s+\Delta s)-\mathbf{t}(s)}{\Delta s}\right| \;=\; \lim_{\Delta s\to 0}\left|\frac{\mathbf{t}(s+\Delta s)-\mathbf{t}(s)}{\Delta s}\right|$$

$$= \lim_{\Delta s\to 0}\frac{\Delta\theta + o(\Delta\theta)}{\Delta s} \;=\; \lim_{\Delta s\to 0}\left[\frac{\Delta\theta}{\Delta s}\left(1+\frac{o(\Delta\theta)}{\Delta\theta}\right)\right]$$

Since $\lim_{\Delta s\to 0}\Delta\theta = 0$, then $\lim_{\Delta s\to 0}\frac{o(\Delta\theta)}{\Delta\theta} = 0$ and $|\kappa| = \lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$.

4.10. Show that a curve $\mathbf{x}=\mathbf{x}(s)$ of class $\geqq 2$ is a straight line if all tangent lines have a common intersection.

Suppose $\mathbf{y}_0$ is a common intersection of the tangent lines $\mathbf{y} = \mathbf{x}(s) + k\mathbf{t}(s)$. Then for each s along the curve, there exists $k = k(s)$ such that

$$\mathbf{y}_0 \;=\; \mathbf{x}(s) + k(s)\,\mathbf{t}(s)$$

Note that along a tangent line $\mathbf{y} = \mathbf{x}(s) + k\mathbf{t}(s)$ we have $k = 0$ if and only if $\mathbf{y}$ is on the curve. Thus if we assume that the curve is simple, which is the case locally, $k(s) \neq 0$ except perhaps for a single value of s, say s_0. Differentiating the above,

$$0 \;=\; \dot{\mathbf{x}} + \dot{k}\mathbf{t} + k\dot{\mathbf{t}} \;=\; (1+\dot{k})\mathbf{t} + k\dot{\mathbf{t}}$$

Multiplying by $\dot{\mathbf{t}}$,
$$0 \;=\; (1+\dot{k})(\mathbf{t}\cdot\dot{\mathbf{t}}) + k(\dot{\mathbf{t}}\cdot\dot{\mathbf{t}})$$

Since $\dot{\mathbf{t}}$ is orthogonal to $\mathbf{t}$, then $0 = k|\dot{\mathbf{t}}|^2$ or $0 = k|\kappa|^2$. Since $k(s) \neq 0$ for $s \neq s_0$, $|\kappa| = 0$ for $s \neq s_0$. But $|\kappa|$ is continuous. Hence $|\kappa| = 0$ for all s, and it follows from Theorem 4.1, page 64, that $\mathbf{x} = \mathbf{x}(t)$ is a straight line.

MOVING TRIHEDRON

4.11. Find a continuous unit principal normal and unit binormal along the curve

$$\mathbf{x} \;=\; (3t-t^3)\mathbf{e}_1 + 3t^2\mathbf{e}_2 + (3t+t^3)\mathbf{e}_3$$

$$\mathbf{x}' \;=\; (3-3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3+3t^2)\mathbf{e}_3$$

$$|\mathbf{x}'| \;=\; 3[(1-t^2)^2 + (2t)^2 + (1+t^2)^2]^{1/2} \;=\; 3\sqrt{2}\,(1+2t^2+t^4)^{1/2} \;=\; 3\sqrt{2}\,(1+t^2)$$

$$\mathbf{t} \;=\; \frac{x'}{|x'|} \;=\; \frac{1}{\sqrt{2}\,(1+t^2)}[(1-t^2)\mathbf{e}_1 + 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3]$$

$$\mathbf{k} \;=\; \dot{\mathbf{t}} \;=\; \frac{\mathbf{t}'}{|\mathbf{x}'|} \;=\; \frac{-2t\mathbf{e}_1 + (1-t^2)\mathbf{e}_2}{3(1+t^2)^3}, \qquad |\mathbf{k}| \;=\; \frac{[(2t)^2 + (1-t^2)^2]^{1/2}}{3(1+t^2)^3} \;=\; \frac{1}{3(1+t^2)^2}$$

Since $\mathbf{k} \neq \mathbf{0}$ for all t, we can choose

$$\mathbf{n} \;=\; \frac{\mathbf{k}}{|\mathbf{k}|} \;=\; \frac{-2t}{1+t^2}\mathbf{e}_1 + \frac{1-t^2}{1+t^2}\mathbf{e}_2$$

and
$$\mathbf{b} \;=\; \mathbf{t}\times\mathbf{n} \;=\; \frac{1}{\sqrt{2}(1+t^2)^2}\det\begin{pmatrix}\mathbf{e}_1 & 1-t^2 & -2t\\ \mathbf{e}_2 & 2t & 1-t^2\\ \mathbf{e}_3 & 1+t^2 & 0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2}[-(1+t^2)(1-t^2)\mathbf{e}_1 - 2t(1+t^2)\mathbf{e}_2 + (1+t^2)^2\mathbf{e}_3]$$

$$= \frac{1}{\sqrt{2}(1+t^2)}[(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3]$$

4.12. Show that along a curve $\mathbf{x} = \mathbf{x}(t)$ the vector $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to $\mathbf{t}$ and $\mathbf{n}$ are $|\mathbf{x}'|'$ and $\kappa|\mathbf{x}'|^2$ respectively.

Differentiating $\mathbf{x}' = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \mathbf{t}s'$ with respect to t, we obtain

$$\mathbf{x}'' = \mathbf{t}s'' + \mathbf{t}'s' = \mathbf{t}s'' + \dot{\mathbf{t}}s'^2 = \mathbf{t}s'' + \mathbf{n}\kappa s'^2$$

where from equation (4.6), $\dot{\mathbf{t}} = \kappa\mathbf{n}$. It follows that $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to $\mathbf{t}$ and $\mathbf{n}$ are $|\mathbf{x}'|' = s''$ and $\kappa|\mathbf{x}'|^2 = \kappa s'^2$ respectively.

4.13. (*a*) If $\mathbf{x}'$ and $\mathbf{x}''$ are linearly independent at a point $\mathbf{x}$ along $\mathbf{x} = \mathbf{x}(t)$, show that the osculating plane at $\mathbf{x}$ is $[(\mathbf{y}-\mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$. (*b*) Use this formula to find the osculating plane to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$.

(*a*) We saw in the above problem that $\mathbf{x}''$ is parallel to the osculating plane and we know that $\mathbf{x}'$, being a multiple of $\mathbf{t}$, is parallel to the osculating plane. Since we are given that $\mathbf{x}'$ and $\mathbf{x}''$ are independent, it follows that $\mathbf{x}' \times \mathbf{x}''$ is a nonzero vector normal to the osculating plane at $\mathbf{x}$. Therefore the equation of the osculating plane at $\mathbf{x}$ is $[(\mathbf{y}-\mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$.

(*b*) $\mathbf{x}' = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$, $\mathbf{x}'' = 2\mathbf{e}_2 + 6t\mathbf{e}_3$

Thus the osculating plane at $t = 1$ is $[(\mathbf{y}-\mathbf{x}(1))\mathbf{x}'(1)\mathbf{x}''(1)] = 0$ or

$$\det\begin{pmatrix} y_1 - 1 & 1 & 0 \\ y_2 - 1 & 2 & 2 \\ y_3 - 1 & 3 & 6 \end{pmatrix} = 0$$

from which

$$(y_1 - 1)6 - (y_2 - 1)6 + (y_3 - 1)2 = 0 \quad \text{or} \quad 3y_1 - 3y_2 + y_3 = 1$$

4.14. Show that the points on the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$ at which the osculating planes pass through a fixed point are confined to a plane.

Let $\mathbf{x}_\lambda = a(\cos t_\lambda)\mathbf{e}_1 + a(\sin t_\lambda)\mathbf{e}_2 + bt_\lambda\mathbf{e}_3$ denote the points on the helix at which the osculating plane passes through the point $\mathbf{y}_0 = y_{01}\mathbf{e}_1 + y_{02}\mathbf{e}_2 + y_{03}\mathbf{e}_3$. It is easily verified that $\mathbf{x}'$ and $\mathbf{x}''$ are independent for all t. Thus the equation of the osculating plane at $\mathbf{x}$ is $[(\mathbf{y}-\mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$. Hence for each λ,

$$[(\mathbf{y}_0 - \mathbf{x}(t_\lambda))\mathbf{x}'(t_\lambda)\mathbf{x}''(t_\lambda)] = 0$$

or

$$\det\begin{pmatrix} y_{01} - a\cos t_\lambda & -a\sin t_\lambda & -a\cos t_\lambda \\ y_{02} - a\sin t_\lambda & a\cos t_\lambda & -a\sin t_\lambda \\ y_{03} - bt_\lambda & b & 0 \end{pmatrix} = 0$$

or, expanding,

$$(by_{02})(a\cos t_\lambda) - (by_{01})(a\sin t_\lambda) + (a^2)(bt_\lambda) = a^2 y_{03}$$

Thus the $\mathbf{x}_\lambda = a(\cos t_\lambda)\mathbf{e}_1 + a(\sin t_\lambda)\mathbf{e}_2 + bt_\lambda\mathbf{e}_3$ lie on the plane $by_{02}x_1 - by_{01}x_2 + a^2x_3 = a^2y_{03}$.

4.15. Prove Theorem 4.3: If $\mathbf{x}(s_0)$ is a point of inflection on an analytic curve $\mathbf{x} = \mathbf{x}(s)$ which is not a straight line, then there exists a continuous principal normal unit vector $\mathbf{n}(s)$ to the curve in a neighborhood of s_0.

Let $\mathbf{x}^{(k)}(s_0)$ be the first nonvanishing derivative of $\mathbf{x}(s)$ at s_0 of order $k > 1$. Since $\mathbf{x} = \mathbf{x}(s)$ is not a straight line, such a nonvanishing derivative exists. In fact $k > 2$, since $\dot{\mathbf{t}}(s_0) = \ddot{\mathbf{x}}(s_0) = \mathbf{0}$ at a point of inflection. Thus we can write

$$\mathbf{x} = \mathbf{x}(s_0) + \dot{\mathbf{x}}(s_0)(s - s_0) + \frac{\mathbf{x}^{(k)}(s_0)}{k!}(s - s_0)^k + \frac{\mathbf{x}^{(k+1)}(s_0)(s - s_0)^{k+1}}{(k+1)!} + \cdots$$

Then $$\mathbf{t} = \dot{\mathbf{x}} = \dot{\mathbf{x}}(s_0) + \frac{\mathbf{x}^{(k)}(s_0)(s-s_0)^{k-1}}{(k-1)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)^{k}}{k!} + \cdots$$

and $$\mathbf{k} = \dot{\mathbf{t}} = \frac{\mathbf{x}^{(k)}(s_0)(s-s_0)^{k-2}}{(k-2)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)^{k-1}}{(k-1)!} + \cdots$$
$$= (s-s_0)^{k-2}\left[\frac{\mathbf{x}^{(k)}(s_0)}{(k-2)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)}{(k-1)!} + \cdots\right] = (s-s_0)^{k-2}\mathbf{w}(s)$$

where $\mathbf{w}(s)$ is analytic at s_0 and $\mathbf{w}(s_0) = \dfrac{\mathbf{x}^{(k)}(s_0)}{(k-2)} \neq \mathbf{0}$. Since $\mathbf{w}(s)$ is certainly continuous at s_0, there exists a neighborhood $S_\delta(s_0)$ such that $\mathbf{w}(s) \neq 0$ for s in $S_\delta(s_0)$. Now consider the vector $\mathbf{n} = \mathbf{w}(s)/|\mathbf{w}(s)|$. For s in $S_\delta(s_0)$, the vector $\mathbf{n}$ is continuous, of unit length, and $\mathbf{k} = \dot{\mathbf{t}}$ is a multiple of $\mathbf{n}$; for,
$$\mathbf{k} = (s-s_0)^{k-2}\mathbf{w}(s) = (s-s_0)^{k-2}|\mathbf{w}(s)|\frac{\mathbf{w}(s)}{|\mathbf{w}(s)|} = (s-s_0)^{k-2}|\mathbf{w}(s)|\mathbf{n} = \kappa(s)n$$
which is the required result. Note that for this n, $\kappa = |\mathbf{k}|$ iff k is even $((s-s_0)^{k-2} \geqq 0)$. Otherwise
$$\kappa = \begin{cases} |\mathbf{k}| & \text{for } s \geqq s_0 \\ -|\mathbf{k}| & \text{for } s < s_0 \end{cases}$$

TORSION

4.16. Using the formulas of Theorems 4.2 and 4.5, find the curvature and torsion along the curve
$$\mathbf{x} = (3t-t^3)\mathbf{e}_1 + 3t^2\mathbf{e}_2 + (3t+t^3)\mathbf{e}_3$$

$$|\kappa| = \frac{|\mathbf{x}'\times\mathbf{x}''|}{|\mathbf{x}'|^3} = \frac{|[(3-3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3+3t^2)\mathbf{e}_3]\times[-6t\mathbf{e}_1 + 6\mathbf{e}_2 + 6t\mathbf{e}_3]|}{|(3-3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3+3t^2)\mathbf{e}_3|^3}$$
$$= \frac{18|(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|}{27|(1-t^2)\mathbf{e}_1 + 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|^3} = \frac{2}{3(1+t^2)^2}$$

$$\tau = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}'\times\mathbf{x}''|^2} = \frac{(\mathbf{x}'\times\mathbf{x}'')\cdot\mathbf{x}'''}{|\mathbf{x}'\times\mathbf{x}''|^2} = \frac{18[(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3]\cdot 6[-\mathbf{e}_1+\mathbf{e}_3]}{18^2|(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|^2} = \frac{2}{3(1+t^2)^2}$$

Observe that along this curve, $|\kappa| = \tau$.

4.17. Show that along a curve $\mathbf{x} = \mathbf{x}(s)$, $\dddot{\mathbf{x}} = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \tau\kappa\mathbf{b}$.

Differentiating $\ddot{\mathbf{x}} = \dot{\mathbf{t}} = \kappa\mathbf{n}$, we have
$$\dddot{\mathbf{x}} = \kappa\dot{\mathbf{n}} + \dot{\kappa}\mathbf{n} = \kappa\frac{d}{dt}(\mathbf{b}\times\mathbf{t}) + \dot{\kappa}\mathbf{n} = \kappa(\mathbf{b}\times\dot{\mathbf{t}} + \dot{\mathbf{b}}\times\mathbf{t}) + \dot{\kappa}\mathbf{n}$$
where we use the fact that $\mathbf{t},\mathbf{n},\mathbf{b}$ is a right-handed orthonormal triplet and hence $\mathbf{n} = \mathbf{b}\times\mathbf{t}$. Again using $\dot{\mathbf{t}} = \kappa\mathbf{n}$ and also $\dot{\mathbf{b}} = -\tau\mathbf{n}$, we have
$$\dddot{\mathbf{x}} = \kappa^2(\mathbf{b}\times\mathbf{n}) - \kappa\tau(\mathbf{n}\times\mathbf{t}) + \dot{\kappa}\mathbf{n} = \kappa^2(-\mathbf{t}) + \kappa\tau\mathbf{b} + \dot{\kappa}\mathbf{n}$$

4.18. Show that along a curve $\mathbf{x} = \mathbf{x}(s)$, $[\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}] = \kappa^2\tau$.

Using the result of the above problem, we have
$$\ddot{\mathbf{x}}\times\dddot{\mathbf{x}} = \dot{\mathbf{t}}\times\dddot{\mathbf{x}} = \kappa\mathbf{n}\times(-\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \tau\kappa\mathbf{b}) = -\kappa^3(\mathbf{n}\times\mathbf{t}) + \kappa\dot{\kappa}(\mathbf{n}\times\mathbf{n}) + \tau\kappa^2(\mathbf{n}\times\mathbf{b}) = \kappa^3\mathbf{b} + \tau\kappa^2\mathbf{t}$$
since $\mathbf{t},\mathbf{n},\mathbf{b}$ is a right-handed orthonormal triplet. Finally,
$$[\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}] = \dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}\times\dddot{\mathbf{x}} = \mathbf{t}\cdot\ddot{\mathbf{x}}\times\dddot{\mathbf{x}} = \kappa^3(\mathbf{t}\cdot\mathbf{b}) + \tau\kappa^2(\mathbf{t}\cdot\mathbf{t}) = \tau\kappa^2$$
since $\mathbf{t}\cdot\mathbf{b} = 0$ and $\mathbf{t}\cdot\mathbf{t} = 1$.

4.19. Prove Theorem 4.5: At a point on a curve $\mathbf{x} = \mathbf{x}(t)$ where $\kappa \neq 0$, the torsion is given by

$$\tau = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \mathbf{x}'\dot{t}, \qquad \ddot{\mathbf{x}} = \frac{d}{ds}(\mathbf{x}'\dot{t}) = \mathbf{x}'\ddot{t} + \left(\frac{d}{ds}\mathbf{x}'\right)\dot{t} = \mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}^2$$

$$\dddot{\mathbf{x}} = \frac{d}{ds}(\mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}^2) = \mathbf{x}'\dddot{t} + \mathbf{x}''\dot{t}\,\ddot{t} + \mathbf{x}''2\dot{t}\,\ddot{t} + \mathbf{x}'''\dot{t}^3 = \mathbf{x}'\dddot{t} + 3\mathbf{x}''\dot{t}\,\ddot{t} + \mathbf{x}'''\dot{t}^3$$

Hence

$$\begin{aligned}
[\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}] &= (\mathbf{x}'\dot{t}) \cdot (\mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}^2) \times (\mathbf{x}'\dddot{t} + 3\mathbf{x}''\dot{t}\,\ddot{t} + \mathbf{x}'''\dot{t}^3) \\
&= (\mathbf{x}'\dot{t}) \cdot [3\ddot{t}^2\dot{t}(\mathbf{x}' \times \mathbf{x}'') + \ddot{t}\dot{t}^3(\mathbf{x}' \times \mathbf{x}''') + \dot{t}^2\dddot{t}(\mathbf{x}'' \times \mathbf{x}') + \dot{t}^5(\mathbf{x}'' \times \mathbf{x}''')] \\
&= \dot{t}^2\ddot{t}^2[\mathbf{x}'\mathbf{x}'\mathbf{x}''] + \ddot{t}\dot{t}^4[\mathbf{x}'\mathbf{x}'\mathbf{x}''] + \dot{t}^3\dddot{t}[\mathbf{x}'\mathbf{x}''\mathbf{x}'] + \dot{t}^6[\mathbf{x}'\mathbf{x}''\mathbf{x}''']
\end{aligned}$$

Since $[\mathbf{x}'\mathbf{x}'\mathbf{x}''] = 0$,

$$[\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}] = \dot{t}^6[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] \quad \text{or} \quad [\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}] = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}'|^6}$$

since $\dot{t} = \dfrac{dt}{ds} = \dfrac{1}{ds/dt} = \dfrac{1}{|\mathbf{x}'|}$. Using the results of the above problem and $\kappa = \dfrac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3}$ (Theorem 4.2), we have

$$\tau = \frac{[\dot{\mathbf{x}}\ddot{\mathbf{x}}\dddot{\mathbf{x}}]}{\kappa^2} = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{\kappa^2|\mathbf{x}'|^6} = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

4.20. Show that a curve is a plane curve if all osculating planes have a common point of intersection.

The equation of the osculating plane at $\mathbf{x}$ is $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{b} = 0$. Thus if $\mathbf{y}_0$ is the common intersection, $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{b} = 0$ for all s. Differentiating and using the fact that $\dot{\mathbf{x}} = \mathbf{t}$ and $\mathbf{b}$ are orthogonal, we have

$$(\mathbf{y}_0 - \mathbf{x}) \cdot \dot{\mathbf{b}} = 0 \quad \text{or} \quad \tau(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{n} = 0$$

since $\dot{\mathbf{b}} = -\tau\mathbf{n}$. Now suppose the curve is not a plane curve. Then from Theorem 4.4, page 70, there exists a point s_0 and hence a neighborhood $S_\delta(s_0)$ where $\tau \neq 0$. Thus in $S_\delta(s_0)$, we have $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{n} = 0$; that is, $\mathbf{y}_0 - \mathbf{x}$ is orthogonal to $\mathbf{n}$. But also, $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{b} = 0$; hence $\mathbf{y}_0 - \mathbf{x}$ is parallel to $\mathbf{t}$. Thus there exists $k = k(s)$ such that for s in $S_\delta(s_0)$, $\mathbf{y}_0 - \mathbf{x} = k\mathbf{t}$ or $\mathbf{y}_0 = \mathbf{x} + k\mathbf{t}$; that is (see Problem 4.10), the point $\mathbf{y}_0$ is also the intersection of the tangent lines. But then in $S_\delta(s_0)$ the curve is a straight line, which is impossible since $\tau \neq 0$ in $S_\delta(s_0)$. Hence $\tau = 0$ for all s and the curve is a plane curve.

4.21. Show that a curve is a general helix if and only if τ/κ is constant where $\kappa \neq 0$ and $\tau = 0$ whenever $\kappa = 0$.

Suppose $\mathbf{x} = \mathbf{x}(s)$ is a general helix with axis $\mathbf{u}$ of unit length and that $\mathbf{t} \cdot \mathbf{u} = \cos\alpha$. Recall (Problem 4.4) that we have excluded the case $\alpha = 0$ which would correspond to a straight line parallel to $\mathbf{u}$. Thus for each s there is a unique unit vector $\mathbf{b}^*(s)$ orthogonal to $\mathbf{t}(s)$ such that $\mathbf{u} = \mathbf{t}\cos\alpha + \mathbf{b}^*\sin\alpha$. Evidently $\mathbf{b}^*(s)$ and also $\mathbf{n}^*(s) = \mathbf{b}^*(s) \times \mathbf{t}(s)$ are continuously differentiable. Differentiating $\mathbf{t} \cdot \mathbf{u} = \cos\alpha$ we obtain $\dot{\mathbf{t}} \cdot (\mathbf{t}\cos\alpha + \mathbf{b}^*\sin\alpha) = 0$ and, since $\dot{\mathbf{t}} \cdot \mathbf{t} = 0$, there follows $\dot{\mathbf{t}} \cdot \mathbf{b}^* = 0$. Thus $\mathbf{n} = \mathbf{n}^*$ and $\mathbf{b} = \mathbf{b}^*$ can be chosen respectively as the principal unit normal and unit binormal along the curve. From

$$\mathbf{0} = \dot{\mathbf{u}} = \dot{\mathbf{t}}\cos\alpha + \dot{\mathbf{b}}\sin\alpha = \kappa\mathbf{n}\cos\alpha - \tau\mathbf{n}\sin\alpha$$

we conclude that $\tau/\kappa = \cot\alpha = \text{constant}$ whenever $\kappa \neq 0$ and that $\tau = 0$ whenever $\kappa = 0$. Conversely, suppose that $\tau/\kappa = \cot\alpha$ whenever $\kappa \neq 0$ and $\tau = 0$ whenever $\kappa = 0$. Then $\kappa\cos\alpha - \tau\sin\alpha = 0$ and hence $\dot{\mathbf{t}}\cos\alpha + \dot{\mathbf{b}}\sin\alpha = \mathbf{0}$ or, integrating, $\mathbf{t}\cos\alpha + \mathbf{b}\sin\alpha = \mathbf{u} = \text{constant}$. Thus $\mathbf{u}$ is of unit length and $\mathbf{t} \cdot \mathbf{u} = \cos\alpha = \text{constant}$; that is, $\mathbf{x} = \mathbf{x}(s)$ is a helix.

4.22. Show that along a regular curve $\mathbf{x} = \mathbf{x}(s)$ of class $\geqq 4$,

$$[\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}\,\mathbf{x}^{(4)}] \;=\; \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$$

and hence $\mathbf{x} = \mathbf{x}(s)$ is a general helix if and only if $[\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}\,\mathbf{x}^{(4)}] = 0$.

Referring to Problem 4.18, $\ddot{\mathbf{x}} \times \dddot{\mathbf{x}} = \kappa^3\mathbf{b} + \tau\kappa^2\mathbf{t}$. Differentiating,

$$\ddot{\mathbf{x}} \times \mathbf{x}^{(4)} \;=\; \left(\frac{d}{ds}\kappa^3\right)\mathbf{b} + \kappa^3\dot{\mathbf{b}} + \left(\frac{d}{ds}\tau\kappa^2\right)\mathbf{t} + \tau\kappa^2\dot{\mathbf{t}}$$

$$= \left(\frac{d}{ds}\kappa^3\right)\mathbf{b} - \tau\kappa^3\mathbf{n} + \left(\frac{d}{ds}\tau\kappa^2\right)\mathbf{t} + \tau\kappa^3\mathbf{n} \;=\; \left(\frac{d}{ds}\kappa^3\right)\mathbf{b} + \left(\frac{d}{ds}\tau\kappa^2\right)\mathbf{t}$$

Then, applying Problem 4.17,

$$\dddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \times \mathbf{x}^{(4)} \;=\; (-\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \tau\kappa\mathbf{b}) \cdot \left[\left(\frac{d}{ds}\kappa^3\right)\mathbf{b} + \left(\frac{d}{ds}\tau\kappa^2\right)\mathbf{t}\right] \;=\; -\kappa^2\frac{d}{ds}\tau\kappa^2 + \tau\kappa\frac{d}{ds}\kappa^3$$

$$= -\kappa^4\dot{\tau} - 2\tau\kappa^3\dot{\kappa} + 3\tau\kappa^3\dot{\kappa} \;=\; -\kappa^3[\kappa\dot{\tau} - \dot{\kappa}\tau] \;=\; -\kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$$

Thus
$$[\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}\,\mathbf{x}^{(4)}] \;=\; -[\dddot{\mathbf{x}}\,\ddot{\mathbf{x}}\,\mathbf{x}^{(4)}] \;=\; \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$$

If $\mathbf{x} = \mathbf{x}(s)$ is a general helix, then everywhere on the curve either $\kappa = 0$ or τ/κ is constant so that $[\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}\,\mathbf{x}^{(4)}] = 0$. Conversely, if $[\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}\,\mathbf{x}^{(4)}] = 0$ and if there are no points of inflection so that $\kappa \neq 0$, then we can conclude that $\kappa/\tau =$ constant and therefore $\mathbf{x} = \mathbf{x}(s)$ is a general helix.

SPHERICAL INDICATRICES

4.23. Show that the tangent to the spherical indicatrix of the tangent to a curve C is parallel to the principal normal to C at corresponding points.

Let $\mathbf{x}_1 = \mathbf{t}(s)$ denote the spherical indicatrix of the tangent to the curve $\mathbf{x} = \mathbf{x}(s)$. But a tangent vector to $\mathbf{x}_1 = \mathbf{t}(s)$ is

$$\frac{d\mathbf{x}_1}{ds} \;=\; \dot{\mathbf{t}}(s) \;=\; \kappa(s)\,\mathbf{n}(s)$$

which gives the required result.

4.24. Show that the curvature of the spherical indicatrix $\mathbf{x}_1 = \mathbf{t}(s)$ of the tangent is $\kappa_1^2 = (\kappa^2 + \tau^2)/\kappa^2$.

From Theorem 4.2, page 64, $|\kappa_1| = \dfrac{|\dot{\mathbf{t}} \times \ddot{\mathbf{t}}|}{|\dot{\mathbf{t}}|^3} = \dfrac{|\ddot{\mathbf{x}} \times \dddot{\mathbf{x}}|}{|\kappa|^3}$. From Problem 4.18,

$$|\kappa_1| \;=\; \frac{|\kappa^3\mathbf{b} + \tau\kappa^2\mathbf{t}|}{|\kappa|^3} \;=\; \frac{|\kappa\mathbf{b} + \tau\mathbf{t}|}{|\kappa|}$$

or
$$\kappa_1^2 \;=\; \frac{|\kappa\mathbf{b} + \tau\mathbf{t}|^2}{\kappa^2} \;=\; \frac{(\kappa\mathbf{b} + \tau\mathbf{t}) \cdot (\kappa\mathbf{b} + \tau\mathbf{t})}{\kappa^2} \;=\; \frac{\kappa^2 + \tau^2}{\kappa^2}$$

4.25. Show that the torsion of the binormal indicatrix $\mathbf{x}_3 = \mathbf{b}(s)$ of a sufficiently regular curve is $\tau_3 = \dfrac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\tau(\kappa^2 + \tau^2)}$.

We use Theorem 4.5, page 70, to find τ_3. From $\dot{\mathbf{b}} = -\tau\mathbf{n}$ we have

$$-\ddot{\mathbf{b}} \;=\; \dot{\tau}\mathbf{n} + \tau\dot{\mathbf{n}} \;=\; \dot{\tau}\mathbf{n} + \tau\frac{d}{ds}(\mathbf{b} \times \mathbf{t}) \;=\; \dot{\tau}\mathbf{n} + \tau[(\dot{\mathbf{b}} \times \mathbf{t}) + (\mathbf{b} \times \dot{\mathbf{t}})]$$

$$= \dot{\tau}\mathbf{n} + \tau[-\tau(\mathbf{n} \times \mathbf{t}) + \kappa(\mathbf{b} \times \mathbf{n})] \;=\; \dot{\tau}\mathbf{n} + \tau^2\mathbf{b} - \kappa\tau\mathbf{t}$$

Then
$$\dot{\mathbf{b}} \times \ddot{\mathbf{b}} \;=\; (\tau\mathbf{n}) \times (\dot{\tau}\mathbf{n} + \tau^2\mathbf{b} - \kappa\tau\mathbf{t}) \;=\; \tau^3(\mathbf{n} \times \mathbf{b}) - \tau^2\kappa(\mathbf{n} \times \mathbf{t}) \;=\; \tau^3\mathbf{t} + \tau^2\kappa\mathbf{b} \;=\; \tau^2(\tau\mathbf{t} + \kappa\mathbf{b})$$

and
$$\dot{\mathbf{b}} \times \dddot{\mathbf{b}} = \frac{d}{ds}(\dot{\mathbf{b}} \times \ddot{\mathbf{b}}) = \frac{d}{ds}\tau^2(\tau\mathbf{t} + \kappa\mathbf{b}) = 2\tau\dot{\tau}(\tau\mathbf{t} + \kappa\mathbf{b}) + \tau^2(\dot{\tau}\mathbf{t} + \tau\dot{\mathbf{t}} + \dot{\kappa}\mathbf{b} + \kappa\dot{\mathbf{b}})$$
$$= 2\tau\dot{\tau}(\tau\mathbf{t} + \kappa\mathbf{b}) + \tau^2(\dot{\tau}\mathbf{t} + \kappa\tau\mathbf{n} + \dot{\kappa}\mathbf{b} - \tau\kappa\mathbf{n})$$
$$= 2\tau\dot{\tau}(\tau\mathbf{t} + \kappa\mathbf{b}) + \tau^2(\dot{\tau}\mathbf{t} + \dot{\kappa}\mathbf{b}) = 3\tau^2\dot{\tau}\mathbf{t} + (2\tau\dot{\tau}\kappa + \tau^2\dot{\kappa})\mathbf{b}$$

Thus
$$[\dot{\mathbf{b}}\ddot{\mathbf{b}}\dddot{\mathbf{b}}] = -\ddot{\mathbf{b}} \cdot (\dot{\mathbf{b}} \times \dddot{\mathbf{b}}) = (\dot{\tau}\mathbf{n} + \tau^2\mathbf{b} - \kappa\tau\mathbf{t}) \cdot [3\tau^2\dot{\tau}\mathbf{t} + (2\tau\dot{\tau}\kappa + \dot{\kappa}\tau^2)\mathbf{b}]$$
$$= \tau^3[-3\kappa\dot{\tau} + 2\kappa\dot{\tau} + \tau\dot{\kappa}] = \tau^3(\tau\dot{\kappa} - \kappa\dot{\tau})$$

and finally
$$\tau_3 = \frac{[\dot{\mathbf{b}}\ddot{\mathbf{b}}\dddot{\mathbf{b}}]}{|\dot{\mathbf{b}} \times \ddot{\mathbf{b}}|^2} = \frac{\tau^3(\tau\dot{\kappa} - \kappa\dot{\tau})}{\tau^4(\kappa^2 + \tau^2)} = \frac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\tau(\kappa^2 + \tau^2)}$$

Supplementary Problems

4.26. Find the intersection of the x_1x_2 plane and the normal plane to the curve $\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$ at the point $t = \pi/2$. *Ans.* $x_1 = -\pi/2,\ x_2 = k,\ x_3 = 0,\ -\infty < k < \infty$

4.27. Find the intersection of the x_1x_2 plane and the tangent line to the curve $\mathbf{x} = (1+t)\mathbf{e}_1 - t^2\mathbf{e}_2 + (1+t^3)\mathbf{e}_3$ at $t = 1$. *Ans.* (4/3, 1/3, 0)

4.28. Show that a curve is a straight line if all tangent lines are parallel.

4.29. Let $\Delta\theta$ be the angle between the unit tangents $t(s+\Delta s)$ and $t(s)$, $\Delta s > 0$. Show that $\lim_{\Delta s \to 0} \Delta\theta = 0$.

4.30. Show that along the plane curve $\mathbf{x} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2$ the curvature is
$$|\kappa| = \frac{|x_1'x_2'' - x_2'x_1''|}{[(x_1')^2 + (x_2')^2]^{3/2}}$$

4.31. Show that a curve is a straight line if $\mathbf{x}'$ and $\mathbf{x}''$ are linearly dependent for all t.

4.32. Find the curvature along the curve $\mathbf{x} = (t - \sin t)\mathbf{e}_1 + (1 - \cos t)\mathbf{e}_2 + t\mathbf{e}_3$.
Ans. $|\kappa| = (1 + 4\sin^4 t/2)^{1/2}/(1 + 4\sin^2 t/2)^{3/2}$

4.33. Find the torsion along the curve $\mathbf{x} = (t - \sin t)\mathbf{e}_1 + (1 - \cos t)\mathbf{e}_2 + t\mathbf{e}_3$.
Ans. $\tau = -1/(1 + 4\sin^4 t/2)$

4.34. Show that $\kappa\tau = |\dot{\mathbf{t}} \cdot \dot{\mathbf{b}}|$.

4.35. Show that the curve $\mathbf{x} = t\mathbf{e}_1 + \dfrac{1+t}{t}\mathbf{e}_2 + \dfrac{1-t^2}{t}\mathbf{e}_3$ lies in a plane.

4.36. Find the most general function $f(t)$ so that the curve $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + f(t)\mathbf{e}_3$ will be a plane curve. *Ans.* $f(t) = A\sin t + B\cos t + C$, $A, B, C =$ constant

4.37. Show that the osculating planes at any three points on the curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1}{2}t^2\mathbf{e}_2 + \frac{1}{3}t^3\mathbf{e}_3$ meet at a point lying in the plane determined by these three points.

4.38. Show that $\mathbf{x} = \mathbf{x}(t)$ is a plane curve if and only if $[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] = 0$.

4.39. Prove the converse of Theorem 4.4, page 70: If $\mathbf{x} = \mathbf{x}(s)$ is a plane curve, then $\tau \equiv 0$.

4.40. Show that a curve is a general helix iff the tangent indicatrix is a circle.

4.41. Show that the tangent to the tangent indicatrix of a curve C is parallel to the tangent to the binormal indicatrix at corresponding points.

4.42. Show that the curvature along the binormal indicatrix is $\kappa_3 = (\kappa^2 + \tau^2)/\kappa^2$.

4.43. Show that the torsion along the tangent indicatrix is $\tau_1 = \dfrac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\kappa(\kappa^2 + \tau^2)}$.

Chapter 5

The Theory of Curves

FRENET EQUATIONS

Theorem 5.1. Along a curve $\mathbf{x} = \mathbf{x}(s)$, the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ satisfy

$$\begin{aligned} \dot{\mathbf{t}} &= \kappa\mathbf{n} \\ \dot{\mathbf{n}} &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \dot{\mathbf{b}} &= -\tau\mathbf{n} \end{aligned} \tag{5.1}$$

The above equations are called the *Serret-Frenet* equations of a curve. They are basic in the development of the theory of curves and should be committed to memory. The first and third equations, (*4.6*) and (*4.14*), have already been derived. To obtain the second we simply differentiate $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. This gives, using the first and third equations,

$$\dot{\mathbf{n}} = \dot{\mathbf{b}} \times \mathbf{t} + \mathbf{b} \times \dot{\mathbf{t}} = -\tau(\mathbf{n} \times \mathbf{t}) + \mathbf{b} \times (\kappa\mathbf{n}) = (-\tau)(-\mathbf{b}) + \kappa(-\mathbf{t}) = -\kappa\mathbf{t} + \tau\mathbf{b}$$

Observe that if we write the Frenet equations as

$$\begin{aligned} \dot{\mathbf{t}} &= 0\mathbf{t} + \kappa\mathbf{n} + 0\mathbf{b} \\ \dot{\mathbf{n}} &= -\kappa\mathbf{t} + 0\mathbf{n} + \tau\mathbf{b} \\ \dot{\mathbf{b}} &= 0\mathbf{t} - \tau\mathbf{n} + 0\mathbf{b} \end{aligned}$$

the coefficients of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ form the matrix

$$\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

INTRINSIC EQUATIONS

As a first consequence of the Frenet equations we will show that a curve is completely determined by its curvature and torsion as functions of a natural parameter. We will prove that if C and C^* are two curves in space such that $\kappa(s) = \kappa^*(s)$ and $\tau(s) = \tau^*(s)$ for all s, then C and C^* are the same except for their position in space. For, given two such curves, let C^* be translated so that for some $s = s_0$ the corresponding points on C^* and C are made to coincide. Let C^* then be rotated about this point so that at s_0 the triads $(\mathbf{t}_0^*, \mathbf{n}_0^*, \mathbf{b}_0^*)$ and $(\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0)$ also coincide. Now by differentiating the product $\mathbf{t} \cdot \mathbf{t}^*$ and using the Frenet equations, we obtain

$$\frac{d}{ds}(\mathbf{t} \cdot \mathbf{t}^*) = \mathbf{t} \cdot \dot{\mathbf{t}}^* + \dot{\mathbf{t}} \cdot \mathbf{t}^* = \mathbf{t} \cdot \kappa^*\mathbf{n}^* + \kappa\mathbf{n} \cdot \mathbf{t}^* = \kappa(\mathbf{t} \cdot \mathbf{n}^* + \mathbf{n} \cdot \mathbf{t}^*)$$

Also,

$$\begin{aligned} \frac{d}{ds}(\mathbf{n} \cdot \mathbf{n}^*) &= \mathbf{n} \cdot \dot{\mathbf{n}}^* + \dot{\mathbf{n}} \cdot \mathbf{n}^* = \mathbf{n} \cdot (-\kappa^*\mathbf{t}^* + \tau^*\mathbf{b}^*) + (-\kappa\mathbf{t} + \tau\mathbf{b}) \cdot \mathbf{n}^* \\ &= -\kappa(\mathbf{n} \cdot \mathbf{t}^* + \mathbf{t} \cdot \mathbf{n}^*) + \tau(\mathbf{n} \cdot \mathbf{b}^* + \mathbf{b} \cdot \mathbf{n}^*) \end{aligned}$$

and

$$\frac{d}{ds}(\mathbf{b} \cdot \mathbf{b}^*) = \mathbf{b} \cdot \dot{\mathbf{b}}^* + \dot{\mathbf{b}} \cdot \mathbf{b}^* = \mathbf{b} \cdot (-\tau^*\mathbf{n}^*) + (-\tau\mathbf{n}) \cdot \mathbf{b}^* = -\tau(\mathbf{b} \cdot \mathbf{n}^* + \mathbf{n} \cdot \mathbf{b}^*)$$

Adding the above, $$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}^* + \mathbf{n}\cdot\mathbf{n}^* + \mathbf{b}\cdot\mathbf{b}^*) = 0$$

Integrating, $$\mathbf{t}\cdot\mathbf{t}^* + \mathbf{n}\cdot\mathbf{n}^* + \mathbf{b}\cdot\mathbf{b}^* = \text{constant}$$

But at s_0, $\mathbf{t}_0 = \mathbf{t}_0^*$, $\mathbf{n}_0 = \mathbf{n}_0^*$, $\mathbf{b}_0 = \mathbf{b}_0^*$; so that $\mathbf{t}_0\cdot\mathbf{t}_0^* = \mathbf{n}_0\cdot\mathbf{n}_0^* = \mathbf{b}_0\cdot\mathbf{b}_0^* = 1$. Thus at s_0, and hence for all s,

$$\mathbf{t}\cdot\mathbf{t}^* + \mathbf{n}\cdot\mathbf{n}^* + \mathbf{b}\cdot\mathbf{b}^* = 3$$

Now two unit vectors, say $\mathbf{t}$ and $\mathbf{t}^*$, have the property that $-1 \leqq \mathbf{t}\cdot\mathbf{t}^* = \cos\measuredangle(\mathbf{t},\mathbf{t}^*) \leqq 1$. Hence if $\mathbf{t}\cdot\mathbf{t}^* + \mathbf{n}\cdot\mathbf{n}^* + \mathbf{b}\cdot\mathbf{b}^* = 3$, then

$$\mathbf{t}\cdot\mathbf{t}^* = 1, \quad \mathbf{n}\cdot\mathbf{n}^* = 1, \quad \mathbf{b}\cdot\mathbf{b}^* = 1$$

Thus for all s, $\mathbf{t} = \mathbf{t}^*$, $\mathbf{n} = \mathbf{n}^*$ and $\mathbf{b} = \mathbf{b}^*$. Finally, since $\mathbf{t} = d\mathbf{x}/ds = \mathbf{t}^* = d\mathbf{x}^*/ds$, it follows that $\mathbf{x}(s) = \mathbf{x}^*(s) + \text{constant}$. But at s_0, $\mathbf{x}(s_0) = \mathbf{x}^*(s_0)$. Hence $\mathbf{x}(s) = \mathbf{x}^*(s)$ for all s, that is, the curves C and C^* coincide. This proves

Theorem 5.2. A curve is defined uniquely by its curvature and torsion as functions of a natural parameter.

The equations $$\kappa = \kappa(s), \quad \tau = \tau(s)$$

which give the curvature and torsion of a curve as functions of s are called the *natural* or *intrinsic* equations of a curve, for they completely define the curve.

Example 5.1:

(*a*) It follows from Theorem 4.1, page 64, and Theorem 4.3, page 66, that the intrinsic equations of a straight line are $\kappa \equiv 0$ and $\tau \equiv 0$.

(*b*) The equations $\kappa = \text{constant} \neq 0$, $\tau = 0$, (see Example 4.3, page 63), are the intrinsic equation of a circle of radius $\rho = 1/|\kappa|$.

(*c*) The intrinsic equations of a circular helix (see Examples 4.4 and 4.9) are

$$\kappa = \text{constant} \neq 0, \quad \tau = \text{constant} \neq 0$$

This helix lies on a circular cylinder of radius $|\kappa|/(\kappa^2+\tau^2)$ and its pitch is equal to $2\pi|\tau|/(\kappa^2+\tau^2)$. It is a right-handed helix if $\tau > 0$ and left-handed if $\tau < 0$.

THE FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM

We observe that the Frenet equations form a system of three vector differential equations of the first order in $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$. It is reasonable to ask, therefore, given arbitrary continuous functions κ and τ, whether or not there exist solutions $\mathbf{t}, \mathbf{n}, \mathbf{b}$ of the Frenet equations, and hence, since $\dot{\mathbf{x}} = \mathbf{t}$, a curve

$$\mathbf{x} = \int \mathbf{t}\,ds + \mathbf{C}$$

with the prescribed curvature and torsion. The answer is in the affirmative and is given by

Theorem 5.3. **Fundamental existence and uniqueness theorem for space curves.**

Let $\kappa(s)$ and $\tau(s)$ be arbitrary continuous functions on $a \leqq s \leqq b$. Then there exists, except for position in space, one and only one space curve C for which $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and s is a natural parameter along C.

The uniqueness of a curve with given torsion and curvature has already been proved (Theorem 5.2). A proof of the existence of such a curve is given in Appendix 1.

In general, solutions to the Frenet equations cannot be obtained by integration. A method is available, however, for reducing the system to that of a first order differential equation, called the Riccati equation, which has been thoroughly investigated. The details may be found in L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn and Co., 1909.

However, in the case of a plane curve, i.e. if $\tau \equiv 0$, integration of the Frenet equations is always possible. For, let ϕ denote the angle that $\mathbf{t}$ makes with the x_1 axis, as shown in Fig. 5-1. Then

$$\mathbf{t} = (\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2 \qquad (5.2)$$

Also, since $\mathbf{n}$ is orthogonal to $\mathbf{t}$, we can write

$$\mathbf{n} = (-\sin\phi)\mathbf{e}_1 + (\cos\phi)\mathbf{e}_2 \qquad (5.3)$$

Differentiating,

$$\dot{\mathbf{t}} = \dot\phi[(-\sin\phi)\mathbf{e}_1 + (\cos\phi)\mathbf{e}_2] = \dot\phi\mathbf{n}$$

and $$\dot{\mathbf{n}} = -\dot\phi[(\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2] = -\dot\phi\mathbf{t}$$

But when $\tau = 0$ the Frenet equations reduce to

$$\dot{\mathbf{t}} = \kappa\mathbf{n}, \qquad \dot{\mathbf{n}} = -\kappa\mathbf{t}$$

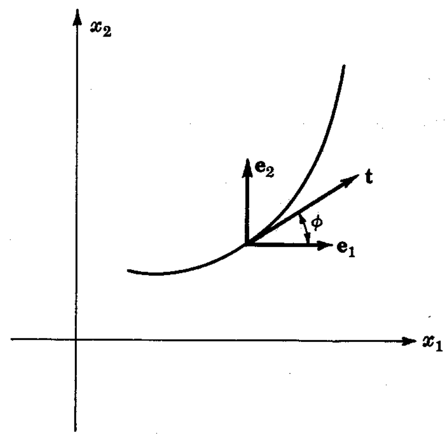


Fig. 5-1

Thus $\mathbf{t}$ and $\mathbf{n}$ above are solutions if $\dot\phi = \kappa$, or

$$\phi = \int \kappa\, ds + C_1 \qquad (5.4)$$

Having determined ϕ, we obtain from (5.2)

$$\mathbf{x} = \int \mathbf{t}\, ds + \mathbf{C}_2 = \int [(\cos\phi(s))\mathbf{e}_1 + (\sin\phi(s))\mathbf{e}_2]\, ds + \mathbf{C}_2 \qquad (5.5)$$

Observe that a change in the constant of integration in (5.4) defines a translation in ϕ and hence a rotation of the curve about the origin. A change in the constant of integration in (5.5) defines a translation of the curve.

Note that if $\kappa \neq 0$ for all s, then $\dot\phi \neq 0$ for all s. This allows us to introduce $\phi = \phi(s)$ as a parameter in (5.5), obtaining

$$\mathbf{x} = \int ((\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2)\frac{ds}{d\phi}\, d\phi + \mathbf{C}_2 = \int \frac{1}{\kappa(\phi)}((\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2)\, d\phi + \mathbf{C}_2 \qquad (5.6)$$

Example 5.2:

The equations $\kappa = 1/s$, $\tau = 0$, $s > 0$ are the intrinsic equations of a logarithmic spiral. For, set $\dot\phi = \kappa = 1/s$. Then $\phi = \log s + C_1$. This gives $s = e^{\phi - C_1}$ and $\kappa = 1/s = e^{-(\phi - C_1)}$. It follows that

$$\begin{aligned}\mathbf{x} = \int \frac{1}{\kappa(\phi)}((\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2)\, d\phi + \mathbf{C}_2 &= \int e^{(\phi - C_1)}((\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2)\, d\phi + \mathbf{C}_2 \\ &= \tfrac{1}{2}e^{(\phi - C_1)}(\cos\phi + \sin\phi)\mathbf{e}_1 \\ &\quad + \tfrac{1}{2}e^{(\phi - C_1)}(\sin\phi - \cos\phi)\mathbf{e}_2 + \mathbf{C}_2 \\ &= \tfrac{1}{2}e^{(\phi - C_1)}[\sqrt{2}(\cos(\phi - \pi/4))\mathbf{e}_1 + 2(\sin(\phi - \pi/4))\mathbf{e}_2] + \mathbf{C}_2\end{aligned}$$

If we choose $C_1 = \pi/4$, $\mathbf{C}_2 = \mathbf{0}$, and set $\phi - \pi/4 = \theta$, we obtain

$$\mathbf{x} = \frac{1}{\sqrt{2}}e^{\theta}[(\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2]$$

which, in polar coordinates, is the logarithmic spiral $r = (1/\sqrt{2})e^{\theta}$.

CANONICAL REPRESENTATION OF A CURVE

Let P be an arbitrary point on a curve C. We suppose that the curve is positioned so that P is at the origin, and we suppose that a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is chosen to coincide with $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ at P. Finally, we suppose that a natural parameter is chosen along C so that $s = 0$ at P. If C is now represented by $\mathbf{x} = \mathbf{x}(s)$, then $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = \mathbf{t}(0) = \mathbf{e}_1$, and $\ddot{\mathbf{x}}(0) = \dot{\mathbf{t}}(0) = \kappa(0)\,\mathbf{n}(0) = \kappa_0 \mathbf{e}_2$. Also, since

$$\dddot{\mathbf{x}} = \frac{d}{ds}\dot{\mathbf{t}} = \frac{d}{ds}\kappa\mathbf{n} = \kappa\dot{\mathbf{n}} + \dot{\kappa}\mathbf{n} = \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot{\kappa}\mathbf{n} = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b}$$

we have
$$\dddot{\mathbf{x}}(0) = -\kappa_0^2\mathbf{e}_1 + \dot{\kappa}_0\mathbf{e}_2 + \kappa_0\tau_0\mathbf{e}_3$$

Finally, since C is of class $\geqq 3$, we can write

$$\begin{aligned}
\mathbf{x}(s) &= \mathbf{x}(0) + \dot{\mathbf{x}}(0)s + \ddot{\mathbf{x}}(0)\frac{s^2}{2!} + \dddot{\mathbf{x}}(0)\frac{s^3}{3!} + \mathbf{o}(s^3) \\
&= \mathbf{e}_1 s + \kappa_0\mathbf{e}_2\frac{s^2}{2!} + (-\kappa_0^2\mathbf{e}_1 + \dot{\kappa}_0\mathbf{e}_2 + \kappa_0\tau_0\mathbf{e}_3)\frac{s^3}{3!} + \mathbf{o}(s^3) \\
&= \left(s - \frac{\kappa_0^2}{6}s^3\right)\mathbf{e}_1 + \left(\frac{\kappa_0}{2}s^2 + \frac{\dot{\kappa}_0}{6}s^3\right)\mathbf{e}_2 + \frac{\kappa_0\tau_0}{6}s^3\mathbf{e}_3 + \mathbf{o}(s^3)
\end{aligned}$$

In terms of the components of $\mathbf{x}(s)$ this becomes

$$x_1 = s - \tfrac{1}{6}\kappa_0^2 s^3 + o(s^3), \quad x_2 = \tfrac{1}{2}\kappa_0 s^2 + \tfrac{1}{6}\dot{\kappa}_0 s^3 + o(s^3), \quad x_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3 + o(s^3)$$

The above equations are called the *canonical representation* of C at P. Its leading terms conveniently describe the behavior of C near P.

Example 5.3:

As shown below, in the canonical representation at P, the projection of $\mathbf{x}$ onto the tangent line is given by the vector $x_1\mathbf{e}_1$. Since x_1 is of first order in s, we see that for the most part a curve lies along its tangent line. The projection of $\mathbf{x}$ onto the principal normal line is the component x_2 and is of second order in s; and the projection onto the binormal is of third order in s. Observe further that if $\kappa_0 \neq 0$, the curve lies on one side of the rectifying plane, since the component x_2, which is like s^2, does not change sign across P. If in addition $\tau_0 \neq 0$, the curve pierces the osculating plane at P, since x_3, which is like s^3, changes sign across P. If $\tau_0 > 0$, as shown in Fig. 5-2(*a*), the triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ rotates as a right-handed screw about P. If $\tau_0 < 0$, then $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ rotates as a left-handed screw about P, as shown in Fig. 5-2(*b*).

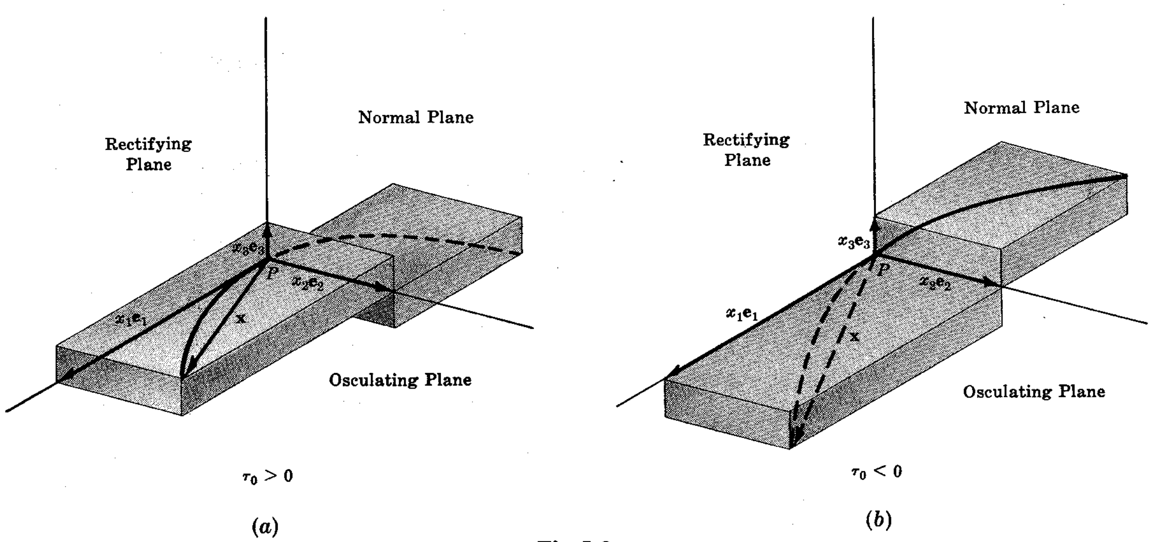


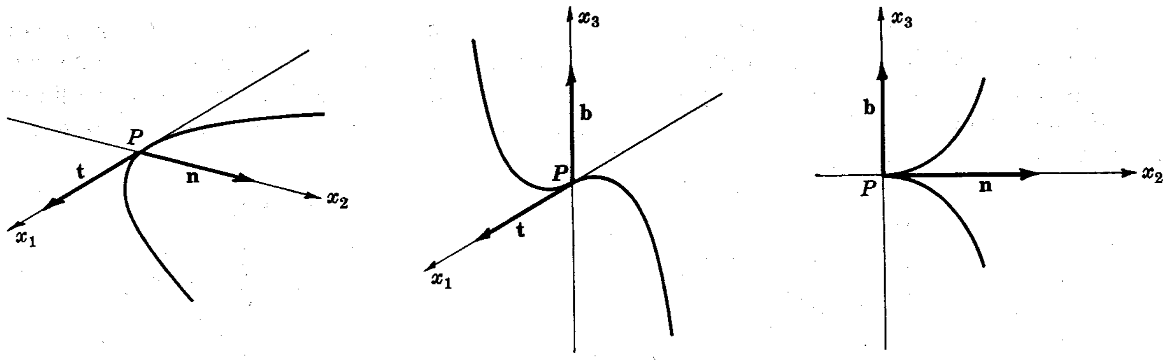
Fig. 5-2

Example 5.4:

Consider the leading terms of the canonical representation of a curve,

$$x_1 = s, \quad x_2 = \tfrac{1}{2}\kappa_0 s^2, \quad x_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3$$

and assume $\kappa_0 > 0$, $\tau_0 > 0$. By eliminating s in the first two equations, we see that in the neighborhood of a point the projection of the curve onto the osculating plane (x_1x_2 plane) is like the parabola $x_2 = \frac{1}{2}\kappa_0 x_1^2$, shown in Fig. 5-3(*a*). The projection onto the rectifying plane is like the cubic $x_3 = \frac{1}{6}\kappa_0\tau_0 x_1^3$ shown in Fig. 5-3(*b*). The projection onto the normal plane is like the curve $x_3^2 = \frac{2}{9}(\tau_0^2/\kappa_0)x_2^3$ shown in Fig. 5-3(*c*).



(*a*) (*b*) (*c*)

Fig. 5-3

INVOLUTES

The tangent lines to a curve C generate a surface called the *tangent surface* of C. A curve C^* which lies on the tangent surface of C and intersects the tangent lines orthogonally is called an *involute* of C.

If C is given by $\mathbf{x} = \mathbf{x}(s)$ and if, as shown in Fig. 5-4, $\mathbf{x}^*$ is a point on an involute C^*, where C^* crosses the tangent line at $\mathbf{x}(s)$, then $\mathbf{x}^* - \mathbf{x}(s)$ is proportional to $\mathbf{t}(s)$. Thus C^* will have a representation of the form $\mathbf{x}^* = \mathbf{x}(s) + k(s)\,\mathbf{t}(s)$. Moreover, on an involute, the tangent vector

$$\frac{d\mathbf{x}^*}{ds} = \dot{\mathbf{x}} + \dot{k}\mathbf{t} + k\dot{\mathbf{t}} = (1+\dot{k})\mathbf{t} + k\kappa\mathbf{n}$$

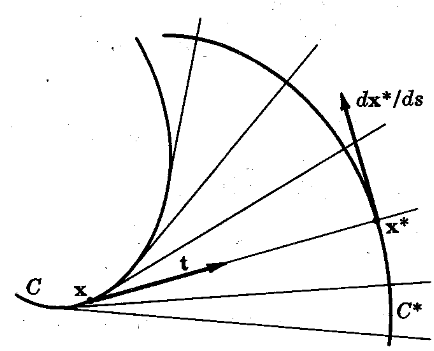


Fig. 5-4

is orthogonal to the tangent vector $\mathbf{t}$ on C; that is,

$$\frac{d\mathbf{x}^*}{ds}\cdot\mathbf{t} = (1+\dot{k})(\mathbf{t}\cdot\mathbf{t}) + k\kappa(\mathbf{n}\cdot\mathbf{t}) = 1 + \dot{k} = 0$$

Integrating gives $k = -s + c$, c = constant. Thus there exists an infinite family of involutes, one for each c, $\mathbf{x}^* = \mathbf{x} + (c-s)\mathbf{t}$.

Note that $\mathbf{x}^*$ is not regular where $\mathbf{x}$ has a point of inflection. For,

$$\frac{d\mathbf{x}^*}{ds} = \frac{d\mathbf{x}}{ds} + (c-s)\frac{d\mathbf{t}}{ds} - \mathbf{t} = (c-s)\dot{\mathbf{t}} = (c-s)\kappa\mathbf{n}$$

and hence $d\mathbf{x}^*/ds = \mathbf{0}$ where $\kappa = 0$. Thus we assume that along C, $\kappa \neq 0$. From this it also follows that $\kappa^* \neq 0$ along the involute of C. For in Problem 5.15, page 97, we show that the curvature of the involute satisfies

$$\kappa^{*2} = \frac{\kappa^2 + \tau^2}{(c-s)^2\kappa^2}$$

Thus $\kappa^* \neq 0$ for $\kappa \neq 0$.

When the tangent line to a curve $\mathbf{x} = \mathbf{x}(s)$ at $\mathbf{x}$ is given by $\mathbf{x}^* = \mathbf{x} + k\mathbf{t}$, $-\infty < k < \infty$, then

$$|d\mathbf{x}^*/dk| = |\mathbf{t}| = 1$$

That is, k is a natural parameter. Also, since $\mathbf{x}^* = \mathbf{x}$ for $k = 0$, it follows that $|k|$ is the distance between the point $\mathbf{x}^*$ on the tangent line and the point $\mathbf{x}$ on C. Consequently the distance between the two involutes $C_1^* : \mathbf{x}^* = \mathbf{x} + (c_1 - s)\mathbf{t}$ and $C_2^* : \mathbf{x}^* = \mathbf{x} + (c_2 - s)\mathbf{t}$ of C, as shown in Fig. 5-5(*a*), remains constant for all s and equal to

$$|(c_1 - s) - (c_2 - s)| = |c_1 - c_2|$$

It also follows from the above that an involute is generated by unrolling a taut string which has been wrapped along C. If, in particular, as shown in Fig. 5-5(*b*), the string is of length c_0 and if a natural parameter s is chosen along C such that s is the distance from the fixed end of the string, then the curve generated by unrolling the string will be the involute $\mathbf{x}^* = \mathbf{x} + (c_0 - s)\mathbf{t}$.

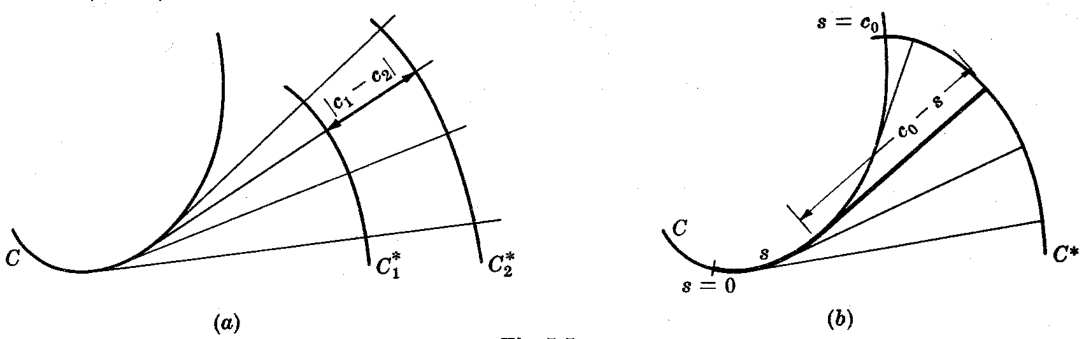


Fig. 5-5

Example 5.5:

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a > 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3, \qquad \left|\frac{d\mathbf{x}}{dt}\right| = (a^2 + b^2)^{1/2}$$

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2 + b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

Also, $s = \int_0^t \left|\frac{d\mathbf{x}}{dt}\right| dt = (a^2 + b^2)^{1/2} t$. Thus the involutes are the curves

$$\mathbf{x}^* = \mathbf{x} + (c - s)\mathbf{t} = [a\cos t - a(c - s)(a^2 + b^2)^{-1/2}\sin t]\mathbf{e}_1 + [a\sin t + a(c - s)(a^2 + b^2)^{-1/2}\cos t]\mathbf{e}_2 + [bt + (c - s)(a^2 + b^2)^{-1/2} b]\mathbf{e}_3$$

or, setting $\gamma = c(a^2 + b^2)^{-1/2}$ and using $t = s(a^2 + b^2)^{-1/2}$,

$$\mathbf{x}^* = a[(\cos t + t\sin t) - \gamma\sin t]\mathbf{e}_1 + a[(\sin t - t\cos t) + \gamma\cos t]\mathbf{e}_2 + b\gamma\mathbf{e}_3$$

Note, as shown in Fig. 5-6, that the involute is a plane curve, confined to the plane $x_3 = b\gamma$.

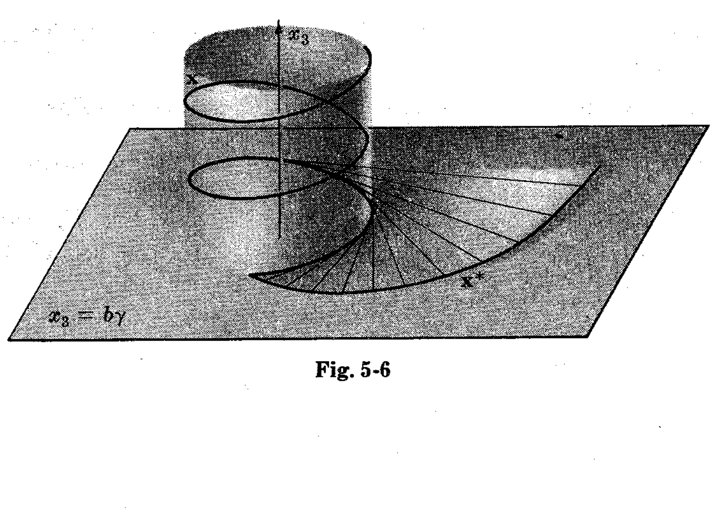


Fig. 5-6

EVOLUTES

If a curve C is an involute of a curve C^*, then by definition C^* is an *evolute* of C. Hence given C, its evolutes are the curves whose tangent lines intersect C orthogonally. If C is given by $\mathbf{x} = \mathbf{x}(s)$ and if $\mathbf{x}^*(s)$ is the point of contact on the evolute to the tangent line which intersects C at $\mathbf{x}(s)$, then

$$\mathbf{x}^*(s) = \mathbf{x}(s) + \alpha(s)\,\mathbf{n}(s) + \beta(s)\,\mathbf{b}(s)$$

For, as shown in Fig. 5-7, $\mathbf{x}^* - \mathbf{x}(s)$ is orthogonal to $\mathbf{t}(s)$ and so it is a linear combination of $\mathbf{n}(s)$ and $\mathbf{b}(s)$. Differentiating the above,

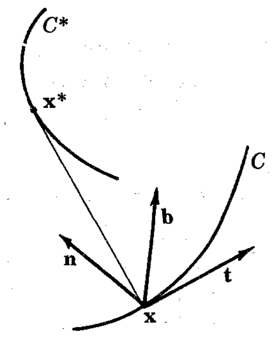


Fig. 5-7

$$\frac{d\mathbf{x}^*}{ds} = \dot{\mathbf{x}} + \dot{\alpha}\mathbf{n} + \alpha\dot{\mathbf{n}} + \dot{\beta}\mathbf{b} + \beta\dot{\mathbf{b}} = \mathbf{t} + \dot{\alpha}\mathbf{n} + \alpha(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot{\beta}\mathbf{b} - \beta\tau\mathbf{n}$$
$$= (1 - \alpha\kappa)\mathbf{t} + (\dot{\alpha} - \beta\tau)\mathbf{n} + (\dot{\beta} + \tau\alpha)\mathbf{b}$$

Now, since $d\mathbf{x}^*/ds$ is also tangent to C^*, it is proportional to $\mathbf{x}^* - \mathbf{x} = \alpha\mathbf{n} + \beta\mathbf{b}$. Hence there exists k such that

$$1 - \alpha\kappa = 0, \quad (\dot{\alpha} - \beta\tau) = k\alpha, \quad \text{and} \quad (\dot{\beta} + \tau\alpha) = k\beta$$

It follows that $\alpha = 1/\kappa$ and, eliminating k in the last two equations,

$$\beta(\dot{\alpha} - \beta\tau) - \alpha(\dot{\beta} + \tau\alpha) = 0$$

or, solving for τ,

$$\tau = \frac{\beta\dot{\alpha} - \alpha\dot{\beta}}{\alpha^2 + \beta^2} = \frac{d}{ds}\operatorname{Cot}^{-1}\frac{\beta}{\alpha}$$

Integrating gives $\beta = \alpha \cot\left[\int \tau\,ds + c\right]$. Since $\alpha = 1/\kappa$, we have $\beta = (1/\kappa)\cot\left[\int \tau\,ds + c\right]$. Thus we have an infinite family of evolutes, one for each choice of c:

$$\mathbf{x}^* = \mathbf{x} + \frac{1}{\kappa}\mathbf{n} + \frac{1}{\kappa}\cot\left(\int \tau\,ds + c\right)\mathbf{b}$$

Note that we must assume that $(\dot{\alpha} - \beta\tau)^2 + (\dot{\beta} + \tau\alpha)^2 \neq 0$ along C. For differentiating $\mathbf{x}^* = \mathbf{x} + \alpha\mathbf{n} + \beta\mathbf{b}$ gives

$$\frac{d\mathbf{x}^*}{ds} = (\dot{\alpha} - \beta\tau)\mathbf{n} + (\dot{\beta} + \tau\alpha)\mathbf{b}$$

so that C^* is not regular where $(\dot{\alpha} - \beta\tau)^2 + (\dot{\beta} + \tau\alpha)^2 = 0$. In particular if C is a plane curve, then $\tau = 0$, $\alpha = \gamma\beta$, $\gamma = \text{constant}$ and $(\dot{\alpha} - \beta\tau)^2 + (\dot{\beta} + \tau\alpha)^2 = (\dot{\kappa}^2/\kappa^4)(1 + \gamma^2)$. Thus for plane curves C we assume that $\dot{\kappa} \neq 0$.

Example 5.6:

If C is a plane curve, then $\tau = 0$ and its evolutes are

$$\mathbf{x}^* = \mathbf{x} + \frac{1}{\kappa}\mathbf{n} + \frac{\gamma}{\kappa}\mathbf{b}, \qquad \gamma = \text{constant}$$

As shown in Fig. 5-8, for $\gamma = 0$ the evolute lies in the same plane as C, the osculating plane of C. In fact, it is the only evolute of C in the same plane and is called the *plane evolute* of C. We observe further that, since $\mathbf{b} = \text{constant}$, the other evolutes lie on a cylinder whose generating lines are perpendicular to the plane of C and pass through the plane evolute of C.

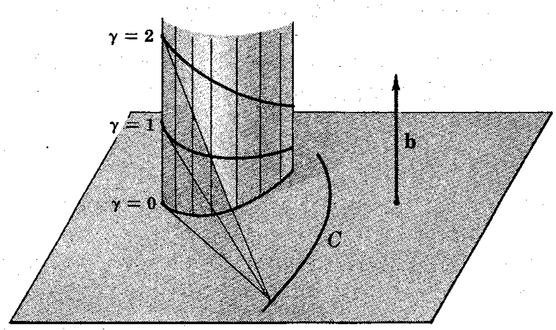


Fig. 5-8

THEORY OF CONTACT

Intuitively, it seems that the planes which intersect a curve C at a point $\mathbf{x}$ and also contain the tangent line at $\mathbf{x}$ have a higher degree of "contact" with C at $\mathbf{x}$ than the planes not containing the tangent line. Also, of all planes containing the tangent line at $\mathbf{x}$, the one with greatest contact seems to be the osculating plane.

In order to investigate the degree to which a curve C can make contact with a surface in general, we suppose C, of sufficiently high class, is given by $\mathbf{x} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$, a surface S is given by $F(x_1, x_2, x_3) = 0$, and that C and S intersect not only at a single point $\mathbf{x}_0 = \mathbf{x}(t_0)$ but also at $n-1$ other points, $\mathbf{x}_1 = \mathbf{x}(t_1), \ldots, \mathbf{x}_{n-1} = \mathbf{x}(t_{n-1})$ in a neighborhood of $\mathbf{x}_0$. Now consider the function

$$f(t) = F(x_1(t),\ x_2(t),\ x_3(t))$$

Clearly,

$$\begin{aligned} f(t_0) &= F(x_1(t_0), x_2(t_0), x_3(t_0)) = 0 \\ f(t_1) &= F(x_1(t_1)\, x_2(t_1),\ x_3(t_1)) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f(t_{n-1}) &= F(x_1(t_{n-1}), x_2(t_{n-1}), x_3(t_{n-1})) = 0 \end{aligned}$$

since C and S intersect at $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$. From Rolle's theorem, it follows that there exist $t_1', t_2', \ldots, t_{n-1}'$, where

$$t_0 \leq t_1' \leq t_1,\ \ t_1 \leq t_2' \leq t_2,\ \ \ldots,\ \ t_{n-2} \leq t_{n-1}' \leq t_{n-1}$$

such that

$$f'(t_1') = f'(t_2') = \cdots = f'(t_{n-1}') = 0$$

But then again from Rolle's theorem, there exist $t_2'', t_3'', \ldots, t_{n-1}''$, where

$$t_1' \leq t_2'' \leq t_2',\ \ \ t_2' \leq t_3'' \leq t_3',\ \ \ \ldots,\ \ \ t_{n-2}' \leq t_{n-1}'' \leq t_{n-1}'$$

such that

$$f''(t_2'') = f''(t_3'') = \cdots = f''(t_{n-1}'') = 0$$

Continuing in this manner, we find that there exist $t_0, t_1', t_2'', \ldots, t_{n-1}^{(n-1)}$ all in a neighborhood of t_0 such that

$$f(t_0) = f'(t_1') = f''(t_2'') = \cdots = f^{(n-1)}(t_{n-1}^{(n-1)}) = 0$$

Now suppose we consider the limit as $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$ approach $\mathbf{x}_0$. That is, we suppose now S is the limiting position, as $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$ approach $\mathbf{x}_0$, of a family of surfaces each of which intersects C at $\mathbf{x}_0$ and $n-1$ points in a neighborhood of $\mathbf{x}_0$. As $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}$ approach $\mathbf{x}_0$, the numbers $t_1', t_2'', \ldots, t_{n-1}^{(n-1)}$ approach t_0, and hence in the limit

$$f(t_0) = f'(t_0) = f''(t_0) = \cdots = f^{(n-1)}(t_0) = 0$$

Thus we are led to the following definition:

> A curve $\mathbf{x} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$ has *n-point contact* with a surface $F(x_1, x_2, x_3) = 0$ at the point corresponding to $t = t_0$ if the function
>
> $$f(t) = F(x_1(t), x_2(t), x_3(t))$$
>
> satisfies $$f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0 \quad \text{but} \quad f^{(n)}(t_0) \neq 0$$

We note that the above definition is independent of the parameterization of the curve. For suppose

$$\mathbf{x} = x_1^*(\theta)\,\mathbf{e}_1 + x_2^*(\theta)\,\mathbf{e}_2 + x_3^*(\theta)\,\mathbf{e}_3$$

is another representation of the curve. Then

$$x_i^*(\theta) = x_i(t(\theta)), \qquad i = 1, 2, 3$$

and
$$\begin{aligned} g(\theta) &= F(x_1^*(\theta), x_2^*(\theta), x_3^*(\theta)) = F(x_1(t(\theta)), x_2(t(\theta)), x_3(t(\theta))) = f(t(\theta)) \\ g'(\theta) &= t' f'(t(\theta)) \\ g''(\theta) &= (t')^2 f''(t(\theta)) + t'' f'(t(\theta)) \end{aligned}$$

In general, $g^{(i)}(\theta)$ will be a linear combination of $f^{(i)}(t(\theta))$ and lower order derivatives

$$g^{(i)}(\theta) = (t')^i f^{(i)}(t(\theta)) + C_1 f^{(i-1)}(t(\theta)) + \cdots + t^{(i)} f'(t(\theta))$$

But if $f^{(i)}(t_0) = f^{(i)}(t(\theta_0)) = 0$, $i = 1, \ldots, n-1$ and $f^{(n)}(t_0) = f^{(n)}(t(\theta_0)) \neq 0$, then $g^{(i)}(\theta_0) = 0$, $i = 1, \ldots, n-1$, and since $t'(\theta_0) \neq 0$, $g^{(n)}(\theta_0) \neq 0$, which is the required result.

Example 5.7:

The equation of an arbitrary plane with unit normal $\mathbf{N}$ passing through the point $\mathbf{x}_0 = \mathbf{x}(s_0)$ on a curve $\mathbf{x} = \mathbf{x}(s)$ is $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{N} = 0$. Now consider the function

$$f(s) = (\mathbf{x}(s) - \mathbf{x}_0) \cdot \mathbf{N}$$

and its derivatives

$$\begin{aligned} f'(s) &= \dot{\mathbf{x}}(s) \cdot \mathbf{N} = \mathbf{t}(s) \cdot \mathbf{N} \\ f''(s) &= \dot{\mathbf{t}}(s) \cdot \mathbf{N} = \kappa(s)\, \mathbf{n}(s) \cdot \mathbf{N} \end{aligned}$$

Clearly, $f(s_0) = (\mathbf{x}(s_0) - \mathbf{x}_0) \cdot \mathbf{N} = 0$. Now,

$$f'(s_0) = \mathbf{t}(s_0) \cdot \mathbf{N} = 0$$

if and only if $\mathbf{N}$ is orthogonal to $\mathbf{t}(s_0)$, the unit tangent at x_0. Thus a plane has at least 2-point contact with a curve iff it contains the tangent line. We have further that if $\kappa(s_0) \neq 0$, then

$$f''(s_0) = \kappa(s_0)\, \mathbf{n}(s_0) \cdot \mathbf{N} = 0$$

iff $\mathbf{N}$ is orthogonal to $\mathbf{n}(s_0)$, the principal normal at $\mathbf{x}_0$. Thus if $\kappa(s_0) \neq 0$, the plane has at least 3-point contact with the curve iff $\mathbf{N}$ is orthogonal to $\mathbf{t}(s_0)$ and $\mathbf{n}(s_0)$, i.e. iff the plane is the osculating plane $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{b}(s_0) = 0$ at x_0. We note also that at a point of inflection, i.e. where $\kappa(s_0) = 0$, all planes containing the tangent line have at least 3-point contact with the curve.

The degree of contact between two curves in space is defined in a similar way. Namely, let a curve C of sufficiently high class be given by

$$\mathbf{x} = x_1(t)\, \mathbf{e}_1 + x_2(t)\, \mathbf{e}_2 + x_3(t)\, \mathbf{e}_3$$

and let a curve Γ be given as the intersection of two surfaces

$$\begin{aligned} F(x_1, x_2, x_3) &= 0 \\ G(x_1, x_2, x_3) &= 0 \end{aligned}$$

Then C has *n-point contact* with Γ at the point corresponding to $t = t_0$ if the functions

$$\begin{aligned} f(t) &\equiv F(x_1(t), x_2(t), x_3(t)) \\ g(t) &\equiv G(x_1(t), x_2(t), x_3(t)) \end{aligned}$$

satisfy
$$\begin{aligned} f(t_0) = f'(t_0) = f''(t_0) = \cdots = f^{(n-1)}(t_0) = 0 \\ g(t_0) = g'(t_0) = g''(t_0) = \cdots = g^{(n-1)}(t_0) = 0 \end{aligned}$$

but $f^{(n)}(t_0) \neq 0$ or $g^{(n)}(t_0) \neq 0$. Accordingly C has n-point contact with Γ iff C has n-point contact with one of the surfaces defining Γ and at least n-point contact with the other.

Example 5.8:

We wish to determine a circle Γ which has at least 3-point contact with a curve $\mathbf{x} = \mathbf{x}(s)$ at a point $\mathbf{x}_0 = \mathbf{x}(s_0)$. Our remarks above suggest that we define Γ as the intersection of a sphere with at least 3-point contact with $\mathbf{x} = \mathbf{x}(s)$ at $\mathbf{x}_0$ and a plane with at least 3-point contact with $\mathbf{x} = \mathbf{x}(s)$ at $\mathbf{x}_0$. Now the equation of an arbitrary sphere with center at $\mathbf{y}_0$ and passing through $\mathbf{x}_0$ is

$$|\mathbf{x} - \mathbf{y}_0|^2 = |\mathbf{x}_0 - \mathbf{y}_0|^2$$

We consider the functions

$$f(s) = |\mathbf{x}(s) - \mathbf{y}_0|^2 - |\mathbf{x}_0 - \mathbf{y}_0|^2 = (\mathbf{x}(s) - \mathbf{y}_0) \cdot (\mathbf{x}(s) - \mathbf{y}_0) - |\mathbf{x}_0 - \mathbf{y}_0|^2$$

$$f'(s) = 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \dot{\mathbf{x}}(s) = 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{t}(s)$$

$$f''(s) = 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \dot{\mathbf{t}}(s) + 2\mathbf{t}(s) \cdot \mathbf{t}(s) = 2\kappa(s)(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{n}(s) + 2$$

Clearly $f(s_0) = 0$. Now, $f'(s_0) = 2(\mathbf{x}_0 - \mathbf{y}_0) \cdot \mathbf{t}_0 = 0$ iff $(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{t}_0 = 0$. Namely, a sphere has at least 2-point contact iff its center, as shown in Fig. 5-9, lies on the normal plane. Continuing, we have $f''(s_0) = 2\kappa_0(\mathbf{x}_0 - \mathbf{y}_0) \cdot \mathbf{n}_0 + 2 = 0$ iff $\kappa_0(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{n}_0 = 1$. Thus a sphere has at least 3-point contact iff its center is on the normal plane and the projection of the vector $\mathbf{y}_0 - \mathbf{x}_0$ onto $\mathbf{n}_0$ equals $1/\kappa_0$. Notice that a sphere with at least 3-point contact does not exist at a point of inflection, i.e. where $\kappa_0 = 0$. Finally, the intersection of such a sphere and the osculating plane at $\mathbf{x}_0$, which, since $\kappa_0 \neq 0$, is the only plane with at least three point contact at $\mathbf{x}_0$, will be a circle Γ of at least 3-point contact at $\mathbf{x}_0$. Note that this circle is unique; for although there exist an infinite number of spheres of at least 3-point contact at x_0, the vectors $\mathbf{y}_0 - \mathbf{x}_0$ all lie on the normal plane and have a constant projection $1/\kappa_0$ onto $\mathbf{n}_0$.

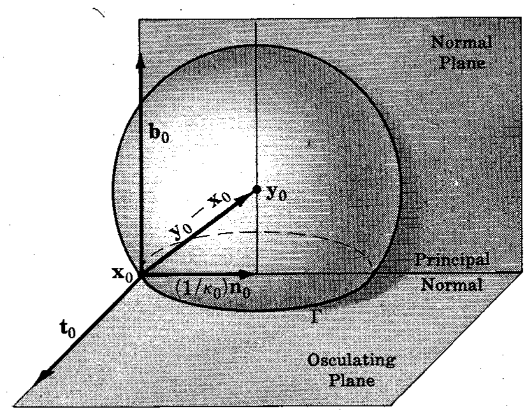


Fig. 5-9

The circle determined in the above example, which has at least 3-point contact with a curve C at $\mathbf{x}$, is called the *osculating circle* to C at $\mathbf{x}$. It lies on the osculating plane, its center is on the principal normal line on the side of the curvature vector to C at $\mathbf{x}$, and its radius is equal to the radius of curvature $\rho = 1/|\kappa|$ at $\mathbf{x}$. The center of the osculating circle is called the *center of curvature,* and its position is $\mathbf{y} = \mathbf{x} + (1/\kappa)\mathbf{n}$.

OSCULATING CURVES AND SURFACES

We recall that of all planes intersecting a curve at a point, the osculating plane is the one with greatest contact. In general, given a family of surfaces S_λ which intersect a curve C at a point $\mathbf{x}$, a member S_0 of S_λ is called an *osculating surface of the family* S_λ to C if the degree of contact of C with S_0 at $\mathbf{x}$ is greater than or equal to the degree of contact of C with any other of the surfaces S_λ. Similarly, a member Γ_0 of a family of curves Γ_λ which intersect C at $\mathbf{x}$ is called an *osculating curve of the family* Γ_λ to C if the degree of contact of C with Γ_0 at $\mathbf{x}$ is greater than or equal to the degree of contact of C with any other of the curves Γ_λ.

In order to investigate osculating surfaces in more detail we suppose C, of sufficiently high class, is given by $\mathbf{x} = x_1(t)\,\mathbf{e}_1 + x_2(t)\,\mathbf{e}_2 + x_3(t)\,\mathbf{e}_3$, and S_λ is an $n-1$ parameter family of surfaces given by

$$F(x_1, x_2, x_3, a_1, a_2, \ldots, a_{n-1}) = 0$$

and that C and S_λ intersect at the point corresponding to $t = t_0$. We consider again the function

$$f(t, a_1, \ldots, a_{n-1}) = F(x_1(t), x_2(t), x_3(t), a_1, a_2, \ldots, a_{n-1})$$

Clearly $f(t_0, a_1, \ldots, a_{n-1}) = 0$, since it is assumed that the family of surfaces intersects C at $x(t_0)$. Now consider the equations

$$\frac{\partial}{\partial t} f(t_0, a_1, \ldots, a_{n-1}) = 0$$

$$\frac{\partial^2}{\partial t^2} f(t_0, a_1, \ldots, a_{n-1}) = 0$$

$$\frac{\partial^{(n-1)}}{\partial t^{(n-1)}} f(t_0, a_1, \ldots, a_{n-1}) = 0$$

This is a system of $n-1$ equations in the $n-1$ unknowns $a_1, \ldots, a_{n-1}$. In principle, a solution $a_1 = \alpha_1$, $a_2 = \alpha_2$, $\ldots$, $a_{n-1} = \alpha_{n-1}$ will exist, and then clearly the surface $f(x_1, x_2, x_3, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0$ will have at least n-point contact with C at $\mathbf{x}$. But the osculating surface of the family has contact with C at $\mathbf{x}$ of degree greater than or equal the degree of contact of any other member of the family. A similar argument holds for curves. Thus in general, the osculating surface (curve) of an $n-1$ parameter family of surfaces (curves) which intersect a curve C at a point $\mathbf{x}$ has at least n-point contact with C at $\mathbf{x}$.

Example 5.9:

The family of all planes passing through the point corresponding to $\mathbf{x}_0 = \mathbf{x}(s_0)$ on the curve $\mathbf{x} = \mathbf{x}(s)$ is a 2-parameter family of surfaces $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{N} = 0$. (The three components of $\mathbf{N}$ are related by the equation $|\mathbf{N}| = 1$.) If $\kappa_0 \neq 0$, the osculating plane at $\mathbf{x}_0$ is the unique osculating plane of this family. As has been shown in Example 5.7, it has at least 3-point contact with the curve at $\mathbf{x}_0$, and any other plane in this case has less than 3-point contact at $\mathbf{x}_0$.

Example 5.10:

The family of all spheres through $\mathbf{x}_0$ on the curve $\mathbf{x} = \mathbf{x}(s)$ is a three parameter (the components of the center $\mathbf{y}_0$) family of spheres

$$|\mathbf{x} - \mathbf{y}_0|^2 = |\mathbf{x}_0 - \mathbf{y}_0|^2$$

If we assume κ_0 and τ_0 are different from zero, the osculation sphere of this family will be unique and have at least 4-point contact with the curve at $\mathbf{x}_0$. The center and radius of this sphere are determined as follows. We consider the functions

$$\begin{aligned}
f(s) &= |\mathbf{x}(s) - \mathbf{y}|^2 - |\mathbf{x}_0 - \mathbf{y}_0|^2 = (\mathbf{x}(s) - \mathbf{y}_0) \cdot (\mathbf{x}(s) - \mathbf{y}_0) - |\mathbf{x}_0 - \mathbf{y}_0|^2 \\
f'(s) &= 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \dot{\mathbf{x}}(s) = 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{t}(s) \\
f''(s) &= 2(\mathbf{x}(s) - \mathbf{y}_0) \cdot \dot{\mathbf{t}}(s) + 2\mathbf{t}(s) \cdot \mathbf{t}(s) = 2\kappa(s)\,(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{n}(s) + 2 \\
f'''(s) &= 2\dot{\kappa}(s)\,(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{n}(s) + 2\kappa(s)\,\mathbf{t}(s) \cdot \mathbf{n}(s) + 2\kappa(s)(\mathbf{x}(s) - \mathbf{y}_0) \cdot \dot{\mathbf{n}}(s) \\
&= 2\dot{\kappa}(s)\,(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{n}(s) + 2\kappa(s)\,(\mathbf{x}(s) - \mathbf{y}_0)[-\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)] \\
&= 2\dot{\kappa}(s)\,(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{n}(s) - 2\kappa^2(s)\,(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{t}(s) + 2\kappa(s)\tau(s)(\mathbf{x}(s) - \mathbf{y}_0) \cdot \mathbf{b}(s)
\end{aligned}$$

Clearly, $f(s_0) = 0$. Also, $f'(s_0) = 0$ iff $(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{t}_0 = 0$, and $f''(s_0) = 0$ iff $(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{n}_0 = 1/\kappa_0$. Finally, using both of these conditions, we have in addition that $f'''(s_0) = 0$ iff

$$-2\dot{\kappa}_0/\kappa_0 - 2\kappa_0\tau_0(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{b}_0 = 0$$

or

$$(\mathbf{y}_0 - \mathbf{x}_0) \cdot \mathbf{b}_0 = -\dot{\kappa}_0/\kappa_0^2\tau_0$$

Thus as shown in Fig. 5-10, if $\kappa_0 \neq 0$ and $\tau_0 \neq 0$ there exists a unique osculating sphere with at least 4-point contact with the curve, whose center $\mathbf{y}_0$ is such that the components of the vector $\mathbf{y}_0 - \mathbf{x}_0$ with respect to the vectors $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ are $0, 1/\kappa_0, -\dot{\kappa}_0/\kappa_0^2\tau_0$ respectively. It is clear that the radius of this sphere is

$$|\mathbf{y}_0 - \mathbf{x}_0| = \sqrt{\left(\frac{1}{\kappa_0}\right)^2 + \left(\frac{\dot{\kappa}_0}{\kappa_0^2\tau_0}\right)^2}$$

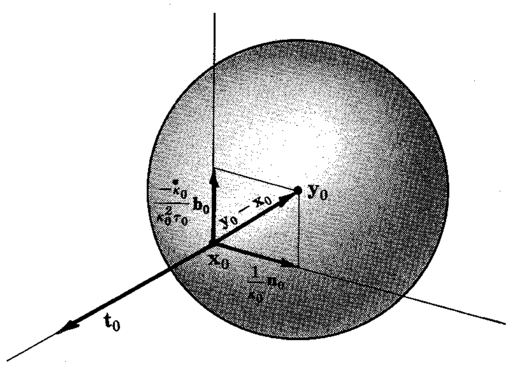


Fig. 5-10

The sphere determined in the example above is called the *osculating sphere* to C at $\mathbf{x}$. The center of the sphere is called the *center of spherical curvature* and its position is

$$\mathbf{y} = \mathbf{x} + \frac{1}{\kappa}\mathbf{n} - \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}$$

It is sometimes convenient to introduce the *radius of torsion* $\sigma = 1/\tau$. In terms of σ and the radius of curvature $\rho = 1/|\kappa|$ the above becomes, for $\kappa > 0$,

$$\mathbf{y} = \mathbf{x} + \rho\mathbf{n} + \dot{\rho}\sigma\mathbf{b}$$

Finally, suppose again that $F(x_1, x_2, x_3, a_1, a_2, \ldots, a_{n-1}) = 0$ is an $n-1$ parameter family of surfaces which intersect a curve $\mathbf{x}(t) = x_1(t)\,\mathbf{e}_1 + x_2(t)\,\mathbf{e}_2 + x_3(t)\,\mathbf{e}_3$ at the point corresponding to $t = t_0$. Note that in principle an $n-1$ parameter family can be made to satisfy $n-1$ conditions, so that in general a member of the family can be found which in addition intersects the curve at the $n-1$ points, say, corresponding to $t_1, \ldots, t_{n-1}$ in a neighborhood of t_0. Now consider the limit as $t_1, \ldots, t_{n-1}$ approach t_0. If a limit surface S exists and is given by $F(x_1, x_2, x_3) = 0$, then it follows by an argument similar to the one on page 87 that the function

$$f(t) = F(x_1(t), x_2(t), x_3(t))$$

satisfies

$$f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0$$

That is, the limit surface S has at least n-point contact with the curve at t_0. A similar argument holds for an $n-1$ parameter family of curves. Thus in general if the osculating surface (curve) of an $n-1$ parameter family of surfaces (curves) is unique and has n-point contact with C at $\mathbf{x}$, then it is the limit of those surfaces (curves) of the family which pass through $n-1$ neighboring points on C as they approach $\mathbf{x}$.

Example 5.11:

The osculating circle to a curve C at a point $\mathbf{x}$ is the limit of the circles which pass through $\mathbf{x}$ and two neighboring points on C as they approach $\mathbf{x}$. Similarly, the osculating sphere is the limit of the spheres which pass through $\mathbf{x}$ and three neighboring points on C as they approach $\mathbf{x}$.

Solved Problems

INTRINSIC EQUATIONS. FUNDAMENTAL THEOREM

5.1. Determine the intrinsic equations of the catenary

$$\mathbf{x} = a(\cosh(t/a))\,\mathbf{e}_1 + t\mathbf{e}_2, \qquad a = \text{constant}$$

Since it is a plane curve, $\tau \equiv 0$. It remains to find κ as a function of s. We compute

$$\mathbf{x}' = \sinh(t/a)\mathbf{e}_1 + \mathbf{e}_2, \qquad |\mathbf{x}'| = [\sinh^2(t/a) + 1]^{1/2} = \cosh(t/a)$$

$$\mathbf{x}'' = (1/a)(\cosh(t/a))\,\mathbf{e}_1, \qquad \mathbf{x}' \times \mathbf{x}'' = -(1/a)(\cosh(t/a))\,\mathbf{e}_3$$

From Theorem 4.2, page 64,

$$\kappa^2 = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot (\mathbf{x}' \times \mathbf{x}'')}{(\mathbf{x}' \cdot \mathbf{x}')^3} = \frac{(1/a^2)\cosh^2(t/a)}{(\sinh^2(t/a) + 1)^3} = \frac{1}{a^2\cosh^4(t/a)}$$

Also

$$s = \int_0^t |\mathbf{x}'|\,dt = \int_0^t \cosh(t/a)\,dt = a\sinh(t/a)$$

Hence

$$s^2 + a^2 = a^2\sinh^2(t/a) + a^2 = a^2\cosh^2(t/a)$$

Eliminating t gives $\kappa = a/(s^2 + a^2)$, which is the required result.

5.2. Determine the intrinsic equations of the epicycloid

$$\mathbf{x} \;=\; \left[(r_0+r_1)\cos\theta \;-\; r_1\cos\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_1 \;+\; \left[(r_0+r_1)\sin\theta \;-\; r_1\sin\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_2$$

We compute

$$\mathbf{x}' \;=\; \frac{d\mathbf{x}}{d\theta} \;=\; \left[-(r_0+r_1)\sin\theta \;+\; (r_0+r_1)\sin\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_1 + \left[(r_0+r_1)\cos\theta \;-\; (r_0+r_1)\cos\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_2$$

$$\mathbf{x}'' \;=\; \frac{d\mathbf{x}'}{d\theta} \;=\; \left[-(r_0+r_1)\cos\theta \;+\; \frac{(r_0+r_1)^2}{r_1}\cos\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_1 + \left[-(r_0+r_1)\sin\theta \;+\; \frac{(r_0+r_1)^2}{r_1}\sin\left(\frac{r_0+r_1}{r_1}\theta\right)\right]\mathbf{e}_2$$

$$\mathbf{x}'\times\mathbf{x}'' \;=\; \frac{(r_0+r_1)^2(r_0+2r_1)}{r_1}\left[1 - \left(\sin\theta\sin\left(\frac{r_0+r_1}{r_1}\theta\right) + \cos\theta\cos\left(\frac{r_0+r_1}{r_1}\theta\right)\right)\right]\mathbf{e}_3$$

$$= \frac{(r_0+r_1)^2(r_0+2r_1)}{r_1}[1-\cos(r_0/r_1)\theta]\mathbf{e}_3$$

$$(\mathbf{x}'\times\mathbf{x}'')\cdot(\mathbf{x}'\times\mathbf{x}'') \;=\; \frac{(r_0+r_1)^4(r_0+2r_1)^2}{r_1^2}[1-\cos(r_0/r_1)\theta]^2$$

$$\mathbf{x}'\cdot\mathbf{x}' \;=\; 2(r_0+r_1)^2\left[1 - \left(\sin\theta\sin\left(\frac{r_0+r_1}{r_1}\theta\right) + \cos\theta\cos\left(\frac{r_0+r_1}{r_1}\theta\right)\right)\right]$$

$$= 2(r_0+r_1)^2[1-\cos(r_0/r_1)\theta]$$

$$\kappa^2 \;=\; \frac{(\mathbf{x}'\times\mathbf{x}'')\cdot(\mathbf{x}'\times\mathbf{x}'')}{(\mathbf{x}'\cdot\mathbf{x}')^3} \;=\; \frac{(r_0+2r_1)^2}{8r_1^2(r_0+r_1)^2[1-\cos(r_0/r_1)\theta]}$$

$$= \frac{(r_0+2r_1)^2}{16r_1^2(r_0+r_1)^2\sin^2(r_0/2r_1)\theta}$$

Also $\quad s = \int_{\pi/2}^{\theta}\left|\frac{d\mathbf{x}}{d\theta}\right|d\theta \;=\; \frac{4r_1(r_0+r_1)}{r_0}\cos(r_0/2r_1)\theta, \qquad s^2 \;=\; \frac{16r_1^2(r_0+r_1)^2}{r_0^2}\cos^2(r_0/2r_1)\theta$

Eliminating θ in the equation for κ^2 and s^2 gives

$$\frac{s^2}{A^2}+\frac{1}{\kappa^2B^2} = 1 \qquad\text{or}\qquad \frac{s^2}{A^2}+\frac{\rho^2}{B^2} = 1$$

where $A = \dfrac{4r_1(r_0+r_1)}{r_0}$ and $B = \dfrac{4r_1(r_0+r_1)}{r_0+2r_1}$, which is the required result. Observe that $A > B$. For a hypocycloid, $B > A$ (see Problem 3.28, page 60).

5.3. Determine the curve whose intrinsic equations are

$$\kappa \;=\; (1/2as)^{1/2}, \qquad \tau \;=\; 0, \qquad a>0,\; s>0$$

We set $\dot\phi = \kappa = (1/2as)^{1/2}$. Integrating gives $\phi = (2s/a)^{1/2}$ or $s = a\phi^2/2$, so that $\kappa = 1/a\phi$. It follows from equation (5.6) that the curve is

$$\mathbf{x} \;=\; a\int\phi[(\cos\phi)\mathbf{e}_1 + (\sin\phi)\mathbf{e}_2]\,d\phi \;=\; a(\cos\phi+\phi\sin\phi)\mathbf{e}_1 + a(\sin\phi-\phi\cos\phi)\mathbf{e}_2$$

5.4. Show that the curve whose intrinsic equations are $\kappa = \sqrt{2}/(s^2+4)$, $\tau = \sqrt{2}/(s^2+4)$ is a general helix on a cylinder whose cross section is a catenary.

From Problem 4.21, page 77, the curve is a general helix, since $\kappa/\tau = 1 =$ constant. Now let $\mathbf{x} = \mathbf{x}(s)$ be a natural representation of the helix, the unit vector $\mathbf{u}$ its axis, and $\alpha = \measuredangle(\mathbf{t},\mathbf{u})$. Also from Problem 4.21, $\kappa/\tau = \tan\alpha$; thus $\alpha = \pi/4$. To find the intrinsic equations of the projection $\mathbf{x}^* = \mathbf{x}(s) - (\mathbf{x}(s)\cdot\mathbf{u})\mathbf{u}$ onto the plane through the origin and perpendicular to $\mathbf{u}$, as shown in Fig. 5-11, we compute

$$\frac{d\mathbf{x}^*}{ds} = \mathbf{t} - (\mathbf{t}\cdot\mathbf{u})\mathbf{u} = \mathbf{t} - (\cos\alpha)\mathbf{u} = \mathbf{t} - \mathbf{u}/\sqrt{2}$$

$$\left|\frac{d\mathbf{x}^*}{ds}\right| = [(\mathbf{t} - \mathbf{u}/\sqrt{2})\cdot(\mathbf{t} - \mathbf{u}/\sqrt{2})]^{1/2} = 1/\sqrt{2}$$

A natural parameter along the projection is

$$s^* = \int_0^s \left|\frac{d\mathbf{x}^*}{ds}\right| ds = s/\sqrt{2}$$

From Problem 4.6, page 72, we have

$$\kappa^* = \kappa/\sin^2\alpha = 2\sqrt{2}/(s^2+4)$$

Hence $\kappa^* = \sqrt{2}/(s^{*2}+2)$ is the intrinsic equation of the projection. It follows from Problem 5.1 that the projection is a catenary.

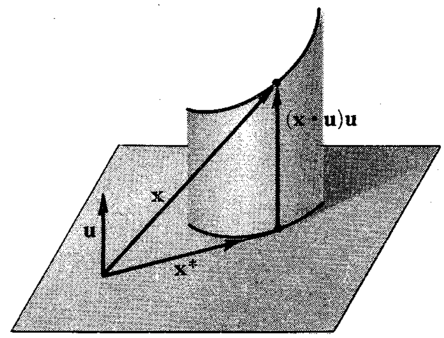


Fig. 5-11

5.5. Show that for a curve lying on a sphere of radius a and such that the torsion τ is never 0, the following equation is satisfied

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot\kappa}{\kappa^2\tau}\right)^2 = a^2$$

Let $\mathbf{x} = \mathbf{x}(s)$ lie on the sphere with center $\mathbf{y}_0$ and radius a. Then for all s,

$$(\mathbf{x}(s) - \mathbf{y}_0)\cdot(\mathbf{x}(s) - \mathbf{y}_0) = a^2$$

Differentiating, $\quad 2(\mathbf{x}-\mathbf{y}_0)\cdot\dot{\mathbf{x}} = 0 \quad$ or $\quad (\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{t} = 0$

Differentiating again,

$$(\mathbf{x}-\mathbf{y}_0)\cdot\dot{\mathbf{t}} + \dot{\mathbf{x}}\cdot\mathbf{t} = 0 \quad \text{or} \quad \kappa(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{n} + 1 = 0$$

Note it follows that $\kappa \neq 0$ and $(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{n} = -1/\kappa$. Finally, differentiating again,

$$\dot{\mathbf{x}}\cdot\mathbf{n} + (\mathbf{x}-\mathbf{y}_0)\cdot\dot{\mathbf{n}} = \dot\kappa/\kappa^2 \quad \text{or} \quad (\mathbf{x}-\mathbf{y}_0)\cdot(-\kappa\mathbf{t}+\tau\mathbf{b}) = \dot\kappa/\kappa^2$$

Using $(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{t} = 0$, we have, where $\tau \neq 0$, $(\mathbf{x}-\mathbf{y}_0)\cdot\mathbf{b} = \dot\kappa/\kappa^2\tau$. Thus the components of $\mathbf{x}-\mathbf{y}_0$ with respect to $\mathbf{t},\mathbf{n},\mathbf{b}$ are $0, -1/\kappa, \dot\kappa/\kappa^2\tau$. Hence

$$\mathbf{x} - \mathbf{y}_0 = \frac{-1}{\kappa}\mathbf{n} + \frac{\dot\kappa}{\kappa^2\tau}\mathbf{b}$$

But on the sphere,

$$(\mathbf{x}-\mathbf{y}_0)\cdot(\mathbf{x}-\mathbf{y}_0) = \left(-\frac{1}{\kappa}\mathbf{n} + \frac{\dot\kappa}{\kappa^2\tau}\mathbf{b}\right)\cdot\left(-\frac{1}{\kappa}\mathbf{n} + \frac{\dot\kappa}{\kappa^2\tau}\mathbf{b}\right) = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot\kappa}{\kappa^2\tau}\right)^2 = a^2$$

5.6. Show that if the position vector $\mathbf{x}$ of a plane curve $\mathbf{x} = \mathbf{x}(s)$ makes a constant angle α with the tangent $\mathbf{t} = \mathbf{t}(s)$, then $\mathbf{x} = \mathbf{x}(s)$ is a logarithmic spiral.

We have $\mathbf{x}\cdot\mathbf{t} = (\cos\alpha)|\mathbf{x}|$. Also, since $\mathbf{n}\perp\mathbf{t}$, $|\mathbf{x}\cdot\mathbf{n}| = |\sin\alpha|\,|\mathbf{x}|$ and we can choose the direction of $\mathbf{n}$ so that $\mathbf{x}\cdot\mathbf{n} = (-\sin\alpha)|\mathbf{x}|$. Differentiating the first gives

$$\dot{\mathbf{x}}\cdot\mathbf{t} + \mathbf{x}\cdot\dot{\mathbf{t}} = (\cos\alpha)\frac{\mathbf{x}\cdot\mathbf{t}}{|\mathbf{x}|} \quad \text{or} \quad 1 + \kappa\mathbf{x}\cdot\mathbf{n} = \cos^2\alpha$$

Differentiating again, $\quad \dot\kappa\mathbf{x}\cdot\mathbf{n} + \kappa\dot{\mathbf{x}}\cdot\mathbf{n} + \kappa\mathbf{x}\cdot\dot{\mathbf{n}} = 0$

Since $\tau = 0$, $\dot{\mathbf{n}} = -\kappa\mathbf{t}$; thus $\dot\kappa\mathbf{x}\cdot\mathbf{n} - \kappa^2\mathbf{x}\cdot\mathbf{t} = 0$ and

$$|\mathbf{x}|(-\dot\kappa\sin\alpha - \kappa^2\cos\alpha) = 0 \quad \text{or} \quad \dot\kappa = -(\cot\alpha)\kappa^2$$

This differential equation for κ can be solved by separating variables, giving

$$\kappa = \frac{1}{(\cot\alpha)s + c}$$

which is the intrinsic equation of a logarithmic spiral (see Example 5.2 and Problem 5.29).

Another method consists in introducing $\phi = \measuredangle(\mathbf{t}, \mathbf{e}_1)$, the polar angle $\theta = \measuredangle(\mathbf{x}, \mathbf{e}_1) = \phi - \alpha$, and $r = |\mathbf{x}|$, as shown in Fig. 5-12. Then $\mathbf{x} \cdot \mathbf{t} = (\cos\alpha)r$ and $\mathbf{x} \cdot \mathbf{n} = (-\sin\alpha)r$. Differentiating the latter gives

$$-\sin\alpha\frac{dr}{d\theta} = \frac{d}{ds}(\mathbf{x}\cdot\mathbf{n})\frac{ds}{d\phi}\frac{d\phi}{d\theta} = (\dot{\mathbf{x}}\cdot\mathbf{n} + \mathbf{x}\cdot\dot{\mathbf{n}})(1/\kappa) = (\mathbf{x}\cdot\dot{\mathbf{n}})(1/\kappa)$$

where we used $d\phi/ds = \kappa$ and $d\phi/d\theta = 1$. Then

$$-\sin\alpha\frac{dr}{d\theta} = -(\mathbf{x}\cdot\mathbf{t}) = -(\cos\alpha)r$$

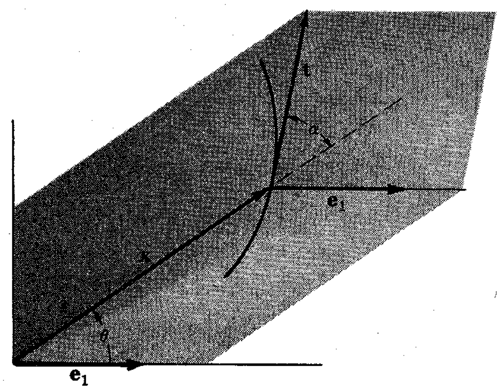


Fig. 5-12

where we used $\dot{\mathbf{n}} = -\kappa\mathbf{t}$. Integrating $dr/d\theta = r\cot\alpha$ gives $r = e^{(\cot\alpha)\theta + C}$, a logarithmic spiral.

5.7. Let a curve C be defined by

$$\mathbf{x} = a\int \mathbf{g}(t) \times \mathbf{g}'(t)\,dt, \qquad a = \text{constant} \neq 0$$

where $\mathbf{g}(t)$ is a vector function satisfying $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\mathbf{g}'\mathbf{g}''] \neq 0$. Show that $\kappa \neq 0$ and $\tau = 1/a$.

$$\mathbf{x}' = a(\mathbf{g}\times\mathbf{g}')$$
$$\mathbf{x}'' = a(\mathbf{g}\times\mathbf{g}'') + a(\mathbf{g}'\times\mathbf{g}') = a(\mathbf{g}\times\mathbf{g}'')$$
$$\mathbf{x}''' = a(\mathbf{g}\times\mathbf{g}''') + a(\mathbf{g}'\times\mathbf{g}'')$$

Since $|\mathbf{g}| = 1$, $\mathbf{g} \perp \mathbf{g}'$; and using the identity in Example 1.19, page 8,

$$\mathbf{x}'\cdot\mathbf{x}' = a^2(\mathbf{g}\times\mathbf{g}')\cdot(\mathbf{g}\times\mathbf{g}') = a^2(\mathbf{g}'\cdot\mathbf{g}')(\mathbf{g}\cdot\mathbf{g}) = a^2|\mathbf{g}'|^2$$

From identity $[\mathbf{F}_2]$, page 10,

$$\mathbf{x}'\times\mathbf{x}'' = a^2(\mathbf{g}\times\mathbf{g}')\times(\mathbf{g}\times\mathbf{g}'') = a^2\{[\mathbf{g}\mathbf{g}'\mathbf{g}'']\mathbf{g} - [\mathbf{g}'\mathbf{g}]\mathbf{g}''\} = a^2[\mathbf{g}\mathbf{g}'\mathbf{g}'']\mathbf{g}$$

But from Theorem 4.2, $\kappa = \dfrac{|\mathbf{x}'\times\mathbf{x}''|}{|\mathbf{x}'|^3}$ and so $\kappa = \dfrac{|[\mathbf{g}\mathbf{g}'\mathbf{g}'']|}{a|\mathbf{g}'|^3} \neq 0$; also

$$[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] = (\mathbf{x}'\times\mathbf{x}'')\cdot\mathbf{x}''' = a^2[\mathbf{g}\mathbf{g}'\mathbf{g}'']\mathbf{g}\cdot[a(\mathbf{g}\times\mathbf{g}''') + a(\mathbf{g}'\times\mathbf{g}'')] = a^3[\mathbf{g}\mathbf{g}'\mathbf{g}'']^2$$

From Theorem 4.5, page 70,

$$\tau = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{(\mathbf{x}'\times\mathbf{x}'')\cdot(\mathbf{x}'\times\mathbf{x}'')} = \frac{a^3[\mathbf{g}\mathbf{g}'\mathbf{g}'']^2}{a^4[\mathbf{g}\mathbf{g}'\mathbf{g}'']^2} = \frac{1}{a} = \text{constant} \neq 0$$

5.8. Prove the converse of the preceding problem: If $\tau = \text{constant} = 1/a$ along a curve, it can be expressed in the form $\mathbf{x} = a\int \mathbf{g}(t)\times\mathbf{g}'(t)\,dt$ where $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\mathbf{g}'\mathbf{g}''] \neq 0$.

From Frenet's equations,

$$\mathbf{t} = \mathbf{n}\times\mathbf{b} = -\mathbf{b}\times\mathbf{n} = -\mathbf{b}\times(-\dot{\mathbf{b}}/\tau) = a\mathbf{b}\times\dot{\mathbf{b}}$$

Thus $\mathbf{x} = \int \mathbf{t}\,ds = a\int \mathbf{b}\times\dot{\mathbf{b}}\,ds$. From $\dot{\mathbf{b}} = -\tau\mathbf{n} = -(1/a)\mathbf{n}$ and $\ddot{\mathbf{b}} = -(1/a)\dot{\mathbf{n}} = -(1/a)(-\kappa\mathbf{t} + \tau\mathbf{b})$, it follows that $\mathbf{b}, \dot{\mathbf{b}}$ and $\ddot{\mathbf{b}}$ are linearly independent or $[\mathbf{b}\dot{\mathbf{b}}\ddot{\mathbf{b}}] \neq 0$.

5.9. Show that the difference in the length s of a sufficiently small arc of a curve and the length $|\mathbf{PQ}|$ of the corresponding chord is of the order s^3.

We use the canonical representation

$$x_1 = s - \tfrac{1}{6}\kappa_0^2 s^3 + o(s^3), \quad x_2 = \tfrac{1}{2}\kappa_0 s^2 + \tfrac{1}{6}\dot{\kappa}_0 s^3 + o(s^3), \quad x_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3 + o(s^3)$$

Squaring each component,

$$x_1^2 = s^2 - \tfrac{1}{3}\kappa_0^2 s^4 + o(s^4), \quad x_2^2 = \tfrac{1}{4}\kappa_0^2 s^4 + o(s^4), \quad x_3^2 = o(s^4)$$

Now

$$\begin{aligned} |\mathbf{PQ}| &= (x_1^2 + x_2^2 + x_3^2)^{1/2} = [s^2 - \tfrac{1}{12}\kappa_0^2 s^4 + o(s^4)]^{1/2} \\ &= [(s - \tfrac{1}{24}\kappa_0^2 s^3 + o(s^3))^2]^{1/2} = s - \tfrac{1}{24}\kappa_0^2 s^3 + o(s^3) \end{aligned}$$

Thus $|\mathbf{PQ}| - s = -\dfrac{\kappa_0^2}{24}s^3 + o(s^3)$ and the difference is of order s^3.

5.10. If the principal normal lines of a curve C are the same as the binormal lines of a curve C^*, show that along C,

$$\alpha(\kappa^2 + \tau^2) = \kappa, \quad \alpha = \text{constant}$$

Let C be given by $\mathbf{x} = \mathbf{x}(s)$ and let $\mathbf{x}^*$ denote a point on C^* where the binormal equals the principal normal at $\mathbf{x}(s)$, as shown in Fig. 5-13. Then $\mathbf{x}^* - \mathbf{x}(s)$ is parallel to $\mathbf{n}(s)$, so that

$$\mathbf{x}^* = \mathbf{x}(s) + \alpha(s)\,\mathbf{n}(s)$$

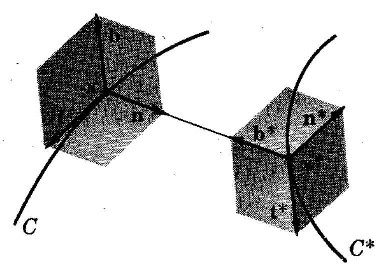


Fig. 5-13

We consider the tangent vector

$$\begin{aligned} \frac{d\mathbf{x}^*}{ds} &= \frac{d}{ds}(\mathbf{x} + \alpha\mathbf{n}) = (\dot{\mathbf{x}} + \dot{\alpha}\mathbf{n} + \alpha\dot{\mathbf{n}}) \\ &= (\mathbf{t} + \dot{\alpha}\mathbf{n} - \alpha\kappa\mathbf{t} + \alpha\tau\mathbf{b}) = (1 - \alpha\kappa)\mathbf{t} + \dot{\alpha}\mathbf{n} + \alpha\tau\mathbf{b} \end{aligned}$$

Since $\dfrac{d\mathbf{x}^*}{ds} \perp \mathbf{b}^*$, also $\dfrac{d\mathbf{x}^*}{ds} \perp \mathbf{n}$. Thus

$$0 = \mathbf{n}\cdot\frac{d\mathbf{x}^*}{ds} = (1 - \alpha\kappa)(\mathbf{t}\cdot\mathbf{n}) + \dot{\alpha}(\mathbf{n}\cdot\mathbf{n}) + \alpha\tau(\mathbf{b}\cdot\mathbf{n}) = \dot{\alpha}$$

and so $\alpha = \text{constant}$, and $d\mathbf{x}^*/ds = (1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}$. Now

$$\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds^*} = \frac{d\mathbf{x}^*}{ds}\frac{ds}{ds^*} = [(1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{ds}{ds^*}$$

and

$$\begin{aligned} \frac{d\mathbf{t}^*}{ds^*} &= \frac{d}{ds}[(1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{ds}{ds^*} + [(1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{d^2s}{ds^{*2}} \\ &= [-\alpha\dot{\kappa}\mathbf{t} + (1 - \alpha\kappa)\dot{\mathbf{t}} + \alpha\dot{\tau}\mathbf{b} + \alpha\tau\dot{\mathbf{b}}]\frac{ds}{ds^*} + [(1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{d^2s}{ds^{*2}} \\ &= [-\alpha\dot{\kappa}\mathbf{t} + [(1 - \alpha\kappa) - \alpha\tau^2]\mathbf{n} + \alpha\dot{\tau}\mathbf{b}]\frac{ds}{ds^*} + [(1 - \alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{d^2s}{ds^{*2}} \end{aligned}$$

But $\dfrac{d\mathbf{t}^*}{ds^*} \perp \mathbf{b}^*$, since $\dfrac{d\mathbf{t}^*}{ds^*} = \kappa^*\mathbf{n}^*$. Thus $\dfrac{d\mathbf{t}^*}{ds^*} \perp \mathbf{n}$ and

$$0 = \mathbf{n}\cdot\frac{d\mathbf{t}^*}{ds^*} = [(1 - \alpha\kappa)\kappa - \alpha\tau^2]\frac{ds}{ds^*}$$

Since $ds/ds^* \neq 0$ (otherwise $s = \text{constant}$), we have $(1 - \alpha\kappa)\kappa - \alpha\tau^2 = 0$ or $\alpha(\kappa^2 + \tau^2) = \kappa$.

5.11. Two curves C and C^* are called *Bertrand curves* if they have common principal normal lines, as shown in Fig. 5-14 below.

(*a*) Show that the distance between corresponding points on two Bertrand curves is constant.

(*b*) Show that the angle between corresponding tangent lines on two Bertrand curves is constant.

(*a*) Let C be given by $\mathbf{x} = x(s)$, and let $\mathbf{x}^*$ denote the point on C^* where the principal normal line is also the principal normal line at $\mathbf{x}(s)$. Then $\mathbf{x}^* - \mathbf{x}(s)$ is proportional to $\mathbf{n}(s)$, so that

$$\mathbf{x}^* = \mathbf{x}(s) + \alpha(s)\,\mathbf{n}(s)$$

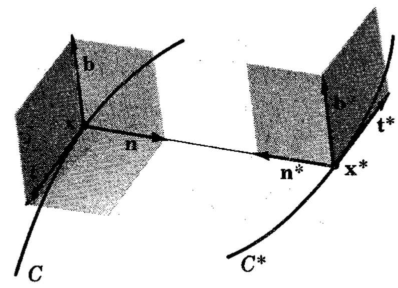


Fig. 5-14

Differentiating,

$$\begin{aligned} d\mathbf{x}^*/ds &= \dot{\mathbf{x}} + \dot{\alpha}\mathbf{n} + \alpha\dot{\mathbf{n}} \\ &= \mathbf{t} + \dot{\alpha}\mathbf{n} + \alpha(-\kappa\mathbf{t} + \tau\mathbf{b}) \\ &= (1-\alpha\kappa)\mathbf{t} + \dot{\alpha}\mathbf{n} + \alpha\tau\mathbf{b} \end{aligned}$$

But $d\mathbf{x}^*/ds$ is tangent to C^*, hence orthogonal to $\mathbf{n}^*$ and $\mathbf{n}$. Thus $\mathbf{n}\cdot(d\mathbf{x}^*/ds) = \dot{\alpha} = 0$ and α = constant, and so the distance between corresponding points is $|\mathbf{x}^* - \mathbf{x}(s)| = |\alpha\,\mathbf{n}(s)| = |\alpha|$ = constant.

(*b*) Let $\mathbf{t}$ and $\mathbf{t}^*$ be the unit tangent vectors on $\mathbf{x} = \mathbf{x}(s)$ and $\mathbf{x}^* = \mathbf{x} + \alpha\mathbf{n}$ respectively and consider

$$\begin{aligned} \frac{d}{ds}(\mathbf{t}^*\cdot\mathbf{t}) &= \frac{d\mathbf{t}^*}{ds}\cdot\mathbf{t} + \mathbf{t}^*\cdot\dot{\mathbf{t}} = \left(\frac{d\mathbf{t}^*}{ds^*}\cdot\mathbf{t}\right)\frac{ds^*}{ds} + \mathbf{t}^*\cdot\kappa\mathbf{n} \\ &= \kappa^*\frac{ds^*}{ds}(\mathbf{n}^*\cdot\mathbf{t}) + \kappa(\mathbf{t}^*\cdot\mathbf{n}) \end{aligned}$$

Since $\mathbf{n}^* = \pm\mathbf{n}$, $\mathbf{n}^* \perp \mathbf{t}$ and $\mathbf{n} \perp \mathbf{t}^*$. Thus $\dfrac{d}{ds}(\mathbf{t}^*\cdot\mathbf{t}) = 0$ and so $\mathbf{t}^*\cdot\mathbf{t}$ = constant.

5.12. If $\tau \neq 0$ along C, show that C is a Bertrand curve (i.e. there exists a curve C^* such that C and C^* are Bertrand curves) iff there are constants γ and α such that $\kappa + \gamma\tau = 1/\alpha$.

Suppose C is given by $\mathbf{x} = \mathbf{x}(s)$, along C, $\tau \neq 0$, and $\kappa + \gamma\tau = 1/\alpha$. Define C^* by

$$\mathbf{x}^* = \mathbf{x}(s) + \alpha\,\mathbf{n}(s)$$

To show that C and C^* are Bertrand curves, we compute

$$\frac{d\mathbf{x}^*}{ds} = (\dot{\mathbf{x}} + \alpha\dot{\mathbf{n}}) = \mathbf{t} - \alpha\kappa\mathbf{t} + \alpha\tau\mathbf{b} = (1-\alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b} = \alpha\tau\gamma\mathbf{t} + \alpha\tau\mathbf{b} = \alpha\tau(\gamma\mathbf{t} + \mathbf{b})$$

$$\left|\frac{d\mathbf{x}^*}{ds}\right| = |\alpha\tau|(\gamma^2+1)^{1/2}$$

$$\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds}\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| = \pm(\gamma^2+1)^{-1/2}(\gamma\mathbf{t} + \mathbf{b})$$

$$\frac{d\mathbf{t}^*}{ds} = \pm(\gamma^2+1)^{-1/2}(\gamma\dot{\mathbf{t}} + \dot{\mathbf{b}}) = \pm(\gamma^2+1)^{-1/2}(\gamma\kappa - \tau)\mathbf{n}$$

$$\frac{d\mathbf{t}^*}{ds^*} = \frac{d\mathbf{t}^*}{ds}\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| = \pm\frac{(\gamma\kappa - \tau)}{|\alpha\tau|(\gamma^2+1)}\mathbf{n} = \kappa^*\mathbf{n}^*$$

It follows that $\mathbf{n} = \pm\mathbf{n}^*$, which proves the curves are Bertrand curves.

Conversely, suppose C, given by $\mathbf{x} = \mathbf{x}(s)$, is a Bertrand curve and $\tau \neq 0$ along C. From the preceding problem, C^* can be represented by

$$\mathbf{x}^* = \mathbf{x}(s) + \alpha\,\mathbf{n}(s), \qquad \alpha = \text{constant} \neq 0$$

We compute

$$\frac{d\mathbf{x}^*}{ds} = \mathbf{t} + \alpha\dot{\mathbf{n}} = (1-\alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}$$

$$\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds}\frac{ds}{ds^*} = [(1-\alpha\kappa)\mathbf{t} + \alpha\tau\mathbf{b}]\frac{ds}{ds^*}$$

Now, from the preceding problem, $\mathbf{t}\cdot\mathbf{t}^*$ = constant = $\cos\beta$. Then

$$\mathbf{t}\cdot\mathbf{t}^* = (1-\alpha\kappa)\frac{ds}{ds^*} = \cos\beta$$

Since $\mathbf{t}^*$ is a unit vector in the plane of $\mathbf{t}$ and $\mathbf{b}$, and $\mathbf{t}\cdot\mathbf{t}^* = \cos\beta$, we have $\mathbf{b}\cdot\mathbf{t}^* = \pm\sin\beta$. Thus

$$\mathbf{b}\cdot\mathbf{t}^* = \alpha\tau\frac{ds}{ds^*} = \pm\sin\beta$$

Since $\alpha \neq 0$, $\tau \neq 0$ and $ds/ds^* \neq 0$, then $\sin\beta \neq 0$. Finally, eliminating ds/ds^*, we have $\pm(1-\alpha\kappa)\sin\beta = \alpha\tau\cos\beta$ or $\kappa \pm \tau\cot\beta = 1/\alpha$ or $\kappa + \gamma\tau = 1/\alpha$, where $\gamma = \pm\cot\beta$.

5.13. If $\tau \neq 0$ along C, show that more than one curve C^* exists such that C and C^* are Bertrand curves, iff C is a circular helix.

Suppose C is a circular helix. Then along C, $\tau = \text{constant} \neq 0$ and $\kappa = \text{constant} \neq 0$. Thus for *any* $\alpha \neq 0$ one can find $\gamma = \text{constant} \neq 0$ such that $\kappa + \gamma\tau = 1/\alpha$. It follows from the preceding problem that C has an infinite number of Bertrand partners.

Conversely, suppose C, C^* and C^{**} are Bertrand curves and along C, $\tau \neq 0$. Then the preceding problem shows that there exist constants $\gamma, \gamma^*, \alpha, \alpha^*$ ($\alpha \neq \alpha^*$, otherwise $C^* = C^{**}$) such that

$$\kappa + \gamma\tau = 1/\alpha \quad \text{and} \quad \kappa + \gamma^*\tau = 1/\alpha^*$$

Thus $\gamma \neq \gamma^*$ and $\tau = (1/\alpha^* - 1/\alpha)/(\gamma^* - \gamma) = \text{constant}$ and solving the above equations gives $\kappa = (\gamma^*/\alpha - \gamma/\alpha^*)(\gamma^* - \gamma) = \text{constant}$. Since $\kappa = \text{constant}$ and $\tau = \text{constant}$ are the intrinsic equations of C, the curve C is a circular helix.

INVOLUTES AND EVOLUTES

5.14. Find the equation of the involute of the circle $\mathbf{x} = a(\cos\theta)\mathbf{e}_1 + a(\sin\theta)\mathbf{e}_2$, $a > 0$. The involute is generated by unwinding a string, starting at $\theta = 0$ as shown in Fig. 5-15.

If s is a natural parameter along the circle such that $s = 0$ when $\theta = 0$, then the equation of the involute is $\mathbf{x}^* = \mathbf{x} - s\mathbf{t}$. Clearly $s = a\theta$, the arc length. Also, $\mathbf{t} = \frac{d\mathbf{x}}{d\theta}\Big/\left|\frac{d\mathbf{x}}{d\theta}\right| = (-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2$. Thus

$$\begin{aligned}\mathbf{x}^* &= (a(\cos\theta)\mathbf{e}_1 + a(\sin\theta)\mathbf{e}_2) \\ &\quad - a\theta((-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2) \\ &= a(\cos\theta + \theta\sin\theta)\mathbf{e}_1 + a(\sin\theta - \theta\cos\theta)\mathbf{e}_2\end{aligned}$$

is the required result.

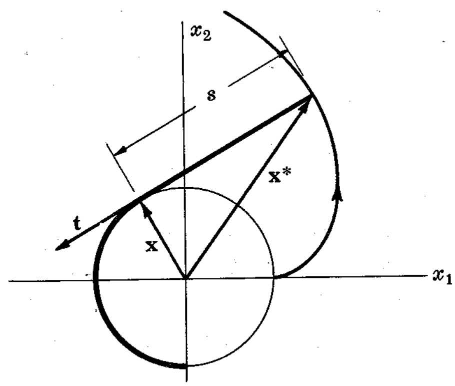


Fig. 5-15

5.15. Show that the curvature of the involute $\mathbf{x}^* = \mathbf{x} + (c - s)\mathbf{t}$ of $\mathbf{x} = \mathbf{x}(s)$ is given by

$$\kappa^{*2} = \frac{\kappa^2 + \tau^2}{(c-s)^2\kappa^2}$$

$$\frac{d\mathbf{x}^*}{ds} = \dot{\mathbf{x}} - \mathbf{t} + (c-s)\dot{\mathbf{t}} = (c-s)\kappa\mathbf{n}, \qquad \left|\frac{d\mathbf{x}^*}{ds}\right| = |(c-s)\kappa|$$

$$\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds}\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| = \text{sign}\,[(c-s)\kappa]\mathbf{n}$$

$$\frac{d\mathbf{t}^*}{ds} = \text{sign}\,[(c-s)\kappa]\dot{\mathbf{n}} = \text{sign}\,[(c-s)\kappa](-\kappa\mathbf{t} + \tau\mathbf{b})$$

$$\frac{d\mathbf{t}^*}{ds^*} = \frac{d\mathbf{t}^*}{ds}\Big/\left|\frac{d\mathbf{x}^*}{ds}\right| = \frac{-\kappa\mathbf{t} + \tau\mathbf{b}}{(c-s)\kappa}$$

Thus

$$\kappa^{*2} = \left|\frac{d\mathbf{t}^*}{ds^*}\right|^2 = \frac{\kappa^2 + \tau^2}{(c-s)^2\kappa^2}$$

5.16. Show that the unit binormal of the involute $\mathbf{x}^* = \mathbf{x} + (c - s)\mathbf{t}$ of $\mathbf{x} = \mathbf{x}(s)$ is

$$\mathbf{b}^* = \frac{\kappa\mathbf{b} + \tau\mathbf{t}}{|(c-s)\kappa|\kappa^*}$$

From the preceding problem

$$\mathbf{t}^* = \text{sign}\,[(c-s)\kappa]\mathbf{n} \quad \text{and} \quad \frac{d\mathbf{t}^*}{ds^*} = \frac{-\kappa\mathbf{t} + \tau\mathbf{b}}{(c-s)\kappa} = \kappa^*\mathbf{n}^*$$

Hence

$$\mathbf{n}^* = \frac{-\kappa\mathbf{t} + \tau\mathbf{b}}{(c-s)\kappa\kappa^*} \quad \text{and} \quad \mathbf{b}^* = \mathbf{t}^* \times \mathbf{n}^* = \frac{\mathbf{n} \times (-\kappa\mathbf{t} + \tau\mathbf{b})}{|(c-s)\kappa|\kappa^*} = \frac{\kappa\mathbf{b} + \tau\mathbf{t}}{|(c-s)\kappa|\kappa^*}$$

5.17. Show that two involutes of a plane curve are Bertrand curves (see Problem 5.11).

As shown in Fig. 5-16, if C_1^* and C_2^* are involutes of the plane curve C at corresponding points along a tangent line L of C, the tangent lines of C_1^* and C_2^* are both perpendicular to L. Hence their principal normal lines at these points coincide and in fact are L, which proves that C_1^* and C_2^* are Bertrand curves.

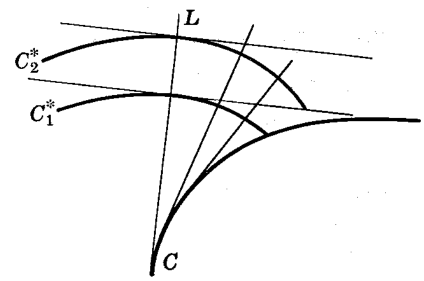


Fig. 5-16

5.18. Prove that the principal normal line of an evolute C^* of a curve C is parallel to the tangent line of C.

By definition C^* is an evolute of C if C is an involute of C^*. Thus the problem is equivalent to showing that the tangent line of an involute C of a curve C^* is parallel to the normal line of C^*. But this follows from the equation

$$\mathbf{t}^* = \text{sign}\,|(c-s)\kappa|\,\mathbf{n} = \pm\mathbf{n}$$

derived in Problem 5.15.

5.19. Show that the intersection of the tangent surface of a curve C and the normal plane to C at a point P where $\tau_0, \kappa_0 \neq 0$ is a curve which has a cusp at P, as shown in Fig. 5-17.

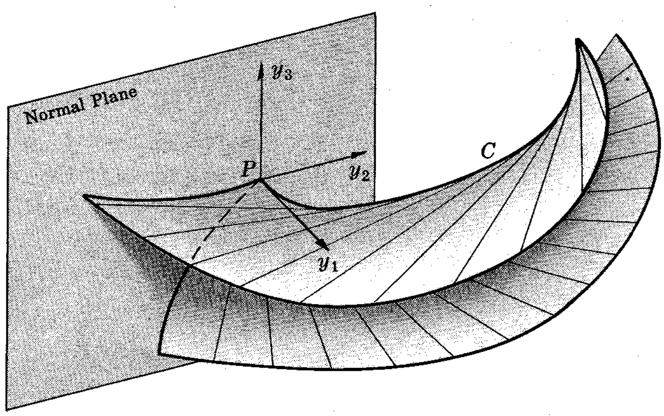


Fig. 5-17

We use the leading terms

$$x_1 = s, \quad x_2 = \tfrac{1}{2}\kappa_0 s^2, \quad x_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3$$

of the canonical representation of C near P, in which case the tangent surface $\mathbf{y} = \mathbf{x} + k\mathbf{t}$, $-\infty < k < \infty$ near P is like

$$y_1 = s + k, \quad y_2 = \tfrac{1}{2}\kappa_0 s^2 + k\kappa_0 s, \quad y_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3 + \tfrac{1}{2}k\kappa_0\tau_0 s^2$$

The normal plane at P is the y_2y_3 plane, i.e. $y_1 = 0$. Then along the intersection, $y_1 = s + k = 0$ or $s = -k$. Thus near P the intersection is like

$$y_2 = \tfrac{1}{2}\kappa_0 s^2 - \kappa_0 s^2 = -\tfrac{1}{2}\kappa_0 s^2, \quad y_3 = \tfrac{1}{6}\kappa_0\tau_0 s^3 - \tfrac{1}{2}\kappa_0\tau_0 s^3 = -\tfrac{1}{3}\kappa_0\tau_0 s^3$$

or $y_2 = -\tfrac{1}{2}\kappa_0^{1/3}(3/\tau_0)^{2/3}y_3^{2/3}$, which is the cusp shown in Fig. 5-17.

THEORY OF CONTACT, OSCULATING SURFACES

5.20. Show that the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ has 6-point contact with the paraboloid $x_1^2 + x_3^2 - x_2 = 0$ at the origin $(t = 0)$.

We consider

$$f(t) = (t^2) + (t^3)^2 - t^2 = t^6$$

$$f'(t) = 6t^5, \quad f''(t) = 30t^4, \quad f'''(t) = 120t^3, \quad f^{(4)}(t) = 360t^2, \quad f^{(5)}(t) = 720t, \quad f^{(6)}(t) = 720$$

Clearly $f^{(i)}(0) = 0$, $i = 1, 2, 3, 4, 5$ and $f^{(6)}(0) \neq 0$. Hence the curve has 6-point contact with the paraboloid at $t = 0$.

5.21. Show that the osculating plane has at least 4-point contact with a curve at P iff either the curvature or the torsion vanishes at P.

The osculating plane to the curve $\mathbf{x} = \mathbf{x}(s)$ at $s = s_0$ is $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{b}_0 = 0$. We consider

$$f(s) = (\mathbf{x}(s) - \mathbf{x}_0) \cdot \mathbf{b}_0$$

$$f'(s) = \dot{\mathbf{x}} \cdot \mathbf{b}_0 = \mathbf{t} \cdot \mathbf{b}_0, \quad f''(s) = \dot{\mathbf{t}} \cdot \mathbf{b}_0 = \kappa\mathbf{n} \cdot \mathbf{b}_0$$

$$f'''(s) = \dot{\kappa}\mathbf{n} \cdot \mathbf{b}_0 + \kappa\dot{\mathbf{n}} \cdot \mathbf{b}_0 = \dot{\kappa}\mathbf{n} \cdot \mathbf{b}_0 - \kappa^2\mathbf{t} \cdot \mathbf{b}_0 + \kappa\tau\mathbf{b} \cdot \mathbf{b}_0$$

Clearly $$f(s_0) = 0, \quad f'(s_0) = \mathbf{t}_0 \cdot \mathbf{b}_0 = 0, \quad f''(s_0) = \kappa_0\mathbf{n}_0 \cdot \mathbf{b}_0 = 0$$

$$f'''(s_0) = \dot{\kappa}_0\mathbf{n}_0 \cdot \mathbf{b}_0 - \kappa_0^2\mathbf{t}_0 \cdot \mathbf{b}_0 + \kappa_0\tau_0\mathbf{b}_0 \cdot \mathbf{b}_0 = \kappa_0\tau_0 = 0$$

iff $\kappa_0 = 0$ or $\tau_0 = 0$, which is the required result.

5.22. Show that the locus of centers of curvature of a curve is an evolute of the curve iff the curve is a plane curve.

The locus of centers of curvature of the curve $\mathbf{x} = \mathbf{x}(s)$ is the curve $\mathbf{y} = \mathbf{x} + (1/\kappa)\mathbf{n}$ and therefore is the unique plane evolute of $\mathbf{x} = \mathbf{x}(s)$. (See Example 5.6.) On the other hand the evolutes of $\mathbf{x} = \mathbf{x}(s)$ are of the form

$$\mathbf{x}^* = \mathbf{x} + (1/\kappa)\mathbf{n} + (1/\kappa)\left[\cot\left(\int \tau\, ds + c\right)\right]\mathbf{b}$$

Since $\mathbf{n}$ and $\mathbf{b}$ are linearly independent, $\mathbf{y}$ can be equal to some $\mathbf{x}^*$ iff for some c,

$$\cot\left(\int \tau\, ds + c\right) = 0 \qquad \text{or} \qquad \int \tau\, ds = \pi/2 - c = \text{constant}$$

Hence $\tau \equiv 0$ and so $\mathbf{x} = \mathbf{x}(s)$ is a plane curve.

5.23. If $\dot{\kappa}$ does not change sign between two points on a plane curve, show that the difference of the radii of curvature at the points is equal to the arc length between the corresponding points on the locus of centers of curvature of the curve.

The locus of centers of curvature is $\mathbf{y} = x + (1/\kappa)\mathbf{n}$. Differentiating,

$$\frac{d\mathbf{y}}{ds} = \dot{\mathbf{x}} - \frac{\dot{\kappa}}{\kappa^2}\mathbf{n} + \frac{1}{\kappa}\dot{\mathbf{n}} = \mathbf{t} - \frac{\dot{\kappa}}{\kappa^2}\mathbf{n} + \frac{1}{\kappa}(-\kappa\mathbf{t}) = \frac{-\dot{\kappa}}{\kappa^2}\mathbf{n}$$

where we used $\tau \equiv 0$ since the curve is plane. Now suppose $\dot{\kappa} \geqq 0$ between the points $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$, $(s_1 < s_2)$. Then for $s_1 \leqq s \leqq s_2$,

$$\left|\frac{d\mathbf{y}}{ds}\right| = \left|\frac{-\dot{\kappa}}{\kappa^2}\right| = \frac{\dot{\kappa}}{\kappa^2}$$

and the length of the corresponding arc of the locus of centers of curvature is

$$\int_{s_1}^{s_2} \left|\frac{d\mathbf{y}}{ds}\right| ds = \int_{s_1}^{s_2} \frac{\dot{\kappa}}{\kappa^2}\, ds = \int_{s_1}^{s_2} \frac{d}{ds}\left(\frac{1}{\kappa}\right) ds = \frac{1}{\kappa(s_2)} - \frac{1}{\kappa(s_1)} = \rho(s_2) - \rho(s_1)$$

which is the required result.

5.24. **Show that tangent lines to the locus of the centers of spherical curvature of a curve are parallel to the binormal lines of the curve at corresponding points.**

The locus of centers of spherical curvature is

$$\mathbf{y} \quad = \quad \mathbf{x} + (1/\kappa)\mathbf{n} - \frac{\dot\kappa}{\kappa^2\tau}\mathbf{b}$$

A tangent vector to the locus is

$$\frac{d\mathbf{y}}{ds} \quad = \quad \dot{\mathbf{x}} - \frac{\dot\kappa}{\kappa^2}\mathbf{n} + (1/\kappa)\dot{\mathbf{n}} - \frac{d}{ds}\left(\frac{\dot\kappa}{\kappa^2\tau}\right)\mathbf{b} - \frac{\dot\kappa}{\kappa^2\tau}\dot{\mathbf{b}}$$

$$= \quad \mathbf{t} - \frac{\dot\kappa}{\kappa^2}\mathbf{n} + (1/\kappa)(-\kappa\mathbf{t} + \tau\mathbf{b}) - \frac{d}{ds}\left(\frac{\dot\kappa}{\kappa^2\tau}\right)\mathbf{b} - \frac{\dot\kappa}{\kappa^2\tau}(-\tau\mathbf{n}) \quad = \quad \left[\frac{\tau}{\kappa} - \frac{d}{ds}\left(\frac{\dot\kappa}{\kappa^2\tau}\right)\right]\mathbf{b}$$

from which the required result follows.

Supplementary Problems

5.25. Find the intrinsic equations of the curve $\mathbf{x} = e^t(a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + b\mathbf{e}_3)$.

Ans. $\kappa = \dfrac{a\sqrt{2}}{s(2a^2+b^2)^{1/2}}, \quad \tau = \dfrac{b}{s(2a^2+b^2)^{1/2}}$

5.26. Find the intrinsic equations of the hypocycloid

$$\mathbf{x} \quad = \quad \left[(r_0 - r_1)\cos\theta - r_1\cos\left(\frac{r_0 - r_1}{r_1}\theta\right)\right]\mathbf{e}_1 + \left[(r_0 - r_1)\sin\theta - r_1\sin\left(\frac{r_0 - r_1}{r_1}\theta\right)\right]\mathbf{e}_2$$

Ans. $\dfrac{s^2}{A^2} + \dfrac{1}{\kappa^2 B^2} = 1, \quad \tau = 0, \quad A < B$

5.27. If C is a plane curve, show that there is always a curve C^* such that C and C^* are Bertrand curves.

5.28. Determine the curve whose intrinsic equations are

$$\kappa \;=\; \frac{1}{as+b}, \qquad \tau \;=\; 0, \qquad s > 0,\; a > 0$$

Ans. Logarithmic spiral $r = ce^{\gamma\theta}$.

5.29. Find the equation of the tangent surface to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$.
Ans. $\mathbf{x} = (t+k)\mathbf{e}_1 + (t^2 + 2kt)\mathbf{e}_2 + (t^3 + 3kt^3)\mathbf{e}_3, \quad -\infty < k < \infty$

5.30. Show that all involutes of a circle are congruent.

5.31. If two curves have the same binormal lines at corresponding points, show that the curves are plane curves.

5.32. Show that a curve $\mathbf{x} = \mathbf{x}(s)$ of class $\geqq 4$ satisfies the differential equation

$$\mathbf{x}^{(4)} - \left(\frac{2\dot\kappa}{\kappa} + \frac{\dot\tau}{\tau}\right)\dddot{\mathbf{x}} + \left(\kappa^2 + \tau^2 + \frac{\dot\kappa\dot\tau}{\kappa\tau} + \frac{2\dot\kappa^2 - \kappa\ddot\kappa}{\kappa^2}\right)\ddot{\mathbf{x}} + \kappa^2\left(\frac{\dot\kappa}{\kappa} - \frac{\dot\tau}{\tau}\right)\dot{\mathbf{x}} \quad = \quad 0$$

(*Hint.* Compute $\dot{\mathbf{x}} = \mathbf{t}$, $\ddot{\mathbf{x}} = \dot{\mathbf{t}} = \kappa\mathbf{n}$, $\dddot{\mathbf{x}} = \dot\kappa\mathbf{n} + \kappa\dot{\mathbf{n}} = \dot\kappa\mathbf{n} - \kappa^2\mathbf{t} + \kappa\tau\mathbf{b}$ and $\mathbf{x}^{(4)} = \cdots$ as linear combinations of $\mathbf{n}, \mathbf{t}, \mathbf{b}$ and substitute.)

5.33. If a curve lies on the surface of a sphere, show that $\dfrac{d}{ds}\left(\dfrac{\dot{\kappa}}{\kappa^2\tau}\right) + \dfrac{\tau}{\kappa} = 0.$

5.34. (*a*) Show that the projection of a helix on a cone of revolution onto a plane perpendicular to the axis of the cone is a logarithmic spiral.

(*b*) Show that the intrinsic equations of a helix on a cone of revolution are

$$\kappa = 1/as, \qquad \tau = 1/bs, \qquad a, b = \text{constant}$$

5.35. Show that the torsion of the involute $\mathbf{x}^* = \mathbf{x} + (c-s)\mathbf{t}$ to the curve $\mathbf{x} = \mathbf{x}(s)$ is

$$\tau^* = \frac{(\dot{\kappa}\tau - \kappa\dot{\tau})}{|(c-s)\kappa|\,(\kappa^2+\tau^2)}$$

5.36. Show that the evolutes of a plane curve are helixes.

5.37. Show that $\cot\left[\int \tau\,ds + c\right]$ is the ratio of the torsion of an evolute to its curvature. (*Hint*: Use the relation in Problem 5.35 between a curve C and an involute C^*, or, equivalently, between a curve C^* and its evolute C.)

5.38. Show that the locus of the centers of curvature of a circular helix is a coaxial helix of the same pitch and that the locus of centers of curvature of the locus of centers of curvature is the original helix.

5.39. Prove that the product of the torsion of a circular helix and the torsion of the locus of centers of the helix at corresponding points is equal to κ^2.

5.40. Integrate the Frenet equations for the case of constant curvature and torsion.

5.41. If a helix lies on a sphere, show that its projection on a plane perpendicular to its axis is an arc of an epicycloid. (*Hint*: Show that the intrinsic equation of the projection is $\dfrac{s^2}{A^2} + \dfrac{1}{\kappa^2 B^2} = 1$, $A > B$. See Problem 5.2.)

5.42. Show that the projection of a helix on a paraboloid of revolution onto a plane perpendicular to its axis is an involute of a circle.

5.43. Prove that the product of the torsions of two Bertrand curves is constant.

5.44. If a curve C is defined by

$$\mathbf{x} = a\int \mathbf{g}(t)\,dt + b\int \mathbf{g}(t)\times\mathbf{g}'(t)\,dt$$

where $|\mathbf{g}(t)| = 1$ and $|\mathbf{g}'(t)| = 1$, show that C is a Bertrand curve.

Chapter 6

Elementary Topology in Euclidean Spaces

INTRODUCTION

The concept of a surface in the theory of differential geometry is much more involved than that of a curve. For example, the most general curve can be given in its entirety by a single regular parametric representation, but a surface as simple as a sphere requires at least two different regular parametric representations to describe it completely. A curve might or might not include its endpoints or boundary points; whereas, for example, the upper hemisphere with its boundary, the great circle, will not be considered an allowable surface, but the upper hemisphere without its boundary is allowable. In order to define a surface, some elementary concepts of topology are required.

It is convenient to have a single symbol denote either the Euclidean line, plane or three dimensional space. Thus by "Euclidean space E" we mean E^1, E^2 or E^3.

OPEN SETS

Observe in Fig. 6-1 that every point in the interior of a circle in the plane can be enclosed in a spherical neighborhood which is also contained in the interior. A set with this property is said to be *open*; that is, a set S in Euclidean space E is open if, for every P in S, there exists a spherical neighborhood $S(P)$ of P which is completely contained in S.

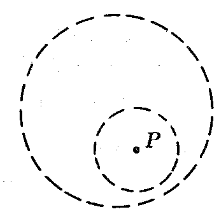


Fig. 6-1

Example 6.1:

(*a*) An open interval $a < x < b$ is an open set in E^1. The interval $a < x \leq b$ is not open; for every $S(b)$ will contain points $x' > b$, and hence not in $a < x \leq b$.

(*b*) The half plane $x_1 > 0$ in E^2 is open. But the set consisting of the half plane $x_1 > 0$ and the x_1 axis shown in Fig. 6-2 is not open. For if P is a point on the x_1 axis to the left of the origin, every $S(P)$ will contain points left of the origin which are not on the x_1 axis and hence not in the set.

(*c*) A spherical neighborhood of a point is itself open. In E^1, it is an open finite interval; in E^2, it is the interior of a circle, called an *open disk*; and in E^3, it is the interior of a sphere, called an *open sphere*.

(*d*) Euclidean space E itself is open. Also the null set $\emptyset$ is open; for otherwise there would be a point P in $\emptyset$ such that every $S(P)$ contains points not in $\emptyset$. But there is no P in $\emptyset$.

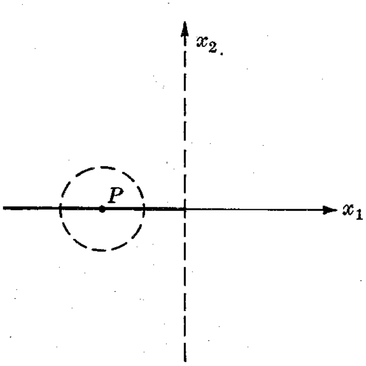


Fig. 6-2

Note. The open disk is an open set in E^2, but it is not open when considered as a subset of a plane in E^3; for every neighborhood of a point on a plane in E^3 contains points off the plane. Thus openness is a *relative* property of a set, depending upon the space in which the set is considered to lie.

If $\{O_\alpha\}$ is any family of open sets, finite or infinite, then the union $\bigcup_\alpha O_\alpha$ is open. For let P be in $\bigcup_\alpha O_\alpha$, then P is in some O_{α_0}. Since O_{α_0} is open, there is an $S(P)$ in O_{α_0}. But then $S(P)$ is in $\bigcup_\alpha O_\alpha$. Thus for any P in $\bigcup_\alpha O_\alpha$ there exists an $S(P)$ in $\bigcup_\alpha O_\alpha$. Hence $\bigcup_\alpha O_\alpha$ is open.

If $\{O_i\}$, $i = 1, \ldots, n$, is a *finite* family of open sets, then also the intersection $\bigcap_i O_i$ is open. For let P be in $\bigcap_i O_i$. Then P is in each O_i. Since the O_i are open, there exist $S_{\epsilon_i}(P)$ in each O_i. Now let $\epsilon = \min_i(\epsilon_i)$. Then $S_\epsilon(P)$ is contained in O_i for all i. Thus $S_\epsilon(P)$ is in $\bigcap_i O_i$ and $\bigcap_i O_i$ is open.

Note that the intersection of an infinite number of open sets need not be open. For example, as shown in Fig. 6-3, the intersection of the infinite family of concentric open disks of radius $1 + 1/n$, $n = 1, 2, \ldots$, about a point P in E^2 is the open disk of radius 1 about P together with its boundary, the circle of radius 1. This is not an open set.

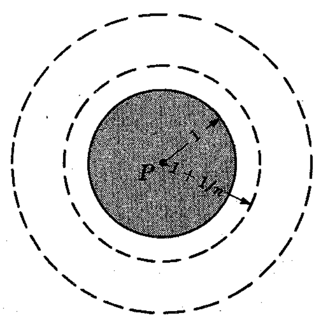


Fig. 6-3

As a consequence of the above we have

Theorem 6.1. Open sets in E have the following properties:

(a) E is open; $\emptyset$ is open.

(b) If O_α are open, then $\bigcup_\alpha O_\alpha$ is open.

(c) If O_i, $i = 1, \ldots, n$, are open, then $\bigcap_i O_i$ is open.

Also, let P and Q be distinct points in E. Clearly by taking ϵ_1 and ϵ_2 sufficiently small, say $\epsilon_1 = \epsilon_2 = \frac{1}{2}|\mathbf{PQ}|$, the neighborhoods $S_{\epsilon_1}(P)$ and $S_{\epsilon_2}(Q)$ are disjoint. Since neighborhoods are open sets, we have

Theorem 6.2. If P and Q are distinct points in E, there exist open sets O_P and O_Q containing P and Q respectively such that $O_P \cap O_Q = \emptyset$.

CLOSED SETS. LIMIT POINTS

A set S in E is *closed* if the set of points not in S is open, i.e. S is closed if its complement S^C is open.

Example 6.2:

(a) The closed interval $a \leq x \leq b$ in E^1 is closed, for its complement is the union of the open sets $x > b$ and $x < a$. The interval $a < x \leq b$ is neither open nor closed.

(b) The set of rational points in the x_1x_2 plane, i.e. the set S of points (p, q) where p and q are rational numbers, is neither open nor closed. Since every neighborhood of a rational number contains an irrational number, every $S(p, q)$ contains points not in S. Hence S is not open. Also, since every neighborhood of an irrational number contains rational numbers, the complement of S is not open. Hence S is not closed.

(c) A set in E consisting of a single point is closed. Also, a set in E consisting of any finite number of points is closed.

(d) E is closed since $\emptyset$ is open. $\emptyset$ is closed since E is open.

(e) The following are examples of closed sets:
   (i) The open disk in E^2 together with its boundary, called the *closed disk*.
   (ii) The open sphere in E^3 together with its boundary, called the *closed sphere*.
   (iii) The sphere in E^3 itself, since its complement is the union of its open interior and exterior.

(iv) A *torus* in E^3 is the surface in E^3, shown in Fig. 6-4, which is obtained by revolving a circle about a line not passing through the circle. A torus in E^3 is closed.

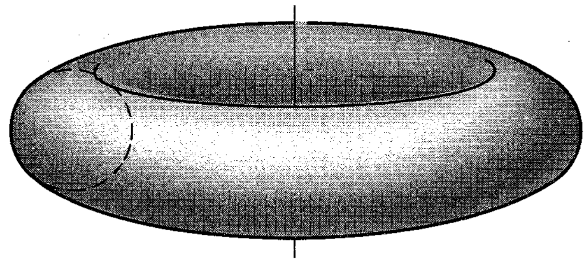


Fig. 6-4

A point P is said to be an *accumulation point* or *limit point* of a set S in E if every deleted spherical neighborhood $S'(P)$ of P contains at least one point of S. We recall that the deleted spherical neighborhood $S'(P)$ of P consists of $S(P)$ without the point P.

Example 6.3:

(*a*) Let S be an open disk in E^2, shown in Fig. 6-5. Clearly every point in S is a limit point of S, for every deleted neighborhood of such a point contains points of S. The points on the circle itself, although not in S, are also limit points of S, since every deleted neighborhood of a point on the circle will have a nonempty intersection with S.

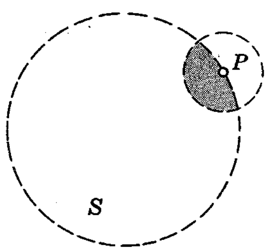


Fig. 6-5

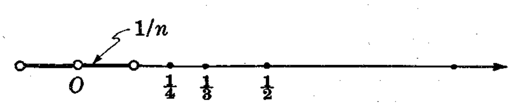


Fig. 6-6

(*b*) Let S be the infinite set of points $1, 1/2, 1/3, \ldots, 1/n, \ldots$ on the x axis shown in Fig. 6-6. Here O is a limit point of S since every deleted neighborhood of O contains at least one point of S. It can be shown that O is the only limit point of the set.

Now suppose a set S in E has the property that every limit point of S is in S. Let P be an arbitrary point not in S, i.e. let P be a point in S^C. Clearly P is not a limit point of S, since S contains its limit points. Thus there exists some full $S(P)$ which contains no point of S. It follows that S^C is open since, for an arbitrary P in S^C, we found an $S(P)$ in S^C. But if S^C is open, then S is closed. Thus if a set contains its limit points, it is closed.

The converse, which is also true, is proved in Problem 6.5. Thus we have

Theorem 6.3. A set in E is closed if and only if it contains its limit points.

The *closure* of a set S, denoted by $\hat{S}$, is the set consisting of S and the set of limit points of S. As a solved problem we will show that $\hat{S}$ is the smallest closed set containing S; that is: (*a*) $\hat{S}$ is closed; (*b*) if T is closed and $S \subseteq T$, then $\hat{S} \subseteq T$.

Example 6.4:

(*a*) We refer to the set in Example 6.1(*b*) consisting of the half plane $x_1 > 0$ and the x_1 axis. This set is neither open nor closed. The points on the x_2 axis are limit points of the set which do not belong to the set. If we include these limit points, that is, if we consider the set consisting of the half plane $x_1 \geqq 0$ and the x_1 axis, we have a closed set which is the closure of the given set.

(*b*) Since every deleted neighborhood of every point in the x_1x_2 plane contains a rational point, every point in E^2 is a limit point of the set of rational points in E^2. Thus the closure of the set of rational points in E^2 is E^2.

Finally, a set in E is *bounded* if it is contained in some neighborhood of a point. Thus in E^1, S is bounded if and only if it lies in a finite open interval; in E^2, S is bounded if and only if it lies in an open disk; and in E^3, S is bounded if and only if it is contained in an open sphere.

Example 6.5:

(*a*) The set of points $1, 1/2, 1/3, \ldots$ in E^1 is bounded. It lies in the interval $0 < x < 2$.

(*b*) The rational points (p, q) in the x_1x_2 plane are not bounded.

(*c*) A finite set of points in E is clearly bounded.

CONNECTED SETS

As shown in Fig. 6-7, let a set S consist of two disjoint closed disks in E^2. Since there is a nonzero distance between the disks, there exist open sets O_1 and O_2 whose union contains S such that their respective intersections with S are nonempty and disjoint. In general, a set S in E is said to be *disconnected* if, as above, there exist open sets O_1 and O_2 such that (*a*) $S \subseteq O_1 \cup O_2$ (O_1 and O_2 *cover* S), (*b*) $O_1 \cap S \neq \emptyset$, $O_2 \cap S \neq \emptyset$ (O_1 and O_2 each have nonempty intersections with S), and (*c*) $(O_1 \cap S) \cap (O_2 \cap S) = O_1 \cap O_2 \cap S = \emptyset$. (The intersections of O_1 and O_2 with S are disjoint.) A nonempty set S is said to be *connected* if it is not disconnected.

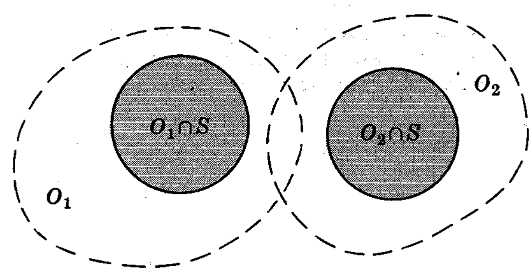


Fig. 6-7

Example 6.6:

(*a*) It follows from (*b*) and (*c*) above that a disconnected set must have at least two points. Thus a set consisting of a single point is connected. On the other hand, a finite set in E consisting of two or more points is disconnected.

(*b*) In Problem 6.14, page 115, we prove that the only connected sets in E^1 are the intervals. (We include a single point, $a \leq x \leq a$.)

(*c*) The following are examples of connected sets in E^3: (i) a line segment; (ii) an open sphere; (iii) a torus; (iv) a torus together with its interior, called the *solid* or *closed torus*.

An open and connected set in E is called a *domain*.

Example 6.7:

(*a*) The following are examples of domains:
  (i) The open set between two concentric spheres in E^3.
  (ii) The half plane $x_1 > a$ in E^2.
  (iii) The interior of a torus, or *open torus*.

(*b*) The following are not domains:
  (i) The solid torus in E^3 (connected but not open).
  (ii) Two disjoint open disks in E^2 (open but not connected).

A nonempty set S in E is said to be *arcwise connected* if any two points in S can be connected by a continuous arc completely contained in S. To be precise, S is arcwise connected if, given any two points in S, say, $\mathbf{x}_1$ and $\mathbf{x}_2$, there exists a continuous function $\mathbf{x}(t)$, defined in $0 \leq t \leq 1$, such that (*a*) $\mathbf{x}(t)$ is in S for all t and (*b*) $\mathbf{x}(0) = \mathbf{x}_1$, $\mathbf{x}(1) = \mathbf{x}_2$.

Example 6.8:

(*a*) A set consisting of a single point $\mathbf{x}_1$ is arcwise connected. $\mathbf{x}(t) = \text{constant} = \mathbf{x}_1$ will do.

(*b*) Clearly E itself is arcwise connected, for the linear function $\mathbf{x}(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)$, $0 \leq t \leq 1$, is a straight line connecting any $\mathbf{x}_1$ and $\mathbf{x}_2$.

(*c*) It can be shown that in E^1 a set is arcwise connected if and only if it is an interval. Thus in E^1 the connected and arcwise connected sets are the same.

If a set S in E is arcwise connected, then it is connected. For suppose S is arcwise connected but disconnected. Then there exist open sets O_1 and O_2 containing S and having nonempty and disjoint intersections with S, as shown in Fig. 6-8.

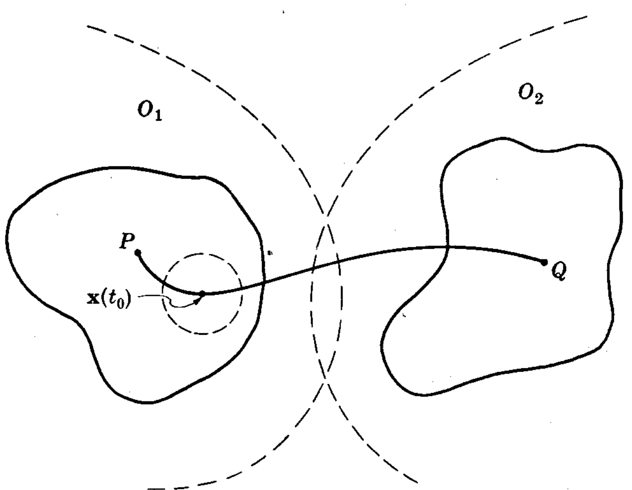


Fig. 6-8

Let P belong to $S \cap O_1$ and let Q belong to $S \cap O_2$ and let $\mathbf{x} = \mathbf{x}(t)$, $0 \leq t \leq 1$, be a continuous arc from P to Q in S. Now consider the real-valued function $f(t)$ on the interval $0 \leq t \leq 1$ defined by

$$f(t) = \begin{cases} 1, & \text{if } \mathbf{x}(t) \text{ is in } S \cap O_1 \\ -1, & \text{if } \mathbf{x}(t) \text{ is in } S \cap O_2 \end{cases}$$

Since $\mathbf{x}(t)$ is in S and $S \subseteq O_1 \cup O_2$, it follows that f is defined for all t in $0 \leq t \leq 1$. It is also single-valued, since $S \cap O_1$ and $S \cap O_2$ are disjoint. Now we wish to show that $f(t)$ is continuous for all t. We consider $t = t_0$ and suppose $\mathbf{x}(t_0)$ is in, say, $S \cap O_1$ as shown in the figure. Then $\mathbf{x}(t_0)$ is in O_1, and, since O_1 is open, there is an $S_\epsilon(\mathbf{x}(t_0))$ also in O_1. Since $\mathbf{x}(t)$ is continuous at t_0, there is an $S_\delta(t_0)$, such that $\mathbf{x}(t)$ is in $S_\epsilon(\mathbf{x}(t_0))$, and hence $S \cap O_1$ for t in $S_\delta(t_0)$. But then $f(t) \equiv 1$ for t in $S_\delta(t_0)$. It follows that $f(t)$ is continuous at t_0. The argument is similar if $\mathbf{x}(t_0)$ is in $S \cap O_2$. Thus $f(t)$ is continuous for all $0 \leq t \leq 1$. But this is impossible, for a continuous function which is 1 at $t = 0$ and -1 at $t = 1$ must take on all values between -1 and 1 by a theorem of elementary calculus, which is not the case with $f(t)$. Thus we have

Theorem 6.4. If a set S in E is arcwise connected, then it is connected.

Although the converse of the above is true in E^1, it is not true for E in general, as shown in Problem 6.20. That is, there are connected sets in E^2 which are not arcwise connected. However, if S is connected and open in E, then it is arcwise connected in E. Namely, in Problem 6.13 we prove

Theorem 6.5. A domain is arcwise connected.

COMPACT SETS

An *open covering* of a set S in E is a family of open sets whose union contains S. A *subcovering* is a subset of the open covering with the same property, and a *finite covering* is an open covering consisting of a finite number of sets. Clearly, for every set in E there exists an open covering, namely the family consisting of only the set E itself.

Now a set S in E is *compact* if for *every* open covering $\{O_\alpha\}$ of S there exists a finite subcovering $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$.

Example 6.9:

Every finite set of points is compact. For, let $S = \{P_1, \ldots, P_n\}$ be a finite set and let $\{O_\alpha\}$ be an arbitrary open covering of S; i.e. the O_α are open and $S \subseteq \bigcup_\alpha O_\alpha$. Now for each P_i in S select an O_{α_i} of the covering which contains P_i. Clearly then $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$ is a finite subcovering of S_1. Since $\{O_\alpha\}$ was arbitrary, S is compact.

Example 6.10:

Let S be the infinite set $\{1, 1/2, 1/3, \ldots\}$ in E^1 shown in Fig. 6-9. This set is not compact, for we can exhibit an open covering of S which has no finite subcovering. Namely, let $O_1 = \{\frac{1}{2} < x < 2\}$ and let O_n denote the open interval $\dfrac{1}{n+1} < x < \dfrac{1}{n-1}$ for $n \geqq 2$. Clearly O_n contains $1/n$ and so the infinite family $\{O_n\}$, $n = 1, \ldots$, is an open covering of S. But note that each O_n contains *only* the point $1/n$ of S and so no *finite* subcovering could contain all of S.

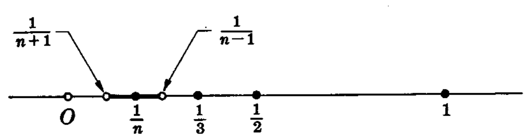


Fig. 6-9

Now suppose S is a compact set. For each P in S select an arbitrary $S(P)$. The collection of these neighborhoods is an open covering of S, and, since S is compact, there exists a finite subcovering $S(P_1), S(P_2), \ldots, S(P_n)$. But the union of such a finite set of neighborhoods is clearly a bounded set. Since S is contained in the union, S is bounded. Thus a compact set is bounded.

A compact set S is also closed. For, let Q be an arbitrary point in its complement S^C. For each P in S there exists an $S(P)$ and an $S^P(Q)$ such that $S(P) \cap S^P(Q) = \emptyset$. Clearly the family $\{S(P)\}$ is an open covering of S, and, since S is compact, there exists a finite subcovering $\{S(P_1), S(P_2), \ldots, S(P_n)\}$. Now let $O = \bigcap_i S^{P_i}(Q)$ be the intersection of the corresponding neighborhoods of Q. Note that O contains Q and is open. But also

$$O \cap (\bigcup_j S(P_j)) \;=\; \bigcup_j (O \cap S(P_j)) \;=\; \bigcup_j (\bigcap_i S^{P_i}(Q) \cap S(P_j))$$

But $S^{P_j}(Q) \cap S(P_j) = \emptyset$. Hence $O \cap (\bigcup_j S(P_j)) = \emptyset$. Since the $S(P_j)$ cover S, it follows that $O \cap S = \emptyset$. Thus $O \subseteq S^C$. But O is open and contains Q. Hence there exists $S(Q) \subseteq S^C$. Thus for an arbitrary Q in S^C there exists a neighborhood of Q in S^C. It follows that S^C is open, and hence S is closed.

Thus a compact set is closed and bounded. The converse of this is also true in E and is known as the Heine-Borel theorem. The reader is referred to a text in advanced calculus for a proof. Hence we have

Theorem 6.6. A set in E is compact if and only if it is closed and bounded.

CONTINUOUS MAPPINGS

Let E and F be Euclidean spaces and S a subset of E. Let f be a mapping of S into F; i.e. to each P in S there is assigned a point $f(P)$ in F. The mapping f is *continuous* at a point P_0 in S if, as indicated in Fig. 6-10 below, for *every* neighborhood $S(f(P_0))$ in F, there exists a neighborhood $S(P_0)$ in E such that $f(P)$ is in $S(f(P_0))$ for all P in $S(P_0) \cap S$. Or, equivalently, f is continuous at P_0 if for every $S(f(P_0))$ there exists an $S(P_0)$ such that

$f(S(P_0) \cap S) \subseteq S(f(P_0))$. The mapping f is said to be continuous on S, or simply continuous, if it is continuous at each point in S.

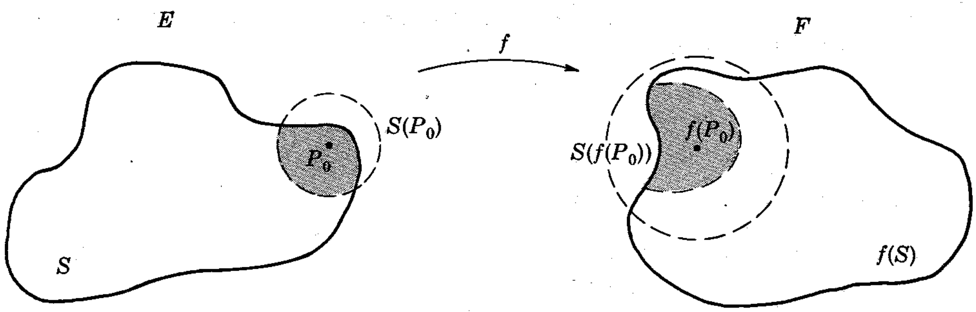


Fig. 6-10

Example 6.11:

(a) The constant mapping $f(P) = Q_0$, which assigns to each P of a set S in E a fixed point Q_0 in F, is continuous on S. For, let P_0 be an arbitrary point in S and let $S(f(P_0)) = S(Q_0)$ be an arbitrary neighborhood of $f(P_0)$. But for all P in S, and hence for all P in any $S(P_0)$ of P_0, we have $f(P) = Q_0$ in $S(Q_0)$. Thus f is continuous at P_0. Since P_0 is an arbitrary point in S, f is continuous on S.

(b) Let S be a sphere in E^3 and L a tangent plane as shown in Fig. 6-11. Suppose for each P on S, $f(P)$ is the orthogonal projection of P onto L. This mapping is continuous for all P_0 in S. For, given $S_\epsilon(f(P_0))$, select $\delta = \epsilon$. Then for P in $S_\delta(P_0) \cap S$, the projection $f(P)$ is in $S_\epsilon(f(P_0))$.

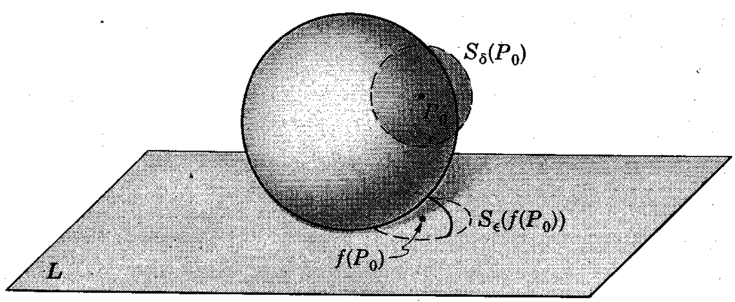


Fig. 6-11

(c) The polar equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

define a mapping f of the half strip $r \geq 0$, $0 \leq \theta \leq \pi/2$, in the $r\theta$ plane into the xy plane. This mapping is continuous at each (r_0, θ_0). Consider first $r_0 \neq 0$ and let $S_\epsilon(f(r_0, \theta_0)) = S_\epsilon(x_0, y_0)$ be an arbitrary neighborhood of $f(r_0, \theta_0)$. As shown in Fig. 6-12, choose $\Delta r > 0$ and $\Delta\theta > 0$ sufficiently small so that the points (x, y) with polar coordinates (r, θ), where $0 < r_0 - \Delta r < r < r_0 + \Delta r$ and $\theta_0 - \Delta\theta < \theta < \theta_0 + \Delta\theta$, are in $S_\epsilon(x_0, y_0)$. Then for all (r, θ) in the open rectangular region R given by

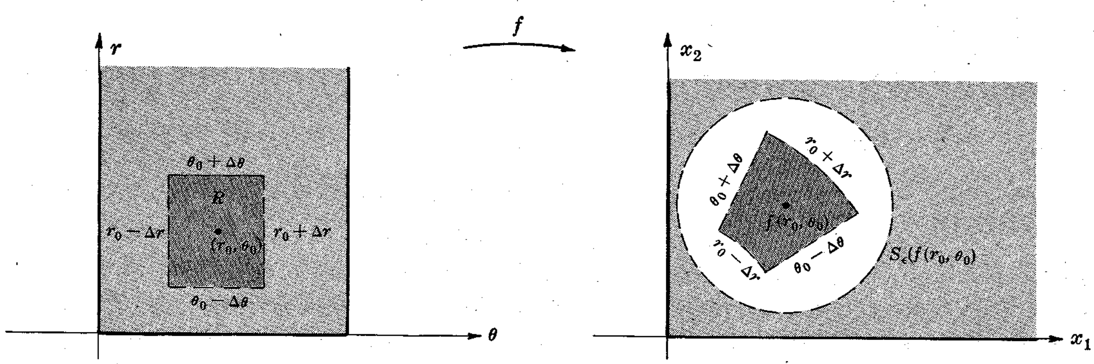


Fig. 6-12

$r_0 - \Delta r < r < r_0 + \Delta r$, $\theta_0 - \Delta\theta < \theta < \theta_0 + \Delta\theta$, we have $f(r,\theta)$ in $S_\epsilon(x_0, y_0)$. But now, select any $S_\delta(r_0, \theta_0)$ in R. Then for (r,θ) in $S_\delta(r_0,\theta_0)$, we have $f(r,\theta)$ in $S_\epsilon(x_0,y_0)$. Since $S_\epsilon(x_0,y_0)$ is arbitrary, f is continuous at (r_0,θ_0). For the case $r_0 = 0$ (we note that $f(0,\theta_0) = (0,0)$) choose $\delta = \epsilon$. Then for (r,θ) in $S_\delta(0,\theta_0)$, we have $0 \leq r < \delta$, or $0 \leq r < \epsilon$; hence $f(r,\theta) = (x,y)$ is in $S_\epsilon(0,0)$. Thus f is continuous on $0 \leq r$, $0 \leq \theta \leq \pi/2$.

(*d*) The equations

$$x = u, \quad y = v, \quad z = \begin{cases} u^2 + v^2, & \text{for } u \geq 0 \\ 1, & \text{for } u < 0 \end{cases}$$

define a mapping of the uv plane into E^3. As shown in Fig. 6-13 the set of assigned points is a surface consisting of a half plane ($u < 0$) and a half paraboloid ($u \geq 0$). The mapping is continuous for $u < 0$, $u > 0$, and the two points $(0,1)$ and $(0,-1)$ where the plane and paraboloid intersect. The mapping is not continuous for the remaining points on the v axis.

For example consider the point $(0,\frac{1}{2})$. Here $f(0,\frac{1}{2}) = (0,\frac{1}{2},\frac{1}{4})$ is a point on the boundary of the paraboloid. Now let ϵ be so small that $S_\epsilon(f(0,\frac{1}{2}))$ does not intersect the plane; $\epsilon \leq \frac{3}{4}$ will do. Now if f were continuous at $(0,\frac{1}{2})$, there would exist $S_\delta(0,\frac{1}{2})$ such that for every (u,v) in $S_\delta(0,\frac{1}{2})$, $f(u,v)$ would lie in $S_\epsilon(f(0,\frac{1}{2}))$. But in every $S_\delta(0,\frac{1}{2})$ there exist (u,v) where $u < 0$. For these (u,v), $f(u,v)$ is on the plane and hence not in $S_\epsilon(f(0,\frac{1}{2}))$. Thus f is not continuous at $(0,\frac{1}{2})$.

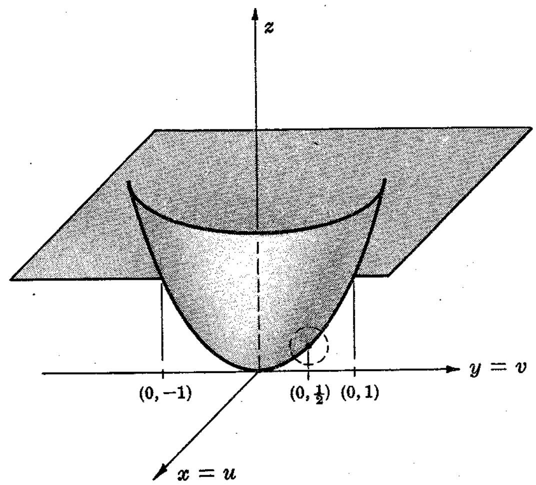


Fig. 6-13

Let E and F be Euclidean spaces. Let S be a connected set in E and f a continuous mapping of S into F. We will show that the image $f(S)$ is a connected set in F. For suppose otherwise. Then there are open sets O_1 and O_2 such that $f(S) \subseteq O_1 \cup O_2$, $f(S) \cap O_1 \neq \emptyset$, $f(S) \cap O_2 \neq \emptyset$, and $f(S) \cap O_1 \cap O_2 = \emptyset$.

Let A_1 denote the set $f(S) \cap O_1$ and A_2 the set $f(S) \cap O_2$, and let P be a point in S such that $f(P)$ is in A_1 and Q a point in S such that $f(Q)$ is in A_2. Such points exist since A_1 and A_2 are not empty. Since A_1 is in O_1, and O_1 is open, there exists $S(f(P))$ in O_1. Since f is continuous, there exists $S(P)$ in E such that $f(S(P))$ is in $S(f(P))$ and hence in O_1. Now let $W_1 = \bigcup_P S(P)$ for all such P in S such that $f(P)$ is in A_1. Similarly, let $W_2 = \bigcup S(Q)$ where $f(Q)$ belongs to A_2. Here $f(S(Q))$ is in O_2 for all Q. Now clearly W_1 and W_2 are open in E, since they are a union of open sets (neighborhoods). Also $S \subseteq W_1 \cup W_2$ and $S \cap W_1 \neq \emptyset$ and $S \cap W_2 \neq \emptyset$. Finally, $S \cap W_1$ and $S \cap W_2$ are disjoint. For suppose that P^* belongs to $S \cap W_1 \cap W_2$. Since P^* belongs to both W_1 and W_2, there are points P in W_1 and Q in W_2 such that P^* belongs to both $S(P)$ and $S(Q)$. Thus $f(P^*)$ belongs to both O_1 and O_2. But also P^* belongs to S and so $f(P^*)$ belongs to $f(S)$, which is impossible since $O_1 \cap O_2 \cap f(S) = \emptyset$. Thus we have

Theorem 6.7. The image of a connected set under a continuous mapping is connected.

Also, in Problem 6.24, page 118, we prove

Theorem 6.8. The image of a compact set under a continuous mapping is compact.

We recall from calculus that a continuous real-valued function $f(t)$ defined on a closed interval I takes on its absolute maximum and absolute minimum. That is, there exist t_1 and t_2 in I such that $f(t) \leq f(t_1)$ for all t in I and $f(t) \geq f(t_2)$ for all t in I. The same is true for real-valued continuous functions defined on compact sets in general. For let f be a real-valued continuous function defined on a compact set C or, equivalently, let f be a continuous mapping of C into E^1. Then from Theorem 6.8 the image $f(C)$ in E^1 is also compact. Since $f(C)$ is compact, it is closed and bounded. Since it is bounded, it has a supremum M and infimum m. It remains to show that M and m belong to $f(C)$, in which case there exist P_1 and P_2 in C such that $f(P) \leqq M = f(P_1)$ and $f(P) \geqq m = f(P_2)$ for all P in C and hence f takes on its maximum and minimum in C. But if $M \notin f(C)$; then $M \in [f(C)]^C$. Since $f(C)$ is closed, $[f(C)]^C$ is open. But then there exists an open interval $M - \epsilon < M < M + \epsilon$ in $[f(C)]^C$. In particular, there exists an M_1 in $M - \epsilon < M < M + \epsilon$ such that $f(P) \leqq M_1 < M$ for all P in C. But this is impossible since M is the least of the upper bounds of $f(C)$. Similarly one proves that $m \in f(C)$. Thus

Theorem 6.9. A continuous real-valued function on a compact set takes on its absolute maximum and absolute minimum.

Finally, suppose E, F and G are Euclidean spaces, S a subset of E, T a subset of F, f a mapping of S into F such that $f(S) \subseteq T$, and g a mapping of T into G. Then for all P in S, there is a composite mapping $g \circ f$ of S into G defined by $(g \circ f)(P) = g(f(P))$ as shown in Fig. 6-14.

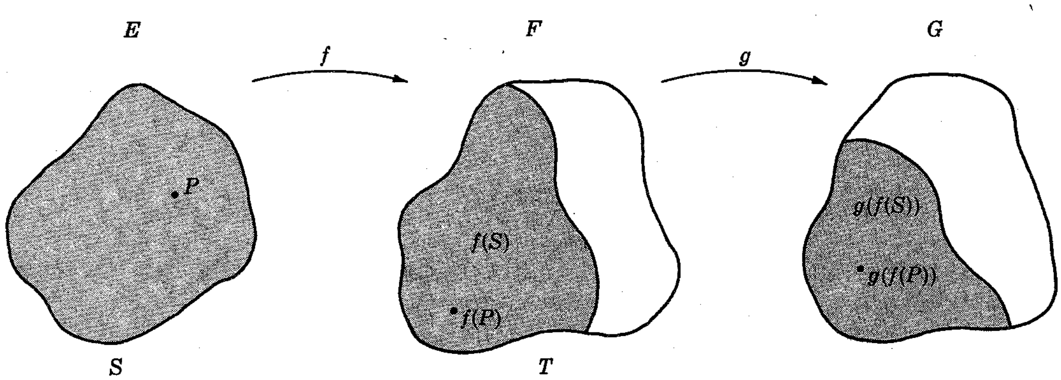


Fig. 6-14

As a solved problem we will show that if f and g are continuous, then $g \circ f$ is continuous. Namely

Theorem 6.10. A continuous mapping composed with a continuous mapping is a continuous mapping.

HOMEOMORPHISMS

Let f be a continuous mapping of a set S (in E) into F. If f is one-to-one, there exists an inverse mapping f^{-1} of the image $f(S)$ of f back onto S. If f^{-1} is continuous on $f(S)$, then f is called a *1-1 bicontinuous* mapping of S into F.

A 1-1 bicontinuous mapping of a set S (in E) *onto* a set T (in F) is called a *topological mapping* or *homeomorphism* of S onto T. Clearly a 1-1 bicontinuous mapping of S (in E) into F determines a homeomorphism of S onto the image $f(S)$. Also if f is a homeomorphism of S onto T, then f^{-1} is a homeomorphism of T onto S. Finally, a set S in E is said to be *topologically equivalent* or *homeomorphic* to a set T in F if there exists a homeomorphism of S onto T.

Intuitively one thinks of a homeomorphism as a mapping in which neighboring points remain neighboring points. Thus two figures are topologically equivalent if there exists an *elastic* motion which will make one figure coincide with the other.

Example 6.12:

(*a*) The equations

$$x_1 = \cos t, \quad x_2 = \sin t, \qquad 0 \leqq t < 2\pi$$

define a continuous and one-to-one mapping f of the interval $0 \leqq t < 2\pi$ onto the circle of radius one about the origin in the x_1x_2 plane. The mapping, however, is not bicontinuous, for the inverse mapping f^{-1} which maps the circle back onto the interval is not continuous at $(1,0)$. As shown in Fig. 6-15, every neighborhood of $(1,0)$ contains points (those below the x_1 axis) which are mapped by f^{-1} into points near 2π. Thus if we consider, say, the $\frac{1}{2}$ neighborhood of $f^{-1}(1,0) = 0$, no neighborhood of $(1,0)$ can be found whose images are all contained in $S_{1/2}(f^{-1}(1,0))$. Note that if the mapping f is restricted to the interval $0 \leqq t \leqq \pi$, as shown in Fig. 6-16, then f *is* a homeomorphism onto the semicircle. Thus the line segment and semicircle are topologically equivalent.

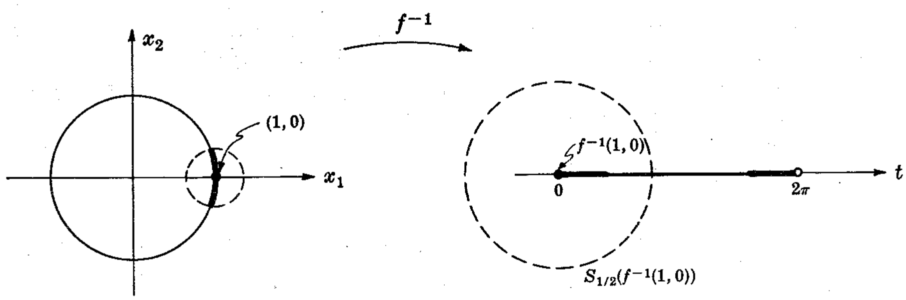


Fig. 6-15

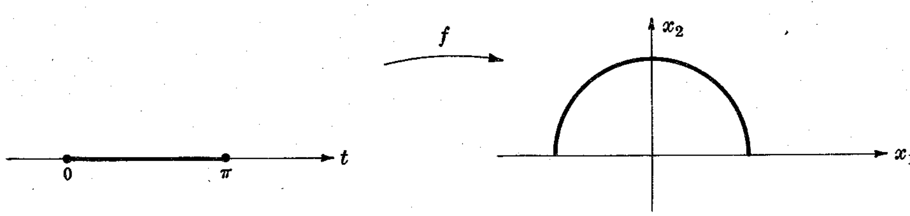


Fig. 6-16

(*b*) An important class of two-sided surfaces in E^3 is called the *spheres with n handles*. These surfaces are obtained by cutting $2n$ holes in a sphere and bending n different tubes so that their ends fit in these holes. A sphere with 3 handles is shown in Fig. 6-17. Observe that these surfaces are closed and bounded (compact) sets of points in E^3. It can be shown that every compact two-sided surface, as we shall soon define a surface, is topologically equivalent to a sphere with some number of handles. The torus, for example, is topologically equivalent to the sphere with one handle.

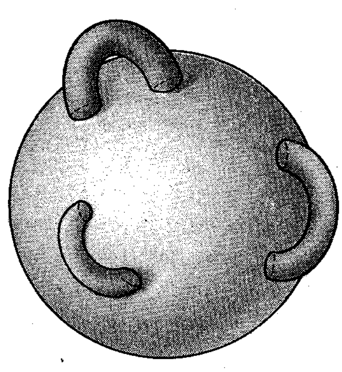


Fig. 6-17

Solved Problems

OPEN SETS. CLOSED SETS

6.1. Determine which of the following sets is (i) open, (ii) closed, (iii) bounded. Find (iv) its set of limit points and (v) its closure.

(*a*) The integers $\{0, \pm 1, \pm 2, \ldots\}$ along E^1.

(*b*) All (x, y) in the xy plane such that $xy \neq 0$.

(*c*) The set $\{1 + \frac{1}{2}, -(1 + \frac{1}{3}), (1 + \frac{1}{4}), -(1 + \frac{1}{5}), \ldots, (-1)^n(1 + \frac{1}{n}), \ldots\}$ in E^1.

(*d*) A torus in E^3.

(*a*) The set is closed, since its complement is a union of open sets $n - 1 < x < n$, $n = 0, \pm 1, \pm 2, \ldots$. The set is clearly not bounded. The set has no limit points; for given any real number x_0, one can find a deleted neighborhood of x_0 which will not contain an integer. Its closure is itself.

(*b*) This set is the xy plane less the x and y axes. The set is open, for any point (x_0, y_0) in the set will have a minimum nonzero distance $d = \min\{|x_0|, |y_0|\}$ to the x or y axis and therefore can be enclosed in a neighborhood which is entirely in the set. It is not bounded. Every point in the set is a limit point of the set. But also the points on the x and y axes are limit points. Thus the set of limit points is the whole xy plane and its closure is the whole xy plane.

(*c*) This set is neither open nor closed. As shown in Fig. 6-18, there is a point in the set, $(1 + \frac{1}{2})$, such that every neighborhood of it contains points not in the set. Hence the set is not open. On the other hand, every neighborhood of the number 1 in the complement of the set contains a number of the set (that is, a point not in the complement). Thus the complement is not open, so that the set is not closed. The set is bounded, since it is contained in the interval $-2 < x < 2$. Its limit points are the numbers 1 and -1. These numbers together with the set make up its closure.

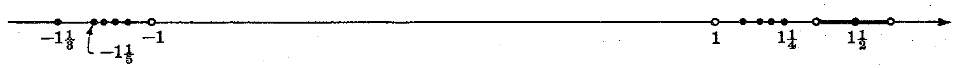


Fig. 6-18

(*d*) The surface of a torus is closed, for every point in its complement will have a nonzero distance to the torus and hence is contained in a neighborhood which is entirely contained in the complement. Thus the complement is open; so the surface is closed. The torus is bounded. Every point on the torus is a limit point, since every deleted neighborhood of a point on the torus has a nonempty intersection with the torus. There are no other limit points, since a closed set contains its limit points. The closure of the torus is the torus.

6.2. State the negation of "S is an open set in E".

S is not an open set in E if there exists a point P in S such that every neighborhood of P contains a point not in S.

6.3. State the negation of "P is a limit point of S".

P is not a limit point of S if there exists a deleted neighborhood of P which contains no point of S, or, equivalently, P is not a limit point of S if there exists a deleted neighborhood of P which is contained in the complement of S.

6.4. Prove that the union of a finite number of closed sets in E is closed.

One of DeMorgan's laws for sets states that the complement of a union of sets is the intersection of the complements, i.e.

$$(\bigcup_i S_i)^C = \bigcap_i (S_i^C)$$

Thus if S_i, $i = 1, \ldots, n$, is a finite number of closed sets, then each S_i^C is open. Since the intersection of a finite number of open sets is open, it follows from DeMorgan's theorem that $\bigcap_i S_i^C = (\bigcup_i S_i)^C$ is open. Hence $\bigcup_i S_i$ is closed, which is the required result. Note that the union of an infinite number of closed sets need not be closed. For example, the union of the closed intervals $-1 + 1/n \leqq x \leqq 1 - 1/n$, $n = 1, 2, \ldots$, in E^1 is the open interval $-1 < x < 1$.

6.5. Prove that a closed set contains its limit points.

Suppose otherwise. That is, suppose S is closed, P is a limit point of S, but P does not belong to S. Since S is closed, S^C is open. Since P belongs to S^C and S^C is open, there exists a neighborhood $S(P)$ of P contained in S^C. That is, there is an $S(P)$ which does not contain points of S. But this is impossible, since P is a limit point of S and every deleted neighborhood must contain a point of S, which proves the proposition.

6.6. If P is a limit point of S, show that every neighborhood of P contains an infinite number of points of S.

Suppose the contrary. That is, suppose P is a limit point of S and there exists a neighborhood of P containing only a finite number of points of S distinct from P, say $Q_1, Q_2, \ldots, Q_n$. Note that there is at least one such Q, since P is a limit point of S. Now let ϵ be the distance between P and the closest Q_i. That is, let $\epsilon = \min\{d(P,Q_1), d(P,Q_2), \ldots, d(P,Q_n)\}$. Then $S_\epsilon(P)$ contains no points of S distinct from P. But this is impossible, since P is a limit point of S. Thus the proposition is proven.

6.7. Prove that the closure of a set is a closed set.

By Theorem 6.3 it suffices to show that the closure $\hat{S}$ of a set S contains all of its limit points. For this purpose suppose that P is a limit point of $\hat{S}$. Then every neighborhood $S(P)$ of P contains at least one point Q in $\hat{S}$ with $Q \neq P$. Either Q itself belongs to S or else Q is a limit point of S and then every neighborhood $S(Q)$ of Q contains a point Q^* in S. Since we may choose $S(Q)$ so that $S(Q)$ lies in $S(P)$ and $P \notin S(Q)$, we can insure that $Q^* \in S(P)$ and $Q^* \neq P$. Thus in either case $S(P)$ contains a point of S distinct from P and P is a limit point of S as required.

6.8. A point P in S is called an *interior point* of S if there exists a spherical neighborhood of P completely contained in S. The set of interior points of S is called the *interior* of S. Note that every point of an open set is an interior point of the set. Thus an open set is equal to its interior. Prove in general that the interior of a set is an open set.

Let T denote the interior of S and let P be an arbitrary point in T. Since P is an interior point of S, there exists a neighborhood $S(P)$ contained in S as shown in Fig. 6-19. Let Q be a point in $S(P)$. Since $S(P)$ is open, there exists $S^*(Q)$ contained in $S(P)$ and hence in S. It follows that Q is also in T, the interior of S. Since Q is an arbitrary point in $S(P)$, all of $S(P)$ is in T. Since P is an arbitrary point in T, it follows that T is open.

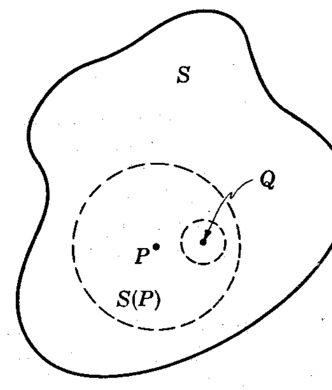


Fig. 6-19

CONNECTED SETS. COMPACT SETS

6.9. Determine which of the following sets are (*a*) connected, (*b*) compact.

(i) The infinite open wedge-shaped region between two intersecting lines in the plane shown in Fig. 6-20. (ii) The set of points (x_1, x_2, x_3) in E^3 such that $x_1 \neq 0$. (iii) The solid torus.

(i) Any two points in the wedge can be connected by a straight line segment. Hence the set is arcwise connected and therefore, by Theorem 6.4, connected. The wedge is not closed and therefore, from Theorem 6.6, not compact.

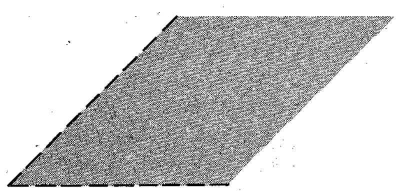


Fig. 6-20

(ii) This set consists of the two half spaces $x_1 > 0$ and $x_1 < 0$. They themselves are nonempty, disjoint, open sets; hence the set is not connected. The set is not closed, so it is not compact.

(iii) The solid torus is arcwise connected and therefore connected. It is closed and bounded and hence compact.

6.10. Show that the set of points (x_1, x_2) in E^2 satisfying

$$x_2 = \begin{cases} \sin 1/x_1, & \text{for } 0 < x_1 \leq 1 \\ 0, & \text{for } \quad x_1 = 0 \end{cases}$$

is connected.

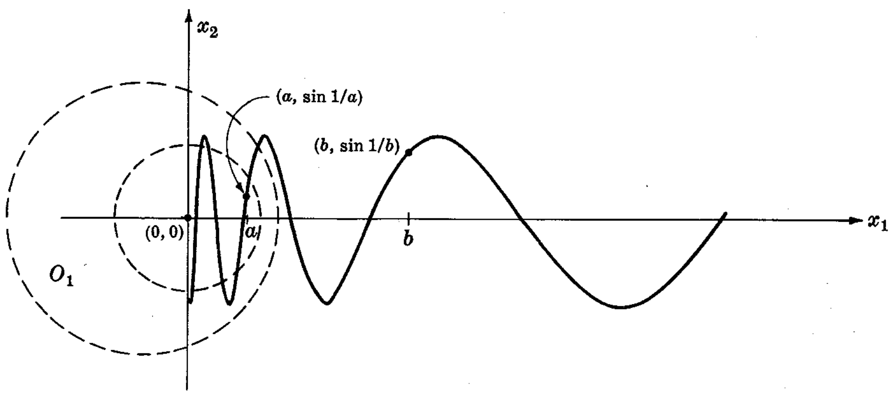


Fig. 6-21

Suppose this set S is not connected. Then there exist open sets O_1 and O_2 which contain S and have nonempty disjoint intersections with S. Now suppose the origin $(0,0)$ belongs to $S \cap O_1$, and say the point $(b, \sin 1/b)$, $0 < b \leq 1$, is in $S \cap O_2$, as shown in Fig. 6-21. Since O_1 is open, there exists an $S(0,0)$ which is contained in O_1. But a neighborhood of the origin will contain a point $(a, \sin 1/a)$ of S, where $0 < a < b$. Now consider the subset S^* of S consisting of the points (x_1, x_2) where $a \leq x_1 \leq b$. Certainly $S^* \subseteq O_1 \cup O_2$, since $S \subseteq O_1 \cup O_2$. Also $S^* \cap O_1 \neq \emptyset$ and $S^* \cap O_2 \neq \emptyset$, since $(a, \sin 1/a)$ is in O_1 and $(b, \sin 1/b)$ is in O_2. Finally $S^* \cap O_1$ and $S^* \cap O_2$ are disjoint, since $S \cap O_1$ and $S \cap O_2$ are disjoint. Thus S^* is also disconnected. But this is a contradiction, for S^* is arcwise connected by the continuous curve $x_2 = \sin 1/x_1$, $0 < a \leq x_1 \leq b$, and hence connected. Thus S is connected. Note that S is not arcwise connected, as shown in Problem 6.20.

6.11. Show directly from the definition, not using Theorem 6.6, that an open disk in E^2 is not compact.

We want to construct an open covering of the given set S which does not contain a finite subcovering. Suppose the given disk is of radius r. Consider the infinite family of concentric open disks $\{O_n\}$ of radii $r - 1/n$, $n = 2, 3, \ldots$. Clearly $\{O_n\}$ covers S, in fact $S = \bigcup_n O_n$. But the union of any finite subset of the O_n will have a maximum radius $r - 1/N$, less than r, and hence will not cover all of S.

6.12. Prove that the closure $\hat{S}$ of a connected set S is connected.

Suppose $\hat{S}$ is not connected; then there are open sets O_1 and O_2 which contain $\hat{S}$ and have nonempty disjoint intersections with $\hat{S}$. Let P be a point in $\hat{S} \cap O_1$. Now P may or may not be in S. If not, it is a limit point of S, in which case there exists, since O_1 is open, an $S(P)$ in O_1 which does contain a point P^* of S. The same argument gives a point Q^* of S in $S \cap O_2$. But this implies that S itself is not connected, for clearly O_1 and O_2 contain S since S is a subset of $\hat{S}$. Also $S \cap O_1 \neq \emptyset$ and $S \cap O_2 \neq \emptyset$, since P^* is in $S \cap O_1$ and Q^* is in $S \cap O_2$. Finally $S \cap O_1$ and $S \cap O_2$ are disjoint, since $\hat{S} \cap O_1$ and $\hat{S} \cap O_2$ are disjoint. But this is impossible, since S is connected. Hence $\hat{S}$ is connected, which is the required result.

6.13. Prove Theorem 6.5: A domain (open connected set) is arcwise connected.

Suppose D is a domain which is not arcwise connected, and suppose P_0 and Q_0 are two points in D which cannot be connected. Let A denote the set of points in D which can be connected to P_0 and let B denote the set of points in D which cannot be connected to P_0. Clearly A and B form a partition of D; that is, $D = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$. If it can be shown that both A and B are open, this would imply that D is not connected, which is a contradiction. To show that A is open, let P^* be an arbitrary point in A and let $S(P^*)$ be a neighborhood of P^* contained in D as shown in Fig. 6-22. Such a neighborhood exists, since D is open. But, as shown in the figure, every point P in $S(P^*)$ can be joined to P^* with a straight line segment. Hence every point in $S(P^*)$ can be joined to P_0, since P^* can be joined to P_0. Thus $S(P^*)$ is contained in A. Hence A is open.

Similarly let Q^* be in B and $S(Q^*)$ in D. Since every point Q in $S(Q^*)$ can be joined to Q^*, every point in $S(Q^*)$ cannot be joined to P_0, for otherwise Q^* could be joined to P_0. Thus B is also open, which gives the required result.

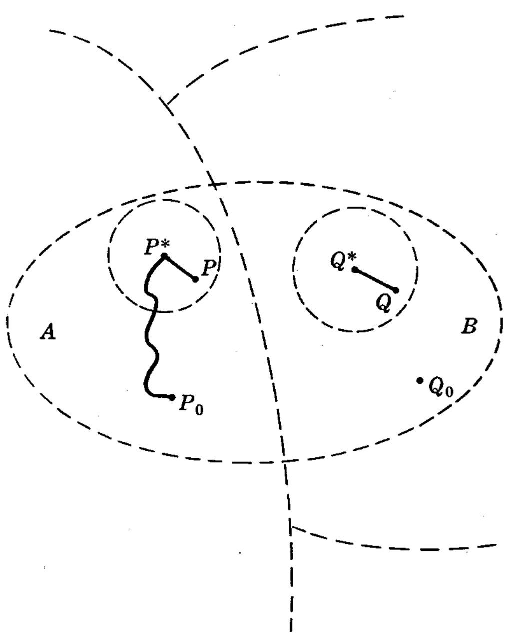


Fig. 6-22

6.14. Prove that the only connected sets in E^1 are the intervals.

Certainly the intervals are arcwise connected and hence connected. Conversely, let S be a connected set in E^1. We suppose S is bounded from above and b is the supremum of S. Also suppose S is bounded from below and a is the infimum of S. We wish to show that any c such that $a < c < b$, is in S. If not, consider the open sets $O_1 = \{x \mid x < c\}$ and $O_2 = \{x \mid x > c\}$. Certainly, $S \subseteq O_1 \cup O_2$. Also $S \cap O_1 \neq \emptyset$, for otherwise c would be an upper bound of S less than b, which is impossible. Similarly $S \cap O_2 \neq \emptyset$. Finally $O_1 \cap S$ and $O_2 \cap S$ are disjoint, since O_1 and O_2 are disjoint. But this is impossible, since S is connected. Thus every c such that $a < c < b$, is in S. Since no number greater than b is in S and no number less than a is in S, it follows that S is a finite interval. The case where S is not bounded from above or from below is left as an exercise for the reader.

6.15. Prove that S is a connected set in E^2 if and only if S is a connected set when regarded as a set in a plane in E^3.

Suppose S is connected in E^2 but not connected in E^3; then there exist open sets O_1 and O_2 in E^3 such that $S = O_1 \cup O_2$, $O_1 \cap S \neq \emptyset$, $O_2 \cap S \neq \emptyset$, $(O_1 \cap S) \cap (O_2 \cap S) = \emptyset$. Now let A be the intersection of O_1 and the plane containing S, and B the intersection of O_2 and the plane. Note that $S \subseteq A \cup B$, $A \cap S \neq \emptyset$, $B \cap S \neq \emptyset$, and $(A \cap S) \cap (B \cap S) = \emptyset$.

If it can be shown that A and B are open sets in the plane, we would have that S is disconnected in E^2, which is a contradiction. To prove that A is open, let P be a point in A. Then P is in O_1. Let $S^*(P)$ be a neighborhood of P in O_1 (in E^3). Then the intersection of $S^*(P)$ and the plane containing A is a neighborhood $S(P)$ in A. Thus A is open in the plane. Similarly B is open, which proves that if S is connected as a set in E^2, then it is connected as a set in E^3.

Conversely, suppose S is connected as a set in E^3, and suppose it is disconnected in the plane. Then there exist open sets A and B in the plane such that $S \subseteq A \cup B$, $S \cap A \neq \emptyset$, $S \cap B \neq \emptyset$, and $(S \cap A) \cap (S \cap B) = \emptyset$. Now since A is open in the plane, for each P in A there exists an $S(P)$ in the plane which is contained in A. Let $S^*(P)$ be the neighborhood of P in E^3 whose intersection with the plane is $S(P)$ and let $O_1 = \bigcup_P S^*(P)$. Similarly, for the points Q in B let $O_2 = \bigcup_Q S^*(Q)$. It follows that O_1 and O_2 are open sets in E^3 such that $S \subseteq O_1 \cup O_2$, $O_1 \cap S = A$, $A \neq \emptyset$, $O_2 \cap S = B$, $B \neq \emptyset$, and $(O_1 \cap S) \cap (O_2 \cap S) = \emptyset$. But this is impossible, since S is connected in E^3, which proves the proposition.

CONTINUOUS MAPPINGS. HOMEOMORPHISMS

6.16. Let f be a mapping of a set of points S in E into the x axis. If f is continuous at a point P_0 in S, and if $f(P_0) > 0$, show that there exists an $S_\delta(P_0)$ such that $f(P) > 0$ for P in $S_\delta(P_0) \cap S$.

As shown in Fig. 6-23, let $\epsilon = \frac{1}{2}f(P_0)$ and consider $S_\epsilon(f(P_0))$, i.e. the interval $\frac{1}{2}f(P_0) < x < \frac{3}{2}f(P_0)$. Note that $\frac{1}{2}f(P_0) > 0$. Since f is continuous at P_0, there is an $S_\delta(P_0)$ such that $f(P)$ is in $S_\epsilon(f(P_0))$. Hence $f(P) > 0$ for all P in $S_\delta(P_0)$, which gives the required result.

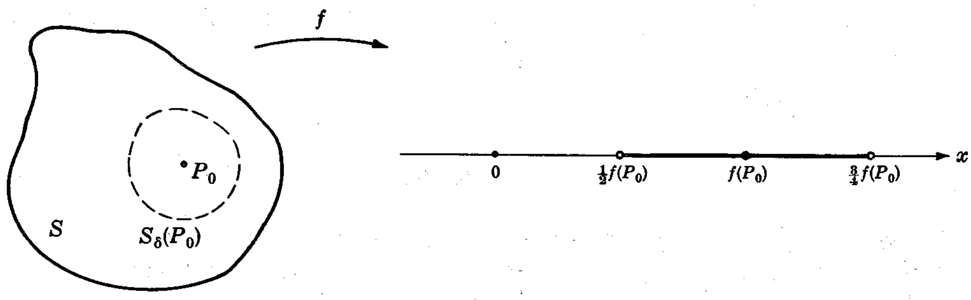


Fig. 6-23

6.17. Let f be a one-to-one continuous mapping of S (in E) into F and let P_0 in S be a limit point of S. Prove that $f(P_0)$ is a limit point of the image $f(S)$.

Let $S(f(P_0))$ be an arbitrary neighborhood of $f(P_0)$ as shown in Fig. 6-24. Since f is continuous on S, there exists an $S(P_0)$ such that $f(P)$ is in $S(f(P_0))$ for P in $S(P_0) \cap S$. Since P_0 is a limit point of S, there is a $Q \neq P_0$ in S and in $S(P_0)$. Hence $f(Q)$ is in $S(f(P_0))$. Also $f(Q) \neq f(P_0)$, since f is one-to-one. Thus for an arbitrary $S(f(P_0))$ there exists $f(Q) \neq f(P_0)$ such that $f(Q)$ is in $S(f(P_0))$. It follows that $f(P_0)$ is a limit point of $f(S)$.

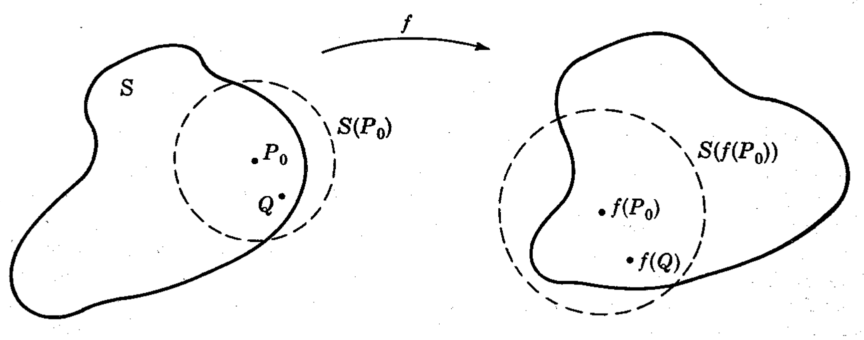


Fig. 6-24

6.18. Show that the mapping of the uv plane into the xy plane given by the equations

$$x = u + v, \qquad y = u - v$$

is one-to-one and continuous.

Clearly the mapping is one-to-one onto the xy plane with inverse $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$, obtained by solving for u and v in terms of x and y. Now observe in Fig. 6-25 below that the mapping takes the coordinate lines $u = c =$ constant onto the family of parallel lines $x + y = 2c$ and the family $v = k =$ constant onto the orthogonal family $x - y = 2k$. Thus given a neighborhood $S_\epsilon(x_0, y_0)$ of (x_0, y_0) in the xy plane, select Δu and Δv sufficiently small such that the rectangle with sides $x + y = 2(u_0 + \Delta u)$, $x + y = 2(u_0 - \Delta u)$, $x - y = 2(v_0 + \Delta v)$, $x - y = 2(v_0 - \Delta v)$ is contained in $S_\epsilon(x_0, y_0)$. Now select δ such that $S_\delta(u_0, v_0)$ is contained in rectangle $u_0 - \Delta u < u < u_0 + \Delta u$, $v_0 - \Delta v < v < v_0 + \Delta v$. But then for (u, v) in $S_\delta(u_0, v_0)$ we have (x, y) in $S_\epsilon(x_0, y_0)$. Since (u_0, v_0) is arbitrary, it follows that the mapping is continuous on the uv plane.

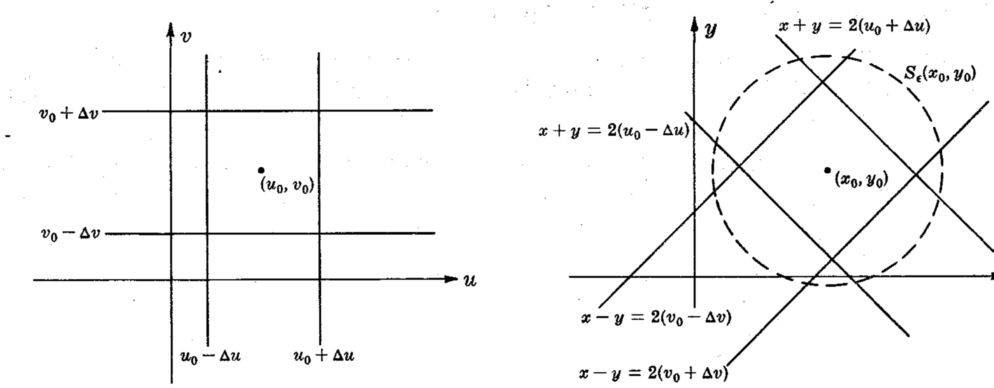


Fig. 6-25

6.19. Let f be a mapping of a set S (in E) into F. Show that f is continuous at a point P_0 in S if and only if for every open set O^* in F containing $f(P_0)$, there exists an open set O in E containing P_0 such that $f(P)$ is in O^* for all P in $O \cap S$.

Suppose f is continuous at P_0 and O^* is an open set containing $f(P_0)$. Since O^* is open, there is an $S(f(P_0))$ contained in O^*. Since f is continuous, there is an $S(P_0)$ such that $f(P)$ is in $S(f(P_0))$ and hence O^* for P in $S(P_0) \cap S$. But $S(P_0)$ itself is an open set containing P_0, which completes the proof. To prove the converse, suppose $S(f(P_0))$ is an arbitrary neighborhood of $f(P_0)$. Since $S(f(P_0))$ is an open set, by hypothesis there is an open set O in E containing P_0 such that $f(P)$ is in $S(f(P_0))$ for P in $O \cap S$. Since O is open and contains P_0, there is an $S(P_0)$ contained in O. It follows that $f(P)$ is in $S(f(P_0))$ for P in $S(P_0) \cap S$. Thus f is continuous at P_0, which is the required result.

6.20. Prove that the set of points (x_1, x_2) in E^2 such that

$$x_2 = \begin{cases} \sin 1/x_1, & \text{for } 0 < x_1 \leq 1 \\ 0, & \text{for } \quad x_1 = 0 \end{cases}$$

is not arcwise connected. (See Problem 6.10.)

Suppose this set S were arcwise connected and $\mathbf{x} = \mathbf{x}(t)$, $0 \leq t \leq 1$, was a continuous arc connecting $(0,0)$ and $(1, \sin 1)$. We wish to show first that $\mathbf{x}(t)$ passes through every point on S, or, equivalently, if R denotes the image of the mapping $\mathbf{x} = \mathbf{x}(t)$ into S, then $R = S$. For if not, there is a point $(b, \sin 1/b)$ on S, where $0 < b < 1$, which is not contained in R. As shown in Fig. 6-26, it follows that the open sets $x_1 < b$ and $x_1 > b$ in the x_1x_2 plane cover R and have nonempty disjoint intersections with R, implying R is disconnected, which would contradict Theorem 6.7 since $[0,1]$ is connected. Thus $R = S$. But now it follows that S is compact, since it is the continuous image of a compact set $0 \leq t \leq 1$ (Theorem 6.8). But then S must be closed, which it is not, since it does not contain all its limit points. For example, every deleted neighborhood of $(0, \frac{1}{2})$ contains a point of S. Hence $(0, \frac{1}{2})$ is a limit point of S. But $(0, \frac{1}{2})$ does not belong to S. Thus S is not arcwise connected and the proposition is proved.

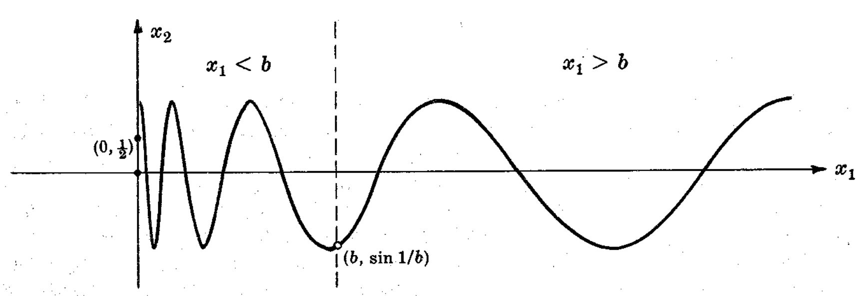


Fig. 6-26

6.21. **Let f be a continuous mapping of an open set O (in E) into F. Let O^* be any open set in F. Show that the set of points P in O such that $f(P)$ is in O^* is an open set in E.**

Let S be the set of points P in O such that $f(P)$ is in O^*. Let P_0 be in S. Since O^* is open, there is an $S(f(P_0))$ contained in O^*. Since f is continuous on O, there exists an $S_\delta(P_0)$ such that for P in $S_\delta(P_0) \cap O$, we have $f(P)$ in $S(f(P_0))$ and hence in O^*. Now if $S_\delta(P_0)$ is contained in O, then for *all* P in $S_\delta(P_0)$ we have $f(P)$ in O^*, so that $S_\delta(P_0)$ is in S. However, if $S_\delta(P_0)$ is not contained in O, since O *is open*, there exists some smaller $S_\gamma(P_0)$ in O contained in $S_\delta(P_0)$, as shown in Fig. 6-27. But then for all P in $S_\gamma(P_0)$, we have $f(P)$ in O^* and hence $S_\gamma(P_0)$ is in S. In any case, for an arbitrary P_0 in S there exists $S(P_0)$ in S. Hence S is open, which is the required result.

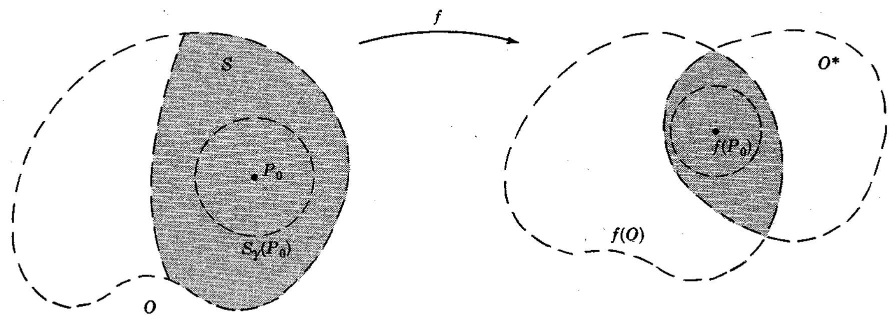


Fig. 6-27

6.22. **Let f be a mapping of a set S (in E) into F such that if C is any closed set in F, the set of points P in S for which $f(P)$ is in C is a closed set in E. Show that f is then continuous on S.**

Note that it follows that S itself must be closed, since $f(S)$ is in the closed set F. Now let P_0 be in S and suppose f is not continuous at P_0. Then there exists an $S(f(P_0))$ such that every $S(P_0)$ contains a point P such that $f(P)$ is not in $S(f(P_0))$; or, since $f(P_0)$ is in $S(f(P_0))$, every deleted neighborhood $S'(P_0)$ contains a point P such that $f(P)$ is in $[S(f(P_0))]^C$. Now let S^* be the set of points P in S such that $f(P)$ is in $[S(f(P_0))]^C$. We note that $[S(f(P_0))]^C$ is closed. Thus by hypothesis S^* is closed. But we have shown that P_0 is a limit point of S^*. Hence P_0 is in S^*. But this is impossible, since $f(P_0)$ is in $S(f(P_0))$, which completes the proof.

6.23. **Prove Theorem 6.9: If f is a continuous mapping of a set S (in E) into F and g is a continuous mapping of the image $f(S)$ (in F) into G, show that the mapping $(g \circ f)(P) = g(f(P))$ is a continuous mapping of S into G.**

Let $S((g \circ f)(P_0)) = S(g(f(P_0)))$ be an arbitrary neighborhood of $(g \circ f)(P_0)$. Since g is continuous on $f(S)$, there exists an $S(f(P_0))$ such that $g(Q)$ is in $S(g(f(P_0)))$ for Q in $S(f(P_0)) \cap f(S)$. Since f is continuous on S, there exists an $S(P_0)$ such that $f(P)$ is in $S(f(P_0))$ for P in $S(P_0) \cap S$. Hence for P in $S(P_0) \cap S$, we have $f(P)$ in $S(f(P_0)) \cap f(S)$ and hence $g(f(P))$ in $S(g(f(P_0)))$. Thus $g \circ f$ is continuous at P_0, which is the required result.

6.24. **Prove Theorem 6.8: If f is a continuous mapping of a compact set S (in E) into F, then the image of f is compact.**

Let $\{O_\alpha^*\}$ be an arbitrary open covering of $f(S)$. For each point Q in S, let O_Q^* denote an open set of the family $\{O_\alpha^*\}$ containing $f(Q)$. Since f is continuous on S, it follows from Problem 6.19 that there exists an open set O_Q in E, containing Q, such that for P in $O_Q \cap S$, we have $f(P)$ in O_Q, or, equivalently, $f(O_Q)$ is contained in O_Q^*. The family $\{O_Q\}$, since it contains every Q in S, is an open covering of S. Since S is compact, there exists a finite subcovering $\{O_{Q_1}, O_{Q_2}, \ldots, O_{Q_n}\}$. Since $f(O_{Q_i})$ is contained in $O_{Q_i}^*$ for $i = 1, \ldots, n$, and $\{O_{Q_i}\}$ is a covering of S, it follows that $\{O_{Q_1}^*, O_{Q_2}^*, \ldots, O_{Q_n}^*\}$ is a finite subcovering of $f(S)$. Thus $f(S)$ is compact.

6.25. If f is a continuous and 1-1 mapping of a compact set S (in E) into F, prove that f is a homeomorphism of S onto its image.

It remains to show that f^{-1} is a continuous mapping of the image of f (in F) into E. Let C be any closed set in E and let S^* be the set of points $f(P)$ in $f(S)$ such that $f^{-1}(f(P)) = P$ is in C. As shown in Fig. 6-28, it follows that $f^{-1}(S^*) = C \cap S$. Now since the compact set S is closed and C is closed, $f^{-1}(S^*)$ is closed. Since a closed subset of a compact set is compact (Problem 6.39), $f^{-1}(S^*)$ is compact. Since the image of a compact set under a continuous mapping is compact, it follows that $f(f^{-1}(S^*)) = S^*$ is compact. Since a compact set is closed, S^* is closed. Thus, given an arbitrary closed set C in E, the set S^* of points P in F such that $f^{-1}(P)$ is in C is closed. It follows from Problem 6.20 that f^{-1} is continuous on the image $f(S)$.

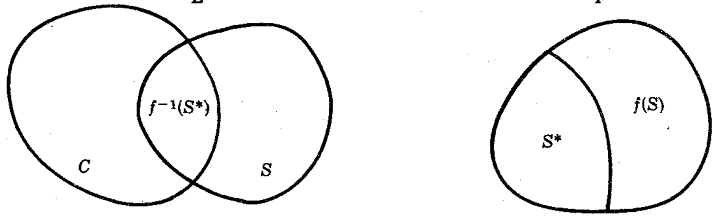


Fig. 6-28

Supplementary Problems

6.26. Determine which of the following sets are (i) open, (ii) closed, (iii) bounded, (iv) connected, (v) compact.

(a) Two disjoint closed finite intervals in E^1.
(b) Two open disjoint disks in E^2.
(c) The complement of two open disjoint disks in E^2.
(d) Two disjoint closed spheres in E^3.
(e) The complement of two disjoint closed spheres in E^3.
(f) A torus in E^3.
(g) The complement of a torus in E^3.

Ans. (a) Closed, bounded, not connected, compact.
(b) Open, bounded, not connected, not compact.
(c) Closed, not bounded, connected, not compact.
(d) Closed, bounded, not connected, compact.
(e) Open, not bounded, connected, not compact.
(f) Closed, bounded, connected, compact.
(g) Open, not bounded, not connected, not compact.

6.27. Prove that the intersection of any number of closed sets in E is closed.

6.28. Prove that P is a limit point of a set S in E if and only if every open set containing P contains a point of S other than P.

6.29. If P is a limit point of S in E, show that every open set containing P contains an infinite number of points of S.

6.30. Show that a set consisting of a finite number of points in E is bounded.

6.31. Show that the union of a finite number of spherical neighborhoods in E is bounded.

6.32. Show that the set of limit points of a set in E is a closed set.

6.33. If T is a closed set containing a set S in E, show that T contains the closure $\hat{S}$ of S.

6.34. A point P is said to be an *exterior* point of a set S if P is an interior point of the complement of S. See Problem 6.8. Prove that the set of exterior points of S, called the *exterior* of S, is an open set.

6.35. A point P is said to be a *boundary point* of S if P is neither an interior nor exterior point of S. Prove that the set of boundary points of S, called the *boundary* of S, is a closed set.

6.36. Prove that E is connected.

6.37. Prove that S is a compact set in E^2 if and only if S is a compact set when regarded as a subset of a plane in E^3.

6.38. Show directly from the definition (not using Theorem 6.6) that E is not compact.

6.39. Show directly (not using Theorem 6.6) that a closed subset of a compact set in E is compact.

6.40. Prove that every infinite subset S^* of a compact set S in E has a limit point in S.

6.41. Prove that $x = u,\ y = v,\ z = f(u,v)$ defines a one-to-one bicontinuous mapping of a set S (in E^2) into E^3 if $f(u,v)$ is continuous on S.

6.42. Show that $x = au + bv + c,\ y = du + ev + f,\ ae - bd \neq 0,$ defines a one-to-one bicontinuous mapping of the uv plane onto the xy plane.

6.43. Let f be a mapping of a set S (in E) into F. Define: f is discontinuous at the point P_0 in S.
Ans. f is discontinuous at P_0 if there exists a neighborhood $S(f(P_0))$ such that for every $S(P_0)$, there is a point P in $S(P_0)$ for which $f(P)$ is in $[S(f(P_0))]^C$.

6.44. Show that the mapping

$$x = u, \quad y = v, \quad z = \begin{cases} u^2 + v^2, & \text{for } u \geqq 0 \\ 1, & \text{for } u < 0 \end{cases}$$

is (*a*) continuous at $(-1,-1)$, (*b*) continuous at $(1,1)$, (*c*) continuous at $(1,0)$, (*d*) discontinuous at $(2,0)$.

6.45. Show that the perimeter of a square is topologically equivalent to a circle by exhibiting a 1-1 bicontinuous mapping of the square onto the circle.

6.46. Let f be a mapping of a set S (in E) into F which preserves distances between points; i.e. for any P and Q in S, $d(P,Q) = d(f(P), f(Q))$. Show that f is a homeomorphism of S onto its image.

6.47. Let f be a continuous mapping of a connected set S (in E) into the x axis. If $f(P) = a$ and $f(Q) = b$, $a < b$, and c is any number in the interval $a < x < b$, show that there exists a P_0 in S such that $f(P_0) = c$.

6.48. If f is a continuous mapping of a compact set S (in E) into F, show directly (not using Theorems 6.6 and 6.8) that the image $f(S)$ is a bounded set in F.

6.49. Let f be a continuous mapping of a closed set C (in E) into F. Let C^* be any closed set in F. Show that the set of points P in E for which $f(P)$ is in C^* is a closed set in E.

6.50. Let f be a mapping of a set S (in E) into F such that if O is any open set in F, the set of points P in S for which $f(P)$ is in O is open. Show that f is continuous on S.

6.51. Show that the property of being topologically equivalent is an equivalence relation between sets in Euclidean spaces.

Chapter 7

Vector Functions of a Vector Variable

VECTOR FUNCTIONS

Let **f** be a mapping of a set of vectors V (in E) into F. Since **f** assigns a vector $\mathbf{f}(\mathbf{x})$ in F to each vector **x** in V, **f** is called a *vector valued function of a vector variable.*

Now suppose the dimension of E is n and the dimension of F is m. Here m and n are any one of the numbers 1, 2 or 3. If a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is fixed in E and $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, we will denote the vector assigned to **x** by $\mathbf{f}(x_1, \ldots, x_n)$. Here **f** assigns a vector to a set of real numbers and as such is called a *vector valued function of* $x_1, \ldots, x_n$. If a basis $(\mathbf{g}_1, \ldots, \mathbf{g}_m)$ is chosen in F then for each **x**, **f** is a linear combination

$$\mathbf{f}(\mathbf{x}) = f_1(\mathbf{x})\mathbf{g}_1 + \cdots + f_m(\mathbf{x})\mathbf{g}_m$$

In this case **f** is associated with m real-valued functions, $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ of **x**, the components of **f** with respect to $(\mathbf{g}_1, \ldots, \mathbf{g}_m)$. With respect to both bases **f** is represented by m real-valued functions $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n$, the components of $\mathbf{f}(x_1, \ldots, x_n)$.

Example 7.1:

(*a*) Let $\mathbf{n}(\mathbf{x})$ be the outward unit normal at a point **x** on the surface of a sphere S of radius r about the origin in E^3, as shown in Fig. 7-1. Then $\mathbf{n} = \mathbf{x}/r$ is a vector function of **x** from the set $|\mathbf{x}| = r$ (in E^3) into E^3. If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a basis in E^3 and $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, then **n** is given by

$$\mathbf{n} = \frac{\mathbf{x}}{r} = \frac{x_1}{r}\mathbf{e}_1 + \frac{x_2}{r}\mathbf{e}_2 + \frac{x_3}{r}\mathbf{e}_3, \qquad |\mathbf{x}| = r$$

The components of **n** are the real valued functions of (x_1, x_2, x_3),

$$n_1 = x_1/r, \quad n_2 = x_2/r, \quad n_3 = x_3/r$$

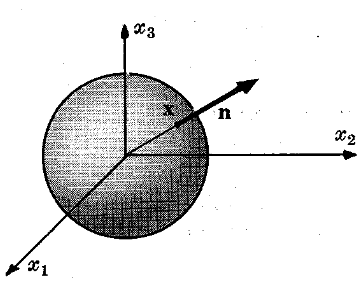


Fig. 7-1

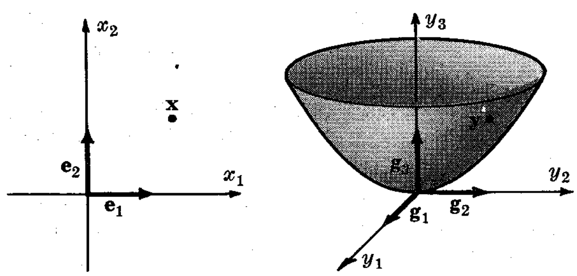


Fig. 7-2

(*b*) Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ and $\mathbf{y} = y_1\mathbf{g}_1 + y_2\mathbf{g}_2 + y_3\mathbf{g}_3$. The equation

$$\mathbf{y} = (x_1 + x_2)\mathbf{g}_1 + (x_1 - x_2)\mathbf{g}_2 + (x_1^2 + x_2^2)\mathbf{g}_3$$

determines a mapping of E^2 into E^3, as shown in Fig. 7-2. The components of **y** are

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2, \quad y_3 = x_1^2 + x_2^2$$

By eliminating x_1 and x_2, we find that the range of the mapping is the elliptic paraboloid

$$y_3 = \tfrac{1}{2}(y_1^2 + y_2^2)$$

LINEAR FUNCTIONS

A vector function $\mathbf{f}$ on E into F is said to be *linear,* if for all $\mathbf{a}$ and $\mathbf{b}$ in E,

$$\text{(i)} \quad \mathbf{f}(\mathbf{a}+\mathbf{b}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})$$

$$\text{(ii)} \quad \mathbf{f}(k\mathbf{a}) = k\mathbf{f}(\mathbf{a}) \qquad k = \text{scalar}$$

If $\mathbf{f}$ is linear and $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ then, using the above,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = \mathbf{f}(x_1\mathbf{e}_1) + \cdots + \mathbf{f}(x_n\mathbf{e}_n) = x_1\mathbf{f}(\mathbf{e}_1) + \cdots + x_n\mathbf{f}(\mathbf{e}_n)$$

If in addition $(\mathbf{g}_1, \ldots, \mathbf{g}_m)$ is a basis in F and the vectors $\mathbf{f}(\mathbf{e}_i)$ are

$$\begin{aligned} \mathbf{f}(\mathbf{e}_1) &= a_{11}\mathbf{g}_1 + a_{21}\mathbf{g}_2 + \cdots + a_{m1}\mathbf{g}_m \\ \mathbf{f}(\mathbf{e}_2) &= a_{12}\mathbf{g}_1 + a_{22}\mathbf{g}_2 + \cdots + a_{m2}\mathbf{g}_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{f}(\mathbf{e}_n) &= a_{1n}\mathbf{g}_1 + a_{2n}\mathbf{g}_2 + \cdots + a_{mn}\mathbf{g}_m \end{aligned}$$

then $\mathbf{f}(\mathbf{x})$ can be expressed as

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= x_1(a_{11}\mathbf{g}_1 + \cdots + a_{m1}\mathbf{g}_m) + x_2(a_{12}\mathbf{g}_1 + \cdots + a_{m2}\mathbf{g}_m) + \cdots + x_n(a_{1n}\mathbf{g}_1 + \cdots + a_{mn}\mathbf{g}_m) \\ &= (a_{11}x_1 + \cdots + a_{1n}x_n)\mathbf{g}_1 + (a_{21}x_1 + \cdots + a_{2n}x_n)\mathbf{g}_2 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n)\mathbf{g}_m \end{aligned}$$

Thus the components of a linear function $\mathbf{f}(\mathbf{x})$ are real-valued linear homogeneous functions of $(x_1, \ldots, x_n)$,

$$\begin{aligned} f_1(\mathbf{x}) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ f_2(\mathbf{x}) &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(\mathbf{x}) &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

Note that the coefficients a_{ij} depend on the bases in E and F.

The converse of the above is also true. In Problem 7.5 we will show that if the components of a vector function $\mathbf{f}$ are linear homogeneous functions of $(x_1, \ldots, x_n)$, then $\mathbf{f}$ is linear.

The matrix of coefficients

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is called the *matrix representation* of $\mathbf{f}$ with respect to the bases $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and $(\mathbf{g}_1, \ldots, \mathbf{g}_m)$. The rank of the matrix, i.e. the order of largest nonvanishing minor of the matrix, can be shown to be independent of the bases and is called the *rank* of $\mathbf{f}$.

Note that, if $\mathbf{f}$ is linear,

$$\mathbf{f}(0) = \mathbf{f}(0-0) = \mathbf{f}(0) - \mathbf{f}(0) = 0$$

Thus if $\mathbf{f}$ is a linear mapping of E into F, the origin in E is always mapped into the origin in F.

Now suppose $\mathbf{f}$ is a linear mapping of E^3 into E^3. In Problem 7.3 we will show that the mapping is 1-1 and onto if and only if the images of the basis vectors

$$\begin{aligned} \mathbf{f}(\mathbf{e}_1) &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 \\ \mathbf{f}(\mathbf{e}_2) &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 \\ \mathbf{f}(\mathbf{e}_3) &= a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 \end{aligned}$$

are independent. But from Theorem 1.5 the above vectors are independent if and only if

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0$$

that is, if and only if the rank of the matrix representation of **f** is 3. Thus a linear mapping **f** of E^3 into E^3 is 1-1 and onto if and only if the rank of **f** equals 3.

If $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, $\mathbf{f}(\mathbf{e}_3)$ are dependent but, say, $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are independent, then all linear combinations of $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, $\mathbf{f}(\mathbf{e}_3)$ lie in the plane through the origin containing $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$. We leave as an exercise for the reader to show that this is the case if and only if the rank of **f** is 2.

Finally, if no two of the vectors $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, $\mathbf{f}(\mathbf{e}_3)$ are independent but, say, $\mathbf{f}(\mathbf{e}_1) \neq \mathbf{0}$, then **f** maps E^3 onto the line through the origin containing $\mathbf{f}(\mathbf{e}_1)$. This is the case if and only if the rank of **f** is 1.

Thus we have

Theorem 7.1. Let **f** be a linear mapping of E^3 into E^3. Then

(i) **f** is 1-1 and onto if and only if the rank of **f** equals 3.

(ii) **f** maps E^3 onto a plane in E^3 if and only if the rank of **f** equals 2.

(iii) **f** maps E^3 onto a line in E^3 if and only if the rank of **f** equals 1.

We will also prove

Theorem 7.2. Let **f** be a linear mapping of E^2 into E^3. Then

(i) **f** maps E^2 1-1 onto a plane in E^3 if and only if the rank of **f** equals 2.

(ii) **f** maps E^2 onto a line in E^3 if and only if the rank of **f** equals 1.

Example 7.2:

(*a*) The equations

$$x_1 = 2u - v$$
$$x_2 = u + v$$
$$x_3 = -u + v$$

define a linear mapping of E^2 into E^3. The matrix representation is

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Since

$$\det\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = 3 \neq 0$$

the mapping is of rank 2. It is easily verified by eliminating u and v that the mapping takes E^2 onto the plane $2x_1 - x_2 + 3x_3 = 0$.

(*b*) The equations

$$y_1 = 2x_1 + x_2 - 2x_3$$
$$y_2 = x_1 + x_2 - x_3$$
$$y_3 = -x_1 + x_3$$

define a linear mapping of E^3 into E^3. Since

$$\det\begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = 0$$

but $$\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1 \neq 0$$

the rank of the mapping is 2. Here the mapping takes E^3 onto the plane $y_1 - y_2 + y_3 = 0$.

CONTINUITY AND LIMITS

Recall that an ϵ-spherical neighborhood of a vector $\mathbf{x}_0$ in E, denoted by $S_\epsilon(\mathbf{x}_0)$, consists of the vectors $\mathbf{x}$ satisfying $|\mathbf{x}-\mathbf{x}_0| < \epsilon$. As shown in Fig. 7-3, $S_\epsilon(\mathbf{x}_0)$ consists of the vectors whose distance from $\mathbf{x}_0$ is less than ϵ. Recall further that a function $\mathbf{f}$ from V (in E) into F is continuous at $\mathbf{x}_0$ in V if for every spherical neighborhood $S(\mathbf{f}(\mathbf{x}_0))$ of $\mathbf{f}(\mathbf{x}_0)$, there exists $S(\mathbf{x}_0)$ such that $\mathbf{f}(\mathbf{x})$ is in $S(\mathbf{f}(\mathbf{x}_0))$ for $\mathbf{x}$ in $S(\mathbf{x}_0) \cap V$. Thus $\mathbf{f}$ is continuous at $\mathbf{x}_0$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \epsilon$ for $\mathbf{x}$ in V and $|\mathbf{x}-\mathbf{x}_0| < \delta$.

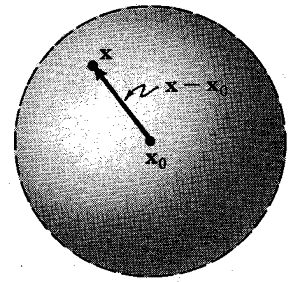


Fig. 7-3

Example 7.3:

Consider the function $\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{x}/|\mathbf{x}|, & \text{if } \mathbf{x} \neq \mathbf{0} \\ \mathbf{0}, & \text{if } \mathbf{x} = \mathbf{0} \end{cases}$. For $\mathbf{x}_0 \neq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| &= \left|\frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|}\right| \leq \left|\frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{x}}{|\mathbf{x}_0|}\right| + \left|\frac{\mathbf{x}}{|\mathbf{x}_0|} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|}\right| \\ &\leq |\mathbf{x}|\frac{\big||\mathbf{x}_0| - |\mathbf{x}|\big|}{|\mathbf{x}|\,|\mathbf{x}_0|} + \frac{1}{|\mathbf{x}_0|}|\mathbf{x}-\mathbf{x}_0| \leq \frac{2|\mathbf{x}-\mathbf{x}_0|}{|\mathbf{x}_0|} \end{aligned}$$

where we used the triangle inequalities $\big||\mathbf{a}| - |\mathbf{b}|\big| \leq |\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. Thus for $|\mathbf{x}-\mathbf{x}_0| < \frac{1}{2}\epsilon|\mathbf{x}_0|$, $|\mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\mathbf{x})| < \epsilon$ and so $\mathbf{f}$ is continuous for all $\mathbf{x}_0 \neq \mathbf{0}$. At $\mathbf{x}_0 = \mathbf{0}$, however, $\mathbf{f}(\mathbf{x})$ is not continuous since $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{0})| = |\mathbf{x}|/|\mathbf{x}| = 1$ which can not be less than ϵ if $\epsilon \leq 1$.

As in the case of vector functions of a single variable, we have

Theorem 7.3. The function $\mathbf{f} = f_1\mathbf{g}_1 + \cdots + f_m\mathbf{g}_m$ is continuous at $\mathbf{x}$ if and only if each component f_i, $i = 1, \ldots, m$ is continuous at $\mathbf{x}$.

Theorem 7.4. If $\mathbf{f}$, $\mathbf{g}$ and h are continuous at $\mathbf{x}$, then $|\mathbf{f}|$, $\mathbf{f}+\mathbf{g}$, $h\mathbf{f}$, $\mathbf{f}\cdot\mathbf{g}$ and $\mathbf{f}\times\mathbf{g}$ are continuous at $\mathbf{x}$.

A function $\mathbf{f}$ from V (in E) into F has a *limit* $\mathbf{L}$ at $\mathbf{x}_0$, denoted by $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}$, if for every $S(\mathbf{L})$ there is a deleted $S'(\mathbf{x}_0)$ such that $\mathbf{f}(\mathbf{x})$ is in $S(\mathbf{L})$ for $\mathbf{x}$ in $S'(\mathbf{x}_0) \cap V$. Thus $\mathbf{f}$ has a limit $\mathbf{L}$ at $\mathbf{x}_0$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{L}| < \epsilon$ for $\mathbf{x}$ in V and $0 < |\mathbf{x}-\mathbf{x}_0| < \delta$. Here $\mathbf{x}_0$ need not be in V; however, it is assumed that $\mathbf{x}_0$ is a limit vector of V, that is, every $S(\mathbf{x}_0)$ contains a vector in V other than $\mathbf{x}_0$. In Problem 7.10 we show that if $\mathbf{x}_0$ is a limit vector of V and $\mathbf{f}$ is continuous at $\mathbf{x}_0$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$. Also if $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ and $\mathbf{f}(\mathbf{x}_0) = \mathbf{L}$, then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

Finally, as in the study of vectors of a single variable, we introduce the Landau notation $\mathbf{o}$ and $\mathbf{O}$. Namely let a scalar function $g(\mathbf{x})$ be different from zero in some deleted neighborhood of $\mathbf{x}_0$. A function $\mathbf{f}(\mathbf{x})$ is "small oh" of $g(\mathbf{x})$ at $\mathbf{x}_0$, denoted by $\mathbf{f}(\mathbf{x}) = \mathbf{o}(g(\mathbf{x}))$, if $\mathbf{f}(\mathbf{x})/g(\mathbf{x}) \to \mathbf{0}$ as $\mathbf{x} \to \mathbf{x}_0$. The function $\mathbf{f}(\mathbf{x})$ is "big oh" of $g(\mathbf{x})$ at $\mathbf{x}_0$, denoted by $\mathbf{f}(\mathbf{x}) = \mathbf{O}(g(\mathbf{x}))$, if $\mathbf{f}(\mathbf{x})/g(\mathbf{x})$ is bounded at $\mathbf{x}_0$; i.e. if there exists a scalar $M > 0$ such that $|\mathbf{f}(\mathbf{x})/g(\mathbf{x})| \leq M$ in some neighborhood of $\mathbf{x}_0$.

Example 7.4:

Let $\mathbf{f}(\mathbf{x}) = x_1^2 x_2 \mathbf{e}_1 + x_1(x_1^2 + x_2^2)\mathbf{e}_2$. Then $\mathbf{f}(\mathbf{x}) = \mathbf{o}(|\mathbf{x}|^2)$. For, introducing polar coordinates, $x_1 = |\mathbf{x}| \cos\theta$, $x_2 = |\mathbf{x}| \sin\theta$, we have

$$|\mathbf{f}(\mathbf{x})/|\mathbf{x}|^2| \;=\; \big|\,|\mathbf{x}|(\cos^2\theta \sin\theta)\mathbf{e}_1 + |\mathbf{x}|(\cos\theta)\mathbf{e}_2\big| \;=\; |\mathbf{x}|\,|(\cos^2\theta\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2| \;\leq\; |\mathbf{x}|2 \;<\; \epsilon$$

for $|\mathbf{x}| < \epsilon/2$. Thus $\mathbf{f}(\mathbf{x})/|\mathbf{x}|^2 \to \mathbf{0}$ as $\mathbf{x} \to \mathbf{0}$ and hence $\mathbf{f}(\mathbf{x}) = \mathbf{o}(|\mathbf{x}|^2)$. Also $\mathbf{f}(\mathbf{x}) = \mathbf{O}(|\mathbf{x}|^3)$, since

$$|\mathbf{f}(\mathbf{x})/|\mathbf{x}|^3| \;=\; |(\cos^2\theta\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2| \;\leq\; 2$$

That is, $\mathbf{f}(\mathbf{x})/|\mathbf{x}|^3$ is bounded at $\mathbf{x} = \mathbf{0}$ and thus $\mathbf{f}(\mathbf{x}) = \mathbf{O}(|\mathbf{x}|^3)$.

Our vector functions will usually be defined on open sets. Recall that V is open if for every $\mathbf{x}_0$ in V there exists a neighborhood $S(\mathbf{x}_0)$ of vectors also in V.

DIRECTIONAL DERIVATIVES

Let $\mathbf{f}$ be defined on an open set V (in E), let $\mathbf{x}_0$ be in V, and let $\mathbf{u}_0$ be a nonzero vector in E. The *directional derivative of* $\mathbf{f}$ *at* $\mathbf{x}_0$ *in the direction of* $\mathbf{u}_0$ is the vector

$$D_{\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0) \;=\; \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + h\mathbf{u}_0) - \mathbf{f}(\mathbf{x}_0)}{h}$$

whenever the limit exists. If we consider $\mathbf{f}$ as a function of h along $\mathbf{x} = \mathbf{x}_0 + h\mathbf{u}_0$, i.e. if we introduce the function $\mathbf{F}(h) = \mathbf{f}(\mathbf{x}_0 + h\mathbf{u}_0)$, then

$$D_{\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0) \;=\; \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + h\mathbf{u}_0) - \mathbf{f}(\mathbf{x}_0)}{h} \;=\; \lim_{h\to 0}\frac{\mathbf{F}(h) - \mathbf{F}(0)}{h} \;=\; \mathbf{F}'(0)$$

Namely $D_{\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0)$ is the derivative of $\mathbf{F}(h) = \mathbf{f}(\mathbf{x}_0 + h\mathbf{u}_0)$ at $h = 0$.

As shown in Fig. 7-4, let $\mathbf{f}$ be a mapping of E^2 into E^3. Observe that $\mathbf{x} = \mathbf{x}_0 + h\mathbf{u}_0$ is a line through $\mathbf{x}_0$ parallel to $\mathbf{u}_0$ and that $\mathbf{y} = \mathbf{F}(h) = \mathbf{f}(\mathbf{x}_0 + h\mathbf{u}_0)$, the image of the line, is a parametric representation of a curve on the surface $\mathbf{y} = \mathbf{f}(\mathbf{x})$. It follows that the directional derivative $D_{\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0) = \mathbf{F}'(0)$ is a vector which is tangent to the curve $\mathbf{y} = \mathbf{F}(h)$ at the point $\mathbf{f}(\mathbf{x}_0)$.

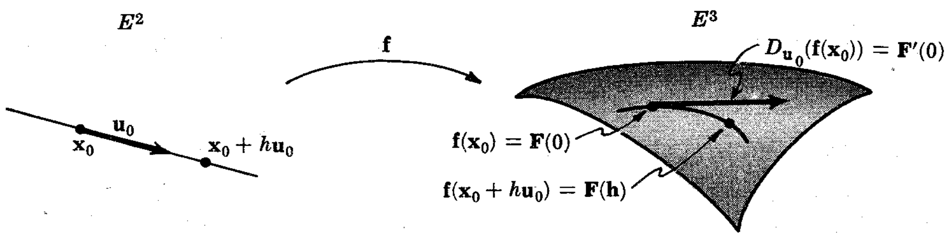


Fig. 7-4

Example 7.5:

(*a*) Consider the function

$$\mathbf{f}(\mathbf{x}) \;=\; x_1\mathbf{g}_1 + x_2\mathbf{g}_2 + (x_1^2 + x_2^2)\mathbf{g}_3$$

Here $\quad \mathbf{F}(h) \;=\; \mathbf{f}(\mathbf{x} + h\mathbf{u}) \;=\; (x_1 + hu_1)\mathbf{g}_1 + (x_2 + hu_2)\mathbf{g}_2 + [(x_1 + hu_1)^2 + (x_2 + hu_2)^2]\mathbf{g}_3$

$$\mathbf{F}'(h) \;=\; u_1\mathbf{g}_1 + u_2\mathbf{g}_2 + [2u_1(x_1 + hu_1) + 2u_2(x_2 + hu_2)]\mathbf{g}_3$$

and $\quad \mathbf{F}'(0) \;=\; u_1\mathbf{g}_1 + u_2\mathbf{g}_2 + (2u_1x_1 + 2u_2x_2)\mathbf{g}_3$

Thus the derivative of $\mathbf{f}$ exists at every $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ in any direction $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ and is given by

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = u_1\mathbf{g}_1 + u_2\mathbf{g}_2 + 2(u_1x_1 + u_2x_2)\mathbf{g}_3$$

(b) A function can have a derivative in one direction and not in another. For consider the real-valued function on E^2,

$$f(\mathbf{x}) = f(x_1, x_2) = \begin{cases} x_1 + x_2, & \text{if } x_1 = 0 \text{ or } x_2 = 0 \\ 1, & \text{otherwise} \end{cases}$$

At $\mathbf{x} = \mathbf{0}$ and in the direction $\mathbf{u} = \mathbf{e}_1$,

$$F(h) = f(h\mathbf{u}) = f(h, 0) = h, \quad \text{and} \quad F'(h) = 1$$

Thus $D_{\mathbf{e}_1}f(\mathbf{0}) = F'(0) = 1$. Also, at $\mathbf{x} = \mathbf{0}$ in the direction $\mathbf{u} = \mathbf{e}_2$,

$$F(h) = f(h\mathbf{u}) = f(0, h) = h, \quad \text{and} \quad F'(h) = 1$$

Hence $D_{\mathbf{e}_2}f(\mathbf{0}) = 1$. In any other direction, however, i.e. for $\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_1$, where $u_1 \neq 0$ and $u_2 \neq 0$, we have

$$F(h) = f(h\mathbf{u}) = f(hu_1, hu_2) = \begin{cases} 0, & \text{if } h = 0 \\ 1, & \text{otherwise} \end{cases}$$

which does not have a derivative at $h = 0$. Thus f has a derivative only in the directions $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{u} = \mathbf{e}_2$ at $\mathbf{x} = \mathbf{0}$.

Now suppose a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is fixed in E. The derivative in the direction of a basis vector $\mathbf{e}_k$ at $\mathbf{x}$, i.e. $D_{\mathbf{e}_k}\mathbf{f}(\mathbf{x})$, is called the *partial derivative of* $\mathbf{f}$ *with respect to the* k*th component of* $\mathbf{x}$. If we write

$$\mathbf{f}(\mathbf{x}) = f_1(x_1, \ldots, x_n)\mathbf{g}_1 + \cdots + f_m(x_1, \ldots, x_n)\mathbf{g}_m$$

then

$$\begin{aligned} D_{\mathbf{e}_k}\mathbf{f}(\mathbf{x}) &= \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{x} + h\mathbf{e}_k) - \mathbf{f}(\mathbf{x})}{h} = \lim_{h\to 0} \frac{\mathbf{f}(x_1, \ldots, x_k + h, \ldots, x_n) - \mathbf{f}(x_1, \ldots, x_n)}{h} \\ &= \lim_{h\to 0} \frac{f_1(x_1, \ldots, x_k + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}\mathbf{g}_1 \\ &\qquad + \cdots + \lim_{h\to 0} \frac{f_m(x_1, \ldots, x_k + h, \ldots, x_n) - f_m(x_1, \ldots, x_m)}{h}\mathbf{g}_m \\ &= \frac{\partial f_1}{\partial x_k}\mathbf{g}_1 + \frac{\partial f_2}{\partial x_k}\mathbf{g}_2 + \cdots + \frac{\partial f_m}{\partial x_k}\mathbf{g}_m \end{aligned}$$

Thus $D_{\mathbf{e}_k}\mathbf{f}(\mathbf{x})$ is a vector whose components are the partial derivatives $\partial f_i/\partial x_k$ at $\mathbf{x}$ of the components of $\mathbf{f}$. The derivative $D_{\mathbf{e}_k}\mathbf{f}$ is also denoted by $D_k\mathbf{f}$ or $\partial\mathbf{f}/\partial x_k$.

Example 7.6:

Let $\mathbf{f}(u, v) = ue^v\mathbf{g}_1 + (u^2 + v^2)\mathbf{g}_2 + uv\mathbf{g}_3$. Then

$$\frac{\partial\mathbf{f}}{\partial u} = \frac{\partial}{\partial u}(ue^v)\mathbf{g}_1 + \frac{\partial}{\partial u}(u^2 + v^2)\mathbf{g}_2 + \frac{\partial}{\partial u}(uv)\mathbf{g}_3 = e^v\mathbf{g}_1 + 2u\mathbf{g}_2 + v\mathbf{g}_3$$

and

$$\frac{\partial\mathbf{f}}{\partial v} = \frac{\partial}{\partial v}(ue^v)\mathbf{g}_1 + \frac{\partial}{\partial v}(u^2 + v^2)\mathbf{g}_2 + \frac{\partial}{\partial v}(uv)\mathbf{g}_3 = ue^v\mathbf{g}_1 + 2v\mathbf{g}_2 + u\mathbf{g}_3$$

DIFFERENTIABLE FUNCTIONS

If a function of a single variable has a derivative at a point, then it is continuous there. However, such is not the case for a function of a vector variable. As shown in Problem 7.14, page 140, $\mathbf{f}$ may have a derivative in every direction at $\mathbf{x}_0$ and not be continuous at $\mathbf{x}_0$. The reason for this is that a directional derivative at a point depends upon the function only along a line near the point and not in a complete neighborhood of the point.

If a function of a single variable has a derivative at a point, it can be linearly approximated in a neighborhood of the point. It is this property that we use to define differentiability for a function of vector variable:

A function $\mathbf{f}$, defined on an open set V (in E) into F, is said to be *differentiable* at $\mathbf{x}_0$ in V if there exists a *linear* function $\mathbf{L}_{\mathbf{x}_0}(\mathbf{v})$ mapping $\mathbf{v}$ in E into F such that

$$\mathbf{f}(\mathbf{x}_0+\mathbf{v}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{L}_{\mathbf{x}_0}(\mathbf{v}) + \mathbf{R}(\mathbf{x}_0, \mathbf{v})$$

where $(\mathbf{R}(\mathbf{x}_0, \mathbf{v})/|\mathbf{v}|) \to \mathbf{0}$ as $\mathbf{v} \to \mathbf{0}$.

Now suppose $\mathbf{f}$ is differentiable at $\mathbf{x}_0$. Then for any nonzero vector $\mathbf{u}$ and scalar h sufficiently small,

$$\mathbf{f}(\mathbf{x}_0+h\mathbf{u}) - \mathbf{f}(\mathbf{x}_0) = \mathbf{L}_{\mathbf{x}_0}(h\mathbf{u}) + \mathbf{R}(\mathbf{x}_0, h\mathbf{u})$$

Since $\mathbf{L}_{\mathbf{x}_0}$ is linear, $\mathbf{L}_{\mathbf{x}_0}(h\mathbf{u}) = h\mathbf{L}_{\mathbf{x}_0}(\mathbf{u})$. Thus, dividing by h,

$$\frac{\mathbf{f}(\mathbf{x}_0+h\mathbf{u}) - \mathbf{f}(\mathbf{x}_0)}{h} = \mathbf{L}_{\mathbf{x}_0}(\mathbf{u}) + \frac{\mathbf{R}(\mathbf{x}_0, h\mathbf{u})}{h}$$

In Problem 7.12 we will show that $(\mathbf{R}(\mathbf{x}_0, h\mathbf{u})/h) \to \mathbf{0}$ as $h \to 0$. It follows that the derivative of $\mathbf{f}$ exists at $\mathbf{x}_0$ in every direction $\mathbf{u}$ and is given by

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}_0) = \lim_{h\to 0}\frac{\mathbf{f}(\mathbf{x}_0+h\mathbf{u}) - \mathbf{f}(\mathbf{x}_0)}{h} = \lim_{h\to 0}\mathbf{L}_{\mathbf{x}_0}(\mathbf{u}) + \lim_{h\to 0}\frac{\mathbf{R}(\mathbf{x}_0, h\mathbf{u})}{h} = \mathbf{L}_{\mathbf{x}_0}(\mathbf{u})$$

Thus

Theorem 7.5. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, then $\mathbf{f}$ has a derivative in every direction at $\mathbf{x}_0$.

Also, in Problem 7.19, page 143, we prove

Theorem 7.6. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

Note. It follows from the fact that $D_{\mathbf{v}}\mathbf{f}(\mathbf{x}_0) = \mathbf{L}_{\mathbf{x}_0}(\mathbf{v})$, that $\mathbf{L}_{\mathbf{x}_0}$ is unique. In particular it is uniquely determined by its values at a basis, which are the partial derivatives $\mathbf{L}_{\mathbf{x}_0}(\mathbf{e}_i) = D_{\mathbf{e}_i}\mathbf{f}(\mathbf{x}_0) = \dfrac{\partial\mathbf{f}}{\partial x_i}(\mathbf{x}_0)$.

The linear function $\mathbf{L}_{\mathbf{x}_0}$ is called the *differential of* $\mathbf{f}$ *at* $\mathbf{x}_0$ and shall be denoted by $d\mathbf{f}(\mathbf{x}_0)$. The value of $d\mathbf{f}(\mathbf{x}_0)$ at $\mathbf{v}$ shall be denoted by $d\mathbf{f}(\mathbf{x}_0)(\mathbf{v})$ instead of $L_{\mathbf{x}_0}(\mathbf{v})$. If $\mathbf{f}$ is differentiable at each $\mathbf{x}$ of an open set V, then $\mathbf{f}$ is called a *differentiable function.* The *differential* of a differentiable function $\mathbf{f}$ is the function $d\mathbf{f}$ whose value at each $\mathbf{x}$ is $d\mathbf{f}(\mathbf{x})$.

Now suppose that $\mathbf{f}$ is a differentiable function from V (in E) into F. At each $\mathbf{x}$, $d\mathbf{f}$ is linear in $\mathbf{v}$; thus

$$d\mathbf{f}(\mathbf{v}) = d\mathbf{f}(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) = v_1 d\mathbf{f}(\mathbf{e}_1) + \cdots + v_n d\mathbf{f}(\mathbf{e}_n)$$

We saw above that $d\mathbf{f}(\mathbf{e}_i) = L(\mathbf{e}_i) = D_{\mathbf{e}_i}\mathbf{f} = \partial\mathbf{f}/\partial x_i$. Thus

$$d\mathbf{f}(\mathbf{v}) = v_1\frac{\partial\mathbf{f}}{\partial x_1} + \cdots + v_n\frac{\partial\mathbf{f}}{\partial x_n} \tag{7.2}$$

Recall that if $\mathbf{f} = f_1\mathbf{g}_1 + \cdots + f_m\mathbf{g}_m$, then

$$\frac{\partial\mathbf{f}}{\partial x_j} = \frac{\partial f_1}{\partial x_j}\mathbf{g}_1 + \cdots + \frac{\partial f_m}{\partial x_j}\mathbf{g}_m \qquad j = 1, \ldots, n$$

Thus equation (7.2) becomes

$$\begin{aligned} d\mathbf{f}(\mathbf{v}) &= v_1\left(\frac{\partial f_1}{\partial x_1}\mathbf{g}_1 + \cdots + \frac{\partial f_m}{\partial x_1}\mathbf{g}_m\right) + \cdots + v_n\left(\frac{\partial f_1}{\partial x_n}\mathbf{g}_1 + \cdots + \frac{f_m}{x_n}\mathbf{g}_m\right) \\ &= \left(\frac{\partial f_1}{\partial x_1}v_1 + \cdots + \frac{\partial f_1}{\partial x_n}v_n\right)\mathbf{g}_1 + \cdots + \left(\frac{\partial f_m}{\partial x_1}v_1 + \cdots + \frac{f_m}{x_n}v_n\right)\mathbf{g}_m \end{aligned}$$

The components of $d\mathbf{f}(\mathbf{v})$ are the differentials

$$df_1(\mathbf{v}) = \frac{\partial f_1}{\partial x_1}v_1 + \cdots + \frac{\partial f_1}{\partial x_n}v_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$df_m(\mathbf{v}) = \frac{\partial f_m}{\partial x_1}v_1 + \cdots + \frac{\partial f_m}{\partial x_n}v_n$$

and the matrix representation of $d\mathbf{f}$ as a linear function is the m by n matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{7.3}$$

The above matrix is called the *Jacobian matrix* of $\mathbf{f}$ with respect to the bases $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and $(\mathbf{g}_1, \ldots, \mathbf{g}_m)$. If $m = n$, the determinant of the Jacobian matrix of $\mathbf{f}$ is called simply the *Jacobian* of $\mathbf{f}$ and is denoted by $\mathbf{J}(\mathbf{f})$ or $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$, i.e. $\mathbf{J}(\mathbf{f}) = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = \det\left(\frac{\partial f_i}{\partial x_j}\right)$.

It is easy to check that the *coordinate* functions $x_1(\mathbf{x}) = x_1, \ldots, x_n(\mathbf{x}) = x_n$ are differentiable functions from E into E^1 and that their differentials $dx_1, \ldots, dx_n$, which are linear functions from E into E^1, satisfy $dx_i(\mathbf{v}) = v_i$, $i = 1, \ldots, n$, for any vector $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$ in E. Hence from equation (*7.2*),

$$d\mathbf{f}(\mathbf{v}) = v_1\frac{\partial \mathbf{f}}{\partial x_1} + \cdots + v_n\frac{\partial \mathbf{f}}{\partial x_n} = dx_1(\mathbf{v})\frac{\partial \mathbf{f}}{\partial x_1} + \cdots + dx_n(\mathbf{v})\frac{\partial \mathbf{f}}{\partial x_n}$$

which establishes the formula

$$d\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_1}dx_1 + \cdots + \frac{\partial \mathbf{f}}{\partial x_n}dx_n \tag{7.4}$$

In practice we will interpret formula (*7.4*) in a slightly different manner. We interpret dx_i not as the differential of the coordinate function x_i on E but as the ith component of a typical vector $d\mathbf{x}$ in E, and we interpret $d\mathbf{f}$ as the image of $d\mathbf{x}$ under the action of the differential of $\mathbf{f}$. Then (*7.4*) expresses the image $d\mathbf{f}$ in terms of the components dx_i of the generic vector $d\mathbf{x}$ in E.

Example 7.7:

Consider the function from E^3 into E^3,

$$\mathbf{f}(\mathbf{x}) = (2x_1 - x_2)\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + (x_2^2 - x_3^2)\mathbf{e}_3$$

The differential of $\mathbf{f}$ is

$$\begin{aligned} d\mathbf{f} &= \frac{\partial \mathbf{f}}{\partial x_1}dx_1 + \frac{\partial \mathbf{f}}{\partial x_2}dx_2 + \frac{\partial \mathbf{f}}{\partial x_3}dx_3 \\ &= (2\mathbf{e}_1 + x_3\mathbf{e}_2)\,dx_1 + (-\mathbf{e}_1 + 2x_2\mathbf{e}_3)\,dx_2 + (x_1\mathbf{e}_2 - 2x_3\mathbf{e}_3)\,dx_3 \end{aligned}$$

The Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ x_3 & 0 & x_1 \\ 0 & 2x_2 & -2x_3 \end{pmatrix}$$

and the Jacobian of $\mathbf{f}$ at $\mathbf{x}$ is

$$J(\mathbf{f})(\mathbf{x}) \;=\; \det\left(\frac{\partial f_i}{\partial x_j}\right) \;=\; -2(2x_1x_2 + x_3^2)$$

Recall that if $\mathbf{f}$ is a mapping of E^2 into E^3, then the directional derivative $D_{d\mathbf{x}}\mathbf{f}(\mathbf{x}_0) = d\mathbf{f}(\mathbf{x}_0)$ is a vector which is tangent to a curve on the surface $\mathbf{y} = \mathbf{f}(\mathbf{x})$ through $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. In particular, if the rank of $d\mathbf{f}$ as a linear function of $d\mathbf{x}$ at $\mathbf{x}_0$ is 2, i.e. if the rank of the Jacobian matrix $(\partial f_i/\partial x_j)$ at $\mathbf{x}_0$ is 2, it follows from Theorem 7.2(i) that $d\mathbf{f}(\mathbf{x}_0)$ is a mapping of the vectors $d\mathbf{x}$ in E^2 one-to-one onto a plane as shown in Fig. 7-5. The plane through $\mathbf{y}_0$ parallel to the vectors $d\mathbf{f}(\mathbf{x}_0)$ is called the tangent plane to $\mathbf{y} = \mathbf{f}(\mathbf{x})$ at $\mathbf{y}_0$. The equation of the tangent plane is $\mathbf{y} = \mathbf{y}_0 + d\mathbf{f}(\mathbf{x}_0)$.

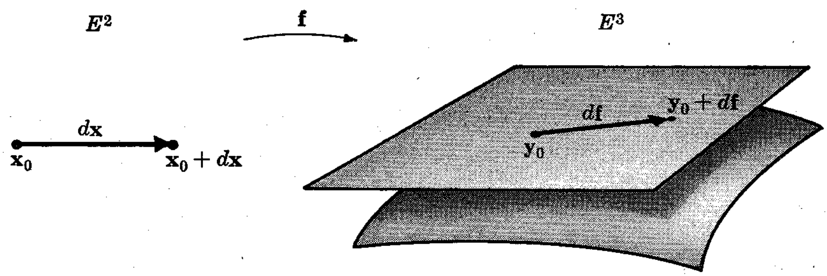


Fig. 7-5

Example 7.8:

Consider the mapping of E^2 into E^3 given by

$$\mathbf{y} \;=\; (u-v)\mathbf{g}_1 + (u+v)\mathbf{g}_2 + (u^2+v^2)\mathbf{g}_3$$

The differential

$$d\mathbf{y} \;=\; \frac{\partial \mathbf{y}}{\partial u}du + \frac{\partial \mathbf{y}}{\partial v}dv \;=\; (\mathbf{g}_1 + \mathbf{g}_2 + 2u\mathbf{g}_3)\,du \;+\; (-\mathbf{g}_1 + \mathbf{g}_2 + 2v\mathbf{g}_3)\,dv$$
$$\;=\; (du - dv)\mathbf{g}_1 \;+\; (du + dv)\mathbf{g}_2 \;+\; (2u\,du + 2v\,dv)\mathbf{g}_3$$

At $u = 1,\; v = -1,$ we have $\mathbf{y}_0 = 2\mathbf{g}_1 + 2\mathbf{g}_3$, and

$$d\mathbf{y}_0 \;=\; (du - dv)\mathbf{g}_1 + (du + dv)\mathbf{g}_2 + (2du - 2dv)\mathbf{g}_3$$

Hence the tangent plane at $\mathbf{y}_0$ is

$$\mathbf{y} \;=\; \mathbf{y}_0 + d\mathbf{y}_0 \;=\; (2 + du - dv)\mathbf{g}_1 + (du + dv)\mathbf{g}_2 + (2 + 2du - 2dv)\mathbf{g}_3$$

or

$$y_1 \;=\; 2 + du - dv, \quad y_2 \;=\; du + dv, \quad y_3 \;=\; 2 + 2du - 2dv$$

or, eliminating du and dv, $2y_1 - y_3 = 2$.

As in the case of derivatives of functions of a single variable, we have

Theorem 7.7. Let $\mathbf{f}$ and $\mathbf{g}$ be vector functions from an open set V (in E) into F, and h a scalar function on V. If $\mathbf{f}$, $\mathbf{g}$ and h are differentiable at $\mathbf{x}$, then $\mathbf{f}+\mathbf{g}$, $h\mathbf{f}$, $\mathbf{f}\cdot\mathbf{g}$ and $\mathbf{f}\times\mathbf{g}$ are differentiable at $\mathbf{x}$ and

(i) $d(\mathbf{f}+\mathbf{g}) \;=\; d\mathbf{f} + d\mathbf{g}$

(ii) $dh\mathbf{f} \;=\; h d\mathbf{f} + (dh)\mathbf{f}$

(iii) $d(\mathbf{f}\cdot\mathbf{g}) \;=\; \mathbf{f}\cdot d\mathbf{g} + (d\mathbf{f})\cdot\mathbf{g}$

(iv) $d(\mathbf{f}\times\mathbf{g}) \;=\; \mathbf{f}\times d\mathbf{g} + (d\mathbf{f})\times\mathbf{g}$

Finally, if $\mathbf{f}$ is differentiable at each $\mathbf{x}$ in V, then $d\mathbf{f}$ is a function of two vector variables, $\mathbf{x}$ and $d\mathbf{x}$. If $d\mathbf{f}$ is continuous in $\mathbf{x}$ and $d\mathbf{x}$, then $\mathbf{f}$ is said to be *continuously differentiable* in V

If $\mathbf{f}$ is continuously differentiable in V, then the $\partial\mathbf{f}/\partial x_i$ are continuous in V, and hence all partial derivatives $\partial f_i/\partial x_j$ are continuous in V. The converse is also true. That is,

Theorem 7.8. A vector function $\mathbf{f}$ on an open set V (in E) into F is continuously differentiable in V if and only if with respect to bases in E and F all partial derivatives $\partial f_i/\partial x_j$, $i = 1, \ldots, m$; $j = 1, \ldots, n$ are continuous in V.

COMPOSITE FUNCTIONS. CHAIN RULE

Let E, F and G be Euclidean spaces. Let $\mathbf{f}$ be a function from a set V (in E) into F, and $\mathbf{g}$ a function from a set U (in F) into G. For all $\mathbf{x}$ in V such that $\mathbf{f}(\mathbf{x})$ is in U, the *composite* function $\mathbf{g}\circ\mathbf{f}$ of $\mathbf{x}$ in E into G is defined by $(\mathbf{g}\circ\mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$.

Example 7.9:

(*a*)
$$\mathbf{f}(\mathbf{x}) = (x_1 - x_2)\mathbf{g}_1 + (x_1^2 + x_2^2)\mathbf{g}_2 + (x_1x_2)\mathbf{g}_3$$

takes $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ (in E^2) into E^3; and

$$\mathbf{g}(\mathbf{y}) = (y_1 + y_2 + y_3)\mathbf{g}_1 + y_1y_2\mathbf{g}_2 + (y_2^2 - y_3^2)\mathbf{g}_3$$

takes $\mathbf{y} = y_1\mathbf{g}_1 + y_2\mathbf{g}_2 + y_3\mathbf{g}_3$ in E^3 into E^3. If we set

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = (x_1 - x_2)\mathbf{g}_1 + (x_1^2 + x_2^2)\mathbf{g}_2 + x_1x_2\mathbf{g}_3$$

so that
$$y_1 = x_1 - x_2, \quad y_2 = x_1^2 + x_2^2, \quad y_3 = x_1x_2$$

we see that the composite function $\mathbf{g}(\mathbf{f}(\mathbf{x}))$, which is a function taking $\mathbf{x}$ (in E^2) into E^3, is

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = [(x_1 - x_2) + (x_1^2 + x_2^2) + x_1x_2]\mathbf{g}_1 + (x_1 - x_2)(x_1^2 + x_2^2)\mathbf{g}_2 + [(x_1^2 + x_2^2)^2 - (x_1x_2)^2]\mathbf{g}_3$$

(*b*) The equations
$$\mathbf{x} = \mathbf{f}(t) = (t^2 + 1)\mathbf{e}_1 + t\mathbf{e}_2, \quad \text{or} \quad x_1 = t^2 + 1, \quad x_2 = t$$

define a mapping of E^1 into E^2 and represent a curve in E^2. The equation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) = x_1\mathbf{g}_1 + x_2\mathbf{g}_2 + (x_1^2 + x_2^2)\mathbf{g}$$

defines a mapping of E^2 into E^3 and represents a surface in E^3. The composite mapping

$$\mathbf{y} = \mathbf{g}(\mathbf{f}(t)) = (t^2 + 1)\mathbf{g}_1 + t\mathbf{g}_2 + [(t^2 + 1)^2 + t^2]\mathbf{g}_3 = (t^2 + 1)\mathbf{g}_1 + t\mathbf{g}_2 + (t^4 + 3t^2 + 1)\mathbf{g}_3$$

is a mapping of E^1 into E^3 and represents a curve on the surface $\mathbf{y} = \mathbf{g}(\mathbf{x})$ in E^3, as shown in Fig. 7-6.

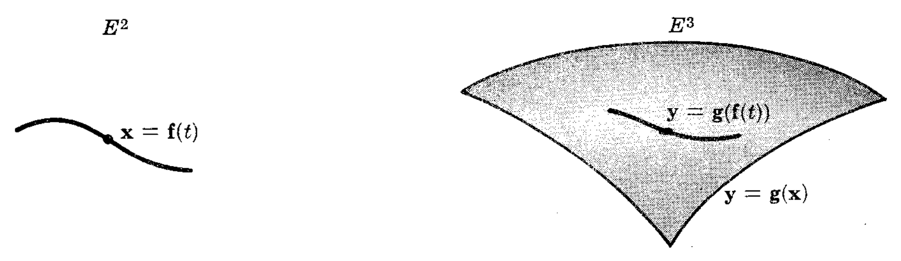


Fig. 7-6

If $\mathbf{f}$ is a linear function from E into F and $\mathbf{g}$ is a linear function from F into G, then the composite function $\mathbf{h} = \mathbf{g}\circ\mathbf{f}$ is a linear function from E into G. For, let $\mathbf{a}$ and $\mathbf{b}$ be vectors in E; then

$$\mathbf{h}(\mathbf{a}+\mathbf{b}) = \mathbf{g}(\mathbf{f}(\mathbf{a}+\mathbf{b})) = \mathbf{g}[\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})] = \mathbf{g}(\mathbf{f}(\mathbf{a})) + \mathbf{g}(\mathbf{f}(\mathbf{b})) = \mathbf{h}(\mathbf{a}) + \mathbf{h}(\mathbf{b})$$

Also, $$\mathbf{h}(k\mathbf{a}) = \mathbf{f}(\mathbf{g}(k\mathbf{a})) = \mathbf{f}(k\mathbf{g}(\mathbf{a})) = k\mathbf{f}(\mathbf{g}(\mathbf{a})) = k\mathbf{h}(\mathbf{a})$$

Thus $\mathbf{h}$ is linear on E. Suppose further that bases are fixed in E, F and G, and that the components of $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{z} = \mathbf{g}(\mathbf{y})$ are given by the linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

and

$$\begin{aligned} z_1 &= b_{11}y_1 + b_{12}y_2 + \cdots + b_{1m}y_m \\ z_2 &= b_{21}y_1 + b_{22}y_2 + \cdots + b_{2m}y_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_r &= b_{r1}y_1 + b_{r2}y_2 + \cdots + b_{rm}y_m \end{aligned}$$

or, in short,

$$y_k = \sum_{j=1}^{n} a_{kj}x_j, \qquad k = 1, \ldots, m$$

$$z_i = \sum_{k=1}^{m} b_{ik}y_k, \qquad i = 1, \ldots, r$$

Substituting, we find that the components of $\mathbf{z} = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ are

$$z_i = \sum_{k=1}^{m} b_{ik}\left(\sum_{j=1}^{n} a_{kj}x_j\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{m} b_{ik}a_{kj}\right) x_j, \qquad i = 1, \ldots, r$$

If (a_{ij}) and (b_{ij}) are the matrix representations of $\mathbf{f}$ and $\mathbf{g}$ respectively, we recognize that the matrix representation $(c_{ij}) = \left(\sum_{k=1}^{m} b_{ik}a_{kj}\right)$ of the composite function is the matrix product $(c_{ij}) = (b_{ij})(a_{ij})$.

Example 7.10:

The equations

$$\begin{aligned} y_1 &= 2x_1 + x_2 \\ y_2 &= -x_1 + 2x_2 \\ y_3 &= x_1 - x_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 &= y_1 - y_2 + y_3 \\ z_2 &= 2y_1 + y_2 - y_3 \end{aligned}$$

define linear mappings of E^2 into E^3 and back again into E^2. The composite mapping is a linear mapping of E^2 into E^2 given by

$$\begin{aligned} z_1 &= (2x_1 + x_2) - (-x_1 + 2x_2) + (x_1 - x_2) = 4x_1 - 2x_2 \\ z_2 &= 2(2x_1 + x_2) + (-x_1 + 2x_2) - (x_1 - x_2) = 2x_1 + 5x_2 \end{aligned}$$

Observe that the matrix representation of $\mathbf{z}$ as a function of $\mathbf{x}$ is the product

$$\begin{pmatrix} 4 & -2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & -1 \end{pmatrix}$$

If $\mathbf{f}$ is continuous at $\mathbf{x}$, and $\mathbf{g}$ is continuous at $\mathbf{y} = \mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$. Namely,

Theorem 7.9. The composite of two continuous functions is continuous.

The same is true for differentiable functions. This, together with the *chain rule*, i.e. the rule for determining the derivatives of the composite functions, is given in the following important theorem:

Theorem 7.10. If $\mathbf{f}$ is differentiable at $\mathbf{x}$ with differential $\mathbf{L}_\mathbf{x}(\mathbf{v})$ and $\mathbf{g}$ is differentiable at $\mathbf{y} = \mathbf{f}(\mathbf{x})$ with differential $\mathbf{M}_\mathbf{y}(\mathbf{u})$, then $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ with differential equal to the composite of the differentials

$$\mathbf{H}_\mathbf{x}(\mathbf{v}) = (\mathbf{M} \circ \mathbf{L})_\mathbf{x}(\mathbf{v}) = \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{L}_\mathbf{x}(\mathbf{v}))$$

We recall that the matrix representation of the differential of $\mathbf{f}$ is the Jacobian matrix $(\partial f_i/\partial x_j)$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, and the matrix representation of the differential of $\mathbf{g}$ is $(\partial g_i/\partial y_j)$, $i = 1, \ldots, r$; $u = 1, \ldots, m$. It follows that the matrix representation of the differential of $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is the product

$$\left(\frac{\partial h_i}{\partial x_j}\right) = \left(\frac{\partial g_i}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial x_j}\right) = \left(\sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}\right)$$

Thus the formula

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial g_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial g_i}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial y_m}\frac{\partial y_m}{\partial x_j}, \qquad i = 1, \ldots, r; \; j = 1, \ldots, n \tag{7.5}$$

which is the chain rule for computing the partial derivatives of the components of composite functions. If $\mathbf{h} = h_1\mathbf{e}_1 + \cdots + h_n\mathbf{e}_r$, then

$$\begin{aligned}
\frac{\partial \mathbf{h}}{\partial x_j} &= \frac{\partial h_1}{\partial x_i}\mathbf{e}_1 + \cdots + \frac{\partial h_r}{\partial x_i}\mathbf{e}_r \\
&= \left(\frac{\partial g_1}{\partial y_i}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial g_1}{\partial y_m}\frac{\partial y_m}{\partial x_j}\right)\mathbf{e}_1 + \cdots + \left(\frac{\partial g_r}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial g_r}{\partial y_m}\frac{\partial y_m}{\partial x_j}\right)\mathbf{e}_r \\
&= \left(\frac{\partial g_1}{\partial y_1}\mathbf{e}_1 + \cdots + \frac{\partial g_r}{\partial y_1}\mathbf{e}_r\right)\frac{\partial y_1}{\partial x_j} + \cdots + \left(\frac{\partial g_1}{\partial y_m}\mathbf{e}_1 + \cdots + \frac{\partial g_r}{\partial y_m}\mathbf{e}_r\right)\frac{\partial y_m}{\partial x_j}
\end{aligned}$$

Thus

$$\frac{\partial \mathbf{h}}{\partial x_j} = \frac{\partial \mathbf{g}}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial \mathbf{g}}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial \mathbf{g}}{\partial y_m}\frac{\partial y_m}{\partial x_j} \tag{7.6}$$

which is the chain rule for the partial derivatives of the composite vector function itself.

Example 7.11.

(a) Let $\mathbf{y} = (x_1 + x_2)\mathbf{e}_1 + (x_1 - x_2)\mathbf{e}_2 + (x_1^2 + x_2^2)\mathbf{e}_3$, $x_1 = t^2 + 1$, $x_2 = \sin t$. Then

$$\begin{aligned}
\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{y}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{y}}{\partial x_2}\frac{dx_2}{dt} &= (\mathbf{e}_1 + \mathbf{e}_2 + 2x_1\mathbf{e}_3)(2t) + (\mathbf{e}_1 - \mathbf{e}_2 + 2x_2\mathbf{e}_3)\cos t \\
&= (2t + \cos t)\mathbf{e}_1 + (2t - \cos t)\mathbf{e}_2 + (4t^3 + 4t + 2\sin t\cos t)\mathbf{e}_3
\end{aligned}$$

(b) Let $\mathbf{y} = u(\cos v)\mathbf{e}_1 + u(\sin v)\mathbf{e}_2 + v\mathbf{e}_3$, $u = \theta + \phi$, $v = \theta\phi$. Then

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial \theta} = \frac{\partial \mathbf{y}}{\partial u}\frac{\partial u}{\partial \theta} + \frac{\partial \mathbf{y}}{\partial v}\frac{\partial v}{\partial \theta} &= (\cos v)\mathbf{e}_1 + (\sin v)\mathbf{e}_2 + (-u(\sin v)\mathbf{e}_1 + u(\cos v)\mathbf{e}_2 + \mathbf{e}_3)\phi \\
&= (\cos(\theta\phi) - \phi(\theta + \phi)\sin(\theta\phi))\mathbf{e}_1 + (\sin(\theta\phi) + \phi(\theta + \phi)\cos(\theta\phi))\mathbf{e}_2 + \phi\mathbf{e}_3
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial \phi} = \frac{\partial \mathbf{y}}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial \mathbf{y}}{\partial v}\frac{\partial v}{\partial \phi} &= (\cos v)\mathbf{e}_1 + (\sin v)\mathbf{e}_2 + (-u(\sin v)\mathbf{e}_1 + u(\cos v)\mathbf{e}_2 + \mathbf{e}_3)\theta \\
&= (\cos(\theta\phi) - \theta(\theta + \phi)\sin(\theta\phi))\mathbf{e}_1 + (\sin(\theta\phi) + \theta(\theta + \phi)\cos(\theta\phi))\mathbf{e}_2 + \theta\mathbf{e}_3
\end{aligned}$$

FUNCTIONS OF CLASS C^m. TAYLOR'S FORMULA

Suppose $\mathbf{f}$ has a derivative in a fixed direction $\mathbf{u}$ at each $\mathbf{x}$ in an open set V in E. Then $D_{\mathbf{u}}\mathbf{f}$ itself is a function of $\mathbf{x}$ in V and we can consider its derivative in a direction $\mathbf{v}$ at $\mathbf{x}$. Namely, $D_{\mathbf{v}}(D_{\mathbf{u}}\mathbf{f})(\mathbf{x})$, if it exists, is called a *second order directional derivative of* $\mathbf{f}$ *at* $\mathbf{x}$ and is denoted by $D^2_{\mathbf{vu}}\mathbf{f}(\mathbf{x})$.

If $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a basis in E, we recall that

$$D_{\mathbf{e}_i}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_i} = \frac{\partial f_1}{\partial x_i}\mathbf{g}_1 + \frac{\partial f_2}{\partial x_i}\mathbf{g}_2 + \cdots + \frac{\partial f_m}{\partial x_i}\mathbf{g}_m$$

Hence

$$D^2_{\mathbf{e}_j\mathbf{e}_i}\mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x_j\,\partial x_i} = \frac{\partial^2 f_1}{\partial x_j\,\partial x_i}\mathbf{g}_1 + \frac{\partial^2 f_2}{\partial x_j\,\partial x_i}\mathbf{g}_2 + \cdots + \frac{\partial^2 f_m}{\partial x_j\,\partial x_i}\mathbf{g}_m$$

That is, the components of the second order derivatives in the direction of the basis are the second order partial derivatives of the components.

Higher order derivatives are defined in the same way. For example, $D_{\mathbf{w}}(D^2_{\mathbf{vu}}\mathbf{f})(\mathbf{x})$ is a third order directional derivative at $\mathbf{x}$ and is denoted by $D^3_{\mathbf{wvu}}f(\mathbf{x})$. Here again the derivatives in the direction of the basis are the partial derivatives,

$$D^3_{\mathbf{e}_k\mathbf{e}_j\mathbf{e}_i}\mathbf{f} \;=\; \frac{\partial^3\mathbf{f}}{\partial x_k\,\partial x_j\,\partial x_i} \;=\; \frac{\partial^3 f_1}{\partial x_k\,\partial x_j\,\partial x_i}\mathbf{g}_1 + \cdots + \frac{\partial^3 f_m}{\partial x_k\,\partial x_j\,\partial x_i}\mathbf{g}_m$$

If $\mathbf{f}$ is continuous in V, we say $\mathbf{f}$ belongs to class C^0 in V. If with respect to a basis in E, the derivatives $\frac{\partial\mathbf{f}}{\partial x_1}, \frac{\partial\mathbf{f}}{\partial x_2}, \ldots, \frac{\partial\mathbf{f}}{\partial x_n}$ are all continuous in V, then $\mathbf{f}$ is said to belong to class C^1 in V. In general, $\mathbf{f}$ is of class C^m in V if all mth order derivatives $\frac{\partial^m\mathbf{f}}{\partial x_{i_1}\ldots\partial x_{i_m}}$ exist and are continuous in V.

Note that it follows from Theorems 7.6 and 7.8 that functions in class C^1 are in class C^0. But then functions in class C^2 are in class C^1. In general, functions of class C^m are in class C^{m-1}.

Example 7.12.

The real-valued function $f(\mathbf{x}) = |\mathbf{x}|^{8/3}$ belongs to C^2 for all $\mathbf{x}$ in E but not to C^3 for all $\mathbf{x}$ in E. For,

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{8}{3}|\mathbf{x}|^{5/3}\frac{\partial}{\partial x_i}|\mathbf{x}| = \frac{8}{3}|\mathbf{x}|^{5/3}\,|\mathbf{x}|^{-1}x_i = \frac{8}{3}|\mathbf{x}|^{2/3}x_i = \frac{8}{3}|\mathbf{x}|^{5/3}\cos\measuredangle(\mathbf{x},\mathbf{e}_i)$$

where we used $x_i = |\mathbf{x}|\cos\measuredangle(\mathbf{x},\mathbf{e}_i)$. Similarly,

$$\frac{\partial^2 \mathbf{f}(\mathbf{x})}{\partial x_j\,\partial x_i} = \frac{8}{3}|\mathbf{x}|^{2/3}\delta_{ij} + \frac{16}{9}|\mathbf{x}|^{2/3}\cos\measuredangle(\mathbf{x},\mathbf{e}_j)\cos\measuredangle(\mathbf{x},\mathbf{e}_i)$$

A third order derivative, however, will contain the term

$$\frac{64}{27}|\mathbf{x}|^{-1/3}\cos\measuredangle(\mathbf{x},\mathbf{e}_k)\cos\measuredangle(\mathbf{x},\mathbf{e}_j)\cos\measuredangle(\mathbf{x},\mathbf{e}_i)$$

which is not bounded at $\mathbf{x} = \mathbf{0}$.

Note that it is possible that $\frac{\partial\mathbf{f}}{\partial x_1\,\partial x_2} \neq \frac{\partial\mathbf{f}}{\partial x_2\,\partial x_1}$. However, this will not happen if $\mathbf{f}$ is of class C^2. In general, if $\mathbf{f}$ is of class C^m, then in any mth order partial derivative only the number of differentiations with respect to each variable matters and not their order. For a proof of this we refer the reader to a text in advanced calculus.

Now suppose $\mathbf{f}$ is of class C^2 in V. Then from Theorem 7.8, $\mathbf{f}$ is differentiable in V; and from equation *(7.2)*,

$$D_{\mathbf{u}}\mathbf{f} = \frac{\partial\mathbf{f}}{\partial x_1}u_1 + \cdots + \frac{\partial\mathbf{f}}{\partial x_n}u_n$$

But also $D_{\mathbf{u}}\mathbf{f}$ is differentiable in V. Hence

$$\begin{aligned} D^2_{\mathbf{vu}}\mathbf{f} &= D_{\mathbf{v}}(D_{\mathbf{u}}\mathbf{f}) = \left(\frac{\partial}{\partial x_1}D_{\mathbf{u}}\mathbf{f}\right)v_1 + \cdots + \left(\frac{\partial}{\partial x_n}D_{\mathbf{u}}\mathbf{f}\right)v_n \\ &= \left[\frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_1}u_1 + \cdots + \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_n}u_n\right]v_1 + \cdots + \left[\frac{\partial^2\mathbf{f}}{\partial x_n\,\partial x_1}u_1 + \cdots + \frac{\partial^2\mathbf{f}}{\partial x_n\,\partial x_n}u_n\right]v_n \end{aligned}$$

or,

$$D^2_{\mathbf{vu}}\mathbf{f} = \sum_{i,j=1}^{n}\frac{\partial^2\mathbf{f}}{\partial x_i\,\partial x_j}v_iu_j \qquad (7.7)$$

Similar formulas hold for higher order derivatives. For example,

$$D^3_{\mathbf{wvu}}\mathbf{f} = \sum_{i,j,k=1}^{n}\frac{\partial^3\mathbf{f}}{\partial x_i\,\partial x_j\,\partial x_k}w_iv_ju_k$$

Thus we have

Theorem 7.11. If $\mathbf{f}$ belongs to class C^m in V, then all mth order directional derivatives are continuous in V, independent of the order of differentiation, and are given by

$$D^m_{\mathbf{u}_1\cdots\mathbf{u}_m}\mathbf{f} = \sum_{i_1,\ldots,i_m=1}^{n} \frac{\partial^m \mathbf{f}}{\partial x_{i_1}\cdots\partial x_{i_m}}(u_1)_{i_1}(u_2)_{i_2}\cdots(u_m)_{i_m}$$

Example 7.13.

Suppose $$\mathbf{f}(\mathbf{x}) = x_1x_2\mathbf{g}_1 + (x_1^3+x_2^3)\mathbf{g}_2 + x_1^3x_2\mathbf{g}_3$$

Then $$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial x_1}u_1 + \frac{\partial \mathbf{f}}{\partial x_2}u_2 = (x_2\mathbf{g}_1 + 3x_1^2\mathbf{g}_2 + 3x_1^2x_2\mathbf{g}_3)u_1 + (x_1\mathbf{g}_1 + 3x_2^2\mathbf{g}_2 + x_1^3\mathbf{g}_3)u_2$$

$$\begin{aligned} D^2_{\mathbf{u}^2}\mathbf{f}(\mathbf{x}) &= \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_1}u_1^2 + \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}u_1u_2 + \frac{\partial^2\mathbf{f}}{\partial x_2\,\partial x_1}u_2u_1 + \frac{\partial^2\mathbf{f}}{\partial x_2\,\partial x_2}u_2^2 \\ &= (6x_1\mathbf{g}_2 + 6x_1x_2\mathbf{g}_3)u_1^2 + 2(\mathbf{g}_1 + 3x_1^2\mathbf{g}_3)u_1u_2 + 6x_2\mathbf{g}_2u_2^2 \end{aligned}$$

$$\begin{aligned} D^2_{\mathbf{uv}}\mathbf{f}(\mathbf{x}) &= \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_1}u_1v_1 + \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}u_1v_2 + \frac{\partial^2\mathbf{f}}{\partial x_2\,\partial x_1}u_2u_1 + \frac{\partial^2\mathbf{f}}{\partial x_2\,\partial x_2}u_2v_2 \\ &= (6x_1\mathbf{g}_2 + 6x_1x_2\mathbf{g}_3)u_1v_1 + (\mathbf{g}_1 + 3x_1^2\mathbf{g}_3)(u_1v_2 + v_1u_2) + 6x_2\mathbf{g}_2u_2v_2 \end{aligned}$$

$$\begin{aligned} D^3_{\mathbf{u}^2\mathbf{v}}\mathbf{f}(\mathbf{x}) &= \frac{\partial^3\mathbf{f}}{\partial x_1\,\partial x_1\,\partial x_1}u_1^2v_1 + \frac{\partial^3\mathbf{f}}{\partial x_1\,\partial x_1\,\partial x_2}u_1u_1v_2 + \frac{\partial^3\mathbf{f}}{\partial x_1\,\partial x_2\,\partial x_1}u_1u_2v_1 + \frac{\partial^3\mathbf{f}}{\partial x_2\,\partial x_1\,\partial x_1}u_2u_1v_1 \\ &\quad + \frac{\partial^3\mathbf{f}}{\partial x_1\,\partial x_2\,\partial x_2}u_1u_2v_2 + \frac{\partial^3\mathbf{f}}{\partial x_2\,\partial x_1\,\partial x_2}u_2u_1v_2 + \frac{\partial^3\mathbf{f}}{\partial x_2\,\partial x_2\,\partial x_1}u_2u_2v_1 + \frac{\partial^3\mathbf{f}}{\partial x_2\,\partial x_2\,\partial x_2}u_2u_2v_2 \\ &= \frac{\partial^3\mathbf{f}}{\partial x_1^3}u_1^2v_1 + \frac{\partial^3\mathbf{f}}{\partial x_1^2\,\partial x_2}(u_1^2v_2 + 2u_1u_2v_1) + \frac{\partial^3\mathbf{f}}{\partial x_1\,\partial x_2^2}(2u_1u_2v_2 + u_2^2v_1) + \frac{\partial^3\mathbf{f}}{\partial x_2^3}u_2^2v_2 \\ &= (6\mathbf{g}_2 + 6x_2\mathbf{g}_3)u_1^2v_1 + 6x_1\mathbf{g}_3(u_1^2v_2 + 2u_1u_2v_1) + 6\mathbf{g}_2u_2^2v_2 \end{aligned}$$

Theorem 7.12. *Taylor's Formula.* If $\mathbf{f}$ is of class C^m in a neighborhood of $\mathbf{x}_0$, then for $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + D_{(\mathbf{x}-\mathbf{x}_0)}\mathbf{f}(\mathbf{x}_0) + \cdots + \frac{1}{m!}D^m_{(\mathbf{x}-\mathbf{x}_0)^m}\mathbf{f}(\mathbf{x}_0) + \mathbf{R}_m(\mathbf{x},\mathbf{x}_0)$$

where $\mathbf{R}_m(\mathbf{x},\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|^m \to \mathbf{0}$ as $\mathbf{x}\to\mathbf{x}_0$.

Example 7.14.

Let $\mathbf{f}(x_1,x_2) = (x_1^2+x_2^3)\mathbf{e}_1 + x_1^3x_2\mathbf{e}_2$. Then

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{x}) = \frac{\partial\mathbf{f}}{\partial x_1}v_1 + \frac{\partial\mathbf{f}}{\partial x_2}v_2 = (2x_1\mathbf{e}_1 + 3x_1^2x_2\mathbf{e}_2)v_1 + (3x_2^2\mathbf{e}_1 + x_1^3\mathbf{e}_2)v_2$$

$$D^2_{\mathbf{v}^2}\mathbf{f}(\mathbf{x}) = \frac{\partial^2\mathbf{f}}{\partial x_1^2}v_1^2 + 2\frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}v_1v_2 + \frac{\partial^2\mathbf{f}}{\partial x_2^2}v_2^2 = (2\mathbf{e}_1 + 6x_1x_2\mathbf{e}_2)v_1^2 + 6x_1^2\mathbf{e}_2v_1v_2 + 6x_2\mathbf{e}_1v_2^2$$

At $\mathbf{x}_0 = \mathbf{e}_1 - \mathbf{e}_2$, we have $\mathbf{x}-\mathbf{x}_0 = (x_1-1)\mathbf{e}_1 + (x_2+1)\mathbf{e}_2$, $\mathbf{f}(\mathbf{x}_0) = -\mathbf{e}_2$, $D_{(\mathbf{x}-\mathbf{x}_0)}\mathbf{f}(\mathbf{x}_0) = (2\mathbf{e}_1 - 3\mathbf{e}_2)(x_1-1) + (3\mathbf{e}_1+\mathbf{e}_2)(x_2+1)$, $D^2_{(\mathbf{x}-\mathbf{x}_0)^2}\mathbf{f}(\mathbf{x}_0) = (2\mathbf{e}_1 - 6\mathbf{e}_2)(x_1-1)^2 + 6\mathbf{e}_2(x_1-1)(x_2+1) - 6\mathbf{e}_1(x_2+1)^2$. Thus

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}_0) + D_{(\mathbf{x}-\mathbf{x}_0)}\mathbf{f}(\mathbf{x}_0) + \tfrac{1}{2}D^2_{(\mathbf{x}-\mathbf{x}_0)^2}\mathbf{f}(\mathbf{x}_0) + \mathbf{o}(|\mathbf{x}-\mathbf{x}_0|^2) \\ &= -\mathbf{e}_2 + (2\mathbf{e}_1 - 3\mathbf{e}_2)(x_1-1) + (3\mathbf{e}_1+\mathbf{e}_2)(x_2+1) + (\mathbf{e}_1 - 3\mathbf{e}_2)(x_1-1)^2 \\ &\qquad + 3\mathbf{e}_2(x_1-1)(x_2+1) - 3\mathbf{e}_1(x_2+1)^2 + \mathbf{o}((x_1-1)^2 + (x_2+1)^2) \end{aligned}$$

This result can also be obtained directly by expanding $\mathbf{f}$ in powers of (x_1-1) and (x_2+1) as follows. We have

$$\begin{aligned} x_1^2 &= [(x_1-1)+1]^2 = (x_1-1)^2 + 2(x_1-1) + 1 \\ x_2^3 &= [(x_2+1)-1]^3 = (x_2+1)^3 - 3(x_2+1)^2 + 3(x_2+1) - 1 \\ x_1^3x_2 &= [(x_1-1)+1]^3[(x_2+1)-1] = [(x_1-1)^3 + 3(x_1-1)^2 + 3(x_1-1) + 1][(x_2+1)-1] \end{aligned}$$

and so

$$\mathbf{f}(\mathbf{x}) = (x_1^2 + x_2^3)\mathbf{e}_1 + x_1^3 x_2 \mathbf{e}_2 = \{(x_1-1)^2 + 2(x_1-1) + 1 - 3(x_2+1)^2 + 3(x_2+1) - 1\}\mathbf{e}_1$$
$$+ \{3(x_1-1)(x_2+1) + (x_2+1) - 3(x_1-1)^2 - 3(x_1-1) - 1\}\mathbf{e}_2$$
$$+ \{(x_2+1)^3\mathbf{e}_1 + [(x_1-1)^3(x_2+1) + 3(x_1-1)^2(x_2-1)]\mathbf{e}_2\}$$

Note that the last term is indeed $\mathbf{o}((x_1-1)^2 + (x_2+1)^2)$ and the result agrees with the first result. The second method for obtaining the Taylor expansion can always be applied when the components of $\mathbf{f}$ are polynomials, but in general it is necessary to apply the formula of Theorem 7.12.

INVERSE FUNCTION THEOREM

Let $\mathbf{f}$ be a function of $\mathbf{x}$ from an open set V in E^3 into E^3. In general, such a function will not be 1-1. However, suppose that at a point $\mathbf{x}_0$ in V we have the Jacobian $J(\mathbf{f}) = \det(\partial f_i/\partial x_j) \neq 0$. We recall that $(\partial f_i/\partial x_j)$ is the matrix representation of the differential $d\mathbf{f}$ as a linear function. Since $\det(\partial f_i/\partial x_j) \neq 0$ at $\mathbf{x}_0$, the rank of $d\mathbf{f}(\mathbf{x}_0)$ is 3. Hence from Theorem 7.1(i), $d\mathbf{f}(\mathbf{x}_0)$ is 1-1. But $\mathbf{f}(\mathbf{x})$ is approximated by $\mathbf{f}(\mathbf{x}_0) + d\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)$ near $\mathbf{x}_0$. Thus we expect that $\mathbf{f}$ will be 1-1 at least in some neighborhood of $\mathbf{x}_0$. This result is part of the following important theorem, called the *inverse function theorem*:

Theorem 7.13. Let $\mathbf{f}$ be a function of $\mathbf{x}$ of class C^m $(m \geqq 1)$ on an open set V (in E) into E. Let the Jacobian $J(\mathbf{f}) = \det(\partial f_i/\partial x_j) \neq 0$ at a point $\mathbf{x}_0$ in V. Then there exists a neighborhood $S(\mathbf{x}_0)$ contained in V such that:

(i) The restriction of $\mathbf{f}$ to $S(x_0)$ is 1-1.

(ii) The image $\mathbf{f}(S(\mathbf{x}_0))$ of $S(\mathbf{x}_0)$ is open.

(iii) The inverse $\mathbf{f}^{-1}$ of $\mathbf{f}$ is of class C^m on $\mathbf{f}(S(\mathbf{x}_0))$. (See Fig. 7-7.)

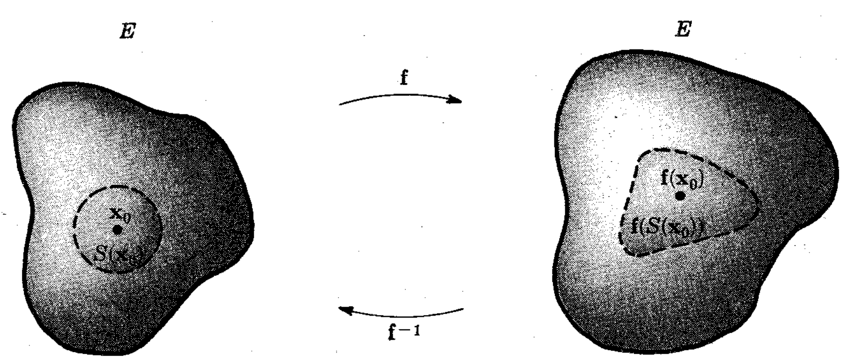


Fig. 7-7

Here again we refer the reader to a text in advanced calculus for a proof.

Example 7.15.

The equations $y_1 = x_1 \cos x_2, \quad y_2 = x_1 \sin x_2, \quad (x_1 > 0)$

define a mapping of the right half of the x_1x_2 plane onto the y_1y_2 plane less its origin, as indicated in Fig. 7-8(a) below. For $x_1 > 0$,

$$J(\mathbf{f}) = \det\begin{pmatrix}\frac{\partial y_i}{\partial x_j}\end{pmatrix} = \det\begin{pmatrix}\cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2\end{pmatrix} = x_1 \neq 0$$

Hence about any point (x_1, x_2) in the right half plane there will be an $S(x_1, x_2)$ in which the mapping is 1-1 and onto an open set in the y_1y_2 plane. The inverse mapping is given by

$$x_1 = (y_1^2 + y_2^2)^{1/2}, \quad x_2 = \tan^{-1}(y_2/y_1)$$

where the appropriate branch of the arc tangent must be taken. Note that the mapping of the right half plane itself is not 1-1 although $\det(\partial y_i/\partial x_j) \neq 0$ at each point, since just the half strip $0 \leq x_2 < 2\pi$ covers all of the y_1y_2 plane less the origin as shown in Fig. 7-8(*b*). Thus the inverse function theorem provides us with a *local* inverse but in general not a *global* inverse.

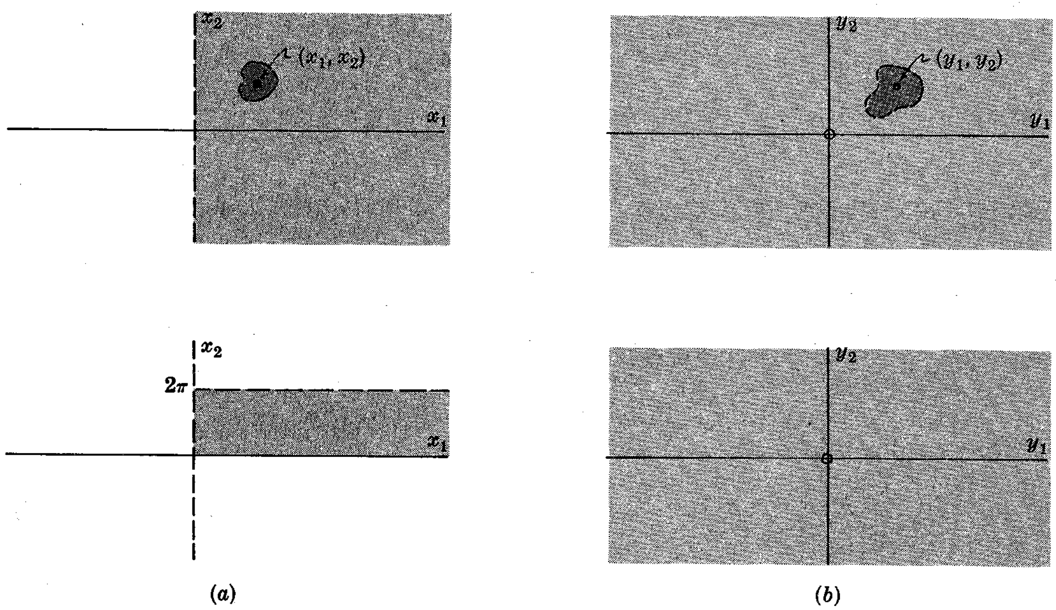


Fig. 7-8

Solved Problems

VECTOR FUNCTIONS. LINEAR FUNCTIONS

7.1. Show that

$$\mathbf{y} = (\sin x_1 \cos x_2)\mathbf{g}_1 + (\sin x_1 \sin x_2)\mathbf{g}_2 + (\cos x_1)\mathbf{g}_3$$

defines a mapping of the region $0 \leq x_1 \leq \pi$, $0 \leq x_2 < 2\pi$ in E^2 onto the sphere of radius 1 about the origin in E^3. Is the mapping 1-1?

$$|\mathbf{y}|^2 = y_1^2 + y_2^2 + y_3^2 = \sin^2 x_1 \cos^2 x_2 + \sin^2 x_1 \sin^2 x_2 + \cos^2 x_1 = \sin^2 x_1 + \cos^2 x_1 = 1$$

Thus the mapping is into the sphere. To show that this mapping is onto, for fixed $\mathbf{y}$ satisfying $|\mathbf{y}|^2 = y_1^2 + y_2^2 + y_3^2$, let

$$x_1 = \text{Cos}^{-1} y_3 \quad \text{and} \quad x_2 = \begin{cases} \text{Cos}^{-1} \dfrac{y_1}{\sqrt{y_1^2 + y_2^2}} & \text{for } y_2 > 0 \\ 2\pi - \text{Cos}^{-1} \dfrac{y_1}{\sqrt{y_1^2 + y_2^2}} & \text{for } y_2 < 0 \\ 0 & \text{for } y_2 = y_1 = 0 \\ 0 & \text{for } y_2 = 0 \text{ and } y_1 > 0 \\ \pi & \text{for } y_2 = 0 \text{ and } y_1 < 0 \end{cases}$$

In every case $\cos x_1 = y_3$. If $y_2 \neq 0$, then

$$\sin x_1 \cos x_2 = \sqrt{1-y_3^2}\,\frac{y_1}{\sqrt{y_1^2+y_2^2}} = y_1$$

If $y_2 > 0$, then

$$\sin x_1 \sin x_2 = \sqrt{1-y_3^2}\,\frac{|y_2|}{\sqrt{y_1^2+y_2^2}} = |y_2| = y_2$$

If $y_2 < 0$, then

$$\sin x_1 \sin x_2 = \sqrt{1-y_3^2}\,\frac{-|y_2|}{\sqrt{y_1^2+y_2^2}} = -|y_2| = y_2$$

If $y_1 = y_2 = 0$, then $|y_3| = 1$. This implies $x_1 = 0$ or π, and so

$$\sin x_1 \cos x_2 = 0 = y_1, \qquad \sin x_1 \sin x_2 = 0 = y_2$$

If $y_2 = 0$ and $y_1 > 0$, then $x_2 = 0$ and so

$$\sin x_1 \cos x_2 = \sin x_1 = \sqrt{1-y_3^2} = y_1, \qquad \sin x_1 \sin x_2 = 0 = y_2$$

If $y_2 = 0$ and $y_1 < 0$, then $x_2 = \pi$ and so

$$\sin x_1 \cos x_2 = -\sin x_1 = -\sqrt{1-y_3^2} = -|y_1| = y_1, \qquad \sin x_1 \sin x_2 = 0 = y_2$$

Thus $\sin x_1 \cos x_2 = y_1$, $\sin x_1 \sin x_2 = y_2$ and $\cos x_3 = y_3$ is established in every case and the mapping is onto the sphere.

7.2. Determine the rank and image of the linear mapping

$$y_1 = 2x_1 + x_2$$
$$y_2 = -4x_1 - 2x_2$$
$$y_3 = -2x_1 - x_2$$

The matrix representation of the mapping is

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \\ -2 & -1 \end{pmatrix}$$

Since $\det\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = 0$, $\det\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = 0$ and $\det\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} = 0$, the rank of the matrix is 1. From the first two equations we obtain $2y_1 + y_2 = 0$, and from the last two equations we obtain $y_2 - 2y_3 = 0$. Thus the image of the mapping is the intersection of the planes $2y_1 + y_2 = 0$ and $y_2 - 2y_3 = 0$.

7.3. Show that a linear function from E^3 into E^3 is 1-1 and onto if and only if the images of a basis are linearly independent.

Suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a basis and $\mathbf{f}$ is a linear function from E^3 into E^3 such that $\mathbf{f}(\mathbf{e}_1), \mathbf{f}(\mathbf{e}_2), \mathbf{f}(\mathbf{e}_3)$ are independent. Then $(\mathbf{f}(\mathbf{e}_1), \mathbf{f}(\mathbf{e}_2), \mathbf{f}(\mathbf{e}_3))$ is also a basis and *every* $\mathbf{b}$ can be written as

$$\mathbf{b} = b_1\,\mathbf{f}(\mathbf{e}_1) + b_2\,\mathbf{f}(\mathbf{e}_2) + b_3\,\mathbf{f}(\mathbf{e}_3)$$

Now define $\mathbf{a} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. Then

$$\mathbf{f}(\mathbf{a}) = \mathbf{f}(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) = b_1\,\mathbf{f}(\mathbf{e}_1) + b_2\,\mathbf{f}(\mathbf{e}_2) + b_3\,\mathbf{f}(\mathbf{e}_3) = \mathbf{b}$$

Hence $\mathbf{f}$ is onto E^3. To show that $\mathbf{f}$ is 1-1, suppose that $\mathbf{f}(\mathbf{a}') = \mathbf{f}(\mathbf{a})$. Then

$$\mathbf{0} = \mathbf{f}(\mathbf{a}') - \mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{a}' - \mathbf{a}) = \mathbf{f}((a_1' - a_1)\mathbf{e}_1 + (a_2' - a_2)\mathbf{e}_2 + (a_3' - a_3)\mathbf{e}_3)$$
$$= (a_1' - a_1)\,\mathbf{f}(\mathbf{e}_1) + (a_2' - a_2)\,\mathbf{f}(\mathbf{e}_2) + (a_3' - a_3)\,\mathbf{f}(\mathbf{e}_3)$$

Since $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, $\mathbf{f}(\mathbf{e}_3)$ are independent, it follows that $a_1' - a_1 = 0$, $a_2' - a_2 = 0$, $a_3' - a_3 = 0$. Thus $\mathbf{a}' = \mathbf{a}$ and $\mathbf{f}$ is 1-1. Conversely if f is 1-1, then $\mathbf{f}(\mathbf{e}_1)$, $\mathbf{f}(\mathbf{e}_2)$, $\mathbf{f}(\mathbf{e}_3)$ are independent. For otherwise, there exist b_1, b_2, b_3 not all zero and hence $\mathbf{a} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \neq 0$ such that

$$\mathbf{0} = b_1\,\mathbf{f}(\mathbf{e}_1) + b_2\,\mathbf{f}(\mathbf{e}_2) + b_3\,\mathbf{f}(\mathbf{e}_3) = \mathbf{f}(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) = \mathbf{f}(\mathbf{a})$$

But for every linear function also $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, which is impossible since $\mathbf{f}$ is 1-1. Thus the proposition is proved.

7.4. Prove Theorem 7.2(i): A linear mapping $\mathbf{f}$ of E^2 into E^3 is 1-1 and onto a plane in E^3 if and only if the rank of $\mathbf{f}$ equals 2.

Suppose $(\mathbf{e}_1, \mathbf{e}_2)$ is a basis in E^2 and the vectors $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are linearly independent in E^3. Since $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1\,\mathbf{f}(\mathbf{e}_1) + x_2\,\mathbf{f}(\mathbf{e}_2)$, every $\mathbf{x}$ in E^2 is mapped onto the plane generated by $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$, as shown in Fig. 7-9. But also every $\mathbf{b}$ in the plane can be uniquely written as a combination of $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$. It follows, as in the problem above, that $\mathbf{f}$ maps E^2 1-1 onto a plane in E^3. Conversely, if $\mathbf{f}$ is a 1-1 mapping onto a plane, then, as in the problem above, $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are linearly independent. We leave the proof of this for the reader. Thus the mapping is 1-1 onto a plane if and only if $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are independent.

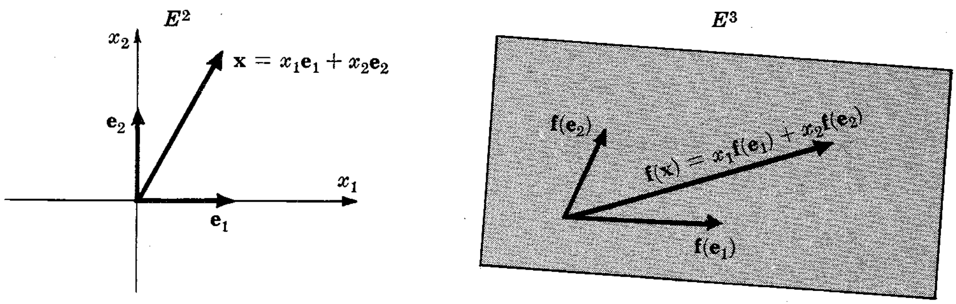


Fig. 7-9

It remains to show that $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are independent if and only if the rank of $\mathbf{f}$ equals 2. Suppose

$$\mathbf{f}(\mathbf{e}_1) = a_{11}\mathbf{g}_1 + a_{21}\mathbf{g}_2 + a_{31}\mathbf{g}_3 \quad \text{and} \quad \mathbf{f}(\mathbf{e}_2) = a_{12}\mathbf{g}_1 + a_{22}\mathbf{g}_2 + a_{32}\mathbf{g}_3$$

We recall that $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are independent if and only if

$$\mathbf{f}(\mathbf{e}_1) \times \mathbf{f}(\mathbf{e}_2) = (a_{21}a_{32} - a_{31}a_{22})\mathbf{g}_1 + (a_{12}a_{31} - a_{11}a_{32})\mathbf{g}_2 + (a_{11}a_{22} - a_{12}a_{21})\mathbf{g}_3 \neq \mathbf{0}$$

But the components of the above are the three 2×2 determinants of the matrix representation of $\mathbf{f}$,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Thus $\mathbf{f}(\mathbf{e}_1)$ and $\mathbf{f}(\mathbf{e}_2)$ are independent if and only if at least one of three 2×2 minors of the matrix representation of $\mathbf{f}$ is different from zero, that is, if and only if the rank of $\mathbf{f}$ equals 2, which proves the theorem.

7.5. If the components of a vector function $\mathbf{f}(\mathbf{x})$ are linear homogeneous functions of $(x_1, \ldots, x_n)$, show that $\mathbf{f}$ is a linear function of $\mathbf{x}$.

It is given that

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}x_j\right]\mathbf{g}_i$$

Then for all $\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$ and $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$, we have

$$\mathbf{f}(\mathbf{u}+\mathbf{v}) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}(u_j + v_j)\right]\mathbf{g}_i = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}u_j\right]\mathbf{g}_i + \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}v_j\right]\mathbf{g}_i = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})$$

and

$$\mathbf{f}(k\mathbf{u}) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}ku_j\right]\mathbf{g}_i = k\sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}u_j\right]\mathbf{g}_i = k\mathbf{f}(\mathbf{u})$$

Thus $\mathbf{f}$ is linear, which is the required result.

CONTINUITY AND LIMITS

7.6. Evaluate $\lim_{\mathbf{x}\to\mathbf{0}} [(x_1^2+x_2^2)\mathbf{e}_1 - x_1x_2\mathbf{e}_2]$.

Since $\mathbf{f}(\mathbf{x}) = (x_1^2+x_2^2)\mathbf{e}_1 - x_1x_2\mathbf{e}_2$ is continuous at $\mathbf{x}=\mathbf{0}$, $\lim_{\mathbf{x}\to\mathbf{0}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0})$. Hence

$$\lim_{\mathbf{x}\to\mathbf{0}} [(x_1^2+x_2^2)\mathbf{e}_1 - x_1x_2\mathbf{e}_2] = \mathbf{0}$$

7.7. Show that the real-valued function

$$f(\mathbf{x}) = \begin{cases} x_1x_2^2/(x_1^2+x_2^2), & \text{if } \mathbf{x}\neq\mathbf{0} \\ 0, & \text{if } \mathbf{x}=\mathbf{0} \end{cases}$$

is continuous at $\mathbf{x}=\mathbf{0}$.

We write $x_1 = |\mathbf{x}|\cos\measuredangle(\mathbf{x},\mathbf{e}_1)$, $x_2 = |\mathbf{x}|\cos\measuredangle(\mathbf{x},\mathbf{e}_2)$. Then

$$|f(\mathbf{x})-f(\mathbf{0})| = \frac{\left||\mathbf{x}|^3\cos\measuredangle(\mathbf{x},\mathbf{e}_1)\cos^2\measuredangle(\mathbf{x},\mathbf{e}_2)\right|}{|\mathbf{x}|^2} \leq |\mathbf{x}|\,|\cos\measuredangle(\mathbf{x},\mathbf{e}_1)|\,|\cos^2\measuredangle(\mathbf{x},\mathbf{e}_2)| \leq |\mathbf{x}| < \epsilon$$

for $|\mathbf{x}| < \delta = \epsilon$. It follows that $f(\mathbf{x})$ is continuous at $\mathbf{0}$.

7.8. Show that the real-valued function

$$f(\mathbf{x}) = \begin{cases} x_1x_2^2/(x_1^2+x_2^4), & \text{if } \mathbf{x}\neq\mathbf{0} \\ 0, & \text{if } \mathbf{x}=\mathbf{0} \end{cases}$$

is not continuous at $\mathbf{x}=\mathbf{0}$.

We consider the vectors near $\mathbf{0}$ of the form $\mathbf{x} = h^2\mathbf{e}_1 + h\mathbf{e}_2$, $h\neq 0$. Then $x_1 = h^2$ and $x_2 = h$ and

$$|f(\mathbf{x})-f(\mathbf{0})| = \frac{h^4}{(h^4+h^4)} - 0 = \frac{1}{2}$$

which cannot be less than ϵ for $\epsilon \leq \frac{1}{2}$.

7.9. A function $\mathbf{f}$ is said to be *bounded* at $\mathbf{x}_0$ if there exist scalars $M>0$ and $\delta>0$ such that $|\mathbf{f}(\mathbf{x})| \leq M$ for $|\mathbf{x}-\mathbf{x}_0|<\delta$. Show that $\mathbf{f}$ is bounded at $\mathbf{x}_0$ if $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

Select any $\epsilon>0$. Since $\mathbf{f}$ is continuous at $\mathbf{x}_0$, there exists $\delta>0$ such that $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|<\epsilon$ for $|\mathbf{x}-\mathbf{x}_0|<\delta$. Using the inequality $|\mathbf{a}|-|\mathbf{b}| \leq |\mathbf{a}-\mathbf{b}|$, it follows that $|\mathbf{f}(\mathbf{x})| \leq |\mathbf{f}(\mathbf{x}_0)|+\epsilon = M$ for $|\mathbf{x}-\mathbf{x}_0|<\delta$, which proves the proposition.

7.10. If $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$, show that $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

Since $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$, given $\epsilon>0$, there is a $\delta>0$ such that $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|<\epsilon$ for $\mathbf{x}$ in the deleted neighborhood $0<|\mathbf{x}-\mathbf{x}_0|<\delta$. But at $\mathbf{x}=\mathbf{x}_0$ we have $\mathbf{f}(\mathbf{x})=\mathbf{f}(\mathbf{x}_0)$. Hence $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|<\epsilon$ for $\mathbf{x}$ in the complete neighborhood $|\mathbf{x}-\mathbf{x}_0|<\delta$. That is, $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

7.11. Show that $\mathbf{f} = f_1\mathbf{g}_1 + \cdots + f_m\mathbf{g}_m$ is continuous at $\mathbf{x}_0$ if and only if each f_i is continuous at $\mathbf{x}_0$.

Suppose $\mathbf{f}$ is continuous at $\mathbf{x}_0$. Then for $|\mathbf{x}-\mathbf{x}_0|<\delta$, $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|<\epsilon$. But then for $|\mathbf{x}-\mathbf{x}_0|<\delta$,

$$|\mathbf{f}_i(\mathbf{x})-\mathbf{f}_i(\mathbf{x}_0)| = |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|\,|\cos\measuredangle(\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0),\mathbf{g}_i)| \leq |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| < \epsilon$$

Hence f_i is continuous at $\mathbf{x}_0$. Conversely, suppose each f_i, $i=1,\ldots,m$, is continuous at $\mathbf{x}_0$. Then for $|\mathbf{x}-\mathbf{x}_0|<\delta_i$, $i=1,\ldots,m$, $|f_i(\mathbf{x})-f_i(\mathbf{x}_0)|<\epsilon/m$. It follows that for $|\mathbf{x}-\mathbf{x}_0|<\delta=\min(\delta_1,\ldots,\delta_m)$, we have

$$\begin{aligned}|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| &= |(f_1(\mathbf{x})-f_1(\mathbf{x}_0))\mathbf{g}_1+\cdots+(f_m(\mathbf{x})-f_m(\mathbf{x}_0))\mathbf{g}_m|\\ &\leqq |f_1(\mathbf{x})-f_1(\mathbf{x}_0)|+\cdots+|f_m(\mathbf{x})-f_m(\mathbf{x}_0)| < m\epsilon/m = \epsilon\end{aligned}$$

which proves the proposition.

7.12. If $\lim\limits_{\mathbf{v}\to\mathbf{0}} \dfrac{\mathbf{R}(\mathbf{v})}{|\mathbf{v}|} = 0$, show that $\lim\limits_{h\to 0} \dfrac{\mathbf{R}(h\mathbf{u}_0)}{h} = \mathbf{0}$ $(\mathbf{u}_0 \neq \mathbf{0})$.

Since $(\mathbf{R}(\mathbf{v})/|\mathbf{v}|) \to 0$ as $\mathbf{v}\to\mathbf{0}$, given $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{R}(\mathbf{v})/|\mathbf{v}|| = |\mathbf{R}(\mathbf{v})|/|\mathbf{v}| < \epsilon/|\mathbf{u}_0|$ for $0 < |\mathbf{v}| < \delta$. But then

$$|\mathbf{R}(h\mathbf{u}_0)/h| = (|\mathbf{R}(h\mathbf{u}_0)|/|h\mathbf{u}_0|)\,|\mathbf{u}_0| < (\epsilon/|\mathbf{u}_0|)\,|\mathbf{u}_0| = \epsilon$$

for $0 < |h\mathbf{u}_0| < \delta$ or $0 < |h| < \delta/|\mathbf{u}_0|$. Thus $(\mathbf{R}(h\mathbf{u}_0)/h) \to \mathbf{0}$ as $h \to 0$.

7.13. If $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{x}_0$, show that $\mathbf{f}\cdot\mathbf{g}$ is continuous at $\mathbf{x}_0$.

We consider

$$\begin{aligned}|\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\cdot\mathbf{g}(\mathbf{x}_0)| &\leqq |\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}_0)| + |\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}_0) - \mathbf{f}(\mathbf{x}_0)\cdot\mathbf{g}(\mathbf{x}_0)|\\ &= |\mathbf{f}(\mathbf{x})|\,|\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{x}_0)| + |\mathbf{g}(\mathbf{x}_0)|\,|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)|\end{aligned}$$

Since $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{x}_0$ and in particular $\mathbf{f}$ is bounded at $\mathbf{x}_0$, for every $\epsilon > 0$ there exist $\delta_1, \delta_2, \delta_3$ and $M > 0$ such that

$$\begin{aligned}&|\mathbf{f}(\mathbf{x})| \leqq M && \text{for } |\mathbf{x}-\mathbf{x}_0| < \delta_1\\ &|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| < \frac{\epsilon}{2\,|\mathbf{g}(\mathbf{x}_0)|} && \text{for } |\mathbf{x}-\mathbf{x}_0| < \delta_2\\ &|\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{x}_0)| < \frac{\epsilon}{2M} && \text{for } |\mathbf{x}-\mathbf{x}_0| < \delta_3\end{aligned}$$

Then for $|\mathbf{x}-\mathbf{x}_0| < \delta = \min(\delta_1, \delta_2, \delta_3)$,

$$|\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\cdot\mathbf{g}(\mathbf{x}_0)| < M\frac{\epsilon}{2M} + |\mathbf{g}(\mathbf{x}_0)|\frac{\epsilon}{2\,|\mathbf{g}(\mathbf{x}_0)|} = \epsilon$$

and so $\mathbf{f}\cdot\mathbf{g}$ is continuous at $\mathbf{x}_0$.

DIRECTIONAL DERIVATIVES. DIFFERENTIABLE FUNCTIONS

7.14. Show that the real-valued function

$$f(\mathbf{x}) = \begin{cases} x_1x_2^2/(x_1^2+x_2^4), & \text{if } \mathbf{x}\neq\mathbf{0}\\ 0, & \text{if } \mathbf{x}=\mathbf{0}\end{cases}$$

has a derivative in every direction at $\mathbf{x} = \mathbf{0}$. Note that f is not continuous at $\mathbf{x} = \mathbf{0}$, as shown in Problem 7.8.

Let $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ be a nonzero vector such that $u_1 \neq 0$. Then

$$D_{\mathbf{u}}f(\mathbf{0}) = \lim_{h\to 0}\frac{f(\mathbf{0}+h\mathbf{u}) - f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{u_1u_2^2}{(u_1^2+h^2u_2^4)} = \frac{u_2^2}{u_1}$$

For $u_1 = 0$, i.e. for $\mathbf{u} = u_2\mathbf{e}_2$, we have

$$D_{\mathbf{u}}f(\mathbf{0}) = \lim_{h\to 0}\frac{f(\mathbf{0}+h\mathbf{u}) - f(\mathbf{0})}{h} = \lim_{h\to 0} 0 = 0$$

Thus f has a derivative in every direction at $\mathbf{0}$. Note that $D_{\mathbf{u}}f(\mathbf{0})$ as a function of $\mathbf{u}$ is not linear. This is in contrast to the case when $\mathbf{f}$ is differentiable, for then $D_{\mathbf{u}}\mathbf{f} = d\mathbf{f}(\mathbf{u})$ is linear in $\mathbf{u}$.

7.15. Find the derivative of

$$\mathbf{f}(\mathbf{x}) = (x_1 \sin x_2)\mathbf{e}_1 + (x_2 \sin x_1)\mathbf{e}_2$$

in the direction $\mathbf{u}_0 = (1/\sqrt{5})\mathbf{e}_1 + (2/\sqrt{5})\mathbf{e}_2$ at $\mathbf{x}_0 = (\pi/2)\mathbf{e}_1 + (\pi/4)\mathbf{e}_2$.

Since the partial derivatives of $\mathbf{f}$ are continuous, it follows from Theorem 7.8 that $\mathbf{f}$ is differentiable. Thus we can use equation (*7.2*), obtaining

$$\begin{aligned} D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) &= d\mathbf{f}(\mathbf{x})(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial x_1}u_1 + \frac{\partial \mathbf{f}}{\partial x_2}u_2 \\ &= ((\sin x_2)\mathbf{e}_1 + (x_2 \cos x_1)\mathbf{e}_2)u_1 + ((x_1 \cos x_2)\mathbf{e}_1 + (\sin x_1)\mathbf{e}_2)u_2 \end{aligned}$$

Hence
$$D_{\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0) = \frac{1}{\sqrt{10}}\mathbf{e}_1 + \left(\frac{\pi}{2\sqrt{2}}\mathbf{e}_1 + \mathbf{e}_2\right)\frac{2}{\sqrt{5}} = \frac{2+\pi}{2\sqrt{10}}\mathbf{e}_1 + \frac{2}{\sqrt{5}}\mathbf{e}_2$$

7.16. For each of the following functions on E^2 into E^3 determine (i) the set V on which $\mathbf{f}$ is continuously differentiable, (ii) the differential of $\mathbf{f}$ on V, (iii) the rank of $d\mathbf{f}$ on V.

(*a*) $\mathbf{f}(\mathbf{x}) = (x_1 \cos x_2)\mathbf{g}_1 + (x_1 \sin x_2)\mathbf{g}_2 + x_2\mathbf{g}_3$

(*b*) $\mathbf{f}(\mathbf{x}) = |x_1 + x_2|\mathbf{g}_1 + |x_1 - x_2|\mathbf{g}_2 + \mathbf{g}_3$

(*a*) (i) $\mathbf{f}(\mathbf{x}) = (x_1 \cos x_2)\mathbf{g}_1 + (x_1 \sin x_2)\mathbf{g}_2 + x_2\mathbf{g}_3$ has continuous derivatives for all $\mathbf{x}$. Hence $\mathbf{f}$ is continuously differentiable on all of E^2.

(ii) The differential of $\mathbf{f}$ is

$$\begin{aligned} d\mathbf{f} &= \frac{\partial \mathbf{f}}{\partial x_1}dx_1 + \frac{\partial \mathbf{f}}{\partial x_2}dx_2 \\ &= [(\cos x_2)\mathbf{g}_1 + (\sin x_2)\mathbf{g}_2]\,dx_1 + [(-x_1 \sin x_2)\mathbf{g}_1 + (x_1 \cos x_2)\mathbf{g}_2 + \mathbf{g}_3]\,dx_2 \end{aligned}$$

(iii) The matrix of the differential is

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{\mathbf{x}} = \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2 \\ 0 & 1 \end{pmatrix}$$

Since

$$\left[\det\begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2 \end{pmatrix}\right]^2 + \left[\det\begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ 0 & 1 \end{pmatrix}\right]^2 + \left[\det\begin{pmatrix} \sin x_2 & x_1 \cos x_2 \\ 0 & 1 \end{pmatrix}\right]^2$$
$$= x_1^2 + \cos^2 x_2 + \sin^2 x_2 = x_1^2 + 1 > 0$$

it follows that the rank of $d\mathbf{f}$ is 2 for all $\mathbf{x}$ in E^2.

(*b*) (i) The function $\mathbf{f}(\mathbf{x}) = |x_1 + x_2|\mathbf{g}_1 + |x_1 - x_2|\mathbf{g}_2 + \mathbf{g}_3$ has continuous derivatives provided that $x_1 + x_2 \neq 0$ and $x_1 - x_2 \neq 0$. That is, $\mathbf{f}$ is continuously differentiable in the four open sets

$$\begin{aligned} A &= \{x_1 + x_2 > 0 \text{ and } x_1 - x_2 > 0\} \\ B &= \{x_1 + x_2 < 0 \text{ and } x_1 - x_2 > 0\} \\ C &= \{x_1 + x_2 < 0 \text{ and } x_1 - x_2 < 0\} \\ D &= \{x_1 + x_2 > 0 \text{ and } x_1 - x_2 < 0\} \end{aligned}$$

shown in Fig. 7-10.

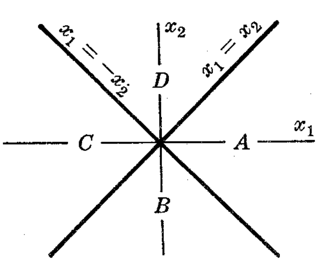


Fig. 7-10

(ii) For $\mathbf{x}$ in A, $|x_1 + x_2| = x_1 + x_2$, $|x_1 - x_2| = x_1 - x_2$ and the differential is

$$d\mathbf{f} = (\mathbf{g}_1 + \mathbf{g}_2)\,dx_1 + (\mathbf{g}_1 - \mathbf{g}_2)\,dx_2$$

For $\mathbf{x}$ in B, $|x_1 + x_2| = -x_1 - x_2$, $|x_1 - x_2| = x_1 - x_2$ and

$$d\mathbf{f} = (-\mathbf{g}_1 + \mathbf{g}_2)\,dx_1 - (\mathbf{g}_1 + \mathbf{g}_2)\,dx_2$$

For x in C, $|x_1+x_2| = -x_1 - x_2$, $|x_1-x_2| = -x_1+x_2$ and

$$d\mathbf{f} = -(\mathbf{g}_1+\mathbf{g}_2)\,dx_1 + (-\mathbf{g}_1+\mathbf{g}_2)\,dx_2$$

For x in D, $|x_1+x_2| = x_1 + x_2$, $|x_1-x_2| = -x_1+x_2$ and

$$d\mathbf{f} = (\mathbf{g}_1-\mathbf{g}_2)\,dx_1 + (\mathbf{g}_1+\mathbf{g}_2)\,dx_2$$

(iii) The matrix representations of $d\mathbf{f}$ are

$$\overset{A}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \qquad \overset{B}{\begin{pmatrix} -1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \qquad \overset{C}{\begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}} \qquad \overset{D}{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}}$$

On each set, the rank of $d\mathbf{f}$ equals 2.

7.17. Let $\mathbf{f} = f_1\mathbf{g}_1 + f_2\mathbf{g}_2 + f_3\mathbf{g}_3$ be a vector function from E^2 into E^3 which is differentiable at $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Show that the rank of $d\mathbf{f}$ at $\mathbf{x}$ equals 2 if and only if $\dfrac{\partial \mathbf{f}}{\partial x_1} \times \dfrac{\partial \mathbf{f}}{\partial x_2} \neq \mathbf{0}$.

$$\frac{\partial \mathbf{f}}{\partial x_1} \times \frac{\partial \mathbf{f}}{\partial x_2} = \left(\frac{\partial f_1}{\partial x_1}\mathbf{g}_1 + \frac{\partial f_2}{\partial x_1}\mathbf{g}_2 + \frac{\partial f_3}{\partial x_1}\mathbf{g}_3\right) \times \left(\frac{\partial f_1}{\partial x_2}\mathbf{g}_1 + \frac{\partial f_2}{\partial x_2}\mathbf{g}_2 + \frac{\partial f_3}{\partial x_2}\mathbf{g}_3\right)$$

$$= \left(\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_2} - \frac{\partial f_3}{\partial x_1}\frac{\partial f_2}{\partial x_2}\right)\mathbf{g}_1 + \left(\frac{\partial f_3}{\partial x_1}\frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_2}\right)\mathbf{g}_2 + \left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}\right)\mathbf{g}_3$$

Observe that the components of $\dfrac{\partial \mathbf{f}}{\partial x_1} \times \dfrac{\partial \mathbf{f}}{\partial x_2}$, except for a sign, are the three 2×2 minors of the matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\[2ex] \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \\[2ex] \dfrac{\partial f_3}{\partial x_1} & \dfrac{\partial f_3}{\partial x_2} \end{pmatrix}$$

It follows that $\dfrac{\partial \mathbf{f}}{\partial x_1} \times \dfrac{\partial \mathbf{f}}{\partial x_2} \neq \mathbf{0}$ if and only if one of the 2×2 minors does not vanish, i.e. if and only if the rank of $d\mathbf{f}$ equals 2.

7.18. Show that if $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, there exists an $M \geq 0$ such that $\dfrac{d\mathbf{f}(\mathbf{x}_0)(\mathbf{v})}{|\mathbf{v}|} \leq M$ for all $\mathbf{v} \neq \mathbf{0}$.

From equation (7.2), page 127,

$$\frac{|d\mathbf{f}(\mathbf{x}_0)(\mathbf{v})|}{|\mathbf{v}|} = \left| v_1 \frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{x}_0) + \cdots + v_n \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}_0) \right| \Big/ |\mathbf{v}|$$

$$\leq \left| \cos \measuredangle(\mathbf{v}, \mathbf{e}_1) \frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{x}_0) + \cdots + \cos \measuredangle(\mathbf{v}, \mathbf{e}_n) \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}_0) \right|$$

$$\leq \left| \frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{x}_0) \right| + \cdots + \left| \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}_0) \right| \leq M$$

which is the required result.

7.19. Prove that $\mathbf{f}(\mathbf{x})$ is continuous at $\mathbf{x}_0$ if $\mathbf{f}(\mathbf{x})$ is differentiable at $\mathbf{x}_0$.

Since $\mathbf{f}$ is differentiable at $\mathbf{x}_0$,

$$\mathbf{f}(\mathbf{x}_0+\mathbf{v}) = \mathbf{f}(\mathbf{x}_0) + d\mathbf{f}(\mathbf{x}_0)(\mathbf{v}) + \mathbf{R}(\mathbf{x}_0, \mathbf{v})$$

or, if we let $\mathbf{v} = \mathbf{x}-\mathbf{x}_0$,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + d\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)$$

where $(\mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|) \to \mathbf{0}$ as $\mathbf{x}\to\mathbf{x}_0$. It follows that for $\mathbf{x}\neq\mathbf{x}_0$,

$$\begin{aligned}|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| &= |d\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)|\\ &\leq \frac{|d\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|}{|\mathbf{x}-\mathbf{x}_0|}|\mathbf{x}-\mathbf{x}_0| + \frac{|\mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)|}{|\mathbf{x}-\mathbf{x}_0|}|\mathbf{x}-\mathbf{x}_0| \leq (M_1+M_2)\,|\mathbf{x}-\mathbf{x}_0|\end{aligned}$$

where we used the result of the preceding problem and the fact that $\mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$ is also bounded in a deleted neighborhood of $\mathbf{x}_0$ since $(\mathbf{R}(\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|)\to\mathbf{0}$ as $\mathbf{x}\to\mathbf{x}_0$. It follows that $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| < \epsilon$ for $0 < |\mathbf{x}-\mathbf{x}_0| < \epsilon/(M_1+M_2)$. Namely $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$ and hence $\mathbf{f}(\mathbf{x})$ is continuous at $\mathbf{x}_0$.

7.20. Prove Theorem 7.7(iii): If $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{x}$, then $\mathbf{f}\cdot\mathbf{g}$ is differentiable at $\mathbf{x}$ and $d(\mathbf{f}\cdot\mathbf{g}) = \mathbf{f}\cdot d\mathbf{g} + (d\mathbf{f})\cdot\mathbf{g}$.

Consider the function of $\mathbf{v}$,

$$\begin{aligned}\mathbf{R}(\mathbf{x},\mathbf{v}) &= \mathbf{f}(\mathbf{x}+\mathbf{v})\cdot\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\cdot d\mathbf{g}(\mathbf{x})(\mathbf{v}) - d\mathbf{f}(\mathbf{x})(\mathbf{v})\cdot\mathbf{g}(\mathbf{x})\\ &= \mathbf{f}(\mathbf{x}+\mathbf{v})\cdot[\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) - d\mathbf{g}(\mathbf{x})(\mathbf{v})]\\ &\qquad + [\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}(\mathbf{x})(\mathbf{v})]\cdot\mathbf{g}(\mathbf{x}) + [\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})]\cdot d\mathbf{g}(\mathbf{x})(\mathbf{v})\end{aligned}$$

Now, since $\mathbf{f}$ and $\mathbf{g}$ are differentiable,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + d\mathbf{f}(\mathbf{x})(\mathbf{v}) + \mathbf{R}_1(\mathbf{x},\mathbf{v}) \quad\text{and}\quad \mathbf{g}(\mathbf{x}+\mathbf{v}) = \mathbf{g}(\mathbf{x}) + d\mathbf{g}(\mathbf{x})(\mathbf{v}) + \mathbf{R}_2(\mathbf{x},\mathbf{v})$$

where $\lim_{\mathbf{v}\to\mathbf{0}} \frac{|\mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = \lim_{\mathbf{v}\to\mathbf{0}} \frac{|\mathbf{R}_2(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$. Thus, substituting in the above,

$$\begin{aligned}\frac{|\mathbf{R}(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} &= |\mathbf{f}(\mathbf{x}+\mathbf{v})\cdot\mathbf{R}_2(\mathbf{x},\mathbf{v}) + \mathbf{g}(\mathbf{x})\cdot\mathbf{R}_1(\mathbf{x},\mathbf{v}) + [\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})]\cdot d\mathbf{g}(\mathbf{x})(\mathbf{v})\,|/|\mathbf{v}|\\ &\leq |\mathbf{f}(\mathbf{x}+\mathbf{v})|\frac{|\mathbf{R}_2(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} + |\mathbf{g}(\mathbf{x})|\frac{|\mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} + |\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})|\frac{|d\mathbf{g}(\mathbf{x})(\mathbf{v})|}{|\mathbf{v}|}\end{aligned}$$

But $\mathbf{f}(\mathbf{x}+\mathbf{v})$ is bounded for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$ since $\mathbf{f}$ is continuous at $\mathbf{x}$ and $|d\mathbf{g}(\mathbf{x})(\mathbf{v})/\mathbf{v}|$ is bounded for $\mathbf{v}\neq\mathbf{0}$ by Problem 7.18. Also $\lim_{\mathbf{v}\to\mathbf{0}}\frac{|\mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$ and $\lim_{\mathbf{v}\to\mathbf{0}}\frac{|\mathbf{R}_2(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$ as stated above and $\lim_{\mathbf{v}\to\mathbf{0}}|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})| = 0$ since $\mathbf{f}$ is continuous at $\mathbf{x}$, and so $\lim_{\mathbf{v}\to\mathbf{0}}\frac{|\mathbf{R}(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$. Thus

$$\mathbf{f}(\mathbf{x}+\mathbf{v})\cdot\mathbf{g}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) + \mathbf{f}(\mathbf{x})\cdot d\mathbf{g}(\mathbf{x})(\mathbf{v}) + d\mathbf{f}(\mathbf{x})(\mathbf{v})\cdot\mathbf{g}(\mathbf{x}) + \mathbf{R}(\mathbf{x},\mathbf{v})$$

where $\lim_{\mathbf{v}\to\mathbf{0}}\frac{\mathbf{R}(\mathbf{x},\mathbf{v})}{|\mathbf{v}|} = \mathbf{0}$. It follows that $\mathbf{f}\cdot\mathbf{g}$ is differentiable at $\mathbf{x}$ and $d(\mathbf{f}\cdot\mathbf{g}) = \mathbf{f}\cdot d\mathbf{g} + (d\mathbf{f})\cdot\mathbf{g}$, which proves the theorem.

7.21. *Mean Value Theorem.* Let f be a real-valued function of $\mathbf{x}$ on an open set in E and let f be differentiable at every point along the line $\mathbf{x}^* = \mathbf{x}_0 + t\mathbf{v}_0$, $0\leq t\leq 1$, as shown in Fig. 7-11. Show that there exists a t_0, $0<t_0<1$, such that $f(\mathbf{x}_0+\mathbf{v}_0) = f(\mathbf{x}_0) + D_{\mathbf{v}_0}f(\mathbf{x}_0+t_0\mathbf{v}_0)$.

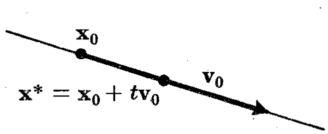


Fig. 7-11

We want to show that the function $F(t) = f(\mathbf{x}_0+t\mathbf{v}_0)$ has a derivative at each $0\leq t\leq 1$ and then apply the mean value theorem for a function of a single real variable. Since f is differentiable along $\mathbf{x}_0+t\mathbf{v}_0$, $df(\mathbf{x}_0+t\mathbf{v}_0)(\mathbf{v}) = D_{\mathbf{v}}f(\mathbf{x}_0+t\mathbf{v}_0)$ and is linear in $\mathbf{v}$. Thus

$$\frac{F(t+h)-F(t)}{h} = \frac{f(\mathbf{x}_0+t\mathbf{v}_0+h\mathbf{v}_0) - f(\mathbf{x}_0+t\mathbf{v}_0)}{h} = \frac{D_{h\mathbf{v}_0}f(\mathbf{x}_0+t\mathbf{v}_0) + R(\mathbf{x}_0+t\mathbf{v}_0, h\mathbf{v}_0)}{h}$$

$$= D_{\mathbf{v}_0}f(\mathbf{x}_0+t\mathbf{v}_0) + \frac{R(\mathbf{x}_0+t\mathbf{v}_0, h\mathbf{v}_0)}{h}$$

where $\lim_{h\mathbf{v}_0\to 0} \frac{R(\mathbf{x}_0+t\mathbf{v}_0, h\mathbf{v}_0)}{|h\mathbf{v}_0|} = 0$ and hence from Problem 7.12, $\lim_{h\to 0} \frac{R(\mathbf{x}_0+t\mathbf{v}_0, h\mathbf{v}_0)}{h} = 0$. It follows that $F'(t)$ exists at each $0 \leqq t \leqq 1$ and is given by

$$F'(t) = \lim_{h\to 0}\frac{F(t+h)-F(t)}{h} = D_{\mathbf{v}_0}f(\mathbf{x}_0+t\mathbf{v}_0)$$

From the mean value theorem for a real-valued function of a real variable, it follows that there exists a t_0, $0 < t_0 < 1$, such that $F(1) = F(0) + F'(t_0)$ or,

$$f(\mathbf{x}_0+\mathbf{v}_0) = f(\mathbf{x}_0) + D_{\mathbf{v}_0}f(\mathbf{x}_0+t_0\mathbf{v}_0)$$

COMPOSITE FUNCTIONS. CHAIN RULE

7.22. Let $\mathbf{y} = (x_1^2+x_2^2)\mathbf{e}_1 + x_1x_2^2\mathbf{e}_2, \quad \mathbf{x} = (u_1\cos u_2)\mathbf{e}_1 + (u_1\sin u_2)\mathbf{e}_2$

Find: (*a*) The partial derivatives $\partial y_1/\partial u_1$, $\partial y_2/\partial u_1$, $\partial y_1/\partial u_2$ and $\partial y_2/\partial u_2$ as functions of $\mathbf{u}$. (*b*) The differential $d\mathbf{y}$ as a function of $\mathbf{u}$.

(*a*) $$\frac{\partial y_1}{\partial u_1} = \frac{\partial y_1}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial x_2}{\partial u_1} = 2x_1\cos u_2 + 2x_2\sin u_2 = 2u_1\cos^2 u_2 + 2u_1\sin^2 u_2 = 2u_1$$

$$\frac{\partial y_2}{\partial u_1} = \frac{\partial y_2}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial y_2}{\partial x_2}\frac{\partial x_2}{\partial u_1} = x_2^2\cos u_2 + 2x_1x_2\sin u_2$$
$$= u_1^2\sin^2 u_2\cos u_2 + 2u_1^2\cos u_2\sin^2 u_2 = 3u_1^2\cos u_2\sin^2 u_2$$

$$\frac{\partial y_1}{\partial u_2} = \frac{\partial y_1}{\partial x_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial y_1}{\partial x_2}\frac{\partial x_2}{\partial u_2} = -2x_1u_1\sin u_2 + 2x_2u_1\cos u_2$$
$$= -2u_1^2\cos u_2\sin u_2 + 2u_1^2\cos u_2\sin u_2 = 0$$

$$\frac{\partial y_2}{\partial u_2} = \frac{\partial y_2}{\partial x_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial y_2}{\partial x_2}\frac{\partial x_2}{\partial u_2} = -x_2^2u_1\sin u_2 + 2x_1x_2u_1\cos u_2$$
$$= -u_1^3\sin^3 u_2 + 2u_1^3\cos^2 u_2\sin u_2$$
$$= u_1^3(\sin u_2)(2\cos^2 u_2 - \sin^2 u_2)$$

Another method:

$$\mathbf{y} = (u_1^2\cos^2 u_2 + u_1^2\sin^2 u_2)\mathbf{e}_1 + (u_1^3\cos u_2\sin^2 u_2)\mathbf{e}_2 = u_1^2\mathbf{e}_1 + (u_1^3\cos u_2\sin^2 u_2)\mathbf{e}_2$$

$$\frac{\partial y_1}{\partial u_1} = 2u_1, \qquad \frac{\partial y_2}{\partial u_1} = 3u_1^2\cos u_2\sin^2 u_2, \qquad \frac{\partial y_1}{\partial u_2} = 0$$

$$\frac{\partial y_2}{\partial u_2} = u_1^3(2\cos^2 u_2\sin u_2 - \sin^3 u_2) = u_1^3(\sin u_2)(2\cos^2 u_2 - \sin^2 u_2)$$

(*b*) $$d\mathbf{y} = \frac{\partial\mathbf{y}}{\partial u_1}du_1 + \frac{\partial\mathbf{y}}{\partial u_2}du_2 = \left(\frac{\partial y_1}{\partial u_1}\mathbf{e}_1 + \frac{\partial y_2}{\partial u_1}\mathbf{e}_2\right)du_1 + \left(\frac{\partial y_1}{\partial u_2}\mathbf{e}_1 + \frac{\partial y_2}{\partial u_2}\mathbf{e}_2\right)du_2$$
$$= (2u_1\mathbf{e}_1 + (3u_1^2\cos u_2)\mathbf{e}_2)\,du_1 + u_1^3\sin u_2(2\cos^2 u_2 - \sin^2 u^2)\mathbf{e}_2\,du_2$$

7.23. If $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{w} = \mathbf{g}(\mathbf{y})$ are differentiable mappings of E into E, show that the Jacobian of the composite mapping $\mathbf{w} = \mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ at $\mathbf{x}$ is the product of the Jacobian of $\mathbf{g}$ at $\mathbf{f}(\mathbf{x})$ with the Jacobian of $\mathbf{f}$ at $\mathbf{x}$, i.e.,

$$\frac{\partial(w_1\cdots w_n)}{\partial(x_1\cdots x_n)} = \frac{\partial(w_1\cdots w_n)}{\partial(y_1\cdots y_n)}\frac{\partial(y_1\cdots y_n)}{\partial(x_1\cdots x_n)}$$

We recall that the determinant of a product of matrices is the product of the determinants; i.e. if $(c_{ij}) = (a_{ij})(b_{ij})$, then $\det(c_{ij}) = \det(a_{ij})\det(b_{ij})$. Since the Jacobian matrix of the composite function $\mathbf{w} = \mathbf{h}(\mathbf{x})$ is $\left(\frac{\partial w_i}{\partial x_j}\right) = \left(\frac{\partial w_i}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial x_j}\right)$, it follows that $\det\left(\frac{\partial w_i}{\partial x_j}\right) = \det\left(\frac{\partial w_i}{\partial y_j}\right)\det\left(\frac{\partial y_i}{\partial x_j}\right)$, which gives the required result.

7.24. Let $\mathbf{y} = \mathbf{f}(\mathbf{x})$ be a mapping of class C^1 from E^2 into E^3 such that the rank of $d\mathbf{f}$ equals 2 for all $\mathbf{x}$, and let $\mathbf{x} = \mathbf{x}(t)$ be a regular curve of class C^1 in E^2. Show that

(a) $\mathbf{y} = \mathbf{f}(\mathbf{x}(t))$ is a regular curve of class C^1 in E^3.

(b) $\left|\frac{d\mathbf{y}}{dt}\right|^2 = \left|\frac{\partial \mathbf{y}}{\partial x_1}\right|^2\left(\frac{dx_1}{dt}\right)^2 + 2\left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\frac{\partial \mathbf{y}}{\partial x_2}\right)\frac{dx_1}{dt}\frac{dx_2}{dt} + \left|\frac{\partial \mathbf{y}}{\partial x_2}\right|^2\left(\frac{dx_2}{dt}\right)^2$

(a) It follows from Theorem 7.10 that $\mathbf{y} = \mathbf{f}(\mathbf{x}(t))$ belongs to C^1 and that

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{y}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{y}}{\partial x_2}\frac{dx_2}{dt} = D_{d\mathbf{x}/dt}\mathbf{f}$$

It remains to show that $d\mathbf{y}/dt \neq \mathbf{0}$. Since the rank of $d\mathbf{f}$ equals 2, it follows from Theorem 7.2(i) that at each t the linear function $D_{\mathbf{v}}\mathbf{f}$ of $\mathbf{v}$ is 1-1. Hence since $d\mathbf{x}/dt \neq 0$, $d\mathbf{y}/dt = D_{d\mathbf{x}/dt}\mathbf{f} \neq 0$.

(b) By expanding the above,

$$\left|\frac{d\mathbf{y}}{dt}\right|^2 = \left(\frac{d\mathbf{y}}{dt}\cdot\frac{d\mathbf{y}}{dt}\right) = \left(\frac{\partial \mathbf{y}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{y}}{\partial x_2}\frac{dx_2}{dt}\right)\cdot\left(\frac{\partial \mathbf{y}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{y}}{\partial x_2}\frac{dx_2}{dt}\right)$$

$$= \left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\frac{\partial \mathbf{y}}{\partial x_1}\right)\left(\frac{dx_1}{dt}\right)^2 + 2\left(\frac{\partial \mathbf{y}}{\partial x_1}\cdot\frac{\partial \mathbf{y}}{\partial x_2}\right)\frac{dx_1}{dt}\frac{dx_2}{dt} + \left(\frac{\partial \mathbf{y}}{\partial x_2}\cdot\frac{\partial \mathbf{y}}{\partial x_2}\right)\left(\frac{dx_2}{dt}\right)^2$$

which gives the required result.

7.25. Prove Theorem 7.10: If $\mathbf{f}$ is differentiable at $\mathbf{x}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x})$, then the composite function $\mathbf{h} = \mathbf{g}\circ\mathbf{f}$ is differentiable at $\mathbf{x}$ and its differential $\mathbf{H}_\mathbf{x} = \mathbf{M}_{\mathbf{f}(\mathbf{x})}\circ\mathbf{L}_\mathbf{x}$, where $\mathbf{L}_\mathbf{x}$ is the differential of $\mathbf{f}$ at $\mathbf{x}$ and $\mathbf{M}_{\mathbf{f}(\mathbf{x})}$ is the differential of $\mathbf{g}$ at $\mathbf{f}(\mathbf{x})$.

It is required to show that

$$\mathbf{h}(\mathbf{x}+\mathbf{v}) = \mathbf{h}(\mathbf{x}) + (\mathbf{M}_{\mathbf{f}(\mathbf{x})}\circ\mathbf{L}_\mathbf{x})(\mathbf{v}) + \mathbf{R}(\mathbf{x},\mathbf{v})$$

where $\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{x},\mathbf{v})}{|\mathbf{v}|} = \mathbf{0}$, or, equivalently,

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) = \mathbf{g}(\mathbf{f}(\mathbf{x})) + \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{L}_\mathbf{x}(\mathbf{v})) + \mathbf{R}(\mathbf{x},\mathbf{v})$$

where $\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{x},\mathbf{v})}{|\mathbf{v}|} = \mathbf{0}$. Now, since $\mathbf{f}$ is differentiable at $\mathbf{x}$,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + \mathbf{L}_\mathbf{x}(\mathbf{v}) + \mathbf{R}_1(\mathbf{x},\mathbf{v})$$

where $\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{R}_1(\mathbf{x},\mathbf{v})}{|\mathbf{v}|} = \mathbf{0}$. Also, since $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x})$,

$$\mathbf{g}(\mathbf{f}(\mathbf{x})+\mathbf{u}) = \mathbf{g}(\mathbf{f}(\mathbf{x})) + \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{u}) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})$$

where $\lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}$. Now let

$$\mathbf{u} = \mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = \mathbf{L}_\mathbf{x}(\mathbf{v}) + \mathbf{R}_1(\mathbf{x},\mathbf{v})$$

Then, substituting in the above and using the fact that $\mathbf{M}$ is linear, we have

$$\begin{aligned}\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) &= \mathbf{g}(\mathbf{f}(\mathbf{x})) + \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{L}_\mathbf{x}(\mathbf{v}) + \mathbf{R}_1(\mathbf{x},\mathbf{v})) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u}) \\ &= \mathbf{g}(\mathbf{f}(\mathbf{x})) + \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{L}_\mathbf{x}(\mathbf{v})) + \mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{R}_1(\mathbf{x},\mathbf{v})) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})\end{aligned}$$

Thus it remains to show that

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{|\mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{R}_1(\mathbf{x},\mathbf{v})) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{v}|} = 0$$

From Problem 7.18, there exists a constant K such that

$$|\mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{R}_1(\mathbf{x},\mathbf{v}) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u}))| \leq K|\mathbf{R}_1(\mathbf{x},\mathbf{v}) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})| \leq K|\mathbf{R}_1(\mathbf{x},\mathbf{v})| + K|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|$$

But $\lim_{\mathbf{v}\to\mathbf{0}} \frac{|\mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$ since $\mathbf{f}$ is differentiable at $\mathbf{x}$. On the other hand $\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u}) = \mathbf{0}$ when $\mathbf{u} = \mathbf{0}$, and otherwise

$$\frac{|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{v}|} = \frac{|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{u}|}\frac{|\mathbf{u}|}{|\mathbf{v}|}$$

Now $\dfrac{|\mathbf{u}|}{|\mathbf{v}|} = \dfrac{|\mathbf{L_x}(\mathbf{v}) + \mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|}$ is bounded for $\mathbf{v}$ in a neighborhood of $\mathbf{0}$ because of Problem 7.18 and because $\lim\limits_{\mathbf{v}\to\mathbf{0}} \dfrac{|\mathbf{R}_1(\mathbf{x},\mathbf{v})|}{|\mathbf{v}|} = 0$ since $\mathbf{f}$ is differentiable at $\mathbf{x}$. Since $\mathbf{u}\to\mathbf{0}$ as $\mathbf{v}\to\mathbf{0}$ and since $\dfrac{|\mathbf{u}|}{|\mathbf{v}|}$ is bounded and $\lim\limits_{\mathbf{u}\to\mathbf{0}} \dfrac{|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{u}|} = 0$ because $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x})$,

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{v}|} = \lim_{\mathbf{u}\to\mathbf{0}} \frac{|\mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})|}{|\mathbf{v}|} = 0$$

Putting all this together, we conclude that

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{M}_{\mathbf{f}(\mathbf{x})}(\mathbf{R}_1(\mathbf{x},\mathbf{v})) + \mathbf{R}_2(\mathbf{f}(\mathbf{x}),\mathbf{u})}{|\mathbf{v}|} = \mathbf{0}$$

which proves the theorem.

FUNCTIONS OF CLASS C^m. INVERSE FUNCTION THEOREM

7.26. Find the following derivatives of $\mathbf{f}(\mathbf{x}) = (x_1^2 + x_2^2)\mathbf{g}_1 + x_1 e^{x_2}\mathbf{g}_2 + x_2 e^{x_1}\mathbf{g}_3$ at the point $\mathbf{x}_0 = \mathbf{e}_1 - \mathbf{e}_2$: (*a*) $\partial^2\mathbf{f}/\partial x_1\,\partial x_2$, (*b*) $\partial^3\mathbf{f}/\partial x_1^2\,\partial x_2$, (*c*) $D^2_{\mathbf{v}_0\mathbf{u}_0}\mathbf{f}$ in the directions $\mathbf{u}_0 = (\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{v}_0 = (\mathbf{e}_1 - 2\mathbf{e}_2)$.

(*a*)
$$\frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2} = \frac{\partial}{\partial x_1}\left[\frac{\partial\mathbf{f}}{\partial x_2}\right] = \frac{\partial}{\partial x_1}[2x_2\mathbf{g}_1 + x_1 e^{x_2}\mathbf{g}_2 + e^{x_1}\mathbf{g}_3] = e^{x_2}\mathbf{g}_2 + e^{x_1}\mathbf{g}_3$$

$$\frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}(\mathbf{x}_0) = e^{-1}\mathbf{g}_2 + e\mathbf{g}_3$$

(*b*)
$$\frac{\partial^3\mathbf{f}}{\partial x_1^2\,\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}\right) = e^{x_1}\mathbf{g}_3, \qquad \frac{\partial^3\mathbf{f}}{\partial x_1^2\,\partial x_2}(\mathbf{x}_0) = e\mathbf{g}_3$$

(*c*)
$$\begin{aligned} D^2_{\mathbf{vu}}\mathbf{f}(\mathbf{x}) &= \frac{\partial^2\mathbf{f}}{\partial x_1^2}u_1v_1 + \frac{\partial^2\mathbf{f}}{\partial x_1\,\partial x_2}(u_1v_2 + u_2v_1) + \frac{\partial^2\mathbf{f}}{\partial x_2^2}u_2v_2 \\ &= (2\mathbf{g}_1 + x_2e^{x_1}\mathbf{g}_3)u_1v_1 + (e^{x_2}\mathbf{g}_2 + e^{x_1}\mathbf{g}_3)(u_1v_2 + u_2v_1) + (2\mathbf{g}_1 + x_1e^{x_2}\mathbf{g}_2)u_2v_2 \end{aligned}$$

$$D^2_{\mathbf{v}_0\mathbf{u}_0}\mathbf{f}(\mathbf{x}_0) = (2\mathbf{g}_1 - e\mathbf{g}_3) + (e^{-1}\mathbf{g}_2 + e\mathbf{g}_3)(-1) + (2\mathbf{g}_1 + e^{-1}\mathbf{g}_2)(-2) = -(2\mathbf{g}_1 + 3e^{-1}\mathbf{g}_2 + 2e\mathbf{g}_3)$$

7.27. Let $\mathbf{w} = \mathbf{g}(y_1, y_2)$, $y_1 = y_1(x_1, x_2)$, $y_2 = y_2(x_1, x_2)$. Show that

$$\begin{aligned} \frac{\partial^2\mathbf{w}}{\partial x_1\,\partial x_2} &= \frac{\partial^2\mathbf{w}}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial^2\mathbf{w}}{\partial y_1\,\partial y_2}\left[\frac{\partial y_2}{\partial x_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\right] \\ &\quad + \frac{\partial^2\mathbf{w}}{\partial y_2^2}\frac{\partial y_2}{\partial x_1}\frac{\partial y_2}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1\,\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_2}\frac{\partial^2 y_2}{\partial x_1\,\partial x_2} \end{aligned}$$

$$\frac{\partial\mathbf{w}}{\partial x_2} = \frac{\partial\mathbf{w}}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_2}\frac{\partial y_2}{\partial x_2}$$

$$\begin{aligned} \frac{\partial^2\mathbf{w}}{\partial x_1\,\partial x_2} &= \frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{w}}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{w}}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_2}\frac{\partial y_2}{\partial x_2}\right) \\ &= \left[\frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{w}}{\partial y_1}\right)\right]\frac{\partial y_1}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1\,\partial x_2} + \left[\frac{\partial}{\partial x_1}\left(\frac{\partial\mathbf{w}}{\partial y_2}\right)\right]\frac{\partial y_2}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_2}\frac{\partial^2 y_2}{\partial x_1\,\partial x_2} \\ &= \left[\frac{\partial^2\mathbf{w}}{\partial y_1^2}\frac{\partial y_1}{\partial x_1} + \frac{\partial^2\mathbf{w}}{\partial y_2\,\partial y_1}\frac{\partial y_2}{\partial x_1}\right]\frac{\partial y_1}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1\,\partial x_2} \\ &\quad + \left[\frac{\partial^2\mathbf{w}}{\partial y_1\,\partial y_2}\frac{\partial y_1}{\partial x_1} + \frac{\partial^2\mathbf{w}}{\partial y_2^2}\frac{\partial y_2}{\partial x_1}\right]\frac{\partial y_2}{\partial x_2} + \frac{\partial\mathbf{w}}{\partial y_2}\frac{\partial^2 y_2}{\partial x_1\,\partial x_2} \end{aligned}$$

which gives the required result.

7.28. Show that the function

$$\mathbf{f}(\mathbf{x}) = (e^{x_1}\cos x_2)\mathbf{e}_1 + (e^{x_1}\sin x_2)\mathbf{e}_2$$

satisfies the conditions of the inverse function theorem on E^2 but is not 1-1 on E^2.

Clearly $\mathbf{f}$ is of class C^1 on E^2. Also

$$J(\mathbf{f})(\mathbf{x}) = \det\begin{pmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{pmatrix} = e^{2x_1} \neq 0$$

for all $\mathbf{x}$. Since $\mathbf{f}(\mathbf{x}+2\pi\mathbf{e}_2) = \mathbf{f}(\mathbf{x})$, the function is not 1-1 on E^2.

7.29. Suppose that the mapping $y_1 = f_1(x_1, x_2)$, $y_2 = f_2(x_1, x_2)$ satisfies the condition of the inverse function theorem and suppose that the inverse mapping is given by $x_1 = F_1(y_1, y_2)$, $x_2 = F_2(y_1, y_2)$. If $J = \partial(f_1 f_2)/\partial(x_1 x_2)$, show that

$$\frac{\partial F_1}{\partial y_1} = \frac{1}{J}\frac{\partial f_2}{\partial x_2}, \quad \frac{\partial F_2}{\partial y_1} = -\frac{1}{J}\frac{\partial f_2}{\partial x_1}, \quad \frac{\partial F_1}{\partial y_2} = -\frac{1}{J}\frac{\partial f_1}{\partial x_2}, \quad \frac{\partial F_2}{\partial y_2} = \frac{1}{J}\frac{\partial f_1}{\partial x_1}$$

From the chain rule

$$1 = \frac{\partial y_1}{\partial y_1} = \frac{\partial f_1}{\partial x_1}\frac{\partial F_1}{\partial y_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial F_2}{\partial y_1} \qquad 0 = \frac{\partial y_2}{\partial y_1} = \frac{\partial f_2}{\partial x_1}\frac{\partial F_1}{\partial y_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial F_2}{\partial y_1}$$

$$0 = \frac{\partial y_1}{\partial y_2} = \frac{\partial f_1}{\partial x_1}\frac{\partial F_1}{\partial y_2} + \frac{\partial f_1}{\partial x_2}\frac{\partial F_2}{\partial y_2} \qquad 1 = \frac{\partial y_2}{\partial y_2} = \frac{\partial f_2}{\partial x_1}\frac{\partial F_1}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial F_2}{\partial y_2}$$

The quantities $\partial F_1/\partial y_1$ and $\partial F_2/\partial y_1$ are solved for in the first and third equations, obtaining

$$\frac{\partial F_1}{\partial y_1} = \frac{\det\begin{pmatrix} 1 & \frac{\partial f_1}{\partial x_2} \\ 0 & \frac{\partial f_2}{\partial x_2}\end{pmatrix}}{\det\begin{pmatrix} \frac{\partial f_1}{x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}} = \frac{1}{J}\frac{\partial f_2}{\partial x_2}, \qquad \frac{\partial F_2}{\partial y_1} = \frac{\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & 1 \\ \frac{\partial f_2}{\partial x_1} & 0\end{pmatrix}}{\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}} = -\frac{1}{J}\frac{\partial f_2}{\partial x_1}$$

similarly for $\partial F_1/\partial y_2$ and $\partial F_2/\partial y_2$, which completes the problem.

Supplementary Problems

7.30. Show that $\mathbf{y} = \cos x_1 \mathbf{g}_1 + \sin x_1 \mathbf{g}_2 + x_2 \mathbf{g}_2$ maps the half strip $0 \leqq x_1 < 2\pi$, $-\infty < x_2 < \infty$ 1-1 onto the circular cylinder of radius 1 about the y_3 axis.

7.31. Show that $\mathbf{y} = \sinh x_1 \sin x_2 \mathbf{g}_1 + \sinh x_1 \cos x_2 \mathbf{g}_2 + \sinh x_1 \mathbf{g}_3$ maps the half strip $x_1 \geqq 0$, $0 < x_2 < 2\pi$ onto a circular cone about the y_3 axis.

7.32. Show that the rank of the linear mapping

$$y_1 = 2x_1 + x_2 - x_3, \quad y_2 = x_1 - x_2 - 2x_3, \quad y_3 = 3x_1 + 3x_2$$

is equal to 2 and determine the plane onto which E^3 is mapped.

7.33. Prove Theorem 7.1(ii): A linear mapping **f** of E^3 into E^3 maps E^3 onto a plane in E^3 if and only if the rank of **f** equals 2.

7.34. Prove Theorem 7.2(ii): A linear mapping **f** of E^2 into E^3 maps E^2 onto a line in E^3 if and only if the rank of **f** equals 1.

7.35. Show that $\mathbf{f}(\mathbf{x}) = \begin{cases} x_1 \sin 1/x_2 + x_2 \sin 1/x_1, & \text{if } x_1x_2 \neq 0 \\ 0, & \text{if } x_1x_2 = 0 \end{cases}$ is continuous at $\mathbf{x} = \mathbf{0}$.

7.36. If **f** is a linear function of **x**, show that there exists an $M > 0$ such that $|\mathbf{f}(\mathbf{x})| < M|\mathbf{x}|$ for all **x**.

7.37. If **f** and **g** are continuous at **x**, show that $\mathbf{f}+\mathbf{g}$ is continuous at **x**.

7.38. If **f** is continuous on a compact set V, show that there exists an $M \geqq 0$ such that $|\mathbf{f}(\mathbf{x})| \leqq M$ for **x** in **V**.

7.39. Show that $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = L_i$, $i = 1,\ldots,m$, where $\mathbf{f} = f_1\mathbf{g}_1 + \cdots + f_m\mathbf{g}_m$ and $\mathbf{L} = L_1\mathbf{g}_1 + \cdots + L_m\mathbf{g}_m$.

7.40. If $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}) = \mathbf{M}$, show that $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}) = \mathbf{L}\cdot\mathbf{M}$.

7.41. If **f** is a linear function on E, show that **f** is continuous on E.

7.42. Find the derivative of $\mathbf{f}(\mathbf{x}) = x_1x_2\mathbf{e}_1 + (x_1^2 + x_2^2)\mathbf{e}_2$ in the direction $\mathbf{u}_0 = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$ at $\mathbf{x}_0 = \mathbf{e}_1 + \mathbf{e}_2$.
Ans. 0

7.43. Determine (*a*) the set of points (x_1, x_2) where $\mathbf{f}(\mathbf{x}) = (x_1^2 + x_2^2)^{-1}\mathbf{e}_1 + (x_1^2 - x_2^2)^{-1}\mathbf{e}_2$ is continuously differentiable and (*b*) the Jacobian of **f**. *Ans.* (*a*) $x_1 \neq \pm x_2$, (*b*) $-8x_1x_2(x_1^2 + x_2^2)^{-2}(x_1^2 - x_2^2)^{-2}$

7.44. Determine the set of points (x_1, x_2, x_3) where the rank of

$$\mathbf{f}(\mathbf{x}) = (x_1 \sin x_2 \cos x_3)\mathbf{e}_1 + (x_1 \sin x_2 \sin x_3)\mathbf{e}_2 + (x_1 \cos x_2)\mathbf{e}_3$$

equals 3. *Ans.* $x_1^2 \sin x_2 \neq 0$

7.45. Find the equation of the plane tangent to the surface in E^3 defined by

$$\mathbf{y} = (x_1^2 + x_2^2)\mathbf{e}_1 - x_1x_2^2\mathbf{e}_2 + x_2x_1^2\mathbf{e}_3$$

at the point corresponding to $\mathbf{x}_0 = \mathbf{e}_1 + \mathbf{e}_2$.
Ans. $y_1 = 2 + 2v_1 + 2v_2$, $y_2 = -1 - v_1 - 2v_2$, $y_3 = 1 + 2v_1 + v_2$

7.46. Show that the Jacobian of a function **f** on E^3 is given by the triple product

$$J(\mathbf{f}) = \left[\frac{\partial \mathbf{f}}{\partial x_1}\,\frac{\partial \mathbf{f}}{\partial x_2}\,\frac{\partial \mathbf{f}}{\partial x_3}\right]$$

7.47. If **f** and **g** are differentiable at **x**, show that
(*a*) $\mathbf{f}+\mathbf{g}$ is differentiable at **x** and $d(\mathbf{f}+\mathbf{g}) = d\mathbf{f} + d\mathbf{g}$,
(*b*) $\mathbf{f}\times\mathbf{g}$ is differentiable at **x** and $d(\mathbf{f}\times\mathbf{g}) = (d\mathbf{f})\times\mathbf{g} + \mathbf{f}\times d\mathbf{g}$.

7.48. Let $\begin{cases} w_1 = y_1 + y_2 + y_3 \\ w_2 = y_1y_2 \\ w_3 = y_1y_3 \end{cases}$ and $\begin{cases} y_1 = x_1x_2x_3 \\ y_2 = e^{x_2} \\ y_3 = e^{x_3} \end{cases}$. Determine $\dfrac{\partial(w_1w_2w_3)}{\partial(x_1x_2x_3)}$.
Ans. $x_1x_2^2x_3^2e^{x_2+x_3}(x_1x_2x_3 - e^{x_2} - e^{x_3})$

7.49. Find the following derivatives of $\mathbf{f}(\mathbf{x}) = (x_1 + x_2)^2\mathbf{e}_1 + x_1 \sin x_2 \mathbf{e}_2$ at $\mathbf{x}_0 = \mathbf{e}_1 + \pi\mathbf{e}_2$:
(a) $\dfrac{\partial^2 \mathbf{f}}{\partial x_1 \partial x_2}$, (b) $\dfrac{\partial^3 \mathbf{f}}{\partial x_1 \partial x_2^2}$, (c) $D_{\mathbf{v}_0\mathbf{u}_0}\mathbf{f}$; $\mathbf{u}_0 = (\mathbf{e}_1 + 2\mathbf{e}_2)$ and $\mathbf{v}_0 = (\mathbf{e}_1 - \mathbf{e}_2)$

Ans. (a) $2\mathbf{e}_1 - \mathbf{e}_2$, (b) $\mathbf{0}$, (c) $-\mathbf{e}_2$

7.50. If $\mathbf{w} = (y_1 + y_2)\mathbf{e}_1 + e^{y_1+y_2}\mathbf{e}_2$ and $\mathbf{y} = x_1 x_2^2 \mathbf{e}_1 + x_1^2 x_2 \mathbf{e}_2$, find $\partial^2 w/\partial x_1 \partial x_2$ using the chain rule.
Ans. $(2x_2 + 2x_1)\mathbf{e}_1 + e^{y_1+y_2}[(x_1^2 + 2x_1x_2)(x_2^2 + 2x_1x_2) + (2x_2 + 2x_1)]\mathbf{e}_2$

7.51. Show that $\mathbf{f}(\mathbf{x}) = (x_1^2 - x_2^2)\mathbf{e}_1 + x_1x_2\mathbf{e}_2$ satisfies the conditions of the inverse function theorem for all $\mathbf{x}$ except $\mathbf{x} = \mathbf{0}$ but is not 1-1 on this set.

7.52. Show that the function

$$\mathbf{f}(\mathbf{x}) = \frac{x_1}{1 + x_1 + x_2 + x_3}\mathbf{g}_1 + \frac{x_2}{1 + x_1 + x_2 + x_3}\mathbf{g}_2 + \frac{x_3}{1 + x_1 + x_2 + x_3}\mathbf{g}_3$$

satisfies the conditions of the inverse function theorem for all $\mathbf{x}$ in E^3 for which $1 + x_1 + x_2 + x_3 \neq 0$. Show that $\mathbf{f}$ is 1-1 where defined, and find $\mathbf{f}^{-1}(\mathbf{x})$ explicitly.

Ans. $$\mathbf{f}^{-1}(\mathbf{x}) = \frac{x_1}{1 - x_1 - x_2 - x_3}\mathbf{g}_1 + \frac{x_2}{1 - x_1 - x_2 - x_3}\mathbf{g}_2 + \frac{x_3}{1 - x_1 - x_2 - x_3}\mathbf{g}_3$$

7.53. If $\mathbf{f}$ satisfies the conditions of the inverse function theorem on an open set V in E, show that $J(\mathbf{f}^{-1})J(\mathbf{f}) = 1$ where $\mathbf{f}$ is 1-1.

7.54. Find the first three terms of the expansion of $\mathbf{f}(\mathbf{x}) = (x_1 \sin x_2)\mathbf{e}_1 + (x_2 \cos x_1)\mathbf{e}_2$ about $\mathbf{x}_0 = \pi\mathbf{e}_2$.
Ans. $\pi\mathbf{e}_2 + \mathbf{e}_2(x_2 - \pi) - \frac{1}{2}\pi\mathbf{e}_2 x_1^2 - \mathbf{e}_1 x_1(x_2 - \pi)$

Chapter 8

Concept of a Surface

REGULAR PARAMETRIC REPRESENTATIONS

Intuitively we think of a surface as a set of points in space which resembles a portion of a plane in the neighborhood of each of its points. This will be the case if the surface is the image of a sufficiently regular mapping of a set of points in the plane into E^3. Since we want to apply the methods of calculus, we assume that the mapping is at least of class C^1. Also, in order to insure that the surface has a tangent plane at each point, we assume that the rank of the Jacobian matrix of the mapping is two at each point. Thus we are led to the following definition:

A regular parametric representation of class C^m $(m \geqq 1)$ of a set of points S in E^3 is a mapping $\mathbf{x} = \mathbf{f}(u, v)$ of an open set U in the uv plane onto S such that

(i) $\mathbf{f}$ is of class C^m in U.

(ii) If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a basis in E^3 and $\mathbf{f}(u, v) = f_1(u, v)\mathbf{e}_1 + f_2(u, v)\mathbf{e}_2 + f_3(u, v)\mathbf{e}_3$, then for all (u, v) in U,

$$\text{rank}\begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{pmatrix} = 2$$

We recall that $\mathbf{f}$ is of class C^m in U if all partial derivatives of $\mathbf{f}$ of order m or less are continuous in U. We also recall that the rank of a matrix is the order of the largest nonvanishing minor of the matrix. Thus the rank of the above matrix is 2 if and only if at least one of the three 2×2 minors of the matrix is different from zero.

As in the case of curves, the variables u and v are called parameters. Also, a parametric representation will be denoted by $\mathbf{x} = \mathbf{x}(u, v)$ and its partial derivatives by

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}, \quad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}, \quad \mathbf{x}_{uu} = \frac{\partial^2 \mathbf{x}}{\partial u^2}, \quad \mathbf{x}_{uv} = \frac{\partial^2 \mathbf{x}}{\partial v\, \partial u}, \quad \text{etc.}$$

Note that strictly speaking a parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ is a *mapping*. However, sometimes we will speak rather loosely and identify the mapping with its *image*, a set of points in E^3. Thus we might say that P is a point on the parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ when P is a point on the image of $\mathbf{x} = \mathbf{x}(u, v)$, or we might say that the parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ is contained in a set of points S in E^3 when the image of $\mathbf{x} = \mathbf{x}(u, v)$ is a subset of S.

Now suppose that $\mathbf{x} = \mathbf{x}(u, v)$ is a regular parametric representation of S defined on U, as indicated in Fig. 8-1. Observe that the image of the coordinate line $v = v_0$ in U will be a curve $\mathbf{x} = \mathbf{x}(u, v_0)$ on S along which u is a parameter. This curve is called the *u-parameter curve*, $v = v_0$. Similarly, the image of the coordinate line $u = u_0$ is the curve $\mathbf{x} = \mathbf{x}(u_0, v)$ on S, called the *v-parameter curve*, $u = u_0$. Thus the parametric representation covers S with two families of curves, the image of the coordinate lines $v =$ constant and $u =$ constant.

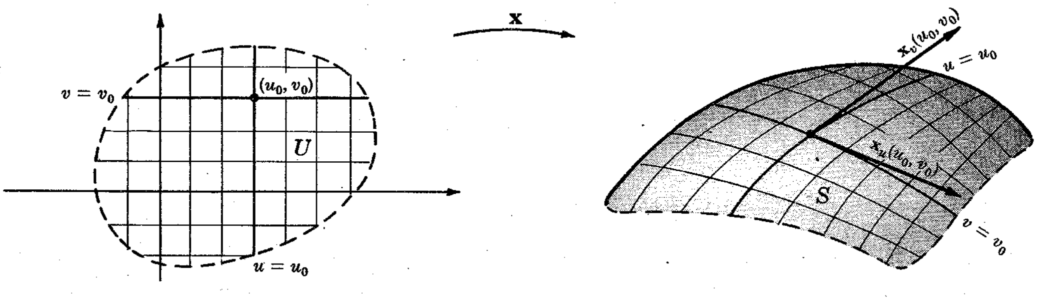


Fig. 8-1

We recall further that $\mathbf{x}_u(u_0, v_0)$ is the derivative of $\mathbf{x}$ at (u_0, v_0) in the direction of the u axis. Hence $\mathbf{x}_u(u_0, v_0)$ is a vector which is tangent to the u-parameter curve at $\mathbf{x}(u_0, v_0)$ in the direction of increasing u. Similarly, $\mathbf{x}_v(u_0, v_0)$ is a vector tangent to the v-parameter curve at $\mathbf{x}(u_0, v_0)$ in the direction of increasing v.

Finally we note that the three components of the vector product

$$\mathbf{x}_u \times \mathbf{x}_v = \det\begin{pmatrix} \mathbf{e}_1 & \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \mathbf{e}_2 & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \\ \mathbf{e}_3 & \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

$$= \left(\frac{\partial x_2}{\partial u}\frac{\partial x_3}{\partial v} - \frac{\partial x_3}{\partial u}\frac{\partial x_2}{\partial v}\right)\mathbf{e}_1 + \left(\frac{\partial x_3}{\partial u}\frac{\partial x_1}{\partial v} - \frac{\partial x_1}{\partial u}\frac{\partial x_3}{\partial v}\right)\mathbf{e}_2 + \left(\frac{\partial x_1}{\partial u}\frac{\partial x_2}{\partial v} - \frac{\partial x_2}{\partial u}\frac{\partial x_1}{\partial v}\right)\mathbf{e}_3$$

differ from the 2×2 minors of the Jacobian matrix of $\mathbf{x}$ at most by a sign. Hence the rank of the Jacobian matrix of $\mathbf{x}$ is two if and only if $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. Thus a mapping $\mathbf{x} = \mathbf{x}(u, v)$ of an open set U onto S is a regular parametric representation of class C^m of S if and only if

(i) $\mathbf{x}$ belongs to class C^m $(m \geq 1)$ on U.

(ii) $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ for all (u, v) in U.

Example 8.1.

(*a*) The equation

$$\mathbf{x} = (u + v)\mathbf{e}_1 + (u - v)\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$$

defines a mapping of the uv plane onto the elliptic paraboloid $x_3 = \frac{1}{2}(x_1^2 + x_2^2)$ shown in Fig. 8-2. Clearly $\mathbf{x}$ has continuous partial derivatives of all orders. Also, for all (u, v),

$$|\mathbf{x}_u \times \mathbf{x}_v| = \left|\det\begin{pmatrix} \mathbf{e}_1 & 1 & 1 \\ \mathbf{e}_2 & 1 & -1 \\ \mathbf{e}_3 & 2u & 2v \end{pmatrix}\right|$$

$$= [4 + 8(u^2 + v^2)]^{1/2} \neq 0$$

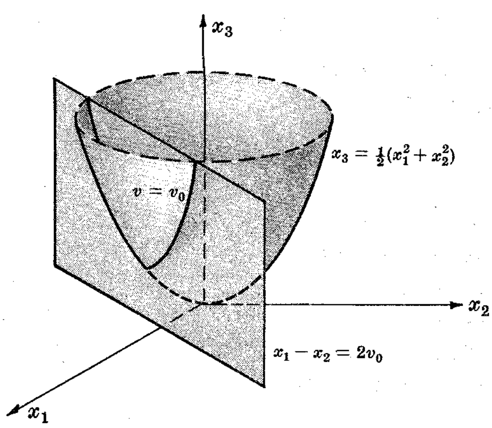


Fig. 8-2

Thus $\mathbf{x}$ is a regular parametric representation of the paraboloid of class C^∞. By eliminating u from the components $x_1 = u + v_0$, $x_2 = u - v_0$ of $\mathbf{x}$, we find that the u-parameter curve, $v = v_0$, is the intersection of the paraboloid and the vertical plane $x_1 - x_2 = 2v_0$. Similarly, the v-parameter curve, $u = u_0$, is the intersection of the paraboloid and the vertical plane $x_1 + x_2 = 2u_0$.

(*b*) We recall (see Problem 7.1, page 136) that

$$\mathbf{x} = (\cos\theta \sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3$$

defines a mapping of the $\theta\phi$ plane onto the unit sphere $|\mathbf{x}| = x_1^2 + x_2^2 + x_3^2 = 1$. Again, $\mathbf{x}$ has partial derivatives of all orders. Note however, that the mapping is not regular along the coordinate lines $\phi = \pm\pi n$, $n = 0, 1, \ldots$, where

$$|\mathbf{x}_\theta \times \mathbf{x}_\phi| = \det\begin{pmatrix} \mathbf{e}_1 & -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \mathbf{e}_2 & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \mathbf{e}_3 & 0 & -\sin\phi \end{pmatrix}$$

$$= |(-\cos\theta\sin^2\phi)\mathbf{e}_1 - (\sin\theta\sin^2\phi)\mathbf{e}_2 - (\sin\phi\cos\phi)\mathbf{e}_3| = |\sin\phi| = 0$$

If the mapping is restricted to the infinite strip $-\infty < \theta < \infty$, $0 < \phi < \pi$, it will be a regular parametric representation of class C^∞ of the sphere punctured at the north and south poles, as shown in Fig. 8-3.

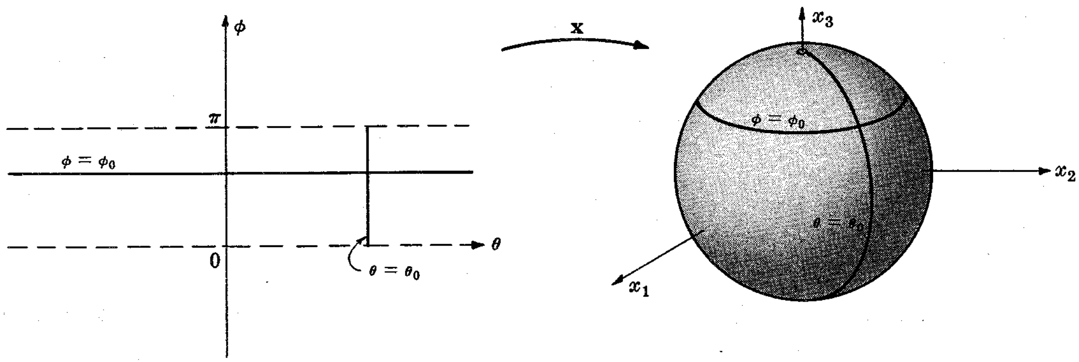


Fig. 8-3

The family of θ-parameter curves, $\phi = \phi_0$, are called the *parallels of latitude*. They are the intersections of the sphere and the family of horizontal planes $x_3 = \cos\phi_0$. The family of ϕ-parameter curves, $\theta = \theta_0$, are called *meridians of longitude*. They are the intersections of the sphere and the family of planes through the x_3 axis, $x_1 \sin\theta_0 - x_2\cos\theta_0 = 0$. Note that the parallels of longitude and the meridians of longitude intersect at right angles, since

$$\mathbf{x}_\theta \cdot \mathbf{x}_\phi = ((-\sin\theta\sin\phi)\mathbf{e}_1 + (\cos\theta\sin\phi)\mathbf{e}_2) \cdot ((\cos\theta\cos\phi)\mathbf{e}_1 + (\sin\theta\cos\phi)\mathbf{e}_2 - (\sin\phi)\mathbf{e}_3) = 0$$

(*c*) A *cylinder* is a surface generated by a line L as it moves parallel to itself along a curve C. As indicated in Fig. 8-4, if C is given by $\mathbf{y} = \mathbf{y}(u)$, and $\mathbf{g}$ is a unit vector in the direction of L, then the cylinder is represented by $\mathbf{x} = \mathbf{y}(u) + v\mathbf{g}$. Clearly, $\mathbf{x}$ is a regular representation of class C^m if $\mathbf{y}$ is of class C^m and $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{y}' \times \mathbf{g} \neq \mathbf{0}$ for all u. The u-parameter curves are *translations* of C in the direction of $\mathbf{g}$. The v-parameter curves are the copies of L and are called the *rulings* of the cylinder.

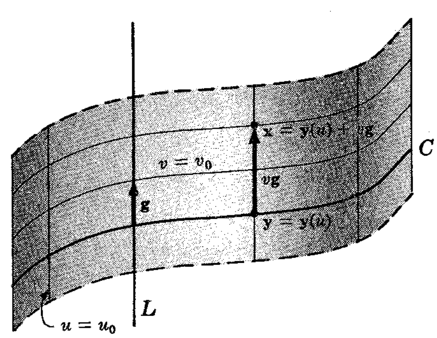


Fig. 8-4

COORDINATE PATCHES

Suppose $\mathbf{x} = \mathbf{x}(u, v)$ is a regular parametric representation of class C^m of S defined on U as shown in Fig. 8-5 and suppose $\theta = \theta(u, v)$, $\phi = \phi(u, v)$ is a mapping of class C^m of U into the $\theta\phi$ plane such that at each (u, v) the Jacobian $\partial(\theta, \phi)/\partial(u, v) \neq 0$. Now, in general, such a mapping $\theta = \theta(u, v)$, $\phi = \phi(u, v)$ is not 1-1. However, it follows from the inverse function theorem that this is the case at least locally. That is, for every (u_0, v_0) in U there exists an open set W containing (u_0, v_0) such that $\theta = \theta(u, v)$, $\phi = \phi(u, v)$ maps W 1-1 onto an open set W^* and such that its inverse $u = u(\theta, \phi)$, $v = v(\theta, \phi)$ is of class C^m on W^*. Now consider the composite mapping $\mathbf{x} = \mathbf{x}^*(\theta, \phi) = \mathbf{x}(u(\theta, \phi), v(\theta, \phi))$ of W^* into S. It follows from the chain rule that $\mathbf{x}^*(\theta, \phi)$ is of class C^m. Also,

$$\mathbf{x}^*_\theta \times \mathbf{x}^*_\phi \;=\; (\mathbf{x}_u u_\theta + \mathbf{x}_v v_\theta) \times (\mathbf{x}_u u_\phi + \mathbf{x}_v v_\phi) \;=\; (\mathbf{x}_u \times \mathbf{x}_v)(u_\theta v_\phi - v_\theta u_\phi) \;=\; (\mathbf{x}_u \times \mathbf{x}_v)\frac{\partial(u, v)}{\partial(\theta, \phi)} \;\neq\; 0$$

since $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ and the Jacobian $\dfrac{\partial(u, v)}{\partial(\theta, \phi)} = \left[\dfrac{\partial(\theta, \phi)}{\partial(u, v)}\right]^{-1} \neq 0$. Thus $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is a regular parametric representation of class C^m, but, note, only of a *part* of S. Since it would be much too restrictive to consider only those changes of parameter $u = u(\theta, \phi)$, $v = v(\theta, \phi)$ which are 1-1 over all of U, we are led to define a surface in terms of a collection of such partial representations of the surface, instead of a single representation of the whole surface. Namely, we define:

A coordinate patch of class C^m $(m \geq 1)$ in S is a mapping $\mathbf{x} = \mathbf{x}(u, v)$ of an open set U into S such that

(i) $\mathbf{x}$ is of class C^m on U.

(ii) $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ for all (u, v) in U.

(iii) $\mathbf{x}$ is 1-1 and bicontinuous on U.

Thus a coordinate patch is a regular parametric representation of a part of S, which is 1-1 and bicontinuous.

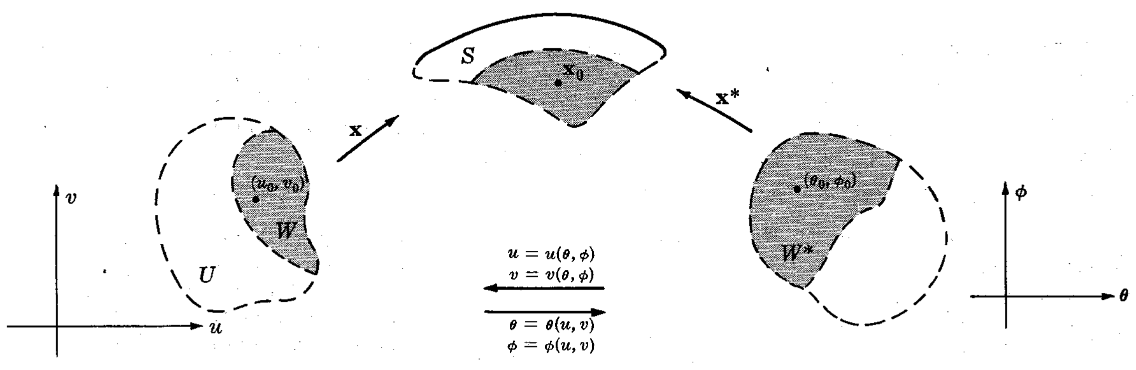


Fig. 8-5

Example 8.2.

(a)
$$\mathbf{x} \;=\; u\mathbf{e}_1 + v\mathbf{e}_2 + \sqrt{1 - (u^2 + v^2)}\,\mathbf{e}_3, \qquad u^2 + v^2 < 1$$

defines a mapping of the unit disk $u^2 + v^2 < 1$ onto the upper hemisphere of the unit sphere $|\mathbf{x}| = 1$. Clearly $\mathbf{x}$ is of class C^∞. Also,

$$|\mathbf{x}_u \times \mathbf{x}_v| \;=\; \left|\frac{u}{[1 - (u^2 + v^2)]^{1/2}}\mathbf{e}_1 + \frac{v}{[1 - (u^2 + v^2)]^{1/2}}\mathbf{e}_2 + \mathbf{e}_3\right| \;=\; [1 + (u^2 + v^2)]^{-1/2} \;\neq\; 0$$

for all (u, v). Thus $\mathbf{x}$ is a regular parametric representation of the hemisphere of class C^∞. The mapping is also 1-1; for $\mathbf{x}(u, v) = \mathbf{x}(u', v')$ implies $(u, v) = (u', v')$, since $x_1 = u$ and $x_2 = v$. The mapping is also bicontinuous; for clearly $\mathbf{x}$ is continuous, and the inverse mapping, which is the projection $u = x_1$, $v = x_2$, is continuous. Thus the mapping is a coordinate patch of class C^∞ on the sphere.

(*b*) Consider the cylinder generated by a vertical line as it moves along the curve in the x_1x_2 plane whose polar equation is $r = \sin 2\theta$ for θ in the interval $0 < \theta < 3\pi/4$, as shown in Fig. 8-6. Note that the cylinder does not quite intersect itself, since we excluded the endpoint $\theta = 0$. It is easily verified that

$$\mathbf{x} = (\sin 2\theta \cos\theta)\mathbf{e}_1 + (\sin 2\theta \sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$$

$(0 < \theta < 3\pi/4,\ -\infty < u < \infty)$ is a regular parametric representation of the cylinder of class C^∞ and is 1-1. The inverse mapping, however, is not continuous, since any neighborhood of a point on the x_3 axis, i.e. where $\theta = \pi/2$, will include points of the cylinder near the edge, $\theta = 0$. Thus this representation is not a coordinate patch on the cylinder. The restriction of $\mathbf{x}$ to (*a*) $0 < \theta < \pi/2$, $-\infty < u < \infty$ and (*b*) $\pi/4 < \theta < 3\pi/4$, $-\infty < u < \infty$ defines two coordinate patches which cover the cylinder.

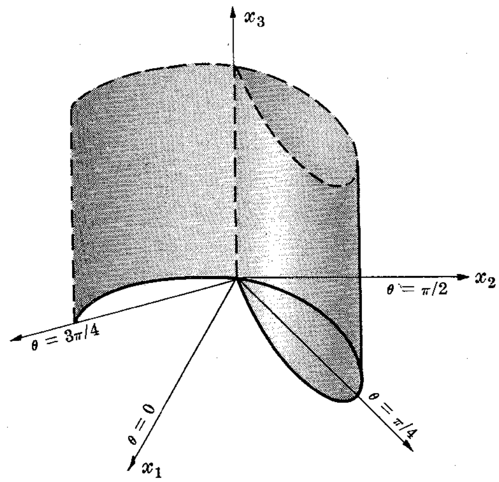


Fig. 8-6

In Problem 8.7 we show that functions of the form $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$, or $\mathbf{x} = u\mathbf{e}_1 + f(u,v)\mathbf{e}_2 + v\mathbf{e}_3$, or $\mathbf{x} = f(u,v)\mathbf{e}_1 + u\mathbf{e}_2 + v\mathbf{e}_3$ define coordinate patches of class C^m simply if $f(u,v)$ is a function of class C^m. These patches are called *Monge* patches and are very useful in the study of surfaces. In fact we will prove that if a set S can be represented by a regular parametric representation of class C^m, then for every P_0 in S there exists a Monge patch of class C^m in S containing P_0. For suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a regular parametric representation of class C^m of S defined on U and (u_0, v_0) is a point in U which maps into P_0, as shown in Fig. 8-7. Since $\mathbf{x}(u,v)$ is regular, at least one of the 2×2 minors of the Jacobian matrix of $\mathbf{x}$ is not zero at (u_0, v_0). Without loss of generality we can assume that

$$\det\begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v}\end{pmatrix} \neq 0$$

at (u_0, v_0). Now consider the mapping $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ of U into the x_1x_2 plane defined by the first two components of $\mathbf{x}$. Clearly this mapping is of class C^m on U, since $\mathbf{x}$ is of class C^m on U. Also its Jacobian, the above determinant, is different from zero in some neighborhood of (u_0, v_0), since it is continuous and different from zero at (u_0, v_0). But then it follows from the inverse function theorem that there exists an open set W in U containing (u_0, v_0) in which the mapping is 1-1 and has an inverse $u = u(x_1, x_2)$, $v = v(x_1, x_2)$ which is of class C^m on an open set W^* in the x_1x_2 plane. But then the composite mapping of W^* into S,

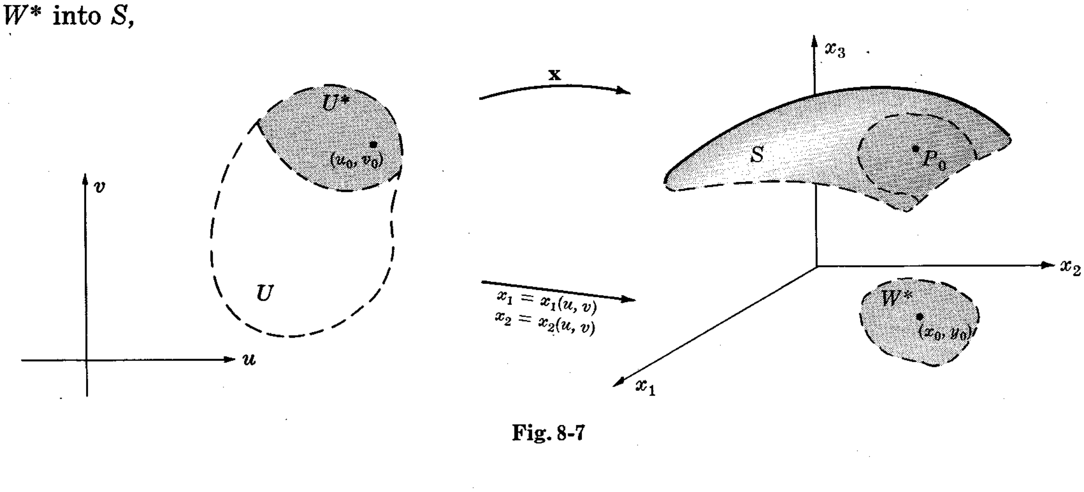


Fig. 8-7

$$\begin{aligned}\mathbf{x} &= x_1(u(x_1,x_2),\, v(x_1,x_2))\mathbf{e}_1 + x_2(u(x_1,x_2),\, v(x_1,x_2))\mathbf{e}_2 + x_3(u(x_1,x_2),\, v(x_1,x_2))\mathbf{e}_3 \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3(u(x_1,x_2),\, v(x_1,x_2))\mathbf{e}_3\end{aligned}$$

is a Monge patch of class C^m in S defined on W^* whose image contains P_0. Thus

Theorem 8.1. If a set S in E^3 has a regular parametric representation of class C^m, then for every point P in S there exists a Monge patch of class C^m in S containing P.

DEFINITION OF A SIMPLE SURFACE

Let S be a set of points in E^3 for which there exists a collection $\mathcal{B}$ of coordinate patches of class C^m $(m \geq 1)$ on S satisfying

(i) $\mathcal{B}$ covers S, i.e. for every point P in S there exists a coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ in $\mathcal{B}$ containing P.

(ii) Every coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ in $\mathcal{B}$ is the intersection of an open set O in E^3 with S, as shown in Fig. 8-8.

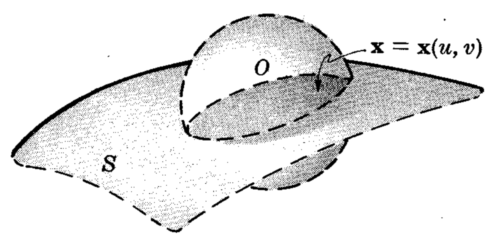


Fig. 8-8

Then S together with the totality of coordinate patches of class C^m in S is a *simple surface of class C^m in E^3*.

A set of patches $\mathcal{B}$ on S satisfying (i) and (ii) above is called a *basis* or a *coordinate patch representation* of S. Thus if a basis of class C^m can be found for a set of points S in E^3, then S together with the set of all patches of class C^m on S is a simple surface of class C^m.

Note that it is not unusual to refer to the set of points S itself as the surface. However, strictly speaking, the surface consists of S together with all patches of the given class on S.

Since a function of class C^m is also of class C^j for $j \leq m$, a basis of class C^m is a basis of class C^j, $j \leq m$. Thus a simple surface of class C^m can always be extended to a simple surface of class C^j, $j \leq m$, by adjoining all coordinate patches of class C^j.

Example 8.3.

(*a*) The upper hemisphere (not including the equator) of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ is a simple surface of class C^∞, since as a basis we can take the Monge patch of class C^∞,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \sqrt{1 - x_1^2 - x_2^2}\,\mathbf{e}_3, \quad x_1^2 + x_2^2 < 1$$

which covers the hemisphere and is the intersection of the hemisphere and the open set E^3 itself.

(*b*) A simple surface does not have a boundary. For example, the upper hemisphere of $x_1^2 + x_2^2 + x_3^2 = 1$ including the equator, is not a simple surface. For suppose otherwise, and let $P(x_{1_0}, x_{2_0}, 0)$ be a point on the equator and $\mathbf{x} = \mathbf{x}(u, v)$ a patch on the hemisphere containing P. Since $\mathbf{x} = \mathbf{x}(u, v)$ is a regular representation containing P, it follows from the previous theorem that there exists a Monge patch containing P of the form

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \sqrt{1 - x_1^2 - x_2^2}\,\mathbf{e}_3$$

and which is defined on an open set W in the x_1x_2 plane, where $x_1^2 + x_2^2 \leq 1$ for (x_1, x_2) in W, as shown in Fig. 8-9. But every neighborhood of the point $(x_{1_0}, x_{2_0}, 0)$ in W contains points not in W, which is impossible, since W is open.

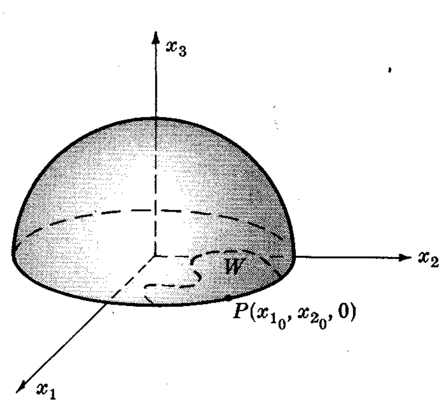


Fig. 8-9

(c) The whole sphere, however, is again a simple surface of class C^∞. As a basis we can take the six Monge patches

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \pm \sqrt{1-x_1^2-x_2^2}\,\mathbf{e}_3, \qquad \mathbf{x} = x_1\mathbf{e}_1 \pm \sqrt{1-x_1^2-x_3^2}\,\mathbf{e}_2 + x_3\mathbf{e}_3,$$
$$\mathbf{x} = \pm\sqrt{1-x_2^2-x_3^2}\,\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

These hemispheres clearly cover the sphere and each is the intersection of the sphere and an appropriate open half space of E^3. For example, the patch $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \sqrt{1-x_1^2-x_2^2}\,\mathbf{e}_3$ is the intersection of the sphere and the half space $x_3 > 0$.

(d) A simple surface cannot intersect itself. For example, consider the cylinder shown in Fig. 8-10 and suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing a point P on the intersection. Since $\mathbf{x} = \mathbf{x}(u,v)$ is a 1-1 bicontinuous mapping of an open set in the plane, it can be on only one piece of the intersection. On the other hand, every open set in E^3 containing P must include points on both pieces of the intersection. Thus no patch containing P can be the intersection of an open set in E^3 with the cylinder.

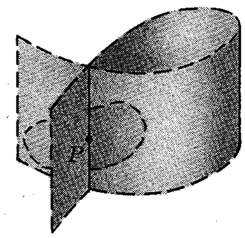


Fig. 8-10

Let P be a point on a simple surface S of class C^m and let $\mathbf{x} = \mathbf{x}(u,v)$ be an arbitrary coordinate patch of class C^m on S defined on an open set U and containing P as shown in Fig. 8-11. Note that the patch is not a connected set in E^3 if U is not connected. However, let (u,v) be the point in U which is mapped into P and let $S(u,v)$ be a spherical neighborhood of (u,v) contained in U. $S(u,v)$ exists since U is open. But then the restriction of $\mathbf{x}$ to $S(u,v)$ is a patch on S of class C^m which is connected and which contains P. Namely, we have

Theorem 8.2. For every point P on a simple surface S of class C^m there exists a connected patch of class C^m on S containing P.

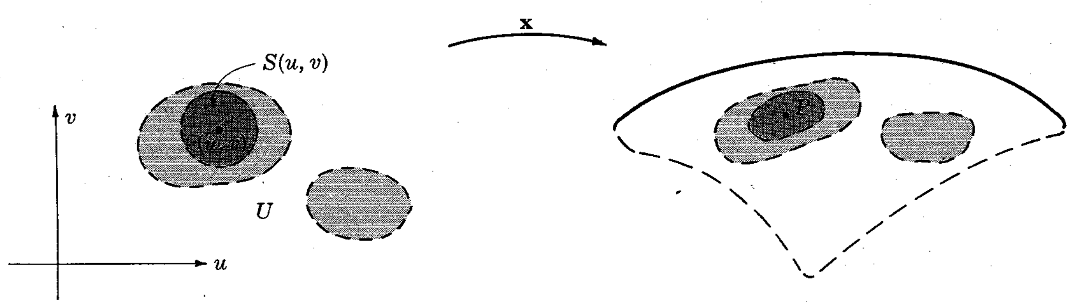


Fig. 8-11

Now suppose that $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ are two patches on a simple surface S of class C^m with a nonempty intersection G. Let W be the set in the uv plane which $\mathbf{x}$ maps onto G and W^* the set in the $\theta\phi$ plane which $\mathbf{x}^*$ maps onto G, as shown in Fig. 8-12. Since both $\mathbf{x}$ and $\mathbf{x}^*$ are 1-1, there exists a 1-1 parameter transformation $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ of W onto W^* such that for all (u,v) in W^* we have $\mathbf{x}(u,v) = \mathbf{x}^*(\theta(u,v), \phi(u,v))$. In Problem 8.16, page 166, we will show that W is open, $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ is of class C^m, and for all (u,v) the Jacobian $\partial(\theta,\phi)/\partial(u,v) \neq 0$.

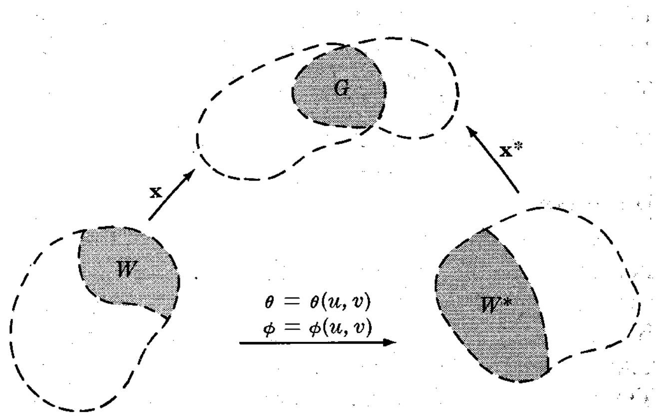


Fig. 8-12

A mapping $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ of class C^m $(m \geqq 1)$ of an open set W in the uv plane into the $\theta\phi$ plane which is 1-1 and such that for all (u,v) in W the Jacobian $\partial(\theta,\phi)/\partial(u,v) \neq 0$ is called an *allowable parameter transformation.* Note that it follows directly from the inverse mapping theorem that if $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ is an allowable parameter transformation of class C^m with image W^*, then W^* is open, the inverse mapping $u = u(\theta,\phi)$, $v = v(\theta,\phi)$ is of class C^m on W^*, and for all (θ,ϕ) in W^* the Jacobian $\dfrac{\partial(u,v)}{\partial(\theta,\phi)} = \left[\dfrac{\partial(\theta,\phi)}{\partial(u,v)}\right]^{-1} \neq 0$. Namely, the inverse of an allowable coordinate transformation is an allowable coordinate transformation. As a result of the above we have

Theorem 8.3. On the intersection of two coordinate patches on a simple surface of class C^m, the parameters are related by allowable coordinate transformations of class C^m.

Example 8.4.

The equations

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + \sqrt{1-u^2-v^2}\,\mathbf{e}_3, \qquad u^2+v^2 > 1$$

and

$$\mathbf{x} = (\cos\theta\sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3, \qquad 0<\theta<\pi,\ 0<\phi<\pi$$

define coordinate patches of class C^∞ on the sphere $x_1^2+x_2^2+x_3^2 = 1$. Their intersection is half the upper hemisphere as shown in Fig. 8-13. Along the intersection we have the parameter transformation $u = \cos\theta\sin\phi$ and $v = \sin\theta\sin\phi$, where $0<\theta<\pi$ and $0<\phi<\pi/2$. This mapping is 1-1, of class C^∞, and, since $0<\phi<\pi/2$,

$$\partial(u,v)/\partial(\theta,\phi) = \sin\phi\cos\phi \neq 0$$

The inverse transformation is

$$\phi = \cos^{-1}\sqrt{1-u^2-v^2}, \qquad \theta = \cos^{-1} u/(u^2+v^2)$$

where here $u^2+v^2 < 1$ and $v > 0$.

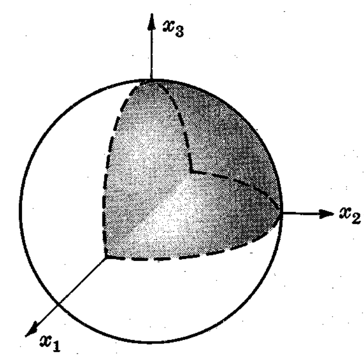


Fig. 8-13

A property defined in terms of a given basis of a surface may or may not be independent of the choice of the particular basis. If it is independent of the basis we say that the property is a property of the surface. In particular, a local property is a property of the surface if and only if it is independent of an allowable parameter transformation.

TANGENT PLANE AND NORMAL LINE

Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a simple surface of class C^m and let $u = u(t)$, $v = v(t)$ be a regular curve C of class C^m in the parameter plane. Now consider the image $\mathbf{x} = \mathbf{y}(t) = \mathbf{x}(u(t), v(t))$ of C on the surface. Clearly $\mathbf{y}(t)$ is a function of class C^m since it is a composite of functions of class C^m. Also, for all t, the tangent vector $d\mathbf{y}/dt \neq \mathbf{0}$. For, suppose otherwise. Since $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ for all (u,v), it follows that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent. Thus if $\dfrac{d\mathbf{y}}{dt} = \mathbf{x}_u\dfrac{du}{dt} + \mathbf{x}_v\dfrac{dv}{dt} = \mathbf{0}$ at some t, then $du/dt = 0$ and $dv/dt = 0$ at t. But this is impossible, since C is regular. Thus every regular curve $u = u(t)$, $v = v(t)$ of class C^m in the parameter plane maps onto a regular curve $\mathbf{x} = \mathbf{y}(t) = \mathbf{x}(u(t), v(t))$ of class C^m on the surface.

Suppose now we start with a regular curve $\mathbf{x} = \mathbf{y}(t)$ of class C^m on the surface. It may not be possible to find a single coordinate patch containing the complete curve. However, consider any connected part of the curve which is contained in a coordinate patch $\mathbf{x} = \mathbf{x}(u,v)$. Since $\mathbf{x}(u,v)$ is 1-1, there exists a unique curve C, $u = u(t)$, $v = v(t)$, in the parameter plane such that $\mathbf{y}(t) = \mathbf{x}(u(t), v(t))$. It can be shown that C is also regular and of class C^m. Thus locally every regular curve $\mathbf{x} = \mathbf{y}(t)$ of class C^m on the surface is the image of a unique regular curve $u = u(t)$, $v = v(t)$ of class C^m in the parameter plane of a patch $\mathbf{x} = \mathbf{x}(u,v)$.

Now a nonzero vector $\mathbf{T}$ is *tangent* to a surface S at a point P if there is a regular curve $\mathbf{x} = \mathbf{y}(t)$ on S through P such that $\mathbf{T} = d\mathbf{y}/dt$. If $\mathbf{x} = \mathbf{x}(u, v)$ is a patch containing P and $\mathbf{x} = \mathbf{y}(t) = \mathbf{x}(u(t), v(t))$, then $\dfrac{d\mathbf{y}}{dt} = \mathbf{x}_u \dfrac{du}{dt} + \mathbf{x}_v \dfrac{dv}{dt}$. Thus the tangents to the surface at P are all linearly dependent upon the two linearly independent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ at P as shown in Fig. 8-14.

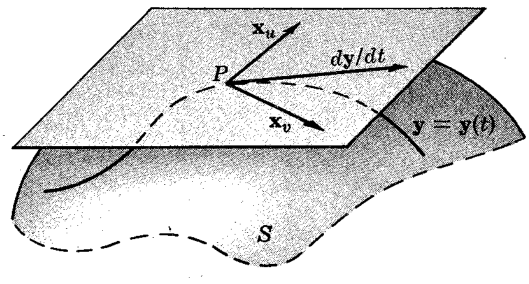


Fig. 8-14

We recall that $\mathbf{x}_u$ and $\mathbf{x}_v$ are tangent to the u-parameter and v-parameter curves respectively. Also every nonzero vector linearly dependent upon $\mathbf{x}_u$ and $\mathbf{x}_v$ is the tangent vector $d\mathbf{y}/dt$ of some curve through P. The proof of this is left to the reader as an exercise.

The plane through P parallel to $\mathbf{x}_u$ and $\mathbf{x}_v$ at P is called the *tangent plane* to S at P. It follows from the above that it is independent of the patch containing P and that a nonzero vector $\mathbf{T}$ is tangent to S at P if and only if it is parallel to the tangent plane at P. Clearly the tangent plane at a point $\mathbf{x}$ on a patch $\mathbf{x} = \mathbf{x}(u, v)$ is given by

$$\mathbf{y} = \mathbf{x} + h\mathbf{x}_u + k\mathbf{x}_v \qquad -\infty < h,\ k < \infty \tag{8.1}$$

Now, at each point P on the surface there are two unit vectors, of opposite sense, each perpendicular to the tangent plane at P. We will see that it is not always possible to select one of them at every point so that they vary continuously through out the whole surface. However, on any one patch $\mathbf{x} = \mathbf{x}(u, v)$ our convention is to choose the one which makes a right-handed system with $\mathbf{x}_u$ and $\mathbf{x}_v$ at P. Namely, we take the vector $\mathbf{N} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$. Clearly $\mathbf{N}$ is of unit length, perpendicular to the tangent plane at P, since it is perpendicular to $\mathbf{x}_u$ and $\mathbf{x}_v$, and varies continuously through the patch since $\mathbf{x}_u$ and $\mathbf{x}_v$ are at least of class C^0 and $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. The vector $\mathbf{N}$ is called a *unit normal vector* to the surface at P.

If $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is any other patch containing P, then on the intersection,

$$\mathbf{x}^*_\theta \times \mathbf{x}^*_\phi = (\mathbf{x}_u u_\theta + \mathbf{x}_v v_\theta) \times (\mathbf{x}_u u_\phi + \mathbf{x}_v v_\phi) = (\mathbf{x}_u \times \mathbf{x}_v)(u_\theta v_\theta - u_\phi v_\phi) = (\mathbf{x}_u \times \mathbf{x}_v)\frac{\partial(u, v)}{\partial(\theta, \phi)}$$

so that
$$\mathbf{N}^* = \frac{\mathbf{x}^*_\theta \times \mathbf{x}^*_\phi}{|\mathbf{x}^*_\theta \times \mathbf{x}^*_\phi|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\frac{\partial(u, v)}{\partial(\theta, \phi)} \Big/ \left|\frac{\partial(u, v)}{\partial(\theta, \phi)}\right| = \mathbf{N}\,\text{sign}\left(\frac{\partial(u, v)}{\partial(\theta, \phi)}\right)$$

Thus $\mathbf{N}^*$ will have the same sense as $\mathbf{N}$ at P if and only if $\partial(u, v)/\partial(\theta, \phi) > 0$ at P.

In any case, the straight line through P perpendicular to the tangent plane at P is independent of the coordinate patch and is called the *normal line* to the surface at P. It follows that the normal line at $\mathbf{x}$ is given by

$$\mathbf{y} = \mathbf{x} + k\mathbf{N}, \qquad -\infty < k < \infty \tag{8.2}$$

Example 8.5.

As a circle revolves about a fixed line in the plane of the circle, it generates a torus. Suppose that the circle is initially in the x_1x_3 plane with center on the x_1 axis, at a distance b from the origin, and its radius is a ($a < b$). Now consider the circle after it has been rotated through an angle θ about the x_3 axis, as shown in Fig. 8-15. If $\mathbf{u}$ is the vector from the origin to the center of the circle and $\mathbf{r}$ is the radius vector of the circle, then $\mathbf{u} = (b\cos\theta)\mathbf{e}_1 + (b\sin\theta)\mathbf{e}_2$ and $\mathbf{r} = (a\sin\phi\cos\theta)\mathbf{e}_1 + (a\sin\phi\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$, where ϕ is the angle that $\mathbf{r}$ makes with the x_3 axis. It follows that

$$\mathbf{x} = \mathbf{u} + \mathbf{r} = (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

where $-\infty < \theta < \infty$ and $-\infty < \phi < \infty$. Clearly $\mathbf{x}$ is of class C^∞; also

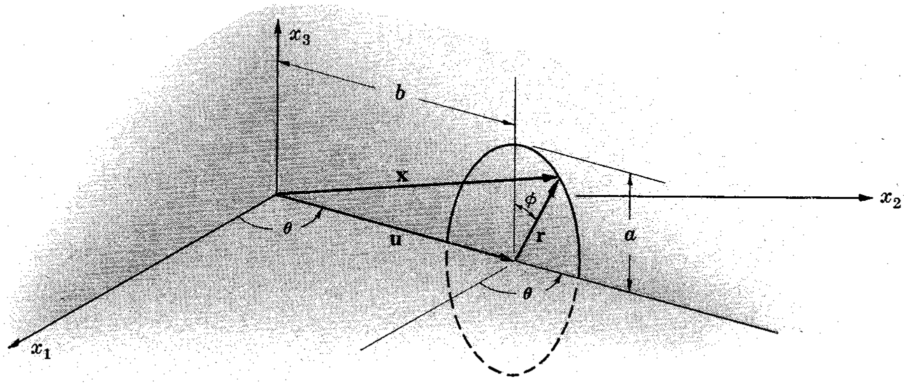


Fig. 8-15

$$\mathbf{x}_\theta = -(b + a \sin\phi)(\sin\theta)\mathbf{e}_1 + (b + a\sin\phi)(\cos\theta)\mathbf{e}_2$$

$$\mathbf{x}_\phi = (a\cos\phi\cos\theta)\mathbf{e}_1 + (a\cos\phi\sin\theta)\mathbf{e}_2 - (a\sin\phi)\mathbf{e}_3$$

and

$$|\mathbf{x}_\theta \times \mathbf{x}_\phi| = |a(b + a\sin\phi)((-\sin\phi\cos\theta)\mathbf{e}_1 - (\sin\phi\sin\theta)\mathbf{e}_2 - (\cos\phi)\mathbf{e}_3)| = a(b + a\sin\phi) \neq 0$$

for all (θ, ϕ). Thus $\mathbf{x}$ is a regular parametric representation of the torus of class C^∞. The torus is a simple surface. As a basis we can take a sufficient number of overlapping parts of the above representation which are 1-1 and cover the torus. For example, it is easily verified that a basis is obtained by restricting $\mathbf{x}$ to the following three open sets in the $\theta\phi$ plane: (a) $0 < \theta < 2\pi,\ 0 < \phi < 2\pi$, (b) $-\pi < \theta < \pi,\ -\pi < \phi < \pi$ and (c) $-(1/2)\pi < \theta < (3/2)\pi,\ -(1/2)\pi < \phi < (3/2)\pi$. Every regular curve in the (θ, ϕ) parameter plane will map onto a regular curve on the torus. For example, the coordinate lines $\theta =$ constant map into the ϕ-parameter curves on the surface which are the copies of the circle which generates the torus. The coordinate lines $\phi =$ constant map into the θ-parameter curves on the surface which are the circles on the torus generated by a fixed point on the generating circle as it revolved about the x_3 axis. Note that $\mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0$. Hence the parameter curves on the surface intersect orthogonally. Also of interest is the image on the torus of the straight line $\theta = t,\ \phi = kt,\ k =$ positive integer, in the parameter plane. This is the curve

$$\mathbf{x} = (b + a\sin kt)(\cos t)\mathbf{e}_1 + (b + a\sin kt)(\sin t)\mathbf{e}_2 + (\cos kt)\mathbf{e}_3$$

It is a helix on the torus which wraps around the torus exactly k times as shown in Fig. 8-16. A unit normal vector on the torus is the vector

$$\mathbf{N} = \frac{\mathbf{x}_\theta \times \mathbf{x}_\phi}{|\mathbf{x}_\theta \times \mathbf{x}_\phi|} = -(\sin\phi\cos\theta)\mathbf{e}_1 - (\sin\phi\sin\theta)\mathbf{e}_2 - (\cos\phi)\mathbf{e}_3$$

Here $\mathbf{N}$ varies continuously throughout the surface. It is directed into the torus opposite to the direction of the radius vector of the generating circle,

$$\mathbf{r} = (a\sin\phi\cos\theta)\mathbf{e}_1 + (a\sin\phi\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

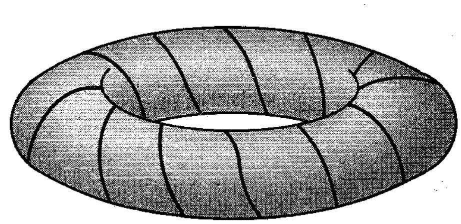


Fig. 8-16

TOPOLOGICAL PROPERTIES OF SIMPLE SURFACES

A simple surface is *connected* if it is connected as a point set in E^3. Thus S is connected if there do not exist open sets O_1 and O_2 in E^3 which cover S and have nonempty and disjoint intersections with S. In Problem 8.23, page 168, we prove

Theorem 8.4. If S is a connected simple surface and if P and Q are arbitrary points on S, then there exists a regular arc connecting P and Q. Thus a connected surface is arcwise connected by regular arcs.

In Problem 8.22 we also prove the interesting and very important

Theorem 8.5. Let S and T be simple surfaces such that S is closed, T is connected, and S is contained in T. Then, as point sets in E^3, S equals T.

Thus a closed simple surface is complete, in the sense that it cannot be a proper subset of a connected simple surface.

A simple surface is *compact* if it is a compact point set in E^3. Thus S is compact if every open covering of S has a finite subcovering; or, equivalently, S is compact if it is closed and bounded. Thus a compact surface does not have open edges. It must be finite in size and closed up everywhere like a sphere or a torus.

Although orientability is also a topological property of a simple surface and can be defined using the notions of continuity, etc., we will use the differential structure (coordinate patches) on the surface to define it. Namely, a simple surface S is *orientable* if there exists a basis for S such that if $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ are any two overlapping patches of the basis, then everywhere on the intersection $\partial(u, v)/\partial(\theta, \phi) > 0$. Recall that the normals $\mathbf{N}$ and $\mathbf{N}^*$ are equal at a point if and only if $\partial(u, v)/(\theta, \phi) > 0$. Thus S is orientable if and only if a unit normal can be defined on the surface which varies continuously throughout the surface.

Intuitively, an *oriented* surface is an orientable surface on which one of the two normal directions has been specified so that it varies continuously throughout the surface. To be precise, let S be a set of points in E^3 and let $\mathcal{F}$ be a collection of coordinate patches on S of class C^m satisfying

(i) There exists a set of coordinate patches in $\mathcal{F}$ which forms a basis for S.

(ii) If $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ are any two overlapping patches in $\mathcal{F}$, then on the intersection $\partial(u, v)/\partial(\theta, \phi) > 0$.

(iii) $\mathcal{F}$ is maximal. That is, if a coordinate patch on S not in $\mathcal{F}$ is adjoined to $\mathcal{F}$, then property (ii) fails.

The point set S together with the collection $\mathcal{F}$ is an *oriented simple surface* of class C^m.

Note that if a simple surface is orientable, then it can be oriented by adjoining to a basis satisfying (ii) all patches which preserve property (ii). Also a *connected* orientable surface S can be oriented in one of two and only two ways; that is, the coordinate patches on S belong to one of two sets, $\mathcal{F}_1$ and $\mathcal{F}_2$, each satisfying (i) (ii) and (iii) above. The proof of this is left to the reader as an exercise.

Finally, we define an *elementary* surface to be a simple surface for which there exists a basis consisting of a single coordinate patch. It follows that an elementary surface is homeomorphic to an open set in the plane and is orientable.

Example 8.6.

The sphere and torus are examples of connected, compact and orientable simple surfaces. The elliptic paraboloid and the plane are examples of connected elementary surfaces. The Moebius strip shown in Fig. 8-17 is not orientable. As indicated in the figure, a normal vector which is continued around the surface returns in the opposite direction.

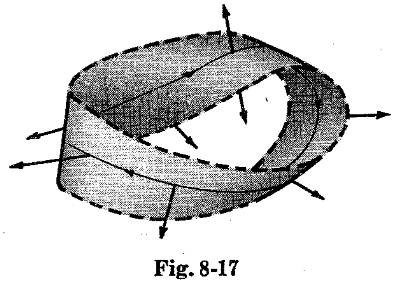


Fig. 8-17

Note. Unless stated to the contrary, we assume that our surfaces are connected. Thus by a "surface of class C^m" we mean a "connected simple surface of class C^m".

Solved Problems

REGULAR PARAMETRIC REPRESENTATIONS

8.1. Show that $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is a regular parametric representation of class C^m if $f(u,v)$ is of class C^m.

$\mathbf{x}$ is of class C^m, since f is of class C^m. Also,

$$\mathbf{x}_u = \mathbf{e}_1 + f_u\mathbf{e}_3, \qquad \mathbf{x}_v = \mathbf{e}_2 + f_v\mathbf{e}_3$$

and $$|\mathbf{x}_u \times \mathbf{x}_v| = |-f_u\mathbf{e}_1 - f_v\mathbf{e}_2 + \mathbf{e}_3| = [f_u^2 + f_v^2 + 1]^{1/2} \neq 0$$

which gives the required result.

8.2. Show that $\mathbf{x} = (a \sin\phi \cos\theta)\mathbf{e}_1 + b(\sin\phi \sin\theta)\mathbf{e}_2 + c(\cos\phi)\mathbf{e}_3$, $a, b, c > 0$, $-\infty < \theta < \infty$, $0 < \phi < \pi$, is a regular parametric representation of class C^∞ of the ellipsoid $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$ punctured at $(0,0,c)$ and $(0,0,-c)$, as shown in Fig. 8-18. Describe the θ- and ϕ-parameter curves.

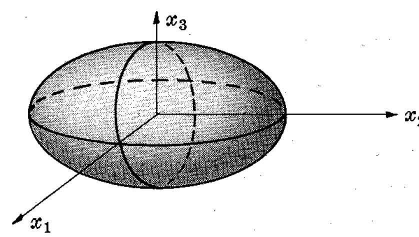


Fig. 8-18

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = \sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi = 1$$

Also, $$\mathbf{x}_\theta = (-a \sin\phi \sin\theta)\mathbf{e}_1 + (b \sin\phi \cos\theta)\mathbf{e}_2$$
$$\mathbf{x}_\phi = (a \cos\phi \cos\theta)\mathbf{e}_1 + (b \cos\phi \sin\theta)\mathbf{e}_2 - (c \sin\phi)\mathbf{e}_3$$

and $$|\mathbf{x}_\theta \times \mathbf{x}_\phi| = |(-bc \sin^2\phi \cos\theta)\mathbf{e}_1 - (ac \sin^2\phi \sin\theta)\mathbf{e}_2 - (ab \sin\phi \cos\phi)\mathbf{e}_3|$$
$$= |\sin\phi|\,[(a^2 \sin^2\theta + b^2 \cos^2\theta)c^2 \sin^2\phi + a^2b^2 \cos^2\phi]^{1/2}$$

We can assume $0 < a \leqq b \leqq c$. Then

$$|\mathbf{x}_\theta \times \mathbf{x}_\phi| \geqq |\sin\phi|\,[(a^2 \sin^2\theta + a^2 \cos^2\theta)a^2 \sin^2\phi + a^4 \cos^2\phi]^{1/2} \geqq |\sin\phi|\,a^2 \neq 0$$

for $0 < \phi < \pi$. Also $\mathbf{x}$ is of class C^∞. Thus $\mathbf{x}$ is a regular parametric representation of class C^∞. The θ-parameter curves (ϕ = constant) are the intersections of the ellipsoid and the family of horizontal planes $x_3 = c \cos\phi$, $0 < \phi < \pi$. These are the ellipses $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1 - \cos^2\phi$. The ϕ-parameter (θ = constant) curves are the intersection of the ellipsoid and the family of half planes $x_1 = aR\cos\theta$, $x_2 = bR\sin\theta$, $R > 0$. These are semi-ellipses. For, let the intersection of such a half plane with the x_1x_2 plane be parameterized by the distance t from the origin. Then

$$x_1 = \frac{a(\cos\theta)t}{d}, \quad x_2 = \frac{b(\sin\theta)t}{d} \quad \text{where } d^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

Hence the intersection is the semi-ellipse $\dfrac{t^2}{d^2} + \dfrac{x_3}{c^2} = 1$, $t > 0$.

8.3. A *surface of revolution* S is obtained by revolving a plane curve C about a line L in its plane. C is called the *profile* curve and L the axis of S. The different positions of C are called the *meridians* of S and the circles generated by each point on C are called the *parallels* of S.

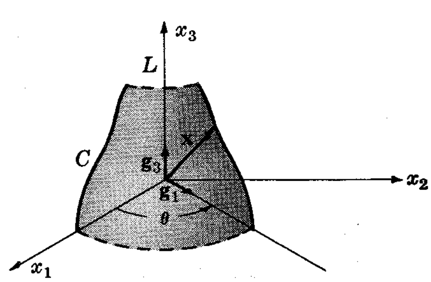


Fig. 8-19

If $x_1 = f(t)$, $x_3 = g(t)$, $a < t < b$, is a regular curve C of class C^m in the x_1x_3 plane and $f > 0$, show that $\mathbf{x} = (f(t)\cos\theta)\mathbf{e}_1 + (f(t)\sin\theta)\mathbf{e}_2 + g(t)\mathbf{e}_3$,

$-\infty < \theta < \infty$, is a regular parametric representation of class C^m of the surface obtained by revolving C about the x_3 axis. Also show that the t-parameter curves (meridians) and the θ-parameter curves (parallels) intersect orthogonally.

Let $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ be a basis obtained by rotating $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ through an angle θ about the x_3 axis as shown in Fig. 8-19. The position vector of a point on the profile curve $x_1 = f(t)$, $x_3 = g(t)$ when it is in the plane containing $\mathbf{g}_1$ and $\mathbf{g}_3$ is $\mathbf{x} = f(t)\,\mathbf{g}_1 + g(t)\,\mathbf{g}_3$. But $\mathbf{g}_1 = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2$ and $\mathbf{g}_3 = \mathbf{e}_3$. Thus $\mathbf{x} = (f(t)\cos\theta)\mathbf{e}_1 + (f(t)\sin\theta)\mathbf{e}_2 + g(t)\,\mathbf{e}_3$. $\mathbf{x}$ is of class C^m, since f and g are of class C^m. Also,

$$\mathbf{x}_t = (f'\cos\theta)\mathbf{e}_1 + (f'\sin\theta)\mathbf{e}_2 + g'\mathbf{e}_3, \qquad \mathbf{x}_\theta = (-f\sin\theta)\mathbf{e}_1 + (f\cos\theta)\mathbf{e}_2$$

and

$$|\mathbf{x}_t \times \mathbf{x}_\theta| = |(-g'f\cos\theta)\mathbf{e}_1 - (g'f\sin\theta)\mathbf{e}_2 + f'f\mathbf{e}_3| = f\sqrt{(g')^2 + (f')^2} \neq 0$$

since $f > 0$ and $x_1 = f(t)$, $x_3 = g(t)$ is regular. Thus $\mathbf{x}$ is regular and of class C^m. Also $\mathbf{x}_t \cdot \mathbf{x}_\theta = 0$ and so t- and θ-parameter curves intersect orthogonally.

8.4. A *ruled surface* is a surface generated by a one parameter family of lines. The various positions of the generating lines are called the *rulings* of the surface. Let $\mathbf{y} = \mathbf{y}(u)$ be a regular curve of class C^m and let $\mathbf{g}(u)$ be a nonzero vector of class C^m along $\mathbf{y} = \mathbf{y}(u)$. Show that $\mathbf{x} = y(u) + v\mathbf{g}(u)$ is a regular parametric representation of a ruled surface of class C^m provided that $\mathbf{y}$ and $\mathbf{g}$ are of class C^m and $(\mathbf{y}' + v\mathbf{g}') \times \mathbf{g} \neq 0$ for all (u, v). A parameterization of the surface in the above form is called a parameterization in *ruled form*. The curve $\mathbf{y} = \mathbf{y}(t)$ is called the *base curve* of the parameterization. Note that the cylinder (Example 8.1(c), page 152) is a ruled surface with parallel rulings, i.e. $\mathbf{g}$ = constant.

As shown in the adjacent Fig. 8-20, a general point on the surface is $\mathbf{x} = \mathbf{y}(u) + v\mathbf{g}(u)$ where v is a parameter along the rulings. Also $\mathbf{x}$ is of class C^m, since $\mathbf{y}$ and $\mathbf{g}$ are of class C^m, and, since

$$\mathbf{x}_u = \frac{d\mathbf{y}}{du} + v\frac{d\mathbf{g}}{du}, \qquad \mathbf{x}_v = \mathbf{g}$$

$\mathbf{x}$ is regular if and only if for all (u, v),

$$\mathbf{x}_u \times \mathbf{x}_v = \left(\frac{d\mathbf{y}}{du} + v\frac{d\mathbf{g}}{du}\right) \times \mathbf{g} \neq 0$$

which is the required result. Observe that the v-parameter curves along which u = constant in the above representation are the rulings themselves.

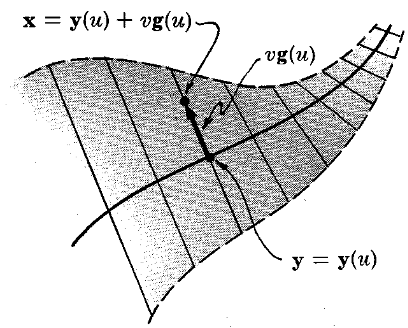


Fig. 8-20

8.5. Show that the hyperbolic paraboloid $x_3 = x_1^2 - x_2^2$ is a doubly ruled surface; that is, it can be generated by two different families of lines. Find parametric representations of the surface in ruled form representing both rulings.

Observe that $x_3 = (x_1 - x_2)(x_1 + x_2)$. Thus the intersection of the plane $x_1 - x_2 = u_0$ and the surface is the straight line determined by the planes $x_1 - x_2 = u_0$ and $x_3 = u_0(x_1 + x_2)$. Hence we are led to set $x_1 - x_2 = u$, $x_1 + x_2 = v$; so that $x_1 = \frac{1}{2}(u+v)$, $x_2 = \frac{1}{2}(u-v)$, and $x_3 = uv$. This gives the representation

$$\mathbf{x} = \tfrac{1}{2}(u+v)\mathbf{e}_1 + \tfrac{1}{2}(u-v)\mathbf{e}_2 + uv\mathbf{e}_3$$

in which the u-parameter curves v = constant and v-parameter curves u = constant are straight lines. Thus the hyperboloid is a doubly ruled surface. Rewriting $\mathbf{x}$ in the form

$$\mathbf{x} = (\tfrac{1}{2}u\mathbf{e}_1 + \tfrac{1}{2}u\mathbf{e}_2) + v(\tfrac{1}{2}\mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2 + u\mathbf{e}_3) = \mathbf{y}(u) + v\mathbf{g}(u)$$

gives a representation in ruled form, where $\mathbf{y} = \frac{1}{2}u\mathbf{e}_1 + \frac{1}{2}u\mathbf{e}_2$, the u-parameter curve $v = 0$, is the base curve and $\mathbf{g} = (\frac{1}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2 + u\mathbf{e}_3)$ is in the direction of the v-parameter curve at u. Similarly,

$$\mathbf{x} = (\tfrac{1}{2}v\mathbf{e}_1 - \tfrac{1}{2}v\mathbf{e}_2) + u(\tfrac{1}{2}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + v\mathbf{e}_3)$$

is a representation in ruled form in which $\mathbf{y} = \frac{1}{2}v\mathbf{e}_1 - \frac{1}{2}v\mathbf{e}_2$, the v-parameter curve $u = 0$, is the base curve and $\mathbf{g} = (\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + v\mathbf{e}_3)$ is in the direction of the u-parameter curve at v.

8.6. A *right conoid* is a ruled surface with rulings parallel to a plane P and passing through a line L perpendicular to the plane. The line L is called the axis of the conoid. If L is taken to be the x_3 axis, as shown in Fig. 8-21, show that the conoid has a parameter representation of the form

$$\mathbf{x} = (v \cos\theta(u))\mathbf{e}_1 + (v \sin\theta(u))\mathbf{e}_2 + u\mathbf{e}_3$$

where the function θ is the angle that the ruling makes with the x_1x_3 plane. Show that the representation is regular and of class C^m provided that $\theta(u)$ is of class C^m.

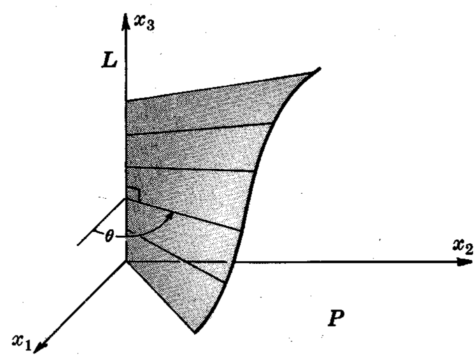


Fig. 8-21

As a base curve for the surface we take the x_3 axis, $\mathbf{y} = u\mathbf{e}_3$. Since the rulings are parallel to the x_1x_2 plane, a unit vector in the direction of the rulings as function of u can be written in the form

$$\mathbf{g} = (\cos\theta(u))\mathbf{e}_1 + (\sin\theta(u))\mathbf{e}_2$$

Hence
$$\mathbf{x} = \mathbf{y} + v\mathbf{g} = (v\cos\theta(u))\mathbf{e}_1 + (v\sin\theta(u))\mathbf{e}_2 + u\mathbf{e}_3$$

is a representation of the surface in ruled form. Here

$$\mathbf{x}_u = (-v\theta'\sin\theta(u))\mathbf{e}_1 + (v\theta'\cos\theta(u))\mathbf{e}_2 + \mathbf{e}_3, \qquad \mathbf{x}_v = (\cos\theta(u))\mathbf{e}_1 + (\sin\theta(u))\mathbf{e}_2$$

$$|\mathbf{x}_u \times \mathbf{x}_v| = |(-\sin\theta(u))\mathbf{e}_1 + (\cos\theta(u))\mathbf{e}_2 - v\theta'\mathbf{e}_3| = [v^2\theta'^2 + 1]^{1/2} \neq 0$$

for all (v, θ). Thus $\mathbf{x}$ is a regular representation of class C^m provided simply that $\theta(u)$ is of class C^m. Note that if $\theta' \neq 0$, the function $\theta(u)$ has an inverse and the surface has a representation of the form

$$\mathbf{x} = (v\cos\theta)\mathbf{e}_1 + (v\sin\theta)\mathbf{e}_2 + u(\theta)\,\mathbf{e}_3$$

SIMPLE SURFACES

8.7. Show that $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\,\mathbf{e}_3$ is a coordinate patch of class C^m if $f(u,v)$ is of class C^m.

From Problem 8.1, $\mathbf{x}$ is a regular parametric representation of class C^m. It remains to show that $\mathbf{x}$ is 1-1 and its inverse is continuous. Since $x_1 = u$, $x_2 = v$, it follows that $\mathbf{x}(u,v) = \mathbf{x}(u',v')$ implies $u = u'$ and $v = v'$. Hence $\mathbf{x}$ is 1-1. The inverse mapping is $u = x_1$, $v = x_2$ and is continuous. Thus $\mathbf{x}$ is a coordinate patch of class C^m.

8.8. Show that the mapping $\mathbf{x} = u^2\mathbf{e}_1 + uv\mathbf{e}_2 + v^2\mathbf{e}_3$ is a coordinate patch of class C^∞ on the first quadrant $u > 0$, $v > 0$.

Clearly $\mathbf{x}$ is of class C^∞ and

$$|\mathbf{x}_u \times \mathbf{x}_v| = |2v^2\mathbf{e}_1 - 4uv\mathbf{e}_2 + 2u^2\mathbf{e}_3| = 2\sqrt{v^4 + 4u^2v^2 + u^4} \neq 0$$

for $u > 0$ and $v > 0$. Since $x_1 = u^2$, $x_2 = uv$, it follows that $\mathbf{x}(u,v) = \mathbf{x}(u',v')$ implies $u^2 = u'^2$ and $uv = u'v'$. Since $u > 0$ and $u' > 0$, it follows first that $u = u'$ and then $v = v'$. Thus the mapping is 1-1. The inverse is $u = \sqrt{x_1}$, $v = x_2/\sqrt{x_1}$. Since $u > 0$, we have $x_1 > 0$ and the inverse is defined and continuous. Thus the mapping is a coordinate patch of class C^∞.

8.9. Show that the hyperbolic paraboloid $x_3 = x_1^2/a^2 - x_2^2/b^2$ is a simple surface of class C^∞.

The mapping

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + (x_1^2/a^2 - x_2^2/b^2)\mathbf{e}_3$$

is a Monge patch (hence a proper coordinate patch) of class C^∞ which covers the surface and is the intersection of the surface and the open set E^3.

8.10. Surfaces are often represented implicitly, i.e. as the set of points S in E^3 which satisfy an equation of the form $f(x_1, x_2, x_3) = c$, c = constant. As a consequence of the implicit function theorem which can be found in any text in advanced calculus, S together with all coordinate patches of class C^m in S is a simple surface provided that f is of class C^m and at least one of the partial derivatives $f_{x_1}, f_{x_2}, f_{x_3}$ is different from zero at each point in S. Using these criteria, determine the values of c for which $x_1^2 - 2x_1 + x_2x_3 = c$ is a simple surface.

$f_{x_1} = 2x_1 - 2$, $f_{x_2} = x_3$, $f_{x_3} = x_2$. These vanish simultaneously if and only if $x_1 = 1$, $x_2 = 0$, $x_3 = 0$. But this is a point satisfying $x_1^2 - 2x_1 + x_2x_3 = c$ if and only if $c = -1$. Thus $x_1^2 - 2x_1 + x_2x_3 = c$ is a simple surface for all $c \neq -1$, which is the required result.

8.11. The *quadratic* surfaces are defined by equations of the form

$$f = \sum_{ij=1}^{3} a_{ij}x_ix_j + \sum_{i=1}^{3} b_ix_i + c = 0$$

By a rotation and translation of the coordinates it can be shown that the nontrivial ones can be brought into one of the following six sketched in Fig. 8-22.

(1) Ellipsoid: $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$

(2) Hyperboloid (one sheet): $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 1$

(3) Hyperboloid (two sheets): $\dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 1$

(4) Elliptic paraboloid: $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} - x_3 = 0$

(5) Hyperbolic paraboloid: $\dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} - x_3 = 0$

(6) Quadric cone: $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 0$, $(x_1, x_2, x_3) \neq (0, 0, 0)$

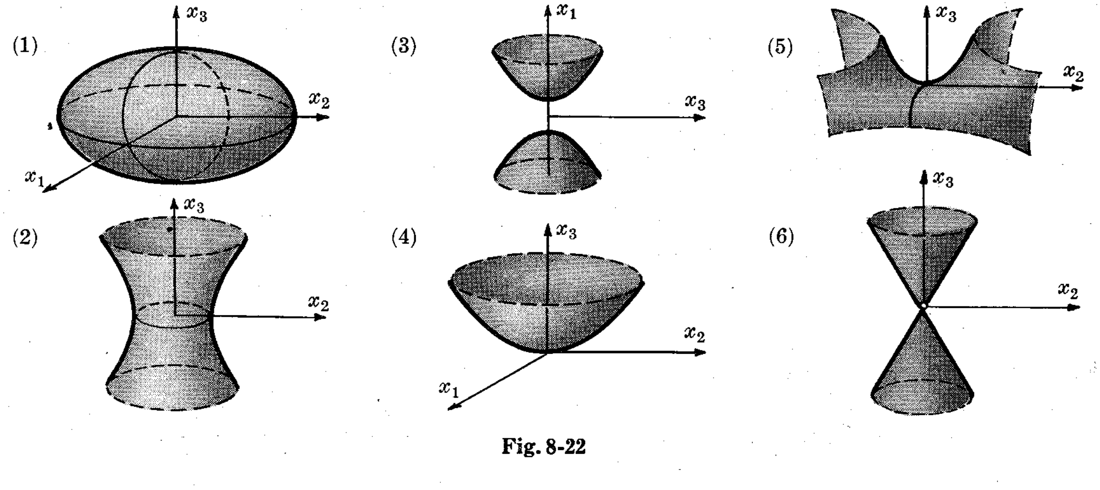


Fig. 8-22

Using the criteria in Problem 8.9, show that each of the above is a simple surface of class C^∞.

For $f = \frac{x_1^2}{a^2} \pm \frac{x_2^2}{b^2} \pm \frac{x_3^2}{c^2}$ we have $f_{x_1} = \frac{2x_1}{a^2}$, $f_{x_2} = \pm\frac{2x_2}{b^2}$, $f_{x_3} = \pm\frac{2x_3}{c^2}$. These all vanish together if and only if $(x_1, x_2, x_3) = (0,0,0)$. But the origin is not on the surfaces (1), (2) and (3) above and has been removed from (6). For the remaining two, $f = \frac{x_1^2}{a^2} \pm \frac{x_2^2}{b^2} - x_3$, so that $f_{x_3} = -1 \neq 0$. Finally f is of class C^∞ in all cases. Hence each is a simple surface of class C^∞.

8.12. Prove that a regular parametric representation is locally 1-1 and bicontinuous. That is, if $\mathbf{x} = \mathbf{x}(u,v)$ is a regular parametric representation defined on an open set U, prove that for every (u,v) in U there exists a neighborhood $S(u,v)$ on which $\mathbf{x}$ is 1-1 and bicontinuous.

Recall in the proof of Theorem 8.1 that since the rank of the Jacobian matrix of $\mathbf{x}$ is two, we can assume that at each (u,v) there exists a 1-1 mapping $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ of class $m \geqq 1$ with inverse of the same class defined on an open set W containing (u,v) and a Monge patch $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + f(x_1,x_2)\mathbf{e}_3$ defined on the image of W such that $\mathbf{x}$, restricted to W, is the composite mapping

$$\mathbf{x} \quad = \quad x_1(u,v)\mathbf{e}_1 + x_2(u,v)\mathbf{e}_2 + f(x_1(u,v), x_2(u,v))\mathbf{e}_3$$

Namely on W, $\mathbf{x}$ is a composite of two 1-1 and bicontinuous mappings and hence it is itself 1-1 and bicontinuous, which proves the proposition.

8.13. It can be shown that if $\mathbf{x} = \mathbf{x}(u,v)$ is a coordinate patch on a simple surface S and P is a point on $\mathbf{x} = \mathbf{x}(u,v)$, then there exists a spherical neighborhood $S(P)$ in E^3 such that the intersection of $S(P)$ with the surface S is contained in the patch $\mathbf{x} = \mathbf{x}(u,v)$. Use this result to prove that every patch on S is the intersection of S with an open set in E^3. From this it follows that any set of patches which covers a simple surface S is a basis for S.

Let G be the image of an arbitrary patch $\mathbf{x} = \mathbf{x}(u,v)$ on S. For each P in G there is $S(P)$ such that $S(P) \cap S \subset G$. Let $O = \bigcup_P S(P)$. Note that O is open since it is a union of open sets. Now suppose Q is a point in G. Since $Q \in S(Q)$, it follows that $Q \in O = \bigcup_P S(P)$. Also $Q \in S$. Hence $Q \in S \cap O$. Thus $G \subset S \cap O$. Conversely, suppose $Q \in S \cap O$. Since $S \cap O = S \cap [\bigcup_P S(P)] = \bigcup_P [S \cap S(P)]$, it follows that Q belongs to some $S \cap S(P)$. But $S \cap S(P) \subset G$. Hence $Q \in G$. Thus $S \cap O \subset G$. It follows the $G = S \cap O$; which is the required result; namely, every patch in S is the intersection of S with an open set in E^3.

8.14. Let $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ be coordinate patches on a simple surface S defined on open sets U and U^* respectively and having overlapping images G and G^* on S. Let W and W^* be the subsets of U and U^* respectively which map onto $G \cap G^*$. Show that W and W^* are open sets in their respective parameter planes.

Let (u_0, v_0) be a point in W with image P_0 in $G \cap G^*$. Let $S_\epsilon(P_0)$ and $S_\delta(P_0)$ be neighborhoods of P_0 such that $S_\epsilon(P_0) \cap S \subset G$ and $S_\delta(P_0) \cap S \subset G^*$. Suppose $\epsilon \leqq \delta$. Then $S_\epsilon(P_0) \cap S \subset G \cap G^*$. Since $\mathbf{x}(u,v)$ is continuous, there exists $S_{\delta_1}(u_0,v_0)$ such that for all (u,v) in $S_{\delta_1} \cap U$ we have $\mathbf{x}(u,v)$ in $S_\epsilon(P_0)$ and hence in $G \cap G^*$. But U is open. Thus for sufficiently small δ_2 we have $\mathbf{x}(u,v)$ in $G \cap G^*$ for all (u,v) in $S_{\delta_2}(u_0,v_0)$. Namely, (u,v) is in W for all (u,v) in $S_{\delta_2}(u_0,v_0)$. Since (u_0,v_0) is an arbitrary point in W it follows that W is open. A similar argument proves that W^* is open.

8.15. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a simple surface S of class C^m defined on an open set U with image G and let $\mathbf{x} = \mathbf{x}^*(x_1,x_2) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3^*(x_1,x_2)\mathbf{e}_3$ be a Monge patch on S defined on an open set V^* with image G^* contained in G, as shown in Fig. 8-23.

Show that there exists an open set W in U and a 1-1 mapping $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ of class C^m of W onto V^*, where $\partial(x_1,x_2)/\partial(u,v) \neq 0$ for all (u,v) in U^*, such that on W $\mathbf{x} = \mathbf{x}(u,v)$ is the composite mapping $\mathbf{x} = \mathbf{x}^*(x_1(u,v), x_2(u,v))$.

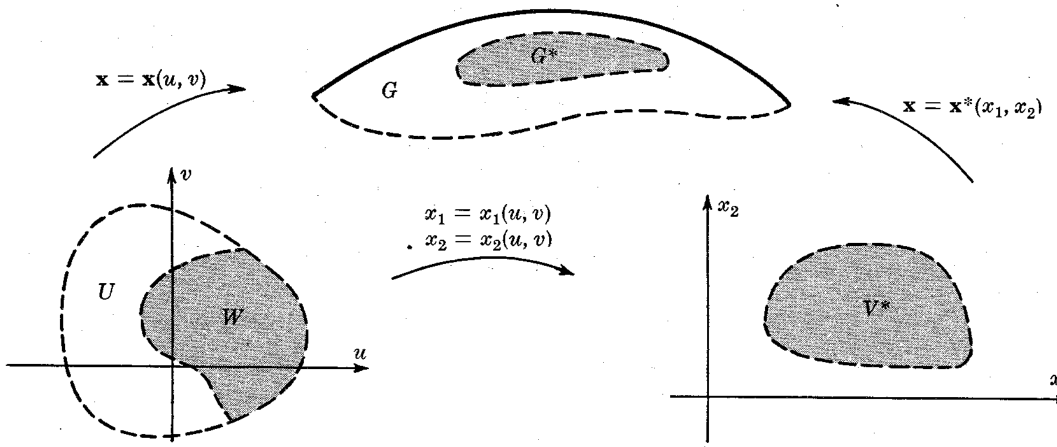


Fig. 8-23

From the preceding problem there exists an open set W in U such that $\mathbf{x}$ maps U^* onto $G \cap G^* = G^*$. Since $\mathbf{x}$ and $\mathbf{x}^*$ are both 1-1, there is a 1-1 mapping $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ of W onto V^* such that on W $\mathbf{x}(u,v) = \mathbf{x}^*(x_1(u,v), x_2(u,v))$. It remains to show that $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ are of class C^m and $\partial(x_1,x_2)/\partial(u,v) \neq 0$. But $x_1 = x_1(u,v)$ and $x_2 = x_2(u,v)$ are simply the first two components of $\mathbf{x} = \mathbf{x}(u,v)$. Since $\mathbf{x}(u,v)$ is of class C^m it follows that $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ is of class C^m. Finally, since the rank of the Jacobian matrices of $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(x_1,x_2)$ is two at each point, the differential of both functions at each point is a 1-1 linear mapping of the vectors in their respective planes onto a plane in E^3. We recall that the differential of a composite function is the composite linear mappings of the individual mappings. Thus at each point the differential of $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ is a 1-1 linear mapping of the vectors in the uv plane onto the vectors in the x_1x_2 plane. Thus the rank of the Jacobian matrix of $x_1 = x_1(\theta,\phi)$, $x_2 = x_2(\theta,\phi)$ is also two at each point; that is, $\det\begin{pmatrix} \partial x_1/\partial u & \partial x_1/\partial v \\ \partial x_2/\partial u & \partial x_2/\partial v \end{pmatrix} \neq 0$ for all (u,v) in W, which is the required result.

8.16. Prove Theorem 8.3. That is, on the intersection of two overlapping coordinate patches $\mathbf{x} = \mathbf{x}(u,v)$, $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ on a simple surface S of class C^m, show that the parameters are related by a 1-1 transformation $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ of class C^m defined on an open set and such that $\partial(\theta,\phi)/\partial(u,v) \neq 0$ for all (u,v).

From Problem 8.14, there exist open sets W and W^* in the uv and $\theta\phi$ planes respectively which map onto the intersection of the patches and a 1-1 mapping $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ of W onto W^* such that $\mathbf{x}(u,v) = \mathbf{x}^*(\theta(u,v), \phi(u,v))$. It remains to show that $\theta(u,v)$ and $\phi(u,v)$ are of class C^m and $\partial(\theta,\phi)/\partial(u,v) \neq 0$ for all (u,v) in W.

Let (u_0,v_0) be a point in W with image P in S. From Theorem 8.2 there exists a Monge patch $\mathbf{x} = \mathbf{x}(x_1,x_2) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3(x_1,x_2)\mathbf{e}_3$ in S containing P. From the preceding problem there exists a 1-1 mapping $x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$ of an open set V containing (u_0,v_0) onto an open set V^{**} in the x_1x_2 plane which is of class C^m in V and $\partial(x_1,x_2)/\partial(u,v) \neq 0$. Also there exists a 1-1 mapping $x_1 = x_1(\theta,\phi)$, $x_2 = x_2(\theta,\phi)$ of an open set V^* in the $\theta\phi$ plane containing $(\theta(u_0,v_0), \phi(u_0,v_0))$ onto V^{**} which is of class C^m on V^* and $\partial(x_1,x_2)/\partial(\theta,\phi) \neq 0$. From the inverse function theorem, the inverse mapping $\theta = \theta(x_1,x_2)$, $\phi = \phi(x_1,x_2)$ is of class C^m and $\partial(\theta,\phi)/\partial(x_1,x_2) \neq 0$. Thus $\theta = \theta(u,v) = \theta(x_1(u,v), x_2(u,v))$ and $\phi = \phi(u,v) = \phi(x_1(u,v), x_2(u,v))$ is of class C^m on V and $\dfrac{\partial(\theta,\phi)}{\partial(u,v)} = \dfrac{\partial(\theta,\theta)}{\partial(x_1,x_2)}\dfrac{\partial(x_1,x_2)}{\partial(u,v)} \neq 0$. Since V contains (u_0,v_0) and (u_0,v_0) is an arbitrary point in W, it follows that $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ is of class C^m in W and $\partial(\theta,\phi)/\partial(u,v) \neq 0$ in W.

TANGENT PLANE AND NORMAL LINE

8.17. Find the equation of the tangent plane and normal line to the surface represented by

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$$

at the point corresponding to $u = 1, v = 1$.

$$\mathbf{x}(1,1) = \mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{x}_u(1,1) = \mathbf{e}_1 + 2\mathbf{e}_3, \quad \mathbf{x}_v(1,1) = \mathbf{e}_2 - 2\mathbf{e}_3$$

Thus $$\mathbf{y} = \mathbf{x}(1,1) + h\mathbf{x}_u(1,1) + k\mathbf{x}_v(1,1) = (1+h)\mathbf{e}_1 + (1+k)\mathbf{e}_2 + 2(h-k)\mathbf{e}_3$$

is the tangent plane at $\mathbf{x}(1,1)$.

$$\mathbf{N} = \frac{\mathbf{x}_u(1,1) \times \mathbf{x}_v(1,1)}{|\mathbf{x}_u(1,1) \times \mathbf{x}_v(1,1)|} = 1/3(-2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$$

Thus $$\mathbf{y} = \mathbf{x}(1,1) + k\mathbf{N}(1,1) = (1 - \tfrac{2}{3}k)\mathbf{e}_1 + (1 + \tfrac{2}{3}k)\mathbf{e}_2 + \tfrac{1}{3}k\mathbf{e}_3$$

or, if we let $t = k/3$,

$$\mathbf{y} = (1-2t)\mathbf{e}_1 + (1+2t)\mathbf{e}_2 + t\mathbf{e}_3, \qquad -\infty < t < \infty$$

is the equation of the normal line at $\mathbf{x}(1,1)$.

8.18. Show that the image of the curve $\theta = \log t$, $\phi = 2\,\mathrm{Tan}^{-1}t$, $t > 0$, on the sphere

$$\mathbf{x} = (\cos\theta\sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3$$

intersects the meridians, ϕ-parameter curves, at a constant angle of $\pi/4$. Observe that as t varies from 0 to 1 to ∞, the angle θ varies from $-\infty$ to 0 to ∞ and the angle ϕ from 0 to $\pi/2$ to π. Thus the curve spirals around north and south poles as shown in Fig. 8-24.

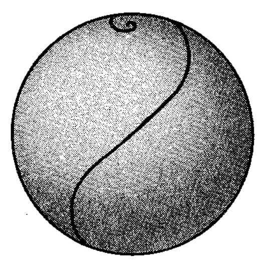


Fig. 8-24

$$\mathbf{x}_\theta = (-\sin\theta\sin\phi)\mathbf{e}_1 + (\cos\theta\sin\phi)\mathbf{e}_2, \qquad \mathbf{x}_\phi = (\cos\theta\cos\phi)\mathbf{e}_1 + (\sin\theta\cos\phi)\mathbf{e}_2 - (\sin\phi)\mathbf{e}_3$$

$$\mathbf{x}_\theta \cdot \mathbf{x}_\theta = \sin^2\phi, \quad \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0, \quad \mathbf{x}_\phi \cdot \mathbf{x}_\phi = 1$$

$$\frac{d\theta}{dt} = \frac{1}{t}, \qquad \frac{d\phi}{dt} = \frac{2}{1+t^2}$$

The cosine of the angle α between the tangent $\dfrac{d\mathbf{y}}{dt} = \mathbf{x}_\theta\dfrac{d\theta}{dt} + \mathbf{x}_\phi\dfrac{d\phi}{dt}$ to the curve on the surface and the tangent $\mathbf{x}_\phi$ to the ϕ-parameter curve is

$$\cos\alpha = \frac{(d\mathbf{y}/dt)\cdot\mathbf{x}_\phi}{|d\mathbf{y}/dt|\,|\mathbf{x}_\phi|} = \frac{d\phi}{dt}\left[\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2\right]^{-1/2}$$

$$= \frac{2}{1+t^2}\left[\frac{\sin^2(2\,\mathrm{Tan}^{-1}t)}{t^2} + \frac{4}{(1+t^2)^2}\right]^{-1/2}$$

$$= \frac{2}{1+t^2}\left[\frac{4t^2}{(1+t^2)^2}\frac{1}{t^2} + \frac{4}{(1+t^2)^2}\right]^{-1/2} = 1/\sqrt{2}\ ,$$

Thus $\alpha = \pi/4 =$ constant, which is the required result.

8.19. Show that the tangent plane is the same at all points along a ruling of (a) a cylinder, (b) a branch of the tangent surface of a curve.

(a) The cylinder is represented in ruled form as

$$\mathbf{x} = \mathbf{y}(t) + v\mathbf{g}, \quad \mathbf{g} = \text{constant} \neq 0, \quad \mathbf{y}' \times \mathbf{g} \neq \mathbf{0}$$

Here $$\mathbf{x}_t = \mathbf{y}', \quad \mathbf{x}_v = \mathbf{g}, \quad \mathbf{x}_t \times \mathbf{x}_u = \mathbf{y}' \times \mathbf{g}, \quad \text{and} \quad \mathbf{N} = \frac{\mathbf{y}' \times \mathbf{g}}{|\mathbf{y}' \times \mathbf{g}|}$$

Since $\mathbf{N}$ is independent of the parameter v along a ruling, the required result follows.

(*b*) The tangent surface to a curve $\mathbf{x} = \mathbf{y}(s)$ with no points of inflection is defined by the representation

$$\mathbf{x} = \mathbf{y}(s) + v\mathbf{t}(s)$$

This surface is not regular where $v = 0$, i.e. along the curve $\mathbf{x} = \mathbf{y}(s)$, and we speak of it as two branches corresponding to $v > 0$ and $v < 0$. For $v > 0$,

$$\mathbf{x}_s = \mathbf{t} + v\dot{\mathbf{t}}, \quad \mathbf{x}_v = \mathbf{t}, \quad \mathbf{x}_s \times \mathbf{x}_v = (\mathbf{t} + v\dot{\mathbf{t}}) \times \mathbf{t} = v\dot{\mathbf{t}} \times \mathbf{t}$$

and $\mathbf{N} = \dfrac{\mathbf{x}_s \times \mathbf{x}_v}{|\mathbf{x}_s \times \mathbf{x}_v|} = \dfrac{\dot{\mathbf{t}} \times \mathbf{t}}{|\dot{\mathbf{t}} \times \mathbf{t}|}$ is independent of the parameter v along a ruling. A similar computation is valid for $v < 0$.

TOPOLOGICAL PROPERTIES OF SIMPLE SURFACES

8.20. If $f(\mathbf{x})$ is continuous at all $\mathbf{x}$ in E^3, show that the set of points M in E^3 satisfying $f(\mathbf{x}) = c = \text{constant}$ is closed.

Suppose $\mathbf{x}_0$ is in the complement M^C of M. Then $f(\mathbf{x}_0) = c^* \neq c$. Since f is continuous at $\mathbf{x}_0$ and defined for all $\mathbf{x}$, there is a $\delta > 0$ such that for all $\mathbf{x}$ in $S_\delta(\mathbf{x}_0)$,

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \tfrac{1}{2}|c - c^*|, \quad \text{or} \quad |f(\mathbf{x}) - c^*| < \tfrac{1}{2}|c - c^*|$$

It follows that $f(\mathbf{x}) \neq c$ for all $\mathbf{x}$ in $S_\delta(\mathbf{x}_0)$. Thus all $\mathbf{x}$ in $S_\delta(\mathbf{x}_0)$ are also in M^C. Since $\mathbf{x}_0$ is an arbitrary point in M^C, it follows that M^C is open. Hence M is closed, which is the required result.

8.21. Using Problem 8.20, determine which of the following surfaces are compact: (*a*) $x_1^2 + x_2^2 x_3^2 = 1$, (*b*) $x_1^2 + x_2^4 + x_3^6 = 1$.

(*a*) From Problem 8.20, the set of points satisfying $x_1^2 + x_2^2 x_3^2 = 1$ is closed. However, it is not bounded. For let $x_1 = 1/\sqrt{2}$. Then the equation becomes $x_2^2 = 1/2x_3^2$. But this can be satisfied for arbitrarily large x_2 for sufficiently small x_3. Thus the surface is not compact.

(*b*) Again the set of points satisfying $x_1^2 + x_2^4 + x_3^6 = 1$ is closed. But here it is also bounded. In fact it is bounded by $|x_1| \leq 1$, $|x_2| \leq 1$, $|x_3| \leq 1$.

8.22. Prove Theorem 8.5: If S and T are simple surfaces such that S is closed, T is connected, and S is contained in T, then S, as point sets in E^3, equals T.

Suppose $S \neq T$. Let M^* be the set of points in T not in S. Note $M^* \neq \emptyset$, $M^* \cup S = T$, and $M^* \cap S = \emptyset$. Now let P be an arbitrary point in S. We want to show that there is an open set O_P in E^3 containing P such that $O_P \cap T \subset S$. Let G be the image of a patch in S containing P. Then there is an open set O_P such that $O_P \cap S = G$. But G is also a patch in T. Thus $O_P \cap T = G \subset S$. Now let Q be a point in M^*. Since S is closed and $Q \notin S$, there is a neighborhood $S(Q)$ such that $S(Q) \cap S = \emptyset$. Hence $S(Q) \cap T \subset M^*$. Let $O_1 = \bigcup_P O_P$ and $O_2 = \bigcup_Q S(Q)$. Then O_1 and O_2 are open and both have nonempty disjoint intersections with T, namely S and M^*. But this is impossible, since T is connected, which proves the theorem.

8.23. Prove Theorem 8.4: If S is a connected simple surface, then S is arcwise connected by regular arcs.

Suppose S is connected, but P and Q are points in S which cannot be connected by a regular arc. Let M_1 be the set of points in S which can be joined to P, and M_2 those which cannot be joined to P. Note $M_2 = \emptyset$. Now let $P^* \in M_1$. We want to show that there is an open set O_{P^*} containing P^* such that $O_{P^*} \cap S \subset M_1$. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch containing P^*, C a regular arc from P to P^*, and (u_0, v_0) the point in the parameter plane which maps into P^*. In the parameter plane, C is a regular curve $u = u(t)$, $v = v(t)$ with an endpoint at (u_0, v_0), as shown in Fig. 8-25 below. Now let $S(u_0, v_0)$ be a neighborhood of (u_0, v_0) on which $\mathbf{x}$ is defined. Note that $S(u_0, v_0)$ exists since $\mathbf{x}$ is defined on an open set. It is easy to see that $u = u(t)$, $v = v(t)$ can be continued as a regular arc to any point (u, v) in $S(u_0, v_0)$. But then the mapping $\mathbf{x}$ restricted to $S(u_0, v_0)$ is a patch on S all of whose points can be joined to P^* and hence to P. But as a patch in S it is the intersection of an open set O_{P^*} with S. Thus there is an open set O_{P^*} in E^3 containing P^* such that $O_{P^*} \cap S \subset M_1$. A similar argument will prove that if Q^* is a point in M_2 then there exists an open set O_{Q^*} containing Q^* such that $O_{Q^*} \cap S \subset M_2$. Now let $O_1 = \bigcup_{P^*} O_{P^*}$ and $O_2 = \bigcup_{Q^*} O_{Q^*}$. The sets O_1 and O_2 are open, cover S, and have nonempty and disjoint intersections with S, namely M_1 and M_2. But this is impossible since S is connected, which proves the theorem.

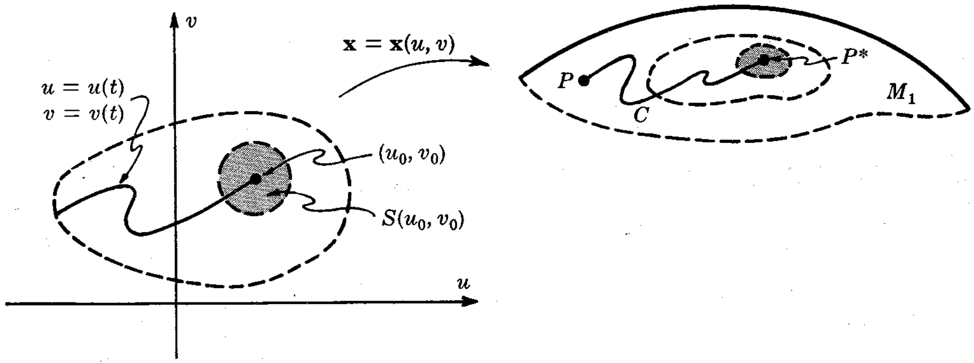


Fig. 8-25

Supplementary Problems

8.24. Show that $\mathbf{x} = \frac{1}{2}(u+v)\mathbf{e}_1 + \frac{1}{2}(u-v)\mathbf{e}_2 + uv\mathbf{e}_3$ is a regular parametric representation of class C^∞ of the hyperbolic paraboloid $x_3 = x_1^2 - x_2^2$. Describe its u- and v-parameter curves.

8.25. Find a regular parametric representation of the right cylinder through the ellipse $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1$ in the x_1x_2 plane.

8.26. Find a basis of coordinate patches for the ellipsoid $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$.

8.27. If G and G^* are the images of two coordinate patches on a simple surface S, show that there exists a patch in S with image equal to the intersection $G \cap G^*$.

8.28. Show that the unit normal to a surface of revolution $\mathbf{x} = f(u)(\cos\theta)\mathbf{e}_1 + f(u)(\sin\theta)\mathbf{e}_2 + g(u)\,\mathbf{e}_3$, $f > 0$ is $\mathbf{N} = [(-g'\cos\theta)\mathbf{e}_1 - (g'\sin\theta)\mathbf{e}_2 + f'\mathbf{e}_3]/[(f')^2 + (g')^2]^{1/2}$.

8.29. Show that the hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ is a doubly ruled surface.

8.30. Find the equation of the tangent plane and normal line to $\mathbf{x} = (u+v)\mathbf{e}_1 + (u-v)\mathbf{e}_2 + uv\mathbf{e}_3$ at $u = 1,\ v = -1$.

8.31. Show that the restriction of the representation of the torus

$$\mathbf{x} = (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

to each of the following three open sets is a basis for the torus. (a) $0 < \theta < 2\pi,\ 0 < \phi < 2\pi$, (b) $-\pi < \theta < \pi,\ -\pi < \phi < \pi$, (c) $-\frac{1}{2}\pi < \theta < \frac{3}{2}\pi,\ -\frac{1}{2}\pi < \phi < \frac{3}{2}\pi$.

8.32. Find the representation in ruled form of the ruled surface with base curve $\mathbf{y} = u\mathbf{e}_1 + u^2\mathbf{e}_2 + u^3\mathbf{e}_3$, $u > 0$, and rulings in the direction of $\mathbf{g} = (\cos u)\mathbf{e}_1 + (\sin u)\mathbf{e}_2$. Show that the representation is regular and of class C^∞.

8.33. A *cone* is a ruled surface in which the rulings pass through a fixed point $\mathbf{p}$ called the vertex. If $\mathbf{g}(u)$ is a nonzero vector in the direction of the rulings, show that $\mathbf{x} = \mathbf{p} + v\mathbf{g}(u)$ is a regular parametric representation of the cone of class C^m if $v \neq 0$, $\mathbf{g}$ is of class C^m, and $\mathbf{g} \times \mathbf{g}' \neq 0$ for all u.

8.34. Show that the tangent plane is constant along a ruling of a cone.

8.35. A right helicoid is a right conoid (see Problem 8.6, page 163) in which the rulings rotate with constant speed about the axis. Show that a right helicoid with the x_3 axis as its axis has a representation of the form

$$\mathbf{x} = (v \cos\theta)\mathbf{e}_1 + (v \sin\theta)\mathbf{e}_2 + (a + b\theta)\mathbf{e}_3, \quad b \neq 0$$

8.36. As a point P moves along a ruling of a right helicoid (see Problem 8.35), show that the unit normal turns about the ruling such that the angle it makes with the axis varies from 0 to $\pi/2$ at ∞. Show that the tangent of this angle is proportional to the distance to the axis.

8.37. Show that the tangent plane along a ruling of the tangent surface of a curve coincides with the osculating plane to the curve at the point on the curve through which the ruling passes.

8.38. The Moebius strip can be represented as the ruled surface $\mathbf{x} = \mathbf{y}(\theta) + v\mathbf{g}(\theta)$, $-\frac{1}{2} < v < \frac{1}{2}$, where

$$\mathbf{y}(\theta) = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2$$

and

$$\mathbf{g}(\theta) = (\sin\tfrac{1}{2}\theta \cos\theta)\mathbf{e}_1 + (\sin\tfrac{1}{2}\theta \sin\theta)\mathbf{e}_2 + (\cos\tfrac{1}{2}\theta)\mathbf{e}_3$$

Verify that the unit normal changes its sense as it is continued around the circle $\mathbf{y} = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2$ once. See Fig. 8-26.

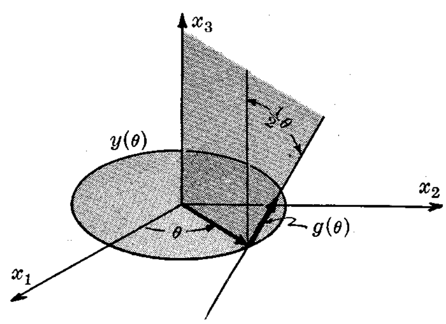


Fig. 8-26

8.39. If the circle $\mathbf{x} = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2$ is removed from the Moebius strip in Problem 8.38, prove that the resulting surface is connected and orientable.

8.40. If $\mathbf{x} = \mathbf{y}(t)$ is a regular curve of class C^m on a patch $\mathbf{x} = \mathbf{x}(u, v)$ of class C^m, show that its image $u = u(t)$, $v = v(t)$ is a regular curve of class C^m in the parameter plane.

8.41. Show that the cone $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 0$, $(x_1, x_2, x_3) \neq (0, 0, 0)$, is an elementary surface of class C^∞.

8.42. If $\mathbf{T}$ is a nonzero vector parallel to the tangent plane of a simple surface at a point P, show that there is a curve $\mathbf{x} = \mathbf{y}(t)$ on the surface through P such that $\mathbf{T} = d\mathbf{y}/dt$ at P.

8.43. Determine which of the following surfaces is compact: (*a*) $x_1^2 - x_2^4 + x_3^6 = 1$, (*b*) $x_1^2 - 2x_1 + x_2^2 + x_3^4 = 1$. *Ans.* (*a*) Not compact, (*b*) Compact.

8.44. Let C be a regular arc of class C^1 in the uv plane with an endpoint at (u_0, v_0), and let $S(u_0, v_0)$ be an arbitrary neighborhood of (u_0, v_0). Show that C can be continued as a regular arc of class C^1 in $S(u_0, v_0)$ to any point (u^*, v^*) in $S(u_0, v_0)$.

8.45. If S is a connected orientable simple surface, prove that there exists one and only one partition of the collection $\mathcal{F}$ of all patches in S into nonempty disjoint sets $\mathcal{F}_1$ and $\mathcal{F}_2$ such that for each $\mathcal{F}_i$, $i = 1, 2$: (i) $\mathcal{F}_i$ is a basis. (ii) If $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ are in $\mathcal{F}_i$, then $\partial(u, v)/\partial(\theta, \phi) > 0$ on the intersection. (iii) $\mathcal{F}_i$ is maximal, i.e. if any patch in S is adjoined to $\mathcal{F}_i$, then property (ii) fails.

Chapter 9

First and Second Fundamental Forms

FIRST FUNDAMENTAL FORM

We recall that a curve in E^3 is uniquely determined by two local invariant quantities, curvature and torsion, as functions of arc length. Similarly, a surface in E^3 is uniquely determined by certain local invariant quantities called the first and second fundamental forms.

Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch on a surface of class ≥ 1. We recall that the differential of the mapping $\mathbf{x} = \mathbf{x}(u, v)$ at (u, v) is a 1-1 linear mapping $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ of the vectors (du, dv) in the uv plane onto the vectors $\mathbf{x}_u\,du + \mathbf{x}_v\,dv$ parallel to the tangent plane at $\mathbf{x}(u, v)$, as shown in Fig. 9-1. Note that we are using the symbols du, dv both for the differentials of the coordinate functions in the uv plane and for the components of a typical vector in the uv plane. In the same spirit, we denote the image $\mathbf{x}_u\,du + \mathbf{x}_v\,dv$ simply by $d\mathbf{x}$. We recall further that $d\mathbf{x}$ has the property that

$$\mathbf{x}(u + du,\, v + dv) \;=\; \mathbf{x}(u, v) \;+\; d\mathbf{x} \;+\; \mathbf{o}((du^2 + dv^2)^{1/2})$$

Thus the vector $d\mathbf{x}$ is a first order approximation to the vector $\mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v)$ from the point $\mathbf{x}(u, v)$ on the patch to the neighboring point $\mathbf{x}(u + du, v + dv)$.

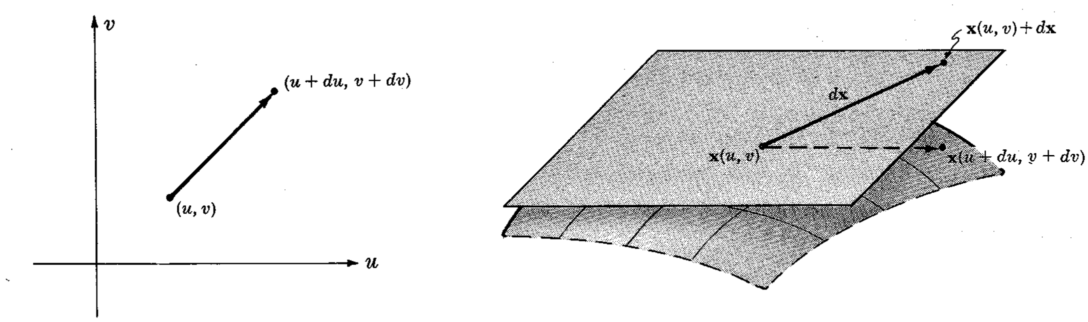


Fig. 9-1

We now consider the quantity

$$\begin{aligned} \mathrm{I} \;&=\; d\mathbf{x}\cdot d\mathbf{x} \;=\; (\mathbf{x}_u\,du + \mathbf{x}_v\,dv)\cdot(\mathbf{x}_u\,du + \mathbf{x}_v\,dv) \\ &=\; (\mathbf{x}_u\cdot\mathbf{x}_u)du^2 \;+\; 2(\mathbf{x}_u\cdot\mathbf{x}_v)du\,dv \;+\; (\mathbf{x}_v\cdot\mathbf{x}_v)dv^2 \;=\; E\,du^2 \;+\; 2F\,du\,dv \;+\; G\,dv^2 \end{aligned}$$

where we set
$$E \;=\; \mathbf{x}_u\cdot\mathbf{x}_u, \quad F \;=\; \mathbf{x}_u\cdot\mathbf{x}_v, \quad G \;=\; \mathbf{x}_v\cdot\mathbf{x}_v \tag{9.1}$$

The function $\mathrm{I} = d\mathbf{x}\cdot d\mathbf{x} = E\,du^2 + 2F\,du\,dv + G\,dv^2$ is called the *first fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$. It is a homogeneous function of second degree in du and dv with coefficients E, F and G, called the *first fundamental coefficients,* which are functions of u and v and vary from point to point on the coordinate patch. Thus the first fundamental form I is the quadratic form defined on vectors (du, dv) in the uv plane by

$$\mathrm{I}(du, dv) \;=\; E\,du^2 \;+\; F\,du\,dv \;+\; G\,dv^2$$

Recall that $d\mathbf{x}$ at (u,v) is the principal part of the vector from the point $\mathbf{x}(u,v)$ on the patch to the point $\mathbf{x}(u+du, v+dv)$. This suggests that in some sense I depends only on the surface and not on the particular representation. In fact this is true in the following sense. Suppose that $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is another coordinate patch containing a neighborhood of $\mathbf{x}(u,v)$. Then the parameter transformation $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ has a differential at (u,v) which maps the vector (du, dv) into the vector $(d\theta, d\phi)$ given by $d\theta = \theta_u\,du + \theta_v\,dv$, $d\phi = \phi_u\,du + \phi_v\,dv$. The statement that I is independent of the representation is interpreted to mean that I and I* agree under this correspondence; that is, $\mathrm{I}(du,dv) = \mathrm{I}^*(d\theta, d\phi)$. This can be verified analytically using the chain rule:

$$\begin{aligned}
\mathrm{I}^*(d\theta, d\phi) &= |d\mathbf{x}^*|^2 = |(\mathbf{x}^*_\theta\,d\theta + \mathbf{x}^*_\phi\,d\phi)|^2 \\
&= |\mathbf{x}^*_\theta(\theta_u\,du + \theta_v\,dv) + \mathbf{x}^*_\phi(\phi_u\,du + \phi_v\,dv)|^2 \\
&= |(\mathbf{x}^*_\theta\theta_u + \mathbf{x}^*_\phi\phi_u)\,du + (\mathbf{x}^*_\theta\theta_v + \mathbf{x}^*_\phi\phi_v)\,dv|^2 \\
&= |\mathbf{x}_u\,du + \mathbf{x}_v\,dv|^2 = |d\mathbf{x}|^2 = \mathrm{I}(du,dv)
\end{aligned}$$

The first fundamental coefficients themselves are not invariant under a parameter transformation, but instead they transform as follows:

$$\begin{aligned}
E &= \mathbf{x}_u\cdot\mathbf{x}_u = (\mathbf{x}^*_\theta\theta_u + \mathbf{x}^*_\phi\phi_u)\cdot(\mathbf{x}^*_\theta\theta_u + \mathbf{x}^*_\phi\phi_u) \\
&= \mathbf{x}^*_\theta\cdot\mathbf{x}^*_\theta\theta_u^2 + 2\mathbf{x}^*_\theta\cdot\mathbf{x}^*_\phi\theta_u\phi_u + \mathbf{x}^*_\phi\cdot\mathbf{x}^*_\phi\phi_u^2 \qquad (9.2)\\
&= E^*\theta_u^2 + 2F^*\theta_u\phi_u + G^*\phi_u^2
\end{aligned}$$

and similarly,

$$\begin{aligned}
F &= E^*\theta_u\theta_v + F^*(\theta_u\phi_v + \phi_u\theta_v) + G^*\phi_u\phi_v \\
G &= E^*\theta_v^2 + 2F^*\theta_v\phi_v + G^*\phi_v^2
\end{aligned} \qquad (9.3)$$

Finally, we note that the first fundamental form is *positive definite*. That is, $\mathrm{I} \geqq 0$, and $\mathrm{I} = 0$ if and only if $du = 0$ and $dv = 0$. For, clearly $\mathrm{I} = |d\mathbf{x}|^2 \geqq 0$, and, since $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent, $\mathrm{I} = |d\mathbf{x}|^2 = |\mathbf{x}_u\,du + \mathbf{x}_v\,dv|^2 = 0$ if and only if $du = 0$ and $dv = 0$.

Since I is positive definite its coefficients must satisfy $E > 0$, $G > 0$ and $EG - F^2 > 0$. This can also be verified directly. For again, since $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent, $\mathbf{x}_u \neq 0$, $\mathbf{x}_v \neq 0$, and hence $E = \mathbf{x}_u\cdot\mathbf{x}_u = |\mathbf{x}_u|^2 > 0$ and $G = \mathbf{x}_v\cdot\mathbf{x}_v = |\mathbf{x}_v|^2 > 0$. Also, using the vector identity [$\mathbf{F}_5$], we find that

$$EG - F^2 = (\mathbf{x}_u\cdot\mathbf{x}_u)(\mathbf{x}_v\cdot\mathbf{x}_v) - (\mathbf{x}_u\cdot\mathbf{x}_v)(\mathbf{x}_u\cdot\mathbf{x}_v) = (\mathbf{x}_u\times\mathbf{x}_v)\cdot(\mathbf{x}_u\times\mathbf{x}_v) = |\mathbf{x}_u\times\mathbf{x}_v|^2 \qquad (9.4)$$

But at every point $\mathbf{x}_u\times\mathbf{x}_v \neq \mathbf{0}$, and so $EG - F^2 > 0$.

Example 9.1.

Consider the surface given by

$$\mathbf{x} = (u+v)\mathbf{e}_1 + (u-v)\mathbf{e}_2 + uv\mathbf{e}_3$$

Here $\mathbf{x}_u = \mathbf{e}_1 + \mathbf{e}_2 + v\mathbf{e}_3$, $\mathbf{x}_v = \mathbf{e}_1 - \mathbf{e}_2 + u\mathbf{e}_3$, $E = \mathbf{x}_u\cdot\mathbf{x}_u = 2 + v^2$, $F = \mathbf{x}_u\cdot\mathbf{x}_v = uv$, $G = \mathbf{x}_v\cdot\mathbf{x}_v = 2 + u^2$

and $\qquad \mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2 = (2+v^2)\,du^2 + 2uv\,du\,dv + (2+u^2)\,dv^2$

Observe that at all (u,v) we have $E > 0$, $G > 0$, and $EG - F^2 = 4 + 2u^2 + 2v^2 > 0$. If we introduce $\theta = u + v$, $\phi = u - v$, then the surface is also represented by

$$\mathbf{x} = \theta\mathbf{e}_1 + \phi\mathbf{e}_2 + \tfrac{1}{4}(\theta^2 - \phi^2)\mathbf{e}_3$$

Here $\mathbf{x}_\theta = \mathbf{e}_1 + \frac{1}{2}\theta\mathbf{e}_3$, $\mathbf{x}_\phi = \mathbf{e}_2 - \frac{1}{2}\phi\mathbf{e}_3$, $E^* = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = 1 + \frac{1}{4}\theta^2$, $F^* = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = -\frac{1}{4}\theta\phi$, $G^* = \mathbf{x}_\phi\cdot\mathbf{x}_\phi = 1 + \frac{1}{4}\phi^2$.

Observe that at the point corresponding to $u = 1$, $v = 1$ we have $E = 3$, $F = 1$, $G = 3$. But at the same point $\theta = 2$, $\phi = 0$, and $E^* = 2$, $F^* = 0$, $G^* = 1$. The first fundamental coefficients are not invariant quantities.

ARC LENGTH AND SURFACE AREA

The first fundamental coefficients play a basic role in calculating arc length, angles, and surface area. For, let $\mathbf{x} = \mathbf{x}(u(t), v(t))$, $a \leq t \leq b$, be a regular arc on a patch $\mathbf{x} = \mathbf{x}(u, v)$. We recall that its length is the integral

$$s = \int_a^b \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_a^b \left(\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{x}}{dt}\right)^{1/2} dt = \int_a^b \left[\left(\mathbf{x}_u\frac{du}{dt} + \mathbf{x}_v\frac{dv}{dt}\right)\cdot\left(\mathbf{x}_u\frac{du}{dt} + \mathbf{x}_v\frac{dv}{dt}\right)\right]^{1/2} dt$$

Hence expanding,
$$s = \int_a^b \left[E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2\right]^{1/2} dt \qquad (9.5)$$

Thus arc length on the surface is the integral of the square root of the first fundamental form.

Suppose further that $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ and $\delta\mathbf{x} = \mathbf{x}_u\,\delta u + \mathbf{x}_v\,\delta v$ are two vectors parallel to the tangent plane at a point $\mathbf{x}$. If α is the angle between $d\mathbf{x}$ and $\delta\mathbf{x}$, then

$$\begin{aligned}\cos\alpha &= \frac{d\mathbf{x}\cdot\delta\mathbf{x}}{|d\mathbf{x}|\,|\delta\mathbf{x}|} = \frac{(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)\cdot(\mathbf{x}_u\,\delta u + \mathbf{x}_v\,\delta v)}{|\mathbf{x}_u\,du + \mathbf{x}_v\,dv|\,|\mathbf{x}_u\,\delta u + \mathbf{x}_v\,\delta v|} \\ &= \frac{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{[E\,du^2 + 2F\,du\,dv + G\,dv^2]^{1/2}\,[E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2]^{1/2}}\end{aligned} \qquad (9.6)$$

In particular, if β is the angle between the u- and v-parameter curves at $\mathbf{x}$, i.e. the angle between $\mathbf{x}_u$ and $\mathbf{x}_v$, then

$$\cos\beta = \frac{\mathbf{x}_u\cdot\mathbf{x}_v}{|\mathbf{x}_u|\,|\mathbf{x}_v|} = \frac{\mathbf{x}_u\cdot\mathbf{x}_v}{\sqrt{\mathbf{x}_u\cdot\mathbf{x}_u}\sqrt{\mathbf{x}_v\cdot\mathbf{x}_v}} = \frac{F}{\sqrt{EG}} \qquad (9.7)$$

As a result of the above we have

Theorem 9.1. (*a*) The tangent vectors $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ and $\delta\mathbf{x} = \mathbf{x}_u\,\delta u + \mathbf{x}_v\,\delta v$ are perpendicular if and only if

$$E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0$$

(*b*) The u- and v-parameter curves at a point are perpendicular if and only if $F = 0$.

Example 9.2.

Consider the image on the unit sphere

$$\mathbf{x} = (\cos\theta\sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3$$

of the curve $\theta = \log\cot(\pi/4 - t/2)$, $\phi = \pi/2 - t$, $0 \leq t \leq \pi/2$. As shown in Fig. 9-2, the curve starts at the equator and winds like a spiral about the north pole. To determine its length we compute

$$\mathbf{x}_\theta = (-\sin\theta\sin\phi)\mathbf{e}_1 + (\cos\theta\sin\phi)\mathbf{e}_2$$

$$\mathbf{x}_\phi = (\cos\theta\cos\phi)\mathbf{e}_1 + (\sin\theta\cos\phi)\mathbf{e}_2 - (\sin\phi)\mathbf{e}_3$$

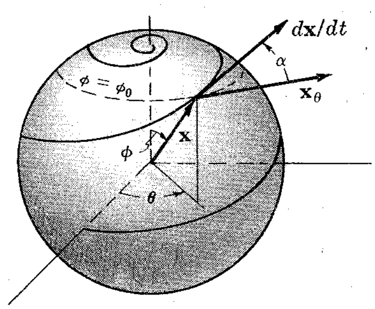


Fig. 9-2

$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = \sin^2\phi$, $F = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0$, $G = \mathbf{x}_\phi\cdot\mathbf{x}_\phi = 1$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2\cot(\pi/4 - t/2)} = \frac{1}{2\sin(\pi/4 - t/2)\cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)} \quad \text{and} \quad \frac{d\phi}{dt} = -1$$

Thus along the curve, where $\phi = \pi/2 - t$,

$$\mathrm{I} = E\left(\frac{d\theta}{dt}\right)^2 + 2F\frac{d\theta}{dt}\frac{d\phi}{dt} + G\left(\frac{d\phi}{dt}\right)^2 = \frac{\sin^2\phi}{\sin^2(\pi/2 - t)} + 1 = 2$$

and

$$s = \int_0^{\pi/2}\sqrt{\mathrm{I}}\,dt = \int_0^{\pi/2}\sqrt{2}\,dt = \pi/\sqrt{2}$$

Note that the curve makes a constant angle α with the parallels $\phi = \text{constant}$. For, using $\phi = \pi/2 - t$,

$$\cos\alpha = \cos\measuredangle\left(\frac{d\mathbf{x}}{dt}, \mathbf{x}_\theta\right) = \frac{\left(\mathbf{x}_\theta \frac{d\theta}{dt} + \mathbf{x}_\phi \frac{d\phi}{dt}\right)\cdot \mathbf{x}_\theta}{\left|\mathbf{x}_\theta \frac{d\theta}{dt} + \mathbf{x}_\phi \frac{d\phi}{dt}\right| |\mathbf{x}_\theta|} = \frac{E\frac{d\theta}{dt} + F\frac{d\theta}{dt}}{\sqrt{\mathrm{I}}\sqrt{E}}$$

$$= \left(\frac{\sin^2\phi}{\sin(\pi/2 - t)}\right)\left(\frac{1}{\sqrt{2}}\frac{1}{\sin\phi}\right) = \frac{1}{\sqrt{2}} \qquad 0 \leqq t \leqq \pi/2$$

Now suppose ΔR is a small region on a patch bounded by neighboring v-parameter curves, u and $u + du$, and u-parameter curves, v and $v + dv$, as indicated in Fig. 9-3. As a first approximation to the area of ΔR we take the area of the parallelogram whose sides are the vectors $\Delta\mathbf{x}_1 = \mathbf{x}_u\, du$ and $\Delta\mathbf{x}_2 = \mathbf{x}_v\, dv$. Assuming $du > 0$ and $dv > 0$, this is the quantity

$$\Delta s = |\Delta\mathbf{x}_1 \times \Delta\mathbf{x}_2| = |\mathbf{x}_u \times \mathbf{x}_v|\, du\, dv = \sqrt{EG - F^2}\, du\, dv$$

Thus we are led to define the area of a region R on $\mathbf{x} = \mathbf{x}(u, v)$ as the double integral, if it exists,

$$A = \iint_W \sqrt{EG - F^2}\, du\, dv \tag{9.8}$$

where W is the set of points in the parameter plane which maps onto R.

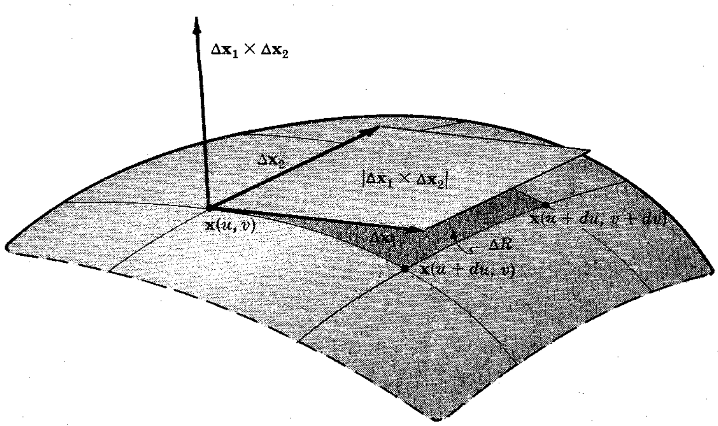


Fig. 9-3

Note that on an oriented surface the above definition of area is independent of the representation of R. For let $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ be any other patch containing R and such that $\partial(\theta,\phi)/\partial(u,v) > 0$ for all (u, v) in W. It can easily be computed from equations (*9.2*) and (*9.3*) that

$$EG - F^2 = (E^*G^* - F^{*2})[\partial(\theta,\phi)/\partial(u,v)]^2 \tag{9.9}$$

and so it follows from the transformation theorem for multiple integrals that

$$A = \iint_W \sqrt{EG - F^2}\, du\, dv = \iint_W \sqrt{E^*G^* - F^{*2}}\, \frac{\partial(\theta,\phi)}{\partial(u,v)}\, du\, dv$$

$$= \iint_{W^*} \sqrt{E^*G^* - F^{*2}}\, d\theta\, d\phi = A^*$$

where W^* is the image of W in the $\theta\phi$ plane, which is the required result.

Example 9.3.

Consider the torus (see Example 8.5, page 158)

$$\mathbf{x} \;=\; (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 \;+\; (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 \;+\; (a\cos\phi)\mathbf{e}_3$$

Here $E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = (b + a\sin\phi)^2$, $F = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0$, $G = \mathbf{x}_\phi\cdot\mathbf{x}_\phi = a^2$ and its surface area

$$S \;=\; \iint_{\substack{0\le\theta\le 2\pi\\ 0\le\phi\le 2\pi}} \sqrt{EG - F^2}\,d\theta\,d\phi \;=\; \int_0^{2\pi}\left[\int_0^{2\pi} a(b + a\sin\phi)\,d\phi\right]d\theta \;=\; 4\pi^2 ab$$

SECOND FUNDAMENTAL FORM

We now suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on a surface of class $\geqq 2$. Then at each point on the patch there is a unit normal $\mathbf{N} = \dfrac{\mathbf{x}_u\times\mathbf{x}_v}{|\mathbf{x}_u\times\mathbf{x}_v|}$, which is a function of u and v of class C^1 with differential $d\mathbf{N} = \mathbf{N}_u\,du + \mathbf{N}_v\,dv$. Observe in Fig. 9-4 that $d\mathbf{N}$ is orthogonal to $\mathbf{N}$ since it is parallel to the tangent plane of the spherical image of $\mathbf{N}$. This also follows from $0 = d(1) = d(\mathbf{N}\cdot\mathbf{N}) = 2d\mathbf{N}\cdot\mathbf{N}$. Thus $d\mathbf{N}$ is a vector parallel to the tangent plane at $\mathbf{x}$ as shown in Fig. 9-4(*b*).

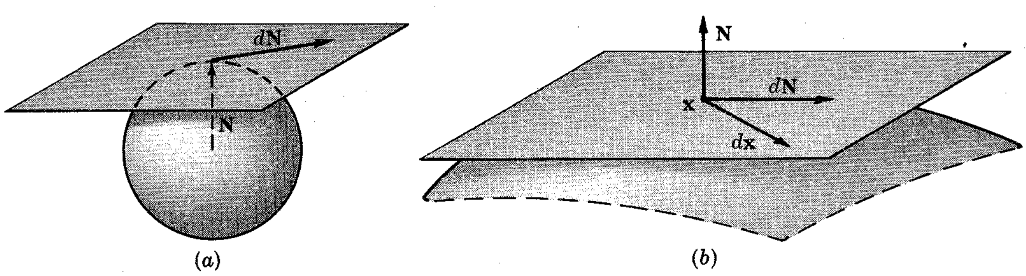


Fig. 9-4

Now consider the quantity

$$\begin{aligned}\mathrm{II} \;&=\; -d\mathbf{x}\cdot d\mathbf{N} \;=\; -(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)\cdot(\mathbf{N}_u\,du + \mathbf{N}_v\,dv)\\ &=\; -\mathbf{x}_u\cdot\mathbf{N}_u\,du^2 - (\mathbf{x}_u\cdot\mathbf{N}_v + \mathbf{x}_v\cdot\mathbf{N}_u)\,du\,dv - \mathbf{x}_v\cdot\mathbf{N}_v\,dv^2 \;=\; L\,du^2 + 2M\,du\,dv + N\,dv^2\end{aligned}$$

where
$$L \;=\; -\mathbf{x}_u\cdot\mathbf{N}_u, \quad M \;=\; -\tfrac{1}{2}(\mathbf{x}_u\cdot\mathbf{N}_v + \mathbf{x}_v\cdot\mathbf{N}_u), \quad N \;=\; -\mathbf{x}_v\cdot\mathbf{N}_v \tag{9.10}$$

The function $\mathrm{II} = -d\mathbf{x}\cdot d\mathbf{N} = L\,du^2 + 2M\,du\,dv + N\,dv^2$ is called the *second fundamental form* of $\mathbf{x} = \mathbf{x}(u,v)$. Here again II is a homogeneous function of second degree in du and dv with coefficients L, M and N, called the *second fundamental coefficients*, which are continuous functions of u and v.

It is easily shown that II is invariant in the same sense that I is invariant under a parameter transformation which preserves the direction of $\mathbf{N}$; otherwise II changes its sign. Also its coefficients transform in the same way as the first fundamental coefficients. That is, if $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ is any other patch on the surface, then at a point on the intersection where $\partial(\theta,\phi)/\partial(u,v) > 0$, we have

$$\begin{aligned}L \;&=\; L^*\theta_u^2 + 2M^*\theta_u\phi_u + N^*\phi_u^2\\ M \;&=\; L^*\theta_u\theta_v + M^*(\phi_u\theta_v + \theta_u\phi_v) + N^*\phi_u\phi_v\\ N \;&=\; N^*\theta_v^2 + 2M^*\theta_v\phi_v + N^*\phi_v^2\end{aligned} \tag{9.11}$$

The proof of the above properties of II is left to the reader as an exercise.

Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are perpendicular to $\mathbf{N}$ for all (u,v), we have

$$0 \;=\; (\mathbf{x}_u\cdot\mathbf{N})_u \;=\; \mathbf{x}_{uu}\cdot\mathbf{N} + \mathbf{x}_u\cdot\mathbf{N}_u, \quad 0 \;=\; (\mathbf{x}_u\cdot\mathbf{N})_v \;=\; \mathbf{x}_{uv}\cdot\mathbf{N} + \mathbf{x}_u\cdot\mathbf{N}_v,$$
$$0 \;=\; (\mathbf{x}_v\cdot\mathbf{N})_u \;=\; \mathbf{x}_{vu}\cdot\mathbf{N} + \mathbf{x}_v\cdot\mathbf{N}_u, \quad\text{and}\quad 0 \;=\; (\mathbf{x}_v\cdot\mathbf{N})_v \;=\; \mathbf{x}_{vv}\cdot\mathbf{N} + \mathbf{x}_v\cdot\mathbf{N}_v$$

Hence
$$\mathbf{x}_{uu}\cdot\mathbf{N} \;=\; -\mathbf{x}_u\cdot\mathbf{N}_u, \quad \mathbf{x}_{uv}\cdot\mathbf{N} \;=\; -\mathbf{x}_u\cdot\mathbf{N}_v \;=\; -\mathbf{x}_v\cdot\mathbf{N}_u, \quad\text{and}\quad \mathbf{x}_{vv}\cdot\mathbf{N} \;=\; -\mathbf{x}_v\cdot\mathbf{N}_v$$

This gives alternative expressions for L, M, and N. Namely from *(9.10)*,

$$L = \mathbf{x}_{uu}\cdot\mathbf{N}, \quad M = \mathbf{x}_{uv}\cdot\mathbf{N}, \quad N = \mathbf{x}_{vv}\cdot\mathbf{N} \tag{9.12}$$

Also we have

$$\text{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 = \mathbf{x}_{uu}\cdot\mathbf{N}\,du^2 + 2\mathbf{x}_{uv}\cdot\mathbf{N}\,du\,dv + \mathbf{x}_{vv}\cdot\mathbf{N}\,dv^2 = d^2\mathbf{x}\cdot\mathbf{N} \tag{9.13}$$

where $d^2\mathbf{x} = \mathbf{x}_{uu}\,du^2 + 2\mathbf{x}_{uv}\,du\,dv + \mathbf{x}_{vv}\,dv^2$ is the second order derivative of $\mathbf{x}$ at (u,v) in the direction du, dv.

Example 9.4.

Consider the surface represented by

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$$

Here $\mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 - 2v\mathbf{e}_3, \quad \mathbf{x}_{uu} = 2\mathbf{e}_3, \quad \mathbf{x}_{uv} = \mathbf{0}, \quad \mathbf{x}_{vv} = -2\mathbf{e}_3$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = (4u^2 + 4v^2 + 1)^{-1/2}(-2u\mathbf{e}_1 + 2v\mathbf{e}_2 + \mathbf{e}_3)$$

Thus the second fundamental coefficients are

$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = 2(4u^2+4v^2+1)^{-1/2}, \quad M = \mathbf{x}_{uv}\cdot\mathbf{N} = 0, \quad N = \mathbf{x}_{vv}\cdot\mathbf{N} = -2(4u^2+4v^2+1)^{-1/2}$$

and the second fundamental form is

$$\text{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 = 2(4u^2+4v^2+1)^{-1/2}(du^2 - dv^2)$$

Suppose P is a point on a surface of class $\geqq 2$, Q is a point in the neighborhood of P and $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P and Q. Let $d = \mathbf{PQ}\cdot\mathbf{N}$ be the projection of $\mathbf{PQ}$ onto the unit normal $\mathbf{N}$ at P, as shown in Fig. 9-5. Observe that d is positive or negative depending on whether Q is on one or the other side of the tangent plane at P and that $|d|$ is the perpendicular distance from Q to the tangent plane at P. Now suppose P and Q are the points $\mathbf{x}(u,v)$ and $\mathbf{x}(u+du, v+dv)$ respectively. Taylor's theorem gives

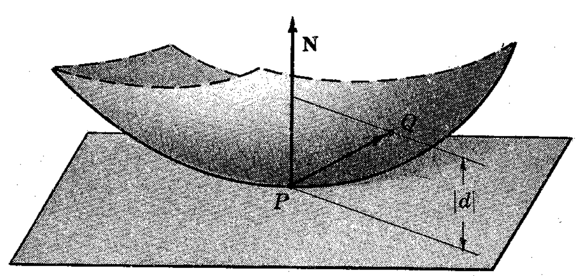


Fig. 9-5

$$\mathbf{x}(u+du, v+dv) = \mathbf{x}(u,v) + d\mathbf{x} + \tfrac{1}{2}d^2\mathbf{x} + \mathbf{o}(du^2+dv^2)$$

Thus

$$\begin{aligned} d &= \mathbf{PQ}\cdot\mathbf{N} = (\mathbf{x}(u+du, v+dv) - \mathbf{x}(u,v))\cdot\mathbf{N} \\ &= [d\mathbf{x} + \tfrac{1}{2}d^2\mathbf{x} + \mathbf{o}(du^2+dv^2)]\cdot\mathbf{N} \\ &= d\mathbf{x}\cdot\mathbf{N} + \tfrac{1}{2}d^2\mathbf{x}\cdot\mathbf{N} + o(du^2+dv^2) \end{aligned}$$

But $d\mathbf{x}\cdot\mathbf{N} = 0$, since $d\mathbf{x}$ is parallel to the tangent plane at P. Hence from *(9.13)*,

$$d = \tfrac{1}{2}d^2\mathbf{x}\cdot\mathbf{N} + o(du^2+dv^2) = \tfrac{1}{2}\text{II} + o(du^2+dv^2)$$

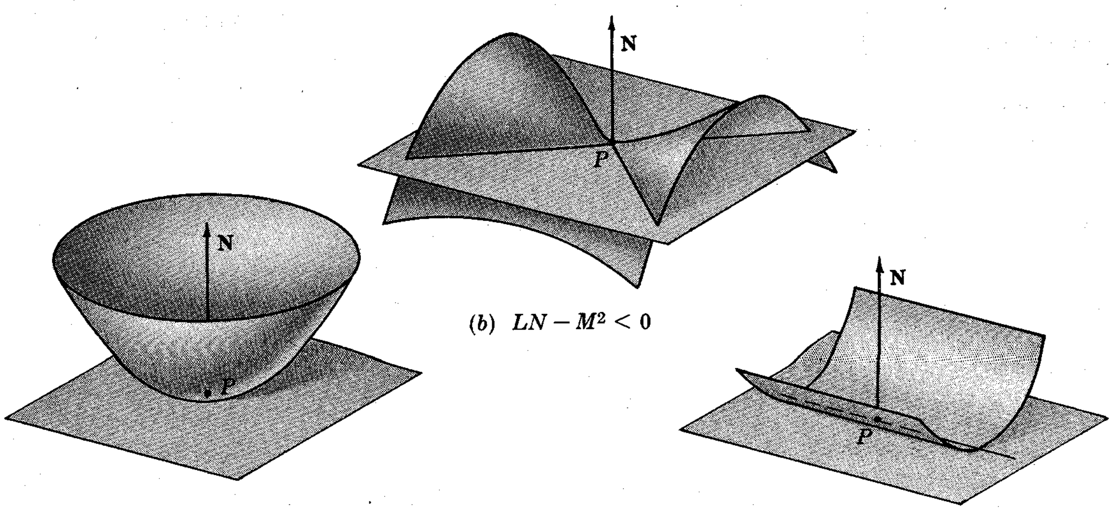
Thus II is the principal part of twice the projection of $\mathbf{PQ}$ onto $\mathbf{N}$ and $|\text{II}|$ is the principal part of twice the perpendicular distance from Q onto the tangent plane at P.

The function

$$\delta = \tfrac{1}{2}\text{II} = \tfrac{1}{2}(L\,du^2 + 2M\,du\,dv + N\,dv^2)$$

is called the *osculating paraboloid* at P. The nature of this paraboloid determines qualitatively the nature of the surface in the neighborhood of P. We distinguish four cases, depending upon the discriminant $LN - M^2$.

(i) *Elliptic case*: A point is called an *elliptic* point if $LN - M^2 > 0$. In this case δ as a function of du and dv is an elliptic paraboloid as shown in Fig. 9-6(*a*). Observe that δ maintains the same sign for all (du, dv). In the neighborhood of an elliptic point the surface lies on one side of the tangent plane at the point and is shaped like the figure.



(*b*) $LN - M^2 < 0$

(*a*) $LN - M^2 > 0$

(*c*) $LN - M^2 = 0$, $L^2 + M^2 + N^2 \neq 0$

Fig. 9-6

(ii) *Hyperbolic case*: A point is called a *hyperbolic* point if $LN - M^2 < 0$. In this case δ as a function of (du, dv) is a hyperbolic paraboloid as shown in Fig. 9-6(*b*). Here there are two distinct lines in the tangent plane through P which divide the tangent plane into four sections in which δ is alternately positive and negative. On the two lines, $\delta = 0$. In the neighborhood of a hyperbolic point the surface lies on both sides of the tangent plane as in the figure.

(iii) *Parabolic case*: A point is called a *parabolic* point if $LN - M^2 = 0$ and $L^2 + M^2 + N^2 \neq 0$, i.e. if $LN - M^2 = 0$ and the coefficients L, M and N are not all zero. In this case δ as a function of (du, dv) is a parabolic cylinder, as shown in Fig. 9-6(*c*). Here there is a single line in the tangent plane through P along which $\delta = 0$, otherwise δ maintains the same sign. It is to be noted that in the neighborhood of a parabolic point the surface itself may lie on both sides of the tangent plane. See Problem 9.8.

(iv) *Planar case*: A point is called a *planar* point if $L = M = N = 0$. Here $\delta = 0$ for all (du, dv). In this case the degree of *contact* of the surface and the tangent plane is of higher order than in the preceding cases.

We expect from the above geometric discussion that the property of a point on a surface being elliptic, hyperbolic, parabolic or planar is independent of the representation of the surface. This can be verified analytically. For if $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is any other patch on the surface, it can be shown from equations (*9.11*) that at each point on the intersection

$$L^*N^* - M^{*2} = \left(\frac{\partial(u,v)}{\partial(\theta,\phi)}\right)^2 (LN - M^2)$$

Since $\partial(u,v)/\partial(\theta,\phi) \neq 0$, it follows that $L^*N^* - M^{*2}$ is positive, negative, or zero together with $LN - M^2$. It also follows from equations (*9.11*) and their corresponding inverse equations that $L = M = N = 0$ if and only if $L^* = M^* = N^* = 0$.

Example 9.5.

We refer to the torus

$$\mathbf{x} = (b + a \sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3 \qquad b > a$$

Here

$$\mathbf{x}_{\theta\theta} = -(b + a\sin\phi)(\cos\theta)\mathbf{e}_1 - (b + a\sin\phi)(\sin\theta)\mathbf{e}_2$$
$$\mathbf{x}_{\theta\phi} = -(a\cos\phi\sin\theta)\mathbf{e}_1 + (a\cos\phi\cos\theta)\mathbf{e}_2$$
$$\mathbf{x}_{\phi\phi} = -(a\sin\phi\cos\theta)\mathbf{e}_1 - (a\sin\phi\sin\theta)\mathbf{e}_2 - (a\cos\phi)\mathbf{e}_3$$
$$\mathbf{N} = (-\cos\theta\sin\phi)\mathbf{e}_1 - (\sin\theta\sin\phi)\mathbf{e}_2 - (\cos\phi)\mathbf{e}_3$$
$$L = \mathbf{x}_{\theta\theta}\cdot\mathbf{N} = (b + a\sin\phi)\sin\phi \qquad M = \mathbf{x}_{\theta\phi}\cdot\mathbf{N} = 0 \qquad N = \mathbf{x}_{\phi\phi}\cdot\mathbf{N} = a$$

and

$$LN - M^2 = a(b + a\sin\phi)\sin\phi$$

Observe that the second fundamental coefficients depend only on ϕ. They are constant along a parallel $\phi = \phi_0$. Since $0 < a < b$, $a(b + a\sin\phi) > 0$. Hence the sign of $LN - M^2$ agrees with the sign of $\sin\phi$. Thus $LN - M^2 > 0$ for $0 < \phi < \pi$, $LN - M^2 = 0$ for $\phi = 0$ or $\phi = \pi$, and $LN - M^2 < 0$ for $\pi < \phi < 2\pi$. Thus as shown in Fig. 9-7, the points towards the outside of the torus $(0 < \phi < \pi)$ are elliptic. Here the surface lies on one side of the tangent plane in the neighborhood of each point. The points on the inside of the torus $(\pi < \phi < 2\pi)$ are hyperbolic. Here the surface lies on both sides of the tangent plane in the neighborhood of each point. The top and bottom parallels of torus, $\phi = 0$ and $\phi = \pi$ are parabolic points.

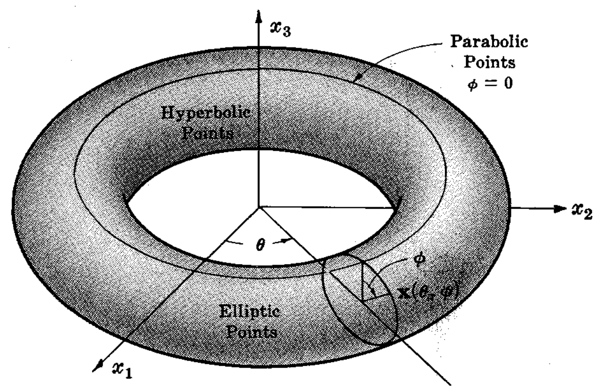


Fig. 9-7

The nature of the surface at a planar point is described by the next higher order terms of the expansion of $\mathbf{x}(u, v)$. Namely, suppose $\mathbf{x} = \mathbf{x}(u, v)$ is of class C^3 and $P = \mathbf{x}(u, v)$ is a planar point; then

$$\mathbf{x}(u + du, v + dv) = \mathbf{x}(u, v) + d\mathbf{x} + \tfrac{1}{2}d^2\mathbf{x} + \tfrac{1}{6}d^3\mathbf{x} + \mathbf{o}[(du^2 + dv^2)^{3/2}]$$

and

$$\begin{aligned} d &= [\mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v)]\cdot\mathbf{N} \\ &= d\mathbf{x}\cdot\mathbf{N} + \tfrac{1}{2}d^2\mathbf{x}\cdot\mathbf{N} + \tfrac{1}{6}d^3\mathbf{x}\cdot\mathbf{N} + o[(du^2 + dv^2)^{3/2}] \\ &= \tfrac{1}{2}\mathrm{II} + \tfrac{1}{6}d^3\mathbf{x}\cdot\mathbf{N} + o[(du^2 + dv^2)^{3/2}] \end{aligned}$$

Since $L = M = N = 0$ at P, we have $\mathrm{II} = 0$. Hence the surface is approximated by

$$\begin{aligned} \delta = \tfrac{1}{6}d^3\mathbf{x}\cdot\mathbf{N} &= \tfrac{1}{6}[\mathbf{x}_{uuu}\cdot\mathbf{N}\,du^3 + 3\mathbf{x}_{uuv}\cdot\mathbf{N}\,du^2\,dv + 3\mathbf{x}_{uvv}\cdot\mathbf{N}\,du\,dv^2 + \mathbf{x}_{vvv}\cdot\mathbf{N}\,dv^3] \\ &= \tfrac{1}{6}[A\,du^3 + B\,du^2\,dv + C\,du\,dv^2 + D\,dv^3] \end{aligned}$$

If the cubic $Ax^3 + Bx^2 + Cx + D$ has three distinct roots, then there are three lines in the tangent plane through P which divide the tangent plane at P into six sectors on which δ is alternately positive and negative. In this case δ as a function of (du, dv) describes a *monkey saddle* as shown in Fig. 9-8. Variations will occur depending on the nature of the roots of $Ax^3 + Bx^2 + Cx + D$.

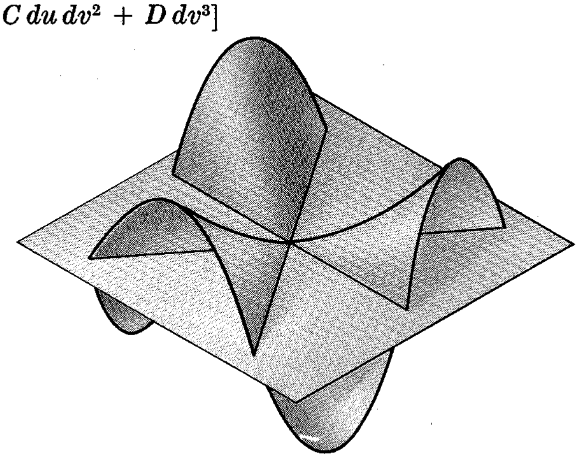


Fig. 9-8

Example 9.6.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^3 + v^3 + u^4)\mathbf{e}_3$$

Here

$$\mathbf{x}_u = \mathbf{e}_1 + (3u^2+4u^3)\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 3v^2\mathbf{e}_3, \quad \mathbf{x}_{uu} = (6u+12u^2)\mathbf{e}_3, \quad \mathbf{x}_{uv} = \mathbf{0}, \quad \mathbf{x}_{vv} = 6v\mathbf{e}_3$$

$$\mathbf{N} = [(3u^2+4u^3)^2 + 9v^4 + 1]^{-1/2}(-(3u^2+4u^3)\mathbf{e}_1 - 3v^2\mathbf{e}_2 + \mathbf{e}_3)$$

$$\mathbf{x}_{uuu} = (6+24u)\mathbf{e}_3, \quad \mathbf{x}_{uuv} = \mathbf{0}, \quad \mathbf{x}_{uvv} = \mathbf{0}, \quad \mathbf{x}_{vvv} = 6\mathbf{e}_3$$

At $u=0$, $v=0$, we have $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = \mathbf{0}$ so that $L = M = N = 0$ and the point is planar. The surface is described at this point by

$$\delta = \tfrac{1}{6}(\mathbf{x}_{uuu}\cdot\mathbf{N}\,du^3 + 3\mathbf{x}_{uuv}\cdot\mathbf{N}\,du^2\,dv + 3\mathbf{x}_{uvv}\cdot\mathbf{N}\,du\,dv^2 + \mathbf{x}_{vvv}\cdot\mathbf{N}\,dv^3)$$

At $u=0$, $v=0$ this is the function

$$\begin{aligned}\delta &= \tfrac{1}{6}[6\mathbf{e}_3\cdot\mathbf{e}_3\,du^3 + 6\mathbf{e}_3\cdot\mathbf{e}_3\,dv^3] \\ &= du^3 + dv^3 \\ &= (du+dv)(du^2 - du\,dv + dv^2)\end{aligned}$$

Here there is a single line, $du + dv = 0$, in the tangent plane along which $\delta = 0$. The second factor $(du^2 - du\,dv + dv^2)$ is positive definite for all real (du, dv). In the neighborhood of this point the surface is shaped as shown in Fig. 9-9.

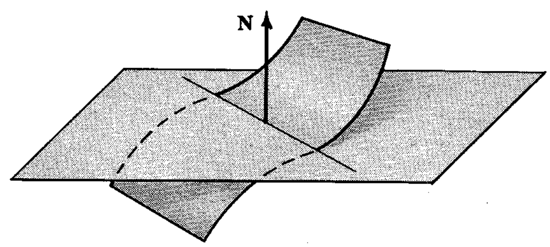


Fig. 9-9

NORMAL CURVATURE

Let P be a point on a surface of class $\geqq 2$, $\mathbf{x} = \mathbf{x}(u,v)$ a patch containing P, and $\mathbf{x} = \mathbf{x}(u(t), v(t))$ a regular curve C of class C^2 through P. The *normal curvature vector* to C at P, denoted by $\mathbf{k}_n$, is the vector projection of the curvature vector $\mathbf{k}$ of C at P onto the normal $\mathbf{N}$ at P; i.e.,

$$\mathbf{k}_n = (\mathbf{k}\cdot\mathbf{N})\mathbf{N} \tag{9.14}$$

Note that $\mathbf{k}_n$ is independent of the sense of $\mathbf{N}$. It is also independent of the sense of C since $\mathbf{k}$ is independent of the sense of C.

The component of $\mathbf{k}_n$ in the direction of $\mathbf{N}$ is called the *normal curvature* of C at P and is denoted by κ_n. That is,

$$\kappa_n = \mathbf{k}\cdot\mathbf{N} \tag{9.15}$$

Here the sign of κ_n depends upon the sense of $\mathbf{N}$; but it is independent of the sense of C.

We recall that the unit tangent to C at P is the vector $\mathbf{t} = \dfrac{d\mathbf{x}}{ds} = \dfrac{d\mathbf{x}}{dt}\Big/\left|\dfrac{d\mathbf{x}}{dt}\right|$, and the curvature vector is $\mathbf{k} = \dfrac{d\mathbf{t}}{ds} = \dfrac{d\mathbf{t}}{dt}\Big/\left|\dfrac{d\mathbf{x}}{dt}\right|$. Thus using the fact that $\mathbf{t}$ is perpendicular to $\mathbf{N}$ along the curve, so that $0 = \dfrac{d}{dt}(\mathbf{t}\cdot\mathbf{N}) = \dfrac{d\mathbf{t}}{dt}\cdot\mathbf{N} + t\cdot\dfrac{d\mathbf{N}}{dt}$, we have

$$\begin{aligned}\kappa_n &= \mathbf{k}\cdot\mathbf{N} = \frac{d\mathbf{t}}{dt}\cdot\mathbf{N}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -\mathbf{t}\cdot\frac{d\mathbf{N}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| \\ &= -\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{N}}{dt}\Big/\left|\frac{dx}{dt}\right|^2 = -\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{N}}{dt}\Big/\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{x}}{dt} \\ &= -\left(\mathbf{x}_u\frac{du}{dt} + \mathbf{x}_v\frac{dv}{dt}\right)\cdot\left(\mathbf{N}_u\frac{du}{dt} + \mathbf{N}_v\frac{dv}{dt}\right)\Big/\left(\mathbf{x}_u\frac{du}{dt} + \mathbf{x}_v\frac{dv}{dt}\right)\cdot\left(\mathbf{x}_u\frac{du}{dt} + \mathbf{x}_v\frac{dv}{dt}\right)\end{aligned}$$

Hence

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2} \tag{9.16}$$

Observe that κ_n as function of du/dt and dv/dt depends only upon the ratio $(du/dt)/(dv/dt)$, i.e. *the direction of the tangent line to C at P*. Otherwise κ_n is a function of the first and second fundamental coefficients which depend only upon P. Thus

Theorem 9.2. All curves through a point P on a patch which are tangent to the same line through P have the same normal curvature at P.

Suppose now C is a curve with a continuous principal normal $\mathbf{n}$ at P, and suppose the sense of $\mathbf{n}$ along C is chosen so that $0 \leq \measuredangle(\mathbf{n}, \mathbf{N}) \leq \pi/2$ at P. From equation (9.15),

$$\kappa_n = \mathbf{k}\cdot\mathbf{N} = \dot{\mathbf{t}}\cdot\mathbf{N} = \kappa(\mathbf{n}\cdot\mathbf{N}) = \kappa\cos\alpha \tag{9.17}$$

where $\alpha = \measuredangle(\mathbf{n}, \mathbf{N})$. Since κ_n depends only upon the direction of the tangent to C and $\cos\alpha$ is determined by the direction of the principal normal line to C, it follows that if $\cos\alpha \neq 0$, the curvature κ of C at P is uniquely determined by the osculating plane to C, as shown in Fig. 9-10.

Note that $\cos\alpha = 0$ if and only if $\mathbf{n}$ is parallel to the tangent plane at P or, equivalently, if and only if the osculating and tangent planes coincide. Thus we have

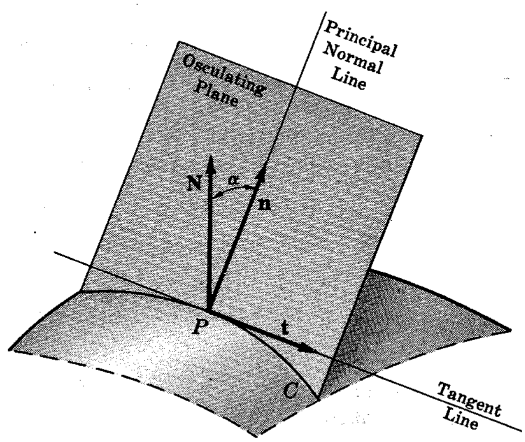


Fig. 9-10

Theorem 9.3. All curves on a surface through a point P which have the same osculating plane through P have the same curvature κ at P provided that the osculating plane is not tangent to the surface.

It follows from the above theorem that except for curves whose osculating plane is tangent to the surface, all possible value for the curvature of a curve through P can be obtained by considering the curves cut out by passing planes through P. In particular suppose C is a curve through P cut out by a plane containing $\mathbf{N}$, i.e. suppose C is a *normal section* of the patch. Then $\mathbf{n}\cdot\mathbf{N} = 1$ and from (9.17) we have $\kappa_n = \kappa$. Hence

Theorem 9.4. The curvature of a normal section of a patch at a point P is equal to the normal curvature of the section at P.

Since the normal curvature to C at P depends only upon P and the direction of the tangent line to C at P, we can speak of the normal curvature at P in the direction $du:dv$, $du^2 + dv^2 \neq 0$, and write, from (9.16),

$$\kappa_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{\mathrm{II}}{\mathrm{I}} \tag{9.18}$$

Here $du:dv$ are the *direction numbers* of the line in the tangent plane parallel to vector $\mathbf{x}_u\,du + \mathbf{x}_v\,dv$. Direction numbers $du:dv$ and $du':dv'$ determine the same line if and only if they are proportional, i.e. if and only if there exists $\lambda \neq 0$ such that $du = \lambda\,du'$ and $dv = \lambda\,dv'$.

From equation (9.18) it follows that κ_n is invariant (in the same sense as I and II) under a parametric transformation which preserves the sense of $\mathbf{N}$ and that κ_n changes sign under a parametric transformation which reverses the sense of $\mathbf{N}$. Also, since I is positive definite, it follows that κ_n is positive, negative, or zero together with II. If P is an elliptic point, $\kappa_n \neq 0$ and maintains the same sign for all $du:dv$. If P is hyperbolic, κ_n is positive, negative, or zero, depending on $du:dv$. If P is parabolic, κ_n maintains the same sign and is zero for the one direction for which $\mathrm{II} = 0$. At a planar point, $\kappa_n = 0$ in all directions.

Example 9.7.

Consider the sphere of radius a,

$$\mathbf{x} = (a\cos\theta\sin\phi)\mathbf{e}_1 + (a\sin\theta\sin\phi)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3 \qquad 0 < \theta < 2\pi,\ 0 < \phi < \pi$$

Here
$$\mathbf{x}_\theta = -(a\sin\theta\sin\phi)\mathbf{e}_1 + (a\cos\theta\sin\phi)\mathbf{e}_2$$
$$\mathbf{x}_\phi = (a\cos\theta\cos\phi)\mathbf{e}_1 + (a\sin\theta\cos\phi)\mathbf{e}_2 - (a\sin\phi)\mathbf{e}_3$$
$$\mathbf{x}_{\theta\theta} = -(a\cos\theta\sin\phi)\mathbf{e}_1 - (a\sin\theta\sin\phi)\mathbf{e}_2$$
$$\mathbf{x}_{\theta\phi} = -(a\sin\theta\cos\phi)\mathbf{e}_1 + (a\cos\theta\cos\phi)\mathbf{e}_2$$
$$\mathbf{x}_{\phi\phi} = -(a\cos\theta\sin\phi)\mathbf{e}_1 - (a\sin\theta\sin\phi)\mathbf{e}_2 - (a\cos\phi)\mathbf{e}_3$$
$$\mathbf{N} = -(\cos\theta\sin\phi)\mathbf{e}_1 - (\sin\theta\sin\phi)\mathbf{e}_2 - (\cos\phi)\mathbf{e}_3$$
$$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = a^2\sin^2\phi,\quad F = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0,\quad G = \mathbf{x}_\phi\cdot\mathbf{x}_\phi = a^2$$
$$L = \mathbf{x}_{\theta\theta}\cdot\mathbf{N} = a\sin^2\phi,\quad M = \mathbf{x}_{\theta\phi}\cdot\mathbf{N} = 0,\quad N = \mathbf{x}_{\phi\phi}\cdot\mathbf{N} = a$$
$$\kappa_n = \frac{L\,d\theta^2 + 2M\,d\theta\,d\phi + 2N\,d\phi^2}{E\,d\theta^2 + 2F\,d\theta\,d\phi + G\,d\phi^2} = \frac{a\sin^2\phi\,d\theta^2 + a\,d\phi^2}{a^2\sin^2\phi\,d\theta^2 + a^2\,d\phi^2} = 1/a$$

Here $\kappa_n = \text{constant} = 1/a$ at every point and in every direction. This is consistent with the well-known fact that every normal section of the sphere at every point intersects the sphere in a great circle with radius a and curvature $1/a$.

Example 9.8.

Consider the surface $\quad \mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$

Here $\quad \mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3,\quad \mathbf{x}_v = \mathbf{e}_2 - 2v\mathbf{e}_3,\quad \mathbf{x}_{uu} = 2\mathbf{e}_3,\quad \mathbf{x}_{uv} = \mathbf{0},\quad \mathbf{x}_{vv} = -2\mathbf{e}_3$
$$\mathbf{N} = (4u^2+4v^2+1)^{-1/2}(-2u\mathbf{e}_1 + 2v\mathbf{e}_2 + \mathbf{e}_3),\quad E = 1+4u^2,\quad F = -4uv,\quad G = 1+4v^2$$
$$L = 2(4u^2+4v^2+1)^{-1/2},\quad M = 0,\quad N = -2(4u^2+4v^2+1)^{-1/2},\quad LN - M^2 = -4(4u^2+4v^2+1)^{-1}$$

Observe that every point is hyperbolic. In particular, at the origin we have $E=1$, $F=0$, $G=1$, $L=2$, $M=0$, $N=-2$ so that $\kappa_n = \dfrac{2(du^2-dv^2)}{du^2+dv^2}$. If we assume $du^2+dv^2=1$ and set $du=\cos\theta$, $dv=\sin\theta$ we have $\kappa_n = 2(\cos^2\theta - \sin^2\theta) = 2\cos 2\theta$. Thus as shown in Fig. 9-11, κ_n varies between positive and negative values in the four intervals $-\pi/4 \le \theta \le \pi/4$, $\pi/4 \le \theta \le 3\pi/4$, $3\pi/4 \le \theta \le 5\pi/4$, $5\pi/4 \le \theta \le 7\pi/4$. The curvature κ of a normal section through the origin will vary as κ_n itself, between -2 and 2.

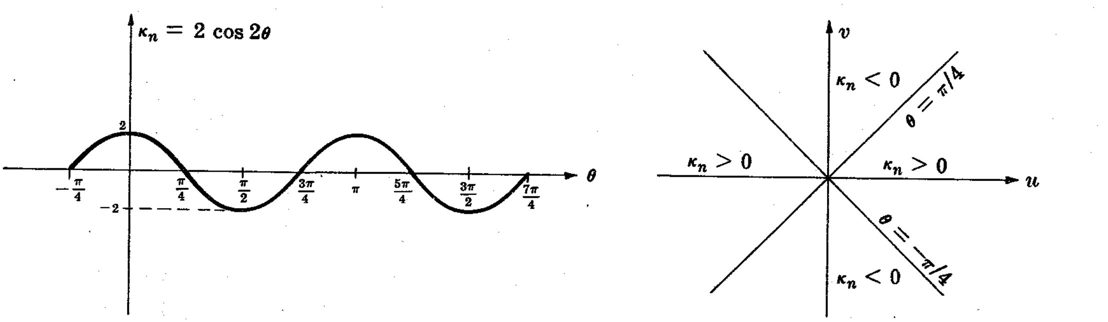


Fig. 9-11

PRINCIPAL CURVATURES AND DIRECTIONS

We want to investigate the normal curvature at point P on a surface in detail. Because of the invariant properties of κ_n, we can assume that a neighborhood of P is represented by a Monge patch of the form $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ such that $\mathbf{x}_u = \mathbf{e}_1$ and $\mathbf{x}_v = \mathbf{e}_2$ at P. This is accomplished by positioning the surface so that P is at the origin and so that the tangent plane coincides with the x_1x_2 plane as shown in Fig. 9-12.

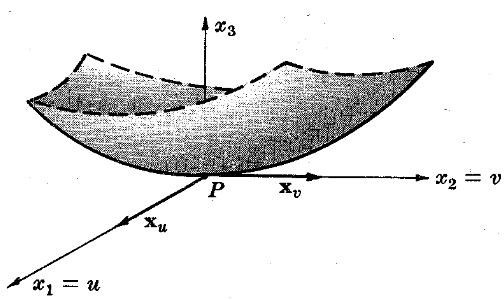


Fig. 9-12

It follows that $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$, and

$$\kappa_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{du^2 + dv^2}$$

Since κ_n depend only on the ratio du/dv, we can assume that $du^2 + dv^2 = 1$ and set $du = \cos\theta$ and $dv = \sin\theta$. Thus

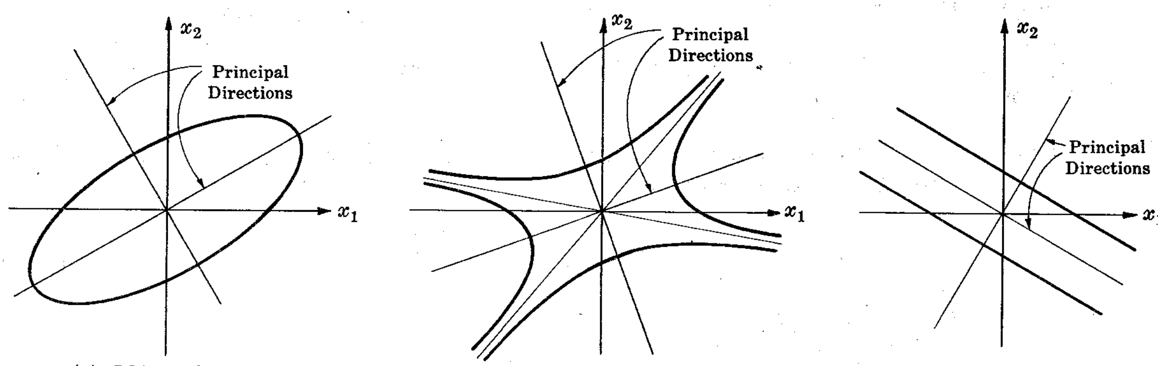
$$\kappa_n = L\cos^2\theta + 2M\cos\theta\sin\theta + N\sin^2\theta$$

Finally we let $|\kappa_n| = 1/r^2$ and set $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, obtaining

$$\pm 1 = Lx_1^2 + 2Mx_1x_2 + Nx_2^2 \tag{9.19}$$

The above equation determines a conic section in the x_1x_2 plane, called *Dupin's indicatrix*, such that the distance r from a point (x_1, x_2) to the origin is the reciprocal of the square root of $|\kappa_n|$ in the direction $\cos\theta : \sin\theta$.

If P is an elliptic point $(LN - M^2 > 0)$, the indicatrix is an ellipse as shown in Fig. 9-13(*a*). If P is a hyperbolic point $(LN - M^2 < 0)$, the indicatrix consists of a pair of conjugate hyperbolas (see Fig. 9-13(*b*)). Along one of the hyperbolas κ_n is positive and along the other κ_n is negative. The common asymptotes correspond to the directions for which $\kappa_n = 0$. In the parabolic case $(LN - M^2 = 0,\ L^2 + N^2 + M^2 \neq 0)$ the quantity $Lx_1^2 + 2Mx_1x_2 + Nx_2^2$ factors and the indicatrix is a pair of parallel lines as shown in Fig. 9-13(*c*), whose directions are also in the direction in which $\kappa_n = 0$. In the planar case $(L = M = N = 0)$ the indicatrix does not exist.



(*a*) $LN - M^2 = 0$ (*b*) $LN - M^2 = 0$ (*c*) $LN - M^2 = 0$, $L^2 + M^2 + N^2 \neq 0$

Fig. 9-13

Recalling that $|\kappa_n| = 1/r^2$, we see that if the indicatrix exists and is not a circle, then κ_n takes on distinct maximum and minimum values, say κ_1 and κ_2, in two *perpendicular* directions, the axes of the indicatrix. At an elliptic point, assuming $\kappa_n > 0$, we see that the maximum κ_1 is taken on in the direction of the minor axis of the indicatrix, i.e. the minimum distance to the origin, and the minimum κ_2 is taken on in the direction of the major axis. At a hyperbolic point the maximum κ_1 is positive and is taken on in the direction of the axis of the hyperbola along which $\kappa_n > 0$. The minimum κ_2 is negative and is taken on in the direction of the axis of the hyperbola along which $\kappa_n < 0$. At a parabolic point, assuming $\kappa_n \geq 0$, the maximum κ_1 is taken on in the direction perpendicular to the parallel lines, and the minimum $\kappa_2 = 0$ is taken on in the direction of the parallel lines of the indicatrix. The two perpendicular directions for which the values of κ_n take on maximum and minimum values are called the *principal directions*, and the corresponding normal curvatures, κ_1 and κ_2, are called the *principal curvatures*.

It remains to consider the elliptic point where the indicatrix is a circle and the planar point where the indicatrix does not exist. At the elliptic point $\kappa_n = \text{constant} \neq 0$ and all directions are called principal directions. Similarly at a planar point $\kappa_n = \text{constant} = 0$ and again all directions are principal directions. A point on the surface at which $\kappa_n = constant$ *is called an umbilical point. The elliptic case is called an elliptic umbilical point.* The planar point is also called a *parabolic umbilical point.*

Example 9.9.

(*a*) From Example 9.7, at every point on a sphere of radius a, the normal curvature $\kappa_n = \text{constant} = 1/a$. Every point on a sphere is an elliptic umbilical point and every direction is a principal direction.

(*b*) The equation of a plane is $\mathbf{x} = \mathbf{a} + \mathbf{b}u + \mathbf{c}v$, $\mathbf{a}, \mathbf{b}, \mathbf{c} = \text{constant}$. Here $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = \mathbf{0}$. Hence $L = M = N = 0$. Every point on a plane is a planar or parabolic umbilical point. Every direction on a plane is a principal direction.

(*c*) From Example 9.8, the normal curvature at the origin of

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$$

is $\kappa_n = \dfrac{2(du^2 - dv^2)}{du^2 + dv^2}$. For $du^2 + dv^2 = 1$, $du = \cos\theta$, $dv = \sin\theta$, $r^2 = 1/|\kappa_n|$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we obtain the indicatrix $\pm 1 = 2(x_1^2 - x_2^2)$. These are conjugate hyperbolas, as shown in Fig. 9-14. Here the maximum value of κ_n is 2 and is taken on in the direction of the x_1 axis. The minimum value of κ_n is -2 and is taken on in the direction of the x_2 axis. The directions of the x_1 and x_2 axes are the principal directions.

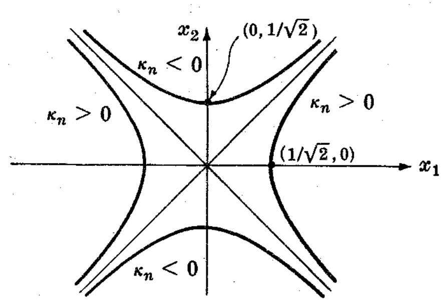


Fig. 9-14

Suppose now $\mathbf{x} = \mathbf{x}(u, v)$ is an arbitrary patch on the surface containing P. In Problem 9.16 we prove

Theorem 9.5. A real number κ is a principal curvature at P in the direction $du : dv$ if and only if κ, du and dv satisfy

$$\begin{aligned}(L - \kappa E)\,du + (M - \kappa F)\,dv &= 0\\(M - \kappa F)\,du + (N - \kappa G)\,dv &= 0\end{aligned} \qquad (9.20)$$

where $du^2 + dv^2 \neq 0$.

The above is a homogeneous system of equations and will have a nontrivial solution du, dv if and only if

$$\det\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} = 0$$

or, expanding,

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

In Problem 9.14 we show that the discriminant of the above equation is greater than or equal to zero. Thus the equation has either two distinct real roots κ_1 and κ_2, the principal curvatures at a nonumbilical point, or a single real root κ with multiplicity two, the curvature at an umbilical point. Thus we have

Theorem 9.6. A number κ is a principal curvature if and only if κ is a solution of the equation

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0 \qquad (9.21)$$

Finally we note that at an umbilical point every direction is a principal direction. But this can only be the case if and only if all coefficients of equations (*9.20*) vanish. Thus we have

Theorem 9.7. A point is an umbilical point if and only if the fundamental coefficients are proportional, in which case the normal curvature

$$\kappa = \frac{L}{E} = \frac{M}{F} = \frac{N}{G} \tag{9.22}$$

GAUSSIAN AND MEAN CURVATURE

If we divide equation (*9.21*) by $EG - F^2$, we can write it in the form

$$\kappa^2 - 2H\kappa + K = 0$$

where

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN + GL - 2FM}{2(EG - F^2)} \tag{9.23}$$

is the average of the roots κ_1 and κ_2 and is called the *mean curvature* at P and

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} \tag{9.24}$$

is the product of the roots and is called the *Gaussian curvature* at P.

Since the normal curvature κ_n of a curve at most changes sign with a change in orientation of the surface, the extreme values of κ_n remain extreme values and, at most, both change sign with a change in orientation (the maximum becoming the minimum, etc.). It follows that the Gaussian curvature $K = \kappa_1\kappa_2$ is an invariant property of the surface, independent of its representation. Note also that $EG - F^2 > 0$. Thus the sign of K agrees with the sign of $LN - M^2$. Namely,

Theorem 9.8. A point on a surface is elliptic if and only if $K > 0$; hyperbolic if and only if $K < 0$; parabolic or planar if and only if $K = 0$.

Example 9.10.

(*a*) From Example 9.9(*a*) we see that at each point on a sphere of radius a, the Gaussian curvature $K = \text{constant} = 1/a^2$. The mean curvature is $H = \pm 1/a$ depending on the orientation of the sphere.

(*b*) From Example 9.9(*b*) we see that at each point on a plane the Gaussian curvature $K = 0$ and the mean curvature $H = 0$.

(*c*) Consider the torus

$$\mathbf{x} = (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

From Example 9.5, page 178, $E = (b + a\sin\phi)^2$, $F = 0$, $G = a^2$, $L = (b + a\sin\phi)\sin\phi$ $M = 0$, $N = a$. From equation (*9.21*), the principal curvatures are the roots of

$$a^2(b + a\sin\phi)^2\kappa^2 - [a(b + a\sin\phi)^2 + a^2(b + a\sin\phi)\sin\phi]\kappa + a(b + a\sin\phi)\sin\phi = 0$$

From the quadratic formula,

$$\kappa = \frac{(b + 2a\sin\phi) \pm b}{2a(b + a\sin\phi)}$$

or $\kappa_1 = \dfrac{2b + 2a\sin\phi}{2a(b + a\sin\phi)} = \dfrac{1}{a}$ which is the maximum curvature, and $\kappa_2 = \dfrac{\sin\phi}{b + a\sin\phi}$ which is the minimum curvature. Observe that κ_1 is the same at every point and is equal to the curvature of the circle generating the torus. The minimum curvature κ_2 varies with ϕ along a meridian. It takes on its maximum $1/(b + a)$ on the outside parallel $\phi = \pi/2$. It is zero on the parallels $\phi = 0$ and $\phi = \pi$, and it takes on its minimum $-1/(b - a)$ on the inside parallel $\phi = -\pi/2$. The Gaussian curvature is $K = \kappa_1\kappa_2 = \dfrac{\sin\phi}{a(b + a\sin\phi)}$.

LINES OF CURVATURE

Suppose again that P is a point on a patch $\mathbf{x} = \mathbf{x}(u, v)$ on a surface of class $\geqq 2$. In Problem 9.24, page 197, we prove

Theorem 9.9. A direction $du:dv$ is a principal direction at P if and only if du and dv satisfy

$$(EM-LF)\,du^2 + (EN-LG)\,du\,dv + (FN-MG)\,dv^2 = 0 \qquad (9.25)$$

At a nonumbilical point the above can be shown to factor into two linear equations of the form $A\,du + B\,dv = 0$ for the two perpendicular principal directions.

Note that since the normal curvature κ_n of a curve is invariant except for sign, it follows that the directions in which κ_n takes on its extreme value, i.e. the principal directions, are also invariant. In particular if $\mathbf{x}^*(\theta,\phi)$ is any other patch containing P, then $d\theta:d\phi$ is a principal direction if and only if $d\theta = \theta_u\,du + \theta_v\,dv$ and $d\phi = \phi_u\,du + \phi_v\,dv$, where $du:dv$ is a principal direction at P with respect to $\mathbf{x} = \mathbf{x}(u,v)$.

Now a curve on a surface whose tangent at each point is along a principal direction is called a *line of curvature.* It follows from the above that a curve is a line of curvature if and only if at each point the direction of its tangent satisfies (*9.25*) for some patch $\mathbf{x} = \mathbf{x}(u,v)$ containing the point. It follows that (*9.25*) can be regarded as a differential equation for two families of lines of curvature. From an existence and uniqueness theorem for differential equations, solutions to (*9.25*) exist if the coefficients are of class C^1. Thus we have

Theorem 9.10. In the neighborhood of a nonumbilical point on a surface of class $\geqq 3$ there exist two orthogonal families of lines of curvature.

Example 9.11.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$$

It can be computed that $E = 1+4u^2$, $F = 4uv$, $G = 1+4v^2$, $L = 2(4u^2+4v^2+1)^{-1/2}$, $M = 0$, $N = 2(4u^2+4v^2+1)^{-1/2}$. From (*9.25*) we obtain, after dividing by $-8(4u^2+4v^2+1)^{-1/2}$,

$$uv\,du^2 + (v^2-u^2)\,du\,dv - uv\,dv^2 = 0$$

or $(u\,du + v\,dv)(v\,du - u\,dv) = 0$, or

$$u\,du + v\,dv = 0, \qquad v\,du - u\,dv = 0$$

The solutions to the first equation is the family of circles $u^2+v^2 = r^2$, and the solutions to the second equation is the family of lines through the origin, $u = bv$. The images of those curves on the surface are the lines of curvature, as shown in Fig. 9-15. Note that at the origin $u=0$, $v=0$, we have $E=1$, $F=0$, $G=1$, $L=2$, $M=0$, $N=2$. Namely, the first and second fundamental coefficients are proportional and we have an umbilical point, with every direction a principal direction.

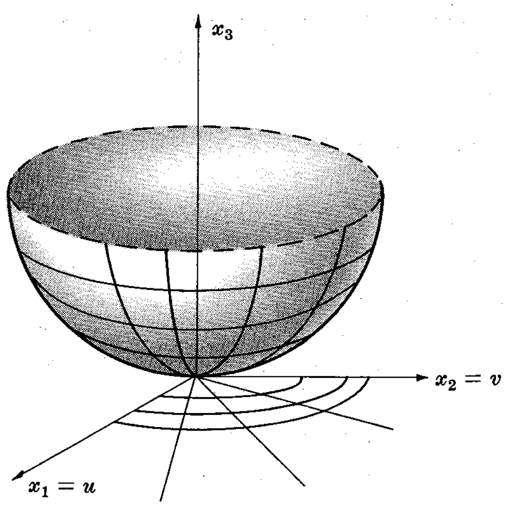


Fig. 9-15

As a consequence of Theorem 9.10, if a surface is sufficiently smooth, one can introduce a coordinate patch of class C^2 in the neighborhood of a nonumbilical point P such that the u and v parameter curves are themselves the lines of curvature. In many problems however it is sufficient to have a coordinate patch of class C^2 containing P such that the u and v parameter curves are in the direction of the principal direction simply at P. In Problem 9.21 we prove

Theorem 9.11. For every point P on a surface of class $\geqq 2$ there exists a coordinate patch containing P such that the directions of the u and v parameter curves at P are principal directions.

Now suppose that the directions of the u- and v-parameter curves at a nonumbilical point P on a patch are principal directions. Since $\mathbf{x}_u = 1\mathbf{x}_u + 0\mathbf{x}_v$ is tangent to the u-parameter curves and $\mathbf{x}_v = 0\mathbf{x}_u + 1\mathbf{x}_v$ is tangent to the v-parameter curves, it follows that at P equation (*9.25*) must be satisfied for $du = 1$, $dv = 0$ and $dv = 1$, $du = 0$. Substituting, we find that

$$FN - MG = 0 \quad \text{and} \quad EM - LF = 0$$

We recall further that the principal directions at a nonumbilical point are orthogonal. Hence from Theorem 9.1, page 173, $F = 0$ at P and so $MG = EM = 0$. Finally, since I is positive definite, $E > 0$ and so $M = 0$. Thus $F = M = 0$ at P. The converse is also true. Namely,

Theorem 9.12. The directions of the u- and v-parameter curves at a nonumbilical point on a patch are in the direction of the principal directions if and only if $F = M = 0$ at the point.

As a consequence of the above we also have

Corollary: The u- and v-parameter curves on a patch without umbilical points are lines of curvature if and only if at every point on the patch $F = M = 0$.

If the directions of the u- and v-parameter curves at a point P on a patch are principal directions, we obtain simple expressions for the principal curvatures. Namely, suppose first that P is a nonumbilical point. Then, from Theorem 9.12 above, $F = M = 0$ at P and equations (*9.20*) for the principal curvatures and directions at P reduce to $(L - \kappa E)\,du = 0$ and $(N - \kappa G)\,dv = 0$. Since $du = 1$, $dv = 0$ are the direction numbers for the u-parameter curve through P, it follows that the principal curvature in this direction is $\kappa_1 = L/E$. The principal curvature in the direction of the v-parameter curve is $\kappa_2 = N/G$ and is obtained by using $du = 0$ and $dv = 1$. If P is an umbilical point then in any case from Theorem 9.7, $\kappa = L/E = M/F = N/G$. Thus we have proved

Theorem 9.13. If the directions of the u- and v-parameter curves at a point P on a patch are principal directions, then the principal curvatures at P are given by

$$\kappa_1 = L/E \quad \text{and} \quad \kappa_2 = N/G$$

Corollary: If the u- and v-parameter curves on a patch are lines of curvature, then at each point the principal curvatures are given by

$$\kappa_1 = L/E \quad \text{and} \quad \kappa_2 = N/G$$

Example 9.12.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$$

From Example 9.11 the lines of curvature are the images of the circles $u^2 + v^2 = r^2$ and lines $u = bv$. The lines of curvature can be introduced as parameter lines by the parameter transformation $u = r\cos\theta$, $v = r\sin\theta$. This gives the representation

$$\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + r^2\mathbf{e}_3, \qquad r > 0$$

It can be computed that here the first and second fundamental coefficients have the simpler form $E = r^2$, $F = 0$, $G = 1 + 4r^2$, $L = -2r^2(1 + 4r^2)^{-1/2}$, $M = 0$, $N = -2(1 + 4r^2)^{-1/2}$ and the principal curvatures are $\kappa_1 = L/E = -2(1 + 4r^2)^{-1/2}$ and $\kappa_2 = N/G = -2(1 + 4r^2)^{-3/2}$.

RODRIGUES' FORMULA

Suppose $du : dv$ is a principal direction at a point P on a patch and κ the corresponding principal curvature. Then from equations (*9.10*), (*9.12*) and (*9.20*) we obtain

$$(-\mathbf{N}_u \cdot \mathbf{x}_u - \kappa \mathbf{x}_u \cdot \mathbf{x}_u)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_u - \kappa \mathbf{x}_v \cdot \mathbf{x}_u)\,dv = 0$$
$$(-\mathbf{N}_u \cdot \mathbf{x}_v - \kappa \mathbf{x}_u \cdot \mathbf{x}_v)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_v - \kappa \mathbf{x}_v \cdot \mathbf{x}_v)\,dv = 0$$

or

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_u = 0$$

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_v = 0$$

or

$$(d\mathbf{N} + \kappa\,d\mathbf{x}) \cdot \mathbf{x}_u = 0, \qquad (d\mathbf{N} + \kappa\,d\mathbf{x}) \cdot \mathbf{x}_v = 0$$

But $d\mathbf{N} + \kappa\,d\mathbf{x}$ is parallel to the tangent plane at P since $d\mathbf{N}$ and $d\mathbf{x}$ are; and the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent. Hence $d\mathbf{N} + \kappa\,d\mathbf{x} = \mathbf{0}$ or $d\mathbf{N} = -\kappa\,d\mathbf{x}$. Thus, in the direction of a principal direction, the vector $d\mathbf{N}$ is parallel to $d\mathbf{x}$ and is given by $d\mathbf{N} = -\kappa\,d\mathbf{x}$, where κ is the principal curvature in that direction. The converse of this is proved in Problem 9.23, page 196. Thus

Theorem 9.14. The direction $du:dv$ is a principal direction at a point on a patch if and only if for some scalar κ, $d\mathbf{N} = \mathbf{N}_u\,du + \mathbf{N}_v\,dv$ and $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ satisfy

$$d\mathbf{N} = -\kappa\,d\mathbf{x} \tag{9.26}$$

When such is the case κ is the principal curvature in the direction $du:dv$.

The above formula, which completely characterizes the principal directions, is called *Rodrigues' formula,* and should be committed to memory.

ASYMPTOTIC LINES — CONJUGATE FAMILIES OF CURVES

A direction at a point on a patch for which

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0 \tag{9.27}$$

is called an *asymptotic direction.* Since $\kappa_n = \mathrm{II}/\mathrm{I}$ and I is positive definite, the asymptotic directions are also the directions in which $\kappa_n = 0$. At an elliptic point there are no asymptotic directions; at a hyperbolic point there are two distinct asymptotic directions; at a parabolic point there is one asymptotic direction; and at a planar point every direction is asymptotic.

A curve on a surface which is tangent to an asymptotic direction at every point is called an *asymptotic line.* Thus a curve on a surface is an asymptotic line if and only if the direction of the tangent line to the curve at each point satisfies (*9.27*) for some patch $\mathbf{x} = \mathbf{x}(u, v)$ containing the point. At a hyperbolic point this equation has two real distinct factors of the form $A\,du + B\,dv = 0$, which can be regarded as first order differential equations for the asymptotic lines. As in the case of the lines of curvature, we have

Theorem 9.15. In a neighborhood of a hyperbolic point on a surface of class $\geqq 3$ there exist two distinct families of asymptotic lines.

If the u- and v-parameter curves on a patch are themselves asymptotic lines, then at each point (*9.27*) must be satisfied for $du = 1, dv = 0$ and $du = 0, dv = 1$. Hence $L = N = 0$. The converse is also true. Namely,

Theorem 9.16. The u- and v-parameter curves on a patch are asymptotic lines if and only if at each point $L = N = 0$.

Example 9.13.

Consider the surface

$$\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + (\log r)\mathbf{e}_3, \qquad r > 0$$

It can be computed that

$$L = -r/(1+r^2)^{1/2}, \quad M = 0, \quad N = 1/r(1+r^2)^{1/2}$$

Multiplying (*9.27*) by $(1+r^2)^{1/2}$ gives the differential equation $-r\,d\theta^2 + dr^2/r = 0$ or $d\theta + dr/r = 0$, $d\theta - dr/r = 0$. These have solutions $\theta + u = \log r$ and $\theta + v = -\log r$, where u and v are the constants of integration. The images of these curves on the surface are two families of asymptotic lines. Solving for θ and r in terms of u and v gives $\theta = -(u+v)/2$, $r = e^{(u-v)/2}$. Then

$$\mathbf{x} = e^{(u-v)/2}(\cos\tfrac{1}{2}(u+v))\mathbf{e}_1 + e^{(u-v)/2}(\sin\tfrac{1}{2}(u+v))\mathbf{e}_2 + \tfrac{1}{2}(u-v)\mathbf{e}_3$$

which is a parameterization of the surface by asymptotic lines.

Now we recall that at each point on a curve on a patch the normal curvature

$$\kappa_n = \mathrm{II}/\mathrm{I} = \mathbf{k}\cdot\mathbf{N}$$

where $\mathbf{k}$ is the curvature vector of the curve. It follows that a curve is an asymptotic line if and only if at each point $\mathbf{k}\cdot\mathbf{N} = 0$; i.e. if and only if $\mathbf{k} = 0$, or $\mathbf{k}$ is perpendicular to $\mathbf{N}$. Thus we have

Theorem 9.17. A curve on a surface is an asymptotic line if and only if every point on the curve is either a point of inflection $(\mathbf{k} = 0)$ or the osculating plane at the point is tangent to the surface.

Since along a straight line $\mathbf{k} \equiv 0$, we have

Corollary: A straight line on a surface is an asymptotic line.

For asymptotic lines which are not straight lines we will prove in Problem 9.30, page 199,

Theorem 9.18. (Beltrami-Enneper). At each point on an asymptotic line which is not a straight line the torsion satisfies

$$\tau^2 = -K$$

where K is the Gaussian curvature at the point.

A direction $\delta u : \delta v$ at a point on a patch is said to be *conjugate* to a direction $du : dv$ if

$$d\mathbf{x}\cdot\delta\mathbf{N} = 0 \tag{9.28}$$

where $d\mathbf{x} = \mathbf{x}_u\,du + \mathbf{x}_v\,dv$ and $\delta\mathbf{N} = \mathbf{N}_u\,\delta u + \mathbf{N}_v\,\delta v$. Expanding and using (*9.10*), we find that the above is equivalent to

$$L\,du\,\delta u + M(du\,\delta v + dv\,\delta u) + N\,dv\,\delta v = 0 \tag{9.29}$$

Note. Because of symmetry, it follows that $\delta u : \delta v$ is also conjugate to $du : dv$, so that we can speak of the conjugate directions $du : dv$ and $\delta u : \delta v$. Also note that an *asymptotic direction is self-conjugate.*

Given an arbitrary direction $du : dv$, (*9.29*) is a linear equation

$$(L\,du + M\,dv)\,\delta u + (M\,du + N\,dv)\,\delta v = 0$$

for $\delta u : \delta v$. In Problem 9.29 we will prove that this equation has a unique solution $\delta u : \delta v$ if $LN - M^2 \neq 0$. Namely,

Theorem 9.19. At an elliptic or hyperbolic point on a surface every direction has a unique conjugate direction.

Two families of curves on a surface are said to be *conjugate families* of curves if the direction of their tangents are conjugate at each point. Given a one parameter family of curves $f(u,v) = C_1$ in the parameter plane, equation (*9.29*) can be regarded as a first order differential equation for a conjugate family $g(u,v) = C_2$.

If the u- and v-parameter curves on a patch are themselves conjugate families of curves, then (*9.29*) must be satisfied at each point by $du = 1$, $dv = 0$ and $\delta u = 0$, $\delta v = 1$. Substituting into (*9.29*) gives $M = 0$. The converse is also true. That is, we have

Theorem 9.20. The u- and v-parameter curves on a patch are conjugate families of curves if and only if at each point $M = 0$.

Note, as a consequence of the above theorem and the corollary to Theorem 9.12 we have

***Corollary*:** The u- and v-parameter curves on a patch without umbilical points are orthogonal and conjugate families of curves if and only if they are lines of curvature.

Example 9.14.

Consider the surface (see Example 9.11)

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$$

Here $L = N = 2(u^2+v^2+1)^{-1/2}$, $M = 0$. Observe that the surface is already represented by conjugate families of parameter curves, since $M = 0$ everywhere. Now consider the family of curves in the parameter plane given by $f(u,v) = u^2+v^2 = C_1^2$. Since $df = f_u\,du + f_v\,dv = 2u\,du + 2v\,dv = 0$ we have $du:dv = f_v : -f_u = -v:u$. Using (*9.29*) and dividing by $2(1+u^2+v^2)^{-1/2}$ gives $-v\,\delta u + u\,\delta v = 0$, which has as solutions the family of lines $u = C_2 v$. The two families of curves $u^2+v^2 = C_1^2$ and $u = C_2 v$ define two conjugate families of curves on the surface which are in fact the lines of curvature.

Solved Problems

FIRST FUNDAMENTAL FORM. ARC LENGTH. SURFACE AREA

9.1. Show that the first fundamental form on the surface of revolution

$$\mathbf{x} = f(t)(\cos\theta)\mathbf{e}_1 + f(t)(\sin\theta)\mathbf{e}_2 + g(t)\,\mathbf{e}_3$$

is

$$\mathrm{I} = f^2\,d\theta^2 + (f'^2+g'^2)\,dt^2$$

$$\mathbf{x}_\theta = -(f\sin\theta)\mathbf{e}_1 + (f\cos\theta)\mathbf{e}_2, \qquad \mathbf{x}_t = (f'\cos\theta)\mathbf{e}_1 + (f'\sin\theta)\mathbf{e}_2 + g'\mathbf{e}_3$$

$$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = f^2, \qquad F = \mathbf{x}_\theta\cdot\mathbf{x}_t = 0, \qquad G = \mathbf{x}_t\cdot\mathbf{x}_t = f'^2+g'^2$$

from which the result follows.

9.2. Find the length of the arc $u = e^{\theta(\cot\beta)/\sqrt{2}}$, $\theta = \theta$, $0 \le \theta \le \pi$, $\beta =$ constant, on the cone

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$$

$$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = u^2, \quad F = \mathbf{x}_\theta\cdot\mathbf{x}_u = 0, \quad G = \mathbf{x}_u\cdot\mathbf{x}_u = 2, \quad du/d\theta = u(\cot\beta)/\sqrt{2}, \quad d\theta/d\theta = 1.$$

$$\begin{aligned} S &= \int_0^\pi \left[E\left(\frac{d\theta}{d\theta}\right)^2 + 2F\frac{d\theta}{d\theta}\frac{du}{d\theta} + G\left(\frac{du}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \int_0^\pi [u^2 + (\cot^2\beta)u^2]^{1/2}\,d\theta = \sqrt{1+\cot^2\beta}\int_0^\pi u\,d\theta \\ &= \sqrt{1+\cot^2\beta}\int_0^\pi e^{\theta(\cot\beta)/\sqrt{2}}\,d\theta = \frac{\sqrt{2}}{\cos\beta}(e^{\pi(\cot\beta)/\sqrt{2}} - 1) \end{aligned}$$

which is the required result.

9.3. Show that the curve in Problem 9.2 above intersects the generating lines $\theta = \text{constant}$ on the cone at a constant angle β.

$$\cos \measuredangle\left(\frac{d\mathbf{x}}{d\theta}, \mathbf{x}_u\right) = \frac{(d\mathbf{x}/d\theta)\cdot\mathbf{x}_u}{|d\mathbf{x}/d\theta|\,|\mathbf{x}_u|} = \frac{(\mathbf{x}_\theta(d\theta/d\theta) + \mathbf{x}_u(du/d\theta))\cdot\mathbf{x}_u}{|\mathbf{x}_\theta(d\theta/d\theta) + \mathbf{x}_u(du/d\theta)|\,|\mathbf{x}_u|} = \frac{F + G(du/d\theta)}{\sqrt{E + F(du/d\theta) + G(du/d\theta)^2}\sqrt{G}}$$

$$= \frac{\sqrt{2}\,(du/d\theta)}{\sqrt{u^2 + 2(du/d\theta)^2}} = \frac{u\cot\beta}{\sqrt{u^2 + u^2\cot^2\beta}} = \frac{\cot\beta}{\sqrt{1+\cot^2\beta}} = \cos\beta$$

which is the required result.

9.4. If the first fundamental form on a patch is of the form $\text{I} = du^2 + f(u,v)\,dv^2$, prove that the v-parameter curves cut off equal segments from all u-parameter curves. Note, since $F = 0$, the parameter curves are also orthogonal. In this case the v-parameter curves are said to be *parallel*, as shown in Fig. 9-16.

The distance along a u-parameter curve $u = u$, $v = v_0$ between $u = u_1$ and $u = u_2$, $u_1 < u_2$, is

$$d = \int_{u_1}^{u_2} [(du/du)^2 + f(u, v_0)(dv/du)^2]^{1/2}\,du$$

Since $du/du = 1$ and $dv/du = 0$, we have

$$d = \int_{u_1}^{u_2} du = u_2 - u_1$$

which shows that d is the same for each $v = v_0$.

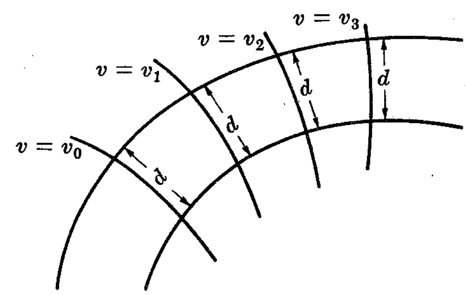


Fig. 9-16

9.5. Show that for a surface represented by $f(x_1, x_2, x_3) = C$, we have $f_{x_1}dx_1 + f_{x_2}dx_2 + f_{x_3}dx_3 = 0$ where $d\mathbf{x} = (dx_1, dx_2, dx_3)$ is an arbitrary tangent vector to the surface at $P(x_1, x_2, x_3)$.

Suppose $\mathbf{x} = x_1(t)\,\mathbf{e}_1 + x_2(t)\,\mathbf{e}_2 + x_3(t)\,\mathbf{e}_3$ is a regular curve on the surface through $P(x_1, x_2, x_3)$. Then for all t, $f(x_1(t), x_2(t), x_3(t)) = C$. Hence $df/dt = f_{x_1}(dx_1/dt) + f_{x_2}(dx_2/dt) + f_{x_3}(dx_3/dt) = 0$. Namely, if $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is an arbitrary vector in the tangent plane at P, then $f_{x_1}dx_1 + f_{x_2}dx_2 + f_{x_3}dx_3 = 0$.

9.6. Show that the parameter transformation $u = \sinh t$, $\phi = \theta$, $-\infty < t < \infty$, $0 < \theta < 2\pi$, between the points in the parameter plane of the surface of revolution M_1,

$$\mathbf{x} = (\cosh t\cos\theta)\mathbf{e}_1 + (\cosh t\sin\theta)\mathbf{e}_2 + t\mathbf{e}_3, \qquad 0 < \theta < 2\pi$$

and the points in the parameter plane of the right conoid M_2,

$$\mathbf{y} = (u\cos\phi)\mathbf{e}_1 + (u\sin\phi)\mathbf{e}_2 + \phi\mathbf{e}_3, \qquad 0 < \phi < 2\pi$$

determines a 1-1 correspondence between the points on the surfaces themselves such that the first fundamental forms agree on corresponding vectors.

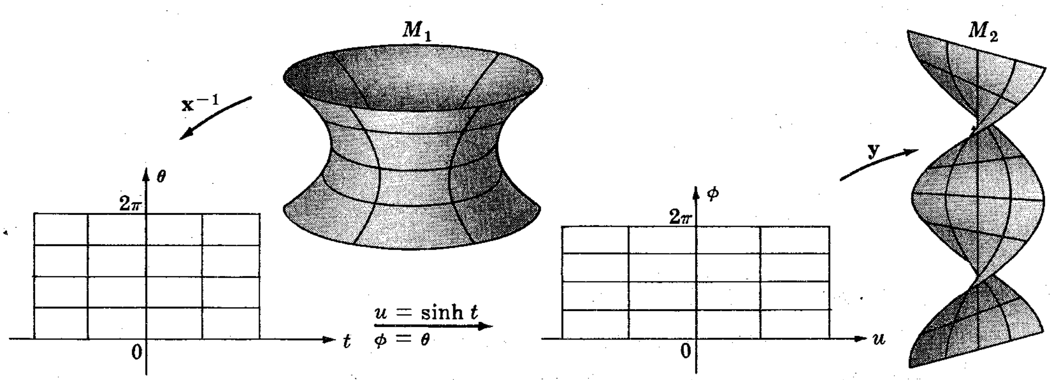


Fig. 9-17

Since $\mathbf{x}$ and $\mathbf{y}$ are both 1-1 mappings of their corresponding domains onto the surfaces M_1 and M_2 and $u = \sinh t$, $\phi = \theta$, defines a 1-1 correspondence between the domains, it follows that the composite mapping $\mathbf{x}^{-1}$ followed by $u = \sinh t$, $\phi = \theta$, followed by $\mathbf{y}$ is a 1-1 mapping of the surface M_1 onto M_2.

Now, at the point $\mathbf{x}(t,\theta)$ on M_1 we have $E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = \cosh^2 t$, $F = \mathbf{x}_\theta \cdot \mathbf{x}_t = 0$, $G = \mathbf{x}_t \cdot \mathbf{x}_t = \cosh^2 t$ and $\mathrm{I} = \cosh^2 t(d\theta^2 + dt^2)$. At $\mathbf{y}(u,\phi)$ on M_2, $E^* = \mathbf{y}_\phi \cdot \mathbf{y}_\phi = u^2+1$, $F^* = \mathbf{y}_\phi \cdot \mathbf{y}_u = 0$, $G^* = \mathbf{y}_u \cdot \mathbf{y}_u = 1$ and $\mathrm{I}^* = (u^2+1)\,d\phi^2 + du^2$, but $u = \sinh t$, $\phi = \theta$, $d\phi = d\theta$, $du = \cosh t\,dt$. Hence

$$\mathrm{I}^* = (\sinh^2 t + 1)\,d\theta^2 + \cosh^2 t\,dt^2 = \cosh^2 t(d\theta^2 + dt^2) = \mathrm{I}$$

which is the required result.

SECOND FUNDAMENTAL FORM

9.7. Show that the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^3)\mathbf{e}_3$$

is elliptic where $v > 0$, hyperbolic where $v < 0$, and parabolic for $v = 0$.

$$\mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3, \qquad \mathbf{x}_v = \mathbf{e}_2 + 3v^2\mathbf{e}_3$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = (4u^2+9v^4+1)^{-1/2}(-2u\mathbf{e}_1 - 3v^2\mathbf{e}_2 + \mathbf{e}_3)$$

$$\mathbf{x}_{uu} = 2\mathbf{e}_3, \quad \mathbf{x}_{uv} = 0, \quad \mathbf{x}_{vv} = 6v\mathbf{e}_3$$

$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = 2(4u^2+9v^4+1)^{-1/2}, \qquad M = \mathbf{x}_{uv}\cdot\mathbf{N} = 0$$

$$N = \mathbf{x}_{vv}\cdot\mathbf{N} = 6v(4u^2+9v^4+1)^{-1/2}, \qquad LN - M^2 = \frac{12v}{(4u^2+9v^4+1)}$$

Since $(4u^2+9v^2+1) > 0$ for all (u,v), we have $LN - M^2 > 0$ for $v > 0$, $LN - M^2 < 0$ where $v < 0$, and $LN - M^2 = 0$ where $v = 0$. Thus the surface is elliptic for $v > 0$, hyperbolic for $v < 0$, and since $L \neq 0$ for all (u,v), parabolic for $v = 0$.

9.8. Show that the surface in Problem 9.7 lies on both sides of the tangent plane in every neighborhood of the parabolic point $P(0,0)$.

At $u = 0$, $v = 0$ we have $\mathbf{x}_u = \mathbf{e}_1$ and $\mathbf{x}_v = \mathbf{e}_2$; hence the tangent plane at $P(0,0)$ is the coordinate plane $x_3 = 0$. The v-parameter curve $u = 0$ is the cubic $\mathbf{x} = v\mathbf{e}_2 + v^3\mathbf{e}_3$ which lies on both sides of the tangent plane in every neighborhood of $P(0,0)$ as shown in Fig. 9-18.

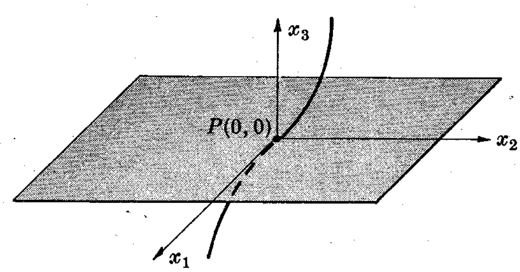


Fig. 9-18

9.9. Prove that all points on the tangent surface of a curve are parabolic.

The tangent surface to the curve $\mathbf{y} = \mathbf{y}(s)$ is $\mathbf{x} = \mathbf{y}(s) + u\mathbf{t}(s)$. Here

$$\mathbf{x}_s = \dot{\mathbf{y}} + u\dot{\mathbf{t}} = \mathbf{t} + u\kappa\mathbf{n}, \qquad \mathbf{x}_u = \mathbf{t}$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_s}{|\mathbf{x}_u \times \mathbf{x}_s|} = \frac{u\kappa\mathbf{b}}{|u\kappa|}, \qquad \mathbf{x}_{ss} = \dot{\mathbf{t}} + u\kappa\dot{\mathbf{n}} + u\dot{\kappa}\mathbf{n} = -u\kappa^2\mathbf{t} + (u\dot{\kappa} + \kappa)\mathbf{n} + u\kappa\tau\mathbf{b}$$

$$\mathbf{x}_{su} = \dot{\mathbf{t}} = \kappa\mathbf{n}, \quad \mathbf{x}_{uu} = 0, \quad L = \mathbf{x}_{ss}\cdot\mathbf{N} = |u\kappa|\tau, \quad M = \mathbf{x}_{su}\cdot\mathbf{N} = 0, \quad N = \mathbf{x}_{uu}\cdot\mathbf{N} = 0$$

Hence at all points $LN - M^2 = 0$, which is the required result.

9.10. Show that every point on the surface of revolution

$$\mathbf{x} = f(t)(\cos\theta)\mathbf{e}_1 + f(t)(\sin\theta)\mathbf{e}_2 + t\mathbf{e}_3, \qquad f(t) > 0$$

is a parabolic point if and only if the surface is a right circular cylinder $f = a$, or a cone $f = at + b$, $a = \text{constant} \neq 0$, $b = \text{constant}$.

It can be computed that $L = \dfrac{-f}{[1+f'^2]^{1/2}}$, $M = 0$, $N = \dfrac{f''}{[1+f'^2]^{1/2}}$. Hence

$$LN - M^2 = \frac{-ff''}{1+f'^2}$$

Since $f > 0$, $LN - M^2 = 0$ if and only if $f'' = 0$. That is, if and only if $f = a \neq 0$ or $f = at + b$, $a \neq 0$, which completes the proof.

9.11. Prove that a surface lies on one side of the tangent plane in a neighborhood of an elliptic point.

Let $P = \mathbf{x}(u,v)$ be an elliptic point, and $Q = \mathbf{x}(u+du, v+dv)$ a neighboring point. We recall that $d = \mathbf{PQ}\cdot\mathbf{N} = \frac{1}{2}\text{II} + o(du^2+dv^2)$, where $\frac{1}{2}\text{II} = \frac{1}{2}(L\,du^2 + 2M\,du\,dv + N\,dv^2)$ is an elliptic paraboloid in (du, dv) maintaining the same sign for all (du, dv) and equal to zero if and only if $du = 0$, $dv = 0$. Without loss of generality we can assume $\frac{1}{2}\text{II} \geqq 0$. Now let $du = r\cos\theta$, $dv = r\sin\theta$ and consider the quantity

$$\frac{1}{2}\frac{\text{II}}{du^2+dv^2} = \frac{1}{2}\frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{du^2+dv^2} = \frac{1}{2}(L\cos^2\theta + 2M\cos\theta\sin\theta + N\sin^2\theta)$$

But this is just $\frac{1}{2}\text{II}$ evaluated on the unit circle $du = \cos\theta$, $dv = \sin\theta$. Since $\frac{1}{2}\text{II}$ is continuous and greater than zero on the circle, $\frac{1}{2}\text{II}/(du^2+dv^2)$ takes on a minimum $m > 0$. Now select ϵ such that $\dfrac{o(du^2+dv^2)}{du^2+dv^2} < m$ for $du^2+dv^2 < \epsilon^2$. But then

$$\frac{d}{du^2+dv^2} = \frac{1}{2}\frac{\text{II}}{du^2+dv^2} + \frac{o(du^2+dv^2)}{du^2+dv^2} > 0 \qquad \text{for} \quad 0 < du^2+dv^2 < \epsilon^2$$

It follows that $d \geqq 0$ or Q lies on the same side of the tangent plane for $du^2+dv^2 < \epsilon^2$, which completes the proof.

NORMAL CURVATURE. GAUSSIAN AND MEAN CURVATURE

9.12. Find the normal curvature vector $\mathbf{k}_n$ and normal curvature κ_n of the curve $u = t^2$, $v = t$ on the surface $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$ at the point $t = 1$.

$E = 1 + 4u^2$, $F = 4uv$, $G = 1 + 4v^2$, $\mathbf{N} = (4u^2+4v^2+1)^{-1/2}(-2u\mathbf{e}_1 - 2v\mathbf{e}_2 + \mathbf{e}_3)$, $L = 2(4u^2+4v^2+1)^{-1/2}$, $M = 0$, $N = 2(4u^2+4v^2+1)^{-1/2}$, $du/dt = 2t$, $dv/dt = 1$.

At $t = 1$: $u = 1$, $v = 1$, $E = 5$, $F = 4$, $G = 5$, $\mathbf{N} = -1/3[2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3]$, $L = 2/3$, $M = 0$, $N = 2/3$, $du/dt = 2$, $dv/dt = 1$. Hence

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2} = \frac{10}{123}$$

and $\mathbf{k}_n = \kappa_n\mathbf{N} = -\dfrac{10}{369}[2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3]$, which are the required results.

9.13. Let L be a tangent line to a surface at a point P in a direction in which $\kappa_n \neq 0$ (i.e. a nonasymptotic direction). Prove that the osculating circles of all curves through P tangent to L lie on a sphere.

Let C be a curve through P tangent to L. Since $\kappa_n = \mathbf{k}\cdot\mathbf{N} \neq 0$, it follows that $\mathbf{k} \neq 0$ and we can write $\kappa_n = \kappa(\mathbf{n}\cdot\mathbf{N}) = \kappa\cos\alpha$ where, by selecting $\mathbf{N}$ so that $\kappa_n > 0$ and $\mathbf{n}$ in the direction of $\mathbf{k}$, we have $\kappa_n > 0$, $\kappa > 0$, and $0 \leqq \alpha = \measuredangle(\mathbf{n},\mathbf{N}) \leqq \pi/2$. Now let $\rho = 1/\kappa$, $R = 1/\kappa_n$; then $\rho = R\cos\alpha$ where R = constant and ρ is the radius of curvature of the osculating circle to C. It follows that the osculating circle is the intersection of the osculating plane and the sphere of radius R tangent to the tangent plane at P as shown in Fig. 9-19, which is the required result.

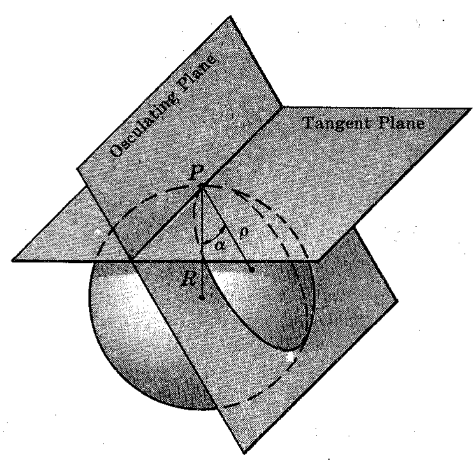


Fig. 9-19

9.14. Prove that the discriminant of the equation

$$(EG-F^2)\kappa^2 - (EN+GL-2FM)\kappa + (LN-M^2) = 0$$

is greater than or equal to zero, and equal to zero if and only if $L/E = M/F = N/G$.

The discriminant is $(EN+GL-2FM)^2 - 4(EG-F^2)(LN-M^2)$, which can be shown to be identically equal to

$$4\left(\frac{EG-F^2}{E^2}\right)(EM-FL)^2 + \left[EN-GL-\frac{2F}{E}(EM-FL)\right]^2$$

Hence the discriminant is greater than or equal to zero. Since $EG-F^2>0$, the above is zero if and only if $EM-FL=0$ and $EN-GL-\frac{2F}{E}(EM-FL)=0$, or if and only if $EM-FL=0$ and $EN-GL=0$ or if and only if $L/E=M/F=N/G$.

9.15. Prove that at every point P on a surface there is a paraboloid such that the normal curvature of the surface at P in any direction is the same as the paraboloid.

Suppose the surface is translated and rotated so that P is at the origin and the tangent plane is the x_1x_2 plane. Then a neighborhood of P can be represented by

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$$

where $\mathbf{x}(0,0)=\mathbf{0}$, $\mathbf{x}_u(0,0)=\mathbf{e}_1$, $\mathbf{x}_v(0,0)=\mathbf{e}_2$. From Taylor's theorem,

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + \tfrac{1}{2}(au^2+2buv+cv^2)\mathbf{e}_3 + \mathbf{o}(u^2+v^2)$$

where $\mathbf{x}_{uu}(0,0) = a\mathbf{e}_3$, $\mathbf{x}_{uv}(0,0) = b\mathbf{e}_3$ and $\mathbf{x}_{vv}(0,0) = c\mathbf{e}_3$, $\mathbf{N}(0,0) = \mathbf{e}_3$, and hence $\kappa_n = \dfrac{a\,du^2+2b\,du\,dv+c\,dv^2}{du^2+dv^2}$. The surface represented by the approximation

$$\mathbf{x}^* = u\mathbf{e}_1 + v\mathbf{e}_2 + \tfrac{1}{2}(au^2+2buv+cv^2)\mathbf{e}_3$$

is a paraboloid tangent to the x_1x_2 plane at $u=0$, $v=0$ and such that $\kappa_n^* = \kappa_n$, which is the required result.

9.16. Prove Theorem 9.5. That is, prove that κ_0 is a principal curvature with principal direction $du_0 : dv_0$ if and only if κ_0, du_0, dv_0 satisfy

$$\begin{aligned}(L-\kappa_0E)\,du_0 + (M-\kappa_0F)\,dv_0 &= 0\\ (M-\kappa_0F)\,du_0 + (N-\kappa_0G)\,dv_0 &= 0\end{aligned} \qquad (a)$$

Suppose κ_0 is a principal curvature with associated principal direction $du_0 : dv_0$. Recall that the principal curvatures are the maximum or minimum values of the normal curvature κ_n. Thus if

$$\kappa_n = \frac{\text{II}}{\text{I}} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

takes on a maximum or minimum κ_0 at (du_0, dv_0), then, from calculus, the partial derivatives

$$\left.\frac{\partial\kappa_n}{\partial\,du}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\frac{\partial\kappa_n}{\partial\,dv}\right|_{(du_0,dv_0)} = 0$$

or, differentiating,

$$\left.\frac{\text{I}\,\text{II}_{du} - \text{II}\,\text{I}_{du}}{\text{I}^2}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\frac{\text{I}\,\text{II}_{dv} - \text{II}\,\text{I}_{dv}}{\text{I}^2}\right|_{(du_0,dv_0)} = 0$$

Multiplying by I gives

$$\left.\text{II}_{du} - \frac{\text{II}}{\text{I}}\text{I}_{du}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\text{II}_{dv} - \frac{\text{II}}{\text{I}}\text{I}_{dv}\right|_{(du_0,dv_0)} = 0$$

But $(\text{II}/\text{I})|_{(du_0,dv_0)} = \kappa_n|_{(du_0,dv_0)} = \kappa_0$. Hence

$$(\text{II}_{du} - \kappa_0\text{I}_{du})|_{(du_0,dv_0)} = 0 \quad\text{and}\quad (\text{II}_{dv} - \kappa_0\text{I}_{dv})|_{(du_0,dv_0)} = 0$$

Since $\text{II}_{du} = 2L\,du + 2M\,dv$ and $\text{I}_{du} = 2E\,du + 2F\,dv$, etc.,

$$\begin{aligned}(L\,du_0 + M\,dv_0) - \kappa_0(E\,du_0 + F\,dv_0) &= 0\\ (M\,du_0 + N\,dv_0) - \kappa_0(F\,du_0 + G\,dv_0) &= 0\end{aligned}$$

which gives the required result. Now conversely, suppose that κ_0, du_0, dv_0, $du_0^2 + dv_0^2 \neq 0$, satisfy equation (*a*) above. Then κ_0 together with the principal curvatures must satisfy

$$\det\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} = 0$$

or, expanding,

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0 \qquad (b)$$

Now suppose P is an umbilical point with curvature κ. Since κ is taken in every direction, the coefficients of (*a*) must all be zero, i.e. $\kappa = E/L = M/F = N/G$. But then it follows from Problem 9.14 that equation (*b*) has a single root with multiplicity two and therefore $\kappa = \kappa_0$. Thus κ_0 is the principal curvature, and every direction including $du_0 : dv_0$ is a principal direction. If P is a nonumbilical point, κ_0 must be one of the two distinct roots of (*b*), i.e. one of the two principal curvatures at a nonumbilical point with principal direction $du_0 : dv_0$, which proves the theorem.

9.17. Show that the curvature κ at a point P on the curve C of intersection of two surfaces satisfies

$$\kappa^2 \sin^2\alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\cos\alpha$$

where κ_1 and κ_2 are the normal curvatures of the surfaces in the direction of C at P, and α is the angle between the normals to the surfaces at P.

From equation (*9.15*), page 179,

$$\kappa_1\mathbf{N}_2 = \kappa(\mathbf{n}\cdot\mathbf{N}_1)\mathbf{N}_2 \quad \text{and} \quad \kappa_2\mathbf{N}_1 = \kappa(\mathbf{n}\cdot\mathbf{N}_2)\mathbf{N}_1$$

Subtracting and using the vector identity of Theorem 1.8, page 10,

$$\kappa_1\mathbf{N}_2 - \kappa_2\mathbf{N}_1 = \kappa[(\mathbf{n}\cdot\mathbf{N}_1)\mathbf{N}_2 - (\mathbf{n}\cdot\mathbf{N}_2)\mathbf{N}_1] = \kappa(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}$$

It follows further, using the identity $[\mathbf{F}_1]$, page 10, that

$$\begin{aligned}(\kappa_1\mathbf{N}_2 - \kappa_2\mathbf{N}_1)\cdot(\kappa_1\mathbf{N}_2 - \kappa_2\mathbf{N}_1) &= \kappa^2[(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}]\cdot[(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}] \\ &= \kappa^2[(\mathbf{N}_1\times\mathbf{N}_2)\cdot(\mathbf{N}_1\times\mathbf{N}_2) - ((\mathbf{N}_1\times\mathbf{N}_2)\cdot\mathbf{n})^2] \\ &= \kappa^2(\mathbf{N}_1\times\mathbf{N}_2)\cdot(\mathbf{N}_1\times\mathbf{N}_2)\end{aligned}$$

where we used the fact that $\mathbf{N}_1\times\mathbf{N}_2$ is a vector parallel to the tangent to the curve and hence $(\mathbf{N}_1\times\mathbf{N}_2)\cdot\mathbf{n} = 0$. Expanding,

$$\kappa_1^2(\mathbf{N}_2\cdot\mathbf{N}_2) - 2\kappa_1\kappa_2(\mathbf{N}_1\cdot\mathbf{N}_2) + \kappa_2^2(\mathbf{N}_1\cdot\mathbf{N}_1) = \kappa^2|\mathbf{N}_1\times\mathbf{N}_2|^2$$

or

$$\kappa_1^2 - 2\kappa_1\kappa_2\cos\alpha + \kappa_2^2 = \kappa^2\sin^2\alpha$$

9.18. Prove that at each point on a patch,

$$\mathbf{N}_u\times\mathbf{N}_v = K(\mathbf{x}_u\times\mathbf{x}_v)$$

where K is the Gaussian curvature at the point.

Since $\mathbf{N}$ is a function of unit length, $\mathbf{N}_u$ and $\mathbf{N}_v$ are orthogonal to $\mathbf{N}$ and hence parallel to the tangent plane. It follows that we can write $\mathbf{N}_u = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{N}_v = c\mathbf{x}_u + d\mathbf{x}_v$ where a, b, c, d are to be determined. Note that

$$\mathbf{N}_u\times\mathbf{N}_v = (a\mathbf{x}_u + b\mathbf{x}_v)\times(c\mathbf{x}_u + d\mathbf{x}_v) = (ad - bc)(\mathbf{x}_u\times\mathbf{x}_v)$$

Thus it remains to show that

$$ad - cb = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = K$$

From the above and (*9.10*), page 175, we have

$$\mathbf{x}_u\cdot\mathbf{N}_u = a\mathbf{x}_u\cdot\mathbf{x}_u + b\mathbf{x}_u\cdot\mathbf{x}_v = aE + bF = -L$$

Similarly,
$$\mathbf{x}_v \cdot \mathbf{N}_u = aF + bG = -M$$
$$\mathbf{x}_u \cdot \mathbf{N}_v = cE + dF = -M$$
$$\mathbf{x}_v \cdot \mathbf{N}_v = cF + dG = -N$$

These equations can be written as the matrix product
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

Hence
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \det\begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

or
$$ad - cb = \frac{LN - M^2}{EG - F^2} = K$$

which completes the proof.

LINES OF CURVATURE

9.19. Determine the principal directions to $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$ at $u = 1,\ v = 1$ and verify Rodrigues' formula in each direction.

From Example 9.11, page 185, $E = 1 + 4u^2$, $F = 4uv$, $G = 1 + 4v^2$, $L = 2(4u^2 + 4v^2 + 1)^{-1/2}$, $M = 0$, $N = 2(4u^2 + 4v^2 + 1)^{-1/2}$. At $u = 1$, $v = 1$, we have $E = 5$, $F = 4$, $G = 5$, $L = 2/3$, $M = 0$, $N = 2/3$. From equation (*9.25*), page 185, the principal directions at $u = 1$, $v = 1$ are the solutions of
$$-\tfrac{8}{3}du^2 + \tfrac{8}{3}dv^2 = 0 \quad \text{or} \quad (du + dv)(du - dv) = 0$$

Hence $du_1 : dv_1 = 1 : -1$ and $du_2 : dv_2 = 1 : 1$. We have further,
$$\mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2v\mathbf{e}_3$$
$$\mathbf{N} = (4u^2 + 4v^2 + 1)^{-1/2}(-2u\mathbf{e}_1 - 2v\mathbf{e}_2 + \mathbf{e}_3)$$
$$\mathbf{N}_u = (4u^2 + 4v^2 + 1)^{-3/2}[-(8v^2 + 2)\mathbf{e}_1 + 8uv\mathbf{e}_2 - 4u\mathbf{e}_3]$$
$$\mathbf{N}_v = (4u^2 + 4v^2 + 1)^{-3/2}[8uv\mathbf{e}_1 - (8u^2 + 2)\mathbf{e}_2 - 4v\mathbf{e}_3]$$

At $u = 1$, $v = 1$,
$$\mathbf{x}_u = \mathbf{e}_1 + 2\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2\mathbf{e}_3, \quad \mathbf{N}_u = \tfrac{1}{27}(-10\mathbf{e}_1 + 8\mathbf{e}_2 - 4\mathbf{e}_3), \quad \mathbf{N}_v = \tfrac{1}{27}(8\mathbf{e}_1 - 10\mathbf{e}_2 - 4\mathbf{e}_3)$$
$$d\mathbf{N}_1 = \mathbf{N}_u\,du_1 + \mathbf{N}_v\,dv_1 = \tfrac{1}{27}(-18\mathbf{e}_1 + 18\mathbf{e}_2) = -\tfrac{18}{27}(\mathbf{e}_1 - \mathbf{e}_2), \quad d\mathbf{x}_1 = \mathbf{x}_u\,du_1 + \mathbf{x}_v\,dv_1 = \mathbf{e}_1 - \mathbf{e}_2$$

Thus $d\mathbf{N}_1 = -\frac{18}{27}d\mathbf{x}_1$. Also,
$$d\mathbf{N}_2 = \mathbf{N}_u\,du_2 + \mathbf{N}_v\,dv_2 = \tfrac{1}{27}[-2\mathbf{e}_1 - 2\mathbf{e}_2 - 8\mathbf{e}_3] = -\tfrac{2}{27}[\mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3]$$
$$d\mathbf{x}_2 = \mathbf{x}_u\,du_2 + \mathbf{x}_v\,dv_2 = \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3$$

Hence $d\mathbf{N}_2 = -\frac{2}{27}d\mathbf{x}_2$, verifying Rodrigues' formula.

9.20. Prove that the solutions to $A\,du^2 + 2B\,du\,dv + G\,dv^2 = 0$ form orthogonal families of curves on a patch if and only if $EC - 2FB + GA = 0$.

Suppose
$$A\,du^2 + 2B\,du\,dv + C^2\,dv^2 = (A'\,du + B'\,dv)(C'\,du + D'\,dv)$$

Then one of the family of curves is the solution to $A'\,du + B'\,dv = 0$ and the other is the solution to, say, $C'\,\delta u + D'\,\delta v = 0$. Along the first, $du : dv = B' : -A'$; and along the second, $\delta u : \delta v = D' : -C'$. But from Theorem 9.1, page 173, the families are orthogonal if and only if
$$E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = EB'D' - F(B'C' + D'A') + GA'C'$$
$$= EC - 2FB + GA = 0$$

9.21. Prove Theorem 9.11. That is, prove that if P is a point on a surface of class ≥ 2, then there exists a coordinate patch containing P such that the directions of the parameter lines are principal directions.

It is sufficient to consider the case where P is a nonumbilical point. For otherwise every direction is a principal direction and any patch containing P will suffice. Now suppose $\mathbf{x} = \mathbf{x}(u, v)$ is an arbitrary patch containing the nonumbilical point P and suppose $du_1 : dv_1$ and $du_2 : dv_2$ are the principal directions at P. Consider the linear parameter transformation $u = du_1\,\theta + du_2\,\phi$, $v = dv_1\,\theta + dv_2\,\phi$. Note that $\dfrac{\partial(u,v)}{\partial(\theta,\phi)} = \det\begin{pmatrix} du_1 & du_2 \\ dv_1 & dv_2 \end{pmatrix} \neq 0$, since the directions $du_1 : dv_1$ and $du_2 : dv_2$ are distinct. Thus the parameter transformation is an allowable transformation of class C^∞. It follows that $\mathbf{x} = \mathbf{x}^*(\theta, \phi) = \mathbf{x}(u(\theta,\phi), v(\theta,\phi))$ is a patch of class C^2 containing P and that

$$\mathbf{x}_\theta = \mathbf{x}_u(\partial u/\partial\theta) + \mathbf{x}_v(\partial v/\partial\theta) = \mathbf{x}_u\,du_1 + \mathbf{x}_v\,dv_1$$

$$\mathbf{x}_\phi = \mathbf{x}_u(\partial u/\partial\phi) + \mathbf{x}_v(\partial u/\partial\phi) = \mathbf{x}_u\,du_2 + \mathbf{x}_v\,dv_2$$

Namely the θ and ϕ parameter curves at P are in the direction of the principal directions.

9.22. *Euler's Theorem.* Prove that the normal curvature at a point on a surface of class ≥ 2 in the direction of a tangent line L is given by

$$\kappa_n = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

where κ_1 and κ_2 are the principal curvatures at P and α is the angle between L and a tangent line in the principal direction corresponding to κ_1.

The theorem is clearly true if P is an umbilical point where $\kappa_1 = \kappa_2 = \kappa_n$. Otherwise let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch containing P such that the u- and v-parameter curves are in the direction of the principal directions. Then from Theorem 9.12, page 186, $F = M = 0$ and the normal curvature in any direction $du : dv$ is $\kappa_n = \dfrac{L\,du^2 + N\,dv^2}{E\,du^2 + G\,dv^2}$. From Theorem 9.13 it follows that the principal curvatures are $\kappa_1 = L/E$ and $\kappa_2 = N/G$ respectively. Substituting,

$$\kappa_n = \kappa_1 \frac{E\,du^2}{E\,du^2 + G\,dv^2} + \kappa_2 \frac{G\,dv^2}{E\,du^2 + G\,dv^2}$$

Now if α and β are the angles between an arbitrary tangent line L with direction numbers $du : dv$ and the tangent lines in the principal directions $1:0$ and $0:1$ respectively, we have from equation (*9.6*), page 173,

$$\cos\alpha = \frac{E\,du}{\sqrt{E\,du^2 + G\,dv^2}\sqrt{E}} \quad\text{and}\quad \cos\beta = \frac{G\,dv^2}{\sqrt{E\,du^2 + G\,dv}\sqrt{G}}$$

Squaring and substituting,

$$\kappa_n = \kappa_1 \cos_2^2\alpha + \kappa_2 \cos^2\beta$$

But the principal directions are perpendicular, i.e. $\beta = \pi/2 - \alpha$. Hence $\kappa_n = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$.

9.23. Complete the proof of Theorem 9.14. That is, prove that $du : dv$ is a principal direction at a point on a patch if for some scalar κ

$$d\mathbf{N} = -\kappa\,d\mathbf{x}$$

in the direction $du : dv$.

From $d\mathbf{N} = -\kappa\,d\mathbf{x}$ we have $(d\mathbf{N} + \kappa\,d\mathbf{x}) \cdot \mathbf{x}_u = 0$ and $(d\mathbf{N} + \kappa\,d\mathbf{x}) \cdot \mathbf{x}_v = 0$ or

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_u = 0$$

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_v = 0$$

or

$$(-\mathbf{N}_u \cdot \mathbf{x}_u - \kappa\mathbf{x}_u \cdot \mathbf{x}_u)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_u - \kappa\mathbf{x}_v \cdot \mathbf{x}_u)\,dv = 0$$

$$(-\mathbf{N}_u \cdot \mathbf{x}_v - \kappa\mathbf{x}_u \cdot \mathbf{x}_v)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_v - \kappa\mathbf{x}_v \cdot \mathbf{x}_v)\,dv = 0$$

or
$$(L-\kappa E)\,du + (M-\kappa F)\,dv = 0$$
$$(M-\kappa F)\,du + (N-\kappa G)\,dv = 0$$

From Theorem 9.5, page 183, it follows that κ is a principal curvature and $du:dv$ is the corresponding principal direction.

9.24. Prove Theorem 9.9. That is, prove that $du:dv$ is a principal direction at a point on a patch if and only if du and dv satisfy

$$(EM-LF)\,du^2 + (EN-LG)\,du\,dv + (FN-MG)\,dv^2 = 0$$

From Theorem 9.5, $du:dv$ is a principal direction if and only if for some κ,

$$(L-\kappa E)\,du + (M-\kappa F)\,dv = 0$$
$$(M-\kappa F)\,du + (N-\kappa G)\,dv = 0$$

or
$$(L\,du+M\,dv) - \kappa(E\,du+F\,dv) = 0$$
$$(M\,du+N\,dv) - \kappa(F\,du+G\,dv) = 0$$

But the above can have a nontrivial solution $(1,-\kappa)$ if and only if

$$\det\begin{pmatrix} L\,du+M\,dv & E\,du+F\,dv \\ M\,du+N\,dv & F\,du+G\,dv \end{pmatrix} = 0$$

or, expanding,

$$(EM-LF)\,du^2 + (EN-LG)\,du\,dv + (FN-MG)\,dv^2 = 0$$

9.25. If two surfaces intersect at a constant angle and if the curve of intersection is a line of curvature on one of the surfaces, prove that it is a line of curvature on the other surface.

Since the surfaces, say M_1 and M_2, intersect at a constant angle, along the curve of intersection $N_1 \cdot N_2 = \text{constant}$. Hence

$$0 = \frac{d}{dt}(\mathbf{N}_1\cdot\mathbf{N}_2) = \left(\frac{d}{dt}\mathbf{N}_1\right)\cdot\mathbf{N}_2 + \mathbf{N}_1\cdot\frac{d}{dt}\mathbf{N}_2$$

Assuming the curve of intersection is a line of curvature along M_1, then (Rodrigues' formula) $\dfrac{d\mathbf{N}_1}{dt} = -\kappa_1\dfrac{d\mathbf{x}}{dt}$. Hence

$$-\kappa_1\frac{d\mathbf{x}}{dt}\cdot\mathbf{N}_2 + \mathbf{N}_1\cdot\frac{d\mathbf{N}_2}{dt} = 0$$

But $d\mathbf{x}/dt$ is orthogonal to $\mathbf{N}_2$, i.e. $(d\mathbf{x}/dt)\cdot\mathbf{N}_2 = 0$. Thus $\mathbf{N}_1\cdot(d\mathbf{N}_2/dt) = 0$. Hence $d\mathbf{N}_2/dt$ is orthogonal to $\mathbf{N}_1$. But $d\mathbf{N}_2/dt$ is also orthogonal to $\mathbf{N}_2$, since $\mathbf{N}_2$ is of unit length. It follows that $d\mathbf{N}_2/dt$ is parallel to $d\mathbf{x}/dt$. Namely, there exists κ_2 such that $d\mathbf{N}_2/dt = -\kappa_2(d\mathbf{x}/dt)$. Hence the curve of intersection is also a line of curvature along M_2.

9.26. *Third Fundamental Form.* The third fundamental form is defined by $\text{III} = d\mathbf{N}\cdot d\mathbf{N}$. Prove that $\text{III} - 2H\,\text{II} + K\,\text{I} = 0$ where H and K are the mean and Gaussian curvatures respectively.

It is easily verified that III is invariant, in the same sense as I. Note that II and H change sign with a change in orientation. Thus it is sufficient to consider a fixed point P and a coordinate patch containing P such that the directions of the u- and v-parameter curves at P are principal directions. From Rodrigues' formula,

$$\mathbf{N}_u = -\kappa_1\mathbf{x}_u \quad\text{and}\quad \mathbf{N}_v = -\kappa_2\mathbf{x}_v$$

where κ_1 and κ_2 are the principal curvatures. It follows that for arbitrary (du, dv),

$$d\mathbf{N} = \mathbf{N}_u\,du + \mathbf{N}_v\,dv = -\kappa_1\mathbf{x}_u\,du - \kappa_2\mathbf{x}_v\,dv$$
$$= -\kappa_1\mathbf{x}_u\,du - \kappa_1\mathbf{x}_v\,dv + \kappa_1\mathbf{x}_v\,dv - \kappa_2\mathbf{x}_v\,dv = -\kappa_1\,d\mathbf{x} + (\kappa_1-\kappa_2)\mathbf{x}_v\,dv$$

or
$$d\mathbf{N} + \kappa_1\,d\mathbf{x} = (\kappa_1-\kappa_2)\mathbf{x}_v\,dv$$

Also, $$d\mathbf{N} \;=\; -\kappa_1 \mathbf{x}_u\,du + \kappa_2 \mathbf{x}_u\,du - \kappa_2 \mathbf{x}_u\,du - \kappa_2 \mathbf{x}_v\,dv \;=\; -\kappa_2\,d\mathbf{x} + (\kappa_2 - \kappa_1)\mathbf{x}_u\,du$$

or $$d\mathbf{N} + \kappa_2\,d\mathbf{x} \;=\; (\kappa_2 - \kappa_1)\mathbf{x}_u\,du$$

Thus $$(d\mathbf{N} + \kappa_1\,d\mathbf{x})\cdot(d\mathbf{N} + \kappa_2\,d\mathbf{x}) \;=\; (\kappa_1 - \kappa_2)(\kappa_2 - \kappa_1)\,du\,dv\,\mathbf{x}_u \cdot \mathbf{x}_v$$

Since the lines of curvature are orthogonal at P, we have $\mathbf{x}_u \cdot \mathbf{x}_v = 0$. Then

$$(d\mathbf{N} + \kappa_1\,d\mathbf{x})\cdot(d\mathbf{N} + \kappa_2\,d\mathbf{x}) \;=\; 0$$

or $$d\mathbf{N}\cdot d\mathbf{N} + (\kappa_1 + \kappa_2)\,d\mathbf{N}\cdot d\mathbf{x} + \kappa_1\kappa_2\,d\mathbf{x}\cdot d\mathbf{x} \;=\; 0$$

Hence $\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$.

9.27. Show that the principal directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions to

$$\det\begin{pmatrix} dx_1 & f_{x_1} & df_{x_1} \\ dx_2 & f_{x_2} & df_{x_2} \\ dx_3 & f_{x_3} & df_{x_3} \end{pmatrix} \;=\; 0, \qquad f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 \;=\; 0$$

If $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is tangent, then $f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 = 0$. Hence $\mathbf{G} = f_{x_1}\mathbf{e}_1 + f_{x_2}\mathbf{e}_2 + f_{x_3}\mathbf{e}_3$ is normal, and $\mathbf{N} = \mathbf{G}/|\mathbf{G}|$. Thus

$$d\mathbf{N} \;=\; \frac{d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}$$

where $d\mathbf{G} = df_{x_1}\,\mathbf{e}_1 + df_{x_2}\,\mathbf{e}_2 + df_{x_3}\mathbf{e}_3$. Now suppose $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is in a principal direction. Then from Rodrigues' formula, $d\mathbf{x}$ is parallel to $d\mathbf{N}$. Hence the three vectors $d\mathbf{x}$, $\mathbf{N}$ and $d\mathbf{N}$ are dependent, or

$$0 \;=\; [d\mathbf{x}\,\mathbf{N}\,d\mathbf{N}] \;=\; \left[d\mathbf{x}\,\frac{\mathbf{G}}{|\mathbf{G}|}\left(\frac{d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}\right)\right] \;=\; \frac{1}{|\mathbf{G}|^2}[d\mathbf{x}\,\mathbf{G}\,d\mathbf{G}] - \frac{\mathbf{G}\cdot d\mathbf{G}}{|\mathbf{G}|^4}[d\mathbf{x}\,\mathbf{G}\mathbf{G}]$$

But $[d\mathbf{x}\,\mathbf{G}\mathbf{G}] = 0$; hence $[d\mathbf{x}\,\mathbf{G}\,d\mathbf{G}] = 0$. The required result follows.

ASYMPTOTIC LINES — CONJUGATE FAMILIES OF CURVES

9.28. Prove that the parameter curves on the surface $\mathbf{x} = \mathbf{x}_1(u) + \mathbf{x}_2(v)$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are arbitrary, are conjugate families of curves.

$\mathbf{x}_u = \mathbf{x}_1'(u)$, $\mathbf{x}_v = \mathbf{x}_2'(v)$, $\mathbf{x}_{uv} \equiv 0$. Hence $M = \mathbf{x}_{uv}\cdot\mathbf{N} \equiv 0$. Thus from Theorem 9.20, page 189, the parameter curves are conjugate families.

9.29. Prove Theorem 9.20; that is, prove that at an elliptic or hyperbolic point every direction has a unique conjugate direction.

From equation (*9.29*), page 188, $\delta u : \delta v$ is conjugate to $du : dv$ if and only if

$$(L\,du + M\,dv)\,\delta u + (M\,du + N\,dv)\,\delta v \;=\; 0$$

The above equation has a unique solution $\delta u : \delta v$, $\delta u^2 + \delta v^2 \neq 0$, if and only if the coefficients do not both vanish. Thus, given $du : dv$, there exists a unique $\delta u : \delta v$ if and only if

$$(L\,du + M\,dv)^2 + (M\,du + N\,dv)^2 \;\neq\; 0$$

or $$(L^2 + M^2)\,du^2 + 2(LM + MN)\,du\,dv + (M^2 + N^2)\,dv^2 \;\neq\; 0$$

But this equation is different from zero for all $du : dv$, $du^2 + dv^2 \neq 0$, if and only if its discriminant

$$(L^2 + M^2)(M^2 + N^2) - (LM + MN)^2 \;>\; 0$$

or, expanding, $(LN - M^2)^2 > 0$ or $LN - M^2 \neq 0$. Thus every direction $du : dv$ has a unique conjugate direction, $\delta u : \delta v$ if and only if $LN - M^2 \neq 0$, which proves the theorem and in fact also its converse.

9.30. **Prove Theorem 9.18; that is, prove that the torsion along an asymptotic line which is not a straight line satisfies $\tau^2 = -K$.**

From Theorem 9.17, page 188, the osculating plane at each point on the asymptotic line is tangent to the surface. Hence along the curve the binormal $\mathbf{b} = \pm\mathbf{N}$. It follows that $\mathbf{b} = \pm(d\mathbf{N}/ds) = -\tau\mathbf{n}$ and $(d\mathbf{N}/ds)\cdot(d\mathbf{N}/ds) = \tau^2(\mathbf{n}\cdot\mathbf{n}) = \tau^2$. From Problem 9.26, $\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$. But along an asymptotic line $\mathrm{II} = 0$, $\mathrm{III} = (d\mathbf{N}/ds)\cdot(d\mathbf{N}/ds) = \tau^2$ and $\mathrm{I} = (d\mathbf{x}/ds)\cdot(d\mathbf{x}/ds) = 1$. Hence $\tau^2 + K = 0$ or $\tau^2 = -K$.

9.31. **Show that the asymptotic directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions of**

$$dx_1\,df_{x_1} + dx_2\,df_{x_2} + dx_3\,df_{x_3} = 0, \qquad f_{x_1}dx_1 + f_{x_2}dx_2 + f_{x_3}dx_3 = 0$$

From Problem 9.27, $\mathbf{G} = f_{x_1}\mathbf{e}_1 + f_{x_2}\mathbf{e}_2 + f_{x_3}\mathbf{e}_3$ is normal to the surface. $\mathbf{N} = \dfrac{\mathbf{G}}{|\mathbf{G}|}$ and $d\mathbf{N} = \dfrac{d\mathbf{G}}{|\mathbf{G}|} - \dfrac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}$. Hence if $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is in an asymptotic direction,

$$0 = \mathrm{II} = -d\mathbf{x}\cdot d\mathbf{N} = -\frac{d\mathbf{x}\cdot d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})(d\mathbf{x}\cdot\mathbf{G})}{|\mathbf{G}|^3}$$

But $d\mathbf{x}\cdot\mathbf{G} = 0$. Thus $d\mathbf{x}\cdot d\mathbf{G} = 0$, or $dx_1\,df_{x_1} + dx_2\,df_{x_2} + dx_3\,df_{x_3} = 0$.

9.32. **Use the above to find the asymptotic curves on the surface $f = x_3 - x_1 \sin x_2 = 0$, $-\pi/2 < x_2 < \pi/2$, $x_1 > 0$.**

$f_{x_1} = -\sin x_2$, $f_{x_2} = -x_1\cos x_2$, $f_{x_3} = 1$, $df_{x_1} = -\cos x_2\,dx_2$, $df_{x_2} = -\cos x_2\,dx_1 + x_1\sin x_2\,dx_2$, $df_{x_3} = 0$. The equation

$$df_{x_1}dx_1 + df_{x_2}dx_2 + df_{x_3}dx_3 = -2\cos x_2\,dx_1\,dx_2 + x_1\sin x_2\,dx_2^2 = 0$$

has factors $dx_2 = 0$ and $-2\cos x_2\,dx_1 + x_1\sin x_2\,dx_2 = 0$ which have solutions $x_2 = C$ and $x_1 = K\sec^{1/2}x_2$. Substituting the first into $x_3 - x_1\sin x_2 = 0$ gives the family of straight lines $x_1 = t$, $x_2 = C$, $x_3 = t\sin C$, $t > 0$. The second gives the family of curves $x_1 = K\sec^{1/2}u$, $x_2 = u$, $x_3 = K\sec^{1/2}u\sin u$, $-\pi/2 < u < \pi/2$.

Supplementary Problems

9.33. Show that the parameter curves on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\,\mathbf{e}_3$ are an orthogonal family of curves if and only if $f_u f_v \equiv 0$.

9.34. Show that the first fundamental form on the tangent surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{t}(s)$ to the curve $\mathbf{y} = \mathbf{y}(s)$ is $\mathrm{I} = (1 + u^2\kappa^2)\,ds^2 + ds\,du + du^2$.

9.35. Show that the first fundamental form of the surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{b}(s)$ generated by the binormal $\mathbf{b}(s)$ of a curve $\mathbf{y} = \mathbf{y}(s)$ is $\mathrm{I} = (1 + u^2\tau^2)\,ds^2 + du^2$.

9.36. Find the length of the arc $\theta = \displaystyle\int_{\pi/4}^{t} \frac{1}{\sin\tau}\,d\tau$, $\phi = t$, $\pi/4 \le t \le \pi/2$, on the sphere $\mathbf{x} = (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_1 + (\cos\phi)\mathbf{e}_3$. *Ans.* $\sqrt{2}\,\pi/4$

9.37. Show that the surface area on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is given by the integral $A = \displaystyle\iint_W \sqrt{1 + f_u^2 + f_v^2}\,du\,dv$.

9.38. Prove that on the intersection of two coordinate patches $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ on a surface, $EG - F^2 = (E^*G^* - F^{*2})\left[\dfrac{\partial(\theta,\phi)}{\partial(u,v)}\right]^2$.

9.39. Show that the curves on the surface $\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + (a\theta + b)\mathbf{e}_3$ satisfying $(u^2 + a^2)\,d\theta^2 - du^2 = 0$ are orthogonal families of curves.

9.40. Show that the θ-parameter curves on the surface $\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + f(\theta)\mathbf{e}_3$ are parallel.

9.41. Show that the second fundamental form on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is

$$\mathrm{II} = (f_u^2 + f_v^2 + 1)^{-1/2}[f_{uu}\,du^2 + 2f_{uv}\,du\,dv + f_{vv}\,dv^2]$$

9.42. Show that all points on the general cylinder $\mathbf{x} = \mathbf{y}(s) + u\mathbf{g}$, $\mathbf{g} =$ constant, are parabolic or planar.

9.43. If $\mathbf{x} = \mathbf{x}(u,v)$, $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ are patches on a surface such that on the intersection $\partial(\theta,\phi)/\partial(u,v) > 0$, show that the second fundamental coefficients transform as follows:

$$\begin{aligned} L &= L^*\theta_u^2 + 2M^*\theta_u\phi_u + M^*\phi_u^2 \\ M &= L^*\theta_u\theta_v + M^*(\theta_u\phi_v + \phi_u\theta_v) + N^*\phi_u\phi_v \\ N &= L^*\theta_v^2 + 2M^*\theta_v\phi_v + N^*\phi_v^2 \end{aligned}$$

9.44. Show that the Gauss and mean curvatures on $\mathbf{x} = (u+v)\mathbf{e}_1 + (u-v)\mathbf{e}_2 + uv\mathbf{e}_3$ at $u = 1,\ v = 1$ are $K = 1/16$ and $H = 1/8\sqrt{2}$.

9.45. Show that the mean curvature is zero at every point on the surface of revolution

$$\mathbf{x} = (\cosh u\cos\theta)\mathbf{e}_1 + (\cosh u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$$

9.46. Show that the principal curvatures of the surface $x_1\sin x_3 - x_2\cos x_3 = 0$ are $\pm 1/(x_1^2 + x_2^2 + 1)$.

9.47. Show that the lines of curvature on the surface $\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$ are the images of $\log(u + \sqrt{u^2+1}) - v = C$ and $\log(u + \sqrt{u^2+1}) + v = K$.

9.48. Find the principal curvatures and principal directions on

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (4u^2 + v^2)\mathbf{e}_3$$

at $u = 0,\ v = 0$ using Dupin's indicatrix.

9.49. Prove that the family of curves on a patch which are orthogonal to the family of curves given by $A\,du + B\,dv = 0$ are given by the solutions to $(EB - FA)\,du + (FB - GA)\,dv = 0$.

9.50. The parameter curves on the surface

$$\mathbf{x} = e^{(u-v)/2}\left(\cos\frac{u+v}{2}\right)\mathbf{e}_1 + e^{(u-v)/2}\left(\sin\frac{u+v}{2}\right)\mathbf{e}_2 + \left(\frac{u-v}{2}\right)\mathbf{e}_3$$

are asymptotic lines. Verify that along the u-parameter curve $v = 0$, the torsion satisfies $\tau^2 = -K$.

9.51. Show that the asymptotic lines on the surface $x_3 - x_1^4 + x_2^4 = 0$ are the intersections of the surface and the families of cylinders $x_1^2 + x_2^2 = C$ and $x_1^2 - x_2^2 = K$.

9.52. Prove that the directions of curvature bisect the asymptotic directions.

9.53. Show that the mean curvature is zero on a surface whose asymptotic lines are orthogonal families of curves.

9.54. If a sphere or a plane intersects a surface at a constant angle, prove that curve of intersection is a line of curvature.

9.55. Prove that the sum of the normal curvatures at a point on a surface in any pair of orthogonal directions is constant.

9.56. If a surface has a one-parameter family of plane asymptotic curves other than straight lines, prove that the surface is a plane.

9.57. Suppose R is a region on a patch on a surface. The endpoints of the unit normals in R form a set R' on the unit sphere called the spherical image of R. Show that the ratio of the area of R' to the area of R tends to $|K|$ at a point P when R shrinks down to the point P. *Hint.* Problem 9.18, page 194.

9.58. If $K \neq 0$ at a point P on a surface, show that there is a neighborhood of P in which the points can be put into a 1-1 correspondence with the spherical image of the neighborhood (see Problem 9.57).

Chapter 10

Theory of Surfaces
Tensor Analysis

GAUSS-WEINGARTEN EQUATIONS

The Gauss-Weingarten equations for surfaces are analogous to the Frenet equations for curves. We recall that the Frenet equations express the vectors $\dot{\mathbf{t}}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{b}}$ as linear combinations of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ with coefficients depending upon κ and τ. Similarly the Gauss-Weingarten equations express the derivatives of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ as linear combinations of these vectors with coefficients which are functions of the first and second fundamental coefficients.

We assume that $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2. Then $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ are functions of class C^1 and have continuous derivatives $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, $\mathbf{N}_u$ and $\mathbf{N}_v$. Since the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ are linearly independent, we can write

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + \alpha_{11}\mathbf{N} \\
\mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + \alpha_{12}\mathbf{N} \\
\mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + \alpha_{22}\mathbf{N} \\
\mathbf{N}_u &= \beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v + \gamma_1\mathbf{N} \\
\mathbf{N}_v &= \beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v + \gamma_2\mathbf{N}
\end{aligned} \tag{10.1}$$

where the coefficients Γ_{ij}^k, α_{ij}, β_i^j, γ_i are to be determined.

Since $\mathbf{N}$ is of unit length, $\mathbf{N}_u$ and $\mathbf{N}_v$ are orthogonal to $\mathbf{N}$. Hence

$$\begin{aligned}
0 &= \mathbf{N}_u \cdot \mathbf{N} = \beta_1^1 \mathbf{x}_u \cdot \mathbf{N} + \beta_1^2 \mathbf{x}_v \cdot \mathbf{N} + \gamma_1 \mathbf{N}\cdot\mathbf{N} \\
0 &= \mathbf{N}_v \cdot \mathbf{N} = \beta_2^1 \mathbf{x}_u \cdot \mathbf{N} + \beta_2^2 \mathbf{x}_v \cdot \mathbf{N} + \gamma_2 \mathbf{N}\cdot\mathbf{N}
\end{aligned}$$

But $\mathbf{x}_u \cdot \mathbf{N} = \mathbf{x}_v \cdot \mathbf{N} = 0$ and $\mathbf{N}\cdot\mathbf{N} = 1$. Thus $\gamma_1 = \gamma_2 = 0$. It follows further that

$$\begin{aligned}
-L &= \mathbf{x}_u \cdot \mathbf{N}_u = \beta_1^1 \mathbf{x}_u \cdot \mathbf{x}_u + \beta_1^2 \mathbf{x}_u \cdot \mathbf{x}_v = \beta_1^1 E + \beta_1^2 F \\
-M &= \mathbf{x}_v \cdot \mathbf{N}_u = \beta_1^1 \mathbf{x}_v \cdot \mathbf{x}_u + \beta_1^2 \mathbf{x}_v \cdot \mathbf{x}_v = \beta_1^1 F + \beta_1^2 G \\
-M &= \mathbf{x}_u \cdot \mathbf{N}_v = \beta_2^1 \mathbf{x}_u \cdot \mathbf{x}_u + \beta_2^2 \mathbf{x}_u \cdot \mathbf{x}_v = \beta_2^1 E + \beta_2^2 F \\
-N &= \mathbf{x}_v \cdot \mathbf{N}_v = \beta_2^1 \mathbf{x}_v \cdot \mathbf{x}_u + \beta_2^2 \mathbf{x}_v \cdot \mathbf{x}_v = \beta_2^1 F + \beta_2^2 G
\end{aligned}$$

Solving the first two equations for β_1^1 and β_1^2 and the second two for β_2^1 and β_2^2 gives

$$\beta_1^1 = \frac{MF - LG}{EG - F^2}, \quad \beta_1^2 = \frac{LF - ME}{EG - F^2}, \quad \beta_2^1 = \frac{NF - MG}{EG - F^2}, \quad \beta_2^2 = \frac{MF - NE}{EG - F^2} \tag{10.2}$$

Continuing, we obtain

$$\begin{aligned}
L &= \mathbf{x}_{uu} \cdot \mathbf{N} = \Gamma_{11}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{11}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{11} \mathbf{N}\cdot\mathbf{N} = \alpha_{11} \\
M &= \mathbf{x}_{uv} \cdot \mathbf{N} = \Gamma_{12}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{12}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{12} \mathbf{N}\cdot\mathbf{N} = \alpha_{12} \\
N &= \mathbf{x}_{vv} \cdot \mathbf{N} = \Gamma_{22}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{22}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{22} \mathbf{N}\cdot\mathbf{N} = \alpha_{22}
\end{aligned}$$

Thus

$$\alpha_{11} = L, \quad \alpha_{12} = M, \quad \alpha_{22} = N \tag{10.3}$$

It remains to determine the Γ_{ij}^k. In Problem 10.3, page 216, we prove that they are given by

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} & \Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG-F^2)} & \Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} \\
\Gamma_{11}^2 &= \frac{2EF_u - EE_v + FE_u}{2(EG-F^2)} & \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG-F^2)} & \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}
\end{aligned} \tag{10.4}$$

Thus we have

Theorem 10.1. On a patch $\mathbf{x} = \mathbf{x}(u,v)$ on a surface of class $\geqq 2$ the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, $\mathbf{N}$ and their derivatives satisfy

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + L\mathbf{N} \\
\mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + M\mathbf{N} \\
\mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + N\mathbf{N} \\
\mathbf{N}_u &= \beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v \\
\mathbf{N}_v &= \beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v
\end{aligned} \tag{10.5}$$

where the coefficients β_i^j and Γ_{ij}^k are given by equations (*10.2*) and (*10.4*).

The first three of the above equations are called the *Gauss* equations and the last two the *Weingarten* equations. The quantities Γ_{ij}^k are called the *Christoffel symbols of the second kind.* Observe in (*10.4*) that the Γ_{ij}^k depend only upon the first fundamental coefficients and their derivatives as compared to the β_i^j which depend on both the first and second fundamental coefficients. We also define $\Gamma_{21}^1 = \Gamma_{12}^1$ and $\Gamma_{21}^2 = \Gamma_{12}^2$. Thus $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $i,j,k = 1,2$.

Example 10.1.

We want to verify equations (*10.2*) and (*10.4*) for the surface of revolution

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + g(u)\mathbf{e}_3, \qquad u > 0$$

Here

$$\mathbf{x}_u = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2 + g'\mathbf{e}_3, \qquad \mathbf{x}_\theta = -(u\sin\theta)\mathbf{e}_1 + (u\cos\theta)\mathbf{e}_2$$

$$\mathbf{N} = -(1+g'^2)^{-1/2}(g'(\cos\theta)\mathbf{e}_1 + g'(\sin\theta)\mathbf{e}_2 - \mathbf{e}_3)$$

$$\mathbf{x}_{uu} = g''\mathbf{e}_3, \qquad \mathbf{x}_{u\theta} = -(\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2, \qquad \mathbf{x}_{\theta\theta} = -(u\cos\theta)\mathbf{e}_1 - (u\sin\theta)\mathbf{e}_2$$

$$E = \mathbf{x}_u\cdot\mathbf{x}_u = 1 + g'^2, \qquad F = \mathbf{x}_u\cdot\mathbf{x}_\theta = 0, \qquad G = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = u^2$$

$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = g''(1+g'^2)^{-1/2}, \qquad M = \mathbf{x}_{u\theta}\cdot\mathbf{N} = 0, \qquad N = \mathbf{x}_{\theta\theta}\cdot\mathbf{N} = ug'(1+g'^2)^{-1/2}$$

Using equations (*10.2*) and (*10.4*) it can be computed that

$$\beta_1^1 = -g''(1+g'^2)^{-3/2}, \qquad \beta_1^2 = \beta_2^1 = 0, \qquad \beta_2^2 = -u^{-1}g'(1+g'^2)^{-1/2}$$

$$\Gamma_{11}^1 = g'g''/(1+g'^2), \qquad \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \qquad \Gamma_{12}^2 = u^{-1}, \qquad \Gamma_{22}^1 = -u/(1+g'^2)$$

It follows that

$$\begin{aligned}
\Gamma_{11}^1\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_\theta + L\mathbf{N} &= g'g''/(1+g'^2)\mathbf{x}_u + g''(1+g'^2)^{-1/2}\mathbf{N} = g''\mathbf{e}_3 = \mathbf{x}_{uu} \\
\Gamma_{12}^1\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_\theta + M\mathbf{N} &= u^{-1}\mathbf{x}_\theta = -(\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2 = \mathbf{x}_{uv} \\
\Gamma_{22}^1\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_\theta + N\mathbf{N} &= -u/(1+g'^2)\mathbf{x}_u + ug'(1+g'^2)^{-1/2}\mathbf{N} = -(u\cos\theta)\mathbf{e}_1 - (u\sin\theta)\mathbf{e}_2 = \mathbf{x}_{\theta\theta} \\
\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_\theta &= -g''(1+g'^2)^{-3/2}\mathbf{x}_u = -g''(1+g'^2)^{-3/2}((\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2 + g'\mathbf{e}_3) = \mathbf{N}_u \\
\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_\theta &= -u^{-1}g'(1+g'^2)^{-1/2}\mathbf{x}_\theta = g'(1+g'^2)^{-1/2}((\sin\theta)\mathbf{e}_1 - (\cos\theta)\mathbf{e}_2) = \mathbf{N}_\theta
\end{aligned}$$

which is the required result.

THE COMPATIBILITY EQUATIONS AND THE THEOREM OF GAUSS

Given functions E, F, G, L, M and N of u and v of sufficiently high class, we want to investigate whether or not there exists a surface $\mathbf{x} = \mathbf{x}(u,v)$ for which E, F, G, L, M and N are the first and second fundamental coefficients. In general, the answer is in the negative unless certain "compatibility" conditions are satisfied. These conditions arise from the fact that if $\mathbf{x}(u,v)$ is a function of class C^3, then the third order mixed partial derivatives of $\mathbf{x}$ are independent of the order of differentiation; i.e.,

$$(\mathbf{x}_u)_{uv} = (\mathbf{x}_u)_{vu}, \quad (\mathbf{x}_v)_{uv} = (\mathbf{x}_v)_{vu} \tag{10.6}$$

In Problem 10.28, page 224, we prove

Theorem 10.2. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class $\geqq 2$ such that the coefficients of the Gauss-Weingarten equations are of class C^1. Then the mixed derivatives $\mathbf{x}_{uuv}, \mathbf{x}_{uvu}, \mathbf{x}_{vuv}, \mathbf{x}_{vvu}$ exist and satisfy *(10.6)* above if and only if the first and second fundamental coefficients satisfy the *compatibility* equations

$$\begin{aligned} L_v - M_u &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \\ M_v - N_u &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12} \end{aligned} \tag{10.7}$$

and

$$\begin{aligned} LN - M^2 = {} & F[(\Gamma^2_{22})_u - (\Gamma^2_{12})_v + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}] + E[(\Gamma^1_{22})_u - (\Gamma^1_{12})_v \\ & + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}] \end{aligned} \tag{10.8}$$

The compatibility equations can be written in various analytical forms. In this form the first two equations *(10.7)* are called the Mainardi-Codazzi equations.

Equation *(10.8)* is of particular interest. We recall that the Γ^k_{ij} depend only upon the first fundamental coefficients and their derivatives. Hence $LN - M^2$ depends only on E, F, G and their derivatives. But then the Gaussian curvature $K = (LN - M^2)/(EG - F^2)$, which was originally defined in terms of the second fundamental form, *depends only upon the coefficients of the first fundamental form.* This is one of the most important results of the theory of surfaces and we will see that it has many important consequences. Thus we have

Theorem 10.3. **Theorema Egregium of Gauss.** The Gaussian curvature on a surface of class $\geqq 3$ is a function only of the coefficients of the first fundamental form (and their derivatives).

THE FUNDAMENTAL THEOREM OF SURFACES

Theorem 10.4. **Fundamental Theorem of Surfaces.** Let E, F and G be functions of u and v of class C^2 and let L, M and N be functions of u and v of class C^1 all defined on an open set containing (u_0, v_0) such that for all (u,v),

(i) $EG - F^2 > 0,\ E > 0,\ G > 0$

(ii) E, F, G, L, M, N satisfy the compatibility equations *(10.7)* and *(10.8)*.

Then there exists a patch $\mathbf{x} = \mathbf{x}(u,v)$ of class C^3 defined in a neighborhood of (u_0, v_0) for which E, F, G, L, M, N are the first and second fundamental coefficients. The surface represented by $\mathbf{x} = \mathbf{x}(u,v)$ is unique except for position in space.

A proof of the existence of the surface with the given functions E, F, G, L, M, N as first and second fundamental coefficients is given in Appendix 2. We now prove uniqueness. We suppose there are two patches, $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(u,v)$, defined in an open connected set U containing (u_0, v_0) and such that for all (u,v) the coefficients $E = E^*$, $F = F^*$, $G = G^*$, $L = L^*$, $M = M^*$ and $N = N^*$. We suppose that the surface represented by $\mathbf{x} = \mathbf{x}^*(u,v)$ is translated and then rotated so that the point corresponding to $\mathbf{x}^*(u_0, v_0)$ coincides with $\mathbf{x}(u_0, v_0)$ and that tangent vectors $\mathbf{x}^*_u(u_0, v_0)$ and $\mathbf{x}^*_v(u_0, v_0)$ coincide with $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ respectively. This is possible since the lengths of the vectors $\mathbf{x}^*_u$ and $\mathbf{x}^*_v$ and the angle between them are determined by E^*, F^* and G^* which agree with

E, F and G at (u_0, v_0). Now let $u = u(t)$, $v = v(t)$ be a regular arc connecting (u_0, v_0) with an arbitrary point (u, v) in U, and consider the functions $\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$, $\mathbf{x}_u(t) = \mathbf{x}_u(u(t), v(t))$ and $\mathbf{x}_v(t) = \mathbf{x}_v(u(t), v(t))$. Differentiating,

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}, \quad \frac{d\mathbf{x}_u}{dt} = \mathbf{x}_{uu} \frac{du}{dt} + \mathbf{x}_{uv} \frac{dv}{dt}, \quad \frac{d\mathbf{x}_v}{dt} = \mathbf{x}_{vu} \frac{du}{dt} + \mathbf{x}_{vv} \frac{dv}{dt}$$

Using the first three equations of *(10.5)* and the fact that $\mathbf{N} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}}$, we obtain

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}$$

$$\frac{d\mathbf{x}_u}{dt} = \left(\Gamma^1_{11} \frac{du}{dt} + \Gamma^1_{12} \frac{dv}{dt}\right)\mathbf{x}_u + \left(\Gamma^2_{11} \frac{du}{dt} + \Gamma^2_{12} \frac{dv}{dt}\right)\mathbf{x}_v + \left(L\frac{du}{dt} + M\frac{dv}{dt}\right)(\mathbf{x}_u \times \mathbf{x}_v)(EG - F^2)^{-1/2}$$

$$\frac{d\mathbf{x}_v}{dt} = \left(\Gamma^1_{12} \frac{du}{dt} + \Gamma^1_{22} \frac{dv}{dt}\right)\mathbf{x}_u + \left(\Gamma^2_{12} \frac{du}{dt} + \Gamma^2_{22} \frac{dv}{dt}\right)\mathbf{x}_v + \left(M\frac{du}{dt} + N\frac{dv}{dt}\right)(\mathbf{x}_u \times \mathbf{x}_v)(EG - F^2)^{-1/2}$$

or

$$\frac{d\mathbf{x}}{dt} = a(t)\mathbf{x}_u + b(t)\mathbf{x}_v$$

$$\frac{d\mathbf{x}_u}{dt} = c(t)\mathbf{x}_u + d(t)\mathbf{x}_v + e(t)\mathbf{x}_u \times \mathbf{x}_v \qquad (10.9)$$

$$\frac{d\mathbf{x}_v}{dt} = f(t)\mathbf{x}_u + g(t)\mathbf{x}_v + h(t)\mathbf{x}_u \times \mathbf{x}_v$$

where $\quad a(t) = \dfrac{du}{dt}, \quad b(t) = \dfrac{dv}{dt}, \quad c(t) = \Gamma^1_{11}\dfrac{du}{dt} + \Gamma^1_{12}\dfrac{dv}{dt}, \quad$ etc.

We now regard the above equations as a system of first order ordinary differential equations for the functions $\mathbf{x}(t)$, $\mathbf{x}_u(t)$ and $\mathbf{x}_v(t)$. We note also that the coefficients $a(t)$, $b(t)$, etc., depend only upon the curve $u(t), v(t)$ and the first and second fundamental coefficients E, F, G, L, M, N and the derivatives of E, F and G along $u = u(t)$, $v = v(t)$. Since $E = E^*$, $F = F^*$, $G = G^*$, $L = L^*$, $M = M^*$ and $N = N^*$ for all (u, v), the corresponding functions $\mathbf{x}^*(t) = \mathbf{x}^*(u(t), v(t))$, $\mathbf{x}^*_u(t) = \mathbf{x}^*_u(u(t), v(t))$, and $\mathbf{x}^*_v(t) = \mathbf{x}^*_v(u(t), v(t))$ along the patch $\mathbf{x} = \mathbf{x}^*(u, v)$ satisfy the same system of equations *(10.9)*. Also initially $\mathbf{x}(t_0) = \mathbf{x}(u_0, v_0) = \mathbf{x}^*(u_0, v_0) = \mathbf{x}^*(t_0)$, $\mathbf{x}_u(t_0) = \mathbf{x}_u(u_0, v_0) = \mathbf{x}^*_u(u_0, v_0) = \mathbf{x}^*(t_0)$ and $\mathbf{x}_v(t_0) = \mathbf{x}_v(u_0, v_0) = \mathbf{x}^*_v(u_0, v_0) = \mathbf{x}^*_v(t_0)$. It follows from the uniqueness theorem for ordinary differential equations that $\mathbf{x}(t) = \mathbf{x}^*(t)$ everywhere along the curve $u = u(t)$, $v = v(t)$. Hence the patches $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(u, v)$ coincide, which proves the theorem.

Example 10.2.

We want to determine the surface whose fundamental coefficients are $E = 1$, $F = 0$, $G = \sin^2 u$, $L = 1$, $M = 0$ and $N = \sin^2 u$, $0 < u < \pi$. From equations *(10.2)* and *(10.4)* we obtain $\beta^1_1 = -1$, $\beta^2_1 = \beta^1_2 = 0$, $\beta^2_2 = -1$, $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{22} = 0$, $\Gamma^2_{12} = \cot u$, $\Gamma^1_{22} = -\sin u \cos u$. Thus the Gauss-Weingarten equations are

$$\mathbf{x}_{uu} = \mathbf{N}, \quad \mathbf{x}_{uv} = (\cot u)\mathbf{x}_v, \quad \mathbf{x}_{vv} = -(\sin u \cos u)\mathbf{x}_u + (\sin^2 u)\mathbf{N}, \quad \mathbf{N}_u = -\mathbf{x}_u, \quad \mathbf{N}_v = -\mathbf{x}_v$$

From the first and fourth equations we obtain $\mathbf{x}_{uuu} = -\mathbf{x}_u$. Integrating gives

$$\mathbf{x} = \mathbf{a}(v) \sin u + \mathbf{b}(v) \cos u + \mathbf{c}(v)$$

From the second equation we obtain

$$\mathbf{x}_{uv} = \mathbf{a}' \cos u - \mathbf{b}' \sin u = (\cot u)\mathbf{x}_v = \mathbf{a}' \cos u + \mathbf{b}' \cos u \cot u + \mathbf{c}' \cot u$$

Thus $\mathbf{b}'(\sin u + \cos u \cot u) = -\mathbf{c}' \cot u$ or $\mathbf{b}' = -\mathbf{c}' \cos u$. Since $\mathbf{b}'$ and $\mathbf{c}'$ are functions only of v, it follows that $\mathbf{b}' = \mathbf{c}' = \mathbf{0}$. Hence $\mathbf{b} =$ constant and $\mathbf{c} =$ constant. From the first and third equations we obtain further that

$$\mathbf{x}_{vv} = \mathbf{a}'' \sin u = -(\sin u \cos u)\mathbf{x}_u + (\sin^2 u)\mathbf{x}_{uu} = -\mathbf{a} \sin u$$

Hence $\mathbf{a}'' = -\mathbf{a}$ and $\mathbf{a} = \mathbf{d}\cos v + \mathbf{e}\sin v$, where $\mathbf{d} = \text{constant}$ and $\mathbf{e} = \text{constant}$. Thus

$$\mathbf{x} = \mathbf{d}\cos v \sin u + \mathbf{e}\sin v \sin u + \mathbf{b}\cos u + \mathbf{c}$$

It remains to show that the vectors **d**, **e** and **b** form an orthonormal set. For then

$$\begin{aligned}|\mathbf{x}-\mathbf{c}|^2 &= (d\cdot d)\cos^2 v\sin^2 u + 2(d\cdot e)\cos v\sin v\sin^2 u \\ &\quad + 2(d\cdot b)\cos v\sin u\cos u + (e\cdot e)\sin^2 v\sin^2 u \\ &\qquad + 2(e\cdot b)\sin v\sin u\cos u + (b\cdot b)\cos^2 u \\ &= \cos^2 v\sin^2 u + \sin^2 v\sin^2 u + \cos^2 u = 1\end{aligned}$$

That is, **x** is on a sphere of radius 1 about **c**. In order to show that **d**, **e** and **b** form an orthonormal set, we observe that

$$\mathbf{x}_v\cdot\mathbf{x}_v = G = \sin^2 u = (\mathbf{d}\cdot\mathbf{d})\sin^2 v\sin^2 u - 2(\mathbf{d}\cdot\mathbf{e})\sin v\cos v\sin^2 u + (\mathbf{e}\cdot\mathbf{e})\cos^2 v\sin^2 u$$

or

$$1 = (\mathbf{d}\cdot\mathbf{d})\sin^2 v - 2(\mathbf{d}\cdot\mathbf{e})\sin v\cos v + (\mathbf{e}\cdot\mathbf{e})\cos^2 v$$

Hence $\mathbf{e}\cdot\mathbf{e} = 1$, $\mathbf{d}\cdot\mathbf{e} = 0$, $\mathbf{d}\cdot\mathbf{d} = 1$. Using this, we obtain further that

$$\mathbf{x}_v\cdot\mathbf{x}_u = F = 0 = (\mathbf{b}\cdot\mathbf{d})\sin v\sin^2 u - (\mathbf{e}\cdot\mathbf{b})\cos v\sin^2 u$$

Hence $\mathbf{b}\cdot\mathbf{d} = \mathbf{e}\cdot\mathbf{b} = 0$. Finally

$$\mathbf{x}_u\cdot\mathbf{x}_u = E = 1 = \cos^2 v\cos^2 u + \sin^2 v\cos^2 u + (\mathbf{b}\cdot\mathbf{b})\sin^2 u$$

Thus $\mathbf{b}\cdot\mathbf{b} = 1$, which is the required result. Accordingly the surface with the given fundamental coefficients is a sphere of radius one.

SOME THEOREMS ON SURFACES IN THE LARGE

We want to prove that the sphere is the only connected and closed surface of class $\geqq 3$ all points of which are spherical umbilical points. Suppose then that S is a surface of class $\geqq 3$ which is connected, closed and that every point on S is a spherical umbilical point. Let P be an arbitrary point on S and $\mathbf{x} = \mathbf{x}(u,v)$ a connected patch on S containing P. We recall that at a spherical umbilical point the normal curvature $\kappa = \text{constant} \neq 0$ in every direction and so every direction on the patch is a principal direction. Thus every curve on the patch and in particular the parameter curves are lines of curvature. It follows from Rodrigues' formula that $\mathbf{N}_u = -\kappa\mathbf{x}_u$ and $\mathbf{N}_v = -\kappa\mathbf{x}_v$. Note that at a fixed point κ is constant in every direction but it is not known a priori that κ is constant from point to point on the patch. To prove this we use the fact that $\mathbf{x}(u,v)$ is of class C^3 and compute $\mathbf{N}_{uv} = -\kappa\mathbf{x}_{uv} - \kappa_v\mathbf{x}_u$ and $\mathbf{N}_{vu} = -\kappa\mathbf{x}_{vu} - \kappa_u\mathbf{x}_v$. Subtracting gives $\kappa_v\mathbf{x}_u - \kappa_u\mathbf{x}_v = 0$. But at each point $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent. Thus $\kappa_v = 0$ and $\kappa_u = 0$. Hence $\kappa = \text{constant}$ on the patch. Thus every point on S belongs to a patch on which $\kappa = \text{constant} \neq 0$. Now consider P to be fixed and Q to be any other point on S. Since S is connected, there exists a regular arc $\Gamma: \mathbf{x} = \mathbf{x}(t)$ on S connecting P and Q. Since every point on S is a spherical umbilical point, every curve on S is a line of curvature. Thus again from Rodrigues' formula along Γ, $d\mathbf{N}/dt = -\kappa(d\mathbf{x}/dt)$ where we note that $\kappa = \text{constant}$ since at each point on Γ, $\kappa = \text{constant}$ on a patch containing the point. It follows by integrating that

$$\mathbf{N} = -\kappa\mathbf{x} + \mathbf{C}, \quad \text{or} \quad |\mathbf{x} - \mathbf{C}/\kappa| = 1/|\kappa| \qquad (\mathbf{C} = \text{constant})$$

Thus **x** and in particular Q is on the sphere Σ of radius $1/|\kappa|$ with center at $\mathbf{C}/\kappa$. Since Q is arbitrary it follows that S is on Σ. Since Σ is connected and S is closed it follows from Theorem 8.5, page 159, that $S = \Sigma$; this proves

Theorem 10.5. The only connected and closed surfaces of class $\geqq 3$ of which all points are spherical umbilical points are spheres.

Similarly one can prove

Theorem 10.6. The only connected and closed surfaces of class $\geqq 2$ of which all points are planar points are planes.

The proof of the above theorem is left to the reader as an exercise (Problem 10.38).

In Problem 10.10 we also prove

Theorem 10.7. **(Liebmann).** The only connected and compact surfaces of sufficiently high class with constant Gaussian curvature are spheres.

We note that as a result of the above theorem we have a particularly important property of spheres which will be considered in greater detail in the next chapter. Namely, suppose there exists a continuous mapping $\mathbf{f}$ of a surface Σ onto a surface S which is locally 1-1 and preserves the first fundamental form, i.e. for each point P on Σ there exists a patch $\mathbf{x} = \mathbf{x}(u,v)$ containing P such that $\mathbf{f}$ is a 1-1 mapping of $\mathbf{x} = \mathbf{x}(u,v)$ onto a patch $\mathbf{x} = \mathbf{x}^*(u,v)$ on S and such that first fundamental coefficients agree at corresponding points. Note that as shown in Problem 9.6, page 190, the two surfaces need not be the same since a surface is uniquely determined by both its first *and* second fundamental coefficients. However, if Σ is a sphere then S must be a sphere of the same radius. For a sphere Σ has constant Gaussian curvature equal to $K = 1/R^2$, where R is the radius of Σ, and since the Gaussian curvature is a function only of the first fundamental coefficients it follows that S also has constant Gaussian curvature equal to $K = 1/R^2$. Also since the sphere Σ is connected and compact and the mapping $\mathbf{f}$ of Σ onto S is continuous, it follows that S is connected and compact. But then it follows from the above theorem that S is also a sphere and, moreover, its radius is $1/K^2 = R$.

NOTATION

The formalism of the theory of surfaces can be greatly simplified with the use of tensors and tensor notation. However, this will require a change in notation. The components of a vector will be denoted by superscripts in place of subscripts. Thus a vector in E^3 is $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$, a point in the parameter plane is (u^1, u^2), and a patch is $\mathbf{x} = \mathbf{x}(u^1, u^2)$.

Also the partial derivatives of $\mathbf{x}$ shall be denoted by

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}, \quad \mathbf{x}_{12} = \frac{\partial^2 \mathbf{x}}{\partial u^1\,\partial u^2}, \quad \mathbf{x}_{22} = \frac{\partial^2 \mathbf{x}}{\partial u^2\,\partial u^2}, \quad \text{etc.}$$

It follows that a tangent vector is $d\mathbf{x} = \mathbf{x}_1\,du^1 + \mathbf{x}_2\,du^2$ and the first fundamental form is

$$\begin{aligned} \mathrm{I} &= d\mathbf{x}\cdot d\mathbf{x} = \mathbf{x}_1\cdot\mathbf{x}_1\,du^1\,du^1 + 2\mathbf{x}_1\cdot\mathbf{x}_2\,du^1\,du^2 + \mathbf{x}_2\cdot\mathbf{x}_2\,du^2\,du^2 \\ &= g_{11}\,du^1\,du^1 + g_{12}\,du^1\,du^2 + g_{21}\,du^2\,du^1 + g_{22}\,du^2\,du^2 = \sum_{ik} g_{ik}\,du^i\,du^k \end{aligned} \tag{10.10}$$

where $g_{11} = \mathbf{x}_1\cdot\mathbf{x}_1 = E$, $g_{12} = g_{21} = \mathbf{x}_1\cdot\mathbf{x}_2 = F$ and $g_{22} = \mathbf{x}_2\cdot\mathbf{x}_2 = G$ denote the first fundamental coefficients and $i,k = 1,2$.

We also use g to denote the discriminant of I; i.e.,

$$g = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{11}g_{22} - g_{21}g_{12} = EG - F^2$$

and introduce the quantities

$$g^{11} = g_{22}/g, \quad g^{12} = g^{21} = -g_{12}/g = -g_{21}/g, \quad g^{22} = g_{11}/g \tag{10.11}$$

In Problem 10.12, page 220, we show that

$$\sum_k g_{ik}g^{kj} = \delta_i^j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{10.12}$$

That is, the matrix (g^{ij}) is the inverse of (g_{ij}), and the product

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finally, the differential of the normal is the vector $d\mathbf{N} = \mathbf{N}_1\,du^1 + \mathbf{N}_2\,du^2$ and the second fundamental form is

$$\begin{aligned} \mathrm{II} = -d\mathbf{x}\cdot d\mathbf{N} &= -\mathbf{x}_1\cdot\mathbf{N}_1\,du^1\,du^1 - \mathbf{x}_1\cdot\mathbf{N}_2\,du^1\,du^2 - \mathbf{x}_2\cdot\mathbf{N}_1\,du^2\,du^1 - \mathbf{x}_2\cdot\mathbf{N}_2\,du^2\,du^2 \\ &= b_{11}\,du^1\,du^1 + b_{12}\,du^1\,du^2 + b_{21}\,du^2\,du^1 + b_{22}\,du^2\,du^2 = \sum_{ik} b_{ik}\,du^i\,du^k \end{aligned} \tag{10.13}$$

where

$$\begin{aligned} b_{11} &= -\mathbf{x}_1\cdot\mathbf{N}_1 = \mathbf{x}_{11}\cdot\mathbf{N} = L \\ b_{12} &= b_{21} = -\mathbf{x}_2\cdot\mathbf{N}_1 = -\mathbf{x}_1\cdot\mathbf{N}_2 = \mathbf{x}_{12}\cdot\mathbf{N} = \mathbf{x}_{21}\cdot\mathbf{N} = M \\ b_{22} &= -\mathbf{x}_2\cdot\mathbf{N}_2 = \mathbf{x}_{22}\cdot\mathbf{N} = N \end{aligned} \tag{10.14}$$

now denote the second fundamental coefficients. We also define

$$b = \det(b_{ij}) = (b_{11}b_{22} - b_{12}b_{21}) = LN - M^2 \tag{10.15}$$

The summation convention to be used is as follows: Consider the sum

$$\sum_{\alpha=1}^{3} a_{i\alpha}b^\alpha = a_{i1}b^1 + a_{i2}b^2 + a_{i3}b^3$$

Observe on the left that α appears in the product $a_{i\alpha}b^\alpha$ exactly once as a subscript and once as a superscript. When this is the case we omit the Σ sign and write simply $a_{i\alpha}b^\alpha$. Thus

$$a_{i\alpha}b^\alpha = \sum_\alpha a_{i\alpha}b^\alpha = a_{i1}b^1 + a_{i2}b^2 + a_{i3}b^3$$

The index α which is summed over is called a *summation* or *dummy* index. A dummy index can always be changed in a computation. That is,

$$a_{i\alpha}b^\alpha = a_{i\beta}b^\beta = a_{i\gamma}b^\gamma$$

The index i is called a *free* index. It cannot be changed. Finally in a derivative $\partial\theta^i/\partial u^j$ or $\mathbf{x}_j = \partial\mathbf{x}/\partial u^j$, the index i is considered a superscript and j a subscript.

Example 10.3.

(*a*) Let $f = f(x^1, x^2, x^3)$ and $x^i = x^i(u^1, u^2)$, $i = 1, 2, 3$. The chain rule for the derivatives of f with respect to the u^i is

$$\frac{\partial f}{\partial u^i} = \frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial u^i} + \frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial u^i} + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial u^i} = \sum_{\alpha=1}^{3}\frac{\partial f}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial u^i}, \qquad i = 1, 2$$

Since α occurs as "subscript" in $\dfrac{\partial f}{\partial x^\alpha}$ and as a "superscript" in $\dfrac{\partial x^\alpha}{\partial u^i}$, this can be written as

$$\frac{\partial f}{\partial u^i} = \frac{\partial f}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial u^i}$$

(*b*) Let $S = a_{\alpha\beta}x^\alpha x^\beta$, $\alpha, \beta = 1, 2, 3$. Since both α and β occur once as a subscript and once as a superscript, S is the double sum

$$\begin{aligned} S &= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} a_{\alpha\beta}x^\alpha x^\beta \\ &= a_{11}x^1x^1 + a_{12}x^1x^2 + a_{13}x^1x^3 + a_{21}x^2x^1 + a_{22}x^2x^2 + a_{23}x^2x^3 + a_{31}x^3x^1 + a_{32}x^3x^2 + a_{33}x^3x^3 \end{aligned}$$

Example 10.4.

Using the summation convention, we can write a tangent vector as $d\mathbf{x} = \mathbf{x}_\alpha\,du^\alpha$ and the first fundamental form as $\mathrm{I} = \mathbf{x}_\alpha\cdot\mathbf{x}_\beta\,du^\alpha\,du^\beta$. The differential of the normal is $d\mathbf{N} = \mathbf{N}_\alpha\,du^\alpha$ and the second fundamental form is $\mathrm{II} = -\mathbf{x}_\alpha\cdot\mathbf{N}_\beta\,du^\alpha\,du^\beta = \mathbf{x}_{\alpha\beta}\cdot\mathbf{N}\,du^\alpha\,du^\beta = b_{\alpha\beta}\,du^\alpha\,du^\beta$. The Gauss-Weingarten equations (*10.5*) are written as

$$\mathbf{x}_{11} = \Gamma^\alpha_{11}\mathbf{x}_\alpha + b_{11}\mathbf{N}, \quad \mathbf{x}_{12} = \Gamma^\alpha_{12}\mathbf{x}_\alpha + b_{12}\mathbf{N}, \quad \mathbf{x}_{22} = \Gamma^\alpha_{22}\mathbf{x}_\alpha + b_{22}\mathbf{N}, \quad N_1 = \beta^\alpha_1\mathbf{x}_\alpha, \quad N_2 = \beta^\alpha_2\mathbf{x}_\alpha$$

or simply

$$\mathbf{x}_{ij} = \Gamma^\alpha_{ij}\mathbf{x}_\alpha + b_{ij}\mathbf{N}, \quad \mathbf{N}_i = \beta^\alpha_i\mathbf{x}_\alpha, \qquad i, j = 1, 2$$

ELEMENTARY MANIFOLDS

Since tensors have a wide variety of applications in addition to surfaces in E^3, we will introduce a generalization of the concept of an elementary surface as follows: We assume we have an abstract collection M of objects P called "points" which can be put into a 1-1 correspondence with a set S of n-tuples of real numbers $(u^1, u^2, \ldots, u^n)$ called the coordinates of P. This correspondence $P(u^1, \ldots, u^n)$ between the points in M and the set of n-tuples S is called a *coordinate system* on M. A coordinate system $P(u^1, \ldots, u^n)$ on M is analogous to a patch $\mathbf{x} = \mathbf{x}(u^1, u^2)$ on a surface, which determines a 1-1 correspondence with the points on the patch and a set of points in the u^1u^2 plane. Any other coordinate system $P(\bar{u}^1, \ldots, \bar{u}^n)$ on M defined on a set of n-tuples $\bar{S}$ will determine a 1-1 correspondence $(u^1, \ldots, u^n) \leftrightarrow (\bar{u}^1, \ldots, \bar{u}^n)$ between the two sets of n-tuples S and $\bar{S}$ called a *coordinate transformation*. This correspondence can be written as $\bar{u}^i = \bar{u}^i(u^1, \ldots, u^n)$, $i = 1, \ldots, n$, with inverse equations $u^i = u^i(\bar{u}^1, \ldots, \bar{u}^n)$. Here the $\bar{u}^i$ are real valued functions on S and the u^i are real valued functions on $\bar{S}$. These equations correspond to the parameter transformation $\bar{u}^1 = \bar{u}^1(u^1, u^2)$, $\bar{u}^2 = \bar{u}^2(u^1, u^2)$ with inverse $u^1 = u^1(\bar{u}^1, \bar{u}^2)$, $u^2 = u^2(\bar{u}^1, \bar{u}^2)$ which exist on the intersection of two patches on a surface.

We assume that the sets S of n-tuples of real numbers on which the coordinate systems are defined are *open*. A set S of n-tuples of real numbers $(u^1, \ldots, u^n)$ is open if for every $(u_0^1, \ldots, u_0^n)$ in S there exists a real $\epsilon > 0$ such that all $(u^1, \ldots, u^n)$ satisfying $\left[\sum_{i=1}^{n} (u^i - u_0^i)^2\right]^{1/2} < \epsilon$ are in S.

We also assume that the transformations $\bar{u}^i = \bar{u}^i(u^1, \ldots, u^n)$ and their inverses $u^i = u^i(\bar{u}^1, \ldots, \bar{u}^n)$ have continuous partial derivatives $\dfrac{\partial \bar{u}^i}{\partial u^j}$ and $\dfrac{\partial u^i}{\partial \bar{u}^j}$, $i, j = 1, \ldots, n$, and non-vanishing Jacobians $\dfrac{\partial(\bar{u}^1, \ldots, \bar{u}^n)}{\partial(u^1, \ldots, u^n)} = \det\left(\dfrac{\partial \bar{u}^i}{\partial u^j}\right) \neq 0$ and $\dfrac{\partial(u^1, \ldots, u^n)}{\partial(\bar{u}^1, \ldots, \bar{u}^n)} = \det\left(\dfrac{\partial u^i}{\partial \bar{u}^j}\right) \neq 0$. In Problem 10.17, page 221, we prove that since the $\bar{u}_i$ and u_i are functions which are inverse to each other, we have

$$\frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta_i^j \tag{10.16}$$

A real valued function $\bar{u}^i(u^1, \ldots, u^n)$ defined on S is continuous at $(u_0^1, \ldots, u_0^n)$ if given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\left[\sum_i (\bar{u}^i(u^1, \ldots, u^n) - \bar{u}^i(u_0^1, \ldots, u_0^n))^2\right]^{1/2} < \epsilon$ for $\left[\sum_i (u^i - u_0^i)^2\right]^{1/2} < \delta$. The function $\bar{u}^i(u^1, \ldots, u^n)$ is continuous on S if it is continuous at each $(u_0^1, \ldots, u_0^n)$ in S. The partial derivative $\partial \bar{u}^i/\partial u^j$ at $(u_0^1, \ldots, u_0^n)$ is defined by the limit

$$\frac{\partial \bar{u}^i}{\partial u^j}(u_0^1, \ldots, u_0^n) = \lim_{k \to 0} \frac{\bar{u}^i(u_0^1, \ldots, u_0^j + k, \ldots, u_0^n) - \bar{u}^i(u_0^1, \ldots, u_0^n)}{k}$$

The underlying collection of points M together with the totality of allowable coordinate systems as defined above is called an *elementary coordinate manifold of n dimensions.*

Example 10.5.

(*a*) A regular simple (nonintersecting) curve in E^3 is an elementary coordinate manifold of dimension 1.

(*b*) An elementary (covered by a single patch) surface in E^3 is an elementary coordinate manifold of 2 dimensions.

(*c*) E^3 itself together with all coordinate systems $P(x^1, x^2, x^3)$ defined on E^3 for which the coordinate transformations $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3)$, $i = 1, 2, 3$, and their inverses $x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, $i = 1, 2, 3$, are of class C^1 with Jacobians $\dfrac{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)}{\partial(x^1, x^2, x^3)} \neq 0$ and $\dfrac{\partial(x^1, x^2, x^3)}{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)} \neq 0$, is an elementary coordinate manifold of 3 dimensions.

TENSORS

A tensor T at a point P on a coordinate manifold can be thought of as a certain geometric "object" attached to P with the following properties:

(i) With respect to each coordinate system $P(u^1, \ldots, u^n)$ on the manifold, T is represented by a set of scalars C called the *components of T with respect to the coordinate system* $P(u^1, \ldots, u^n)$.

(ii) If $P(\bar{u}^1, \ldots, \bar{u}^n)$ is any other coordinate system on the manifold, the components $\bar{C}$ of T with respect to $P(\bar{u}^1, \ldots, \bar{u}^n)$ are related to the components C by certain transformation laws which depend upon the coordinate transformation $\bar{u}^i = \bar{u}^i(u^1, \ldots, u^n)$, $i = 1, \ldots, n$, and its inverse $u^i = u^i(\bar{u}^1, \ldots, \bar{u}^n)$.

An important example of a tensor at a point P on a surface is the one whose components with respect to a patch $\mathbf{x} = \mathbf{x}(u^1, u^2)$ containing P are the first fundamental coefficients g_{ij}, $i, j = 1, 2$. If $\mathbf{x} = \mathbf{x}^*(\bar{u}^1, \bar{u}^2)$ is another patch containing P with first fundamental coefficients $\bar{g}_{ij}$, then it follows from equations (*9.2*) and (*9.3*) that the g_{ij} are related to the $\bar{g}_{ij}$ by the transformation law

$$\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}, \qquad \alpha, \beta = 1, 2 \tag{10.17}$$

This tensor is called the *covariant metric* tensor of the surface at P.

As for the metric tensor considered above, the tensor T will usually depend on the point P in the manifold, the components of $T(P)$ being given by functions of the coordinates $(u^1, \ldots, u^n)$ of P. The function which assigns to each point P in the manifold the tensor $T(P)$ is called a *tensor field.*

Tensors are classified according to their transformation laws as follows:

(i) A tensor is called a *contravariant tensor of rank 1* or a *contravariant vector* if it has n components $A^1, \ldots, A^n$ which transform in accordance with the law

$$\bar{A}^i = A^\alpha \frac{\partial \bar{u}^i}{\partial u^\alpha}, \qquad \alpha, i = 1, \ldots, n \tag{10.18}$$

(ii) A tensor is called a *covariant tensor of rank 1* or a *covariant vector* if it has n components $A_1, \ldots, A_n$ which transform in accordance with the law

$$\bar{A}_i = A_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^i}, \qquad \alpha, i = 1, \ldots, n \tag{10.19}$$

(iii) A tensor is called a *contravariant tensor of rank 2* if it has n^2 components A^{ij} $i, j = 1, \ldots, n$, which transform in accordance with the law

$$\bar{A}^{ij} = A^{\alpha\beta} \frac{\partial \bar{u}^i}{\partial u^\alpha} \frac{\partial \bar{u}^j}{\partial u^\beta}, \qquad \alpha, \beta, i, j = 1, \ldots, n \tag{10.20}$$

(iv) A tensor is called a *covariant tensor of rank 2* if it has n^2 components A_{ij}, $i, j = 1, \ldots, n$, which transform in accordance with the law

$$\bar{A}_{ij} = A_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \tag{10.21}$$

The above four tensors are special cases of

(v) A tensor is called a *mixed tensor, contravariant of order r and covariant of order s with weight N,* if it has n^{r+s} components $A^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, $i_k, j_m = 1, \ldots, n$, which transform in accordance with the law

$$\bar{A}^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \left[\det\left(\frac{\partial u_i}{\partial \bar{u}_j}\right)\right]^N A^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial \bar{u}^{i_r}}{\partial u^{\alpha_r}} \frac{\partial u^{\beta_1}}{\partial \bar{u}^{j_1}} \cdots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} \tag{10.22}$$

If the exponent N of the Jacobian $\det(\partial u^i/\partial \bar{u}^j)$ is zero, the tensor is called an *absolute* tensor. If $s=0$, the tensor is purely contravariant. If $r=0$, the tensor is purely covariant. The sum $r+s$ is called the *rank* of the tensor. Note that the tensors (i), (ii), (iii) and (iv) above are absolute tensors. We also define

(vi) A scalar is a tensor of rank zero.

Finally we note that the transformation law of a general tensor is *transitive*. For example, consider the transformation law $\bar{\bar{A}}^i = \bar{A}^\alpha \dfrac{\partial \bar{\bar{u}}^i}{\partial \bar{u}^\alpha}$ which expresses the relation between the components of a contravariant vector in the $P(\bar{\bar{u}}^1, \ldots, \bar{\bar{u}}^n)$ and $P(\bar{u}^1, \ldots, \bar{u}^n)$ coordinate systems. From equation *(10.18)* it follows by substitution and the chain rule that

$$\bar{\bar{A}}^i = \left(A^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right)\frac{\partial \bar{\bar{u}}^i}{\partial \bar{u}^\alpha} = A^\beta\left(\frac{\partial \bar{\bar{u}}^i}{\partial \bar{u}^\alpha}\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right) = A^\beta \frac{\partial \bar{\bar{u}}^i}{\partial u^\beta}$$

which is exactly the transformation law relating the components of the vector with respect to the $P(\bar{\bar{u}}^1, \ldots, \bar{\bar{u}}^n)$ and $P(u^1, \ldots, u^n)$ coordinate systems.

Example 10.6.

(*a*) Let $f(x^1, x^2, x^3)$ be a real valued function of class C^1 defined on an open set U in E^3. If $x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, $i = 1, 2, 3,$ is an allowable coordinate transformation, then it follows from the chain rule that at each point

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^i}, \qquad i = 1, 2, 3$$

Thus from equation *(10.19)* we see that the $\partial f/\partial x^i$, $i = 1,2,3,$ are the components of a covariant vector field on U called the *gradient* of f.

(*b*) It follows from equations *(10.17)* and *(10.21)* that the covariant metric tensor, i.e. the tensor field on a surface whose components are the first fundamental coefficients g_{ik}, is an absolute covariant tensor field of rank 2.

(*c*) We recall (see equation *(9.11)*, page 175) that the second fundamental coefficients b_{ij} transform in the same way as the g_{ij}, i.e. on the intersection of two patches $\mathbf{x} = \mathbf{x}(u^1, u^2)$ and $\mathbf{x} = \mathbf{x}^*(\bar{u}^1, \bar{u}^2)$ we have

$$\bar{b}_{ij} = b_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}$$

Thus the second fundamental coefficients are also the components of an absolute covariant tensor field on the surface of rank 2.

(*d*) Let $\bar{u}^i = \bar{u}^i(u^1, \ldots, u^n)$ be an allowable coordinate transformation on a coordinate manifold of dimension n and consider at a point the sum $\delta^\beta_\alpha \dfrac{\partial \bar{u}^j}{\partial u^\beta}\dfrac{\partial u^\alpha}{\partial \bar{u}^i}$. Since the only contribution occurs when $\alpha = \beta$, we have

$$\delta^\beta_\alpha \frac{\partial \bar{u}^j}{\partial u^\beta}\frac{\partial u^\alpha}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta^j_i = \bar{\delta}^j_i$$

where we used equation *(10.16)*. It follows from equation *(10.22)* that the Kronecker delta δ^j_i is a mixed absolute tensor of rank 2, covariant of order 1 and contravariant of order 1. Note that this is a tensor whose components are the same with respect to every coordinate system on the manifold.

(*e*) At a point on the intersection of two patches on a surface let $A^{ij} = g^{\alpha\beta}\dfrac{\partial \bar{u}^i}{\partial u^\alpha}\dfrac{\partial \bar{u}^j}{\partial u^\beta}$, where the g^{ij} are defined in equation *(10.11)*, and consider the sum

$$\begin{aligned}
\bar{g}_{ik}A^{kj} &= g_{\gamma\sigma}\frac{\partial u^\gamma}{\partial \bar{u}^i}\frac{\partial u^\sigma}{\partial \bar{u}^k}g^{\alpha\beta}\frac{\partial \bar{u}^k}{\partial u^\alpha}\frac{\partial \bar{u}^j}{\partial u^\beta} = \left(\frac{\partial u^\sigma}{\partial \bar{u}^k}\frac{\partial \bar{u}^k}{\partial u^\alpha}\right)g_{\gamma\sigma}g^{\alpha\beta}\frac{\partial u^\gamma}{\partial \bar{u}_i}\frac{\partial \bar{u}^j}{\partial u^\beta} \\
&= \delta^\sigma_\alpha g_{\gamma\sigma}g^{\alpha\beta}\frac{\partial u^\gamma}{\partial \bar{u}_i}\frac{\partial \bar{u}^j}{\partial u^\beta} = g_{\gamma\alpha}g^{\alpha\beta}\frac{\partial u^\gamma}{\partial \bar{u}^i}\frac{\partial \bar{u}^j}{\partial u^\beta} = \delta^\beta_\gamma\frac{\partial u^\gamma}{\partial \bar{u}^i}\frac{\partial \bar{u}^j}{\partial u^\beta} \\
&= \frac{\partial u^\beta}{\partial \bar{u}^i}\frac{\partial u^j}{\partial u^\beta} = \delta^j_i = \bar{\delta}^j_i
\end{aligned}$$

Since $\bar{g}_{ik}A^{kj} = \delta_i^j$ it follows that

$$\bar{g}^{ij} = A^{ij} = g^{\alpha\beta}\frac{\partial \bar{u}^i}{\partial u^\alpha}\frac{\partial \bar{u}^j}{\partial u^\beta}$$

Thus from equation (*10.20*) we see that the g^{ij} are the components of an absolute contravariant tensor field on the surface of rank 2, called the *contravariant metric tensor.*

(*f*) Consider the 27 scalars e^{ijk}, $i, j, k = 1, 2, 3$, defined as follows:

$$e^{ijk} = \begin{cases} 0 & \text{if two indices } i, j, k \text{ are the same} \\ 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

For example $e^{112} = 0$, $e^{212} = 0$, $e^{123} = 1$, $e^{213} = -1$, $e^{231} = 1$. We recall that

$$\det(a_i^j) = \det\begin{pmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{pmatrix} = \sum \pm a_i^1 a_j^2 a_k^3$$

where $+$ is taken when i, j, k is an even permutations of $1, 2, 3$ and $-$ is taken when i, j, k is an odd permutation of $1, 2, 3$. But then it follows from the definition of e^{ijk} that we can write

$$\det(a_i^j) = \sum_{\alpha\beta\gamma} e^{\alpha\beta\gamma} a_\alpha^1 a_\beta^2 a_\gamma^3 = e^{\alpha\beta\gamma} a_\alpha^1 a_\beta^2 a_\gamma^3$$

where we sum over all α, β, γ. We note further that

$$e^{\alpha\beta\gamma} a_\alpha^p a_\beta^q a_\gamma^r = \begin{cases} +\det(a_i^j) & \text{if } p, q, r \text{ is an even permutation of } 1, 2, 3 \\ -\det(a_i^j) & \text{if } p, q, r \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

Thus
$$e^{pqr}\det(a_i^j) = e^{\alpha\beta\gamma} a_\alpha^p a_\beta^q a_\gamma^r$$

Now let $\bar{u}^i = \bar{u}^i(u^1, u^2, u^3)$ be a coordinate transformation on a coordinate manifold of 3 dimensions. Then it follows from the above that

$$e^{pqr}\det\left(\frac{\partial \bar{u}^j}{\partial u^i}\right) = e^{\alpha\beta\gamma}\frac{\partial \bar{u}^p}{\partial u^\alpha}\frac{\partial \bar{u}^q}{\partial u^\beta}\frac{\partial \bar{u}^r}{\partial u^\gamma}$$

or
$$\bar{e}^{pqr} = e^{pqr} = \left[\det\left(\frac{\partial \bar{u}^j}{\partial u^k}\right)\right]^{-1} e^{\alpha\beta\gamma}\frac{\partial \bar{u}^p}{\partial u^\alpha}\frac{\partial \bar{u}^q}{\partial u^\beta}\frac{\partial \bar{u}^r}{\partial u^\gamma} = \det\left(\frac{\partial u^j}{\partial \bar{u}^i}\right) e^{\alpha\beta\gamma}\frac{\partial \bar{u}^p}{\partial u^\alpha}\frac{\partial \bar{u}^q}{\partial u^\beta}\frac{\partial \bar{u}^r}{\partial u^\gamma}$$

It follows that the e^{ijk} are the components of a contravariant tensor of rank 3 and weight 1. We note that the quantities e_{ijk}, $i, j, k = 1, 2, 3$, where $e_{ijk} = e^{ijk}$, are the components of a covariant tensor of rank 3 and weight -1. The proof of this is left to the reader as an exercise.

A set of components of a tensor is said to be *symmetric* with respect to two contravariant indices (superscripts) or with respect to two covariant indices (subscripts) if the components remain the same when the indices are interchanged. For example A_{pq}^{ijk} is symmetric with respect to the first and third contravariant indices if $A_{pq}^{ijk} = A_{pq}^{kji}$ for all i and k.

The component of a tensor is said to be *skew-symmetric* with respect to two contravariant indices or with respect to two covariant indices if the algebraic signs of the components change when the indices are interchanged. Thus A_{pq}^{ijk} is skew-symmetric with respect to the first and third contravariant indices if $A_{pq}^{ijk} = -A_{pq}^{kji}$ for all i, k.

A tensor is said simply to be symmetric if it is symmetric with respect to all pairs of contravariant and all pairs of covariant indices. If the tensor is skew-symmetric with respect to all pairs of contravariant and all pairs of covariant indices it is said to be skew-symmetric.

In Problem 10.19 we show that if the components of a tensor with respect to one coordinate system are symmetric with respect to a pair of indices, then the components of the tensor are symmetric with respect to the same indices in any other coordinate system. Thus the property of being symmetric with respect to a pair of indices is a property of the tensor. The same is true for skew-symmetry.

Example 10.7.

(*a*) Both the covariant and contravariant metric tensors are symmetric tensors, since $g^{ij} = g^{ji}$, $i,j = 1,2$, and $g_{ij} = g_{ji}$, $i,j = 1,2$.

(*b*) The Kronecker delta can be extended as follows:

$$\delta^{pq}_{ij} = \begin{cases} 1 & \text{if } i=p,\ j=q, \text{ and } i\neq j \\ -1 & \text{if } i=q,\ j=p, \text{ and } i\neq j \\ 0 & \text{otherwise} \end{cases} \qquad (i,j,p,q = 1,\ldots,n)$$

In Problem 10.14, page 220, we show that the δ^{pq}_{ij} are the components of an absolute tensor of rank 4, covariant of order 2 and contravariant of order 2. It is clearly skew-symmetric with respect to its contravariant indices. For if $i=p,\ j=q$ and $i\neq j$, $\delta^{qp}_{ij} = 1$ and $\delta^{pq}_{ij} = -1$; hence $\delta^{pq}_{ij} = -\delta^{qp}_{ij}$. If $i=q,\ j=p$ and $i\neq j$, then $\delta^{pq}_{ij} = -1$, $\delta^{qp}_{ij} = 1$ and $\delta^{pq}_{ij} = -\delta^{qp}_{ij}$. Otherwise $\delta^{pq}_{ij} = 0$, $\delta^{qp}_{ij} = 0$ and $\delta^{pq}_{ij} = -\delta^{qp}_{ij}$. Similarly it is skew-symmetric with respect to its covariant indices.

(*c*) The above can be generalized as follows:

$$\delta^{j_1\cdots j_m}_{i_1\cdots i_m} = \begin{cases} 1 & \text{if the } i_1,\ldots,i_m \text{ are distinct and } j_1,\ldots,j_m \text{ is an even permutation of } i_1,\ldots,i_m \\ -1 & \text{if the } i_1,\ldots,i_m \text{ are distinct and } j_1,\ldots,j_m \text{ is an odd permutation of } i_1,\ldots,i_m \\ 0 & \text{otherwise} \end{cases}$$

For example,

$$\delta^{1245}_{1122} = 0, \quad \delta^{1134}_{1234} = 0, \quad \delta^{5678}_{1234} = 0, \quad \delta^{2134}_{1234} = -1, \quad \delta^{2314}_{1234} = +1, \quad \delta^{3465}_{3456} = -1, \quad \delta^{4356}_{4356} = 1$$

It can be shown that the $\delta^{j_1\cdots j_m}_{i_1\cdots i_m}$ are the components of a skew-symmetric absolute tensor covariant of order m and contravariant of order m.

TENSOR ALGEBRA

(*a*) *Addition.* Let $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ and $B^{i_1\cdots i_r}_{j_1\cdots j_s}$ be the components of two tensors A and B contravariant and covariant of the same order and of the same weight. In Problem 10.18, page 221, we show that the sum

$$C^{i_1\cdots i_r}_{j_1\cdots j_s} = A^{i_1\cdots i_r}_{j_1\cdots j_s} + B^{i_1\cdots i_r}_{j_1\cdots j_s}$$

obtained by adding corresponding components of A and B are the components of a tensor C contravariant and covariant of the same orders and of the same weight as A and B. The tensor C is called the *sum* of A and B.

(*b*) *Outer product of tensors.* If the components $B^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ of a tensor B of rank $p+q$ are multiplied by the component $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ of a tensor A of rank $r+s$, the result will be a set of $n^{r+s+p+q}$ numbers

$$C^{i_1\cdots i_r\alpha_1\cdots\alpha_p}_{j_1\cdots j_s\beta_1\cdots\beta_q} = A^{i_1\cdots i_r}_{j_1\cdots j_s} B^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$$

It is easily verified that the $C^{i_1\cdots\alpha_p}_{j_1\cdots\beta_q}$ are the components of a tensor C, contravariant of order $r+p$, covariant of order $s+q$, and of weight N_1+N_2, where N_1 is the weight of A and N_2 is the weight of B. The tensor C is called the *outer product* of A and B. As a special case A can be a tensor of rank 0, i.e. A = scalar.

(c) *Contraction.* Let $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ be the components of a tensor A contravariant of order r, covariant of order s, and of weight N. The set of n^{r+s-2} scalars

$$B^{i_2\cdots i_r}_{j_2\cdots j_s} \;=\; A^{\alpha i_2\cdots i_r}_{\alpha j_2\cdots j_s}$$

obtained by identifying the first contravariant index with the first covariant index and summing, can be shown to be the components of a tensor B, contravariant of order $r-1$, covariant of order $s-1$ and of weight N. The tensor B is said to be a contraction of the tensor A. Such a contracted tensor can be formed for any choice of one covariant and one contravariant index.

Example 10.8.

(a) Let A^{ij} be the components of an arbitrary contravariant tensor of rank 2, and let $B^{ij} = A^{ji}$. Note that the B^{ij} are also the components of a contravariant tensor of rank 2, since

$$B^{\alpha\beta}\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)^N \frac{\partial \bar{u}^i}{\partial u^\alpha}\frac{\partial \bar{u}^j}{\partial u^\beta} \;=\; A^{\beta\alpha}\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)^N \frac{\partial \bar{u}^i}{\partial u^\alpha}\frac{\partial \bar{u}^j}{\partial u^\beta} \;=\; A^{\beta\alpha}\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)^N \frac{\partial \bar{u}^j}{\partial u^\beta}\frac{\partial \bar{u}^i}{\partial u^\alpha} \;=\; \bar{A}^{ji} = \bar{B}^{ij}$$

Now consider the tensors defined by

$$C^{ij} \;=\; \tfrac{1}{2}(A^{ij}+B^{ij})$$

$$D^{ij} \;=\; \tfrac{1}{2}(A^{ij}-B^{ij})$$

Here $$C^{ji} \;=\; \tfrac{1}{2}(A^{ji}+B^{ji}) \;=\; \tfrac{1}{2}(B^{ij}+A^{ij}) \;=\; C^{ij}$$

and $$D^{ji} \;=\; \tfrac{1}{2}(A^{ji}-B^{ji}) \;=\; \tfrac{1}{2}(B^{ij}-A^{ij}) \;=\; -D^{ij}$$

Thus C^{ij} is symmetric and D^{ij} is skew-symmetric. But $C^{ij}+D^{ij} = A^{ij}$ and so every contravariant tensor of rank 2 is the sum of a symmetric and a skew-symmetric contravariant tensor of rank 2.

(b) Let $e^{11} = e_{11} = 0$, $e^{12} = e_{12} = 1$, $e^{21} = e_{21} = -1$ and $e^{22} = e_{22} = 0$. Here e^{ij} and e_{ij} are the two dimensional analogs of e^{ijk} and e_{ijk} in Example 10.6(*f*). Now consider the outer product

$$A^{ij}_{pq} \;=\; e^{ij}e_{pq}$$

Observe that if the p, q are distinct, then $A^{ij}_{pq} = 1$ if $i = p$ and $j = q$, and $A^{ij}_{pq} = -1$ if $i = q$ and $j = p$. Otherwise $A^{ij}_{pq} = 0$. Namely from Example 10.7(*b*),

$$A^{ij}_{pq} \;=\; e^{ij}e_{pq} \;=\; \delta^{ij}_{pq}$$

We see further that the contraction

$$\delta^{\alpha j}_{\alpha q} \;=\; e^{\alpha j}e_{\alpha q} \;=\; e^{1j}e_{1q} + e^{2j}e_{2q} \;=\; \begin{Bmatrix} 1 & \text{if } j = q \\ 0 & \text{otherwise} \end{Bmatrix} \;=\; \delta^j_q$$

APPLICATIONS OF TENSORS TO THE EQUATIONS OF SURFACE THEORY

We consider the Gauss equations

$$\mathbf{x}_{ij} \;=\; \Gamma^\alpha_{ij}\mathbf{x}_\alpha + b_{ij}\mathbf{N} \qquad (\alpha, i, j = 1,2) \tag{10.23}$$

By taking the scalar product of the Gauss equations with $\mathbf{x}_k$, we obtain

$$\mathbf{x}_{ij}\cdot\mathbf{x}_k \;=\; \Gamma^\alpha_{ij}(\mathbf{x}_\alpha\cdot\mathbf{x}_k) \;=\; \Gamma^\alpha_{ij}g_{\alpha k}$$

The quantities $\Gamma_{ijk} \equiv (\mathbf{x}_{ij}\cdot\mathbf{x}_k)$ are called the *Christoffel symbols of the first kind.* Using $g_{i\alpha}g^{\alpha j} = \delta^j_i$, it follows further that

$$\Gamma_{ij\beta}g^{\beta k} \;=\; \Gamma^\alpha_{ij}g_{\alpha\beta}g^{\beta k} \;=\; \Gamma^\alpha_{ij}\delta^k_\alpha \;=\; \Gamma^k_{ij}$$

Thus the Christoffel symbols of the first kind are related to the Christoffel symbols of the second kind by the equations

$$\Gamma_{ijk} \;=\; g_{k\alpha}\Gamma^\alpha_{ij} \quad\text{and}\quad \Gamma^k_{ij} \;=\; g^{k\alpha}\Gamma_{ij\alpha} \tag{10.24}$$

In Problem 10.24, page 223, we prove that the Γ_{ijk} are given by

$$\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right] \qquad (10.25)$$

and hence the Γ^k_{ij} are given by

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{k\alpha}\left[\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha}\right] \qquad (10.26)$$

We note that the Christoffel symbols are *not* the components of a tensor as previously defined. Their transformation laws involve the second derivatives of the parameter transformation. In Problem 10.27, page 223, we prove

Theorem 10.8. The Christoffel symbols transform in accordance with the laws

$$\bar{\Gamma}^k_{ij} = \left[\Gamma^\gamma_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^i \partial \bar{u}^j}\right]\frac{\partial \bar{u}^k}{\partial u^\gamma}$$

$$\bar{\Gamma}_{ijk} = \left[\Gamma_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i \partial \bar{u}^j}\right]\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

We consider next the Weingarten equations

$$\mathbf{N}_i = \beta^\alpha_i \mathbf{x}_\alpha, \qquad \alpha, i = 1,2 \qquad (10.27)$$

By taking the scalar product with $\mathbf{x}_j$, we find that the second fundamental coefficients b_{ij} satisfy

$$-b_{ij} = \mathbf{N}_i \cdot \mathbf{x}_j = \beta^\alpha_i \mathbf{x}_\alpha \cdot \mathbf{x}_j = \beta^\alpha_i g_{\alpha j}$$

If we define $b^j_i = b_{i\alpha}g^{\alpha j}$, it follows that

$$b^j_i = b_{i\gamma}g^{\gamma j} = -\beta^\alpha_i g_{\alpha\gamma}g^{\gamma j} = -\beta^\alpha_i \delta^j_\alpha = -\beta^j_i$$

Thus the Weingarten equations can be written as

$$\mathbf{N}_i = -b^\alpha_i \mathbf{x}_\alpha, \qquad i = 1,2 \qquad (10.28)$$

where the b^j_i and b_{ij} are related by

$$b^j_i = g^{\alpha j}b_{i\alpha} \quad \text{and} \quad b_{ij} = g_{\alpha j}b^\alpha_i \qquad (10.29)$$

Here the b_{ij} are the components of an absolute covariant tensor of rank 2 and the b^j_i are the components of an absolute tensor of rank 2 contravariant of order 1 and covariant of order 1.

We now define the *Riemann symbols of the second kind,*

$$R_{mijk} = b_{ik}b_{jm} - b_{ij}b_{km} \qquad (10.30)$$

and the associated *Riemann symbols of the first kind,*

$$R^p_{ijk} = g^{\alpha p}R_{\alpha ijk} \qquad (10.31)$$

We see that the R_{mijk} are the components of an absolute covariant tensor of rank 4 and that the R^p_{ijk} are the components of an absolute tensor of rank 4 covariant of order 3 and contravariant of order 1, called the *covariant Riemann curvature tensor* and *mixed Riemann curvature tensor* respectively. From equations (*10.29*) and (*10.31*) it follows that

$$R^p_{ijk} = g^{\alpha p}(b_{ik}b_{j\alpha} - b_{ij}b_{k\alpha}) = b_{ik}b^p_j - b_{ij}b^p_k \qquad (10.32)$$

Observe in (*10.30*) that the R_{mijk} are skew-symmetric with respect to the first two indices and the last two indices, i.e.,

$$R_{imjk} = -R_{mijk} \quad \text{and} \quad R_{mikj} = -R_{mijk} \qquad (10.33)$$

Thus $R_{imjk} = 0$ whenever the first two or last two indices are the same. Thus only four of the components are different from zero. They are

$$R_{1212} = R_{2121} = b_{22}b_{11} - b_{12}b_{21} = LN - M^2 = b \qquad (10.34)$$

and

$$R_{1221} = R_{2112} = b_{12}b_{21} - b_{22}b_{11} = -(LN - M^2) = -b \qquad (10.35)$$

Although the curvature tensors were defined in terms of the coefficients of the second fundamental form, they can in fact be expressed only in terms of the coefficients of the first fundamental form, i.e. the metric tensors, and their derivatives. In Problem 10.29, page 224, we prove

Theorem 10.9.
$$R_{mijk} = \frac{\partial}{\partial u^j}\Gamma_{ikm} - \frac{\partial}{\partial u^k}\Gamma_{ijm} + \Gamma^{\alpha}_{ij}\Gamma_{mk\alpha} - \Gamma^{\alpha}_{ik}\Gamma_{mj\alpha}$$

Since the Christoffel symbols depend only upon the metric tensors and their derivatives, it follows that the same is true for the curvature tensors. We note that this is equivalent to Gauss's theorem, since from equation (10.34) the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{b}{g} = \frac{R_{1212}}{g}$$

Solved Problems

THEORY OF SURFACES

10.1. Show that the Gauss-Weingarten equations for a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ are

$$g\mathbf{x}_{uu} = pr\mathbf{x}_u + qr\mathbf{x}_v + rg^{1/2}\mathbf{N}$$
$$g\mathbf{x}_{uv} = ps\mathbf{x}_u + qs\mathbf{x}_v + sg^{1/2}\mathbf{N}$$
$$g\mathbf{x}_{vv} = pt\mathbf{x}_u + qt\mathbf{x}_v + tg^{1/2}\mathbf{N}$$

$$g^{3/2}\mathbf{N}_u = (spq - rq^2 - r)\mathbf{x}_u + (rpq - sp^2 - s)\mathbf{x}_v$$
$$g^{3/2}\mathbf{N}_v = (tpq - sq^2 - s)\mathbf{x}_u + (spq - tp^2 - t)\mathbf{x}_v$$

where $p = f_u,\ q = f_v,\ r = f_{uu},\ s = f_{uv},\ t = f_{vv},\ g = 1 + p^2 + q^2$.

$$\mathbf{x}_u = \mathbf{e}_1 + p\mathbf{e}_3,\quad \mathbf{x}_v = \mathbf{e}_2 + q\mathbf{e}_3,\quad \mathbf{x}_{uu} = r\mathbf{e}_3,\quad \mathbf{x}_{uv} = s\mathbf{e}_3,\quad \mathbf{x}_{vv} = t\mathbf{e}_3$$
$$E = \mathbf{x}_u\cdot\mathbf{x}_u = 1 + p^2,\quad F = \mathbf{x}_u\cdot\mathbf{x}_v = pq,\quad G = \mathbf{x}_v\cdot\mathbf{x}_v = 1 + q^2$$
$$EG - F^2 = 1 + p^2 + q^2 = g,\quad \mathbf{N} = \mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = -(p\mathbf{e}_1 + q\mathbf{e}_2 - \mathbf{e}_3)/g^{1/2}$$
$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = r/g^{1/2},\quad M = \mathbf{x}_{uv}\cdot\mathbf{N} = s/g^{1/2},\quad N = \mathbf{x}_{vv}\cdot\mathbf{N} = t/g^{1/2}$$
$$E_u = 2pr,\quad E_v = 2ps,\quad F_u = ps + qr,\quad F_v = pt + qs,\quad G_u = 2qs,\quad G_v = 2qt$$

From equations (10.2) and (10.4) we obtain

$$\Gamma^1_{11} = pr/g \qquad \Gamma^1_{12} = ps/g \qquad \Gamma^1_{22} = pt/g$$
$$\Gamma^2_{11} = qr/g \qquad \Gamma^2_{12} = qs/g \qquad \Gamma^2_{22} = qt/g$$

$$\beta^1_1 = (spq - rq^2 - r)/g^{3/2} \qquad \beta^1_2 = (tpq - sq^2 - s)/g^{3/2}$$
$$\beta^2_1 = (rpq - sp^2 - s)/g^{3/2} \qquad \beta^2_2 = (spq - tp^2 - t)/g^{3/2}$$

from which the result follows.

10.2. Using the Weingarten equations, show that

$$\text{III} - 2H\text{II} + K\text{I} = 0$$

where the third fundamental form $\text{III} = d\mathbf{N}\cdot d\mathbf{N}$, H is the mean curvature and K is the Gaussian curvature.

Using equation *(10.2)*, page 201, we obtain

$$\begin{aligned}
\mathbf{N}_u\cdot\mathbf{N}_u &= (\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\cdot(\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\\
&= \frac{(MF-LG)^2E}{(EG-F^2)^2} + \frac{2(MF-LG)(LF-ME)F}{(EG-F^2)^2} + \frac{(LF-ME)^2G}{(EG-F^2)^2}\\
&= \frac{(-2LMF+L^2G+EM^2)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)L-(LN-M^2)E}{EG-F^2}\\
&= 2HL - KE
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbf{N}_u\cdot\mathbf{N}_v &= (\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\cdot(\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\\
&= \frac{(MF-LG)(NF-MG)E}{(EG-F^2)^2} + \frac{(NF-MG)(LF-ME)F}{(EG-F^2)^2}\\
&\quad + \frac{(MF-LG)(MF-NE)F}{(EG-F^2)^2} + \frac{(LF-ME)(MF-NE)G}{(EG-F^2)^2}\\
&= \frac{(MEN-M^2F+LGM-FLN)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)M}{EG-F^2} - \frac{(LN-M^2)F}{EG-F^2}\\
&= 2HM - KF
\end{aligned}$$

Also

$$\begin{aligned}
\mathbf{N}_v\cdot\mathbf{N}_v &= (\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\cdot(\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\\
&= \frac{(NF-MG)^2E}{(EG-F^2)^2} + \frac{2(NF-MG)(MF-NE)F}{(EG-F^2)^2} + \frac{(MF-NE)^2G}{(EG-F^2)^2}\\
&= \frac{(EN^2-2MFN+M^2G)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)N}{EG-F^2} - \frac{(LN-M^2)G}{EG-F^2}\\
&= 2HN - KG
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{III} &= d\mathbf{N}\cdot d\mathbf{N} = (\mathbf{N}_u\,du + \mathbf{N}_v\,dv)\cdot(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) = \mathbf{N}_u\cdot\mathbf{N}_u\,du^2 + 2\mathbf{N}_u\cdot\mathbf{N}_v\,du\,dv + \mathbf{N}_u\cdot\mathbf{N}_v\,dv^2\\
&= (2HL-KE)\,du^2 + 2(2HM-KF)\,du\,dv + (2HN-KG)\,dv^2\\
&= 2H(L\,du^2 + 2M\,du\,dv + N\,dv^2) - K(E\,du^2 + 2F\,du\,dv + G\,dv^2) = 2H\text{II} - K\text{I}
\end{aligned}$$

which gives the required result.

10.3. Prove that the Christoffel symbols Γ_{ij}^k are given by equations *(10.4)*, page 202.

Observe that

$$\mathbf{x}_u\cdot\mathbf{x}_{uu} = \tfrac{1}{2}(\mathbf{x}_u\cdot\mathbf{x}_u)_u = \tfrac{1}{2}E_u,\quad \mathbf{x}_u\cdot\mathbf{x}_{uv} = \tfrac{1}{2}(\mathbf{x}_u\cdot\mathbf{x}_u)_v = \tfrac{1}{2}E_v$$
$$\mathbf{x}_v\cdot\mathbf{x}_{vv} = \tfrac{1}{2}(\mathbf{x}_v\cdot\mathbf{x}_v)_v = \tfrac{1}{2}G_v,\quad \mathbf{x}_v\cdot\mathbf{x}_{uv} = \tfrac{1}{2}(\mathbf{x}_v\cdot\mathbf{x}_v)_u = \tfrac{1}{2}G_u$$

Also using the above,

$$F_u = (\mathbf{x}_u\cdot\mathbf{x}_v)_u = \mathbf{x}_{uu}\cdot\mathbf{x}_v + \mathbf{x}_u\cdot\mathbf{x}_{uv} = \mathbf{x}_{uu}\cdot\mathbf{x}_v + \tfrac{1}{2}E_v$$
$$F_v = (\mathbf{x}_u\cdot\mathbf{x}_v)_v = \mathbf{x}_{uv}\cdot\mathbf{x}_v + \mathbf{x}_u\cdot\mathbf{x}_{vv} = \tfrac{1}{2}G_u + \mathbf{x}_u\cdot\mathbf{x}_{vv}$$

Hence

$$\mathbf{x}_v\cdot\mathbf{x}_{uu} = F_u - \tfrac{1}{2}E_v,\quad \mathbf{x}_u\cdot\mathbf{x}_{vv} = F_v - \tfrac{1}{2}G_u$$

Now from the Gauss equations and the above,

$$\begin{aligned}
\tfrac{1}{2}E_u &= \mathbf{x}_u\cdot\mathbf{x}_{uu} = \Gamma_{11}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{11}^1E + \Gamma_{11}^2F\\
F_u - \tfrac{1}{2}E_v &= \mathbf{x}_v\cdot\mathbf{x}_{uu} = \Gamma_{11}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{11}^1F + \Gamma_{11}^2G\\
\tfrac{1}{2}E_v &= \mathbf{x}_u\cdot\mathbf{x}_{uv} = \Gamma_{12}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{12}^1E + \Gamma_{12}^2F\\
\tfrac{1}{2}G_u &= \mathbf{x}_v\cdot\mathbf{x}_{uv} = \Gamma_{12}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{12}^1F + \Gamma_{12}^2G\\
F_v - \tfrac{1}{2}G_u &= \mathbf{x}_u\cdot\mathbf{x}_{vv} = \Gamma_{22}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{22}^1E + \Gamma_{22}^2F\\
\tfrac{1}{2}G_v &= \mathbf{x}_v\cdot\mathbf{x}_{vv} = \Gamma_{22}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{22}^1F + \Gamma_{22}^2G
\end{aligned}$$

Solving the first two equations for Γ_{11}^1 and Γ_{11}^2, the second two equations for Γ_{12}^1 and Γ_{12}^2, and the last two equations for Γ_{22}^1 and Γ_{22}^2, we obtain

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} \qquad \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)} \qquad \Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v + FE_u}{2(EG-F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

which are the required results.

10.4. Prove that $K(EG-F^2)^2 = [\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2$.

$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = \mathbf{x}_{uu}\cdot\mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = [\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|$$

$$M = \mathbf{x}_{uv}\cdot\mathbf{N} = [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|, \qquad N = \mathbf{x}_{vv}\cdot\mathbf{N} = [\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|$$

Thus
$$LN - M^2 = \frac{[\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2}{|\mathbf{x}_u\times\mathbf{x}_v|^2}$$

Also
$$|\mathbf{x}_u\times\mathbf{x}_v|^2 = (\mathbf{x}_u\times\mathbf{x}_v)\cdot(\mathbf{x}_u\times\mathbf{x}_v) = (\mathbf{x}_u\cdot\mathbf{x}_u)(\mathbf{x}_v\cdot\mathbf{x}_v) - (\mathbf{x}_u\cdot\mathbf{x}_v)^2 = EG - F^2$$

Hence
$$K = \frac{LN-M^2}{EG-F^2} = \frac{[\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2}{(EG-F^2)^2}$$

10.5. Using the result of the above problem, prove that

$$K(EG-F^2)^2 = (F_{uv} - \tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu})(EG-F^2) + \det\begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

Note that this is a direct proof of Gauss' Theorem.

We note that

$$[\mathbf{abc}][\mathbf{def}] = \det\begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{pmatrix} \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= \det\left(\begin{bmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{bmatrix}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right) = \det\begin{pmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{d} \\ \mathbf{a}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{e} \\ \mathbf{a}\cdot\mathbf{f} & \mathbf{b}\cdot\mathbf{f} & \mathbf{c}\cdot\mathbf{f} \end{pmatrix}$$

Thus from Problem 10.4 and the computations from Problem 10.3, we have

$$K(EG-F^2)^2 = \det\begin{pmatrix} \mathbf{x}_{uu}\cdot\mathbf{x}_{vv} & \mathbf{x}_u\cdot\mathbf{x}_{vv} & \mathbf{x}_v\cdot\mathbf{x}_{vv} \\ \mathbf{x}_{uu}\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_u \\ \mathbf{x}_{uu}\cdot\mathbf{x}_v & \mathbf{x}_u\cdot\mathbf{x}_v & \mathbf{x}_v\cdot\mathbf{x}_v \end{pmatrix} - \det\begin{pmatrix} \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} & \mathbf{x}_u\cdot\mathbf{x}_{uv} & \mathbf{x}_v\cdot\mathbf{x}_{uv} \\ \mathbf{x}_{uv}\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_u \\ \mathbf{x}_{uv}\cdot\mathbf{x}_v & \mathbf{x}_u\cdot\mathbf{x}_v & \mathbf{x}_v\cdot\mathbf{x}_v \end{pmatrix}$$

$$= \det\begin{pmatrix} \mathbf{x}_{uu}\cdot\mathbf{x}_{vv} & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

Since both determinants have the common minor $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, it follows that

$$K(EG-F^2)^2 = (\mathbf{x}_{uu}\cdot\mathbf{x}_{vv} - \mathbf{x}_{uv}\cdot\mathbf{x}_{uv})(EG-F^2) + \det\begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

In Problem 10.3 we obtained $\mathbf{x}_{uu}\cdot\mathbf{x}_v = F_u - \frac{1}{2}E_v$ and $\mathbf{x}_{uv}\cdot\mathbf{x}_v = \frac{1}{2}G_u$. Hence

$$(F_u - \tfrac{1}{2}E_v)_v = (\mathbf{x}_{uu}\cdot\mathbf{x}_v)_v = \mathbf{x}_{uuv}\cdot\mathbf{x}_v + \mathbf{x}_{uu}\cdot\mathbf{x}_{vv}, \qquad (\tfrac{1}{2}G_u)_u = (\mathbf{x}_{uv}\cdot\mathbf{x}_v)_u = \mathbf{x}_{uvu}\cdot\mathbf{x}_v + \mathbf{x}_{uv}\cdot\mathbf{x}_{uv}$$

Subtracting,

$$\mathbf{x}_{uu}\cdot\mathbf{x}_{vv} - \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} = (F_u - \tfrac{1}{2}E_v)_v - \tfrac{1}{2}G_{uu} = F_{uv} - \tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu}$$

which gives the required result.

10.6. If the parameter curves on a patch are lines of curvature, show that the Codazzi-Mainardi equations (*10.7*) take the form

$$\frac{\partial\kappa_1}{\partial v} = \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1), \qquad \frac{\partial\kappa_2}{\partial u} = \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)$$

where κ_1 and κ_2 are the principal curvatures.

When the parameter curves are lines of curvature, $F = M = 0$. Thus equations (*10.7*) reduce to

$$L_v = L\Gamma^1_{12} - N\Gamma^2_{11} = \frac{LGE_v}{2EG} + \frac{NEE_v}{2EG} = \tfrac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right)$$

and

$$N_u = -L\Gamma^1_{22} + N\Gamma^2_{12} = \frac{LGG_u}{2EG} + \frac{NEG_u}{2EG} = \tfrac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right)$$

or

$$\left(\frac{L}{E}\right)_v = \frac{E_v}{2E}\left(\frac{N}{G} - \frac{L}{E}\right) \quad \text{and} \quad \left(\frac{N}{G}\right)_u = \frac{G_u}{2G}\left(\frac{L}{E} - \frac{N}{G}\right)$$

But from Theorem 9.13, page 186, if the parameter curves are lines of curvature, $\kappa_1 = L/E$ and $\kappa_2 = N/G$, which gives the required result.

10.7. Prove that there does not exist a compact surface in E^3 of class $\geqq 2$ with Gaussian curvature $K \leqq 0$.

Suppose otherwise, i.e. suppose S is a compact surface of class $\geqq 2$ with $K \leqq 0$ at each point. Now consider the real-valued function $f(P) = |\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$, where $\mathbf{x}$ is the point P. We leave as an exercise for the reader to show that f is continuous on S. Hence from Theorem 6.9, page 110, f takes on its maximum, say, $f(P_0) = |\mathbf{x}_0|^2 = r^2$ at some point P_0 on S. Note that $r^2 > 0$. Otherwise, we would have $f \equiv 0$ on S, since $f \geqq 0$ and r^2 is its maximum. But then S would consist of a single point $\mathbf{x} = \mathbf{0}$, which is impossible. Now let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on S containing P_0 such that the u and v parameter curves have principal directions at P_0. Since $f(P) = f(\mathbf{x}(u,v))$ has a maximum at P_0,

$$\partial f/\partial u = 2\mathbf{x}\cdot\mathbf{x}_u = 0 \quad \text{and} \quad \partial f/\partial v = 2\mathbf{x}\cdot\mathbf{x}_v = 0$$

at P_0. Also

$$\partial^2 f/\partial u^2 = 2\mathbf{x}_u\cdot\mathbf{x}_u + 2\mathbf{x}\cdot\mathbf{x}_{uu} \leqq 0 \quad \text{and} \quad \partial^2 f/\partial v^2 = 2\mathbf{x}_v\cdot\mathbf{x}_v + 2\mathbf{x}\cdot\mathbf{x}_{vv} \leqq 0$$

at P_0. From the first two equations above we obtain that $\mathbf{x}$ is orthogonal to $\mathbf{x}_u$ and $\mathbf{x}_v$ at P_0. Hence $\mathbf{N} = \pm\mathbf{x}/|\mathbf{x}| = \pm\mathbf{x}/r$ at P_0. We may assume that the sense of $\mathbf{N}$ is such that $\mathbf{N} = \mathbf{x}/r$. Substituting into the second two equations above, we obtain that $\mathbf{x}_u\cdot\mathbf{x}_u + r\mathbf{N}\cdot\mathbf{x}_{uu} \leqq 0$ and $\mathbf{x}_v\cdot\mathbf{x}_v + r\mathbf{N}\cdot\mathbf{x}_{vv} \leqq 0$, or $E + rL \leqq 0$ and $G + rN \leqq 0$, or $L/E \leqq -1/r < 0$ and $N/G \leqq -1/r < 0$ at P_0. Since the u and v parameter curves have principal directions at P_0, it follows from Theorem 9.11, page 185, that $\kappa_1 = L/E$ and $\kappa_2 = N/G$, and hence from the above $K = \kappa_1\kappa_2 = LN/EG \geqq 1/r^2 > 0$ at P_0, which is again impossible since $K \leqq 0$ on S. Thus the proposition is proved.

10.8. Prove that $f(P) = [\kappa_1(P) - \kappa_2(P)]^2$ is a continuous function on a surface.

Recall that the principal curvatures at a point P on a surface S depends on the orientation of the patch containing P, changing sign when there is a change in the sense of $\mathbf{N}$. Thus unless S is orientable, it may not be possible to define $\kappa_1(P)$ and $\kappa_2(P)$ themselves as continuous functions throughout S. Note however that f is independent of a change in sign of both κ_1 and κ_2 and hence is an intrinsic property of S, independent of the patch containing P.

To prove that f is continuous at a point P_0, we suppose that $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P_0. Since κ_1 and κ_2 are continuous functions of the first and second fundamental coefficients, $f(P) = f(\mathbf{x}(u,v))$ is a continuous function of u and v. Thus given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|f(\mathbf{x}(u,v)) - f(\mathbf{x}(u_0,v_0))| < \epsilon$ for (u,v) in $S_{\delta_1}(u_0,v_0)$. From Problem 8.13, page 165, the image M of $S_{\delta_1}(u_0,v_0)$ on S is the intersection of an open set O in E^3 with S. It follows that there exists an $S_\delta(\mathbf{x}_0)$ in E^3 such that $S_\delta(\mathbf{x}_0) \cap S$ is contained in M. Thus for $\mathbf{x}$ in $S_\delta(\mathbf{x}_0) \cap S$ we have $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. This shows that f is continuous at P_0, which proves the proposition.

10.9. Prove *Hilbert's lemma*: If at a point P_0 on a surface of sufficiently high class, (i) $\kappa_1(P_0)$ is a local maximum, (ii) $\kappa_2(P_0)$ local minimum, (iii) $\kappa_1(P_0) > \kappa_2(P_0)$, then $K(P_0) \leqq 0$.

Since $\kappa_1(P_0) \neq \kappa_2(P_0)$, P_0 is not an umbilical point. Thus from Theorem 9.10, page 185, there exists a patch $\mathbf{x} = \mathbf{x}(u,v)$ containing P_0 for which the parameter curves are lines of curvatures. From Problem 10.6 it follows that

$$\frac{\partial \kappa_1}{\partial v} = \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial u} = \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)$$

Differentiating,

$$\frac{\partial^2 \kappa_1}{\partial v^2} = \frac{1}{2}\left(\frac{EE_{vv} - E_v^2}{E^2}\right)(\kappa_2 - \kappa_1) + \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1)_v$$

$$\frac{\partial^2 \kappa_2}{\partial u^2} = \frac{1}{2}\left(\frac{GG_{uu} - G_u^2}{G^2}\right)(\kappa_1 - \kappa_2) + \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)_u$$

Since κ_1 and κ_2 are extreme values at P_0, $\partial\kappa_1/\partial v = \partial\kappa_2/\partial u = 0$ at P_0. Also $\kappa_1 \neq \kappa_2$ at P_0. Thus from the first two equations above, $E_v = G_u = 0$ at P_0. Substituting into the second two equations above, it follows that

$$\frac{\partial^2 \kappa_1}{\partial v^2} = \frac{1}{2}\frac{E_{vv}}{E}(\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial u^2} = \frac{1}{2}\frac{G_{uu}}{G}(\kappa_1 - \kappa_2)$$

Since κ_1 is a maximum at P_0, $\partial^2\kappa_1/\partial v^2 \leqq 0$ at P_0. Also $\kappa_1 > \kappa_2$ at P_0 and $E > 0$. Hence from the first equation above, $E_{vv} \geqq 0$ at P_0. Since κ_2 is a minimum at P_0, $\partial^2\kappa_2/\partial u^2 \geqq 0$ at P_0. Also $G > 0$. Hence $G_{uu} \geqq 0$ at P_0. Finally, since the parameter curves are lines of curvature, we have $F = M = 0$. Also at P_0, $E_v = 0$ and $G_u = 0$. From Problem 10.5 it follows that at P_0

$$K = -\frac{1}{2}\frac{E_{vv} + G_{uu}}{EG}$$

Since $E_{vv} \geqq 0$ and $G_{uu} \geqq 0$ it follows that $K \leqq 0$, which is the required result.

10.10. Prove Theorem 10.7: The only connected and compact surfaces of sufficiently high class with constant Gaussian curvature are spheres.

Suppose S is a connected and compact surface with $K =$ constant. From Problem 10.7, not every point on S can have $K \leqq 0$. Hence we can assume $K =$ constant > 0. Now if we can show that every point on S is a spherical umbilical point, then it would follow from Theorem 10.5, page 205, that S is a sphere and we are finished. In order to show that every point on S is a spherical umbilical point, we consider the function $f(P) = [(\kappa_1(P) - \kappa_2(P)]^2$. From Problem 10.8, $f(P)$ is continuous on S. Since S is compact, f takes on an absolute maximum at some point P_0 on S. Now suppose $f > 0$ at P_0. Since f is continuous at P_0, $f > 0$ in some neighborhood $S(P_0)$. Since $f = (\kappa_1 - \kappa_2)^2 > 0$ in $S(P_0)$, $\kappa_1 \neq \kappa_2$ in $S(P_0)$. Also κ_1 and κ_2 have the same sign in $S(P_0)$ since $K = \kappa_1\kappa_2 > 0$ in $S(P_0)$. Thus we can assume $\kappa_1 > \kappa_2 > 0$ in $S(P_0)$. Since $\kappa_1 - \kappa_2 > 0$ in $S(P_0)$ and $(\kappa_1 - \kappa_2)^2$ has a maximum at P_0, it follows that $\kappa_1 - \kappa_2$ has a local maximum at P_0. Since $K = \kappa_1\kappa_2 =$ constant > 0, κ_2 decreases when κ_1 increases and it follows that κ_1 has a local maximum at P_0 and κ_2 has a local minimum at P_0. Thus we see that if $f > 0$ at P_0, then (i) κ_1 has a local maximum at P_0, (ii) κ_2 has a local minimum at P_0 and (iii) $\kappa_1 > \kappa_2$ at P_0. It follows from Problem 10.9 that $K \leqq 0$ at P_0. But this is impossible since $K > 0$ on S. Thus f is not positive at P_0. But f takes on its maximum at P_0 and $f(P) \geqq 0$ for all P. Hence $f \equiv 0$ on S. It follows that $\kappa_1 = \kappa_2$ at each P on S. Since the principal curvatures are extreme values of the normal curvature at P and since $K > 0$, it follows that the normal curvature $\kappa =$ constant $\neq 0$ at each P. Namely, every point on S is a spherical umbilical point and hence S is a sphere.

TENSORS

10.11. If $v^i = a^i_\alpha u^\alpha$ and $w^i = b^i_\alpha v^\alpha$, show that $w^i = b^i_\alpha a^\alpha_\beta u^\beta$.

We write $v^\alpha = \sum_\beta a^\alpha_\beta u^\beta$. Hence

$$w^i = \sum_\alpha b^i_\alpha v^\alpha = \sum_\alpha b^i_\alpha \sum_\beta a^\alpha_\beta u^\beta = \sum_\alpha \sum_\beta b^i_\alpha a^\alpha_\beta u^\beta = b^i_\alpha a^\alpha_\beta u^\beta$$

10.12. Show that $g_{i\alpha} g^{\alpha j} = \delta^j_i$, $\alpha, i, j = 1, 2$, where the $g^{\alpha j}$ are defined in equation *(10.11)*.

$$g_{1\alpha} g^{\alpha 1} = g_{11} g^{11} + g_{12} g^{21} = g_{11} g_{22}/g - g_{12} g_{12}/g = g/g = 1$$

$$g_{1\alpha} g^{\alpha 2} = g_{11} g^{12} + g_{12} g^{22} = -g_{11} g_{12}/g + g_{12} g_{11}/g = 0$$

$$g_{2\alpha} g^{\alpha 1} = g_{21} g^{11} + g_{22} g^{21} = g_{21} g_{22}/g - g_{22} g_{12}/g = 0$$

$$g_{2\alpha} g^{\alpha 2} = g_{21} g^{12} + g_{22} g^{22} = -g_{21} g_{21}/g + g_{22} g_{11}/g = g/g = 1$$

Hence

$$g_{i\alpha} g^{\alpha j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \delta^j_i$$

10.13. Show that $\delta^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$, where the δ^{pq}_{ij} are defined in Example 10.7(*b*), page 212.

Let $A^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$. Clearly $A^{pq}_{ij} = 0$ if $i = j$. Now suppose $i \neq j$ and $p \neq i$. Then $\delta^p_i = 0$ and $A^{pq}_{ij} = 0$ unless $p = j$ and $q = i$, in which case $A^{pq}_{ij} = -1$. If $i \neq j$ and $p = i$, then $\delta^p_j = 0$. Hence $A^{pq}_{ij} = 0$ unless also $q = j$, in which case $A^{pq}_{ij} = 1$. Thus

$$A^{pq}_{ij} = \left\{ \begin{array}{rl} 1 & \text{if } i \neq j,\ p = i,\ q = j \\ -1 & \text{if } i \neq j,\ q = i,\ p = j \\ 0 & \text{otherwise} \end{array} \right\} = \delta^{pq}_{ij}$$

which is the required result.

10.14. Show that the δ^{pq}_{ij} in the above problem are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

$$\begin{aligned}
\delta^{\gamma\sigma}_{\alpha\beta} \frac{\partial \bar{u}^p}{\partial u^\gamma} \frac{\partial \bar{u}^q}{\partial u^\sigma} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} &= (\delta^\gamma_\alpha \delta^\sigma_\beta - \delta^\gamma_\beta \delta^\sigma_\alpha) \frac{\partial \bar{u}^p}{\partial u^\gamma} \frac{\partial \bar{u}^q}{\partial u^\sigma} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \\
&= \left(\delta^\gamma_\alpha \frac{\partial \bar{u}^p}{\partial u^\gamma}\right)\left(\delta^\sigma_\beta \frac{\partial \bar{u}^q}{\partial u^\sigma}\right)\left(\frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) - \left(\delta^\gamma_\beta \frac{\partial \bar{u}^p}{\partial u^\gamma}\right)\left(\delta^\sigma_\alpha \frac{\partial \bar{u}^q}{\partial u^\sigma}\right)\left(\frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) \\
&= \frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} - \frac{\partial \bar{u}^p}{\partial u^\beta} \frac{\partial \bar{u}^q}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \\
&= \left(\frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i}\right)\left(\frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) - \left(\frac{\partial \bar{u}^p}{\partial u^\beta} \frac{\partial u^\beta}{\partial \bar{u}^j}\right)\left(\frac{\partial \bar{u}^q}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i}\right) \\
&= \delta^p_i \delta^q_j - \delta^p_j \delta^q_i = \delta^{pq}_{ij} = \bar{\delta}^{pq}_{ij}
\end{aligned}$$

Thus δ^{pq}_{ij} are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

Another Method. The products $\delta^p_i \delta^q_j$ and $\delta^p_j \delta^q_i$ are the outer products of mixed absolute tensors covariant of order 1 and contravariant of order 1. Hence they are the components of absolute tensors covariant of order 2 and contravariant of order 2. It follows that the differences

$$\delta^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$$

are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

10.15. Show that the contraction A_α^α of the components A_i^j of an absolute mixed tensor is a scalar invariant.

Since $\bar{A}_i^j = A_\alpha^\beta \dfrac{\partial u^\alpha}{\partial \bar{u}^i}\dfrac{\partial \bar{u}^j}{\partial u^\beta}$, we have $\bar{A}_\gamma^\gamma = A_\alpha^\beta \dfrac{\partial u^\alpha}{\partial \bar{u}^\gamma}\dfrac{\partial \bar{u}^\gamma}{\partial u^\beta}$. From equation *(10.16)*, page 208, $\dfrac{\partial u^\alpha}{\partial \bar{u}^\gamma}\dfrac{\partial \bar{u}^\gamma}{\partial u^\beta} = \delta_\beta^\alpha$. Hence $\bar{A}_\gamma^\gamma = A_\alpha^\beta\,\delta_\beta^\alpha = A_\alpha^\alpha$, which is the required result.

10.16. If $\bar{A}_{ij}^{pq}$ are the components of a tensor contravariant of order 2, covariant of order 2, and of weight N, show that the contraction $A_{\alpha j}^{\alpha q}$ are the components of a mixed tensor contravariant of order 1, covariant of order 1, and of weight N.

Since $\bar{A}_{ij}^{pq} = \left[\det\left(\dfrac{\partial u_i}{\partial \bar{u}_j}\right)\right]^N A_{\alpha\beta}^{\gamma\sigma}\dfrac{\partial \bar{u}^p}{\partial u^\gamma}\dfrac{\partial \bar{u}^q}{\partial u^\sigma}\dfrac{\partial u^\alpha}{\partial \bar{u}^i}\dfrac{\partial u^\beta}{\partial \bar{u}^j}$, we have

$$\begin{aligned}\bar{A}_{\eta j}^{\eta q} &= \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A_{\alpha\beta}^{\gamma\sigma}\frac{\partial \bar{u}^\eta}{\partial u^\gamma}\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\alpha}{\partial \bar{u}^\eta}\frac{\partial u^\beta}{\partial \bar{u}^j}\\ &= \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A_{\alpha\beta}^{\gamma\sigma}\delta_\gamma^\alpha\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\beta}{\partial \bar{u}^j} = \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A_{\alpha\beta}^{\alpha\sigma}\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\beta}{\partial \bar{u}^j}\end{aligned}$$

That is, the $A_{\alpha j}^{\alpha q}$ transform as an absolute mixed tensor of weight N, which is the required result.

10.17. If $\bar{u}^i = \bar{u}^i(u^1,\ldots,u^n)$, $i = 1,\ldots,n$, is an allowable coordinate transformation on a coordinate manifold of n dimensions and $u^i = u^i(\bar{u}^1,\ldots,\bar{u}^n)$ is its inverse, show that $\dfrac{\partial \bar{u}^j}{\partial u^\alpha}\dfrac{\partial u^\alpha}{\partial \bar{u}^i} = \delta_i^j$.

From the chain rule,

$$\frac{\partial \bar{u}^j}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^1}\frac{\partial u^1}{\partial \bar{u}^i} + \frac{\partial u^j}{\partial u^2}\frac{\partial u^2}{\partial \bar{u}^i} + \cdots + \frac{\partial \bar{u}^j}{\partial u^n}\frac{\partial u^n}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}$$

But $\dfrac{\partial \bar{u}^j}{\partial \bar{u}^i} = \begin{Bmatrix}1 & \text{if } i = j\\ 0 & \text{otherwise}\end{Bmatrix} = \delta_i^j$. Thus $\dfrac{\partial \bar{u}^j}{\partial u^\alpha}\dfrac{\partial u^\alpha}{\partial \bar{u}^i} = \delta_i^j$.

10.18. If $A_{j_1\cdots j_s}^{i_1\cdots i_r}$ and $B_{j_1\cdots j_s}^{i_1\cdots i_r}$ are the components of two tensors A and B, contravariant and covariant of the same orders and of the same weight, show that

$$C_{j_1\cdots j_s}^{i_1\cdots i_r} = A_{j_1\cdots j_s}^{i_1\cdots i_r} + B_{j_1\cdots j_s}^{i_1\cdots i_r}$$

are the components of a tensor, contravariant and covariant of the same orders and of the same weight as A and B.

$$\begin{aligned}\bar{C}_{j_1\cdots j_s}^{i_1\cdots i_r} &= \bar{A}_{j_1\cdots j_s}^{i_1\cdots i_r} + \bar{B}_{j_1\cdots j_s}^{i_1\cdots i_r}\\ &= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} + \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N B_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}\\ &= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N [A_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r} + B_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}]\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}\\ &= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N C_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}\end{aligned}$$

which is the required result.

10.19. If the components $A_{j_1\cdots j_s}^{i_1\cdots i_r}$ of a tensor are symmetric with respect to, say, the i_1 and i_2 indices, show that the transformed components

$$\bar{A}_{j_1\cdots j_s}^{i_1\cdots i_r} = \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

are also symmetric with respect to i_1 and i_2.

$$\bar{A}^{i_2 i_1 \cdots i_r}_{j_1 \cdots j_s} = \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \frac{\partial \bar{u}^{i_2}}{\partial u^{\alpha_1}} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_2}} \cdots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

$$= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_2 \alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_2}} \frac{\partial \bar{u}^{i_2}}{\partial u^{\alpha_1}} \cdots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} = \bar{A}^{i_1 i_2 \cdots i_r}_{j_1 \cdots j_s}$$

10.20. If the quantity $J = C^{ij}A_iB_j$ is a scalar invariant with respect to the components A_i and B_j of any two covariant vectors, show that the C^{ij} are the components of an absolute contravariant tensor of rank 2.

It is given that the sum $C^{ij}A_iB_j = \bar{C}^{\alpha\beta}\bar{A}_\alpha \bar{B}_\beta$ for any A_i and B_j. Hence

$$C^{ij}A_iB_j = \bar{C}^{\alpha\beta}\bar{A}_\alpha\bar{B}_\beta = \bar{C}^{\alpha\beta}A_\gamma \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} B_\sigma \frac{\partial u^\sigma}{\partial \bar{u}^\beta} = \bar{C}^{\alpha\beta} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta} A_\gamma B_\sigma$$

Identifying coefficients gives $C^{ij} = \bar{C}^{\alpha\beta}\frac{\partial u^i}{\partial \bar{u}^\alpha}\frac{\partial u^j}{\partial \bar{u}^\beta}$ which shows that the C^{ij} are the components of an absolute tensor of rank 2.

10.21. If A_{ij} and B_{ij} are the components of symmetric tensors, and if x_i and y_i are components of contravariant vectors such that

$$(A_{ij} - \kappa_1 B_{ij})x^i = 0, \quad (A_{ij} - \kappa_2 B_{ij})y^i = 0, \qquad i, j = 1, \ldots, n, \ \kappa_1 \neq \kappa_2$$

show that $A_{ij}x^ix^j = B_{ij}x^ix^j = 0$ and that κ_1 is a scalar invariant.

Since $(A_{ij} - \kappa_1 B_{ij})x^i = 0$ for all j, we have $(A_{ij} - \kappa_1 B_{ij})x^iy^j = 0$. Similarly from the second equation $(A_{ij} - \kappa_2 B_{ij})y^ix^j = 0$, or, since the A_{ij} and B_{ij} are symmetric, $(A_{ij} - \kappa_2 B_{ij})x^iy^j = 0$. Subtracting, we obtain $(\kappa_1 - \kappa_2)B_{ij}x^iy^j = 0$. Since $\kappa_1 \neq \kappa_2$, it follows that $B_{ij}x^iy^j = 0$ and hence $A_{ij}x^iy^j = 0$. To show that κ_1 is a scalar invariant, we suppose that $\bar{z}^i$ are the components of an arbitrary contravariant vector and we consider the sum

$$(\bar{A}_{ij} - \kappa_1 \bar{B}_{ij})\bar{x}^i\bar{z}^j = (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} x^\gamma \frac{\partial \bar{u}^i}{\partial u^\gamma} z^\sigma \frac{\partial \bar{u}^j}{\partial u^\sigma}$$

$$= (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\gamma z^\sigma \left(\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial \bar{u}^i}{\partial u^\gamma}\right)\left(\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial \bar{u}^j}{\partial u^\sigma}\right)$$

$$= (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\gamma z^\sigma \delta^\alpha_\gamma \delta^\beta_\sigma = (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\alpha z^\beta$$

Since $(A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\alpha = 0$ for all β, $(\bar{A}_{ij} - \kappa_1\bar{B}_{ij})\bar{x}^i\bar{z}^j = 0$. But the $\bar{z}^i$ are arbitrary. Hence $(\bar{A}_{ij} - \kappa_1\bar{B}_{ij})\bar{x}^i = 0$ for all j. Thus κ_1 is a scalar invariant.

APPLICATIONS OF TENSORS

10.22. Show that the components du^i, $i = 1, 2$, of a tangent vector $d\mathbf{x} = \mathbf{x}_\alpha du^\alpha$ transform as the components of a contravariant vector. They are called the *contravariant components* of $d\mathbf{x}$.

Suppose $\bar{u}^i = \bar{u}^i(u^1, u^2)$, $i = 1, 2$, is an allowable parameter transformation with inverse $u^i = u^i(\bar{u}^1, \bar{u}^2)$, $i = 1, 2$. Then from the chain rule, $\mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial u^\alpha} = \frac{\partial \mathbf{x}}{\partial \bar{u}^1}\frac{\partial \bar{u}^1}{\partial u^\alpha} + \frac{\partial \mathbf{x}}{\partial \bar{u}^2}\frac{\partial \bar{u}^2}{\partial u^\alpha} = \frac{\partial \mathbf{x}}{\partial \bar{u}^i}\frac{\partial \bar{u}^i}{\partial u^\alpha}$. It follows that $d\mathbf{x} = \mathbf{x}_\alpha\, du^\alpha = \frac{\partial \mathbf{x}}{\partial \bar{u}^i}\frac{\partial \bar{u}^i}{\partial u^\alpha}du^\alpha = \frac{\partial \mathbf{x}}{\partial \bar{u}^i}d\bar{u}^i$. Hence $d\bar{u}^i = du^\alpha \frac{\partial \bar{u}^i}{\partial u^\alpha}$, which is the required result.

10.23. Show that $\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$.

Differentiating $g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ with respect to u^k, $\partial g_{ij}/\partial u^k = \mathbf{x}_{ik}\cdot\mathbf{x}_j + \mathbf{x}_i\cdot\mathbf{x}_{jk}$. By definition, $\Gamma_{ijk} = \mathbf{x}_{ij}\cdot\mathbf{x}_k$. Hence $\partial g_{ij}/\partial u^k = \Gamma_{ikj} + \Gamma_{jki}$.

10.24. Show that $\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right]$.

From the above problem, $\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{jik} + \Gamma_{kij}$, $\frac{\partial g_{ki}}{\partial u^j} = \Gamma_{kji} + \Gamma_{ijk}$, and $\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$. Since $\Gamma_{ijk} = \Gamma_{jik}$ for all i, j, k, it follows that $\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2\Gamma_{ijk}$.

10.25. Show that $\frac{\partial g}{\partial u^i} = 2g\Gamma^\alpha_{\alpha i}$.

$$\begin{aligned}\frac{\partial g}{\partial u^i} &= \frac{\partial}{\partial u^i}(g_{11}g_{22} - (g_{12})^2) = \frac{\partial g_{11}}{\partial u^i}g_{22} + g_{11}\frac{\partial g_{22}}{\partial u^i} - 2g_{12}\frac{\partial g_{12}}{\partial u^i} \\ &= g\left[g^{11}\frac{\partial g_{11}}{\partial u^i} + g^{22}\frac{\partial g_{22}}{\partial u^i} + 2g^{12}\frac{\partial g_{12}}{\partial u^i}\right] = gg^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial u^i}\end{aligned}$$

where we used equation *(10.11)*, page 206. From Problem 10.23,

$$\frac{\partial g}{\partial u^i} = gg^{\alpha\beta}(\Gamma_{\alpha i\beta} + \Gamma_{\beta i\alpha}) = g(g^{\alpha\beta}\Gamma_{\alpha i\beta} + g^{\alpha\beta}\Gamma_{\beta i\alpha}) = g(\Gamma^\alpha_{\alpha i} + \Gamma^\beta_{\beta i}) = 2g\Gamma^\alpha_{\alpha i}$$

where we used $\Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij\alpha}$, and where we replaced the dummy index β by α.

10.26. Show that $R_{mijk} = g_{\alpha m}R^\alpha_{ijk}$.

From *(10.31)*, page 214, $g_{\alpha m}R^\alpha_{ijk} = g_{\alpha m}g^{\beta\alpha}R_{\beta ijk} = \delta^\beta_m R_{\beta ijk} = R_{mijk}$.

10.27. Show that the Christoffel symbols of the first kind transform in accordance with the law

$$\bar{\Gamma}_{ijk} = \left\{\Gamma_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\right\}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

We recall that the g_{jk} are the components of a covariant tensor of rank 2. Thus $\bar{g}_{jk} = g_{\beta\gamma}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k}$. Differentiating with respect to $\bar{u}^i$,

$$\begin{aligned}\frac{\partial \bar{g}_{jk}}{\partial \bar{u}^i} &= \frac{\partial g_{\beta\gamma}}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\beta\gamma}\frac{\partial^2 u^\beta}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\beta\gamma}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial^2 u^\gamma}{\partial \bar{u}^i\,\partial \bar{u}^k} \\ &= \frac{\partial g_{\beta\gamma}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^j}\end{aligned}$$

where we used the chain rule $\frac{\partial g_{\beta\gamma}}{\partial \bar{u}^i} = \frac{\partial g_{\beta\gamma}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}$, changed a few dummy indices, and used the fact that $g_{\alpha\beta} = g_{\beta\alpha}$. Similarly

$$\frac{\partial \bar{g}_{ki}}{\partial \bar{u}^j} = \frac{\partial g_{\gamma\alpha}}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k}\frac{\partial u^\alpha}{\partial \bar{u}^i} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^j\,\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^i} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^j\,\partial \bar{u}^i}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

and

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{u}^k} = \frac{\partial g_{\gamma\alpha}}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^i}\frac{\partial u^\alpha}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^k\,\partial \bar{u}^i}\frac{\partial u^\gamma}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^k\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^i}$$

It follows from equation *(10.25)*, page 214, that

$$\begin{aligned}\bar{\Gamma}_{ijk} &= \frac{1}{2}\left[\frac{\partial \bar{g}_{jk}}{\partial \bar{u}^i} + \frac{\partial \bar{g}_{ki}}{\partial \bar{u}^j} - \frac{\partial \bar{g}_{ij}}{\partial \bar{u}^k}\right] \\ &= \frac{1}{2}\left[\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right]\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} \\ &= \left\{\Gamma_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\right\}\frac{\partial u^\gamma}{\partial \bar{u}^k}\end{aligned}$$

10.28. Prove Theorem 10.2: Let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch on a surface of class $\geqq 2$ such that the coefficients of the Gauss-Weingarten equations are of class C^1. Then the mixed derivatives $\mathbf{x}_{uuv}, \mathbf{x}_{uvu}, \mathbf{x}_{vuv}, \mathbf{x}_{vvu}$ exist and satisfy equation (*10.6*) if and only if the first and second fundamental coefficients satisfy the compatibility equations (*10.7*) and (*10.8*).

Since it is assumed the coefficients of the Gauss equations $\mathbf{x}_{ij} = \Gamma^\alpha_{ij}\mathbf{x}_\alpha + b_{ij}\mathbf{N}$ are of class C^1, we can compute the third order derivatives.

$$\begin{aligned}\frac{\partial \mathbf{x}_{ij}}{\partial u^k} = \mathbf{x}_{ijk} &= (\Gamma^\alpha_{ij})_k\mathbf{x}_\alpha + \Gamma^\alpha_{ij}\mathbf{x}_{\alpha k} + (b_{ij})_k\mathbf{N} + b_{ij}\mathbf{N}_k \\ &= (\Gamma^\alpha_{ij})_k\mathbf{x}_\alpha + \Gamma^\alpha_{ij}[\Gamma^\beta_{\alpha k}\mathbf{x}_\beta + b_{\alpha k}\mathbf{N}] + (b_{ij})_k\mathbf{N} + b_{ij}(-b^\alpha_k\mathbf{x}_\alpha) \\ &= [(\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - b_{ij}b^\alpha_k]\mathbf{x}_\alpha + [\Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k]\mathbf{N}\end{aligned}$$

Observe that we used the Weingarten equation $\mathbf{N}_i = -b^\alpha_i\mathbf{x}_\alpha$. Similarly

$$\mathbf{x}_{ikj} = [(\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ik}b^\alpha_j]\mathbf{x}_\alpha + [\Gamma^\alpha_{ik}b_{\alpha j} + (b_{ik})_j]\mathbf{N}$$

Now the third order derivatives are independent of the order of differentiation if and only if $\mathbf{x}_{ijk} = \mathbf{x}_{ikj},\ i, k, j = 1, 2,$ or, if and only if

$$\begin{aligned}\mathbf{x}_{ijk} - \mathbf{x}_{ikj} &= [(\Gamma^\alpha_{ij})_k - (\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ij}b^\alpha_k + b_{ik}b^\alpha_j]\mathbf{x}_\alpha \\ &\quad + [\Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k - \Gamma^\alpha_{ik}b_{\alpha j} - (b_{ik})_j]\mathbf{N} = 0\end{aligned}$$

Since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{N}$ are independent, the equation is equivalent to

(*a*) $(\Gamma^\alpha_{ij})_k - (\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ij}b^\alpha_k + b_{ik}b^\alpha_j = 0, \qquad \alpha, i, j, k = 1, 2$

(*b*) $\Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k - \Gamma^\alpha_{ik}b_{\alpha j} - (b_{ik})_j = 0, \qquad i, j, k = 1, 2$

We consider first equation (*b*). Observe that the equation is obviously satisfied if $j = k$. Also the left hand side simply changes sign if j and k are interchanged. Thus (*b*) is equivalent to the two equations obtained by taking $i = 1,\ j = 1,\ k = 2$ and $i = 2,\ j = 1,\ k = 2$:

$$(b_{11})_2 - (b_{12})_1 = \Gamma^\alpha_{12}b_{\alpha 1} - \Gamma^\alpha_{11}b_{\alpha 2}, \qquad (b_{21})_2 - (b_{22})_1 = \Gamma^\alpha_{22}b_{\alpha 1} - \Gamma^\alpha_{21}b_{\alpha 2}$$

If we expand the right hand sides of the above and use $b_{11} = L,\ b_{12} = b_{21} = M,\ b_{22} = N,\ u = u^1$ and $v = u^2$ we obtain the Mainardi-Codazzi equations (*10.7*):

$$\begin{aligned}L_v - M_u &= \Gamma^1_{12}L + (\Gamma^2_{12} - \Gamma^1_{11})M - \Gamma^2_{11}N \\ M_v - N_u &= \Gamma^1_{22}L + (\Gamma^2_{22} - \Gamma^1_{12})M - \Gamma^2_{12}N\end{aligned}$$

We now consider equation (*a*) above which we can write, using equation (*10.32*), as

(*c*) $R^\alpha_{ijk} = (\Gamma^\alpha_{ik})_j - (\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}, \qquad \alpha, i, j, k = 1, 2$

From equation (*10.31*) and Problem 10.26, the above is equivalent to

$$R_{pijk} = g_{\alpha p}R^\alpha_{ijk} = g_{\alpha p}(\Gamma^\alpha_{ik})_j - g_{\alpha p}(\Gamma^\alpha_{ij})_k + g_{\alpha p}\Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - g_{\alpha p}\Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

From the skew-symmetric properties of R_{pijk} (see equation (*10.33*)) and equations (*10.34*) and (*10.35*), it follows that the above equation is equivalent to the single equation,

$$R_{1212} = g_{\alpha 1}(\Gamma^\alpha_{22})_1 - g_{\alpha 1}(\Gamma^\alpha_{21})_2 + g_{\alpha 1}\Gamma^\beta_{22}\Gamma^\alpha_{\beta 1} - g_{\alpha 1}\Gamma^\beta_{21}\Gamma^\alpha_{\beta 2}$$

or expanding and collecting terms,

$$\begin{aligned}R_{1212} &= g_{11}\{(\Gamma^1_{22})_1 - (\Gamma^1_{21})_2 + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22}\} \\ &\quad + g_{21}\{(\Gamma^2_{22})_1 - (\Gamma^2_{21})_2 + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{12}\}\end{aligned}$$

If we use $g_{11} = E,\ g_{21} = F,\ u^1 = u,\ u^2 = v,\ \Gamma^j_{12} = \Gamma^j_{21},$ and, from equation (*10.34*), $R_{1212} = LM - N^2$, we obtain

$$\begin{aligned}LM - N^2 &= E\{(\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}\} \\ &\quad + F\{(\Gamma^2_{22})_u - (\Gamma^2_{12})_v + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}\}\end{aligned}$$

which is the third of the compatibility equations (*10.8*).

10.29. Prove Theorem 10.9: $R_{mijk} = (\Gamma_{ikm})_j - (\Gamma_{ijm})_k + \Gamma^\alpha_{ij}\Gamma_{mk\alpha} - \Gamma^\alpha_{ik}\Gamma_{mj\alpha}$.

From Problem 10.28, equation (*c*),

$$R^\alpha_{ijk} = (\Gamma^\alpha_{ik})_j - (\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

From Problem 10.26,

$$R_{mijk} = g_{\alpha m}R^\alpha_{ijk} = g_{\alpha m}(\Gamma^\alpha_{ik})_j - g_{\alpha m}(\Gamma^\alpha_{ij})_k + g_{\alpha m}\Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - g_{\alpha m}\Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

Now $\quad g_{\alpha m}(\Gamma^{\alpha}_{ik})_j \;=\; (g_{\alpha m}\Gamma^{\alpha}_{ik})_j - (g_{\alpha m})_j\Gamma^{\alpha}_{ik} \;=\; (\Gamma_{ikm})_j - (\Gamma_{\alpha jm} + \Gamma_{mj\alpha})\Gamma^{\alpha}_{ik}$

where we used equation (*10.24*) and Problem 10.23. Similarly

$$g_{\alpha m}(\Gamma^{\alpha}_{ij})_k \;=\; (\Gamma_{ijm})_k - (\Gamma_{\alpha km} + \Gamma_{mk\alpha})\Gamma^{\alpha}_{ij}$$

Also $g_{\alpha m}\Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j} = \Gamma^{\beta}_{ik}g_{\alpha m}\Gamma^{\alpha}_{\beta j} = \Gamma^{\beta}_{ik}\Gamma_{\beta jm}$ and similarly $g_{\alpha m}\Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} = \Gamma^{\beta}_{ij}\Gamma_{\beta km}$. Thus substituting in the above

$$R_{mijk} \;=\; (\Gamma_{ikm})_j - \Gamma^{\alpha}_{ik}\Gamma_{\alpha jm} - \Gamma^{\alpha}_{ik}\Gamma_{mj\alpha} - (\Gamma_{ijm})_k + \Gamma^{\alpha}_{ij}\Gamma_{\alpha km} + \Gamma^{\alpha}_{ij}\Gamma_{mk\alpha} + \Gamma^{\beta}_{ik}\Gamma_{\beta jm} - \Gamma^{\beta}_{ij}\Gamma_{\beta km}$$

which gives the required result.

Supplementary Problems

THEORY OF SURFACES

10.30. Obtain the Christoffel symbols Γ^k_{ij} for the cylinder $\mathbf{x} = \mathbf{y}(u) + v\mathbf{g}$, $\mathbf{g} = \text{constant}$, $|\mathbf{g}| = 1$.
Ans. $\Gamma^1_{11} = \mathbf{y}'\cdot\mathbf{y}''/|\mathbf{y}'\times\mathbf{g}|^2$, $\Gamma^2_{11} = (g\cdot y')(y'\cdot y'')/|y'\times g|^2$, otherwise $\Gamma^k_{ij} = 0$.

10.31. Verify that the functions $E = 1 + 4u^2$, $F = -4uv$, $G = 1 + 4v^2$, $L = 2(4u^2 + 4v^2 + 1)^{-1/2}$, $M \doteq 0$, $N = -2(4u^2 + 4v^2 + 1)^{-1/2}$ satisfy the compatibility conditions, equations (*10.7*) and (*10.8*), page 203.

10.32. Using the Weingarten equations, prove that $\mathbf{N}_u \times \mathbf{N}_v = (EG - F^2)K\mathbf{N}$.

10.33. Solve the Gauss-Weingarten equations for the surface whose fundamental coefficients are $E = 1$, $F = 0$, $G = 1$, $L = -1$, $M = 0$, $N = 0$. *Ans.* Circular cylinder of radius 1.

10.34. Derive Rodrigues' formula from the Weingarten equations.

10.35. If the parameter curves on a patch are orthogonal, prove that

$$K \;=\; -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right]$$

10.36. Prove that $f(P) = \mathbf{x}\cdot\mathbf{x}$ is a continuous function of a point P on a surface.

10.37. Prove that the principal curvatures $\kappa_1(P)$ and $\kappa_2(P)$ are continuous functions of a point P on an oriented surface.

10.38. Prove Theorem 10.6: The only connected and closed surfaces of class $\geqq 2$ of which all points are planar points are planes.

10.39. Prove that the spheres are the only connected compact surfaces with positive Gaussian curvature and constant mean curvature.

TENSORS

10.40. If A_i and B_j are the components of two covariant vectors, show that the outer product $C_{ij} = A_iB_j$ are the components of a covariant tensor of rank 2.

10.41. Show that $\det\begin{pmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{pmatrix} \;=\; e^{ijk}a^1_i a^2_j a^3_k$ where e^{ijk} is defined in Example 10.6, page 210.

10.42. Show that $\delta^{\alpha\beta}_{ij}A_{\alpha\beta} \;=\; A_{ij} - A_{ji}$.

10.43. If A^{ij} are the components of an absolute contravariant tensor and $A_{i\alpha}A^{\alpha j} = \delta_i^j$, show that $A^{\alpha j}$ are the components of an absolute covariant tensor. The two tensors are said to be *reciprocal.*

10.44. If A^{ij} and A_{ij} are the components of reciprocal symmetric tensors, and if x_i are the components of a covariant vector, show that $A_{ij}x^i x^j = A^{ij}x_i x_j$ where $x_i = A^{i\alpha}x_\alpha$.

10.45. Show that the quantities $e_{ijk} = e^{ijk}$, where the e^{ijk} are defined in Example 10.6(*f*), page 211, are the components of a covariant tensor of rank 3 and weight -1.

10.46. Let $\epsilon_{11} = 0$, $\epsilon_{12} = \sqrt{g}$, $\epsilon_{21} = -\sqrt{g}$, $\epsilon_{22} = 0$, where $g = g_{11}g_{22} - (g_{12})^2$. Show that the ϵ_{ij}, $i, j = 1, 2$, are the components of a skew-symmetric covariant tensor such that $\bar{\epsilon}_{11} = 0$, $\bar{\epsilon}_{12} = \sqrt{\bar{g}}$, $\epsilon_{21} = -\sqrt{\bar{g}}$, $\epsilon_{22} = 0$.

10.47. Let $\epsilon^{ij} = \epsilon_{\alpha\beta}\, g^{i\alpha} g^{j\beta}$ where $\epsilon_{\alpha\beta}$ is defined in the preceding problem. Show that $\epsilon^{11} = 0$, $\epsilon^{12} = 1/\sqrt{g}$, $\epsilon^{21} = -1/\sqrt{g}$, $\epsilon^{22} = 0$.

10.48. Show that $b_i^\beta\, b_{\beta j} - b_j^\beta\, b_{\beta i} = 0$, $i, j = 1, 2$.

10.49. Prove that $$\bar{\Gamma}_{ij}^k = \left[\Gamma_{\alpha\beta}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^i\,\partial \bar{u}^j}\right]\frac{\partial \bar{u}^k}{\partial u^\gamma}.$$

10.50. Prove that $$\frac{\partial g^{ij}}{\partial u^k} = -g^{\alpha j}\Gamma_{\alpha k}^i - g^{\alpha i}\Gamma_{\alpha k}^j.$$

10.51. Show that
$$R_{112}^1 = R_{221}^2 = -R_{121}^1 = -R_{212}^2 = F\frac{LN - M^2}{EG - F^2}$$
$$R_{212}^1 = -R_{221}^1 = G\frac{LN - M^2}{EG - F^2}$$
$$R_{121}^2 = -R_{121}^2 = E\frac{LN - M^2}{EG - F^2}$$
and otherwise $R_{ijk}^p = 0$.

10.52. Prove that $$R_{ijk}^p = \frac{\partial \Gamma_{ik}^p}{\partial u^j} - \frac{\partial \Gamma_{ij}^p}{\partial u^k} + \Gamma_{ik}^\alpha \Gamma_{\alpha j}^p - \Gamma_{ij}^\alpha \Gamma_{\alpha k}^p.$$

10.53. Prove that $$\frac{1}{2}\frac{\partial \log g}{\partial u^1} = \Gamma_{11}^1 + \Gamma_{12}^2, \quad \frac{1}{2}\frac{\partial \log g}{\partial u^2} = \Gamma_{12}^1 + \Gamma_{22}^2.$$

Chapter 11

Intrinsic Geometry

MAPPINGS OF SURFACES

Let S be a surface of class C^m, S^* a surface of class C^n and $\mathbf{f}$ a mapping of S into S^* as shown in Fig. 11-1. If for every coordinate patch $\mathbf{x} = \mathbf{x}(u,v)$ on S with domain U, the composite mapping $\mathbf{x}^* = \mathbf{x}^*(u,v) = \mathbf{f}(\mathbf{x}(u,v))$ of U into S^* is a regular parametric representation of class C^r $(r \leq \text{Min}\,(m,n))$, then $\mathbf{f}$ is called a *regular differentiable mapping of S into S^* of class C^r*. We recall that $\mathbf{x}^* = \mathbf{x}^*(u,v)$ is a regular parametric representation of class C^r if

(i) $\mathbf{x}^*$ belongs to C^r on U,

(ii) $\mathbf{x}_u^* \times \mathbf{x}_v^* \neq \mathbf{0}$ for all (u,v) in U.

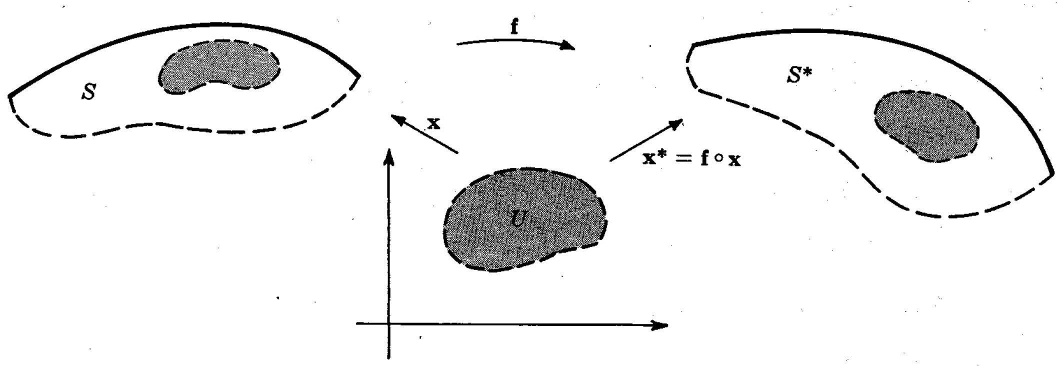


Fig. 11-1

In Problem 11.3, page 247, we prove that if a mapping $\mathbf{f}$ of S into S^* has the property that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation of class C^r for every patch $\mathbf{x} = \mathbf{x}(u,v)$ of a basis for S, then $\mathbf{f}$ has the property that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation of class C^r for all patches on S. Thus in applying the above definition it is sufficient to check $\mathbf{f}$ on enough patches to cover S.

Example 11.1.

(*a*) Let S be the sphere of radius 1 with center at $\mathbf{x}_0 = \mathbf{e}_3$ punctured at the north pole and let S^* be the x_1x_2 plane as shown in Fig. 11-2. Let $\mathbf{f}$ be the mapping that projects a point $\mathbf{x}$ on the sphere into S^* along a line through the north pole. This mapping $\mathbf{f}$ is called the *stereographic projection* of S onto S^*. It is easily shown that $\mathbf{x}^* = \dfrac{2}{2-x_3}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$. For a basis for S we can use two patches obtained from

$$\mathbf{x} = (\cos\theta\sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi + 1)\mathbf{e}_3$$

$(0 < \phi < \pi)$ when $0 < \theta < 3\pi/2$ and $\pi/2 < \theta < 5\pi/2$, and the Monge patch

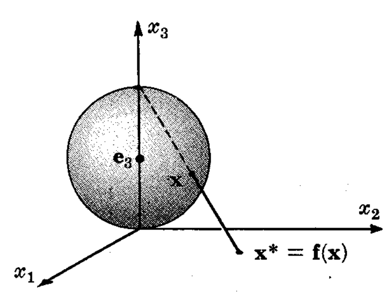


Fig. 11-2

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + (1 - \sqrt{1 - x_1^2 - x_2^2})\mathbf{e}_3, \qquad x_1^2 + x_2^2 < 1$$

to cover the south pole. On the first two patches,

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}(\theta, \phi)) = \frac{2}{1 - \cos\phi}((\cos\theta \sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2), \qquad 0 < \phi < \pi$$

Here $\mathbf{x}^*$ is of class C^∞ and

$$|\mathbf{x}_\theta^* \times \mathbf{x}_\phi^*| = 4\sin\phi/(1 - \cos\phi)^2 \neq 0$$

On the Monge patch we have

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}(x_1, x_2)) = \frac{2}{1 + (1 - x_1^2 - x_2^2)^{1/2}}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$$

Again, $\mathbf{x}^*$ is of class C^∞ and

$$|\mathbf{x}_{x_1}^* \times \mathbf{x}_{x_2}^*| = 4/(1 - x_1^2 - x_2^2)^{1/2}[1 + (1 - x_1^2 - x_2^2)^{1/2}]^2 \neq 0$$

Thus the stereographic projection of the punctured sphere onto the plane is a regular differential mapping of class C^∞.

(b) Let S be the x_1x_2 plane and S^* the circular cylinder of radius 1 about the x_3 axis. The function

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}) = (\cos x_1)\mathbf{e}_1 + (\sin x_1)\mathbf{e}_2 + x_2\mathbf{e}_3$$

defines a mapping of S onto S^* which wraps the plane around the cylinder such that the lines $x_1 = \text{constant}$ map onto the rulings of the cylinder and the lines $x_2 = \text{constant}$ map onto the cross-sectional circles of the cylinder. Here $\mathbf{x} = \theta\mathbf{e}_1 + \phi\mathbf{e}_2$ is a patch covering S on which

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}(\theta, \phi)) = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2 + \phi\mathbf{e}_3$$

is clearly of class C^∞ and

$$|x_\theta^* \times x_\phi^*| = |(\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2| = 1 \neq 0$$

Thus $\mathbf{f}$ is a regular differentiable mapping of class C^∞ of the plane onto the cylinder.

We recall (Problem 8.12, page 165) that a regular parametric representation is locally 1-1 and bicontinuous. Thus if $\mathbf{f}$ is a regular differentiable mapping of S into S^*, P a point on S, and $\mathbf{x} = \mathbf{x}(u, v)$ a coordinate patch on S containing P such that P is the image of (u, v), then there exists a neighborhood $S(u, v)$ of (u, v) on which $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is 1-1 and bicontinuous and hence a patch on S^* containing $\mathbf{f}(P)$ as shown in Fig. 11-3. Since the restriction of $\mathbf{x} = \mathbf{x}(u, v)$ to $S(u, v)$ is a coordinate patch on S, we have

Theorem 11.1. If $\mathbf{f}$ is a regular differentiable mapping of a surface S into a surface S^*, then for every point P on S there exists a coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ on S containing P such that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a coordinate patch on S^*.

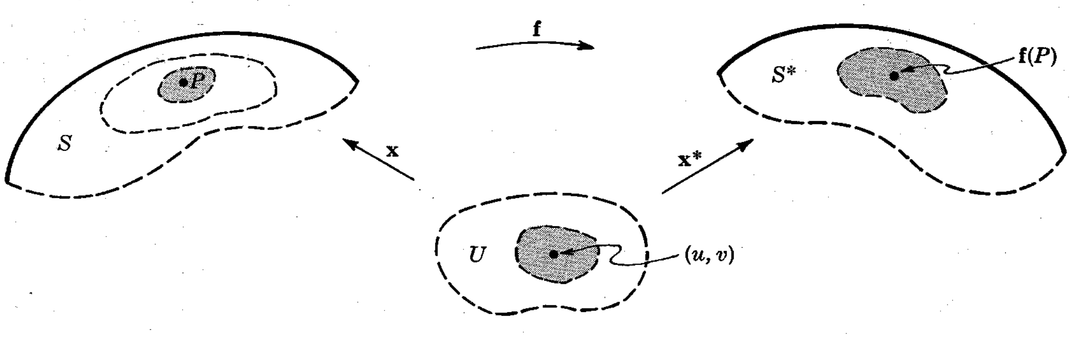


Fig. 11-3

We note further that for every patch $\mathbf{x} = \mathbf{x}(u, v)$ on S, the mapping $\mathbf{f}$ is the composite mapping $\mathbf{x}^* \circ \mathbf{x}^{-1}$ where $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$. Since the composite of two 1-1 and bicontinuous mappings is 1-1 and bicontinuous, we have

***Corollary*:** A regular differentiable mapping is locally 1-1 and bicontinuous. That is, if **f** is a regular differentiable mapping of a surface S into a surface S^*, then for every point P on S there exists a coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ on S containing P on which **f** is a 1-1 and bicontinuous mapping into S^*.

Finally we have

Theorem 11.2. If **f** is a regular differentiable mapping of class C^r of S into S^* and $\mathbf{x} = \mathbf{x}(t)$ is a regular curve C of class C^r on S, then $\mathbf{x}^* = \mathbf{x}^*(t) = \mathbf{f}(\mathbf{x}(t))$ is a regular curve of class C^r on S^*.

The proof of the above is left to the reader as an exercise.

Note. Unless stated to the contrary, by a "mapping of class C^m" of a surface S into a surface T we shall mean a "regular differentiable mapping of class C^m" of S into T.

ISOMETRIC MAPPINGS. INTRINSIC GEOMETRY

A 1-1 mapping **f** of a surface S onto a surface S^* is called an *isometric mapping* or *isometry* if the length of an arbitrary regular arc $\mathbf{x} = \mathbf{x}(t)$ on S is equal to the length of its image $\mathbf{x}^* = \mathbf{x}^*(t) = \mathbf{f}(\mathbf{x}(t))$ on S^*. In Problem 11.5 we show that if **f** is an isometry from S onto S^*, then $\mathbf{f}^{-1}$ is an isometry from S^* into S.

If there exists an isometry from S onto S^*, then S and S^* are said to be *isometric*. It is intuitively clear that if a sheet of paper is bent into various shapes smoothly and without stretching, the resulting surfaces are all isometric as shown in Fig. 11-4.

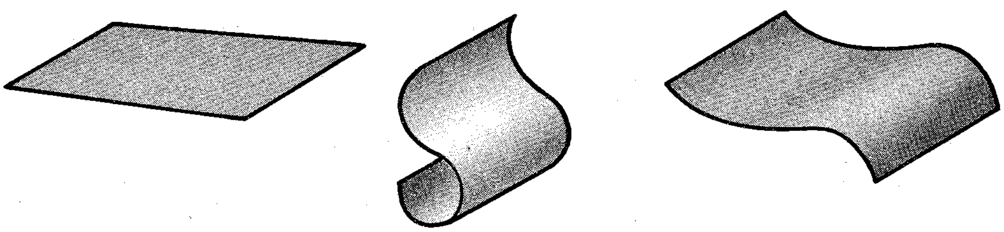


Fig. 11-4

Now suppose **f** is a 1-1 mapping of S onto S^* such that the fundamental coefficients E, F and G on every patch $\mathbf{x} = \mathbf{x}(u, v)$ on S are equal to the fundamental coefficients E^*, F^* and G^* along its image $\mathbf{x}^* = \mathbf{x}^*(u, v) = \mathbf{f}(\mathbf{x}(u, v))$. Then **f** is an isometry. For suppose $\mathbf{x} = \mathbf{x}(t)$, $a \leqq t \leqq b$, is an arbitrary arc C on S. In general C may not lie in any one patch on S. However, since C is compact (the continuous image of the compact interval $a \leqq t \leqq b$), it will consist of a finite number of arcs C_i, $t_i \leqq t \leqq t_{i+1}$, $i = 0, \ldots, n-1$, placed end to end such that each C_i is on some patch $\mathbf{x}_i = \mathbf{x}_i(u, v)$. Now we recall that the length of an arc on a patch is the integral of the square root of the first fundamental form. Thus the length $L(C)$ of C is given by

$$L(C) \;=\; \sum_i L(C_i) \;=\; \sum_i \left[\int_{t_i}^{t_i+1} \sqrt{E_i\left(\frac{du}{dt}\right)^2 + 2F_i\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G_i\left(\frac{dv}{dt}\right)^2}\, dt\right]$$

But it is given that for all i, $E_i = E_i^*$, $F_i = F_i^*$ and $G_i = G_i^*$ where E_i^*, F_i^*, G_i^* are the fundamental coefficients on $\mathbf{x}_i^* = \mathbf{f}(\mathbf{x}_i(u, v))$. Hence

$$\begin{aligned} L(C) \;&=\; \sum_i \left[\int_{t_i}^{t_i+1} \sqrt{E_i^*\left(\frac{du}{dt}\right)^2 + 2F_i^*\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G_i^*\left(\frac{dv}{dt}\right)^2}\, dt\right] \\ &=\; \sum_i L(C_i^*) \;=\; L(C^*) \end{aligned}$$

Thus the length of any arc C on S is equal to the length of its image C^* on S^*. Hence $\mathbf{f}$ is an isometry. In Problem 11.9, page 249, we prove the converse. Thus

Theorem 11.3. A 1-1 mapping $\mathbf{f}$ of S onto S^* is an isometry if and only if on every patch $\mathbf{x} = \mathbf{x}(u, v)$ on S the first fundamental coefficients

$$E = E^*, \quad F = F^* \quad \text{and} \quad G = G^*$$

where E^*, F^* and G^* are the first fundamental coefficients on its image $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$.

Example 11.2.

Let S be the surface of revolution

$$\mathbf{x} \;=\; (\cos\theta\cosh v)\mathbf{e}_1 + (\sin\theta\cosh v)\mathbf{e}_2 + v\mathbf{e}_3, \qquad 0 < \theta < 2\pi, \; -\infty < v < \infty$$

and S^* the right conoid

$$\mathbf{x}^* \;=\; (u\cos\phi)\mathbf{e}_1 + (u\sin\phi)\mathbf{e}_2 + \phi\mathbf{e}_3, \qquad 0 < \phi < 2\pi, \; -\infty < u < \infty$$

Let $\mathbf{f}$ be the mapping which takes the point $\mathbf{x}(\theta, v)$ on S into the point $\mathbf{x}^*(\phi, u)$ on S^*, where $\phi = \theta$ and $u = \sinh v$. Namely $\mathbf{f}$ is the composite mapping $\mathbf{x}^{-1}$ followed by $\phi = \theta$, $u = \sinh v$, followed by $\mathbf{x}^*$, as shown in Fig. 11-5. For any part of the strip $0 < \theta < 2\pi$, $-\infty < v < \infty$, on which $\mathbf{x}$ is a patch (bicontinuous), its image on S^*

$$\mathbf{x}^* \;=\; \mathbf{x}^{**}(\theta, v) \;=\; \mathbf{f}(\mathbf{x}(\theta, v)) \;=\; \mathbf{x}^*(\theta, \sinh v) \;=\; (\sinh v\cos\theta)\mathbf{e}_1 + (\sinh v\sin\theta)\mathbf{e}_2 + \theta\mathbf{e}_3$$

is of class C^∞ and $\mathbf{x}^{**}_\theta \times \mathbf{x}^{**}_v = \cosh^2 v \neq 0$. Thus $\mathbf{f}$ is regular and of class C^∞. Also $\mathbf{f}$ is 1-1 and onto. We leave the proof of this as an exercise for the reader. Finally, it is easily computed that

$$\begin{aligned} E &= \mathbf{x}_\theta \cdot \mathbf{x}_\theta = \cosh^2 v = \mathbf{x}^{**}_\theta \cdot \mathbf{x}^{**}_\theta = E^{**} \\ F &= \mathbf{x}_\theta \cdot \mathbf{x}_v = 0 = \mathbf{x}^{**}_\theta \cdot \mathbf{x}^{**}_v = F^{**} \\ G &= \mathbf{x}_v \cdot \mathbf{x}_v = \cosh^2 v = \mathbf{x}^{**}_v \mathbf{x}^{**}_v = G^{**} \end{aligned}$$

Thus from the theorem above, $\mathbf{f}$ is an isometry from S onto S^*.

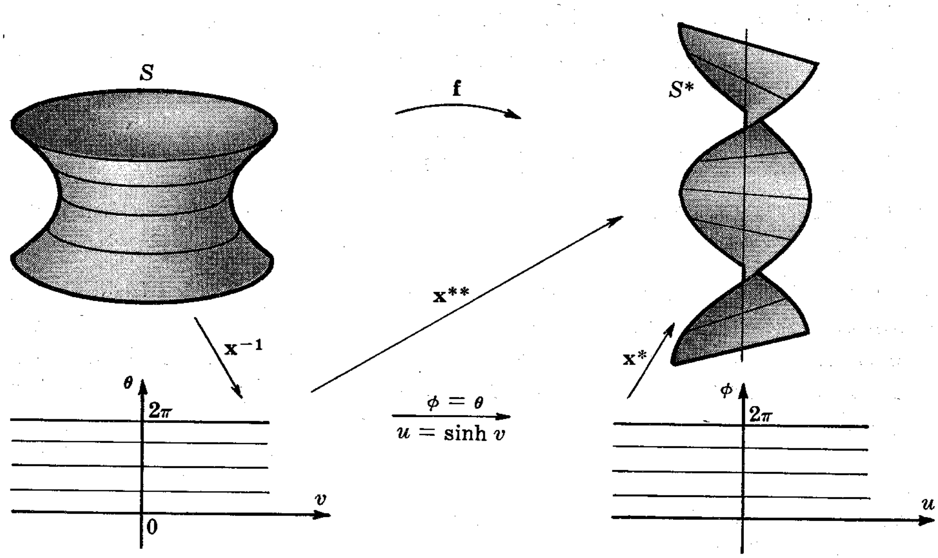


Fig. 11-5

Consider the mapping of the plane around the cylinder in Example 11.1(*b*). Intuitively we see that the image of every curve on the plane is a curve of equal length on the cylinder. The mapping however is not 1-1 and hence not an isometry. The plane and the cylinder however are *locally isometric*. Namely, we define a *locally isometric mapping* of S into S^* to be a mapping of S into S^* which preserves the length of arcs but is not necessarily 1-1 and onto.

An important aspect of geometry is the study of the properties of a surface which remain invariant under a given class of 1-1 mappings. For example, a topological property of a surface is a property which remains invariant under a 1-1 bicontinuous (topological) mapping. Compactness is an example of a topological property of a surface. A property of a surface which remains invariant under an isometry of the surface is called an *intrinsic* property of the surface. The totality of intrinsic properties of the surface is called the *intrinsic geometry* of the surface.

It follows from Theorem 11.3 above that a property of a surface is an intrinsic property if and only if it depends only on the first fundamental form. Thus we see that the Gaussian curvature is an intrinsic property of the surface.

Let P and Q be two points on a surface S. We define *the intrinsic distance from P to Q*, denoted by $D(P,Q)$, to be the infimum (greatest lower bound) of the lengths $L(C)$ of all possible regular arcs C on S joining P to Q. It is clear that the intrinsic distance between two points on a surface always exists since the set of real numbers $L(S)$ is not empty (S is connected and hence arcwise connected) and is bounded from below by the Euclidean distance between P and Q, $|P-Q|$. Clearly the intrinsic distance between two points on a surface is an intrinsic property of the surface. In Problem 11.6, page 248, we prove

Theorem 11.4. (i) $D(P,Q) = D(Q,P)$
(ii) $D(P,R) \leqq D(P,Q) + D(Q,R)$
(iii) $D(P,Q) \geqq 0, \quad D(P,Q) = 0 \quad \text{iff} \quad P = Q$

Now, given P and Q, if there exists a regular arc C joining P and Q whose length is equal to the intrinsic distance between P and Q, then C is called an *arc of minimum length between P and Q*.

It follows from the definition of the infimum that if C is an arc of minimum length between P and Q, then its length $L(C)$ satisfies:

(i) $L(C) \leqq L(C')$, where C' is any other arc joining P and Q. ($L(C) = D(P,Q)$ is a lower bound.)

(ii) For an arbitrary $\epsilon > 0$, there exists a regular arc C' joining P and Q such that $L(C) + \epsilon > L(C')$. ($L(C) = D(P,Q)$ is the greatest of the lower bounds.)

Again it is clear that the arcs of minimum length between points belong to the intrinsic geometry of the surface.

In the plane, $D(P,Q)$ is the Euclidean distance and an arc of minimum length always exists, is unique and is the straight line segment between P and Q. In general, however, as shown in Example 11.3(a) below, there need not exist an arc of minimum length between two given points on a surface or, as shown in Example 11.3(b), such an arc if it exists need not be unique.

Example 11.3.

(a) Let S be the xy plane less the origin and let $P = (0,1)$ and $Q = (0,-1)$ as shown in Fig. 11-6. Clearly, for any $\epsilon > 0$, there exists an arc joining P and Q of length less than $2+\epsilon$. One could take the segment of a circle between P and Q with center at $(R,0)$, $R > 0$, for sufficiently large R. Also the length of any arc between P and Q must be greater than or equal to 2. Thus $D(P,Q) = 2$. On the other hand there does not exist an arc in S joining P and Q with length equal to 2, since the origin must be avoided. Thus there does not exist an arc of minimum length between P and Q on S.

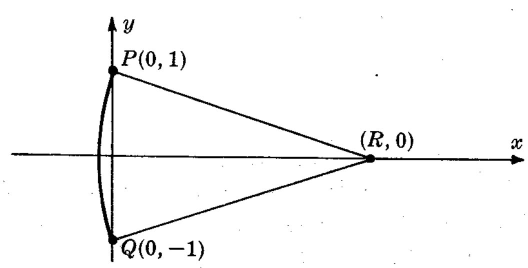


Fig. 11-6

(*b*) Let S be a sphere. Intuitively we see that every great circle joining the north and south poles is an arc of minimum length between the poles. Thus we see that there may exist an infinite number of different arcs of minimum length between two points on a surface.

GEODESIC CURVATURE

Suppose C is an arc of minimum length between two points on a surface S. If P is any point on C and Q a neighboring point on C, then intuitively the part of the arc between P and Q is also an arc of minimum length between P and Q. Also it seems that the orthogonal projection C^* of the part of C between P and Q onto the tangent plane of S at P, as shown in Fig. 11-7, is an arc of minimum length on the tangent plane between P and the projection Q^* of Q onto the tangent plane. But then C^* must be a straight line or, equivalently, a curve with zero curvature. Thus as candidates for the arcs of minimum length we are led to consider those curves where the curvature vector of the orthogonal projection of the curve onto the tangent plane is zero.

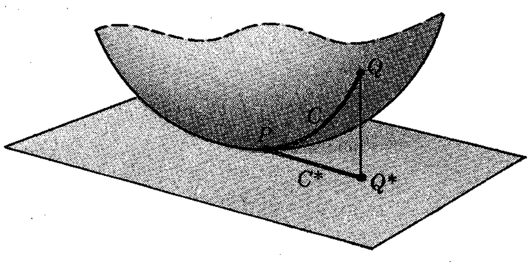


Fig. 11-7

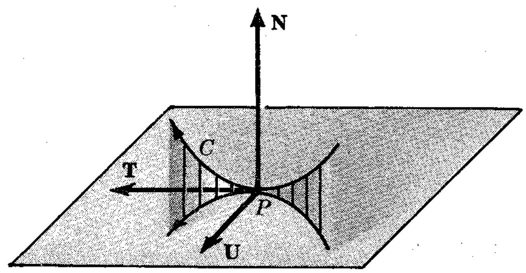


Fig. 11-8

The curvature vector at P of the projection of a curve C onto the tangent plane at P is called the *geodesic curvature vector of C at P* and is denoted by $\mathbf{k}_g$. In order to calculate $\mathbf{k}_g$, we suppose that S is a surface of class $\geqq 2$, $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P, and $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ is a natural representation of C of class C^2. Let $\mathbf{T}$, for the moment, denote the unit tangent vector to C at P and let $\mathbf{U}$ be the vector in the tangent plane at P such that $(\mathbf{T}, \mathbf{U}, \mathbf{N})$ is a right-handed orthonormal triad as shown in Fig. 11-8. Without loss of generality we can assume that P is the origin. Then the projection of C onto the tangent plane at P is given by $\mathbf{x}^* = (\mathbf{x}\cdot\mathbf{T})\mathbf{T} + (\mathbf{x}\cdot\mathbf{U})\mathbf{U}$. Differentiating,

$$d\mathbf{x}^*/ds = (\dot{\mathbf{x}}\cdot\mathbf{T})\mathbf{T} + (\dot{\mathbf{x}}\cdot\mathbf{U})\mathbf{U} = (\mathbf{t}\cdot\mathbf{T})\mathbf{T} + (\mathbf{t}\cdot\mathbf{U})\mathbf{U}$$

$$d^2\mathbf{x}^*/ds^2 = (\dot{\mathbf{t}}\cdot\mathbf{T})\mathbf{T} + (\dot{\mathbf{t}}\cdot\mathbf{U})\mathbf{U} = (\mathbf{k}\cdot\mathbf{T})\mathbf{T} + (\mathbf{k}\cdot\mathbf{U})\mathbf{U}$$

$$\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds}\bigg/\left|\frac{d\mathbf{x}^*}{ds}\right| \quad \text{and} \quad \mathbf{k}^* = \frac{d\mathbf{t}^*}{ds^*} = \frac{d\mathbf{t}^*}{ds}\bigg/\left|\frac{d\mathbf{x}^*}{ds}\right| = \frac{\left(\dfrac{d\mathbf{x}^*}{ds}\cdot\dfrac{d\mathbf{x}^*}{ds}\right)\dfrac{d^2\mathbf{x}^*}{ds^2} - \left(\dfrac{d\mathbf{x}^*}{ds}\cdot\dfrac{d^2\mathbf{x}^*}{ds^2}\right)\dfrac{d\mathbf{x}^*}{ds}}{\left|\dfrac{d\mathbf{x}^*}{ds}\right|^4}$$

Now at P, $\mathbf{t} = \mathbf{T}$; thus at P,

$$\frac{d\mathbf{x}^*}{ds} = (\mathbf{T}\cdot\mathbf{T})\mathbf{T} + (\mathbf{T}\cdot\mathbf{U})\mathbf{U} = \mathbf{T}, \quad \left|\frac{d\mathbf{x}^*}{ds}\right| = 1, \quad \frac{d^2\mathbf{x}^*}{dx^2} = (\mathbf{k}\cdot\mathbf{T})\mathbf{T} + (\mathbf{k}\cdot\mathbf{U})\mathbf{U} = (\mathbf{k}\cdot\mathbf{U})\mathbf{U}$$

and hence $\mathbf{k}^* = \mathbf{k}_g = (\mathbf{k}\cdot\mathbf{U})\mathbf{U} - (\mathbf{k}\cdot\mathbf{U})(\mathbf{U}\cdot\mathbf{T})\mathbf{T}$. But $\mathbf{U}$ is orthogonal to $\mathbf{T}$. Thus the formula

$$\mathbf{k}_g = (\mathbf{k}\cdot\mathbf{U})\mathbf{U} \tag{11.1}$$

As a consequence of (*11.1*) we see that $\mathbf{k}_g$ is in fact the orthogonal projection onto the tangent plane of the curvature vector $\mathbf{k}$ of C at P. Since $\mathbf{k}$ is orthogonal to $\mathbf{T}$, its orthogonal projection onto the tangent plane is simply its projection $(\mathbf{k}\cdot\mathbf{U})\mathbf{U}$ onto $\mathbf{U}$. Thus we have

Theorem 11.5. The geodesic curvature $\mathbf{k}_g$ of a curve C at P is the vector projection of the curve vector $\mathbf{k}$ of C at P onto the tangent plane at P.

From equation (*11.1*) and the fact that $\mathbf{k}\cdot\mathbf{T} = 0$, we can write

$$\mathbf{k} = \mathbf{k}_g + \mathbf{k}_n = (\mathbf{k}\cdot\mathbf{U})\mathbf{U} + (\mathbf{k}\cdot\mathbf{N})\mathbf{N} \tag{11.2}$$

where we recall $\mathbf{k}_n = (\mathbf{k}\cdot\mathbf{N})\mathbf{N}$ is the normal curvature vector of C at P. Thus $\mathbf{k}_g$ is independent of the orientation of the surface S and the curve C, since $\mathbf{k}$ and $\mathbf{k}_n$ are.

The scalar function κ_g defined by $\mathbf{k}_g = \kappa_g\mathbf{U}$ is called the *geodesic curvature* of C at P. It follows from equation (*11.2*) that $\kappa_g = \mathbf{k}\cdot\mathbf{U}$. Also, since $\mathbf{U}$ was chosen such that $(\mathbf{T},\mathbf{U},\mathbf{N})$, i.e. $(\mathbf{t},\mathbf{U},\mathbf{N})$, is a right-handed orthonormal triplet, we have $\mathbf{U} = \mathbf{N}\times\mathbf{t}$. Hence $\kappa_g = \mathbf{k}\cdot\mathbf{U} = \mathbf{k}\cdot(\mathbf{N}\times\mathbf{t})$. Thus the formula

$$\kappa_g = [\mathbf{t}\mathbf{k}\mathbf{N}] \quad\text{or}\quad \kappa_g = [\dot{\mathbf{x}}\ddot{\mathbf{x}}\mathbf{N}] \tag{11.3}$$

Note that κ_g depend both on the orientation of S (the sense of $\mathbf{N}$) and on the orientation of C (the sense of $\mathbf{t}$).

Contrary to κ_n, which depends on both the first and second fundamental coefficients, the geodesic curvature κ_g depends only on the first fundamental coefficients (and their derivatives) and is hence an intrinsic property of the surface. This can be shown by finding κ_g explicitly in terms of E, F and G. In Problem 11.13, page 250, we prove that

$$\kappa_g = \left[\Gamma_{11}^2\left(\frac{du}{ds}\right)^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)\left(\frac{du}{ds}\right)^2\frac{dv}{ds} + (\Gamma_{22}^2 - 2\Gamma_{12}^1)\frac{du}{ds}\left(\frac{dv}{ds}\right)^2 - \Gamma_{22}^1\left(\frac{dv}{ds}\right)^3 + \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right]\sqrt{EG-F^2} \tag{11.4}$$

and hence

Theorem 11.6. The geodesic curvature along a curve on a surface is an intrinsic property of the surface.

Note that along the u-parameter curves $v = \text{constant}$, $dv/ds = 0$ and $du/ds = 1/\sqrt{E}$; and that along the v-parameter curves $v = \text{constant}$, $du/ds = 0$ and $dv/ds = 1/\sqrt{G}$. Thus for the geodesic curvature of the parameter curves, equation (*11.4*) becomes

$$\begin{aligned}(\kappa_g)_{v=\text{constant}} &= \Gamma_{11}^2\left(\frac{du}{ds}\right)^3\sqrt{EG-F^2} = \Gamma_{11}^2\frac{\sqrt{EG-F^2}}{E\sqrt{E}}\\ (\kappa_g)_{u=\text{constant}} &= -\Gamma_{22}^1\left(\frac{dv}{ds}\right)^3\sqrt{EG-F^2} = -\Gamma_{22}^1\frac{\sqrt{EG-F^2}}{G\sqrt{G}}\end{aligned} \tag{11.5}$$

If in addition the parameter curves are orthogonal, then $F = 0$, $\Gamma_{11}^2 = -\tfrac{1}{2}E_v/G$ and $\Gamma_{22}^1 = -\tfrac{1}{2}G_u/E$. Thus

$$(\kappa_g)_{v=\text{constant}} = -\frac{E_v}{2E\sqrt{G}}, \quad (\kappa_g)_{u=\text{constant}} = \frac{G_u}{2G\sqrt{E}} \tag{11.6}$$

Example 11.4.

Consider the paraboloid $\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + r^2\mathbf{e}_3$, $0 < r < \infty$, $-\infty < \theta < \infty$. Here

$$\mathbf{x}_r = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2 + 2r\mathbf{e}_3, \quad \mathbf{x}_\theta = (-r\sin\theta)\mathbf{e}_1 + (r\cos\theta)\mathbf{e}_2$$

$$E = \mathbf{x}_r\cdot\mathbf{x}_r = 1 + 4r^2, \quad F = \mathbf{x}_r\cdot\mathbf{x}_\theta = 0, \quad G = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = r^2$$

$$\mathbf{N} = \frac{\mathbf{x}_r\times\mathbf{x}_\theta}{|\mathbf{x}_r\times\mathbf{x}_\theta|} = (1+4r^2)^{-1/2}(-2r(\cos\theta)\mathbf{e}_1 - 2r(\sin\theta)\mathbf{e}_2 + \mathbf{e}_3)$$

The θ-parameter curve $r = r_0$ is

$$\mathbf{x} = (r_0\cos\theta)\mathbf{e}_1 + (r_0\sin\theta)\mathbf{e}_2 + r_0^2\mathbf{e}_3$$

Along this curve,

$$\mathbf{x}' = (-r_0\sin\theta)\mathbf{e}_1 + (r_0\cos\theta)\mathbf{e}_2, \quad |\mathbf{x}'| = r_0$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2, \quad \mathbf{t}' = (-\cos\theta)\mathbf{e}_1 - (\sin\theta)\mathbf{e}_2$$

$$\begin{aligned}
\mathbf{k} &= \mathbf{t}'/|\mathbf{x}'| = r_0^{-1}((-\cos\theta)\mathbf{e}_1 - (\sin\theta)\mathbf{e}_2)\\
\mathbf{N} &= (1+4r_0^2)^{-1/2}(-2r_0(\cos\theta)\mathbf{e}_1 - 2r_0(\sin\theta)\mathbf{e}_2 + \mathbf{e}_3)\\
\mathbf{U} &= \mathbf{N}\times\mathbf{t} = (1+4r_0^2)^{-1/2}((-\cos\theta)\mathbf{e}_1 - (\sin\theta)\mathbf{e}_2 - 2r_0\mathbf{e}_3)\\
\mathbf{k}_g &= (\mathbf{k}\cdot\mathbf{U})\mathbf{U} = r_0^{-1}(1+4r_0^2)^{-1}((-\cos\theta)\mathbf{e}_1 - (\sin\theta)\mathbf{e}_2 - 2r_0\mathbf{e}_3)\\
\kappa_g &= \mathbf{k}\cdot\mathbf{U} = (1/r_0)(1+4r_0^2)^{-1/2}
\end{aligned}$$

and as we expect, κ_g is independent of θ. Note that the above agrees with formula (11.6),

$$(\kappa_g)_{r=r_0} = \left.\frac{G_r}{2G\sqrt{E}}\right|_{r=r_0} = \left.\frac{2r}{2r^2\sqrt{1+4r^2}}\right|_{r=r_0} = (1/r_0)(1+4r_0^2)^{-1/2}$$

GEODESICS

Recall that our candidates for arcs of minimum length were the curves along which the geodesic curvature vector vanishes. A curve C along which $\mathbf{k}_g = \mathbf{0}$ is called a *geodesic line* or simply a geodesic. Along a straight line, $\mathbf{k} \equiv \mathbf{0}$ and hence $\mathbf{k}_g = (\mathbf{k}\cdot\mathbf{U})\mathbf{U} = \mathbf{0}$. If C is not a straight line, then it follows from $\mathbf{k} = \mathbf{k}_g + \mathbf{k}_n$ that $\mathbf{k}_g = \mathbf{0}$ if and only if $\mathbf{k} = \mathbf{k}_n = (\mathbf{k}\cdot\mathbf{N})\mathbf{N}$, or if and only if the osculating plane, which we recall is the plane parallel to $\mathbf{k}$ and $\mathbf{t}$, contains the normal line to S. Thus

Theorem 11.7. All straight lines on a surface are geodesics. A curve not a straight line is a geodesic if and only if the osculating plane of the curve is perpendicular to the tangent plane to the surface at each point.

Note that an asymptotic curve is a straight line or a curve along which the osculating plane and tangent plane to the surface coincide, while a geodesic is a straight line or a curve along which the osculating plane is perpendicular to the tangent plane. Also, a curve is an asymptotic curve if $\mathbf{k}_n = \mathbf{0}$; a curve is a geodesic if $\mathbf{k}_g = \mathbf{0}$.

In Problem 11.14, page 251, we prove

Theorem 11.8. A natural representation of a curve $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ of class C^2 on a patch $\mathbf{x} = \mathbf{x}(u, v)$ of class C^2 is a geodesic if and only if $u(s)$ and $v(s)$ satisfy

$$\begin{aligned}
\frac{d^2u}{ds^2} + \Gamma_{11}^1\left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^1\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^1\left(\frac{dv}{ds}\right)^2 &= 0\\
\frac{d^2v}{ds^2} + \Gamma_{11}^2\left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^2\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^2\left(\frac{dv}{ds}\right)^2 &= 0
\end{aligned} \tag{11.7}$$

Thus we are led to consider as a candidate for the geodesic through an arbitrary point $\mathbf{x}(u_0, v_0)$ in an arbitrary direction $(du/ds)_0 : (dv/ds)_0$ the curve $\mathbf{x}(u(s), v(s))$, where $u(s), v(s)$ are the solution to equations (11.7) satisfying the initial conditions

$$u(0) = u_0, \quad v(0) = v_0, \quad \frac{du}{ds}(0) = \left(\frac{du}{ds}\right)_0, \quad \frac{dv}{ds}(0) = \left(\frac{dv}{ds}\right)_0$$

If in equations (11.7) the Γ_{ij}^k are of class C^1, then from the theory of differential equations there will indeed exist a unique solution $u(s), v(s)$ in a neighborhood of $s = 0$ satisfying the given initial conditions. In general, however, s will not represent arc length along $\mathbf{x} = \mathbf{x}(u(s), v(s))$ and hence it does not follow directly from the theorem above that $\mathbf{x} = \mathbf{x}(u(s), v(s))$ is a geodesic. In Problem 11.15, page 251, however, we prove that if $(du/ds)_0 : (dv/ds)_0$ is chosen so that initially

$$\left|\frac{d\mathbf{x}}{ds}\right|^2 = E\left(\frac{du}{ds}\right)_0^2 + 2F\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G\left(\frac{dv}{ds}\right)_0^2 = 1$$

then $|d\mathbf{x}/ds|^2 = 1$ for all s, i.e. s equals arc length, and hence from the theorem above $\mathbf{x} = \mathbf{x}(u(s), v(s))$ is the unique geodesic through $\mathbf{x}(u_0, v_0)$ in the given direction $(du/ds)_0 : (dv/ds)_0$.

Clearly direction numbers $(du/ds)_0:(dv/ds)_0$ satisfying the above equation can be found for any direction $du_0:dv_0$ by setting $(du/ds)_0 = du_0/\lambda$ and $(dv/ds)_0 = dv_0/\lambda$, where $\lambda = E_0\,du_0^2 + 2F_0\,du_0\,dv_0 + G\,dv_0^2$. Finally, the Γ^k_{ij} will be of class C^1 if $\mathbf{x} = \mathbf{x}(u,v)$ is of class C^3. Thus

Theorem 11.9. In the neighborhood of a point P on a surface of class $\geqq 3$ there exists one and only one geodesic through P in any given direction. The geodesic is of class C^3.

Example 11.5.

(a) *Plane.* On a plane, $\mathbf{k} = \mathbf{k}_g$. Hence a curve is a geodesic if and only if it is a straight line. Through every point clearly passes a geodesic in every direction.

(b) *Sphere.* Since the osculating planes along a geodesic are parallel to $\mathbf{N}$, the planes must all pass through the center of the sphere. But from Problem 4.20, page 77, a curve whose osculating planes pass through a fixed point lies on a plane. Thus a geodesic on a sphere is a *great circle,* and conversely. Again it is clear that through any point passes a geodesic in every direction.

(c) *General Cylinder.* Suppose the rulings of the cylinder are in the direction of the constant unit vector $\mathbf{g}$. Since C is a geodesic if and only if $\mathbf{k} = \mathbf{k}_n = (\mathbf{k}\cdot\mathbf{N})\mathbf{N}$ and since on the cylinder $\mathbf{N}\cdot\mathbf{g} = 0$, it follows that C is a geodesic if and only if $\mathbf{k}\cdot\mathbf{g} = 0$ or $\dot{\mathbf{t}}\cdot\mathbf{g} = 0$ or, integrating, $\mathbf{t}\cdot\mathbf{g} = \text{constant}$. Thus the geodesics on a cylinder are general helixes. These include the rulings themselves where $\mathbf{t}\cdot\mathbf{g} = \pm 1$ and the cross sections of the cylinder where $\mathbf{t}\cdot\mathbf{g} = 0$.

Finally, suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on a surface such that the u- and v-parameter curves are orthogonal and such that the first fundamental coefficients depend on only one of the parameters. Then the geodesics can always be found by quadratures. Namely, in Problem 11.17, page 252, we prove

Theorem 11.10. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class $\geqq 2$ such that $E = E(u)$, $F = 0$ and $G = G(u)$. Then

(i) The v-parameter curves $u = \text{constant}$ are geodesics.

(ii) The u-parameter curve $v = v_0$ is a geodesic if and only if $G_u(u_0) = 0$.

(iii) A curve of the form $\mathbf{x} = \mathbf{x}(u, v(u))$ is a geodesic if and only if

$$v \quad = \quad \pm\int \frac{C\sqrt{E}}{\sqrt{G}\sqrt{G-C^2}}\,du \qquad (C = \text{constant})$$

GEODESIC COORDINATES

It is often convenient to be able to introduce coordinates on a surface where the parameter curves have some special properties. A coordinate patch where the parameter curves are orthogonal and one of the families of parameter curves are geodesics is called a *set of geodesic coordinates.*

Geodesic coordinates can be introduced on a surface in an infinite number of ways. For let $\mathbf{x} = \mathbf{x}(v)$, $a \leqq v \leqq b$, be an arbitrary arc C_0 of class C^2 on a surface of class $\geqq 3$. From Theorem 11.9, through each point $\mathbf{x}(v_0)$ on C_0 there passes a unique geodesic $\mathbf{x} = \mathbf{x}(u, v_0)$ perpendicular to C_0 along which u equals arc length and such that $\mathbf{x}(0, v_0) = \mathbf{x}(v_0)$. See Fig. 11-9.

In Problem 11.20, page 253, we prove that for a sufficiently small ϵ, the function $\mathbf{x} = \mathbf{x}(u,v)$, $-\epsilon < u < \epsilon$, $a < v < b$, is a regular parametric representation of class C^2.

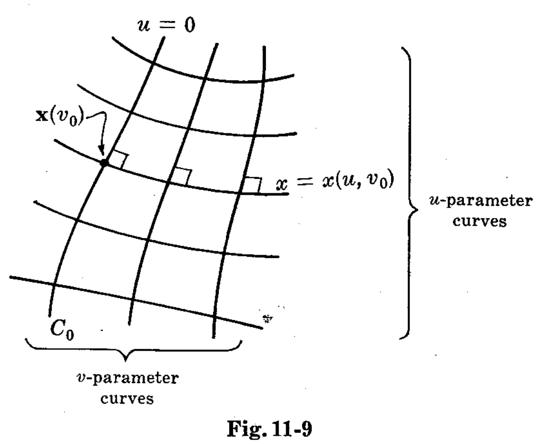


Fig. 11-9

It remains to show that the u- and v-parameter curves are orthogonal. Consider the derivative

$$F_u = (\mathbf{x}_u \cdot \mathbf{x}_v)_u = \mathbf{x}_{uu} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{uv} \qquad (11.8)$$

Since u is arc length, $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$. Hence $E_v = 2\mathbf{x}_{uv} \cdot \mathbf{x}_u = 0$. Also, since u is arc length, $\mathbf{x}_u$ is the unit tangent to the u-parameter curves and $\mathbf{x}_{uu}$ is the curvature vector to the u-parameter curves. But along a geodesic the curvature vector is in the direction of the normal to the surface. Thus also $\mathbf{x}_{uu} \cdot \mathbf{x}_v = 0$. Hence from equation (*11.8*), $F_u = 0$. Thus $F =$ constant at all points along the geodesic $v =$ constant. But the geodesic $v =$ constant is orthogonal to C_0. That is, $F = 0$ for $u = 0$ and for every v. Hence $F \equiv 0$. It follows that the u- and v-parameters are orthogonal and hence $\mathbf{x} = \mathbf{x}(u, v)$ is a set of geodesic coordinates. Thus we have

Theorem 11.11. Let $\mathbf{x} = \mathbf{x}(v)$, $a \leqq v \leqq b$, be an arbitrary arc of class C^2 on a surface S of class $\geqq 3$. Then there exists a set of geodesic coordinates $\mathbf{x} = \mathbf{x}(u, v)$, $-\epsilon < u < \epsilon$, $a < v < b$, on S of class C^2 such that $\mathbf{x}(0, v) = \mathbf{x}(v)$ and such that the u-parameter curves are natural representations of geodesics.

Note that in the above construction the distances along the geodesics included between any two orthogonal trajectories are equal since the parameter along each geodesic is a natural parameter. But this is the case for any set of geodesic coordinates. For let $\mathbf{x} = \mathbf{x}(u, v)$ be a set of geodesic coordinates where the u-parameter curves are geodesics. Since the u- and v-parameter curves are orthogonal, equation (*11.6*) applies and so $(\kappa_g)_{v=\text{constant}} = -E_v/2E\sqrt{G} = 0$. Hence $E_v = 0$ or $E = E(u)$. Also $F = 0$. Thus the first fundamental form of $\mathbf{x} = \mathbf{x}(u, v)$ is of the form

$$\mathrm{I} = E(u)\,du^2 + G(u, v)\,dv^2$$

Now along a geodesic $v =$ constant, $dv = 0$. Hence the arc length along a geodesic between the orthogonal trajectory $u = u_1$ and $u = u_2$ is

$$S = \int_{u_1}^{u_2} \sqrt{\mathrm{I}} = \int_{u_1}^{u_2} \sqrt{E(u)}\,du$$

which is independent of v and hence of the geodesic.

It follows from the discussion above that arc length can always be introduced as a parameter along the geodesics of a set of geodesic coordinates by introducing the parameter transformation

$$u^* = \int_{u_1}^{u} \sqrt{E(t)}\,dt, \qquad v = v$$

If this is done or, equivalently, if $\mathbf{x} = \mathbf{x}(u, v)$ is a set of geodesic coordinates such that u is a natural parameter along the geodesics, then $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$ and the first fundamental form is of the form

$$\mathrm{I} = du^2 + G(u, v)\,dv^2 \qquad (11.9)$$

Finally, directly from the formula

$$K = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right] \qquad (11.10)$$

for the Gaussian curvature when $F = 0$ derived in Problem 10.35 we obtain

Theorem 11.12. If $\mathbf{x} = \mathbf{x}(u, v)$ is a set of geodesic coordinates on a surface of class $\geqq 3$ such that the u-parameter curves are geodesics and u is a natural parameter, then

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2} \qquad (11.11)$$

Example 11.6.

$\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + r^2\mathbf{e}_3$, $1 < r < 3$, $-\infty < \theta < \infty$, is a set of geodesic coordinates on the paraboloid $x_3 = x_1^2 + x_2^2$. For it is easily verified that $E = \mathbf{x}_r\cdot\mathbf{x}_r = 1 + 4r^2$, $F = \mathbf{x}_r\cdot\mathbf{x}_\theta = 0$, $G = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = r^2$. Thus the parameter curves are orthogonal and since, from equation (11.6), $(\kappa_g)_{\theta=\text{constant}} = -E_\theta/2E\sqrt{G} = 0$, the r-parameter curves $\theta = \text{constant}$ are geodesics. If we introduce the parameter $r^* = \int_2^r \sqrt{E}\,d\rho = \int_2^r \sqrt{1+4\rho^2}\,d\rho$ in place of r, the patch $\mathbf{x}^* = \mathbf{x}^*(\theta, r^*) = \mathbf{x}(\theta, r(r^*))$ will be a set of geodesic coordinates with r^* a natural parameter. Then

$$E^* = \mathbf{x}^*_{r^*}\cdot\mathbf{x}^*_{r^*} = (\mathbf{x}_r\cdot\mathbf{x}_r)(dr/dr^*)^2 = (\mathbf{x}_r\cdot\mathbf{x}_r)/(dr^*/dr)^2 = 1$$

$$F^* = \mathbf{x}^*_{r^*}\cdot\mathbf{x}^*_\theta = (\mathbf{x}_r\cdot\mathbf{x}_\theta)(dr/dr^*) = 0$$

$$G^* = \mathbf{x}^*_\theta\cdot\mathbf{x}^*_\theta = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = r^2$$

and

$$K = -\frac{1}{\sqrt{G^*}}\frac{\partial^2\sqrt{G^*}}{\partial r^{*2}} = -\frac{1}{r}\frac{d^2r}{dr^{*2}} = 4/(1+4r^2)$$

where $r = r(r^*)$.

GEODESIC POLAR COORDINATES

Let P be a point on a surface of class $\geqq 3$ and let $\mathbf{g}_1$ and $\mathbf{g}_2$ be arbitrarily chosen orthonormal vectors parallel to the tangent plane at P, as shown in Fig. 11-10. From Theorem 11.9, for each real θ_0 there exists a unique naturally represented geodesic $\mathbf{x} = \mathbf{x}(r, \theta_0)$ through P in the direction of the tangent vector $(\cos\theta_0)\mathbf{g}_1 + (\sin\theta_0)\mathbf{g}_2$. In Problem 11.21, page 254, we prove that there exists an $\epsilon > 0$ such that for $0 < r < \epsilon$, $\mathbf{x} = \mathbf{x}(r,\theta)$ is a regular parametric representation of class C^2 and hence a set of geodesic coordinates, called a set of *geodesic polar coordinates* at P. We also prove that for $0 < r < \epsilon$, $0 \leqq \theta < 2\pi$, $\mathbf{x} = \mathbf{x}(r,\theta)$ is a 1-1 mapping onto a deleted neighborhood of P. Thus a unique geodesic joins P with every point in the neighborhood of P. The θ-parameter curves, $r = \text{constant}$, are called *geodesic circles* and the corresponding value of r is called the *radius* of the geodesic circle.

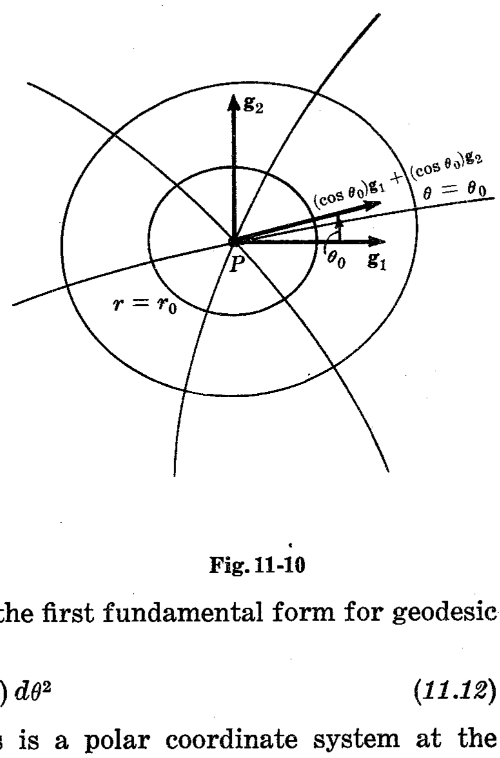


Fig. 11-10

As in the case of geodesic coordinates in general, the first fundamental form for geodesic polar coordinates $\mathbf{x} = \mathbf{x}(r,\theta)$, $r > 0$, is of the form

$$\mathrm{I} = dr^2 + G(r,\theta)\,d\theta^2 \qquad (11.12)$$

A simple example of geodesic polar coordinates is a polar coordinate system at the origin of the x_1x_2 plane. It is given by $\mathbf{x} = r\cos\theta\mathbf{e}_1 + r\sin\theta\mathbf{e}_2$, $r > 0$, $-\infty < \theta < \infty$. Clearly $E = \mathbf{x}_r\cdot\mathbf{x}_r = 1$, $F = \mathbf{x}_r\cdot\mathbf{x}_\theta = 0$ and $G = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = r^2$ and hence $\mathrm{I} = dr^2 + r^2\,d\theta^2$. In Problem 11.23, page 255, we prove that the behavior of $G(r,\theta)$ for small r in the general case is like polar coordinates in the plane. In particular we prove the interesting

Theorem 11.13. If $\mathbf{x} = \mathbf{x}(r,\theta)$ is a set of geodesic polar coordinates at a point P on a surface of sufficiently high class, then

$$\sqrt{G(r,\theta)} = r - \tfrac{1}{6}K(P)r^3 + R(r,\theta) \qquad (11.13)$$

where $\lim_{r\to 0}(R(r,\theta)/r^3) = 0$ and $K(P)$ is the Gaussian curvature at P.

Example 11.7.

The parameter representation

$$\mathbf{x} = (a\cos\theta\sin(r/a))\mathbf{e}_1 + (a\sin\theta\sin(r/a))\mathbf{e}_2 + (a\cos(r/a))\mathbf{e}_3 \qquad a > 0,\ 0 < r < \pi/2a,\ -\infty < \theta < \infty$$

is a set of geodesic polar coordinates at the north pole of the sphere of radius a about the origin as shown in Fig. 11-11. The r-parameter curves $\theta = \theta_0$ are the great circles around the north pole and the θ-parameter curves $r = r_0$ are the parallels of latitude near the pole. It is easily computed that $E = \mathbf{x}_r \cdot \mathbf{x}_r = 1$, $F = \mathbf{x}_r \cdot \mathbf{x}_\theta = 0$, $G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = a^2 \sin^2(r/a)$ and so $\text{I} = dr^2 + a^2\sin^2(r/a)\,d\theta^2$. Since $\sin x = x - \frac{1}{6}x^3 + o(x^3)$, $\sqrt{G} = a\sin(r/a) = r - r^3/6a^2 + o(r^3)$, which agrees with equation (*11.13*).

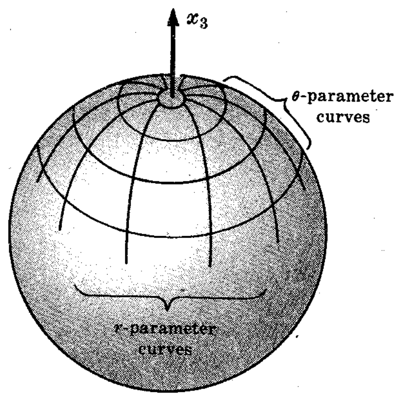


Fig. 11-11

As a consequence of Theorem 11.13 we obtain some very interesting and important interpretations of the Gaussian curvature. For along the geodesic circle $r = \text{constant}$, we have $dr = 0$. Thus its circumference is given by the integral

$$C(r) = \int_0^{2\pi} \sqrt{G(r,\theta)}\,d\theta = 2\pi r - \tfrac{1}{3}K(P)\pi r^3 + o(r^3) \tag{11.14}$$

Hence
$$K(P) = \frac{3}{\pi}\left(\frac{2\pi r - C(r)}{r^3}\right) + o(1)$$

or, since $K(P)$ is independent of r,

$$K(P) = \lim_{r\to 0} \frac{3}{\pi}\left(\frac{2\pi r - C(r)}{r^3}\right) \tag{11.15}$$

Thus we see the Gaussian curvature again expressed in terms of intrinsic properties of the surface.

Finally, recalling the formula $\iint_R \sqrt{EG - F^2}\,du\,dv$ for surface area, we find that the surface area enclosed by a geodesic circle is

$$A(r) = \int_0^r \int_0^{2\pi} \sqrt{G}\,d\theta\,dr = \pi r^2 - (\pi/12)K(P)r^4 + o(r^4)$$

Hence as above,
$$K(P) = \lim_{r\to 0} \frac{12}{\pi}\left(\frac{\pi r^2 - A(r)}{r^4}\right) \tag{11.16}$$

Thus we have

Theorem 11.14. The Gaussian curvature at a point P on a surface of class ≥ 3 is given by

$$K(P) = \lim_{r\to 0} \frac{3}{\pi}\left(\frac{2\pi r - C(r)}{r^3}\right) \quad \text{or} \quad K(P) = \lim_{r\to 0} \frac{12}{\pi}\left(\frac{\pi r^2 - A(r)}{r^4}\right)$$

where r, $C(r)$, and $A(r)$ are respectively the radius, circumference, and surface area enclosed by a geodesic circle about P.

ARCS OF MINIMUM LENGTH

Let P and Q be points on a surface close enough such that there exists a set of geodesic polar coordinates $\mathbf{x} = \mathbf{x}(r, \theta)$ at P containing Q, as shown in Fig. 11-12. Then we will prove that the geodesic $\theta = \text{constant}$ containing Q is the unique arc of minimum length between P and Q.

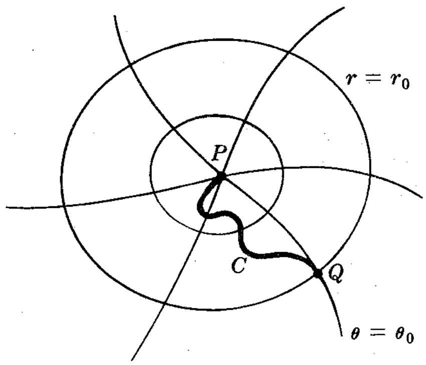


Fig. 11-12

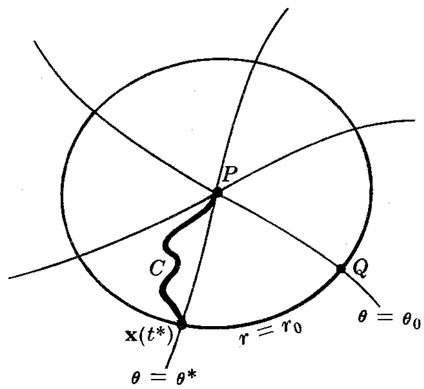


Fig. 11-13

We suppose that Q is on the geodesic circle $r = r_0$ and on the geodesic $\theta = \theta_0$ and that $\mathbf{x} = \mathbf{x}(t)$, $a \leqq t \leqq b$, is an arc C on S connecting P and Q. For the moment we assume that C is contained in $\mathbf{x} = \mathbf{x}(r, \theta)$. Then the length $L(C)$ of C is given by the integral

$$L(C) \quad = \quad \int_a^b \sqrt{(dr/dt)^2 + G(r, \theta)(d\theta/dt)^2}\, dt$$

Since $G > 0$,

$$L(C) \quad \geqq \quad \int_a^b \sqrt{(dr/dt)^2}\, dt \quad \geqq \quad \int_a^b (dr/dt)\, dt \quad = \quad \int_0^{r_0} dr \quad = \quad r_0$$

But r_0 is the length of the geodesic $\theta = \theta_0$ between P and Q, and the equality sign above holds if and only if $d\theta/dt = 0$ or $\theta = \text{constant}$. Thus the geodesic $\theta = \theta_0$ is the unique arc of minimum length between P and Q among arcs contained in $\mathbf{x} = \mathbf{x}(r, \theta)$. But now we can show that it is in fact the shortest of all regular arcs on S between P and Q. For let $\mathbf{x} = \mathbf{x}(t)$, $a \leqq t \leqq b$, be an arc C from P to Q which leaves $\mathbf{x} = \mathbf{x}(r, \theta)$ as shown in Fig. 11-13. Then it can be shown that at some point $\mathbf{x} = \mathbf{x}(t^*)$, where $t^* < t$, C intersects the geodesic circle $r = r_0$ at, say, θ^* and is contained in $\mathbf{x} = \mathbf{x}(r, \theta)$ for $a \leqq t \leqq t^*$. Now let C^* denote the part of C on the interval $a \leqq t \leqq t^*$. Then C^* is a regular arc in $\mathbf{x} = \mathbf{x}(r, \theta)$ connecting P and the point P^* corresponding to (r_0, θ^*). The above argument applied to the points P and P^* gives $L(C^*) \geqq r_0$. But $L(C) > L(C^*)$. Hence $L(C) > r_0$, the length of the geodesic $\theta = \theta_0$ connecting P and Q. Thus

Theorem 11.15. If P and Q are points on a surface such that there exists a set of geodesic polar coordinates at P containing Q, then there exists a unique arc of minimum length between P and Q given by the geodesic joining P and Q.

Using the above theorem we now prove that if C is an arc of minimum length between any two points P_1 and P_2 on a surface of class C^3, then C is a geodesic. We suppose that C is given by $\mathbf{x} = \mathbf{x}(t)$, $a \leqq t \leqq b$, and that P is an arbitrary point on C different from P_1 and P_2. Since S is of class C^3, there exists a set of geodesic polar coordinates $\mathbf{x} = \mathbf{x}(r, \theta)$ at P. Now let $\epsilon > 0$ be chosen, sufficiently small, so that the points P_ϵ and $P_{-\epsilon}$, which correspond to $t + \epsilon$ and $t - \epsilon$, lie in $\mathbf{x} = \mathbf{x}(r, \theta)$, as shown in Fig. 11-14. Let C_ϵ denote the part of C between P and P_ϵ. We want to show that C is an arc of minimum length between P and P_ϵ. For suppose otherwise, i.e. suppose Γ_ϵ is a regular arc between P and

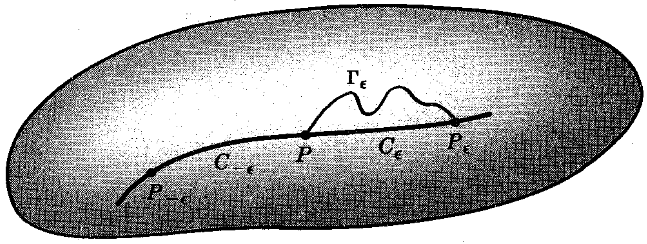


Fig. 11-14

P_ϵ such that the length $L(\Gamma)$ is strictly less than $L(C_\epsilon)$. In particular let $L(\Gamma_\epsilon) + \delta = L(C_\epsilon)$. Now consider the arc Γ obtained by replacing C_ϵ on C by Γ_ϵ. Note that this arc may not be regular at P and P_ϵ. However, it can be shown that such an arc can be rounded off at P and P_ϵ to give an arc C^* whose length differs from the length of Γ by at most $\delta/2$. Since $L(\Gamma) + \delta = L(C)$, it follows that C^* is a regular arc between P_1 and P_2 such that

$$L(C^*) + \delta/2 \leqq L(C) \quad \text{or} \quad L(C^*) < L(C)$$

which is impossible, since C is a regular arc of minimum length between P_1 and P_2. Thus C_ϵ is an arc of minimum length between P and P_ϵ. For the same reason $C_{-\epsilon}$, the part of C between $P_{-\epsilon}$ and P, is an arc of minimum length between $P_{-\epsilon}$ and P.

But then as a consequence of the preceding theorem, $C_{-\epsilon}$ and C_ϵ are geodesic curves, say $\theta = \theta_\epsilon$ and $\theta = \theta_{-\epsilon}$ from P to P_ϵ and $P_{-\epsilon}$ respectively, as shown in Fig. 11-15. But C is regular at P; hence $\theta_{-\epsilon} = \theta_\epsilon + \pi$. Namely, near P, C is the unique geodesic through P in the direction θ. Since P is an arbitrary point on C, the theorem is proved. Thus

Theorem 11.16. If C is an arc of minimum length between two points on a surface of class $\geqq 3$, then C is a geodesic.

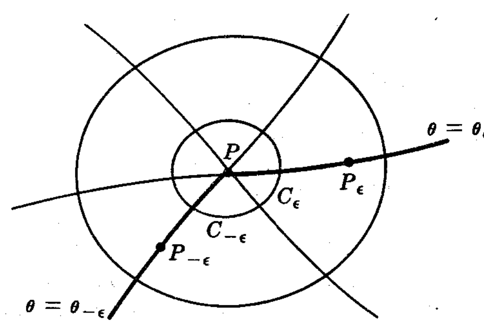


Fig. 11-15

SURFACES WITH CONSTANT GAUSSIAN CURVATURE

Recall that the Gaussian curvature K is invariant under isometric mappings. Thus at corresponding points of two isometric surfaces, the Gaussian curvatures are the same. The converse of this in general is not true, as shown in Problem 11.28. However, if two surfaces have the same *constant* Gaussian curvature, then we will prove that any two sufficiently small neighborhoods of the surfaces are isometric. For suppose that $\mathbf{x} = \mathbf{x}(r, \theta)$ is a set of geodesic polar coordinates at a point on a surface of constant Gaussian curvature K. From equations (*11.11*) and (*11.13*), it follows that $\sqrt{G} = \sqrt{\mathbf{x}_\theta \cdot \mathbf{x}_\theta}$ satisfies the second order linear differential equation with constant coefficients

$$\frac{\partial^2}{\partial r^2}\sqrt{G} + K\sqrt{G} = 0 \qquad (11.17)$$

with initial conditions
$$\lim_{r \to 0} \sqrt{G} = 0, \quad \lim_{r \to 0} (\partial\sqrt{G}/\partial r) = 1 \qquad (11.18)$$

If $K = 0$, then the general solution to equation (*11.17*) is $\sqrt{G} = rC_1(\theta) + C_2(\theta)$. Applying the initial condition (*11.18*) gives $C_1 = 1$ and $C_2 = 0$, and so $G = r^2$. Thus the first fundamental coefficients of $\mathbf{x} = \mathbf{x}(r, \theta)$ must be $E = 1$, $F = 0$ and $G = r^2$. If $K > 0$, then the general solution to (*11.17*) is $\sqrt{G} = C_1(\theta)\sin(r\sqrt{K}) + C_2(\theta)\cos(r\sqrt{K})$. Here the initial conditions (*11.18*) imply that $C_1 = 1/\sqrt{K}$ and $C_2 = 0$. Thus the first fundamental coefficients of $\mathbf{x} = \mathbf{x}(r,\theta)$ in this case must be $E = 1$, $F = 0$, $G = (1/K)\sin^2(r\sqrt{K})$. Finally, if $K < 0$, then the general solution to (*11.17*) is

$$\sqrt{G} \;=\; C_1(\theta)\sinh(r\sqrt{-K}) \;+\; C_2(\theta)\cosh(r\sqrt{-K})$$

Applying the initial condition gives $C_1 = 1/\sqrt{-K}$ and $C_2 = 0$. Thus in this case $E = 1$, $F = 0$, $G = (1/\sqrt{-K})\sinh r\sqrt{-K}$.

Thus we see that the first fundamental coefficients of a set of geodesic polar coordinates at any point P on a surface S of constant K are uniquely determined and depend only on K. But then it is clear that if P^* is any point on any other surface of the same constant Gaussian curvature K and $\mathbf{x}^* = \mathbf{x}^*(r,\theta)$ is a set of geodesic polar coordinates at P^*, then the mapping $\mathbf{f}$ defined by $\mathbf{f}(\mathbf{x}(r,\theta)) = \mathbf{x}^*(r,\theta)$ is an isometric mapping of a neighborhood of P onto a neighborhood of P^*. Thus

Theorem 11.17. (Minding). Any two sufficiently small neighborhoods of surfaces of class $\geqq 3$ having the same constant Gaussian curvature are isometric.

A sphere is an example of a surface of constant positive Gaussian curvature. In Example 11.8 we construct a surface of constant negative curvature called the *pseudosphere.*

Example 11.8.

Let C be the curve in the x_1x_3 plane which starts at the point $x_1 = a$, $x_3 = 0$, $a > 0$, with the property that the segment of the tangent between the point of tangency P and the x_3 axis has length a as shown in Fig. 11-16. C is called a *tractrix* and is clearly given by the solution of the differential equation

$$dx_3/dx_1 \;=\; -\sqrt{a^2 - x_1^2}/x_1, \qquad x_3(a) \;=\; 0$$

The ***pseudosphere*** of ***pseudoradius*** a is the surface of revolution generated by C as it revolves about the x_3 axis. It follows that the pseudosphere is represented by

$$\mathbf{x} \;=\; (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + f(r)\mathbf{e}_3, \qquad 0 < r < a,\; -\infty < \theta < \infty$$

where $f'(r) = -\sqrt{a^2 - r^2}/r$. Here $E = \mathbf{x}_r \cdot \mathbf{x}_r = 1 + (f')^2 = a^2/r^2$, $F = \mathbf{x}_r \cdot \mathbf{x}_\theta = 0$ and $G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = r^2$; hence from equation (*11.10*), page 236, $K = -1/a^2$. Thus the pseudosphere of pseudoradius a has constant Gaussian curvature equal to $-1/a^2$.

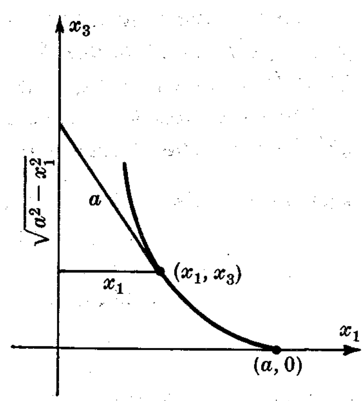


Fig. 11-16

Recall that the direction chosen to correspond to $\theta = 0$ on a set of geodesic coordinates is entirely arbitrary. Thus from Theorem 11.17, neighborhoods of two different points on a surface of constant Gaussian curvature can be mapped isometrically onto each other in such a way that any given direction at one of the points can be made to correspond to any chosen direction through the other point. Intuitively this means that a geometric figure on a surface of constant Gaussian curvature can be translated and rotated freely without changing lengths of curves. It follows that the surfaces of constant positive and negative curvature lead to models of elliptic and hyperbolic non-Euclidean geometries. These models identify the geodesics of the surfaces with the "straight lines" of the geometry. The principal difference in the plane ($K = 0$), elliptic ($K > 0$) and hyperbolic ($K < 0$) geometries lies in the "parallel" axiom. In Euclidean geometry this axiom states that through any point not on a given straight line there exists a unique parallel (nonintersecting) line. In elliptic geometry (the sphere) no parallel lines can be drawn, since any two "straight lines" (great circles) always intersect. In hyperbolic geometry there exist infinitely many parallel lines to a given "straight line" as shown in the example below.

Example 11.9. *Hyperbolic Plane.*

We consider the interior of the circle of radius 2 in the x_1x_2 plane as a coordinate manifold of two dimensions, and with respect to a polar coordinate system (r, θ) at the origin we define the metric tensor

$$g_{11} = E = 1/(1-r^2/4)^2, \quad g_{12} = g_{21} = F = 0, \quad g_{22} = G = r^2/(1-r^2/4)^2$$

From equation (*11.10*) it is easily computed that $K = -1$. This "surface" of constant Gaussian curvature $K = -1$ is called the *hyperbolic plane.* Referring to Theorem 11.10 we find that the geodesics consist of the r-parameter curves, i.e. the Euclidean straight lines through the origin, and the curves

$$\theta = \pm\int \frac{C\sqrt{E}\,dr}{\sqrt{G}\sqrt{G-C^2}} = \pm\int \frac{C(1-r^2/4)\,dr}{r\sqrt{r^2-C^2(1-r^2/4)^2}} \qquad (C = \text{constant})$$

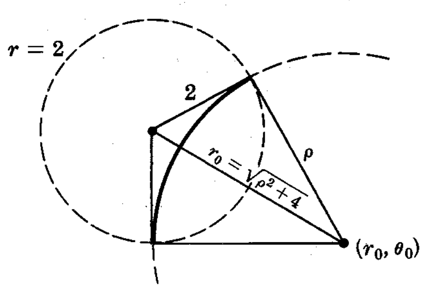
If we let $u = a(1+r^2/4)/r$, where $a = C/\sqrt{1+C^2}$, then

$$1-u^2 = [r^2 - C^2(1-r^2/4)^2]/r^2(1+C^2), \qquad du = -a(1-r^2/4)\,dr/r^2$$

and
$$\theta = \pm\int \frac{C(1-r^2/4)\,dr}{r\sqrt{r^2-C^2(1-r^2/4)^2}} = \mp\int \frac{du}{\sqrt{1-u^2}}$$

Integrating gives
$$\theta - \theta_0 = \cos^{-1} u \quad \text{or} \quad u = a(1+r^2/4)/r = \cos(\theta-\theta_0)$$

or $r^2 + r_0^2 - 2r_0 r\cos(\theta-\theta_0) = \rho^2$, where we let $r_0 = 2/a$ and $\rho^2 = r_0^2 - 4$. This is the equation in polar coordinates of the circle with center at (r_0, θ_0) and radius ρ, as shown in Fig. 11-17(*a*). Observe that $r_0^2 = \rho^2 + 4 > 4$; hence the center lies outside the boundary $r = 2$ and intersects $r = 2$ orthogonally. Thus the "straight lines" (geodesics) of the hyperplane are the straight lines through the origin and the circles which intersect the boundary $r = 2$ orthogonally. As shown in Fig. 11-17(*b*), through a point P, not on the "straight line" C, there exist infinitely many "parallel lines" to C.



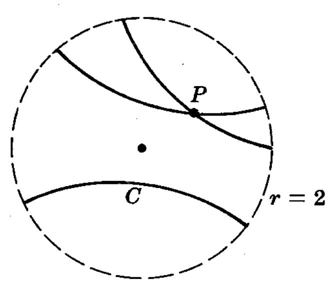


Fig. 11-17

GAUSS-BONNET THEOREM

A finite sequence of regular arcs C_i, $i = 1, \ldots, n$, of class C^m placed end to end, as shown in Fig. 11-18, is called a *Jordan arc C* of class C^m. It is intuitively clear that a Jordan arc C has a single continuous representation $\mathbf{x} = \mathbf{x}(t)$, $t_0 \leq t \leq t_n$, such that its components C_i are represented by $\mathbf{x} = \mathbf{x}(t)$ on sub-intervals, $t_{i-1} \leq t \leq t_i$. It is also clear that a Jordan arc is rectifiable and that its length is equal to the sum of the lengths of its components. We leave the proofs of the above as exercises for the reader.

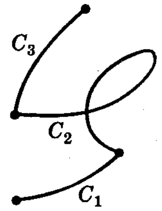


Fig. 11-18

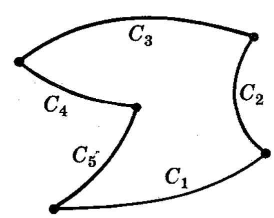


Fig. 11-19

If the end points of a Jordan arc are equal, the Jordan arc is said to be *closed*. A *simple closed* Jordan arc or *curvilinear polygon* is a closed Jordan arc which has no multiple points other than its end points as shown in Fig. 11-19 above. A regular component of a curvilinear polygon is called an *edge* of the polygon and the point between two edges is called a *vertex* of the polygon.

If C: $u=u(t)$, $v=v(t)$ is a curvilinear polygon in the plane, it can be shown (Jordan Curve theorem) that C is the boundary of a bounded simply connected domain D called its *interior*. (A point set D in Euclidean space E is *simply connected* if every closed curvilinear polygon in D can be contracted continuously into a point without leaving D.) Also, if $P(u,v)$ and $Q(u,v)$ are differentiable functions on an open set U in the plane containing a curvilinear polygon C and its interior D and if C has a *positive orientation* around D, i.e. if a small positive rotation of a tangent vector to C points into D, then it can be shown that (Green's theorem)

$$\oint_C \left(P\frac{du}{dt} + Q\frac{dv}{dt}\right)dt \;=\; \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv \tag{11.19}$$

where R is the closed set $D \cup C$.

Now suppose $\mathbf{x}=\mathbf{x}(u,v)$ is a coordinate patch on a surface S defined on an open set U. Clearly a curve C: $\mathbf{x}=\mathbf{x}(t)=\mathbf{x}(u(t),v(t))$ is a curvilinear polygon on the patch if and only if $u=u(t), v=v(t)$ is a curvilinear polygon in U. If C is a curvilinear polygon on the patch, we define the *interior* W of C on $\mathbf{x}=\mathbf{x}(u,v)$ to be the image of the interior of $u=u(t), v=v(t)$ in U; also we say C has positive orientation on the patch if $u=u(t), v=v(t)$ has a positive orientation in the parameter plane. Finally we note that it can be shown that the interior W of C on the patch is a simply connected subset of the patch if and only if the interior of $u=u(t), v=v(t)$ is contained in U as shown in Fig. 11-20.

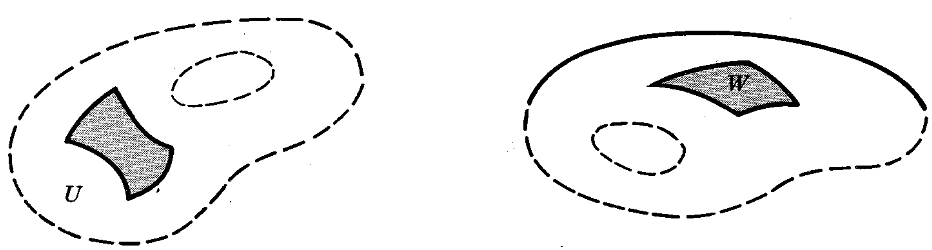


Fig. 11-20

Now suppose that $\mathbf{x}=\mathbf{x}(u,v)$ is a patch on a surface of class ≥ 3 such that the parameter curves are orthogonal, and suppose that $\mathbf{x}=\mathbf{x}(s)=\mathbf{x}(u(s),v(s))$ is a natural representation of a curvilinear polygon C of class C^2 on $\mathbf{x}=\mathbf{x}(u,v)$ which has a positive orientation and whose interior is simply connected. Let

$$\mathbf{g}_1 = \mathbf{x}_u/|\mathbf{x}_u| = \mathbf{x}_u/\sqrt{E} \quad\text{and}\quad \mathbf{g}_2 = \mathbf{x}_v/|\mathbf{x}_v| = \mathbf{x}_v/\sqrt{G}$$

be the unit vectors along C in the direction of the u- and v-parameter respectively and $\theta(s)$ the piecewise differentiable function defined by $\mathbf{t}=(\cos\theta)\mathbf{g}_1+(\sin\theta)\mathbf{g}_2$ where $\mathbf{t}$ is the unit tangent along C as shown in Fig. 11-21. Note that $\theta(s)$ has a jump at each vertex P_i of C equal to an angle α_i, where $-\pi<\alpha_i<\pi$. The angle α_i is called the *exterior* angle of C at P_i.

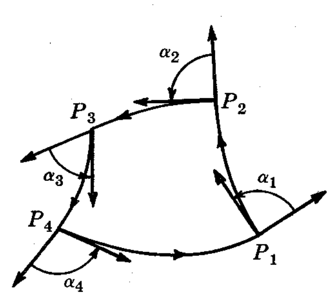


Fig. 11-21

Now in Problem 11.19, page 253, we will prove (*Liouville's formula*) that along each edge of C the geodesic curvature is given by

$$\kappa_g = d\theta/ds + \kappa_1 \cos\theta + \kappa_2 \sin\theta \tag{11.20}$$

where κ_1 and κ_2 are the geodesic curvatures of the parameter curves $v = \text{constant}$ and $u = \text{constant}$ respectively. Thus

$$\int_C \kappa_g\, ds = \int_C \frac{d\theta}{ds}\, ds + \int_C (\kappa_1 \cos\theta + \kappa_2 \sin\theta)\, ds$$

Now $$\cos\theta = \mathbf{t}\cdot\frac{\mathbf{x}_u}{\sqrt{E}} = \left(\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right)\cdot\frac{\mathbf{x}_u}{\sqrt{E}} = \frac{\mathbf{x}_u\cdot\mathbf{x}_u}{\sqrt{E}}\frac{du}{ds} = \sqrt{E}\frac{du}{ds} \tag{11.21a}$$

and $$\sin\theta = \mathbf{t}\cdot\frac{\mathbf{x}_v}{\sqrt{G}} = \left(\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right)\cdot\frac{\mathbf{x}_v}{\sqrt{G}} = \frac{\mathbf{x}_v\cdot\mathbf{x}_v}{\sqrt{G}}\frac{dv}{ds} = \sqrt{G}\frac{dv}{ds} \tag{11.21b}$$

where we used the fact that the parameter curves are orthogonal and so $\mathbf{x}_u\cdot\mathbf{x}_v = 0$. Thus, substituting in the above integral,

$$\int_C \kappa_g\, ds = \int_C d\theta + \int_C \left(\kappa_1\sqrt{E}\frac{du}{ds} + \kappa_2\sqrt{G}\frac{dv}{ds}\right) ds$$

From Green's theorem [(*11.19*), page 243] it follows that

$$\int_C \kappa_g\, ds = \int_C d\theta + \iint_{R'} \left[\frac{\partial}{\partial u}(\kappa_2\sqrt{G}) - \frac{\partial}{\partial v}(\kappa_1\sqrt{E})\right] du\, dv$$

where R' is the interior and boundary of $u = u(s)$, $v = v(s)$ in the plane. From equation (*11.6*), page 233,

$$\begin{aligned}\int_C \kappa_g\, ds &= \int_C d\theta + \iint_{R'} \left[\frac{\partial}{\partial u}\frac{G_u}{2\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{2\sqrt{EG}}\right] du\, dv \\ &= \int_C d\theta + \iint_{R'} \frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right]\sqrt{EG}\, du\, dv\end{aligned}$$

and hence from the formula (*11.10*) for the Gaussian curvature,

$$\int_C \kappa_g\, ds = \int_C d\theta - \iint_{R'} K\sqrt{EG}\, du\, dv$$

Thus the formula $$\int_C \kappa_g\, ds = \int_C d\theta - \iint_R K\, ds$$

where R is the union of C and its interior on S. It remains to evaluate $\int_C d\theta$. Since C is a simple curve, it can be shown that the total change in θ around C is 2π. A proof of this can be found in a textbook on topology. Since $\int_C d\theta$ measures the change in θ along the edges, we have $\int_C d\theta + \sum_i \alpha_i = 2\pi$. Thus

Theorem 11.18. **Gauss-Bonnet Formula.** Let C be a curvilinear polygon of class C^2 on a patch on a surface of class ≥ 3. Suppose that C has a positive orientation and suppose that its interior on the patch is simply connected. Then

$$\int_C \kappa_g\, ds + \iint_R K\, dS = 2\pi - \sum_i \alpha_i \tag{11.22}$$

where κ_g is the geodesic curvature along C, R is the union of C and its interior, K the Gaussian curvature and α_i are the exterior angles of C.

Example 11.10.

Suppose that C consists of three geodesics, forming a *geodesic triangle*. Since $\kappa_g = 0$ on C, the Gauss-Bonnet formula becomes $\iint_R K\,ds = 2\pi - \sum_i \alpha_i$. If $\beta_i = \pi - \alpha_i$, $i = 1, 2, 3$, denotes the *interior* angles of the triangle as shown in Fig. 11-22, then

$$\iint_R K\,ds = \sum_{i=1}^{3} \beta_i - \pi$$

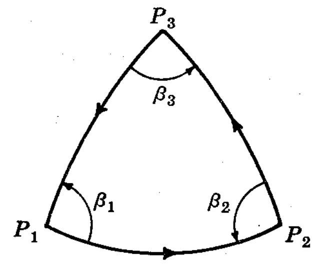


Fig. 11-22

For a sphere of radius a, $K = 1/a^2$, and the formula becomes

$$\sum_{i=1}^{3} \beta_i = \pi + A/a^2$$

where A is the area of the geodesic triangle. If $K = -1$, a pseudosphere, then $\sum_{i=1}^{3} \beta_i = \pi - A/a^2$. Thus the sum of the interior angles of a geodesic triangle is greater than, less than, or equal to π, depending on whether the Gaussian curvature is positive, negative, or zero.

We now suppose that S is a compact (closed and bounded) and orientable surface. It can be shown that such a surface can be covered by a finite number of regions R_i, $i = 1, \ldots, n$, where each R_i consists of a curvilinear polygon C_i with its interior W_i, such that if any two of the R_i overlap they do so on either a single common edge or a common vertex as shown in Fig. 11-23. The covering R_i, $i = 1, \ldots, n$, is called a *polygonal decomposition* of S. In particular if an orientation is selected, then there exists a polygonal decomposition with positively oriented polygons such that on any two overlapping edges the orientations are opposite as shown in Fig. 11-24.

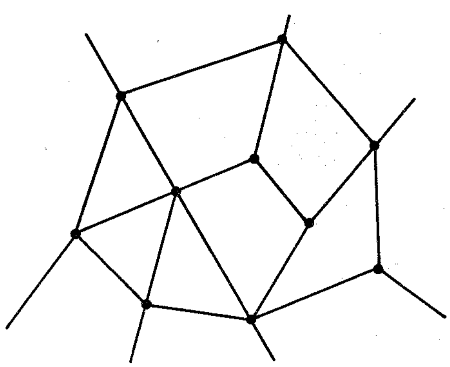


Fig. 11-23

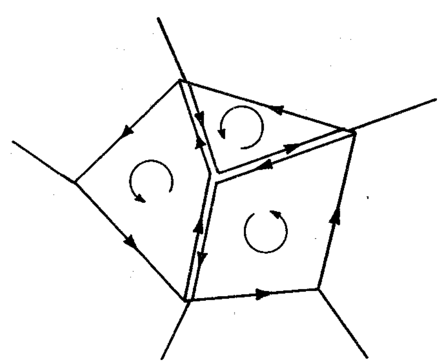


Fig. 11-24

Now suppose that the Gauss-Bonnet formula is applied to each polygon C_i of such a covering. This gives

$$\int_{C_i} \kappa_g\,ds + \iint_{R_i} K\,ds = 2\pi - k_i\pi + \sum_j \beta_{ij}$$

where k_i is the number of edges of C_i and $\beta_{ij} = \pi - \alpha_{ij}$ is the *interior angle* at a vertex as in the example above. Suppose further that the above equation is summed over all polygons C_i. Since each edge is covered twice in the sum with opposite orientations, we have $\sum_i \left[\int_{C_i} \kappa_g\,ds\right] = 0$. Thus $\iint_S K\,dA = 2\pi \sum_{i=1}^{n} 1 - \pi \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} \sum_j \beta_{ij}$. Since each edge occurs twice in the sum $\sum_{i=1}^{n} k_i$ and since each vertex contributes 2π to the sum $\sum_{i=1}^{n} \sum_j \beta_{ij}$, we have finally the formula

$$\iint_S K\,dA = 2\pi(a_2 - a_1 + a_0) \qquad (11.23)$$

where $a_2 = n$ is the total number of polygons, a_1 the total number of edges and a_0 the total number of vertices of the decomposition. The integral $\iint_S K\,dS$ is called the *total curvature* of S.

As a first consequence of the above equation we see that the integer $\chi = a_2 - a_1 + a_0$ depends only on the surface and not on the polygonal decomposition of the surface. It is called the *Euler characteristic* of the surface. But also the Euler characteristic is invariant under 1-1 bicontinuous mappings of surfaces. For it is intuitively clear that a polygonal decomposition of a surface S will map into a polygonal decomposition of its image with the same number of polygons, edges and vertices. But then it follows that for such compact orientable surfaces the total curvature $\iint_S K\,dS$ is in fact a topological invariant!

We state equation (*11.23*) formally as

Theorem 11.19. **Gauss-Bonnet Theorem.** If S is an orientable compact surface of class C^3, then

$$\iint_S K\,dS = 2\pi\chi(s)$$

where K is the Gaussian curvature of S and $\chi(s)$ is the Euler characteristic of S.

Example 11.11.

(*a*) Observe in Fig. 11-25 a polygonal decomposition of a sphere consisting of 4 polygons, 6 edges and 4 vertices. Thus the Euler characteristic of the sphere is two, and from equation (*11.23*) the total curvature of the sphere is 4π.

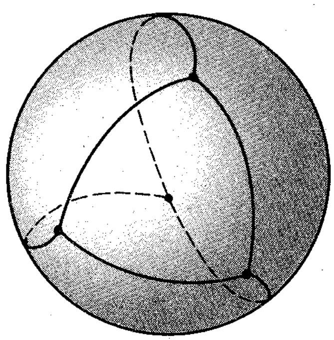


Fig. 11-25

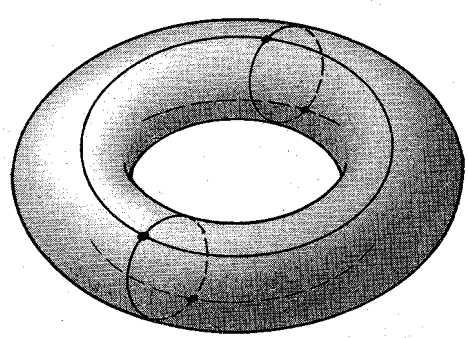


Fig. 11-26

(*b*) In Fig. 11-26 we observe a polygonal decomposition of the torus consisting of 4 polygons, 8 sides and 4 vertices. Thus the Euler characteristic of a torus is 0. The total curvature is also 0.

(*c*) The torus is an example of a sphere with one handle. See page 258. As illustrated in Fig. 11-27, a sphere with two handles is obtained by attaching a torus to a sphere with one handle along a polygon. This has the effect of reducing the total number of polygons of both surfaces by 2, the total number of edges by the number of edges of the polygon, and the total number of vertices by the number of vertices of the polygon. Since the number of edges and vertices of a polygon are the same, the Euler characteristic of the sphere with two handles is -2. In general the Euler characteristic of the surface obtained by adding a handle to a surface with $p-1$ handles is 2 less than the Euler characteristic of the original surface. Thus the formula $\chi = 2(1-p)$ for the Euler characteristic of a surface with p handles.

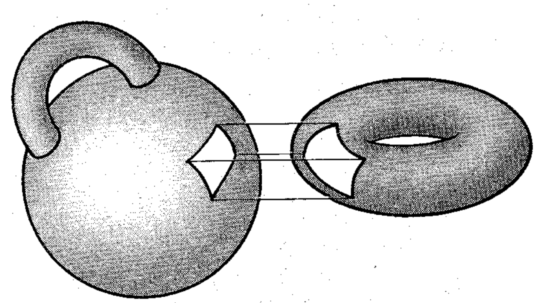


Fig. 11-27

Solved Problems

MAPPINGS OF SURFACES

11.1. Prove that a regular differentiable mapping **f** of a surface S into a surface S^* is a continuous mapping of S into S^*.

Suppose $\mathbf{f}(P_0) = P_0^*$ is a point on S^* and $S_\epsilon(P_0^*)$ is an arbitrary neighborhood of P_0^*. From the corollary on page 229, there exists a patch D on S containing P_0 on which **f** is continuous. Thus there exists a neighborhood $S_{\delta_1}(P_0)$ such that $\mathbf{f}(P) \in S_\epsilon(P_0^*)$ for P in $S_{\delta_1}(P_0) \cap D$. Now, from Problem 8.13, for any point P_0 on a patch D on a surface S there exists a neighborhood $S_{\delta_2}(P_0)$ such that $S_{\delta_2}(P_0) \cap S \subset D$. Thus for P in $S_\delta(P_0) \cap S$, where $\delta = \min(\delta_1, \delta_2)$, we have P in $S_{\delta_1}(P_0) \cap D$ and hence $\mathbf{f}(P)$ in $S_\epsilon(P_0^*)$. It follows that **f** is continuous at P_0. Since P_0 is arbitrary, **f** is a continuous mapping of S into S^*.

11.2. Prove that if **f** is a regular differentiable mapping of S into S^* and **g** is a regular differentiable mapping of S^* into S^{**}, then the composite mapping $\mathbf{g} \circ \mathbf{f}$ is a regular differentiable mapping of S into S^{**}.

Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on S defined on U. It is required to show that $\mathbf{x}^{**} = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u,v))$ is a regular parametric representation on S^{**}. Since **f** is a regular differentiable mapping of S into S^*, it follows from Theorem 11.1 that there exists a neighborhood $S(u,v)$ of each (u,v) in U such that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a coordinate patch on S^* for (u,v) in $S(u,v)$. Since **g** is a regular differentiable mapping of S^* into S^{**}, then $\mathbf{x}^{**} = \mathbf{g}(\mathbf{x}^*(u,v)) = \mathbf{g}(\mathbf{f}(\mathbf{x}(u,v))) = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u,v))$ is a regular parametric representation on S^{**} for (u,v) in $S(u,v)$. Since (u,v) is an arbitrary point in U, $\mathbf{x}^{**} = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u,v))$ is a regular parametric representation on S^{**} defined on U, which is the required result.

11.3. Let **f** be a mapping of S into S^* such that for every coordinate patch $\mathbf{x} = \mathbf{x}(u,v)$ of a basis of patches for S the mapping $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation of class C^r. Prove that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation of class C^r for all coordinate patches $\mathbf{x} = \mathbf{x}(u,v)$ on S and hence **f** is a regular differentiable mapping of class C^r of S into S^*.

Let $\mathbf{x} = \mathbf{x}(u,v)$ be an arbitrary patch on S defined on U. It is required to show that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation on S^*. Let (u,v) be an arbitrary point in U and let P be the point on $\mathbf{x} = \mathbf{x}(u,v)$ corresponding to (u,v). Let $\mathbf{x} = \mathbf{y}(\theta,\phi)$ be a patch of the basis which contains P. From Theorem 8.3, page 157, the intersection of $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{y}(\theta,\phi)$ is a patch on S defined on an open set W containing (u,v) on which $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ is an allowable parameter transformation. Since $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v)) = \mathbf{f}(\mathbf{y}(\theta(u,v), \phi(u,v)))$, where $\mathbf{f}(\mathbf{y}(\theta,\phi))$ is a regular parametric representation, it follows that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation defined on W. Since (u,v) is an arbitrary point in U, it follows that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ is a regular parametric representation defined on U.

11.4. If **f** is a 1-1 regular differentiable mapping of a surface S onto a surface S^*, prove that $\mathbf{f}^{-1}$ is a regular differentiable mapping of S^* onto S.

Let $\mathbf{x}^* = \mathbf{x}^*(u,v)$ be a patch on S^* defined on U. It is required to show that $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u,v))$ is a regular parametric representation on S. As in the preceding problems, it is sufficient to show that $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u,v))$ is a regular parametric representation for some neighborhood of an arbitrary point (u,v) in U. We let P^* denote the image of (u,v) under $\mathbf{x}^* = \mathbf{x}^*(u,v)$ and P the image of P^* under $\mathbf{f}^{-1}$. Now let $\mathbf{x} = \mathbf{x}(\theta,\phi)$ be a patch on S containing P. Since **f** is a regular differentiable mapping of S into S^*, $\mathbf{x}^* = \mathbf{x}^*(\theta,\phi) = \mathbf{f}(\mathbf{x}(\theta,\phi))$ is a regular parametric representation on S^* which contains P^*. From Theorem 11.1, page 228, we can assume that $\mathbf{x}^* = \mathbf{x}^*(\theta,\phi)$ is a patch. From Theorem 8.3, page 157, the intersection of the patches $\mathbf{x}^* = \mathbf{x}^*(u,v)$ and $\mathbf{x}^* = \mathbf{x}^*(\theta,\phi)$ on S^* is a patch which contains P^* such that $\theta = \theta(u,v)$, $\phi = \phi(u,v)$ is an allowable parameter transformation. But then on the intersection, $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u,v)) = \mathbf{x}(\theta(u,v), \phi(u,v))$ is a regular parametric representation on S, which completes the proof.

ISOMETRIC MAPPINGS

11.5. If $\mathbf{f}$ is an isometry of a surface S onto a surface S^*, prove that $\mathbf{f}^{-1}$ is an isometry of S^* onto S.

From Problem 11.4, $\mathbf{f}^{-1}$ is a regular differential mapping of S^* on S. Thus for any arc C^* on S^*, $\mathbf{f}^{-1}(C^*)$ is a regular arc on S and, since $\mathbf{f}$ is an isometric mapping of S into S^*,

$$L(\mathbf{f}^{-1}(C)) = L(\mathbf{f}(\mathbf{f}^{-1}(C^*)) = L(C^*)$$

where $L(C^*)$ is the length of C^*, which proves that $\mathbf{f}^{-1}$ is an isometry from S^* onto S.

11.6. Prove Theorem 11.4: The intrinsic distance $D(P,Q)$ from a point P to a point Q on a surface satisfies

(i) $D(P,Q) = D(Q,P)$

(ii) $D(P,R) \leqq D(P,Q) + D(Q,R)$

(iii) $D(P,Q) \geqq 0, \quad D(P,Q) = 0$ iff $P = Q$.

(i) Since the length $L(C)$ of a regular arc C from P to Q is independent of the orientation of C, the set of numbers $L(C)$ for all regular arcs C from P to Q is independent of the orientation of C. Thus $D(P,Q)$, which is the infimum of the numbers $L(C)$, is independent of the orientation of C. Hence $D(P,Q) = D(Q,P)$.

(ii) Since $D(P,Q)$ is the infimum of the lengths of the arcs from P to Q, for arbitrary $\epsilon > 0$ there exists a regular arc C_1 from P to Q such that $L(C_1) \leqq D(P,Q) + \epsilon$. For the same reason there exists a regular arc C_2 from Q to R such that $L(C_2) \leqq D(Q,R) + \epsilon$. Now the arc obtained by joining C_2 to C_1 will in general have a "corner" at Q and hence not be a regular arc from P to R. However, it can be shown that there exists a regular arc from P to R which at worst is slightly longer. Namely, there exists an arc C from P to R such that $L(C) \leqq L(C_1) + L(C_2) + \epsilon$. But then it follows that

$$D(P,R) \leqq L(C) \leqq L(C_1) + L(C_2) + \epsilon \leqq D(P,Q) + D(Q,R) + 3\epsilon$$

Since ϵ is arbitrary, we have $D(P,R) \leqq D(P,Q) + D(Q,R)$.

(iii) Since $L(C) \geqq 0$, for any arc C from P to Q, $D(P,Q) \geqq 0$. If $P = Q$, for any $\epsilon > 0$, there exists a regular arc C from P to Q such that $L(C) \leqq \epsilon$. Since $D(P,Q) \leqq L(C) \leqq \epsilon$ for arbitrary ϵ, it follows that $D(P,Q) = 0$. Conversely, suppose $D(P,Q) = 0$. Then for an arbitrary $\epsilon > 0$ there exists an arc C from P to Q such that $L(C) \leqq D(P,Q) + \epsilon = \epsilon$. But the Euclidean distance $|P - Q| \leqq L(C)$. Since ϵ is arbitrary, $|P - Q| = 0$ or $P = Q$.

11.7. Let $\mathbf{f}$ be a local isometric mapping of a surface S into a surface S^*. Prove that for any two points P and Q on S, the intrinsic distance $D(P,Q) = D(\mathbf{f}(P), \mathbf{f}(Q))$.

Since $D(P,Q)$ is the infimum of the lengths of the arcs between P and Q, given an arbitrary $\epsilon > 0$, there exists an arc C joining P and Q such that its length $L(C)$ satisfies $L(C) \leqq D(P,Q) + \epsilon$. Now let $C^* = \mathbf{f}(C)$. Since $\mathbf{f}$ is a local isometry, $L(C^*) = L(C)$. Thus $D(\mathbf{f}(P), \mathbf{f}(Q)) \leqq L(C^*) = L(C) \leqq D(P,Q) + \epsilon$. Since ϵ is arbitrary, the required result follows.

11.8. Let $\mathbf{y} = \mathbf{y}(s,v) = \mathbf{x}(s) + v\mathbf{t}(s)$, $v > 0$, be a branch of the tangent surface of a natural represented curve $\mathbf{x} = \mathbf{x}(s)$ without points of inflection (see Problem 8.19, page 167). Prove that a neighborhood of every point on the tangent surface can be mapped isometrically onto a subset of the plane.

From the fundamental theorem for curves, there exists a natural representation of a curve $\mathbf{x}^* = \mathbf{x}^*(s)$ in the x_1x_2 plane such that along $\mathbf{x}^* = \mathbf{x}^*(s)$ the curvature $\kappa^*(s)$ is equal to the curvature $\kappa(s)$ along $\mathbf{x} = \mathbf{x}(s)$. Given a point P on the tangent surface and a patch $\mathbf{y} = \mathbf{y}(s,v)$ containing P, define the mapping $\mathbf{f}$ of the patch into the plane by $\mathbf{y}^* = \mathbf{f}(\mathbf{y}(s,v)) = \mathbf{f}(\mathbf{x}(s) + v\mathbf{t}(s)) = \mathbf{x}^*(s) + v\mathbf{t}^*(s)$. Here $\mathbf{y}_s^* = \dot{\mathbf{x}}^* + v\dot{\mathbf{t}} = \mathbf{t}^* + v\kappa^*\mathbf{n}^*$ and $\mathbf{y}_v^* = \mathbf{t}^*$ are continuous and $|\mathbf{y}_s^* \times \mathbf{y}_v^*| = v\kappa^* \neq 0$, since $v > 0$ and $\kappa^* = \kappa \neq 0$; hence $\mathbf{f}$ is a regular differentiable mapping. From Theorem 11.1, page 228, we can assume that the patch containing P is small enough so that $\mathbf{f}$ is 1-1. Now along the patch $\mathbf{y} = \mathbf{y}(s,v)$ we have

$$E = \mathbf{y}_s \cdot \mathbf{y}_s = (\mathbf{t} + v\kappa\mathbf{n}) \cdot (\mathbf{t} + v\kappa\mathbf{n}) = 1 + v^2\kappa^2$$

$$F = \mathbf{y}_s \cdot \mathbf{y}_v = (\mathbf{t} + v\kappa\mathbf{n}) \cdot (\mathbf{t}) = 1, \qquad G = \mathbf{y}_v \cdot \mathbf{y}_v = \mathbf{t} \cdot \mathbf{t} = 1$$

On $y^* = f(y(s,v)) = x^*(s) + vt^*(s)$ we have

$$E^* = y_s^* \cdot y_s^* = 1 + v^2(\kappa^*)^2, \quad F^* = y_s^* \cdot y_v^* = 1, \quad G^* = y_v^* \cdot y_v^* = 1$$

But $\kappa = \kappa^*$. Hence $E = E^*$, $F = F^*$, $G = G^*$. From Theorem 11.3, page 230, it follows that f is an isometry.

11.9. If f is an isometry from S onto S^* and $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on S, prove that $E = E^*$, $F = F^*$, $G = G^*$, where E, F and G are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and E^*, F^* and G^* are the first fundamental coefficients on $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$.

Let (u,v) be an arbitrary point in the domain of $\mathbf{x} = \mathbf{x}(u,v)$. Let $u = u(t)$, $v = v(t)$, $a \leqq t \leqq b$, be an arbitrary arc through (u,v) and let C_τ and C_τ^* be the images on S and S^* respectively of $u = u(t)$, $v = (t)$ on the interval $a \leqq t \leqq \tau$. Since f is an isometry of S onto S^*,

$$L(C_\tau) = \int_a^\tau \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt = L(C_\tau^*)$$

$$= \int_a^\tau \sqrt{E^*\left(\frac{du}{dt}\right)^2 + 2F^*\frac{du}{dt}\frac{dv}{dt} + G^*\left(\frac{dv}{dt}\right)^2}\, dt$$

But the above is valid for all τ. Hence for all t, and in particular at (u,v),

$$E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 = E^*\left(\frac{du}{dt}\right)^2 + 2F^*\frac{du}{dt}\frac{dv}{dt} + G^*\left(\frac{dv}{dt}\right)^2$$

But the curve $u = u(t)$, $v = v(t)$ through (u,v) is arbitrary. Hence the above equation is also valid at (u,v) for all du/dt, dv/dt. It follows that $E = E^*$, $F = F^*$ and $G = G^*$ at (u,v). Since (u,v) is arbitrary, the result follows.

11.10. A regular differentiable mapping f of a surface S into a surface S^* is said to be *conformal* if for every patch $\mathbf{x} = \mathbf{x}(u,v)$ on S there exists a function $\lambda(u,v) > 0$ such that for all (u,v)

$$E = \lambda E^*, \quad F = \lambda F^* \quad \text{and} \quad G = \lambda G^*$$

where E, F and G and E^*, F^* and G^* are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ respectively. Prove that a conformal mapping preserves the angle between intersecting oriented curves. By the angle between two intersecting oriented curves $\mathbf{x} = \mathbf{x}(t)$ and $\xi = \xi(\tau)$ we mean the angle $\theta = \measuredangle(\mathbf{x}', \xi')$, between their tangents at the point of intersection.

Suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P and $\mathbf{x} = \mathbf{x}(u(t), v(t))$ and $\xi = \mathbf{x}(\eta(\tau), \zeta(\tau))$ are two curves on S intersecting at P with tangents $\mathbf{x}' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and $\xi' = \mathbf{x}_u \eta' + \mathbf{x}_v \zeta'$ at P respectively. If $\theta = \measuredangle(\mathbf{x}', \xi')$, then from equation (9.6), page 172,

$$\cos\theta = \frac{Eu'\eta' + F(u'\zeta' + v'\eta') + Gv'\zeta'}{[E(u')^2 + 2Fu'v' + G(v')^2]^{1/2}[E(\eta')^2 + 2F\eta'\zeta' + G(\zeta')^2]^{1/2}}$$

On the other hand if θ^* is the angle between the tangents $\mathbf{x}^{*\prime} = \mathbf{x}_u^* u' + \mathbf{x}_v^* v'$ and $\xi^{*\prime} = \mathbf{x}_u^* \eta' + \mathbf{x}_v^* \zeta'$ of the images $\mathbf{x}^* = \mathbf{x}^*(u(t), v(t))$ and $\xi^* = \mathbf{x}^*(\eta(\tau), \zeta(\tau))$ of the curves on S^*, then

$$\cos\theta^* = \frac{E^*u'\eta' + F^*(u'\zeta' + v'\eta') + G^*v'\zeta'}{[E^*(u')^2 + 2F^*u'v' + G^*(v')^2]^{1/2}[E^*(\eta')^2 + 2F^*\eta'\zeta' + G^*(\zeta')^2]^{1/2}}$$

But at corresponding points $E = \lambda E^*$, $F = \lambda F^*$ and $G = \lambda G^*$. Hence $\cos\theta = \cos\theta^*$ or $\theta = \theta^*$, which completes the proof.

11.11. Two surfaces S and S^* are said to be *applicable* if there exists a continuous family $\mathbf{f}_\lambda$, $0 \leqq \lambda \leqq 1$, of mappings of S into E^3 such that (i) $\mathbf{f}_0(S) = S$, (ii) $\mathbf{f}_1(S) = S^*$, (iii) for all λ the mappings $\mathbf{f}_\lambda$ are isometric mappings of S onto $\mathbf{f}_\lambda(S)$. Intuitively S and S^* are applicable if S can be continuously and isometrically bent onto S^*. If S and S^* are applicable, we say that S^* can be obtained from S by *bending*. A property of a surface which is invariant under such a continuous family of isometries is called a *bending invariant.* Clearly if S and S^* are applicable, then they are isometric. The converse is not true in general. Prove that a neighborhood of every point on a branch of the tangent surface of a curve can be bent onto the plane (see Problem 11.8).

Let $\mathbf{y} = \mathbf{y}(s,v) = \mathbf{x}(s) + v\mathbf{t}(s)$, $v > 0$, be the tangent surface of a curve $\mathbf{x} = \mathbf{x}(s)$ without points of inflection. From the fundamental theorem of curves for each λ, $0 \leqq \lambda \leqq 1$, there exists a curve $\mathbf{x} = \mathbf{x}_\lambda(s)$, with curvature $\kappa(s)$ and torsion $(1-\lambda)\tau(s)$, where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion along $\mathbf{x} = \mathbf{x}(s)$. Note that $\mathbf{x} = \mathbf{x}_0(s) = \mathbf{x}(s)$ and that $\mathbf{x} = \mathbf{x}_1(s)$ is a plane curve since its torsion is zero. It can also be shown that $\mathbf{x}_\lambda(s)$ is continuous in λ. Now consider the family of mappings $\mathbf{f}_\lambda$ of the tangent surface defined by $\mathbf{x} = \mathbf{f}_\lambda(\mathbf{y}(s,v)) = \mathbf{x}_\lambda(s) + v\mathbf{t}_\lambda(s)$, $v > 0$. As in Problem 11.8 it is easily verified that for each λ, $\mathbf{f}_\lambda$ is a regular differentiable mapping of a patch on the tangent surface onto its image. Clearly $\mathbf{f}_0(\mathbf{y}(s,v)) = \mathbf{y}(s,v)$ and $\mathbf{f}_1(\mathbf{y}(s,v))$ is a subset of the plane generated by the tangents to the plane curve $\mathbf{x} = \mathbf{x}_1(s)$. Finally, for each λ, $E_\lambda = (\mathbf{x}_\lambda)_s \cdot (\mathbf{x}_\lambda)_s = 1 + v^2\kappa^2 = E_0$, $F_\lambda = (\mathbf{x}_\lambda)_s \cdot (\mathbf{x}_\lambda)_v = 1 = F$, and $G_\lambda = (\mathbf{x}_\lambda)_v \cdot (\mathbf{x}_\lambda)_v = 1 = E$, which proves the proposition.

GEODESICS

11.12. Determine the geodesics on the right circular cone

$$\mathbf{x} = (u \sin\alpha \cos\theta)\mathbf{e}_1 + (u\sin\alpha\sin\theta)\mathbf{e}_2 + (u\cos\alpha)\mathbf{e}_3 \qquad \alpha = \text{constant},\ 0 < \alpha < \pi/2,\ u > 0$$

by solving equations *(11.7)*, page 234.

$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_\theta = 0$, $G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = u^2\sin^2\alpha$; $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = \Gamma^2_{22} = 0$, $\Gamma^1_{22} = -u\sin^2\alpha$, $\Gamma^2_{12} = 1/u$. Thus the second of equations *(11.7)* is $\dfrac{d^2\theta}{ds^2} = -(2/u)\dfrac{du}{ds}\dfrac{d\theta}{ds}$. Setting $\phi = \dfrac{d\theta}{ds}$ gives $\dfrac{1}{\phi}\dfrac{d\phi}{ds} = \dfrac{-2}{u}\dfrac{du}{ds}$. Hence $\log\phi = -2\log u + K$ or $\phi = \dfrac{d\theta}{ds} = C/u^2\sin^2\alpha$, where $C = e^K\sin^2\alpha$. Since s is arc length,

$$1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_\theta\frac{d\theta}{ds}\right|^2 = E\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{d\theta}{ds} + G\left(\frac{d\theta}{ds}\right)^2$$

or $1 = \left(\dfrac{du}{ds}\right)^2 + u^2\sin^2\alpha\left(\dfrac{d\theta}{ds}\right)^2$. Substituting $d\theta/ds = C/u^2\sin^2\alpha$ gives

$$du/ds = \sqrt{u^2\sin^2\alpha - C^2}/u\sin\alpha$$

Hence $\qquad du/d\theta = (1/C)u\sin\alpha\sqrt{u^2\sin^2\alpha - C^2} \qquad$ or $\qquad u = A\sec[(\sin\alpha)\theta + B]$

where A = constant, B = constant.

11.13. Show that the geodesic curvature κ_g of a naturally represented curve $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ of class C^2 on a patch $\mathbf{x} = \mathbf{x}(u,v)$ of class C^2 is given by

$$\kappa_g = \left[\Gamma^2_{11}\left(\frac{du}{ds}\right)^3 + (2\Gamma^2_{12} - \Gamma^1_{11})\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right) + (\Gamma^2_{22} - 2\Gamma^1_{12})\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2 - \Gamma^1_{22}\left(\frac{dv}{ds}\right)^3 + \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right]\sqrt{EG-F^2}$$

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}, \quad \mathbf{k} = \frac{d\mathbf{t}}{ds} = \mathbf{x}_{uu}\left(\frac{du}{ds}\right)^2 + 2\mathbf{x}_{uv}\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + \mathbf{x}_{vv}\left(\frac{dv}{ds}\right)^2 + \mathbf{x}_u\frac{d^2u}{ds^2} + \mathbf{x}_v\frac{dv^2}{ds^2}$$

Thus from equation *(11.3)*, page 233,

$$\kappa_g = [\mathbf{tkN}] = [\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}]\left(\frac{du}{ds}\right)^3 + (2[\mathbf{x}_u\mathbf{x}_{uv}\mathbf{N}] + [\mathbf{x}_v\mathbf{x}_{uu}\mathbf{N}])\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right)$$
$$+([\mathbf{x}_u\mathbf{x}_{vv}\mathbf{N}] + 2[\mathbf{x}_v\mathbf{x}_{uv}\mathbf{N}])\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2 + [\mathbf{x}_v\mathbf{x}_{vv}\mathbf{N}]\left(\frac{dv}{ds}\right)^3 + [\mathbf{x}_u\mathbf{x}_v\mathbf{N}]\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right)$$

Now from the Gauss equation $\mathbf{x}_{uu} = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L\mathbf{N}$ on page 202, we obtain

$$[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}] = \Gamma^1_{11}[\mathbf{x}_u\mathbf{x}_u\mathbf{N}] + \Gamma^2_{11}[\mathbf{x}_u\mathbf{x}_v\mathbf{N}] + L[\mathbf{x}_u\mathbf{NN}] = \Gamma^2_{11}[\mathbf{x}_u\mathbf{x}_v\mathbf{N}]$$

But $[\mathbf{x}_u\mathbf{x}_v\mathbf{N}] = (\mathbf{x}_u\times\mathbf{x}_v)\cdot(\mathbf{x}_u\times\mathbf{x}_v)/|\mathbf{x}_u\times\mathbf{x}_v| = |\mathbf{x}_u\times\mathbf{x}_v| = \sqrt{EG-F^2}$. Hence $[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}] = \Gamma^2_{11}\sqrt{EG-F^2}$. Similarly, $[\mathbf{x}_v\mathbf{x}_{uu}\mathbf{N}] = -\Gamma^1_{11}\sqrt{EG-F^2}$, $[\mathbf{x}_u\mathbf{x}_{uv}\mathbf{N}] = \Gamma^2_{12}\sqrt{EG-F^2}$, $[\mathbf{x}_v\mathbf{x}_{uv}\mathbf{N}] = -\Gamma^1_{12}\sqrt{EG-F^2}$, $[\mathbf{x}_u\mathbf{x}_{vv}\mathbf{N}] = \Gamma^2_{22}\sqrt{EG-F^2}$, $[\mathbf{x}_v\mathbf{x}_{vv}\mathbf{N}] = -\Gamma^1_{22}\sqrt{EG-F^2}$. Substituting in the equation for κ_g above gives the required result.

11.14. Prove Theorem 11.8: A natural representation of a curve $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ of class C^2 on a patch $\mathbf{x} = \mathbf{x}(u, v)$ of class C^2 is a geodesic if and only if $u(s)$ and $v(s)$ satisfy

$$\frac{d^2u}{ds^2} + \Gamma^1_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^1_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

$$\frac{d^2v}{ds^2} + \Gamma^2_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^2_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

Recall that $\kappa_g = \mathbf{k}\cdot\mathbf{U}$ where $\mathbf{t}, \mathbf{U}, \mathbf{N}$ is an orthonormal triad. Thus $\mathbf{x} = \mathbf{x}(s)$ is a geodesic iff $\mathbf{k}\cdot\mathbf{U} = 0$. Since $\mathbf{k}$ is always orthogonal to $\mathbf{t}$, it follows that $\mathbf{x} = \mathbf{x}(s)$ is a geodesic if and only if $\mathbf{k}\cdot\mathbf{x}_u = 0$ and $\mathbf{k}\cdot\mathbf{x}_v = 0$. From $\mathbf{t} = \mathbf{x}_u\cdot(du/ds) + \mathbf{x}_v\cdot(dv/ds)$ we obtain

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \mathbf{x}_{uu}\left(\frac{du}{ds}\right)^2 + 2\mathbf{x}_{uv}\frac{du}{ds}\frac{dv}{ds} + \mathbf{x}_{vv}\left(\frac{dv}{ds}\right)^2 + \mathbf{x}_u\frac{d^2u}{ds^2} + \mathbf{x}_v\frac{d^2v}{ds^2}$$

Hence $\mathbf{x} = \mathbf{x}(s)$ is a geodesic if and only if

$$\mathbf{k}\cdot\mathbf{x}_u = (\mathbf{x}_{uu}\cdot\mathbf{x}_u)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_{uv}\cdot\mathbf{x}_u)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_{vv}\cdot\mathbf{x}_u)\left(\frac{dv}{ds}\right)^2 + (\mathbf{x}_u\cdot\mathbf{x}_u)\frac{d^2u}{ds^2} + (\mathbf{x}_u\cdot\mathbf{x}_v)\frac{d^2v}{ds^2} = 0$$

$$\mathbf{k}\cdot\mathbf{x}_v = (\mathbf{x}_{uu}\cdot\mathbf{x}_v)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_{uv}\cdot\mathbf{x}_v)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_{vv}\cdot\mathbf{x}_v)\left(\frac{dv}{ds}\right)^2 + (\mathbf{x}_u\cdot\mathbf{x}_v)\frac{d^2u}{ds^2} + (\mathbf{x}_v\cdot\mathbf{x}_v)\frac{d^2v}{ds^2} = 0$$

Solving for d^2u/ds^2 and d^2v/ds^2 and using the vector identity $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$ gives the equivalent equations

$$(EG-F^2)\frac{d^2u}{ds^2} = (\mathbf{x}_v\times\mathbf{x}_{uu})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_v\times\mathbf{x}_{uv})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_v\times\mathbf{x}_{vv})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\left(\frac{dv}{ds}\right)^2$$

$$(EG-F^2)\frac{d^2v}{ds^2} = (\mathbf{x}_u\times\mathbf{x}_{uu})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_u\times\mathbf{x}_{uv})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_u\times\mathbf{x}_{vv})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\left(\frac{dv}{ds}\right)^2$$

Using $\mathbf{N} = \mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = \mathbf{x}_u\times\mathbf{x}_v/\sqrt{EG-F^2}$ and the expressions for $[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}]$, etc., in the preceding problem gives the required equations.

11.15. If $u(s)$ and $v(s)$ are solutions to the differential equations in the preceding problem such that at some point $s = s_0$, $E_0\left(\frac{du}{ds}\right)_0^2 + 2F_0\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G_0\left(\frac{dv}{ds}\right)_0^2 = 1$, prove that s is a natural parameter along the curve $\mathbf{x} = \mathbf{x}(u(s), v(s))$.

From the preceding problem, $u(s), v(s)$ is a solution to the differential equations if and only if the vector $\frac{d\mathbf{t}}{ds}$, where $\mathbf{t} = \frac{d\mathbf{x}}{ds} = \mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}$, satisfies $\frac{d\mathbf{t}}{ds}\cdot\mathbf{x}_u = 0$ and $\frac{d\mathbf{t}}{ds}\cdot\mathbf{x}_v = 0$, or, equivalently $\frac{d\mathbf{t}}{ds}\cdot\mathbf{U} = 0$ for *every* tangent vector $\mathbf{U}$. But then $\frac{d}{ds}|\mathbf{t}|^2 = \frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}) = 2\frac{d\mathbf{t}}{ds}\cdot\mathbf{t} = 0$ since $\mathbf{t}$ is a tangent vector. Hence integrating gives $|\mathbf{t}|^2 = C =$ constant. But at $s = s_0$,

$$|\mathbf{t}_0|^2 = \left|(\mathbf{x}_u)_0\left(\frac{du}{ds}\right)_0 + (\mathbf{x}_v)_0\left(\frac{dv}{ds}\right)_0\right|^2 = E_0\left(\frac{du}{ds}\right)_0^2 + 2F_0\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G_0\left(\frac{dv}{ds}\right)_0^2 = 1$$

Thus $|\mathbf{t}|^2 = |d\mathbf{x}/ds|^2 = 1$ for all s, which completes the proof.

11.16. If $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ is a natural representation of a geodesic on a patch $\mathbf{x} = \mathbf{x}(u, v)$ such that $E = E(u)$, $F = 0$ and $G = G(u)$, prove that $\sqrt{G}\cos\theta = C =$ constant, where θ is the angle between the geodesic and the v-parameter curves $u =$ constant, i.e. $\theta = \measuredangle(\mathbf{t}, \mathbf{x}_v)$.

It is easily computed from equation (*10.4*), page 202, that $\Gamma^2_{11} = 0$, $\Gamma^2_{12} = G_u/2G$, $\Gamma^2_{22} = 0$. Hence the second of the equations (*11.7*) is $\frac{d^2v}{ds^2} + \frac{G_u}{G}\frac{du}{ds}\frac{dv}{ds} = 0$. Since $\frac{d}{ds}\left(G\frac{dv}{ds}\right) = G\frac{d^2v}{ds^2} + G_u\frac{du}{ds}\frac{dv}{ds}$, this is equivalent to $\frac{d}{ds}\left(G\frac{dv}{ds}\right) = 0$. Hence $G\frac{dv}{ds} = C = \text{constant}$. Using $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, we find that

$$G\frac{dv}{ds} = (\mathbf{x}_v \cdot \mathbf{x}_v)\frac{dv}{ds} = \left(\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right)\cdot \mathbf{x}_v = \mathbf{t}\cdot\mathbf{x}_v = |\mathbf{t}|\,|\mathbf{x}_v| \cos\measuredangle(\mathbf{t},\mathbf{x}_v) = \sqrt{G}\cos\theta$$

Thus $\sqrt{G}\cos\theta = C = \text{constant}$.

11.17. Prove Theorem 11.10: If $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on a surface of class $\geqq 2$ such that $E = E(u)$, $F = 0$ and $G = G(u)$, then

(i) the u-parameter curves $v = \text{constant}$ are geodesics,

(ii) the v-parameter curve $u = u_0$ is a geodesic iff $G_u(u_0) = 0$,

(iii) a curve $\mathbf{x} = \mathbf{x}(u, v(u))$ is a geodesic iff $v = \pm\int \frac{C\sqrt{E}}{\sqrt{G}\sqrt{G-C^2}}\,du$, $C = \text{constant}$.

(i) From equation *(11.6)*, page 233, $(\kappa_g)_{v=\text{constant}} = -\frac{E_v}{2E\sqrt{G}} = 0$. Hence the u-parameter curves $v = \text{constant}$ are geodesics.

(ii) Again from equation *(11.6)*, $(\kappa_g)_{u=u_0} = \frac{G_u(u_0)}{2G(u_0)\sqrt{E(u_0)}}$. Hence $u = u_0$ is a geodesic if and only if $G_u(u_0) = 0$.

(iii) As in the preceding problem, $G\frac{dv}{ds} = C = \text{constant}$. Also, since s equals arc length, $1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right|^2 = E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2$. Substituting $\frac{dv}{ds} = \frac{C}{G}$ gives $1 = E\left(\frac{du}{ds}\right)^2 + \frac{C^2}{G}$ or $\frac{du}{ds} = \pm\frac{\sqrt{G-C^2}}{\sqrt{E}\sqrt{G}}$. Thus $\frac{dv}{du} = \frac{dv}{ds}\Big/\frac{du}{ds} = \pm\frac{C\sqrt{E}}{\sqrt{G}\sqrt{G-C^2}}$, from which the required result follows.

11.18. A patch $\mathbf{x} = \mathbf{x}(u,v)$ is called a *Liouville* patch if $E = G = U + V$ and $F = 0$, where U is a function only of u and V is a function only of v. If $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ is a natural representation of a geodesic on such a patch, prove that

$$U\sin^2\theta - V\cos^2\theta = C, \qquad C = \text{constant}$$

where θ is the angle between the geodesic and the u-parameter curves, $v = \text{constant}$; i.e. $\theta = \measuredangle(\mathbf{t}, \mathbf{x}_u)$.

Here $\frac{U'}{2(U+V)} = \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12}$ and $\frac{V'}{2(U+V)} = \Gamma^1_{12} = -\Gamma^2_{11} = \Gamma^2_{22}$. Substituting into equation *(11.4)*, page 233, and setting $\kappa_g = 0$,

$$-V'\left(\frac{du}{ds}\right)^3 + U'\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right) - V'\frac{du}{ds}\left(\frac{dv}{ds}\right)^2 + U'\left(\frac{dv}{ds}\right)^3 + 2(U+V)\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right) = 0$$

or

$$\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]\left(U'\frac{dv}{ds} - V'\frac{du}{ds}\right) + 2(U+V)\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right) = 0$$

This is equivalent to

$$\frac{d}{ds}\left[\frac{U(dv/ds)^2 - V(du/ds)^2}{(du/ds)^2 + (dv/ds)^2}\right] = 0$$

Hence

$$(a) \qquad \left[U\left(\frac{dv}{ds}\right)^2 - V\left(\frac{du}{ds}\right)^2\right]\Big/\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right] = C = \text{constant}$$

Now $\cos\theta = \frac{\mathbf{t}\cdot\mathbf{x}_u}{|\mathbf{t}|\,|\mathbf{x}_u|} = \frac{(\mathbf{x}_u\,du/ds + \mathbf{x}_v\,dv/ds)\cdot\mathbf{x}_u}{|\mathbf{x}_u|} = |\mathbf{x}_u|\frac{du}{ds}$. Similarly $\sin\theta = \frac{\mathbf{t}\cdot\mathbf{x}_v}{|\mathbf{x}_v|} = |\mathbf{x}_v|\frac{dv}{ds}$. Since s equals arc length,

$$1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right|^2 = (U+V)\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$$
$$= |\mathbf{x}_v|^2\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right] = |\mathbf{x}_u|^2\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$$

Thus $\cos^2\theta = \left(\frac{du}{ds}\right)^2 \Big/ \left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$ and $\sin^2\theta = \left(\frac{dv}{ds}\right)^2 \Big/ \left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$. Substituting in (a) above gives the required result.

11.19. *Liouville's Formula.* Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class ≥ 2 such that the u- and v-parameter curves are orthogonal and let $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ be a naturally represented curve C on the patch of class C^2. Let $\mathbf{g}_1 = \mathbf{x}_u/|\mathbf{x}_u|$ and $\mathbf{g}_2 = \mathbf{x}_v/|\mathbf{x}_v|$ be the unit vectors in the direction of the parameter curves and let $\theta = \theta(s)$ be the function along C defined by $\mathbf{t} = (\cos\theta)\mathbf{g}_1 + (\sin\theta)\mathbf{g}_2$, where $\mathbf{t}$ is the unit tangent along C. Prove that the geodesic curvature of C is given by

$$\kappa_g = d\theta/ds + \kappa_1\cos\theta + \kappa_2\sin\theta$$

where κ_1 is the geodesic curvature of the u-parameter curve and κ_2 is the geodesic curvature of the v-parameter curve.

Differentiating $\mathbf{g}_1$ along C and using equations (11.21), page 244, gives

$$\frac{d\mathbf{g}_1}{ds} = \frac{\partial\mathbf{g}_1}{\partial u}\frac{du}{ds} + \frac{\partial\mathbf{g}_1}{\partial v}\frac{dv}{ds} = \frac{d\mathbf{g}_1}{ds_1}\frac{ds_1}{du}\frac{du}{ds} + \frac{d\mathbf{g}_1}{ds_2}\frac{ds_2}{dv}\frac{dv}{ds}$$
$$= \frac{d\mathbf{g}_1}{ds_1}|\mathbf{x}_u|\frac{du}{ds} + \frac{d\mathbf{g}_1}{ds_2}|\mathbf{x}_v|\frac{dv}{ds} = \frac{d\mathbf{g}_1}{ds_1}\cos\theta + \frac{d\mathbf{g}_1}{ds_2}\sin\theta$$

where s_1 is arc length along the u-parameter curves and s_2 is arc length along the v-parameter curves. Similarly, $\frac{d\mathbf{g}_2}{ds} = \frac{d\mathbf{g}_2}{ds_1}\cos\theta + \frac{d\mathbf{g}_2}{ds_1}\sin\theta$. Differentiating $\mathbf{t}$ along C and using the above gives

$$\mathbf{k} = \dot{\mathbf{t}} = (\cos\theta)\frac{d\mathbf{g}_1}{ds} - (\sin\theta)\mathbf{g}_1\frac{d\theta}{ds} + (\sin\theta)\frac{d\mathbf{g}_2}{ds} + (\cos\theta)\mathbf{g}_2\frac{d\theta}{ds}$$
$$= \frac{d\mathbf{g}_1}{ds_1}\cos^2\theta + \left(\frac{d\mathbf{g}_1}{ds_2} + \frac{d\mathbf{g}_2}{ds_1}\right)\cos\theta\sin\theta + \frac{d\mathbf{g}_2}{ds_2}\sin^2\theta + \mathbf{U}\frac{d\theta}{ds}$$

where $\mathbf{U} = -\mathbf{g}_1\sin\theta + \mathbf{g}_2\cos\theta$. From equation (11.2), page 233, and the fact that $\mathbf{g}_1\cdot\frac{d\mathbf{g}_1}{ds_1} = \mathbf{g}_1\cdot\frac{d\mathbf{g}_1}{ds_2} = \mathbf{g}_2\cdot\frac{d\mathbf{g}_2}{ds_1} = \mathbf{g}_2\cdot\frac{d\mathbf{g}_2}{ds_2} = 0$, we have

$$\kappa_g = \mathbf{k}\cdot\mathbf{U} = \mathbf{k}\cdot(-\mathbf{g}_1\sin\theta + \mathbf{g}_2\cos\theta)$$
$$= \left(\mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_1}\right)\cos^3\theta + \left(\mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_2}\right)\cos^2\theta\sin\theta$$
$$- \left(\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_1}\right)\sin^2\theta\cos\theta - \left(\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_2}\right)\sin^3\theta + \frac{d\theta}{ds}$$

Finally we observe that the geodesic curvature along the u-parameter curves is given by $\kappa_1 = \mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_1}$ and the curvature along the v-parameter curve is $\kappa_2 = -\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_2}$ and since $\mathbf{g}_1\cdot\mathbf{g}_2 = 0$ also $\kappa_1 = -\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_1}$ and $\kappa_2 = \mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_2}$. Thus

$$\kappa_g = d\theta/ds + \kappa_1\cos^3\theta + \kappa_2\cos^2\theta\sin\theta + \kappa_1\sin^2\theta\cos\theta + \kappa_2\sin^3\theta$$

which gives the required result.

GEODESIC COORDINATES

11.20. Let $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, be a curve C of class C^2 on a surface of class ≥ 3. Let $\mathbf{x} = \mathbf{x}(s,t)$ be the family of naturally represented geodesics issuing from C and orthogonal to C, i.e. $\mathbf{x}(0,t) = \mathbf{x}(t)$ and $\mathbf{x}_s(0,t)\cdot\dot{\mathbf{x}}(t) = 0$. Prove that there exists an $\epsilon > 0$ such that $\mathbf{x} = \mathbf{x}(s,t)$ is a regular parametric representation of class C^2 for $-\epsilon < s < \epsilon$ and $a < t < b$.

Let P be an arbitrary point on C and $\mathbf{x} = \mathbf{x}(u, v)$ a patch containing P. Let $u = u_0(t)$, $v = v_0(t)$ be the curve in the parameter plane which maps onto C in a neighborhood of P and let $u = u(s, t)$, $v = v(s, t)$ be the family of curves which map onto the geodesics $\mathbf{x} = \mathbf{x}(s, t)$. From Theorem 11.8, page 234, for all t, $u(s, t)$ and $v(s, t)$ are the unique solutions to the differential equations *(11.7)*, page 234, satisfying the initial conditions

$$u(0, t) = u_0(t), \quad v(0, t) = v_0(t), \quad u_s(0, t) = \xi(t), \quad v_s(0, t) = \eta(t)$$

where the $\xi(t)$, $\eta(t)$ are of class C^1 and are uniquely determined by the equations

$$\text{(i)} \quad E\xi^2 + 2F\xi\eta + G\eta^2 = 1$$

$$\text{(ii)} \quad E\xi \frac{du_0}{dt} + F\left(\xi \frac{dv_0}{dt} + \eta \frac{du_0}{dt}\right) + G\eta \frac{dv_0}{dt} = 0$$

$$\text{(iii)} \quad \det \begin{pmatrix} \xi & du_0/dt \\ \eta & dv_0/dt \end{pmatrix} > 0$$

Equation (i) states that initially $|\mathbf{x}_s(0, t)| = 1$; equation (ii) states that $\cos \measuredangle(\mathbf{x}_s(0, v), d\mathbf{x}/dt) = 0$, i.e. the geodesics cut C orthogonally; and equation (iii) determines the orientation of the geodesics $\mathbf{x} = \mathbf{x}(s, t)$. Note that the determinant is different from zero since the geodesics cut C orthogonally. But then from the theory on the dependence on initial conditions of solutions to differential equations, it follows that the functions $u(s, t)$ and $v(s, t)$ have continuous second order derivatives in some neighborhood of C. Also the Jacobian $\partial(u, v)/\partial(s, t)$ is different from zero in a neighborhood of C since it is continuous and at $(0, t)$

$$\left.\frac{\partial(u, v)}{\partial(s, t)}\right|_{(0,t)} = \det \begin{pmatrix} u_s(0, t) & u_t(0, t) \\ v_s(0, t) & v_t(0, t) \end{pmatrix} = \det \begin{pmatrix} \xi & du_0/dt \\ \eta & dv_0/dt \end{pmatrix} \neq 0$$

Thus in a neighborhood of the point P the function $\mathbf{x} = \mathbf{x}(s, t) = \mathbf{x}(u(s, t), v(s, t))$ is a regular parametric representation of class C^2. Since C is compact, there exists an $\epsilon > 0$ such that $\mathbf{x} = \mathbf{x}(s, t)$ is a regular parametric representation of class C^2 for $-\epsilon < s < \epsilon$, $a < t < b$.

11.21. Prove that there exists an $\epsilon > 0$ such that a geodesic polar coordinate system $\mathbf{x} = \mathbf{x}(r, \theta)$ at a point P on a surface of class $\geqq 3$ is a regular parametric representation of class C^2 for $0 < r < \epsilon$, $-\infty < \theta < \infty$, mapping $0 < r < \epsilon$, $0 \leqq \theta < 2\pi$, one-to-one onto a deleted neighborhood of P.

Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch containing P such that $(0, 0)$ maps into P and such that at P, $\mathbf{x}_u = \mathbf{g}_1$ and $\mathbf{x}_v = \mathbf{g}_2$, where $\mathbf{g}_1$ and $\mathbf{g}_2$ are the orthogonormal tangent vectors with respect to which θ is measured. Note that at P, $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$. Now with respect to the patch $\mathbf{x} = \mathbf{x}(u, v)$ consider the differential equations *(11.7)*

$$(a) \qquad \begin{aligned} u'' + \Gamma^1_{11}(u')^2 + \Gamma^1_{12}u'v' + \Gamma^1_{22}(u')^2 &= 0 \\ v'' + \Gamma^2_{11}(u')^2 + \Gamma^2_{12}u'v' + \Gamma^1_{22}(v')^2 &= 0 \end{aligned}$$

with initial conditions

$$(b) \qquad u(0) = 0, \quad v(0) = 0, \quad u'(0) = \xi, \quad v'(0) = \eta$$

From the theory of differential equations, for all ξ, η there exists a unique solution $u(t; \xi, \eta)$, $v = (t; \xi, \eta)$ in a neighborhood of $t = 0$ which has continuous second order derivatives with respect to t, ξ, η. Since the equations are linear homogeneous in the second order derivatives and products of two first derivatives, it follows that for any solution $u(t; \xi, \eta)$, $v(t; \xi, \eta)$, the functions $u(s; \rho\xi, \rho\eta)$, $v(s, \rho\xi, \rho\eta)$ where $t = \rho s$, is also a solution to the differential equation for small ρs and satisfies the initial conditions $u|_{s=0} = u|_{t=0} = 0$, $v|_{s=0} = v|_{t=0} = 0$, $u_s|_{s=0} = u_t|_{t=0} = \rho\xi$ and $v_s|_{s=0} = v_t|_{t=0} = \rho\eta$. Hence $u(t; \xi, \eta) = u(s; \rho\xi, \rho\eta)$ and $v(t; \xi, \eta) = v(s; \rho\xi, \rho\eta)$. In particular, for $s = 1$ we have $u(t; \xi, \eta) = u(1; \rho\xi, \rho\eta)$ and $v(t; \xi, \eta) = v(1; \rho\xi, \rho\eta)$. We now set $x = \rho\xi$, $y = \rho\eta$ and consider the parameter transformation $u = u^*(x, y)$, $v = v^*(x, y)$, where $u^*(x, y) = u(1; x, y)$ and $v^*(x, y) = v(1; x, y)$. The above maps a neighborhood of the origin in the xy plane into a neighborhood of the origin in the uv plane. From the differential equations and initial conditions we see that $u^*(0, 0) = 0$ and $v^*(0, 0) = 0$. Also, at $t = 0$, $x = 0$, $y = 0$ and all ξ, η we have

$$\xi = u_t = u^*_x x_t + u^*_y y_t = u^*_x \xi + u^*_y \eta, \qquad \eta = v_t = v^*_x x_t + v^*_y y_t = v^*_x \xi + v^* \eta$$

Hence $u^*_x = 1$, $u^*_y = 0$, $v^*_x = 0$ and $v^*_y = 1$ and so the Jacobian

$$\left.\frac{\partial(u^*, v^*)}{\partial(x, y)}\right|_{(0,0)} = \det \begin{pmatrix} u^*_x & u^*_y \\ v^*_x & v^*_y \end{pmatrix}_{(0,0)} = 1$$

Since the Jacobian is continuous, it is different from zero in a neighborhood of $(0,0)$. Thus $u = u^*(x,y)$, $v = v^*(x,y)$ is an allowable coordinate transformation of class C^2 mapping a neighborhood of the origin of the xy plane 1-1 onto a neighborhood of the origin of the uv plane. We now consider the mapping $\mathbf{x} = \mathbf{x}^*(x,y) = \mathbf{x}(u^*(x,y), v^*(x,y))$. It is a coordinate patch of class C^2 on the surface in a neighborhood of P mapping $(0,0)$ into P, called a set of *Riemann normal coordinates* at P. It is easily verified that at P, $\mathbf{x}_x^* = \mathbf{x}_u$, $\mathbf{x}_y^* = \mathbf{x}_v$; and so at P, $E^* = 1, F^* = 0, G^* = 1$.

Finally we set $\xi = \cos\phi$ and $\xi = \sin\phi$, so that $x = \rho\cos\phi$ and $y = \rho\sin\phi$, and consider the function $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi) = \mathbf{x}^*(\rho\cos\phi, \rho\sin\phi)$. Clearly $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi)$ is a regular parametric representation of class C^2 for $0 < \rho < \epsilon$ and $-\infty < \phi < \infty$ and maps $0 < \rho < \epsilon$, $0 \leqq \phi < 2\pi$ one-to-one onto a deleted neighborhood of P since for these ρ, ϕ, $x = \rho\cos\phi$, $y = \rho\sin\phi$ is one-to-one in a deleted neighborhood of the origin of the xy plane. It remains to show that in fact $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi)$ is the given geodesic polar coordinate system, i.e. $\mathbf{x}^{**}(r,\theta) \equiv \mathbf{x}(r,\theta)$. For a fixed ϕ_0 we have

$$\mathbf{x} = \mathbf{x}^{**}(\rho,\phi_0) = \mathbf{x}(u^*(\rho\cos\phi_0, \rho\sin\phi_0), v^*(\rho\cos\phi_0, \rho\sin\phi_0))$$

where $$u^*(\rho\cos\phi_0, \rho\sin\phi_0) = u(1; \rho\cos\phi_0, \rho\sin\phi_0) = u(\rho; \cos\phi_0, \sin\phi_0)$$

and similarly $v^*(\rho\cos\phi_0, \rho\sin\phi_0) = v(\rho, \cos\phi_0, \sin\phi_0)$. But u and v are solutions of the differential equations (a) satisfying the initial condition (b) where initially,

$$E(u')^2 + 2Fu'v' + G(v')^2 = \xi^2 + \eta^2 = \sin\phi_0^2 + \cos\phi_0^2 = 1$$

It follows that $\mathbf{x} = \mathbf{x}^*(\rho,\phi_0)$ is the naturally represented geodesic through P in the direction of the tangent vector $\mathbf{x}_u\cos\phi_0 + \mathbf{x}_v\sin\phi_0 = \mathbf{g}_1\cos\phi_0 + \mathbf{g}_2\cos\phi_0$. Since these geodesics are unique it follows that $\mathbf{x}^{**}(r,\theta) \equiv \mathbf{x}(r,\theta)$, which completes the proof.

11.22. Prove that all partial derivatives of the first fundamental coefficients of a set of Riemann normal coordinates at P vanish at P.

Let $\mathbf{x} = \mathbf{x}(x,y)$ be a set of Riemann normal coordinates at P. Then for each θ_0 and $x = r\cos\theta_0$, $y = r\sin\theta_0$, the curve $\mathbf{x} = \mathbf{x}(r) = \mathbf{x}(x(r,\theta_0), y(r,\theta_0))$ is a naturally represented geodesic through P. Hence $x(r,\theta_0)$, $y(r,\theta_0)$ satisfy

$$\ddot{x} + \Gamma_{11}^1(\dot{x})^2 + 2\Gamma_{12}^1\dot{x}\dot{y} + \Gamma_{22}^1(\dot{y})^2 = 0$$

$$\ddot{y} + \Gamma_{11}^2(\dot{x})^2 + 2\Gamma_{12}^2\dot{x}\dot{y} + \Gamma_{22}^2(\dot{y})^2 = 0$$

Since $\dot{x} = \frac{d}{dr}(r\cos\theta_0) = \cos\theta_0$, $\ddot{x} = 0$, $\dot{y} = \sin\theta_0$ and $\ddot{y} = 0$, it follows that

$$\Gamma_{11}^1\cos^2\theta_0 + 2\Gamma_{12}^1\cos\theta_0\sin\theta_0 + \Gamma_{22}^1\sin^2\theta_0 = 0$$

$$\Gamma_{11}^2\cos^2\theta_0 + 2\Gamma_{12}^2\cos\theta_0\sin\theta_0 + \Gamma_{22}^2\sin^2\theta_0 = 0$$

But at P the above is true for all θ_0. Hence at P, $\Gamma_{ij}^k = 0$ for all $i,j,k = 1,2$. We recall further that $E = G = 1$, $F = 0$ at P. Hence from equations (*10.4*), page 202, at P,

$$\Gamma_{11}^1 = \tfrac{1}{2}E_x = 0 \qquad \Gamma_{12}^1 = \tfrac{1}{2}E_y = 0 \qquad \Gamma_{22}^1 = \tfrac{1}{2}(2F_y - G_x) = 0$$

$$\Gamma_{11}^2 = \tfrac{1}{2}(2F_x - E_y) = 0 \qquad \Gamma_{12}^2 = \tfrac{1}{2}G_x = 0 \qquad \Gamma_{22}^2 = \tfrac{1}{2}G_y = 0$$

from which the required result follows.

11.23. Prove Theorem 11.13: If $\mathbf{x} = \mathbf{x}(r,\theta)$ is a set of geodesic polar coordinates at a point P on a surface of sufficiently high class, then

$$\sqrt{G} = r - \tfrac{1}{6}K(P)r^3 + R(r,\theta)$$

where $\lim_{r\to 0}(R(r,\theta)/r^3) = 0$ and $K(P)$ is the Gaussian curvature at P.

Let $\mathbf{x} = \mathbf{x}^*(x,y)$ be Riemann normal coordinates at P. Then for $r > 0$, and any wedge $\theta_1 < \theta < \theta_2$, where, say, $\theta_2 - \theta_1 \leqq \frac{3}{2}\pi$, $x = r\cos\theta$, $y = r\sin\theta$ is an allowable coordinate transformation and from equations (*9.3*), page 172,

$$G = E^*(x_\theta)^2 + 2F^*x_\theta y_\theta + G^*y_\theta^2 = r^2(E^*\sin^2\theta - 2F^*\sin\theta\cos\theta + G^*\cos^2\theta)$$

From the preceding problem $E^* = G^* = 1$, $F^* = 0$, $E^*_x = E^*_y = F^*_x = F^*_y = G^*_x = G^*_y = 0$ at P. Using this, it is easily calculated from the above that

(a) $$\lim_{r\to 0} \sqrt{G} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial\sqrt{G}}{\partial r} = 1$$

Also from equation *(11.11)*, page 232, we have $\frac{\partial^2\sqrt{G}}{\partial r^2} = -K\sqrt{G}$. Differentiating gives $\frac{\partial^3\sqrt{G}}{\partial r^3} = -K\frac{\partial\sqrt{G}}{\partial r} - \frac{\partial K}{\partial r}\sqrt{G}$. Thus, using (a) above,

(b) $$\lim_{r\to 0} \frac{\partial^2\sqrt{G}}{\partial r^2} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial^3\sqrt{G}}{\partial r^3} = -K(P)$$

For each θ we can write

$$\sqrt{G} = (\sqrt{G})_0 + \left(\frac{\partial\sqrt{G}}{\partial r}\right)_0 r + \frac{1}{2}\left(\frac{\partial^2\sqrt{G}}{\partial r^2}\right)_0 r^2 + \frac{1}{6}\left(\frac{\partial^3\sqrt{G}}{\partial r^3}\right)_0 r^3 + R(r,\theta)$$

Hence from equations (a) and (b) above,

$$\sqrt{G} = r - K(P)r^3 + R(r,\theta)$$

where $\lim_{r\to 0} (R(r,\theta)/r^3) = 0$, which completes the proof.

SURFACES OF CONSTANT GAUSSIAN CURVATURE

11.24. Suppose R is a region on a patch $\mathbf{x} = \mathbf{x}(u,v)$ on a surface of sufficiently high class. The endpoints of the unit normal $\mathbf{N}$ in R form a set R' on the unit sphere called the spherical image of R. Show that the ratio of the area of R' to the area of R tends to K at a point P where R shrinks down to P.

From equation *(9.8)*, page 174, the element of surface area on the patch is

$$dR = \sqrt{EG - F^2}\, du\, dv = |\mathbf{x}_u \times \mathbf{x}_v|\, du\, dv$$

and the element of surface area on the spherical image is $dR' = |\mathbf{N}_u \times \mathbf{N}_v|\, du\, dv$. From Problem 9.18, page 194, $\mathbf{N}_u \times \mathbf{N}_v = K|\mathbf{x}_u \times \mathbf{x}_v|$. Thus $dR'/dR = K$, which proves the proposition.

11.25. A ruled surface (see Problem 8.4, page 183) is called a *developable* surface if the tangent plane is constant along each ruling. Prove that the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}(s)| = 1$, where $\mathbf{y} = \mathbf{y}(s)$ is a naturally represented base curve, is a developable if and only if $[\dot{\mathbf{y}}\mathbf{g}\dot{\mathbf{g}}] = 0$.

The tangent plane at a point on the ruling $\mathbf{x} = \mathbf{x}(s_0, v)$ is spanned by the vectors $\mathbf{x}_s(s_0,v) = \dot{\mathbf{y}}(s_0) + v\dot{\mathbf{g}}(s_0)$ and $\mathbf{x}_v(s_0,v) = \mathbf{g}(s_0)$. At $v = 0$ these are the vectors $\mathbf{x}_s(s_0,0) = \dot{\mathbf{y}}(s_0)$ and $\mathbf{x}_v(s_0,0) = \mathbf{g}(s_0) = \mathbf{x}_v(s_0,v)$. It follows that the tangent plane is the same along a ruling if and only if the three vectors $\dot{\mathbf{y}} + v\dot{\mathbf{g}}$, $\mathbf{g}$ and $\dot{\mathbf{y}}$ are dependent, i.e. if and only if

$$0 = [(\dot{\mathbf{y}} + v\dot{\mathbf{g}})\mathbf{g}\,\dot{\mathbf{y}}] = \dot{\mathbf{y}} \times (\dot{\mathbf{y}} + v\dot{\mathbf{g}}) \cdot \mathbf{g} = v\dot{\mathbf{y}} \times \dot{\mathbf{g}} \cdot \mathbf{g} = -v[\dot{\mathbf{y}}\mathbf{g}\dot{\mathbf{g}}]$$

which proves the proposition.

11.26. Let $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}(s)| = 1$, $a < s < b$, be a developable surface. Prove that the interval $a < s < b$ can be subdivided into subintervals $s_{i-1} < s < s_i$ such that on each of the subdivisions the surface is either a plane, a cylinder, a cone, or the tangent surface of a curve.

From the preceding problem the vectors $\dot{\mathbf{y}}$, $\mathbf{g}$ and $\dot{\mathbf{g}}$ are dependent. Thus there exist $k_1(s)$, $k_2(s)$ and $k_3(s)$, where $k_1^2 + k_2^2 + k_3^2 \neq 0$, such that $k_1\dot{\mathbf{y}} + k_2\mathbf{g} + k_3\dot{\mathbf{g}} = 0$. Now suppose $k_1 \equiv 0$ in an interval $s_{i-1} < s < s_i$. Then $k_2\mathbf{g} + k_3\dot{\mathbf{g}} = 0$ where $k_2^2 + k_3^2 \neq 0$. Since $|\mathbf{g}| = 1$, $\mathbf{g}\cdot\dot{\mathbf{g}} = 0$. Thus $0 = (k_2\mathbf{g} + k_3\dot{\mathbf{g}})\cdot\mathbf{g} = k_2|\mathbf{g}|^2$. Hence $k_2 \equiv 0$. But then $k_3 \neq 0$. Hence $\dot{\mathbf{g}} \equiv 0$ or $\mathbf{g} = \text{constant}$. It follows that in this case the surface is a portion of a plane or a cylinder. Now suppose $k_1 \neq 0$ in $s_{i-1} < s < s_i$. Then we can write $\dot{\mathbf{y}} = c_1\mathbf{g} + c_2\dot{\mathbf{g}}$ where $c_1 = -k_2/k_1$ and $c_2 = -k_3/k_1$. Now let $\mathbf{y}^* = \mathbf{y} - c_2\mathbf{g}$. Then $\dot{\mathbf{y}}^* = \dot{\mathbf{y}} - c_2\dot{\mathbf{g}} - \dot{c}_2\mathbf{g} = c_3\mathbf{g}$ where $c_3 = c_1 - \dot{c}_2$. If $c_3 \equiv 0$ in $s_{i-1} < s < s_i$, then $\mathbf{y}^* = \text{constant} = \mathbf{y}^*_0$. Hence the surface is of the form $\mathbf{x} = \mathbf{y}^*_0 + (v + c_2)\mathbf{g}$. But this is either the equation of a cone or part of a plane. Finally we have the case where $\mathbf{y}^* = c_3\mathbf{g}$ and $c_3 \neq 0$ in some subinterval of $s_{i-1} < s < s_i$. Then $\dot{\mathbf{y}} = \mathbf{y}^*/c_3$ and so the surface is of the form $\mathbf{x} = \mathbf{y} + v\mathbf{g} = \mathbf{y}^* + u\dot{\mathbf{y}}^*$, where $u = (v + c_2)/c_3$, which is the tangent surface of the curve $\mathbf{x} = \mathbf{y}^*(s)$.

11.27. Prove that a surface of sufficiently high class without planar points has constant zero Gaussian curvature if and only if a neighborhood of every point on the surface is a developable surface. *Note.* It then follows from Theorem 11.17, page 241, that a neighborhood of every point on a surface of sufficiently high class without planar points can be mapped isometrically onto a plane if and only if a neighborhood of every point on the surface is a developable surface.

It is easily verified directly that a plane, cylinder, cone, or tangent surface of a curve has constant zero Gaussian curvature. Thus it follows from the preceding problem that if a neighborhood of every point on a surface is a developable surface, then $K = 0$ on the surface.

Another interesting proof of this, is to consider the spherical image of a developable surface. Since the tangent plane is constant along the family of rulings of the surface, its spherical image is either a point (in the case of a plane) or a line. Now let P be a point on the surface, R a region containing the point and R' the spherical image of R. From Problem 11.24 the Gaussian curvature at P in absolute value is equal to the ratio of the area of R' to the area of R as R shrinks down to P. But the spherical image of a developable surface is at best a line. Thus for all R the area of R' is zero. Hence $K = 0$.

Now suppose that $K \equiv 0$ on the surface. Let P be a point on the surface and $\mathbf{x} = \mathbf{x}(u, v)$ a patch containing P. Since $K = \dfrac{LN - M^2}{EG - F^2} \equiv 0$, we have $LN - M^2 \equiv 0$. Since there are no planar points on the patch, it follows that each point is a parabolic point with a single asymptotic direction $du : dv$ satisfying $\text{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 = (\sqrt{L}\,du + \sqrt{N}\,dv)^2 = 0$ where we used the fact that $\text{II} = 0$ has a single real root and so can factor into a square. The above equation provides a one-parameter family of asymptotic lines in the neighborhood of P which we suppose to have been chosen as the u-parameter curves $v = \text{constant}$. Note that at a parabolic point the asymptotic direction coincides with a principal direction and so the u-parameter curves are also lines of curvature. Now along these curves we have $dv = 0$. Thus from the above differential equation, $\sqrt{L}\,du = 0$. Since $du \neq 0$, $L = 0$. But $LN - M^2 = 0$. Hence also $M = 0$. Since the u-parameter curves cover a neighborhood of P, $L \equiv M \equiv 0$ in the neighborhood. From the Weingarten equations *(10.5)*, page 202, it follows that $\mathbf{N}_u = \mathbf{0}$ and thus $\mathbf{N}$ is constant along each u-parameter curve. It remains to show that the u-parameter curves are straight lines. Since they are also lines of curvature, we have, from the Rodrigues formula, $\mathbf{N}_v = -\kappa\mathbf{x}_u$. Since $\mathbf{N}_u = \mathbf{0}$, $\kappa = 0$ and so the neighborhood of P is a developable surface with the u-parameter curves as rulings along which the tangent plane is constant.

11.28. Prove that at (u, θ) the surface

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + (\log u)\mathbf{e}_3, \qquad u > 0$$

has the same Gaussian curvature as the surface

$$\mathbf{x}^* = (u^*\cos\theta^*)\mathbf{e}_1 + (u^*\sin\theta^*)\mathbf{e}_2 + u^*\mathbf{e}_3, \qquad u^* > 0$$

at $u^* = u$ and $\theta^* = \theta$ but the surfaces are not isometric.

We leave as an exercise for the reader to verify that on $\mathbf{x}$ we have $E = (1 + 1/u^2)$, $F = 0$, $G = u^2$ and $K = -1/(1+u^2)^2$; and on $\mathbf{x}^*$ we have $E^* = 1$, $F^* = 0$, $G^* = 1 + (u^*)^2$ and $K^* = -1/(1+(u^*)^2)^2$. Thus the surfaces have the same Gaussian curvature at corresponding points. Now suppose the surfaces are isometric. Then there exists a parameter transformation $\theta^* = \theta^*(\theta, u)$, $u^* = u^*(\theta, u)$ such that at corresponding points $E = E^*$, $F = F^*$, $G = G^*$ and $K = K^*$. From $K = K^*$ we obtain $1 + (u^*)^2 = 1 + (u)^2$. Then $u^* = \pm u$ or $u^* = \sqrt{-2 - u^2}$. Using the transformation properties of the first fundamental coefficients (equations *(9.2)* and *(9.3)*, page 172) and assuming $u^* = \pm u$, it is easily computed that the parameter transformation must also satisfy

$$(a)\ 1 + (1+u^2)(\theta^*_u)^2 = 1 + 1/u^2, \quad (b)\ \theta^*_u\theta^*_\theta = 0 \quad \text{and} \quad (c)\ (1+u^2)\theta^2_\theta = u^2$$

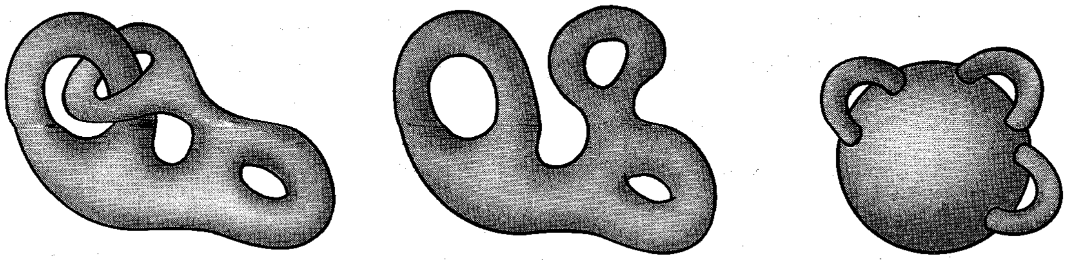
Since u^* is independent of θ, $u^*_\theta \equiv 0$. Since we must have $\partial(\theta^*, u^*)/\partial(\theta, u) \neq 0$, we must have $\theta^*_\theta \neq 0$. Hence from equation *(b)*, $\theta^*_u \equiv 0$. But then it is impossible to satisfy *(a)* and the theorem is proved.

GAUSS-BONNET THEOREM

11.29. Determine the total curvature of an ellipsoid.

The ellipsoid is homeomorphic to the sphere. Hence the total curvature of the ellipsoid is equal to the total curvature of sphere which is 4π.

11.30. Determine the total curvature of the surface shown in Fig. 11-28(*a*).



(*a*) (*b*) (*c*)

Fig. 11-28

As shown in Fig. 11-28(*b*) and (*c*), the surface is homeomorphic to a sphere with 3 handles. It follows from the formula $\chi = 2(1-p)$, where p is the number of handles of the surface, (see Example 11.11, page 246), that the total curvature of the surface is $2\pi\chi = -8\pi$.

11.31. Determine directly all terms of the Gauss-Bonnet formula (*11.22*), page 244, for the image of the polygon with edges C_1': $\theta = t$, $\phi = \pi/4$, $0 \leq t \leq \pi/2$; C_2': $\theta = \pi/2$, $\phi = t$, $\pi/4 \leq t \leq \pi/2$; C_3': $\theta = \pi/2 - t$, $\phi = \pi/2$, $0 \leq t \leq \pi/2$; C_4': $\theta = 0$, $\phi = \pi/2 - t$, $0 \leq t \leq \pi/4$ on the sphere of radius one

$$\mathbf{x} = (\cos\theta \sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3$$

See Fig. 11-29.

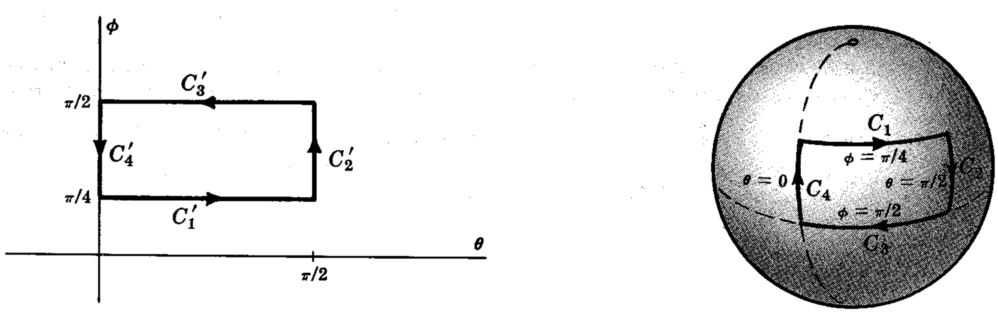


Fig. 11-29

Here $E = \sin^2\phi$, $F = 0$, $G = 1$ and the Gaussian curvature $K = 1$. Hence

(*a*) $$\iint_R K\,dS = \iint_{R'} \sqrt{EG-F^2}\,d\theta\,d\phi = \int_0^{\pi/2}\left[\int_{\pi/4}^{\pi/2} \sin\phi\,d\phi\right]d\theta = \pi\sqrt{2}/4$$

From equation (*11.6*), page 233, $(K_g)_{\phi=\text{constant}} = -\cot\phi$ and so

$$\int_{C_1} \kappa_g\,dS = -\int_0^{\pi/2} \cot(\pi/4)\sqrt{E\left(\frac{d\theta}{dt}\right)^2}\,dt = -\int_0^{\pi/2} \cos(\pi/4)\,dt = -\pi\sqrt{2}/4$$

Since C_2, C_3 and C_4 are geodesics, $\int_{C_2} \kappa_g\,ds = \int_{C_3} \kappa_g\,ds = \int_{C_4} \kappa_g\,ds = 0$. Thus

(*b*) $$\int_C \kappa_g\,ds = \int_{C_1} \kappa_g\,ds = -\pi\sqrt{2}/4$$

Finally, since the parameter curves are orthogonal,

(*c*) $$\sum_{i=1}^{4} \alpha_i = 4(\pi/2) = 2\pi$$

11.32. Prove that a surface has constant zero Gaussian curvature if in the neighborhood of each point there exist two families of geodesics which intersect at a constant angle.

Let P be an arbitrary point on the surface and C a quadrilateral made up of geodesics and containing P in its interior. Applying the Gauss-Bonnet formula gives $\iint_R K\,dS = 2\pi - \sum_{i=1}^{4} \alpha_i$. Since the geodesics intersect at a constant angle, $\sum_i \alpha_i = 2\pi$. Thus $\iint_R K\,dS = 0$. Since R can be chosen arbitrarily small, it follows that $K(P) = 0$, which proves the proposition.

11.33. Prove that on a surface with $K < 0$, a geodesic cannot have a multiple point as shown in Fig. 11-30(*a*), nor can two geodesics have more than one intersection as shown in Fig. 11-30(*b*), assuming the geodesics bound a simply connected domain.

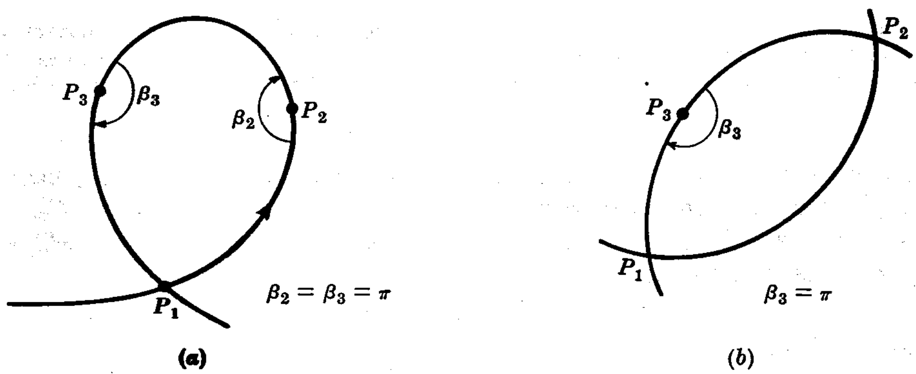


Fig. 11-30

Suppose geodesics having either of the two properties exist. By adding additional vertices to the geodesics, as shown in the figure, both cases become geodesic triangles, for which the Gauss-Bonnet formula is $\iint_R K\,dS = \sum_{i=1}^{3} \beta_i - \pi$ where β_i are the interior angles of the triangles. But in both cases $\sum_{i=1}^{3} \beta_i > \pi$, which is impossible since $K < 0$.

Supplementary Problems

11.34. If S is a compact surface and f is a regular differentiable mapping of S onto a surface S^*, prove that S^* is compact.

11.35. Prove Theorem 11.2: If f is a regular differentiable mapping of class C^r of S into S^* and $\mathbf{x} = \mathbf{x}(t)$ is a regular curve C of class C^r on S, then $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(t))$ is a regular curve of class C^r on S^*.

11.36. Show that the stereographic projection of a sphere onto a plane (see Example 11.1, page 227) is a conformal mapping.

11.37. Prove that a 1-1 regular differentiable mapping of a surface S onto a surface S^* is a 1-1 bicontinuous (topological) mapping of S onto S^*.

11.38. Prove that a mapping f of a surface S into a surface S^* is a local isometry if and only if for every patch $\mathbf{x} = \mathbf{x}(u,v)$ on S we have $E = E^*$, $F = F^*$ and $G = G^*$ where E, F, G are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and E^*, G^*, F^* are the first fundamental coefficients on $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$.

11.39. Show that the differential equations of the geodesics on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ are

$$(1+p^2+q^2)\ddot{u} + pr\dot{u}^2 + 2ps\dot{u}\dot{v} + pt\dot{v}^2 = 0$$
$$(1+p^2+q^2)\ddot{v} + qr\dot{u}^2 + 2qs\dot{u}\dot{v} + qt\dot{v}^2 = 0$$

where $p = f_u$, $q = f_v$, $v = f_{uu}$, $s = f_{uv}$, $t = f_{vv}$.

11.40. Find the geodesics on the plane by solving equations (*11.7*), page 234, in polar coordinates.

11.41. Show that the solutions to the equation

$$d\theta = \frac{Ca\,dr}{r\sqrt{r^2 - C^2}\sqrt{a^2 - (r-b)^2}}$$

where $r = b + a\sin\phi$, are geodesics on the torus

$$\mathbf{x} = (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

11.42. Show that the geodesics on a Liouville surface (see Problem 11.18) are given by

$$\int (U - C)^{-1/2}\,du \pm \int (V + C)^{-1/2}\,dv = \text{constant}, \qquad C = \text{constant}$$

11.43. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class $\geqq 3$ such that the parameter curves are orthogonal. Prove that

$$K = \frac{d(\kappa_g)_1}{ds_1} - \frac{d(\kappa_g)_2}{ds_2} - (\kappa_g)_1^2 - (\kappa_g)_2^2$$

where $(\kappa_g)_1$ and $(\kappa_g)_2$ are the geodesic curvatures along the u- and v-parameter curves respectively and s_1 and s_2 are natural parameters along the respective u- and v-parameter curves.

11.44. Let P and Q be two points on a geodesic $v = \text{constant}$ of a coordinate patch of geodesic coordinates $\mathbf{x} = \mathbf{x}(u,v)$. Prove that of all regular arcs on the patch joining P to Q, the geodesic containing P and Q is the one of shortest length.

11.45. Prove that the surface of revolution

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + f(v)\mathbf{e}_3$$

where $u = C_1\cos(v/a) + C_2\sin(v/a)$ and $f(v) = \int \sqrt{1 - (du/dv)^2}\,dv$ is a surface of constant positive Gaussian curvature $K = 1/a^2$ for all C_1, C_2. For what values of C_1 and C_2 is the surface a sphere? *Ans.* $C_1 = a$, $C_2 = 0$ or $C_1 = 0$, $C_2 = a$.

11.46. Prove that the surface of revolution

$$\mathbf{x} = u(\cos\theta)\mathbf{e}_1 + u(\sin\theta)\mathbf{e}_2 + f(v)\mathbf{e}_3$$

where $u = C_1e^{v/a} + C_2e^{-v/a}$ and $f(v) = \int \sqrt{1 - (du/dv)^2}\,dv$ is a surface constant negative Gaussian curvature $K = -1/a^2$.

11.47. Prove that $\mathbf{x} = 2(\tanh(r/2)\cos\theta)\mathbf{e}_1 + 2(\tan(r/2)\sin\theta)\mathbf{e}_2$ is a set of geodesic polar coordinates at the origin of the hyperbolic plane. See Example 11.9, page 242.

11.48. Determine the intrinsic distance from the origin to a point P in the hyperbolic plane. *Ans.* $D(0,P) = \tanh(|P|/2)$.

11.49. Let $\mathbf{x} = \mathbf{y}(s)$ be a naturally represented curve C without points of inflection. Prove that the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\,\mathbf{n}(s)$, where $\mathbf{n}$ is the unit principal normal to C, is a developable surface if and only if $\mathbf{x} = \mathbf{y}(s)$ is a plane curve.

11.50. Prove that a curve $\mathbf{x} = \mathbf{y}(s)$ on orientable surface S is a line of curvature on S if and only if the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\mathbf{N}(s)$, where $\mathbf{N}$ is the normal to S, is a developable surface.

11.51. Let $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}| = 1$, be a developable surface such that $\mathbf{g} \cdot \mathbf{y} = 0$ and let $\phi(s) = \measuredangle(\mathbf{g}, \mathbf{n})$ where $\mathbf{n}$ is the principal normal to $\mathbf{x} = \mathbf{y}(s)$. Prove that $\phi = -\tau$ where τ is the torsion along $\mathbf{x} = \mathbf{y}(s)$.

11.52. If S is a surface with constant Gaussian curvature $K \neq 0$, prove that the area of a geodesic polygon is determined by its interior angles.

11.53. Let C_n, $n = 1, 2, \ldots$, be an infinite sequence of geodesic triangles which shrink to a point P as $n \to \infty$. Prove that $K(P) = \lim_{n \to \infty} \dfrac{\sum_{i=1}^{3} \beta_{in} - \pi}{A_n}$ where A_n is the area of C_n, and β_{in} are its interior angles.

11.54. Let S be a sphere with p handles. Prove that there exists a point P on S such that (a) $K(P) > 0$ if $p = 0$, (b) $K(P) = 0$ if $p = 1$, (c) $K(P) < 0$ if $p > 1$.

11.55. Let S be a surface with Gaussian curvature $K < 0$. Let P_1, P_2, P_3, P_4 be the vertices of a geodesic quadrilateral with simply connected interior such that the lengths of two opposite sides P_1P_2 and P_3P_4 are equal and perpendicular to the third side P_2P_3. Prove that the interior angles at P_1 and P_4 are acute.

11.56. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a set of geodesic coordinates such that the u-parameters are natural representations of geodesics. Prove that if C is a naturally represented geodesic on the patch and $\theta(s)$ is the function defined by $\mathbf{t} = (\cos\theta)\mathbf{g}_1 + (\sin\theta)\mathbf{g}_2$ where $\mathbf{g}_1$ and $\mathbf{g}_2$ are unit vectors in the direction of the u- and v-parameter curves respectively and $\mathbf{t}$ is the unit tangent to C, then $\dfrac{d\theta}{ds} + \dfrac{\partial\sqrt{G}}{\partial u}\dfrac{dv}{ds} = 0$ along C.

11.57. Using the results of the preceding problem, derive the Gauss-Bonnet formula for a geodesic triangle by taking one of the vertices as the center of a set of geodesic polar coordinates.

Appendix I

Existence Theorem for Curves.

Let $\kappa(s)$ and $\tau(s)$ be continuous functions of a real variable s for $0 \leqq s \leqq a$. Then there exists a curve $\mathbf{x} = \mathbf{x}(s)$, $0 \leqq s \leqq a$, of class C^2 for which κ is the curvature, τ is the torsion and s is a natural parameter.

Proof: We consider the system of nine scalar differential equations

$$\dot{t}_i = \kappa n_i, \quad \dot{n}_i = -\kappa t_i + \tau b_i, \quad \dot{b}_i = -\tau n_i \qquad (i = 1, 2, 3)$$

with initial conditions $t_1(0) = 1$, $t_2(0) = 0$, $t_3(0) = 0$, $n_1(0) = 0$, $n_2(0) = 1$, $n_3(0) = 0$, $b_1(0) = 0$, $b_2(0) = 0$, $b_3(0) = 1$. This is a system of linear homogeneous differential equations with continuous coefficients and, from an existence and uniqueness theorem for an initial value problem of this type, there exist unique solutions $t_i(s)$, $n_i(s)$, $b_i(s)$, $i = 1, 2, 3$, of class C^1 for $0 \leqq s \leqq a$ satisfying the given initial conditions.

Now let $\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$, $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. We want to show that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is a right-handed orthonormal triplet for all s. To this end we consider a system of differential equations for the quantities $\mathbf{t}\cdot\mathbf{t}, \mathbf{t}\cdot\mathbf{n}, \mathbf{t}\cdot\mathbf{b}, \mathbf{n}\cdot\mathbf{n}, \mathbf{n}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$. Clearly $\dot{\mathbf{t}} = \kappa\mathbf{n}$, $\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}$, $\dot{\mathbf{b}} = -\tau\mathbf{n}$ and $\mathbf{t}(0) = \mathbf{e}_1$, $\mathbf{n}(0) = \mathbf{e}_2$, $\mathbf{b}(0) = \mathbf{e}_3$. Using this it is easily computed that the functions $\mathbf{t}\cdot\mathbf{t}, \mathbf{t}\cdot\mathbf{n}, \mathbf{t}\cdot\mathbf{b}, \mathbf{n}\cdot\mathbf{n}, \mathbf{n}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ satisfy the linear homogeneous differential equations

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}) = 2\kappa(\mathbf{t}\cdot\mathbf{n}) \qquad \frac{d}{ds}(\mathbf{n}\cdot\mathbf{n}) = -2\kappa(\mathbf{t}\cdot\mathbf{n}) + 2\tau(\mathbf{b}\cdot\mathbf{n})$$

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{n}) = \kappa(\mathbf{n}\cdot\mathbf{n}) - \kappa(\mathbf{t}\cdot\mathbf{t}) + \tau(\mathbf{t}\cdot\mathbf{b}) \qquad \frac{d}{ds}(\mathbf{n}\cdot\mathbf{b}) = -\kappa(\mathbf{t}\cdot\mathbf{b}) + \tau(\mathbf{b}\cdot\mathbf{b}) - \tau(\mathbf{n}\cdot\mathbf{n})$$

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{b}) = \kappa(\mathbf{n}\cdot\mathbf{b}) - \tau(\mathbf{t}\cdot\mathbf{n}) \qquad \frac{d}{ds}(\mathbf{b}\cdot\mathbf{b}) = -2\tau(\mathbf{n}\cdot\mathbf{b})$$

with initial conditions $(\mathbf{t}\cdot\mathbf{t})(0) = 1$, $(\mathbf{t}\cdot\mathbf{b})(0) = 0$, $(\mathbf{t}\cdot\mathbf{n})(0) = 0$, $(\mathbf{n}\cdot\mathbf{n})(0) = 1$, $(\mathbf{n}\cdot\mathbf{b})(0) = 0$, $(\mathbf{b}\cdot\mathbf{b})(0) = 1$. But the solution to an initial value problem of this type is *unique* and it is easily verified by substitution that a solution is given by the functions $(\mathbf{t}\cdot\mathbf{t})^* \equiv 1$, $(\mathbf{t}\cdot\mathbf{n})^* \equiv 0$, $(\mathbf{t}\cdot\mathbf{b})^* \equiv 0$, $(\mathbf{n}\cdot\mathbf{n})^* \equiv 1$, $(\mathbf{n}\cdot\mathbf{b})^* \equiv 0$, $(\mathbf{b}\cdot\mathbf{b})^* \equiv 1$. Hence indeed $\mathbf{t}\cdot\mathbf{t} = 1$, $\mathbf{t}\cdot\mathbf{n} = 0$, $\mathbf{t}\cdot\mathbf{b} = 0$, $\mathbf{n}\cdot\mathbf{n} = 1$, $\mathbf{n}\cdot\mathbf{b} = 0$, $\mathbf{b}\cdot\mathbf{b} = 1$ for all s. Thus $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is an orthonormal triad for all s. It is also a right-handed triplet since $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are continuous and $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is a right-handed system initially.

We now define the curve $\mathbf{x} = \mathbf{x}(s) = \int_0^s \mathbf{t}(\sigma)\,d\sigma$. Clearly $\mathbf{x}(s)$ is of class C^2. Also $|\dot{\mathbf{x}}| = |\mathbf{t}| = 1$. Hence s is a natural parameter. Since $\dot{\mathbf{t}} = \kappa\mathbf{n}$ where $|\mathbf{n}| = 1$, it follows that κ is its curvature. Finally, since $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ and

$$\dot{\mathbf{b}} = \dot{\mathbf{t}}\times\mathbf{n} + \mathbf{t}\times\dot{\mathbf{n}} = \kappa(\mathbf{n}\times\mathbf{n}) + (-\kappa(\mathbf{t}\times\mathbf{t})) + \tau(\mathbf{t}\times\mathbf{n}) = \tau\mathbf{b}$$

where $|\mathbf{b}| = 1$, it follows that τ is the torsion of the curve, which proves the theorem.

Appendix II

Existence Theorem for Surfaces.

Let E, F and G be functions of u and v of class C^2 and L, M and N functions of u and v of class C^1 defined on an open set containing (u_0, v_0) such that

(i) $EG - F^2 > 0,\ E > 0,\ G > 0$

(ii) E, F, G, L, M, N satisfy the compatibility equations *(10.7)* and *(10.8)*.

Then there exists a regular parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ of class C^3 defined in a neighborhood of (u_0, v_0) for which E, F, G, L, M and N are the first and second fundamental coefficients.

Proof. We consider the scalar system of partial differential equations

$$\begin{aligned}
(U_i)_u &= \Gamma^1_{11}U_i + \Gamma^2_{11}V_i + LN_i & (V_i)_v &= \Gamma^1_{22}U_i + \Gamma^2_{22}V_i + NN_i \\
(U_i)_v &= \Gamma^1_{12}U_i + \Gamma^2_{12}V_i + MN_i & (N_i)_u &= \beta^1_1 U_i + \beta^2_1 V_i \qquad (i = 1, 2, 3)\\
(V_i)_u &= \Gamma^1_{12}U_i + \Gamma^2_{12}V_i + MN_i & (N_i)_v &= \beta^1_2 U_i + \beta^2_2 V_i
\end{aligned}$$

with initial condition

$$\begin{aligned}
U_1(u_0, v_0) &= \sqrt{E_0} & V_1(u_0, v_0) &= F_0/\sqrt{E_0} & N_1(u_0, v_0) &= 0\\
U_2(u_0, v_0) &= 0 & V_2(u_0, v_0) &= \sqrt{E_0G_0 - F_0^2}/\sqrt{E_0} & N_2(u_0, v_0) &= 0\\
U_3(u_0, v_0) &= 0 & V_3(u_0, v_0) &= 0 & N_3(u_0, v_0) &= 1
\end{aligned}$$

$(E_0 = E(u_0, v_0),\ F_0 = F(u_0, v_0),\ G_0 = G(u_0, v_0))$ where the Christoffel symbols Γ^k_{ij} are given in equation *(10.4)*, page 202, and the β^j_i in equation*(10.2)*. This is a system of eighteen first order linear homogeneous partial differential equations with coefficients of class C^1 for the nine functions $U_i(u, v), V_i(u, v), N_i(u, v),\ i = 1, 2, 3.$ Since in addition the compatibility conditions for these equations are satisfied, it follows from an existence and uniqueness theorem for a system of this type that there exists a unique solution $U_i(u, v)$, $V_i(u, v)$ and $N_i(u, v)$, $i = 1, 2, 3,$ of class C^2 in a neighborhood of (u_0, v_0) satisfying the given initial conditions.

Now let $\mathbf{U} = U_1\mathbf{e}_1 + U_2\mathbf{e}_2 + U_3\mathbf{e}_3$, $\mathbf{V} = V_1\mathbf{e}_1 + V_2\mathbf{e}_2 + V_3\mathbf{e}_3$ and $\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3$. It is easily computed from the initial conditions that at (u_0, v_0) we have $\mathbf{U}\cdot\mathbf{U} = E$, $\mathbf{U}\cdot\mathbf{V} = F$, $\mathbf{V}\cdot\mathbf{V} = G$, $\mathbf{U}\cdot\mathbf{N} = 0$, $\mathbf{V}\cdot\mathbf{N} = 0$, $\mathbf{N}\cdot\mathbf{N} = 1$. We want to show that this is the case for all (u, v) in a neighborhood of (u_0, v_0). As in the case of the existence theorem for curves, we consider a system of differential equations for the quantities $\mathbf{U}\cdot\mathbf{U}, \mathbf{U}\cdot\mathbf{V}, \mathbf{V}\cdot\mathbf{V}, \mathbf{U}\cdot\mathbf{N}, \mathbf{V}\cdot\mathbf{N}$ and $\mathbf{N}\cdot\mathbf{N}$. It is easily computed from the first system of differential equations above that these functions satisfy the partial differential equations

$$\begin{aligned}
(\mathbf{U}\cdot\mathbf{U})_u &= 2\Gamma^1_{11}(\mathbf{U}\cdot\mathbf{U}) + 2\Gamma^2_{11}(\mathbf{U}\cdot\mathbf{V}) + 2L(\mathbf{U}\cdot\mathbf{N})\\
(\mathbf{U}\cdot\mathbf{U})_v &= 2\Gamma^1_{12}(\mathbf{U}\cdot\mathbf{U}) + 2\Gamma^2_{12}(\mathbf{U}\cdot\mathbf{V}) + 2M(\mathbf{U}\cdot\mathbf{N})\\
(\mathbf{U}\cdot\mathbf{V})_u &= \Gamma^1_{12}(\mathbf{U}\cdot\mathbf{U}) + (\Gamma^1_{11} + \Gamma^2_{12})(\mathbf{U}\cdot\mathbf{V}) + \Gamma^2_{11}(\mathbf{V}\cdot\mathbf{V}) + M(\mathbf{U}\cdot\mathbf{N}) + L(\mathbf{V}\cdot\mathbf{N})\\
(\mathbf{U}\cdot\mathbf{V})_v &= \Gamma^1_{22}(\mathbf{U}\cdot\mathbf{U}) + (\Gamma^1_{12} + \Gamma^2_{22})(\mathbf{U}\cdot\mathbf{V}) + \Gamma^2_{12}(\mathbf{V}\cdot\mathbf{V}) + N(\mathbf{U}\cdot\mathbf{N}) + M(\mathbf{V}\cdot\mathbf{N})
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{V}\cdot\mathbf{V})_u &= 2\Gamma^1_{12}(\mathbf{U}\cdot\mathbf{V}) + 2\Gamma^2_{12}(\mathbf{V}\cdot\mathbf{V}) + 2M(\mathbf{V}\cdot\mathbf{N})\\
(\mathbf{V}\cdot\mathbf{V})_v &= 2\Gamma^1_{22}(\mathbf{U}\cdot\mathbf{V}) + 2\Gamma^2_{22}(\mathbf{V}\cdot\mathbf{V}) + 2N(\mathbf{V}\cdot\mathbf{N})\\
(\mathbf{V}\cdot\mathbf{N})_u &= \beta^1_1(\mathbf{U}\cdot\mathbf{V}) + \beta^2_1(\mathbf{V}\cdot\mathbf{V}) + \Gamma^1_{12}(\mathbf{U}\cdot\mathbf{N}) + \Gamma^2_{12}(\mathbf{V}\cdot\mathbf{N}) + M(\mathbf{N}\cdot\mathbf{N})\\
(\mathbf{V}\cdot\mathbf{N})_v &= \beta^1_2(\mathbf{U}\cdot\mathbf{V}) + \beta^2_2(\mathbf{V}\cdot\mathbf{V}) + \Gamma^1_{22}(\mathbf{U}\cdot\mathbf{N}) + \Gamma^2_{22}(\mathbf{V}\cdot\mathbf{N}) + N(\mathbf{N}\cdot\mathbf{N})\\
(\mathbf{U}\cdot\mathbf{N})_u &= \beta^1_1(\mathbf{U}\cdot\mathbf{U}) + \beta^2_1(\mathbf{U}\cdot\mathbf{V}) + \Gamma^1_{11}(\mathbf{U}\cdot\mathbf{N}) + \Gamma^2_{11}(\mathbf{V}\cdot\mathbf{N}) + L(\mathbf{N}\cdot\mathbf{N})\\
(\mathbf{U}\cdot\mathbf{N})_v &= \beta^1_2(\mathbf{U}\cdot\mathbf{U}) + \beta^2_2(\mathbf{U}\cdot\mathbf{V}) + \Gamma^1_{12}(\mathbf{U}\cdot\mathbf{N}) + \Gamma^2_{12}(\mathbf{V}\cdot\mathbf{N}) + M(\mathbf{N}\cdot\mathbf{N})\\
(\mathbf{N}\cdot\mathbf{N})_u &= 2\beta^1_1(\mathbf{U}\cdot\mathbf{N}) + 2\beta^2_1(\mathbf{V}\cdot\mathbf{N})\\
(\mathbf{N}\cdot\mathbf{N})_v &= 2\beta^1_2(\mathbf{U}\cdot\mathbf{N}) + 2\beta^2_2(\mathbf{V}\cdot\mathbf{N})
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
(\mathbf{U}\cdot\mathbf{U})(u_0,v_0) &= E_0 \qquad & (\mathbf{U}\cdot\mathbf{V})(u_0,v_0) &= F_0 \qquad & (\mathbf{V}\cdot\mathbf{V})(u_0,v_0) &= G_0\\
(\mathbf{U}\cdot\mathbf{N})(u_0,v_0) &= 0 & (\mathbf{V}\cdot\mathbf{N})(u_0,v_0) &= 0 & (\mathbf{N}\cdot\mathbf{N})(u_0,v_0) &= 1
\end{aligned}
$$

This is again a linear homogeneous first order system of partial differential equations. Since we know that these equations possess solutions of class C^2, it follows that the compatibility conditions for this system are also satisfied. Thus from the existence and uniqueness theorem for an initial valued problem of this type, we know that a solution satisfying the initial conditions is *unique*. But it is easily verified by substitution that a solution of the equations satisfying the initial conditions is given by the functions $(\mathbf{U}\cdot\mathbf{U})^* = E$, $(\mathbf{U}\cdot\mathbf{V})^* = F$, $(\mathbf{V}\cdot\mathbf{V})^* = G$, $(\mathbf{U}\cdot\mathbf{N})^* \equiv 0$, $(\mathbf{V}\cdot\mathbf{N})^* \equiv 0$, $(\mathbf{N}\cdot\mathbf{N})^* \equiv 1$. Hence $\mathbf{U}\cdot\mathbf{U} = E$, $\mathbf{U}\cdot\mathbf{V} = F$, $\mathbf{V}\cdot\mathbf{V} = G$, $\mathbf{U}\cdot\mathbf{N} = 0$, $\mathbf{V}\cdot\mathbf{N} = 0$ and $\mathbf{N}\cdot\mathbf{N} = 1$ for all (u,v) in a neighborhood of (u_0,v_0).

Now we define the candidate for the surface in terms of a solution to the partial differential equations $\mathbf{x}_u = \mathbf{U}$ and $\mathbf{x}_v = \mathbf{V}$. Since $\mathbf{U}_v = \mathbf{V}_u$, a solution $\mathbf{x} = \mathbf{x}(u,v)$ of class C^3 exists and in fact can be given by $\mathbf{x} = \mathbf{x}(u,v) = \int_{u_0}^{u} U(\xi,v)\,d\xi + \int_{v_0}^{v} V(u_0,\eta)\,d\eta$. It remains to show that $\mathbf{x} = \mathbf{x}(u,v)$ is regular, that is, $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$, and that its first and second fundamental coefficients are E, F, G, L, M and N. Since $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{U}\times\mathbf{V}$ is continuous and $(\mathbf{x}_u \times \mathbf{x}_v)(u_0,v_0) = (\mathbf{U}\times\mathbf{V})(u_0,v_0) = \sqrt{E_0G_0 - F_0^2}\,\mathbf{e}_3 \neq \mathbf{0}$ initially, it follows that $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ in some neighborhood of (u_0,v_0). It is also clear that $\mathbf{x}_u\cdot\mathbf{x}_u = \mathbf{U}\cdot\mathbf{U} = E$, $\mathbf{x}_u\cdot\mathbf{x}_v = \mathbf{U}\cdot\mathbf{V} = F$ and $\mathbf{x}_v\cdot\mathbf{x}_v = G$. Finally from the first system of partial differential equations and the fact that $\mathbf{U}\cdot\mathbf{N} = 0$, $\mathbf{V}\cdot\mathbf{N} = 0$ and $\mathbf{N}\cdot\mathbf{N} = 1$, it follows directly that $\mathbf{x}_{uu}\cdot\mathbf{N} = \mathbf{U}_u\cdot\mathbf{N} = L$, $\mathbf{x}_{uv}\cdot\mathbf{N} = \mathbf{V}_v\cdot\mathbf{N} = M$ and $\mathbf{x}_{vv}\cdot\mathbf{N} = \mathbf{V}_v\cdot\mathbf{N} = N$. Hence the second fundamental coefficients of $\mathbf{x} = \mathbf{x}(u,v)$ are L, M and N, which completes the proof of the theorem.

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